Characteristic Currents for Metrized Vector Bundles on Berkovich Spaces

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Introduction

I shall do my best to modernize my language and notations, but I am well aware of my shortcomings in that respect; I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial. But please allow me to assume that the characteristic is not 2.

(André Weil)

In algebraic geometry, specifically in intersection theory, it is often necessary to work with varieties which are proper. For example one compactifies the affine line $\mathbf{A}^1_k$ over a field by adding a point at infinity to obtain the proper curve $\mathbf{P}^1_k = \mathbf{A}^1_k \cup \{\infty\}$. The scheme $\operatorname{Spec}(\mathbf{Z})$ behaves very much like the affine line over a field, so the fundamental idea of Arakelov geometry is that in order to develop an intersection theory for varieties over $\operatorname{Spec}(\mathbf{Z})$ one should compactify $\operatorname{Spec}(\mathbf{Z})$ to a space $\widehat{\operatorname{Spec}(\mathbf{Z})} = \operatorname{Spec}(\mathbf{Z}) \cup \{\infty\}$ and develop a geometry over this base space. To make this idea precise one has to mix algebraic geometry over $\operatorname{Spec}(\mathbf{Z})$ with differential geometry over the completion of the algebraic closure of the fraction field of $\mathbf{Z}$, namely the metrically complete field $\mathbf{C}$ of complex numbers. This was first carried out by Arakelov for the case of arithmetic surfaces [Ara74] and later applied with great success by Faltings, leading for example to his celebrated proof of the Mordell conjecture [Fal83]. The theory has been further advanced and extended to arbitrary dimensions by Gillet and Soulé as well as many others.

Another important technique in arithmetic geometry is to work locally at the primes of $\mathbf{Z}$, i.e. to work over the rings $\mathbf{Z}_p$ of p-adic integers. Even though much simpler, the schemes $\mathrm{Spec}(\mathbf{Z}_p)$ are still 1-dimensional affine schemes so they still do not behave like proper schemes over a field and it is desirable to have a non-archimedean Arakelov geometry for rings like $\mathbf{Z}_p$ (complete valuation rings of rank 1). One such theory has been proposed by Bloch, Gillet and Soulé which makes however strong assumptions about resolution of singularities in mixed characteristics and is furthermore purely algebraic so that it does not allow the use of the tools of analysis [BGS95]. It would be desirable to have a differential geometry over $\mathbf{C}_p$, the completion of the algebraic closure of the fraction field of $\mathbf{Z}_p$ (in contrast to the archimedean situation, the algebraic closure of $\mathbf{Q}_p$ is not metrically complete).

In the following we fix a complete valuation ring K° of rank 1 with fraction field K, which is then a so-called non-archimedean field. A more analytic approach to non-archimedean Arakelov geometry is made possible by two new developments: First the introduction of Berkovich K-analytic spaces by Berkovich [Ber90; Ber93]. These serve as a non-archimedean analogue of the complex analytic spaces to which techniques from differential geometry are applied in the archimedean setting. In particular, every proper algebraic K-variety X gives rise to a non-archimedean Berkovich analytification X^{an} which is a compact, Hausdorff, path-connected topological space together with a structure sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ of analytic functions.

The second recent development is the introduction of smooth real-valued differential forms and currents on Berkovich analytic spaces due to Chambert-Loir–Ducros and Gubler. In [CD12] Chambert-Loir and Ducros developed a differential calculus on Berkovich analytic spaces by associating to every Berkovich analytic space V a sheaf of bigraded differential $\mathbf{R}$-algebras $\{\mathscr{A}_V^{p,q}\}_{0\leq p,q\leq \dim(V)}$ with differentials d', d'' resembling the sheaves of (p,q)-forms with holomorphic and anti-holomorphic derivatives $\partial, \overline{\partial}$ in complex differential geometry. Their theory is based on super-forms and super-currents introduced by Lagerberg [Lag12] as well as ideas and techniques from the young field of tropical geometry. For analytifications of algebraic varieties, Gubler gave a more concrete description of the sheaves $\mathscr{A}_{X^{\mathrm{an}}}^{\bullet,\bullet}$ [Gub16]. One of the main results of [CD12] is a non-archimedean analogue of the Poincaré–Lelong formula which is a basic ingredient for the study of first Chern currents $[c_1(\overline{L})]$ for continuously metrized line bundles $\overline{L}$ on Berkovich spaces as well as first Chern forms $c_1(\overline{L})$ in the case of a smooth metric.

The purpose of this thesis is to contribute to the study of differential geometry of Berkovich spaces with a view towards non-archimedean Arakelov geometry and more explicitly to extend the theory of Chern forms and currents to metrized vector bundles of higher rank. After some preliminary chapters, we pursue in Chapter 4 a detailed study of the notion of a vector bundle E on a Berkovich K-analytic space V. As is customary in algebraic flavors of geometry we define a vector bundle to be a locally free sheaf of modules. However we show that just as in differential geometry we can speak about the *fiber vector space* $E(x)$ over a point $x \in V$ and about the total space $\mathrm{Tot}(E) \to V$ which is a fiber bundle in the analytic category with typical fiber $\mathbf{A}^{r,\mathrm{an}}$ where r is the rank of E. Both of these notions appear in our definition of a *continuous metric* on E. In addition to the total space we also construct the *projective bundle* $P(E) \to V$ parametrizing lines in E as well as the tautological line bundle $\mathcal{O}_E(-1)$ on $P(E)$ and its dual $\mathcal{O}_E(1)$.

In Chapter 5 we study *continuous metrics* on vector bundles on a Berkovich K-analytic space V. A *metric* on a vector bundle E is a family $\{\|-\|_x\}_{x\in V}$ of vector space norms $\|-\|_x$ on the fiber vector spaces $E(x)$ as x varies in V. Such a metric extends canonically to a map $\mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$ and we call the metric *continuous* if this induced map is continuous with respect to the Berkovich topology of $\mathrm{Tot}(E)$. We show that the metric also defines an induced *Fubini-Study metric* $\|-\|_{\mathrm{FS}}$ on the line bundle $\mathcal{O}_E(1)$. In Proposition 5.13 we show that the metric $\|-\|$ is continuous if and only if the metric $\|-\|_{\mathrm{FS}}$ is continuous. In general our philosophy is to reduce as many constructions and properties of (metrized) vector bundles as possible to constructions and properties of the associated line bundle $\mathcal{O}_E(1)$. Note that while the study of metrized line bundles on Berkovich spaces goes back to [Gub98], metrics on vector bundles have been defined in two different ways in [CD12] and [CM20]. We explain in Paragraph 5.31 how our definition relates to the other definitions in the literature. For line bundles all notions agree with the classical one. In view of the philosophy alluded to above, it makes sense to define a *pseudo-metric* on E to be a metric on $\mathcal{O}_E(1)$ (Paragraph 5.32) and to work with pseudo-metrics rather than metrics as appropriate.

As for line bundles, a formal K°-model $(\mathfrak{X}, \mathfrak{E})$ for a vector bundle E induces a *formal metric* $\|-\|_{\mathfrak{E}}$ on E which is continuous. We investigate the construction $\mathfrak{E} \mapsto \|-\|_{\mathfrak{E}}$ in Chapter 6 and show that it is compatible with all natural constructions of vector bundles.

In Chapter 7 we review the sheaves $\mathcal{A}^{\bullet,\bullet}$ and $\mathcal{D}^{\bullet,\bullet}$ of smooth forms and currents in the sense of [CD12]. Based on these we introduce the Bott–Chern cohomology groups $\hat{H}^p_{\mathcal{D}}(V)$ of a K-analytic space V as the group of d'- and d''-closed currents of degree (p,p) modulo the subgroup of currents of the form $d'd''T$ for a current $T \in \mathcal{D}^{p-1,p-1}(V)$. The Poincaré-Lelong equation

$$[c_1(\overline{L})] = \delta_{[\mathrm{div}(s)]} + d'd''[-\log\|s(-)\|]$$

for a continuously metrized line bundle L and a regular meromorphic section

$s \in \Gamma(V, L)$ shows that the class of $[c_1(\overline{L})]$ in the Bott–Chern cohomology group does not depend on the metric.

If $\overline{L}$ is a smoothly metrized line bundle, then Chambert-Loir and Ducros construct $c_1(\overline{L})$ as a smooth $(1,1)$-form while if the metric is merely continuous, one only gets a current $[c_1(\overline{L})]$. Even though currents can usually not be multiplied, Chambert-Loir and Ducros developed a non-archimedean analogue of Bedford–Taylor theory which allowed them to construct for continuously metrized line bundles $\overline{L}_1, \dots, \overline{L}_r$ which are locally approachable a current $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$. Here $\overline{L}_k$ is called *locally approachable* if every point of V has an open neighborhood U over which L_k admits a nowhere-vanishing section $s \in \Gamma(U, L)$ such that the function $-\log\|s(-)\|$ is a difference of two functions which are uniform limits of smooth plurisubharmonic functions on U. We show in Proposition 8.11 that the class of the current $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ in Bott–Chern cohomology is independent of the metrics.

In Chapter 9 we make the following construction: Let $\overline{E}$ be a locally approachably pseudo-metrized vector bundle, i.e. suppose that the metrized line bundle $\mathcal{O}_{\overline{E}}(1) := (\mathcal{O}_E(1), \|-\|_{\mathrm{FS}})$ is locally approachable in the sense of [CD12] and let $i \in \mathbf{N}$. On the projective bundle $p : P(E) \to V$ we can use non-archimedean Bedford–Taylor theory to form the current $[c_1(\mathcal{O}_{\overline{E}}(1))^{e+i}]$ where $e+1$ is the rank of E. Then the push-forward $[s_i(\overline{E})] := p_*[c_1(\mathcal{O}_{\overline{E}}(1))^{e+i}] \in \mathcal{D}_V^{i,i}(V)$ is our definition of the *i-th Segre current* of $\overline{E}$. More generally, if $\overline{E}_1, \dots, \overline{E}_r$ are locally approachably pseudo-metrized vector bundles, we can define the product

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$$

by first pulling all the line bundles $\mathcal{O}_{\overline{E}_k}(1)$ back to the fibered product $P := P(E_1) \times_V \cdots \times_V P(E_r)$ and pushing down to V only after forming the product of all the involved first Chern currents of line bundles on P. This allows us to define polynomial expressions in the Segre currents and in particular to define Chern currents $[c_i(\overline{E})]$ of locally approachably pseudo-metrized vector bundles. Forgetting about the metrics, we get well-defined classes in the Bott–Chern cohomology of V.

Besides the Bott–Chern cohomology groups which seem to be the natural home for our characteristic classes there exist also the Dolbeault cohomology groups $H_{\mathcal{A}}^{p,q}(V)$ of d''-closed (p,q)-forms modulo d''-exact forms and similarly there are Dolbeault cohomology groups $H_{\mathcal{D}}^{p,q}(V)$ of currents. These groups have been studied extensively by Jell [Jel16]. For a smooth K-variety X, Liu constructed in [Liu20] a tropical cycle class map $\mathrm{cl}_{\mathrm{trop}} : \mathrm{CH}^p(X) \to H_{\mathcal{A}}^{p,p}(X^{\mathrm{an}})$. In Appendix A we review the construction of Liu and show how to use his arguments to get a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^p(X) \arrow[r, "\delta"] \arrow[d, "\mathrm{cl}_{\mathrm{trop}}"'] & \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}}) \arrow[d] \\
H^{p,p}_{\mathscr{A}}(X^{\mathrm{an}}) \arrow[r, "{[-]}"'] & H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}}).
\end{tikzcd}$$

Here the map $\delta\colon \mathrm{CH}^p(X) \to \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})$ maps the class of a cycle Z to the class of the current of integration δ_Z. The lower map is induced by the canonical map $[-]\colon \mathscr{A}^{\bullet,\bullet} \to \mathscr{D}^{\bullet,\bullet}$ associating to a smooth form α the current $\alpha \wedge \delta_X$ and the map on the right sends the class in $\hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})$ represented by a current T to the class in $H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}})$ which is represented by the same current T. The commutativity of the diagram implies in particular that for smooth varieties our characteristic classes in $\hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})$ and the characteristic classes in $H^{p,p}_{\mathscr{A}}(X^{\mathrm{an}})$ constructed by means of Liu's tropical cycle class map have the same image in the Dolbeault cohomology of currents $H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}})$.

It would be very desirable if the Segre current

$$[s_i(\overline{E})] = p_*[c_1(\mathscr{O}_{\overline{E}}(1))^{e+i}]$$

associated to a pseudo-metrized vector bundle $\overline{E}$ was represented by a smooth form $s_i(\overline{E}) \in \mathscr{A}^{i,i}(V)$ (under appropriate smoothness conditions for the pseudo-metric). Philosophically, this means that $s_i(\overline{E})$ should be given by the fiber integral of the form $c_1(\mathscr{O}_{\overline{E}}(1))^{e+i}$ along the fiber bundle $p\colon P(E) \to V$. Unfortunately, we are still lacking a theory of fiber integrals in non-archimedean versions of differential geometry. In Chapter 10 we say that the *i-th Segre form exists* for $\overline{E}$ if there exists a smooth form β such that $[s_i(f^*\overline{E})] = [f^*\beta]$ for every morphism $f\colon V' \to V$ of analytic spaces.

A basic ingredient in classical Arakelov theory is the existence of Green currents for subvarieties of an algebraic variety X. While Gubler and Künnemann have shown the existence of Green (δ-)currents for complete intersections in [GK17], the existence of Green currents for arbitrary subvarieties is still open in the non-archimedean setting. In Proposition 10.7 we show that if X is a smooth algebraic K-variety and if Segre forms exist for all vector bundles on X, then every subvariety of X admits a Green current.

If $\overline{E}_1, \dots, \overline{E}_r$ are formally metrized vector bundles on V and $F(X_1, \dots, X_r)$ is a polynomial such that the current $[F(c_{i_1}(\overline{E}_1), \dots, c_{i_r}(\overline{E}_r))]$ has bi-degree (n, n) then it is in fact a discrete signed Radon measure. In the case where $V = X^{\mathrm{an}}$ is the analytification of a projective algebraic K-variety and K is algebraically closed, we can give a concrete description in terms of intersection numbers on the special fiber of a formal model of X (Corollary 11.10). These results both generalize and follow from results of [CD12] about Monge–Ampère measures introduced first in [Cha06]. If $F(X_1, \dots, X_r)$ is a numerically non-negative polynomial for nef vector bundles and $\overline{E}$ is a semi-positive formally metrized vector bundle of rank r then $[F(c_1(\overline{E}), \dots, c_r(\overline{E}))]$ is a positive measure by Corollary 11.13.

Over the last decade a number of variations on the theme of differential forms and currents on Berkovich spaces have been developed. Among those are for example the δ-forms of [GK17], the weakly smooth forms based on harmonic tropicalization maps of [GJR21] and the δ-forms of [Mih23a]. All of these satisfy similar formal properties as the smooth forms and currents of [CD12]. Since our theory of characteristic forms and currents relies only on these formal properties, one can replace smooth forms and currents by any of these other theories. We outline how to obtain a theory of characteristic δ-current and δ-forms in the sense of [Mih23a] in Chapter 12.

Let us briefly comment on why the complex analytic approach using Chern–Weil theory to define Chern (or, equivalently, Segre) forms of metrized vector bundles seems to fail in non-archimedean geometry. Let X be a complex manifold and let $\overline{E} = (E, h)$ be a holomorphic vector bundle E on X together with a smooth Hermitian metric h. We can consider the spaces $\mathscr{A}^p(X, E) = \Gamma(X, \bigwedge^p T^*X \otimes E)$ of smooth differential p-forms with values in E; in particular for $p = 0$ we obtain the space $\mathscr{A}^0(X, E) = \Gamma(X, \mathscr{C}_X^\infty \otimes E)$ of smooth sections of E. There exists a unique unitary connection

$$\nabla : \mathscr{A}^0(X, E) \to \mathscr{A}^1(X, E)$$

whose $(0, 1)$-part $\nabla^{0,1}$ agrees with the Dolbeault operator $\overline{\partial}_E$. Its curvature ∇^2 can be regarded as an element of $\mathscr{A}^{1,1}(X, \operatorname{End}(E))$. This curvature form induces by setting

$$\operatorname{ch}(\overline{E}) \coloneqq \operatorname{tr}_E \exp\Big(\frac{-1}{2\pi i}\nabla^2\Big) \in \bigoplus_{p \geq 0} \mathscr{A}^{p,p}(X)$$

the *Chern character form* associated to $\overline{E}$. The *Chern forms* $c_i(\overline{E})$ are then obtained as certain polynomial expressions in the components of $\operatorname{ch}(\overline{E})$. We refer to [Sou+92, Sec. IV.2] for details on this approach.

In non-archimedean geometry on the other hand it is already unclear what the space $\mathscr{A}^0(V, E)$ of smooth sections of a vector bundle E on a K-analytic space V should be. In the case of a complex manifold the definition $\mathscr{A}^0(X, E) \coloneqq \Gamma(X, \mathscr{C}_X^\infty \otimes E)$ makes sense because via the embedding $\mathscr{O}_X \hookrightarrow \mathscr{C}_X^\infty$ we can regard $\mathscr{C}_X^\infty$ as a sheaf of algebras over the sheaf of holomorphic functions $\mathscr{O}_X$ and form the tensor product with the sheaf of $\mathscr{O}_X$-modules E. For a non-archimedean analytic space V we also have a sheaf $\mathscr{C}_V^\infty$ of smooth functions, defined as real-valued functions which are locally smooth combinations of functions of the form $-\log|f|$ for f a nowhere-vanishing analytic function, but there is no natural ring homomorphism $\mathscr{O}_V \to \mathscr{C}_V^\infty$. The only map we have is the map given by tropicalization

$$\mathscr{O}_V^\times \to \mathscr{C}_V^\infty, \qquad f \mapsto -\log|f|$$

which is not at all similar to a ring homomorphism because of the involved absolute value. For similar reasons it is unclear how to make sense of smooth vector bundles, like the tangent bundle, on a non-archimedean K-analytic space. An approach to characteristic forms via connections seems therefore difficult to realize in the non-archimedean setting.

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Finally I want to thank my family—my brother, my parents and my grand-parents—for always supporting me in all possible ways. I dedicate this thesis to my grand-father Hans Langmaack. It was when I first opened some of his old mathematics books and could not understand any of those cryptic words and symbols that I first got seriously interested in mathematics.

November, 2023
Regensburg, Germany *Jakob Werner*

Notation and Conventions

0.1 Number Sets. We denote by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ the familiar sets of *natural numbers* (including 0), *integers*, *rational numbers*, *real numbers* and *complex numbers*. We write $\mathbf{R}_{\geq 0}$, resp. $\mathbf{R}_{>0}$ for the set of *non-negative*, resp. *positive* real numbers. If p is a prime number, we denote by $\mathbf{Z}_p$ the ring of *p-adic integers* and by $\mathbf{Q}_p$ the field of *p-adic rational numbers*.

0.2 Tuples. If S is some set, then we sometimes denote *tuples* $(s_1, \dots, s_r) \in S^r$ of elements of S by $\underline{s} = (s_1, \dots, s_r)$. This applies in particular if $\underline{e} = (e_1, \dots, e_r)$ is a *basis* of a vector space over some field or if $\underline{s} = (s_1, \dots, s_r)$ is a *frame* (a trivializing family of sections) of a vector bundle over some scheme or analytic space.

Another common situation where we use this notation is when $i_1, \dots, i_r \in \mathbf{N}$ are natural numbers, usually serving as *indices*, so that $\underline{i} = (i_1, \dots, i_r)$. In this case we denote by

$$|i| \coloneqq i_1 + \cdots + i_r \in \mathbf{N}$$

the sum of the indices.

0.3 Point-Set Topology. In general we follow the conventions of [Bou98] for point-set topology. In particular, the definitions of *compactness*, *local compactness* and *paracompactness* for a topological space all include the Hausdorff property. A map $f : T' \to T$ of topological spaces is called *proper* if for every topological space Z the induced map $T' \times Z \to T \times Z$ is closed [Bou98, Chap. 1, § 10.1, Def. 1]. Equivalently, by [Bou98, Chap. 1, § 10.2, Thm. 1], it is closed and has quasi-compact fibers.

0.4 Non-archimedean Fields. A *non-archimedean field* is a field K equipped with a non-trivial complete non-archimedean absolute value $|-| : K \to \mathbf{R}_{\geq 0}$. We denote by $K^\circ := \{ a \in K \mid |a| \leq 1 \}$ the *valuation ring* of the non-archimedean field K, equipped with the induced topology (which agrees with the $K^{\circ\circ}$-adic topology, where $K^{\circ\circ}$ denotes the maximal ideal of K°). Furthermore we denote by $\widetilde{K} := K^\circ / K^{\circ\circ}$ the *residue field* of K.

We usually denote non-archimedean fields with capital letters $K, L, \ldots$, whereas we reserve lowercase letters $k, l, \ldots$ for fields without additional metric structure.

An *extension* of non-archimedean fields is a field homomorphism $i \colon K \hookrightarrow L$, where K and L are non-archimedean fields, such that pull-back of the absolute value of L along i coincides with the absolute value of K. We often suppress the homomorphism i from the notation and write simply L/K for an extension of non-archimedean fields.

0.5 Varieties. If k is a field, a *variety* over k is a separated, integral, finite type k-scheme. We do not assume that k is geometrically integral, proper or smooth, unless we explicitly state it. Sometimes we say *algebraic variety* instead of *variety* in order to emphasize the contrast to a situation where also Berkovich analytic spaces play a role.

Over any field k and for any integer n we have n-dimensional *affine space* $\mathbf{A}^n_k := \operatorname{Spec}(k[T_1, \ldots, T_n])$ as well as the n-dimensional *projective space* $\mathbf{P}^n_k := \operatorname{Proj}(k[T_0, \ldots, T_n])$.

0.6 Chow Groups. If X is a scheme of finite type over a field k, we denote by $\mathrm{CH}_d(X) = Z_d(X)/R_d(X)$ the *Chow group* generated by prime cycles (closed subvarieties) of dimension d. In our main reference for intersection theory [Ful98], this group is denoted by $A_d(X)$.

If X is equidimensional of dimension n (e.g. a variety), we write $\mathrm{CH}^p(X) := \mathrm{CH}_{n-p}(X)$ where n is the dimension of X.

0.7 K-Analytic Spaces. By a *K-analytic space* we always mean a K-analytic space in the sense of [Ber93]. When dealing with smooth forms and currents on an analytic space, we usually assume all K-analytic spaces to be good, topologically Hausdorff, boundaryless and equidimensional (Chapters 7 to 12 and Appendix A). When dealing with formal models, we assume K-analytic spaces to be strict and paracompact (Chapters 3, 6 and 11). Note that the analytification of an algebraic variety possesses all of the above properties. We state our assumptions explicitly at the beginning of each chapter.

If V is a K-analytic space, we denote the underlying topological space with its Berkovich topology again by V. The G-topological space with the same underlying set, equipped with the G-topology of all analytic domains is denoted by V_G. The respective structure sheaves are denoted by $\mathcal{O}_V$ and $\mathcal{O}_{V_G}$ respectively. In Chapter 2 we give a recollection of the theory of Berkovich analytic spaces.

0.8 Admissible Formal Schemes. We review the notion of *admissible formal schemes* over the valuation ring K° in Chapter 3. Note that we always assume admissible formal schemes to be *quasi-paracompact*, i.e. to admit a locally finite covering by affine admissible formal schemes. With this convention we are following e.g. [Gub07]. It allows us to consider the generic fiber $\mathfrak{V}_\eta$ as a Berkovich K-analytic space (Paragraph 3.2).

0.9 Projective Bundles. Let E be a vector bundle on a scheme X. We follow the convention of [Ful98] and denote by

$$P(E) \coloneqq \operatorname{Proj}(\operatorname{Sym}(E^\vee))$$

the *projective bundle* parametrizing lines in E, in contrast to Grothendieck's bundle

$$\mathbf{P}(E) \coloneqq \operatorname{Proj}(\operatorname{Sym}(E))$$

parametrizing line quotients of E. We denote by $\mathcal{O}_E(1)$ the canonical relatively ample line bundle on $P(E)$ (not on $\mathbf{P}(E)$) and by $\mathcal{O}_E(-1)$ its dual. If $p\colon P(E) \to X$ denotes the canonical projection morphism, then we have the universal embedding

$$\mathcal{O}_E(-1) \hookrightarrow p^*E.$$

Given a vector bundle E on a K-analytic space V, we construct in Proposition 4.34 similarly a K-analytic space $P(E)$ with a canonical projection map $p\colon P(E) \to V$, a canonical line bundle $\mathcal{O}_E(-1)$ on $P(E)$ and a canonical embedding $\mathcal{O}_E(-1) \hookrightarrow p^*E$.

1. Norms

In this chapter we review the notion of a *norm* on a K-vector space where K is a non-archimedean field which will be fixed throughout the chapter. In particular we discuss constructions with norms such as *duals* (Paragraph 1.8), *direct sums* (Paragraph 1.10) and *tensor products* (Paragraph 1.11). More details on the constructions and their properties can be found in [CM20] and [BE21]. It should be remarked that we assume norms to satisfy the non-archimedean triangle property, so that some care has to be taken when comparing our treatment with that of [CM20].

1.1 Norms. Let E be a K-vector space. A *norm* on E is a map $\|-\| : E \to \mathbf{R}_{\geq 0}$ satisfying:

(i) $\|v\| = 0 \iff v = 0$ for $v \in E$.

(ii) $\|v + w\| \leq \max(\|v\|, \|w\|)$ for $v, w \in E$.

(iii) $\|av\| = |a|\|v\|$ for $a \in K, v \in E$.

The pair $(E, \|-\|)$ is called a *normed K-vector space*. We often use the notation $\overline{E} = (E, \|-\|)$ to denote a normed K-vector space. If $\overline{F}$ is another normed K-vector space, we often do not distinguish the norms of $\overline{E}$ and $\overline{F}$ notationally.

We regard K itself as a normed K-vector space with the absolute value $|-|$ as the norm.

1.2 Bounded Operators. Let $\overline{E}, \overline{F}$ be two normed K-vector spaces. A *bounded operator* $\alpha : \overline{E} \to \overline{F}$ is a K-linear map $\alpha : E \to F$ such that there exists a constant

$C \in \mathbf{R}_{\geq 0}$ satisfying

$$\|\alpha(v)\| \leq C\|v\|$$

for all $v \in E$. The smallest such constant is denoted by

$$\begin{aligned}\|\alpha\| &:= \inf\{\, C \in \mathbf{R}_{\geq 0} \mid \|\alpha(v)\| \leq C\|v\| \text{ for all } v \in E \,\} \\ &= \sup\{\, \|\alpha(v)\| \mid v \in E \text{ with } \|v\| \leq 1 \,\}\end{aligned}$$

and is called the *(operator) norm* of α. The operator α is called *contractive* if $\|\alpha\| \leq 1$.

Note that if $\alpha\colon \overline{E} \to \overline{F}$ and $\beta\colon \overline{F} \to \overline{G}$ are bounded operators, then their composition is bounded with

$$\|\beta \circ \alpha\| \leq \|\beta\| \cdot \|\alpha\|.$$

There is a category of normed K-vector spaces with bounded operators as morphisms and also a category of normed K-vector spaces with contractive operators as morphisms. We will refer to these as the *bounded*, resp. the *contractive* category of normed K-vector spaces.

1.3 Equivalent Norms. Let E be a K-vector space and let $\|-\|_1$, $\|-\|_2$ be two norms on E. The norms $\|-\|_1$ and $\|-\|_2$ are called *equivalent* if the operators $\mathrm{id}\colon (E, \|-\|_1) \to (E, \|-\|_2)$ and $\mathrm{id}\colon (E, \|-\|_2) \to (E, \|-\|_1)$ are bounded. In other words, $\|-\|_1$ and $\|-\|_2$ are equivalent if there exists some $C \in \mathbf{R}_{>0}$ such that

$$C^{-1}\|v\|_2 \leq \|v\|_1 \leq C\|v\|_2$$

for all $v \in E$.

1.4 Proposition. *Let $\overline{E}, \overline{F}$ be two normed K-vector spaces. If E is finite-dimensional, then every linear operator $\alpha\colon \overline{E} \to \overline{F}$ is bounded.*

In particular, any two norms on a finite-dimensional K-vector space are equivalent.

Proof. The second statement is [BE21, Prop. 1.6]. By the second statement, we may assume that $E = K^r$ with the maximum norm in the first statement, in which case the claim is easy to check. ∎

1.5 Corollary. *If $\overline{E}$ is a finite-dimensional normed K-vector space, then it is complete with respect to the induced metric and every K-linear subspace is closed.* ∎

1.6 Internal Homs. Let $\overline{E}, \overline{F}$ be two normed K-vector spaces. We write $\mathrm{Hom}(\overline{E}, \overline{F})$ for the normed K-vector space of all bounded operators with the operator norm as the norm. Note that if E is finite-dimensional, then the underlying vector space of $\mathrm{Hom}(\overline{E}, \overline{F})$ is just the space $\mathrm{Hom}(E, F)$ of all linear operators by Proposition 1.4.

1.7 Isomorphisms. Let $\overline{E}, \overline{F}$ be two normed K-vector spaces. By an *isomorphism* $\alpha : \overline{E} \xrightarrow{\sim} \overline{F}$ we mean an isomorphism in the bounded category of normed K-vector spaces. By an *isometric isomorphism* we mean an isomorphism in the contractive category. An isometric isomorphism satisfies automatically

$$\|\alpha(v)\| = \|v\|$$

for all $v \in E$.

If there are natural (functorial in $\overline{G}$) isomorphisms (of normed K-vector spaces or just of the underlying sets)

$$\mathrm{Hom}(\overline{E}, \overline{G}) \xrightarrow{\sim} \mathrm{Hom}(\overline{F}, \overline{G}), \tag{1.7.1}$$

then by the Yoneda lemma, these are induced by a unique isomorphism $\alpha : \overline{F} \xrightarrow{\sim} \overline{E}$. If the isomorphisms in Eq. (1.7.1) are isometric, then the corresponding isomorphism $\overline{F} \xrightarrow{\sim} \overline{G}$ is isometric. Indeed, we obtain α by setting $\overline{G} := \overline{E}$ and applying the isomorphism (1.7.1) to $\mathrm{id}_{\overline{E}}$. But as we have $\|\mathrm{id}_{\overline{E}}\| \leq 1$ and the isomorphism (1.7.1) is contractive, we get also $\|\alpha\| \leq 1$. By a symmetric argument we get also $\|\alpha^{-1}\| \leq 1$.

There is of course a similar principle regarding natural isomorphisms

$$\mathrm{Hom}(\overline{G}, \overline{E}) \xrightarrow{\sim} \mathrm{Hom}(\overline{G}, \overline{F}),$$

which is however even more trivial, because one can simply plug in $\overline{G} := K$.

1.8 Duals. Let $\overline{E}$ be a normed K-vector space. Then we denote by

$$\overline{E}^{\vee} := \mathrm{Hom}(\overline{E}, K)$$

the *dual* normed K-vector space. If the norm on E is denoted by $\|-\|$, then we often write $\|-\|^{\vee}$ for the operator norm on $\overline{E}^{\vee}$. Note that if E is finite-dimensional, then the underlying vector space of $\overline{E}^{\vee}$ agrees with the dual vector space $E^{\vee}$ of E by Proposition 1.4.

1.9 Proposition. *Let $\overline{E}$ be a finite-dimensional normed K-vector space. Then there is a natural isometric isomorphism*

$$\overline{E} \xrightarrow{\sim} (\overline{E}^{\vee})^{\vee}.$$

Proof. This is shown in [CM20, Cor. 1.2.12]. ∎

1.10 Direct Sums. Let $\overline{E}$, $\overline{F}$ be two normed K-vector spaces. We denote by $\overline{E} \oplus \overline{F}$ the normed K-vector space which is given by the direct sum $E \oplus F$ of the underlying vector spaces together with the direct sum norm

$$\|(v, w)\| := \max(\|v\|, \|w\|).$$

If the norm of $\overline{E}$ is denoted $\|-\|_1$ and the norm of $\overline{F}$ is denoted $\|-\|_2$, then we often write $\|-\|_1 \oplus \|-\|_2$ for the direct sum norm on $\overline{E} \oplus \overline{F}$.

Given a third normed K-vector space $\overline{G}$, it is easy to see that there are natural isometric isomorphisms

$$\mathrm{Hom}(\overline{E} \oplus \overline{F}, \overline{G}) \cong \mathrm{Hom}(\overline{E}, \overline{G}) \oplus \mathrm{Hom}(\overline{F}, \overline{G}) \tag{1.10.1}$$

and

$$\mathrm{Hom}(\overline{G}, \overline{E} \oplus \overline{F}) \cong \mathrm{Hom}(\overline{G}, \overline{E}) \oplus \mathrm{Hom}(\overline{G}, \overline{F}). \tag{1.10.2}$$

1.11 Tensor Products. Let $\overline{E}$, $\overline{F}$ be two normed K-vector spaces. We denote by $\overline{E} \otimes \overline{F}$ the normed K-vector space which is given by the tensor product $E \otimes F$ of the underlying vector spaces together with the norm

$$\|t\| = \inf\Big\{ \max_{i=1,\dots,n} \|v_i\|\|w_i\| \;\Big|\; v_i \in E, w_i \in F, t = \sum_{i=1}^{n} v_i \otimes w_i \Big\}.$$

It follows from [Gru66, Sect. 3.2, Thm. 1] that this semi-norm is indeed a norm.

If $\overline{E}, \overline{F}, \overline{G}$ are normed K-vector spaces, then we denote by $\mathrm{Bil}(\overline{E}, \overline{F}; \overline{G})$ the normed K-vector space of bounded bilinear maps $E \times F \to G$. Here a bilinear map $\beta\colon E \times F \to G$ is called *bounded* if there exists a constant $C \in \mathbf{R}_{\geq 0}$ such that $\|\beta(v, w)\| \leq C\|v\|\|w\|$ for all $v \in E$ and $w \in F$. The smallest such constant is the *norm* $\|\beta\|$ of β. It is easy to check that there are natural isometric isomorphisms

$$\mathrm{Hom}(\overline{E} \otimes \overline{F}, \overline{G}) \cong \mathrm{Bil}(\overline{E}, \overline{F}; \overline{G}) \cong \mathrm{Hom}(\overline{E}, \mathrm{Hom}(\overline{F}, \overline{G})). \tag{1.11.1}$$

Note that our tensor product norm does not agree with the π-tensor product norm defined in [CM20, Def. 1.1.52] (which is not even a norm in our sense, because it does not satisfy the non-archimedean triangle inequality).

If the norm of $\overline{E}$ is denoted $\|-\|_1$ and the norm of $\overline{F}$ is denoted $\|-\|_2$, then we often write $\|-\|_1 \otimes \|-\|_2$ for the tensor product norm on $\overline{E} \otimes \overline{F}$.

1.12 Proposition. *Let $\overline{E}$, $\overline{F}$, $\overline{G}$ be normed K-vector spaces. There are natural isometric isomorphisms as follows:*

(i) $\overline{E} \otimes K \cong \overline{E}$.

(ii) $\overline{E} \otimes \overline{F} \cong \overline{F} \otimes \overline{E}$.

(iii) $(\overline{E} \otimes \overline{F}) \otimes \overline{G} \cong \overline{E} \otimes (\overline{F} \otimes \overline{G})$.

(iv) $(\overline{E} \oplus \overline{F}) \otimes \overline{G} \cong \overline{E} \otimes \overline{G} \oplus \overline{F} \otimes \overline{G}$.

Proof. These follow easily from using the universal properties Eq. (1.11.1) and Eq. (1.10.1) as well as the Yoneda principle from Paragraph 1.7. ∎

1.13 Subspaces. Let $\overline{E}$ be a normed K-vector space and $i\colon U \hookrightarrow E$ be the inclusion of a K-linear subspace $U \subset E$, or more generally just any injective K-linear map of vector spaces. Then there is an induced norm on U given by

$$\|v\| = \|i(v)\|$$

for $v \in U$. We call it the *subspace norm* induced by i.

1.14 Scalar Extension. Let $\overline{E}$ be a normed K-vector space and suppose that L/K is an extension of non-archimedean fields. We can view L as a normed K-vector space by viewing the absolute value as the norm. The tensor product norm from Paragraph 1.11 on $E \otimes_K L$ is then in fact a norm on the L-vector space $E \otimes_K L$. We denote the resulting normed L-vector space by $(\overline{E})_L$ and call it the *scalar extension* of $\overline{E}$ along L/K. If the norm on $\overline{E}$ is denoted by $\|-\|$, then we sometimes denoted the scalar extension norm on $(\overline{E})_L$ by $\|-\|_L$.

Given a normed L-vector space $\overline{F}$, it is easy to show that there is a natural isometric isomorphism

$$\mathrm{Hom}_L(\overline{E}_L, \overline{F}) \cong \mathrm{Hom}_K(\overline{E}, \overline{F}).$$

1.15 Proposition. *Let L/K be an extension of non-archimedean fields. Let $\overline{E}, \overline{F}$ be two normed K-vector spaces. There are natural isometric isomorphisms as follows:*

(i) $(K)_L \cong L$ *(here K and L are considered as normed vector spaces with respect to their absolute values).*

(ii) $(\overline{E} \oplus \overline{F})_L \cong (\overline{E})_L \oplus (\overline{F})_L$.

(iii) $(\overline{E} \otimes_K \overline{F})_L \cong (\overline{E})_L \otimes_L (\overline{F})_L$.

If L'/L is a further extension of non-archimedean fields, then there is a natural isomorphism

(iv) $(\overline{E} \otimes_K L) \otimes_L L' \cong E \otimes_K L'$.

Proof. These isomorphisms follow easily from the various universal properties discussed above. ∎

1.16 Proposition. *Let $\overline{E}$ be a normed K-vector space and let L/K be an extension of non-archimedean fields. Then $\|-\|_L$ restricts to the original norm $\|-\|$ under the canonical injective map $E \hookrightarrow E_L$, $v \mapsto v \otimes 1$, i.e. the diagram*

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, "\|-\|"'] & E_L \arrow[d, "\|-\|_L"] \\ & \mathbf{R}_{\geq 0} \end{tikzcd}$$

commutes.

Proof. This is shown in [BE21, Prop. 1.25 (i)]. ∎

1.17 Lemma. *Let $\overline{F} = (F, \|-\|')$ and $\overline{E} = (E, \|-\|)$ be two normed K-vector spaces and let $i\colon \overline{F} \hookrightarrow \overline{E}$ be an isometric embedding. In other words, we can think of $\overline{F}$ as being equipped with the subspace norm induced from $\overline{E}$.*

(i) *If F is one-dimensional and L/K is an extension of non-archimedean fields then the embedding $i_L\colon \overline{F}_L \to \overline{E}_L$ is again isometric.*

(ii) *If $\overline{G}$ is a one-dimensional normed K-vector space then the induced embedding $\overline{F} \otimes \overline{G} \to \overline{E} \otimes \overline{G}$ is again isometric.*

Proof. For the first statement we note that we have a canonical commutative diagram

$$\begin{tikzcd} \overline{F} \arrow[r, hook] \arrow[d, hook] & \overline{F}_L \arrow[d, hook] \\ \overline{E} \arrow[r, hook] & \overline{E}_L \end{tikzcd}$$

where the horizontal maps are isometric by Proposition 1.16 and the left vertical map is isometric by assumption. The claim is that the right vertical map is also isometric, or equivalently that the restriction of the norm of $\overline{E}_L$ to F_L agrees with the norm of $\overline{F}_L$. Since F_L is a one-dimensional L-vector space, both of these norms are determined by their value on an arbitrary non-zero element, in particular it suffices to compare them on $F \subset F_L$. But the fact that their restrictions to F agree follows from the fact that $\overline{F} \to \overline{E} \to \overline{E}_L$ is isometric.

Let us treat the second statement. We write $\|-\|''$ for the norm of $\overline{G}$. We fix a non-zero element $g \in G$ with norm $r := \|g\|'' \in \mathbf{R}_{>0}$. On F we introduce a norm $\|-\|'_r$ by setting $\|f\|'_r := r \cdot \|f\|'$ for $f \in F$. We claim that the isomorphism of K-vector spaces

$$\alpha_F\colon (F, \|-\|'_r) \to \overline{F} \otimes \overline{G}, \qquad f \mapsto f \otimes g$$

is isometric. It is clearly contractive. To see that the inverse isomorphism given by $f \otimes ag \mapsto af$ for $f \in F$ and $a \in K$ is also contractive it suffices by the universal property of the normed tensor product to note that the bilinear map $(f, ag) \mapsto af$ is bounded with norm ≤ 1 which is easy to verify.

Similarly we introduce a norm $\|-\|_r$ on E and a map

$$\alpha_E \colon (E, \|-\|_r) \to \overline{E} \otimes \overline{G}, \qquad e \mapsto e \otimes g$$

which is isometric for the same reason. The maps fit into a commutative diagram

$$\begin{tikzcd} (F, \|-\|'_r) \arrow[r, "\alpha_F", "\cong"'] \arrow[d, hook] & \overline{F} \otimes \overline{G} \arrow[d, hook] \\ (E, \|-\|_r) \arrow[r, "\cong", "\alpha_E"'] & \overline{E} \otimes \overline{G}. \end{tikzcd}$$

Since the horizontal isomorphisms are isometric and the map on the left is obviously isometric, the same is true for the map on the right. ∎

1.18 Orthonormal Bases. Let $\overline{E}$ be a finite-dimensional normed K-vector space and let $e_1, \dots, e_r$ be a basis for E. It is called an *orthonormal basis* if the isomorphism

$$K^r \to \overline{E}, \qquad (a_1, \dots, a_r) \mapsto \sum_{i=1}^{r} a_i e_i$$

is isometric. Here K^r is equipped with the direct sum metric.

1.19 Proposition. *Let $\overline{E}$, $\overline{F}$ be finite-dimensional normed K-vector spaces and let L/K be an extension of non-archimedean fields.*

(i) If $e_1, \dots, e_r$ is an orthonormal basis of $\overline{E}$, then the dual basis $e_1^\vee, \dots, e_r^\vee$ is an orthonormal basis of $\overline{E}^\vee$.

(ii) If $e_1, \dots, e_r$ is an orthonormal basis of $\overline{E}$ and $e'_1, \dots, e'_{r'}$ is an orthonormal basis of $\overline{F}$, then $(e_1, 0), \dots, (e_r, 0), (0, e'_1), \dots, (0, e'_{r'})$ is an orthonormal basis of $\overline{E} \oplus \overline{F}$.

(iii) If $e_1, \dots, e_r$ is an orthonormal basis of $\overline{E}$ and $e'_1, \dots, e'_{r'}$ is an orthonormal basis of $\overline{F}$, then $e_1 \otimes e'_1, \dots, e_r \otimes e'_{r'}$ is an orthonormal basis of $\overline{E} \otimes \overline{F}$.

(iv) If $e_1, \dots, e_r$ is an orthonormal basis of $\overline{E}$, then $e_1 \otimes 1, \dots, e_r \otimes 1$ is an orthonormal basis for $(\overline{E})_L$.

Proof. These statements all follow easily from the various natural isomorphisms discussed above. Let us prove the first statement as an example. Using the isometric isomorphism $\overline{E} \cong K^r$ induced from the orthonormal basis, we get an isometric isomorphism

$$\overline{E}^\vee \cong (K^r)^\vee \cong (K^\vee)^r \cong K^r.$$

Under this isomorphism, the standard basis on the right hand side corresponds to the dual basis $e_1^\vee, \dots, e_r^\vee$ on the left hand side, which is thus an orthonormal basis. ∎

1.20 Diagonalizable Norms. Let E be a finite-dimensional K-vector space, let $e_1, \dots, e_r$ be a basis for E and let $\phi_1, \dots, \phi_r \in \mathbf{R}_{>0}$ be positive real numbers. Then we call the norm $\|-\|_{\underline{e},\underline{\phi}}$ satisfying

$$\|a_1 e_1 + \dots + a_r e_r\|_{\underline{e},\underline{\phi}} = \max_{i=1,\dots,r} |a_i| \phi_i$$

for $a_1, \dots, a_r \in K$ the *diagonalizable norm* associated to $\underline{e}$ and $\underline{\phi}$.

A norm $\|-\|$ on E is called *diagonalizable* if there exists a basis $e_1, \dots, e_r$ and a family of positive real numbers $\phi_1, \dots, \phi_r \in \mathbf{R}_{>0}$ such that $\|-\| = \|-\|_{\underline{e},\underline{\phi}}$. If $\|-\|$ is a diagonalizable norm and $e_1, \dots, e_r$ is a basis of E, then $e_1, \dots, e_r$ is called an *orthogonal basis* for $\|-\|$ if there exist $\phi_1, \dots, \phi_r \in \mathbf{R}_{>0}$ such that $\|-\| = \|-\|_{\underline{e},\underline{\phi}}$. For properties of diagonalizable norms see [BE21, Sec. 1].

1.21 The Space of Norms. Let E be a finite-dimensional K-vector space and let $\|-\|, \|-\|'$ be two norms on E. By Proposition 1.4, $\|-\|$ and $\|-\|'$ are equivalent and hence the quantity

$$d(\|-\|, \|-\|') := \sup_{0 \neq v \in E} |\log\|v\| - \log\|v\|'|$$

is finite. This equips the set of all norms on E with the structure of a complete metric space [BE21, Prop. 1.8].

1.22 Lemma. *Let E be a finite-dimensional K-vector space and let $\|-\|, \|-\|'$ be two norms on E. Let L/K be an extension of non-archimedean fields. Denote by $\|-\|_L, \|-\|'_L$ the respective scalar extension norms on $E_L = E \otimes_K L$ as defined in Paragraph 1.14. Then we have*

$$d(\|-\|_L, \|-\|'_L) = d(\|-\|, \|-\|').$$

Proof. This is proved in [BE21, Prop. 1.25. (ii)]. ∎

1.23 Lemma. *Let E be a finite-dimensional K-vector space and let $\|-\|, \|-\|'$ be two norms on E. Then we have*

$$d(\|-\|^\vee, \|-\|'^\vee) = d(\|-\|, \|-\|').$$

Proof. This is proved in [CM20, Prop. 1.1.43]. ∎

2. Berkovich Analytic Spaces

Throughout this chapter we fix a non-archimedean field K. We review some basic notions related to *K-analytic spaces* in the sense of [Ber93] for the convenience of the reader and to fix terminology and notation. All results proved here are probably well-known, but since we could not track down proofs in the literature, we provide them here.

2.1 Berkovich Analytic Spaces. In [Ber93, Sec. 1.2, p. 22] Berkovich introduces the category of Φ_K-analytic spaces where Φ is a class of affinoid spaces satisfying some stability properties [Ber93, Sec. 1.2, p. 16]. If Φ is the class of all affinoid spaces, then a Φ_K-analytic space is simply called a *K-analytic space*. If Φ is the class of all strictly affinoid spaces, then a Φ_K-analytic space is called a *strictly K-analytic space*. The category of strictly K-analytic spaces is a full subcategory of the category of all K-analytic spaces by [Tem04, Cor. 4.10].

By [Ber93, Prop. 1.4.1], the category of (strictly) K-analytic spaces admits all fiber products.

2.2 G-topological Spaces. Let X be a set. A *G-topology* on X is given by a collection of distinguished subsets of X, called *admissible open subsets*, as well as, for each admissible open subset $U \subset X$, a collection of distinguished set-theoretic coverings $U = \bigcup\{U_i\}_{i\in I}$ by other admissible open subsets $U_i \subset X$, called *admissible coverings*, in such a way that the following axioms are satisfied:

(i) The intersection $U \cap V$ of two admissible open subsets $U, V \subset X$ is again admissible open.

(ii) For each admissible open subset $U \subset X$, the trivial covering $\{U\}$ of U is admissible.

(iii) If U is an admissible open subset of X, $\{U_i\}_{i \in I}$ is an admissible covering of U, and if for each $i \in I$, the family $\{V_{ij}\}_{j \in J_i}$ is an admissible covering of U_i, then the family $\{V_{ij}\}_{i \in I, j \in J_i}$ is an admissible covering of U.

(iv) If $U, V \subset X$ are two admissible open subsets of X and $V \subset U$, and if $\{U_i\}_{i \in I}$ is an admissible covering of U, then the family $\{V \cap U_i\}_{i \in I}$ is an admissible covering of V.

We call the G-topology *saturated* if the following additional properties are satisfied:

(v) The subsets $\varnothing$ and X of X are admissible.

(vi) Let $U \subset X$ be an admissible open subset, $\{U_i\}_{i \in I}$ an admissible covering of U and let $V \subset U$ be any subset. If every intersection $V \cap U_i$, $i \in I$ is admissible open, then V is admissible open.

(vii) Let $\{U_i\}_{i \in I}$ be a set-theoretic covering of an admissible open subset $U \subset X$. If $\{U_i\}_{i \in I}$ admits a refinement which is admissible, then $\{U_i\}_{i \in I}$ itself is an admissible covering.

The set X together with the G-topology is called a *G-topological space*. The theory of G-topological spaces can be found in [BGR84, Sec. 9.1]. Conditions (v) to (vii) are called (G_0) to (G_2) in [BGR84, Sec. 9.1.2].

2.3 Sheaves. There is a notion of presheaves and sheaves on a G-topological space, generalizing the theory of sheaves on a topological space. Also for G-topological spaces, there exists a sheafification functor left adjoint to the forgetful functor from sheaves to presheaves [BGR84, Prop. 9.2.2/4]. A pair $(X, \mathcal{O}_X)$ where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of rings on X is called a *ringed G-topological space*. There is a category of sheaves of $\mathcal{O}_X$-modules on a ringed G-topological space $(X, \mathcal{O}_X)$ admitting direct sums, kernels, cokernels and tensor products as usual. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed G-topological spaces (i.e. a G-continuous map $f : X \to Y$ together with a morphism of sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$), then the functor $\mathcal{F} \mapsto f_* \mathcal{F}$ from sheaves of $\mathcal{O}_X$-modules to sheaves of $\mathcal{O}_Y$-modules admits a left adjoint f^* which commutes with direct sums, cokernels and tensor products. The theory of sheaves on G-topological spaces is generalized by the theory of sheaves on sites [Sta23, Chap. 00UZ, Chap. 03A4].

2.4 Quasi-nets. Let X be a topological space, $U \subset X$ a subset and $\{U_i\}_{i \in I}$ a family of subsets $U_i \subset U$. The family $\{U_i\}_{i \in I}$ is called a *quasi-net* on U if the following holds: For every $x \in U$, there exists a finite subset $J_x \subset I$ such that $x \in \bigcap_{i \in J_x} U_i$ and such that $\bigcup_{i \in J_x} U_i$ is a neighborhood of x in U. Note that a quasi-net is, in particular, a set-theoretic covering of U.

2.5 The Topology and the G-Topology of an Analytic Space. Let V be a K-analytic space. By definition [Ber93, Sec. 1.2, p. 17], V has an underlying topological space (which we often denote by V as well). Any morphism of K-analytic spaces induces a continuous map of the underlying topological spaces.

In addition, the underlying set of V also carries a canonical G-topology which we will simply call the *G-topology of V*. We write V_G for the G-topological space obtained in this way. The admissible open subsets of the G-topology of a Berkovich analytic space are given by the *analytic domains*. Here, a subset $U \subset V$ is called an *analytic domain* if there exists a quasi-net $\{U_i\}_{i \in I}$ on U consisting of affinoid domains in V. If $U \subset V$ is an analytic domain, then a set-theoretic covering $U = \bigcup_{i \in I} U_i$ by analytic domains $U_i \subset V$ is admissible for the G-topology of V if and only if it is a quasi-net. See [Ber93, Sec. 1.3] for details.

If $U \subset V$ is an analytic domain and $U = \bigcup_{i \in I} U_i$ is an admissible covering by analytic domains in the G-topology of V, then we also say that the family $\{U_i\}_{i \in I}$ forms a *G-covering* of U. By construction, every K-analytic space V admits a G-covering $V = \bigcup_{i \in I} V_i$ by affinoid domains.

Note that if $\mathscr{M}(A)$ is a K-affinoid space, then each point of $\mathscr{M}(A)$ has a fundamental neighborhood of affinoid (Weierstraß) domains. In the language of [Ber93, Sec. 1.2, p.16] this means that the class Φ of all K-affinoid spaces is *dense*. It follows that every open subset of a K-analytic space V is an analytic domain and every covering of an open subset by open subsets is a G-covering or in other words that the identity map is a morphism of G-topological spaces $\pi_V \colon V_G \to V$ [Ber93, Sec. 1.3, p. 25].

There is a canonical sheaf of rings on V_G denoted by $\mathscr{O}_{V_G}$ called the *structure sheaf* of V [Ber93, Sec. 1.3, p. 25]. It is given by

$$\mathscr{O}_{V_G}(U) = \mathrm{Hom}(U, \mathbf{A}_K^{1,\mathrm{an}})$$

for an analytic domain $U \subset V$. Here $\mathbf{A}_K^{1,\mathrm{an}}$ denotes the analytification of the algebraic K-variety $\mathbf{A}_K^1$ in the sense of Paragraph 2.16. By [Ber93, Sec. 1.3, p. 27], a morphism $f \colon V' \to V$ of K-analytic spaces induces a morphism of ringed G-topological spaces, again denoted by $f \colon (V', \mathscr{O}_{V'_G}) \to (V, \mathscr{O}_{V_G})$.

Sometimes we consider also the sheaf $\mathscr{O}_V := (\pi_V)_* \mathscr{O}_{V_G}$ on the topological space V. By construction, there is a canonical morphism of ringed G-topological spaces $\pi_V \colon (V_G, \mathscr{O}_{V_G}) \to (V, \mathscr{O}_V)$ given by the identity map $\mathrm{id} \colon \mathscr{O}_V \to (\pi_V)_* \mathscr{O}_{V_G}$. We remark that the sheaf $\mathscr{O}_V$ is usually only interesting if V is good (see Paragraph 2.15 for this notion).

2.6 Scalar Extension. Let L/K be an extension of non-archimedean fields. Then there is a functor $V \mapsto V \hat{\otimes}_K L$ from the category of K-analytic spaces to the category of L-analytic spaces given by $\mathscr{M}(A) \hat{\otimes}_K L := \mathscr{M}(A \hat{\otimes}_K L)$ for K-affinoid algebras A and extended to the general case by gluing using [Ber93, Prop. 1.3.3], see [Ber93, Sec. 1.4, p. 30]. From the construction of fibered products in [Ber93, Prop. 1.4.1] one sees that the scalar extension functor preservers fibered products.

2.7 Analytic Spaces over K. An *analytic space over K* is a pair (V,L), where L/K is a non-archimedean extension field and V is an L-analytic space. We call L the *field of definition* of (V,L). A *morphism* $f\colon (V',L') \to (V,L)$ of analytic spaces over K is a pair consisting of a morphism $L \hookrightarrow L'$ of non-archimedean extension fields of K and a morphism $\widetilde{f}\colon V' \to V \hat{\otimes}_L L'$ of L'-analytic spaces. If $f\colon (V',L') \to (V,L)$ and $f'\colon (V'',L'') \to (V',L')$ are morphisms of analytic spaces over K then their composition is given by the composition $L \hookrightarrow L' \hookrightarrow L''$ of field extensions and the composition

$$V'' \xrightarrow{\widetilde{f'}} V' \hat{\otimes}_{L'} L'' \xrightarrow{\widetilde{f}\hat{\otimes}_{L'}L''} (V \hat{\otimes}_L L') \hat{\otimes}_{L'} L'' \xrightarrow{\sim} V \hat{\otimes}_L L''.$$

Be cautious that *K-analytic spaces* and *analytic spaces over K* are not the same thing. The former form a full subcategory of the latter via the embedding $V \mapsto (V,K)$. For details, see [Ber93, Sec. 1.4, p. 30].

Let (V,L) be an analytic space over K and let $i\colon L \hookrightarrow L'$ be an extension of non-archimedean extension fields of K. We can regard $(V \hat{\otimes}_L L', L')$ as an analytic space over K. There is a canonical morphism of analytic spaces over K

$$\pi_{L'/L}\colon (V \hat{\otimes}_L L', L') \to (V,L)$$

given by the field extension $i\colon L \hookrightarrow L'$ and the identity morphism $\mathrm{id}\colon V \hat{\otimes}_L L' \to V \hat{\otimes}_L L'$ of L'-analytic spaces.

If $f\colon (V',L') \to (V,L)$ is any morphism of analytic spaces over K, given by the embedding $i\colon L \hookrightarrow L'$ and the morphism $\widetilde{f}\colon V' \to V \hat{\otimes}_L L'$ of L'-analytic spaces, then it factors canonically as

$$(V',L') \xrightarrow{\widetilde{f}} (V' \hat{\otimes}_L L', L') \xrightarrow{\pi_{L'/L}} (V,L).$$

2.8 The Functor from Analytic Spaces over K to G-topological Spaces. Next we describe a functor from the category of analytic spaces over K to ringed G-topological spaces given on objects by sending a pair (V,L) to the underlying G-topological space of the L-analytic space V in the sense of Paragraph 2.5. We start by sketching how the morphism $\pi_{L'/L}\colon (V \hat{\otimes}_L L', L') \to (V,L)$ induces a morphism of the underlying G-topological spaces, again denoted by

$$\pi_{L'/L}\colon V \hat{\otimes}_L L' \to V. \tag{2.8.1}$$

If $V = \mathcal{M}(A)$ is an L-affinoid space, then we define $\pi_{L'/L} : \mathcal{M}(A \hat{\otimes}_L L') \to \mathcal{M}(A)$ to be the map induced by the canonical morphism $A \to A \hat{\otimes}_L L'$ of Banach rings. We omit here the verification that it is G-continuous and induces a morphism of the structure sheaves. In the general case the map (2.8.1) is obtained by gluing.

If $f : (V', L') \to (V, L)$ is an arbitrary morphism of analytic spaces over L, then by Paragraph 2.7 it factors as

$$(V', L') \xrightarrow{\tilde{f}} (V' \hat{\otimes}_L L', L') \xrightarrow{\pi_{L'/L}} (V, L).$$

By Paragraph 2.5 the morphism $\tilde{f} : V' \to V' \hat{\otimes}_L L'$ of L'-analytic spaces induces a morphism of the underlying ringed G-topological spaces and we define the underlying morphism $f : V' \to V$ of ringed G-topological spaces to be the composition

$$V' \xrightarrow{\tilde{f}} V' \hat{\otimes}_L L' \xrightarrow{\pi_{L'/L}} V.$$

It is easy to verify that this assignment is functorial. Note that the induced map of sets $f : V' \to V$ is also continuous with respect to the Berkovich topology.

2.9 Remark. From now on we usually write simply V instead of (V, L) for an analytic space over K since the field of definition can usually be inferred from the context. Similarly, we do not distinguish notationally between a morphism $f : V' \to V$ of analytic spaces over K and the underlying morphism of ringed G-topological spaces.

2.10 Characters Corresponding to Points. Let V be a K-analytic space and let $x \in V$ be a point. By [Ber93, Sec. 1.4, p. 30] there is a canonical associated extension of non-archimedean fields $\mathcal{H}(x)/K$ together with a morphism of analytic spaces over K

$$i_x : \mathcal{M}(\mathcal{H}(x)) \to V$$

(or more precisely $(\mathcal{M}(\mathcal{H}(x)), \mathcal{H}(x)) \to (V, K)$) such that the underlying morphism of G-topological spaces maps the unique point of $\mathcal{M}(\mathcal{H}(x))$ to x.

We see in particular that any point $x \in V$ arises as the image of a morphism $\mathcal{M}(L) \to V$ for some non-archimedean extension field L/K. Note that if the image of $\mathcal{M}(L) \to V$ is $x \in V$ then there is an induced embedding $\mathcal{H}(x) \hookrightarrow L$ of non-archimedean extension fields of K such that the diagram

$$\begin{tikzcd} \mathcal{M}(L) \arrow[r] \arrow[dr] & \mathcal{M}(\mathcal{H}(x)) \arrow[d, "i_x"] \\ & V \end{tikzcd}$$

commutes. If $\mathcal{M}(L') \to V$ is another such morphism mapping to the same point $x \in V$, then there exists a third non-archimedean extension field L'' extending both L and L' such that the diagram

$$\begin{tikzcd} \mathcal{M}(L'') \arrow[r] \arrow[d] & \mathcal{M}(L) \arrow[d] \\ \mathcal{M}(L') \arrow[r] & V \end{tikzcd}$$

commutes. (To see this, note that by [Gru66, § 3, Thm. 1], $L \otimes_{\mathcal{H}(x)} L'$ injects into $L \hat{\otimes}_{\mathcal{H}(x)} L'$ so that in particular the latter Banach ring is non-zero and hence has non-empty Berkovich spectrum. Choose L'' to be the completed residue field of some point of $\mathcal{M}(L \hat{\otimes}_{\mathcal{H}(x)} L')$.)

This shows that there is a canonical bijection of sets

$$\operatorname*{colim}_{L/K} \operatorname{Hom}(\mathcal{M}(L), V) \xrightarrow{\sim} V \tag{2.10.1}$$

where L/K runs over the non-archimedean extension fields of K. Note here that the index category is not small, so a priori the colimit need not exist, but our argument shows that in this case it does indeed exist and is given by the set on the right hand side.

2.11 Fibers. Let $f : V' \to V$ be a morphism of K-analytic spaces and let $x \in V$ be a point. Suppose that $x' \in V'$ is given as the image of a morphism $\mathcal{M}(L) \to V'$ where L/K is a non-archimedean extension field of K. If $f(x') = x$ then by Paragraph 2.10 there is an induced embedding $\mathcal{H}(x) \hookrightarrow L$ of non-archimedean extension fields of K such that the diagram

$$\begin{tikzcd} \mathcal{M}(L) \arrow[r] \arrow[d] & \mathcal{M}(\mathcal{H}(x)) \arrow[d, "i_x"] \\ V' \arrow[r, "f"'] & V \end{tikzcd} \tag{2.11.1}$$

commutes. This shows that every point $x' \in f^{-1}(x)$ arises as the image of a morphism $\mathcal{M}(L) \to V'$ where $L/\mathcal{H}(x)$ is a non-archimedean extension field and the resulting diagram Eq. (2.11.1) commutes. By the same argument as in Paragraph 2.10 we see that there is a canonical bijection of sets

$$\operatorname*{colim}_{L/\mathcal{H}(x)} \operatorname{Hom}_V(\mathcal{M}(L), V') \xrightarrow{\sim} f^{-1}(x) \tag{2.11.2}$$

where the colimit runs over all non-archimedean extension fields of $\mathcal{H}(x)$ and $\operatorname{Hom}_V(\mathcal{M}(L), V')$ denotes the set of morphisms *over* V, i.e. making (2.11.1) commutative.

The preimage $f^{-1}(x)$ carries the structure of a $\mathscr{H}(x)$-analytic space which can be described as follows: Recall from Paragraph 2.10 that we have the morphism of analytic spaces over K

$$i_x : (\mathscr{M}(\mathscr{H}(x)), \mathscr{H}(x)) \to (V, K)$$

which is given by the field extension $\mathscr{H}(x)/K$ and a morphism of $\mathscr{H}(x)$-analytic spaces $\mathscr{M}(\mathscr{H}(x)) \to V \otimes_K \mathscr{M}(\mathscr{H}(x))$. Furthermore, the morphism $f : V' \to V$ induces a morphism $V' \hat{\otimes}_K \mathscr{H}(x) \to V \hat{\otimes}_K \mathscr{H}(x)$ of $\mathscr{H}(x)$-analytic spaces and by Paragraph 2.1 we can form the fibered product of $\mathscr{H}(x)$-analytic spaces

$$V'_x := (V' \hat{\otimes}_K \mathscr{H}(x)) \times_{V \hat{\otimes}_K \mathscr{H}(x)} \mathscr{M}(\mathscr{H}(x))$$

to obtain a $\mathscr{H}(x)$-analytic space called the *fiber of f over x*. There is a canonical identification of topological spaces $V'_x \cong f^{-1}(x)$, where $f^{-1}(x)$ is equipped with the subspace topology [Ber93, Sec. 1.4, p. 30].

2.12 Lemma. *Let V be a K-analytic space, let $V = \bigcup_{i \in I} V_i$ be a G-covering by analytic domains and let Y be a topological space. If $f : V \to Y$ is a map of the underlying sets such that the restriction to each of the V_i is continuous with respect to the Berkovich topology, then f is continuous.*

Proof. Let $x \in V$ and let $f(x) \in W \subset Y$ be an open neighborhood of $f(x)$. There exists a finite subset $J \subset I$ such that $x \in \bigcap_{i \in J} V_i$ and such that $\bigcup_{i \in J} V_i$ is a neighborhood of x in V. Hence there exists an open subset U of V such that $x \in U \subset \bigcup_{i \in J} V_i$.

As $f|_{V_i}$ is continuous for each i, there exist open neighborhoods U_i of x in V such that $f(U_i \cap V_i) \subset W$. It follows that $U \cap \bigcap_{i \in J} U_i$ is an open neighborhood of x in V which is mapped into W by f. ∎

2.13 Compact and Paracompact K-analytic Spaces. Let V be a K-analytic space. It is called *compact* if its underlying topological space is compact (in particular Hausdorff) and *paracompact* if its underlying topological space is paracompact. Recall that "paracompact" means that V is Hausdorff and every open covering of V admits a locally finite open refinement.

2.14 Lemma. *Let V be a K-analytic space.*

(i) Any locally finite covering $V = \bigcup_{i \in I} V_i$ by closed analytic domains is a G-covering.

(ii) If V is compact, then any G-covering $V = \bigcup_{i \in I} V_i$ of V by analytic domains admits a finite sub-covering (which is a G-covering if the V_i are closed). In particular, any compact analytic space admits a finite G-covering by affinoid domains.

(iii) *Let V be a paracompact K-analytic space. Then any G-covering $V = \bigcup_{i\in I} V_i$ of V by analytic domains admits a locally finite G-covering by affinoid domains as a refinement. In particular, any paracompact K-analytic space admits a locally finite G-covering by affinoid domains.*

Proof. In order to prove the first statement, we consider an arbitrary point $x \in V$. By local finiteness, there exists an open neighborhood U of $x \in V$ such that the set $J := \{ i \in I \mid U \cap V_i \neq \varnothing \}$ is finite. Let $J_1 := \{ i \in J \mid x \in V_i \}$ and $J_2 := \{ i \in J \mid x \notin V_i \}$. Then $x \in \bigcap_{i\in J_1} V_i$ and $x \in U \setminus (\bigcup_{i\in J_2} V_i) \subset \bigcup_{i\in J_1} V_i$ proves that the latter set is a neighborhood of $x \in V$. This proves (i).

Next we assume that V is compact and that $V = \bigcup_{i\in I} V_i$ is a G-covering by analytic domains. By the definition of a G-covering, any point $x \in V$ has a neighborhood of the form $V_{i_1} \cup \cdots \cup V_{i_n}$ with $i_1, \dots, i_n \in I$. Therefore the topological interiors of these finite unions form an open covering of V. As V is compact, there exists a finite sub-covering of this open covering, and in particular V is covered by finitely many of the V_i.

If we start with a G-covering $V = \bigcup_{i\in I} V_i$ by affinoid domains and pick a finite sub-covering, we see that V can be covered by finitely many affinoid domains. Since V is compact and in particular Hausdorff, and affinoid spaces are compact, the V_i are closed in V, so the finite covering is indeed a G-covering. This proves (ii).

Now let us assume that V is paracompact and that $V = \bigcup_{i\in I} V_i$ is a G-covering by analytic domains. By the first part of this lemma it is enough to find a locally finite refinement consisting of affinoid domains because it will automatically be a G-covering. In fact, by the second part of this lemma, it is enough to find a locally finite refinement consisting of compact analytic domains. Recall that K-analytic spaces are always locally compact by [Tem15, Rem. 4.1.2.3]. Hence, by [Bou98, Chap. 1, § 9.10, Thm. 5], V can be written as a disjoint union $V = \bigcup_{j\in I} U_j$ of open σ-compact subspaces. (Recall that a space is *σ-compact* if it is locally compact and can be written as a countable union of compact subsets.) It is enough to show for each of the σ-compact spaces U_j that the G-covering $U_j = \bigcup_{i\in I}(U_j \cap V_i)$ can be refined to a locally finite covering consisting of compact analytic domains. Hence, we may assume that V is σ-compact. Furthermore, writing each V_i as a G-union of compact analytic domains, we may assume from the start that all the V_i are already compact.

By [Bou98, Chap. 1, §9.9, Prop. 15], there is a sequence $\{U_n\}_{n\in\mathbf{N}}$ of relatively compact open subsets of V which cover V and such that $\overline{U_n} \subset U_{n+1}$ for each n. For each $n \in \mathbf{N}$, we denote by K_n the compact set $\overline{U_n} \setminus U_{n-1}$ (we set $U_0 := \varnothing$). The open set $U_{n+1} \setminus \overline{U_{n-2}}$ is a neighborhood of K_n by construction. By [Tem15, Fact 4.3.1.1], any point in a K-analytic space has a fundamental system of neighborhoods consisting of compact analytic domains. Hence, using that the V_i form a G-covering, given any point $x \in K_n$, there exists a compact analytic domains $W_x^{(n)}$

which is a neighborhood of $x \in V$ and such that $W_x^{(n)} \subset (\bigcup_{i \in J_x} V_i) \cap (U_{n+1} \setminus \overline{U_{n-2}})$ where $J_x \subset I$ is some finite subset. Since K_n is compact, a finite number of the $W_x^{(n)}$ cover K_n, i.e. we have $K_n \subset \bigcup_{x \in A_n} W_x^{(n)}$ where $A_n \subset K_n$ is a finite set of points. Now let Λ be the set of all triples (n, x, i) where $n \in \mathbf{N}$, $x \in A_n$ and $i \in J_x$. Then we claim that $V = \bigcup_{(n,x,i) \in \Lambda} (W_x^{(n)} \cap V_i)$ and that this is a locally finite covering of V.

Now we consider an arbitrary point $z \in V$ and consider the minimal $n \in \mathbf{N}$ such that $z \in U_n$. By minimality, $z \notin U_{n-1}$ and hence $z \in K_n$. It follows from $K_n \subset \bigcup_{x \in A_n} W_x^{(n)}$ that $z \in W_x^{(n)}$ for some $x \in A_n$. Finally it follows from $W_x^{(n)} \subset \bigcup_{i \in J_x} V_i$ that $z \in W_x^{(n)} \cap V_i$ for some $i \in J_x$. This proves that we have indeed a covering. To prove local finiteness, we consider the neighborhood $T := U_n \setminus \overline{U_{n-2}}$ of z and note that it only meets the finitely many sets $W_x^{(m)} \cap V_i$ with $(m, x, i) \in \Lambda$ and $n - 2 \leq m \leq n + 1$. ∎

2.15 Good *K*-Analytic Spaces. We already remarked in the proof of Lemma 2.14 that any point $x \in V$ of a K-analytic space has a fundamental system of neighborhoods consisting of compact analytic domains. If any point $x \in V$ has a fundamental system of neighborhoods consisting of affinoid domains, then V is called *good*. In [Ber90] a notion of K-analytic spaces different from the one in [Ber93] was introduced. The category of good K-analytic spaces in the sense of [Ber93] is equivalent to the category of K-analytic spaces from [Ber90] by [Ber93, Sec. 1.5].

2.16 Analytification. If X is a locally finite type K-scheme, then there is an associated K-analytic space X^{an} called the *analytification* of X. The analytification of a finite type K-scheme is always good. There is a canonical morphism

$$(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$$

of locally ringed spaces. Composing with the morphism $\pi_{X^{\mathrm{an}}} : (X_G^{\mathrm{an}}, \mathcal{O}_{X_G^{\mathrm{an}}}) \to (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ from Paragraph 2.5, we obtain a morphism of G-topological spaces

$$\pi_X : (X_G^{\mathrm{an}}, \mathcal{O}_{X_G^{\mathrm{an}}}) \to (X, \mathcal{O}_X).$$

The analytification functor is compatible with fibered products and change of base field. Details can be found in [Ber90, Sec. 3.4].

2.17 Lemma. *(i) If V is a K-analytic space and L/K is an extension of non-archimedean fields, then the canonical base-change morphism*

$$\pi_{L/K} : V \hat{\otimes}_K L \to V$$

of Paragraph 2.8 is topologically proper and surjective.

(ii) *If V, V' are two K-analytic spaces and if we denote by $|-|$ the underlying topological space of a K-analytic space, then the canonical map*

$$|V \times V'| \to |V| \times |V'|$$

is proper.

Proof. Let us start with the first statement. Since properness can be checked locally with respect to an open covering of the base space and a K-analytic space admits a basis of open paracompact subsets by [Ber93, Rem. 1.2.4], we may assume that V is paracompact. In that case, by Lemma 2.14, V admits a locally finite G-covering by affinoid domains. By [Bou98, Chp. 1, § 10.1, Prop. 3], we can now assume that $V = \mathcal{M}(A)$ is affinoid. In that case, $V \hat{\otimes}_K L = \mathcal{M}(A \hat{\otimes}_K L)$ is a compact Hausdorff space, so the map $V \hat{\otimes}_K L \to V$ is proper.

As for the surjectivity, if $x \in V$, then the fiber of $\pi_{L/K}$ over x is given by the Berkovich spectrum $\mathcal{M}(\mathcal{H}(x) \hat{\otimes}_K L)$ which is non-empty as by [Gru66, § 3, Thm. 1], $\mathcal{H}(x) \otimes_K L$ injects into $\mathcal{H}(x) \hat{\otimes}_K L$ so that, in particular, the latter ring is non-zero.

To prove properness of the map $|V \times V'| \to |V| \times |V'|$, we can similarly reduce to the case where both V and V' are K-affinoid, in which case the statement is trivial. ∎

2.18 Flat Morphisms. We recall from [Duc18, Sec. 4.1] the notion of *flatness*. Let $f: V' \to V$ be a morphism of K-analytic spaces, $\mathcal{F}$ a coherent sheaf on V and $y \in V$ a point with $x := f(y) \in V$. First assume that V, V' are good. Then we say that $\mathcal{F}$ is *naively V-flat* at y if the stalk $\mathcal{F}_y$ is a flat $\mathcal{O}_{V,x}$-module [Duc18, § 4.1.1].

We say that $\mathcal{F}$ is *V-flat* at y if for every cartesian commutative diagram

$$\begin{tikzcd} W' \arrow[r, "p_2"] \arrow[d, "p_1"'] & W \arrow[d] \\ V' \arrow[r, "f"'] & V \end{tikzcd}$$

of good K-analytic spaces and every $z \in W'$ lying over $y \in V'$ the sheaf $p_1^*\mathcal{F}$ on W' is naively W-flat at z [Duc18, Def. 4.1.2].

Finally, if V, V' are not assumed to be good, we say that $\mathcal{F}$ is *V-flat* at y if there exists an affinoid domain U' of V' containing y and an affinoid domain U of V containing $f(U')$ such that the coherent sheaf $\mathcal{F}|_{U'}$ is U-flat at y [Duc18, Def. 4.1.8].

We say that the coherent sheaf $\mathcal{F}$ is *V-flat* if it is V-flat at every point $y \in V'$. We say that the morphism $f: V' \to V$ is flat if the coherent sheaf $\mathcal{O}_{V'_G}$ is V-flat.

2.19 Interior and Boundary. Recall from [Ber90, Def. 2.5.7] the notion of *relative interior* $\operatorname{Int}(V'/V) \subset V'$ of a morphism of K-affinoid spaces $f : V' \to V$. If $f : V' \to V$ is more generally a morphism of K-analytic spaces, then a point $y \in V'$ with $x := f(y) \in V$ is called *interior* with respect to f if for any affinoid domain $U \subset V$ containing x, there exists an affinoid domain $U' \subset f^{-1}(U)$ such that U' is a neighborhood of y in $f^{-1}(U)$ and $y \in \operatorname{Int}(U'/U)$ [Tem15, Def. 4.2.4.1]. We write $\operatorname{Int}(V'/V)$ for the set of interior points of V' with respect to V. The morphism $f : V' \to V$ is called *boundaryless* if $\operatorname{Int}(V'/V) = V'$.

2.20 Proposition (Berkovich, Ducros)**.** *If $f : V' \to V$ is a flat and boundaryless morphism of K-analytic spaces then it is topologically open.*

Proof. By [Ber93, Lem. 1.1.1 (i)] a subset of V is open if and only if its intersection with every affinoid domain W is open in W. It follows that it is enough to show that for every affinoid domain $W \subset V$ the morphism $f : f^{-1}(W) \to W$ is topologically open. By [Duc18, § 4.1.12] if $W \subset V'$ is an affinoid domain then $f : f^{-1}(W) \to W$ is flat. By [Tem15, Fact 4.2.4.3 (ii)] if $W \subset V'$ is an affinoid domain then $f : f^{-1}(W) \to W$ is boundaryless. It follows that we may assume that V is a K-affinoid space and in particular good.

Now let $U' \subset V'$ be an open subset; we want to show that $f(U') \subset V$ is open. Let $x \in f(U')$ be a point. We will show that $f(U')$ is a neighborhood of x in V. Choose $y \in V'$ such that $x = f(y)$. Let $W \subset V$ be an affinoid neighborhood of x in V. Since f is boundaryless, there exists an affinoid domain $W' \subset f^{-1}(W)$ such that W' is a neighborhood of y in $f^{-1}(W)$ and $y \in \operatorname{Int}(W'/W)$. Again by [Duc18, § 4.1.12] the coherent sheaf $\mathcal{O}_{W'_G}$ on W' is W-flat. Furthermore, the support of $\mathcal{O}_{W'_G}$ is all of W', so by [Duc18, Thm. 9.2.3] applied to the morphism $f : W' \subset f^{-1}(W) \to W$, the image $f(W')$ is a neighborhood of x in W. Since W is a neighborhood of x in V it follows that $f(W')$ and in particular also $f(U')$ is a neighborhood of x in V. ∎

2.21 Proper Morphisms. A morphism $f : V' \to V$ of K-analytic spaces is called *proper* if it is boundaryless and the preimage of every compact analytic domain of V is compact in V' [Tem15, Def. 4.2.4.1 (ii)].

2.22 Remark. Note that a map between general topological spaces such that preimages of compact sets are compact need not be proper in the sense of [Bou98, Chap. 1, § 10.1, Def. 1], even though this is true if the source is Hausdorff and the target is locally compact. However in the context of K-analytic spaces, properness in the sense of Paragraph 2.21 is enough to imply topological properness by Lemma 2.23 below.

2.23 Lemma. *A proper morphism $f : V' \to V$ of K-analytic spaces is topologically proper.*

Proof. Note that by [Ber93, Rem. 1.2.4], V admits an open covering by paracompact open subsets and by Lemma 2.14 (iii) every paracompact K-analytic space admits a locally finite G-covering by affinoid domains. By [Bou98, Chap. 1, § 10.1, Prop. 3] topological properness of a map can be checked after restricting to the covering sets of an open covering of the base or of a locally finite closed covering of the base. Since properness of a morphism of K-analytic spaces is stable under pull-back by [Tem15, Fact 4.2.4.3 (iii)] we may assume that V is a K-affinoid space and in particular Hausdorff. By the definition of properness, $V' = f^{-1}(V)$ is a compact K-analytic space and by [Bou98, Chap. 1, § 10.1, Cor. 2] it is topologically proper over V. ∎

3. Formal Geometry

Throughout this chapter, K denotes a non-archimedean field. We recall some facts and constructions related to *admissible formal schemes* over the valuation ring K°. Vector bundles on admissible formal schemes serve as *models* of vector bundles on Berkovich spaces. We will focus on vector bundles on admissible formal schemes and their associated metrics in Chapter 6.

3.1 Admissible Formal Schemes. A topological K°-algebra $\mathscr{A}$ is called *admissible* if it is topologically finitely generated (i.e. topologically isomorphic to a quotient of the Tate algebra

$$K^\circ\{T_1, \dots, T_n\} = \Big\{ \sum_{i \in \mathbf{N}^n} a_i T^i \mid a_i \in K^\circ, \lim_{|i| \to \infty} |a_i| = 0 \Big\}$$

over K°) and flat over K°.

An *admissible formal K°-scheme* is a formal scheme over K° which admits a locally finite covering by open subsets isomorphic to $\mathrm{Spf}(\mathscr{A})$ for admissible K°-algebras $\mathscr{A}$. Note that these are called *quasi-paracompact* admissible formal schemes in [Bos14, p. 204]. We include the quasi-paracompactness in the definition so that we can speak about the generic fiber as a Berkovich K-analytic space. With this convention we are following e.g. [Gub07, § 2.6].

3.2 Generic Fiber. There is a functor $\mathfrak{V} \mapsto \mathfrak{V}_\eta$ from admissible formal schemes over K° to strictly K-analytic spaces given on affine admissible formal K°-schemes by

$$\mathrm{Spf}(\mathscr{A})_\eta \coloneqq \mathscr{M}(A \otimes_{K^\circ} K)$$

and extended to arbitrary admissible formal K°-schemes by gluing. Note that $\mathfrak{V}_\eta$ is always paracompact, so by [Ber93, Thm. 1.6.1], we can equivalently view it as a quasi-paracompact, quasi-separated rigid K-analytic space. By [Bos14, Thm. 8.4/3], the generic fiber functor induces an equivalence between the category of admissible formal schemes, localized at the class of admissible blow-ups and the category of paracompact strictly K-analytic spaces.

3.3 Remark. Let $\mathfrak{V}$ be an admissible formal K°-scheme. The generic fiber $\mathfrak{V}_\eta$ is not a subspace of $\mathfrak{V}$ in a precise sense (e.g. the sense of G-topological spaces).

However, if we denote by $\mathscr{C}(\mathfrak{V})$ the site associated to $\mathfrak{V}$ (with objects given by the open subsets of $\mathfrak{V}$), and by $\mathscr{C}((\mathfrak{V}_\eta)_G)$ the site associated to $(\mathfrak{V}_\eta)_G$ (with objects given by the analytic domains of $\mathfrak{V}_\eta$), then we do have a canonical morphism of sites ([Sta23, Def. 00X1])

$$i\colon \mathscr{C}((\mathfrak{V}_\eta)_G) \to \mathscr{C}(\mathfrak{V}).$$

It is given by mapping an open subset $\mathfrak{U} \subset \mathfrak{V}$ to the analytic domain $\mathfrak{U}_\eta \hookrightarrow \mathfrak{V}_\eta$. Viewing $\mathscr{C}((\mathfrak{V}_\eta)_G)$ and $\mathscr{C}(\mathfrak{V})$ as ringed sites, with structure sheaves given by the structure sheaf of $\mathfrak{V}_\eta$, resp. the structure sheaf of $\mathfrak{V}$, we can even view i as a morphism of ringed sites. The morphism between the structure sheaves is locally given by the ring homomorphisms

$$\mathscr{A} \to \mathscr{A} \otimes_{K^\circ} K, \qquad f \mapsto f \otimes 1.$$

3.4 Lemma. *Let $\mathscr{A}$ be an admissible K°-algebra. Then the canonical ring homomorphism $\mathscr{A} \to \mathscr{A} \otimes_{K^\circ} K$ maps $\mathscr{A}$ into the subring $(\mathscr{A} \otimes_{K^\circ} K)^\circ$ of power-bounded elements.*

Proof. Choosing a surjective homomorphism $K^\circ\{T_1, \dots, T_r\} \twoheadrightarrow \mathscr{A}$, we get a surjective homomorphism $K\{T_1, \dots, T_r\} \twoheadrightarrow \mathscr{A} \otimes_{K^\circ} K$ fitting into a commutative diagram

$$\begin{tikzcd}
K^\circ\{T_1, \dots, T_r\} \arrow[r] \arrow[d, two heads] & K\{T_1, \dots, T_r\} \arrow[d, two heads] \\
\mathscr{A} \arrow[r] & \mathscr{A} \otimes_{K^\circ} K.
\end{tikzcd}$$

Since $K^\circ\{T_1, \dots, T_n\}$ maps onto the power-bounded elements of $K\{T_1, \dots, T_r\}$ and homomorphisms of K-affinoid algebras map power-bounded elements to power-bounded elements, we see that $\mathscr{A}$ is mapped into the power-bounded elements of $\mathscr{A} \otimes_{K^\circ} K$. ∎

3.5 Special Fiber and Reduction Map. There is a functor $\mathfrak{B} \mapsto \widetilde{\mathfrak{B}}$ from admissible formal K°-schemes to $\widetilde{K}$-schemes of locally finite type given for affine admissible formal K°-schemes by

$$\widetilde{\mathrm{Spf}(\mathscr{A})} := \mathrm{Spec}(\mathscr{A} \otimes_{K^\circ} \widetilde{K}) = \mathrm{Spec}(\mathscr{A}/K^{\circ\circ}\mathscr{A})$$

and extended to arbitrary admissible formal K°-schemes by gluing. The $\widetilde{K}$-scheme $\widetilde{\mathfrak{B}}$ is called the *special fiber* of the admissible formal K°-scheme $\mathfrak{B}$.

For every admissible formal scheme $\mathfrak{B}$, there is a canonical map of sets

$$\mathrm{red}\colon \mathfrak{B}_\eta \to \widetilde{\mathfrak{B}}.$$

It is given locally by

$$\mathscr{M}(\mathscr{A} \otimes_{K^\circ} K) \to \mathrm{Spec}(\mathscr{A}/K^{\circ\circ}\mathscr{A}), \qquad x \mapsto \{\, f \in \mathscr{A} \mid |(f \otimes 1)(x)| < 1 \,\}/K^{\circ\circ}\mathscr{A}.$$

The reduction map is functorial in $\mathfrak{B}$, i.e. if $\mathfrak{f}\colon \mathfrak{B}' \to \mathfrak{B}$ is a morphism of admissible formal K°-schemes, then the induced diagram

$$\begin{tikzcd} \mathfrak{B}'_\eta \arrow[r] \arrow[d, "\mathrm{red}"'] & \mathfrak{B}_\eta \arrow[d, "\mathrm{red}"] \\ \widetilde{\mathfrak{B}'} \arrow[r] & \widetilde{\mathfrak{B}} \end{tikzcd}$$

commutes.

3.6 Formal Models. If V is a paracompact strictly K-analytic space, then an admissible formal K°-scheme $\mathfrak{B}$ together with a fixed isomorphism $\mathfrak{B}_\eta \xrightarrow{\sim} V$ is called a *formal model* for V. It follows from [Bos14, Thm. 8.4/3] that any paracompact strictly K-analytic space admits a formal model. If $\mathfrak{B}$, $\mathfrak{B}'$ are two formal models for V, then a morphism of models from $\mathfrak{B}'$ to $\mathfrak{B}$ is a morphism $\mathfrak{f}\colon \mathfrak{B}' \to \mathfrak{B}$ of formal K°-schemes whose restriction to the generic fiber is the identity $\mathrm{id}_V\colon V \to V$. The category of formal models for V is directed [GM19, § 2.2].

If X is a K-scheme of finite type, then a *formal model* for X is by definition a formal model for X^{an}.

If $\mathfrak{B}$ is a formal model for the paracompact strictly K-analytic space V and $\mathfrak{U} \hookrightarrow \mathfrak{B}$ is the embedding of an open subset of $\mathfrak{B}$, then $\mathfrak{U}_\eta \hookrightarrow \mathfrak{B}_\eta$ is an analytic domain embedding. We denote the corresponding analytic domain of V under the isomorphism $\mathfrak{B}_\eta \xrightarrow{\sim} V$ by $\mathfrak{U} \cap V$. By construction $\mathfrak{U}$ is a formal model for $\mathfrak{U} \cap V$. In particular, we have $\mathfrak{B} \cap V = V$.

By Remark 3.3 above, the assignment $\mathfrak{U} \mapsto \mathfrak{U} \cap V$ defines a morphism of sites $i\colon \mathscr{C}(V_G) \to \mathscr{C}(\mathfrak{V})$ which is part of a canonical morphism of ringed sites

$$i\colon (\mathscr{C}(V_G), \mathscr{O}_{V_G}) \to (\mathscr{C}(\mathfrak{V}), \mathscr{O}_{\mathfrak{V}}).$$

If $\mathfrak{U} \subset \mathfrak{V}$ is an open subset, we use the notation

$$\Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{V}}) \to \Gamma(\mathfrak{U} \cap V, \mathscr{O}_{V_G}), \qquad f \mapsto f|_{\mathfrak{U}\cap V},$$

for the canonical ring homomorphism.

If $x \in \mathfrak{U} \cap V$, then we write $f(x) \coloneqq f|_{V\cap\mathfrak{U}}(x) \in \mathscr{H}(x)$. Note that by Lemma 3.4 we have $|f(x)| \leq 1$ for all $x \in \mathfrak{U} \cap V$ and $f \in \Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{V}})$.

3.7 Lemma. *Assume that K is algebraically closed. Let V be a reduced paracompact strictly K-analytic space and $\mathfrak{V}$ a formal model for V. Then there exists a formal model $\mathfrak{V}'$ and a morphism of formal models $\mathfrak{f}\colon \mathfrak{V}' \to \mathfrak{V}$ such that the special fiber $\widetilde{\mathfrak{V}'}$ of $\mathfrak{V}'$ is a reduced scheme.*

Proof. The proof is essentially an application of [BL86, Lem. 1.1]. If $\mathfrak{V} = \mathrm{Spf}(\mathscr{A})$ is affine, then we define the formal scheme $\mathfrak{V}'$ by $\mathfrak{V}' = \mathrm{Spf}((\mathscr{A} \otimes_{K^\circ} K)^\circ)$. By Lemma 3.4, we have a canonical morphism $\mathfrak{f}\colon \mathfrak{V}' \to \mathfrak{V}$. In general, we construct the formal scheme $\mathfrak{V}'$ and the morphism $\mathfrak{f}\colon \mathfrak{V}' \to \mathfrak{V}$ by gluing from the affine case. In the notation of [BL86, Sec. 1], we have $\mathfrak{V}' = (\mathfrak{V}^{\text{f-an}})^{\text{f-sch}}$.

We want to argue that $\mathfrak{V}'$ is an admissible formal scheme with reduced special fiber and that the canonical morphism $\mathfrak{f}\colon \mathfrak{V}' \to \mathfrak{V}$ induces an isomorphism on the generic fibers. For this we can assume that $\mathfrak{V} = \mathrm{Spf}(\mathscr{A})$ is affine.

By definition, $\mathfrak{V}^{\text{f-an}}$ is the formal analytic variety associated to the strictly K-affinoid algebra $\mathscr{A} \otimes_{K^\circ} K$. Since $V = \mathscr{M}(\mathscr{A} \otimes_{K^\circ} K)$ is reduced, the algebra $\mathscr{A} \otimes_{K^\circ} K$ is reduced and since K is algebraically closed, it is distinguished by [BGR84, Thm. 6.4.3/1]. By [BL86, Lem. 1.1], $(\mathfrak{V}^{\text{f-an}})^{\text{f-sch}}$ is an admissible formal K°-scheme with reduced special fiber. The morphism on generic fibers induced by $\mathfrak{f}$ is given by the canonical map

$$\mathscr{A} \otimes_{K^\circ} K \to (\mathscr{A} \otimes_{K^\circ} K)^\circ \otimes_{K^\circ} K,$$

which is obviously an isomorphism. ∎

3.8 Algebraic Models. Let X be a proper K-scheme. An *algebraic K°-model* for X is a proper flat K°-scheme $\mathscr{X}$ together with a fixed isomorphism $\mathscr{X}_\eta \xrightarrow{\sim} X$. Here $\mathscr{X}_\eta \coloneqq X \times_{K^\circ} \mathrm{Spec}(K)$ denotes the generic fiber of $\mathscr{X}$. In that case the formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along the special fiber is a formal model for X [Bos14, Exmp. 7.2/4].

4. Vector Bundles on Berkovich Spaces

Throughout this section, V will denote a K-analytic space. We define vector bundles on V as locally free sheaves in the G-topology. We discuss operations of pull-back, change of base field, tensor product, dual bundle etc. for vector bundles. Particularly important will be the construction of the total space of a vector bundle as defined in Paragraph 4.6 because of its use in our definition of continuous metrics as well as the construction of the projective bundle and the tautological line bundle as defined in Paragraph 4.28 because of their relation with the Fubini-Study metric.

4.1 Vector Bundles. A *vector bundle* of rank r on V is a G-locally free sheaf of $\mathcal{O}_{V_G}$-modules on V_G of rank r. In other words, it is an $\mathcal{O}_{V_G}$-module E such that there exists a G-covering $V = \bigcup_{i \in I} V_i$ by analytic domains such that $E|_{V_i} \cong \mathcal{O}^r_{(V_i)_G}$ for all $i \in I$ where $E|_{V_i}$ denotes the restriction of the sheaf E to $(V_i)_G$. We do not consider vector bundles of non-constant rank. If $U \subset V$ is an analytic domain and $s_1, \dots, s_r \in \Gamma(U, E)$ are sections inducing an isomorphism $E|_U \cong \mathcal{O}^r_{U_G}$, then we call $(s_1, \dots, s_r)$ a *frame* for E over U.

A *morphism* of vector bundles $\alpha\colon E \to F$ on V is a morphism of sheaves of $\mathcal{O}_{V_G}$-modules.

Recall from Paragraph 2.3 that there is a canonical morphism $\pi_V\colon (V_G, \mathcal{O}_{V_G}) \to (V, \mathcal{O}_V)$ of ringed G-topological spaces. It follows from [Ber93, Prop. 1.3.4 (iii)] that if V is good, then this morphism induces an equivalence from the category of locally free $\mathcal{O}_V$-modules to the category of locally free $\mathcal{O}_{V_G}$-modules, i.e. the

Berkovich topology can be used instead of the G-topology in the definition of a vector bundle. In particular, if V is good and E is a vector bundle on V, then there exists a covering $V = \bigcup_{i\in I} U_i$ by open subsets U_i such that $E|_{U_i}$ admits a frame for each $i \in I$.

4.2 Pull-backs. Let $f : V' \to V$ be a morphism of K-analytic spaces (or more generally of analytic spaces over K) and let E be a vector bundle on V. By Paragraph 2.5 (resp. Paragraph 2.8), f induces a morphism $f : (V'_G, \mathcal{O}_{V'_G}) \to (V_G, \mathcal{O}_{V_G})$ of ringed G-topological spaces. The sheaf f^*E defined in Paragraph 2.3 is then a vector bundle on V' which we call the *pull-back* of E along f.

The unit morphism $E \to f_* f^* E$ of the adjunction between f_* and f^* gives us for every analytic domain $U \subset V$ a map

$$f^* : \Gamma(U, E) \to \Gamma(f^{-1}(U), f^*E), \qquad s \mapsto f^*s.$$

It is clear that if $s_1, \dots, s_r \in \Gamma(U, E)$ form a frame of E over U, then $f^*s_1, \dots, f^*s_r$ form a frame for f^*E over $f^{-1}U$.

If $g : V'' \to V'$ is another morphism of analytic spaces over K, then by [Sta23, Lem. 03D8] there is a canonical isomorphism

$$(fg)^*E \xrightarrow{\sim} g^* f^* E \tag{4.2.1}$$

which we often use to identify $(fg)^*E$ and g^*f^*E.

4.3 Vector Bundles over a Point. Let $V = \mathcal{M}(K)$ be a point. Then a vector bundle on V is simply a finite-dimensional K-vector space. More precisely, the assignment $E \mapsto \Gamma(V, E)$ is an equivalence of categories from the category of vector bundles on V to the category of finite-dimensional K-vector spaces.

If L/K is an extension of non-archimedean fields, then the pull-back functor for vector bundles along the morphism $f : \mathcal{M}(L) \to \mathcal{M}(K)$ of analytic spaces over K corresponds to the scalar extension $E \mapsto E \otimes_K L$ of finite dimensional K-vector spaces. Under this identification the unit map $E \to f_* f^* E$ of Paragraph 4.2 corresponds to the embedding $E \to E \otimes_K L$, $v \mapsto v \otimes 1$.

Indeed, by construction the push-forward functor f_* corresponds to restriction of scalars and the scalar extension functor for vector spaces with the specified unit map is left adjoint to this functor.

4.4 Fiber Vector Spaces. If $x \in V$ is a point, then we have by Paragraph 2.10 a canonical morphism of analytic spaces over K which we denote by

$$i_x : \mathcal{M}(\mathcal{H}(x)) \to V.$$

If E is a vector bundle on V, then it pulls back to the vector bundle i_x^*E on $\mathcal{M}(\mathcal{H}(x))$ which can be identified with the finite-dimensional $\mathcal{H}(x)$-vector space

$$E(x) := \Gamma(\mathcal{M}(\mathcal{H}(x)), i_x^*E)$$

by Paragraph 4.3. The finite-dimensional $\mathscr{H}(x)$-vector space $E(x)$ is called the *fiber* of E over x.

Obviously, a morphism $\alpha\colon E \to F$ of vector bundles induces for each $x \in V$ an $\mathscr{H}(x)$-linear map $\alpha(x)\colon E(x) \to F(x)$.

If $s \in \Gamma(V, E)$ is a section, then we write $s(x) := i_x^* s \in E(x)$.

4.5 Lemma. *Let $f\colon V' \to V$ be a morphism of analytic spaces over K, $x' \in V'$ and let E be a vector bundle on V. Then there is a canonical morphism of completed residue fields $\mathscr{H}(f(x')) \to \mathscr{H}(x')$ and a canonical isomorphism*

$$(f^*E)(x') \cong E(f(x')) \otimes_{\mathscr{H}(f(x'))} \mathscr{H}(x').$$

If $s \in \Gamma(V, E)$ is a section, then under this isomorphism the element $(f^*s)(x') \in (f^*E)(x')$ corresponds to the element $s(f'(x)) \otimes 1 \in E(f(x')) \otimes_{\mathscr{H}(f(x'))} \mathscr{H}(x')$.

Proof. The morphism $\mathscr{M}(\mathscr{H}(x')) \to V' \to V$ maps the unique point of $\mathscr{M}(\mathscr{H}(x'))$ to $f(x')$ so according to Paragraph 2.10 we have a commutative diagram

$$\begin{tikzcd} \mathscr{M}(\mathscr{H}(x')) \arrow[r] \arrow[d, "i_{x'}"'] & \mathscr{M}(\mathscr{H}(f(x'))) \arrow[d, "i_{f(x')}"] \\ V' \arrow[r, "f"'] & V. \end{tikzcd}$$

The vector bundle on $\mathscr{M}(\mathscr{H}(x'))$ corresponding to $E(f(x'))$ is obtained by pulling back E along the composition in this commutative diagram.

Pulling back to $\mathscr{M}(\mathscr{H}(f(x')))$ yields $E(f(x'))$ so the claim follows from Paragraph 4.3. ∎

4.6 Total Spaces. Let E be a vector bundle on V. A *total space* of E is a triple $(\mathrm{Tot}(E), \pi_E, \Phi)$ where $\mathrm{Tot}(E)$ is a K-analytic space, $\pi_E\colon \mathrm{Tot}(E) \to V$ is a morphism of K-analytic spaces and $\Phi = \{\Phi_h\}_h$ is a family of natural bijections

$$\Phi_h\colon \mathrm{Hom}_V(W, \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(W, h^*E) \tag{4.6.1}$$

for every K-analytic space W and every morphism of K-analytic spaces $h\colon W \to V$. Here the left hand side denotes the set of morphisms $l\colon W \to \mathrm{Tot}(E)$ over V, i.e. satisfying $\pi_E \circ l = h$. We require that the family Φ is functorial in h, i.e. given $h'\colon W' \to V$ and a morphism $g\colon W' \to W$ satisfying $h \circ g = h'$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_V(W, \mathrm{Tot}(E)) \arrow[r, "\Phi_h"] \arrow[d, "g^*"'] & \Gamma(W, h^*E) \arrow[d, "g^*"] \\ \mathrm{Hom}_V(W', \mathrm{Tot}(E)) \arrow[r, "\Phi_{h'}"'] & \Gamma(W', (h')^*E) \end{tikzcd}$$

is commutative. Here the map g^* on the left hand side is given by $l \mapsto l \circ g$ while the map g^* on the right maps a section $s \in \Gamma(W, h^*E)$ to the section $g^*s \in \Gamma(W', g^*h^*E) = \Gamma(W', (h')^*E)$.

Note that by Yoneda's Lemma a total space $(\mathrm{Tot}(E), \pi_E, \Phi)$ is unique up to unique isomorphism. We prove in Proposition 4.12 below that every vector bundle on a K-analytic space admits a total space. Furthermore we show in Proposition 4.14 that the universal property (4.6.1) holds more generally for a morphism $h\colon W \to V$ of analytic spaces over K.

4.7 Total Space over a Point. Let E be an r-dimensional K-vector space with basis $e_1, \dots, e_r$ regarded as a vector bundle over $\mathscr{M}(K)$. Consider the analytic affine space $\mathbf{A}_K^{r,\mathrm{an}}$ with its structural morphism $\pi\colon \mathbf{A}_K^{r,\mathrm{an}} \to \mathscr{M}(K)$. We construct a family $\Phi_{\underline{e}}$ of bijections $\Phi_{\underline{e},h}\colon \mathrm{Hom}(W, \mathbf{A}_K^{r,\mathrm{an}}) \xrightarrow{\sim} \Gamma(W, h^*E)$ for every morphism of K-analytic spaces $h\colon W \to \mathscr{M}(K)$ such that the triple $(\mathbf{A}_K^{r,\mathrm{an}}, \pi, \Phi_{\underline{e}})$ is a total space for E.

For any such h, the pull-back h^*E admits a frame $h^*e_1, \dots, h^*e_r \in \Gamma(W, h^*E)$ which in turn induces an isomorphism $\mathscr{O}_{W_G}(W)^r \xrightarrow{\sim} \Gamma(W, h^*E)$. We let $\Phi_{\underline{e},h}$ be the composition of the chain of bijections

$$\mathrm{Hom}(W, \mathbf{A}_K^{r,\mathrm{an}}) \xrightarrow{\sim} \mathrm{Hom}(W, \mathbf{A}_K^{1,\mathrm{an}})^r = \mathscr{O}_{W_G}(W)^r \xrightarrow{\sim} \Gamma(W, h^*E).$$

For the first bijection we have used that $\mathbf{A}_K^{r,\mathrm{an}}$ is a product of r copies of $\mathbf{A}_K^{1,\mathrm{an}}$ in the category of K-analytic spaces (because the analytification functor commutes with fibered products). It is easy to verify that the family $\Phi_{\underline{e}}$ is functorial.

4.8 Total Space of a Pull-Back. Let $f\colon V' \to V$ be a morphism of K-analytic spaces and let E be a vector bundle on V with a total space $(\mathrm{Tot}(E), \pi_E, \Phi)$. Form the fibered product

$$\begin{tikzcd} \mathrm{Tot}(E) \times_V V' \arrow[r, "f'"] \arrow[d, "\pi'"'] & \mathrm{Tot}(E) \arrow[d, "\pi_E"] \\ V' \arrow[r, "f"'] & V. \end{tikzcd}$$

We construct a family $\Phi_{V'}$ of bijections $\Phi_{V',h'}\colon \mathrm{Hom}_{V'}(W, \mathrm{Tot}(E) \times_V V') \xrightarrow{\sim} \Gamma(W, (h')^*f^*E)$ for every morphism $h'\colon W \to V'$ such that the triple $(\mathrm{Tot}(E) \times_V V', \pi', \Phi_{V'})$ is a total space for E.

For any such h' we can view W as a K-analytic space over V via the composition $h \coloneqq f \circ h'$. Then the universal property of the fibered product provides us with a bijection

$$\mathrm{Hom}_{V'}(W, \mathrm{Tot}(E) \times_V V') \xrightarrow{\sim} \mathrm{Hom}_V(W, \mathrm{Tot}(E))$$

(given by composition with f'). We let $\Phi_{V',h'}$ be the composition

$$\mathrm{Hom}_{V'}(W, \mathrm{Tot}(E) \times_V V') \xrightarrow{\sim} \mathrm{Hom}_V(W, \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(W, h^*E) = \Gamma(W, (h')^* f^* E).$$

It is easy to verify that the family $\Phi_{V'}$ is functorial in h'.

4.9 Total Space of a Trivial Vector Bundle. Let E be a trivial vector bundle on V with a frame $s_1, \ldots, s_r$. The frame $\underline{s}$ corresponds to an identification of E with a pull-back of a K-vector space with basis $e_1, \ldots, e_r$, regarded as a vector bundle on $\mathscr{M}(K)$, along the structural morphism $V \to \mathscr{M}(K)$. By Paragraphs 4.7 and 4.8 the projection $\pi : V \times \mathbf{A}_K^{r,\mathrm{an}} \to V$ carries canonically the structure $\Phi_{\underline{s}}$ of a total space for E.

4.10 Lemma. *Let $f : V' \to V$ be a morphism of K-analytic spaces, let E be a vector bundle on V with pull-back f^*E to V'. Let $(\mathrm{Tot}(E), \pi_E, \Phi)$ and $(\mathrm{Tot}(f^*E), \pi_{f^*E}, \Phi')$ be total spaces for E and f^*E respectively. Denote by $f' : \mathrm{Tot}(f^*E) \to \mathrm{Tot}(E)$ the morphism corresponding to $\mathrm{id} : \mathrm{Tot}(f^*E) \to \mathrm{Tot}(f^*E)$ under the chain of bijections*

$$\begin{aligned}(\Phi')^{-1} \circ \Phi : \mathrm{Hom}_V(\mathrm{Tot}(f^*E), \mathrm{Tot}(E)) &\xrightarrow{\sim} \Gamma(\mathrm{Tot}(f^*E), \pi_{f^*E}^*(f^*E)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{V'}(\mathrm{Tot}(f^*E), \mathrm{Tot}(f^*E)).\end{aligned}$$

Here we regard $\mathrm{Tot}(f^*E)$ as a space over V via the map $f \circ \pi_{f^*E} : \mathrm{Tot}(f^*E) \to V$ in the first line. Then the diagram

$$\begin{tikzcd} \mathrm{Tot}(f^*E) \arrow[r, "f'"] \arrow[d, "\pi_{f^*E}"'] & \mathrm{Tot}(E) \arrow[d, "\pi_E"] \\ V' \arrow[r, "f"'] & V \end{tikzcd} \qquad (4.10.1)$$

is a cartesian commutative diagram of K-analytic spaces.

Proof. The fact that the diagram commutes is just a translation of the fact that f' is a morphism of K-analytic spaces over V. To show that the diagram is cartesian, suppose that we are given $h : W \to V'$ and $l : W \to \mathrm{Tot}(E)$ such that $f \circ h = \pi_E \circ l$. By the universal property of $\mathrm{Tot}(E)$, l corresponds to a section $s \in \Gamma(W, h^* f^* E)$. By the universal property of $\mathrm{Tot}(f^*E)$, the section s corresponds to a morphism $l' : W \to \mathrm{Tot}(f^*E)$ satisfying $\pi_{f^*E} \circ l' = h$. In fact, also $f' \circ l' = l$ holds. This follows from sending $\mathrm{id} \in \mathrm{Hom}_{V'}(\mathrm{Tot}(f^*E), \mathrm{Tot}(f^*E))$ around the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{V'}(\mathrm{Tot}(f^*E), \mathrm{Tot}(f^*E)) & \longrightarrow & \mathrm{Hom}_{V'}(W, \mathrm{Tot}(f^*E)) \\ \downarrow & & \downarrow \\ \Gamma(\mathrm{Tot}(f^*E), \pi_{f^*E}^*(f^*E)) & & \Gamma(W, h^*f^*E) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_V(\mathrm{Tot}(f^*E), \mathrm{Tot}(E)) & \longrightarrow & \mathrm{Hom}_V(W, \mathrm{Tot}(E)). \end{array}$$

The lower and the upper vertical maps are given by composition with $l' : W \to \mathrm{Tot}(f^*E)$. The commutativity of the diagram follows from the functoriality condition in Paragraph 4.6.

We have shown how to construct from a pair (h, l) satisfying $f \circ h = \pi_E \circ l$ a morphism l' satisfying $\pi_{f^*E} \circ l' = h$ and $f' \circ l' = l$. It is easy to verify that our construction is inverse to the operation $l' \mapsto (\pi_{f^*E} \circ l', f' \circ l')$ which proves that the diagram (4.10.1) is cartesian. ∎

4.11 Gluing Total Space Structures. Let E be a vector bundle on E, $\pi : T \to V$ a morphism of K-analytic spaces and let $U \subset V$ be an analytic domain in V. Applying Paragraph 4.8 to the embedding $U \hookrightarrow V$ we see that if Φ is a family of bijections such that (T, π, Φ) is a total space for E then there is an induced family $\Phi|_U$ such that the triple $(\pi^{-1}(U), \pi|_U, \Phi|_U)$ is a total space for $E|_U$ (here $\pi|_U$ denotes the induced morphism $\pi^{-1}(U) \to U$). In this way, the assignment

$$U \mapsto \{\, \Phi \mid (\pi^{-1}(U), \pi|_U, \Phi) \text{ is a total space for } E|_U \,\}$$

is a G-presheaf of sets on V. We claim that it is in fact a sheaf. So let $V = \bigcup_{i \in I} V_i$ be a G-covering of V by analytic domains and for each $i \in I$ let Φ_i be a family of bijections such that each $(\pi^{-1}(U_i), \pi|_{U_i}, \Phi_i)$ is a total space for $E|_{V_i}$ and such that $\Phi_i|_{V_{ij}} = \Phi_j|_{V_{ij}}$ for $i, j \in I$. We sketch here how to construct the unique total space structure Φ such that (T, π, Φ) is a total space for E and such that $\Phi|_{V_i} = \Phi_i$.

In the following we write $T_i := \pi^{-1}(V_i)$ and $\pi_i : T_i \to V_i$ for the restriction of π. Suppose that $h : W \to V$ is a morphism of K-analytic spaces and let $s \in \Gamma(W, h^*E)$ be a section. For each $i \in I$ we denote $W_i := h^{-1}(V_i)$ and we denote by $h_i : W_i \to h_i$ the restriction of h. By restriction we obtain for each $i \in I$ a section $s_i \in \Gamma(W_i, h_i^*(E|_{V_i}))$. This induces a morphism $l_i := \Phi_{i,h_i}^{-1}(s_i) : W_i \to T_i$. Using that the Φ_i agree on the overlaps one checks that the morphisms

$$W_i \xrightarrow{l_i} T_i \hookrightarrow T$$

agree on the overlaps, so by [Ber93, Prop. 1.3.2] they glue to a morphism of K-analytic spaces $\Phi^{-1}(s): W \to T$. The construction gives a map

$$\Phi^{-1}: \Gamma(W, h^*E) \to \mathrm{Hom}_V(W, T)$$

and by a similar argument one constructs the inverse map. Functoriality in h is easy to verify as is the fact that restriction to V_i gives back the Φ_i.

4.12 Proposition. *Every vector bundle E on the K-analytic space V admits a (unique up to unique isomorphism) total space* $(\mathrm{Tot}(E), \pi_E, \Phi)$*.*

Proof. For trivial E we constructed the total space in Paragraph 4.9. In the following we choose for every trivial vector bundle on every K-analytic space a fixed total space. Next we treat the case where V is paracompact and E is arbitrary. By Lemma 2.14 (iii) there exists a locally finite G-covering by affinoid domains $V = \bigcup_{i \in I} V_i$ such that the restrictions $E|_{V_i}$ are trivial. We denote $V_{ij} := V_i \cap V_j$ for $i, j \in I$. Note that since a paracompact space is in particular Hausdorff, the compact set V_j is closed in V, so V_{ij} is closed in V_i. By the case treated above, $\mathrm{Tot}(E|_{V_i})$ exists, we denote the structural map by $\pi_i: \mathrm{Tot}(E|_{V_i}) \to V_i$. Similarly, $\mathrm{Tot}(E|_{V_{ij}})$ exists, we denote the structural map by $\pi_{ij}: \mathrm{Tot}(E|_{V_{ij}}) \to V_{ij}$. Note that $E|_{V_{ij}}$ is just the pull-back of $E|_{V_i}$ along the embedding $\iota_{ij}: V_{ij} \hookrightarrow V_i$. Hence, by Lemma 4.10 we have a canonical cartesian commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}(E|_{V_{ij}}) \arrow[r, hook, "\iota'_{ij}"] \arrow[d, "\pi_{ij}"'] & \mathrm{Tot}(E|_{V_i}) \arrow[d, "\pi_i"] \\
V_{ij} \arrow[r, hook, "\iota_{ij}"'] & V_i.
\end{tikzcd}$$

The fact that the diagram is cartesian means that ι'_{ij} induces an isomorphism $\sigma_{ij}: \mathrm{Tot}(E|_{V_{ij}}) \xrightarrow{\sim} T_{ij} := \pi_i^{-1}(V_{ij}) \subset \mathrm{Tot}(E|_{V_i})$. As the preimage of the closed analytic domain $V_{ij} \subset V_i$ under π_i, the subset $T_{ij} \subset \mathrm{Tot}(E|_{V_i})$ is a closed analytic domain. Since the covering $\{V_i\}_{i \in I}$ is locally finite, for each $i \in I$ the set of $j \in I$ with $T_{ij} \neq \emptyset$ is finite. Furthermore the isomorphisms

$$\nu_{ij}: T_{ij} \xrightarrow{\sigma_{ij}{}^{-1}} \mathrm{Tot}(E|_{V_{ij}}) \xrightarrow{\sigma_{ji}} T_{ji}$$

satisfy the cocycle condition, which allows by [Ber93, Prop. 1.3.3 (b)] to glue the spaces $\mathrm{Tot}(E|_{V_i})$ to a space $\mathrm{Tot}(E)$ which is covered by the spaces $\mathrm{Tot}(E|_{V_i})$ for $i \in I$.

By construction, the maps

$$\mathrm{Tot}(E|_{V_i}) \xrightarrow{\pi_i} V_i \hookrightarrow V$$

are compatible with the gluing data, so by [Ber93, Prop. 1.3.2] they induce a map $\pi\colon \mathrm{Tot}(E) \to V$ which restricts on each V_i to the map $\pi_i\colon \mathrm{Tot}(E|_{V_i}) \to V_i$. The total space structures Φ_i making $(\mathrm{Tot}(E|_{V_i}), \pi_i, \Phi_i)$ a total space for $E|_{V_i}$ agree on the overlaps so by Paragraph 4.11 they glue to a total space structure Φ making $(\mathrm{Tot}(E), \pi_E, \Phi)$ a total space for E.

It remains to treat the case where V is not necessarily paracompact. In the following we fix for every vector bundle on any K-analytic space and any restriction of such a vector bundle to an analytic domain a total space. By [Ber93, Rem. 1.2.4] we can find a covering $V = \bigcup_{i\in I} V_i$ by open paracompact subsets. We already know that the vector bundles $E|_{V_i}$ and $E|_{V_{ij}}$ admit total spaces, so we can repeat the argument from the paracompact case. The only difference is that in the gluing process we have to invoke [Ber93, Prop. 1.3.3 (a)] rather than [Ber93, Prop. 1.3.3 (b)] in order to glue along open subsets. ∎

4.13 Change of Base Field. Let E be a vector bundle on V with total space $(\mathrm{Tot}(E), \pi_E, \Phi)$ and let L/K be an extension of non-archimedean fields. Recall from Paragraph 2.7 that it induces a morphism $\pi_{L/K}\colon V \hat{\otimes}_K L \to V$ of analytic spaces over K and in particular a morphism of the underlying ringed G-topological spaces. We call the pull-back

$$E \otimes_K L \coloneqq \pi_{L/K}^* E$$

the *scalar extension* of E along L/K. It is a vector bundle on $V \hat{\otimes}_K L$.

We consider now the scalar extension of the morphism $\pi_E\colon \mathrm{Tot}(E) \to V$ along L/K, namely

$$\pi_E \hat{\otimes}_K L\colon \mathrm{Tot}(E) \hat{\otimes}_K L \to V \hat{\otimes}_K L$$

and sketch how to construct a family Φ_L of bijections

$$\Phi_{L,h}\colon \mathrm{Hom}_{V\hat{\otimes}_K L}(W, \mathrm{Tot}(E) \hat{\otimes}_K L) \xrightarrow{\sim} \Gamma(W, h^*(E \otimes_K L))$$

for every morphism of L-analytic spaces $h\colon W \to V \hat{\otimes}_K L$ such that the triple

$$(\mathrm{Tot}(E) \hat{\otimes}_K L, \pi_E \hat{\otimes}_K L, \Phi_L)$$

is a total space for $E \otimes_K L$.

First, assume that E is trivial with a frame $s_1, \dots, s_r$. By Paragraph 4.9 the projection $\pi\colon \mathbf{A}_K^{r,\mathrm{an}} \times_K V \to V$ is a total space with the family of bijections $\Phi_{\underline{s}}$. By the uniqueness of total spaces there is a unique isomorphism $\sigma_{\underline{s}}\colon \mathrm{Tot}(E) \to$

$\mathbf{A}_K^{r,\mathrm{an}} \times V$ compatible with the projection maps and total space structures. By scalar extension we obtain an isomorphism

$$\mathrm{Tot}(E) \hat{\otimes}_K L \xrightarrow{\sim} (\mathbf{A}_K^{r,\mathrm{an}} \times_K V) \hat{\otimes}_K L \xrightarrow{\sim} \mathbf{A}_L^{r,\mathrm{an}} \times_L (V \hat{\otimes}_K L). \tag{4.13.1}$$

The frame $\pi_{L/K}^* \underline{s}$ for $E \otimes_K L$ provides the projection $\mathbf{A}_L^{r,\mathrm{an}} \times (V \hat{\otimes}_K L) \to V \hat{\otimes}_K L$ with the structure of a total for $E \otimes_K L$. Pulling it back along the isomorphism (4.13.1) we obtain a total space structure Φ_L on $\mathrm{Tot}(E) \hat{\otimes}_K L$. We omit here the straight-forward verification that Φ_L is independent of the chosen frame and compatible with restriction to analytic domains of V.

If E is not necessarily trivial, we can pick a G-covering $V = \bigcup_{i \in I} V_i$ of V by analytic domains such that each $E|_{V_i}$ is trivial. For each $i \in I$ the restriction $(\pi_E^{-1}(V_i), \pi_E|_{V_i}, \Phi|_{V_i})$ is a total space for $E|_{V_i}$ so by the case above it induces a total space structure $\Phi_{i,L}$ for the scalar extension $\pi_E^{-1}(V_i) \hat{\otimes}_K L$ which identifies canonically with $(\pi_E \hat{\otimes}_K L)^{-1}(V_i \hat{\otimes}_K L)$. It is easy to check that these structures are compatible with restriction so by Paragraph 4.11 we obtain a total space structure Φ_L for the morphism $\mathrm{Tot}(E) \hat{\otimes}_K L \to V \hat{\otimes}_K L$.

4.14 Proposition. *Let E be a vector bundle on V and let $(\mathrm{Tot}(E), \pi_E, \Phi)$ be a total space for V. Then the functorial family of bijections*

$$\Phi_h : \mathrm{Hom}_V(W, \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(W, h^*E)$$

for $h : W \to V$ a morphism of K-analytic spaces, extends uniquely to a functorial family of bijections

$$\Phi_h : \mathrm{Hom}_V(W, \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(W, h^*E) \tag{4.14.1}$$

where $h : W \to V$ is a allowed to be morphism of analytic spaces over K.

Proof. Indeed, uniqueness follows from Yoneda's Lemma, because the family (4.14.1) is completely determined by its component at $h = \pi_E$ which is a morphism of K-analytic spaces. We describe how to construct a natural bijection (4.14.1) where $h : W \to V$ or more precisely $h : (W, L) \to (V, K)$ is a morphism of analytic spaces over K given by a morphism of L-analytic spaces $\tilde{h} : W \to V \hat{\otimes}_K L$. Here L/K is a non-archimedean field extension of K.

Let $l \in \mathrm{Hom}_V(W, \mathrm{Tot}(E))$ be a morphism of analytic spaces over K such that $\pi_E \circ l = h$. By definition (Paragraph 2.7), l is given by a morphism $\tilde{l} : W \to \mathrm{Tot}(E) \hat{\otimes}_K L$ of L-analytic spaces such that $(\pi_E \hat{\otimes}_K L) \circ \tilde{l} = \tilde{h}$. Recall from Paragraph 4.13 that the triple $(\mathrm{Tot}(E) \hat{\otimes}_K L, \pi_E \hat{\otimes}_K L, \Phi_L)$ is a total space for $E \otimes_K L = \pi_{L/K}^* E$, so we obtain a section $\Phi_L(\tilde{l}) \in \Gamma(W, \tilde{h}^*(\pi_{L/K}^* E))$. Recall from Paragraph 2.7 that $\pi_{L/K} \circ \tilde{h} = h$ so in fact $\tilde{h}^* \pi_{L/K}^* E = h^* E$. We have thus described a map

$$\Phi_h : \mathrm{Hom}_V(W, \mathrm{Tot}(E)) \to \Gamma(W, h^*E), \qquad l \mapsto \Phi_L(\tilde{l})$$

and it is straight-forward to verify that it is bijective, functorial with respect to h and agrees with Φ_h in the original sense if $L = K$. ∎

4.15 Remark. From now on we fix for every vector bundle E on V a total space $(\mathrm{Tot}(E), \pi_E, \Phi)$. We often keep the family Φ implicit and suppress it from the notation.

4.16 Proposition. *Let E be a vector bundle on V and let $x \in V$ be a point. There is a canonical bijection of sets*

$$\Psi_x \colon \operatorname*{colim}_{L/\mathscr{H}(x)} E(x) \otimes_{\mathscr{H}(x)} L \xrightarrow{\sim} \pi_E^{-1}(x).$$

Proof. By Paragraph 2.11 we have a canonical bijection

$$\operatorname*{colim}_{L/\mathscr{H}(x)} \mathrm{Hom}_V(\mathscr{M}(L), \mathrm{Tot}(E)) \xrightarrow{\sim} \pi_E^{-1}(x)$$

so it only remains to establish for a non-archimedean field extension $L/\mathscr{H}(x)$ a natural bijection of sets

$$\mathrm{Hom}_V(\mathscr{M}(L), \mathrm{Tot}(E)) \xrightarrow{\sim} E(x) \otimes_{\mathscr{H}(x)} L.$$

This follows from the fact that by Proposition 4.14 we have a canonical bijection

$$\Phi_h \colon \mathrm{Hom}_V(\mathscr{M}(L), \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(\mathscr{M}(L), h^*E)$$

where h denotes the composition $h \colon \mathscr{M}(L) \to \mathscr{M}(\mathscr{H}(x)) \to V$. But the pull-back of E to $\mathscr{M}(\mathscr{H}(x))$ has global sections $E(x)$ by Paragraph 4.4 whereas the further pull-back to $\mathscr{M}(L)$ has global sections $E(x) \otimes_{\mathscr{H}(x)} L$ by Paragraph 4.3. ∎

4.17 Remark. From the proof of Proposition 4.16 we see that the bijection

$$\Psi_x \colon \operatorname*{colim}_{L/\mathscr{H}(x)} E(x) \otimes_{\mathscr{H}(x)} L \xrightarrow{\sim} \pi_E^{-1}(x)$$

is given explicitly as follows: An element in the colimit is given by a non-archimedean field extension $L/\mathscr{H}(x)$ and a vector $v \in E(x) \otimes_{\mathscr{H}(x)} L$. We can interpret v as a global section of the vector bundle E pulled back along $\mathscr{M}(L) \to \mathscr{M}(\mathscr{H}(x)) \to V$, so it induces a morphism $\mathscr{M}(L) \to \mathrm{Tot}(E)$ by Proposition 4.14. The image of this morphism is the desired point in $\pi_E^{-1}(x) \subset \mathrm{Tot}(E)$.

4.18 Sections. Let E be a vector bundle on V and let $s \in \Gamma(V, E)$ be a section of the sheaf E. Under the bijection

$$\Phi_{\mathrm{id}_V} \colon \mathrm{Hom}_V(V, \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(V, E),$$

it corresponds to a morphism $\tilde{s} \colon V \to \mathrm{Tot}(E)$ over V. The fact that $\tilde{s}$ is a morphism over V means that $\pi_E \circ \tilde{s} = \mathrm{id}_V$, i.e. $\tilde{s} \colon V \to \mathrm{Tot}(E)$ is a *section* of the projection $\pi_E \colon \mathrm{Tot}(E) \to V$.

4.19 Lemma. *Let $f\colon V' \to V$ be a morphism of K-analytic spaces, E a vector bundle on V and consider the cartesian commutative diagram*

$$\begin{tikzcd} \mathrm{Tot}(f^*E) \arrow[r, "f'"] \arrow[d, "\pi_{f^*E}"'] & \mathrm{Tot}(E) \arrow[d, "\pi_E"] \\ V' \arrow[r, "f"'] & V \end{tikzcd}$$

of Lemma 4.10. Let $s \in \Gamma(V,E)$ be a section. We can consider the section $f^*s \in \Gamma(V', f^*E)$ and the induced morphism $\widetilde{f^*s}\colon V' \to \mathrm{Tot}(f^*E)$. Then the diagram

$$\begin{tikzcd} \mathrm{Tot}(f^*E) \arrow[r, "f'"] & \mathrm{Tot}(E) \\ V' \arrow[r, "f"'] \arrow[u, "\widetilde{f^*s}"] & V \arrow[u, "\tilde{s}"'] \end{tikzcd}$$

commutes, i.e. we have $\tilde{s} \circ f = f' \circ \widetilde{f^*s}$.

Proof. First consider the commutative diagram

$$\begin{tikzcd} \Gamma(V,E) \arrow[r, "\Phi_{\mathrm{id}}^{-1}"] \arrow[d, "f^*"'] & \mathrm{Hom}_V(V, \mathrm{Tot}(E)) \arrow[d, "f^*"] \\ \Gamma(V', f^*E) \arrow[r, "\Phi_f^{-1}"'] & \mathrm{Hom}_V(V', \mathrm{Tot}(E)) \end{tikzcd}$$

obtained from the functoriality condition in Paragraph 4.6 applied to the morphism $f\colon V' \to V$. Sending $s \in \Gamma(V,E)$ around the diagram we see that the isomorphism

$$\Phi_f^{-1}\colon \Gamma(V', f^*E) \xrightarrow{\sim} \mathrm{Hom}_V(V', \mathrm{Tot}(E))$$

maps $f^*s \in \Gamma(V', f^*E)$ to the morphism $\tilde{s} \circ f\colon V \to \mathrm{Tot}(E)$.

Let us view $\pi_{f^*E}\colon \mathrm{Tot}(f^*E) \to V'$ and $\mathrm{id}_{V'}\colon V' \to V'$ as K-analytic spaces over V'. The morphism $\widetilde{f^*s}\colon V' \to \mathrm{Tot}(f^*E)$ satisfies $\pi_{f^*E} \circ \widetilde{f^*s} = \mathrm{id}_{V'}$ so we can apply the functoriality condition of the total space $(\mathrm{Tot}(f^*E), \pi_{f^*E}, \Phi')$ to obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{V'}(\mathrm{Tot}(f^*E), \mathrm{Tot}(f^*E)) \arrow[r, "(\widetilde{f^*s})^*"] \arrow[d, "\Phi'_{\pi_{f^*E}}"', "\cong"] & \mathrm{Hom}_{V'}(V', \mathrm{Tot}(f^*E)) \arrow[d, "\Phi'_{\mathrm{id}_{V'}}", "\cong"'] \\ \Gamma(\mathrm{Tot}(f^*E), \pi_{f^*E}^*(f^*E)) \arrow[r, "(\widetilde{f^*s})^*"'] & \Gamma(V', f^*E). \end{tikzcd}$$

Similarly, if we view $f \circ \pi_{f^*E} : \operatorname{Tot}(f^*E) \to V$ and $f : V' \to V$ as K-analytic spaces over V then we have $f \circ \pi_{f^*E} \circ \widetilde{f^*s} = f$ and hence we obtain a commutative diagram

$$\begin{tikzcd}
\Gamma(\operatorname{Tot}(f^*E), \pi^*_{f^*E}(f^*E)) \arrow[r, "(\widetilde{f^*s})^*"] \arrow[d, "\Phi^{-1}_{f\circ\pi_{f^*E}}"', "\cong"] & \Gamma(V', f^*E) \arrow[d, "\cong"', "\Phi_f^{-1}"] \\
\operatorname{Hom}_V(\operatorname{Tot}(f^*E), \operatorname{Tot}(E)) \arrow[r, "(\widetilde{f^*s})^*"'] & \operatorname{Hom}_V(V', \operatorname{Tot}(E)).
\end{tikzcd}$$

Pasting the diagrams together we obtain

$$\begin{tikzcd}
\operatorname{Hom}_{V'}(\operatorname{Tot}(f^*E), \operatorname{Tot}(f^*E)) \arrow[r, "(\widetilde{f^*s})^*"] \arrow[d, "\Phi'_{\pi_{f^*E}}"', "\cong"] & \operatorname{Hom}_{V'}(V', \operatorname{Tot}(f^*E)) \arrow[d, "\cong"', "\Phi'_{\mathrm{id}_{V'}}"] \\
\Gamma(\operatorname{Tot}(f^*E), \pi^*_{f^*E}(f^*E)) \arrow[r] \arrow[d, "(\Phi'_{f\circ\pi_{f^*E}})^{-1}"', "\cong"] & \Gamma(V', f^*E) \arrow[d, "\cong"', "\Phi_f^{-1}"] \\
\operatorname{Hom}_V(\operatorname{Tot}(f^*E), \operatorname{Tot}(E)) \arrow[r, "(\widetilde{f^*s})^*"'] & \operatorname{Hom}_V(V', \operatorname{Tot}(E))
\end{tikzcd}$$

We claim that sending $\mathrm{id} \in \operatorname{Hom}_V(\operatorname{Tot}(f^*E), \operatorname{Tot}(f^*E))$ around the lower left corner we obtain $f' \circ \widetilde{f^*s} \in \operatorname{Hom}_V(V', \operatorname{Tot}(E))$ while sending it around the upper right corner we obtain $\tilde{s} \circ f \in \operatorname{Hom}_V(V', \operatorname{Tot}(E))$ so that both morphisms must be equal.

Sending $\mathrm{id} \in \operatorname{Hom}_V(\operatorname{Tot}(f^*E), \operatorname{Tot}(f^*E))$ to the lower left corner we obtain the morphism $f' : \operatorname{Tot}(f^*E) \to \operatorname{Tot}(E)$ by its construction in Lemma 4.10. Sending it further to the lower right corner we obtain $f' \circ \widetilde{f^*s}$.

Sending $\mathrm{id} \in \operatorname{Hom}_V(\operatorname{Tot}(f^*E), \operatorname{Tot}(f^*E))$ to the upper right corner instead we obtain $\widetilde{f^*s} \in \operatorname{Hom}_{V'}(V', \operatorname{Tot}(f^*E))$; by definition this is the morphism corresponding to the section $f^*s \in \Gamma(V', f^*E)$. We have argued above that under the map $\Gamma(V', f^*E) \to \operatorname{Hom}_V(V', \operatorname{Tot}(E))$, the section f^*s is indeed mapped to $\tilde{s} \circ f$ which finishes the proof. ∎

4.20 Lemma. *Let E be a vector bundle on E and let $x \in V$ be a point. Consider a section $s \in \Gamma(V, E)$ and the induced element in the fiber vector space $s(x) \in E(x)$ according to Paragraph 4.4. Under the map*

$$E(x) \hookrightarrow \operatorname*{colim}_{L/\mathscr{H}(x)} E(x) \otimes_{\mathscr{H}(x)} L \xrightarrow{\sim} \pi_E^{-1}(x),$$

given by Proposition 4.16 the element $s(x) \in E(x)$ is mapped to the point in the fiber $\tilde{s}(x) \in \pi_E^{-1}(x) \subset \operatorname{Tot}(E)$.

Proof. Recall that by definition, $E(x) = \Gamma(\mathscr{M}(\mathscr{H}(x)), i_x^*E)$ and $s(x) = i_x^* s \in E(x)$. By the concrete description of Remark 4.17 we see how to map $s(x)$ to a point in $\pi_E^{-1}(x)$: The section $s(x) \in \Gamma(\mathscr{M}(\mathscr{H}(x)), i_x^*E)$ induces a morphism $\mathscr{M}(\mathscr{H}(x)) \to \mathrm{Tot}(E)$ whose image is the desired point in the fiber. On the other hand, $\tilde{s}(x)$ is the unique point in the image of the composition

$$\mathscr{M}(\mathscr{H}(x)) \xrightarrow{i_x} V \xrightarrow{\tilde{s}} \mathrm{Tot}(E).$$

So the claim follows by sending the section $s \in \Gamma(V, E)$ around the commutative diagram

$$\begin{tikzcd} \Gamma(V,E) \arrow[r, "i_x^*"] \arrow[d, "\cong"'] & \Gamma(\mathscr{M}(\mathscr{H}(x)), i_x^*E) \arrow[d, "\cong"] \\ \mathrm{Hom}_V(V, \mathrm{Tot}(E)) \arrow[r, "i_x^*"'] & \mathrm{Hom}_V(\mathscr{M}(\mathscr{H}(x)), \mathrm{Tot}(E)). \end{tikzcd}$$

∎

4.21 Functoriality. Let $\alpha : E \to F$ be a morphism of vector bundles on V. For every K-analytic spaces $h : W \to V$ over V the morphism α induces a map

$$\Gamma(W, h^*E) \to \Gamma(W, h^*E)$$

and hence by the universal property of total spaces a map

$$\mathrm{Hom}_V(W, \mathrm{Tot}(E)) \to \mathrm{Hom}_V(W, \mathrm{Tot}(F)) \tag{4.21.1}$$

which is functorial in W. By Yoneda's Lemma this map is induced by a unique morphism $\mathrm{Tot}(\alpha) : \mathrm{Tot}(E) \to \mathrm{Tot}(F)$ of K-analytic spaces over V. Concretely, $\mathrm{Tot}(\alpha)$ is obtained as the image of $\mathrm{id}_{\mathrm{Tot}(E)}$ in Eq. (4.21.1) when $W = \mathrm{Tot}(E)$ is plugged in.

4.22 Tensor Products. Let E, F be two vector bundles on V. Then the tensor product of $\mathcal{O}_{V_G}$-modules $E \otimes F$ is again a vector bundle on V by [Sta23, Lem. 03L6 (2)]. If $s \in \Gamma(V, E)$ and $s' \in \Gamma(V, F)$ are sections, then there is an induced section $s \otimes s' \in \Gamma(V, E \otimes E')$.

If $f : V' \to V$ is a morphism of analytic spaces over K, then according to [Sta23, Lem. 03EL] there is a canonical identification

$$f^*(E \otimes F) = f^*E \otimes f^*F$$

of vector bundles on V'.

In particular, if $x \in V$ is a point then there is a canonical identification

$$(E \otimes F)(x) \cong E(x) \otimes_{\mathscr{H}(x)} F(x). \tag{4.22.1}$$

Under this identification we have

$$(s \otimes s')(x) = s(x) \otimes s'(x) \in (E \otimes F)(x) \cong E(x) \otimes_{\mathscr{H}(x)} F(x). \tag{4.22.2}$$

If $U \subset V$ is an analytic domain such that E admits a frame $s_1, \dots, s_r \in \Gamma(U, E)$ over U and F admits a frame $s'_1, \dots, s'_{r'} \in \Gamma(U, F)$ over U, then the sections

$$s_i \otimes s'_j \in \Gamma(U, E \otimes F), \quad 1 \leq i \leq r,\ 1 \leq j \leq r'$$

form a frame for $E \otimes F$ over U.

4.23 Direct Sums. Let E, F be two vector bundles on V. Then the direct sum $E \oplus F$ of $\mathscr{O}_{V_G}$-modules is again a vector bundle. The projections $\mathrm{pr}_1 : E \oplus F \to E$ and $\mathrm{pr}_2 : E \oplus F \to F$ induce morphisms $\mathrm{Tot}(\mathrm{pr}_1) : \mathrm{Tot}(E \oplus F) \to \mathrm{Tot}(E)$ and $\mathrm{Tot}(\mathrm{pr}_2) : \mathrm{Tot}(E \oplus F) \to \mathrm{Tot}(F)$ over V which induce an identification

$$\mathrm{Tot}(E \oplus F) \cong \mathrm{Tot}(E) \times_V \mathrm{Tot}(F).$$

If $s \in \Gamma(V, E)$ and $s' \in \Gamma(V, F)$ are sections, then there is an induced section $(s, s') \in \Gamma(V, E \oplus F)$. If $x \in V$ is a point, then there is a canonical identification $(E \oplus F)(x) \cong E(x) \oplus F(x)$. Under this identification, we have

$$(s, s')(x) = (s(x), s'(x)) \in (E \oplus F)(x) \cong E(x) \oplus F(x). \tag{4.23.1}$$

Indeed, this follows from pull-back along $\mathscr{M}(\mathscr{H}(x)) \to V$ being a left adjoint functor.

If $U \subset V$ is an analytic domain where E admits a frame $s_1, \dots, s_r \in \Gamma(U, E)$ and F admits a frame $s'_1, \dots, s'_{r'} \in \Gamma(U, F)$, then the sections

$$(s_1, 0), \dots, (s_r, 0), (0, s'_1), \dots, (0, s'_{r'}) \in \Gamma(U, E \oplus F)$$

form a frame for $E \oplus F$.

4.24 Duals. Let E be a vector bundle on V. Then the dual sheaf $E^\vee$ given by

$$\Gamma(U, E^\vee) = \mathrm{Hom}_{\mathscr{O}_{V_G}\text{-mod}}(E|_U, \mathscr{O}_{V_G}|_U)$$

is again a vector bundle. If $s_1, \dots, s_r \in \Gamma(U, E)$ is a trivialization, then there is an induced trivialization

$$s_1^\vee, \dots, s_r^\vee \in \Gamma(U, E^\vee)$$

determined uniquely by

$$s_i^\vee(s_j) = \delta_{ij} \in \mathscr{O}_{V_G}(U).$$

4.25 Analytification of Algebraic Vector Bundles. Let X be a locally finite type K-scheme and let E be a vector bundle on X. We define the *analytification* of E to be the pull-back of E along the morphism of ringed G-topological spaces $\pi_X\colon (X_G^{\mathrm{an}}, \mathcal{O}_{X_G^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ of Paragraph 2.16,

$$E^{\mathrm{an}} := \pi_X^* E.$$

This is a line bundle on the K-analytic space X^{an}.

We define the *total space* of E

$$\mathrm{Tot}(E) := \mathrm{Spec}(\mathrm{Sym}(E^\vee))$$

to be the relative spectrum of the quasi-coherent algebra $\mathrm{Sym}(E^\vee)$. It is a K-scheme of locally finite type and comes with a canonical structure morphism $\pi_E\colon \mathrm{Tot}(E) \to X$ of K-schemes. Note that in the notation of [GW20, Def. 11.2] we have $\mathrm{Tot}(E) = \mathbf{V}(E^\vee)$. Hence by [GW20, Prop. 11.4] there is for every morphism $h\colon T \to X$ of K-schemes a natural bijection

$$\Phi_h\colon \mathrm{Hom}_X(T, \mathrm{Tot}(E)) \xrightarrow{\sim} \Gamma(T, h^* E).$$

The family $\Phi = \{\Phi_h\}_h$ induces naturally a family Φ^{an} such that the triple

$$(\mathrm{Tot}(E)^{\mathrm{an}}, \pi_E^{\mathrm{an}}, \Phi^{\mathrm{an}})$$

is a total space for E^{an}, i.e. analytification is compatible with total spaces. The construction is completely analogous to Paragraph 4.13.

The analytification operation $E \mapsto E^{\mathrm{an}}$ is also compatible with pull-backs, tensor products, direct sums and duals.

4.26 Remark. In the case of a line bundle L on an algebraic K-variety X, the total space $\mathrm{Tot}((L^{\mathrm{an}})^\vee) = \mathrm{Tot}(L^\vee)^{\mathrm{an}}$ has also been considered by Yanbo Fang in [Fan23] in the context of metrized line bundles.

4.27 Vector Sub-Bundles. Let E be a vector bundle on V. A *vector sub-bundle* of E is a sub-sheaf of $\mathcal{O}_{V_G}$-modules $F \subset E$ such that F is a vector bundle (i.e. G-locally free of constant rank) and the quotient E/F is also G-locally free. There exists then a G-covering of V by affinoid domains V_i such that on $V_i = \mathcal{M}(A)$, the exact sequence

$$0 \to F \to E \to E/F \to 0$$

corresponds to an exact sequence $0 \to M' \to M \to M'' \to 0$ of free A-modules. This sequence is then necessarily split and the rank of M'' is the rank of E minus the rank of F. This shows that the embedding $F \hookrightarrow E$ is G-locally split and that the quotient E/F is G-locally free of constant rank.

Note that if $F \subset E$ is a vector sub-bundle and $h\colon W \to V$ is a morphism of analytic spaces over K, then since the embedding $F \hookrightarrow E$ is G-locally split, the induced morphism of G-sheaves $h^*F \to h^*E$ is again G-locally split and in particular it is injective. Since pull-back is compatible with quotients, the quotient $h^*E/h^*F = h^*(E/F)$ is again G-locally free so $h^*F \subset h^*E$ is again a vector sub-bundle.

In particular, if $W = \mathscr{M}(\mathscr{H}(x))$ for a point $x \in V$ and $h = i_x\colon \mathscr{M}(\mathscr{H}(x)) \to V$, we see that $F(x)$ is a vector sub-space of $E(x)$.

If $L \subset E$ is a vector sub-bundle of rank 1, we say that F is a *line sub-bundle*.

4.28 Projective Bundles. Let E be a vector bundle on V. A *projective bundle* for E is given by a morphism of K-analytic spaces

$$p_E\colon P(E) \to V$$

together with a line sub-bundle $\mathscr{O}_E(-1) \subset p_E^*E$ such that for every morphism of K-analytic spaces $h\colon W \to V$ and every line sub-bundle $L \subset h^*E$ there exists a unique morphism $l\colon V \to P(E)$ such that $p_E \circ l = h$ and $L = l^*\mathscr{O}_E(-1)$ as sub-sheaves of h^*E. In other words, for every morphism $h\colon W \to V$ of K-analytic spaces, the natural map

$$\operatorname{Hom}_V(W, P(E)) \to \{\,\text{Line sub-bundles of } h^*E\,\}, \qquad l \mapsto l^*\mathscr{O}_E(-1)$$

is a bijection.

We call $\mathscr{O}_E(-1)$ the *tautological line bundle* on $P(E)$ and often denote by

$$i\colon \mathscr{O}_E(-1) \to p_E^*E$$

the canonical embedding. Furthermore we denote by $\mathscr{O}_E(1) \coloneqq \mathscr{O}_E(-1)^\vee$ its dual line bundle on $P(E)$.

4.29 Projective Bundle over a Point. Consider the analytification

$$p\colon \mathbf{P}_K^{e,\mathrm{an}} \to \mathscr{M}(K)$$

of the morphism of algebraic K-varieties $\mathbf{P}_K^e \to \operatorname{Spec}(K)$ and the analytification $\mathscr{O}_{\mathbf{P}_K^e}(-1)^{\mathrm{an}} \subset \mathscr{O}_{\mathbf{P}_K^{e,\mathrm{an}}}^{e+1}$ of the line bundle $\mathscr{O}_{\mathbf{P}_K^e}(-1) \subset \mathscr{O}_{\mathbf{P}_K^e}^{e+1}$ on $\mathbf{P}_K^e$. Here we denote by $\mathscr{O}_{\mathbf{P}_K^{e,\mathrm{an}}}$ the structure G-sheaf on $\mathbf{P}_K^{e,\mathrm{an}}$ and skip the index "G". We claim that the data thus specified provide a projective bundle of the vector bundle on $\mathscr{M}(K)$ given by the $(e+1)$-dimensional vector space K^{e+1}. So let W be any K-analytic space and let $L \subset \mathscr{O}_{W_G}^{e+1}$ be a line sub-bundle. We sketch how to construct from this a morphism $W \to \mathbf{P}_K^{e,\mathrm{an}}$ of K-analytic spaces.

Denote by $T_0, \dots, T_e$ the homogeneous coordinates of $\mathbf{P}_K^{e+1}$, so that $\mathbf{P}_K^{e+1}$ is covered by the open affine subsets

$$U_i = \operatorname{Spec}(K[T_0/T_i, \dots, \widehat{T_i/T_i}, \dots, T_e/T_i]) \xrightarrow{\sim} \mathbf{A}_K^e$$

as $i \in \{0, \dots, e\}$ with affine coordinate functions $T_0/T_i, \dots, \widehat{T_i/T_i}, \dots, T_e/T_i$.

G-locally on W we can find a nowhere-vanishing section $s \in \Gamma(W, L)$. Denote by $(f_0, \dots, f_e) = s \in \Gamma(W, L) \subset \mathcal{O}_{W_G}(W)^{e+1}$ its components. We remarked in Paragraph 4.27 that for any point $y \in W$, the fiber vector space $L(y)$ lies injective in $\mathcal{H}(y)^{e+1}$. This shows that by shrinking W further, we may assume that one of the functions f_i, where $i \in \{0, \dots, e\}$, is nowhere-vanishing. The functions $f_0/f_i, \dots, \widehat{f_i/f_i}, \dots, f_e/f_i$ define a map

$$(f_0/f_i, \dots, \widehat{f_i/f_i}, \dots, f_e/f_i) \colon W \to \mathbf{A}_K^{e,\mathrm{an}} \xrightarrow{\sim} U_i^{\mathrm{an}} \subset \mathbf{P}_K^{e,\mathrm{an}}.$$

One checks that this map is independent of the choices made and thus glues to a map $W \to \mathbf{P}_K^{e,\mathrm{an}}$.

4.30 Lemma. *Let $f \colon V' \to V$ be a morphism of K-analytic spaces and let E be a vector bundle on V which admits a projective bundle $p_E \colon P(E) \to V$. Form the fibered product*

$$\begin{tikzcd} P(E) \times_V V' \arrow[r, "\tilde{f}"] \arrow[d, "p'_E"'] & P(E) \arrow[d, "p_E"] \\ V' \arrow[r, "f"'] & V. \end{tikzcd}$$

Then the projection $p'_E \colon P(E) \times_V V' \to V'$ together with the line sub-bundle $\tilde{f}^* \mathcal{O}_E(-1) \subset \tilde{f}^* p_E^* E = (p'_E)^* f^* E$ is a projective bundle for $f^* E$.

Proof. We sketch how to construct for a morphism of K-analytic spaces $W \to V'$ an inverse to the natural map

$$\operatorname{Hom}_{V'}(W, P(E) \times_V V') \to \{\text{Line sub-bundles of } h^* f^* E\}. \tag{4.30.1}$$

Let $L \subset h^* f^* E$ be a line sub-bundle of $h^* f^* E$. Regarding W as a K-analytic space over V via the map $f \circ h$, we get a morphism $l \colon W \to P(E)$ over V such that $L = l^* \mathcal{O}_E(-1)$ as a line sub-bundle of $h^* f^* E$. The fact that l is a morphism over V means that $p_E \circ l = f \circ h$ so it induces a map $l' \colon W \to P(E) \times_V V'$ satisfying $p'_E \circ l' = h$ and $\tilde{f} \circ l' = l$.

It is straight-forward to verify that this construction provides an inverse to the map (4.30.1). ∎

4.31 Projective Bundle of a Trivial Vector Bundle. Similarly to Paragraph 4.9 it follows from Paragraph 4.29 and Lemma 4.30 that a trivial vector bundle E on V with frame $s_0, \dots, s_e$ admits the projection $p\colon V \times \mathbf{P}_K^{e,\mathrm{an}} \to V$ as a projective bundle with $\mathcal{O}_E(-1) \subset p^*E$ given as follows: If $q\colon V \times \mathbf{P}_K^{e,\mathrm{an}} \to \mathbf{P}_K^{e,\mathrm{an}}$ denotes the second projection, then we set

$$\mathcal{O}_E(-1) \coloneqq q^*\mathcal{O}_{\mathbf{P}_K^e}(-1)^{\mathrm{an}} \subset q^*\mathcal{O}_{\mathbf{P}_K^{e,\mathrm{an}}}^{e+1} \xrightarrow{\sim} p^*E.$$

4.32 Lemma. *Let $f\colon V' \to V$ be a morphism of K-analytic spaces, let E be a vector bundle on V with pull-back f^*E to V'. Let $(P(E), p_E, \mathcal{O}_E(-1))$ as well as $(P(f^*E), p_{f^*E}, \mathcal{O}_{f^*E}(-1))$ be projective bundles for E and for f^*E respectively. The line sub-bundle*

$$\mathcal{O}_{f^*E}(-1) \subset p_{f^*E}^*(f^*E) = (f \circ p_{f^*E})^*E$$

induces then a unique morphism $\widetilde{f}\colon P(f^*E) \to P(E)$ over V such that as line sub-bundles of $p_{f^*E}f^*E = \widetilde{f}^*p_E^*E$ we have

$$\widetilde{f}^*\mathcal{O}_E(-1) = \mathcal{O}_{f^*E}(-1).$$

We claim that the commutative diagram

$$\begin{array}{ccc} P(f^*E) & \xrightarrow{\widetilde{f}} & P(E) \\ {\scriptstyle p_{f^*E}}\downarrow & & \downarrow{\scriptstyle p_E} \\ V' & \xrightarrow[f]{} & V \end{array}$$

is a cartesian diagram of K-analytic spaces.

Proof. The proof is similar to Lemma 4.10 so we omit it here. ∎

4.33 Gluing Projective Bundles. Let E be a vector bundle on V and let $p\colon P \to V$ be a morphism of K-analytic spaces. If $O \subset p^*E$ is a line sub-bundle such that the triple (P, p, O) is a projective bundle for E then it follows from Lemma 4.30 that for an analytic domain $U \subset V$ the triple $(p^{-1}(U), p|_U, O|_{p^{-1}(U)})$ is a projective bundle for $E|_U$ where $p|_U$ denotes the map $p^{-1}(U) \to U$ induced by p. In this way the assignment

$$U \mapsto \{\, O \mid (p^{-1}(U), p|_U, O) \text{ is a projective bundle for } E|_U \,\}$$

is a G-presheaf of sets on V. Similarly to Paragraph 4.11 one sees that it forms in fact a sheaf. (Note that by [Sta23, Lem. 04TR] sheaves can be glued on an arbitrary site, in particular on a G-topological space.)

4.34 Proposition. *Every vector bundle E on the K-analytic space V admits a (unique up to unique isomorphism) projective bundle* $(P(E), p_E, \mathcal{O}_E(-1))$.

Proof. Using the results of 4.30, 4.31, 4.32 and 4.33 this can be shown *mutatis mutandis* like the analogous result for total spaces in Proposition 4.12. ∎

4.35 Remark. From now on we fix for every vector bundle E on a K-analytic space a projective bundle $(P(E), p_E, \mathcal{O}_E(-1))$.

4.36 Projective Bundle Under Base Change. Let E be a vector bundle on V with projective bundle $p_E\colon P(E) \to V$ and tautological line bundle $\mathcal{O}_E(-1) \subset p_E^*E$ and let L/K be an extension of non-archimedean fields. The commutative diagram

$$\begin{tikzcd} P(E) \hat{\otimes}_K L \arrow[r, "\pi_{L/K}"] \arrow[d, "p_E \hat{\otimes}_K L"'] & P(E) \arrow[d, "p_E"] \\ V \hat{\otimes}_K L \arrow[r, "\pi_{L/K}"'] & V \end{tikzcd}$$

implies together with Eq. (4.2.1) that we have a canonical identification

$$p_E^*E \otimes_K L = (p_E \hat{\otimes}_K L)^*(E \otimes_K L).$$

The scalar extension $p_E \hat{\otimes}_K L\colon P(E) \hat{\otimes}_K L \to V \hat{\otimes}_K L$ of p_E together with the line sub-bundle

$$\mathcal{O}_E(-1) \otimes_K L \subset p_E^*E \otimes_K L = (p_E \hat{\otimes}_K L)^*(E \otimes_K L)$$

is a projective bundle for $E \otimes_K L$.

Indeed, by Paragraph 4.33 we may assume that E is trivial. Then $E \otimes_K L$ is also trivial and we may use Paragraph 4.31 to compare $P(E \otimes_K L)$ and $P(E) \hat{\otimes}_K L$.

4.37 Lemma. *Let E be a vector bundle on V with projective bundle $p\colon P(E) \to V$. We denote by $i\colon \mathcal{O}_E(-1) \to p^*E$ the embedding of the tautological line sub-bundle of p^*E. Recall from Lemma 4.10 that there is an induced morphism $p'\colon \mathrm{Tot}(p^*E) \to \mathrm{Tot}(E)$. The composition*

$$\sigma_E\colon \mathrm{Tot}(\mathcal{O}_E(-1)) \xrightarrow{\mathrm{Tot}(i)} \mathrm{Tot}(p^*E) \xrightarrow{p'} \mathrm{Tot}(E)$$

satisfies $\sigma_E^{-1}(\{0\}) = \{0\}$, where $\{0\}$ denotes the zero section of a vector bundle, and restricts to an isomorphism

$$\sigma_E\colon \mathrm{Tot}(\mathcal{O}_E(-1)) \setminus \{0\} \xrightarrow{\sim} \mathrm{Tot}(E) \setminus \{0\}.$$

Furthermore, the morphism σ_E fits into the commutative diagram

$$\begin{tikzcd} \mathrm{Tot}(\mathcal{O}_E(-1)) \arrow[r, "\sigma_E"] \arrow[d, "\pi_{\mathcal{O}_E(-1)}"'] & \mathrm{Tot}(E) \arrow[d, "\pi_E"] \\ P(E) \arrow[r, "p"'] & V. \end{tikzcd}$$

Proof. This can be checked locally on V so we can assume that E is trivial. In that case, E arises as the pull-back of the trivial vector bundle K^{e+1} along the structural morphism $V \to \mathcal{M}(K)$. It is easy to check that the statement of the lemma is stable under pull-back, so we may reduce to the case $V = \mathcal{M}(K)$ and $E = K^{e+1}$. Then the composition in question is the analytification of the morphism

$$\mathrm{Tot}(\mathcal{O}_{\mathbf{P}^e_K}(-1)) \to \mathbf{P}^e_K \times \mathbf{A}^{e+1}_K \to \mathbf{A}^{e+1}_K$$

of algebraic varieties which is well-known from algebraic geometry to realize the total space of the line bundle $\mathcal{O}_{\mathbf{P}^e_K}(-1)$ as the blow-up of $\mathbf{A}^{e+1}_K$ in the origin. ∎

4.38 Twisting. Let E be a vector bundle on V and let L be a line bundle on V. On $P(E)$ we have the tautological line bundle $\mathcal{O}_E(-1) \subset p_E^* E$. We obtain an induced line sub-bundle $\mathcal{O}_E(-1) \otimes p_E^* L \subset p_E^* E \otimes p_E^* L = p_E^*(E \otimes L)$. Hence we obtain a unique morphism of K-analytic spaces $\tau_{E,L} \colon P(E) \to P(E \otimes L)$ over V such that $p_{E\otimes L} \circ \tau_{E,L} = p_E$ and

$$\tau_{E,L}^* \mathcal{O}_{E\otimes L}(-1) = \mathcal{O}_E(-1) \otimes p_E^* L$$

as line sub-bundles of

$$\tau_{E,L}^* p_{E\otimes L}^*(E \otimes L) = p_E^*(E \otimes L) = p_E^* E \otimes p_E^* L.$$

4.39 Lemma. *Let E be a vector bundle on V and let L be a line bundle on V. The canonical morphism $\tau = \tau_{E,L} \colon P(E) \to P(E \otimes L)$ of K-analytic spaces over V is an isomorphism.*

Proof. It follows from the definitions that for any morphism $h \colon W \to V$ of K-analytic spaces the diagram

$$\begin{tikzcd} \mathrm{Hom}_V(W, P(E)) \arrow[r, "\cong"] \arrow[d, "\tau_*"'] & \{\text{Line sub-bundles of } h^*E\} \arrow[d] \\ \mathrm{Hom}_V(W, P(E \otimes L)) \arrow[r, "\cong"'] & \{\text{Line sub-bundles of } h^*(E \otimes L)\} \end{tikzcd}$$

commutes. Here the horizontal morphisms are the bijections from the definition of a projective bundle in Paragraph 4.28 given by $l \mapsto l^*\mathscr{O}_E(-1)$ and $l \mapsto l^*\mathscr{O}_{E\otimes L}(-1)$ respectively. The map on the left is given by $l \mapsto \tau \circ l$ while the map on the right is given by $M \mapsto M \otimes h^*L$.

Since the map on the right is obviously a bijection, the same must be true for the map on the left. By Yoneda's lemma, the map τ must be an isomorphism. ∎

4.40 Analytification of Projective Bundles. Let X be a K-scheme of locally finite type and let E be a vector bundle on E. We define the *projective bundle of lines* of E to be the relative Proj-construction

$$P(E) \coloneqq \operatorname{Proj}(\operatorname{Sym}(E^\vee))$$

of the graded quasi-coherent algebra $\operatorname{Sym}(E^\vee)$. It comes with a canonical structure morphism $p_E\colon P(E) \to X$. In the notation of [GW20, Sec. 8.8, Sec. 13.8] we have $P(E) = \mathbf{P}(E^\vee)$. There is a canonical quotient line bundle of $p_E^*E^\vee$ or equivalently a canonical line sub-bundle $\mathscr{O}_E(-1) \subset p_E^*E$ which satisfies a universal property which is similar to the one of Paragraph 4.28.

The analytification $p_E^{\mathrm{an}}\colon P(E)^{\mathrm{an}} \to X^{\mathrm{an}}$ together with the line sub-bundle $\mathscr{O}_E(-1)^{\mathrm{an}} \subset (p_E^*E)^{\mathrm{an}} = (p_E^{\mathrm{an}})^*E^{\mathrm{an}}$ is a projective bundle for E^{an}. The proof is completely analogous to Paragraph 4.36.

4.41 Proposition. *Let E be a vector bundle on V. The projection from the projective bundle $p_E\colon P(E) \to V$ is a flat and proper morphism of K-analytic spaces and in particular topologically proper and open.*

Proof. By [Tem15, Fact 4.2.4.3 (i)], the class of proper morphisms of K-analytic spaces is G-local on the base and the same holds for flat morphisms by [Duc18, § 4.1.12]. We may therefore assume that E is trivial so that $p_E\colon P(E) \to V$ is given by the projection to the first factor $V \times \mathbf{P}_K^{e,\mathrm{an}} \to V$, i.e. the pull-back of the structural morphism $\mathbf{P}_K^{e,\mathrm{an}} \to \mathscr{M}(K)$ along the structural morphism $V \to \mathscr{M}(K)$. By [Tem15, Fact 4.2.4.3 (i)], the class of proper morphisms of K-analytic spaces is stable under pull-back and the same holds for flat morphisms by [Duc18, § 4.1.9]. It is therefore enough to show that $\mathbf{P}_K^{e,\mathrm{an}} \to \mathscr{M}(K)$ is proper and flat. Properness is due to the GAGA statement [Ber90, Prop. 3.4.7] whereas flatness follows from [Duc18, Lem. 4.1.13] which shows that every morphism to $\mathscr{M}(K)$ is flat.

Now topological properness follows from Lemma 2.23. Since proper morphisms are in particular boundaryless, the openness follows from Proposition 2.20. ∎

5. Continuous Metrics

Throughout this chapter, K will be a non-archimedean field and V will denote a K-analytic space. In Paragraph 5.1 we introduce *metrics* on vector bundles over V. We study constructions with metrics like *pull-backs* (Paragraph 5.5), *duals* (Paragraph 5.11), *Fubini-Study metrics* (Paragraph 5.12), *direct sums* (Paragraph 5.14) and *tensor products* (Paragraph 5.15). Fiberwise, these constructions are given by the constructions with norms discussed in Chapter 1. We have a notion of *continuity* of metrics (Paragraph 5.1) for which we prove several permanence properties. In particular we show in Proposition 5.13 that a metric on a vector bundle E is continuous if and only if the induced Fubini-Study metric on $\mathcal{O}_E(1)$ is continuous. In Paragraph 5.31 we compare our notion of metrized vector bundles with alternatives introduced in [CD12; CM20]. In Paragraph 5.32 we define *pseudo-metrics*, a weakening of the notion of a metric which is sufficient for our construction of characteristic currents in Chapter 9.

5.1 Metrics. Let E be a vector bundle on V. A *metric* on E is a family $\{\|-\|_x\}_{x\in V}$, where each $\|-\|_x : E(x) \to \mathbf{R}_{\geq 0}$ is a norm on the finite-dimensional $\mathscr{H}(x)$-vector space $E(x)$.

Recall from Proposition 4.16 that for every point $x \in V$ we have a canonical bijection of sets

$$\operatorname*{colim}_{L/\mathscr{H}(x)} E(x) \otimes_{\mathscr{H}(x)} L \xrightarrow{\sim} \pi_E^{-1}(x).$$

If $x \in V$ is a point and $L/\mathscr{H}(x)$ is a non-archimedean field extension, then by Paragraph 1.14, we have the scalar extension norm $\|-\|_{x,L} : E(x) \otimes_{\mathscr{H}(x)} L \to \mathbf{R}_{\geq 0}$.

Using Proposition 1.16, these maps glue to a map of sets

$$\|-\|_{\mathrm{Tot},x}\colon \pi_E^{-1}(x) \cong \operatorname*{colim}_{L/\mathscr{H}(x)} E(x) \otimes_{\mathscr{H}(x)} L \to \mathbf{R}_{\geq 0}$$

Letting x vary, we get a map of sets

$$\|-\|_{\mathrm{Tot}}\colon \mathrm{Tot}(E) \cong \coprod_{x\in V} \pi_E^{-1}(x) \to \mathbf{R}_{\geq 0}$$

induced from the metric $\|-\|$.

We call the metric $\|-\|$ *continuous* if the map $\|-\|_{\mathrm{Tot}}\colon \mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$ is continuous with respect to the Berkovich topology on $\mathrm{Tot}(E)$. Since continuity of functions is a G-local condition by Lemma 2.12, continuous metrics on E form a sheaf of sets for the G-topology of V and in particular for the Berkovich topology of V.

We call the pair $\overline{E} = (E, \|-\|)$ a *metrized vector bundle* on V. We denote by $\overline{E}(x) \coloneqq (E(x), \|-\|_x)$ the fiber vector space $E(x)$ together with its norm $\|-\|_x$ given by the metric $\|-\|$.

If X is a finite type scheme over K and E is a vector bundle on X, then by a metric on E, we mean a metric on E^{an}.

5.2 Continuously Diagonalizable Metrics. Let E be a trivial vector bundle on V with frame $s_1, \dots, s_r \in \Gamma(V, E)$. Let $\phi_1, \dots, \phi_r\colon V \to \mathbf{R}_{>0}$ be continuous positive functions on V. Then there is a metric $\|-\|_{\underline{s},\underline{\phi}}$ on E such that for $x \in V$ the norm $\|-\|_{\underline{s},\underline{\phi},x}$ on $E(x)$ is diagonalizable in the sense of Paragraph 1.20 with $s_1(x), \dots, s_r(x)$ as an orthogonal basis and such that $\|s_i(x)\|_{\underline{s},\underline{\phi},x} = \phi_i(x)$. This means that for $a_1, \dots, a_r \in \mathscr{H}(x)$ we have

$$\|a_1 s_1(x) + \cdots + a_r s_r(x)\|_{\underline{s},\underline{\phi},x} = \max_{i=1,\dots,r} |a_i| \phi_i(x).$$

We can describe the map $\|-\|_{\underline{s},\underline{\phi},\mathrm{Tot}}$ as follows: By Paragraph 4.9 the frame $s_1, \dots, s_r$ induces an identification $\mathrm{Tot}(E) = V \times \mathbf{A}_K^{r,\mathrm{an}}$ such that the projection $\pi_E\colon \mathrm{Tot}(E) \to V$ is given by the projection to the first factor. Denote by $T_1, \dots, T_r\colon \mathrm{Tot}(E) = V \times \mathbf{A}_K^{r,\mathrm{an}} \to \mathbf{A}_K^{r,\mathrm{an}} \to \mathbf{A}_K^{1,\mathrm{an}}$ the coordinate functions. We can also view $T_i \in \mathscr{O}(\mathrm{Tot}(E))$. It follows from [BE21, Prop. 1.25 (iv)] that under this identification, the map $\|-\|_{\underline{s},\underline{\phi},\mathrm{Tot}}\colon \mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$ is given by

$$\|-\|_{\underline{s},\underline{\phi},\mathrm{Tot}}\colon \mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}, \qquad v \mapsto \max_{i=1,\dots,r} \phi_i(\pi_E(v))|T_i(v)|,$$

which is continuous by the definition of the Berkovich topology.

In the case where $\phi_1, \dots, \phi_r \equiv 1$ on V we simply write $\|-\|_{\underline{s}}$ instead of $\|-\|_{\underline{s},\underline{\phi}}$ and call it the *orthonormal* metric associated to the frame $\underline{s}$.

5.3 Sections. Let $\overline{E}$ be a metrized vector bundle on V and let $s \in \Gamma(V, E)$ be a section. Then there is an induced function

$$\|s(-)\| : V \to \mathbf{R}_{\geq 0}, \qquad x \mapsto \|s(x)\|_x,$$

where $s(x) \in E(x)$ is defined as in Paragraph 4.4.

It follows from Lemma 4.20 that in the notation of Paragraph 4.18 we have $\|s(x)\| = \|\tilde{s}(x)\|_{\mathrm{Tot}}$. In particular, we see that if the metric $\|-\|$ is continuous, then the function $\|s(-)\|$ is continuous for every section $s \in \Gamma(V, E)$.

5.4 Metrics on Line Bundles. Let $\overline{L}$ be a metrized line bundle on V. Suppose that $U \subset V$ is an analytic domain and that $s \in \Gamma(U, L)$ is a G-local nowhere-vanishing section of L. Then for each $x \in U$, the $\mathscr{H}(x)$-vector space $L(x)$ is one-dimensional with basis $s(x)$. It follows that the norm $\|-\|_x$ is diagonalizable with orthogonal basis $s(x)$. In particular, we see that the restriction of the metric $\|-\|$ to U is completely determined by the function $\|s(-)\| : U \to \mathbf{R}_{>0}$. If the function $\|s(-)\|$ is continuous, then the metric $\|-\|$ is continuous on U by Paragraph 5.2.

We see that for line bundles, continuous metrics can be equivalently described by associating to a trivialization $\{(V_i, s_i)\}_{i \in I}$ of L a family of continuous positive functions $\|s_i(-)\| : V_i \to \mathbf{R}_{>0}$ satisfying

$$\|s_i(-)\| = |(s_i/s_j)(-)| \|s_j(-)\|$$

on the overlaps $V_i \cap V_j$.

5.5 Pull-backs. Let $\overline{E}$ be a metrized vector bundle on V and let $f : V' \to V$ be a morphism of K-analytic spaces. Given $x' \in V'$, we have an identification $(f^*E)(x') \cong E(f(x')) \otimes_{\mathscr{H}(f(x'))} \mathscr{H}(x')$ by Lemma 4.5. Denoting by $f^*\|-\|_{x'}$ the scalar extension norm induced by $\|-\|_{f(x')}$ on $(f^*E)(x')$, we get a metric $f^*\|-\|$ on f^*E. It is easy to see using Proposition 1.15 (iv) that the diagram

$$\begin{tikzcd} & & \mathbf{R}_{\geq 0} \\ \mathrm{Tot}(f^*E) \arrow[r, "f'"'] \arrow[urr, "f^*\|-\|_{\mathrm{Tot}}"] & \mathrm{Tot}(E) \arrow[ur, "\|-\|_{\mathrm{Tot}}"'] & \end{tikzcd}$$

commutes. In particular we see that if $\|-\|$ is continuous, then $f^*\|-\|$ is continuous.

We write $f^*\overline{E} := (f^*E, f^*\|-\|)$.

5.6 Lemma. *Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle on V and let $f : V' \to V$ be a morphism of K-analytic spaces. Let $s \in \Gamma(V, E)$ be a section. Then we have a* *commutative diagram*

$$\begin{tikzcd} V' \arrow[r, "f"] \arrow[dr, "f^*\|f^*s(-)\|"'] & V \arrow[d, "\|s(-)\|"] \\ & \mathbf{R}_{\geq 0}. \end{tikzcd}$$

Proof. Recall from Paragraph 5.3 that the function $\|s(-)\|$ agrees with the composition

$$V \xrightarrow{\tilde{s}} \mathrm{Tot}(E) \xrightarrow{\|-\|_{\mathrm{Tot}}} \mathbf{R}_{\geq 0}$$

and similarly for the function $f^*\|f^*s(-)\|$. Hence it suffices to note that the diagram

$$\begin{tikzcd} V' \arrow[r, "f"] \arrow[d, "\widetilde{f^*s}"'] & V \arrow[d, "\tilde{s}"] \\ \mathrm{Tot}(f^*E) \arrow[r, "f'"'] \arrow[dr, "f^*\|-\|_{\mathrm{Tot}}"'] & \mathrm{Tot}(E) \arrow[d, "\|-\|_{\mathrm{Tot}}"] \\ & \mathbf{R}_{\geq 0} \end{tikzcd}$$

commutes by the commutative diagrams in Lemma 4.19 and Paragraph 5.5. ∎

5.7 Change of Fields. Let $\overline{E}$ be a metrized vector bundle on V and assume that L/K is an extension of non-archimedean fields. Recall from Paragraph 4.13 that we write $E \otimes_K L := \pi_{L/K}^* E$ where $\pi_{L/K} \colon V \hat{\otimes}_K L \to V$ denotes the canonical base-change morphism. By Lemma 4.5, we have a canonical identification $(E \otimes_K L)(x') \cong E(\pi_{L/K}(x')) \otimes_{\mathscr{H}(\pi_{L/K}(x'))} \mathscr{H}(x')$ for $x' \in V \hat{\otimes}_K L$. Similarly to Paragraph 5.5 above, we get a metric $\|-\| \otimes_K L$ on $E \otimes_K L$. By Paragraph 4.13 we have an identification $\mathrm{Tot}(E \otimes_K L) = \mathrm{Tot}(E) \hat{\otimes}_K L$. In particular, we have a canonical map

$$\pi_{L/K} \colon \mathrm{Tot}(E \otimes_K L) = \mathrm{Tot}(E) \hat{\otimes}_K L \to \mathrm{Tot}(E).$$

One checks easily that the diagram

$$\begin{tikzcd} & & \mathbf{R}_{\geq 0} \\ \mathrm{Tot}(E \otimes_K L) \arrow[r] \arrow[urr, "(\|-\| \otimes_K L)_{\mathrm{Tot}}"] & \mathrm{Tot}(E) \arrow[ur, "\|-\|_{\mathrm{Tot}}"'] & \end{tikzcd}$$

commutes.

We write $\overline{E} \otimes_K L := (E \otimes_K L, \|-\| \otimes_K L)$.

5.8 Lemma. *Let the situation be as in Paragraph 5.7. Then the metric $\|-\|$ is continuous if and only if the metric $\|-\| \otimes_K L$ is continuous.*

Proof. It follows from Lemma 2.17 (i) that the canonical map $\pi_{L/K} \colon \operatorname{Tot}(E \hat{\otimes}_K L) \to \operatorname{Tot}(E)$ is topologically proper and surjective so that $\operatorname{Tot}(E)$ is equipped with the quotient topology with respect to this map. The claim now follows from the commutative diagram in Paragraph 5.7. ∎

5.9 Sub-Bundles. If $\overline{E} = (E, \|-\|)$ is a metrized vector bundle on V and $F \subset E$ is a vector sub-bundle in the sense of Paragraph 4.27, then there is an induced metric $\|-\||_F$ on F given as follows: For each $x \in V$, we have by Paragraph 4.27 an induced embedding $F(x) \subset E(x)$ and we take as a norm on $F(x)$ the restriction of $\|-\|_x$ with respect to this embedding, as in Paragraph 1.13. It follows from Lemma 5.10 below that if F is of rank 1, i.e. a line sub-bundle, and the norm $\|-\|$ on E is continuous then the restricted norm $\|-\||_F$ is continuous on F.

5.10 Lemma. *Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle on V and let $F \subset E$ be a line sub-bundle. If we denote by $i \colon \operatorname{Tot}(F) \hookrightarrow \operatorname{Tot}(E)$ the canonical inclusion morphism and by $\operatorname{Tot}(i) \colon \operatorname{Tot}(F) \to \operatorname{Tot}(E)$ the induced morphism according to Paragraph 4.21 then the diagram*

$$\begin{tikzcd}
 & & \mathbf{R}_{\geq 0} \\
\operatorname{Tot}(F) \arrow[r, hook, "\operatorname{Tot}(i)"'] \arrow[urr, "(\|-\||_F)_{\mathrm{Tot}}"] & \operatorname{Tot}(E) \arrow[ur, "\|-\|_{\mathrm{Tot}}"'] &
\end{tikzcd}$$

is commutative.

Proof. We write $\|-\|' := \|-\||_F$ for the restricted norm. By definition of the restricted norm, for every point $x \in V$ the induced embedding $i(x) \colon \overline{F}(x) \hookrightarrow \overline{E}(x)$ is isometric. Let $v \in \operatorname{Tot}(F)$ be a point over $x := \pi_F(v) \in V$. Under the identification $\pi_F^{-1}(x) = \operatorname{colim}_{L/K} F(x) \otimes_{\mathscr{H}(x)} L$ it is given by a non-archimedean field extension $L/\mathscr{H}(x)$ and a vector $v \in F(x) \otimes_{\mathscr{H}(x)} L$ and we have $\|v\|'_{\mathrm{Tot}} = \|v\|'_{x,L}$ where $\|-\|'_{x,L}$ denotes the scalar extension norm of $\|-\|'_x$ to $F(x) \otimes_{\mathscr{H}(x)} L$. The point $\operatorname{Tot}(i)(v) \in \pi_E^{-1}(x) \subset \operatorname{Tot}(E)$ is given by the same field extension $L/\mathscr{H}(x)$ and the image $(i(x) \otimes L)(v) \in E(x) \otimes_{\mathscr{H}(x)} L$. Since by Lemma 1.17 (i) the embedding $i(x) \otimes L \colon \overline{F}(x) \otimes_{\mathscr{H}(x)} L \to \overline{E}(x) \otimes_{\mathscr{H}(x)} L$ is isometric, we have

$$\|\operatorname{Tot}(i)(v)\|_{\mathrm{Tot}} = \|(i(x) \otimes L)(v)\|_{x,L} = \|v\|'_{x,L} = \|v\|'_{\mathrm{Tot}}.$$ ∎

5.11 Dual Metrics. Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle on V. There is a metric $\|-\|^\vee$ on $E^\vee$ given as follows: For $x \in V$, there is a canonical identification

$E^\vee(x) \cong E(x)^\vee$ of $\mathscr{H}(x)$-vector spaces and we take on $E^\vee(x)$ the dual norm induced by $\|-\|_x$ as in Paragraph 1.8. We write $\overline{E}^\vee := (E^\vee, \|-\|^\vee)$.

If $\overline{E} = \overline{L}$ is a metrized line bundle and the metric is continuous, then the dual metric on $L^\vee$ is again continuous. Indeed, if s is a local nowhere-vanishing section of L, then there is an induced nowhere-vanishing section s^{-1} of $L^\vee$ and we have

$$\|s^{-1}(x)\|^\vee = \|s(x)\|^{-1}$$

for $x \in V$.

5.12 Fubini-Study Metrics. Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle on V. Denote by $p\colon P(E) \to V$ its projective bundle with the tautological line subbundle $\mathscr{O}_E(-1) \subset p^*E$. Using Paragraphs 5.5 and 5.9, we get a canonical metric on $\mathscr{O}_E(-1)$ by restricting the pull-back metric $p^*\|-\|$ to $\mathscr{O}_E(-1)$, which we call the *dual Fubini-Study metric* associated to $\|-\|$ and which we denote by $\|-\|^\vee_{\mathrm{FS}}$. We write $\mathscr{O}_{\overline{E}}(-1) := (\mathscr{O}_E(-1), \|-\|^\vee_{\mathrm{FS}})$.

The dual metric of $\|-\|^\vee_{\mathrm{FS}}$ on $\mathscr{O}_E(1)$ is called the *Fubini-Study metric* associated to $\|-\|$ and is denoted by $\|-\|_{\mathrm{FS}}$. We write $\mathscr{O}_{\overline{E}}(1) := (\mathscr{O}_E(1), \|-\|_{\mathrm{FS}}) = \mathscr{O}_{\overline{E}}(-1)^\vee$. It follows from Paragraphs 5.5, 5.9 and 5.11 that if $\|-\|$ is continuous, then so are $\|-\|^\vee_{\mathrm{FS}}$ and $\|-\|_{\mathrm{FS}}$. In fact, the converse also holds by Proposition 5.13 below.

5.13 Proposition. *Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle on V. Then the following are equivalent:*

(i) The metric $\|-\|$ is continuous.

(ii) The induced dual Fubini-Study metric $\|-\|^\vee_{\mathrm{FS}}$ on $\mathscr{O}_E(-1)$ is continuous.

(iii) The induced Fubini-Study metric $\|-\|_{\mathrm{FS}}$ on $\mathscr{O}_E(1)$ is continuous.

Proof. The equivalence of (ii) and (iii) follows from Paragraph 5.11. We have already noted in Paragraph 5.12 that (i) implies (ii), so it remains to prove that if $\|-\|^\vee_{\mathrm{FS}}$ is continuous, then $\|-\|$ is continuous. It follows from the commutative diagrams in Paragraph 5.5 and Lemma 5.10 that the diagram

$$\begin{tikzcd}
 & & & \mathbf{R}_{\geq 0} \\
\mathrm{Tot}(\mathscr{O}_E(-1)) \arrow[r, "\mathrm{Tot}(i)"'] \arrow[urrr, bend left, "(\|-\|^\vee_{\mathrm{FS}})_{\mathrm{Tot}}"] & \mathrm{Tot}(p^*E) \arrow[r, "p'"'] \arrow[urr, "p^*\|-\|_{\mathrm{Tot}}"] & \mathrm{Tot}(E) \arrow[ur, "\|-\|_{\mathrm{Tot}}"'] &
\end{tikzcd}$$

commutes. By Lemma 4.37, the lower composition restricts to an isomorphism $\mathrm{Tot}(\mathscr{O}_E(-1)) \setminus \{0\} \xrightarrow{\sim} \mathrm{Tot}(E) \setminus \{0\}$. We can therefore conclude that the map $\|-\|_{\mathrm{Tot}}\colon \mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$ is continuous in all points of $\mathrm{Tot}(E) \setminus \{0\}$. It remains to show that it is also continuous in the points of the zero section. To do so, we may work G-locally on V by Lemma 2.12 and hence we may assume that V is compact and that E is trivialized by sections $s_1, \dots, s_r \in \Gamma(V, E)$. Since $s_1, \dots, s_r$ are nowhere-vanishing, we can conclude from the continuity on $\mathrm{Tot}(E) \setminus \{0\}$ that the functions $\|s_i(-)\| : V \to \mathbf{R}_{>0}$ are continuous as in Paragraph 5.3. Let $S_i := \sup_{x\in V} \|s_i(x)\|$. We denote by $T_i \in \mathcal{O}(\mathrm{Tot}(E))$ the coordinate functions induced by the frame $s_1, \dots, s_r$ (as in Paragraph 5.2).

Now let $\epsilon > 0$ and consider the domain

$$U = \{ v \in \mathrm{Tot}(E) \mid |T_i(v)| \le \epsilon/S_i \}.$$

Since U is a neighborhood of all points in the zero section, it is enough to show that U is mapped into the interval $[0, \epsilon]$ by $\|-\|_{\mathrm{Tot}}$. Suppose that $v \in U \cap \pi_E^{-1}(x)$ is given as $v \in E(x) \otimes_{\mathscr{H}(x)} L$ in the colimit description of Paragraph 5.1. Write $v = a_1 s_1(x) + \cdots + a_r s_r(x)$ with $a_i \in L$. Then $|a_i| = |T_i(v)| \le \epsilon/S_i$ and we see that

$$\|v\|_{\mathrm{Tot}} \le \max_{i=1,\dots,r} |a_i| \|s_i(x)\| \le \max_{i=1,\dots,r} (\epsilon/S_i) \cdot S_i = \epsilon.$$

This finishes the proof. ∎

5.14 Direct Sums. Let $\overline{E} = (E, \|-\|)$, $\overline{F} = (F, \|-\|')$ be metrized vector bundles on V. Given $x \in V$, we have by Paragraph 4.23 an identification $(E \oplus F)(x) \cong E(x) \oplus F(x)$. Equipping the fiber $(E \oplus F)(x)$ with the direct sum norm induced from $\|-\|_x$ and $\|-\|'_x$, we get a metric $\|-\| \oplus \|-\|'$ on $E \oplus F$. Denoting by $\mathrm{pr}_1 : E \oplus F \to E$ and $\mathrm{pr}_2 : E \oplus F \to F$ the two natural projections, it follows from Proposition 1.15 (ii) that the map $(\|-\| \oplus \|-\|')_{\mathrm{Tot}} : \mathrm{Tot}(E \oplus F) \to \mathbf{R}_{\ge 0}$ is given by

$$v \mapsto \max\{ \|\mathrm{Tot}(\mathrm{pr}_1)(v)\|_{\mathrm{Tot}}, \|\mathrm{Tot}(\mathrm{pr}_2)(v)\|'_{\mathrm{Tot}} \}.$$

In particular, we see that if $\|-\|$ and $\|-\|'$ are continuous, then $\|-\| \oplus \|-\|'$ is again continuous.

We write $\overline{E} \oplus \overline{F} := (E \oplus F, \|-\| \oplus \|-\|')$.

5.15 Tensor Products. Let $\overline{E} = (E, \|-\|)$, $\overline{F} = (F, \|-\|')$ be metrized vector bundles on V. Given $x \in V$, we have by Paragraph 4.22 an identification $(E \otimes F)(x) \cong E(x) \otimes_{\mathscr{H}(x)} F(x)$. Equipping the fiber $(E \otimes F)(x)$ with the tensor product norm induced from $\|-\|_x$ and $\|-\|'_x$, we get a metric $\|-\| \otimes \|-\|'$ on $E \otimes F$. We write $\overline{E} \otimes \overline{F} := (E \otimes F, \|-\| \otimes \|-\|')$.

If the metrics $\|-\|$ and $\|-\|'$ are continuous and either E or F is a line bundle, then $\|-\| \otimes \|-\|'$ is again continuous, we show this in Corollary 5.20 below.

For vector bundles of arbitrary rank it seems difficult to prove that the tensor product of continuous norms remains continuous.

5.16 Lemma. *Let $f: V' \to V$ be a morphism of K-analytic spaces and let $\overline{E} = (E, \|-\|)$ and $\overline{F} = (F, \|-\|')$ be two metrized vector bundles on V. On the vector bundle*

$$f^*(E \otimes F) = f^*E \otimes f^*F$$

we have an equality of metrics

$$f^*(\|-\| \otimes \|-\|') = f^*\|-\| \otimes f^*\|-\|'$$

and hence we have an identification of metrized vector bundles

$$f^*(\overline{E} \otimes \overline{F}) = f^*\overline{E} \otimes f^*\overline{F}.$$

Proof. Let $x' \in V'$ be a point with image $x := f(x') \in V$. Viewing the fiber vector spaces at x' as normed $\mathscr{H}(x')$-vector spaces, we have

$$f^*(\overline{E} \otimes \overline{F})(x') = (\overline{E}(x) \otimes \overline{F}(x)) \otimes_{\mathscr{H}(x)} \mathscr{H}(x')$$

and

$$(f^*\overline{E} \otimes f^*\overline{F})(x') = (\overline{E}(x) \otimes_{\mathscr{H}(x)} \mathscr{H}(x')) \otimes_{\mathscr{H}(x')} (\overline{F}(x) \otimes_{\mathscr{H}(x)} \mathscr{H}(x'))$$

By Proposition 1.15 (iii) the respective metrics agree under the canonical identification of the underlying vector spaces. ∎

5.17 Lemma. *Let $g: V'' \to V'$ and $f: V' \to V$ be morphisms of K-analytic spaces and let $\overline{E}$ be a metrized vector bundle on V. Then the canonical identification*

$$(f \circ g)^*E = g^*f^*E$$

of vector bundles on V'' is an identity of metrized vector bundles

$$(f \circ g)^*\overline{E} = g^*f^*\overline{E}.$$

Proof. This follows similarly to Lemma 5.16 from Proposition 1.15 (iv). ∎

5.18 Lemma. *Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle on V and let $F \subset E$ be a vector sub-bundle equipped with the induced norm $\|-\||_F$ according to Paragraph 5.9.*

(i) Assume that $\overline{F}$ has rank 1. If $f: V' \to V$ is a morphism of K-analytic spaces, then we have an equality

$$f^*(\|-\||_F) = (f^*\|-\|)|_{f^*F}$$

of metrics on the line sub-bundle $f^*F \subset f^*E$.

(ii) If $\overline{G} = (G, \|-\|')$ is a metrized line bundle on V, then we have an equality

$$\|-\||_F \otimes \|-\|' = (\|-\| \otimes \|-\|')|_{F \otimes G}$$

of metrics on the vector sub-bundle $F \otimes G \subset E \otimes G$.

Proof. By considering the fiber vector spaces these statements follow from the two statements of Lemma 1.17. ∎

5.19 Proposition. *Let $\overline{E} = (E, \|-\|)$ be a metrized vector bundle and $\overline{L} = (L, \|-\|')$ a metrized line bundle on V. Denote by $p_E : P(E) \to V$ the projective bundle of E and by $\tau = \tau_{E,L} : P(E) \xrightarrow{\sim} P(E \otimes L)$ the canonical isomorphism of Paragraph 4.38. We write $\|-\|_{\mathrm{FS}}^{\vee}$ both for the dual Fubini-Study metric on $\mathcal{O}_{\overline{E}}(-1)$ and on $\mathcal{O}_{\overline{E} \otimes \overline{L}}(-1)$ induced from $\overline{E}$ and $\overline{E} \otimes \overline{L}$ respectively. Then on the line bundle*

$$\tau^* \mathcal{O}_{E \otimes L}(-1) = \mathcal{O}_E(-1) \otimes p_E^* L$$

on $P(E)$ we have an equality of metrics

$$\tau^* \|-\|_{\mathrm{FS}}^{\vee} = \|-\|_{\mathrm{FS}}^{\vee} \otimes p_E^* \|-\|'$$

and hence we have an identity of metrized line bundles

$$\tau^* \mathcal{O}_{\overline{E} \otimes \overline{L}}(-1) = \mathcal{O}_{\overline{E}}(-1) \otimes p_E^* \overline{L}.$$

Proof. By Lemmas 5.16 and 5.17 we have an equality of metrized vector bundles

$$\tau^* p_{E \otimes L}^* (\overline{E} \otimes \overline{L}) = p_E^* (\overline{E} \otimes \overline{L}) = p_E^* \overline{E} \otimes p_E^* \overline{L}$$

on $P(E)$. By definition, $\mathcal{O}_{\overline{E} \otimes \overline{L}}(-1)$ carries the sub-bundle metric induced from the embedding $\mathcal{O}_{E \otimes L}(-1) \subset p_{E \otimes L}^* (\overline{E} \otimes \overline{L})$. By Lemma 5.18 (i) the pull-back $\tau^* \mathcal{O}_{\overline{E} \otimes \overline{L}}(-1)$ carries the sub-bundle metric induced from the embedding

$$\tau^* \mathcal{O}_{E \otimes L}(-1) \subset \tau^* p_{E \otimes L}^* (\overline{E} \otimes \overline{L}) = p_E^* (\overline{E} \otimes \overline{L}).$$

Similarly, by definition, $\mathcal{O}_{\overline{E}}(-1)$ carries the sub-bundle metric induced from the embedding $\mathcal{O}_E(-1) \subset p_E^* \overline{E}$. So by Lemma 5.18 (ii) the tensor product $\mathcal{O}_{\overline{E}}(-1) \otimes p_E^* \overline{L}$ carries the sub-bundle metric induced from the embedding

$$\mathcal{O}_{\overline{E}}(-1) \otimes p_E^* \overline{L} \subset p_E^* \overline{E} \otimes p_E^* \overline{L} = p_E^* (\overline{E} \otimes \overline{L}).$$

It follows that as metrized line sub-bundles of $p_E^* (\overline{E} \otimes \overline{L})$ both line bundles agree. ∎

5.20 Corollary. *Let $\overline{E}$ be a continuously metrized vector bundle and $\overline{L}$ a continuously metrized line bundle on V. Then the twisted metrized vector bundle $\overline{E} \otimes \overline{L}$ is continuously metrized.*

Proof. If E is a line bundle, this is well-known and easy to show using the description of continuously metrized line bundles in Paragraph 5.4. The general case is reduced to this case by Proposition 5.13 and Proposition 5.19. ∎

5.21 Lemma. *Let $f\colon V' \to V$ be a morphism of K-analytic spaces and let E be a vector bundle on V. Denote by $\widetilde{f}\colon P(f^*E) \to P(E)$ the induced morphism of Lemma 4.32 determined by the equality*

$$\widetilde{f}^* \mathcal{O}_E(-1) = \mathcal{O}_{f^*E}(-1)$$

of line sub-bundles of $\widetilde{f}^* p_E^* E = p_{f^*E}^* f^* E$. If we equip $\widetilde{f}^* \mathcal{O}_E(-1)$ with the pull-back metric of the Fubini-Study metric on $\mathcal{O}_{\overline{E}}(-1)$ along $\widetilde{f}$ and if we equip $\mathcal{O}_{f^*E}(-1)$ with the Fubini-Study metric induced from $f^*\overline{E}$ then the above equality is in fact an equality of metrized line bundles

$$\widetilde{f}^* \mathcal{O}_{\overline{E}}(-1) = \mathcal{O}_{f^*\overline{E}}(-1).$$

Proof. Since the embedding $\mathcal{O}_{\overline{E}}(-1) \subset p_E^*\overline{E}$ is by construction isometric, it follows from Lemma 5.18 (i) that the embedding $\widetilde{f}^* \mathcal{O}_{\overline{E}}(-1) \subset \widetilde{f}^* p_E^* \overline{E} = p_{f^*E}^* f^* \overline{E}$ is isometric. The second identity of metrized vector bundles follows from the commutative diagram in Lemma 4.32 and Lemma 5.17.

Since the metric of $\mathcal{O}_{f^*\overline{E}}(-1)$ is by definition the sub-bundle metric induced from the inclusion $\mathcal{O}_{f^*\overline{E}}(-1) \subset p_E^* f^* \overline{E}$ this proves the claim. ∎

5.22 Convergence of Metrics. Let E be a vector bundle and let $\|-\|$, $\|-\|'$ be two metrics on E. For each $x \in V$, we have the distance $d(\|-\|_x, \|-\|'_x) \in \mathbf{R}_{\geq 0}$ as defined in Paragraph 1.21. This defines a function

$$d(\|-\|, \|-\|')\colon V \to \mathbf{R}_{\geq 0}, \qquad x \mapsto d(\|-\|, \|-\|')(x) := d(\|-\|_x, \|-\|'_x).$$

By Proposition 5.26 below, the function $d(\|-\|, \|-\|')$ is continuous if the metrics $\|-\|$ and $\|-\|'$ are continuous.

If $\{\|-\|_n\}_{n \in \mathbf{N}}$ is a sequence of metrics on E, then we say that it *converges (locally uniformly)* to a metric $\|-\|$ if the functions $d(\|-\|_n, \|-\|)\colon V \to \mathbf{R}_{\geq 0}$ converge locally uniformly to 0 as $n \to \infty$. This defines a topology on the set of all metrics by stipulating that a set Z of metrics is closed if and only if for every sequence $\{\|-\|_n\}_{n \in \mathbf{N}}$ of metrics in Z converging to a metric $\|-\|$ it follows that $\|-\| \in Z$. If V is compact, then the subspace of all continuous metrics is metrizable by the absolute distance

$$d_{\text{abs}}(\|-\|, \|-\|') := \sup_{x \in V} d(\|-\|, \|-\|')(x).$$

5.23 Lemma. *Let E be a vector bundle on V and let $\|-\|$, $\|-\|'$ be two metrics on E. Denote by $p: P(E) \to V$ the associated projective bundle and by $\pi : \operatorname{Tot}(E) \to V$ its total space. Then for all $x \in V$ we have*

$$\begin{aligned} d(\|-\|, \|-\|')(x) &= \sup_{v \in \pi^{-1}(x) \setminus \{0\}} |\log \|v\|_{\mathrm{Tot}} - \log \|v\|'_{\mathrm{Tot}}| \\ &= \sup_{y \in p^{-1}(x)} d(\|-\|^{\vee}_{\mathrm{FS}}, \|-\|'^{\vee}_{\mathrm{FS}})(y) \\ &= \sup_{y \in p^{-1}(x)} d(\|-\|_{\mathrm{FS}}, \|-\|'_{\mathrm{FS}})(y). \end{aligned}$$

Proof. Let us start by proving the first equality. Recall the colimit description of $\pi^{-1}(x)$ from Proposition 4.16. If $v \in \pi^{-1}(x) \setminus \{0\}$ is given as $v \in E(x) \otimes_{\mathscr{H}(x)} L$ for some field extension $L/\mathscr{H}(x)$, then $\|v\|_{\mathrm{Tot}}$ is by definition $\|v\|_{x,L}$ where $\|-\|_{x,L}$ is the scalar extension norm of $\|-\|_x$ along $L/\mathscr{H}(x)$. Hence the first equality follows from Lemma 1.22.

To prove the second equality, we recall that we have a commutative diagram

$$\begin{tikzcd} & & & \mathbf{R}_{\geq 0} \\ \operatorname{Tot}(\mathscr{O}_E(-1)) \arrow[r, "\operatorname{Tot}(i)"'] \arrow[urrr, bend left, "(\|-\|^{\vee}_{\mathrm{FS}})_{\mathrm{Tot}}"] \arrow[dr, "\pi_{\mathscr{O}_E(-1)}"'] & \operatorname{Tot}(p^*E) \arrow[r, "p'"'] \arrow[urr, "p^*\|-\|_{\mathrm{Tot}}"] \arrow[d, "\pi_{p^*E}"'] & \operatorname{Tot}(E) \arrow[ur, "\|-\|_{\mathrm{Tot}}"'] \arrow[d, "\pi"] & \\ & P(E) \arrow[r, "p"'] & V & \end{tikzcd}$$

and that the composition $p' \circ \operatorname{Tot}(i)$ restricts to an isomorphism $\operatorname{Tot}(\mathscr{O}_E(-1)) \setminus \{0\} \xrightarrow{\sim} \operatorname{Tot}(E) \setminus \{0\}$. It follows that

$$\begin{aligned} &\sup_{v \in \pi^{-1}(x) \setminus \{0\}} |\log \|v\|_{\mathrm{Tot}} - \log \|v\|'_{\mathrm{Tot}}| \\ = &\sup_{y \in p^{-1}(x)} \sup_{w \in \pi^{-1}_{\mathscr{O}_E(-1)}(y) \setminus \{0\}} |\log \|w\|^{\vee}_{\mathrm{FS,Tot}} - \log \|w\|'^{\vee}_{\mathrm{FS,Tot}}|. \end{aligned} \tag{5.23.1}$$

Applying the first equality of this lemma to $\|-\|^{\vee}_{\mathrm{FS}}$ and $\|-\|'^{\vee}_{\mathrm{FS}}$, we see that

$$\sup_{w \in \pi^{-1}_{\mathscr{O}_E(-1)}(y) \setminus \{0\}} |\log \|w\|^{\vee}_{\mathrm{FS,Tot}} - \log \|w\|'^{\vee}_{\mathrm{FS,Tot}}| = d(\|-\|^{\vee}_{\mathrm{FS}}, \|-\|'^{\vee}_{\mathrm{FS}})(y).$$

Plugging this into Eq. (5.23.1), we get the second equality.

The third equality follows from Lemma 1.23. ∎

5.24 Corollary. *Let E be a vector bundle on V and let $\{\|-\|_n\}_{n \in \mathbf{N}}$ be a sequence of metrics on E and let $\|-\|$ also be a metric on E. Then the sequence $\{\|-\|_n\}_{n \in \mathbf{N}}$*

converges locally uniformly to $\|-\|$ *if and only if the sequence* $\{\|-\|_{n,\mathrm{FS}}\}_{n\in\mathbf{N}}$ *of metrics on* $\mathcal{O}_E(1)$ *converges locally uniformly to* $\|-\|_{\mathrm{FS}}$.

If V *is compact, then the assignment* $\|-\| \mapsto \|-\|_{\mathrm{FS}}$ *provides an isometric embedding of metric spaces from the space of continuous metrics on* E *to the space of continuous metrics on* $\mathcal{O}_E(1)$. ∎

In order to prove Proposition 5.26 below, we need the following point-set topological lemma. Recall from [Bou98, Chap. 1, § 10.2, Thm. 1] that a map $p\colon P \to X$ of topological spaces is proper if and only if it is closed and has quasi-compact fibers.

5.25 Lemma. *Let* $p\colon P \to X$ *be a surjective, proper and open map of topological spaces and let* $\phi\colon P \to \mathbf{R}$ *be a continuous function. Then the induced function*

$$\widetilde{\phi}\colon X \to \mathbf{R}, \qquad x \mapsto \sup_{y\in p^{-1}(x)} \phi(y)$$

is well-defined and continuous.

Proof. Since the fibers $p^{-1}(x)$ are quasi-compact, the function ϕ attains a supremum (in fact, a maximum) on $p^{-1}(x)$, so the map $\widetilde{\phi}$ is well-defined.

Let $x_0 \in X$ with $s_0 := \widetilde{\phi}(x_0) = \sup_{y\in p^{-1}(x)} \phi(y) \in \mathbf{R}$. Let $\epsilon > 0$ and consider the standard neighborhood $U_\epsilon = \{\, t \in \mathbf{R} \mid |s_0 - t| < \epsilon \,\}$ of $s_0 \in \mathbf{R}$. We will construct a neighborhood U_0 of $x_0 \in X$ such that $\widetilde{\phi}(x) \in U_\epsilon$ for all $x \in U_0$.

Consider an arbitrary $a \in p^{-1}(x_0) \subset P$ and the open neighborhood

$$V(a) = \{\, y \in P \mid |\phi(y) - \phi(a)| < \epsilon/2 \,\}$$

of a in P. We have $p^{-1}(x_0) \subset \bigcup_{a\in p^{-1}(x_0)} V(a)$, so by compactness, there exists a finite subset $A \subset p^{-1}(x_0)$ such that $p^{-1}(x_0) \subset \bigcup_{a\in A} V(a)$. Now let $U_0 \subset X$ be an open neighborhood of x_0 such that

(i) $p^{-1}(U_0) \subset \bigcup_{a\in A} V(a)$.

(ii) for $x \in U_0$ and $a \in A$, we have $p^{-1}(x) \cap V(a) \neq \varnothing$.

For example, we can take

$$U_0 := X \setminus p\big(P \setminus \bigcup_{a\in A} V(a)\big) \cap \bigcap_{a\in A} p(V(a)).$$

This set is open since p is both open and closed.

We claim that this U_0 works. Indeed, given $x_1 \in U_0$ let $s_1 := \widetilde{\phi}(x_1) \in \mathbf{R}$. We write $s_0 = \phi(y_0)$, $s_1 = \phi(y_1)$ for some $y_0 \in p^{-1}(x_0)$, $y_1 \in p^{-1}(x_1)$. We have to show that $|s_0 - s_1| < \epsilon$. By (i), we have $y_0 \in V(a_0)$, $y_1 \in V(a_1)$ for some $a_0, a_1 \in A$. We know that $p^{-1}(x_0) \cap V(a_1) \neq \varnothing$ (because a_1 is contained in this set) and $p^{-1}(x_1) \cap V(a_0) \neq \varnothing$ by (ii). Hence we can pick some $b_0 \in p^{-1}(x_1) \cap V(a_0)$ and $b_1 \in p^{-1}(x_0) \cap V(a_1)$. Then we have

$$\begin{aligned} s_0 - s_1 &= \phi(y_0) - s_1 \\ &= \phi(y_0) - \phi(b_0) + \underbrace{\phi(b_0) - s_1}_{\leq 0 \text{ by def. of } s_1} \\ &\leq |\phi(y_0) - \phi(b_0)| \\ &\leq \underbrace{|\phi(y_0) - \phi(a_0)|}_{\leq \epsilon/2 \text{ since } y_0 \in V(a_0)} + \underbrace{|\phi(a_0) - \phi(b_0)|}_{\leq \epsilon/2 \text{ since } b_0 \in V(a_0)} \\ &\leq \epsilon. \end{aligned}$$

By a symmetric argument, we also get $s_1 - s_0 \leq \epsilon$. ∎

5.26 Proposition. *Let E be a vector bundle on V and let $\|-\|$, $\|-\|'$ be two continuous metrics on E. Then the function*

$$d(\|-\|, \|-\|') : V \to \mathbf{R}_{\geq 0}$$

is continuous.

Proof. First we note that the claim is true when $E = L$ is a line bundle. In that case, we may work G-locally on V and assume that L is trivialized by a section $s \in \Gamma(V, L)$. Then one checks easily that

$$d(\|-\|, \|-\|')(x) = |\log\|s(x)\| - \log\|s(x)\|'|$$

which is continuous.

For the general case, we note that if $p : P(E) \to V$ denotes the projective bundle associated to E, then by Lemma 5.23 we have

$$d(\|-\|, \|-\|')(x) = \sup_{y \in p^{-1}(x)} \phi(y)$$

where $\phi : P(E) \to \mathbf{R}$ is the function $\phi = d(\|-\|_{\mathrm{FS}}, \|-\|'_{\mathrm{FS}})$ which is continuous by the line bundle case above.

Hence the result follows from Lemma 5.25 because the projective bundle $p : P(E) \to V$ is topologically proper and open by Proposition 4.41. ∎

5.27 Proposition. *Let E be a vector bundle on V, let $\{\|-\|_n\}_{n \in \mathbf{N}}$ be a family of metrics on E converging locally uniformly to a metric $\|-\|$. If each of the metrics $\|-\|_n$ is continuous, then $\|-\|$ is continuous.*

Proof. For line bundles, this is easy. Indeed, the claim may be checked locally so we may assume that our line bundle is trivial. In that case, the metrics can be identified with functions on V and the claim is reduced to the fact that uniform limits of continuous functions are continuous. The general case now follows from Proposition 5.13 and Corollary 5.24. ∎

5.28 Metrized Vector Bundles over a Point. Let $V = \mathcal{M}(K) = \{*\}$ be a point. A vector bundle on V can be identified with the K-vector space $E(*)$ and a metric on E is simply a norm on the K-vector space $E(*)$. More precisely, the assignment $\|-\| \mapsto \|-\|_*$ is an isometric bijection from the space of metrics on E to the space of norms on $E(*)$.

5.29 Proposition. *Let E be a vector bundle over $\mathcal{M}(K)$. Then any metric on E is continuous.*

Proof. It follows from [BE21, Thm. 1.19] that any norm on $E(*)$ can be approximated by diagonalizable norms. By Proposition 5.27 it suffices to prove that metrics on E corresponding to diagonalizable norms on $E(*)$ are continuous. This is a special case of Paragraph 5.2. ∎

5.30 Constant Metrics. Let E be a finite-dimensional K-vector space. By Paragraph 4.3 we can regard it as a vector bundle on $\mathcal{M}(K)$. We denote by $E \otimes_K \mathcal{O}_V$ the pull-back of E under the structural morphism $V \to \mathcal{M}(K)$. If $\|-\|$ is a metric on E, then we denote by $\|-\|_{\mathrm{const}}$ the pull-back metric of $\|-\|$ on $E \otimes_K \mathcal{O}_V$. Metrics of this form are called *constant*. By Paragraph 5.5 and Proposition 5.29, constant metrics are continuous.

Concretely, if $x \in V$, then the fiber vector space $(E \otimes_K \mathcal{O}_V)(x)$ is isomorphic to $E \otimes_K \mathcal{H}(x)$ by Lemma 4.5 and the norm $\|-\|_{\mathrm{const},x}$ on $E(x)$ corresponds to the scalar extension norm $\|-\|_{\mathcal{H}(x)}$ on $E \otimes_K \mathcal{H}(x)$.

Note that the property of being a constant metric is not preserved under automorphisms of $E \otimes_K \mathcal{O}_V$ but only under automorphisms coming from automorphisms of E.

5.31 Comparison with [CD12] and [CM20]. In [CM20], Chen and Moriwaki consider metrics on the Berkovich analytification of algebraic vector bundles. Let X be a locally finite type K-scheme and E a vector bundle on X. A metric on E in the sense of Paragraph 5.1 is precisely a metric on E in the sense of [CM20, Def. 2.1.8] which has the additional property that all norms $\|-\|_x$ satisfy the non-archimedean triangle inequality. Let us refer to the latter as *non-archimedean* metrics in the sense of [CM20]. The metric is *continuous* in the sense of [CM20, Def. 2.1.8] if for each (Zariski-)local algebraic section s of E the function $\|s^{\mathrm{an}}(-)\|$ is continuous with respect to the Berkovich topology. Hence by Paragraph 5.3 a continuous metric in our sense is a continuous non-archimedean metric in the sense of [CM20]. If L is a line bundle, then every metric in the sense of [CM20] is non-archimedean and every continuous metric in the sense of [CM20] is continuous in our sense by Paragraph 5.4. I do not know if all non-archimedean continuous metrics in the sense of [CM20] are continuous in our sense for higher rank vector bundles.

Chambert-Loir and Ducros consider metrics on vector bundles over arbitrary K-analytic spaces in [CD12, Def. 6.2.2], defined as certain continuous maps $\mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$. If $\|-\|$ is a continuous metric in the sense of Paragraph 5.1, then the associated map $\|-\|_{\mathrm{Tot}} \colon \mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$ is a metric in the sense of [CD12]. Conversely, if $\|-\| \colon \mathrm{Tot}(E) \to \mathbf{R}_{\geq 0}$ is a metric in the sense of [CD12], then we get an associated metric $\|-\|'$ in the sense of Paragraph 5.1 which is given by the norms

$$\|-\|'_x \colon E(x) \subset \pi^{-1}(x) \subset \mathrm{Tot}(E) \xrightarrow{\|-\|} \mathbf{R}_{\geq 0}$$

The metric $\|-\|'$ will have the property that for local sections s the functions $\|s(-)\|$ are continuous (in particular it will be continuous in the sense of [CM20] in the algebraic case) but I do not know if it is continuous in the sense of Paragraph 5.1 and if the associated map $\|-\|'_{\mathrm{Tot}}$ is equal to the original metric $\|-\|$ in the sense of [CD12].

We can summarize the situation in the following commutative diagram:

$$\begin{tikzcd}
\{\text{Metrics}\} \arrow[rr, equal] & & \{\text{N.a. [CM20]-metrics}\} \\
\{\text{Cont. Metrics}\} \arrow[u, hook] \arrow[rr, hook] \arrow[dr, hook] & & \{\text{Cont. n.a. [CM20]-metrics}\} \arrow[u, hook] \\
& \{\text{[CD12]-metrics}\}. \arrow[ur] &
\end{tikzcd}$$

For line bundles all arrows in the lower triangle become bijections.

5.32 Pseudo-Metrics. If E is a vector bundle on V then by a *pseudo-metric* on E we mean a metric on the corresponding line bundle $\mathcal{O}_E(1)$ on $P(E)$. Note that via the injective mapping $\|-\| \mapsto \|-\|_{\mathrm{FS}}$ the set of metrics on E forms a subset of the set of all pseudo-metrics. We use the notation $\overline{E} = (E, \|-\|)$ to denote a pseudo-metrized vector bundle. Here $\|-\|$ is a metric on $\mathcal{O}_E(1)$. If $\overline{E} = (E, \|-\|)$ is a pseudo-metrized line bundle then we write $\mathcal{O}_{\overline{E}}(1) \coloneqq (\mathcal{O}_E(1), \|-\|)$ and $\mathcal{O}_{\overline{E}}(-1) \coloneqq \mathcal{O}_{\overline{E}}(1)^\vee$. To keep a uniform notation for metrics and pseudo-metrics we also write $\|-\|_{\mathrm{FS}}$ for the metric on $\mathcal{O}_E(1)$ given by a pseudo-metric on E.

We call a pseudo-metrized vector bundle $\overline{E} = (E, \|-\|)$ *continuous* if $\mathcal{O}_{\overline{E}}(1)$ is continuously metrized, i.e. if $\|-\|$ is a continuous metric on $\mathcal{O}_E(1)$. By Proposition 5.13 a metric is continuous if and only if it is continuous as a pseudo-metric.

Let $f: V' \to V$ be a morphism of K-analytic spaces and let $\overline{E}$ be a pseudo-metrized vector bundle on V. We denote by $f^*\overline{E}$ the unique pseudo-metrized vector bundle with underlying vector bundle f^*E such that the equality of line sub-bundles of $p_{f^*E}f^*E$

$$\mathcal{O}_{f^*E}(-1) = \widetilde{f}^*\mathcal{O}_E(-1)$$

of Lemma 4.32 is in fact an equality of metrized line bundles

$$\mathcal{O}_{f^*\overline{E}}(-1) = \widetilde{f}^*\mathcal{O}_{\overline{E}}(-1).$$

By Lemma 5.21 this is compatible with the pull-back of metrized vector bundles as defined above.

Similarly if $\overline{E}$ is a pseudo-metrized vector bundle on V and $\overline{L}$ is a line bundle on E then we denote by $\overline{E} \otimes \overline{L}$ the unique pseudo-metrized vector bundle with underlying vector bundle $E \otimes L$ such that the equality of line sub-bundles of $p_E^*(E \otimes L)$

$$\tau^*\mathcal{O}_{E\otimes L}(-1) = \mathcal{O}_E(-1) \otimes p_E^*L$$

of Paragraph 4.38 is an equality of metrized line bundles

$$\tau^*\mathcal{O}_{\overline{E}\otimes\overline{L}}(-1) = \mathcal{O}_{\overline{E}}(-1) \otimes p_E^*\overline{L}.$$

By Proposition 5.19 this is compatible with the twisting of metrized vector bundles with metrized line bundles as defined above.

Finally let $\overline{E}$ be a pseudo-metrized vector bundle on V and let L/K be an extension of non-archimedean fields. It follows from Paragraph 4.36 that we can make the identifications $P(E \otimes_K L) = P(E) \hat{\otimes}_K L$ and $\mathcal{O}_{E\otimes_K L}(-1) = \mathcal{O}_E(-1) \otimes_K L$. We define $\overline{E} \otimes_K L$ to be the unique pseudo-metrized vector bundle on $V \hat{\otimes}_K L$ with underlying vector bundle $E \otimes_K L$ such that we have

$$\mathcal{O}_{\overline{E}\otimes_K L}(-1) = \mathcal{O}_{\overline{E}}(-1) \otimes_K L$$

as metrized line bundles on $P(E \otimes_K L) = P(E) \hat{\otimes}_K L$. It is easy to show that this is compatible with the scalar extension of metrized vector bundles as defined in Paragraph 5.7.

Note that direct sums, duals and general tensor products do not have an obvious analogue for pseudo-metrized vector bundles because the projective bundles of direct sums, duals and tensor products cannot be expressed in terms of the projective bundles of the original vector bundles.

5.33 Smooth Functions. A function $f: V \to \mathbf{R}$ is called *smooth* if for every point $x \in V$ there exists an open neighborhood $U \subset V$ of x, nowhere-vanishing analytic functions $g_1, \dots, g_r \in \mathcal{O}_V^\times(U)$ and a smooth function $\varphi\colon \mathbf{R}^n \to \mathbf{R}$ such that

$$f \equiv \varphi(-\log|g_1|, \dots, -\log|g_r|)$$

identically on U. This definition is due to [CD12, § 3.1.3].

Smooth functions form a sheaf of $\mathbf{R}$-algebras in the Berkovich topology of V which we denote by $\mathscr{C}_V^\infty$.

5.34 Smoothly Metrized Line Bundles. Let $\overline{L} = (L, \|-\|)$ be a metrized line bundle on V. We call $\overline{L}$ and the metric $\|-\|$ *smooth* if for every analytic domain $U \subset V$ and every G-local section $s \in \Gamma(U, L)$ the function

$$-\log\|s(-)\|\colon U \to \mathbf{R}$$

is smooth in the sense of Paragraph 5.33. In [CD12, Def. 6.2.4], Chamber-Loir and Ducros gave a definition of smooth metrics on vector bundles; by [CD12, Rem. 6.2.5] it specializes to the definition above in the case of line bundles. Every smooth metric is continuous.

It is easy to see that tensor products and duals of smoothly metrized line bundles are again smoothly metrized.

5.35 Smooth (Pseudo-)Metrics on Vector Bundles. Let $\overline{E} = (E, \|-\|)$ be a (pseudo-)metrized vector bundle on V. We call it *smoothly (pseudo-)metrized* if the metrized line bundle $\mathcal{O}_{\overline{E}}(1)$ is smooth.

Note that the argument contained in the proof of Proposition 5.13 shows that a metrized vector bundle $\overline{E} = (E, \|-\|)$ is smoothly metrized if and only if the map

$$\|-\|_{\mathrm{Tot}}\colon \mathrm{Tot}(E) \setminus \{0\} \to \mathbf{R}_{>0}$$

is smooth, i.e. if and only if $\overline{E}$ is smoothly metrized in the sense of [CD12, Def. 6.2.4].

5.36 Existence of Continuous and Smooth Metrics. If V is paracompact, then by [CD12, Prop. 6.2.13] every vector bundle E on V admits a formal model and hence a formal metric, which is in particular continuous in the sense of Paragraph 5.1. (We introduce formal models and the associated metrics in Paragraph 6.1.)

Smooth metrics behave best for good K-analytic spaces so let us assume that V is good. If V is additionally paracompact then by [CD12, Prop. 6.2.6] every line bundle on V admits a smooth metric.

Applying this to $\mathcal{O}_E(1)$ we see that every vector bundle on a good paracompact K-analytic space admits a smooth pseudo-metric. It is unknown whether vector bundles of higher rank admit smooth metrics (as opposed to smooth pseudo-metrics), cf. the remark after [CD12, Prop. 6.2.6].

6. Models and Model Metrics

Throughout this chapter K will be a non-archimedean field, V will denote a paracompact strictly K-analytic space and X will be a proper K-scheme. We introduce *formal models* of vector bundles on V in Paragraph 6.1 as well as their associated formal metrics in Paragraph 6.4. We show that model metrics are compatible with all the common operations on vector bundles. In Paragraph 6.13 we introduce a notion of *semipositive metrics* on vector bundles.

6.1 Formal Models. Let $\mathfrak{V}$ be a formal model for V. A *vector bundle* on $\mathfrak{V}$ is a locally free sheaf $\mathfrak{E}$ of $\mathscr{O}_{\mathfrak{V}}$-modules of constant rank. Recall from Paragraph 3.6 that there is a canonical morphism of ringed sites $(\mathscr{C}(V_G), \mathscr{O}_{V_G}) \to (\mathscr{C}(\mathfrak{V}), \mathscr{O}_{\mathfrak{V}})$. We denote the pull-back of $\mathfrak{E}$ along this morphism of ringed sites by $\mathfrak{E}|_V$ and call it the *generic fiber* of $\mathfrak{E}$. It is a vector bundle on V.

Now let E be a vector bundle on V. A *formal K°-model* $(\mathfrak{V}, \mathfrak{E})$ for (V, E) consists of a formal K°-model $\mathfrak{V}$ for V and a vector bundle $\mathfrak{E}$ on $\mathfrak{V}$ together with a fixed isomorphism $\mathfrak{E}|_V \xrightarrow{\sim} E$.

In that case we have for every open $\mathfrak{U} \subset \mathfrak{V}$ a canonical $\Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{V}})$-linear map

$$\Gamma(\mathfrak{U}, \mathfrak{E}) \to \Gamma(V \cap \mathfrak{U}, \mathfrak{E}|_V), \qquad s \mapsto s|_{V \cap \mathfrak{U}}.$$

If $x \in V \cap \mathfrak{U}$ is a point, then we write $s(x) := s|_{V\cap\mathfrak{U}}(x) \in E(x)$. If $s_1, \dots, s_r \in \Gamma(\mathfrak{U}, \mathfrak{E})$ is a local frame for $\mathfrak{E}$, then $s_1|_{V\cap\mathfrak{U}}, \dots, s_r|_{V\cap\mathfrak{U}}$ is a frame for E over $V \cap \mathfrak{U}$.

By [CD12, Prop. 6.2.13], a formal K°-model for (V, E) always exists.

If E is a vector bundle on the proper K-scheme X, then by a *formal K°-model* $(\mathfrak{X}, \mathfrak{E})$ for (X, E), we mean a formal K°-model for $(X^{\mathrm{an}}, E^{\mathrm{an}})$.

6.2 Remark. Just as for vector bundles on Berkovich spaces, there are operations of pull-backs, duals, directs sums, tensor products, projective bundles, etc. for vector bundles on formal schemes. The operation $(\mathfrak{V}, \mathfrak{E}) \mapsto (\mathfrak{V}_\eta, \mathfrak{E}|_{\mathfrak{V}_\eta})$ is compatible with all of these constructions.

6.3 Algebraic Models. Let E be a vector bundle on X. An *algebraic K°-model* for (X, E) consists of an algebraic K°-model $\mathcal{X}$ for X and a vector bundle $\mathcal{E}$ on $\mathcal{X}$ together with a fixed isomorphism $\mathcal{E}|_X \xrightarrow{\sim} E$. In that case the formal completion along the special fiber $(\widehat{\mathcal{X}}, \widehat{\mathcal{E}})$ is a formal model for (X, E).

6.4 Formal Metrics. Let E be a vector bundle on V and let $(\mathfrak{V}, \mathfrak{E})$ be a formal model for (V, E). Then there is an associated metric $\|-\|_{\mathfrak{E}}$ on E which we all the *formal metric* associated to $(\mathfrak{V}, \mathfrak{E})$ and which is constructed as follows: Let $\mathfrak{U} \subset \mathfrak{V}$ be a formal open subscheme such that $\mathfrak{E}$ admits a frame $s_1, \dots, s_r \in \Gamma(\mathfrak{U}, \mathfrak{E})$ over $\mathfrak{U}$. We write $U := V \cap \mathfrak{U} \subset V$. We equip $E|_U$ with the orthonormal metric induced by this frame: $\|-\|_{\mathfrak{E}}|_U := \|-\|_{\underline{s}|_U}$ (see Paragraph 5.2 for this notation). By Lemma 6.5 below, the metrics $\|-\|_{\mathfrak{E}}|_U$ do not depend on the choice of the frame and glue to a well-defined continuous metric $\|-\|_{\mathfrak{E}}$ on E.

6.5 Lemma. *Let $(\mathfrak{V}, \mathfrak{E})$ be a formal K°-model for (V, E) and assume that the vector bundle $\mathfrak{E}$ is trivial. Let $s_1, \dots, s_r \in \Gamma(\mathfrak{V}, \mathfrak{E})$ and $t_1, \dots, t_r \in \Gamma(\mathfrak{V}, \mathfrak{E})$ be two different frames for $\mathfrak{E}$. We get associated frames $\underline{s}|_V$ and $\underline{t}|_V$ for E over V. Then the orthonormal metrics $\|-\|_{\underline{s}|_V}$ and $\|-\|_{\underline{t}|_V}$ coincide.*

Proof. Write $s_i = f_{i1}t_1 + \dots + f_{ir}t_r$ for some functions $f_{ij} \in \Gamma(\mathfrak{V}, \mathcal{O}_{\mathfrak{V}})$. Note that $E(x)$ admits $s_1(x), \dots, s_r(x) \in E(x)$ as a basis. Given $v = a_1 s_1(x) + \dots + a_r s_r(x) \in E(x)$ with $a_i \in \mathcal{H}(x)$, we have

$$\|v\|_{\underline{s}|_V, x} = \max(|a_1|, \dots, |a_r|).$$

On the other hand, we have

$$\|v\|_{\underline{t}|_V, x} \leq \max(|a_1| \|s_1(x)\|_{\underline{t}|_V, x}, \dots, |a_r| \|s_r(x)\|_{\underline{t}|_V, x}).$$

For each i, we have

$$\begin{aligned} \|s_i(x)\|_{\underline{t}|_V, x} &= \|f_{i1}(x)t_1(x) + \dots + f_{ir}(x)t_r(x)\|_{\underline{t}|_V, x} \\ &= \max(|f_{i1}(x)|, \dots, |f_{ir}(x)|) \\ &\overset{3.6}{\leq} 1. \end{aligned}$$

Putting things together, we get $\|v\|_{\underline{t}|_V, x} \leq \|v\|_{\underline{s}|_V, x}$. By symmetry we get equality. ∎

6.6 Lemma. *Let $f\colon V' \to V$ be a morphism of paracompact strictly K-analytic spaces and let $\mathfrak{V}$ be a formal K°-model for V. Then there exists a formal K°-model $\mathfrak{V}'$ for V' and a morphism of admissible formal K°-schemes $\mathfrak{f}\colon \mathfrak{V}' \to \mathfrak{V}$ whose restriction to the generic fiber is f.*

Now assume that E is a vector bundle on V and that $\mathfrak{E}$ is a formal model for E defined on $\mathfrak{V}$. Then $\mathfrak{f}^*\mathfrak{E}$ is a formal model for f^*E and we have

$$f^*\|{-}\|_{\mathfrak{E}} = \|{-}\|_{\mathfrak{f}^*\mathfrak{E}}.$$

In particular, the pull-back of a formal metric is again a formal metric.

Proof. The first claim follows from [Bos14, Lem. 8.4/4 (c)]. By Remark 6.2, the generic fiber of $\mathfrak{f}^*\mathfrak{E}$ is f^*E.

To compare the metrics, we choose a formal open subset $\mathfrak{U} \subset \mathfrak{V}$ where $\mathfrak{E}$ admits a frame $s_1, \dots, s_r \in \Gamma(\mathfrak{U}, \mathfrak{E})$. There is an induced frame $\mathfrak{f}^*s_1, \dots, \mathfrak{f}^*s_r \in \Gamma(\mathfrak{f}^{-1}(\mathfrak{U}), \mathfrak{f}^*\mathfrak{E})$. Let $U \coloneqq V \cap \mathfrak{U} \subset V$ and $U' \coloneqq V' \cap \mathfrak{f}^{-1}(\mathfrak{U}) \subset V'$. It is easy to check that we have

$$(\mathfrak{f}^*s_i)|_{U'} = f^*(s_i|_U) \in \Gamma(U', f^*E). \tag{6.6.1}$$

For $x' \in U'$, we have an identification $(f^*E)(x') = E(f(x')) \otimes_{\mathscr{H}(f(x'))} \mathscr{H}(x')$ by Lemma 4.5. The norm $f^*\|{-}\|_{\mathfrak{E},x}$ is the scalar-extension norm of the metric $\|{-}\|_{\mathfrak{E},f(x')}$ on $E(f(x'))$. Since the norm $\|{-}\|_{\mathfrak{E},f(x')}$ on $E(f(x'))$ admits

$$s_1(f(x')), \dots, s_r(f(x'))$$

as an orthonormal basis, it follows from Proposition 1.19 (iv) that the norm $f^*\|{-}\|_{\mathfrak{E},x'}$ has

$$s_i(f(x')) \otimes 1 \overset{4.5}{=} (f^*(s_i|_U))(x'), \qquad i = 1, \dots, r$$

as an orthonormal basis. On the other hand, the norm $\|{-}\|_{\mathfrak{f}^*\mathfrak{E}}$ has by definition the vectors $(\mathfrak{f}^*s_i)(x')$ as an orthonormal basis. Hence, by Eq. (6.6.1), both metrics agree.

To prove the last claim, let $\|{-}\| = \|{-}\|_{\mathfrak{E}}$ be any formal metric. Choosing a model for the morphism f as in the first claim of this lemma and using that $f^*\|{-}\| = \|{-}\|_{\mathfrak{f}^*\mathfrak{E}}$, we see that $f^*\|{-}\|$ is again a formal metric. ∎

6.7 Corollary. *Let E be a vector bundle on V and let $(\mathfrak{V}, \mathfrak{E})$ be a formal model for (V, E). Let $\mathfrak{f}\colon \mathfrak{V}' \to \mathfrak{V}$ be a morphism of models for V. Then $\mathfrak{f}^*\mathfrak{E}$ is again a model for E and we have*

$$\|{-}\|_{\mathfrak{E}} = \|{-}\|_{\mathfrak{f}^*\mathfrak{E}}.$$ ∎

6.8 Lemma. *Let E, E' be two vector bundles on V and let $(\mathfrak{V}, \mathfrak{E})$, $(\mathfrak{V}, \mathfrak{E}')$ be two formal K°-models for (V, E) and (V, E') defined over the same model $\mathfrak{V}$ for V. Then $\mathfrak{E} \otimes \mathfrak{E}'$ is a formal model for $E \otimes E'$ and we have*

$$\|-\|_{\mathfrak{E}} \otimes \|-\|_{\mathfrak{E}'} = \|-\|_{\mathfrak{E}\otimes\mathfrak{E}'}. \tag{6.8.1}$$

In particular, the metrized tensor product of two formally metrized vector bundles is again formally metrized.

Proof. The generic fiber of $\mathfrak{E} \otimes \mathfrak{E}'$ is $E \otimes E'$ by Remark 6.2. Let $\mathfrak{U} \subset \mathfrak{V}$ be a formal open subscheme such that $\mathfrak{E}|_{\mathfrak{U}}$ admits a frame $s_1, \ldots, s_r \in \Gamma(\mathfrak{U}, \mathfrak{E})$ and $\mathfrak{E}'|_{\mathfrak{U}}$ admits a frame $s'_1, \ldots, s'_{r'} \in \Gamma(\mathfrak{U}, \mathfrak{E}')$. There is an induced frame $s_1 \otimes s'_1, \ldots, s_r \otimes s'_{r'} \in \Gamma(\mathfrak{U}, \mathfrak{E} \otimes \mathfrak{E}')$. Write $U \coloneqq V \cap \mathfrak{U} \subset V$. It is easy to check that we have

$$(s_i \otimes s'_j)|_U = s_i|_U \otimes s'_j|_U \in \Gamma(U, E \otimes E').$$

For $x \in U$, one checks using Proposition 1.19 (iii) that an orthonormal basis in $(E \otimes E')(x) \cong E(x) \otimes_{\mathscr{H}(x)} E'(x)$ for both $\|-\|_{\mathfrak{E}} \otimes \|-\|_{\mathfrak{E}'}$ and for $\|-\|_{\mathfrak{E}\otimes\mathfrak{E}'}$ is given by

$$(s_i \otimes s'_j)(x) \overset{(4.22.2)}{=} s_i(x) \otimes s'_j(x), \qquad i \in \{1, \ldots, r\},\ j \in \{1, \ldots, r'\}.$$

To prove the last claim, let $\|-\| = \|-\|_{\mathfrak{E}}$ and $\|-\|' = \|-\|_{\mathfrak{E}'}$ be two formal metrics on E resp. E', where $\mathfrak{E}$ and $\mathfrak{E}'$ are formal models for E and E' which are defined on two *a priori* different formal models $\mathfrak{V}$ and $\mathfrak{V}'$ for V. By Paragraph 3.6, formal K°-models for V form a directed category, so we find a formal K° model dominating both $\mathfrak{V}$ and $\mathfrak{V}'$. Pulling back $\mathfrak{E}$ and $\mathfrak{E}'$ to this model and using Corollary 6.7, we may assume that $\mathfrak{V} = \mathfrak{V}'$. In that case Eq. (6.8.1) shows that $\|-\| \otimes \|-\|'$ is again a formal metric. ∎

6.9 Lemma. *Let E, E' be two vector bundles on V and let $(\mathfrak{V}, \mathfrak{E})$, $(\mathfrak{V}, \mathfrak{E}')$ be models for (V, E) and (V, E') defined over the same model $\mathfrak{V}$ for V. Then $\mathfrak{E} \oplus \mathfrak{E}'$ is a formal model for $E \oplus E'$ and we have*

$$\|-\|_{\mathfrak{E}} \oplus \|-\|_{\mathfrak{E}'} = \|-\|_{\mathfrak{E}} \oplus \|-\|_{\mathfrak{E}'}.$$

In particular, the metrized direct sum of two formally metrized vector bundles is again formally metrized.

Proof. The proof is completely analogous to the proof of Lemma 6.8. ∎

6.10 Lemma. *Let E be a vector bundle on V and let $(\mathfrak{V}, \mathfrak{E})$ be a model for (V, E). Then $\mathfrak{E}^\vee$ is a formal model for $E^\vee$ and we have*

$$\|-\|_{\mathfrak{E}}^\vee = \|-\|_{\mathfrak{E}^\vee}.$$

In particular, the metrized dual of a formally metrized vector bundle is again formally metrized.

Proof. The proof is also similar to the proofs of Lemmas 6.8 and 6.9. ∎

6.11 Lemma. *Let E be a vector bundle on V and let $(\mathfrak{V}, \mathfrak{E})$ be a formal model for (V, E). Then $(P(\mathfrak{E}), \mathcal{O}_{\mathfrak{E}}(1))$ is a formal model for $(P(E), \mathcal{O}_E(1))$ and we have an equality*

$$\|-\|_{\mathfrak{E},\mathrm{FS}} = \|-\|_{\mathcal{O}_{\mathfrak{E}}(1)}$$

of metrics on the line bundle $\mathcal{O}_E(1)$.

In particular, the Fubini-Study metric associated to a formal metric is again a formal metric.

Proof. Denote the projective bundle of $\mathfrak{E}$ by $\mathfrak{p}\colon P(\mathfrak{E}) \to \mathfrak{V}$. It is easy to check that the generic fiber of $P(\mathfrak{E})$ equals $P(E)$ and that $\mathfrak{p}$ restricts to the projective bundle projection $p\colon P(E) \to V$.

We compare the respective dual metrics on $\mathcal{O}_E(-1)$. By Lemma 6.10, the dual metric of $\|-\|_{\mathcal{O}_{\mathfrak{E}}(1)}$ is the formal metric $\|-\|_{\mathcal{O}_{\mathfrak{E}}(-1)}$. We may argue locally on $\mathfrak{V}$ and assume that there is a frame $s_0, \dots, s_e \in \Gamma(\mathfrak{V}, \mathfrak{E})$. Let $T_0, \dots, T_e$ be the dual frame for $\mathfrak{E}^\vee$. Let $\mathfrak{U}_i \coloneqq D_+(T_i) = \mathfrak{V} \times \mathrm{Spf}(K^\circ\{T_0/T_i, \dots, T_e/T_i\}) \subset P(\mathfrak{E})$ and let us write $U_i \coloneqq P(E) \cap \mathfrak{U}_i$. Over $\mathfrak{U}_i$, the line bundle $\mathcal{O}_{\mathfrak{E}}(-1)$ admits the frame $1/T_i \in \Gamma(\mathfrak{U}_i, \mathcal{O}_{\mathfrak{E}}(-1))$. Hence, $(1/T_i)|_{U_i} \in \Gamma(U_i, \mathcal{O}_E(-1))$ is a frame and we have by definition of formal metrics

$$\|(1/T_i)(y)\|_{\mathcal{O}_{\mathfrak{E}}(-1)} = 1$$

for all $y \in U_i$.

The embedding $i\colon \mathcal{O}_E(-1) \hookrightarrow p^*E$ is given in local coordinates by $(1/T_i)|_{U_i} \mapsto (\mathfrak{p}^* s_i)|_{U_i} = p^*(s_i|_{U_i})$. Hence we also have

$$\|(1/T_i)(y)\|^\vee_{\mathfrak{E},\mathrm{FS}} = p^*\|p^*(s_i|_{U_i})(y)\|_{\mathfrak{E}} \overset{5.6}{=} \|s_i(y)\|_{\mathfrak{E}} = 1$$

for $y \in U_i$. Since the U_i cover $P(E)$, this finishes the proof. ∎

6.12 Change of Base Field. Let L/K be an extension of non-archimedean fields. Then there is an induced extension L°/K° of valuation rings, a scalar extension functor for admissible formal K°-schemes $\mathfrak{V} \mapsto \mathfrak{V} \mathbin{\hat{\otimes}}_{K^\circ} L^\circ$ and a corresponding scalar extension functor $\mathfrak{E} \mapsto \mathfrak{E} \otimes_{K^\circ} L^\circ$ for vector bundles on an admissible formal K°-scheme $\mathfrak{V}$. It is not hard to verify that if the generic fiber of $\mathfrak{V}$ is V, then the generic fiber of $\mathfrak{V} \mathbin{\hat{\otimes}}_{K^\circ} L^\circ$ is $V \mathbin{\hat{\otimes}}_K L$ and if the generic fiber of $\mathfrak{E}$ is E, then the generic fiber of $\mathfrak{E} \otimes_{K^\circ} L^\circ$ is $E \otimes_K L$. Furthermore for the induced model metrics we have $\|-\|_{\mathfrak{E}} \otimes_K L = \|-\|_{\mathfrak{E} \otimes_{K^\circ} L^\circ}$ where the left hand side denotes the scalar extension metric in the sense of Paragraph 5.7. In particular, the base change of a model metric is again a model metric.

See also [GM19, Prop. 2.18] for the case of line bundles.

6.13 Semipositive Metrics. Let $\overline{E}$ be a formally metrized vector bundle on V. We call $\overline{E}$ *semipositive* if the formally metrized line bundle $\mathcal{O}_{\overline{E}^\vee}(1)$ on $P(E^\vee)$ is semipositive in the sense of [GM19, § 3.2], i.e. if there exists a formal model $(\mathfrak{P}, \mathfrak{L})$ for $(P(E^\vee), \mathcal{O}_{E^\vee}(1))$ inducing the Fubini-Study-metric on $\mathcal{O}_{E^\vee}(1)$ which is numerically effective. Here the pair $(\mathfrak{P}, \mathfrak{L})$ is called *numerically effective* if $\deg_{\mathfrak{L}}(C) \geq 0$ for any closed curve C in the special fiber $\widetilde{\mathfrak{P}}$ which is proper over the residue field $\widetilde{K}$.

If $(\mathfrak{V}, \mathfrak{E})$ is a formal model for (V, E), then we call $\mathfrak{E}$ *numerically effective* if the line bundle $\mathcal{O}_{\mathfrak{E}^\vee}(1)$ is a numerically effective line bundle on $P(\mathfrak{E}^\vee)$. It follows immediately that in this case the metric $\|-\|_{\mathfrak{E}}$ is semipositive.

6.14 Proposition. *Let $\overline{E} = (E, \|-\|)$ be a formally metrized vector bundle on V. Then the following are equivalent:*

(i) The metric $\|-\|$ is a semipositive formal metric.

(ii) There exists a numerically effective formal model $(\mathfrak{V}, \mathfrak{E})$ for the pair (V, E) such that $\|-\| = \|-\|_{\mathfrak{E}}$.

(iii) Every formal model $(\mathfrak{V}, \mathfrak{E})$ for (V, E) such that $\|-\| = \|-\|_{\mathfrak{E}}$ is numerically effective.

Proof. This follows immediately from Lemma 6.11 and [GM19, Prop. 3.5]. ∎

7. Smooth Forms and Currents on Berkovich Spaces

Throughout this chapter, K will be a non-archimedean field. We assume that all K-analytic spaces are good, topologically Hausdorff, boundaryless and equidimensional. These assumptions are for example satisfied by the analytification of an algebraic K-variety. We fix a K-analytic space V of dimension n and an algebraic K-variety X of dimension n.

We recall here the most important properties of smooth forms and currents in the sense of [CD12]. In Paragraph 7.10 we introduce the *Bott–Chern cohomology groups* of the K-analytic space V. By Paragraph 8.12 a family of line bundles on V gives rise to a well-defined class in this cohomology group.

7.1 Smooth Forms. In [CD12], Chambert-Loir and Ducros introduced a bigraded sheaf of differential $\mathbf{R}$-algebras $\mathscr{A}_V^{\bullet,\bullet}$, called the sheaf of *smooth differential (super-)forms* on V. By [CD12, § 3.1.3], we have $\mathscr{A}_V^{0,0} = \mathscr{C}_V^\infty$ where $\mathscr{C}_V^\infty$ denotes the sheaf of smooth functions in the sense of Paragraph 5.33. The wedge product is denoted by $\wedge : \mathscr{A}_V^{p,q} \times \mathscr{A}_V^{p',q'} \to \mathscr{A}_V^{p+p',q+q'}$ and the differentials are denoted by $d' : \mathscr{A}_V^{p,q} \to \mathscr{A}_V^{p+1,q}$ and $d'' : \mathscr{A}_V^{p,q} \to \mathscr{A}_V^{p,q+1}$.

If there is no danger of confusion, we simply write $\mathscr{A}^{p,q}(U)$ for the space of differential forms of bi-degree (p,q) on an open subset $U \subset V$.

If $f : V' \to V$ is a morphism of K-analytic spaces, then there is a functorial pull-back map $f^* : \mathscr{A}_V^{\bullet,\bullet} \to f_*\mathscr{A}_{V'}^{\bullet,\bullet}$ of sheaves of differential $\mathbf{R}$-algebras on V. In the case of an open immersion $i : U \hookrightarrow V$ of an open subset of V, the pull-back i^*

coincides with the restriction operation to an open subset.

If X is an algebraic K-variety, we write $\mathscr{A}_X^{\bullet,\bullet}$ rather than $\mathscr{A}_{X^{\mathrm{an}}}^{\bullet,\bullet}$. If $f\colon X' \to X$ is a morphism of algebraic K-varieties, we write $f^*\colon \mathscr{A}_X^{\bullet,\bullet} \to (f^{\mathrm{an}})_*\mathscr{A}_{X'}^{\bullet,\bullet}$ for the pull-back of forms along the analytification of f. See [Gub16] for a more elementary construction of the sheaves $\mathscr{A}_X^{\bullet,\bullet}$.

7.2 Compact Support. If $\alpha \in \mathscr{A}^{p,q}(V)$, then the *support* $\mathrm{Supp}(\alpha)$ consists of all points $x \in V$ such that there exists no neighborhood of x in V on which α vanishes identically. We say that α has *compact support* if $\mathrm{Supp}(\alpha)$ is a compact subset in the Berkovich topology. We write $\mathscr{A}_c^{p,q}(V)$ for the subspace of (p,q)-forms with compact support.

If $U \subset U' \subset V$ are open subsets and $\alpha \in \mathscr{A}^{p,q}(U)$ has compact support, then there is a unique form $\alpha|^{U'} \in \mathscr{A}^{p,q}(U')$ which satisfies $(\alpha|^{U'})|_U = \alpha$ and $(\alpha|^{U'})|_{U'\setminus \mathrm{Supp}(\alpha)} = 0$. From $\mathrm{Supp}(\alpha|^{U'}) = \mathrm{Supp}(\alpha)$ it follows that $\alpha|^{U'}$ has compact support.

7.3 Currents. A *current* of bi-degree (p,q) is an $\mathbf{R}$-linear form T on the space $\mathscr{A}_c^{n-p,n-q}(V)$ which is continuous with respect to a certain topology, similar to the Schwartz topology from the theory of distributions [CD12, § 4.1.1]. We write $\mathscr{D}^{p,q}(V)$ for the space of currents of bi-degree (p,q). The extension by zero map of Paragraph 7.2 gives rise to a restriction operation on currents, turning currents of bi-degree (p,q) into a sheaf $\mathscr{D}_V^{p,q}$ [CD12, § 4.2.5].

Similarly, the differential operators on forms give rise, by duality and an appropriate choice of signs, to differential operators $d'\colon \mathscr{D}_V^{p,q} \to \mathscr{D}_V^{p+1,q}$ and $d''\colon \mathscr{D}_V^{p,q} \to \mathscr{D}_V^{p,q+1}$. Finally, every form $\alpha \in \mathscr{A}^{p,q}(U)$ for an open subset $U \subset V$ gives rise to an operation $\alpha \wedge -\colon \mathscr{D}^{r,s}(U) \to \mathscr{D}^{p+r,q+s}(U)$ defined by

$$\langle \alpha \wedge T, \beta\rangle = (-1)^{(p+q)(r+s)}\langle T, \alpha\wedge\beta\rangle$$

for $\beta \in \mathscr{A}_c^{n-p-r,n-q-s}(U)$.

The above operations provide $\mathscr{D}_V^{\bullet,\bullet}$ with the structure of a sheaf of bigraded differential $\mathscr{A}_V^{\bullet,\bullet}$-modules.

If X is an algebraic K-variety, we write $\mathscr{D}_X^{\bullet,\bullet}$ rather than $\mathscr{D}_{X^{\mathrm{an}}}^{\bullet,\bullet}$.

7.4 Proper Push-Forward of Currents. If $f\colon V' \to V$ is a proper morphism of K-analytic spaces of dimensions n', resp. n and $T \in \mathscr{D}^{p,q}(V')$ is a current, then the current $f_*T \in \mathscr{D}^{p+n-n',q+n-n'}(V)$ is defined by the formula

$$\langle f_*T, \alpha\rangle = \langle T, f^*\alpha\rangle$$

for $\alpha \in \mathscr{A}_c^{n'-p,n'-q}(V)$. The push-forward operation defines a morphism

$$f_*\colon f_*\mathscr{D}_{V'}^{\bullet,\bullet} \to \mathscr{D}_V^{\bullet,\bullet}$$

of bigraded differential $\mathscr{A}_{V}^{\bullet,\bullet}$-modules (where the left hand is a $\mathscr{A}_{V}^{\bullet,\bullet}$-module via the map $\mathscr{A}_{V}^{\bullet,\bullet} \to f_*\mathscr{A}_{V'}^{\bullet,\bullet}$). In particular, the projection formula

$$f_*(f^*\alpha \wedge T) = \alpha \wedge f_*(T)$$

holds. This follows immediately by evaluating both sides on an arbitrary test form $\beta \in \mathscr{A}_c(V)$.

7.5 Integration. There is a distinguished current $\int_V \in \mathscr{D}^{0,0}(V)$ called the *current of integration* over V [CD12, § 4.3.2]. We also write δ_V instead of $\int_V$. By Stokes' Theorem [CD12, Thm. 3.12.1] (and our assumption that all K-analytic spaces are boundaryless) it follows that δ_V is d'- and d''-closed.

If $V = X^{\mathrm{an}}$ is the analytification of an algebraic variety, we also write $\delta_X := \delta_{X^{\mathrm{an}}} \in \mathscr{D}^{0,0}(X^{\mathrm{an}})$. If $Z = \sum_i n_i Z_i$ is a cycle on X, i.e. a formal $\mathbf{Z}$-linear combination of closed subvarieties $Z_i \subset X$, then we write

$$\delta_Z := \sum_i n_i \iota_{i*}\delta_{Z_i},$$

where $\iota_i : Z_i \hookrightarrow X$ is the closed embedding of Z_i into X.

7.6 Proposition. *Let $f : X' \to X$ be a proper morphism of K-varieties and let Z' be a cycle on X'. Then there is an equality*

$$f_*(\delta_{Z'}) = \delta_{f_*(Z')}$$

of currents on X^{an}.

Proof. This is shown in [GK17, Prop. 6.12] even in the sense of δ-currents. ∎

7.7 Lemma. *Let V, W be K-analytic spaces of dimensions n, resp. m. Denote by $\pi_0 : V \times W \to V$ and by $\pi_1 : V \times W \to W$ the canonical projection maps. Let $\alpha \in \mathscr{A}_c^{\bullet,\bullet}(V)$ and $\beta \in \mathscr{A}_c^{\bullet,\bullet}(W)$. Then $\pi_0^*\alpha \wedge \pi_1^*\beta \in \mathscr{A}(V \times W)$ has compact support.*

If furthermore $\alpha \in \mathscr{A}_c^{n,n}(V)$ and $\beta \in \mathscr{A}_c^{m,m}(W)$ then

$$\int_{V\times W} \pi_0^*\alpha \wedge \pi_1^*\beta = \int_V \alpha \cdot \int_W \beta. \tag{7.7.1}$$

Proof. Denoting by $|V|$, $|W|$ the underlying topological spaces of V and W it is easy to see that $\mathrm{Supp}(\pi_0^*\alpha \wedge \pi_1^*\beta)$ is contained in the preimage of $\mathrm{Supp}(\alpha) \times \mathrm{Supp}(\beta)$ under the canonical map $|V \times W| \to |V| \times |W|$. Since this map is proper by Lemma 2.17 (ii) it follows that $\mathrm{Supp}(\pi_0^*\alpha \wedge \pi_1^*\beta)$ is compact.

If α has bi-degree (n, n) and β has bi-degree (m, m), then Eq. (7.7.1) follows from unpacking the definition and using that tropicalizations are compatible with products; a proof in the algebraic case can be found in [Sto21, Prop. 3.4.21]. A proof of a stronger statement regarding δ-forms is given in [Pre23, Thm. 4.2.86]. ∎

7.8 Currents Induced from Forms. There exists a unique morphism of bigraded differential $\mathscr{A}_V^{\bullet,\bullet}$-modules $[-]\colon \mathscr{A}_V^{\bullet,\bullet} \to \mathscr{D}_V^{\bullet,\bullet}$ satisfying $[1] = \delta_V$. Concretely, it is given by

$$\langle [\alpha], \beta \rangle = \langle \alpha \wedge \delta_V, \beta \rangle = \int_V \alpha \wedge \beta$$

for $\alpha \in \mathscr{A}^{p,q}(V)$ and $\beta \in \mathscr{A}_c^{n-p,n-q}(V)$.

7.9 The Poincaré-Lelong Formula. Let f be an invertible meromorphic function on V. Let $U \subset V$ be the locus where f has no zeros or poles. Then the function

$$\log|f(-)|\colon U \to \mathbf{R}$$

is smooth and hence there is an induced current $d'd''[\log|f|] \in \mathscr{D}^{1,1}(U)$. By [CD12, Lem. 4.6.1], this current extends uniquely to a current on V, which we denote again by $d'd''[\log|f|]$.

If f is a regular function which is not a zero-divisor in $\mathscr{O}(V)$, then we denote the current of integration over the vanishing locus of f by $\delta_{\mathrm{div}(f)}$. By additivity and locality of this operation, it extends to all invertible meromorphic functions.

The *Poincaré-Lelong equation* [CD12, Thm. 4.6.5] states that

$$\delta_{\mathrm{div}(f)} = d'd''[\log|f|]$$

as currents on V.

In the case where X is an algebraic variety and f is a non-zero rational function, the above current is also equal to the current of integration over the Weil divisor $[\mathrm{div}(f)]$ associated to f, namely

$$\delta_{[\mathrm{div}(f)]} = d'd''[\log|f|]$$

as currents on X^{an} [Gub16, Prop. 6.10].

7.10 Bott–Chern Cohomology of Currents. For $p \in \mathbf{N}$ we denote by $\hat{H}^p_{\mathscr{D}}(V)$ the group

$$\hat{H}^p_{\mathscr{D}}(V) := \frac{\mathrm{Ker}(d'\colon \mathscr{D}^{p,p}(V) \to \mathscr{D}^{p+1,p}(V)) \cap \mathrm{Ker}(d''\colon \mathscr{D}^{p,p}(V) \to \mathscr{D}^{p,p+1}(V))}{\mathrm{Im}(d'd''\colon \mathscr{D}^{p-1,p-1}(V) \to \mathscr{D}^{p,p}(V))}.$$

Note that the denominator is indeed contained in the numerator because d' and d'' are anti-commuting differentials.

Let X be an algebraic variety. Let $W \subset X$ be a closed sub-variety and f a non-zero rational function on W. Denote by $i\colon W \hookrightarrow X$ the closed embedding. By Proposition 7.6 and the Poincaré-Lelong formula 7.9 we have

$$\delta_{i_*[\mathrm{div}(f)]} = i_*\delta_{[\mathrm{div}(f)]} = i_* d'd''[\log|f|] = d'd'' i_*[\log|f|],$$

so in particular the class of $\delta_{i_*[\mathrm{div}(f)]}$ in $\hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})$ vanishes.

It follows that the map $Z \mapsto \delta_Z$ from cycles on X to currents on X^{an} induces a well-defined *cycle class map*

$$\delta\colon\ \mathrm{CH}^p(X) \to \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}}).$$

If $f\colon X' \to X$ is a proper morphism of K-varieties of dimensions $n+e$, resp. n, then the diagram

$$\begin{tikzcd}
\mathrm{CH}^{p+e}(X') \arrow[r, "\delta"] \arrow[d, "f_*"'] & \hat{H}^{p+e}_{\mathscr{D}}((X')^{\mathrm{an}}) \arrow[d, "f_*"] \\
\mathrm{CH}^p(X) \arrow[r, "\delta"'] & \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})
\end{tikzcd} \tag{7.10.1}$$

is commutative by Proposition 7.6.

7.11 Dolbeault Cohomology. For $p, q \in \mathbf{N}$ we denote by

$$H^{p,q}_{\mathscr{A}}(V) \coloneqq \frac{\mathrm{Ker}(d''\colon\ \mathscr{A}^{p,q}(V) \to \mathscr{A}^{p,q+1}(V))}{\mathrm{Im}(d''\colon\ \mathscr{A}^{p,q-1}(V) \to \mathscr{A}^{p,q}(V))}$$

and by

$$H^{p,q}_{\mathscr{D}}(V) \coloneqq \frac{\mathrm{Ker}(d''\colon\ \mathscr{D}^{p,q}(V) \to \mathscr{D}^{p,q+1}(V))}{\mathrm{Im}(d''\colon\ \mathscr{D}^{p,q-1}(V) \to \mathscr{D}^{p,q}(V))}$$

the *Dolbeault cohomology groups* of forms, resp. of currents associated to V. Note that the map $[-]\colon \mathscr{A} \to \mathscr{D}$ of sheaves induces a natural map

$$[-]\colon\ H^{p,q}_{\mathscr{A}}(V) \to H^{p,q}_{\mathscr{D}}(V).$$

Furthermore, there is a canonical map

$$\hat{H}^p_{\mathscr{D}}(V) \to H^{p,p}_{\mathscr{D}}(V)$$

mapping the class of a current T in the Bott–Chern cohomology to the class of T in the Dolbeault cohomology of currents.

7.12 Liu's Tropical Cycle Class Map. For smooth a smooth algebraic K-variety X, Liu defined in [Liu20] a *tropical cycle class map*

$$\mathrm{cl}_{\mathrm{trop}}\colon\ \mathrm{CH}^p(X) \to H^{p,p}_{\mathscr{A}}(X^{\mathrm{an}})$$

which is compatible with the graded ring structures induced from the intersection product resp. the wedge product and which is natural with respect to pull-back.

We review the construction of $\mathrm{cl}_{\mathrm{trop}}$ in Appendix A. Liu also showed that if Z is a cycle on X, then for every d''-closed form $\alpha \in \mathscr{A}_c^{n-p,n-p}(X^{\mathrm{an}})$, one has

$$\int_{X^{\mathrm{an}}} \mathrm{cl}_{\mathrm{trop}}(Z) \wedge \alpha = \int_{Z^{\mathrm{an}}} \alpha.$$

In fact his proof gives the slightly stronger statement that the image of $\mathrm{cl}_{\mathrm{trop}}(Z)$ in $H_{\mathscr{D}}^{p,p}(X^{\mathrm{an}})$ is represented by the current δ_Z (see Corollary A.22). In other words, the diagram

$$\begin{tikzcd} \mathrm{CH}^p(X) \arrow[r, "\delta"] \arrow[d, "\mathrm{cl}_{\mathrm{trop}}"'] & \hat{H}_{\mathscr{D}}^{p}(X^{\mathrm{an}}) \arrow[d] \\ H_{\mathscr{A}}^{p,p}(X^{\mathrm{an}}) \arrow[r, "{[-]}"'] & H_{\mathscr{D}}^{p,p}(X^{\mathrm{an}}) \end{tikzcd} \tag{7.12.1}$$

is commutative.

7.13 Change of Base Field. Let L/K be an extension of non-archimedean fields. Recall that we assume the K-analytic space V to be boundaryless; in particular it is also strict by [Tem15, Exmp. 4.2.4.2 (ii)]. Let $\varphi : V \to T$ be a moment map to an analytic torus. The base change $T \hat{\otimes}_K L$ is an L-analytic torus. There is a canonical identification of real affine spaces $(T \hat{\otimes}_K L)_{\mathrm{trop}} = T_{\mathrm{trop}}$ such that the diagram

$$\begin{tikzcd} V \hat{\otimes}_K L \arrow[r, "\varphi \hat{\otimes}_K L"] \arrow[d, "\pi_{L/K}"] & T \hat{\otimes}_K L \arrow[r, "\mathrm{trop}"] \arrow[d, "\pi_{L/K}"] & (T \hat{\otimes}_K L)_{\mathrm{trop}} \arrow[d, equal] \\ V \arrow[r, "\varphi"'] & T \arrow[r, "\mathrm{trop}"'] & T_{\mathrm{trop}} \end{tikzcd}$$

commutes. Let P be a compact polytope in T_{trop} containing $\varphi_{\mathrm{trop}}(V)$ and let $\omega \in \mathscr{A}_P^{p,q}(P)$ be a Lagerberg form on P. This gives rise to a form $\alpha = \varphi_{\mathrm{trop}}^*(\omega) \in \mathscr{A}^{p,q}(V)$ and locally on V every (p,q)-form is of this type by definition of the sheaf $\mathscr{A}_V^{p,q}$ [CD12, §3.1.2, § 3.1.9]. If $\alpha = \varphi_{\mathrm{trop}}^*(\omega)$ for $\omega \in \mathscr{A}_P^{p,q}(P)$ then we define

$$\pi_{L/K}^* \alpha := (\varphi \hat{\otimes}_K L)_{\mathrm{trop}}^*(\omega)$$

where on the right hand side, we regard ω as a form on $P \subset (T \hat{\otimes}_K L)_{\mathrm{trop}}$. It is easy to see that this extends to a well-defined morphism of sheaves of bigraded differential $\mathbf{R}$-algebras

$$\pi_{L/K}^* : \mathscr{A}_V^{\bullet,\bullet} \to (\pi_{L/K})_* \mathscr{A}_{V \hat{\otimes}_K L}^{\bullet,\bullet},$$

similarly to the pull-back along a morphism of K-analytic spaces.

Since the map $\pi_{L/K} : V \hat{\otimes}_K L \to V$ is topologically proper, the pull-back of a compactly supported form is again compactly supported and we obtain a push-forward morphism

$$(\pi_{L/K})_* : (\pi_{L/K})_* \mathscr{D}^{\bullet,\bullet}_{V \hat{\otimes}_K L} \to \mathscr{D}^{\bullet,\bullet}_V$$

defined by

$$\langle (\pi_{L/K})_* T, \alpha \rangle = \langle T, \pi^*_{L/K} \alpha \rangle$$

for $T \in \mathscr{D}^{n-p,n-q}(V \hat{\otimes}_K L)$ and $\alpha \in \mathscr{A}^{p,q}_c(V)$. For $T \in \mathscr{D}^{\bullet,\bullet}(V \hat{\otimes}_K L)$ and $\alpha \in \mathscr{A}^{\bullet,\bullet}(V)$ we have the projection formula

$$(\pi_{L/K})_* (\pi^*_{L/K} \alpha \wedge T) = \alpha \wedge (\pi_{L/K})_* T \tag{7.13.1}$$

which is easy to verify.

If $\alpha \in \mathscr{A}^{n,n}_c(V)$ has compact support, then we have

$$\int_{V \hat{\otimes}_K L} \pi^*_{L/K} \alpha = \int_V \alpha.$$

This is stated in [GJR21, Prop. 14.5] for a finite Galois extension L/K but the proof works for arbitrary extensions of non-archimedean fields. It follows that for $\alpha \in \mathscr{A}^{p,q}(V)$ we have

$$(\pi_{L/K})_* [\pi^*_{L/K} \alpha] = [\alpha] \tag{7.13.2}$$

as currents on V.

8. First Chern Forms and Currents for Metrized Line Bundles

Throughout this chapter, K will be a non-archimedean field. As in Chapter 7, all K-analytic spaces are assumed to be good, topologically Hausdorff, boundaryless and equidimensional. We fix a K-analytic space V of dimension n and an algebraic K-variety X of dimension n.

We review the constructions of first Chern forms and currents of metrized line bundles from [CD12]. Using the Bedford–Taylor theory developed in [CD12] one can also define products of first Chern forms of locally approachably metrized line bundles (Paragraph 8.9). The class of $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ in the Bott–Chern cohomology group does not depend on the given metrics (Proposition 8.11). In Proposition 8.14 we see that for an algebraic variety, this class is compatible with the Chern classes of algebraic intersection theory.

8.1 First Chern Forms of Smoothly Metrized Line Bundles. Let $\overline{L}$ be a smoothly metrized line bundle on V in the sense of Paragraph 5.34. Note that since V is assumed to be good, by Paragraph 4.1 every point of V admits an open neighborhood over which L is trivial. Let $U \subset V$ be an open subset over which L is trivial, i.e. over which it admits a nowhere-vanishing section. If $s, s' \in \Gamma(U, L)$ are two nowhere-vanishing sections, then the $(1,1)$-forms $d'd''(-\log\|s(-)\|)$ and $d'd''(-\log\|s'(-)\|)$ differ by the form $d'd''(-\log|f(-)|) = 0$ with $f = s/s'$ a nowhere-vanishing regular function. Hence the form $(1,1)$-form

$$c_1(\overline{L}|_U) := d'd''(-\log\|s(-)\|) \in \mathscr{A}^{1,1}(U)$$

is independent of the choice of the section s. Since it is also local with respect to restriction to a smaller open subset U, there exists a unique form $c_1(\overline{L}) \in \mathscr{A}^{1,1}(V)$ satisfying $c_1(\overline{L})|_U = c_1(\overline{L}|_U)$ for every open subset $U \subset V$ admitting a nowhere-vanishing section of L.

This construction is due to [CD12, § 6.4.1].

8.2 First Chern Currents of Continuously Metrized Line Bundles. Let $\overline{L}$ be a continuously metrized line bundle on V. Similarly to Paragraph 8.1, there exists a unique current

$$[c_1(\overline{L})] \in \mathscr{D}^{1,1}(V)$$

such that for every open subset $U \subset V$ and every nowhere-vanishing section $s \in \Gamma(U, L)$ one has

$$[c_1(\overline{L})]|_U = d'd''[-\log\|s\|].$$

If $\overline{L}$ is smoothly metrized, the symbol $[c_1(\overline{L})]$ can be interpreted both as the first Chern current associated to the continuously metrized line bundle $\overline{L}$ or as the current associated to the smooth form $c_1(\overline{L})$ constructed in Paragraph 8.1. It is easy to see that both interpretations give the same result.

This construction is due to [CD12, § 6.4.1].

8.3 The Poincaré-Lelong Equation. Let $\overline{L}$ be a line bundle on X and let s be a non-zero rational section of L. Denote by $[\mathrm{div}(s)]$ the Weil divisor on X associated to s. Then by [CD12, § 6.4.2] there is an equality of currents

$$[c_1(\overline{L})] = \delta_{[\mathrm{div}(s)]} + d'd''[-\log\|s(-)\|].$$

8.4 Plurisubharmonicity. Let u be a continuous function on V. It is called *plurisubharmonic (or psh)* if the current $d'd''[u]$ is positive in the sense of [CD12, § 5.4.1]. The function is called *locally psh-approachable* if every point of V admits an open neighborhood on which u is a uniform limit of smooth plurisubharmonic functions. It is called *locally approachable* if every point of V admits an open neighborhood on which u is the difference of two locally psh-approachable functions. Lemma 8.6 below shows in particular that smooth functions are locally approachable.

In [CD12, Lem. 5.5.3] it is shown that if V is compact and $u = \varphi^* v$ for a moment map $\varphi : V \to T$ to an analytic torus T and v a smooth function on $\varphi_{\mathrm{trop}}(V)$, then u is plurisubharmonic if and only if the restriction of v to every maximal face of the polyhedral subset $\varphi_{\mathrm{trop}}(V)$ of the tropicalization of T is convex.

8.5 Lemma. *Let $f : V' \to V$ be a morphism of K-analytic spaces and let u be a continuous function on V.*

(i) If u is smooth and plurisubharmonic, then the same is true for f^*u.

(ii) If u is locally psh-approachable, then the same is true for f^*u.

(iii) If u is locally approachable, then the same is true for f^*u.

Proof. Assume that u is smooth and plurisubharmonic. Let $y \in V'$ be a point with $x := f(y) \in V$. By the definition of smooth functions, there exists a compact analytic neighborhood W of x in V and a moment map $\varphi : W \to T$ to an analytic torus such that $u \equiv \varphi^* v$ for some smooth function v on $\varphi_{\mathrm{trop}}(W)$. By [CD12, Lem. 5.5.3] the function v is convex on every maximal face of $\varphi_{\mathrm{trop}}(V)$. Let W' be a compact analytic neighborhood of y contained in $f^{-1}(W)$. Then $f^*u \equiv (\varphi \circ f)^* v$ on W'. Since $(\varphi \circ f)_{\mathrm{trop}}(W') \subset \varphi_{\mathrm{trop}}(W)$, the restriction of v to every maximal face of $(\varphi \circ f)_{\mathrm{trop}}(W')$ is convex. This shows that f^*u is plurisubharmonic in a neighborhood of y. Since plurisubharmonic functions form a sheaf by [CD12, Lem. 5.5.2], f^*u is plurisubharmonic which proves the first statement.

The remaining statements follow immediately from the first one. ∎

8.6 Lemma. *If u is a smooth function on V, then every point of V admits an open neighborhood on which u is the difference of two smooth plurisubharmonic functions.*

Proof. Let x be an arbitrary point of V. There exists a compact analytic neighborhood W of x and a moment map $\varphi : W \to T$ to some analytic torus such that $u \equiv \varphi^* v$ for some smooth function v on the polyhedral subset $\varphi_{\mathrm{trop}}(W)$ of the tropicalization of T. For big enough $a > 0$ the function $v_2 := v + v_1$ with $v_1 = a \cdot \sum x_i^2$ (for some choice of affine coordinates on T_{trop}), will be convex on every maximal face of the compact polyhedral set $\varphi_{\mathrm{trop}}(W)$. Then writing $u = \varphi^* v_2 - \varphi^* v_1$ shows that on a neighborhood of x, the function u is the difference of two smooth plurisubharmonic functions. ∎

8.7 Bedford–Taylor Theory. Denote by $\mathrm{LPSHA}(V)$ the space of locally psh-approachable functions on V. The operation $U \mapsto \mathrm{LPSHA}(U)$ is a sheaf of real cones ($\mathbf{R}_{\geq 0}$-modules) on V. By [CD12, Cor. 5.6.5] there exists a unique symmetric multilinear morphism of sheaves

$$\underbrace{\mathrm{LPSHA}_V \times \cdots \times \mathrm{LPSHA}_V}_{r \text{ times}} \to \mathscr{D}_V^{r,r}, \qquad (u_1, \ldots, u_r) \mapsto [d'd''u_1 \wedge \cdots \wedge d'd''u_r]$$

such that for every open subset $U \subset V$, every form $\alpha \in \mathscr{A}_c^{n-r,n-r}(U)$ and every family $\{u_n^j\}_{n \in \mathbf{N}}$, $j = 1, \ldots, r$ of sequences of smooth plurisubharmonic functions on U converging uniformly to $u_j|_U$, one has

$$\langle [d'd''u_1 \wedge \cdots \wedge d'd''u_r]|_U, \alpha\rangle = \lim_n \int_U d'd''u_n^1 \wedge \cdots \wedge d'd''u_n^r \wedge \alpha.$$

We denote by LA_V the sheaf of locally approachable function on V (functions that are locally a difference of locally psh-approachable functions). It follows that the morphism above extends uniquely to a symmetric multilinear morphism of sheaves of real vector spaces

$$\underbrace{\mathrm{LA}_V \times \cdots \times \mathrm{LA}_V}_{r \text{ times}} \to \mathscr{D}_V^{r,r}, \qquad (u_1, \ldots, u_r) \mapsto [d'd''u_1 \wedge \cdots \wedge d'd''u_r].$$

If $u_1, \ldots, u_{r'}$ are smooth functions and $u_{r'+1}, \ldots, u_r$ are locally approachable, then

$$d'd''u_1 \wedge \cdots \wedge d'd''u_{r'} \wedge [d'd''u_{r'+1} \wedge \cdots \wedge d'd''u_r] = [d'd''u_1 \wedge \cdots \wedge d'd''u_r]. \quad (8.7.1)$$

Using Lemma 8.6 this follows immediately from the definition. In particular in the case $r' = r$ we see that the symbol $[d'd''u_1 \wedge \cdots \wedge d'd''u_r]$ as defined in this paragraph agrees with the old meaning of the current associated to the smooth form defined as the product of the forms $d'd''u_j$.

Note that if f is a nowhere-vanishing analytic function on V, then $u = \log|f|$ is a smooth function satisfying $d'd''u = 0$ and it follows from Eq. (8.7.1) that for locally approachable functions $u_1, \ldots, u_p$, the current $[d'd''u_1 \wedge \cdots \wedge d'd''u_r]$ vanishes as soon as one of the functions u_j has the form $\log|f|$ for an invertible analytic function f.

8.8 Locally Approachable Metrics. A continuously metrized line bundle $\overline{L}$ on V is called *locally approachable* if for every local nowhere-vanishing section $s \in \Gamma(U, L)$ over an open subset $U \subset V$, the function $-\log\|s(-)\|$ is locally approachable on U [CD12, § 6.3.1]. Tensor products and duals of locally approachably metrized line bundles are locally approachably metrized. By Lemma 8.5 pull-backs of locally approachably metrized line bundles are locally approachably metrized.

It follows from the proof of [CD12, Prop. 6.9.2] that formal metrics are always locally approachable.

8.9 Products of First Chern Currents. Let $\overline{L}_1$, ..., $\overline{L}_r$ be locally approachably metrized line bundles on V. Let $U \subset V$ be an open subset on which $L_1, \ldots, L_r$ admit nowhere-vanishing sections $s_1, \ldots, s_r$. If $s'_1, \ldots, s'_r$ is another such family of nowhere-vanishing sections, then the functions $-\log\|s_i(-)\|$ and $-\log\|s'_i(-)\|$

differ by the smooth function $-\log|f(-)|$, with $f = s_i/s_i'$, so it follows from Paragraph 8.7 that the current

$$[c_1(\overline{L}_1|_U) \wedge \cdots \wedge c_1(\overline{L}_r|_U)] := [d'd''(-\log\|s_1(-)\|) \wedge \cdots \wedge d'd''(-\log\|s_r(-)\|)]$$

is independent of the choice of the sections s_i. The construction is obviously compatible with restriction. Since every point of V admits an open neighborhood U trivializing $L_1, \dots, L_r$ by Paragraph 4.1, it follows that there exists a unique current

$$[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)] \in \mathscr{D}^{r,r}(V)$$

such that

$$[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]_U = [c_1(\overline{L}_1|_U) \wedge \cdots \wedge c_1(\overline{L}_r|_U)]$$

for every open subset U trivializing $L_1, \dots, L_r$.

The expression $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ is additive in every factor and local with respect to restriction to open subsets. It is easy to see that $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ is a d'- and d''-closed current.

8.10 Lemma. *Let V, W be K-analytic spaces of dimensions n, resp. m let $\overline{L}_1, \dots, \overline{L}_r$ be locally approachably metrized line bundles on V and let $\overline{M}_1, \dots, \overline{M}_s$ be locally approachably metrized line bundles on W. Denote by $\pi_0 : V \times W \to V$ and by $\pi_1 : V \times W \to W$ the canonical projections. Let $\alpha \in \mathscr{A}_c(V)$ and $\beta \in \mathscr{A}_c(W)$ such that $\pi_0^*\alpha \wedge \pi_1^*\beta$ has bi-degree $(n+m-r-s, n+m-r-s)$. If α has bi-degree $(n-r, n-r)$ and β has bi-degree $(m-s, m-s)$ then*

$$\begin{aligned}&\langle [c_1(\pi_0^*\overline{L}_1) \wedge \cdots \wedge c_1(\pi_0^*\overline{L}_r) \wedge c_1(\pi_1^*\overline{M}_1) \wedge \cdots \wedge c_1(\pi_1^*\overline{M}_s)], \pi_0^*\alpha \wedge \pi_1^*\beta\rangle \\ &= \langle [c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)], \alpha\rangle \cdot \langle [c_1(\overline{M}_1) \wedge \cdots \wedge c_1(\overline{M}_s)], \beta\rangle\end{aligned}$$

while

$$\langle [c_1(\pi_0^*\overline{L}_1) \wedge \cdots \wedge c_1(\pi_0^*\overline{L}_r) \wedge c_1(\pi_1^*\overline{M}_1) \wedge \cdots \wedge c_1(\pi_1^*\overline{M}_s)], \pi_0^*\alpha \wedge \pi_1^*\beta\rangle = 0$$

otherwise.

Proof. Let us first assume that α has bi-degree $(n-r, n-r)$ and β has bi-degree $(m-s, m-s)$. If $u_1, \dots, u_r$ are smooth functions on V and $v_1, \dots, v_s$ are smooth functions on W then

$$\begin{aligned}&\int_{V\times W} d'd''\pi_0^*u_1 \wedge \cdots \wedge d'd''\pi_0^*u_r \wedge d'd''\pi_1^*v_1 \wedge \cdots \wedge d'd''\pi_1^*v_s \wedge \pi_0^*\alpha \wedge \pi_1^*\beta \\ &= \int_V d'd''u_1 \wedge \cdots \wedge d'd''u_r \wedge \alpha \cdot \int_W d'd''v_1 \wedge \cdots \wedge d'd''v_s \wedge \beta\end{aligned}$$

by Lemma 7.7. By an approximation argument we find that

$$\begin{aligned}&\langle [d'd''\pi_0^* u_1 \wedge \cdots \wedge d'd''\pi_0^* u_r \wedge d'd''\pi_1^* v_1 \wedge \cdots \wedge d'd''\pi_1^* v_s], \pi_0^*\alpha \wedge \pi_1^*\beta\rangle \\ &= \langle [d'd'' u_1 \wedge \cdots \wedge d'd'' u_r], \alpha\rangle \cdot \langle [d'd'' v_1 \wedge \cdots \wedge d'd'' v_s], \beta\rangle\end{aligned}$$

for locally approachable functions $u_1, \dots, u_r$ and $v_1, \dots v_s$. This implies the result for line bundles with trivializing sections and the general result can be checked locally.

In the case where α and β do not have the appropriate bi-degree, we can assume without loss of generality that α has bi-degree (p, q) with $p > n - r$. Then for smooth functions $u_1, \dots, u_r$, the form $d'd''u_1 \wedge \cdots \wedge d'd''u_r \wedge \alpha$ on V vanishes for dimension reasons. Then its pull-back to $V \times W$ vanishes as well and hence

$$\langle [d'd''\pi_0^* u_1 \wedge \cdots \wedge d'd''\pi_0^* u_r \wedge d'd''\pi_1^* v_1 \wedge \cdots \wedge d'd''\pi_1^* v_s], \pi_0^*\alpha \wedge \pi_1^*\beta\rangle$$

vanishes. The same argument as above now yields the result also in the case where α is not of degree $(n - r, n - r)$. ∎

8.11 Proposition. *Let $\overline{L}_1, \dots, \overline{L}_r$ be locally approachably metrized line bundles on V. Then the class of the current $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ in the Bott–Chern cohomology group $\hat{H}^r_{\mathscr{D}}(V)$ does not depend on the metrics of the line bundles.*

Proof. By symmetric multilinearity it is enough to show that if the underlying line bundle of $\overline{L}_1$ is trivial, then $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ lies in the image of $d'd''$. Let $s \in \Gamma(V, L_1)$ be a nowhere-vanishing section and denote $u := -\log\|s(-)\|$.

By [CD12, Cor. 5.4.7], there is for every positive current T defined on an open subset $U \subset V$, a current $uT := u|_U T \in \mathscr{D}_V(U)$ such that for every $\alpha \in \mathscr{A}_c(U)$ and every sequence u_n of smooth functions converging uniformly to $u|_U$ on a neighborhood of $\mathrm{supp}(\alpha)$, one has $\langle uT, \alpha\rangle = \lim_n \langle u_n T, \alpha\rangle$. It is easy to verify that the operation $T \mapsto uT$ defines a morphism of sheaves of real cones. Since for locally psh-approachable functions $u_2, \dots, u_r$, the current $[d'd''u_2 \wedge \cdots \wedge d'd''u_r]$ constructed by Bedford–Taylor theory is positive, we get a symmetric multilinear morphism of sheaves of real cones

$$\mathrm{LPSHA}_V \times \cdots \times \mathrm{LPSHA}_V \to \mathscr{D}_V^{r-1,r-1}, \qquad (u_2, \dots, u_r) \mapsto u[d'd''u_2 \wedge \cdots \wedge d'd''u_r].$$

Repeating the arguments of Paragraph 8.7 and Paragraph 8.9, we see that we get a well-defined current

$$u[c_1(\overline{L}_2) \wedge \cdots \wedge c_1(\overline{L}_r)] \in \mathscr{D}_V^{r-1,r-1}(V)$$

such that for every open subset $U \subset V$ where $L_2, \dots, L_r$ admit nowhere-vanishing sections $s_2, \dots, s_r$ we have

$$(u[c_1(\overline{L}_2) \wedge \cdots \wedge c_1(\overline{L}_r)])|_U = u[d'd''u_2 \wedge \cdots \wedge d'd''u_r]$$

where we write $u_j := -\log\|s_j(-)\|$. Unpacking the gluing and limit operations one verifies that

$$d'd''(u[c_1(\overline{L}_2) \wedge \cdots \wedge c_1(\overline{L}_r)]) = [c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$$

which proves the claim. ∎

8.12 The Class in Bott–Chern Cohomology. Let $L_1, \dots, L_r$ be line bundles on V. We define

$$[c_1(L_1) \wedge \cdots \wedge c_1(L_r)]_{\mathrm{BC}} \in \hat{H}^r_{\mathcal{D}}(V)$$

to be the class of the current $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ for any choice of locally approachable metrics on the L_k. This is well-defined by Proposition 8.11.

8.13 First Chern Classes in Intersection Theory. Let X be an algebraic K-variety. Recall from [Ful98, Sec. 2.5] the definition of the homomorphism

$$c_1(L) \cap - \colon \mathrm{CH}^p(X) \to \mathrm{CH}^{p+1}(X)$$

for a line bundle L in terms of divisor intersection. If $L_1, \dots, L_r$ are line bundles on X, then we set

$$c_1(L_1) \cdots c_1(L_r) := c_1(L_1) \cap (c_1(L_2) \cap (\cdots \cap [X])) \in \mathrm{CH}^r(X).$$

The expression $c_1(L_1) \cdots c_1(L_r)$ is multilinear and symmetric in the L_k.

8.14 Proposition. *Let $L_1, \dots, L_r$ be line bundles on the algebraic K-variety X. Then we have*

$$\delta(c_1(L_1) \cdots c_1(L_r)) = [c_1(L_1) \wedge \cdots \wedge c_1(L_r)]_{\mathrm{BC}} \in \hat{H}^r_{\mathcal{D}}(X^{\mathrm{an}}),$$

where $\delta \colon \mathrm{CH}^r(X) \to \hat{H}^r_{\mathcal{D}}(X^{\mathrm{an}})$ denotes cycle class map of Paragraph 7.10.

Proof. It is enough to show that for smoothly metrized line bundles $\overline{L}_1, \dots, \overline{L}_r$ with underlying line bundles $L_1, \dots, L_r$ the class $\delta(c_1(L_1) \cdots c_1(L_r)) \in \hat{H}^r_{\mathcal{D}}(X^{\mathrm{an}})$ is represented by the current $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$.

By an inductive argument it is enough to show that if $[Z] \in \mathrm{CH}^p(X)$ is a cycle class represented by a cycle Z and $\overline{L} = (L, \|{-}\|)$ is a smoothly metrized line bundle on X, then the class $\delta(c_1(L) \cap [Z])$ is represented by the current $c_1(\overline{L}) \wedge \delta_Z$. By linearity we may assume that Z is a subvariety of X. Applying the commutative diagram Eq. (7.10.1) to the embedding $i \colon Z \hookrightarrow X$ we may assume that $Z = X$. Then choosing a non-zero rational section $s \in \Gamma(X, L)$, the class $c_1(L) = c_1(L) \cap [X] \in \mathrm{CH}^1(X)$ is by definition represented by the Weil divisor $[\mathrm{div}(s)]$. Hence $\delta(c_1(L) \cap [X])$ is by definition represented by the current of integration $\delta_{[\mathrm{div}(s)]}$.

On the other hand, we have by the Poincaré-Lelong equation that $c_1(\overline{L}) \cap \delta_X = [c_1(\overline{L})] = \delta_{[\mathrm{div}(s)]} + d'd''[-\log\|s(-)\|]$. ∎

8.15 Lemma. *Let $\overline{L}_1, \dots, \overline{L}_r$ be locally approachably metrized line bundles on V and let L/K be an extension of non-archimedean fields. Then the metrized scalar extensions $\pi_{L/K}^*\overline{L}_i$ are again locally approachably metrized and we have*

$$(\pi_{L/K})_*[c_1(\pi_{L/K}^*\overline{L}_1) \wedge \cdots \wedge c_1(\pi_{L/K}^*\overline{L}_r)] = [c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)] \tag{8.15.1}$$

as currents on V.

Proof. Let us abbreviate $\pi := \pi_{L/K}$. Let $U \subset V$ be an open subset. If $s_i \in \Gamma(U, L_i)$ is a local frame for L_i with $u_i := -\log\|s_i(-)\|$ then $\pi^* s_i$ is a local frame for $\pi^* L_i$ and for the function $u_i' := -\log\|\pi^* s_i(-)\|$ we have

$$u_i' = \pi^* u_i$$

by the definition of the scalar extension metric. An argument as in Lemma 8.5 shows that if u_i is smooth and plurisubharmonic, resp. locally psh-approachable, resp. locally approachable then the same is true for u_i'. This shows in particular that the $\pi^*\overline{L}_i$ are locally approachably metrized.

If the metrics and hence the u_i are smooth, a repeated application of the projection formula Eq. (7.13.1) and Eq. (7.13.2) shows that

$$\pi_*[d'd''u_1' \wedge \cdots \wedge d'd''u_r'] = [d'd''u_1 \wedge \cdots \wedge d'd''u_r].$$

By an approximation argument we get the above formula also if the u_i are merely locally approachable. This shows that Eq. (8.15.1) holds locally on V and hence it holds globally. ∎

9. Characteristic Currents of Metrized Vector Bundles

Throughout this chapter we let K be a non-archimedean field. All K-analytic spaces are assumed to be good, topologically Hausdorff, boundaryless and equidimensional. We fix a K-analytic space V of dimension n and an algebraic variety X of dimension n.

Inspired by the construction of Segre classes of a vector bundle E in [Ful98] in terms of first Chern class of $\mathcal{O}_E(1)$, we define (products of) *Segre currents* of (pseudo-)metrized vector bundles in Paragraph 9.4. We prove some basic relations among the Segre and Chern currents (Propositions 9.5 to 9.8 and 9.10). In Paragraph 9.11 we define polynomial expressions in the Segre currents, allowing us in particular to consider *Chern currents* (Paragraph 9.12). In the algebraic case, Segre and Chern currents refine in a certain sense the Segre and Chern classes of algebraic intersection theory (Paragraph 9.14).

9.1 Locally Approachable Metrics. Recall our notion of pseudo-metrics from Paragraph 5.32. Let $\overline{E} = (E, \|-\|)$ be a continuously (pseudo-)metrized vector bundle on V. It is called *locally approachably (pseudo-)metrized* and the (pseudo-)metric $\|-\|$ is called *locally approachable* if the induced Fubini-Study metric $\|-\|_{\mathrm{FS}}$ on $\mathcal{O}_E(1)$ is locally approachable in the sense of Paragraph 8.8. From the definition of pull-backs and twisting of pseudo-metrized vector bundles in Paragraph 5.32 it follows that if $\overline{E}$ is locally approachable and $\overline{L}$ is a locally approachably metrized line bundle, then $\overline{E}\otimes\overline{L}$ is locally approachable. Similarly it follows that pull-backs of locally approachable (pseudo-)metrics are locally approachable.

Every formal metric on E is locally approachable by Lemma 6.11 and Paragraph 8.8.

9.2 Segre and Chern Classes in Algebraic Intersection Theory. If E is a vector bundle of rank $r = e + 1$ on an algebraic variety X, then the homomorphisms

$$s_i(E) \cap - \colon \mathrm{CH}^{p+i}(X) \to \mathrm{CH}^p(X)$$

are defined by the formula

$$s_i(E) \cap [Z] = p_*(c_1(\mathcal{O}_E(1))^{e+i} \cap p^*[Z]),$$

where $p\colon P(E) \to X$ denotes the projective bundle associated to E [Ful98, Sec. 3.1].

If $E_1, \dots, E_r$ are vector bundles on X and $i_1, \dots, i_r \in \mathbf{N}$ are natural numbers, then we write

$$s_{i_1}(E_1) \cdots s_{i_r}(E_r) \coloneqq s_{i_1}(E_1) \cap (\cdots \cap (s_{i_r}(E_r) \cap [X])) \in \mathrm{CH}^{|i|}(X).$$

Finally, one defines

$$c_i(E) \cap [Z] = C_i(s_0(E), s_1(E), \dots, s_i(E)) \cap [Z],$$

where $C_i \in \mathbf{Z}[X_0, \dots, X_i]$ is a certain universal polynomial [Ful98, Sec. 3.2]. Namely, the C_i are the unique polynomials verifying the equality

$$(C_0 + C_1 T + C_2 T^2 + \cdots)(1 + X_1 T + X_2 T^2 + \cdots) = 1$$

in the ring of formal power series $\mathbf{Z}[X_0, X_1, X_2, \dots][\![T]\!]$.

9.3 Lemma. *Let $E_1, \dots, E_r$ be vector bundles on X and $i_1, \dots, i_r \in \mathbf{N}$ be natural numbers. For each $k \in \{1, \dots, r\}$, we denote by $p_k\colon P(E_k) \to X$ the projection from the respective projective bundle. We let*

$$p\colon P \coloneqq P(E_1) \times_X \cdots \times_X P(E_r) \to X$$

be their fibered product over X and denote by $q_k\colon P \to P(E_k)$ the projections from the fibered product. Then we have

$$s_{i_1}(E_1) \cdots s_{i_r}(E_r) = p_*(c_1(q_1^*\mathcal{O}_{E_1}(1))^{e_1+i_1} \cdots c_1(q_r^*\mathcal{O}_{E_r}(1))^{e_r+i_r}).$$

Proof. This follows from an iterated application of the base change formula [Ful98, Prop. 1.7]. See the proof of [Ful98, Prop. 3.1 (b)] for a similar computation. ∎

9.4 Segre Currents. Let $\overline{E}_1, \dots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles of ranks $e_1 + 1, \dots, e_r + 1$ on V and let $i_1, \dots, i_r \in \mathbf{N}$ be natural numbers. For each $k \in \{1, \dots, r\}$, we denote by $p_k : P(E_k) \to V$ the projection from the respective projective bundle. We let

$$p : P := P(E_1) \times_V \cdots \times_V P(E_r) \to V$$

be their fibered product over V and denote by $q_k : P \to P(E_k)$ the projections from the fibered product. We define

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] \in \mathcal{D}^{|i|,|i|}(V)$$

to be the current

$$p_*[c_1(q_1^* \mathcal{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^* \mathcal{O}_{\overline{E}_r}(1))^{e_r+i_r}].$$

Here $\mathcal{O}_{\overline{E}_k}(1)$ denotes the locally approachably metrized line bundle on $P(E_k)$ given by the line bundle $\mathcal{O}_{E_k}(1)$ together with the induced Fubini-Study metric.

The current $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$ is d'- and d''-closed since the same is true for products of first Chern currents by Paragraph 8.9. It follows from Paragraph 8.12 that the class of $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$ in $\hat{H}^{|i|}_{\mathcal{D}}(V)$ does not depend on the metrics. We denote it by

$$[s_{i_1}(E_1) \wedge \cdots \wedge s_{i_r}(E_r)]_{\mathrm{BC}} \in \hat{H}^{|i|}_{\mathcal{D}}(V).$$

9.5 Proposition. *Let $\overline{E}_1, \dots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V and let $i_1, \dots, i_r \in \mathbf{N}$ be natural numbers. Let $\sigma : \{1, \dots, r\} \to \{1, \dots, r\}$ be a permutation. Then we have*

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = [s_{i_{\sigma(1)}}(\overline{E}_{\sigma(1)}) \wedge \cdots \wedge s_{i_{\sigma(r)}}(\overline{E}_{\sigma(r)})].$$

Proof. This follows from the commutativity of fibered products and of products of first Chern currents of line bundles. ∎

9.6 Proposition. *Let $\overline{E}_1, \dots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V and let $i_1, \dots, i_r \in \mathbf{N}$ be natural numbers. Let $U \subset V$ be an open subset. Then we have*

$$[s_{i_1}(\overline{E}_1|_U) \wedge \cdots \wedge s_{i_r}(\overline{E}_r|_U)] = [s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]|_U.$$

Proof. There is a canonical commutative diagram

$$\begin{array}{ccc} P' = P(E_1|_U) \times_U \cdots \times_U P(E_r|_U) & \xrightarrow{p'} & U \\ \big\downarrow & & \big\downarrow \\ P = P(E_1) \times_V \cdots \times_V P(E_r) & \xrightarrow[p]{} & V \end{array}$$

which is cartesian and hence identifies P' with the open subset $p^{-1}(U) \subset P$. Denoting by $q_k\colon P \to P(E_k)$ and by $q'_k\colon P' \to P(E_k|_U)$ the canonical projections and writing

$$T := [c_1(q_1^* \mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^* \mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}]$$

and

$$T' := [c_1(q_1'^* \mathscr{O}_{\overline{E}_1|_U}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r'^* \mathscr{O}_{\overline{E}_r|_U}(1))^{e_r+i_r}]$$

on verifies that under the identification $P' = p^{-1}(U)$ we have $T' = T|_{P'}$. Hence the result follows from the fact that $p_*\colon p_*\mathscr{D}_P \to \mathscr{D}_V$ is a morphism of sheaves. ∎

9.7 Proposition. *Let $\overline{E}_1, \ldots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V and let $i_1, \ldots, i_r \in \mathbf{N}$ be natural numbers. We consider $\overline{\mathscr{O}}_V$ as the trivial line bundle on V carrying the trivial metric. Let $e_0, i_0 \in \mathbf{N}$ be natural numbers. Then we have*

$$[s_{i_0}(\overline{\mathscr{O}}_V^{e_0+1}) \wedge s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = \begin{cases} [s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] & \text{if } i_0 = 0 \\ 0 & \text{if } i_0 > 0. \end{cases}$$

Proof. For each $k \in \{1, \ldots, r\}$ let $e_k + 1$ be the rank of E_k. We write $\overline{E}_0 := \overline{\mathscr{O}}_V^{e_0+1}$. We denote by $\mathbf{P}^{e_0}$ the e_0-dimensional projective space and by $\overline{\mathscr{O}}_{\mathbf{P}^{e_0}}(1)$ the canonical line bundle on $\mathbf{P}^{e_0}$ carrying the standard Fubini-Study metric. We write

$$p\colon P := P(E_1) \times_V \cdots \times_V P(E_r) \to V$$

and denote by $q_k\colon P \to P(E_k)$ the canonical projections. Let $\mathbf{P}^{e_0} \times P$ be the product over K and denote by

$$\pi_0\colon \mathbf{P}^{e_0} \times P \to \mathbf{P}^{e_0}$$

as well as

$$\pi_1\colon \mathbf{P}^{e_0} \times P \to P$$

the canonical projection maps onto the factors.

There is a canonical identification

$$P' := P(E_0) \times_V P(E_1) \times_V \cdots \times_V P(E_r) \cong \mathbf{P}^{e_0} \times P.$$

Under this identification, the structural map $p'\colon P' \to V$ is given by $p \circ \pi_1\colon \mathbf{P}^{e_0} \times P \to V$. For $k \in \{1, \ldots, r\}$, the canonical projection map $q'_k\colon P' \to P(E_k)$ corresponds to $q_k \circ \pi_1\colon \mathbf{P}^{e_0} \times P \to P(E_k)$. Denoting by $q'_0\colon P' \to P(E_0)$ the canonical projection onto the first factor, we note that the metrized line bundle $q_0'^*\mathcal{O}_{\overline{E}_0}(1)$ on P' corresponds to the metrized line bundle $\pi_0^*\overline{\mathcal{O}}_{\mathbf{P}^{e_0}}(1)$ on $\mathbf{P}^{e_0}\times P$.

By definition we have

$$\begin{aligned}&[s_{i_0}(\overline{\mathcal{O}_V^{e_0+1}})\wedge s_{i_1}(\overline{E}_1)\wedge\cdots\wedge s_{i_r}(\overline{E}_r)]\\ &= p_*\pi_{1*}\big[c_1(\pi_0^*\overline{\mathcal{O}}_{\mathbf{P}^{e_0}}(1))^{e_0+i_0}\wedge c_1(\pi_1^*q_1^*\mathcal{O}_{\overline{E}_1}(1))^{e_1+i_1}\wedge\cdots\wedge c_1(\pi_1^*q_r^*\mathcal{O}_{\overline{E}_r}(1))^{e_r+i_r}\big].\end{aligned}$$

Evaluating this current on a test form $\alpha\in\mathscr{A}_c(V)$ we get

$$\big\langle\big[c_1(\pi_0^*\overline{\mathcal{O}}_{\mathbf{P}^{e_0}}(1))^{e_0+i_0}\wedge c_1(\pi_1^*q_1^*\mathcal{O}_{\overline{E}_1}(1))^{e_1+i_1}\wedge\cdots\wedge c_1(\pi_1^*q_r^*\mathcal{O}_{\overline{E}_r}(1))^{e_r+i_r}\big],\pi_1^*p^*\alpha\big\rangle.$$

Writing $\pi_1^*p^*\alpha$ as $\pi_0^*1\wedge\pi_1 p^*\alpha$ and taking into account Lemma 8.10 we are left to show the equality

$$\langle[c_1(\overline{\mathcal{O}}_{\mathbf{P}^{e_0}}(1))^{e_0}],1\rangle = 1.$$

This follows from the fact that the total mass of the Monge–Ampère measure $[c_1(\overline{\mathcal{O}}_{\mathbf{P}^{e_0}}(1))^{e_0}]$ can be computed as an intersection number by [CD12, Cor. 6.4.4]. ∎

9.8 Proposition. *Let $\overline{E}_0,\dots,\overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V and let $i_1,\dots,i_r\in\mathbf{N}$ be natural numbers. Then we have*

$$[s_0(\overline{E}_0)\wedge s_{i_1}(\overline{E}_1)\wedge\cdots\wedge s_{i_r}(\overline{E}_r)] = [s_{i_1}(\overline{E}_1)\wedge\cdots\wedge s_{i_r}(\overline{E}_r)].$$

Proof. We first prove the special case $[s_0(\overline{E}_0)] = [1]\in\mathscr{D}_V^{0,0}(V)$. This identity can be checked locally on V so we may assume that E_0 is trivial. Since in bidegree $(0,0)$ there are no boundaries it is enough to show $[s_0(\overline{E}_0)] = [1]$ in the cohomology group $\hat{H}^0_{\mathscr{D}}(V)$. Since the class of $[s_0(\overline{E}_0)]$ in $\hat{H}^0_{\mathscr{D}}(V)$ does not depend on the metric, we may assume that E_0 carries the trivial metric with respect to some trivialization, so the result follows from Proposition 9.7.

To show the general case, we consider the bundle

$$p\colon P = P(E_1)\times_V\cdots\times_V P(E_r)\to V$$

with the projection maps $q_k\colon P\to P(E_k)$. Let $e_0+1,\dots,e_r+1$ be the ranks of $E_0,\dots,E_r$.

The bundle $P' := P(E_0)\times_V P(E_1)\times_V\cdots\times_V P(E_r)$ identifies with the bundle $P(p^*E_0)$. We denote by $p'\colon P(p^*E_0)\to P$ the canonical projection map. Then the canonical map $P'\to V$ corresponds to the composition $p\circ p'\colon P(p^*E_0)\to V$. The projection maps $P'\to P(E_k)$ for $1\le k\le r$ correspond to the compositions $q_k\circ p'\colon P'\to P(E_k)$.

By the special case above we know that

$$p'_*[c_1(\mathscr{O}_{p^*\overline{E_0}}(1))^{e_0}] = [1]$$

on P. We claim that this implies

$$\begin{aligned} &p'_*[c_1(\mathscr{O}_{p^*\overline{E_0}}(1))^{e_0} \wedge c_1(p'^*q_1^*\mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(p'^*q_r^*\mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}] \\ = &[c_1(q_1^*\mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^*\mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}] \end{aligned} \tag{9.8.1}$$

Applying p_* to Eq. (9.8.1) gives the result.

To prove Eq. (9.8.1) we note that by an approximation argument we may assume that all pseudo-metrics involved are smooth. In that case we have $c_1(p'^*q_k^*\mathscr{O}_{\overline{E}_k}(1))^{e_k+i_k} = p'^*c_1(q_k^*\mathscr{O}_{\overline{E}_r}(1))^{e_k+i_k}$ and Eq. (9.8.1) follows easily by integrating against a test form. ∎

9.9 Line Bundles. Let $\overline{E}_1, \ldots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles and let $\overline{L}_1, \ldots, \overline{L}_s$ be locally approachably metrized line bundles. Let

$$p\colon P = P(E_1) \times_V \cdots \times_V P(E_r) \to V$$

with projection maps $q_k\colon P \to P(E_k)$. We introduce the notation

$$\begin{aligned} &c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_s) \wedge [s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] \\ := &p_*[c_1(p^*\overline{L_1}) \wedge \cdots \wedge c_1(p^*\overline{L}_s) \wedge c_1(q_1^*\mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^*\mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}]. \end{aligned}$$

If $\overline{L}_1, \ldots, \overline{L}_s$ are smoothly metrized, then $c_1(\overline{L}_1)\wedge\cdots\wedge c_1(\overline{L}_s)\wedge[s_{i_1}(\overline{E}_1)\wedge\cdots\wedge s_{i_r}(\overline{E}_r)]$ actually agrees with the wedge product of the smooth form $c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_s)$ with the current $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$. Indeed, by an approximation argument one can assume that the pseudo-metrics on the E_k are smooth in which case the claim is easy to verify.

Note that we have

$$c_1(\overline{L}) \wedge [s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = -[s_1(\overline{L}) \wedge s_{i_1}(\overline{E}_1) \wedge \cdots \wedge si_r(\overline{E}_r)].$$

Indeed, this follows from the fact that the projective bundle $P(L)$ can be identified with V and under this identification the metrized line bundle $\mathscr{O}_{\overline{L}}(1)$ agrees with $\overline{L}^\vee$.

Together with Proposition 9.8 this describes the Segre currents of metrized line bundles.

9.10 Proposition. *Let $\overline{E}_0, \overline{E}_1, \dots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V and let $\overline{L}$ be a locally approachably metrized line bundle on V. Let $e_0 + 1$ be the rank of E_0. Let $i_0, i_1, \dots, i_r$ be natural numbers. Then we have*

$$\begin{aligned}
&[s_{i_0}(\overline{E}_0 \otimes \overline{L}) \wedge s_{i_1}(\overline{E}_1) \wedge \dots \wedge s_{i_r}(\overline{E}_r)] \\
&= \sum_{k=0}^{i_0} (-1)^{i_0-k} \binom{e_0 + i_0}{e_0 + k} c_1(\overline{L})^{i_0-k} \wedge [s_k(\overline{E}_0) \wedge s_{i_1}(\overline{E}_1) \wedge \dots \wedge s_{i_r}(\overline{E}_r)]
\end{aligned}$$

Proof. Similarly to the proof of Proposition 9.8 it is enough to show the special case

$$[s_i(\overline{E} \otimes \overline{L})] = \sum_{k=0}^{i} (-1)^{i-k} \binom{e+i}{e+k} c_1(\overline{L})^{i-k} \wedge [s_k(\overline{E})] \tag{9.10.1}$$

for a pseudo-metrized vector bundle E of rank $e+1$ and $i \in \mathbf{N}$.

In order to prove Eq. (9.10.1) we denote by $p\colon P(E) \to V$ the projective bundle associated to E. Recall that we have an identification $P(E \otimes L) = P(E)$ with $\mathcal{O}_{\overline{E} \otimes \overline{L}}(1) = \mathcal{O}_{\overline{E}}(1) \otimes p^* \overline{L}^\vee$ by Paragraph 5.32. By definition, we have

$$[s_i(\overline{E} \otimes \overline{L})] = p_*[c_1(\mathcal{O}_{\overline{E} \otimes \overline{L}}(1))^{e+i}].$$

Rewriting $\mathcal{O}_{\overline{E} \otimes \overline{L}}(1)$ as above and using symmetric multilinearity of the product of first Chern currents of line bundles we get

$$[s_i(\overline{E} \otimes \overline{L})] = \sum_{k=0}^{e+i} (-1)^{e+i-k} \binom{e+i}{k} p_*[c_1(p^*\overline{L})^{e+i-k} \wedge c_1(\mathcal{O}_{\overline{E}}(1))^k].$$

For $k < e$ we have $p_*[c_1(p^*\overline{L})^{e+i-k} \wedge c_1(\mathcal{O}_{\overline{E}}(1))^k] = 0$. Indeed, if $\overline{L}$ is smooth, we have

$$p_*[c_1(p^*\overline{L})^{e+i-k} \wedge c_1(\mathcal{O}_{\overline{E}}(1))^k] = c_1(\overline{L})^{e+i-k} \wedge p_*[c_1(\mathcal{O}_{\overline{E}}(1))^k]$$

and $p_*[c_1(\mathcal{O}_{\overline{E}}(1))^k] = 0$ for dimension reasons. If $\overline{L}$ is not smooth, the claim can be reduced to the smooth case by an approximation argument.

Forgetting about the unnecessary summands and substituting the summation index we get

$$[s_i(\overline{E} \otimes \overline{L})] = \sum_{k=0}^{i} (-1)^{i-k} \binom{e+i}{e+k} p_*[c_1(p^*\overline{L})^{i-k} \wedge c_1(\mathcal{O}_{\overline{E}}(1))^{e+k}].$$

We have

$$p_*[c_1(p^*\overline{L})^{i-k} \wedge c_1(\mathcal{O}_{\overline{E}}(1))^{e+k}] = c_1(\overline{L})^{i-k} \wedge [s_k(\overline{E})]$$

by the notation introduced in Paragraph 9.9. This finishes the proof. ∎

9.11 Polynomial Expressions. Let $i_1, \dots, i_r \in \mathbf{N}$ be natural numbers. Let $F(X_1, \dots, X_r) \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree i with respect to the grading determined by $\deg(X_k) = i_k$ for $k = 1, \dots, r$. Let $\overline{E}_1, \dots, \overline{E}_r$ be formally metrized vector bundles on V. We define the current

$$[F(s_{i_1}(\overline{E}_1), \dots, s_{i_r}(\overline{E}_r))] \in \mathscr{D}^{i,i}(V).$$

It suffices to do this when F is a monomial and then to extend $\mathbf{R}$-linearly. For monomials, we use the products of Segre currents as defined in Paragraph 9.4.

We denote by

$$[F(s_{i_1}(E_1), \dots, s_{i_r}(E_r))]_{\mathrm{BC}} \in \hat{H}^i_{\mathscr{D}}(V)$$

the class in the Bott–Chern cohomology.

9.12 Chern Currents. Let $C_i \in \mathbf{Z}[X_0, \dots, X_i] \subset \mathbf{R}[X_0, \dots, X_i]$ be the polynomials satisfying

$$(C_0 + C_1 T + C_2 T^2 + \cdots)(1 + X_1 T + X_2 T^2 + \cdots) = 1$$

as in Paragraph 9.2. The polynomial C_i is homogeneous of degree i if we let $\deg(X_k) = k$ for $k = 0, \dots, i$. Hence we can define the Chern currents of a locally approachably pseudo-metrized vector bundle $\overline{E}$ on V by

$$[c_i(\overline{E})] \coloneqq [C_i(s_0(\overline{E}), \dots, s_i(\overline{E}))] \in \mathscr{D}^{i,i}(V).$$

We denote by

$$[c_i(E)]_{\mathrm{BC}} \in \hat{H}^i_{\mathscr{D}}(V)$$

the class of $[c_i(\overline{E})]$ in $\hat{H}^i_{\mathscr{D}}(V)$.

For example, we have $C_2 = X_1^2 - X_2$ and hence

$$[c_2(\overline{E})] = [s_1(\overline{E}) \wedge s_1(\overline{E})] - [s_2(\overline{E})].$$

We can also define polynomial expressions in the Chern classes. Let $\overline{E}_1, \dots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V, let $i_1, \dots, i_r \in \mathbf{N}$ and let $F \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree i if we let $\deg(X_k) = i_k$. For each $k \in \{1, \dots, r\}$ we denote by

$$C^{(k)}_{i_k} \coloneqq C_{i_k}(X^{(k)}_0, \dots, X^{(k)}_{i_k})$$

the i_k-th Chern polynomial as above (but with disjoint sets of variables for varying k). We can plug these polynomials into F and obtain a polynomial

$$G \coloneqq F(c^{(1)}_{i_1}, \dots, C^{(r)}_{i_r}) \in \mathbf{R}[X^{(1)}_0, \dots, X^{(1)}_{i_1}, \dots, X^{(r)}_0, \dots, X^{(r)}_{i_r}].$$

Then we define

$$[F(c_{i_1}(\overline{E}_1), \dots, c_{i_r}(\overline{E}_r))] := [G(s_0(\overline{E}_r), \dots, s_{i_1}(\overline{E}_1), \dots, s_0(\overline{E}_r), \dots, s_{i_r}(\overline{E}_r))].$$

For example we can make the formal computation

$$[c_1(\overline{E})^2 - c_2(\overline{E})] = [(-s_1(\overline{E}))^2 - (s_1(\overline{E})^2 - s_2(\overline{E}))] = [s_2(\overline{E})].$$

We denote by $[F(c_{i_1}(E_1), \dots, c_{i_r}(E_r))]_{\mathrm{BC}}$ the class in the Bott–Chern cohomology group.

9.13 Remark. In Propositions 9.5, 9.7, 9.8 and 9.10 we worked out some basic relations among Segre currents, which lead immediately to similar relations for the Chern currents. We conjecture that similar analogues to well-known formulas from intersection theory for Chern classes of dual bundles, direct sums and tensor products hold also for Chern currents of dual bundles, directs sums and tensor products with the respective induced metrics, but we cannot prove them currently.

In general, only constructions which can be performed for general pseudo-metrics (cf. Paragraph 5.32) are easy to handle. We do however have formulas for the classes in the Bott–Chern cohomology, see Paragraph 9.14.

9.14 Comparison with Intersection Theory. Let $E_1, \dots, E_r$ be vector bundles on an algebraic K-variety X. It follows immediately from the compatibility (7.10.1) of the map $\delta\colon \mathrm{CH}^\bullet(X) \to \hat{H}^\bullet_{\mathscr{D}}(X^{\mathrm{an}})$ with push-forward as well as Proposition 8.14 and Lemma 9.3 that

$$\delta(s_{i_1}(E_1) \cdots s_{i_r}(E_r)) = [s_{i_1}(E_1) \wedge \cdots \wedge s_{i_r}(E_r)]_{\mathrm{BC}} \in \hat{H}^{|i|}_{\mathscr{D}}(X^{\mathrm{an}}).$$

By linearity we get

$$\delta(F(s_{i_1}(E_1), \dots, s_{i_r}(E_r))) = [F(s_{i_1}(E_1), \dots, s_{i_r}(E_r))]_{\mathrm{BC}}$$

for polynomials in the Segre classes. In particular, we have

$$\delta(c_i(E)) = [c_i(E)]_{\mathrm{BC}}$$

and a similar formula holds for polynomials in the Chern classes.

In particular, all relations among Segre and Chern classes in the Chow group carry over to relations in the Bott–Chern cohomology group. For example, if $\mathscr{E}\colon\ 0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles on X then the Whitney sum formula [Ful98, Thm. 3.2 (e)]

$$c_i(E) = \sum_{j+k=i} c_j(E')c_k(E'') \in \mathrm{CH}^i(X)$$

implies the relation

$$[c_i(E)]_{\mathrm{BC}} = \sum_{j+k=i} [c_j(E') \wedge c_k(E'')]_{\mathrm{BC}} \in \hat{H}^i_{\mathscr{D}}(X^{\mathrm{an}}).$$

9.15 Lemma. *Let L/K be an extension of non-archimedean fields and denote by $\pi := \pi_{L/K} : V \hat{\otimes}_K L \to V$ the canonical base-change morphism. Let $\overline{E}_1, \dots, \overline{E}_r$ be locally approachably pseudo-metrized vector bundles on V and let $F \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree i if we let $\deg(X_k) = i_k$. Then the pull-backs $\pi^*\overline{E}_k$ are locally approachably pseudo-metrized and we have*

$$\pi_*[F(s_{i_1}(\pi^*\overline{E}_1), \dots, s_{i_r}(\pi^*\overline{E}_r))] = [F(s_{i_1}(\overline{E}_1), \dots, s_{i_r}(\overline{E}_r))] \tag{9.15.1}$$

as well as

$$\pi_*[F(c_{i_1}(\pi^*\overline{E}_1), \dots, c_{i_r}(\pi^*\overline{E}_r))] = [F(c_{i_1}(\overline{E}_1), \dots, c_{i_r}(\overline{E}_r))]. \tag{9.15.2}$$

Proof. By Paragraph 4.36 the formation of projective bundles and the tautological line bundles is compatible with base change. By definition of the base-change of pseudo-metrized vector bundles in Paragraph 5.32 the same is true for the formation of the metrized tautological line bundle equipped with the dual Fubini-Study metric. Since by Lemma 8.15 the base-change of locally approachably metrized line bundles remains locally approachable, we see that the $\pi^*\overline{E}_k$ are locally approachably metrized.

The formation of fibered products is compatible with base change by Paragraph 2.6 and the formation of products of first Chern currents of line bundles is compatible with base change by Lemma 8.15. This shows that all ingredients of the construction in Paragraph 9.4 are compatible with base change. By linearity we obtain Eq. (9.15.1). Since polynomials in the Chern currents are defined as certain polynomial expressions in the Segre currents (Paragraph 9.12) we obtain also Eq. (8.15.1). ∎

10. Characteristic Forms of Metrized Vector Bundles

Throughout this chapter, K is a non-archimedean field. All K-analytic spaces are assumed to be good, topologically Hausdorff, boundaryless and equidimensional. We fix a K-analytic space V dimension n and an algebraic K-variety X of dimension n. In Paragraph 10.1 we define what it means for a pseudo-metrized vector bundle $\overline{E}$ on V to admit *Segre forms*. Unfortunately we cannot prove existence of Segre-forms except in trivial cases (the 0-th Segre form for general vector bundles and the 1-st Segre form for line bundles). For this reason the results of this chapter are conditional in that they depend on the existence of Segre forms for the occurring vector bundles.

In Proposition 10.7 we show that existence of Segre forms implies existence of Green currents for cycles on smooth algebraic varieties and in Remark 10.11 we give a concrete construction for these Green currents.

10.1 Segre Forms. Let $\overline{E}$ be a smoothly pseudo-metrized vector bundle of rank $e+1$ and denote by $p\colon P(E) \to V$ the projective bundle. (Recall the notion of smooth pseudo-metrics from Paragraph 5.35.) Let us say that *the i-th Segre form of $\overline{E}$ exists* if there exists a (necessarily uniquely determined) smooth form $\beta \in \mathscr{A}^{i,i}(V)$ such that for every K-analytic space V' and every morphism $f\colon V' \to V$ we have

$$[s_i(f^*\overline{E})] = [f^*\beta].$$

In this case we denote $s_i(\overline{E}) \coloneqq \beta$. In particular, the current associated to the smooth form $s_i(\overline{E})$ agrees with the current $[s_i(\overline{E})]$ as defined in Paragraph 9.4 so the notation is consistent. We say that *all Segre forms of $\overline{E}$ exist* if the i-th Segre form of $\overline{E}$ exists for all $i \in \mathbf{N}$.

Note that if the i-th Segre form of $\overline{E}$ exists and $f : V' \to V$ is a morphism of K-analytic spaces then the i-th Segre form of $f^*\overline{E}$ exists and $s_i(f^*\overline{E}) = f^*s_i(\overline{E})$.

10.2 Remark. The formula $[s_i(\overline{E})] = p_*[c_1(\mathscr{O}_{\overline{E}}(1))^{e+i}]$ shows that, philosophically, $s_i(\overline{E})$ should be given by integration along the fiber of the smooth form $c_1(\mathscr{O}_{\overline{E}}(1))^{e+i}$ along the fiber bundle $p : P(E) \to V$. It is however non-trivial to imitate the construction of fiber integrals from differential geometry in the non-archimedean situation, because for a trivial fiber bundle $P \times V \to V$ there is no reason in non-archimedean geometry that a differential form on $P \times V$ can be written as a sum of differential forms of the form $p_0^*\alpha \wedge p_1^*\beta$ where $p_0 : P \times V \to V$, $p_1 : P \times V \to P$ denote the projections.

For this reason, the existence of Segre forms of smoothly pseudo-metrized vector bundles remains conjectural.

10.3 Remark. From Proposition 9.8 it follows that any smoothly pseudo-metrized vector bundle E admits the 0-th Segre form and that it is given by the unit function

$$s_0(\overline{E}) = 1 \in \mathscr{A}^{0,0}(V).$$

Since the first Chern current $[c_1(\overline{L})]$ of a smoothly metrized line bundle $\overline{L}$ is represented by the first Chern form $c_1(\overline{L})$ and because Chern forms are compatible with pull-back of smoothly metrized line bundles, it follows from Paragraph 9.9 that the first Segre form of $\overline{L}$ exists and is given by

$$s_1(\overline{L}) = -c_1(\overline{L}).$$

10.4 Lemma. *Let $\overline{E}_1, \dots, \overline{E}_r$ be smoothly pseudo-metrized vector bundles over V and let $i_1, \dots, i_r \in \mathbf{N}$ be natural numbers. Assume that for every $k \in \{1, \dots, r\}$, the i_k-th Segre form of $\overline{E}_k$ exists and let $f : V' \to V$ be a morphism of K-analytic spaces. Then the current associated to the smooth form $f^*(s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r))$ agrees with the Segre current*

$$[s_{i_1}(f^*\overline{E}_1) \wedge \cdots \wedge s_{i_r}(f^*\overline{E}_r)].$$

Proof. For simplicity, we only show the claim in the case $f = \mathrm{id}_V$, the general statement can be shown similarly. Furthermore, we assume $r = 2$, the general case follows by induction. So let $\overline{E}$, $\overline{F}$ be two smoothly pseudo-metrized vector bundles of ranks e, resp. f and let $i, j \in \mathbf{N}$. We want to show that the current associated to the form $s_i(\overline{E}) \wedge s_j(\overline{F})$ agrees with $[s_i(\overline{E}) \wedge s_j(\overline{F})]$ as defined in Paragraph 9.4.

For this we consider the cartesian commutative diagram

$$\begin{array}{ccc} P(p_0^* F) & \xrightarrow{\widetilde{p}_0} & P(F) \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle p_1} \\ P(E) & \xrightarrow[p_0]{} & V \end{array}$$

of Lemma 4.32 which shows that we can identify the space $P(E) \times_V P(F)$ with $P(p_0^* F)$. Here, p_0, p_1, p_1' are projections from the respective projective bundles onto the respective base space. By Paragraph 5.32 we have an equality

$$\mathcal{O}_{p_0^* \overline{F}}(1) = \widetilde{p}_0^* \mathcal{O}_{\overline{F}}(1) \tag{10.4.1}$$

of metrized line bundles on $P(p_0^* F)$.

Given $\alpha \in \mathscr{A}_c^{\bullet,\bullet}(V)$ we want to prove

$$\langle [s_i(\overline{E}) \wedge s_j(\overline{F})], \alpha \rangle = \int_V s_i(\overline{E}) \wedge s_j(\overline{F}) \wedge \alpha$$

where the symbol $[s_i(\overline{E}) \wedge s_j(\overline{F})]$ on the left denotes the Segre current in the sense of Paragraph 9.4 and $s_i(\overline{E})$, $s_j(\overline{F})$ on the right denote Segre forms in the sense of Paragraph 10.1.

By definition we have

$$[s_i(\overline{E}) \wedge s_j(\overline{F})] = p_{0*} p'_{1*} [c_1(p_1'^* \mathcal{O}_{\overline{E}}(1))^{e+i} \wedge c_1(\widetilde{p}_0^* \mathcal{O}_{\overline{F}}(1))^{f+j}].$$

Using that first Chern forms of smoothly metrized line bundles are compatible with pull-back and using Eq. (10.4.1) we get

$$\langle [s_i(\overline{E}) \wedge s_j(\overline{F})], \alpha \rangle = \int_{P(p_0^* F)} c_1(\mathcal{O}_{p_0^* \overline{F}}(1))^{f+j} \wedge p_1'^* (c_1(\mathcal{O}_{\overline{E}}(1))^{e+i} \wedge p_0^* \alpha).$$

By definition of the Segre current $[s_j(p_0^* \overline{F})]$ in the sense of Paragraph 9.4 we see

$$\langle [s_i(\overline{E}) \wedge s_j(\overline{F})], \alpha \rangle = \langle [s_j(p_0^* \overline{F})], c_1(\mathcal{O}_{\overline{E}}(1))^{e+i} \wedge p_0^* \alpha \rangle.$$

By the defining property of the Segre form $s_j(\overline{F})$ applied to the morphism p_0, the current $[s_j(p_0^* \overline{F})]$ is associated to the form $p_0^* s_j(\overline{F})$. It follows that

$$\begin{aligned} \langle [s_i(\overline{E}) \wedge s_j(\overline{F})], \alpha \rangle &= \int_{P(E)} p_0^* s_j(\overline{F}) \wedge c_1(\mathcal{O}_{\overline{E}}(1))^{e+i} \wedge p_0^* \alpha \\ &= \int_{P(E)} c_1(\mathcal{O}_{\overline{E}}(1))^{e+i} \wedge p_0^* (s_j(\overline{F}) \wedge \alpha) \\ &= \langle [s_i(\overline{E})], s_j(\overline{F}) \wedge \alpha \rangle. \end{aligned}$$

In the last line, $[s_i(\overline{E})]$ denotes the Segre current in the sense of Paragraph 9.4. By the defining property of the Segre form it is the current associated to the form $s_i(\overline{E})$ and we get

$$\langle [s_i(\overline{E}) \wedge s_j(\overline{F})], \alpha \rangle = \int_V s_i(\overline{E}) \wedge s_j(\overline{F}) \wedge \alpha$$

which we wanted to show. ∎

10.5 Chern Forms. Let $\overline{E}$ be a smoothly pseudo-metrized vector bundle on V and assume that all Segre forms of $\overline{E}$ exist. For $i \in \mathbf{N}$ we define the *i-th Chern form* of $\overline{E}$ by

$$c_i(\overline{E}) := C_i(s_0(\overline{E}), \dots, s_i(\overline{E})) \in \mathscr{A}^{i,i}(V)$$

where $C_i \in \mathbf{Z}[X_0, \dots, X_i]$ denotes the Chern polynomial as in Paragraph 9.12.

Note that it follows from Lemma 10.4 that if $\overline{E}_1, \dots, \overline{E}_r$ are smoothly pseudo-metrized vector bundles all of whose Segre forms exist, $f\colon V' \to V$ is a morphism of K-analytic spaces and $i_1, \dots, i_r$ are natural numbers, then the current associated to the form

$$f^*(c_{i_1}(\overline{E}_1) \wedge \cdots \wedge c_{i_1}(\overline{E}_r))$$

agrees with the current

$$[c_{i_1}(f^*\overline{E}_1) \wedge \cdots \wedge c_{i_r}(f^*\overline{E}_r)]$$

interpreted as a polynomial expression in the Chern currents of $f^*\overline{E}_1, \dots, f^*\overline{E}_r$ in the sense of Paragraph 9.12.

10.6 Green Currents. Let X be an n-dimensional algebraic variety on X and let Z be a p-codimensional cycle on X. A current $g \in \mathscr{D}^{p-1,p-1}(X^{\mathrm{an}})$ is called a *Green current* for Z if there exists a smooth form $\omega \in \mathscr{A}^{p,p}(X^{\mathrm{an}})$ satisfying

$$[\omega] = d'd''g + \delta_Z.$$

One sees that there exists a Green current for Z if and only if there exists a smooth form $\omega \in \mathscr{A}^{p,p}(X^{\mathrm{an}})$ such that the class of $[\omega]$ in $\hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})$ equals $\delta([Z])$ where $\delta\colon \mathrm{CH}^p(X) \to \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}})$ denotes the cycle class map of Paragraph 7.10. In particular the existence of a Green current for Z depends only on the class $[Z] \in \mathrm{CH}^p(X)$. The question whether or not every cycle on X has a Green current is open in the non-archimedean setting.

Note that if $\overline{L}$ is a smoothly metrized line bundle on X and s is a meromorphic section of L with Cartier divisor D then by the Poincaré-Lelong formula of Paragraph 8.3 the current $g_Y := [-\log\|s\|]$ is a Green current for the Weil divisor Y associated to D with smooth form $\omega_Y = c_1(\overline{L})$.

Now assume in addition that Z is a p-codimensional prime cycle such that D intersects Z properly and that $g_Z \in \mathscr{D}^{p-1,p-1}(X^{\mathrm{an}})$ is a Green current for Z. Following [GK17, § 11.2] we define $g_Y \wedge \delta_Z \in \mathscr{D}^{p,p}(X^{\mathrm{an}})$ as the push-forward of $[-\log\|s\||_Z]$ with respect to the inclusion $i_Z\colon Z \hookrightarrow X$. If Z is not prime, we define $g_Y \wedge \delta_Z \in \mathscr{D}^{p,p}(X^{\mathrm{an}})$ by extending linearly from the prime case. Finally we define the $*$-product

$$g_Y * g_Z := g_Y \wedge \delta_Z + \omega_Y \wedge g_Z \in \mathscr{D}^{p,p}(X^{\mathrm{an}}).$$

Then as in [GK17, Prop. 11.4] it follows that $g_Y * g_Z$ is a Green current for the cycle $D \cdot Z$.

10.7 Proposition. *Let X be a smooth algebraic K-variety and assume that for every vector bundle E on X there exists a smooth pseudo-metric $\|-\|$ on E such that all Segre forms for the pseudo-metrized vector bundle $\overline{E} := (E, \|-\|)$ exist. Under this hypothesis, every cycle Z on X admits a Green current.*

Proof. By [Ful98, Exmp. 15.2.16] the Chern character induces an isomorphism from the Grothendieck group of vector bundles on X with rational coefficients onto the Chow group with rational coefficients. In particular, every Chow class is equal to a polynomial expression in the Chern classes of a family of vector bundles, or equivalently a polynomial expression in the Segre classes of a family of vector bundles. By linearity, it is enough to show that for vector bundles $E_1, \dots, E_r$ on X and $i_1, \dots, i_r \in \mathbf{N}$ the class $s_{i_1}(E_1) \cdots s_{i_r}(E_r)$ admits a Green current. By Paragraph 9.14 the class $\delta(s_{i_1}(E_1) \cdots s_{i_r}(E_r))$ is represented by the current $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$ for any choice of locally approachable pseudo-metrics on $E_1, \dots, E_r$. By hypothesis we can choose the pseudo-metrics smooth and such that all Segre forms exist for $\overline{E}_1, \dots, \overline{E}_r$. By Paragraph 10.5 the current $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$ is represented by the smooth form $s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)$, so by the characterization of Paragraph 10.6 the class $s_{i_1}(E_1) \cdots s_{i_r}(E_r)$ admits a Green current as was to be shown. ∎

10.8 Proper Intersection of Divisors. Let X be an n-dimensional algebraic K-variety and let $D_1, \dots, D_r$ be Cartier divisors on X. Following [CM21, Def. 1.3.2] we say that $D_1, \dots, D_r$ *intersect properly* if for any k-element subset J of $\{1, \dots, r\}$,

$$\dim\Big(\bigcap_{j \in J} \mathrm{Supp}(D_j)\Big) \leq n - k.$$

10.9 Lemma. *Let X be a quasi-projective variety and let $L_1, \dots, L_r$ be line bundles on X. Then there exists a projective variety $\widetilde{X}$ together with an open immersion $X \hookrightarrow \widetilde{X}$ and line bundles $\tilde{L}_1, \dots, \tilde{L}_r$ on $\widetilde{X}$ such that the restriction of each $\tilde{L}_k$ to X is isomorphic to L_k.*

Proof. By a diagonal argument one reduces to the case $r = 1$, so let L be a line bundle on X. Let $\overline{X}$ be a projective closure of X. By [EGA1, Cor. 6.9.5], L extends to a coherent sheaf $\mathscr{F}$ on $\overline{X}$. By [RG71, Thm. 5.2.2] there exists an X-admissible blowing-up $\widetilde{X} \to \overline{X}$ (in particular it restricts to an isomorphism over the open subset $X \subset \overline{X}$) such that the strict transform $\tilde{L}$ of $\mathscr{F}$ is a flat $\mathcal{O}_{\widetilde{X}}$-module and restricts to the line bundle L on X. By [Har77, Prop. III.9.2 (e)], $\tilde{L}$ is locally free and since the rank function is locally constant, it must have constant rank 1 on $\widetilde{X}$. ∎

10.10 Lemma. *Let X be a quasi-projective K-variety and let $L_1, \dots, L_r$ be line bundles on X. Then there exist non-zero rational sections s_k of L_k for $k = 1, \dots, r$ such that the divisors $D_1, \dots, D_r$ with $D_k = \operatorname{div}(s_k)$ intersect properly.*

Proof. By Lemma 10.9 we can extend the line bundles to a projective closure of X and we may assume that X is projective. In this case the result is shown in [CM21, Lem. 1.3.7]. ∎

10.11 Remark. Let the setting be as in Proposition 10.7 and assume additionally that X is quasi-projective. Then the Green current of Proposition 10.7 can be constructed explicitly as follows: As in the proof of Proposition 10.7 it is enough to find a Green current for the Chow class $s_{i_1}(E_1) \cdots s_{i_r}(E_r)$. Denoting by

$$p \colon P(E_1) \times_X \cdots \times_X P(E_r) \to X$$

the fibered product of the projective bundles with projections $q_k \colon P \to P(E_k)$, the class $s_{i_1}(E_1) \cdots s_{i_r}(E_r)$ is by Lemma 9.3 equal to

$$p_*(c_1(q_1^*\mathcal{O}_{E_1}(1))^{e_1+i_1} \cdots c_1(q_r^*\mathcal{O}_{E_r}(1))^{e_r+i_r}).$$

Let us choose a smooth pseudo-metric on each of the E_k such that all Segre forms exist for each $\overline{E}_k$. For each of the line bundles

$$L_1, \dots, L_m = \underbrace{q_1^*\mathcal{O}_{E_1}(1), \dots, q_1^*\mathcal{O}_{E_1}(1)}_{e_1+i_1 \text{ times}}, \dots, \underbrace{q_r^*\mathcal{O}_{E_r}(1), \dots, q_r^*\mathcal{O}_{E_r}(1)}_{e_r+i_r \text{ times}}$$

(where $m = |e| + |i|$) choose meromorphic sections $s_1, \dots, s_m$ with Cartier divisors $D_1, \dots, D_m$ and associated Weil divisors $Y_1, \dots, Y_m$ such that the divisors D_k intersect properly. Let $g_{Y_k} \coloneqq [-\log\|s_k\|]$ be the Green current for Y_k with $\omega_{Y_k} = c_1(\overline{L}_k)$. Then $g \coloneqq g_1 * (g_2 * \cdots (\cdots * g_m))$ is a Green current for $Y \coloneqq D_1 \cdot (D_2 \cdot \cdots (\cdots \cdot D_m))$ satisfying

$$[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_m)] = d'd''g + \delta_Y.$$

Applying p_* we get

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = d'd''p_*g + \delta_{p_*Y}$$

This shows that p_*g is a Green current for p_*Y with associated form $s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)$. So we have an explicit Green current for the explicit cycle p_*Y representing the class $s_{i_1}(E_1) \cdots s_{i_r}(E_r)$.

10.12 Remark. Since we cannot currently prove existence of Segre forms, Proposition 10.7 does not give an unconditional proof for the existence of Green currents. However, even if it turns out that Segre forms as defined in Paragraph 10.1 do not exist in general, they might still exist in analogous theories of characteristic currents and forms based on a different class of differential forms and currents, for example the δ-forms of [GK17] or of [Mih23a; Mih23b]. In this case the arguments given above should imply the existence of Green δ-currents in the context of δ-forms. See Chapter 12 for an outline of a theory of characteristic currents and forms based on δ-forms.

11. Measures

Throughout this chapter, K will be a non-archimedean field. All K-analytic spaces are assumed to be good, topologically Hausdorff, boundaryless and equidimensional (and hence in particular strict by [Tem15, Exmp. 4.2.4.2 (ii)]). If a polynomial expression in the Segre or Chern currents of a family of locally approachably (pseudo-)metrized vector bundles on a K-analytic space V has top homogeneous degree n, then it is given by a measure. If $V = X^{\mathrm{an}}$ is the analytification of a proper algebraic variety over an algebraically closed field and all the metrics are formal metrics, one gets a more concrete description which is analogous to the definition of Monge–Ampère measures in [Cha06].

From this description (referring to intersection numbers on the special fiber) we deduce a positivity result for semipositive formally metrized vector bundles.

11.1 Measures. Let V be a K-analytic space. Observe that our assumptions for K-analytic spaces imply that the underlying topological space of V is a locally compact Hausdorff topological space. We denote by $\mathscr{C}_c^0(V)$ the topological $\mathbf{R}$-vector space of compactly supported continuous functions on V equipped with the locally convex topology of uniform convergence on compacta. By a *measure* on V we mean a Radon measure on the underlying topological space of V, i.e. a continuous $\mathbf{R}$-linear functional $\mu\colon \mathscr{C}_c^0(V) \to \mathbf{R}$. Note that a functional $\mu\colon \mathscr{C}_c^0(V) \to \mathbf{R}$ is continuous if and only if for every compact subset K of V there exists a constant M_K such that, for every $f \in \mathscr{C}_c^0(V)$ with $\mathrm{Supp}(f) \subset K$ the bound $|\mu(f)| \le M_K \|f\|_K$ holds, where $\|f\|_K = \sup_{x\in K} |f(x)|$. We denote by $\mathrm{Mea}(V)$ the space of measures on V.

Note that if $U \subset V$ is an open subset, then the map $\mathscr{C}_c^0(U) \to \mathscr{C}_c^0(V)$ given by extension by 0 is continuous and hence there is a dual map

$$\mathrm{Mea}(V) \to \mathrm{Mea}(U), \qquad \mu \mapsto \mu|_U.$$

This shows that the assignment $U \mapsto \mathrm{Mea}(U)$ defines a presheaf of real vector spaces on V.

11.2 Lemma. *Let V be a paracompact K-analytic space and let $V = \bigcup_{i\in I} U_i$ be a locally finite covering of V by open subsets. For each $i \in I$ let $\mu_i \in \mathrm{Mea}(U_i)$ be a measure such that $\mu_i|_{U_i\cap U_j} = \mu_j|_{U_i\cap U_j}$ for all $i, j \in I$. Then there is a unique measure $\mu \in \mathrm{Mea}(V)$ such that $\mu|_{U_i} = \mu_i$ for all $i \in I$.*

Proof. By paracompactness we can pick a partition of unity subordinate to the covering, i.e. a family $\{\phi_i\}_{i\in I}$ of non-negative continuous functions $\phi_i \in \mathscr{C}^0(V)$ such that $\mathrm{Supp}(\phi_i) \subset U_i$ and such that $\sum_{i\in I} \phi_i \equiv 1$ on V. Note that if $f \in \mathscr{C}_c^0(V)$ is a compactly supported function then the compact set $\mathrm{Supp}(f)$ meets only finitely many sets U_i and hence $(\phi_i f)|_{U_i} = 0$ for almost all $i \in I$. Furthermore one checks that $\mathrm{Supp}((\phi_i f)|_{U_i}) = \mathrm{Supp}(\phi_i f)$ is compact so the sum

$$\mu(f) := \sum_{i\in I} \mu_i((\phi_i f)|_{U_i})$$

has well-defined summands almost all of which vanish.

We claim that $\mu : \mathscr{C}_c^0(V) \to \mathbf{R}$ defined in this way is continuous, so let $K \subset V$ be compact. Then the set $I' = \{ i \in I \mid K \cap U_i \neq \varnothing \}$ is finite. For each $i \in I'$ the set $K_i := K \cap \mathrm{Supp}(\phi_i) \subset U_i$ is compact, so there exists a constant M_i such that $|\mu_i(g)| \le M_i \|g\|_{K_i}$ for all $g \in \mathscr{C}_c^0(U_i)$ with $\mathrm{Supp}(g) \subset K_i$, in particular for $g = (\phi_i f)|_{U_i}$ with $f \in \mathscr{C}_c^0(V)$ with $\mathrm{Supp}(f) \subset K$. Then we see that

$$\begin{aligned} |\mu(f)| &= |\sum_{i\in I'} \mu_i((\phi_i f)|_{U_i})| \\ &\le \sum_{i\in I'} |\mu_i((\phi_i f)|_{U_i})| \\ &\le \sum_{i\in I'} M_i \cdot \|(\phi_i f)|_{U_i}\|_{K_i} \\ &\le (\sum_{i\in I'} M_i)\|f\|_K. \end{aligned}$$

This shows that μ is a measure.

To show that $\mu|_{U_j} = \mu_j$, for $j \in I$, let $g \in \mathscr{C}_c^0(U_j)$. The set $I' = \{ i \in I \mid \mathrm{Supp}(g) \cap U_i \neq \varnothing \}$ is finite and by definition we have

$$\begin{aligned} \mu|_{U_j}(g) &= \sum_{i \in I'} \mu_i(\phi_i g) \\ &= \sum_{i \in I'} \mu_j(\phi_i g) \\ &= \mu_j(\sum_{i \in I'} \phi_i g). \end{aligned}$$

To make sense of the calculation, observe that $\mathrm{Supp}(\phi_i g) \subset U_i \cap U_j$ for all $i \in I'$. We have used the hypothesis that the μ_i agree on overlaps. Finally observe that all $i \in I$ such that $\phi_i g$ does not vanish identically on U_j are already contained in I', so we have $\sum_{i \in I'} \phi_i g \equiv \sum_{i \in I} \phi_i g \equiv g$ on U_j. This finishes the proof. ∎

11.3 Measures and Currents. Let V be a K-analytic space and let $\mu : \mathscr{C}_c^0(V) \to \mathbf{R}$ be a measure on V. By restricting μ to the subspace $\mathscr{C}_c^\infty(V) = \mathscr{A}_c^{0,0}(V)$ of compactly supported smooth functions on V we obtain a current $[\mu] \in \mathscr{D}_{0,0}(V)$. The operation

$$\mathrm{Mea}(V) \hookrightarrow \mathscr{D}_{0,0}(V), \qquad \mu \mapsto [\mu]$$

is injective since by the Stone–Weierstraß theorem for Berkovich spaces [CD12, Prop. 3.3.5], smooth functions lie dense in the space of continuous functions. By definition of the presheaf structures for Mea as well as for $\mathscr{D}_{0,0}$ this is defines morphism of presheaves.

Denoting by $\mathrm{Mea}^+(V)$, resp. $\mathscr{D}_{0,0}^+(V)$ the real cones of positive measures, resp. currents, the embedding restricts to a bijection

$$\mathrm{Mea}^+(V) \xrightarrow{\sim} \mathscr{D}_{0,0}^+(V), \qquad \mu \mapsto [\mu]$$

by [CD12, Prop. 5.4.6].

In general, the injection $\mathrm{Mea}(V) \hookrightarrow \mathscr{D}_{0,0}(V)$ allows us to view $\mathrm{Mea}(V)$ as a subset of $\mathscr{D}_{0,0}(V)$, so we say a current $T \in \mathscr{D}_{0,0}(V)$ *is a measure* if it lies in the image of $\mathrm{Mea}(V) \hookrightarrow \mathscr{D}_{0,0}(V)$.

11.4 Proposition. *Let V be a paracompact K-analytic space of dimension n and let $\overline{L}_1, \dots, \overline{L}_n$ be locally approachably metrized line bundles on V. Then the current*

$$[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)] \in \mathscr{D}^{n,n}(V) = \mathscr{D}_{0,0}(V)$$

is a measure on V.

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of V such that on each U_i, all of the line bundles $L_1, \dots, L_n$ are trivial and admit trivializing sections $s_k^{(i)} \in \Gamma(U_i, L_k)$ for

$k \in \{1, \dots, n\}$ such that $-\log\|s_k^{(i)}(-)\| \equiv u_k^{(i)} - v_k^{(i)}$ on U_i where $u_k^{(i)}, v_k^{(i)}$ are locally psh-approachable functions on U_i. This is possible by the definition of locally approachable metrics. By paracompactness, there exists a locally finite refinement of the covering $\{U_i\}_{i\in I}$, so we may assume $\{U_i\}_{i\in I}$ to be locally finite. Recall from Paragraph 11.3 that Mea $\subset \mathscr{D}_{0,0}$ is a sub-presheaf. Then it follows from Lemma 11.2 that it is enough to show that the restriction of $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)]$ to each U_i is a measure. We have

$$[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)]|_{U_i} = [d'd''(u_1^{(i)} - v_1^{(i)}) \wedge \cdots \wedge d'd''(u_n^{(i)} - v_n^{(i)})],$$

so by symmetric multilinearity it is enough to see that for locally psh-approachable functions $w_1^{(i)}, \dots, w_n^{(i)}$ on U_i the current

$$[d'd''w_1^{(i)} \wedge \cdots \wedge d'd''w_n^{(i)}] \tag{11.4.1}$$

is a measure. But the current (11.4.1) is positive by [CD12, Cor. 5.6.5] and so it is a measure by [CD12, Prop. 5.4.6]. ∎

11.5 Monge–Ampère Measures. Let V be a paracompact K-analytic space of dimension n and let $\overline{L}_1, \dots, \overline{L}_n$ be locally approachably metrized line bundles on V. We call the measure

$$\mathrm{MA}(\overline{L}_1, \dots, \overline{L}_n) := [c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)] \in \mathrm{Mea}(V) \subset \mathscr{D}_{0,0}(V) \tag{11.5.1}$$

the *Monge–Ampère measure* associated to the family $\overline{L}_1, \dots, \overline{L}_n$. The construction of such measures in non-archimedean geometry goes back to [Cha06]; see also [Gub07] for a construction in slightly greater generality. The definition (11.5.1) is due to [CD12] (even though it is not explicitly formulated in the cited work).

11.6 Monge–Ampère Measures of Formally Metrized Line Bundles. Let V be a paracompact K-analytic space of dimension n and let $\overline{L}_1, \dots, \overline{L}_n$ be formally metrized line bundles on V. Recall from Paragraph 8.8 that formal metrics are locally approachable. In this case, the measure $\mathrm{MA}(\overline{L}_1, \dots, \overline{L}_n)$ is discrete in the following sense: There exists a closed discrete subset $S \subset V$ and a family $\{\lambda_x\}_{x\in S}$ of real numbers such that

$$\mathrm{MA}(\overline{L}_1, \dots, \overline{L}_r) = \sum_{x\in S} \lambda_x \delta_x.$$

A more refined statement is contained in [CD12, Prop. 6.9.2, Thm. 6.9.3].

We focus here on the case where K is algebraically closed and $V = X^{\mathrm{an}}$ is the analytification of an n-dimensional proper algebraic K-variety. In that case we can choose formal models $(\mathfrak{X}_k, \mathfrak{L}_k)$ for (X, L_k) defining the metrics of the $\overline{L}_k$.

Since formal models for X form a directed category by Paragraph 3.6 and using Corollary 6.7, we can assume that all the $\mathfrak{X}_k$ are given by a single formal model $\mathfrak{X}$ for X. By Lemma 3.7 we can even assume that $\mathfrak{X}$ has reduced special fiber.

Then the Monge–Ampère measure is given by

$$\mathrm{MA}(\overline{L}_1, \dots, \overline{L}_n) = \sum_{Y \in \widetilde{\mathfrak{X}}^{(0)}} \Big(\int_Y c_1(\widetilde{\mathfrak{L}}_1) \cdots c_1(\widetilde{\mathfrak{L}}_n) \Big) \delta_{\xi_Y},$$

where the sum runs over all irreducible components of the special fiber $\widetilde{\mathfrak{X}}$ and δ_{ξ_Y} denotes the Dirac measure supported in the Shilov point $\xi_Y \in X^{\mathrm{an}}$ reducing to the generic point of Y. By [CD12, Prop. 6.4.3] the total mass of the Monge–Ampère measure is equal to the algebraic intersection number

$$\langle [c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)], 1 \rangle = \int_X c_1(L_1) \cdots c_1(L_n).$$

11.7 Theorem. *Let V be a paracompact K-analytic space of dimension n and let $\overline{E}_1, \dots, \overline{E}_r$ be formally metrized vector bundles on V. Let $F \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree n with respect to the grading given by* $\deg(X_k) = i_k$.

Then the current $[F(s_{i_1}(\overline{E}_1), \dots, s_{i_r}(\overline{E}_r))] \in \mathscr{D}^{n,n}(V) = \mathscr{D}_{0,0}(V)$ is a discrete measure. More precisely, there exists a closed discrete subset $S \subset V$ and a family of real numbers $\{\lambda_x\}_{x \in S}$ such that

$$[F(s_{i_1}(\overline{E}_1), \dots, s_{i_r}(\overline{E}_r))] = \sum_{x \in S} \lambda_x \delta_x.$$

Proof. We may assume that F is a monomial, so we consider the current

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)]$$

with $|i| = n$. Let $e_1 + 1, \dots, e_r + 1$ be the ranks of $E_1, \dots, E_r$. Let $p\colon P := P(E_1) \times_V \cdots \times_V P(E_r) \to V$ be the fibered product of the projective bundles and let $q_k\colon P \to P(E_k)$ denote the canonical projection maps. By definition we have

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = p_*[c_1(q_1^* \mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^* \mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}].$$

By Paragraph 11.6 there exists a closed discrete subset $\Sigma \subset P$ and a family $\{\mu_y\}_{y \in \Sigma}$ of real numbers such that

$$[c_1(q_1^* \mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^* \mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}] = \sum_{y \in \Sigma} \mu_y \delta_y.$$

Since $p\colon P \to V$ is topologically proper and open it follows that $S := p(\Sigma)$ is closed and discrete. Furthermore the proper map $p\colon \Sigma \to S$ is finite-to-one because the fibers are both compact and discrete.

It follows that $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = p_*(\sum_{y \in \Sigma} \mu_y \delta_y)$ has the desired form. ∎

11.8 Theorem. *Assume that K is algebraically closed, that X is a proper algebraic K-variety of dimension n and let $\overline{E}_1, \dots, \overline{E}_r$ be formally metrized vector bundles on X. Let $F \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree n with respect to the grading given by* $\deg(X_k) = i_k$.

There exists a formal K°-model $\mathfrak{X}$ of X with reduced special fiber $\widetilde{\mathfrak{X}}$ and formal K°-models $\mathfrak{E}_1, \dots, \mathfrak{E}_r$ for $E_1, \dots, E_r$ defined on $\mathfrak{X}$ such that the metric of $\overline{E}_k$ is induced by $\mathfrak{E}_k$ for $k = 1, \dots, r$ and we have

$$[F(s_{i_1}(\overline{E}_1), \dots, s_{i_r}(\overline{E}_r))] = \sum_{Y \in \widetilde{\mathfrak{X}}^{(0)}} \Big(\int_Y F(s_{i_1}(\widetilde{\mathfrak{E}}_1), \dots, s_{i_r}(\widetilde{\mathfrak{E}}_r)) \Big) \cdot \delta_{\xi_Y}$$

where Y ranges over the irreducible components of $\widetilde{\mathfrak{X}}$ and δ_{ξ_Y} is the Dirac measure supported in the Shilov point $\xi_Y \in X^{\mathrm{an}}$ with reduction the generic point of Y.

The total mass is equal to the algebraic intersection number

$$\langle [F(s_{i_1}(\overline{E}_1), \dots, s_{i_r}(\overline{E}_r))], 1 \rangle = \int_X F(s_{i_1}(E_1), \dots, s_{i_1}(E_r)).$$

Proof. Again, we may assume that F is a monomial, so we want to prove

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] = \sum_Y \Big(s_{i_1}(\widetilde{\mathfrak{E}}_1) \cdots s_{i_r}(\widetilde{\mathfrak{E}}_r) \Big) \cdot \delta_{\xi_Y}.$$

The existence of the models $(\mathfrak{X}, \mathfrak{E}_k)$ follows as in Paragraph 11.6. For each $k \in \{1, \dots, r\}$ we denote by $\mathfrak{p}_k : P(\mathfrak{E}_k) \to \mathfrak{X}$ the projective bundle in the formal category. We let $\mathfrak{P} := P(\mathfrak{E}_1) \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} P(\mathfrak{E}_r) \to \mathfrak{X}$ be their fibered product over $\mathfrak{X}$ and denote by $\mathfrak{q}_k : \mathfrak{P} \to P(\mathfrak{E}_k)$ the projections. We note that $\mathfrak{P}$ is a formal K°-model of $P := P(E_1) \times_X \cdots \times_X P(E_r)$ with special fiber $\widetilde{\mathfrak{P}} = P(\widetilde{\mathfrak{E}}_1) \times_{\widetilde{\mathfrak{X}}} \cdots \times_{\widetilde{\mathfrak{X}}} P(\widetilde{\mathfrak{E}}_r)$. Since this fiber product can also be obtained by taking iterative projective bundles, we see that the special fiber is reduced and furthermore that the assignment

$$Y \mapsto \widetilde{\mathfrak{P}}_Y := P(\widetilde{\mathfrak{E}}_1|_Y) \times_Y \cdots \times_Y P(\widetilde{\mathfrak{E}}_r|_Y)$$

is a bijection from the set of irreducible components of $\widetilde{\mathfrak{X}}$ to the set of irreducible components of $\widetilde{\mathfrak{P}}$.

Note that for each $k \in \{1, \dots, r\}$ the line bundle $\mathfrak{q}_k^* \mathcal{O}_{\mathfrak{E}_k}(1)$ on $\mathfrak{P}$ is a formal K°-model of the line bundle $q_k^* \mathcal{O}_{E_k}(1)$ on P and that by Lemmas 6.6 and 6.11 it induces the metric of the metrized line bundle $q_k^* \mathcal{O}_{\overline{E}_k}(1)$. Furthermore, its restriction to the special fiber $\widetilde{\mathfrak{P}}$ is given by $\tilde{\mathfrak{q}}_k^* \mathcal{O}_{\widetilde{\mathfrak{E}}_k}(1)$.

It follows from Paragraph 11.6 that the zero-dimensional current

$$[c_1(q_1^* \mathcal{O}_{\overline{E}_1}(1))^{e_1 + i_1} \wedge \cdots \wedge c_1(q_r^* \mathcal{O}_{\overline{E}_r}(1))^{e_r + i_r}]$$

on P^{an} is a measure and in fact equal to

$$\sum_Y \Big(\int_{\widetilde{\mathfrak{P}}_Y} c_1(\tilde{\mathfrak{q}}_1^* \mathcal{O}_{\widetilde{\mathfrak{E}}_1}(1))^{e_1+i_1} \cdots c_1(\tilde{\mathfrak{q}}_r^* \mathcal{O}_{\widetilde{\mathfrak{E}}_r}(1))^{e_r+i_r} \Big) \cdot \delta_{\xi_{\widetilde{\mathfrak{P}}_Y}}.$$

It is therefore only left to show that

$$p_*(\delta_{\xi_{\widetilde{\mathfrak{P}}_Y}}) = \delta_{\xi_Y} \tag{11.8.1}$$

and that

$$\int_{\widetilde{\mathfrak{P}}_Y} c_1(\tilde{\mathfrak{q}}_1^* \mathcal{O}_{\widetilde{\mathfrak{E}}_1}(1))^{e_1+i_1} \cdots c_1(\tilde{\mathfrak{q}}_r^* \mathcal{O}_{\widetilde{\mathfrak{E}}_r}(1))^{e_r+i_r} = \int_Y s_{i_1}(\widetilde{\mathfrak{E}}_1) \cdots s_{i_r}(\widetilde{\mathfrak{E}}_r). \tag{11.8.2}$$

For Eq. (11.8.1) it suffices to show that $p^{\mathrm{an}}(\xi_{\widetilde{\mathfrak{P}}_Y}) = \xi_Y$. This follows from the commutative diagram of Paragraph 3.5.

Equation (11.8.2) follows from the fact that $[\widetilde{\mathfrak{P}}_Y] = \tilde{\mathfrak{p}}^*[Y]$ as cycles on $\widetilde{\mathfrak{P}}$ and Lemma 9.3 (applied to the base field $\widetilde{K}$).

Since the total mass is compatible with push-forward of measures, the total mass of $[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E_r})]$ equals the total mass of

$$[c_1(q_1^* \mathcal{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^* \mathcal{O}_{\overline{E}_r}(1))^{e_r+i_r}].$$

By Paragraph 11.6 this measure has total mass

$$\int_P c_1(q_1^* \mathcal{O}_{E_1}(1))^{e_1+i_1} \cdots c_1(q_r^* \mathcal{O}_{E_r}(1))^{e_r+i_r},$$

which by Lemma 9.3 equals $\int_X s_{i_1}(E_1) \cdots s_{i_r}(E_r)$ as claimed. ∎

11.9 Corollary. *Let $\overline{E}_1, \dots, \overline{E}_r$ be formally metrized vector bundles on V. Let $F \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree n with respect to the grading given by $\deg(X_k) = i_k$.*

There exists a closed discrete subset $S \subset V$ and a family of real numbers $\{\lambda_x\}_{x \in S}$ such that

$$[F(c_{i_1}(\overline{E}_1), \dots, c_{i_r}(\overline{E}_r))] = \sum_{x \in S} \lambda_x \delta_x.$$

Proof. This follows from Theorem 11.7 because a polynomial in the Chern currents is by definition a polynomial in the Segre currents of $\overline{E}_1, \dots, \overline{E}_r$. ∎

11.10 Corollary. *Assume that K is algebraically closed, that X is a proper algebraic K-variety of dimension n and let $\overline{E}_1, \dots, \overline{E}_r$ be formally metrized vector bundles on X.*

Let $F \in \mathbf{R}[X_1, \dots, X_r]$ be a polynomial, homogeneous of degree n with respect to the grading given by $\deg(X_k) = i_k$.

There exists a formal K°-model $\mathfrak{X}$ of X with reduced special fiber $\widetilde{\mathfrak{X}}$ and formal K°-models $\mathfrak{E}_1, \dots, \mathfrak{E}_r$ for $E_1, \dots, E_r$ defined on $\mathfrak{X}$ such that the metric of $\overline{E}_k$ is induced by $\mathfrak{E}_k$ for $k = 1, \dots, r$ and we have

$$[F(c_{i_1}(\overline{E}_1), \dots, c_{i_r}(\overline{E}_r))] = \sum_{Y \in \widetilde{\mathfrak{X}}^{(0)}} \left(\int_Y F(c_{i_1}(\widetilde{\mathfrak{E}}_1), \dots, c_{i_r}(\widetilde{\mathfrak{E}}_r)) \right) \cdot \delta_{\xi_Y} \tag{11.10.1}$$

where Y ranges over the irreducible components of $\widetilde{\mathfrak{X}}$ and δ_{ξ_Y} is the Dirac measure supported in the Shilov point $\xi_Y \in X^{\mathrm{an}}$ with reduction the generic point of Y.

The total mass is equal to the algebraic intersection number

$$\langle [F(c_{i_1}(\overline{E}_1), \dots, c_{i_r}(\overline{E}_r))], 1 \rangle = \int_X F(c_{i_1}(E_1), \dots, c_{i_1}(E_r)).$$

Proof. This follows from Theorem 11.8. ∎

11.11 Schur Polynomials. Let $n, r \in \mathbf{N}$ and denote by $\Lambda(n, r)$ the set of all partitions of n by non-negative integers $\leq r$. Thus an element $\lambda \in \Lambda(n, r)$ is specified by a sequence

$$r \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$$

with $\sum \lambda_i = n$. For $\lambda \in \Lambda(n, r)$ the *Schur polynomial* $P_\lambda \in \mathbf{Q}[X_1, \dots, X_r]$ is defined as

$$P_\lambda := \det \begin{pmatrix} X_{\lambda_1} & X_{\lambda_1+1} & \cdots & X_{\lambda_1+n-1} \\ X_{\lambda_2-1} & X_{\lambda_2} & \cdots & X_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \ddots \\ X_{\lambda_n-n+1} & X_{\lambda_n-n+2} & \ddots & X_{\lambda_n} \end{pmatrix},$$

where by convention $X_0 = 1$ and $X_i = 0$ if $i \notin \{0, \dots, r\}$. The first Schur polynomials are

$$\begin{aligned} P_{(1)} &= X_1 \\ P_{(2,0)} &= X_2, \qquad P_{(1,1)} = X_1^2 - X_2 \\ P_{(3,0,0)} &= X_3, \qquad P_{(2,1,0)} = X_1X_2 - X_3 \qquad P_{(1,1,1)} = X_1^3 - 2X_1X_2 + X_3. \end{aligned}$$

The family $\{P_\lambda\}_{\lambda \in \Lambda(n,r)}$ forms a basis for the $\mathbf{Q}$-vector space of polynomials $F \in \mathbf{Q}[X_1, \dots, X_r]$ which are homogeneous with respect to the grading $\deg(X_l) = l$. In other words, every such polynomial F can be written uniquely in the form

$$F = \sum_{\lambda \in \Lambda(n,r)} a_\lambda(F) \cdot P_\lambda \tag{11.11.1}$$

with $a_\lambda(F) \in \mathbf{Q}$.

11.12 Numerically Non-Negative and Positive Polynomials. Let k be a field. Let $F \in \mathbf{Q}[X_1, \ldots, X_r]$ be a polynomial, homogeneous of degree n with respect to the grading $\deg(X_l) = l$. The polynomial F is called *numerically positive for ample vector bundles* (with respect to k) if for every projective algebraic k-variety X of dimension n and every ample vector bundle E of rank r on X the Chern number

$$\int_X F(c_1(E), \ldots, c_r(E))$$

is strictly positive. By [FL83, Thm. 1], the polynomial F is numerically positive for ample vector bundles if and only if $F \neq 0$ and the coefficients $a_\lambda(F)$ in Eq. (11.11.1) satisfy $a_\lambda(F) \geq 0$ for all $\lambda \in \Lambda(n, r)$. In particular, the notion of numerical positivity is independent of the ground field k.

We call F *numerically non-negative for nef vector bundles* (with respect k) if for every projective algebraic k-variety X of dimension n and every nef vector bundle E of rank r on X the Chern number $\int_X F(c_1(E), \ldots c_r(E))$ is non-negative. In the case of the complex ground field $k = \mathbf{C}$, the polynomial F is numerically non-negative if and only if $a_\lambda(F) \geq 0$ for all $\lambda \in \Lambda(n, r)$ [Laz04, Exmp. 8.3.10]. The proof (of a more general fact regarding filtered vector bundles) can be found in [Ful95, Thm.']. In particular, every positive polynomial is non-negative and the only non-negative polynomial which is not positive is the zero polynomial. For general ground fields this result is likely also true and the proof in [Ful95], even though it is concerned with compact Kähler manifolds, should go through, but there does not seem to be a reference in the literature.

11.13 Corollary. *Assume that K is algebraically closed. Let X be an n-dimensional projective K-variety, let $\overline{E}$ formally metrized vector bundle of rank r on E and let $F \in \mathbf{Q}[X_1, \ldots, X_r]$ be a polynomial which is homogeneous with respect to the weights $\deg(X_k) = k$. Suppose that either of the following conditions is satisfied:*

- *(i) The polynomial F is numerically non-negative for nef vector bundles with respect to the residue field $\widetilde{K}$ and $\overline{E}$ is semipositive.*
- *(ii) The polynomial F is numerically positive for ample vector bundles and $\overline{E}$ admits a formal K°-model $(\mathfrak{X}, \mathfrak{E})$ inducing the metric such that $\mathfrak{X}$ has reduced special fiber $\widetilde{\mathfrak{X}}$ and the special fiber $\widetilde{\mathfrak{E}}$ is ample on all irreducible components of $\widetilde{\mathfrak{X}}$.*

Then $[F(c_1(\overline{E}), \ldots, c_r(\overline{E}))]$ is a positive Radon measure.

Proof. Assume that (i) is satisfied. By Corollary 11.10 we can choose a model $(\mathfrak{X}, \mathfrak{E})$ defining the metric of $\overline{E}$ and with reduced special fiber $\widetilde{\mathfrak{X}}$. By Proposition 6.14, $\mathfrak{E}$ is numerically effective which means that its special fiber $\widetilde{\mathfrak{E}}$ is numerically effective. The definition of numerically non-negative polynomials for nef vector bundles shows exactly that the coefficients in Eq. (11.10.1) are non-negative.

If (ii) is true, we have by assumption a formal model $(\mathfrak{X}, \mathfrak{E})$ defining the metric of $\overline{E}$, with reduced special fiber and such that $\widetilde{\mathfrak{E}}$ is ample on all irreducible components of $\widetilde{\mathfrak{X}}$. Then the definition of positive polynomials and Eq. (11.10.1) yield the claim. ∎

11.14 Remark. If it is true that the notion of numerical non-negativity for nef vector bundles is independent of the base field then condition (i) is implied by (ii) in Corollary 11.13.

12. Characteristic Currents Based on δ-Forms

Throughout this chapter K will be a non-archimedean field. We recall the definition of δ-forms and δ-currents from [Mih23a] and sketch how to develop a theory of characteristic δ-currents by replacing the smooth forms of [CD12] by δ-forms. All K-analytic spaces are assumed to be good, topologically Hausdorff, boundaryless, equidimensional and paracompact. We fix an algebraic K-variety X of dimension n and a K-analytic space V of dimension n.

12.1 δ-Forms. In [Mih23a], Mihatsch developed an extension of the theory of smooth forms introduced in [CD12] (based on earlier work by Gubler and Künnemann [GK17]).

By [Mih23a, Def. 4.2] there is a bigraded sheaf of differential $\mathbf{R}$-algebras $\mathscr{B}_V^{\bullet,\bullet}$ on V the sections of which sections are called *δ-forms* on V. By [Mih23a, Thm. 4.5] pull-backs of δ-forms along morphisms of K-analytic spaces are defined.

By [Mih23a, Sec. 4.2] the sheaf $\mathrm{PS}_V^{\bullet,\bullet}$ of piecewise smooth forms is contained in $\mathscr{B}_V^{\bullet,\bullet}$ with equality $\mathscr{B}_V^{0,0} = \mathrm{PS}_V^{0,0}$ in degree $(0,0)$. In particular, $\mathscr{B}_V^{\bullet,\bullet}$ contains the sheaf $\mathscr{A}_V^{\bullet,\bullet}$ of smooth forms in the sense of Chambert-Loir and Ducros.

12.2 δ-Currents. A *δ-current* of degree (p,q) on an open subset V is an $\mathbf{R}$-linear functional $T\colon \mathscr{B}_c^{n-p,n-q}(V) \to \mathbf{R}$ which is continuous with respect to a Schwartz-style topology on the space $\mathscr{B}_c^{n,n}(V)$ of compactly supported δ-forms on V. By a partition of unity argument, δ-currents form a sheaf on X^{an} which we denote by $\mathscr{E}_V^{\bullet,\bullet}$. Similarly to Paragraph 7.3, the sheaf $\mathscr{E}_V^{\bullet,\bullet}$ carries the structure of a bigraded differential $\mathscr{B}_V^{\bullet,\bullet}$-module. Similarly to Paragraph 7.4, a proper morphism $f: V' \to V$ of K-analytic spaces of dimensions n', resp. n induces a morphism of sheaves

$$f_*: f_*\mathscr{E}_{V'}^{\bullet,\bullet} \to \mathscr{E}_V^{\bullet,\bullet}.$$

12.3 Integration. By [Mih23a, Def. 4.10] there is a well-defined integration operator $\int_V: \mathscr{B}_c^{n,n}(V) \to \mathbf{R}$. By [Mih23a, Prop. 4.11] it defines a δ-current $\delta_V := \int_V \in \mathscr{E}^{0,0}(V)$. If $V = X^{\mathrm{an}}$ is the analytification of an algebraic variety, then by linear extension a δ-current δ_Z of degree (p,p) is defined for every algebraic cycle Z of codimension p on X.

If $f: X' \to X$ is a proper morphism of K-varieties and Z' is a cycle on X' then one can show that

$$f_*(\delta_{Z'}) = \delta_{f_*(Z')}.$$

This follows similarly to [GK17, Prop. 6.12], using a version of the Sturmfels–Tevelev formula for skeletons [GJR21, Prop. 8.27].

There exists a unique morphism of bigraded differential $\mathscr{B}_V^{\bullet,\bullet}$-modules

$$[-]: \mathscr{B}_V^{\bullet,\bullet} \to \mathscr{E}_V^{\bullet,\bullet}$$

satisfying $[1] = \delta_V$.

12.4 The Poincaré-Lelong Formula. Let f be an invertible meromorphic function on V. We interpret the terms in the Poincaré-Lelong equation

$$\delta_{\mathrm{div}(f)} = d'd''[\log|f|] \tag{12.4.1}$$

as δ-currents on V similarly to Paragraph 7.9. Then the formula (12.4.1) is a special case of [Mih23a, Thm. 5.5].

12.5 Bott–Chern Cohomology of δ-Currents. For $p \in \mathbf{N}$ we denote by $\hat{H}^p_{\mathscr{E}}(V)$ the group

$$\frac{\mathrm{Ker}(d': \mathscr{E}^{p,p}(V) \to \mathscr{E}^{p+1,p}(V)) \cap \mathrm{Ker}(d'': \mathscr{E}^{p,p}(V) \to \mathscr{E}^{p,p+1}(V))}{\mathrm{Im}(d'd'': \mathscr{E}^{p-1,p-1}(V) \to \mathscr{E}^{p,p}(V))}.$$

If X is an algebraic variety then as in Paragraph 7.10 there is a natural map

$$\delta: \mathrm{CH}^p(X) \to \hat{H}^p_{\mathscr{E}}(X^{\mathrm{an}})$$

given by integration which is compatible with proper push-forward.

12.6 Piecewise Smooth Metrics. Let $\overline{L} = (L, \|-\|)$ be a metrized line bundle on V. The metric $\|-\|$ is called *piecewise smooth* if for every section $s \in \Gamma(U, L)$ over an open subset $U \subset V$ the function $-\log\|s(-)\| : U \to \mathbf{R}$ is piecewise smooth. By [GM19, Prop. 2.10] every formal metric is piecewise linear and in particular piecewise smooth.

12.7 First Chern δ-Forms. Let $\overline{L}$ be a piecewise smoothly metrized line bundle on V. Similarly to Paragraph 8.1 there exists a unique δ-form $c_1(\overline{L}) \in \mathscr{B}^{1,1}(V)$ such that for every open subset $U \subset V$ and nowhere-vanishing section $s \in \Gamma(U, L)$ we have

$$c_1(\overline{L})|_U = d'd''(-\log\|s(-)\|).$$

It is easy to see that if $\overline{L}'$ is another piecewise smooth metrized line bundle then

$$c_1(\overline{L} \otimes \overline{L}') = c_1(\overline{L}) + c_1(\overline{L}').$$

If $f : V' \to V$ is a morphism of K-analytic spaces then

$$c_1(f^*\overline{L}) = f^*c_1(\overline{L}).$$

If $\overline{L}$ is a piecewise smooth metrized line bundle and s is a regular meromorphic section then we have the Poincaré-Lelong formula

$$[c_1(\overline{L})] = \delta_{\operatorname{div}(s)} + d'd''[-\log\|s(-)\|]. \tag{12.7.1}$$

After localizing this follows from Paragraph 12.4 above.

12.8 Products of First Chern δ-Currents. If $\overline{L}_1, \dots, \overline{L}_r$ are piecewise smoothly metrized line bundles on V then we can form the wedge product of δ-forms $c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r) \in \mathscr{B}^{r,r}(V)$ and the associated δ-current

$$[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)] \in \mathscr{E}^{r,r}(V). \tag{12.8.1}$$

It satisfies similar formal properties as the product of first Chern currents of locally approachably metrized line bundles defined in Paragraph 8.9. For example, an analogue of Lemma 8.10 holds also for piecewise smoothly metrized line bundles and δ-forms. In Proposition 12.10 below we give a proof of an analogue of Proposition 8.11 (which is simpler, because no approximation procedures are performed in the definition of the current (12.8.1)).

12.9 Lemma. *Let $\alpha, \alpha' \in \mathscr{B}^{p,q}(V)$, $\beta, \beta' \in \mathscr{B}^{p',q'}(V)$ be d'- and d''-closed δ-forms and suppose that $[\alpha'] = [\alpha] + d'd''S$, $[\beta'] = [\beta] + d'd''T$ for δ-currents $S \in \mathscr{E}^{p-1,q-1}(V)$ and $T \in \mathscr{E}^{p'-1,q'-1}(V)$. Then*

$$[\alpha \wedge \beta] = [\alpha' \wedge \beta'] + d'd''U$$

for some $U \in \mathscr{E}^{p+p'-1,q+q'-1}(V)$.

Proof. It follows from the Leibniz rule that

$$d'd''(\alpha \wedge T) = \alpha \wedge d'd''T.$$

Hence we get

$$[\alpha \wedge \beta'] = [\alpha \wedge \beta] + d'd''(\alpha \wedge T).$$

Similarly we get $[\alpha \wedge \beta'] \equiv [\alpha' \wedge \beta']$ modulo $d'd''\mathscr{E}^{p+q-1,p'+q'-1}(V)$. ∎

12.10 Proposition. *Let $\overline{L}_1, \dots, \overline{L}_r$ be piecewise smoothly metrized line bundles on V. Then the class of the current $[c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_r)]$ in the Bott–Chern cohomology group $\hat{H}^r_{\mathscr{E}}(V)$ does not depend on the metrics of the line bundles.*

Proof. By the Poincaré-Lelong formula (12.7.1) the result is true if $r = 1$. In general it follows by induction from Lemma 12.9. ∎

12.11 Characteristic δ-Currents and δ-Forms. Now all results and constructions of Chapters 9 to 11 have analogues in the context of δ-currents if we replace locally approachably metrized line bundles with piecewise smoothly metrized line bundles. Some of the proofs in Chapter 9 become simpler; notably in Propositions 9.8 and 9.10 and Paragraph 9.9 we can replace the approximation arguments by direct algebraic manipulation.

Explicitly, assume that $\overline{E}_1, \dots, \overline{E}_r$ are piecewise smoothly pseudo-metrized vector bundles on X. This means that $\overline{E}_k = (E_k, \|{-}\|_k)$ where E_k is a vector bundle on V and $\|{-}\|_k$ is a piecewise smooth metric on $\mathscr{O}_{E_k}(1)$. Consider the fibered product

$$p\colon\ P \coloneqq P(E_1) \times_V \cdots \times_V P(E_r) \to V$$

with projection maps $q_k : P \to P(E_k)$. Given $i_1, \dots, i_r \in \mathbf{N}$ we get d'- and d''-closed δ-currents

$$[s_{i_1}(\overline{E}_1) \wedge \cdots \wedge s_{i_r}(\overline{E}_r)] \coloneqq p_*([c_1(q_1^*\mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^*\mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}])$$

as elements of $\mathscr{E}^{|i|,|i|}(V)$ if we interpret

$$[c_1(q_1^*\mathscr{O}_{\overline{E}_1}(1))^{e_1+i_1} \wedge \cdots \wedge c_1(q_r^*\mathscr{O}_{\overline{E}_r}(1))^{e_r+i_r}]$$

in the sense of Paragraph 12.8.

Then we also get

$$[c_{i_1}(\overline{E}_1) \wedge \cdots \wedge c_{i_r}(\overline{E}_r)] \in \mathscr{E}^{|i|,|i|}(V)$$

by mimicking the definition in Paragraph 9.12.

We can define *Segre δ-forms* of piecewise smoothly pseudo-metrized vector bundles similarly to Paragraph 10.1 and prove existence in degree 0 and for line bundles as in Remark 10.3.

12.12 Remark. It is probably possible to develop a Bedford–Taylor theory for δ-currents analogous to [CD12, Chap. 5] which would allow to define characteristic δ-currents for more general (pseudo-)metrics than just piecewise smooth (pseudo-)metrics by an approximation process. In an algebraic context, positivity notions of piecewise smoothly metrized line bundles have been investigated in [GK16, Sec. 8].

A. Liu's Tropical Cycle Class Map

The purpose of this appendix is to review the construction of Liu's cycle class map from [Liu20] and to give a proof of the commutativity of the diagram (7.12.1). Before turning to the constructions of Liu, we recall the definitions of cohomology presheaves in Paragraph A.1 and cohomology with support in Paragraph A.3.

We make explicit some of the constructions and arguments which are only implicitly present in [Liu20] in order to extract the stronger statement of Theorem A.21 from the proof of [Liu20, Thm. 3.9].

Throughout the appendix, K denotes a non-archimedean field (although the metric structure is not always used). All K-analytic spaces are assumed to be good, paracompact, Hausdorff, equidimensional and boundaryless. Note that boundaryless K-analytic spaces are strict by [Tem15, Exmp. 4.2.4.2 (ii)] so the results and constructions of [Liu20] apply. We fix a K-analytic space V and a smooth separated finite type K-scheme X.

A.1 Cohomology Presheaves. Let T be a topological space. We write $\mathrm{Sh}(T)$ for the category of abelian sheaves on T and $\mathrm{D}(T) \coloneqq \mathrm{D}(\mathrm{Sh}(T))$ for its derived category. We write

$$i\colon \mathrm{Sh}(T) \to \mathrm{PSh}(T)$$

for the inclusion functor from the category of sheaves on T to the category of presheaves on T. This functor is left exact and we denote its derived functors by

$$\underline{H}^p(-) \coloneqq R^p i\colon \mathrm{D}(T) \to \mathrm{PSh}(T).$$

Concretely, if $\mathcal{F}$ is a sheaf on T, then the presheaf $\underline{H}^p(\mathcal{F})$ maps an open subset $U \subset T$ to the sheaf cohomology group $H^p(U, \mathcal{F})$ by [Sta23, Lem. 01ER].

A.2 Lemma. *Let T be a topological space, $\mathcal{F} \in \mathrm{D}(T)$ and $p \in \mathbf{N}$. Assume that there exists a basis $\mathfrak{B}$ for the topology of T such that for every open subset $U \in \mathfrak{B}$ and every $q < p$, we have*

$$H^q(U, \mathcal{F}) = 0.$$

Then the presheaf $\underline{H}^p(\mathcal{F})$ given by

$$U \mapsto H^p(U, \mathcal{F})$$

is a sheaf.

Proof. Since the sheaf property can be checked for open subsets of $\mathfrak{B}$, we can restrict our attention to the site (giving rise to the same sheaf category) defined by the open subsets of $\mathfrak{B}$, so that we may actually assume that $\underline{H}^q(\mathcal{F}) = 0$ for $q < p$. We denote by $\check{H}^p(\mathcal{U}, -)$ the Čech cohomology functor for presheaves with respect to a covering $\mathcal{U}$ of an open subset $U \subset V$. Then the sheaf condition for $\underline{H}^p(\mathcal{F})$ can be expressed as the natural isomorphism

$$\check{H}^0(\mathcal{U}, \underline{H}^p(\mathcal{F})) \cong H^p(U, \mathcal{F}).$$

Using the given vanishing conditions, this follows from the Čech-to-cohomology spectral sequence

$$\check{H}^i(\mathcal{U}, \underline{H}^j(\mathcal{F})) \Rightarrow H^{i+j}(U, \mathcal{F})$$

[Sta23, Lem. 015N]. ∎

A.3 Cohomology with Support. Let T be a topological space and let $Z \subset T$ be a closed subset. Given a sheaf $\mathcal{F}$ on T, we denote by

$$\Gamma_Z(\mathcal{F}) := \{\, s \in \Gamma(T, \mathcal{F}) \mid \mathrm{Supp}(s) \subset Z \,\}$$

the group of sections with support in Z. The functor Γ_Z is left exact and its derived functors are denoted by

$$H^p_Z(T, -) := R^i\Gamma_Z \colon \mathrm{D}(T) \to \mathrm{Ab}.$$

The inclusion $\Gamma_Z \hookrightarrow \Gamma$ induces a natural (functorial in $\mathcal{F}$) morphism

$$H^p_Z(T, \mathcal{F}) \to H^p(T, \mathcal{F}).$$

Note that if we denote by $i = i_Z \colon Z \hookrightarrow T$ the inclusion map, then Γ_Z can be expressed as the composition

$$\mathrm{Sh}(T) \xrightarrow{i^!} \mathrm{Sh}(Z) \xrightarrow{i_!} \mathrm{Sh}(T) \xrightarrow{\Gamma} \mathrm{Ab}$$

where $i_!$ and $i^!$ denote the exceptional direct and inverse image functors [Ive86, Def. II.6.1, Prop. II.6.6]. For a closed embedding, the functor $i_!$ coincides with the push-forward functor i_* by [Ive86, Prop. II.6.9 (i)] and hence maps injective sheaves to injective sheaves by [Ive86, Cor. II.4.13]. Also the exceptional inverse image functor $i^!$ maps injective sheaves to injective sheaves by [Ive86, Prop. II.6.8]. It follows from [GM03, Thm. III.7.1] that $R\Gamma_Z$ factors as

$$\mathrm{D}(T) \xrightarrow{i^!} \mathrm{D}(Z) \xrightarrow{i_!} \mathrm{D}(T) \xrightarrow{R\Gamma} \mathrm{D}(\mathrm{Ab}),$$

where by abuse of notation, we denote the right derived functors of $i_!$ and $i^!$ by $i_!$, resp. $i^!$ again. It follows that we have

$$H^p_Z(T, \mathcal{F}) = H^p(T, i_{Z!} i^!_Z \mathcal{F})$$

for $\mathcal{F} \in \mathrm{D}(T)$. In particular, the association

$$U \mapsto H^p_Z(U, \mathcal{F}) := H^p(U, i_{Z!} i^!_Z \mathcal{F})$$

defines a presheaf on T.

A.4 Remark. In [Liu20] Liu does not use the notation $H^p_Z(T, \mathcal{F})$ and only writes $H^p(T, i_{Z!} i^!_Z \mathcal{F})$. We write $H^p_Z(T, \mathcal{F})$ in order to get simpler formulas.

A.5 The Gysin Exact Sequence. Let T be a topological space, $Z \subset T$ a closed subset and $\mathcal{F}$ a sheaf on T. Then there is a canonical long exact sequence

$$\cdots \longrightarrow H^p_Z(T, \mathcal{F}) \longrightarrow H^p(T, \mathcal{F}) \longrightarrow H^p(T \setminus Z, \mathcal{F}) \xrightarrow{\delta} H^{p+1}_Z(T, \mathcal{F}) \longrightarrow \cdots,$$

functorial in (T, Z) and $\mathcal{F}$ by [Ive86, Prop. II.9.2, Prop. II.9.7].

A.6 Milnor K-Theory. Let R be a commutative ring. Then the *p-th Milnor K-theory* $K^{p,M}(R)$ of R is the group generated by the formal symbols $\{f_1, \dots, f_p\}$ with $f_1, \dots, f_p \in R^\times$, modulo the Steinberg relations

(i) $\{f_1, \dots, f_i f_i', \dots, f_p'\} = \{f_1, \dots, f_i, \dots, f_p\} + \{f_1, \dots, f_i', \dots, f_p\}$

(ii) $\{f_1, \dots, f, \dots, 1 - f, \dots, f_p\} = 0.$

If $(T, \mathcal{O}_T)$ is a ringed G-topological space, then we define the *p-th sheaf of rational Milnor K-theory* $\mathcal{K}^p_T$ to be the sheaf associated to the presheaf which maps an admissible open $U \subset T$ to $K^{p,M}(\mathcal{O}_X(U)) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Here we follow [Liu20, Def. 2.1] with the notation.

A.7 The Gersten Complex. Recall that we fixed the smooth separated finite type K-scheme X. For each non-negative integer p there is a canonical flasque resolution

$$0 \longrightarrow \mathscr{K}_X^p \longrightarrow \mathscr{K}_X^{p,0} \xrightarrow{d} \mathscr{K}_X^{p,1} \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{K}_X^{p,p} \longrightarrow 0,$$

where

$$\mathscr{K}_X^{p,q} = \bigoplus_{x \in X^{(q)}} i_{x*} K^{p-q,M}(k(x)) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

In particular, we have canonical isomorphisms

$$H^q(X, \mathscr{K}_X^p) \cong \frac{\operatorname{Ker}(d \colon \mathscr{K}_X^{p,q}(X) \to \mathscr{K}_X^{p,q+1}(X))}{\operatorname{Im}(d \colon \mathscr{K}_X^{p,q-1}(X) \to \mathscr{K}_X^{p,q}(X))},$$

see also [Liu20, Eq. (2.2)].

A.8 The Universal Cycle Class Map. By [Liu20, Lem. 2.2 (i)] we have

$$H^p(X, \mathscr{K}_X^p) \cong \operatorname{Coker}(d \colon \mathscr{K}_X^{p,p-1}(X) \to \mathscr{K}_X^{p,p}(X)) \cong \mathrm{CH}^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

The inverse isomorphism is denoted by

$$\mathrm{cl}_{\mathrm{univ}} \colon \mathrm{CH}^p(X)_{\mathbf{Q}} \xrightarrow{\sim} H^p(X, \mathscr{K}_X^p)$$

and is called the *universal cycle class map*.

A.9 Lemma. *Let $Z \subset X$ be an integral closed subscheme of codimension p. Then the presheaf $\underline{H}^p(i_{Z!} i_Z^! \mathscr{K}_X^p)$ given by*

$$U \mapsto H_Z^p(U, \mathscr{K}_X^p)$$

is a sheaf.

Proof. This follows from [Liu20, Lem. 2.2 (iii)] and Lemma A.2. ∎

A.10 Lifting $\mathrm{cl}_{\mathrm{univ}}(Z)$. Let $Z \subset X$ be a smooth, integral closed subscheme of codimension p. We want to construct a preimage $c(Z) \in H_Z^p(X, \mathscr{K}_X^p)$ of $\mathrm{cl}_{\mathrm{univ}}(Z)$ under the canonical map $H_Z^p(X, \mathscr{K}_X^p) \to H^p(X, \mathscr{K}_X^p)$.

We choose a finite affine open covering $\{U_\alpha\}$ of X and for each α a regular sequence $f_{\alpha 1}, \dots, f_{\alpha p} \in \mathscr{O}_X(U_\alpha)$ such that $Z \cap U_\alpha$ is defined by the ideal $\langle f_{\alpha 1}, \dots, f_{\alpha p} \rangle$. Let $U_{\alpha i}$ be the non-vanishing locus of $f_{\alpha i}$. Then $\{U_{\alpha i}\}_{i=1,\dots,p}$ is an open covering of

$U_\alpha \setminus Z$ and the element $\{f_{\alpha 1}, \dots, f_{\alpha p}\} \in \mathcal{K}_X^p(\bigcap_{i=1}^p U_{\alpha i})$ defines a $(p-1)$-Čech cocycle with respect to this covering and hence gives an element in $H^{p-1}(U_\alpha \setminus Z, \mathcal{K}_X^p)$. We denote its image under the connecting homomorphism

$$\delta : H^{p-1}(U_\alpha \setminus Z, \mathcal{K}_X^p) \to H_Z^{p-1}(U_\alpha, \mathcal{K}_X^p)$$

by $c(Z)_\alpha \in H_Z^p(U_\alpha, \mathcal{K}_X^p)$.

Recall from Lemma A.9 that the assignment $U \mapsto H_Z^p(U, \mathcal{K}_X^p)$ is a sheaf on X. It follows from [Liu20, Lem. 2.3] that the $c(Z)_\alpha$ glue to an element $c(Z) \in H_Z^p(X, \mathcal{K}_X^p)$. By [Liu20, Lem. 2.4] it is indeed a preimage of $\mathrm{cl}_{\mathrm{univ}}(Z)$.

A.11 Dolbeault Cohomology. Recall that we fixed the K-analytic space V. Since we assumed V to be in particular paracompact, by partition of unity the complex

$$0 \longrightarrow \mathcal{A}_V^{p,0} \xrightarrow{d''} \mathcal{A}_V^{p,1} \xrightarrow{d''} \cdots \tag{A.11.1}$$

consists of flasque sheaves, so that we have

$$H^q(V, \mathcal{A}_V^{p,\bullet}) = H_{\mathcal{A}}^{p,q}(V).$$

in the notation of Paragraph 7.11.

Similarly we have the complex $\mathcal{D}_V^{p,\bullet}$ and we have

$$H^q(V, \mathcal{D}_V^{p,\bullet}) = H_{\mathcal{D}}^{p,q}(V).$$

A.12 Remark. By [Jel16, Cor. 4.6], the complex Eq. (A.11.1) is also acyclic except for degree 0. In other words, it is a flasque resolution of $\mathrm{Ker}(d'' : \mathcal{A}_V^{p,0} \to \mathcal{A}_V^{p,1})$, so we can identify

$$H^q(V, \mathcal{A}_V^{p,\bullet}) = H^q(V, \mathrm{Ker}(d'' : \mathcal{A}_V^{p,0} \to \mathcal{A}_V^{p,1})).$$

A.13 The Map τ_V^p. By [Liu20, Def. 3.3] there is a canonical $\mathbf{Q}$-linear map of complexes of sheaves

$$\tau_V^p : \mathcal{K}_V^p \to \mathcal{A}_V^{p,\bullet}$$

(in other words, a map of sheaves $\mathcal{K}_V^p \to \mathrm{Ker}(d'' : \mathcal{A}_V^{p,0} \to \mathcal{A}_V^{p,1})$), given by

$$\tau_V^p(\{f_1, \dots, f_p\}) = d'(-\log|f_1|) \wedge \cdots \wedge d'(-\log|f_p|)$$

for $f_1, \dots, f_p \in \mathcal{O}_V^\times(W)$ for some open subset $W \subset V$.

It follows that there is an induced map

$$\tau_V^p : H^q(V, \mathcal{K}_V^p) \to H^q(V, \mathcal{A}_V^{p,\bullet}) = H_{\mathcal{A}}^{p,q}(V).$$

A.14 Remark. It follows from [Liu20, Cor. 3.6] that the map τ_V^p factors through a canonical rational subspace $H^q(V, \mathcal{F}_V^p) \subset H_{\mathcal{A}}^{p,q}(V)$ which provides $H_{\mathcal{A}}^{p,q}(V)$ with a canonical rational structure. We will not use this.

A.15 Lemma. *Let $Z \subset V$ be a Zariski closed subset of codimension at least p for some $p \geq 0$. Then the presheaf*

$$W \mapsto H_Z^p(W, \mathcal{D}_V^{p,\bullet})$$

is a sheaf on V.

Proof. It follows from [Liu20, Lem. 3.10] that $H_Z^q(W, \mathcal{D}_V^{p,\bullet}) = 0$ for every paracompact open $W \subset V$ and $q < p$. Hence the result follows from Lemma A.2. ∎

A.16. Let $i_Z : Z \hookrightarrow V$ be a Zariski closed subset of codimension at least p for some $p \geq 0$. Following the notation introduced in the proof of [Liu20, Thm. 3.9] we write $\mathcal{D}_V^{p,p,\mathrm{cl}}$ to denote the space of (p,p)-currents which are d''-closed. In the proof of [Liu20, Lem. 3.10], Liu constructs a map

$$H_Z^p(V, \mathcal{D}_V^{p,\bullet}) \to \mathcal{D}_V^{p,p,\mathrm{cl}}(V) \tag{A.16.1}$$

satisfying the following properties:

(i) The map is injective and induces an isomorphism

$$H_Z^p(V, \mathcal{D}_V^{p,\bullet}) \xrightarrow{\sim} \mathrm{Ker}(\mathcal{D}_V^{p,p,\mathrm{cl}}(V) \to \mathcal{D}_V^{p,p,\mathrm{cl}}(V \setminus Z)),$$

where the map in the kernel is given by restriction.

(ii) It is a morphism of sheaves, i.e. if $W \subset V$ is an open subset, then the diagram

$$\begin{tikzcd} H_Z^p(V, \mathcal{D}_V^{p,\bullet}) \arrow[r] \arrow[d] & \mathcal{D}_V^{p,p,\mathrm{cl}}(V) \arrow[d] \\ H_Z^p(W, \mathcal{D}_V^{p,\bullet}) \arrow[r] & \mathcal{D}_V^{p,p,\mathrm{cl}}(W) \end{tikzcd}$$

commutes.

(iii) The diagram

$$\begin{tikzcd} H_Z^p(V, \mathcal{D}_V^{p,\bullet}) \arrow[r] \arrow[d] & \mathcal{D}_V^{p,p,\mathrm{cl}}(V) \arrow[dl] \\ H^p(V, \mathcal{D}_V^{p,\bullet}) & \end{tikzcd}$$

commutes.

Indeed, the first property is the statement of [Liu20, Lem. 3.10], while the remaining two statements can be derived directly from the construction.

A.17. We denote by $\pi : X^{\mathrm{an}} \to X$ the natural projection map. The maps

$$\mathcal{K}_X^p \longrightarrow \pi_* \mathcal{K}_{X^{\mathrm{an}}}^p \xrightarrow{\tau_{X^{\mathrm{an}}}^p} \pi_* \mathcal{A}_X^{p,\bullet} \xrightarrow{[]} \pi_* \mathcal{D}_X^{p,\bullet}$$

induce natural maps

$$H^p(X, \mathcal{K}_X^p) \longrightarrow H^p(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^p) \xrightarrow{\tau_{X^{\mathrm{an}}}^p} H^p(X^{\mathrm{an}}, \mathcal{A}_X^{p,\bullet}) \xrightarrow{[]} H^p(X^{\mathrm{an}}, \mathcal{D}_X^{p,\bullet}).$$

If $Z \subset X$ is an integral closed subscheme of codimension p, then there are similarly maps

$$H_Z^p(X, \mathcal{K}_X^p) \longrightarrow H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^p) \xrightarrow{\tau_{X^{\mathrm{an}}}^p} H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{A}_X^{p,\bullet}) \xrightarrow{[]} H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{D}_X^{p,\bullet}). \tag{A.17.1}$$

These maps are compatible in the sense that the diagram

$$\begin{array}{ccccccc}
H^p(X, \mathcal{K}_X^p) & \longrightarrow & H^p(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^p) & \xrightarrow{\tau_{X^{\mathrm{an}}}^p} & H^p(X^{\mathrm{an}}, \mathcal{A}_X^{p,\bullet}) & \xrightarrow{[]} & H^p(X^{\mathrm{an}}, \mathcal{D}_X^{p,\bullet}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_Z^p(X, \mathcal{K}_X^p) & \longrightarrow & H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^p) & \xrightarrow{\tau_{X^{\mathrm{an}}}^p} & H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{A}_X^{p,\bullet}) & \xrightarrow{[]} & H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{D}_X^{p,\bullet})
\end{array}$$

commutes.

Moreover all maps defined here are morphisms of presheaves on X. For example, if $U \subset X$ is an open subset, then the diagram

$$\begin{array}{ccc}
H_Z^p(X, \mathcal{K}_X^p) & \longrightarrow & H_{Z^{\mathrm{an}}}^p(X^{\mathrm{an}}, \mathcal{D}_X^{p,\bullet}) \\
\downarrow & & \downarrow \\
H_Z^p(U, \mathcal{K}_X^p) & \longrightarrow & H_{Z^{\mathrm{an}}}^p(U^{\mathrm{an}}, \mathcal{D}_X^{p,\bullet})
\end{array}$$

commutes.

A.18 The Tropical Cycle Class Map. In [Liu20, Def. 3.8] the *tropical cycle class map* $\mathrm{cl}_{\mathrm{trop}}$ is defined to be the composition

$$\mathrm{cl}_{\mathrm{trop}} : \mathrm{CH}^p(X)_{\mathbf{Q}} \xrightarrow{\mathrm{cl}_{\mathrm{univ}}} H^p(X, \mathcal{K}_X^p) \longrightarrow H^p(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^p) \xrightarrow{\tau_{X^{\mathrm{an}}}^p} H^p(X^{\mathrm{an}}, \mathcal{A}_X^{p,\bullet}).$$

Note that $\mathrm{cl}_{\mathrm{trop}}$ is functorial in both K and X.

We denote the composition

$$\mathrm{CH}^p(X)_{\mathbf{Q}} \xrightarrow{\mathrm{cl}_{\mathrm{trop}}} H^{p,p}_{\mathscr{A}}(X^{\mathrm{an}}) \xrightarrow{[]} H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}})$$

by $\mathrm{cl}'_{\mathrm{trop}}$. This notation is also introduced *ad hoc* in the proof of [Liu20, Thm. 3.9].

A.19 Lifting $\mathrm{cl}'_{\mathrm{trop}}(Z)$. We follow here Step 1 of the proof of [Liu20, Thm. 3.9]. Let X be a smooth separated scheme of finite type over K of dimension n. Let $Z \subset X$ be a smooth integral closed subscheme of X of codimension p. We choose a finite affine open covering $\{U_\alpha\}$ of X and $f_{\alpha 1}, \dots, f_{\alpha p} \in \mathscr{O}_X(U_\alpha)$ such that $Z \cap U_\alpha$ is defined by the ideal $\langle f_{\alpha 1}, \dots, f_{\alpha p}\rangle$ and such that the induced morphism $(f_{\alpha 1}, \dots f_{\alpha p})\colon U_\alpha \to \mathbf{A}^p_K$ is smooth. Recall that in this situation, we constructed in Paragraph A.10 a class $c(Z) \in H^p_Z(X, \mathscr{K}^p_X)$ mapping to $\mathrm{cl}_{\mathrm{univ}}(Z)$.

We denote by $c \in \mathscr{D}^{p,p,\mathrm{cl}}_X(X^{\mathrm{an}})$ the image of $c(Z)$ under the composite

$$H^p_Z(X, \mathscr{K}^p_X) \longrightarrow H^p_{Z^{\mathrm{an}}}(X^{\mathrm{an}}, \mathscr{D}^{p,\bullet}_X) \longrightarrow \mathscr{D}^{p,p,\mathrm{cl}}_X(X^{\mathrm{an}}), \tag{A.19.1}$$

where the first map is given as in Paragraph A.17 and the second map is given as in Paragraph A.16. Then the cohomology class of c in $H^p(X^{\mathrm{an}}, \mathscr{D}^{p,\bullet}_X)$ agrees with $\mathrm{cl}'_{\mathrm{trop}}(Z)$. Indeed, this follows from the commutative diagram

$$\begin{tikzcd} & H^p_Z(X, \mathscr{K}^p_X) \arrow[r] \arrow[d] & H^p_{Z^{\mathrm{an}}}(X^{\mathrm{an}}, \mathscr{D}^{p,\bullet}_X) \arrow[r] \arrow[d] & \mathscr{D}^{p,p,\mathrm{cl}}_X(X^{\mathrm{an}}) \arrow[dl] \\ \mathrm{CH}^p(X)_{\mathbf{Q}} \arrow[r, "\mathrm{cl}_{\mathrm{univ}}"] \arrow[rr, bend right, "\mathrm{cl}'_{\mathrm{trop}}"'] & H^p(X, \mathscr{K}^p_X) \arrow[r] & H^p(X^{\mathrm{an}}, \mathscr{D}^{p,\bullet}_X) & \end{tikzcd}$$

and Paragraph A.10.

For each α, we denote $c_\alpha := c|_{U_\alpha^{\mathrm{an}}} \in \mathscr{D}^{p,p,\mathrm{cl}}_X(U_\alpha^{\mathrm{an}})$. Note that as the maps in Eq. (A.19.1) are morphisms of sheaves by Paragraphs A.16 and A.17, c_α arises as the image of $c(Z)_\alpha$ under the map

$$H^p_Z(U_\alpha, \mathscr{K}^p_X) \longrightarrow H^p_{Z^{\mathrm{an}}}(U_\alpha^{\mathrm{an}}, \mathscr{D}^{p,\bullet}_X) \longrightarrow \mathscr{D}^{p,p,\mathrm{cl}}_X(U_\alpha^{\mathrm{an}}).$$

A.20 Lemma. *In the situation of Paragraph A.19, the current $c \in \mathscr{D}^{p,p,\mathrm{cl}}$ agrees with the current of integration δ_Z.*

Proof. Since currents form a sheaf, we may restrict to U_α so that it suffices to prove that $c_\alpha = \delta_{Z \cap U_\alpha}$. In other words, we want to show that for each $\omega \in \mathscr{A}_c^{n-p,n-p}(U_\alpha^{\mathrm{an}})$ (where n is the dimension of X), we have

$$\langle c_\alpha, \omega \rangle = \int_{Z \cap U_\alpha} \omega.$$

By construction, c_α arises as the image of $\{f_{\alpha 1}, \dots, f_{\alpha p}\}$ under the composite map

$$H^{p-1}(U_\alpha \setminus Z, \mathscr{K}_X^p) \xrightarrow{\delta} H_Z^p(U_\alpha, \mathscr{K}_X^p) \longrightarrow H_{Z^{\mathrm{an}}}(U_\alpha^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}) \longrightarrow \mathscr{D}_X^{p,p,\mathrm{cl}}(U_\alpha^{\mathrm{an}}).$$

If we pick a form $\theta_\alpha \in \mathscr{A}^{p,p-1,\mathrm{cl}}(U_\alpha^{\mathrm{an}} \setminus Z^{\mathrm{an}})$ representing the class

$$\tau_{X^{\mathrm{an}}}^p(\{f_{\alpha 1}, \dots, f_{\alpha p}\}) \in H^{p-1}(U_\alpha^{\mathrm{an}} \setminus Z^{\mathrm{an}}, \mathscr{A}_X^{p,\bullet})$$

then we see from the commutative diagram

$$\begin{tikzcd}
H^{p-1}(U_\alpha \setminus Z, \mathscr{K}_X^p) \arrow[r, "\delta"] \arrow[d, "\tau_{X^{\mathrm{an}}}^p"'] & H_Z^p(U_\alpha, \mathscr{K}_X^p) \arrow[r] \arrow[d, "\tau_{X^{\mathrm{an}}}^p"] & H_{Z^{\mathrm{an}}}(U_\alpha^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}) \arrow[r] & \mathscr{D}_X^{p,p,\mathrm{cl}}(U_\alpha^{\mathrm{an}}) \\
H^{p-1}(U_\alpha^{\mathrm{an}} \setminus Z^{\mathrm{an}}, \mathscr{A}_X^{p,\bullet}) \arrow[r, "\delta"'] \arrow[d, "{[]}"'] & H_{Z^{\mathrm{an}}}^p(U_\alpha^{\mathrm{an}}, \mathscr{A}_X^{p,\bullet}) \arrow[ur, "{[]}"'] \arrow[d, "{[]}"] & & \\
H^{p-1}(U_\alpha^{\mathrm{an}} \setminus Z^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}) \arrow[r, "\delta"'] & H_{Z^{\mathrm{an}}}^p(U_\alpha^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}) \arrow[uur, equal, bend right] & &
\end{tikzcd}$$

that c_α is obtained as the image of

$$\delta([\theta_\alpha]) \in H_{Z^{\mathrm{an}}}^p(U_\alpha^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}).$$

Denoting its image in $\mathscr{D}_X^{p,p,\mathrm{cl}}(U_\alpha^{\mathrm{an}})$ by $\delta([\theta_\alpha])$ again, what we want to show is the equality

$$\langle \delta([\theta_\alpha]), \omega \rangle = \int_{Z \cap U_\alpha} \omega$$

for $\omega \in \mathscr{A}_c^{n-p,n-p}(U_\alpha^{\mathrm{an}})$. This is precisely the content of Step 3 in the proof of [Liu20, Thm. 3.9]. ∎

A.21 Theorem. *Let $Z \subset X$ be an integral closed subscheme of codimension p. Then the class $\mathrm{cl}'_{\mathrm{trop}}(Z) \in H^p(X^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet})$ is represented by the current $\delta_Z \in \mathscr{D}^{p,p,\mathrm{cl}}(X^{\mathrm{an}})$*

Proof. After base change to a finite extension of K, we may assume that Z is generically smooth over K. Let $Z_{\mathrm{sing}} \subset Z$ be the singular locus, which is a proper closed subscheme of Z and hence a closed subscheme of X of codimension $> p$. Put $U \coloneqq X \setminus Z_{\mathrm{sing}}$ and denote by $j\colon U \hookrightarrow X$ the inclusion.

By the functoriality of the tropical cycle class map, we have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^p(X)_{\mathbf{Q}} \arrow[r, "\mathrm{cl}_{\mathrm{trop}}"] \arrow[d, "j^*"'] & H^p(X^{\mathrm{an}}, \mathscr{A}_X^{p,\bullet}) \arrow[d, "j^*"] \\ \mathrm{CH}^p(U)_{\mathbf{Q}} \arrow[r, "\mathrm{cl}_{\mathrm{trop}}"'] & H^p(U^{\mathrm{an}}, \mathscr{A}_U^{p,\bullet}). \end{tikzcd}$$

Note also that restriction of currents induces a map

$$j^*\colon H^p(X^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}) \to H^p(U^{\mathrm{an}}, \mathscr{D}_U^{p,\bullet})$$

which is an isomorphism by [CD12, Lem. 3.2.5] (compare the proof of [GK17, Prop. 6.5]) and which fits into a commutative diagram

$$\begin{tikzcd} H^p(X^{\mathrm{an}}, \mathscr{A}_X^{p,\bullet}) \arrow[r, "{[]}"] \arrow[d, "j^*"'] & H^p(X^{\mathrm{an}}, \mathscr{D}_X^{p,\bullet}) \arrow[d, "j^*"] \\ H^p(U^{\mathrm{an}}, \mathscr{A}_U^{p,\bullet}) \arrow[r, "{[]}"'] & H^p(U^{\mathrm{an}}, \mathscr{D}_U^{p,\bullet}). \end{tikzcd}$$

It follows that it is enough to show that $\mathrm{cl}_{\mathrm{trop}}(Z \cap U)$ is represented by the current $\delta_{Z\cap U}$ in $H^p(U^{\mathrm{an}}, \mathscr{D}_U^{p,\bullet})$. Replacing X by U and Z by $Z \cap U$, we may therefore assume that Z is smooth. In this case, the result follows from Paragraph A.19 and Lemma A.20. ∎

A.22 Corollary. *Let X be a smooth algebraic K-variety. Then the diagram*

$$\begin{tikzcd} \mathrm{CH}^p(X) \arrow[r, "\delta"] \arrow[d, "\mathrm{cl}_{\mathrm{trop}}"'] & \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}}) \arrow[d] \\ H^{p,p}_{\mathscr{A}}(X^{\mathrm{an}}) \arrow[r, "{[-]}"'] & H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}}) \end{tikzcd}$$

is commutative.

Proof. By linearity, it suffices to consider prime cycles in $\mathrm{CH}^p(X)$ represented by a closed subvariety Z of codimension p. The composition

$$\mathrm{CH}^p(X) \xrightarrow{\mathrm{cl}_{\mathrm{trop}}} H^{p,p}_{\mathscr{A}}(X^{\mathrm{an}}) \xrightarrow{[-]} H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}})$$

is exactly the map $\mathrm{cl}'_{\mathrm{trop}}$ of Paragraph A.18. The composition

$$\mathrm{CH}^p(X) \xrightarrow{\ \delta\ } \hat{H}^p_{\mathscr{D}}(X^{\mathrm{an}}) \longrightarrow H^{p,p}_{\mathscr{D}}(X^{\mathrm{an}})$$

maps Z to the class represented by the current of integration δ_Z. Hence the result follows from Theorem A.21. ∎

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