Deligne's Conjecture on the Critical Values of Hecke L-Functions

Dissertation zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)
der Fakultät für Mathematik
der Universität Regensburg

vorgelegt von
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aus Regensburg
im Jahr 2023

Promotionsgesuch eingereicht am:	4. Oktober 2023
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Abstract. We give a proof of Deligne's conjecture on the critical values of L-functions for arbitrary algebraic Hecke characters. This is achieved by following the general strategy of the proof in the CM-case due to Blasius, [Bla86], and making use of the recently constructed Eisenstein-Kronecker classes of Kings-Sprang [KS19].

Contents

Introduction

Let χ be an algebraic Hecke character of a number field L and of conductor dividing $\mathfrak{f}$. Let T denote the number field generated by the values of χ. For every embedding $\iota : T \hookrightarrow \mathbf{C}$, one can consider the Dirichlet series

$$L(\chi_\iota, s) = \sum_{\substack{0 \neq \mathfrak{a} \subset \mathcal{O}_L \\ (\mathfrak{f},\mathfrak{a})=1}} \frac{(\iota \circ \chi)(\mathfrak{a})}{N\mathfrak{a}^s},$$

which converges for any $s \in \mathbf{C}$ with sufficiently large real part and defines a holomorphic function on its domain of convergence. Let $L(\chi, s) := (L(\chi_\iota, s))_\iota$ denote the $T \otimes \mathbf{C} \cong \mathbf{C}^{\mathrm{Hom}(T,\mathbf{C})}$-valued L-function. The function $L(\chi, s)$ admits a meromorphic continuation and satisfies a functional equation. We say that χ is *critical* if the Γ-factors on both sides of the functional equation have no pole at $s = 0$. For critical values to exist, it is necessary that L is either totally real or contains a CM-field.

For every algebraic Hecke character χ there exists a motive $M(\chi)$ (for absolute Hodge cycles and generated by abelian varieties and Artin motives) defined over L with coefficients in T whose associated L-function coincides with $L(\chi, s)$. Let $RM(\chi)$ denote the restriction of scalars of $M(\chi)$ from L to $\mathbf{Q}$, $RM(\chi)^+_B$ the subspace in the Betti realization fixed by the involution induced by complex conjugation and let $F^\bullet$ denote the Hodge filtration. If χ is critical, the composite

$$I_+ : RM(\chi)^+_B \otimes \mathbf{C} \subset RM(\chi)_B \otimes \mathbf{C} \xrightarrow{I_\infty^{-1}} RM(\chi)_{\mathrm{dR}} \otimes \mathbf{C} \to RM(\chi)_{\mathrm{dR}}/F^0 RM(\chi)_{\mathrm{dR}} \otimes \mathbf{C},$$

where I_∞ denotes the comparison isomorphism between the de Rham and the Betti realization, is an isomorphism between finite free $T \otimes \mathbf{C}$-modules. This allows to define a period

$$c^+(\chi) = \det(I_+) \in (T \otimes \mathbf{C})^\times,$$

where the determinant is calculated with respect to bases of the T-vector spaces $RM(\chi)_B^+$ and $RM(\chi)_{\mathrm{dR}}/F^0 RM(\chi)_{\mathrm{dR}}$. Note that $c^+(\chi)$ is only well-defined up to a factor in $T^\times$. The goal of this thesis is to show the following theorem, which verifies a conjecture of Deligne ([Del79], Conjecture 2.8) for the motive $M(\chi)$:

Theorem (Deligne conjecture for χ, Theorem 11.7)**.** *Let L and T be number fields. Let χ be a critical algebraic Hecke character of L with values in T. Then $L(\chi, 0)$ coincides with $c^+(\chi)$ up to a factor in T.*

In the case where L is totally real, the above result follows from the work of Euler, Siegel, [Sie37], and Klingen, [Kli62]. When L is an imaginary quadratic field, Damerell, [Dam70], showed that the critical L-values agree with certain periods of CM elliptic curves up to an algebraic number. Deligne's conjecture was settled in the imaginary quadratic case by Goldstein-Schappacher in [GS81] and [GS83]. If L is a CM-field, algebraicity results were obtained by Shimura, [Shi75], and Katz, [Kat78]. It was shown by Deligne, [Del79], that the results of Shimura are compatible with his conjecture. In [Bla86], Blasius was able to prove Deligne's conjecture for algebraic Hecke characters of CM-fields. If L contains an imaginary quadratic number field, algebraicity results have previously been obtained by Colmez, [Col89], and more recently by Bergeron-Charollois-Garcia, [BCG23]. For arbitrary number fields L containing a CM-field, Kings-Sprang, [KS19], were able to relate the L-value $L(\chi, 0)$ to periods of abelian varieties with complex multiplication by L up to an algebraic integer. This allowed them to establish Deligne's conjecture up to a factor in $T \otimes \overline{\mathbf{Q}}$. Central to their approach is a novel construction of equivariant coherent cohomology classes attached to CM abelian varieties. For arbitrary L containing a CM-field K, Deligne's conjecture was announced by Harder-Schappacher, [HS85], but details were never published. Roughly speaking, the strategy outlined in loc. cit. is to use results on the Eisenstein cohomology for GL_n, where $n = [L : K]$, to show that the ratio $L(\chi, 0)/L(\chi|_{\mathbb{A}_K^\times}, 0)$ behaves exactly like the corresponding ratio between the c^+-periods. This would reduce the general case to the CM-case established by Blasius. For $n = 2$, the necessary theory is developed in [Har87] and in this case one obtains unconditional results.

In our approach we combine the strategy of Blasius used in [Bla86] but instead of Eisenstein series on Hilbert modular varieties, which are only available in the CM-case, we make use of the Eisenstein-Kronecker classes of Kings-Sprang, [KS19], which allow for a cohomological interpretation of the value $L(\chi, 0)$ for Hecke characters χ of arbitrary totally imaginary fields. We also rely on ideas of Goldstein-Schappacher developed in [GS81].

Outline of the argument. Let us first give a quick summary of the proof presented in this thesis, before we give a slightly more detailed sketch. Let L be a totally imaginary number containing a CM-field and let χ be a critical algebraic Hecke character of L. The classes constructed in [KS19], which allow to access $L(\chi, 0)$, live on abelian varieties with complex multiplication by the field L. In contrast, the motive $M(\chi)$ has L as its field of definition: A prototypical example for such a motive would be the h^1 of a CM abelian variety defined over L. Thus, the question arises, how to relate the motive $M(\chi)$ and the period $c^+(\chi)$ to such abelian varieties with CM by L. It is a seminal insight of Blasius how to establish this connection: From $M(\chi)$ he constructs a motive $M(\Xi)$, referred to as the reflex motive in the following, and recovers the period $c^+(\chi)$ (up to a factor in $T^\times$) in terms of another period of $M(\Xi)$, which we call the reflex period for a moment. The reflex motive $M(\Xi)$ is defined over the reflex field of a certain CM-type of L and the key point, as our notation already suggests, is that it is the motive for a Hecke character Ξ, which essentially is obtained from χ by precomposing with the reflex norm. This makes it possible to give an alternative construction of $M(\Xi)$ in terms of an abelian variety with complex multiplication by L. The construction shows that the Eisenstein-Kronecker classes of Kings-Sprang naturally live in the de Rham realization $M(\Xi)_{\mathrm{dR}}$ and they are admissible for computing the reflex period. By giving an explicit presentation, in terms of the alternative construction of

$M(\Xi)$, of those Betti classes which are needed to calculate the reflex period and slightly extending results of Kings-Sprang, we show that the reflex period coincides with the L-value $L(\chi, 0)$. Deligne's conjecture then follows from Blasius' period relation.

We now give a more detailed outline of the above argument. Let us fix the following notations: For any number field L, we let J_L denote the set of embeddings $L \hookrightarrow \mathbf{C}$ and let I_L denote the free abelian group generated by J_L. For any element $\mu = \sum_\sigma \mu(\sigma)\sigma$ in I_L we let $d(\mu) = \sum_\sigma \mu(\sigma)$ denote its degree. If $\ell \in L$, we write $\ell^\mu := \prod_\sigma \sigma(\ell)^{\mu(\sigma)}$. Moreover, we let L^μ denote the number field generated by the elements ℓ^μ for $\ell \in L$. Let L_f denote the ring of finite adeles of L.

Blasius' period relation. The condition that χ is critical can be entirely expressed in terms of its infinity-type $\chi_a \in I_L$: There exists a CM-type Φ of L, induced from a CM subfield of L, such that χ_a can be split up in the form

$$\chi_a = \beta - \alpha,$$

where α (resp. β) is supported on Φ (resp. $\overline{\Phi}$) and such that $\alpha(\sigma) \geq 1$ and $\beta(\overline{\sigma}) \geq 0$ for all $\sigma \in \Phi$. Let E denote the reflex field of (L, Φ) and Φ^* denote the reflex CM-type of Φ. Then, Φ^* gives rise to a continuous homomorphism $\Phi^* : E_f^\times \to L_f^\times$ and we define a Hecke character (cf. (6.4))

$$\Xi = (\chi \circ \Phi^*)^{-1} \|\cdot\|^{-d(\beta)} \varepsilon_\Phi : E_f^\times \to T^\times,$$

where $\|\cdot\|$ is the idele norm and ε_Φ is a certain sign character attached to Φ (cf. Definition 6.3). Attached to Ξ there is a motive $M(\Xi)$ over E with coefficients in T. For any motive, let I_f denote the comparison isomorphism between the Betti and the finite adelic realization. One can show the following result about the existence of a certain basis of $RM(\Xi)_B$:

Proposition (cf. Proposition 6.6)**.** *For any embedding η of E and any element $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, define the elements $\xi(\tau, \eta) \in T_f^\times$ as in (6.5). There exists a system of T-basis elements $a_\eta \in M(\Xi)_B^\eta$ such that*

$$I_f(a_\eta)^\tau = \xi(\tau, \eta) \cdot I_f(a_{\tau\eta}).$$

The following theorem expresses the period $c^+(\chi)$ in terms of the motive $RM(\Xi)$. For every embedding η of E let $e(\eta) \in T \otimes \mathbf{C}$ denote the idempotent, which singles out the coordinates indexed by the embeddings restricting to η under the identification $T \otimes \mathbf{C} \cong \mathbf{C}^{\mathrm{Hom}(T, \mathbf{C})}$.

Theorem (Blasius, cf. Theorem 6.7)**.** *The subspace $F^{d(\alpha)+d(\beta)} RM(\Xi)_{\mathrm{dR}}$ is a 1-dimensional T-vector space. Let $\{a_\eta\}$ be a system as in the previous proposition. For any non-zero element $\omega \in F^{d(\alpha)+d(\beta)} RM(\Xi)_{\mathrm{dR}}$, one has*

$$I_\infty(\omega) = (2\pi i)^{d(\beta)} c^+(\chi) \cdot \sum_{\eta \in J_E} e(\eta) a_\eta \mod T^\times.$$

We give an exposition on Blasius' proof of this theorem in Appendix A. The result may be seen as the starting point of our proof. It allows to shift the focus from the motive $M(\chi)$ to the motive $M(\Xi)$. We now explain how to give a more geometric construction of $M(\Xi)$.

Relation to CM abelian varieties. Let F/E be a finite extension and A/F an abelian variety of CM-type (L, Φ) with CM-character $\psi : \mathbb{A}_F^\times \to L^\times$. The dual abelian variety $A^\vee$ canonically acquires a CM-structure from A and has CM-character $\overline{\psi}$. We assume that T contains L^α and L^β. Consider the algebraic Hecke character $\psi^\alpha : \mathbb{A}_L^\times \to T^\times$ given by $\psi^\alpha(x) = \psi(x)^\alpha$ for $x \in \mathbb{A}_L^\times$, and similarly $\overline{\psi}^\beta : \mathbb{A}_L^\times \to T^\times$. By comparing the infinity-types, it follows that for sufficiently large F/E, one has

$$\Xi \circ N_{F/E} = \psi^\alpha \overline{\psi}^\beta. \tag{0.1}$$

Such an equality of Hecke characters should have a counterpart on the level of motives. By definition, the motive $h^1(A)$ over F with coefficients in L is a motive for ψ. From this motive, one can construct a motive $h^1(A)^\alpha$ for the Hecke character ψ^α by considering a suitable direct summand in

$$\mathrm{Sym}_{\mathbf{Q}}^{d(\alpha)}(R_{L/\mathbf{Q}} h^1(A)),$$

where $R_{L/\mathbf{Q}}$ denotes restriction of coefficients (cf. Chapter 8). In the same way, one obtains a motive $h^1(A^\vee)^\beta$ for the Hecke character $\overline{\psi}^\beta$. Extending a method of Goldstein-Schappacher,

[GS81], to motives (cf. Theorem 4.1), the above identity (0.1) allows to recover $M(\Xi)$ as a suitable direct summand in
$$R_{F/E}(h^1(A)^\alpha \otimes_T h^1(A^\vee)^\beta),$$
where $R_{F/E}$ denotes restriction of scalars (cf. Chapter 9).

Special values of L-functions. With the above construction of $M(\Xi)$ at hand and in view of Blasius' theorem, the strategy is now as follows:

- Find a presentation for the special basis elements $\{a_\eta \mid \eta \in J_E\}$ in terms of Betti classes of the motive $h^1(A)$. This is done in Section 9.2 of Chapter 9.
- Construct an element in $F^{d(\alpha)+d(\beta)}RM(\Xi)_{\mathrm{dR}}$, which admits a relation to the L-values $L(\chi, 0)$. This is provided by taking a suitable linear combination of the Eisenstein-Kronecker classes constructed by Kings-Sprang, cf. Chapter 10 and the discussion below.

The construction of the basis elements a_η roughly goes as follows: By construction of the motive $h^1(A)^\alpha$, one can canonically attach to every element $\gamma \in h^1_B(A)$ an element $\gamma^{\boldsymbol{\alpha}} \in h^1(A)^\alpha_B$. For every embedding σ of F, we choose an L-basis γ_σ of $h^1_B(A^\sigma)$ and let $\gamma_\sigma^\vee \in h^1_B(A^{\sigma,\vee})$ denote dual basis with respect to the canonical pairing $h^1_B(A^\sigma) \otimes h^1_B(A^{\sigma,\vee}) \to (2\pi i)^{-1}L$. Then, for every $\eta \in J_E$, one defines a_η as the image of
$$\sum_{\sigma \in J_{F,\eta}} t_\sigma \cdot \gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}} \in R_{F/E}(h^1(A)^\alpha \otimes_T h^1(A^\vee)^\beta)^\eta_B$$
in $M(\Xi)^\eta_B$. Here $t_\sigma \in T^\times$ is a certain factor depending on the choice of $\{\gamma_\sigma \mid \sigma \in J_F\}$, cf. Definition 9.4. Let us just remark that the t_σ are similarly defined as the $\xi(\tau, \eta)$. The following theorem shows that the elements a_η constructed above are in fact suitable for applying Blasius' theorem. Let us remark that the first part of the theorem is the more difficult one.

Theorem (cf. Theorem 9.6)**.** *The elements $\{a_\eta\}_{\eta \in J_E}$ form a system of basis elements of $M(\Xi)^\eta_B$. For any $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, one has $I_f(a_\eta)^\tau = \xi(\tau, \eta) \cdot I_f(a_{\tau\eta})$.*

As for the de Rham realization of $M(\Xi)_{\mathrm{dR}}$ we have the following theorem, which is based on results of Kings-Sprang:

Theorem (cf. Theorem 10.21 and Theorem 11.6)**.** *There is a class $\mathcal{EK} \in F^{d(\alpha)+d(\beta)}(h^1(A)^\alpha \otimes h^1(A^\vee)^\beta)_{\mathrm{dR}}$ such that*
$$I_\infty(\mathcal{EK}^\sigma) = t_\sigma \cdot (2\pi i)^{d(\beta)} e(\sigma|_E) \cdot L(\chi, 0) \cdot \gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}}.$$

As the de Rham realization of $RM(\Xi)_{\mathrm{dR}}$ is just $M(\Xi)_{\mathrm{dR}}$ regarded as a T-vector space, the element $\mathcal{EK}$ gives rise to an element in $RM(\Xi)_{\mathrm{dR}}$ which we denote by the same letter. By the above theorem, we have
$$\begin{aligned} I_{\infty, RM(\Xi)}(\mathcal{EK}) &= \sum_{\sigma \in J_F} I_\infty(\mathcal{EK}^\sigma) \\ &= (2\pi i)^{d(\beta)} \cdot L(\chi, 0) \cdot \sum_{\eta \in J_E} e(\eta) a_\eta, \end{aligned}$$
and Blasius' theorem then implies
$$L(\chi, 0) = c^+(\chi) \mod T^\times,$$
as desired.

Overview. Let us give a chapterwise overview of the thesis.

Chapter 1: Types. In this chapter, we recall basic constructions with what we call *types*. These are simply elements in the free abelian group I_L generated by the embeddings $J_L := \mathrm{Hom}(L, \mathbf{C})$ of a number field L. Later on, they will naturally arise as the infinity-types of algebraic Hecke characters. We usually write such elements in the form $\mu = \sum_{\sigma \in J_L} \mu(\sigma)\sigma$. There is an obvious Galois action of $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on I_L. If μ is a type, we define its field of values to be the fixed field of the stabilizer $\{\tau \in G_{\mathbf{Q}} \mid \tau\mu = \mu\}$. If $K \subset L$ is a subfield and $\kappa \in I_K$ is a type on K, we define its induced type $\mathrm{Inf}_{L/K}(\kappa) \in I_L$ by $\mathrm{Inf}_{L/K}(\kappa)(\sigma) = \kappa(\sigma|_K)$ for all $\sigma \in J_L$. We show that for any type $\mu \in I_L$ there exists a unique minimal subfield, the minimal field of definition, from which μ is induced. For a given type $\mu \in I_L$, we define the reflex type μ^*, which is defined on the field of values L^μ, and show some basic properties. This construction plays an important role since the Hodge structure of a motive attached to a Hecke character χ will be determined in terms of the reflex of its infinity-type. We proceed to recall basic facts about CM-fields and CM-types. Finally, we discuss those types which arise as infinity-types of algebraic Hecke character. They are precisely those types $\mu \in I_L$ which are induced from a totally real or CM-subfield and which admit a weight, i.e. $\mu(\sigma) + \mu(\overline{\sigma}) = w$ for a fixed integer $w \in \mathbb{Z}$. The results of this chapter are mostly based on [Bla86], chapter 0. They are entirely elementary and well-known but we write down detailed proofs for the sake of completeness.

Chapter 2: Algebraic Hecke characters. Let T be a number field. An algebraic homomorphisms is a homomorphisms $L^\times \to T^\times$ which is given by $\ell \mapsto \ell^\mu$ for a type $\mu \in I_L$. We define an algebraic Hecke character as a continuous homomorphism $\chi : \mathbb{A}_L^\times \to T^\times$ such that its restriction to the principal ideles $L^\times$ is an algebraic homomorphism. The type in I_L which describes $\chi|_{L^\times}$ is the infinity-type χ_a of χ. Examples include finite abelian characters of L via class field theory or, if L is totally complex, the idele norm. We then give an equivalent definition in terms of ideals. From this perspective an algebraic Hecke character is a homomorphism $\chi : I^{\mathfrak{f}} \to T^\times$ on the group of fractional ideals coprime to a fixed ideal $\mathfrak{f} \subset \mathcal{O}_L$ whose value on a principal ideal (α) with $\alpha \in L^\times$ totally positive and $\alpha \equiv 1 \mod {}^\times\mathfrak{f}$ is given by α^μ for a fixed type μ. This is the classical definition of algebraic Hecke characters and we briefly explain how to pass from one perspective to the other. Finally, we define for every embedding ι of T the Hecke L-function $L_{\mathfrak{f}}(\chi_\iota, s) = \sum_{\mathfrak{a}} \frac{(\iota\circ\chi)(\mathfrak{a})}{N\mathfrak{a}^s}$, where the sum ranges over all integral ideals coprime to $\mathfrak{f}$, and, by putting them together, a $T \otimes \mathbf{C}$-valued L-function $L_{\mathfrak{f}}(\chi, s)$. We also introduce partial L-functions. Again, all results of this chapter are well-known and the main purpose is to collect the facts we need and to fix notation.

Chapter 3: CM-Motives. In this chapter we collect facts about the category of motives that will be used in this work. First, we recall motives for absolute Hodge cycles and that they may be fully faithfully embedded into a category of compatible systems of realizations. We recall the fact that a morphism $\varphi : M \times_L L' \to N \times_L L'$ descends to L if and only if its étale realization φ_f is $\mathrm{Gal}(\overline{L}/L)$-equivariant. We then introduce the cateory CM_L of CM-motives over L, which is the Tannakian category generated by the motives of potentially CM abelian varieties and Artin motives. By using the above descent result and Deligne's theorem that over an algebraically closed field every Hodge cycle on an abelian variety is absolute Hodge, we show that the category CM_L embeds fully faithfull in an even simpler category of compatible systems of realizations. We recall extension and restriction of scalars and of coefficients. We define the L-function of a motive as well as the period $c^+(M)$ and formulate Deligne's conjecture on the critical values of such L-functions. Finally, we discuss the correspondence between motives of rank 1 and algebraic Hecke characters.

Chapter 4: On a result of Goldstein-Schappacher. Let E/F be an elliptic curve over a number field F with CM by an imaginary quadratic field K. A theorem of Goldstein-Schappacher, [GS81] asserts that the Weil restriction $\mathrm{Res}_{F/K}\, E$ acquires complex multiplication if and only if its CM-character factors via the norm $N_{F/K}$ through a Hecke character of K. By a theorem of Shimura, [Shi94] the latter condition is equivalent to the fact that the field $F(E_{\mathrm{tor}})$ generated by the torsion points of E is an abelian extension over K. We give a generalization of this theorem to motives of rank 1. More precisely, let $\psi : F_f^\times \to K^\times$ be an algebraic Hecke character and let $E \subset F$ be a subfield containing the minimal field of definition of ψ_a such that F/E is Galois. Let M be the motive attached to ψ. We show that the restriction of scalars $R_{F/E}M$ acquires coefficients in a commutative semisimple K-algebra over which $R_{F/E}M$ is of rank 1 if and only if F/E is an abelian extension and the Hecke character ψ factors via the norm $N_{F/E}$ through an algebraic Hecke character of E. In analogy to Shimura's theorem, we also prove that this is the case if and only if the Galois representation $R_{F/E}(M)_f$ is abelian. The proof works completely analogously as the one given by Shimura and Goldstein-Schappacher.

Chapter 5: Abelian varieties with complex multiplication. We recall basic facts about abelian varieties with complex mutliplication. We then use the result of Chapter 4 and a Theorem of Casselman to construct CM abelian varieties with favourable properties. We recall the main theorem of complex multiplication for general automorphisms which do not necessarily fix the reflex field.

Chapter 6: Blasius' period relation. In this chapter, we first introduce the Hecke characters we will be interested in throughtout the rest of the thesis. We fix a Hecke character $\chi : L_f^\times \to T^\times$ which is critical with respect to a CM-type Φ on L. We attach to it a Hecke character $\Xi : E_f^\times \to T^\times$ defined on the reflex field E of (L, Φ), essentially by precomposing χ with the reflex norm $\Phi^* : E_f^\times \to L_f^\times$. Let $M(\Xi)$ be a motive for the Hecke character Ξ. We state a theorem of Blasius, which expresses Deligne's period $c^+(\chi)$ in terms of a period of $M(\Xi)$. Finally, we give a slight refinement of the construction of well-behaved CM abelian varieties in Chapter 5 to satisfy a few extra conditions related to our setup.

Chapter 7: Algebraic $\mathbf{T}_{L,F}$-modules. For a number fields L and F, we recall basic facts about algebraic $\mathbf{T}_{L,F}$-modules, i.e. representations of the algebraic torus $\mathbf{T}_{L,F} = \operatorname{Res}_{L/\mathbf{Q}} \mathbf{G}_m \times F$. Any such representation decomposes into eigenspaces indexed by the G_F-orbits of the character group I_L of $\mathbf{T}_{L,F}$. We discuss an imporant, although trivial, example: Any $L \otimes F$-module V gives rise to a $\mathbf{T}_{L,F}$-module and similarly the symmetric powers $\operatorname{Sym}_F^d(V)$ of such a module acquires the structure of a $\mathbf{T}_{L,F}$-module. Given a type $\alpha \in I_L^+$ of degree $d(\alpha)$, we define V^α to be the eigenspace in the $\mathbf{T}_{L,F}$-module $\operatorname{Sym}_F^{d(\alpha)}(V)$ defined above corresponding to the G_F-orbits occuring in the full $G_{\mathbf{Q}}$-orbit of α. We explain how to equip V^α with the structure of an $L^\alpha \otimes F$-module and study basic properties of the assignment $V \mapsto V^\alpha$. The contents of this chapter are elementary and the formalism developed in this chapter is not strictly necessary. In fact, Galois descent alone is completely sufficient. We choose this presentation because it is more clean and it highlights the connection to the results in [KS19], 1.1.

Chapter 8: The motive M^α. Let M be a motive attached to a Hecke character $\psi : F_f^\times \to L^\times$. Let $\alpha \in I_L^+$ be a type. We use the construction of the previous chapter to build a motive M^α from M which corresponds to the Hecke character ψ^α. In each realization, the motive M^α is given by applying the functor $V \mapsto V^\alpha$ to the corresponding category of modules. We describe a canonical way to attach an element $\gamma^{\boldsymbol{\alpha}}$ in the Betti realization of M^α for any given element γ in M_B. Similarly, we construct de Rham classes $\omega^{\boldsymbol{\alpha}}$ for M^α for every element $\omega \in M_{\mathrm{dR}}$. Finally, we express the periods relating the elements $\gamma^{\boldsymbol{\alpha}}$ and $\omega^{\boldsymbol{\alpha}}$ in terms of the period relating γ and ω.

Chapter 9: A motive for Ξ. The infinity-type of a critical Hecke character χ can be canonically decomposed in the form $\chi_a = \beta - \alpha$ with certain types $\alpha, \beta \in I_L^+$. The results of Chapter 5 (resp. its refinement in Chapter 6) show that there is an abelian extension F/E and an abelian variety A/F with complex multiplication by L such that its CM-character ψ satisfies $\Xi \circ N_{F/E} = \psi^\alpha \overline{\psi}^\beta$. Based on this observation, we use the results of Chapter 4 to construct a motive $M(\Xi)$ for the Hecke character Ξ as a suitable direct summand of $R_{F/E}(h^1(A)^\alpha \otimes h^1(A^\vee)^\beta)$. We then investigate its Betti realization more closely. By using the elements $\gamma^{\boldsymbol{\alpha}}$ constructed in Chapter 9, we give an explicit presentation of the elements a_η, $\eta \in J_E$ which are used in the alternative description of $c^+(\chi)$ given in Blasius' Theorem.

Chapter 10: Eisenstein-Kronecker classes. We recall properties of the Eisenstein–Kronecker classes defined by Kings–Sprang in [KS19]. In particular, we formulate results on their behaviour under conjugation. We recall their analytic description in terms of Eisenstein–Kronecker series and the relation to Hecke L-functions. We give an extension of a result in loc. cit., which intimately links the Eisenstein–Kronecker classes with the special values of Hecke L-functions, to include all conjugates of the EK-classes as well.

Chapter 11: Deligne's conjecture for Hecke characters. We show that the Eisenstein-Kronecker classes naturally live in the de Rham realization of $h^1(A)^\alpha \otimes h^1(A^\vee)^\beta$. We define a normalized class $\mathcal{EK}$ as a suitable linear combination. The results of Chapter 8 and Chapter 10 allow to reinterpret the L-value $L(\chi, 0)$ as a certain period of $RM(\Xi)$. The main result of Chapter 9 identifies the Betti class in this period construction with the Betti class occurring in Blasius Theorem. Deligne's conjecture for algebraic Hecke characters follows.

Appendix A: Proof of Blasius' Theorem. Following [Bla86] closely, we give an exposition on the proof of Blasius' Theorem.

Appendix B: Equivariant Cohomology. Basic facts about equivariant sheaf cohomology for a discrete group Γ are established. In particular it is shown that associated with a decomposition of a (ringed) space with Γ-action into a Γ-invariant closed and open part, there is a localization sequence for equivariant cohomology.

Appendix C: Eisenstein-Kronecker classes and base change. In this technical Appendix, we recall the construction of the (equivariant) Eisenstein–Kronecker class and show that its formation is compatible under base change. Our presentation closely follows [KS19] but slightly deviates from it in the following point: Instead of working with the completion $\widehat{\mathcal{P}}$ of the Poincaré bundle, we consider each "finite stage" $\mathcal{P}_n$, given by the restriction of $\mathcal{P}$ to the nth infinitesimal neighbourhood of $A \times A^\vee$ along the unit section $\mathrm{id}_A \times e^\vee$, regarded as a sheaf of modules over $\mathcal{O}_A$. The reason is that the sheaf $\widehat{\mathcal{P}}$ is not quasi-coherent which we believe to cause certain technical problems.

Acknowledgements. I would like to thank my advisor Guido Kings for his constant support. I am very grateful for his patience and the valuable feedback he provided throughout the years. I wish to thank my friends and colleagues Niklas Kipp, Julio de Mello Bezerra, Bernadette Melichar, Kim Nguyen, Lukas Prader, Benedikt Preis, Chiara Sabadin, Johannes Sprang, Florian Strunk, Georg Tamme, Marco Volpe, Jakob Werner and Sebastian Wolf for all the interesting discussions and for making the last years so enjoyable. I am especially grateful to Florian, Johannes and Georg for offering advice on many occasions. I am deeply indebted to Werner Bauer for recognizing my interest in mathematics in the past and for finding a way for me to pursue it. I want to thank my parents and siblings for their unconditional love and support. Finally, I want to express my deepest gratitude to Günter Tamme for everything I have learned from him, the encouragement and all the beautiful memories.

Notation and conventions. We assume throughout this text that number fields are embedded into $\mathbf{C}$. We let $\overline{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. In particular, complex conjugation on $\mathbf{C}$ restricts to an automorphism of $\overline{\mathbf{Q}}$ which we denote by the letter c or $\alpha \mapsto \overline{\alpha}$ for $\alpha \in \overline{\mathbf{Q}}$. For a number field L we let $J_L := \mathrm{Hom}(L, \mathbf{C}) = \mathrm{Hom}(L, \overline{\mathbf{Q}})$ denote the set of embeddings into $\mathbf{C}$ (equivalently, into $\overline{\mathbf{Q}}$) and let $I_L := \mathbb{Z}[J_L]$ denote the free abelian group generated on J_L. We denote the fixed embedding of L into $\mathbf{C}$ (or $\overline{\mathbf{Q}}$) by 1_L. If $K \subset L$ is a subfield and $\sigma \in J_E$, we denote by $J_{L,\sigma}$ the set of those embeddings in J_L which restrict to σ on K. We also write $J_{L/K}$ instead of $J_{L,1_K}$. For any element $\mu = \sum_{\sigma \in J_L} \mu(\sigma)\sigma \in I_L$, we let $d(\mu) = \sum_{\sigma \in J_L} \mu(\sigma) \in \mathbb{Z}$ denote the *degree of* μ. We let $I_L^+ := \{\mu \in I_L \mid \mu(\sigma) \geq 0\}$. For any integer $d \geq 0$, we write $I_L^{+,d} = \{\mu \in I_L^+ \mid d(\mu) = d\}$. For any number field L, we use the notation $\mathbb{A}_L$ to denote its ring of adeles and $\mathbb{A}_L^\times$ for the group of ideles. We denote the ring of finite adeles by L_f and the group of finite ideles by $L_f^\times$. The idele norm of L will be denoted by $|\cdot|_L$. If $|\cdot|_v$ denotes the normalized absolute value on the completion L_v for any place v of L, one has $|(\alpha_v)|_L = \prod_v |\alpha_v|_v$. If $x \in L_f^\times$ is a finite idele, we denote its associated fractional ideal in L by $[x]_L$. If L/K is a Galois extension, we sometimes denote its Galois group by $G_{L/K}$. The absolute Galois group of a field L will be denoted by G_L. We denote the closure of the commutator subgroup $[G_L, G_L]$ by G_L^{c} and let G_L^{ab} denote the Galois group of the maximal abelian extension of L in $\overline{\mathbf{Q}}$. There is an isomorphism $G_L^{\mathrm{ab}} = G_L/G_L^{\mathrm{c}}$. If L is a number field, we use the *geometric* normalization of the Artin homomorphism and denote it by $r_L : \mathbb{A}_L^\times \to G_L^{\mathrm{ab}}$. If L is totally complex, r_L factors through $L_f^\times$ and its kernel is the closure of $L^\times$ embedded diagonally in $L_f^\times$. We let $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \widehat{\mathbb{Z}}^\times$ denote the cyclotomic character, i.e. the character which is characterized by $\sigma(\zeta) = \zeta^{\chi_{\mathrm{cyc}}(\sigma)}$ for every root of unity ζ in $\overline{\mathbf{Q}}$. It satisfies $r_{\mathbf{Q}}(\chi_{\mathrm{cyc}}(\sigma)) = \sigma|_{\mathbf{Q}^{\mathrm{ab}}}$.

Let R be a commutative ring and M an R-module. For an integer $n \geq 0$, we let $\mathrm{Sym}^n M$ denote the nth symmetric powers, i.e. the coinvariants of $M^{\otimes n}$ under the permutation action of the symmetric group S_n, and we let $\mathrm{TSym}^n(M)$ denote the R-module of symmetric tensors in $M^{\otimes n}$, i.e. the S_n-invariants of $M^{\otimes n}$. For $k, l \geq 0$ define

$$S_{k,l} := \{\sigma \in S_{k+l} \mid \sigma(1) < \ldots < \sigma(k),\ \sigma(k+1) < \ldots < \sigma(k+l)\}.$$

For $x \in \mathrm{TSym}^k(M)$ and $y \in \mathrm{TSym}^l(M)$ define $x \cdot y := \sum_{\sigma \in S_{k,l}} \sigma(x \otimes y)$. This turns $\mathrm{TSym}^\bullet(M)$ into a commutative graded R-algebra. For $m \in M$ we set

$$m^{[n]} := m \otimes \ldots \otimes m \in \mathrm{TSym}^n(M).$$

This equip $\mathrm{TSym}^\bullet(M)$ with the structure of a divided power algebra. The inclusion $M = \mathrm{TSym}^1(M) \hookrightarrow \mathrm{TSym}^\bullet(M)$ induces a canonical map

$$\mathrm{Sym}^\bullet(M) \to \mathrm{TSym}^\bullet(M) \tag{0.2}$$

by the universal property of the symmetric algebra. For fixed $n \geq 0$, it is induced by the symmetrisation map $\varphi : M^{\otimes n} \to \mathrm{TSym}^n(M)$, $x \mapsto \sum_{\sigma \in S_n} \sigma(x)$. There is also the canonical map

$$\psi : \mathrm{TSym}^n(M) \subset M^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n(M) \tag{0.3}$$

and one checks that $\varphi \circ \psi$ and $\psi \circ \varphi$ are given by multiplication by $n!$. Hence, if R is a $\mathbf{Q}$-algebra, the map (0.2) is an isomorphism, under which an element of the form $m^{[n]}$ corresponds to $\frac{1}{n!} m^n$ in $\mathrm{Sym}^n(M)$. The isomorphism (0.3) sends $m^{[n]}$ to m^n. The drawback of (0.3) is that it does not define a morphism of graded algebras. Still, let us fix the convention that we identify TSym and Sym via (0.3). The reason is that it saves us to write down denominators in Definition 10.14.

We fix a square root i of -1. This will be convenient from Chapter 9 onwards. All results will be independent of this choice.

1. Types

To large parts of this section, we closely follow [Bla86], section 0. Let $L \subset \mathbf{C}$ denote a number field.

1.1. Basic definitions and the field of values.

Definition 1.1. *A* type on L *is an element* $\mu \in I_L$ *and we write*

$$\mu = \sum_{\sigma \in J_L} \mu(\sigma)\sigma.$$

The subset $|\mu| := \{\sigma \in J_L \mid \mu(\sigma) \neq 0\}$ *of* J_L *is called the* support of μ. *For any extension* L/K *of number fields, we call* $N_{L/K} := \sum_{\sigma \in J_{L/K}} \sigma$ *the* norm type for L/K.

The group of types I_L acts on $L^\times$ by

$$\ell^\mu := \prod_{\sigma \in J_L} \sigma(\ell)^{\mu(\sigma)} \in \overline{\mathbf{Q}}^\times$$

for $\ell \in L^\times$ and $\mu \in I_L$. The absolute Galois group $G_\mathbf{Q}$ acts from the left on the abelian group I_L by

$$(\tau \cdot \mu)(\sigma) = \mu(\tau^{-1}\sigma)$$

where $\tau \in G_\mathbf{Q}$, $\mu \in I_L$ and $\sigma \in J_L$. In fact, this turns I_L into a discrete $G_\mathbf{Q}$-module: If $\mu \in I_L$ and $\tau \in G_\mathbf{Q}$ fixes the Galois closure of L in $\overline{\mathbf{Q}}$, then clearly $\tau \cdot \mu = \mu$, so that the stabilizer $\{\tau \in G_\mathbf{Q} \mid \tau\mu = \mu\}$ is open. Note that for every $\ell \in L^\times$ one has $\ell^{\tau\mu} = \tau(\ell^\mu)$.

Definition 1.2. *Let* $\mu \in I_L$ *be a type on* L. *The* number field L^μ generated by the values of μ *is defined to be the fixed field of the open subgroup* $\{\tau \in G_\mathbf{Q} \mid \tau\mu = \mu\}$ *in* $G_\mathbf{Q}$.

Note that, by definition, for $\tau \in G_\mathbf{Q}$ and $\mu \in I_L$, the type $\tau\mu$ only depends on the restriction $\tau|_{L^\mu}$. Therefore, the following notation makes sense:

Notation 1.3. *Let* T *be a number field containing* L^μ. *For an embedding* $\tau \in J_T$, *we define* $\tau\mu \in I_L$ *to be the type* $\tilde{\tau}\mu$ *where* $\tilde{\tau}$ *denotes any lift of* τ *to an element of* $G_\mathbf{Q}$.

Remark 1.4. The terminology in Definition 1.2 is justified as follows: The field L^μ is the smallest subfield of $\overline{\mathbf{Q}}$ which contains the the values ℓ^μ for all $\ell \in L^\times$. Indeed, let F denote the subfield of $\overline{\mathbf{Q}}$ generated by the elements ℓ^μ for all $\ell \in L^\times$. If $\tau \in G_\mathbf{Q}$ satisfies $\tau\mu = \mu$, then one has $\tau(\ell^\mu) = \ell^{\tau\mu} = \ell^\mu$, so that τ also fixes F. This show $F \subset L^\mu$. Let conversely $\tau \in G_F$. Let $L = \mathbf{Q}(\theta)$ for a primitive element $\theta \in L^\times$. Consider the polynomial $f(X) = \prod_{\sigma \in J_L} (X - \sigma(\theta))^{\mu(\sigma)} \in \overline{\mathbf{Q}}[X]$. Then, for any rational number $a \in \mathbf{Q}$, one has $f(a) = (a - \theta)^\mu \in F$. It follows that f actually has coefficients in F (e.g. by evaluating on sufficiently many distinct points and using that the

Vandermonde matrix is invertible). Thus, the automorphism τ fixes f. On the other hand, one has

$$\tau(f(X)) = \prod_{\sigma\in J_L} (X - \sigma(\theta))^{(\tau\mu)(\sigma)}.$$

and since the conjugates $\sigma(\theta)$ for $\sigma \in J_L$ are pairwise distinct, comparing the orders of the zeroes yields $\tau\mu = \mu$, as desired.

1.2. **Change of fields.** Let L and K be two number fields with $K \subset L$. Then the map

$$\mathrm{Inf}_{L/K} : I_K \to I_L,$$

defined by

$$(\mathrm{Inf}_{L/K}(\kappa))(\sigma) = \kappa(\sigma|_K)$$

for any $\kappa \in I_K$ and $\sigma \in J_L$, is a injective $G_{\mathbf{Q}}$-equivariant homomorphism. Moreover, there is a surjective $G_{\mathbf{Q}}$-equivariant homomorphism $\mathrm{Res}_{L/K} : I_L \to I_K$ given by $\mathrm{Res}_{L/K}(\mu)(\sigma) = \sum_{\sigma'|_K=\sigma} \mu(\sigma')$ if $\mu \in I_L$ and $\sigma \in J_K$. The composites $\mathrm{Inf}_{L/K} \circ \mathrm{Res}_{L/K}$ and $\mathrm{Res}_{L/K} \circ \mathrm{Inf}_{L/K}$ are given by multiplication by $[L : K]$ on I_L and I_K, respectively.

1.3. **Minimal field of definition.** We will sometimes say that an infinity type $\mu \in I_L$ is induced (or extended) from a subfield $K \subset L$ if there exists a type $\kappa \in I_K$ such that

$$\mu = \mathrm{Inf}_{L/K}(\kappa).$$

The following lemma shows that for every type $\mu \in I_L$ there exists a minimal such subfield.

Lemma 1.5. *Let $\mu \in I_L$. There exists a minimal subfield $L_0 \subset L$ such that there exists a type $\mu_0 \in I_{L_0}$ satisfying*

$$\mu = \mathrm{Inf}_{L/L_0}(\mu_0).$$

We call L_0 the minimal field of definition of μ.

Proof. Let L' be a Galois extension over $\mathbf{Q}$ which contains L and let $\mu' = \mathrm{Inf}_{L'/L}(\mu)$. Consider the subgroup

$$H := \{\tau \in \mathrm{Gal}(L'/\mathbf{Q}) \mid \mu'(\sigma\circ\tau) = \mu'(\sigma) \text{ for all } \sigma \in J_{L'}\}$$

and define

$$L_0 := (L')^H.$$

We can define a type $\mu_0 \in I_{L_0}$ as follows: For every $\sigma_0 \in J_{L_0}$ choose an extension $\sigma' \in J_{L'}$ and put

$$\mu_0(\sigma_0) := \mu'(\sigma').$$

This is independent of the choice of the embedding σ' since any other such extension is of the form $\sigma'\tau$ with $\tau \in \mathrm{Gal}(L'/L_0) = H$ and by construction, one has $\mu'(\sigma'\tau) = \mu'(\sigma')$. We now claim that $\mu \in I_L$ is induced from $\mu_0 \in I_{L_0}$. First, one indeed has $L_0 \subset L'$ since for any $\tau \in \mathrm{Gal}(L'/L)$ and $\sigma \in J_{L'}$, one has

$$\mu'(\sigma\circ\tau) = \mu((\sigma\circ\tau)|_L) = \mu(\sigma|_L) = \mu'(\sigma).$$

Now let $\sigma \in J_L$ and $\sigma' \in J_{L'}$ an extension to an embedding of L'. Then σ' clearly is also an extension of $\sigma|_{L_0}$ and we obtain

$$\mathrm{Inf}_{L/L_0}(\mu_0)(\sigma) = \mu_0(\sigma|_{L_0}) = \mu'(\sigma') = \mu(\sigma),$$

i.e. $\mathrm{Inf}_{L/L_0}(\mu_0) = \mu$. Finally, let $K \subset L$ be a subfield such that there exists a type $\kappa \in I_K$ with $\mu = \mathrm{Inf}_{L/K}(\kappa)$. Just as before, if $\tau \in \mathrm{Gal}(L'/K)$ and $\sigma \in J_{L'}$, one has

$$\mu'(\sigma\circ\tau) = \kappa(\sigma\circ\tau|_K) = \kappa(\sigma|_K) = \mu'(\sigma),$$

so that $\tau \in H$. Thus, $L_0 \subset K$. Both $\mathrm{Inf}_{K/L_0}(\mu_0)$ and κ induce μ on L, hence

$$\kappa = \mathrm{Inf}_{K/L_0}(\mu_0),$$

as desired. $\square$

1.4. **Composition of types.** Let $\mu \in I_L$ and let T be a number field containing the field of values L^μ. Recall that, for any embedding $\tau \in J_T$, we can form the type $\tau\mu$ in I_L, cf. Notation 1.3. For any type $\pi \in I_T$, we define

$$\pi \circ \mu := \mu^\pi := \sum_{\tau \in J_T} \pi(\tau) \cdot (\tau\mu) \in I_L.$$

For any $\ell \in L^\times$, one has $\ell^{\pi\circ\mu} = (\ell^\mu)^\pi$ and clearly $T^\pi \subset L^{\pi\circ\mu}$. Moreover, one immediately checks that $d(\pi \circ \mu) = d(\pi)d(\mu)$.

1.5. **The reflex type.** If L is a number field which is Galois over $\mathbf{Q}$ and $\mu \in I_L$, we denote by $\mu^* \in I_L$ the type defined by

$$\mu^*(\sigma) = \mu(\sigma^{-1})$$

for $\sigma \in J_L$ and where we identify J_L with $\mathrm{Gal}(L/\mathbf{Q})$ via the fixed embedding $L \subset \mathbf{C}$. Now let L be an arbitrary number field and $\mu \in I_L$. Let L' denote a Galois extension of $\mathbf{Q}$ containing L. Note that L' also contains the field of values L^μ.

Lemma 1.6. *The minimal subfield of definition of the type*

$$\mathrm{Inf}_{L'/L}(\mu)^* \in I_{L'} \tag{1.1}$$

coincides with the field of values L^μ.

Proof. If $\tau \in \mathrm{Gal}(L'/\mathbf{Q})$ and $\sigma \in J_{L'} \cong \mathrm{Gal}(L'/\mathbf{Q})$, one calculates

$$\begin{aligned}\mathrm{Inf}_{L'/L}(\mu)^*(\sigma\circ\tau) &= \mathrm{Inf}_{L'/L}(\mu)((\sigma\circ\tau)^{-1})\\ &= \mu(\tau^{-1}\circ\sigma^{-1}|_L)\\ &= (\tau\mu)(\sigma^{-1}|_L).\end{aligned}$$

It follows that $\tau \in \mathrm{Gal}(L'/\mathbf{Q})$ satisfies $\mathrm{Inf}_{L'/L}(\mu)^*(\sigma\circ\tau) = \mathrm{Inf}_{L'/L}(\mu)^*(\sigma)$ for all $\sigma \in J_{L'}$ if and only if $\tau\mu = \mu$. This proves the claim. $\square$

Definition 1.7. *The unique type $\mu^* \in I_{L^\mu}$ which induces (1.1) is called the* reflex of μ. *If T is a number field containing L^μ, we call $\mathrm{Inf}_{T/L^\mu}(\mu^*)$ the* reflex of μ relative to T. *We still denote it by μ^* and simply call it the reflex of μ if the field T is clear from the context.*

Concretely this means the following: If $\sigma \in J_T$, then

$$\mu^*(\sigma) = \mu(\widetilde{\sigma}^{-1}|_L)$$

where $\widetilde{\sigma} \in J_{L'}$ is any extension of σ to an embedding of L'. Let us record some further properties of the reflex construction. Let $\mu \in I_L$ be a type and $\tau \in G_{\mathbf{Q}}$. Then there is a type $\mu\circ\tau^{-1} \in I_{L^\tau}$ given by

$$(\mu\circ\tau^{-1})(\sigma) = \mu(\sigma\circ\tau|_L)$$

for all $\sigma \in J_{L^\tau}$. Then, for any $x \in (L^\tau)^\times$, one has $x^{\mu\circ\tau^{-1}} = \tau^{-1}(x)^\mu$.

Lemma 1.8. *Let μ, μ_1 and μ_2 be types of L. Let L_0 be the minimal field of definition of μ and $\mu_0 \in I_{L_0}$ such that $\mu = \mathrm{Inf}_{L/L_0}(\mu_0)$.*

(i) *For every $a \in \mathbb{Z}$, one has $(a\mu)^* = a\mu^*$.*

(ii) *If T is a number field containing both L^{μ_1} and L^{μ_2}, then*

$$(\mu_1+\mu_2)^* = \mu_1^* + \mu_2^*,$$

where every reflex is taken relative to T.

(iii) *If L'/L is an extension of number fields, then*

$$(\mathrm{Inf}_{L'/L}(\mu))^* = \mu^*.$$

(iv) *Let $\tau \in G_{\mathbf{Q}}$. Then, relative to any number field T containing L^μ, one has*

$$(\mu\circ\tau^{-1})^* = \tau\mu^*.$$

(v) *Let T be a number field containing L^μ and $\mu^* \in I_T$ be the reflex of μ relative to T. Then*

$$T^{\mu^*} = L_0 \quad \textit{and} \quad (\mu^*)^* = \mu_0.$$

(vi) *For any* $\tau \in G_{\mathbf{Q}}$*, one has*
$$(\tau\mu)^* = \mu^* \circ \tau^{-1}$$
relative to any number field containing L^τ*.*

Proof. Assertions (i)–(iii) immediately follow from the definition of the reflex type. Note that $(L^\tau)^{\mu\circ\tau^{-1}} = L^\mu$. Let L' denote a Galois extension of $\mathbf{Q}$ containing L (and hence also L^τ and L^μ). Let $\sigma \in J_{L^\mu}$ and let $\widetilde{\sigma} \in J_{L'}$ denote a lift. Then one has
$$\begin{aligned}(\mu\circ\tau^{-1})^*(\sigma) &= (\mu\circ\tau^{-1})(\widetilde{\sigma}^{-1}|_{L^\tau})\\ &= \mu(\widetilde{\sigma}^{-1}\circ\tau|_L)\\ &= \mu((\tau^{-1}\widetilde{\sigma})^{-1}|_L)\\ &= \mu^*(\tau^{-1}\sigma)\\ &= (\tau\mu)(\sigma).\end{aligned}$$
This shows (iv). Hence, an element $\tau \in G_{\mathbf{Q}}$ satisfies $\tau\mu^* = \mu^*$ if and only if $(\mu\circ\tau^{-1})^* = \mu^*$, or equivalently $(\mathrm{Inf}_{L'/L}(\mu)\circ\tau^{-1})^* = \mathrm{Inf}_{L'/L}(\mu)^*$ by (iii). This is equivalent to $\mathrm{Inf}_{L'/L}(\mu)\circ\tau^{-1} = \mathrm{Inf}_{L'/L}(\mu)$, i.e. to $\tau|_{L_0} = \mathrm{id}$ by construction of the minimal field of definition. This shows $T^{\mu^*} = L_0$ and $(\mu^*)^* = \mu$ is a straightforward consequence. Assertion (vi) follows from (iv) and (v). □

1.6. **CM-fields and CM-types.**

Definition 1.9. *A number field K is called a* CM-field *if it satisfies one of the following equivalent conditions:*

(i) *K is an imaginary quadratic extension of a totally real number field.*
(ii) *There exists an automorphism $c_K \neq \mathrm{id}_K$ such that*
$$\sigma\circ c_K = c\circ\sigma$$
for all embeddings $\sigma \in J_K$.
(iii) *$K \subset \overline{\mathbf{Q}}$ is totally imaginary and fixed by all commutators*
$$[\tau, c] = \tau c\tau^{-1}c^{-1} \in G_{\mathbf{Q}}$$
for all $\tau \in G_{\mathbf{Q}}$.

Proof. If K satisfies (i) with totally real subfield F, then there is a unique non-trivial automorphism c_K fixing F satisfying (ii). Conversely, an automorphism c_K as in (ii) is of order 2 and its fixed field must be totally real. Note that K is an imaginary quadratic extension of this field since $c_K \neq \mathrm{id}_K$. Assertion (ii) readily implies (iii) since c_K is necessarily induced by complex conjugation $c \in G_{\mathbf{Q}}$. Conversely, let us assume (iii). It follows that for any $\tau \in G_K$, one has $c\tau c^{-1}|_K = 1_K$, i.e. $cG_Kc^{-1} = G_K$. Passing to fixed fields implies $c(K) = K$, so that complex conjugation c descends to an automorphism of K. It is non-trivial since K is assumed to be totally imaginary. For any $\tau \in G_{\mathbf{Q}}$, we have $\tau^{-1}c^{-1}\tau c|_K = 1_K$, hence $\tau|_K\circ c_K = c\circ\tau|_K$. This shows (ii). □

Remark 1.10. If one drops the conditions that $c_K \neq \mathrm{id}_K$ in (ii) and that K is totally imaginary in (iii) of Definition 1.9, one precisely obtains the number fields which are totally real or a CM-field.

Lemma 1.11. *The composite of CM-fields is a CM-field. In particular, the Galois closure of a CM-field is a CM-field.*

Proof. This follows immediately using the characterization (iii) in Definition 1.9. □

Let L be a number field containing a CM-field. It follows from Lemma 1.11 that L contains a *unique* maximal CM-subfield.

Definition 1.12.

(i) *Let K be a CM-field. A subset $\Phi \subset J_K$ satisfying*
$$\Phi\cup\overline{\Phi} = J_K \quad \textit{and} \quad \Phi\cap\overline{\Phi} = \emptyset$$
is called a CM-type *of K.*

(ii) *Let L be a number field containing a CM-field and let K denote its maximal CM-subfield. A subset $\Phi \subset J_L$ is a* CM-type *of L if there exists a CM-type Φ_K of K such that*

$$\Phi = \{\sigma \in J_L \mid \sigma|_K \in \Phi_K\}.$$

Remark 1.13. Let L be a number field with maximal CM-subfield K and let Φ be a CM-type of L.

(i) Just as in the CM-case, one has

$$\Phi \cup \overline{\Phi} = J_L \quad \text{and} \quad \Phi \cap \Phi = \emptyset.$$

Nevertheless, we are only interested in such subsets which are moreover induced from a CM-subfield.

(ii) We freely identify the CM-type Φ with its characteristic function in I_L and simply denote it by Φ or, if we want to stress that it is a type, by 1_Φ.

Lemma 1.14. *Let L be a number field and $\mu \in I_L$ a type. Assume that μ is induced from a totally real field or a CM-field. Then L^μ is totally real or a CM-field.*

Proof. Let $K \subset L$ be a totally real or CM-subfield such that there exists a type $\kappa \in I_K$ with $\mu = \mathrm{Inf}_{L/K}(\kappa)$. By definition of L^μ, it suffices to show that, for every $\tau \in G_{\mathbf{Q}}$, one has

$$\tau c \tau^{-1} c^{-1} \mu = \mu.$$

Let $\sigma \in J_L$. Then

$$\begin{aligned}(\tau c \tau^{-1} c^{-1} \mu)(\sigma) &= \kappa(\tau^{-1} c^{-1} \tau c \sigma|_K) \\ &= \kappa(\sigma|_K) \\ &= \mu(\sigma),\end{aligned}$$

as $\tau^{-1} c^{-1} \tau c \sigma|_K = \sigma|_K$ by the assumption that K is totally real or a CM-field. □

Definition 1.15. *Let L be a number field containing a CM-field and Φ a CM-type of L. The field of values $E := L^\Phi$ is called the* reflex field *of (L, Φ). The reflex $\Phi^* \in I_E$ of Φ is called the* reflex CM-type *of Φ.*

By Lemma 1.14, the reflex field $E = L^\Phi$ indeed is totally real or a CM-field. As $\overline{\Phi} \neq \Phi$, it is in fact a CM-field. By definition of the reflex, the type Φ^* takes on the values 0 or 1. Moreover, using Lemma 1.8, $(c\Phi)^* = \Phi^* \circ c^{-1} = c\Phi^*$ since E is a CM-field, so that $\Phi^* + c\Phi^* = \Phi^* + (c\Phi)^* = (\Phi + c\Phi)^*_E = (N_{L/\mathbf{Q}})^*_E = N_{E/\mathbf{Q}}$. It follows that Φ^* is actually a CM-type as its name suggests.

1.7. **Hecke character types.** We single out two conditions that a type needs to satisfy if it is the infinity type of an algebraic Hecke character.

Definition 1.16. *Let L be a number field. A type $\mu \in I_L$ is a* Hecke character type *if it satisfies the following two conditions:*

(i) *μ is induced from a subfield of L which is totally real or a CM-field.*

(ii) *μ admits a* weight, *i.e. there exists an integer $w \in \mathbb{Z}$ such that*

$$\mu(\sigma) + \mu(c\sigma) = w$$

for all $\sigma \in J_L$. In other words, $\mu + c\mu = w \cdot N_{L/\mathbf{Q}}$.

Lemma 1.17. *Let $T \supset L^\mu$ be a number field and $\mu^* \in I_T$ denote the reflex of μ relative to T. If ψ is a Hecke character type of weight $w \in \mathbb{Z}$, then so is ψ^*.*

Proof. Lemma 1.14 implies that ψ^* is induced from a totally real field or a CM-field. Let L_0 be the field of definition of μ and $\mu = \mathrm{Inf}_{L/L_0}(\mu_0)$ for $\mu_0 \in I_{L_0}$. By assumption, L_0 is a totally real or a CM-field and one immediately verifies in this case that $\mu_0 \circ c = c\mu_0$. Then

$$\begin{aligned}\mu^* + c\mu^* &= \mu_0^* + c\mu_0^* \\ &= \mu_0^* + (\mu_0 \circ c)^* \\ &= (\mu_0 + c\mu_0)^* \\ &= (w \cdot N_{L_0/\mathbf{Q}})^* \\ &= w \cdot N_{T/\mathbf{Q}},\end{aligned}$$

and the assertion follows. □

Remark 1.18. The Hecke character types on L in the sense of Definition 1.16 are precisely the types which arise as infinity types of algebraic Hecke characters of L. If $\mu \in I_L$ is a Hecke character type and T a number field containing L^μ it is not necessarily true that there exists an algebraic Hecke character of L with values in T whose infinity type is μ. In general, this is only true after enlarging the field T.

There is the following characterization of Hecke character types:

Proposition 1.19. *Let $\mu \in I_L$ be a type. The following conditions are equivalent:*

(i) *μ is a Hecke character type.*
(ii) *There exists a finite index subgroup $\Gamma \subset \mathcal{O}_L^\times$ such that $\mu(\Gamma) = 1$.*

Proof. If μ is a Hecke character type which is induced from a totally real field, it is necessarily of the form $N_{L/\mathbf{Q}}^{w/2}$ for even $w \in \mathbb{Z}$. Clearly, it vanishes on a finite index subgroup of $\mathcal{O}_L^\times$ in this case. Let μ be a Hecke character type which is induced from a CM-field. For any $\ell \in \mathcal{O}_L^\times$ and $\tau \in G_{\mathbf{Q}}$, one has

$$|\tau\mu(\ell)|^2 = \tau\mu(\ell) \cdot \overline{\tau\mu(\ell)} = N_{L/\mathbf{Q}}(\ell)^w = 1.$$

Since $\mu(\ell)$ is an algebraic integer, it follows that $\mu(\ell)$ is a root of unity. Since $\mathcal{O}_L^\times$ is finitely generated by Dirichlet's unit theorem, the image $\mu(\mathcal{O}_L^\times)$ has to be of finite order and hence μ vanishes on a finite index subgroup of $\mathcal{O}_L^\times$. For the converse, we refer the reader to [Sch88], §0, section 3. □

Definition 1.20. *For any finite index subgroup Γ of $\mathcal{O}_L^\times$ we let*

$$\mathrm{HChar}_L(\Gamma) := \{\mu \in I_L^+ \mid \mu(\Gamma) = 1\}$$

denote the set of Hecke character types which are trivial on Γ.

Finally, let us define *critical* Hecke character types:

Definition 1.21. *Let (L, Φ) be a CM-type. A Hecke character type $\mu \in I_L$ is said to be* critical *with respect to Φ if it can be written in the form*

$$\mu = \beta - \alpha,$$

where α is supported on Φ, β is supported on $\overline{\Phi}$ and satisfying

$$\alpha(\sigma) \geq 1 \quad \text{and} \quad \beta(\overline{\sigma}) \geq 0.$$

We say that μ is critical *if it is critical with respect to some CM-type of L. Note that for critical μ, the CM-type Φ is uniquely determined as the set of embeddings of L on which μ takes on negative values. The set of all critical Hecke character types in $\mathrm{HChar}_L(\Gamma)$ is denoted by $\mathrm{Crit}_L(\Gamma)$.*

2. Algebraic Hecke characters

In this section we give two equivalent definitions for algebraic Hecke characters: One in terms of ideles and one in terms of ideals. Afterwards we define the L-function associated to an algebraic Hecke character.

2.1. **Two definitions.** Recall our convention that all number fields are embedded into $\mathbf{C}$. Let L and T be number fields. Recall that $J_L = \mathrm{Hom}(L, \mathbf{C}) = \mathrm{Hom}(L, \overline{\mathbf{Q}})$ denotes the (pointed) set of embeddings of L into $\mathbf{C}$ (resp. $\overline{\mathbf{Q}}$) and that $G_{\mathbf{Q}}$ acts on this set in a canonical way. We first define algebraic homomorphisms which are homomorphisms $L^\times \to T^\times$ that are induced by a homomorphism of algebraic tori:

Definition/Proposition 2.1. *A group homomorphism $\theta : L^\times \to T^\times$ is* algebraic *if the following equivalent conditions are satisfied:*

(i) *There exists a homomorphism of algebraic tori*

$$\theta : \mathrm{Res}_{L/\mathbf{Q}} \mathbf{G}_m \to \mathrm{Res}_{T/\mathbf{Q}} \mathbf{G}_m$$

which coincides with θ on $\mathbf{Q}$-valued points.

(ii) *There exists a type* $\mu = \sum_{\sigma \in J_L} \mu(\sigma)\sigma \in I_L$*, which is invariant under the action of* G_T*, such that*

$$\theta(\ell) = \ell^\mu = \prod_{\sigma \in J_L} \sigma(\ell)^{\mu(\sigma)}$$

for all $\ell \in L^\times$*.*

Proof. A homomorphism $\theta : \mathrm{Res}_{L/\mathbf{Q}} \mathbf{G}_m \to \mathrm{Res}_{T/\mathbf{Q}} \mathbf{G}_m$ uniquely corresponds by adjunction to a homomorphism $\mathrm{Res}_{L/\mathbf{Q}} \mathbf{G}_m \times T \to \mathbf{G}_{m,T}$, which in turn amounts to a G_T-invariant character $\mu : \mathrm{Res}_{L/\mathbf{Q}} \mathbf{G}_m \times \overline{\mathbf{Q}} \to \mathbf{G}_{m,\overline{\mathbf{Q}}}$. Now through the identification of $G_{\mathbf{Q}}$-modules

$$X^*(\mathrm{Res}_{L/\mathbf{Q}} \mathbf{G}_m) = I_L,$$

the equivalence of assertions (i) and (ii) follows. □

We now can give our first definition of algebraic Hecke characters. In this form they arguably fit most naturally in the theory of complex multiplication (cf. [ST68], section 7). They are sometimes referred to as Serre-Tate characters in the literature.

Definition 2.2. *Let* T *be a number field and equip it with the discrete topology in the following. An* algebraic Hecke character with values in T *is a continuous group homomorphism*

$$\chi : \mathbb{A}_L^\times \to T^\times$$

such that its restriction

$$\chi|_{L^\times} : L^\times \to T^\times$$

is an algebraic homomorphism. We refer to this algebraic homomorphism (or its extension to a homomorphism of tori) as the algebraic part of χ *and denote it by* χ_a*. The corresponding element in* $\mathbb{Z}[J_L]$*, which we usually also denote by* χ_a*, will be called the* infinity-type of χ*.*

More generally, if $T = \prod_{i=1}^r T_i$ is a product of number fields, we say that a continuous homomorphism $\chi : \mathbb{A}_L^\times \to T^\times$ is an algebraic Hecke character if each projection to $T_i^\times$ for $i = 1, \ldots, r$ is an algebraic Hecke character in the above sense.

Remark 2.3. If $\chi : \mathbb{A}_L^\times \to T^\times$ is an algebraic Hecke character and the number field L is totally imaginary, then $L_\infty^\times = (L \otimes \mathbf{R})^\times$ is connected, so that $\chi(L_\infty) = 1$ and χ factors as a continuous character $\chi : L_f^\times \to T^\times$ of the finite ideles.

Example 2.4. We collect a few easy but important examples of algebraic Hecke characters:

(1) If $\chi : \mathbb{A}_L^\times \to T^\times$ is an algebraic Hecke character with trivial algebraic part, then it factors through the idele class group $C_L = \mathbb{A}_L^\times / L^\times$ and hence through C_L / C_L^0, where C_L^0 denotes the connected component of the identity. By class field theory and continuity, χ therefore corresponds to a finite abelian character

$$\mathrm{Gal}(L^{\mathrm{ab}}/L) \to T^\times.$$

These Hecke characters will later correspond to abelian Artin motives with coefficients in T.

(2) For a totally imaginary number field L, the idele norm character

$$\|\cdot\|_L : L_f^\times \to \mathbf{Q}^\times, \alpha \mapsto |\alpha|_L$$

is an algebraic Hecke character. Its algebraic part is the inverse of the norm $N_{L/\mathbf{Q}}^{-1}$. This Hecke character will later on correspond to the Tate motive $\mathbf{Q}(1)$ over L.

(3) More generally, if L is an arbitrary number field, the map

$$\|\cdot\|_L : \mathbb{A}_L^\times \to \mathbf{Q}^\times$$

given by

$$\alpha \mapsto N_{L/\mathbf{Q}}(\alpha_\infty)^{-1} \cdot |\alpha|_L = \prod_{v \text{ real}} \mathrm{sgn}(\alpha_v) \cdot |\alpha_f|_L$$

defines an algebraic Hecke character with algebraic part $N_{L/\mathbf{Q}}^{-1}$. We still call it the idele norm character.

We now give the second and more classical definition in terms of ideals. Again, we fix number fields L and T. Let $\mathfrak{f} \subset \mathcal{O}_L$ be an integral ideal. Let $I^{\mathfrak{f}}$ denote the group of fractional ideals of L that are coprime to $\mathfrak{f}$ and denote by $P^{\mathfrak{f}}$ the subgroup of fractional ideals of the form (α), where $\alpha \in L^\times$ satisfies $\alpha \equiv 1 \mod {}^\times \mathfrak{f}$ and α is totally positive.

Definition 2.5. *An* algebraic Hecke character conductor dividing $\mathfrak{f}$ with values in T *is a homomorphism*

$$\chi : I^{\mathfrak{f}} \to T^\times$$

such that there exists an algebraic homomorphism $\chi_a : L^\times \to T^\times$ satisfying

$$\chi((\alpha)) = \chi_a(\alpha)$$

for all $\alpha \in L^\times$ with $\alpha \equiv 1 \mod {}^\times \mathfrak{f}$ and α totally positive.

Let us explain how to pass between Definition 2.2 and Definition 2.5. For more details we refer the reader to [CCO14], 2.4.3. Let $\chi : \mathbb{A}_L^\times \to T^\times$ be an algebraic Hecke character with algebraic part $\chi_a : L^\times \to T^\times$ as in Definition 2.2. By the continuity of χ, there exists an integral ideal $\mathfrak{f} \subset \mathcal{O}_L$ such that $\ker(\chi)$ contains the open subgroup $U_{\mathfrak{f}} := (L_\infty^\times)^0 \times \prod_{v|\mathfrak{f}}(1+\mathfrak{f}\mathcal{O}_{L_v}) \times \prod_{v\nmid\mathfrak{f}\infty} \mathcal{O}_{L_v}^\times$. Let S denote the set of places dividing $\mathfrak{f}$ and the archimedean places and let $\mathbb{A}_L^S$ denote the ring of adeles away from S. Let $[\cdot]_L : \mathbb{A}_L^{S,\times} \to I^{\mathfrak{f}}$ denote the surjective homomorphism which associates to a finite idele α its corresponding fractional ideal $[\alpha]_L$ in L. One checks that, via $[\cdot]_L$, the restriction $\chi : \mathbb{A}_L^{S,\times} \to T^\times$ factors as $\chi' : I^{\mathfrak{f}} \to T^\times$ and that χ' is an algebraic Hecke character of conductor dividing $\mathfrak{f}$ as in Definition 2.5. The smallest ideal $\mathfrak{f}$ (with respect to divisibility) such that $\ker(\chi)$ contains $U_{\mathfrak{f}}$ is called the *conductor of* χ.

Conversely, let $\chi' : I^{\mathfrak{f}} \to T^\times$ be given as in Definition 2.5 and let χ'_a denote its associated algebraic homomorphism. Let the notations be as above. For every $\alpha \in \mathbb{A}_L^\times$ there exist $u \in U_{\mathfrak{f}}$, $y \in L^\times$ and $z \in \mathbb{A}_L^{S,\times}$ such that $\alpha = u \cdot y \cdot z$ by weak approximation. Then $\chi : \mathbb{A}_L^\times \to T^\times$, $\chi(\alpha) := \chi'_a(y)\chi'([z]_L)$ is well-defined and an algebraic Hecke character with algebraic part χ'_a in the sense of Definition 2.2. Let us note that the Hecke character $\|\cdot\|$ corresponds to the inverse of the norm character $N_{L/\mathbf{Q}} : I^{\mathfrak{f}} \to \mathbf{Q}^\times$, which sends an integral ideal $\mathfrak{a}$ to $\#\mathcal{O}_L/\mathfrak{a}$.

Let us recall the following fundamental fact that algebraic Hecke characters admit a weight and the resulting classification:

Proposition 2.6. *Let L and T be number fields and $\chi : \mathbb{A}_L^\times \to T^\times$ an algebraic Hecke character.*

(i) *There exists an integer $w \in \mathbb{Z}$ such that*

$$\chi \cdot \overline{\chi} = \|\cdot\|_L^{-w}$$

We call w the weight *of χ.*

(ii) *There is the following classification of algebraic Hecke characters:*

(a) *If L does not contain a CM-field, then χ is of the form*

$$\chi = \varrho \cdot \|\cdot\|_L^{-w/2},$$

where ϱ is a finite Hecke character (as in Example 2.4, (i)), $\|\cdot\|$ is the idele norm character and $w \in 2\mathbb{Z}$ is an even integer.

(b) *If L contains a CM-field and $K \subset L$ is the maximal one, then χ is of the form*

$$\chi = \varrho \cdot (\varphi \circ N_{L/K}),$$

where ϱ is a finite Hecke character and φ is an algebraic Hecke character of K.

Proof. E.g. by regarding χ as homomorphism $\chi : I^{\mathfrak{f}} \to T^\times$ as in Definition 2.5, one sees that χ vanishes on the finite index subgroup $\{\ell \in \mathcal{O}_L^\times \mid \ell \equiv 1 \mod {}^\times\mathfrak{f}\}$. Proposition 1.19 thus implies that the infinity-type of χ is a Hecke character type. From this assertion (ii) immediately follows. For (i), observe that the character of ideals $\chi \cdot \overline{\chi} \cdot N_{L/\mathbf{Q}}^{-w}$ vanishes on the finite-index subgroup $P^{\mathfrak{f}}$ of $I^{\mathfrak{f}}$ and takes values in $\mathbf{R}_{>0}$. Since $\mathbf{R}_{>0}$ is torsion-free, claim (i) follows. □

Finally let us make the following definition:

Definition 2.7. *An algebraic Hecke character χ is* critical *if its infinity-type χ_a is a critical Hecke character type in the sense of Definition 1.21*

2.2. **Hecke L-functions.** We recall a few facts about Hecke L-functions and fix some definitions. Let $\mathfrak{f} \subset \mathcal{O}_L$ be an integral ideal. Let $\chi : I^{\mathfrak{f}} \to T^{\times}$ be an algebraic Hecke character of conductor dividing $\mathfrak{f}$ (cf. Definition 2.5). For an embedding $\iota \in J_T$, we set $\chi_\iota := \iota \circ \chi$ and define the Hecke L-function

$$L_{\mathfrak{f}}(\chi_\iota, s) := \sum_{\substack{\mathfrak{a} \in I^{\mathfrak{f}} \\ \mathfrak{a} \subset \mathcal{O}_L}} \frac{\chi_\iota(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p} \nmid \mathfrak{f}} (1 - \chi_\iota(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1} \tag{2.1}$$

for all $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \frac{w}{2} + 1$, where w denotes the weight of χ. The L-series $L_{\mathfrak{f}}(\chi, s)$ admits a meromorphic continuation to $\mathbf{C}$ and satisfies a functional equation. For every fractional ideal $\mathfrak{b} \in I^{\mathfrak{f}}$ with associated class $[\mathfrak{b}] \in I^{\mathfrak{f}}/P^{\mathfrak{f}}$, we also define the partial L-function

$$L_{\mathfrak{f}}(\chi_\iota, s, [\mathfrak{b}]) := \sum_{\substack{\mathfrak{a} \in [\mathfrak{b}] \\ \mathfrak{a} \subset \mathcal{O}_L}} \frac{\chi_\iota(\mathfrak{a})}{N\mathfrak{a}^s}$$

for $\mathrm{Re}(s) > \frac{w}{2} + 1$. Again, the function $L_{\mathfrak{f}}(\chi_\iota, s, [\mathfrak{b}])$ admits a meromorphic continuation to $\mathbf{C}$. Finally, we use the identification $T \otimes \mathbf{C} \cong \mathbf{C}^{J_T}$, $t \otimes z \mapsto (\iota(t)z)_{\iota \in J_T}$ to define the $T \otimes \mathbf{C}$-valued L-functions

$$L_{\mathfrak{f}}(\chi, s) := (L_{\mathfrak{f}}(\chi_\iota, s))_{\iota \in J_T}$$

and, if $\mathfrak{b} \in I^{\mathfrak{f}}$,

$$L_{\mathfrak{f}}(\chi, s, [\mathfrak{b}]) := (L_{\mathfrak{f}}(\chi_\iota, s, [\mathfrak{b}]))_{\iota \in J_T}.$$

3. CM-motives

We recollect some facts about motives for absolute Hodge cycles, cf. [Pan94], [DMOS82], I and II §6 and [Jan90], Part 1. We define CM-motives and explain the formalism of [Bla86], 3.1. We recommend the reader, who is only interested in the applications, to consult loc. cit., or to briefly skim through the discussion after Corollary 3.7.

3.1. **Motives for absolute Hodge cycles.** Let L be a subfield of $\mathbf{C}$ and let $\overline{L}$ denote the algebraic closure of L in $\mathbf{C}$. Let X/L be a smooth projective variety. We have the following cohomology theories:

(1) For every embedding $\sigma \in J_L$, we have the Betti cohomology $H^i_\sigma(X) := H^i(X^\sigma(\mathbf{C}), \mathbf{Q})$. It is a finite-dimensional $\mathbf{Q}$-vector space and it is equipped with a rational $\mathbf{Q}$-Hodge structure $H^i_\sigma(X) = \bigoplus_{p,q} H^{p,q}_\sigma$. If σ is a real embedding, complex conjugation induces an involution $F_{\infty,\sigma}$ on $H^i_\sigma(X)$. When $\sigma = 1_L$ is the fixed embedding, we denote it by $H^i_B(X)$.
(2) The algebraic de Rham cohomology $H^i_{\mathrm{dR}}(X) := H^i(X, \Omega^\bullet_{X/L})$. It is a finite dimensional L-vector space and equipped with the finite filtration $F^i := \mathrm{im}(H^i(X, \Omega^{\geq i}_{X/L}) \to H^i(X, \Omega^\bullet_{X/L})$, the so-called Hodge filtration.
(3) The finite adelic cohomology $H^i_f(X) = (\varprojlim_n H^i_{\mathrm{et}}(X_{\overline{L}}, \mathbb{Z}/n\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbf{Q}$. It is a finite free $\mathbf{Q}_f$-module equipped with a $\mathbf{Q}_f$-linear continuous action of G_L.
(4) For every $\sigma \in J_L$, there is a comparison isomorphism
$$I_{\infty,\sigma} : H^i_{\mathrm{dR}}(X) \otimes_{L,\sigma} \mathbf{C} \xrightarrow{\sim} H^i_\sigma(X) \otimes \mathbf{C},$$
such that
$$I_{\infty,\sigma}(F^p H^i_{\mathrm{dR}}(X) \otimes_{L,\sigma} \mathbf{C}) = \bigoplus_{p' \geq p, q} H^{p',q}_\sigma(X).$$
Let us simply write I_∞ if σ is the fixed embedding $L \subset \mathbf{C}$.
(5) For every embedding $\overline{\sigma} \in J_{\overline{L}}$ which restricts to $\sigma \in J_L$, we have a comparison isomorphism
$$I_{f,\overline{\sigma}} : H^i_\sigma(X) \otimes \mathbf{Q}_f \xrightarrow{\sim} H^i_f(X),$$

such that, for every $\rho \in G_L$, the diagram

$$\begin{tikzcd} H^i_\sigma(X)\otimes \mathbf{Q}_f \arrow[r, "I_{f,\overline{\sigma}\rho}"] \arrow[dr, "I_{f,\overline{\sigma}}"'] & H^i_f(X) \arrow[d, "\rho"] \\ & H^i_f(X) \end{tikzcd}$$

commutes. When $\overline{\sigma}$ is the fixed embedding $\overline{L} \subset \mathbf{C}$, we simply denote it by I_f.

(6) We define the $\mathbf{Q}$-rational Hodge structure $\mathbf{Q}(1)_B := 2\pi i\mathbf{Q}$ with $\mathbf{Q}(1)_B \otimes \mathbf{C}$ of pure bidgree $(-1.-1)$, the filtered L-vector space $\mathbf{Q}(1)_{\mathrm{dR}} := L$ with $\mathbf{Q}(1)_{\mathrm{dR}} = F^{-1} \supset F^0 = 0$ and the G_L-representation $\mathbf{Q}(1)_f := (\varprojlim \mu_n(\overline{L})) \otimes_{\mathbb{Z}} \mathbf{Q}$. For every $\sigma \in J_L$, we identify

$$\mathbf{Q}(1)_{\mathrm{dR}} \otimes_{L,\sigma} \mathbf{C} \cong \mathbf{C} \cong 2\pi i\mathbf{Q} \otimes \mathbf{C} = \mathbf{Q}(1)_B \otimes \mathbf{C}$$

via the inclusion $2\pi i\mathbf{Q} \subset \mathbf{C}$. For every embedding $\overline{\sigma} : \overline{L} \hookrightarrow \mathbf{C}$, we identify

$$\mathbf{Q}(1)_B \otimes \mathbf{Q}_f \cong (\varprojlim_n 2\pi i\mathbb{Z}/2\pi in\mathbb{Z}) \otimes \mathbf{Q} \xrightarrow{\sim} (\varprojlim_n \mu_n(\mathbf{C})) \otimes \mathbf{Q} \cong \mathbf{Q}(1)_f$$

via the exponential map and $\overline{\sigma} : \mu_n(\overline{L}) \cong \mu_n(\mathbf{C})$. For every integer $m \in \mathbb{Z}$, we put $H^i_\sigma(X)(m) := H^i_\sigma(X) \otimes \mathbf{Q}(1)_B^{\otimes m}$, $H^i_{\mathrm{dR}}(X)(m) := H^i_{\mathrm{dR}}(X) \otimes_L \mathbf{Q}(1)_{\mathrm{dR}}^{\otimes m}$ and $H^i_f(X)(m) := H^i_f(X) \otimes_{\mathbf{Q}_f} \mathbf{Q}(1)_f^{\otimes m}$ in the respective categories, and we define comparison isomorphisms $I_{\infty,\sigma}$ and $I_{f,\overline{\sigma}}$ for all $\sigma \in J_L$ and $\overline{\sigma} \in J_{\overline{L}}$ by using the above identifications for $\mathbf{Q}(1)$.

Definition 3.1. *Let X/L be a smooth projective variety. Fix an integer $p \geq 0$. An* absolute Hodge cycle *on X is an element*

$$x = (x_\sigma, x_{\mathrm{dR}}, x_f)_{\sigma\in J_L} \in \prod_{\sigma\in J_L} H^{2p}_\sigma(X)(p) \times H^{2p}_{\mathrm{dR}}(X)(p) \times H^{2p}_f(X)(p)$$

satisfying

$$I_{\infty,\sigma}(x_\sigma) = x_{\mathrm{dR}}, \quad I_{f,\overline{\sigma}}(x_\sigma) = x_f, \quad x_{\mathrm{dR}} \in F^0 H^{2p}_{\mathrm{dR}}(X)(p)$$

for all $\sigma \in J_L$ and $\overline{\sigma} \in J_{\overline{L}}$ extending σ. Denote the $\mathbf{Q}$-vector space of absolute Hodge cycles by $C^p_{\mathrm{AH}}(X)$.

One then defines the category M_L of *motives for absolute Hodge cycles* in the usual way: Start with the category whose objects are smooth projective varieties over L, where the morphisms between connected varieties X and Y are given by $\mathrm{Hom}(X,Y) = C^{\dim(X)}(X \times Y)$ and pass to the Karoubian envelope. The usual cohomological grading gives rise to a natural $\mathbb{Z}$-grading. Invert the Lefschetz motive $H^2(\mathbf{P}^1)$ and finally modify the commutativity constraint of the tensor structure, which is induced by the Cartesian product of varieties. For details we refer the reader to [DMOS82], II.6. One can show that M_L is a semisimple Tannakian category over $\mathbf{Q}$. Let $\mathcal{V}_L$ denote the category of smooth projective varieties over L. There is a canonical contravariant functor

$$\mathcal{V}_L^{\mathrm{op}} \to \mathrm{M}_L, \quad X \mapsto h(X)$$

and we let $h(X) = \bigoplus_{i\in\mathbb{Z}} h^i(X)$ denotes its canonical grading.

3.2. **Realization functors.** Define a category $\widetilde{\mathcal{R}}_L$ (cf. [Jan90], Definition 2.1) whose objects are tuples

$$H = \{H_\sigma, H_{\mathrm{dR}}, H_f, I_{\infty,\sigma}, I_{f,\overline{\sigma}}\}_{\sigma\in J_L, \overline{\sigma}\in J_{L,\sigma}},$$

consisting of the formal data (1)-(5) listed above and whose morphisms $\varphi : H \to H'$ are given by tuples $\varphi = (\varphi_\sigma, \varphi_{\mathrm{dR}}, \varphi_f)$ consisting of a morphism $\varphi_\sigma : H_\sigma \to H'_\sigma$ of rational Hodge structures, a filtered morphism $\varphi_{\mathrm{dR}} : H_{\mathrm{dR}} \to H'_{\mathrm{dR}}$ and a G_L-equivariant $\mathbf{Q}_f$-linear map $\varphi_f : H_f \to H'_f$, which are compatible under the isomorphisms $I_{\infty,\sigma}$ and $I_{f,\overline{\sigma}}$ for every $\sigma \in J_L$ and $\overline{\sigma} \in J_{\overline{L},\sigma}$. Similarly, let us define a category $\mathcal{R}_L$, which is exactly defined as $\widetilde{\mathcal{R}}_L$, but it only consists of tuples $\{H_B, H_{\mathrm{dR}}, H_f, I_\infty, I_f\}$ and morphisms $\varphi = (\varphi_B, \varphi_{\mathrm{dR}}, \varphi_f)$. There is an obvious forgetful functor $\widetilde{\mathcal{R}}_L \to \mathcal{R}_L$. There are $\otimes$-structures on $\widetilde{\mathcal{R}}_L$ and $\mathcal{R}_L$, which are defined componentwise.

The above cohomology theories give rise to $\otimes$-functors $(-)_\sigma$, $(-)_{\mathrm{dR}}$ and $(-)_f$ from M_L into the respective target categories and, for every $M \in \mathrm{M}_L$, there are induced comparison isomorphisms $I_{\infty,\sigma} : M_{\mathrm{dR}} \otimes_{L,\sigma} \mathbf{C} \to M_\sigma \otimes \mathbf{C}$ and $I_{f,\overline{\sigma}} : M_\sigma \otimes \mathbf{Q}_f \to M_f$, which satisfy the analogous compatibilities. In fact, we obtain a commutative diagram of $\otimes$-functors

$$\begin{tikzcd} \mathrm{M}_L \arrow[r] \arrow[dr] & \widetilde{\mathcal{R}}_L \arrow[d, "\text{forget}"] \\ & \mathcal{R}_L. \end{tikzcd}$$

For $M \in \mathrm{M}_L$, we denote the Betti realization with respect to the fixed embedding $L \subset \mathbf{C}$ by M_B and we simply write I_f for the comparison isomorphism with respect to the fixed embedding $\overline{L} \subset \mathbf{C}$. For a smooth projective variety X/L, we sometime denote its realizations by $h^i_\sigma(X)$, $h^i_B(X)$, $h^i_{\mathrm{dR}}(X)$ and $h^i_f(X)$, respectively.

Proposition 3.2. *The functor* $\mathrm{M}_L \to \widetilde{\mathcal{R}}_L$ *is fully faithful.*

Proof. This follows by unwinding the definition of M_L, cf. [DMOS82], Ch.II, Theorem 6.7 (g). □

In the following sections, our aim is to show that the functor $\mathrm{M}_L \to \mathcal{R}_L$ becomes fully faithful when restricted to the full subcategory CM_L of CM-motives (cf. Section 3.4 below).

Example 3.3 (Tate twist)**.** There is a motive $\mathbf{Q}(1)$ in M_L whose realizations are given exactly as in (6) at the beginning of Section 3.1. In fact, one can define $\mathbf{Q}(1) = h^2(\mathbf{P}^1)^\vee$. For every integer $n \in \mathbb{Z}$, we set $\mathbf{Q}(n) := \mathbf{Q}(1)^{\otimes n}$. For M a motive in CM_L and $n \in \mathbb{Z}$, we define

$$M(n) := M \otimes \mathbf{Q}(n).$$

Remark 3.4. Of course, there are also the ℓ-adic realizations M_ℓ for every prime ℓ. They are finite dimensional $\mathbf{Q}_\ell$-vector spaces with a continuous linear G_L-action. Each M_ℓ may be recovered from M_f by scalar extension along $\mathbf{Q}_f \to \mathbf{Q}_\ell$. Let $\widetilde{\mathcal{R}}'_L$ and $\mathcal{R}'_L$ denote the categories of realizations analogous to $\widetilde{\mathcal{R}}_L$ and $\mathcal{R}_L$, where one uses the ℓ-adic realizations for all ℓ instead of the finite adelic one. Then the obivous functors $\widetilde{\mathcal{R}}_L \to \widetilde{\mathcal{R}}'_L$ and $\mathcal{R}_L \to \mathcal{R}'_L$ are fully faithful and hence all results of the present and the following sections remain true if one replaces $\widetilde{\mathcal{R}}_L$ by $\widetilde{\mathcal{R}}'_L$ and $\mathcal{R}_L$ by $\mathcal{R}'_L$.

3.3. **Descent.** Let L'/L be an extension of subfields of $\mathbf{C}$. Recall that the association $X \mapsto X \times_L L'$ defines a functor

$$(-) \times_L L' : \mathrm{M}_L \to \mathrm{M}_{L'},$$

the *extension of scalars functor from L to L'*. Note that for $M \in \mathrm{M}_L$, there is a canonical isomorphism $(M \times_L L')_f \cong M_f$ of $\mathbf{Q}_f$-modules, so that $(M \times_L L')_f$ acquires an action of G_L (which extends the action of $G_{L'}$). The following result is convenient for the purpose of defining morphisms:

Proposition 3.5 (Descent)**.** *Let L'/L be an extension in $\overline{L}$. Let $M, N \in \mathrm{M}_L$ and let $\varphi' : M \times_L L' \to N \times_L L'$ be a morphism in $\mathrm{M}_{L'}$ such that the finite adelic realization φ'_f is G_L-equivariant. Then there exists a unique morphism $\varphi : M \to N$ in CM_L such that $\varphi' = \varphi \times_L L'$.*

Proof. This is well-known, but let us recall an argument. Note that the claim follows from the corresponding statement about absolute Hodge cycles (equivalently, one may use Proposition 3.2 and copy the following argument for a compatible system of morphisms $(\varphi_\sigma, \varphi_{\mathrm{dR}}, \varphi_f)_{\sigma \in J_{L'}}$ in place of the absolue Hodge cycle x below): Let X/L be a smooth projective variety and $X' = X \times_L L'$. Let $x = (x_\sigma, x_{\mathrm{dR}}, x_f)_{\sigma \in J_L} \in \prod_{\sigma \in J_{L'}} H^{2p}_\sigma(X')(p) \times H^{2p}_{\mathrm{dR}}(X')(p) \times H^{2p}_f(X')(p)$ be an absolute Hodge cycle on X' such that x_f is G_L-invariant. Note that G_L acts on $H^{2p}_f(X') = H^{2p}_f(X)(p)$. Let $\sigma, \tau \in J_{L'}$ such that $\sigma|_L = \tau|_L$. Then $H^{2p}_\sigma(X')$ and $H^{2p}_\tau(X')(p)$ canonically identify. Choose an embedding $\overline{\sigma} : \overline{L} \to \mathbf{C}$ which restricts to σ. There is an element $\rho \in G_L$ such that $\overline{\sigma}\rho : \overline{L} \hookrightarrow \mathbf{C}$ restricts to τ on L'. By the compatibility of the comparison isomorphisms $I_{f,\overline{\sigma}}$

and $I_{f,\overline{\sigma}\rho}$ for the variety X/L, one has the commutative diagram

$$\begin{array}{ccc} H^{2p}_\sigma(X')(p)\otimes\mathbf{Q}_f & \xrightarrow{=} & H^{2p}_\tau(X')(p)\otimes\mathbf{Q}_f \\ {\scriptstyle I_{f,\overline{\sigma}}}\big\downarrow & & \big\downarrow{\scriptstyle I_{f,\overline{\tau}}} \\ H^{2p}_f(X')(p) & \xrightarrow{\rho} & H^{2p}_f(X')(p), \end{array}$$

which shows that the elements x_σ and x_τ identify in $H^{2p}_\sigma(X')(p)=H^{2p}_\tau(X')(p)$, i.e. the element x_σ only depends on $\sigma|_L$. Now the comparison isomorphism $I_{\infty,\sigma}$ for $X'=X\times_L L'$ only depends on $\sigma|_L$ and it follows that the images of the element $x_{\mathrm{dR}}\in H^{2p}_{\mathrm{dR}}(X')(p)\cong H^{2p}_{\mathrm{dR}}(X)(p)\otimes_L L'$ in $(H^{2p}_{\mathrm{dR}}(X)(p)\otimes_L L')\otimes_{L',\sigma}\mathbf{C}$ agree for all $\sigma\in J_{L'}$ that restrict to the same embedding of L. We may assume that L'/L is Galois. It follows that the element $x_{\mathrm{dR}}\in H^{2p}_{\mathrm{dR}}(X)\otimes_L L'$ is invariant under the action of $\mathrm{Gal}(L'/L)$ and hence comes from an element in $H^{2p}_{\mathrm{dR}}(X)$. Hence, the absolute Hodge cycle $x=(x_\sigma,x_{\mathrm{dR}},x_f)_{\sigma\in J_{L'}}$ on X' is induced by a unique absolute Hodge cycle on X. □

3.4. **CM-Motives.** Now, let

$$\mathrm{CM}_L\subset \mathrm{M}_L$$

denote the full Tannakian subcategory generated by the Artin motives over L, i.e. motives of zero-dimensional varieties over L, and motives of the form $h^1(A)$, where A/L is a potentially CM abelian variety. We remark that $\mathbf{Q}(n)$ for $n\in\mathbb{Z}$ lies in CM_L since $\mathbf{Q}(-1)$ can be obtained as $H^2(E)$ for any potentially CM elliptic curve over L (cf. [Bla86], 3.3). Let us recall the following consequence of Deligne's theorem ([DMOS82], Ch.I, Theorem 2.11) that every Hodge cycle on an abelian variety over an algebraically closed field is absolute Hodge. Let $\mathrm{Hod}_{\mathbf{Q}}$ denote the category of $\mathbf{Q}$-rational Hodge structures.

Theorem 3.6 (Deligne)**.** *Let L be algebraically closed and $\sigma: L\to\mathbf{C}$ a fixed embedding. Then the functor*

$$\mathrm{CM}_L\to\mathrm{Hod}_{\mathbf{Q}},\quad M\mapsto M_\sigma$$

is fully faithful.

Proof. We refer the reader [Sch88], Ch.1, Theorem 2.4.3, where it is explained how the result follows from Deligne's theorem. □

As an immediate consequence of Proposition 3.5 and Theorem 3.6, one obtains:

Corollary 3.7. *The functor $\mathrm{CM}_L\to\mathcal{R}_L$ is fully faithful. In fact, even more is true: For specifying a morphism $\varphi: M\to N$ in CM_L, it is equivalent to give a morphism $\varphi_B: M_B\to N_B$ of $\mathbf{Q}$-rational Hodge structure and a G_L-equivariant homomorphism $\varphi_f: M_f\to N_f$ such that φ_B and φ_f are compatible via the comparison isomorphism I_f.*

To fix notations, let us now recall the formalism from [Bla86], 3.1, which is justified in view of Corollary 3.7: Attached to an object M in CM_L there are the following realizations and comparison isomorphisms:

(1) A finite dimensional $\mathbf{Q}$-vector space M_B, the *Betti realization of* M, equipped with a decomposition
$$M_B\otimes\mathbf{C}=\bigoplus_{p,q\in\mathbb{Z}}M_B^{p,q},$$
(the *Hodge decomposition*) into $\mathbf{C}$-subspaces such that $(M_B^{p,q})^{1\otimes c}=M_B^{q,p}$.

(2) A finite dimensional L-vector space M_{dR}, the *de Rham realization of* M, equipped with a decreasing filtration F^pM_{dR}.

(3) A finite free $\mathbf{Q}_f$-module M_f, the *finite adelic realization of* M, equipped with a continuous linear action of the absolute Galois group G_L. We denote the action by $v\mapsto v^\tau$ for $\tau\in G_L$ and $v\in M_f$.

(4) A $\mathbf{C}$-linear isomorphism $I_\infty: M_{\mathrm{dR}}\otimes_L\mathbf{C}\xrightarrow{\sim}M_B\otimes\mathbf{C}$ satisfying
$$I_\infty(F^pM_{\mathrm{dR}}\otimes\mathbf{C})=\bigoplus_{p'\geq p,q}M_B^{p',q}.$$

(5) A $\mathbf{Q}_f$-linear isomorphism $I_f: M_B\otimes\mathbf{Q}_f\xrightarrow{\sim}M_f$.

We write $M = \{M_B, M_{\mathrm{dR}}, M_f, I_\infty, I_f\}$. A morphisms $\varphi : M \to N$ is a triple $\varphi = (\varphi_B, \varphi_{\mathrm{dR}}, \varphi_f)$, where:

(a) $\varphi_B : M_B \to N_B$ is a $\mathbf{Q}$-linear homomorphism such that $\varphi_B(M_B^{p,q}) \subset N_B^{p,q}$ for all $p, q \in \mathbb{Z}$;
(b) $\varphi_{\mathrm{dR}} : M_{\mathrm{dR}} \to N_{\mathrm{dR}}$ is a L-linear homomorphism satisfying $\varphi_{\mathrm{dR}}(F^p M_{\mathrm{dR}}) \subset F^p N_{\mathrm{dR}}$ for all $p \in \mathbb{Z}$;
(c) $\varphi_f : M_f \to N_f$ is a G_L-equivariant $\mathbf{Q}_f$-linear homomorphism;
(d) $\varphi_f \circ I_f = I_f \circ \varphi_B$;
(e) $I_\infty \circ \varphi_{\mathrm{dR}} = \varphi_B \circ I_\infty$.

A motive is *of weight* $w \in \mathbb{Z}$, if $M_B^{pq} = 0$ unless $p + q \neq w$. To make up for the other embeddings of L, we employ the following formalism: For every $\tau \in G_{\mathbf{Q}}$ and $M \in \mathrm{CM}_L$, there is a motive $M^\tau \in \mathrm{CM}_{L^\tau}$, which is induced by taking the base change along $\tau : L \to L^\tau$ on the level of varieties. There is a τ-semilinear bijection

$$M_{\mathrm{dR}} \to M_{\mathrm{dR}}^\tau, \omega \mapsto \omega^\tau$$

as well as a $\mathbf{Q}_f$-linear bijection

$$M_f \to M_f^\tau, v \mapsto v^\tau.$$

For a morphism $\varphi \in \mathrm{Hom}(M, N)$, the morphism $\varphi^\tau \in \mathrm{Hom}(M^\tau, N^\tau)$ is the unique morphism such that $\varphi_{\mathrm{dR}}^\tau(\omega^\tau) = \varphi_{\mathrm{dR}}(\omega)^\tau$ and $\varphi_f^\tau(v) = \varphi_f(v)^\tau$ for all $\omega \in M_{\mathrm{dR}}$ and $v \in M_f$. The $\mathbf{Q}_f$-linear homomorphism $M_f \to M_f^c$ induced by complex conjugation corresponds via I_f to a $\mathbf{Q}$-linear isomorphism $M_B \to M_B^c$. If $L \subset \mathbf{R}$, this identifies with the involution $F_\infty : M_B \to M_B$. Note that the object M^τ only depends on the restriction $\tau|_L \in J_L$, but not the maps $M_f \to M_f^\tau$. If $\tau \in G_L$, we recover the G_L-action on M_f.

Convention. *From now a motive over L is always understood to be an object in* CM_L.

3.5. **Extension and restriction of scalars.** Let L'/L be an extension of subfields of $\mathbf{C}$. Let M be a motive in CM_L. For the extension of scalars $M \times_L L' \in \mathrm{CM}_{L'}$ from L to L', one has $(M \times_L L')_{\mathrm{dR}} = M_{\mathrm{dR}} \otimes_L L'$, $(M \times_L L')_f = M_f$ with the Galois action restricted to $G_{L'} \subset G_L$ and all the other data is unchanged. Now assume that L'/L is an extension of number fields. Let $J_{L'/L} \subset J_L$ denote the set of embeddings of L which restrict to the fixed embedding $L \subset \overline{\mathbf{Q}}$. For every motive N in $\mathrm{CM}_{L'}$, there exists a motive $R_{L'/L}(N) \in \mathrm{CM}_L$, the *restriction of scalars of N from L' to L*, such that

$$\begin{aligned} R_{L'/L}(N)_B &= \bigoplus_{\sigma \in J_{L'/L}} N_B^\sigma, \\ R_{L'/L}(N)_{\mathrm{dR}} &= N_{\mathrm{dR}} \text{ regarded as a } L\text{-module, and} \\ R_{L'/L}(N)_f &= \bigoplus_{\sigma \in J_{L'/L}} M_f^\sigma, \end{aligned}$$

where an element $\tau \in G_L$ acts via the $\mathbf{Q}_f$-linear maps $\tau : M_f^\sigma \xrightarrow{\sim} M_f^{\tau\sigma}$, for $\sigma \in J_{L'/L}$. One easily checks that this G_L-module is isomorphic to the induced representation of N_f along $G_{L'} \subset G_L$.

Proposition 3.8. *Let L'/L be a finite extension. The functors $(-) \times_L L' : \mathrm{CM}_L \to \mathrm{CM}_{L'}$ and $R_{L'/L} : \mathrm{CM}_{L'} \to \mathrm{CM}_L$ are left and right adjoint to each other.*

Proof. By Corollary 3.7, everything can be checked on realizations. Now the result follows as in [Jan90], Proposition 2.20. □

3.6. **Motives with coefficients.** Let T be a finite-dimensional commutative semisimple $\mathbf{Q}$-algebra, i.e. a finite product of number fields.

Definition 3.9. *A motive M defined over L* with coefficients in T *is an object in* CM_L *together with a ring homomorphism*

$$T \to \mathrm{End}(M).$$

A morphism $\varphi : M \to N$ between motives with coefficients in T is a morphism in CM_L *which is compatible with the structure maps in the obvious way. We denote the category of motives over L with coefficients in T by* $\mathrm{CM}_L(T)$. *We call* $\dim_T M_B$ *the rank of the motive M.*

If $M \in \mathrm{CM}_L(T)$ has coefficients in T, then the realisations M_B, M_{dR} and M_f acquire the structures of finite free T-, $T \otimes L$- and $T_f := T \otimes \mathbf{Q}_f$-modules. Moreover the comparison isomorphism I_∞ (resp. I_f) is $T \otimes \mathbf{C}$-linear (resp. T_f-linear). Let T'/T be an extension of number fields. Let $M \in \mathrm{CM}_L(T)$. There is a motive $M \otimes_T T'$ in $\mathrm{CM}_L(T')$, the *extension of coefficients of M from T to T'*, such that $(M \otimes_T T')_B = M_B \otimes_T T'$, $(M \otimes_T T')_{\mathrm{dR}} = M_{\mathrm{dR}} \otimes_T T'$ and $(M \otimes_T T')_f = M_f \otimes_T T'$. Let N be a motive in $\mathrm{CM}_L(T')$. There exists a motive $R_{T'/T}N$, *the restriction of coefficients of N from T' to T*, such that $(R_{T'/T}N)_B$, $(R_{T'/T}N)_{\mathrm{dR}}$ and $(R_{T'/T}N)_f$ are given by N_B, N_{dR} and N_f regarded as T-, $T \otimes L$- and T_f-modules. For any two motives $M, N \in \mathrm{CM}_L(T)$, one can define a tensor product $M \otimes_T N$, which in terms of realisations is given by $(M \otimes_T N)_B = M_B \otimes_T N_B$, $(M \otimes_T N)_{\mathrm{dR}} = M_{\mathrm{dR}} \otimes_{T \otimes L} N_{\mathrm{dR}}$ and $(M \otimes_T N)_f = M_f \otimes_{T_f} N_f$. Similarly, one can define T-linear symmetric and exterior powers in the category $\mathrm{CM}_L(T)$.

3.7. **Motivic L-functions.** Let L and T be number fields and M an object in $\mathrm{CM}_L(T)$. Let v be a finite place of L. Fix a decomposition group $D_v \subset G_L$ of v and denote the inertia subgroup by I_v. Let $\mathrm{Fr}_v \in D_v$ denote a geometric Frobenius element at v. For any prime p such that $v \nmid p$, we define the polynomial

$$P_v(M,t) := \det_{T_p}(1 - \mathrm{Fr}_v \cdot t | M_p^{I_v}) \in T_p[t],$$

where $T_p := T \otimes \mathbf{Q}_p$. The definition of the L-function of M relies on the following:

Conjecture 3.10 (Independence of p)**.** *The polynomial $P_v(M,t)$ lies in $T[t]$ and is independent of the prime p with $v \nmid p$.*

Definition 3.11. *Assume that Conjecture 3.10 is true for M. For every finite place v of L, let Nv denote the cardinality of its residue field. We define the L-function of M as the Euler product*

$$L(M,s) := \prod_v P_v(M, Nv^{-s})^{-1}.$$

This converges for $\mathrm{Re}(s) \gg 0$ *and on its domain of convergence $L(M,s)$ is a holomorphic function with values in $T \otimes \mathbf{C}$. Moreover, if S is a finite set of places of L, we put $L_S(M,s) := \prod_{v \notin S} P_v(M, Nv^{-s})^{-1}$*

It follows from the definition that for any $n \in \mathbb{Z}$ one has

$$L(M(n), s) = L(M, s+n).$$

In the study of the values of $L(M,s)$ at integers, the above equality allows to restrict the attention to the value at $s = 0$.

3.8. **Critical values and Deligne's conjecture.** The L-function $L(M,s)$ can be completed by certain Γ-factors which depend on the Hodge realization M_B and conjecturally this completed L-function satisfies a functional equation. The motive M is called *critical* if the Γ-factors on both sides of this functional equation do not have a pole at $s = 0$ (cf. [Del79], Definition 1.3). In terms of the Hodge realization of M, this means the following: Let $RM := R_{L/\mathbf{Q}}M$ denote the restriction of scalars of M to $\mathbf{Q}$. Recall that complex conjugation induces a $\mathbf{Q}$-linear involution F_∞ on RM_B.

Definition 3.12. *The motive M is* critical *if it satisfies the following conditions:*

(i) $RM_B^{pq} = 0$ *unless (p,q) satisfies $p = q$, $(p < 0,\ q \geq 0)$ or $(p \geq 0,\ q < 0)$*
(ii) F_∞ *acts trivially on RM_B^{pp} if $p < 0$ and via multiplication by -1 if $p \geq 0$.*

Remark 3.13. One can show that M is critical in the above sense if and only if the Γ-factors on both sides of the conjectural functional equation for $L(M,s)$ do not have a pole at $s = 0$, cf. [Del79], Définition 1.3.

We let RM_B^+ denote the subspace of RM_B which is fixed by F_∞. Consider the composite

$$I_+ : RM_B^+ \otimes \mathbf{C} \subset RM_B \otimes \mathbf{C} \xrightarrow[\sim]{I_\infty^{-1}} RM_{\mathrm{dR}} \otimes \mathbf{C} \twoheadrightarrow RM_{\mathrm{dR}}/F^0 RM_{\mathrm{dR}} \otimes \mathbf{C}.$$

It is shown in [Del79], 1.7 that I_+ is a $T \otimes \mathbf{C}$-linear isomorphism if M is critical. Note that the definition of I_+ given here slightly differs from the one in loc. cit., but both definitions agree when M is critical.

Definition 3.14. *Let M be a critical motive in $\mathrm{CM}_L(T)$. We define*

$$c^+(\chi) := \det(I_\infty) \in (T \otimes \mathbf{C})^\times,$$

where the determinant is calculated with respect to T-bases of RM_B^+ and $RM_{\mathrm{dR}}/F^0 RM_{\mathrm{dR}}$. Note that $c^+(M)$ is only well-defined up to a factor in $T^\times$.

Conjecture 3.15 (Deligne conjecture)**.** *Let M be a critical motive over L with coefficients in T. Then*

$$L(M,0) = c^+(M) \mod T^\times.$$

Note that the L-value $L(M,0)$ can be replaced by the value $L_S(M,0)$ for any finite set of places S without changing the conjecture.

3.9. **Motives of Hecke characters.** Let L and T be number fields and assume for simplicity that L is totally imaginary. Following [Bla86], 3.2 (with references to [Sch88]), we briefly review the tight relationship between motives in $\mathrm{CM}_L(T)$ of rank 1 and algebraic Hecke character $\chi : \mathbb{A}_L^\times \to T^\times$. Recall that, by our assumption on L, any such character factors through $L_f^\times$.

Definition 3.16. *Let $\chi : L_f^\times \to T^\times$ be an algebraic Hecke character. We say that a motive $M \in \mathrm{CM}_L(T)$ is of type (L, χ, T) if M is of rank 1 and for every $\tau \in G_L$ and $\ell \in L_f^\times$ satisfying $r_L(\ell) = \tau|_{L^{\mathrm{ab}}}$, one has*

$$v^\tau = \chi_a(\ell)^{-1}\chi(\ell) \cdot v$$

for all $v \in M_f$.

Let M be a motive of type (L, χ, T). Let $\mathfrak{f} \subset \mathcal{O}_L$ be the exact conductor of χ. For any finite place $v|\mathfrak{f}$, one has $\chi|_{\mathcal{O}_{L_v}^\times} \not\equiv 1$ and hence $M_p^{I_v} = 0$ for any prime p with $v \nmid p$. If $v \nmid \mathfrak{f}$, the inertia group I_v acts trivially on M_p and any geometric Frobenius element Fr_v acts via multiplication by $\chi(\pi_v) \in T^\times$ where $\pi_v \in L_v^\times \subset L_f^\times$ is a uniformizer at v. This establishes Conjecture 3.10 for M and shows

$$L(M,s) = L_{\mathfrak{f}}(\chi, s),$$

where $L_{\mathfrak{f}}(\chi, s)$ is the Hecke L-function of χ defined in (2.1). If χ is of *conductor dividing* $\mathfrak{f}$, so that $\mathfrak{f}$ is not necessarily the exact conductor, and if S denote the finite set of places of L which divide $\mathfrak{f}$, then one has $L_S(M,s) = L_{\mathfrak{f}}(\chi, s)$. In particular, when it comes to Deligne's conjecture, we may enlarge the ideal $\mathfrak{f}$ if necessary.

In the next theorem we summarize a tight relationship between CM-motives and algebraic Hecke characters. In fact, there is a bijection between the isomorphism classes of *rank* 1 CM-motives and algebraic Hecke characters:

Theorem 3.17 (Deligne, Langlands)**.**

(i) *Any two motives of type (L, χ, T) are isomorphic in $\mathrm{CM}_L(T)$.*
(ii) *For every algebraic Hecke character $\chi : L_f^\times \to T^\times$, there exists a motive $M(\chi)$ of type (L, χ, T).*
(iii) *Every motive $M \in \mathrm{CM}_L(T)$ of rank 1 is of type (L, χ, T) for a unique algebraic Hecke character $\chi : L_f^\times \to T^\times$.*

Proof. This is a consequence of a theorem of Deligne, cf. [DMOS82], IV, which identifies the motivic Galois group of the Tannakian category $\mathrm{CM}_{\mathbf{Q}}$ (with respect to Betti realization as fibre functor) with Langlands' Taniyama group. Regarding (ii), we refer the reader to [Mil98], Section 15 for more information. The book [Sch88] contains a proof of the Theorem and, regarding (ii), the motive $M(\chi)$ is constructed in more down-to-earth terms from the cohomology of CM abelian varieties. Assertion (i) is loc. cit., Theorem I.5.1, assertion (ii) is loc. cit., Theorem I.4.1 and (iii) is loc. cit., Theorem I.6.6.1. Let us remark that the proof of the latter relies on loc. cit., Proposition I.1.4, which is deduced as a corollary of a result of Henniart ([Hen82]). This in turn makes use of results of Waldschmidt in transcendental number theory. In [Kha03], Khare gives a purely algebraic proof of [Sch88], Proposition I.1.4. □

The correspondence between T-isomorphism classes of motives of rank 1 in $\mathrm{CM}_L(T)$ with algebraic Hecke characters of L with values in T has the following nice properties:

Proposition 3.18 (cf. [Bla86], Proposition 3.2.1)**.** *Let $\chi : L_f^\times \to T^\times$ be an algebraic Hecke character of weight $w(\chi)$. Let $M(\chi)$ be a motive of type (L,χ,T) and let $\sigma \in G_{\mathbf{Q}}$. Let L'/L and T'/T be finite extensions.*

(i) *$M(\chi)$ is pure of weight $w(\chi)$.*
(ii) *$M(\chi)^\sigma$ is of type $(L^\sigma, \chi\circ\sigma^{-1}, T^\times)$.*
(iii) *$M(\chi)\times_L L'$ is of type $(L', \chi\circ N_{L'/L}, T)$.*
(iv) *$M(\chi)\otimes_T T'$ is of type (L,χ,T').*

Proof. (i) follows from the proposition below. The remaining assertions are clear. □

Remark 3.19. Let $\chi : L_f^\times \to T^\times$ be an algebraic Hecke character and $\alpha \in I_T^+$. Later, we will give a way of constructing a motive $M(\chi)^\alpha$ of type (L,χ^α,T^α) out of $M(\chi)$. This construction is explained in the proof of [Bla86], Proposition 5.5.1. It will play an important role in this thesis.

Let $\chi : L_f^\times \to T^\times$ be an algebraic Hecke character. Let $M(\chi)$ be a motive of type (L,χ,T). The next proposition shows how the Hodge decomposition of $M(\chi)$ is governed by the infinity-type $\chi_a \in I_L$ of χ. For every embedding $\tau \in J_T$, let $e(\tau) \in T\otimes\mathbf{C} = \mathbf{C}^{J_T}$ denote the idempotent corresponding to the embedding τ.

Proposition 3.20 (Hodge decomposition of $M(\chi)$)**.** *Let $\chi^* \in I_T$ denote the reflex of χ_a relative to T. For $\tau \in J_T$ let $M_{B,\tau} = e(\tau)(M_B\otimes\mathbf{C})$. Then*

$$M_{B,\tau} \subset M_B^{\chi^*(\tau),\chi^*(c\tau)}.$$

Proof. This is [Bla86], Proposition 3.2.2, also cf. [Sch88], Remark I.4.2. □

Proposition 3.21 (Critical Hecke characters)**.** *An algebraic Hecke character is critical in the sense of Definition 2.7 if and only if $M(\chi)$ is critical.*

Proof. This follows by using the functional equation for $L(\chi,s)$, which is not a conjecture in this case, or by using Proposition 3.20. □

Example 3.22 (Tate motive)**.** The motive $\mathbf{Q}(n)$ is of type $(L, \|\cdot\|^n, \mathbf{Q})$ since

$$\chi_{\mathrm{cyc}}(\tau) = N_{L/\mathbf{Q}}(\ell)\cdot\|\ell\|_L$$

for any $\tau \in G_L$ and $\ell \in L_f^\times$ satisfying $r_L(\ell) = \tau|_{L^{\mathrm{ab}}}$.

4. On a result of Goldstein-Schappacher

In the following, we generalize a theorem of Shimura (cf. [Shi94], Theorem 7.44) and Goldstein-Schappacher (cf. [GS81], Théorème 4.1) to motives of rank 1. The result gives sufficient and necessary conditions for the restriction of scalars of a motive of rank 1 to acquire coefficients in a commutative semisimple algebra such that it becomes of rank 1 again. The proof is essentially the same as in [Shi94] and [GS81], but we still provide a detailed proof for the convenience of the reader. Another valuable reference is [Den89], Section 2.

Theorem 4.1 (Shimura, Goldstein-Schappacher)**.** *Let F be a number field and M a motive of type (F,ψ,K). Let $E\subset F$ be a subfield such that there exists a type $\varphi_a \in I_E$ with $\psi_a = N^*_{F/E}(\varphi_a)$ and such that F/E is Galois. Then the following conditions are equivalent:*

(i) *The extension F/E is abelian and there is an algebraic Hecke character φ of E such that $\psi = \varphi\circ N_{F/E}$.*
(ii) *The induced representation $R_{F/E}(M)_f$ is abelian.*
(iii) *The restriction of scalars $R_{F/E}M$ acquires coefficients in a commutative semisimple K-algebra $\mathscr{T}$ such that it is of rank 1 over $\mathscr{T}$.*

Proof. In the following, let $G := \mathrm{Gal}(F/E)$. Recall that the induced representation is given by

$$R_{F/E}(M)_f = \bigoplus_{\sigma\in G} M_f^\sigma,$$

where the action of G_E is induced by the K_f-linear isomorphisms $M_f^\sigma \xrightarrow{\tau} M_f^{\tau\sigma}$ for all $\tau\in G_E$. We first show the implication (i) ⇒ (ii): Let $\sigma\in G$. By Proposition 3.18 and the assumption, the motive M^σ is of type (F,ψ,K). We have to show that the (topological) commutator G_E^c acts trivially. As F/E is assumed to be abelian, we have $G_E^c \subset G_F$. Let $\tau \in G_E^c$ and $x \in F_f^\times$ such that $r_F(x) = \tau|_{F^{\mathrm{ab}}}$. The restriction of $R_{F/E}(M)_f$ to G_F is just the direct sum of G_F-representations $\bigoplus_{\sigma\in G} M_f^\sigma$ and each M^σ being of type (F,ψ,K), we obtain that τ acts via multiplication by

$$\psi_a(x)^{-1}\psi(x) \in K_f^\times.$$

By assumption, we have

$$\psi_a(x)^{-1}\psi(x) = \varphi_a(N_{F/E}(x))^{-1}\varphi(N_{F/E}(x)).$$

The character φ and φ_a (viewed as a homomorphism on $E_f^\times$) agree on the kernel $\ker(r_E) = \overline{E^\times}$ of the Artin map. Hence, the claim follows from $r_E(N_{F/E}(x)) = r_F(x)|_{E^{\mathrm{ab}}} = \tau|_{E^{\mathrm{ab}}} = 1$, as τ was lying in the commutator G_E^c.

We now show the converse implication (ii) $\Rightarrow$ (i). Let $\rho : G_F \to \mathrm{Aut}_{K_f}(M_f)$ denote the representation of G_F on the finite adelic realization M_f and let us assume that the induced representation

$$\pi : G_E \to \mathrm{Aut}_{K_f}\Big(\bigoplus_{\sigma\in G} M_f^\sigma\Big)$$

is abelian, i.e. the commutator G_E^c is contained in $\ker(\pi)$. Moreover, since for an element $\tau \in G_E$, the automorphism $\pi(\tau)$ carries the summand indexed by id_F to the one indexed by $\tau|_F$, it follows that $\ker(\pi) \subset G_F$. Therefore, $G_E^c \subset G_F$ and as $G \cong G_E/G_F$ is now a quotient of G_E^{ab}, the extension F/E must be abelian. Moreover, since the G_F-representation M_f is a direct summand of the restriction $\pi|_{G_F}$, we deduce $G_E^c \subset \ker(\pi) \cap G_F \subset \ker(\rho)$, so that

$$\rho \text{ factors through } G_F/G_E^c \cong \mathrm{Gal}(E^{\mathrm{ab}}/F). \tag{4.1}$$

We now use this observation to show that the assignment

$$E^\times \cdot N_{F/E}(F_f^\times) \to K^\times, \quad x \cdot N_{F/E}(y) \mapsto \varphi_a(x)\psi(y) \tag{4.2}$$

is well-defined. Indeed, let us assume that $z \in E^\times$ is simultaneously of the form $z = N_{F/E}(y)$ for $y \in F_f^\times$. Basic properties of the Artin map imply

$$r_F(y)|_{E^{\mathrm{ab}}} = r_E(N_{F/E}(y)) = r_E(z) = 1.$$

Therefore, by our observation (4.1), the element $r_F(y) \in \mathrm{Gal}(F^{\mathrm{ab}}/F)$ acts trivially on M_f. It follows that $\psi_a(y)^{-1}\psi(y) = 1$ since M is of type (F,ψ,K), and hence

$$\varphi_a(z) = \psi_a(y) = \psi(y).$$

Thus, the map (4.2) is well-defined. It is moreover continuous as its kernel contains $N_{F/E}(\ker(\psi))$ which is open by the continuity of ψ and the fact that the idele norm is an open homomorphism. The group $E^\times \cdot N_{F/E}(F_f^\times)$ is of finite index in $E_f^\times$, and just as in [Shi94], Lemma 7.45, the homomorphism (4.2) can be extended to a continuous homomorphism

$$\varphi : E_f^\times \to T_\varphi^\times$$

where T_φ is a number field (which may be obtained by adjoining roots of unity to K).

The implication (iii) $\Rightarrow$ (ii) follows immediately from the fact that the representation factors through $\mathrm{Aut}_{\mathscr{T}_f}(R_{F/E}M)_f$, which is abelian as $R_{F/E}(M)_f$ is a free $\mathscr{T}_f = \mathscr{T} \otimes \mathbf{Q}_f$-module of rank 1. It remains to show that the equivalent statements (i) and (ii) imply assertion (iii), which we will show after we explain the construction of $\mathscr{T}$. □

Construction of $\mathscr{T}$: Assume the the equivalent hypotheses (i) and (ii) of the Theorem 4.1 hold. For each $\sigma \in G$, the motive M^σ is of type (F,ψ,K) since $\psi \circ \sigma^{-1} = \psi$ by (i). Thus, for $\sigma \in G$, the motives M^σ are pairwise K-isomorphic and $\mathrm{Hom}_K(M^\sigma, M)$ is a 1-dimensional K-vector space. We define the ring

$$\mathscr{T} := \bigoplus_{\sigma\in G} \mathrm{Hom}_K(M, M^\sigma)\cdot\sigma$$

consisting of formal linear combinations $\sum_{\sigma\in G}\phi_\sigma\cdot\sigma$ with $\phi_\sigma \in \mathrm{Hom}_K(M, M^\sigma)$, where the multiplication is determined by the rule

$$(\phi_\sigma\sigma)\cdot(\phi_\tau\tau) = (\phi_\sigma^\tau \circ \phi_\tau)\tau\sigma$$

for $\sigma, \tau \in G$ and $\phi_\sigma \in \mathrm{Hom}_K(M, M^\sigma)$, $\phi_\tau \in \mathrm{Hom}_K(M, M^\tau)$. Since M is of type (F, ψ, K), it comes equipped with an isomorphism $K \xrightarrow{\sim} \mathrm{End}_K(M)$, and this equips $\mathscr{T}$ with the structure of a K-algebra. There is a K-algebra homomorphism

$$\mathscr{T} \to \mathrm{End}_K(R_{F/E}M) \tag{4.3}$$

defined as follows: Let $\sum_\sigma \phi_\sigma \sigma \in \mathscr{T}$ with $\phi_\sigma \in \mathrm{Hom}_K(M, M^\sigma)$ for $\sigma \in G$. Then the ϕ_σ induce a morphism

$$M \to \bigoplus_{\sigma \in G} M^\sigma \cong R_{F/E}M \times_E F$$

of motives over F with coefficients in K, and by adjunction (cf. Proposition 3.8(i)) this corresponds to a morphism $R_{F/E}M \to R_{F/E}M$ of motives over E with coefficients in K. It is completely formal to check that this actually defines a K-algebra homomorphism.

Proof of (i),(ii) $\Rightarrow$ (iii). Finally, let us discuss how the equivalent statements (i) and (ii) imply (iii), i.e. that $\mathscr{T}$ is a commutative semisimple K-algebra and that $R_{F/E}M$ equipped with the coefficients (4.3) is of rank 1. Let $n := [F : E]$ and choose a finite place λ of K. Let $V_\lambda = (R_{F/E}M)_\lambda = \bigoplus_{\sigma \in G} M^\sigma_\lambda$. By assumption (ii), the λ-adic realization

$$\pi : G_E \to \mathrm{Aut}_{K_\lambda}(V_\lambda)$$

is an abelian representation of G_E on an n-dimensional K_λ-vector space. Let $\mathcal{I}$ denote the image of the group algebra $K_\lambda[G_E]$ in $\mathrm{End}_{K_\lambda}(V_\lambda)$. By assumption this is a commutative K_λ-subalgebra of $\mathrm{End}_{K_\lambda}(V_\lambda)$.

Lemma 4.2. *For each $\sigma \in G$, let $\widetilde{\sigma} \in G_E$ denote a lift of σ. Then the elements $\pi(\widetilde{\sigma})$ for $\sigma \in G$ form a K_λ-basis of $\mathcal{I}$.*

Proof of the lemma. By definition of the representation π on $V_\lambda = \bigoplus_{\tau \in G} M^\tau_\lambda$, one observes that for each $\sigma \in G$, the automorphism $\pi(\widetilde{\sigma})$ induces an isomorphism

$$\pi(\widetilde{\sigma}) : M_\lambda \xrightarrow{\sim} M^\sigma_\lambda.$$

Let $v \in M_\lambda$ be a K_λ-basis. Then the elements $\pi(\widetilde{\sigma})(v)$ for $\sigma \in G$ form a K_λ-basis of V_λ. Let $a_\sigma \in K_\lambda$ for $\sigma \in G$ such that

$$\sum_{\sigma \in G} a_\sigma \pi(\widetilde{\sigma}) = 0$$

in $\mathcal{I} \subset \mathrm{End}_{K_\lambda}(V_\lambda)$. Evaluating at $v \in M_\lambda \subset V_\lambda$ gives

$$\sum_{\sigma \in G} a_\sigma \pi(\widetilde{\sigma})(v) = 0$$

in V_λ, and by the above observation, it follows that $a_\sigma = 0$ for all $\sigma \in G$. Thus, the elements $\pi(\widetilde{\sigma})$, $\sigma \in G$ are K_λ-linearly independent in $\mathcal{I}$.

It remains to show that the $\pi(\widetilde{\sigma})$, $\sigma \in G$ generate the K_λ-vector space $\mathcal{I}$. Consider an element of the form

$$\pi(\tau) \in \mathcal{I}$$

for $\tau \in G_E$. One has $\tau = \tau' \cdot \widetilde{\tau|_F}$ with $\tau' \in G_F$. Since $\pi|_{G_F}$ is the direct sum of the G_F-representations M^σ_λ for $\sigma \in G$ and since each M^σ is of type (F, ψ, K), the automorphism $\pi(\tau')$ acts via multiplication by an element $a \in K^\times_\lambda$ (namely the λ-component of $\psi_a(x)^{-1}\psi(x)$ where $x \in F^\times_f$ satisfies $r_F(x) = \tau'|_{F^{\mathrm{ab}}}$). Therefore,

$$\pi(\tau) = a \cdot \pi(\widetilde{\tau|_F}),$$

and the claim follows. □

We now show that $\mathcal{I}$ is its own commutant in $\mathrm{End}_{K_\lambda}(V_\lambda)$ and that it is a semisimple K_λ-algebra.

Lemma 4.3. *The K_λ-algebra $\mathcal{I}$ satisfies the following properties:*

(i) *$\mathcal{I}$ is its own commutant in* $\mathrm{End}_{K_\lambda}(V_\lambda)$.
(ii) *$\mathcal{I}$ is semisimple.*

Before we give a proof of this lemma, let us show how it allows to finish the proof of Proposition 4.1. Note that (4.3) induces an injective K_λ-algebra homomorphism $\mathscr{T}_\lambda \to \mathrm{End}_{K_\lambda}(V_\lambda)$ whose image commutes with $\mathcal{I}$. The above lemma now gives that $\mathscr{T}_\lambda = \mathcal{I}$ and hence $\mathscr{T}_\lambda$ is a commutative semisimple K_λ-algebra. It follows that $\mathscr{T}$ is a commutative semisimple K-algebra. Moreover, the λ-adic realization $V_\lambda = \bigoplus_{\sigma\in G} M_\lambda^\sigma$ of $R_{F/E}M$ is generated by any non-zero element in M_λ (regarded as being in the id-component). Thus, there is a surjective $\mathscr{T}_\lambda$-linear homomorphism $\mathscr{T}_\lambda \twoheadrightarrow V_\lambda$ which has to be an isomorphism since both source and target are of the same dimension n over K_λ. It follows that $R_{F/E}M$ is of rank 1 over $\mathscr{T}$. □

Proof of Lemma 4.3. (i) Fix a bijection $G \cong \{1,\ldots,n\}$. There is a simply transitive action of G on $\{1,\ldots,n\}$ induced by the action of G on itself by multiplication. The diagonal action on $\{1,\ldots,n\}$ is still simple and the set of orbits $G\backslash\{1,\ldots,n\}^2$ is in bijection with $\{1,\ldots,n\}$ by sending an element $i \in \{1,\ldots,n\}$ to the orbit of $(1,i)$. We choose a K_λ-basis of V_λ. Together with the fixed bijection $G \cong \{1,\ldots,n\}$ this yields an identification $\mathrm{End}_{K_\lambda}(V_\lambda) \cong M_{n\times n}(K_\lambda)$ and we regard $\mathcal{I}$ as a K_λ-subalgebra. For each $\sigma \in G$ let $\widetilde{\sigma} \in G_E$ denote a lift. Then the matrix representing $\pi(\widetilde{\sigma})$ is of the form

$$P_\sigma = (\beta_{i,\sigma}\cdot\delta_{i\sigma(j)})_{i,j=1,\ldots,n}$$

where $\beta_{i,\sigma} \in K_\lambda^\times$ and $\delta_{i\sigma(j)}$ denotes the Kronecker delta. By Lemma 4.2, the matrices P_σ form a K_λ-basis of $\mathcal{I}$. Let $A = (a_{ij})_{i,j} \in M_{n\times n}(K_\lambda)$ be an element in the commutant of $\mathcal{I}$ in $M_{n\times n}(K_\lambda)$, i.e. A commutes with the P_σ for all $\sigma\in G$. For the entries of A this means that

$$\beta_{\sigma(j),\sigma}\cdot a_{\sigma(i)\sigma(j)} = \beta_{i,\sigma}\cdot a_{ij} \tag{4.4}$$

for all $\sigma \in G$ and $i,j \in \{1,\ldots,n\}$. Note that this condition is also satisfied if A is one of the matrices P_τ for $\tau \in G$ since $\mathcal{I}$ is commutative by assumption. It follows from (4.4) and our initial remark on the action of G on $\{1,\ldots,n\}^2$ that any matrix commuting with all P_σ, $\sigma\in G$ is determined by its first row. We claim that

$$A = \sum_{\sigma\in G} a_{1,\sigma^{-1}(1)}\beta_{1,\sigma}^{-1}\cdot P_\sigma. \tag{4.5}$$

Indeed, the right-hand side lies in $\mathcal{I}$ and therefore also satisfies (4.4), so we only have to compare the first rows. The first row of $a_{1,\sigma^{-1}(1)}\beta_{1,\sigma}P_\sigma$ is $(0,\ldots,0,a_{1,\sigma^{-1}(1)},0,\ldots,0)$ with the only potentially non-zero element in the $\sigma^{-1}(1)$-entry. Thus, the first row of the right-hand side matrix in (4.5) is given by $(a_{11},\ldots,a_{1n})$ and assertion (i) follows.

(ii) Fix an algebraic closure $\overline{K}_\lambda$ of K_λ. Let $\sigma\in G$ and $\widetilde{\sigma}\in G_E$ be a lift. Let N denote the order of σ in the group G. Then $\widetilde{\sigma}^N \in G_F$ and $\pi(\widetilde{\sigma})^N$ acts via multiplication by an element $\beta \in K_\lambda^\times$. Hence, the minimal polynomial of $\pi(\widetilde{\sigma})$ divides the separable polynomial $X^N - \beta$ and therefore $\pi(\widetilde{\sigma})$ is diagonalizable over $\overline{K}_\lambda$. Since the endomorphisms $\pi(\widetilde{\sigma})$ for $\sigma\in G$ commute with each other and they generate $\mathcal{I}$ as a K_λ-vector space by Lemma 4.2, it follows that every element in $\mathcal{I}$ can be simultaneously diagonalized over $\overline{K}_\lambda$, hence $\mathcal{I}\otimes_{K_\lambda}\overline{K}_\lambda$ is semisimple. It follows that $\mathcal{I}$ is a semisimple K_λ-algebra. □

Remark 4.4.

(1) If K is an imaginary quadratic field, F/E a finite extension and E/F an elliptic curve with complex multiplication by K, then applying Theorem 4.1 to $M = h^1(E)$ with coefficients in K, one essentially recovers [GS81], Théorème 4.1. The equivalence between assertions (i) and (ii) is due to Shimura, cf. [Shi94], Theorem 7.44.

(2) Taking $M = h^1(A)$ for an abelian variety A/F of CM-type (K,Φ), the minimal subfield $E\subset F$ satisfying the conditions of the theorem is precisely the reflex field of (K,Φ). In general, the minimal such subfield $E\subset F$ is the minimal subfield of definition of ψ_a, or equivalently the field of values $K^{\psi_a^*}$ where $\psi_a^*\in I_K$ denotes the reflex of ψ_a relative to K.

5. Abelian varieties with complex multiplication

5.1. **CM abelian varieties.** We recall some facts about abelian varieties with complex mutliplication. Let L be a totally imaginary number field of degree $2g$ over $\mathbf{Q}$ which contains a CM-field K. Let F be a number field.

Definition 5.1 ([ST68], section 4)**.** *An* abelian variety with complex multiplication by L *is an abelian variety A/F of dimension g together with an embedding*

$$i : L \hookrightarrow \mathrm{End}^0_F(A) := \mathrm{End}_F(A) \otimes_{\mathbb{Z}} \mathbf{Q},$$

which we then refer to as the CM-structure of A/F.

If (A, i) is an abelian variety with complex multiplication by L over F, the ring

$$L \cap \mathrm{End}_F(A) = i^{-1}(\mathrm{End}_F(A))$$

is an order in L. If it is the maximal order $\mathcal{O}_L$, we say that (A, i) *has complex multiplication by* $\mathcal{O}_L$. We usually drop the embedding i from the notation.

In the following, let $\pi : A \to F$ be an abelian variety with complex multiplication by L. We denote its unit section by e. Let $A^\vee$ denote the dual abelian variety of A and equip it with the CM-structure $L \to \mathrm{End}_F(A) \xrightarrow{\vee} \mathrm{End}_F(A^\vee)$. As it is usual, we define

$$\omega_{A/F} := e^*\Omega^1_{A/F} \cong \pi_*\Omega^1_{A/F}$$

and let $\mathrm{Lie}(A/F) = \omega^\vee_{A/F}$ denote the relative Lie algebra of A/F. We let $H^1_{\mathrm{dR}}(A/F)$ (or sometimes $h^1_{\mathrm{dR}}(A)$) denote the algebraic de Rham cohomology of A/F. We simply write $H^1(A/\mathbf{C})$, $\omega_{A/\mathbf{C}}$ and $\mathrm{Lie}(A/\mathbf{C})$ instead of $H^1_{\mathrm{dR}}(A \times_F \mathbf{C}/\mathbf{C})$, $\omega_{A\times_F\mathbf{C}/\mathbf{C}}$ and $\mathrm{Lie}(A \times_F \mathbf{C}/\mathbf{C})$. Via the CM-structure $\omega_{A/F}$, $\mathrm{Lie}(A/F)$ and $H^1_{\mathrm{dR}}(A/F)$ acquire the structures of $L \otimes F$-modules. There is a canonical isomorphism $R^1\pi_*\mathcal{O}_A \cong \mathrm{Lie}(A^\vee/F)$ and an exact sequence

$$0 \to \omega_{A/F} \to H^1_{\mathrm{dR}}(A/F) \to \mathrm{Lie}(A^\vee/F) \to 0 \tag{5.1}$$

of $L \otimes F$-modules (the "Hodge filtration").

Via the CM-structure, the singular cohomology $H^1(A(\mathbf{C}), \mathbf{Q})$ is an L-vector space and it is 1-dimensional since its $\mathbf{Q}$-dimension is $2g = [L : \mathbf{Q}]$. In particular $H^1(A(\mathbf{C}), \mathbf{C})$ is a free $L \otimes \mathbf{C}$-module of rank 1. By the Hodge decomposition theorem, one has $H^1(A(\mathbf{C}), \mathbf{C}) \cong \omega_{A/\mathbf{C}} \oplus \overline{\omega_{A/\mathbf{C}}}$ with respect to the real structure $H^1(A(\mathbf{C}), \mathbf{R})$. It follows that there exists a CM-type Φ of L such that L acts through the g embeddings in Φ on $\omega_{A/\mathbf{C}}$. We say that A/F *is of CM-type* (L, Φ).

5.2. **Weil pairing.** For any abelian variety A/F with complex multiplication by L, there is a perfect pairing (in fact, an isomorphism)

$$h^1(A) \otimes_L h^1(A^\vee) \to L(-1)$$

of motives over F with coefficients in L which induces the corresponding well-known pairings in each realization: In the de Rham realization one has a canonical pairing $H^1_{\mathrm{dR}}(A/F) \otimes_{L\otimes F} H^1_{\mathrm{dR}}(A^\vee/F) \to L \otimes F$, which is an L-linear version of the pairing given by [BBM82], Théorème 5.1.6. In the finite adelic realization, it induces a perfect G_F-equivariant pairing $\langle\cdot,\cdot\rangle_A : h^1_f(A) \otimes_{L_f} h^1_f(A^\vee) \to L_f(-1)$ which is an dual (and L-linear) version of the usual Weil pairing $V_f(A) \otimes V_f(A^\vee) \to \mathbf{Q}(1)$. For later reference, let us observe that for any $\tau \in G_{\mathbf{Q}}$ the diagram

$$\begin{tikzcd}
h^1_f(A) \otimes h^1_f(A^\vee) \arrow[r, "\langle\cdot,\cdot\rangle_A"] \arrow[d, "\tau"'] & L_f(-1) \arrow[d, "\tau"] \\
h^1_f(A^\tau) \otimes h^1_f(A^{\tau,\vee}) \arrow[r, "\langle\cdot,\cdot\rangle_{A^\tau}"] & L_f(-1)
\end{tikzcd}$$

commutes. Similarly, there is a canonical pairing $h^1_B(A) \otimes_L h^1_B(A^\vee) \to (2\pi i)^{-1}L$ and the above pairings are compatible via the comparison isomorphisms. For further information, we refer the reader to [Del74], 10.2.

5.3. **Serre's tensor construction.** In this section, we collect a few facts about Serre's tensor construction. More details can for example be found in [CCO14], 1.7.4. Let A/F be an abelian variety with CM by $\mathcal{O}_L$. For every finitely generated projective $\mathcal{O}_L$-module $\mathfrak{a}$, there is an abelian variety

$$A(\mathfrak{a}) := A \otimes_{\mathcal{O}_L} \mathfrak{a}$$

defined over F which represents the functor

$$\mathrm{Sch}^{\mathrm{op}}_F \to \mathrm{Ab}, (A \otimes_{\mathcal{O}_L} \mathfrak{a})(T) = A(T) \otimes_{\mathcal{O}_L} \mathfrak{a}.$$

If $\mathfrak{a}, \mathfrak{b}$ are fractional ideals of L with $\mathfrak{a} \subset \mathfrak{b}$, one has a canonical isogeny $A(\mathfrak{a}) \to A(\mathfrak{b})$ of degree $[\mathfrak{b} : \mathfrak{a}]$. In the case where $\mathfrak{a} \subset \mathcal{O}_L$ is an integral ideal, we denote the isogeny induced by the inclusion $\mathcal{O}_L \subset \mathfrak{a}^{-1}$ by

$$[\mathfrak{a}] : A \to A(\mathfrak{a}^{-1}) = A \otimes_{\mathcal{O}_L} \mathfrak{a}^{-1}$$

and define

$$A[\mathfrak{a}] := \ker([\mathfrak{a}]).$$

Since we are working over a base field of characteristic zero, every isogeny is étale. In particular, $[\mathfrak{a}]$ induces an isomorphism

$$[\mathfrak{a}]^* : \omega_{A(\mathfrak{a}^{-1})/F} \xrightarrow{\sim} \omega_{A/F}$$

Similarly, the dual isogeny $[\mathfrak{a}]^\vee$ induces an isomorphism

$$[\mathfrak{a}]_\# := ([\mathfrak{a}]^\vee)^* : \omega_{A/F} \to \omega_{A(\mathfrak{a}^{-1})/F}.$$

5.4. **The associated Hecke character.** As before, let A/F be an abelian variety with CM by L. Let Φ be the CM-type of A/F and let E denote the reflex field of (L, Φ). Then $F \supset E$ (and in particular, F is totally imaginary). Let $\Phi^* \in I_E$ be the reflex CM-type of Φ on E.

Theorem 5.2 (Associated Hecke character)**.** *There is a unique algebraic Hecke character*

$$\psi : F_f^\times \to L^\times$$

of infinity-type $\mathrm{Inf}_{F/E}(\Phi^*)$ *such that the motive* $h^1(A)$ *is of type* (F, ψ, L)*.*

Proof. This is a classical statement in the theory of complex multiplication, cf. [ST68], Theorem 10 or [CCO14], Theorem 2.5.1. □

Let us state a Theorem due to Casselman which achieves the converse.

Theorem 5.3 (Casselman)**.** *Let L be a CM-field of degree $[L : \mathbf{Q}] = 2g$ and Φ be a CM-type on L. Let (E, Φ^*) be the reflex CM-pair of (L, Φ). Let F/E be a finite extension and $\psi : F_f^\times \to L^\times$ an algebraic Hecke character of infinity-type $\mathrm{Inf}_{F/E}(\Phi^*)$. There exists a g-dimensional abelian variety A/F together with a CM-structure $L \hookrightarrow \mathrm{End}^0(A)$ such that A/F is of CM-type (L, Φ) and has associated CM-character ψ. The CM-abelian variety $(A, L \hookrightarrow \mathrm{End}^0(A))$ is unique up to L-linear isogeny over F.*

Proof. [CCO14], Theorem 2.5.2. □

5.5. **An application of Theorem 4.1.** Let L be a totally imaginary number field containing a CM-field. Let F be a number field and let A/F be an abelian variety equipped with a CM-structure $L \hookrightarrow \mathrm{End}^0(A)$. Let $\Phi \in I_L$ denote the CM-type of A and let E be the reflex field of (L, Φ). Then $F \supset E$ and we assume in the following that F/E is an abelian extension. Let $\psi : F_f^\times \to L^\times$ be the CM-character associated to A/F.

Corollary 5.4. *The following assertions are equivalent:*

(i) *There exists an algebraic Hecke character φ of E such that*

$$\psi = \varphi \circ N_{F/E}.$$

(ii) *The extension $F(A_{\mathrm{tor}})/E$ is abelian.*

Here $F(A_{\mathrm{tor}})/F$ is the field extension of F generated by the torsion points of A, i.e. the fixed field of the kernel of the representation of G_F on the rational Tate module $V_f(A)$.

Proof. As the infinity-type of ψ is induced from Φ^* and F/E is abelian by assumption, the equivalence of assertions (i) and (ii) follows from the equivalence of the corresponding assertions (i) and (ii) in Theorem 4.1. Note that $h^1(A)_f$ is the dual representation of $V_f(A)$. □

Casselman's theorem combined with Corollary 5.4 provides a powerful tool to construct CM abelian varieties.

Proposition 5.5. *Let L be a totally imaginary number field containing a CM-field. Let Φ be a CM-type of L and E the reflex field of (L, Φ). Let $\mathfrak{a} \subset \mathcal{O}_L$ be an integral ideal. There exists a finite abelian extension F/E and an abelian variety A/F with complex multiplication by L satisfying the following properties:*

(i) *A has CM by $\mathcal{O}_L$ and is of CM-type (L, Φ).*

(ii) *There is an algebraic Hecke character φ of E such that $\psi = \varphi \circ N_{F/E}$.*
(iii) *There is an isomorphism $\xi : \mathcal{O}_L \cong H_1(A(\mathbf{C}), \mathbb{Z})$.*
(iv) *Every $[\mathfrak{a}]$-torsion point of A is defined over F.*

Proof. Let K denote the maximal CM-subfield of L and Φ_K the CM-type of K from which Φ is induced. Let $\varphi : E_f^\times \to T_\varphi^\times$ be an auxiliary Hecke character with infinity type $\varphi_a = \Phi^*$, which takes values in an extension of K. Such a Hecke character always exists if one chooses T_φ large enough, cf. [Sch88], 0.3. Since φ is continuous, the preimage $\varphi^{-1}(K^\times)$ is an open subgroup of $E_f^\times$ containing $E^\times$. By class field theory, there exists a finite abelian extension F/E such that $N_{F/E}(C_F) \subset \varphi^{-1}(K^\times)/E^\times$ and hence the Hecke character $\psi := \varphi \circ N_{F/E}$ takes values in $K^\times$. By Casselman's Theorem 5.3, there exists an abelian variety B/F of CM-type (K, Φ_K) whose Hecke character is given by ψ. Here B/F with its CM-structure $K \to \mathrm{End}_F^0(B)$ is only determined up to a K-linear isogeny. By using Serre's tensor construction (cf. [CCO14], Proposition 1.7.4.5), we may assume that the CM-order of B/F is given by $\mathcal{O}_K$. Then the $\mathcal{O}_K$-module $H_1(B(\mathbf{C}), \mathbb{Z})$ is invertible and by applying Serre's tensor construction again, we may assume that there is an isomorphism $H_1(B(\mathbf{C}), \mathbb{Z}) \cong \mathcal{O}_K$. The abelian variety $A := B \otimes_{\mathcal{O}_K} \mathcal{O}_L$ acquires CM by $\mathcal{O}_L$, is of CM-type (L, Φ) and its first integral homology is a free $\mathcal{O}_L$-module of rank 1. Note that CM-character of A/F is ψ with values regarded in $L^\times$. By construction A/F satisfies condition (i) of Corollary 5.4, hence the extension $F(A_{\mathrm{tor}})/E$ is abelian. It follows that we may pass to a larger extension F/E, so that every $[\mathfrak{a}]$-torsion point is defined over F and such that F/E still is an abelian extension. □

5.6. **The main theorem of complex multiplication.**

The type transfer. In the following, let L denote a number field containing a CM-field and let Φ be an induced CM-type of L. Let E denote the reflex field of (L, Φ) and let $\Phi^* \in I_E$ be the reflex CM-type. For each embedding $\sigma \in J_L$, we choose a lift $w_\sigma \in G_{\mathbf{Q}}$ such that $cw_\sigma = w_{c\sigma}$. We define the type transfer

$$\mathrm{tr}_\Phi : G_{\mathbf{Q}} \to G_L^{\mathrm{ab}}$$

by the formula

$$\mathrm{tr}_{L,\Phi}(\tau) = \prod_{\sigma \in \Phi} (w_{\tau\sigma}^{-1} \tau w_\sigma)^{-1} \mod G_L^{\mathrm{c}}.$$

Proposition 5.6.

(i) $\mathrm{tr}_{L,\Phi}$ *is a well-defined continuous map which is independent of the choice of lifts w_σ for $\sigma \in J_L$.*
(ii) *For all $\tau_1, \tau_2 \in G_{\mathbf{Q}}$, one has $\mathrm{tr}_{L,\Phi}(\tau_1\tau_2) = \mathrm{tr}_{L,\Phi\circ\tau_2^{-1}}(\tau_1) \cdot \mathrm{tr}_{L,\Phi}(\tau_2)$.*
(iii) *Let $L' \supset L$ be a finite extension and $\Phi' = \mathrm{Inf}_{L'/L}(\Phi)$. Let $\mathrm{tr}_{L'/L} : G_L^{\mathrm{ab}} \to G_{L'}^{\mathrm{ab}}$ denote the usual transfer map. One has $\mathrm{tr}_{L',\Phi'} = \mathrm{tr}_{L'/L} \circ \mathrm{tr}_{L,\Phi}$.*
(iv) *The restriction $\mathrm{tr}_{L,\Phi} : G_E \to G_L^{\mathrm{ab}}$ is a homomorphism and the diagram*

$$\begin{array}{ccc} E_f^\times & \xrightarrow{\Phi^{*-1}} & L_f^\times \\ {\scriptstyle r_K}\downarrow & & \downarrow{\scriptstyle r_L} \\ G_E^{\mathrm{ab}} & \xrightarrow{\mathrm{tr}_{L,\Phi}} & G_L^{\mathrm{ab}} \end{array}$$

is commutative.

Proof. As (i) is usually proven in the literature under the condition that L is a CM-field, let us give an argument. Any other choice of lifts for the $\sigma \in J_L$ is of the form $w'_\sigma = w_\sigma \cdot g_\sigma$ with $g_\sigma \in G_L$. It is compatible with complex conjugation if and only if

$$g_{c\sigma} = g_\sigma \tag{5.2}$$

for all $\sigma \in J_L$. Let $\tau \in G_{\mathbf{Q}}$. Using the lifts w'_σ, the defining expression for $\mathrm{tr}_\Phi(\tau)$ then changes by the factor $\prod_{\sigma\in\Phi}(g_{\tau\sigma}^{-1} g_\sigma)^{-1}$ in G_L^{ab}. It therefore suffices to show

$$\prod_{\sigma \in \tau\Phi} g_\sigma = \prod_{\sigma \in \Phi} g_\sigma$$

in G_L^{ab}. For this we decompose $\tau\Phi = (\Phi \cap \tau\Phi) \sqcup (\overline{\Phi} \cap \tau\Phi)$. Complex conjugation carries $\overline{\Phi} \cap \tau\Phi$ bijectively to $\Phi \cap \overline{\tau\Phi}$ and using (5.2), the claim follows. Part (ii) is straighforward from the definition of $\mathrm{tr}_{L,\Phi}$. For (iii), recall that the usual transfer map $\mathrm{tr}_{L'/L}$ for the finite extension L'/L can be calculated as follows: Let us choose a representative $v_\rho \in G_L$ for every coset $\rho \in G_L/G_{L'}$. Then, for any $\tau \in G_L$, one has

$$\mathrm{tr}_{L'/L}(\tau) = \prod_{\rho \in G_L/G_{L'}} v_{\tau\rho}^{-1} \tau v_\rho \mod G_{L'}^c.$$

Additionally, choose lifts w_σ for every $\sigma \in J_L$ satisfying $cw_\sigma = w_{c\sigma}$. We claim that the set

$$\{w_\sigma \cdot v_\rho \mid \sigma \in J_L, \rho \in G_L/G_{L'}\}$$

forms a choice of lifts for the embeddings of L' which is still compatible with complex conjugation. Indeed, it has the same cardinality as $J_{L'}$ and one checks that, by construction, the restrictions of any two different elements to L' give rise to different embeddings. Hence, these lifts run through the lifted CM-type Φ' of L' if one lets σ runs through Φ. Moreover, observe that for $\tau \in G_{\mathbf{Q}}$ the element

$$w_{\tau\sigma} v_{(w_{\tau\sigma}^{-1} \tau w_\sigma) \cdot \rho}$$

lifts $\tau(w_\sigma v_\rho)|_{L'}$. Therefore, in $G_{L'}^{\mathrm{ab}}$ one has

$$\begin{aligned}
\mathrm{tr}_{L',\Phi'}(\tau)^{-1} &= \prod_{\substack{\sigma \in \Phi \\ \rho \in G_L/G_{L'}}} (w_{\tau\sigma} v_{(w_{\tau\sigma}^{-1}\tau w_\sigma)\cdot\rho})^{-1} \tau(w_\sigma v_\rho) \\
&= \prod_{\substack{\sigma \in \Phi \\ \rho \in G_L/G_{L'}}} v_{(w_{\tau\sigma}^{-1}\tau w_\sigma)\cdot\rho}^{-1} (w_{\tau\sigma}^{-1}\tau w_\sigma) v_\rho \\
&= \prod_{\sigma \in \Phi} \mathrm{tr}_{L'/L}(w_{\tau\sigma}^{-1} \tau w_\sigma) \\
&= \mathrm{tr}_{L'/L}(\mathrm{tr}_{L,\Phi}(\tau))^{-1},
\end{aligned}$$

as desired. By (iii) and the injectivity of the transfer $\mathrm{tr}_{L'/L}$, we reduce assertion (iv) to the case where L is a CM-field and then the statement can be found in [Lan83], Ch.7, Theorem 1.1. Note that the type-transfer V_Φ defined in loc. cit. is the inverse of $\mathrm{tr}_{L,\Phi}$. □

The reciprocity law. If L is a CM-field and $\tau \in G_{\mathbf{Q}}$, we define

$$L(\Phi, \tau) := \{\lambda \in L_f^\times \mid r_L(\lambda) = \mathrm{tr}_{L,\Phi}(\tau) \text{ and } \lambda\lambda^c \chi_{\mathrm{cyc}}(\tau) \in L^\times\}. \tag{5.3}$$

It is shown in [Lan83], Ch.7, Theorem 2.2 that $L(\Phi, \tau)$ is a $L^\times$-coset in $L_f^\times$. If, more generally, L is a totally imaginary field containing a CM-field K and Φ is a type of L which is lifted from a CM-type Φ_K of K, we define

$$L(\Phi, \tau) := L^\times \cdot K(\Phi_K, \tau). \tag{5.4}$$

By Proposition 5.6 (iii) and the compatibility of the Artin map with the transfer, this definition is independent of the choice of (K, Φ_K) and we have

$$r_L(\lambda) = \mathrm{tr}_{L,\Phi}(\tau) \tag{5.5}$$

for any $\lambda \in L(\Phi, \tau)$. We record two properties of the cosets (5.4) for later use:

Proposition 5.7. *Let K be a CM-field and Φ_K a CM-type. Let L be a number field containing K and let Φ be the CM-type lifted from Φ_K. Let E denote the reflex field of (L, Φ).*

(i) *For any $\tau_1, \tau_2 \in G_{\mathbf{Q}}$, one has $L(\Phi, \tau_1\tau_2) = L(\Phi \circ \tau_2^{-1}, \tau_1) \cdot L(\Phi, \tau_2)$.*
(ii) *If $\tau \in G_E$ and $e \in E_f^\times$ satisfies $r_E(e) = \tau|_{E^{\mathrm{ab}}}$, one has $L(\Phi, \tau) = L^\times \cdot \Phi^*(e)^{-1}$.*

Proof. The assertions immediately reduce to the case where L is a CM-field. Statement (i) follows from the definition (5.3) and Proposition 5.6 (ii). Assertion (ii) follows from Proposition 5.6 (iv). □

Theorem 5.8 (Shimura-Taniyama, Tate, Langlands, Deligne)**.** *Let L be a CM-field and Φ a CM-type of L. Let F be a number field and A/F be an abelian variety of CM-type (L, Φ). Let $a \in h^1(A)_B$ be an L-basis. Let $\tau \in G_{\mathbf{Q}}$ and $\lambda \in L(\Phi, \tau)$. Then there exists a unique L-basis $b \in h^1(A^\tau)_B$ such that the diagram*

$$\begin{tikzcd} L_f \arrow[r, "a_f"] \arrow[d, "\lambda"'] & h^1(A)_f \arrow[d, "\tau"] \\ L_f \arrow[r, "b_f"] & h^1(A^\tau)_f \end{tikzcd}$$

commutes. Here a_f and b_f are the L_f-linear isomorphisms corresponding to the bases $I_f(a \otimes 1)$ and $I_f(b \otimes 1)$, respectively.

Proof. This is a dual version (without mention of polarizations) of [Mil98], Theorem 1.3. □

Remark 5.9. Equivalently, if $a \in h^1_B(A)$ and $b \in h^1_B(A^\tau)$ are given L-bases, there exists a unique element $\lambda \in L(\Phi, \tau)$ such that the diagram in Theorem 5.8 commutes.

Let us also state the following equivalent formulation of Theorem 5.8 in terms of complex uniformizations:

Theorem 5.10. *Let L be a CM-field and Φ a CM-type of L. Let F be a number field and A/F be an abelian variety of CM-type (L, Φ). Let $\theta : \mathbf{C}^\Phi/\mathfrak{a} \to A(\mathbf{C})$ be a $\mathcal{O}_L$-linear complex uniformization where $\mathfrak{a}$ is a fractional ideal of L. Let $\tau \in G_{\mathbf{Q}}$ and $\lambda \in L(\Phi, \tau)$. Then there exists a $\mathcal{O}_L$-linear complex uniformization $\theta_\tau : \mathbf{C}^{\tau\Phi}/\mathfrak{a}[\lambda]_L^{-1} \to A^\tau(\mathbf{C})$ such that the diagram*

$$\begin{tikzcd} L/\mathfrak{a} \arrow[r, "\theta"] \arrow[d, "\lambda^{-1}"] & A(\overline{\mathbf{Q}})_{\mathrm{tor}} \arrow[d, "\tau"] \\ L/\mathfrak{a}[\lambda]_L^{-1} \arrow[r, "\theta_\tau"] & A^\tau(\overline{\mathbf{Q}})_{\mathrm{tor}} \end{tikzcd}$$

commutes.

Proof. This is part of [Mil98], Theorem 1.1. In loc. cit. Lemma 1.4 the assertion is shown to be equivalent to Theorem 5.8. □

Recall the primary decomposition of $L/\mathfrak{a}$: For any finite place v, let $\mathfrak{a}_v = \mathfrak{a}\mathcal{O}_{L_v}$. The homomorphism $L/\mathfrak{a} \to \bigoplus_v L_v/\mathfrak{a}_v$, induced by the canonical inclusions $L \hookrightarrow L_v$, is an $\mathcal{O}_L$-linear isomorphism (cf. [Sil94], Lemma II.8.1; one easily checks that the map is given as above). Now an idele $\lambda \in L_f^\times$ induces a map $\lambda : \bigoplus_v L_v/\mathfrak{a}_v \to \bigoplus_v L_v/\mathfrak{a}[\lambda]_L$, $(x_v)_v \mapsto (\lambda_v x_v)_v$ and one defines the homomorphism $\lambda : K/\mathfrak{a} \to K/\mathfrak{a}[\lambda]_L$ such that it makes the diagram

$$\begin{tikzcd} L/\mathfrak{a} \arrow[r, "\sim"] \arrow[d, "\lambda"'] & \bigoplus_v L_v/\mathfrak{a}_v \arrow[d, "\lambda"] \\ L/\mathfrak{a}[\lambda]_L \arrow[r, "\sim"] & \bigoplus_v L_v/\lambda_v\mathfrak{a}_v \end{tikzcd}$$

commute. Also, note that the inclusions $L_v \hookrightarrow L_f$ induce a canonical isomorphism $\bigoplus_v L_v/\mathfrak{a}_v \cong L_f/\mathfrak{a} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}_L}$ such that the map λ on the left-hand side corresponds to multiplication by λ on the right.

Finally let us briefly discuss the compatibility of Theorems 5.8 and 5.10 with Serre's tensor construction. Let L be a totally imaginary field which contains a CM-field K and let $\Phi = \mathrm{Inf}_{L/K}(\Phi_K)$ be a CM-type lifted from a CM-type Φ_K of K. Let F be a number field and B/F an abelian variety of CM-type (K, Φ_K) whose CM-order is the ring of integers $\mathcal{O}_K$. The abelian variety $A := B \otimes_{\mathcal{O}_K} \mathcal{O}_L$ then has complex multiplication by $\mathcal{O}_L$ and is of CM-type (L, Φ).

Corollary 5.11. *As above, let A be an abelian variety which arises by Serre's tensor construction as above. Let $L(\Phi, \tau)$ denote the coset in $L_f^\times$ as defined in (5.4). Then the statements of Theorem 5.8 and Theorem 5.10 holds true verbatim.*

Proof. The analogue of Theorem 5.8 follows immediately from the definition of the coset $L(\Phi, \tau)$ and the canonical isomorphism $h^1_f(B \otimes_{\mathcal{O}_K} \mathcal{O}_L) \cong h^1_f(B) \otimes_{\mathcal{O}_K} \mathcal{O}_L$. As before, the analogue of Theorem 5.10 is equivalent to this. □

Remark 5.12. For a nice generalization of the main theorem of complex multiplication to motives for Hecke characters, we refer the reader to [Bla86], 4.1 and 4.2.

6. Blasius' period relation

In the following, let L be a totally imaginary field containing a CM-field. Let K be the maximal CM-subfield of L and Φ_K a CM-type of K. Let E denote the reflex field of (K,Φ) and let $\Phi^* \in I_E$ denote the reflex CM-type on E. Let $\Phi = \operatorname{Inf}_{L/K}\Phi_K$ be the CM-type of L lifted from Φ_K. Let

$$\chi_a = \beta - \alpha \in I_L$$

with $\alpha \in I_\Phi$ and $\beta \in I_{\overline{\Phi}}$ be a Hecke character type on L, i.e. there exist $\alpha_0 \in I_{\Phi_K}$, $\beta_0 \in I_{\overline{\Phi}_K}$ such that

$$\alpha = \operatorname{Inf}_{L/K}(\alpha_0) \text{ and } \beta = \operatorname{Inf}_{L/K}(\beta_0)$$

and there is an integer $w \in \mathbb{Z}$ such that

$$\beta(\overline{\sigma}) - \alpha(\sigma) = w \text{ for all } \sigma \in \Phi, \tag{6.1}$$

i.e. $c\beta - \alpha = w\Phi$. Moreover, we assume in the following that the type χ_a is *critical with respect to* Φ, i.e. that

$$\alpha(\sigma) \geq 1 \text{ and } \beta(\overline{\sigma}) \geq 0 \text{ for all } \sigma \in \Phi. \tag{6.2}$$

Remark 6.1. Our assumptions on χ_a are analogous to the condition [Bla86], (4.4.4) for $\mu = c\beta$. In the notation of [Bla86], 4.4, we have $\alpha = -w\Phi + \mu$ and $\beta = c\mu$. But note that the condition in loc. cit. is only equivalent to the criticality of χ when the weight w is negative. We do not make any assumptions on the weight.

Lemma 6.2. *Let $\chi_a = \beta - \alpha$ be a Hecke character type as above.*
- (i) *The field L^α is a CM-field containing the reflex field E of (K,Φ_K).*
- (ii) *One has $L^\beta \subset L^\alpha$.*

Proof. Since K is a CM-field, it is clear that $L^\alpha = K^{\alpha_0}$ is either totally real or a CM-field. Any $\tau \in G_{\mathbf{Q}}$ which fixes the type $\alpha_0 \in I_{\Phi_K}$ also has to fix the CM-type Φ_K on which α_0 is supported. It follows that $E \subset L^\alpha$. In particular, L^α is a CM-field. For assertion (ii), consider $\tau \in G_{\mathbf{Q}}$ such that $\tau\alpha = \alpha$. Since α is induced from a CM-field, it follows that τ fixes $c\alpha$ as well. Moreover, by (i), τ fixes the CM-type Φ and hence also its complex conjugate $\overline{\Phi}$. Finally, $\beta = w\overline{\Phi} + c\alpha$ by the weight condition (6.1) and thus $\tau\beta = \beta$, which proves the claim. ◻

Now assume that T is a CM-field containing L^α and that

$$\chi : L_f^\times \to T^\times$$

is an algebraic Hecke character with infinity type χ_a.

Definition 6.3. *Let $\eta \in J_E$. We define the map*

$$\varepsilon_{\eta\Phi} : G_{\mathbf{Q}} \to \{\pm 1\}$$

as follows: For $\tau \in G_{\mathbf{Q}}$, we let $\varepsilon_{\eta\Phi}(\tau) \in \{\pm1\}$ be the sign of the bijection

$$\langle c\rangle\backslash J_L \xrightarrow{\sim} \eta\Phi \xrightarrow{\tau} \tau\eta\Phi \xrightarrow{\sim} \langle c\rangle\backslash J_L,$$

where the first and the last bijection are induced from the inclusions $\Phi \subset J_L$ and $\eta\Phi \subset J_L$.

For $\tau_1, \tau_2 \in G_{\mathbf{Q}}$, one immediately checks that

$$\varepsilon_{\eta\Phi}(\tau_1\tau_2) = \varepsilon_{\tau_2\eta\Phi}(\tau_1)\varepsilon_{\eta\Phi}(\tau_2). \tag{6.3}$$

In particular, the restriction $\varepsilon_\Phi : G_E \to \{\pm1\}$ is a character and we may regard it as a finite Hecke character $\varepsilon_\Phi : E_f^\times \to T^\times$.

Definition 6.4. *Let $\Xi : E_f^\times \to T^\times$ denote the algebraic Hecke character*

$$\Xi = (\chi \circ \Phi^*)^{-1} \|\cdot\|_E^{-d(\beta)} \varepsilon_\Phi. \tag{6.4}$$

Note that the infinity-type of Ξ is given by $\Xi_a = (\Phi^*)^\alpha \cdot (\overline{\Phi}^*)^\beta$.

For any $\eta \in J_E$, we have the coset $L(\eta\Phi, \tau)$, defined in (5.4), associated to the CM-type $\eta\Phi$ of L. For any $\tau \in G_{\mathbf{Q}}$, let us define the elements $\xi(\tau, \eta) \in T_f^\times$ by

$$\xi(\tau, \eta) = \chi_a(\lambda)^{-1}\chi(\lambda)\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)}\varepsilon_{\eta\Phi}(\tau), \tag{6.5}$$

where $\lambda \in L(\eta\Phi, \tau)$ can be chosen arbitrarily.

Proposition 6.5 ([Bla86], Proposition 4.4.1)**.**

(i) *For all $\tau_1, \tau_2 \in G_{\mathbf{Q}}$ and $\eta \in J_E$, one has*

$$\xi(\tau_1\tau_2, \eta) = \xi(\tau_1, \tau_2\eta)\xi(\tau_2, \eta).$$

(ii) *If $\tau \in G_E$ and $e \in E_f^\times$ satisfies $r_E(e) = \tau|_{E^{\mathrm{ab}}}$, one has*

$$\xi(\tau, 1_E) = \Xi_a(e)^{-1}\Xi(e).$$

Proof. The assertions follow readily from the definitions and Proposition 5.7. □

Proposition 6.6. *Let $M(\Xi)$ be a motive of type (E, Ξ, T). For every $\eta \in J_E$ there exists a T-basis $a_\eta \in M(\Xi)_B^\eta$ such that for any $\tau \in G_{\mathbf{Q}}$ the diagram*

$$\begin{tikzcd} T_f \arrow[r, "I_f(a_\eta)"] \arrow[d, "\xi(\tau,\eta)"'] & M(\Xi)_f^\eta \arrow[d, "\tau"] \\ T_f \arrow[r, "I_f(a_{\tau\eta})"] & M(\Xi)_f^{\tau\eta} \end{tikzcd}$$

is commutative, where the element $\xi(\tau, \eta) \in T_f^\times$ is defined as in (6.5).

Proof. This is Corollary 4.4.2 in [Bla86]. Also cf. Corollary A.4 in the appendix. □

Theorem 6.7 (Blasius)**.** *Let Ξ be the Hecke character as defined in (6.4), $M(\Xi)$ a motive of type (E, Ξ, T) and $\{a_\eta \mid \eta \in J_E\}$ as in Proposition 6.6. Then $F^{d(\alpha)+d(\beta)}RM(\Xi)_{\mathrm{dR}}$ is a 1-dimensional T-vector space. If ω is a basis, there exists a unique element $v(\chi) \in (T \otimes \mathbf{C})^\times$ satisfying*

$$I_\infty(\omega) = v(\chi) \cdot \sum_{\eta \in J_E} e(\eta)a_\eta,$$

and one has

$$(2\pi i)^{-d(\beta)}v(\chi) \sim c_+(\chi) \mod T.$$

Proof. This is [Bla86], Proposition 5.3.4 in the case where L is a CM-field. For general L, the result is [Bla97], Theorem H.3, where it is pointed out that the same proof as in the CM-case works if one uses the sign from Definition 6.3 instead of a slightly different sign employed in [Bla86]. We recall the proof in Appendix A, cf. Theorem A.10. □

Finally, for future reference, let us state the following minor extension of Proposition 5.5:

Proposition 6.8. *Let L be a totally imaginary number field containing a CM-field. Let Φ be a CM-type of L and E the reflex field of (L, Φ). Let $\mathfrak{a} \subset \mathcal{O}_L$ be an integral ideal. Then there exists a finite abelian extension F/E and an abelian variety A/F with complex multiplication by L satisfying the following properties:*

1. *A has CM by $\mathcal{O}_L$ and is of CM-type (L, Φ).*
2. *There is an isomorphism $\xi : \mathcal{O}_L \cong H_1(A(\mathbf{C}), \mathbb{Z})$.*
3. *Every $[\mathfrak{a}]$-torsion point of A is defined over F.*
4. *The character $\varepsilon_\Phi : G_E \to \{\pm 1\}$ is trivial on G_F.*
5. *$\Xi \circ N_{F/E} = \psi^\alpha\overline{\psi}^\beta$, where $\psi : F_f^\times \to L^\times$ denotes the CM-character of A/F.*

Proof. By Proposition 5.5, there exists a finite abelian extension F/E and a CM abelian variety A/F which satisfies conditions (1)–(3) and whose CM character ψ is of the form $\psi = \varphi \circ N_{F/E}$ for a Hecke character φ of E. Note that any base change to a larger abelian extension of E still has these properties. Since the character $\varepsilon_\Phi : G_E \to \{\pm 1\}$ is abelian, we may enlarge F so that (4) holds. For (5), let T_φ be a number field containing L and the values of φ. Let $\widetilde{\alpha}, \widetilde{\beta} \in I_{T_\varphi}$ denote lifts of $\alpha, \beta \in I_L$ along the surjection $\mathrm{Res}_{T_\varphi/L} : I_{T_\varphi} \to I_L$ defined in , e.g. choose lifts $\widetilde{\sigma} \in J_{T_\varphi}$ for every $\sigma \in J_L$ and put $\widetilde{\alpha} = \sum_{\sigma\in J_L} \alpha(\sigma)\cdot\widetilde{\sigma}$. Note that for any $\ell \in L^\times$, one has $\ell^{\widetilde{\alpha}} = \ell^\alpha$, where on the left ℓ is regarded as an element in T_φ. It follows that

$$\varphi^{\widetilde{\alpha}}\overline{\varphi}^{\widetilde{\beta}} \circ N_{F/E} = \psi^\alpha\overline{\psi}^\beta \tag{6.6}$$

and this remains true after passing to the base change of A to a larger abelian extension F/E. The infinity-type of $\varphi^{\widetilde{\alpha}}\overline{\varphi}^{\widetilde{\beta}}$ is given by $(\Phi^*)^\alpha\cdot(\overline{\Phi}^*)^\beta$, i.e. it coincides with the infinity-type of Ξ (cf. Definition 6.4). It follows that we may enlarge the finite abelian extension F/E such that $\varphi^{\widetilde{\alpha}}\overline{\varphi}^{\widetilde{\beta}} \circ N_{F/E} = \Xi\circ N_{F/E}$. By (6.6), we obtain $\psi^\alpha\overline{\psi}^\beta = \Xi\circ N_{F/E}$, as desired. □

Remark 6.9. For the proof that condition (5) in the above Proposition can be achieved, it is not enough to observe that $\psi^\alpha\overline{\psi}^\beta$ has the same infinity-type as $\Xi\circ N_{F/E}$. This only shows that we can pass to an abelian extension of F such that the condition (5) holds. We actually need to work with the character φ to ensure that F can be chosen to be abelian *over* E.

7. Algebraic $\mathbf{T}_{L,F}$-modules

7.1. **Definition and decomposition results.** Let L be a number field and F a field of characteristic zero. Fix an algebraic closure $\overline{F}$ as well as an embedding $\overline{\mathbf{Q}} \subset \overline{F}$. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of F. The chosen embedding $\overline{\mathbf{Q}} \subset \overline{F}$ induces a homomorphism

$$G_F \to G_{\mathbf{Q}} \tag{7.1}$$

by restriction. We consider the algebraic torus

$$\mathbf{T}_{L,F} := \mathrm{Res}_{L/\mathbf{Q}}\,\mathbf{G}_m \times_{\mathbf{Q}} F$$

over F. Its character group $X^*(\mathbf{T}_{L,F})$ identifies with the free abelian group I_L of types on L. The canonical action of G_F is induced by the $G_{\mathbf{Q}}$-action on I_L via the homomorphism (7.1). Equivalently, we can regard I_L as the free abelian group on $\mathrm{Hom}(L,\overline{F})$, on which G_F acts in the obvious way.

Definition 7.1. *An* algebraic $\mathbf{T}_{L,F}$-module *or* representation of $\mathbf{T}_{L,F}$ *is an F-vector space V which, for every F-algebra R, is equipped with a R-linear action*

$$\rho_R : \mathbf{T}_{L,F}(R) = (L\otimes_{\mathbf{Q}} R)^\times \to \mathrm{Aut}_R(V\otimes_F R)$$

that is functorial in R.

The representations of algebraic groups of multiplicative type, such as $\mathbf{T}_{L,F}$, can be described as follows:

Proposition 7.2. *Let V be an algebraic $\mathbf{T}_{L,F}$-module. There is a decomposition*

$$V = \bigoplus_{\Theta\in G_F\backslash I_L} V_\Theta \tag{7.2}$$

indexed by the G_F-orbits of the character group $X^*(\mathbf{T}_{L,F})$. Here V_Θ consists of those elements $v\in V$ such that there exists $\mu\in\Theta$ with

$$\rho_{\overline{F}}(l)(v\otimes 1) = l^\mu\cdot(v\otimes 1) \text{ for all } l\in L^\times.$$

Proof. See e.g. [Mil17], Theorem 12.30: One easily shows the Proposition after passing to the separable closure $\overline{F}$ where the torus $\mathbf{T}_{L,F}$ becomes split. The general case then follows by Galois descent. □

Let V be an algebraic $\mathbf{T}_{L,F}$-module. Let F' be a field extension. Fix an algebraic closure $\overline{F'}$ of F' and fix an embedding $\overline{F}\subset\overline{F'}$. Again, there is a canonical homomorphism $G_{F'}\to G_F$ which induces an action on I_L (or we may use the identifications $J_L = \mathrm{Hom}(L,\overline{F}) = \mathrm{Hom}(L,\overline{F'})$). Moreover, the group $G_{F'}$ acts on every G_F-orbit of I_L, which therefore decomposes itself into a disjoint union of $G_{F'}$-orbits. The F'-vector space $V' = V\otimes_F F'$ acquires the structure of an algebraic $\mathbf{T}_{L,F'}$-module in a canonical way. By Lemma 7.2, we have the decomposition

$$V\otimes_F F' = \bigoplus_{\Theta'\in G_{F'}\backslash I_L} V'_{\Theta'}. \tag{7.3}$$

The two decompositions (7.2) and (7.3) are related as follows:

Proposition 7.3. *For every G_F-orbit Θ of I_L, one has*

$$V_\Theta \otimes_F F' = \bigoplus_{\Theta' \in G_{F'} \backslash \Theta} V'_{\Theta'}$$

as F'-subspaces of $V \otimes_F F'$.

Proof. The case where $F' = \overline{F}$ is contained in the proof of Proposition 7.2. The general case is immediate, e.g. by using the explicit description of the subspaces V_Θ in Proposition 7.2. ☐

7.2. A basic example.

The general case. We now discuss an example which is essentially trivial but useful when we later construct certain idempotent endomorphisms in various realizations of a motive. Let $L/\mathbf{Q}$ be a finite extension and F a field of characteristic 0 together with a fixed embedding $\overline{\mathbf{Q}} \subset \overline{F}$. Let V be a $L \otimes_{\mathbf{Q}} F$-module. For any F-algebra R, the extension of scalars $V \otimes_F R$ is a $L \otimes_{\mathbf{Q}} R$-module. In particular, we obtain an action

$$\rho_R : (L \otimes_{\mathbf{Q}} R)^\times \to \mathrm{Aut}_R(V \otimes_F R) \tag{7.4}$$

via R-linear isomorphisms, which is immediately checked to be functorial. In this way, V is equipped with the structure of an algebraic $\mathbf{T}_{L,F}$-module.

More generally, let $d \geq 0$ be an integer. Then the dth symmetric powers $\mathrm{Sym}^d_F(V)$ acquire the structure of an algebraic $\mathbf{T}_{L,F}$-module: For any F-algebra R, we have a functorial action

$$(L \otimes_{\mathbf{Q}} R)^\times \to \mathrm{Aut}_R(\mathrm{Sym}^d_R(V \otimes_F R)), \quad \ell \mapsto \mathrm{Sym}^d_R(\rho_R(\ell))$$

via R-linear isomorphisms on $\mathrm{Sym}^d_F(V) \otimes_F R \cong \mathrm{Sym}^d_R(V \otimes_F R)$. Hence, Proposition 7.2 implies that we have a decomposition

$$\mathrm{Sym}^d_F(V) = \bigoplus_{\Theta \in G_F \backslash I_L} \mathrm{Sym}^d_F(V)_\Theta \tag{7.5}$$

indexed by the orbits of the action of G_F on I_L through the homomorphism $G_F \to G_{\mathbf{Q}}$ given by restriction along $\overline{\mathbf{Q}} \subset \overline{F}$.

The split case. Let us describe the decompositions in the split case. Assume that the Galois closure of L in $\overline{\mathbf{Q}} \subset \overline{F}$ already lies in F, so that $\mathbf{T}_{L,F}$ splits. Note that then G_F acts trivially on I_L, so that $G_F \backslash I_L = I_L$. In this situation, let us fix the following notation:

Definition 7.4. *Let V be an $L \otimes F$-module and assume that F contains the Galois closure of L. We write*

$$V = \bigoplus_{\sigma \in J_L} V(\sigma), \tag{7.6}$$

for the decomposition induced by the canonical isomorphism $L \otimes F = \prod_{\sigma \in J_L} F_\sigma$, where F_σ denotes the L-algebra $\sigma : L \to F$. There are canonical isomorphisms

$$V(\sigma) = V \otimes_{L \otimes F, \sigma} F,$$

where the tensor product is taken with respect to the homomorphism $\sigma : L \otimes F \to F$, $\ell \otimes x \mapsto \sigma(\ell)x$. For any type $\mu \in I^+_L$, we set

$$\mathrm{Sym}^\mu(V) := \bigotimes_{\sigma \in J_L} \mathrm{Sym}^{\mu(\sigma)}_F(V(\sigma)). \tag{7.7}$$

Clearly, an element $\ell \in L^\times$ acts via the embedding σ on V_σ and hence the decomposition (7.2) is given by (7.6) In terms of (7.6), the decomposition (7.5) of $\mathrm{Sym}^d_F(V)$ is then given by

$$\mathrm{Sym}^d_F(V) = \mathrm{Sym}^d_F\Big(\bigoplus_{\sigma \in J_L} V(\sigma)\Big) \cong \bigoplus_{\mu \in I^{+,d}_L} \mathrm{Sym}^\mu(V) \tag{7.8}$$

7.3. **The functor V^α.** In the following, let $\alpha \in I_L^+$ be a type and let $d = d(\alpha)$ denote its degree. Let $\Theta_\alpha = G_{\mathbf{Q}} \cdot \alpha$ denote the $G_{\mathbf{Q}}$-orbit of the type α. For a $L \otimes F$-module V, we define V^α to be the direct summand

$$V^\alpha = \bigoplus_{\Theta \in G_F \backslash \Theta_\alpha} \mathrm{Sym}_F^d(V)_\Theta.$$

of $\mathrm{Sym}_F^d(V)$ in (7.5). This construction is functorial: Any $L\otimes F$-linear homomorphism $f : V \to W$ induces $\mathbf{T}_{L,F}$-equivariant homomorphisms $f : V \to W$ and $\mathrm{Sym}^d(f) : \mathrm{Sym}_F^d(V) \to \mathrm{Sym}_F^d(W)$. In particular, they are compatible with the decompositions (7.2) and hence there is an induced homomorphism

$$f^\alpha : V^\alpha \to W^\alpha.$$

Change of the field F. Let F' be another characteristic 0 field containing F. Fix embeddings $\overline{\mathbf{Q}} \subset \overline{F} \subset \overline{F'}$. The construction $V \mapsto V^\alpha$ behaves as follows under extension of scalars along $F \subset F'$:

Lemma 7.5. *For every $L \otimes_{\mathbf{Q}} F$-module V, there is a natural isomorphism*

$$(V \otimes_F F')^\alpha \cong V^\alpha \otimes_F F'.$$

Proof. By Proposition 7.3, we have the isomorphism

$$\begin{aligned} V^\alpha \otimes_F F' &= \bigoplus_{\Theta \in G_F \backslash \Theta_\alpha} \mathrm{Sym}_F^d(V)_\Theta \otimes_F F' \\ &\cong \bigoplus_{\Theta \in G_F \backslash \Theta_\alpha} \bigoplus_{\Theta' \in G_{F'} \backslash \Theta} \mathrm{Sym}_{F'}^d(V \otimes_F F')_{\Theta'} \\ &\cong \bigoplus_{\Theta' \in G_{F'} \backslash \Theta_\alpha} \mathrm{Sym}_{F'}^d(V \otimes_F F')_{\Theta'} \\ &= (V \otimes_F F')^\alpha \end{aligned}$$

which is immediately checked to be functorial in V. □

Module structure over $L^\alpha \otimes F$. Let us explain how V^α canonically acquires the structure of an $L^\alpha \otimes F$-module. Note that, by definition of the field of values L^α, one has

$$\Theta_\alpha = \{\iota\alpha \mid \iota \in J_{L^\alpha}\}.$$

By Lemma 7.5 and the decomposition (7.8), we have the decomposition

$$V^\alpha \otimes_F \overline{F} = \bigoplus_{\iota \in J_{L^\alpha}} \mathrm{Sym}^{\iota\alpha}(V \otimes_F \overline{F}).$$

We now let an element $t \in L^\alpha$ act on $V^\alpha \otimes_F \overline{F}$ via multiplication by $\iota(t)$ on each summand $\mathrm{Sym}^{\iota\alpha}(V \otimes_F \overline{F})$. For each $\tau \in G_F$, the semilinear isomorphism $\mathrm{id} \otimes \tau$ on $V^\alpha \otimes_F \overline{F}$ induces semilinear maps $\mathrm{id} \otimes \tau : \mathrm{Sym}^{\iota\alpha}(V \otimes_F \overline{F}) \to \mathrm{Sym}^{\tau\iota\alpha}(V \otimes_F \overline{F})$ for every $\iota \in J_{L^\alpha}$ and it is immediate to check that the diagram

$$\begin{tikzcd} \mathrm{Sym}^{\iota\alpha}(V \otimes_F \overline{F}) \arrow[r, "\mathrm{id}\otimes\tau"] \arrow[d, "\iota(t)"'] & \mathrm{Sym}^{\tau\iota\alpha}(V \otimes_F \overline{F}) \arrow[d, "\tau\iota(t)"] \\ \mathrm{Sym}^{\iota\alpha}(V \otimes_F \overline{F}) \arrow[r, "\mathrm{id}\otimes\tau"] & \mathrm{Sym}^{\tau\iota\alpha}(V \otimes_F \overline{F}) \end{tikzcd}$$

commutes. Hence, the action of t on $V^\alpha \otimes_F \overline{F}$ is G_F-equivariant and therefore uniquely descends to an F-linear endomorphism of V^α. Clearly, the assignment $L^\alpha \to \mathrm{End}_F(V^\alpha)$ is a ring homomorphism and we thus have equipped V^α with the structure of an $L^\alpha \otimes F$-module. One also checks that the construction $V \mapsto V^\alpha$ promotes to a functor

$$(-)^\alpha : \mathrm{Mod}_{L\otimes F} \to \mathrm{Mod}_{L^\alpha \otimes F}.$$

8. The motive M^α

The following construction is outlined in the proof of [Bla86], Proposition 5.5.1. First of all let us remark that instead of the finite adelic realization, we may use the ℓ-adic realizations for all primes ℓ by Remark 3.4. The reason for doing this is that it is more convenient to write down the results over a field.

Let F be a totally imaginary number field and $\psi : F_f^\times \to L^\times$ an algebraic Hecke character. Let $\alpha \in I_L^+$ and let $d := d(\alpha)$. Given a motive M of type (F, ψ, L), we want to construct a motive M^α of type $(F, \psi^\alpha, L^\alpha)$. We define M^α as a direct summand in

$$N := \mathrm{Sym}^d(R_{L/\mathbf{Q}} M),$$

where $R_{L/\mathbf{Q}}M$ denotes the restriction of coefficients of M from L to $\mathbf{Q}$, as follows: Applying the functor $(-)^\alpha$ constructed in Section 7.3 to the L-vector space M_B, we obtain a direct summand M_B^α in $N_B = \mathrm{Sym}^d_{\mathbf{Q}}(M_B)$. Similarly, we obtain direct summands M_{dR}^α in $N_{\mathrm{dR}} = \mathrm{Sym}^d_F(M_{\mathrm{dR}})$ and, for each prime number ℓ, a direct summand M_ℓ^α in $N_\ell = \mathrm{Sym}^d_{\mathbf{Q}_\ell}(M_\ell)$. Let P_B, P_{dR} and P_ℓ denote the corresponding idempotent endomorphisms of N_B, N_{dR} and N_ℓ.

Lemma 8.1. *The endomorphism $P_B \in \mathrm{End}(N_B)$ respects the Hodge structure, i.e. for all $p, q \in \mathbb{Z}$, one has*

$$P_B(N_B^{p,q}) \subset N_B^{p,q}.$$

Proof. By Proposition 7.3 and (7.8), we have a decomposition

$$N_B \otimes \mathbf{C} = \bigoplus_{\mu \in I_L^{+,d}} \mathrm{Sym}^\mu(M_B \otimes \mathbf{C}) \tag{8.1}$$

where $\mathrm{Sym}^\mu(M_B \otimes \mathbf{C}) = \bigotimes_{\sigma \in J_L} \mathrm{Sym}^{\mu(\sigma)}_{\mathbf{C}}(M_B \otimes_{L,\sigma} \mathbf{C})$, and P_B is the projector onto the summand indexed by the Galois orbit $G_{\mathbf{Q}}\alpha$. By Proposition 3.20, each $M_B \otimes_{L,\sigma} \mathbf{C}$ is of pure Hodge bidegree $(\psi^*(\sigma), \psi^*(c\sigma))$. Thus, every summand in (8.1) is of pure Hodge bidegree and from this the claim follows. □

Lemma 8.2. *For each prime ℓ, the endomorphism $P_\ell \in \mathrm{End}(N_\ell)$ is G_F-equivariant.*

Proof. One has to show that the direct summand M_ℓ^α is stable under the action of G_F. It suffices to check this after base change to $\overline{\mathbf{Q}}_\ell$. Then, by Lemma 7.3 and (7.8), one has

$$N_\ell \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell = \bigoplus_{\mu \in I_L^{+,d}} \mathrm{Sym}^\mu(M_\ell \otimes \overline{\mathbf{Q}}_\ell),$$

where $\mathrm{Sym}^\mu(M_\ell \otimes \overline{\mathbf{Q}}_\ell) = \bigotimes_{\sigma \in J_L} \mathrm{Sym}^d_{\overline{\mathbf{Q}}_\ell}(M_\ell \otimes_{L \otimes \mathbf{Q}_\ell, \sigma} \overline{\mathbf{Q}}_\ell)$. Since the G_F-representation M_ℓ is L-linear, this is a decomposition of G_F-representations and the claim follows. □

Lemma 8.3. *The comparison isomorphisms $I_\infty : N_{\mathrm{dR}} \otimes_F \mathbf{C} \to N_B \otimes \mathbf{C}$ and $I_\ell : N_B \otimes \mathbf{Q}_\ell \to N_\ell$ are compatible with the projectors P_{dR}, P_B and P_ℓ.*

Proof. Since $I_\infty : N_{\mathrm{dR}} \otimes_F \mathbf{C} \to N_B \otimes \mathbf{C}$ identifies with the dth $\mathbf{C}$-linear symmetric power of the $L \otimes \mathbf{C}$-linear comparison isomorphism $I_{\infty,M}$, it is a $\mathbf{T}_{L,\mathbf{C}}$-equivariant isomorphism and by construction it restricts to the isomorphism $I_{\infty,M}^\alpha : (M_{\mathrm{dR}} \otimes_F \mathbf{C})^\alpha \to (M_B \otimes \mathbf{C})^\alpha$. The desired compatibility follows from Lemma 7.5. The compatibility of I_ℓ with P_B and P_ℓ follows completely analogously since $I_\ell : N_B \otimes \mathbf{Q}_\ell \to N_\ell$ is induced by the $L \otimes \mathbf{Q}_\ell$-linear isomorphism $I_{\ell,M} : M_B \otimes \mathbf{Q}_\ell \to M_\ell$ on dth symmetric powers. □

8.1. Definition of M^α.

The discussion above shows that there is an idempotent endomorphism

$$P \in \mathrm{End}(\mathrm{Sym}^d(M_{\mathbf{Q}}))$$

which agrees with the projectors P_B, P_{dR} and P_ℓ in the respective realizations.

Definition 8.4. *Let M be a motive of type (F, ψ, L) and $\alpha \in I_L^+$ be a type of degree d. We define $M^\alpha = P \cdot \mathrm{Sym}^d(M_{\mathbf{Q}})$.*

Next, we show that M^α is a motive of type $(F, \psi^\alpha, L^\alpha)$. For the proof of the upcoming proposition, the following general observation about algebraic homomorphisms and types is useful: Let $\alpha_\ell : (L \otimes \overline{\mathbf{Q}}_\ell)^\times \to (L^\alpha \otimes \overline{\mathbf{Q}}_\ell)^\times$ denote the homomorphism obtained from the homomorphism $\underline{\alpha} : \mathrm{Res}_{L/\mathbf{Q}} \mathbf{G}_m \to \mathrm{Res}_{L^\alpha/\mathbf{Q}} \mathbf{G}_m$ of algebraic tori corresponding to the type α on $\overline{\mathbf{Q}}_\ell$-valued points. Then, under the identification $L^\alpha \otimes \overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_\ell^{J_{L^\alpha}}$, the homomorphism α_ℓ is given by

$$\alpha_\ell(y) = \Big(\prod_{\sigma \in J_L} \sigma(y)^{\iota\alpha(\sigma)} \Big)_{\iota \in J_{L^\alpha}}, \tag{8.2}$$

where $y \in (L \otimes \overline{\mathbf{Q}}_\ell)^\times$ and $\sigma : L \otimes \overline{\mathbf{Q}}_\ell \to \overline{\mathbf{Q}}_\ell$ denotes the homomorphism given by $l \otimes z \mapsto \sigma(l)z$.

Proposition 8.5. *Let M be a motive of type (F, ψ, L). Then M^α is of type $(F, \psi^\alpha, L^\alpha)$.*

Proof. Recall that each of the realizations M_B^α, M_{dR}^α and M_ℓ^α has the structure of a module over L^α, $L^\alpha \otimes_{\mathbf{Q}} F$ and $L^\alpha \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$, respectively. Moreover, it follows from the proof of Lemma 8.3 that the comparison isomorphism $I_\infty : M_{\mathrm{dR}}^\alpha \otimes_F \mathbf{C} \to M_B^\alpha \otimes \mathbf{C}$ is given, via the identifications of Lemma 7.5, by applying the functor $(-)^\alpha$ to the $L \otimes \mathbf{C}$-linear isomorphism $I_{\infty,M}$. In particular, it is $L^\alpha \otimes \mathbf{C}$-linear. Similarly, the comparison isomorphism $I_\ell : M_B^\alpha \otimes \mathbf{Q}_\ell \to M_\ell^\alpha$ is $L^\alpha \otimes \mathbf{Q}_\ell$-linear. It follows that the motive M^α acquires coefficients in L^α such that the induced L-module structure is given as in Section 7.3 in each realization. Let us now show that M^α is a motive for the Hecke character ψ^α. By Lemma 7.3 and (7.8), one has

$$M_B^\alpha \otimes \overline{\mathbf{Q}} = \bigoplus_{\mu \in G_{\mathbf{Q}} \cdot \alpha} \mathrm{Sym}^\mu(M_B \otimes \overline{\mathbf{Q}})$$

where each summand $\mathrm{Sym}^\mu(M_B \otimes \overline{\mathbf{Q}}) = \bigotimes_{\sigma \in J_L} \mathrm{Sym}^{\mu(\sigma)}(M_B \otimes_{L,\sigma} \overline{\mathbf{Q}})$ is a 1-dimensional $\overline{\mathbf{Q}}$-vector space since M is of rank 1, by assumption. As the orbit $G_{\mathbf{Q}} \cdot \alpha$ is of cardinality $[L^\alpha : \mathbf{Q}]$, it follows that M_B^α is a 1-dimensional L^α-vector space. Hence, M^α is a motive of rank 1. We now show that, for a fixed rational prime ℓ, the G_F-representation M_ℓ^α is determined by the Hecke character ψ^α. To check this, we may pass to the algebraic closure $\overline{\mathbf{Q}}_\ell$. Then, by Proposition 7.3, (7.8) and by using that the $G_{\mathbf{Q}}$-orbit of α is given by $\{\iota\alpha \mid \iota \in J_{L^\alpha}\}$, we have

$$M_\ell^\alpha \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell = \bigoplus_{\iota \in J_{L^\alpha}} \mathrm{Sym}^{\iota\alpha}(M_\ell \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell),$$

where $\mathrm{Sym}^{\iota\alpha}(M_\ell \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell) = \bigotimes_{\sigma \in J_L} \mathrm{Sym}^{\iota\alpha(\sigma)}(M_\ell \otimes_{L \otimes \mathbf{Q}_\ell, \sigma} \overline{\mathbf{Q}}_\ell)$. The module $L^\alpha \otimes \overline{\mathbf{Q}}_\ell$ acts in the obvious way via $L^\alpha \otimes \overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_\ell^{J_{L^\alpha}}$. Let $\tau \in G_F$ and $x \in F_f^\times$ such that $r_F(x) = \tau|_{F^{\mathrm{ab}}}$. Since M is of type (F, ψ, L), the action of τ on $M_\ell \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell$ is given via multiplication by $\psi_{a,\ell}^{-1}(x)\psi(x) \in (L \otimes \overline{\mathbf{Q}}_\ell)^\times$. Thus, τ acts on $M_\ell^\alpha \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell$ via multiplication by

$$\Big(\prod_{\sigma \in J_L} \sigma\big(\psi_{a,\ell}^{-1}(x)\psi(x)\big)^{\iota\alpha(\sigma)} \Big)_{\iota \in J_{L^\alpha}} \in \overline{\mathbf{Q}}_\ell^{J_{L^\alpha}}.$$

By the remark before the proposition, this coincides under $L^\alpha \otimes \overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_\ell^{J_{L^\alpha}}$ with

$$\alpha_\ell(\psi_{a,\ell}^{-1}(x)\psi(x)) = \psi_{a,\ell}^{-\alpha}(x) \cdot \psi^\alpha(x) \in (L^\alpha \otimes \overline{\mathbf{Q}}_\ell)^\times.$$

This actually lies in $(L^\alpha \otimes \mathbf{Q}_\ell)^\times$ as $\psi_{a,\ell}(x)$ lies in $(L \otimes \mathbf{Q}_\ell)^\times$. It follows that M^α is of type $(F, \psi^\alpha, L^\alpha)$. □

8.2. **Construction of elements in M_B^α and M_{dR}^α.** Starting from elements in M_B and M_{dR}, we now explain how to construct elements in M_B^α and M_{dR}^α. In the de Rham realization, it is sufficient for our purposes to restrict to the case where the field of definition F contains the Galois closure of L, so that the torus $\mathbf{T}_{L,F}$ is split. In this case, the de Rham realization becomes easier to manage. More generally, one could define a map $V \to V^\alpha$ for any $L \otimes F$-module V.

Betti realization. As before, let F and L be number fields and M be a motive of type (F, ψ, L). Let $\gamma \in M_B$. We will now construct an element $\gamma^{\boldsymbol{\alpha}} \in M_B^\alpha$ via Galois descent. Recall that

$$M_B^\alpha \otimes \overline{\mathbf{Q}} = \bigoplus_{\iota \in J_{L^\alpha}} \mathrm{Sym}^{\iota\alpha}(M_B \otimes \overline{\mathbf{Q}})$$

where as usual $\mathrm{Sym}^{\iota\alpha}(M_B \otimes \overline{\mathbf{Q}}) = \bigotimes_{\sigma\in J_L} \mathrm{Sym}^{\iota\alpha(\sigma)}_{\overline{\mathbf{Q}}}(M_B \otimes_{L,\sigma} \overline{\mathbf{Q}})$. For every $\iota \in J_{L^\alpha}$, we define the element

$$\gamma^{\iota\alpha} := \bigotimes_{\sigma\in J_L} (\gamma \otimes_\sigma 1_{\overline{\mathbf{Q}}})^{\iota\alpha(\sigma)} \in \mathrm{Sym}^{\iota\alpha}(M_B \otimes \overline{\mathbf{Q}})$$

and we set

$$\gamma^{\boldsymbol{\alpha}} := (\gamma^{\iota\alpha})_{\iota\in J_{L^\alpha}} \in M_B^\alpha \otimes \overline{\mathbf{Q}}.$$

Here, we write $\gamma \otimes_\sigma 1_{\overline{\mathbf{Q}}}$ for $\gamma \otimes 1$ to indicate that it lies in $M_B \otimes_{L,\sigma} \overline{\mathbf{Q}}$. If $\tau \in G_{\mathbf{Q}}$, one has $\tau(\gamma^{\boldsymbol{\alpha}}) = \gamma^{\boldsymbol{\alpha}}$. Indeed, under the action of τ, the element $\gamma^{\iota\alpha}$ gets sent to $\bigotimes_{\sigma\in J_L}(\gamma \otimes_{\tau\sigma} 1)^{\iota\alpha(\sigma)} = \bigotimes_{\sigma\in J_L}(\gamma \otimes_\sigma 1)^{\tau\iota\alpha(\sigma)} = \gamma^{\tau\iota\alpha}$. Taking the sum over all $\iota \in J_{L^\alpha}$ gives the claim. Hence, we obtain an element

$$\gamma^{\boldsymbol{\alpha}} \in M_B^\alpha. \tag{8.3}$$

The reader hopefully takes notice of the bold font in the exponent. It should always be clear from the context, whether $\gamma^{\boldsymbol{\alpha}}$ or γ^α is meant. The latter usually appears as in the form $\gamma^{\iota\alpha}$.

De Rham realization. We assume that F contains the Galois closure of L. Then the torus $\mathbf{T}_{L,F}$ is split and we may regard every embedding of L to take values in F. The de Rham realization of M^α is given by

$$M_{\mathrm{dR}}^\alpha = \bigoplus_{\iota\in J_{L^\alpha}} \mathrm{Sym}^{\iota\alpha}(M_{\mathrm{dR}})$$

where

$$\mathrm{Sym}^{\iota\alpha}(M_{\mathrm{dR}}) = \bigotimes_{\sigma\in J_L} \mathrm{Sym}_F^{\iota\alpha(\sigma)}(M_{\mathrm{dR}} \otimes_{L\otimes F,\sigma} F).$$

Now, if we are given an element $\omega \in M_{\mathrm{dR}}$ and $\iota \in J_{L^\alpha}$, we define the element

$$\omega^{\iota\alpha} := \bigotimes_{\sigma\in J_L} (\omega \otimes_\sigma 1_F)^{\iota\alpha(\sigma)} \in \mathrm{Sym}^{\iota\alpha}(\omega_{A/F})$$

and set

$$\omega^{\boldsymbol{\alpha}} := (\omega^{\iota\alpha})_{\iota\in J_{L^\alpha}} \in M_{\mathrm{dR}}^\alpha. \tag{8.4}$$

Again, there should be no room for confusion between $\omega^{\boldsymbol{\alpha}}$ and ω^α.

Remark 8.6. Let $M = h^1(A)$ for an abelian variety A/F with CM by $\mathcal{O}_L$ such that F contains the Galois closure of L. For $\sigma \in J_L$, the direct summand $h^1_{\mathrm{dR}}(A)(\sigma) := h^1_{\mathrm{dR}}(A) \otimes_{L\otimes F,\sigma} F$ of $h^1_{\mathrm{dR}}(A)$ is denoted by $H^1_{\mathrm{dR}}(A/F)(-\sigma)$ in the notation of [KS19]. The notation $\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))$ is analogous to the notation introduced in [KS19], Definition 1.6. If $\omega \in h^1_{\mathrm{dR}}(A)$ decomposes as $\omega = \sum_\sigma \omega(\sigma)$ with $\omega(\sigma) \in h^1_{\mathrm{dR}}(A)(\sigma)$, the element $\omega^{[\iota\alpha]}$, as defined in loc. cit., gets mapped to $\omega^{\iota\alpha}$ under the identification $\mathrm{TSym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A)) \cong \mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))$ via (0.3).

8.3. Periods of M^α.

Let M be a motive of type (F, ψ, L) and assume that F contains the Galois closure of L. Let $\gamma \in M_B$ be an L-basis and $\omega \in M_{\mathrm{dR}}$. Consider the $L \otimes \mathbf{C}$-linear comparison isomorphism

$$I_{\infty,M} : M_{\mathrm{dR}} \otimes_F \mathbf{C} \to M_B \otimes \mathbf{C}$$

attached to the motive M. There is a unique element $\Omega \in L \otimes \mathbf{C}$ satisfying

$$I_{\infty,M}(\omega) = \Omega \cdot \gamma. \tag{8.5}$$

Notation 8.7. *Let $\alpha \in I_L^+$. For an element $\Omega \in L \otimes \mathbf{C}$ and $\iota \in J_{L^\alpha}$, we define*

$$\Omega^{\iota\alpha} := \prod_{\sigma\in J_L} \Omega(\sigma)^{\iota\alpha(\sigma)},$$

where $\Omega = (\Omega(\sigma))_{\sigma\in J_L}$ under the canonical isomorphism $L \otimes \mathbf{C} = \mathbf{C}^{J_L}$. Moreover, we put

$$\Omega^{\boldsymbol{\alpha}} := (\Omega^{\iota\alpha})_{\iota\in J_{L^\alpha}} \in L^\alpha \otimes \mathbf{C},$$

where again, we use the identification $L^\alpha \otimes \mathbf{C} = \mathbf{C}^{J_{L^\alpha}}$.

Proposition 8.8. *Let M be a motive of type (F, ψ, L) and assume that F contains the Galois closure of L. Let $\gamma \in M_B$ be an L-basis, $\omega \in M_{\mathrm{dR}}$ and $\Omega \in L \otimes \mathbf{C}$ as in (8.5). Then*

$$I_{\infty,M^\alpha}(\omega^{\boldsymbol{\alpha}}) = \Omega^{\boldsymbol{\alpha}} \cdot \gamma^{\boldsymbol{\alpha}}.$$

Proof. According to the canonical isomorphism $L \otimes \mathbf{C} = \mathbf{C}^{J_L}$, the isomorphism $I_{\infty,M}$ decomposes into $\mathbf{C}$-linear isomorphisms $I_{\infty,\sigma} : M_{\mathrm{dR}} \otimes_{F\otimes L,\sigma} \mathbf{C} \to M_B \otimes_{L,\sigma} \mathbf{C}$, which satisfy

$$I_{\infty,\sigma}(\omega \otimes_\sigma 1_{\mathbf{C}}) = \Omega_\sigma \cdot (\gamma \otimes_\sigma 1_{\mathbf{C}}).$$

Since the $L^\alpha \otimes \mathbf{C}$-linear isomorphism I_{∞,M^α} is induced by the $L \otimes \mathbf{C}$-linear isomorphism $I_{\infty,M}$ by applying the functor $(-)^\alpha$, it now follows from the construction of the elements $\gamma^{\boldsymbol{\alpha}}$ and $\omega^{\boldsymbol{\alpha}}$ that $I_{\infty,M^\alpha}(\omega^{\boldsymbol{\alpha}}) = \Omega^{\boldsymbol{\alpha}} \cdot \gamma^{\boldsymbol{\alpha}}$, as desired. □

9. A motive for Ξ

In the following, we work with setup introduced in Section 6: Let L be a totally imaginary number field containing a CM-field. Let Φ be a CM-type of L and let χ_a be a critical infinity-type with respect to Φ, i.e.

$$\chi_a = \beta - \alpha$$

with $\beta \in I^+_{\overline{\Phi}}$, $\alpha \in I^+_\Phi$ such that $\alpha(\sigma) \geq 1$ and $\beta(\overline{\sigma}) \geq 0$ for all $\sigma \in \Phi$. Let T be a CM-field containing L^α and let

$$\chi : L_f^\times \to T^\times$$

be an algebraic Hecke character of infinity-type χ_a. Let E be the reflex field of (L, Φ) and let $\Phi^* \in I_E$ denote the reflex CM-type of Φ. We define the algebraic Hecke character $\Xi : E_f^\times \to T^\times$ by

$$\Xi = (\chi \circ \Phi^*)^{-1} \|\cdot\|_E^{-d(\beta)} \varepsilon_\Phi,$$

as in (6.4), where ε_Φ is the sign defined in Definition 6.3.

9.1. **Construction.** The following construction is based on results of Goldstein-Schappacher ([GS81], §4) and Deninger ([Den89], section 2). In fact, it will be clear that all of the following can be more generally carried out in the context of Theorem 4.1.

Let F/E be a finite abelian extension and A/F an abelian variety of CM-type (L, Φ) satisfying condition (5) of Proposition 6.8, i.e. the CM-character $\psi : F_f^\times \to L^\times$ of A satisfies

$$\Xi \circ N_{F/E} = \psi^\alpha \overline{\psi}^\beta. \tag{9.1}$$

By the constructions and results of Section 8, we have the motives $h^1(A)^\alpha$ and $h^1(A^\vee)^\beta$ of types $(F, \psi^\alpha, L^\alpha)$ and $(F, \overline{\psi}^\beta, L^\beta)$, respectively. Let $h^1(A)^\alpha_T$ and $h^1(A^\vee)^\beta_T$ denote the extension of coefficients to T. Then

$$N := h^1(A)^\alpha_T \otimes h^1(A^\vee)^\beta_T \tag{9.2}$$

is a motive of type $(F, \psi^\alpha \overline{\psi}^\beta, T)$. We now use the results of Section 4 to construct a motive for Ξ out of the motive (9.2). By (9.1), condition (i) of Theorem 4.1 is satisfied and hence the restriction of scalars $R_{F/E}N$ acquires coefficients in the commutative semisimple T-algebra

$$\mathscr{T} = \bigoplus_{\sigma \in G_{F/E}} \mathrm{Hom}_T(N, N^\sigma) \cdot \sigma,$$

such that it is of rank 1 over $\mathscr{T}$. Theorem 3.17 implies that there exists an algebraic Hecke character

$$\widetilde{\Xi} : E_f^\times \to \mathscr{T}^\times$$

such that $R_{F/E}N$ is of type $(E, \widetilde{\Xi}, \mathscr{T})$.

Lemma 9.1. *There is a canonical isomorphism*

$$N \otimes_T \mathscr{T} \xrightarrow{\sim} R_{F/E}N \times_E F$$

of motives over F with coefficients in $\mathscr{T}$.

Proof. Note that $R_{F/E}N \times_E F = \bigoplus_{\sigma \in G_{F/E}} N^\sigma$. The canonical inclusion $N \hookrightarrow \bigoplus_{\sigma \in G_{F/E}} N^\sigma$ then induces a morphism

$$\varphi : N \otimes_T \mathscr{T} \to \bigoplus_{\sigma \in G_{F/E}} N^\sigma$$

and it suffices to check that it is an isomorphism in the Betti realization. Let $\sigma \in G_{F/E}$ and $v_\sigma \in N_B^\sigma$. Choose any non-zero $\theta_\sigma \in \mathrm{Hom}_T(N, N^\sigma)$. Then θ_σ is an isomorphism and there exists

$v \in N_B$ such that $\theta_{\sigma,B}(v) = v_\sigma$. Then one has $\varphi_B(v \otimes \theta_\sigma) = v_\sigma$, so that φ_B is surjective. Since source and target of φ_B are both of rank 1 over $\mathscr{T}$, we deduce that φ_B is an isomorphism. □

The following result is an adaptation of [GS81], Lemme 4.8 to our setting:

Lemma 9.2. *The algebraic Hecke character $\widetilde{\Xi}$ satisfies the following properties:*

(i) $\widetilde{\Xi} \circ N_{F/E} = \psi^\alpha \overline{\psi}^\beta$.

(ii) *As a T-module $\mathscr{T}$ is generated by the values of $\widetilde{\Xi}$.*

(iii) *One has an equality*

$$\{\varepsilon \circ \widetilde{\Xi} \mid \varepsilon \in \mathrm{Hom}_T(\mathscr{T}, \overline{\mathbf{Q}})\} = \{\Xi \cdot \lambda \mid \lambda \in \widehat{G}_{F/E}\}$$

of subsets of $\mathrm{Hom}(E_f^\times, \overline{\mathbf{Q}}^\times)$, *where* $\widehat{G}_{F/E} = \mathrm{Hom}(G_{F/E}, \overline{\mathbf{Q}}^\times)$ *denotes the Pontryagin dual of* $G_{F/E}$.

Proof. In terms of the corresponding Hecke characters, Lemma 9.1 implies that

$$\widetilde{\Xi} \circ N_{F/E} = \psi^\alpha \overline{\psi}^\beta,$$

where we regard $\psi^\alpha \overline{\psi}^\beta$ as having values in $\mathscr{T}$. This shows assertion (i). Note that in particular the infinity types of $\widetilde{\Xi}$ and Ξ agree. For (ii), let $\sigma \in G_{F/E}$ and $e_\sigma \in E_f^\times$ such that $r_E(e_\sigma)|_F = \sigma$. Since $R_{F/E}(N)$ is of type $(E, \widetilde{\Xi}, \mathscr{T})$ and $r_E(e_\sigma)$ carries the id-component N_f to N_f^σ in $R_{F/E}(N)_f$, we may regard $\widetilde{\Xi}(e_\sigma) \in \mathscr{T}^\times$ as a non-zero element in the 1-dimensional T-subspace $\mathrm{Hom}_T(N, N^\sigma)\sigma \subset \mathscr{T}$. From this claim (ii) follows. We now prove assertion (iii). For every $\varepsilon \in \mathrm{Hom}_T(\mathscr{T}, \overline{\mathbf{Q}})$, one has

$$(\varepsilon \circ \widetilde{\Xi}) \circ N_{F/E} = \varepsilon \circ \psi^\alpha \overline{\psi}^\beta = \psi^\alpha \overline{\psi}^\beta$$

since $\psi^\alpha \overline{\psi}^\beta$ takes values in T. It follows that there exists a finite character $\lambda \in \widehat{G}_{F/E}$ such that $\varepsilon \circ \widetilde{\Xi} = \Xi \cdot \lambda$. This establishes one inclusion in (iii). By (ii), both of the sets in question have the same cardinality $[F : E] = [\mathscr{T} : T]$, hence they are equal. □

By taking the trivial character $\lambda = 1$ in Lemma 9.2 (iii), we deduce the existence of a homomorphism $\varepsilon \in \mathrm{Hom}_T(\mathscr{T}, \overline{\mathbf{Q}})$ such that

$$\varepsilon \circ \widetilde{\Xi} = \Xi,$$

where Ξ is regarded to have values in $\overline{\mathbf{Q}}$ via the fixed embedding $T \subset \overline{\mathbf{Q}}$. It follows from 9.2 (ii) that the T-algebra homomorphism ε factors as $\varepsilon : \mathscr{T} \to T$. Since $\mathscr{T}$ is a product of finite field extensions of T, it follows that T is one of the direct factors of $\mathscr{T}$ and that ε is the canonical projection. In other words, there exists a primitive idempotent e_Ξ in $\mathscr{T}$ such that

$$e_\Xi \cdot \mathscr{T} = T \quad \text{and} \quad e_\Xi \cdot \widetilde{\Xi} = \Xi.$$

Hence, the motive

$$M(\Xi) := e_\Xi \cdot R_{F/E}(h^1(A)_T^\alpha \otimes h^1(A^\vee)_T^\beta) \tag{9.3}$$

is of type (E, Ξ, T).

9.2. **The Betti realization of $M(\Xi)$.** For each $\sigma \in J_F$, let $\gamma_\sigma \in h_B^1(A^\sigma)$ be an L-basis. Via the canonical L-linear pairing $h_B^1(A^\sigma) \otimes_L h_B^1(A^{\vee,\sigma}) \to L(-1)_B = (2\pi i)^{-1}L$, we can associate dual basis vectors $\gamma_\sigma^\vee \in h_B^1(A^{\vee,\sigma})$. For each $\eta \in J_E$, the Betti realization of $M(\Xi)^\eta$ is given by

$$M(\Xi)_B^\eta = e_\Xi \cdot \bigoplus_{\sigma \in J_{F,\eta}} h_B^1(A^\sigma)_T^\alpha \otimes h_B^1(A^{\vee,\sigma})_T^\beta. \tag{9.4}$$

Recall the construction $\gamma \mapsto \gamma^{\boldsymbol{\alpha}}$ from (8.3). For each $\sigma \in J_F$ and any L-basis $\gamma_\sigma \in h_B^1(A^\sigma)$, we consider the T-basis

$$\gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}}$$

of $h_B^1(A^\sigma)_T^\alpha \otimes h_B^1(A^{\vee,\sigma})_T^\beta$.

Proposition 9.3. *For $\sigma \in J_F$ and $\tau \in G_{\mathbf{Q}}$, let $\lambda(\tau, \sigma) \in L_f^\times$ denote the unique element satisfying*

$$I_f(\gamma_\sigma)^\tau = \lambda(\tau, \sigma) I_f(\gamma_{\tau\sigma}).$$

Then one has

$$I_f(\gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}})^\tau = \chi_a(\lambda(\tau, \sigma))^{-1} \cdot \chi_{\mathrm{cyc}}(\tau)^{-d(\beta)} \cdot I_f(\gamma_{\tau\sigma}^{\boldsymbol{\alpha}} \otimes (\gamma_{\tau\sigma}^\vee)^{\boldsymbol{\beta}}).$$

Proof. Let $\lambda^\vee(\tau,\sigma)\in L_f^\times$ denote the coefficients obtained from the system of dual basis elements $\{\gamma_\sigma^\vee \mid \sigma\in J_F\}$. By construction of the comparison isomorphism I_f for the motives $h^1(A^\sigma)^\alpha$ and $h^1(A^{\sigma,\vee})^\beta$, it immediately follows that

$$I_f(\gamma_\sigma^{\boldsymbol\alpha}\otimes(\gamma_\sigma^\vee)^{\boldsymbol\beta})^\tau=\lambda(\tau,\sigma)^\alpha\cdot\lambda^\vee(\tau,\sigma)^\beta\cdot I_f(\gamma_{\tau\sigma}^{\boldsymbol\alpha}\otimes(\gamma_{\tau\sigma}^\vee)^{\boldsymbol\beta})$$

and it remains to determine $\lambda^\vee(\tau,\sigma)$ in terms of $\lambda(\tau,\sigma)$. Recall Section 5.2. The Weil pairings

$$\langle\cdot,\cdot\rangle_\sigma: h_f^1(A^\sigma)\times h_f^1(A^{\vee,\sigma})\to L_f(-1)$$

are compatible with the action of $G_{\mathbf Q}$, i.e. for any $\tau\in G_{\mathbf Q}$, $a\in h_f^1(A^\sigma)$ and $b\in h_f^1(A^{\vee,\sigma})$, one has $\langle a,b\rangle_\sigma^\tau=\langle a^\tau,b^\tau\rangle_{\tau\sigma}$, where we use the full $G_{\mathbf Q}$-action on $L_f(-1)$. It follows that

$$\begin{aligned}\langle\gamma_{\sigma,f},\gamma_{\sigma,f}^\vee\rangle_\sigma^\tau&=\langle\gamma_{\sigma,f}^\tau,(\gamma_{\sigma,f}^\vee)^\tau\rangle_{\tau\sigma}\\&=\langle\lambda(\tau,\sigma)\gamma_{\tau\sigma,f},\lambda^\vee(\tau,\sigma)\gamma_{\tau\sigma,f}^\vee\rangle_{\tau\sigma}\\&=\lambda(\tau,\sigma)\lambda^\vee(\tau,\sigma)\cdot\langle\gamma_{\tau\sigma,f},\gamma_{\tau\sigma,f}^\vee\rangle_{\tau\sigma}\end{aligned}$$

On the other hand, we have

$$\langle\gamma_{\sigma,f},\gamma_{\sigma,f}^\vee\rangle_\sigma^\tau=\chi_{\mathrm{cyc}}(\tau)^{-1}\cdot\langle\gamma_{\sigma,f},\gamma_{\sigma,f}^\vee\rangle_\sigma.$$

Since the Weil pairing is of motivic origin and by definition of the dual bases $\gamma_\sigma^\vee$ in terms of the pairing in the Betti realization, the L_f-basis $\langle\gamma_{\sigma,f},\gamma_{\sigma,f}^\vee\rangle_\sigma$ of $L_f(-1)$ is independent of σ. Indeed, if $\langle\cdot,\cdot\rangle_B$ denotes the pairing in the Betti realization, we have $\langle\gamma_{\sigma,f},\gamma_{\sigma,f}^\vee\rangle_\sigma=I_f(\langle\gamma_\sigma,\gamma_\sigma^\vee\rangle_B)$ for all $\sigma\in J_F$ by virtue of the results stated in Section 5.2. By construction, $\langle\gamma_\sigma,\gamma_\sigma^\vee\rangle_B=(2\pi i)^{-1}\in L(-1)_B$ is independent of the embedding σ. Thus

$$\lambda^\vee(\tau,\sigma)=\lambda(\tau,\sigma)^{-1}\chi_{\mathrm{cyc}}(\tau)^{-1}$$

and one calculates

$$\begin{aligned}\lambda(\tau,\sigma)^\alpha\cdot\lambda^\vee(\tau,\sigma)^\beta&=\lambda(\tau,\sigma)^{\alpha-\beta}\chi_{\mathrm{cyc}}(\tau)^{-\beta}\\&=\chi_a(\lambda(\tau,\sigma))^{-1}\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)}.\end{aligned}$$

□

9.3. **Basis elements in $M(\Xi)_B^\eta$.** As before, let F/E be a finite abelian extension and A/F an abelian variety of CM-type (L,Φ) such that its CM-character ψ satisfies $\Xi\circ N_{F/E}=\psi^\alpha\overline{\psi}^\beta$. Let $M(\Xi)$ denote the motive defined in (9.3). Fix a system $\{\gamma_\sigma\mid\sigma\in J_F\}$ of L-bases $\gamma_\sigma\in h_B^1(A^\sigma)$ and, for every $\tau\in G_{\mathbf Q}$ and $\sigma\in J_F$, let

$$\lambda(\tau,\sigma)\in L_f^\times$$

denote the unique elements such that

$$I_f(\gamma_\sigma)^\tau=\lambda(\tau,\sigma)\cdot I_f(\gamma_{\tau\sigma}).$$

Recall the definition of the elements $\xi(\tau,\eta)$ in (6.5). The aim of this section is to construct a system of basis elements $a_\eta\in M(\Xi)_B^\eta$ for $\eta\in J_E$ satisfying

$$I_f(a_\eta)^\tau=\xi(\tau,\eta)\cdot I_f(a_{\tau\eta}).\tag{9.5}$$

Essentially, we define the elements a_η to satisfy (9.5) by design. The more involved part will then be to show that these elements are in fact non-zero.

Definition/Proposition 9.4. *Let $\sigma\in G_{\mathbf Q}$. The element*

$$t_\sigma:=\chi(\lambda(\sigma,1_F))^{-1}\varepsilon_\Phi(\sigma)\in T^\times$$

only depends on the restriction $\sigma|_F$.

Proof. Two element $\sigma,\sigma'\in G_{\mathbf Q}$ with the same restriction to F satisfy $\sigma'=\sigma\tau$ for some $\tau\in G_F$. We have

$$\lambda(\sigma\tau,1_F)=\lambda(\sigma,1_F)\lambda(\tau,1_F)$$

and hence $t_{\sigma\tau}=t_\sigma t_\tau$. It therefore suffices to show $t_\tau=1$. Since $h^1(A)$ is of type (F,ψ,L) and $\psi_a=\Phi^*\circ N_{F/E}$, one has

$$\lambda(\tau,1_F)=\psi(x)\cdot\Phi^*(N_{F/E}(x))^{-1},$$

for any $x \in F_f^\times$ with $r_F(x) = \tau|_{F^{\mathrm{ab}}}$. Hence, one obtains

$$\begin{aligned} t_\tau &= \chi(\lambda(\tau, 1_F))^{-1}\varepsilon_\Phi(\tau) \\ &= \chi_a(\psi(x))^{-1}\chi(\Phi^*(N_{F/E}(x)))\varepsilon_\Phi(\tau) \\ &= \psi(x)^{\alpha-\beta} \cdot \Xi(N_{F/E}(x))^{-1} \cdot \|N_{F/E}(x)\|_E^{-d(\beta)} \cdot \varepsilon_\Phi(N_{F/\mathbf{Q}}(x)) \cdot \varepsilon_\Phi(\tau) \\ &= \psi(x)^{\alpha-\beta} \cdot \psi(x)^{-\alpha}\overline{\psi}(x)^{-\beta} \cdot \|x\|_F^{-d(\beta)} \\ &= \psi(x)^{-\beta}\overline{\psi}(x)^{-\beta} \cdot \|x\|_F^{-d(\beta)} \\ &= 1, \end{aligned}$$

which proves the claim. □

Let us introduce the following notation: Recall (9.4). For each $\sigma \in J_F$, consider the element

$$b(\gamma_\sigma) := e_\Xi \cdot \gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}} \tag{9.6}$$

in $M(\Xi)_B^\sigma$. Proposition 9.3 implies that

$$I_f(b(\gamma_\sigma))^\tau = \chi_a(\lambda(\tau,\sigma))^{-1}\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)} I_f(b(\gamma_{\tau\sigma})) \tag{9.7}$$

for all $\sigma \in J_F$ and $\tau \in G_{\mathbf{Q}}$.

Definition 9.5. *For each $\eta \in J_E$, we define the the element*

$$a_\eta := \sum_{\sigma \in J_{F,\eta}} t_\sigma \cdot b(\gamma_\sigma)$$

in $M(\Xi)_B^\eta$.

We can now prove the main result of this section:

Theorem 9.6. *For every $\eta \in J_E$, the element $a_\eta \in M(\Xi)_B^\eta$ is non-zero. For all $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_E$, one has*

$$I_f(a_\eta)^\tau = \xi(\tau,\eta) \cdot I_f(a_{\tau\eta}). \tag{9.8}$$

Proof. We first show that the elements satisfy (9.8). By (9.7), we have

$$\begin{aligned} I_f(a_\eta)^\tau &= \sum_{\sigma \in J_{F,\eta}} t_\sigma I_f(b(\gamma_\sigma))^\tau \\ &= \sum_{\sigma \in J_{F,\eta}} t_\sigma \chi_a(\lambda(\tau,\sigma))^{-1}\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)} I_f(b(\gamma_{\tau\sigma})). \end{aligned}$$

It now suffices to show that the elements t_σ satisfy

$$t_\sigma \cdot \chi_a(\lambda(\tau,\sigma))^{-1}\chi_{\mathrm{cyc}}(\tau)^{-d(\beta} = t_{\tau\sigma} \cdot \xi(\tau,\eta)$$

for all $\sigma \in J_{F,\eta}$ and $\tau \in G_{\mathbf{Q}}$. This follows essentially by construction: For every $\sigma \in J_F$, we choose an extension $\sigma \in G_{\mathbf{Q}}$. By making use of the equalities

$$\begin{aligned} \lambda(\tau\sigma, 1_F) &= \lambda(\sigma, 1_F)\lambda(\tau,\sigma), \\ \varepsilon_\Phi(\sigma)\varepsilon_{\eta\Phi}(\tau) &= \varepsilon_\Phi(\tau\sigma), \end{aligned}$$

we obtain

$$\begin{aligned} t_{\tau\sigma} \cdot \xi(\tau,\eta) &= \chi(\lambda(\tau\sigma, 1_F))^{-1}\varepsilon_\Phi(\tau\sigma) \cdot \chi_a(\lambda(\tau,\sigma))^{-1}\chi(\lambda(\tau,\sigma))\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)}\varepsilon_{\eta\Phi}(\tau) \\ &= \chi(\lambda(\sigma, 1_F))^{-1}\varepsilon_\Phi(\sigma)\chi_a(\lambda(\tau,\sigma))^{-1}\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)} \\ &= t_\sigma \cdot \chi_a(\lambda(\tau,\sigma))^{-1}\chi_{\mathrm{cyc}}(\tau)^{-d(\beta)}, \end{aligned}$$

and the claim follows.

We now prove that for any $\eta \in J_E$, the element a_η in $M(\Xi)_B^\eta$ is in fact non-zero. For every $\sigma \in J_F$, we choose an extension $\sigma \in G_{\mathbf{Q}}$ and abbreviate $\lambda_\sigma := \lambda(\sigma, 1_F) \in L_f^\times$. Recall that $t_\sigma = \chi(\lambda_\sigma)^{-1}\varepsilon_\Phi(\sigma)$.

Step 1: First, we observe that the claim $a_\eta \neq 0$ is independent of the choice of the basis elements $\{\gamma_\sigma\}_{\sigma \in J_{F,\eta}}$ used in the construction of a_η. Indeed, a different choice $\{\gamma'_\sigma\}_{\sigma \in J_F}$ is of the form

$$\gamma'_\sigma = u_\sigma \cdot \gamma_\sigma \quad \text{with} \quad u_\sigma \in L^\times.$$

Let $\sigma \in G_{\mathbf{Q}}$ denote an extension of $\sigma \in J_F$. We have

$$\begin{aligned} I_f(\gamma'_{1_F})^{\sigma} &= u_{1_F} \cdot I_f(\gamma_{1_F})^{\sigma} \\ &= u_{1_F}\lambda_\sigma \cdot I_f(\gamma_\sigma) \\ &= u_{1_F}u_\sigma^{-1}\lambda_\sigma \cdot I_f(\gamma'_\sigma), \end{aligned}$$

so that the coefficients $\lambda'_\sigma = \lambda'(\sigma, 1_F)$ for the system $\{\gamma'_\sigma\}_\sigma$ are given by

$$\lambda'_\sigma = u_{1_F}u_\sigma^{-1} \cdot \lambda_\sigma.$$

It follows that $\chi(\lambda'_\sigma) = (u_{1_F}u_\sigma^{-1})^{\beta-\alpha} \cdot \chi(\lambda_\sigma)$. On the other hand, one has $b(\gamma'_\sigma) = u_\sigma^{\alpha-\beta} \cdot b(\gamma_\sigma)$, and hence

$$\chi(\lambda'_\sigma)\varepsilon_\Phi(\sigma) \cdot b(\gamma'_\sigma) = u_{1_F}^{\alpha-\beta} \cdot \chi(\lambda_\sigma)\varepsilon_\Phi(\sigma) \cdot b(\gamma_\sigma).$$

This shows

$$a'_\eta = u_{1_F}^{\alpha-\beta} \cdot a_\eta$$

and the claim follows.

Step 2: It suffices to show $a_{1_E} \neq 0$, since for arbitrary $\eta \in J_E$ and any extension $\eta \in G_{\mathbf{Q}}$ of $\eta \in J_E$, one has

$$I_f(a_{1_E})^\eta = \xi(\eta, 1_E) \cdot I_f(a_\eta)$$

by design. Now $\xi(\eta, 1_F) \in E_f^\times$ and I_f is an isomorphism, so that $a_{1_E} \neq 0$ implies $a_\eta \neq 0$ for any $\eta \in J_E$.

Step 3: By Step 1, we may choose the system $\{\gamma_\sigma\}_\sigma$ in the following way: For every $\sigma \in J_{F/E}$ fix an extension $\sigma \in G_E$ and $e_\sigma \in E_f^\times$ such that

$$r_E(e_\sigma) = \sigma|_{E^{\mathrm{ab}}}.$$

Assume that id extends 1_E and that $e_{1_E} = 1$. Recall the $L^\times$-coset $L(\Phi, \sigma)$ defined in section 5.6. Since every extension σ fixes the reflex field, we have

$$L(\Phi, \sigma) = L^\times \cdot \Phi^*(e_\sigma)^{-1}$$

by Proposition 5.7 (ii). Fix an arbitrary L-basis $\gamma_1 \in h^1(A)_B$. For every $\sigma \in J_{F/E}$, the main theorem of complex multiplication (cf. Theorem 5.8) implies the existence of a basis vector $\gamma_\sigma \in h^1(A)_B^\sigma$ satisfying

$$\lambda_\sigma = \Phi^*(e_\sigma)^{-1}. \tag{9.9}$$

On the other hand, the motive $M(\Xi)$ is of type (E, Ξ, T) and $\sigma \in G_E$ and hence

$$I_f(b(\gamma_1))^\sigma = \Xi_a(e_\sigma)^{-1} \cdot \Xi(e_\sigma) \cdot I_f(b(\gamma_1)). \tag{9.10}$$

Note that for the above it is essential that we passed to the direct summand cut out by e_Ξ. Recall that $\Xi_a = (\Phi^*)^{\alpha+c\beta}$. Using (9.7), (9.9) and (9.10), we obtain

$$\begin{aligned} I_f(b(\gamma_\sigma)) &= \lambda_\sigma^{\alpha-\beta}\chi_{\mathrm{cyc}}(\sigma)^{d(\beta)}\Xi_a(e_\sigma)^{-1}\Xi(e_\sigma) \cdot I_f(b(\gamma_1)) \\ &= \Phi^*(e_\sigma)^{\alpha-\beta}\Phi^*(e_\sigma)^{-\alpha-c\beta}\chi_{\mathrm{cyc}}(\sigma)^{d(\beta)}\Xi(e_\sigma) \cdot I_f(b(\gamma_1)) \\ &= N_{E/\mathbf{Q}}(e_\sigma)^{-d(\beta)}\chi_{\mathrm{cyc}}(\sigma)^{d(\beta)}\Xi(e_\sigma) \cdot I_f(b(\gamma_1)) \\ &= \|e_\sigma\|_E^{d(\beta)} \cdot \Xi(e_\sigma) \cdot I_f(b(\gamma_1)), \end{aligned}$$

and hence

$$b(\gamma_\sigma) = \|e_\sigma\|_E^{d(\beta)} \cdot \Xi(e_\sigma) \cdot b(\gamma_1) \tag{9.11}$$

in $M(\Xi)_B$. Therefore, again by (9.9) and by definition of the Hecke character Ξ (cf. Definition 6.4), we obtain

$$\begin{aligned} a_1 &= \sum_{\sigma \in J_{F/E}} t_\sigma \cdot b(\gamma_\sigma) \\ &= \sum_{\sigma \in J_{F/E}} \chi(\Phi^*(e_\sigma))\varepsilon_\Phi(e_\sigma)\|e_\sigma\|_E^{d(\beta)} \cdot \Xi(e_\sigma) \cdot b(\gamma_1) \\ &= [F:E] \cdot b(\gamma_1). \end{aligned}$$

Finally, observe that

$$b(\gamma_1) \neq 0$$

in $M(\Xi)$ as otherwise one has $b(\gamma_\sigma) = 0$ for all $\sigma \in J_{F/E}$ by (9.11). But the elements $b(\gamma_\sigma)$ for $\sigma \in J_{F/E}$ generate $M(\Xi)_B$ as a T-vector space and hence one would have $M(\Xi)_B = 0$, in contradiction to $\dim_T M(\Xi)_B = 1$. □

The theorem provides an explicit presentation, in terms of the CM abelian variety A, of the basis elements used in Blasius' Theorem 6.7, which a priori are rather implicit in nature. This is crucial in our approach: In the next sections, we will turn our attention to the de Rham realization $M(\Xi)_{\mathrm{dR}}$ and construct elements in $F^{d(\alpha)+d(\beta)}RM(\Xi)_{\mathrm{dR}}$ by using Eisenstein-Kronecker classes as defined by Kings-Sprang in [KS19]. The ultimate goal is then to compute the period $v(\chi)$ occuring in Blasius Theorem and for this we need the explicit presentation of Definition 9.5.

10. Eisenstein-Kronecker classes

In this section, we briefly recall properties of the Eisenstein-Kronecker classes ("EK-classes" in short) defined in [KS19]. The construction of these classes and their compatibility under base change can be found in Appendix C. The results of this chapter are largely based on [KS19], sections 2–4. In section 2.4 of [KS19], the EK-classes are further refined and we recall this process in detail. It is at this point where base change by a Galois automorphism $\sigma \in G_{\mathbf{Q}}$ introduces a sign. As it turns out it is precisely the sign from Definition 6.3, cf. Proposition 10.12. Next, we recall how the EK-classes relate to the special values of Hecke L-functions. For this we state several results from [KS19] without proof, in particular the important Theorem 10.17 (cf. [KS19], Corollary 3.23), which expresses the EK-classes analytically in terms of Eisenstein-Kronecker series. For applications to special values of Hecke L-functions, one fixes a CM abelian variety A/F such that $H_1(A(\mathbf{C}), \mathbb{Z}) \cong \mathcal{O}_L$. The main result of this section is Theorem 10.21, which slightly generalizes results of [KS19], 4.2-4.4 to incorporate the conjugates of the abelian variety A/F.

10.1. **CM abelian varietes revisited.** Let L be a totally complex number field and let F be a Galois extension over $\mathbf{Q}$ which contains L. Let $\pi : A \to F$ be an abelian variety of relative dimension $g > 0$ with CM by $\mathcal{O}_L$ and Φ denote the CM-type of A. Let e denote the unit section of A/F. We let $A^\vee/F$ denote the dual abelian variety. As usual, we define $\omega_{A/F} = e^*\Omega^1_{A/F} \cong \pi_*\Omega^1_{A/F}$ and let $\omega^g_{A/F} := \Lambda^g \omega_{A/F}$ denote its top degree exterior power. Moreover, we abbreviate

$$\mathcal{H}_A := H^1_{\mathrm{dR}}(A^\vee/F).$$

We freely identify quasi-coherent modules on $\mathrm{Spec}(F)$ with F-vector spaces. Let $\Gamma \subset \mathcal{O}_L^\times$ be a finite index subgroup. Then Γ acts from the left on A/F via automorphisms. This way the F-vector spaces $\mathrm{Lie}(A/F)$, $\omega_{A^\vee/F}$ and $\mathcal{H}_A$ acquire induced Γ-module structures by functoriality. On $\omega_{A/F}$ we consider the Γ-action dual to the one on $\mathrm{Lie}(A/F)$. In fact, in this way the above modules even become algebraic $\mathbf{T}_{L,F}$-modules one obtains decompositions (cf. [KS19], Proposition 1.11)

$$\mathrm{Lie}(A/F) = \bigoplus_{\sigma\in\Phi} \mathrm{Lie}(A/F)[\sigma], \qquad \omega_{A/F} = \bigoplus_{\sigma\in\Phi} \omega_{A/F}[-\sigma], \tag{10.1}$$

$$\omega_{A^\vee/F} = \bigoplus_{\overline{\sigma}\in\overline{\Phi}} \omega_{A^\vee/F}[\overline{\sigma}], \qquad \mathcal{H}_A = \bigoplus_{\sigma\in J_L} \mathcal{H}_A[\sigma]. \tag{10.2}$$

For a type $\alpha \in I_L^+$, we use the notation of [KS19], Definition 1.6, i.e. we write

$$\mathrm{TSym}^\alpha(\omega_{A/F}) = \bigotimes_{\sigma\in J_L} \mathrm{TSym}^{\alpha(\sigma)}(\omega_{A/F}[-\sigma])$$

and similarly for other $\mathbf{T}_{L,F}$-modules whose decomposition is indexed by elements of J_L. For an element $\omega \in \omega_{A/F}$, we write

$$\omega^{[\alpha]} = \bigotimes_{\sigma\in J_L} \omega(-\sigma)^{[\alpha(\sigma)]}$$

if $\omega = \sum_{\sigma\in J_L} \omega(-\sigma)$ with $\omega(-\sigma) \in \omega_{A/F}[-\sigma]$.

Remark 10.1. Remark about sign conventions is in order.

(1) There is a canonical *left* Γ-action on the dual abelian variety such that an element $\gamma \in \Gamma$ acts via $(\gamma_A^{-1})^\vee$ if γ_A denotes the induced automorphism of A. It induces the same $\mathbf{T}_{L,F}$-module structure on $\omega_{A^\vee/F}$ as defined above, because the two inverses cancel each other out. This Γ-action on the dual abelian variety $A^\vee/F$ does *not* agree with the Γ-action induced by the dual CM-structure (which implicitly uses the commutativity of $\mathcal{O}_L$), but they are inverse to each other.

(2) Note that for $\omega_{A^\vee/F}$, $\mathrm{Lie}(A/F)$ and $\mathcal{H}_A$, on which Γ acts "covariantly", the $\mathbf{T}_{L,F}$-module structure introduced above coincides with the one induced by the $L\otimes F$-module structure as defined in section 7.2. In contrast, the $\mathbf{T}_{L,F}$-module structure on $\omega_{A/F}$ defined above is inverse to the one defined as in loc. cit.

(3) It follows that the induced decompositions are the same up to obvious reindexing: If
$$\omega_{A/F} = \bigoplus_{\sigma\in\Phi} \omega_{A/F}(\sigma)$$
is the decomposition as an $L\otimes F$-module as in (7.6), then for all $\sigma \in \Phi$ one has $\omega_{A/F}(\sigma) = \omega_{A/F}[-\sigma]$. They only differ in their $\mathbf{T}_{L/F}$-module structures, which are inverse to each other.

(4) The expression $\mathrm{TSym}^\alpha(\omega_{A/F})$ defined above is completely analogous to the notation from Definition 7.4, cf. Remark 8.6.

It follows that the Hodge filtration $0 \to \omega_{A^\vee/F} \to \mathcal{H}_A \to \mathrm{Lie}(A/F) \to 0$ splits, i.e.
$$\mathcal{H}_A = \omega_{A^\vee/F} \oplus \mathrm{Lie}(A/F), \tag{10.3}$$
and the via the Hodge decomposition $\mathcal{H}_A \otimes_F \mathbf{C} = \omega_{A^\vee/\mathbf{C}} \oplus \overline{\omega_{A^\vee/\mathbf{C}}}$ (with respect to the real structure induced by the period isomorphism $\mathcal{H}_A \otimes_F \mathbf{C} \cong H^1(A^\vee(\mathbf{C}), \mathbf{C})$), one obtains an induced isomorphism
$$\mathrm{Lie}(A/F) \cong \overline{\omega_{A^\vee/\mathbf{C}}}. \tag{10.4}$$

10.2. **Periods.** First, let us introduce a little bit of notation for the following sections:

Notation 10.2. *For an element $z = (z(\sigma))_{\sigma\in J_L} \in \mathbf{C}^{J_L}$ we write*
$$\overline{z} := (\overline{z(\overline{\sigma})})_{\sigma\in J_L}.$$
Note that this corresponds to the involution $\mathrm{id}\otimes c$ on $L\otimes\mathbf{C}$. Recall from Notation 8.7, that for any type $\alpha \in I_L^+$, we write $z^\alpha := \prod_{\sigma\in J_L} z(\sigma)^{\alpha(\sigma)}$. Once we have fixed a CM-type Φ of L and are given an element $z = (z(\sigma))_{\sigma\in\Phi} \in \mathbf{C}^\Phi$, we write
$$Nz := \prod_{\sigma\in\Phi} |z(\sigma)|^2.$$

For any abelian variety A/F of CM-type (L, Φ), we let $I_{\infty,A} : H^1_{\mathrm{dR}}(A/\mathbf{C}) \xrightarrow{\sim} H^1(A(\mathbf{C}), \mathbf{C})$ denote the period isomorphism and we let
$$\langle\cdot,\cdot\rangle : H^1_{\mathrm{dR}}(A/\mathbf{C}) \times H_1(A(\mathbf{C}), \mathbf{C}) \to L\otimes\mathbf{C}$$
denote the corresponding pairing.

Definition 10.3. *Let A/F be as above. In the following, we identify $L\otimes\mathbf{C} \cong \mathbf{C}^{J_L}$. Let $\omega(A) \in \omega_{A/F}$ and $\omega(A^\vee) \in \omega_{A^\vee/F}$ denote fixed $L\otimes F$-generators. Moreover, let γ be an L-basis of $H^1(A(\mathbf{C}), \mathbf{Q})$ and let $\gamma^\vee$ denote its dual basis with respect to the pairing $H^1(A(\mathbf{C}), \mathbf{Q}) \otimes_L H^1(A^\vee(\mathbf{C}), \mathbf{Q}) \to (2\pi i)^{-1}L$. In this situation, we denote by $\Omega \in \mathbf{C}^\Phi$ and $\Omega^\vee \in \mathbf{C}^{\overline{\Phi}}$ the unique elements such that*
$$I_{\infty,A}(\omega(A)) = \Omega\cdot\gamma \quad \text{and} \quad I_{\infty,A^\vee}(\omega(A^\vee)) = \Omega^\vee\cdot\gamma^\vee.$$

Remark 10.4. Note that the period Ω defined above coincides with period defined in (8.5) for the motive $M = h^1(A)$ and by regarding $\omega(A)$ in $h^1_{\mathrm{dR}}(A)$ via the Hodge filtration $\omega_{A/F} \subset h^1_{\mathrm{dR}}(A)$. Similarly, $\Omega^\vee$ is associated to the motive $h^1(A^\vee)$.

The isomorphism (10.4) (for $A^\vee$ in place of A) induces a pairing
$$\langle\cdot,\cdot\rangle_A : \overline{\omega_{A/\mathbf{C}}} \times \omega_{A^\vee/\mathbf{C}} \to \mathbf{C}^{\overline{\Phi}}.$$

Proposition 10.5. *In the setup and notation of Definition 10.3 one has*

$$\Omega^\vee = \frac{2\pi i \langle \overline{\omega(A)}, \omega(A^\vee)\rangle_A}{\overline{\Omega}}.$$

Proof. This is [KS19], Proposition 4.4. There it is assumed that γ comes from an isomorphism $\mathcal{O}_L \xrightarrow{\sim} H_1(A(\mathbf{C}), \mathbb{Z})$, but this is not used in the proof. □

Similarly as in Section 5.3, we write $\varphi_\# := (\varphi^\vee)^* : \omega_{A^\vee/F} \to \omega_{B^\vee/F}$ for any isogeny $\varphi : A \to B$ over F. Let us also observe that the pairing $\langle\cdot,\cdot\rangle_A : \overline{\omega_{A/\mathbf{C}}} \times \omega_{A^\vee/\mathbf{C}} \to \mathbf{C}^\Phi$ satisfies the following functoriality property:

Proposition 10.6. *Let $\varphi : A \to B$ be an F-isogeny. For any $\omega \in \omega_{B/\mathbf{C}}$ and $\eta \in \omega_{A^\vee/\mathbf{C}}$, one has*

$$\langle \overline{\varphi^*\omega}, \eta\rangle_A = \langle \overline{\omega}, \varphi_\#(\eta)\rangle_B.$$

Proof. Recall that the pairing was induced by the composite $\overline{\omega_{A/\mathbf{C}}} \subset H^1_{\mathrm{dR}}(A/\mathbf{C}) \twoheadrightarrow \mathrm{Lie}(A^\vee/\mathbf{C})$. Observe that the inclusion $\overline{\omega_{A/\mathbf{C}}} \subset H^1_{\mathrm{dR}}(A/\mathbf{C})$ from the Hodge decomposition is compatible with φ^* since φ^* respects the real structure. Since the Hodge filtration is functorial, the diagram

$$\begin{array}{ccc} \overline{\omega_{B/\mathbf{C}}} & \xrightarrow{\sim} & \mathrm{Lie}(B^\vee/\mathbf{C}) \\ {\scriptstyle \overline{\varphi^*}}\downarrow & & \downarrow{\scriptstyle \varphi^\vee_*} \\ \overline{\omega_{A/\mathbf{C}}} & \xrightarrow{\sim} & \mathrm{Lie}(A^\vee/\mathbf{C}) \end{array}$$

commutes and the claim follows. □

Finally, let us recall the notion of an $\mathfrak{a}$-structure of A for a fractional ideal $\mathfrak{a}$ of L.

Definition 10.7. *Let A/F be an abelian variety of CM-type (L, Φ) and let $\mathfrak{a}$ be a fractional ideal of L. An* $\mathfrak{a}$-structure of A *is an isomorphism $\xi_\mathfrak{a} : \mathfrak{a} \xrightarrow{\sim} H_1(A(\mathbf{C}), \mathbb{Z})$. An* L-structure *is an isomorphism $\xi_L : L \xrightarrow{\sim} H_1(A(\mathbf{C}), \mathbf{Q})$, i.e. the choice of an L-basis.*

If A/F admits an $\mathfrak{a}$-structure $\xi_\mathfrak{a}$ and $\mathfrak{b}$ is another fractional ideal of A, the Serre construction $A(\mathfrak{b}) = A \otimes_{\mathcal{O}_L} \mathfrak{b}$ admits an induced $\mathfrak{ab}$-structure because $H_1((A\otimes_{\mathcal{O}_L}\mathfrak{b})(\mathbf{C}), \mathbb{Z}) \cong H_1(A(\mathbf{C}), \mathbb{Z}) \otimes_{\mathcal{O}_L} \mathfrak{b}$. Every $\mathfrak{a}$-structure $\xi_\mathfrak{a}$ induces an L-structure via

$$L \cong \mathfrak{a} \otimes_\mathbb{Z} \mathbf{Q} \xrightarrow{\xi_\mathfrak{a}\otimes\mathbf{Q}} H_1(A(\mathbf{C}), \mathbf{Q}).$$

There also is a canonical isomorphism $\mathrm{Lie}(A(\mathfrak{a})/F) \cong \mathrm{Lie}(A/F) \otimes_\mathbb{Z} \mathfrak{a} \cong \mathrm{Lie}(A/F)$, which induces a canonical isomorphism $\omega_{A(\mathfrak{a})/F} \cong \omega_{A/F}$. If $\omega(A)$ denotes an $L \otimes F$-generator of $\omega_{A/F}$, we let $\omega(A) \otimes \mathfrak{a}$ denote the corresponding generator in $\omega_{A(\mathfrak{a})/F}$ under the above isomorphism.

Proposition 10.8. *Let $\xi : \mathcal{O}_L \xrightarrow{\sim} H_1(A(\mathbf{C}), \mathbb{Z})$ be an $\mathcal{O}_L$-structure.*

(i) *Let $\mathfrak{a}$ be a fractional ideal of L and let $\xi_\mathfrak{a}$ denote the $\mathfrak{a}$-structure induced by ξ. Let $\omega(A)$ be an $L \otimes F$-generator of $\omega_{A/F}$. Then one has*

$$\langle \omega(A) \otimes \mathfrak{a}, \xi_\mathfrak{a}\rangle_{A(\mathfrak{a})} = \langle \omega(A), \xi(1)\rangle_A = \Omega.$$

Moreover, the image of

$$H_1(A(\mathfrak{a}), \mathbb{Z}) \to \mathrm{Lie}(A(\mathfrak{a})/\mathbf{C}) \xrightarrow[\sim]{\omega(A)\otimes\mathfrak{a}} \mathbf{C}^\Phi$$

coincides with $\mathfrak{a}\Omega \subset \mathbf{C}^\Phi$, where $\mathfrak{a}$ is embedded $\mathbf{C}^\Phi$ via the injection $L \hookrightarrow \mathbf{C}^\Phi$, $\ell \mapsto (\sigma(\ell))_{\sigma\in\Phi}$. In particular, the generator $\omega(A) \otimes \mathfrak{a}$ induces a complex uniformization $\mathbf{C}^\Phi/\mathfrak{a}\Omega \cong A(\mathfrak{a})(\mathbf{C})$.

(ii) *Moreover, if $\mathfrak{f}$ and $\mathfrak{b}$ are coprime integral ideals of $\mathfrak{f}$, one has*

$$\omega(A) \otimes \mathfrak{f}\mathfrak{b}^{-1} = [\mathfrak{f}]^*([\mathfrak{b}]^*)^{-1}\omega(A),$$

where $[\mathfrak{f}] : A(\mathfrak{f}\mathfrak{b}^{-1}) \to A(\mathfrak{b}^{-1})$ and $[\mathfrak{b}] : A \to A(\mathfrak{b}^{-1})$ denote the canonical isogenies as defined in Section 5.3.

Proof. This is [KS19], Proposition 4.5. □

10.3. **Eisenstein-Kronecker classes and their conjugates.** Let $\mathfrak{c} \subset \mathcal{O}_L$ be an integral ideal and let $A[\mathfrak{c}] := \ker([\mathfrak{c}]) \subset A$ denote the Γ-invariant closed subscheme of $[\mathfrak{c}]$-torsion points. Define

$$F[D]^{0,\Gamma} := \ker(H^0(A[\mathfrak{c}], \mathcal{O}_{A[\mathfrak{c}]}) \xrightarrow{\mathrm{tr}} F)^\Gamma,$$

where tr denotes the trace homomorphism associated to the finite F-algebra $H^0(A[\mathfrak{c}], \mathcal{O}_{A[\mathfrak{c}]})$. Let $\mathfrak{b} \subset \mathcal{O}_L$ be an integral ideal which is coprime to $\mathfrak{c}$ and let $x \in A[\mathfrak{b}](F)$ be a $[\mathfrak{b}]$-torsion section. Let $a, b \geq 0$ be integers and let $f \in F[D]^{0,\Gamma}$. By using equivariant cohomology and the Poincaré bundle, Kings-Sprang define cohomology classes

$$\mathrm{EK}^{b,a}_{A,\Gamma}(f, x) \in H^{g-1}(\Gamma, \mathrm{TSym}^a(\omega_{A/F}) \otimes \mathrm{TSym}^b(\mathcal{H}_A) \otimes \omega^g_{A/F}), \tag{10.5}$$

cf. Definition C.24. By the formal nature of their construction, one easily verifies that these classes are compatible with conjugation: Let $\sigma \in G_{\mathbf{Q}}$. The canonical σ-semilinear homomorphism

$$(-)^\sigma : \mathrm{TSym}^a(\omega_{A/F}) \otimes \mathrm{TSym}^b(\mathcal{H}_A) \otimes \omega^g_{A/F} \to \mathrm{TSym}^a(\omega_{A^\sigma/F}) \otimes \mathrm{TSym}^b(\mathcal{H}_{A^\sigma}) \otimes \omega^g_{A^\sigma/F}$$

is Γ-equivariant and induces a map on group cohomology, which we denote in the same way.

Lemma 10.9. *One has*

$$\mathrm{EK}^{b,a}_{A,\Gamma}(f, x)^\sigma = \mathrm{EK}^{b,a}_{A^\sigma,\Gamma}(f^\sigma, x^\sigma).$$

Proof. This is Proposition C.26 in the appendix. $\square$

Let $a, b \geq 0$ be non-negative integers. The splitting (10.3) induces a projection

$$\mathrm{TSym}^b(\mathcal{H}_A) \to \mathrm{TSym}^b(\omega_{A^\vee/F}).$$

Recall that $\mathrm{Crit}_L(\Gamma)$ denotes the set of Hecke character types with respect to Γ that are critical. Recall that a type $\mu \in I_L$ is of Hecke character type with respect to Γ if $\mu(\Gamma) = 1$. Also recall the notation of Remark 1.13 (ii). For every $\alpha, \beta \in I^+_L$, the group Γ acts on the 1-dimensional F-vector space $\mathrm{TSym}^\alpha(\omega_{A/F}) \otimes \mathrm{TSym}^\beta(\omega_{A^\vee/F}) \otimes \omega^g_{A/F}$ via $\beta - \alpha - 1_\Phi$. Therefore, one has

$$\begin{aligned}(\mathrm{TSym}^a(\omega_{A/F}) \otimes \mathrm{TSym}^b(\omega_{A^\vee/F}) \otimes \omega^g_{A/F})^\Gamma \cong \\ \bigoplus_{\substack{\beta-\alpha-1_\Phi \in \mathrm{Crit}_L(\Gamma)\\ d(\alpha)=a, d(\beta)=b}} \mathrm{TSym}^\alpha(\omega_{A/F}) \otimes \mathrm{TSym}^\beta(\omega_{A^\vee/F}) \otimes \omega^g_{A/F}\end{aligned} \tag{10.6}$$

Note that the occurrence of $\mathrm{Crit}_L(\Gamma)$ merely comes from our assumption that $a, b \geq 0$; more crucially, $\beta - \alpha - 1_\Phi$ needs to be of Hecke character type. In particular, $(\mathrm{TSym}^a(\omega_{A/F}) \otimes \mathrm{TSym}^b(\omega_{A^\vee/F}) \otimes \omega^g_{A/F})^\Gamma$ is a direct summand on which Γ-acts trivially. Let pr_Γ denote the induced projection map and consider

$$\mathrm{pr}_\Gamma(\mathrm{EK}^{b,a}_\Gamma(f, x)) \in \bigoplus_{\substack{\beta-\alpha-1_\Phi \in \mathrm{Crit}_L(\Gamma)\\ d(\alpha)=a, d(\beta)=b}} H^{g-1}(\Gamma, \mathrm{TSym}^\alpha(\omega_{A/F}) \otimes \mathrm{TSym}^\beta(\omega_{A^\vee/F}) \otimes \omega^g_{A/F}). \tag{10.7}$$

Proposition 10.10. *For $\beta - \alpha - 1_\Phi \in \mathrm{Crit}_L(\Gamma)$, there is a canonical homomorphism*

$$\begin{aligned}&H^{g-1}(\Gamma, \mathrm{TSym}^\alpha(\omega_{A/F}) \otimes \mathrm{TSym}^\beta(\omega_{A^\vee/F}) \otimes \omega^g_{A/F}) \to \\ &\mathrm{TSym}^{\alpha+1_\Phi}(\omega_{A/F}) \otimes \mathrm{TSym}^\beta(\omega_{A^\vee/F}).\end{aligned}$$

Proof. This is [KS19], Proposition 2.24. Let us review their construction: Since the action of Γ on $\mathrm{TSym}^\alpha(\omega_{A/F}) \otimes \mathrm{TSym}^\beta(\omega_{A^\vee/F}) \otimes \omega^g_{A/F}$ is trivial, it suffices (by the universal coefficient theorem) to construct a canonical homomorphism

$$\mathrm{can} : H^{d-1}(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \omega^g_{A/F} \to \mathrm{TSym}^{1_\Phi}(\omega_{A/F})$$

For this, first choose a subgroup $\Gamma' \subset \Gamma$ which is free of finite index; by Dirichlet's unit theorem it is of rank $g - 1$. Choose an *ordering of the CM-type* Φ. This induces an orientation on the norm 1 hypersurface

$$L^1_{\mathbf{R},\Phi} := \left\{ (r_\tau)_{\tau\in\Phi} \in \mathbf{R}^\Phi_{>0} \mid \prod_{\tau\in\Phi} r_\tau = 1 \right\} \subset \mathbf{R}^\Phi_{>0}$$

on which the group Γ' acts freely via $\gamma\cdot(r_\tau)_{\tau\in\Phi} = (|\tau(\gamma)|^2 r_\tau)_{\tau\in\Phi}$. This induces an isomorphism

$$H^{g-1}(\Gamma',\mathbb{Z}) = H^{g-1}(\Gamma'\backslash L^1_{\mathbf{R},\Phi},\mathbb{Z}) \cong \mathbb{Z}. \tag{10.8}$$

The choice of ordering on Φ also determines an isomorphism

$$\omega^g_{A/F} \cong \mathrm{TSym}^{1_\Phi}(\omega_{A/F}) \tag{10.9}$$

whose inverse sends a basis element $\otimes_{\tau\in\Phi}\omega(\tau)$ for $0\neq\omega(\tau)\in\omega_{A/F}(\tau)$ to $\bigwedge_{\tau\in\Phi}\omega(\tau)$ with respect to the chosen ordering. The isomorphisms (10.8) and (10.9) together then give rise to an isomorphism

$$\mathrm{can}' : H^{g-1}(\Gamma',\mathbb{Z})\otimes\omega^g_{A/F} \xrightarrow{\sim} \mathrm{TSym}^{1_\Phi}(\omega_{A/F})$$

which is *independent of the choice of the ordering of* Φ.

In the general case, one shows first that the restriction map

$$\mathrm{res} : H_{g-1}(\Gamma,\mathbb{Z}) \to H_{g-1}(\Gamma',\mathbb{Z})$$

is surjective. Let $\xi'\in H_{g-1}(\Gamma',\mathbb{Z})$ be the generator corresponding to the isomorphism (10.8) and $\xi\in H_{g-1}(\Gamma,\mathbb{Z})$ an element satisfying $\mathrm{res}(\xi)=\xi'$. Then, using the fixed ordering Φ, one obtains a map

$$\mathrm{can} : H^{g-1}(\Gamma,\mathbb{Z})\otimes\omega^g_{A/F} \xrightarrow{\cap\xi} \mathrm{TSym}^{1_\Phi}(\omega_{A/F}) \tag{10.10}$$

and it is shown that the square

$$\begin{tikzcd}
H^{g-1}(\Gamma,\mathbb{Z})\otimes\omega^g_{A/F} \arrow[r, "\cap\xi"] \arrow[d, "\mathrm{res}"'] & \mathrm{TSym}^{1_\Phi}(\omega_{A/F}) \arrow[d, "{[\Gamma:\Gamma']}"] \\
H^{g-1}(\Gamma',\mathbb{Z})\otimes\omega^g_{A/F} \arrow[r, "\cap\xi'"] & \mathrm{TSym}^{1_\Phi}(\omega_{A/F})
\end{tikzcd}$$

is commutative. Since F is a field, the vertical maps in the diagram are isomorphisms and it follows that (10.10) is an isomorphism which is independent of the choice of ξ and the ordering of Φ. □

Definition 10.11. *Let A/F be an abelian variety with CM by $\mathcal{O}_L$ of CM-type Φ. Let $\beta-\alpha-1_\Phi\in\mathrm{Crit}_L(\Gamma)$ be a critical Hecke character type and put $a=d(\alpha)$, $b=d(\beta)$. Let $\mathfrak{f}$ and $\mathfrak{c}$ be coprime integral ideals and let $D:=A[\mathfrak{c}]$ and $x\in U_D(F)$ a $\mathfrak{f}$-torsion section. Let $\Gamma\subset\mathcal{O}_L^\times$ be a finite index subgroup fixing x. Let $\mathfrak{f}\in F[D]^{0,\Gamma}$. Then define*

$$\mathrm{EK}^{\beta,\alpha}_{A,\Gamma}(f,x)\in\mathrm{TSym}^{\alpha+1_\Phi}(\omega_{A/F})\otimes_F\mathrm{TSym}^{\beta}(\omega_{A^\vee/F})$$

as the image of the (α,β)-component of $\mathrm{pr}_\Gamma\,\mathrm{EK}^{b,a}_\Gamma(f,x)$ in (10.7) under the canonical homomorphism of Proposition 10.10.

Proposition 10.12 (Conjugates of EK-classes)**.** *Notation as in Definition 10.11. For any $\sigma\in G_{\mathbf{Q}}$, one has*

$$\mathrm{EK}^{\beta,\alpha}_{A,\Gamma}(f,x)^\sigma = \varepsilon_\Phi(\sigma)\,\mathrm{EK}^{\sigma\beta,\sigma\alpha}_{A^\sigma,\Gamma}(f,x^\sigma).$$

Proof. Let $a=d(\alpha)$ and $b=d(\beta)$. The (α,β)-component of $\mathrm{pr}_\Gamma(\mathrm{EK}^{b,a}_\Gamma(f,x))$ lies in

$$H^{g-1}(\Gamma,\mathrm{Sym}^\alpha(\omega_{A/F})\otimes_F\mathrm{Sym}^\beta(\omega_{A^\vee/F})\otimes\omega^g_{A/F}).$$

By Lemma 10.9, we have

$$\mathrm{EK}^{b,a}_{A,\Gamma}(f,x)^\sigma = \mathrm{EK}^{b,a}_{A^\sigma,\Gamma}(f^\sigma,x^\sigma)$$

and the same holds true after applying pr_Γ. Moreover, note that the (α,β)-component is carried to the $(\sigma\alpha,\sigma\beta)$-component. By construction of the class $\mathrm{EK}^{\beta,\alpha}_{A,\Gamma}(f,x)$, it suffices to show that, for some free subgroup $\Gamma'\subset\Gamma$ of finite index, the diagram

$$\begin{tikzcd}
H^{g-1}(\Gamma',\mathbb{Z})\otimes\omega^g_{A/F} \arrow[r, "\mathrm{can}_A"] \arrow[d, "\mathrm{id}\otimes\sigma"'] & \mathrm{Sym}^{1_\Phi}(\omega_{A/F}) \arrow[d, "\sigma"] \\
H^{g-1}(\Gamma',\mathbb{Z})\otimes\omega^g_{A^\sigma/F} \arrow[r, "\mathrm{can}_{A^\sigma}"] & \mathrm{Sym}^{1_{\sigma\Phi}}(\omega_{A^\sigma/F})
\end{tikzcd}$$

commutes up to the sign $\varepsilon_\Phi(\sigma)$. Choose an ordering of the set of complex places $\langle c\rangle\backslash J_L$. This induces an ordering of every CM-type Φ' of L, in particular on Φ and $\sigma\Phi$. We get induced orientations on $L^1_{\mathbf{R},\Phi}$ and $L^1_{\mathbf{R},\sigma\Phi}$. The canonical map

$$L^1_{\mathbf{R},\Phi}\to L^1_{\mathbf{R},\sigma\Phi}$$

which maps $(r_\tau)_{\tau\in\Phi}$ to the family

$$\sigma\Phi\ni\tau'\mapsto\begin{cases}r_{\tau'} & \text{if } \tau'\in\tau\Phi\cap\Phi\\ r_{\overline{\tau'}} & \text{if } \tau'\in\tau\Phi\cap\overline{\Phi}\end{cases}$$

is a Γ'-equivariant isomorphism. Using the chosen orderings on Φ and $\sigma\Phi$, this isomorphism identifies with the identity map on $(\mathbf{R}^g_{>0})^{N=1}$, and in particular, the two induced isomorphisms

$$H^{g-1}(\Gamma',\mathbb{Z})\to\mathbb{Z}$$

coincide. On the other hand, we observe that with respect to our orderings the diagram

$$\begin{tikzcd} \omega^g_{A/F} \arrow[r] \arrow[d, "\sigma"'] & \mathrm{Sym}^{1_\Phi}(\omega_{A/F}) \arrow[d, "\sigma"] \\ \omega^g_{A^\sigma/F} \arrow[r] & \mathrm{Sym}^{1_{\sigma\Phi}}(\omega_{A^\sigma/F}) \end{tikzcd}$$

commutes up to the sign $\varepsilon_\Phi(\sigma)$. This proves the claim. $\square$

Remark 10.13. As a sanity check, we note that for any $\sigma\in G_{\mathbf{Q}}$ the sign $\varepsilon_\Phi(\sigma)$ only depends on the restriction $\sigma|_F\in J_F$. This follows immediately from of our assumption that F contains the Galois closure of L and the definition of ε_Φ.

Finally, let us define a slight modification of the class $\mathrm{EK}^{\beta,\alpha}_A(f,x)$. As before, let $\mathfrak{f}$ and $\mathfrak{c}$ be coprime integral ideals of L. Let moreover $\mathfrak{b}$ be integral ideals of L which is coprime to $\mathfrak{f}$ and $\mathfrak{c}$. We consider the abelian variety $A(\mathfrak{f}\mathfrak{b}^{-1}):=\mathfrak{f}\mathfrak{b}^{-1}\otimes_{\mathcal{O}_L}A$. Note that $A(\mathfrak{f}\mathfrak{b}^{-1})$ is also of CM-type (L,Φ). Let $D=A(\mathfrak{f}\mathfrak{b}^{-1})[\mathfrak{c}]$ and $f\in F[D]^{0,\Gamma}$ and let $x_\mathfrak{b}\in A(\mathfrak{f}\mathfrak{b}^{-1})$ be a $\mathfrak{f}$-torsion point. Associated to the data $(A(\mathfrak{f}\mathfrak{b}^{-1}),f,x_\mathfrak{b})$, we have the Eisenstein-Kronecker class

$$\mathrm{EK}^{\beta,\alpha-1}_{A(\mathfrak{f}\mathfrak{b}^{-1})}(f,x_\mathfrak{b})\in\mathrm{TSym}^\alpha(\omega_{A(\mathfrak{f}\mathfrak{b}^{-1})/F})\otimes_F\mathrm{TSym}^\beta(\omega_{A(\mathfrak{f}\mathfrak{b}^{-1})^\vee/F})$$

In the following, we denote by $[\mathfrak{b}]:A\to A(\mathfrak{b}^{-1})=A\otimes\mathfrak{b}^{-1}$ and $[\mathfrak{f}]:A(\mathfrak{f}\mathfrak{b}^{-1})\to A(\mathfrak{b}^{-1})$ the canonical isogenies. They induce isomorphisms

$$[\mathfrak{b}]^*([\mathfrak{f}]^*)^{-1}:\omega_{A(\mathfrak{f}\mathfrak{b}^{-1})/F}\xrightarrow{\sim}\omega_{A/F}$$
$$([\mathfrak{b}]_\#)^{-1}[\mathfrak{f}]_\#:\omega_{A(\mathfrak{f}\mathfrak{b}^{-1})^\vee/F}\xrightarrow{\sim}\omega_{A^\vee/F},$$

cf. section 5.3. For the following definition, let us recall our convention, that we use the isomorphism (0.3) to identify TSym and Sym.

Definition 10.14. *We define*

$$\widetilde{\mathrm{EK}}^{\beta,\alpha}_A(f,x_\mathfrak{b})\in\mathrm{Sym}^{\alpha+1_\Phi}(\omega_{A/F})\otimes_F\mathrm{Sym}^\beta(\omega_{A^\vee/F})$$

to be the image of $\mathrm{EK}^{\beta,\alpha}_{A(\mathfrak{f}\mathfrak{b}^{-1})}(f,x_\mathfrak{b})$ *under the isomorphism*

$$\mathrm{TSym}^{\alpha+1_\Phi}([\mathfrak{b}]^*([\mathfrak{f}]^*)^{-1})\,\mathrm{TSym}^\beta([\mathfrak{b}]^{-1}_\#[\mathfrak{f}]_\#)$$

followed by the canonical identification

$$\mathrm{TSym}^{\alpha+1_\Phi}(\omega_{A/F})\otimes_F\mathrm{TSym}^\beta(\omega_{A^\vee/F})\cong\mathrm{Sym}^{\alpha+1_\Phi}(\omega_{A/F})\otimes_F\mathrm{Sym}^\beta(\omega_{A^\vee/F}).$$

While the notation is slightly ambiguous, we hope that it will always be clear from the context on which abelian variety the element f and the torsion point $x_\mathfrak{b}$ are considered. We note that Proposition 10.12 implies that $\widetilde{\mathrm{EK}}^{\beta,\alpha}_A(f,x_\mathfrak{b})^\sigma=\varepsilon_\Phi(\sigma)\cdot\widetilde{\mathrm{EK}}^{\sigma\beta,\sigma\alpha}_{A^\sigma}(f^\sigma,x^\sigma_\mathfrak{b})$ via the canonical isomorphism $A(\mathfrak{f}\mathfrak{b}^{-1})^\sigma\cong A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})$.

10.4. **Eisenstein-Kronecker series.** Let $\omega(A) \in \omega_{A/\mathbf{C}}$ be a fixed $L\otimes\mathbf{C}$-generator. This induces an $L\otimes\mathbf{C}$-linear isomorphism $\mathrm{Lie}(A/\mathbf{C}) \cong \mathbf{C}^{\Phi}$, where L acts diagonally via the embeddings of Φ on $\mathbf{C}^{\Phi}$, and via the exponential sequence

$$0 \to H_1(A(\mathbf{C}),\mathbb{Z}) \to \mathrm{Lie}(A/\mathbf{C}) \xrightarrow{\exp} A(\mathbf{C}) \to 0,$$

the choice of $\omega(A)$ gives rise to a complex uniformization

$$\theta : \mathbf{C}^{\Phi}/\Lambda \xrightarrow{\sim} A(\mathbf{C}).$$

For the following definition, recall the notation from Notation 10.2.

Definition 10.15 ([KS19], Definition 3.19). *Let $\Gamma \subset \mathcal{O}_L^\times$ be a finite index subgroup and let $\beta-\alpha \in \mathrm{Crit}_L(\Gamma)$ be a Hecke character type with respect to Γ which is critical with respect to Φ. For any element $t \in \Lambda\otimes\mathbf{Q}$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \frac{d(\beta)-d(\alpha)}{2} + g$ define*

$$E^{\beta,\alpha}(t,s;\Lambda,\Gamma) := \sum_{\lambda\in\Gamma\backslash(\Lambda+\Gamma t)}{}' \frac{\overline{\lambda}^{\beta}}{\lambda^{\alpha}N(\lambda)^s}. \tag{10.11}$$

Here the summation ranges over all non-zero Γ-cosets of $\Lambda+\Gamma t$.

Remark 10.16. Note that for any non-zero element $\lambda = (\lambda_\sigma)$ in the 1-dimensional L-vector space $\Lambda\otimes\mathbf{Q}$, one has $\lambda_\sigma \neq 0$ for all $\sigma\in\Phi$. Also note that the summands in (10.11) do in fact only depend on the Γ-coset since $\beta-\alpha$ is trivial on Γ and the norm is Γ-invariant. One can show that $\prod_{\sigma\in\Phi}\Gamma(\alpha(\sigma)+s)\cdot E^{\beta,\alpha}(t,s;\Lambda,\Gamma)$ admits an analytic continuation to $\mathbf{C}$ (cf. [KS19], Lemma 3.21).

For any integral ideal $\mathfrak{c} \subset \mathcal{O}_L$, the complex uniformization induces an isomorphism

$$A[\mathfrak{c}](\mathbf{C}) \cong \mathfrak{c}^{-1}\Lambda/\Lambda.$$

Via the Hodge decomposition $H_1(A(\mathbf{C}),\mathbf{C}) \cong \omega_{A^\vee/\mathbf{C}} \oplus \overline{\omega_{A^\vee/\mathbf{C}}}$, the period pairing $H^1_{\mathrm{dR}}(A/\mathbf{C}) \times H_1(A(\mathbf{C}),\mathbf{C}) \to L\otimes\mathbf{C}$ induces a pairing

$$\langle\cdot,\cdot\rangle : \overline{\omega_{A/\mathbf{C}}} \times \omega_{A^\vee/\mathbf{C}} \to \mathbf{C}^{\overline{\Phi}}.$$

The following result, which expresses the classes defined in Definition 10.11 in analytic terms, is the key to relating these classes to special values of Hecke L-functions:

Theorem 10.17 (Kings-Sprang). *Let $\beta-\alpha \in \mathrm{Crit}_L(\Gamma)$. Let $\mathfrak{f}$ and $\mathfrak{c}$ be coprime integral ideals in $\mathcal{O}_L$. Fix $L\otimes\mathbf{C}$-generators $\omega(A)$ and $\omega(A^\vee)$ of $\omega_{A/\mathbf{C}}$ and $\omega_{A^\vee/\mathbf{C}}$. Let Λ be the period lattice associated to $\omega(A)$. Let $x \in \mathfrak{f}^{-1}\Lambda/\Lambda$ and let $f : \mathfrak{c}^{-1}\Lambda/\Lambda \to \mathbf{C}$ be a Γ-invariant function such that $\sum_{t\in\mathfrak{c}^{-1}\Lambda/\Lambda} f(t) = 0$. Then there is an equality*

$$\mathrm{EK}^{\beta,\alpha-1_\Phi}_{A,\Gamma}(f,x) = \frac{(\alpha-1)!}{\langle\overline{\omega(A)},\omega(A^\vee)\rangle_A^{\beta}} \sum_{t\in\Gamma\backslash(\mathfrak{c}^{-1}\Lambda/\Lambda)} f(-t)E^{\beta,\alpha}(t+x,0;\Lambda,\Gamma)\cdot\omega(A)^{[\alpha]}\otimes\omega(A^\vee)^{[\beta]}$$

in $\mathrm{TSym}^{\alpha}(\omega_{A/\mathbf{C}})\otimes\mathrm{TSym}^{\beta}(\omega_{A^\vee/\mathbf{C}})$.

Proof. This is [KS19], Corollary 3.23. □

In our applications, we will make use of specific choices of elements $f \in F[D]^{0,\Gamma}$ in the definition of Eisenstein-Kronecker classes: Let $\mathfrak{c} \subset \mathcal{O}_L$ be an integral ideal such that all $[\mathfrak{c}]$-torsion points of A are defined over F. Then the finite étale F-scheme $D := A[\mathfrak{c}]$ is isomorphic to the coproduct of copies of $\mathrm{Spec}(F)$ indexed by the set $A[\mathfrak{c}](F)$ of F-valued points. Note that $A[\mathfrak{c}](F) = A[\mathfrak{c}](\overline{\mathbf{Q}}) = A[\mathfrak{c}](\mathbf{C})$. One then has $F[D] \cong \mathrm{Map}(A[\mathfrak{c}](F),F)$ and one checks that

$$F[D]^{0,\Gamma} = \{f : A[\mathfrak{c}](F) \to F \mid f \text{ is } \Gamma\text{-equivariant}, \sum_{t\in A[\mathfrak{c}](F)} f(t) = 0\}.$$

We now introduce the elements $f \in F[D]^{0,\Gamma}$ that we later use for our applications:

Definition 10.18. *Let $\mathfrak{c} \subset \mathcal{O}_L$ be an integral ideal and set $D := \ker([\mathfrak{c}])$. Let $\Gamma \subset \mathcal{O}_L^\times$ be a finite index subgroup. Assume that all $[\mathfrak{c}]$-torsion points of A are defined over F. We define the element $f_{[\mathfrak{c}]} \in F[D]^{0,\Gamma}$ as*

$$f_{[\mathfrak{c}]} := N\mathfrak{c}\cdot 1_e - 1_{\ker[\mathfrak{c}]},$$

where 1_X denotes the characteristic function of a Γ-invariant subset $X \subset \ker([\mathfrak{c}])$ in $F[D]^{0,\Gamma}$.

Let $\mathcal{O}_{\mathfrak{f}}^{\times}$ denote the finite index subgroup of $\mathcal{O}_L^{\times}$ consisting of those units which are congruent to 1 modulo $\mathfrak{f}$. From now on, once an ideal $\mathfrak{f}$ is fixed, we will always consider $\Gamma = \mathcal{O}_{\mathfrak{f}}^{\times}$ and we often drop it from the notation and, for example, just write $\mathrm{EK}_A^{\beta,\alpha}(f,x)$ and $E^{\beta,\alpha}(t,s;\Lambda)$.

Proposition 10.19 (Distribution relation)**.** *Let $\mathfrak{f}$, $\mathfrak{b}$ and $\mathfrak{c}$ be pairwise coprime integral ideals. One has*

$$\sum_{t\in\mathcal{O}_{\mathfrak{f}}^{\times}\backslash\mathfrak{f}\mathfrak{b}^{-1}\mathfrak{c}^{-1}/\mathfrak{f}\mathfrak{b}^{-1}} f_{[\mathfrak{c}]}(-t)E^{\beta,\alpha}(1+t,0;\mathfrak{f}\mathfrak{b}^{-1}) = N\mathfrak{c}E^{\beta,\alpha}(1,0;\mathfrak{f}\mathfrak{b}^{-1}) - E^{\beta,\alpha}(1,0;\mathfrak{f}\mathfrak{b}^{-1}\mathfrak{c}^{-1}).$$

Proof. This is [KS19], Proposition 4.10. □

Now, following [KS19], 4.3 closely, we discuss how the Eisenstein-Kronecker series relate to Hecke L-functions. Let $\mathfrak{f} \neq \mathcal{O}_L$ be an integral ideal and set $\Gamma = \mathcal{O}_{\mathfrak{f}}^{\times}$. Let $\chi : I^{\mathfrak{f}} \to T^{\times}$ be an algebraic Hecke character of conductor dividing $\mathfrak{f}$ and let $\iota \in J_T$ be a fixed embedding. Then, for every integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$ with ray class $[\mathfrak{b}] \in \mathcal{I}^{\mathfrak{f}}/P^{\mathfrak{f}}$, there is a bijection

$$\mathcal{O}_{\mathfrak{f}}^{\times}\backslash(1+\mathfrak{f}\mathfrak{b}^{-1}) \xrightarrow{\sim} \{\mathfrak{a}\in[\mathfrak{b}] \mid \mathfrak{a} \text{ integral }\}, \quad y\mapsto y\mathfrak{b}.$$

Using this bijection, it follows that

$$\begin{aligned} L_{\mathfrak{f}}(\chi,s,[\mathfrak{b}]) &= \sum_{\mathfrak{a}\in[\mathfrak{b}],\mathfrak{a}\subset\mathcal{O}_L} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \\ &= \chi(\mathfrak{b})N\mathfrak{b}^{-s}\cdot \sideset{}{'}\sum_{y\in\mathcal{O}_{\mathfrak{f}}^{\times}\backslash(1+\mathfrak{f}\mathfrak{b}^{-1})} \frac{\chi((y))}{Ny^s} \\ &= \chi(\mathfrak{b})N\mathfrak{b}^{-s}E^{\beta,\alpha}(1,s;\mathfrak{f}\mathfrak{b}^{-1},\mathcal{O}_{\mathfrak{f}}^{\times}). \end{aligned}$$

Evaluating at $s=0$ gives

$$L_{\mathfrak{f}}(\chi,0,[\mathfrak{b}]) = \chi(\mathfrak{b})E^{\beta,\alpha}(1,0;\mathfrak{f}\mathfrak{b}^{-1}). \tag{10.12}$$

10.5. **Eisenstein-Kronecker classes and special values of Hecke L-functions.** Let L be a totally imaginary number field and Φ a CM-type of L. Let $\chi : L_f^{\times} \to T^{\times}$ be an algebraic Hecke character whose infinity-type $\chi_a = \beta - \alpha$ is critical with respect to Φ. Let $\mathfrak{f} \neq \mathcal{O}_L$ be an integral ideal such that χ can be regarded as a having conductor dividing $\mathfrak{f}$. Let E denote the reflex field of (L,Φ). Let F/E be a finite extension such that F contains the Galois closure of L and let A/F be an abelian variety of CM-type (L,Φ) whose CM-order is $\mathcal{O}_L$. We assume that $H_1(A(\mathbf{C}),\mathbb{Z})$ is a free $\mathcal{O}_L$-module and we fix an $\mathcal{O}_L$-structure

$$\xi : \mathcal{O}_L \xrightarrow{\sim} H_1(A(\mathbf{C}),\mathbb{Z}).$$

We assume that all $[\mathfrak{f}]$-torsion points are defined over F. For example, one could choose an abelian variety as in Proposition 6.8 for $\mathfrak{a} = \mathfrak{f}$ and consider its base change to a field that contains the Galois closure of L.

Let $\gamma \in h_B^1(A)$ denote the dual of the L-basis $\xi(1) \in H_1(A(\mathbf{C}),\mathbf{Q})$. Let $\gamma^{\vee} \in h_B^1(A^{\vee})$ be the dual basis of γ with respect to the canonical pairing $h_B^1(A)\otimes_L h_B^1(A^{\vee}) \to (2\pi i)^{-1}L$. Let $\omega(A)$ and $\omega(A^{\vee})$ denote $L\otimes F$-generators of $\omega_{A/F}$ and $\omega_{A^{\vee}/F}$ (cf. [KS19], Definition 1.15). Let $\Omega = \langle\omega(A),\gamma\rangle \in \mathbf{C}^{\Phi}$ and $\Omega^{\vee} = \langle\omega(A^{\vee}),\gamma^{\vee}\rangle \in \mathbf{C}^{\overline{\Phi}}$ denote the periods defined Definition 10.3. For any integral ideal $\mathfrak{b}$ which is coprime to $\mathfrak{f}$, the abelian variety $A(\mathfrak{f}\mathfrak{b}^{-1}) := \mathfrak{f}\mathfrak{b}^{-1}\otimes_{\mathcal{O}_L} A$ still has complex mulitplication by $\mathcal{O}_L$. By Proposition 10.8, the differential form $\mathfrak{f}\mathfrak{b}^{-1}\otimes\omega(A) := [\mathfrak{f}]^*([\mathfrak{b}]^*)^{-1}\omega(A)$ on $A(\mathfrak{f}\mathfrak{b}^{-1})$ generates $\omega_{A(\mathfrak{f}\mathfrak{b}^{-1})/F}$ and induces a complex uniformization of $A(\mathfrak{f}\mathfrak{b}^{-1})$ such that the $\mathfrak{f}$-torsion points are given by

$$A(\mathfrak{f}\mathfrak{b}^{-1})(\mathbf{C})[\mathfrak{f}] = \mathfrak{b}^{-1}\Omega/\mathfrak{f}\mathfrak{b}^{-1}\Omega.$$

Definition 10.20. *Let $x_{\mathfrak{b}} \in A(\mathfrak{f}\mathfrak{b}^{-1})[\mathfrak{f}]$ denote the $\mathfrak{f}$-torsion point of $A(\mathfrak{f}\mathfrak{b}^{-1})$ which correspond to the point*

$$\Omega + \mathfrak{f}\mathfrak{b}^{-1}\Omega \in A(\mathfrak{f}\mathfrak{b}^{-1})(\mathbf{C})[\mathfrak{f}].$$

By our assumptions on A/F, the torsion point $x_{\mathfrak{b}}$ is defined over F. Note that $x_{\mathfrak{b}}$ depends on the choice of $\mathcal{O}_L$-structure but not on the choice of $\omega(A)$.

Recall the modified Eisenstein-Kronecker classes from Definition 10.14. The following theorem links these classes to special values of Hecke L-functions. It is essentially a reformulation of the results in [KS19], 4.4 in our setup.

Recall that we have fixed an $\mathcal{O}_L$-structure ξ, which determined elements $\gamma \in h^1_B(A)$ and $h^1_B(A^\vee)$. Now let $\sigma \in J_F$ and choose a fixed extension in $G_{\mathbf{Q}}$, which we also denote by σ. Similarly, we fix the following data corresponding to the abelian variety A^σ:

- Let $\gamma_\sigma \in h^1_B(A^\sigma)$ be an L-basis with dual $\gamma_\sigma^\vee \in h^1_B(A^{\sigma,\vee})$.
- Let $\lambda_\sigma \in L_f^\times$ be the unique idele such that $I_f(\gamma)^\sigma = \lambda_\sigma \cdot I_f(\gamma_\sigma)$.
- Let $\ell_\sigma \in L^\times$ such that $\ell_{\sigma,v}\lambda_{\sigma,v} \equiv 1 \mod \mathfrak{f}_v$ for every finite place v of L. The existence of ℓ_σ is guaranteed by the primary decomposition of $L/\mathfrak{f}$, cf. the discussion after Theorem 5.10. Note that then $\ell_{\sigma,v}\lambda_{\sigma,v} \equiv 1 \mod (\mathfrak{f}\mathfrak{b}^{-1})_v$ for every integral ideal $\mathfrak{b}$ coprime to $\mathfrak{f}$.
- Let $\omega(A^\sigma)$ and $\omega(A^{\vee,\sigma})$ be $L\otimes F$-generators of $\omega_{A^\sigma/F}$ and $\omega_{A^{\sigma,\vee}/F}$, respectively.
- Let $\Omega_\sigma \in \mathbf{C}^{\sigma\Phi}$ and $\Omega_\sigma^\vee \in \mathbf{C}^{\overline{\sigma\Phi}}$ denote the periods associated to γ_σ, $\gamma_\sigma^\vee$ and $\omega(A^\sigma)$, $\omega(A^{\sigma,\vee})$.

Theorem 10.21. *Let $\mathfrak{f} \neq \mathcal{O}_L$, $\mathfrak{b}$ and $\mathfrak{c}$ be pairwise coprime integral ideals of L. Let A/F be as above and morever assume that all of its $[\mathfrak{c}]$-torsion is defined over F. Let χ an algebraic Hecke character of conductor dividing $\mathfrak{f}$ whose infinity-type $\beta-\alpha$ is critical with respect to Φ. Let $x_{\mathfrak{b}} \in A(\mathfrak{f}\mathfrak{b}^{-1})[\mathfrak{f}](F)$ be the torsion point defined in Definition 10.20. Let $D = A(\mathfrak{f}\mathfrak{b}^{-1})[\mathfrak{c}]$ and let $f_{[\mathfrak{c}]} \in F[D]^{0,\Gamma}$ be defined as in Definition 10.18. Let $\iota \in J_{T,\sigma|_E}$ and assume that we are given the data as listed above. Then*

$$\chi_\iota(\mathfrak{b})\cdot \widetilde{\mathrm{EK}}_{A^\sigma}^{\iota\beta,\iota\alpha-1_{\sigma\Phi}}(f_{[\mathfrak{c}]}, x_{\mathfrak{b}}^\sigma) =$$
$$\frac{(\alpha-1)!(2\pi i)^{d(\beta)}}{\Omega_\sigma^{\iota\alpha}(\Omega_\sigma^\vee)^{\iota\beta}}\chi_\iota(\lambda_\sigma)^{-1}\left(N\mathfrak{c}L_{\mathfrak{f}}(\chi_\iota,0,[\ell_\sigma\lambda_\sigma\mathfrak{b}]) - \chi_\iota(\mathfrak{c})^{-1}L_{\mathfrak{f}}(\chi_\iota,0,[\ell_\sigma\lambda_\sigma\mathfrak{b}\mathfrak{c}])\right)\cdot$$
$$\omega(A^\sigma)^{\iota\alpha}\otimes\omega(A^{\vee,\sigma})^{\iota\beta}.$$

In particular, the element on the left hand side only depends on the class $[\mathfrak{b}] \in I^{\mathfrak{f}}/P^{\mathfrak{f}}$.

Remark 10.22. Before we give a proof, let us make a remark on the various dependencies on the input data.

(1) First of all, we note that the statement of the theorem is independent of the choice of extension of σ to $G_{\mathbf{Q}}$. The only part of the data that changes are the λ_σ-elements. But if $\sigma' \in G_{\mathbf{Q}}$ is another extension, one has $\chi(\lambda_{\sigma'}) = \chi(\lambda_\sigma)$ by Definition 9.4 and Remark 10.13.
(2) Note that the right-hand side in the formula of the theorem is independent of the choice of $\omega(A^\sigma)$ and $\omega(A^{\sigma,\vee})$. A different choice also changes the periods Ω_σ and $\Omega_\sigma^\vee$ accordingly. Similarly, it is independent of the choices of γ_σ as both $\chi_\iota(\lambda_\sigma)$ and $\Omega_\sigma^{\iota\alpha}(\Omega_\sigma^\vee)^{\iota\beta}$ vary in the same way.
(3) If $\ell'_\sigma \in L^\times$ is a different choice for ℓ_σ, it follows that $\ell_\sigma\lambda_\sigma\mathfrak{b}$ and $\ell'_\sigma\lambda_\sigma\mathfrak{b}$ differ by an ideal in $P^{\mathfrak{f}}$, cf. the proof below.
(4) Note that both sides of the theorem depend on the choice of ξ. If $\xi' = u\cdot\xi$ for $u \in \mathcal{O}_L^\times$ is a different choice, both sides change by a factor of $u^{\beta-\alpha}$.

Proof. Fix $\sigma \in G_{\mathbf{Q}}$ and simply denote ℓ_σ and λ_σ by ℓ and λ. By our assumptions on A and choice of γ, there exists a uniformization

$$\theta : \mathbf{C}^\Phi/\mathcal{O}_L \xrightarrow{\sim} A(\mathbf{C}),$$

which induces the isomorphism $L \xrightarrow{\sim} H^1(A(\mathbf{C}),\mathbf{Q})$, $1 \mapsto \gamma_1$. By the main theorem of complex multiplication (cf. Corollary 5.11), there exists a complex uniformization

$$\theta_\sigma : \mathbf{C}^{\sigma\Phi}/[\lambda]_L^{-1} \xrightarrow{\sim} A^\sigma(\mathbf{C}),$$

inducing the isomorphism $L \xrightarrow{\sim} H^1(A^\sigma(\mathbf{C}),\mathbf{Q})$, $1 \mapsto \gamma_\sigma$ on rational cohomology and such that the diagram

$$\begin{tikzcd} L/\mathcal{O}_L \arrow[r, "\theta"] \arrow[d, "\lambda^{-1}"] & A_{\mathrm{tor}} \arrow[d, "\sigma"] \\ L/[\lambda]_L^{-1} \arrow[r, "\theta_\sigma"] & A^\sigma_{\mathrm{tor}} \end{tikzcd}$$

is commutative. We also obtain an induced commutative diagram

$$\begin{array}{ccc} L/\mathfrak{f}\mathfrak{b}^{-1} & \xrightarrow{\theta} & A(\mathfrak{f}\mathfrak{b}^{-1})_{\mathrm{tor}} \\ \downarrow{\scriptstyle \lambda^{-1}} & & \downarrow{\scriptstyle \sigma} \\ L/\mathfrak{f}\mathfrak{b}^{-1}[\lambda]_L^{-1} & \xrightarrow{\theta_\sigma} & A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})_{\mathrm{tor}}. \end{array}$$

By definition, the torsion point $x_\mathfrak{b}$ corresponds to the point $1+\mathfrak{f}\mathfrak{b}^{-1} \in L/\mathfrak{f}\mathfrak{b}^{-1}$. Thus, the point $x_\mathfrak{b}^\sigma$ corresponds to

$$\ell + \mathfrak{f}\mathfrak{b}^{-1}[\lambda]_L^{-1} \in L/\mathfrak{f}\mathfrak{b}^{-1}[\lambda]_L^{-1}.$$

Observe that the ideal $\ell\lambda\mathfrak{b}$ is an integral ideal coprime to $\mathfrak{f}$. Indeed, for the integrality it suffices to show $(\ell\lambda\mathfrak{b})_v = \ell_v\lambda_v\mathfrak{b}_v \subset \mathcal{O}_{L_v}$ for any finite place v of L. This is immediate, since by definition of ℓ, one has

$$\ell_v\lambda_v \in 1 + \mathfrak{f}_v\mathfrak{b}_v^{-1}.$$

Now let v be a place with $v|\mathfrak{f}$. Since $\mathfrak{f}$ and $\mathfrak{b}$ are coprime, this implies $v \nmid \mathfrak{b}$, hence we get $\ell_v\lambda_v \in 1+\mathfrak{f}_v \in \mathcal{O}_{L_v}^\times$. This shows $v \nmid \ell\lambda\mathfrak{b}$ and hence $\ell\lambda\mathfrak{b}$ is coprime to $\mathfrak{f}$.

The equality (10.12) with $\ell\lambda\mathfrak{b}$ in place of $\mathfrak{b}$ gives

$$\chi_\iota(\ell\lambda\mathfrak{b})^{-1} \cdot L_\mathfrak{f}(\chi_\iota, 0, [\ell\lambda\mathfrak{b}]) = E^{\iota\beta,\iota\alpha}(1,0;\mathfrak{f}(\ell\lambda\mathfrak{b})^{-1}). \tag{10.13}$$

We set

$$\omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})) := \ell_\sigma^{-1} \cdot [\mathfrak{f}]^*([\mathfrak{b}]^*)^{-1}\omega(A^\sigma)$$
$$\omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})^\vee) := [\mathfrak{f}]_\#^{-1}[\mathfrak{b}]_\#\omega(A^{\sigma,\vee})$$

and abbreviate $\mathfrak{a} := \mathfrak{f}(\ell\lambda\mathfrak{b})^{-1}$. By the above, the period lattice of $A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})$ associated with $\omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1}))$ is then given by

$$\mathfrak{a}\Omega_\sigma.$$

and the $\mathfrak{f}$-torsion point $x_\mathfrak{b}^\sigma$ corresponds to the point $\Omega_\sigma + \mathfrak{a}\Omega_\sigma$ in

$$A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})(\mathbf{C})[\mathfrak{f}] = \mathfrak{f}^{-1}\mathfrak{a}\Omega_\sigma/\mathfrak{a}\Omega_\sigma.$$

The assertion of the Proposition now follows, just as in the Proof of [KS19], Theorem 4.6, from Theorem 10.17. For the convenience of the reader, let us recall the argument in our setting: Applying Theorem 10.17 to the abelian variety $A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})$ and the critical infinity-type $\iota\beta - \iota\alpha$, we obtain that

$$\begin{aligned} \mathrm{EK}^{\iota\beta,\iota\alpha-1}_{A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})}(f_{[\mathfrak{c}]}, x_\mathfrak{b}^\sigma) = & \frac{(\alpha-1)!}{\langle\overline{\omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1}))}, \omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})^\vee)\rangle^{\iota\beta}} \cdot \\ & \sum_{t\in\mathcal{O}_\mathfrak{f}^\times\setminus\mathfrak{c}^{-1}\mathfrak{a}\Omega_\sigma/\mathfrak{a}\Omega_\sigma} f_{[\mathfrak{c}]}(-t)E^{\iota\beta,\iota\alpha}(t+\Omega_\sigma, 0;\mathfrak{a}\Omega_\sigma)\cdot \\ & \omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1}))^{[\iota\alpha]} \otimes \omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})^\vee)^{[\iota\beta]} \end{aligned} \tag{10.14}$$

in $\mathrm{TSym}^{\iota\alpha}(\omega_{A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})/F}) \otimes \mathrm{TSym}^{\iota\beta}(\omega_{A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})^\vee/F})$. Since the pairing $\langle\cdot,\cdot\rangle_A$ behaves well under isogenies (cf. Proposition 10.6), we have

$$\langle\overline{\omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1}))}, \omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})^\vee)\rangle^{\iota\beta} = \ell^{-\iota\beta} \cdot \langle\overline{\omega(A^\sigma)}, \omega(A^{\sigma,\vee})\rangle^{\iota\beta}.$$

Moreover, one has

$$\Omega_\sigma^\vee = \frac{2\pi i\langle\overline{\omega(A^\sigma)}, \omega(A^{\sigma,\vee})\rangle_A}{\overline{\Omega}_\sigma}$$

by Proposition 10.5, so that

$$\langle\overline{\omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1}))}, \omega(A^\sigma(\mathfrak{f}\mathfrak{b}^{-1})^\vee)\rangle^{\iota\beta} = \ell^{-\iota\beta} \cdot \frac{\overline{\Omega}_\sigma^{\iota\beta}\Omega_\sigma^{\vee,\iota\beta}}{(2\pi i)^{d(\beta)}}. \tag{10.15}$$

Moreover, it follows from the definition of the Eisenstein-Kronecker series that

$$E^{\iota\beta,\iota\alpha}((t+1)\Omega_\sigma, 0;\mathfrak{a}\Omega_\sigma) = \frac{\overline{\Omega}_\sigma^{\iota\beta}}{\Omega_\sigma^{\iota\alpha}} \cdot E^{\iota\beta,\iota\alpha}(t+1,0;\mathfrak{a}). \tag{10.16}$$

for all $t \in \mathcal{O}_{\mathfrak{f}}^{\times}\backslash\mathfrak{c}^{-1}\mathfrak{a}/\mathfrak{a}$. The distribution relation Proposition 10.19 gives

$$\begin{aligned}\sum_{t\in\mathcal{O}_{\mathfrak{f}}^{\times}\backslash\mathfrak{c}^{-1}\mathfrak{a}/\mathfrak{a}} f_{[\mathfrak{c}]}(-t)E^{\iota\beta,\iota\alpha}(t+1,0;\mathfrak{a}) = \\ N\mathfrak{c}E^{\iota\beta,\iota\alpha}(1,0;\mathfrak{a}) - E^{\iota\beta,\iota\alpha}(1,0;\mathfrak{c}^{-1}\mathfrak{a})\end{aligned} \tag{10.17}$$

By putting together (10.13), (10.14), (10.15), (10.16) and (10.17), we get the equality

$$\begin{aligned}\mathrm{EK}^{\iota\beta,\iota\alpha-1}_{A^{\sigma}(\mathfrak{f}\mathfrak{b}^{-1})}(f_{[\mathfrak{c}]},x^{\sigma}_{\mathfrak{b}}) = \frac{(\alpha-1)!(2\pi i)^{d(\beta)}}{\Omega_{\sigma}^{\iota\alpha}\Omega_{\sigma}^{\vee,\iota\beta}}\cdot \ell^{-\iota\beta}\chi_{\iota}(\ell\lambda\mathfrak{b})^{-1}\cdot \\ \big(N\mathfrak{c}L_{\mathfrak{f}}(\chi_{\iota},0,[\ell\lambda\mathfrak{b}]) - \chi_{\iota}(\mathfrak{c})^{-1}L_{\mathfrak{f}}(\chi_{\iota},0,[\ell\lambda\mathfrak{b}\mathfrak{c}])\big)\cdot \\ \omega(A^{\sigma}(\mathfrak{f}\mathfrak{b}^{-1}))^{[\iota\alpha]}\otimes\omega(A^{\sigma}(\mathfrak{f}\mathfrak{b}^{-1})^{\vee})^{[\iota\beta]}\end{aligned}$$

Recall the definition of $\omega(A^{\sigma}(\mathfrak{f}\mathfrak{b}^{-1}))$ and observe that

$$\ell^{\iota\alpha-\iota\beta}\chi_{\iota}(\ell\lambda\mathfrak{b})^{-1} = \chi_{\iota}(\mathfrak{b})^{-1}\chi_{\iota}(\lambda)^{-1}$$

where we freely regard χ_{ι} as a character of ideals and ideles. Passing to $\mathrm{Sym}^{\iota\alpha}(\omega_{A^{\sigma}/F})\otimes\mathrm{Sym}^{\iota\beta}(\omega_{A^{\sigma,\vee}/F})$ via the isogenies $[\mathfrak{f}]$ and $[\mathfrak{b}]$ and our identificaiton of TSym with Sym then gives the claim. ☐

Remark 10.23. Note that we do not actually use the main theorem of complex multiplication which is concerned with the explicit description of the $L^{\times}$-coset of $\lambda_{\sigma}\in L_f^{\times}$. We only need that λ_{σ} describes the action of σ on the torsion points, which it essentially does by definition.

11. Deligne's conjecture for Hecke characters

11.1. **The de Rham realization of $h^1(A)^{\alpha}\otimes h^1(A^{\vee})^{\beta}$ in the split setup.** Let L be a totally imaginary number field and let Φ be a CM-type of L. Let E denote the reflex field of (L,Φ). Let F/E be a finite extension such that F contains the Galois closure of L and let A/F be an abelian variety of CM-type (L,Φ). In this case, the de Rham realization of $h^1(A)^{\alpha}_T\otimes_T h^1(A^{\vee})^{\beta}_T$ is particularly convenient. Under the assumptions on F, the torus $\mathbf{T}_{L,F}$ is split and one has

$$h^1(A)^{\alpha}_{\mathrm{dR}} = \bigoplus_{\iota\in J_{L^{\alpha}}}\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A)). \tag{11.1}$$

Proposition 11.1. *Let T be a number field and assume that $L^{\alpha}\subset T\subset F$. Then there is a canonical isomorphism*

$$(h^1(A)^{\alpha}\otimes_{L^{\alpha}}T)_{\mathrm{dR}} = \bigoplus_{\iota\in J_T}\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A)).$$

Proof. The decomposition (11.1) is one of L^{α}-modules, where L^{α} acts on $\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))$ via the embedding $\iota: L^{\alpha}\hookrightarrow F$. By the assumption $T\subset F$, one has the canonical decomposition

$$F\otimes_{\iota,L^{\alpha}}T = \prod_{\iota'\in J_{T,\iota}}F,$$

and hence

$$\begin{aligned}\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))\otimes_{\iota,L^{\alpha}}T &= \mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))\otimes_F(F\otimes_{\iota,L^{\alpha}}T)\\ &= \bigoplus_{\iota'\in J_{T,\iota}}\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A)),\end{aligned}$$

from which the claim follows. ☐

Proposition 11.2. *Let T be a number field and assume that $L^{\alpha}\subset T\subset F$. There is a canonical isomorphism*

$$\big(h^1(A)^{\alpha}_T\otimes_T h^1(A^{\vee})^{\beta}_T\big)_{\mathrm{dR}} = \bigoplus_{\iota\in J_T}\mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))\otimes_F\mathrm{Sym}^{\iota\beta}(h^1_{\mathrm{dR}}(A^{\vee})). \tag{11.2}$$

Proof. For each $\iota \in J_T$, let us temporarily write $V_\iota = \mathrm{Sym}^{\iota\alpha}(h^1_{\mathrm{dR}}(A))$ and $W_\iota = \mathrm{Sym}^{\iota\beta}(h^1_{\mathrm{dR}}(A^\vee))$. We regard V_ι and W_ι as $T \otimes F$-modules via the homomorphism $\iota : T \otimes F \to F$, $t \otimes x \mapsto \iota(t)x$. Then, for $\iota, \iota' \in J_T$, there is a canonical isomorphism

$$V_\iota \otimes_{T\otimes F} W_{\iota'} = V_\iota \otimes_F (F \otimes_{\iota, T\otimes F, \iota'} F) \otimes_F W_{\iota'},$$

and with the observation that

$$F \otimes_{\iota, T\otimes F, \iota'} F = \begin{cases} F & \text{if } \iota = \iota' \\ 0 & \text{if } \iota \neq \iota' \end{cases}$$

the assertion follows. □

Proposition 11.3 (Hodge Filtration)**.** *Let T be a number field and assume that $L^\alpha \subset T \subset F$. Under the canonical identification (11.2), one has*

$$F^{d(\alpha)+d(\beta)}\big(h^1(A)^\alpha_T \otimes_T h^1(A^\vee)^\beta_T\big)_{\mathrm{dR}} = \bigoplus_{\iota \in J_{T,E}} \mathrm{Sym}^{\iota\alpha}(\omega_{A/F}) \otimes_F \mathrm{Sym}^{\iota\beta}(\omega_{A^\vee/F}).$$

Proof. This follows from the constructions of $h^1(A)^\alpha$ and $h^1(A^\vee)^\beta$ and the fact that for the abelian variety A/F, the Hodge filtration is given by $h^1_{\mathrm{dR}}(A) \supset \omega_{A/F} \supset 0$ (and analogously for $A^\vee$). Note that $\mathrm{Sym}^{\iota\alpha}(\omega_{A/F}) \neq 0$ if and only if $\iota|_E = 1_E$ since $\omega_{A/F}$ is of type Φ, i.e. $\omega_{A/F}(\sigma) \neq 0$ if and only if $\sigma \in \Phi$, and $\iota\alpha$ has support Φ if and only if $\iota|_E = 1_E$. □

11.2. **The normalized class $\mathcal{EK}$.** Let L be a totally imaginary number field and let Φ be a CM-type of L. Let $\chi : L^\times_{\mathfrak{f}} \to T^\times$ be an algebraic Hecke character whose infinity-type $\chi_a = \beta - \alpha$ is critical with respect to Φ. We assume $T \supset L^\alpha$. Fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_L$ such that χ is of conductor dividing $\mathfrak{f}$. Note that $\chi \neq N^{-1}$ as χ is critical. Fix an integral ideal $\mathfrak{c}$ which is coprime to $\mathfrak{f}$ and satisfies $\chi(\mathfrak{c}) \neq N\mathfrak{c}^{-1}$, so that $N\mathfrak{c} - \chi(\mathfrak{c})^{-1} \in T^\times$.

Let E be the reflex field of (L, Φ). Let A/F be the base change of an abelian variety of CM-type (L, Φ) as in Proposition 6.8 for $\mathfrak{a} = \mathfrak{f}\mathfrak{c}$ to a finite extension F of E which contains the Galois closure of L and T. Fix an $\mathcal{O}_L$-structure $\xi : \mathcal{O}_L \xrightarrow{\sim} H_1(A(\mathbf{C}), \mathbb{Z})$. Recall from Proposition 11.3 that

$$F^{a+b}(h^1(A)^\alpha_T \otimes_T h^1(A^\vee)^\beta_T)_{\mathrm{dR}} = \bigoplus_{\iota \in J_{T,E}} \mathrm{Sym}^{\iota\alpha}(\omega_{A/F}) \otimes_F \mathrm{Sym}^{\iota\beta}(\omega_{A^\vee/F}).$$

Definition 11.4. *We define the* normalized Eisenstein-Kronecker class *to be the element*

$$\mathcal{EK} := ((\alpha-1)!(N\mathfrak{c} - \chi(\mathfrak{c})^{-1}))^{-1} \cdot \sum_{[\mathfrak{b}] \in I^{\mathfrak{f}}/P^{\mathfrak{f}}} \chi(\mathfrak{b}) \cdot (\widetilde{\mathrm{EK}}_A^{\iota\beta, \iota\alpha-1}(f_{[\mathfrak{c}]}, x_{\mathfrak{b}}))_{\iota \in J_{T/E}}$$

in $F^{a+b}(h^1(A)^\alpha_T \otimes_T h^1(A^\vee)^\beta_T)_{\mathrm{dR}}$, where we let the sum range over a set of integral representatives of $I^{\mathfrak{f}}/P^{\mathfrak{f}}$ which are coprime to $\mathfrak{c}$.

11.3. **Descent to the abelian subfield.** Let F_0/E be a finite abelian extension and A_0/F_0 an abelian variety of CM-type (L, Φ) as in Proposition 6.8 for the ideal $\mathfrak{a} = \mathfrak{f}\mathfrak{c}$. Let $F \supset F_0$ be a number field containing the Galois closure of L as well as T and set

$$A := A_0 \times_{F_0} F.$$

Let $G := \mathrm{Gal}(F/F_0)$. Then

$$(h^1(A_0)^\alpha_T \otimes h^1(A_0)^\beta_T)_{\mathrm{dR}} \cong \Big(\bigoplus_{\iota \in J_{T,E}} \mathrm{Sym}^{\iota\alpha}(\omega_{A/F}) \otimes_F \mathrm{Sym}^{\iota\beta}(\omega_{A^\vee/F}) \Big)^G,$$

where the semilinear action of G is induced by conjugation on $\omega_{A/F}$. We now show that the class $\mathcal{EK}$ on A descends to a class on A_0. Hence, for every CM abelian variety as in Proposition 6.8, we have the class $\mathcal{EK}$ at our disposal.

Proposition 11.5. *For any $\sigma \in G$, one has*

$$\mathcal{EK}^\sigma = \mathcal{EK},$$

so that $\mathcal{EK}$ defines an element in $F^{a+b}(h^1(A_0)^\alpha_T \otimes h^1(A_0)^\beta_T)_{\mathrm{dR}}$.

Proof. By definition of A in terms of Proposition 6.8, every $\mathfrak{f}$-torsion point of A is already defined over F_0 and ε_Φ is trivial on G. Hence, by Proposition 10.12, we have

$$\widetilde{\mathrm{EK}}_A^{\iota\beta,\iota\alpha-1}(f_{[\mathfrak{c}]}, x_{\mathfrak{b}})^\sigma = \widetilde{\mathrm{EK}}_A^{\sigma\iota\beta,\sigma\iota\alpha-1}(f_{[\mathfrak{c}]}, x_{\mathfrak{b}}).$$

for every $\sigma \in G$. This implies that $(\widetilde{\mathrm{EK}}_A^{\iota\beta,\iota\alpha-1}(f_{[\mathfrak{c}]}, x_{\mathfrak{b}}))_{\iota\in J_{T/E}}$, and hence the class $\mathcal{EK}$ is invariant under the action of G. ☐

11.4. **The class $\mathcal{EK}$ and L-values.** We now reformulate Proposition 10.21 in terms of the motive $h^1(A)^{\alpha}_T \otimes h^1(A^\vee)^{\beta}_T$. This reinterprets the L-value $L(\chi, 0)$ as a period.

Let L be a totally imaginary number field and let Φ be a CM-type of L. Let $\chi : L_f^\times \to T^\times$ be an algebraic Hecke character whose infinity-type $\chi_a = \beta - \alpha$ is critical with respect to Φ. We assume $T \supset L^\alpha$. Fix an integral ideal $\mathfrak{f} \neq \mathcal{O}_L$ such that χ is of conductor dividing $\mathfrak{f}$. As before, fix an integral ideal $\mathfrak{c}$ which is coprime to $\mathfrak{f}$ and satisfies $N\mathfrak{c} - \chi(\mathfrak{c})^{-1} \in T^\times$.

Let A/F be of CM-type (L, Φ) satisfying the conditions of Proposition 6.8 for the ideal $\mathfrak{a} = \mathfrak{f}\mathfrak{c}$. As before, fix an $\mathcal{O}_L$-linear isomorphism $\mathcal{O}_L \cong H_1(A(\mathbf{C}), \mathbb{Z})$ and let $\gamma \in h^1_B(A)$ be the corresponding element. Let $\sigma \in J_F$. Let $\gamma_\sigma \in h^1_B(A^\sigma)$ be an L-basis and let $\gamma_\sigma^\vee \in h^1_B(A^{\sigma,\vee})$ be the dual basis with respect to the canonical pairing $h^1_B(A^\sigma) \otimes_L h^1_B(A^{\sigma,\vee}) \to (2\pi i)^{-1} L$. Recall the definition of the elements from Lemma 9.4: For every $\sigma \in J_F$, choose an extension to $G_{\mathbf{Q}}$, which we denote by the same letter. Let $\lambda_\sigma \in L_f^\times$ be the unique element such that $I_f(\gamma)^\sigma = \lambda_\sigma I_f(\gamma_\sigma)$. Then

$$t_\sigma := \chi(\lambda_\sigma)^{-1} \varepsilon_\Phi(\sigma) \in T^\times$$

only depends on $\sigma \in J_F$.

Theorem 11.6. *Let $\sigma \in J_F$ and γ_σ and $\gamma_\sigma^\vee$ as above. Let $\eta = \sigma|_E \in J_E$. One has*

$$I_\infty(\mathcal{EK}) = (2\pi i)^{d(\beta)} \left(L_{\mathfrak{f}}(\chi_\iota, 0)\right)_{\iota\in J_{T,\eta}} \cdot t_\sigma (\gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}})$$

in $(h^1(A)^{\alpha}_T \otimes h^1(A^\vee)^{\beta}_T)_B \otimes \mathbf{C}$.

Proof. Choose $L \otimes F$-generators $\omega(A^\sigma)$ and $\omega(A^{\sigma,\vee})$ of $\omega_{A^\sigma/F}$ and $\omega_{A^{\sigma,\vee}/F}$ and let Ω_σ and $\Omega_\sigma^\vee$ denote the periods as in Definition 10.3. Proposition 8.8 implies

$$I_\infty(\omega(A^\sigma)^{\boldsymbol{\alpha}} \otimes \omega(A^{\sigma,\vee})^{\boldsymbol{\beta}}) = \Omega_\sigma^{\boldsymbol{\alpha}} \Omega_\sigma^{\vee,\boldsymbol{\beta}} \cdot (\gamma_\sigma^{\boldsymbol{\alpha}} \otimes \gamma_\sigma^{\boldsymbol{\beta}}).$$

Now, for each $\sigma \in J_F$ choose an element $\ell_\sigma \in L^\times$ such that $\ell_{\sigma,v}\lambda_{\sigma,v} \equiv 1 \mod \mathfrak{f}_v$ for all finite places v of L. Together with Proposition 10.12 and Theorem 10.21, we obtain

$$\begin{aligned}
I_\infty(\mathcal{EK}^\sigma) &= ((N\mathfrak{c} - \chi(\mathfrak{c})^{-1})(\alpha-1)!)^{-1} \sum_{[\mathfrak{b}]} \chi(\mathfrak{b}) \varepsilon_\Phi(\sigma) \cdot I_\infty\left(\widetilde{\mathrm{EK}}_{A^\sigma}^{\iota\beta,\iota\alpha-1}(f_{[\mathfrak{c}]}, x_{\mathfrak{b}}^\sigma)\right)_{\iota\in J_{T,\eta}} \\
&= (2\pi i)^{d(\beta)} (N\mathfrak{c} - \chi(\mathfrak{c})^{-1})^{-1} \varepsilon_\Phi(\sigma) \chi(\lambda_\sigma)^{-1} \cdot \\
&\sum_{[\mathfrak{b}]} \Big((N\mathfrak{c} L_{\mathfrak{f}}(\chi_\iota, 0, [\ell_\sigma\lambda_\sigma\mathfrak{b}]) - \chi_\iota(\mathfrak{c})^{-1} L_{\mathfrak{f}}(\chi_\iota, 0, [\ell_\sigma\lambda_\sigma\mathfrak{b}\mathfrak{c}])) \Big)_{\iota\in J_{T,\eta}} (\gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}}) \\
&= (2\pi i)^{d(\beta)} \left(L_{\mathfrak{f}}(\chi_\iota, 0)\right)_{\iota\in J_{T,\eta}} \cdot t_\sigma (\gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}}),
\end{aligned}$$

where the first and last equality follow by reindexing. ☐

The main result of this thesis is the following:

Theorem 11.7 (Deligne conjecture for χ)**.** *Let L and T be number fields and $\chi : \mathbb{A}_L^\times \to T^\times$ a critical algebraic Hecke character of conductor dividing $\mathfrak{f}$. Then*

$$L_{\mathfrak{f}}(\chi, 0) \sim c^+(\chi) \mod T.$$

Proof. Since χ is critical, L must be a totally real field or a totally complex field containing a CM-field. In the case where L is totally real, the Theorem follows by classical results due to Euler, Siegel and Klingen. In the totally complex case there exists a CM-type Φ of L such that $\chi_a = \beta - \alpha$ as in Section 6. We may assume that $L^\alpha \subset T$ since Deligne's conjecture is invariant under extension of coefficients (cf. [Del79], Remarque 2.10). Choose a finite abelian extension F/E and an abelian variety A/F of CM-type (L, Φ) satisfying the conditions of Proposition 6.8 for $\mathfrak{a} = \mathfrak{f}\mathfrak{c}$. By Proposition 11.5, we have the normalized Eisenstein-Kronecker class $\mathcal{EK}$ on A.

We can regard $\mathcal{EK}$ as an element in $F^{d(\alpha)+d(\beta)}R_{F/\mathbf{Q}}(h^1(A)^{\alpha}_T \otimes h^1(A^\vee)^{\beta}_T)_{\mathrm{dR}}$ and hence we obtain an element

$$e_\Xi \cdot \mathcal{EK} \in F^{d(\alpha)+d(\beta)}RM(\Xi)_{\mathrm{dR}}.$$

Note that $I_\infty(\mathcal{EK}) = \sum_{\sigma\in J_F} I_\infty(\mathcal{EK}^\sigma)$, where on the left, we regard $\mathcal{EK}$ in $R_{F/\mathbf{Q}}(h^1(A)^{\alpha}_T \otimes h^1(A^\vee)^{\beta}_T)_{\mathrm{dR}}$ and on the right, every $\mathcal{EK}^\sigma$ is regarded as an element in $(h^1(A^\sigma)^{\alpha}_T \otimes h^1(A^{\sigma,\vee})^{\beta}_T)_{\mathrm{dR}}$. Theorem 11.6 therefore implies

$$I_\infty(\mathcal{EK}) = (2\pi i)^{d(\beta)} \cdot L(\chi,0) \cdot \sum_{\eta\in J_E} e(\eta) \sum_{\sigma\in J_{F,\eta}} t_\sigma(\gamma_\sigma^{\boldsymbol{\alpha}} \otimes (\gamma_\sigma^\vee)^{\boldsymbol{\beta}}) \tag{11.3}$$

and applying the idempotent e_Ξ gives

$$I_\infty(e_\Xi \cdot \mathcal{EK}) = (2\pi i)^{d(\beta)} \cdot L(\chi,0) \cdot \sum_{\eta\in J_E} e(\eta) a_\eta,$$

with the elements a_η from Definition 9.5. Now, Deligne's conjecture follows from Theorem 9.6 and Blasius' Theorem 6.7. □

Remark 11.8. Note that we do not have to check that $e_\Xi \cdot \mathcal{EK} \neq 0$. Indeed, if $L(\chi,0)=0$, there is nothing to show and if $L(\chi,0)\neq 0$, then $e_\Xi \cdot \mathcal{EK}$ also does not vanish since no non-zero element in $T\otimes \mathbf{C}$ annihilates $\sum_{\eta\in J_E} e(\eta)a_\eta$ by Theorem 9.6.

Appendix A. Proof of Blasius' Theorem

In this section, we give a proof of Theorem 6.7. In the case where L is a CM-field, the result is proven in [Bla86]. But as it is remarked in [Bla97], Theorem H.3, the proof in [Bla86] works the same in the general case if one uses the signs of Definition 6.3 instead of the sign character $\widetilde{\varepsilon}$ of loc. cit., Theorem 4.5.2. In our treatment, we follow the chapter 4 and 5 of [Bla86] very closely. The only difference is that we rearrange a few results and occasionally fill in some details. Needless to say, no originality is claimed in anything that follows and the reader is advised to consult the original source [Bla86]. Nevertheless, due to the key role that Theorem 6.7 plays in this thesis, we decided to keep this section.

A.1. **Setup.** Let us recall the setup from Section 6: Let L be a totally imaginary number field containing a CM-field. Let Φ be a CM-type of L and let χ_a be a critical infinity-type with respect to Φ, i.e.

$$\chi_a = \beta - \alpha$$

with $\beta \in I_{\overline{\Phi}}^+$, $\alpha \in I_\Phi^+$ such that $\alpha(\sigma) \geq 1$ and $\beta(\overline{\sigma}) \geq 0$ for all $\sigma \in \Phi$. Let T be a CM-field containing L^α and let

$$\chi : L_f^\times \to T^\times$$

be an algebraic Hecke character of infinity-type χ_a. Let E be the reflex field of (L, Φ) and let $\Phi^* \in I_E$ denote the reflex CM-type of Φ. For every $\eta \in J_E$, recall the sign $\varepsilon_{\eta\Phi} : G_{\mathbf{Q}} \to \{\pm 1\}$ from Definition 6.3. Let $\Xi : E_f^\times \to T^\times$ denote the algebraic Hecke character

$$\Xi = (\chi \circ \Phi^*)^{-1} \|\cdot\|_E^{-d(\beta)} \varepsilon_\Phi,$$

as in (6.4). For any $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_E$, we define elements

$$\begin{aligned} \theta(\tau, \eta) &= \chi_a(\lambda)^{-1} \chi(\lambda) \varepsilon_{\eta\Phi}(\tau) \\ \xi(\tau, \eta) &= \chi_a(\lambda)^{-1} \chi(\lambda) \chi_{\mathrm{cyc}}(\tau)^{-d(\beta)} \varepsilon_{\eta\Phi}(\tau) \end{aligned} \tag{A.1}$$

in $T_f^\times$, where λ is an arbitrary element in the coset $L(\eta\Phi, \tau)$ defined in (5.4). They have the following properties (cf. Proposition 6.5):

- For all $\tau_1, \tau_2 \in G_{\mathbf{Q}}$ and $\eta \in J_E$, one has
$$\xi(\tau_1\tau_2, \eta) = \xi(\tau_1, \tau_2\eta)\xi(\tau_2, \eta).$$
- If $\tau \in G_E$ and $e \in E_f^\times$ satisfies $r_E(e) = \tau|_{E^{\mathrm{ab}}}$, one has
$$\xi(\tau, 1_E) = \Xi_a(e)^{-1}\Xi(e).$$

A.2. **The reflex motive.**

Convention. *Throughout the remainder of Appendix A, we adopt the conventions of* [Bla86], *3.3 regarding Tate twists: If M is the motive defined over L and $n \in \mathbb{Z}$ an integer, then $M(n)$ is is the motive with $M(n)_B = M_B$ with Hodge structure $M(n)_B^{p-n,q-n} = M_B^{pq}$, $M(n)_{\mathrm{dR}} = M_{\mathrm{dR}}$ with Hodge filtration $F^{p-n}M(n)_{\mathrm{dR}} = F^p M_{\mathrm{dR}}$, and $M(n)_f = M_f$ with G_L-action twisted by χ_{cyc}^n, where χ_{cyc} denotes the cyclotomic character.*

Let $M(\chi^{-1})$ be a motive of type (L, χ^{-1}, T). We will now describe a way to construct a motive of type (E, Ξ, T) from $M(\chi^{-1})$. Denote by $RM(\chi^{-1})$ the restriction of scalars from L to $\mathbf{Q}$. Let $g := \frac{1}{2}[L : \mathbf{Q}]$. Consider the gth T-linear exterior power

$$N = \Lambda^g RM(\chi^{-1}).$$

There is a canonical isomorphism

$$N \times_{\mathbf{Q}} \overline{\mathbf{Q}} \cong \Lambda^g \Big(\bigoplus_{\sigma \in J_L} M(\chi^{-1})^\sigma \Big).$$

Fix an ordering on every subset $\Delta \subset J_L$ with g elements. With this choice fixed, one obtains an isomorphism

$$\Lambda^g \Big(\bigoplus_{\sigma \in J_L} M(\chi^{-1})^\sigma \Big) \cong \bigoplus_{\substack{\Delta \subset J_L \\ |\Delta| = g}} \bigotimes_{\sigma \in \Delta} M(\chi^{-1})^\sigma.$$

In the Betti realization, for example, this sends an element $\bigotimes_{\sigma\in\Delta}\gamma_\sigma$ on the right-hand side to $\bigwedge_{\sigma\in\Delta}\gamma_\sigma$ with the fixed ordering. For each g-element subset Δ of J_L, we therefore have a projector

$$P^\Delta \in \mathrm{End}(N \times \overline{\mathbf{Q}}),$$

which is independent of the choice of orderings. In order to descend the field of definition, we need to understand the Galois action on the étale realization P_f^Δ.

Lemma A.1. *For each $\tau \in G_{\mathbf{Q}}$, one has*

$$(P_f^\Delta)^\tau = P_f^{\tau\Delta}.$$

Proof. This follows immediately from the fact that, for any motive M (with coefficients), the functor $M \mapsto M^\tau$ as well as the natural $\mathbf{Q}_f$-linear isomorphism $M \to M^\tau$ commutes with direct sums and tensor products. □

For each embedding $\eta \in J_E$, we have the CM-type $\eta\Phi$ whose field of values is given by E^η. By Lemma A.1, the corresponding idempotent endomorphism $P^{\eta\Phi}$ can already be defined on $\Lambda^g RM(\chi^{-1}) \times E^\eta$. Moreover, the projector

$$P := \sum_{\eta\in J_E} P^{\eta\Phi}$$

is defined over $\mathbf{Q}$, and we set

$$M^* := P \cdot \Lambda^g RM(\chi^{-1}). \tag{A.2}$$

We sometimes refer to the motive M^* as the reflex motive of χ. We also set

$$M^*_\eta := P^{\eta\Phi} \cdot (\Lambda^g RM(\chi^{-1}) \times E^\eta).$$

For $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_E$, one has $(M^*_\eta)^\tau = M^*_{\tau\eta}$.

Lemma A.2. *There is a canonical isomorphism*

$$M^* \cong R_{E/\mathbf{Q}}(M^*_{1_E})$$

of motives over $\mathbf{Q}$ with coefficients in T.

Proof. For convenience abbreviate $N := \Lambda^g RM(\chi^{-1})$. If $\widetilde{E}$ denotes a Galois extension of $\mathbf{Q}$ containing E, we have a canonical isomorphism

$$\begin{aligned} R_{E/\mathbf{Q}}(M^*_{1_E}) \times_{\mathbf{Q}} \widetilde{E} &= R_{E/\mathbf{Q}}(P^\Phi \cdot (N \times_{\mathbf{Q}} E)) \times_{\mathbf{Q}} \widetilde{E} \\ &\cong \bigoplus_{\eta\in J_E} (P^\Phi \cdot (N \times_{\mathbf{Q}} E))^\eta \times_{E^\eta} \widetilde{E} \\ &\cong \bigoplus_{\eta\in J_E} P^{\eta\Phi} \cdot (N \times_{\mathbf{Q}} \widetilde{E}) \\ &\cong P \cdot (N \times_{\mathbf{Q}} \widetilde{E}) \\ &= M^* \times_{\mathbf{Q}} \widetilde{E}. \end{aligned}$$

One immediately checks that on finite adelic realizations, this isomorphism is $G_{\mathbf{Q}}$-equivariant. Indeed, in this case the "same isomorphism" also works without passing to $\widetilde{E}$. □

Note that each of the T-vector spaces $M^*_{\eta,B}$, for $\eta \in J_E$, is 1-dimensional. We will now specify a basis element in each $M^*_{\eta,B}$. First, we fix an ordering on the set of complex places $\langle c\rangle\backslash J_L$ of L. This induces an ordering on each CM-type Ψ of L via the bijection

$$\Psi \xrightarrow{\sim} \langle c\rangle\backslash J_L$$

induced by the inclusion. For each $\sigma \in J_L$, we choose a T-basis

$$\gamma_\sigma \in M(\chi^{-1})^\sigma_B \tag{A.3}$$

satisfying $\gamma_\sigma^c = \gamma_{c\sigma}$. For each $\eta \in J_E$ we define

$$\gamma_{\eta\Phi} := \bigwedge_{\sigma\in\eta\Phi} \gamma_\sigma,$$

where the wedge product is taken with respect to the ordering on the CM-type $\eta\Phi$ as above. By construction, for each $\eta \in J_E$, the element $\gamma_{\eta\Phi}$ is a basis of $M^*_{\eta,B}$. For each $\eta \in J_E$ and $\tau \in G_{\mathbf{Q}}$, there is a unique element $c(\tau,\eta) \in T_f^\times$ such that the diagram

$$\begin{tikzcd} T_f \arrow[r, "I_f(\gamma_{\eta\Phi})"] \arrow[d, "c(\tau,\eta)"'] & M^*_{\eta,f} \arrow[d, "\tau"] \\ T_f \arrow[r, "I_f(\gamma_{\tau\eta\Phi})"] & M^*_{\tau\eta,f} \end{tikzcd}$$

is commutative.

Recall the elements $\theta(\tau,\eta)$ defined in (A.1). The following Theorem is the key to one "half" of Blasius' period relation, namely the part which establishes the connection between the reflex motive M^* and the Hecke character Ξ. This is formulated in Corollary A.5 below. Theorem A.3 and its corollaries make up the remainder of this section and only reappear in the proof of Blasius' Theorem A.10. Let us remark that, within this thesis, the proof of the following result is the only instance where the main theorem of complex multiplication for arbitrary automorphisms is invoked.

Theorem A.3 ([Bla86], Theorem 4.5.2)**.** *For each $\eta \in J_E$ and $\tau \in G_{\mathbf{Q}}$, one has*

$$\theta(\tau,\eta) = c(\tau,\eta).$$

Proof. We first construct basis elements $\delta_{\eta\Phi}$ of $M^*_{\eta,f}$, which transform by the factor $\theta(\tau,\eta)$ under conjugation by τ. Afterwards, we show that $I_f(\gamma_{\eta\Phi}) = t \cdot \delta_{\eta\Phi}$ for a constant $t \in T_f^\times$, which is independent of η. Once this is established, the claim follows. For each $\sigma \in J_L$, we choose a lift $\widetilde{\sigma} \in G_{\mathbf{Q}}$ satisfying $\widetilde{c\sigma} = c\widetilde{\sigma}$. Fix an arbitrary T_f-basis $\delta \in M(\chi^{-1})_f$ and set $\delta_\sigma := \delta^{\widetilde{\sigma}} \in M(\chi^{-1})_f^\sigma$. For $\sigma \in J_L$ and $\tau \in G_{\mathbf{Q}}$, we define $\alpha(\tau,\sigma) \in T_f^\times$ by $\delta_\sigma^\tau = \alpha(\tau,\sigma)\delta_{\tau\sigma}$, so that $\delta^{\tau\widetilde{\sigma}} = \alpha(\tau,\sigma)\cdot\delta^{\widetilde{\tau\sigma}}$, i.e. $\delta^{\widetilde{\tau\sigma}^{-1}\tau\widetilde{\sigma}} = \alpha(\tau,\sigma)\cdot\delta$. Let

$$\delta_{\eta\Phi} := \bigwedge_{\sigma\in\eta\Phi} \delta_\sigma \in M^*_{\eta,f}$$

taken with respect to the ordering of CM-types specified above. Recall that $\varepsilon_{\eta\Phi}(\tau)$ is the sign of the permutation $\langle c\rangle\backslash J_L \cong \eta\Phi \xrightarrow{\tau} \tau\eta\Phi \cong \langle c\rangle\backslash J_L$. Hence, one has

$$\begin{aligned} \delta_{\eta\Phi}^\tau &= \bigwedge_{\sigma\in\eta\Phi} \delta_\sigma^\tau \\ &= \varepsilon_{\eta\Phi}(\tau)\cdot \prod_{\sigma\in\eta\Phi} \alpha(\tau,\sigma)\cdot\delta_{\tau\eta\Phi}. \end{aligned}$$

We now want to express the product $\prod_{\sigma\in\eta\Phi}\alpha(\tau,\sigma)$ in more familiar terms. The element $\beta := \prod_{\sigma\in\eta\Phi} \widetilde{\tau\sigma}^{-1}\tau\widetilde{\sigma}$ lies in G_L and by definition of the elements δ_σ for $\sigma \in J_L$, one has

$$\delta^\beta = \prod_{\sigma\in\eta\Phi}\alpha(\tau,\sigma)\cdot\delta.$$

The action of G_L on $M(\chi^{-1})_f$ is abelian and by definition of the type-transfer, one has $\delta^\beta = \delta^{\mathrm{tr}_{L,\eta\Phi}(\tau)^{-1}}$. Let $\lambda \in L(\eta\Phi,\tau)$. By (5.5), we have $r_L(\lambda) = \mathrm{tr}_{L,\eta\Phi}(\tau)$. Since $M(\chi^{-1})$ is of type (L,χ^{-1},T), we obtain $\delta^\beta = \chi_a(\lambda)^{-1}\chi(\lambda)\cdot\delta$, and hence

$$\prod_{\sigma\in\eta\Phi}\alpha(\tau,\sigma) = \chi_a(\lambda)^{-1}\chi(\lambda).$$

This proves

$$\delta_{\eta\Phi}^\tau = \chi_a(\lambda)^{-1}\chi(\lambda)\varepsilon_{\eta\Phi}(\tau)\cdot\delta_{\tau\eta\Phi} = \theta(\tau,\eta)\cdot\delta_{\tau\eta\Phi}.$$

It remains to show that we may replace the basis elements $I_f(\gamma_{\eta\Phi})$, $\eta \in J_E$ by the elements $\delta_{\eta\Phi}$, $\eta \in J_E$. For each $\sigma \in J_L$, let $t_\sigma \in T_f^\times$ denote the unique element satisfying $I_f(\gamma_\sigma) = t_\sigma\cdot\delta_\sigma$, so that

$$I_f(\gamma_{\eta\Phi}) = \prod_{\sigma\in\eta\Phi} t_\sigma\cdot\delta_{\eta\Phi}.$$

We need to show that the factor $\prod_{\sigma\in\eta\Phi} t_\sigma$ is independent of η. Note that by the choice of $\{\gamma_\sigma \mid \sigma \in J_L\}$ and $\{\delta_\sigma \mid \sigma \in J_L\}$, one has

$$t_{c\sigma} = t_\sigma \tag{A.4}$$

for all $\sigma \in J_L$. Since Φ is a CM-type, there is a decomposition

$$\eta\Phi = (\eta\Phi \cap \Phi) \sqcup (\eta\Phi \cap \overline{\Phi}).$$

Now complex conjugation carries $\eta\Phi \cap \overline{\Phi}$ into $\overline{\eta\Phi} \cap \Phi$ and $\Phi = (\eta\Phi \cap \Phi) \sqcup (\overline{\eta\Phi} \cap \Phi)$, so it follows from (A.4) that the product of the elements t_σ indexed by $\eta\Phi$ coincides with the product indexed by Φ. □

Corollary A.4. *There exists a motive $M(\Xi)$ of type (E, Ξ, T) and basis elements $a_\eta \in M(\Xi)^\eta_B$ for $\eta \in J_E$, such that for all $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_E$ the diagram*

$$\begin{tikzcd} T_f \arrow[r, "I_f(a_\eta)"] \arrow[d, "\xi(\tau,\eta)"'] & M(\Xi)^\eta_f \arrow[d, "\tau"] \\ T_f \arrow[r, "I_f(a_{\tau\eta})"] & M(\Xi)^{\tau\eta}_f \end{tikzcd}$$

commutes.

Proof. Note that $\xi(\tau, \eta) = \theta(\tau, \eta) \cdot \chi_{\mathrm{cyc}}(\tau)^{-d(\beta)}$ for all $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_E$. It follows from Theorem A.3 and Proposition 6.5 (iii) that $M^*_{1_E}(-d(\beta))$ is of type (E, Ξ, T). Note that under our conventions the Tate twist does not alter the underlying module of the Betti realisation. The basis elements $\gamma_{\eta\Phi} \in M^*_\eta(-d(\beta))_B = M^*_{1_E}(-d(\beta))^\eta_B$ have the desired properties by Theorem A.3. □

Corollary A.5. *Let $M(\Xi)$ be any motive of type (E, Ξ, T) and let $\{a_\eta \mid \eta \in J_E\}$ be a system of basis elements $a_\eta \in M(\Xi)^\eta_B$ as in Corollary A.4. Then there exists a unique isomorphism*

$$\varphi : RM(\Xi)(d(\beta)) \xrightarrow{\sim} M^*$$

such that

$$\varphi_B(a_\eta) = \gamma_{\eta\Phi}.$$

Proof. Let $\widetilde{E}$ be a Galois extension of $\mathbf{Q}$ containing E. Let $\eta \in J_E$. By the proof of Corollary A.4, the motive $M^*_\eta(-d(\beta)) \times_{E^\eta} \widetilde{E}$ is of type $(\widetilde{E}, \Xi \circ \eta^{-1} \circ N_{\widetilde{E}/E^\eta}, T)$ and hence there exists a unique isomorphism

$$\varphi_\eta : M(\Xi)^\eta(d(\beta)) \times_{E^\eta} \widetilde{E} \xrightarrow{\sim} M^*_\eta \times_{E^\eta} \widetilde{E}$$

of motives over $\widetilde{E}$ and coefficients in T, such that $\varphi_{\eta,B}(a_\eta) = \gamma_{\eta\Phi}$. They assemble into an isomorphism

$$\varphi : RM(\Xi)(d(\beta)) \times \widetilde{E} \xrightarrow{\sim} M^* \times \widetilde{E}.$$

of motives over $\widetilde{E}$ with coefficients in T with $\varphi_B(a_\eta) = \gamma_{\eta\Phi}$ for all $\eta \in J_E$. Theorem A.3 implies that φ_f is $G_{\mathbf{Q}}$-equivariant, and hence φ descends to an isomorphism

$$\varphi : RM(\Xi)(d(\beta)) \xrightarrow{\sim} M^*$$

satisfying $\varphi_B(a_\eta) = \gamma_{\eta\Phi}$ for all $\eta \in J_E$, as desired. □

A.3. **The period relation.** For any subset $A \subset J_T$, we let e_A denote the idempotent in $T \otimes \mathbf{C}$ which singles out the direct factors indexed by A in the canonical decomposition $T \otimes \mathbf{C} = \mathbf{C}^{J_T}$. If $\tau \in J_T$, we write e_τ instead of $e_{\{\tau\}}$. If $\eta \in J_E$ is an embedding of the reflex field E, we simply write $e(\eta)$ for the idempotent which corresponds to the subset $J_{T,\eta} \subset J_T$. We write $(T \otimes \mathbf{C})_\eta := e(\eta) \cdot (T \otimes \mathbf{C})$.

Let $\chi : L^\times_f \to T^\times$ be a critical algebraic Hecke character as in Section A.1 and let $M(\chi)$ be a motive of type (L, χ, T). Let us write $c^+(\chi) := c^+(M(\chi))$ for Deligne's period of the critical motive $M(\chi)$ as in Definition 3.14. Recall that $c^+(\chi)$ is the determinant of the $T \otimes \mathbf{C}$-linear isomorphism

$$I_+ : RM(\chi)^+_B \otimes \mathbf{C} \subset RM(\chi)_B \otimes \mathbf{C} \xrightarrow[\sim]{I_\infty^{-1}} RM(\chi)_{\mathrm{dR}} \otimes \mathbf{C} \twoheadrightarrow RM(\chi)^+_{\mathrm{dR}} \otimes \mathbf{C}, \tag{A.5}$$

where we abbreviate $RM(\chi)^+_{\mathrm{dR}} := RM(\chi)_{\mathrm{dR}}/F^0RM(\chi)_{\mathrm{dR}}$. We now express the period $c_+(\chi)$ in terms of the motive $RM(\chi^{-1})$ which is dual to $RM(\chi)$. The dual of (A.5) identifies with the composite

$$I_+^\vee : F^1RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C} \xrightarrow{I_\infty} RM(\chi^{-1})_B \otimes \mathbf{C} \to RM(\chi^{-1})_B/RM(\chi^{-1})_B^- \otimes \mathbf{C},$$

where $RM(\chi^{-1})_B^-$ denotes the subspace of $RM(\chi^{-1})_B$ on which the involution F_∞ acts by multiplication by -1. It follows that

$$c_+(\chi) = \det(I_+^\vee) \mod T^\times, \tag{A.6}$$

where $\det(I_+^\vee)$ is calculated with respect to bases of the T-vector spaces $F^1RM(\chi^{-1})_{\mathrm{dR}}$ and $RM(\chi^{-1})_B/RM(\chi^{-1})_B^-$. Now, let

$$\pi_+ : RM(\chi^{-1})_B \otimes \mathbf{C} \to I_\infty(F^1RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) = \bigoplus_{p>0} RM(\chi^{-1})_B^{p,q}$$

denote the canonical projection. Let $\{\gamma_\sigma \mid \sigma \in J_L\}$ with $\gamma_\sigma^c = \gamma_{c\sigma}$ be a basis of $RM(\chi^{-1})_B$ as in (A.3). Then

$$\{\gamma_\sigma + \gamma_{c\sigma} \mid \sigma \in \Phi\} \tag{A.7}$$

is a T-basis of $RM(\chi^{-1})_B^+$, and hence

$$\{\pi_+(\gamma_\sigma + \gamma_{c\sigma}) \mid \sigma \in \Phi\} \tag{A.8}$$

is a $T \otimes \mathbf{C}$-basis of $I_\infty(F^1RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$. Define the period

$$p(\chi) \in (T \otimes \mathbf{C})^\times$$

as the determinant of the isomorphism

$$I_\infty : F^1RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C} \to I_\infty(F^1RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$$

calculated with respect to a T-basis of $F^1RM(\chi^{-1})_{\mathrm{dR}}$ and the basis (A.8). Needless to say that $p(\chi)$ is only well-defined up to elements in $T^\times$.

Lemma A.6 ([Bla86], Lemma 5.2.1)**.** *One has*

$$p(\chi) = c_+(\chi) \mod T^\times.$$

Proof. By (A.6), it suffices to show that the determinant of the isomorphism

$$I_\infty(F^1RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) \to RM(\chi^{-1})_B/RM(\chi^{-1})_B^- \otimes \mathbf{C} \tag{A.9}$$

calculated with respect to the basis (A.8) and the image of the basis (A.7) in the quotient $RM(\chi^{-1})_B/RM(\chi^{-1})_B^-$ (or any basis of this T-vector space) lies in $T^\times$. By Lemma A.7 (iii) below, one has

$$\pi_+(\gamma_\sigma) = e_{\sigma\Phi^*} \cdot \gamma_\sigma$$

for all $\sigma \in J_L$, and hence

$$\pi_+(\gamma_\sigma + \gamma_{c\sigma}) = e_{\sigma\Phi^*} \cdot \gamma_\sigma + e_{c\sigma\Phi^*} \cdot \gamma_{c\sigma}.$$

On the other hand, there exist $a_\sigma, b_\sigma \in T \otimes \mathbf{C}$ such that

$$\begin{aligned}\pi_+(\gamma_\sigma + \gamma_{c\sigma}) &= a_\sigma \cdot (\gamma_\sigma + \gamma_{c\sigma}) + b_\sigma \cdot (\gamma_\sigma - \gamma_{c\sigma}) \\ &= (a_\sigma + b_\sigma) \cdot \gamma_\sigma + (a_\sigma - b_\sigma) \cdot \gamma_{c\sigma}.\end{aligned}$$

It follows that $a_\sigma + b_\sigma = e_{\sigma\Phi^*}$ and $a_\sigma - b_\sigma = e_{c\sigma\Phi^*}$. Since $e_{\sigma\Phi^*} + e_{c\sigma\Phi^*} = 1$ as $\sigma\Phi^*$ is a CM-type, one obtains

$$a_\sigma = \frac{1}{2} \quad \text{and} \quad b_\sigma = \frac{1}{2}(e_{\sigma\Phi^*} - e_{c\sigma\Phi^*}).$$

As $\gamma_\sigma - \gamma_{c\sigma} \in RM(\chi^{-1})_B^-$, one has $\pi_+(\gamma_\sigma + \gamma_{c\sigma}) = \frac{1}{2}(\gamma_\sigma + \gamma_{c\sigma})$ in $RM(\chi^{-1})_B/RM(\chi^{-1})_B^-$ and the isomorphism (A.9) has determinant $2^{-g} \in T^\times$. ☐

Lemma A.7 ([Bla86], Lemma 5.3.1)**.** *In the following, let Φ^* denote the reflex of Φ relative to T.*

(i) *For $\sigma \in J_L$ and $\tau \in J_T$, one has $\sigma \in \tau\Phi$ if and only if $\tau \in \sigma\Phi^*$.*

(ii) *If $\chi^* \in I_T$ denotes the reflex of χ_a relative to T, then, for any $\sigma \in J_L$, one has*

$$\sigma\Phi^* = \{\tau \in J_T \mid (\chi^{-1} \circ \sigma^{-1})^*(\tau) > 0\}.$$

Similarly, for any $\tau \in J_T$, one has

$$\tau\Phi = \{\sigma \in J_L \mid (\chi^{-1} \circ \sigma^{-1})^*(\tau) > 0\}.$$

(iii) *For $\sigma \in J_L$ and $\tau \in J_T$, the summand $e_\tau \cdot M(\chi^{-1})^\sigma_B \otimes \mathbf{C}$ is purely of Hodge type $((\chi^{-1} \circ \sigma^{-1})^*(\tau), (\chi^{-1} \circ \sigma^{-1})^*(c\tau))$.*

(iv) *For $\sigma \in J_L$ and $\tau \in J_T$, the inclusion*

$$e_\tau \cdot M(\chi^{-1})^\sigma_B \otimes \mathbf{C} \subset I_\infty(F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$$

holds if and only if $\tau \in \sigma\Phi^*$, or equivalently $\sigma \in \tau\Phi$.

Proof. Let $\widetilde{L}$ be a Galois extension over $\mathbf{Q}$ containing L and T and let $\widetilde{\Phi} = \mathrm{Inf}_{\widetilde{L}/L}(\Phi)$. Let $\widetilde{\sigma}$ and $\widetilde{\tau}$ denote extensions of σ and τ to $\widetilde{L}$. Then, $\sigma \in \tau\Phi$ if and only if $\widetilde{\Phi}(\widetilde{\tau}^{-1}\widetilde{\sigma}) \neq 0$. By definition of the reflex, this is equivalent to $\widetilde{\Phi}^*(\widetilde{\sigma}^{-1}\widetilde{\tau}) \neq 0$, which, just as before, holds if and only if $\tau \in \sigma\Phi^*$. This shows assertion (i). For (ii) observe that

$$(\chi^{-1} \circ \sigma^{-1})^* = -\sigma\chi^* = \sigma\alpha^* - \sigma\beta^*.$$

The type $\sigma\alpha^*$ is supported on $\sigma\Phi^*$ and takes values ≥ 1 by the criticality of χ. Similarly, $\sigma\beta^*$ is supported on $\overline{\Phi^*}$ and takes non-negative values. It follows that an embedding $\tau \in J_T$ satisfies the condition

$$(\chi^{-1} \circ \sigma^{-1})^*(\tau) > 0$$

if and only if $\tau \in \sigma\Phi^*$. The other part of the statement then follows from (i). As $M(\chi^{-1})^\sigma$ is of type $(L^\sigma, \chi^{-1} \circ \sigma^{-1}, T)$, assertion (iii) follows immediately from Proposition 3.20. Since $I_\infty(F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) = \bigoplus_{p>0} RM(\chi^{-1})^{p,q}_B$, statement (iv) follows from (ii) and (iii). $\square$

Proposition A.8 ([Bla86], Proposition 5.3.2)**.**

(i) $\Lambda^g F^1 RM(\chi^{-1})_{\mathrm{dR}} = F^{d(\alpha)} \Lambda^g RM(\chi^{-1})_{\mathrm{dR}}$.

(ii) $\bigwedge_{\sigma \in \Phi} \pi_+(\gamma_\sigma + \gamma_{c\sigma}) = \pi^{d(\alpha), -d(\beta)}\Big(\bigwedge_{\sigma \in \Phi}(\gamma_\sigma + \gamma_{c\sigma})\Big)$.

(iii) *Let $\eta \in J_E$. For any CM-type Ψ of L, one has*

$$\pi^{d(\alpha), -d(\beta)}(\gamma_\Psi) = \begin{cases} e(\eta) \cdot \gamma_{\eta\Phi} & \text{if } \Psi = \eta\Phi \\ 0 & \text{else.} \end{cases}$$

Moreover, $e(\eta) \cdot \gamma_{\eta\Phi}$ is a $(T \otimes \mathbf{C})_\eta$-basis of $e(\eta) \cdot I_\infty(F^{d(\alpha)} \Lambda^g RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$.

Proof. We write

$$\Lambda^g RM(\chi^{-1})_B \otimes \mathbf{C} = \bigoplus_{\tau \in J_T} \Lambda^g\Big(\bigoplus_{\sigma \in J_L} e_\tau \cdot M(\chi^{-1})^\sigma_B \otimes \mathbf{C}\Big),$$

where we use $T \otimes \mathbf{C}$-linear exterior powers on the right-hand side. Let $\tau \in J_T$ and $\Delta \subset J_L$ be a g-element subset. Since any element $v_{\sigma,\tau} \in e_\tau \cdot M(\chi^{-1})^\sigma_B \otimes \mathbf{C}$ is of Hodge type $((\chi^{-1} \circ \sigma^{-1})^*(\tau), (\chi^{-1} \circ \sigma^{-1})^*(c\tau))$ by Lemma A.7 (iii), an element of the form $\bigwedge_{\sigma \in \Delta} v_{\sigma,\tau}$ is of Hodge type (p_Δ, q_Δ), where

$$p_\Delta = \sum_{\sigma \in \Delta} (\chi^{-1} \circ \sigma^{-1})^*(\tau) \quad \text{and} \quad q_\Delta = \sum_{\sigma \in \Delta} (\chi^{-1} \circ \sigma^{-1})^*(c\tau).$$

As $(\chi^{-1} \circ (c\sigma)^{-1})^*(\tau) = (\chi^{-1} \circ \sigma^{-1})^*(c\tau)$, it follows from Lemma A.7 (ii) that the index p_Δ becomes maximal if $\Delta = \tau\Phi$. Let us consider the case $\Delta = \tau\Phi$ and set $p := p_{\tau\Phi}$ and $q := q_{\tau\Phi}$.

Choose an extension $\widetilde{\tau} \in G_{\mathbf{Q}}$ of τ and, for each $\sigma \in \tau\Phi$, an extension $\widetilde{\sigma} \in G_{\mathbf{Q}}$. Then, one has

$$\begin{aligned} p = \sum_{\sigma \in \tau\Phi} (\chi^{-1} \circ \sigma^{-1})^*(\tau) &= - \sum_{\sigma \in \tau\Phi} (\sigma\chi^*)(\tau) \\ &= - \sum_{\sigma \in \tau\Phi} \chi^*(\widetilde{\sigma}^{-1}\tau) \\ &= - \sum_{\sigma \in \tau\Phi} \chi_a(\widetilde{\tau}^{-1}\sigma) \\ &= - \sum_{\sigma \in \Phi} \chi_a(\sigma) \\ &= d(\alpha) \end{aligned}$$

and for weight reasons $q = -wg - p = -d(\chi_a) - d(\alpha) = -d(\beta)$. Now, by Lemma A.7 (iv), the an element $v_{\sigma,\tau}$ as above lies in $I_\infty(F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$ if and only if $\sigma \in \tau\Phi$. Hence

$$\begin{aligned} I_\infty(\Lambda^g F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) &= \Lambda^g I_\infty(F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) \\ &= \big(\Lambda^g RM(\chi^{-1})\big)_B^{p,q} \end{aligned}$$

and by the maximality of p, this coincides with $I_\infty(F^p \Lambda^g RM(\chi^{-1})_{\mathrm{dR}})$. This proves (i) since $p = d(\alpha)$. Moreover, we see that $\Lambda^g(\pi_+) = \pi^{p,q} = \pi^{d(\alpha),-d(\beta)}$ and (ii) follows. For (iii), let Ψ be a CM-type of L and $\tau \in J_T$. By the above, the element

$$e_\tau \cdot \gamma_\Psi = \bigwedge_{\sigma \in \Psi} e_\tau \cdot \gamma_\sigma$$

acquires the Hodge type $(p, q) = (d(\alpha), -d(\beta))$ if and only if $\Psi = \tau\Phi$. Note that $\tau\Phi$ only depends on the restriction $\tau|_E$. It follows that $\pi^{p,q}(\gamma_\Psi) = 0$ if $\Psi \neq \eta\Phi$ for some $\eta \in J_E$, and

$$\begin{aligned} \pi^{p,q}(\gamma_{\eta\Phi}) &= \sum_{\tau \in J_{T,\eta}} e_\tau \cdot \gamma_{\eta\Phi} \\ &= e(\eta) \cdot \gamma_{\eta\Phi}. \end{aligned}$$

Recall that $\{\pi_+(\gamma_\sigma + \gamma_{c\sigma}) \mid \sigma \in \Phi\}$ is a $T \otimes \mathbf{C}$-basis of $I_\infty(F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$. Thus, the right-hand side in (ii) also is a basis. Since $\pi^{p,q}(\gamma_\Delta) \neq 0$ only if $\Delta = \eta\Phi$ for some $\eta \in J_E$, we get

$$\begin{aligned} \pi^{p,q}\Big(\bigwedge_{\sigma \in \Phi} (\gamma_\sigma + \gamma_{c\sigma})\Big) &= \pi^{p,q}\Big(\sum_{\eta \in J_E} \gamma_{\eta\Phi}\Big) \\ &= \sum_{\eta \in J_E} e(\eta) \cdot \gamma_{\eta\Phi}. \end{aligned}$$

Note that, with the orderings we have chosen on the CM-types, no signs have to be introduced while expanding $\bigwedge_{\sigma\in\Phi}(\gamma_\sigma + \gamma_{c\sigma})$ as above. Thus,

$$\sum_{\eta \in J_E} e(\eta) \cdot \gamma_{\eta\Phi}$$

is a $T \otimes \mathbf{C}$-basis of $I_\infty(\Lambda^g F^1 RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) = I_\infty(F^{d(\alpha)} \Lambda^g RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$. By applying an idempotent $e(\eta)$ for a fixed $\eta \in J_E$, we obtain the remaining part of (iii). □

Let $M^* = P \cdot \Lambda^g RM(\chi^{-1})$ be the reflex motive defined in (A.2). Then

$$F^{d(\alpha)} M^*_{\mathrm{dR}}$$

is a 1-dimensional T-vector space: By Proposition A.8 (iii), $I_\infty(F^{d(\alpha)} \Lambda^g RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C})$ is free of rank 1 over $T \otimes \mathbf{C}$ and $\sum_{\eta \in J_E} e(\eta)\gamma_{\eta\Phi}$ is a basis. Since $P_B(\gamma_{\eta\Phi}) = \gamma_{\eta\Phi}$ by construction of the idempotent P, we deduce that

$$\begin{aligned} I_\infty(F^{d(\alpha)} M^*_{\mathrm{dR}} \otimes \mathbf{C}) &= P_B \cdot I_\infty(F^{d(\alpha)} \Lambda^g RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) \\ &= I_\infty(F^{d(\alpha)} \Lambda^g RM(\chi^{-1})_{\mathrm{dR}} \otimes \mathbf{C}) \end{aligned}$$

is a free $T \otimes \mathbf{C}$-module of rank 1 and

$$\sum_{\eta \in J_E} e(\eta)\gamma_{\eta\Phi}$$

is a $T \otimes \mathbf{C}$-basis. Let $\omega \in F^{d(\alpha)} M^*_{\mathrm{dR}}$ be a T-basis. Define the period

$$q(\chi) \in (T \otimes \mathbf{C})^\times$$

to be the unique element satisfying

$$I_\infty(\omega) = q(\chi) \cdot \sum_{\eta \in J_E} e(\eta)\gamma_{\eta\Phi}.$$

Proposition A.9 ([Bla86], Proposition 5.3.3). *One has*

$$q(\chi) = c_+(\chi) \mod T^\times.$$

Proof. This follows from the definition of the period $p(\chi)$, Lemma A.6 and Proposition A.8. □

Theorem A.10 ([Bla86], Proposition 5.3.4, [Bla97], Theorem H.3). *Let Ξ be the Hecke character as defined in (6.4), $M(\Xi)$ a motive of type (E, Ξ, T) and $\{a_\eta \mid \eta \in J_E\}$ a system of basis elements $a_\eta \in M(\Xi)^\eta_B$ satisfying*

$$I_f(a_\eta)^\tau = \xi(\tau, \eta) \cdot I_f(a_{\tau\eta})$$

for all $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_E$.

(i) *The T-vector space $F^{d(\alpha)+d(\beta)} RM(\Xi)_{\mathrm{dR}}$ is 1-dimensional. If ω is a basis element in $F^{d(\alpha)+d(\beta)} RM(\Xi)_{\mathrm{dR}}$, there exists a unique element $v(\chi) \in (T \otimes \mathbf{C})^\times$ satisfying*

$$I_\infty(\omega) = v(\chi) \cdot \sum_{\eta \in J_E} e(\eta) a_\eta.$$

(ii) *One has*

$$(2\pi i)^{-d(\beta)} v(\chi) \sim c_+(\chi) \mod T^\times.$$

Proof. By Corollary A.5, there is an isomorphism

$$\varphi : RM(\Xi)(d(\beta)) \xrightarrow{\sim} M^*$$

satisfying $\varphi_B(a_\eta) = \gamma_{\eta\Phi}$ for all $\eta \in J_E$. Together with the discussion after Proposition A.8, it follows that the T-vector space $F^{d(\alpha)+d(\beta)} RM(\Xi)_{\mathrm{dR}} \cong F^{d(\alpha)} M^*_{\mathrm{dR}}$ is 1-dimensional and that $\sum_{\eta \in J_E} e(\eta) a_\eta$ is a $T \otimes \mathbf{C}$-basis of $I_\infty(F^{d(\alpha)+d(\beta)} RM(\Xi)_{\mathrm{dR}} \otimes \mathbf{C})$. This implies (i). Assertion (ii) then follows from $I_\infty(RM(\Xi)) = (2\pi i)^{d(\beta)} I_\infty(RM(\Xi)(d(\beta)))$ and Proposition A.9. □

Appendix B. Equivariant cohomology

Let Γ be a discrete group and let $(X, \mathcal{R}_X)$ be a ringed space. Denote the category of $\mathcal{R}_X$-modules by $\mathrm{Mod}(\mathcal{R}_X)$. A *left Γ-action on* $(X, \mathcal{R}_X)$ is a group homomorphism

$$\Gamma \to \mathrm{Aut}(X, \mathcal{R}_X).$$

The image of an element $\gamma \in \Gamma$ under this map will again be denoted by γ. For every $\gamma \in \Gamma$, there are isomorphisms of sheaves of rings

$$\gamma^{-1}\mathcal{R}_X \to \mathcal{R}_X. \tag{B.1}$$

Definition B.1. *Let $(X, \mathcal{R}_X)$ be a fixed ringed space with a left Γ-action. A* Γ-equivariant $\mathcal{R}_X$-module on X *is a $\mathcal{R}_X$-module $\mathcal{F}$ together with a family of $\mathcal{R}_X$-linear isomorphisms*

$$a_\gamma : \gamma^*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \tag{B.2}$$

satisfying $a_{\gamma_1\gamma_2} = a_{\gamma_2} \circ \gamma_2^*(a_{\gamma_1})$ for all $\gamma_1, \gamma_2 \in \Gamma$. We will often suppress the data (B.2) from the notation and simply say that $\mathcal{F}$ is an equivariant $\mathcal{R}_X$-module on X.

A homomorphism $f : \mathcal{F} \to \mathcal{G}$ of Γ-equivariant $\mathcal{R}_X$-modules is a morphism of $\mathcal{R}_X$-modules such that f is comptatible with the Γ-actions, i.e. the diagram

$$\begin{tikzcd} \gamma^*\mathcal{F} \arrow[r, "\gamma^*(f)"] \arrow[d, "a_{\mathcal{F},\gamma}"'] & \gamma^*\mathcal{G} \arrow[d, "a_{\mathcal{G},\gamma}"] \\ \mathcal{F} \arrow[r, "f"] & \mathcal{G} \end{tikzcd}$$

commutes for every $\gamma \in \Gamma$.

We denote the category of Γ-equivariant $\mathcal{R}_X$-modules with Γ-equivariant homomorphisms by $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$.

The structure of a Γ-equivariant $\mathcal{R}_X$-module on a $\mathcal{R}_X$-module $\mathcal{F}$ is equivalent to specifying isomorphisms

$$b_\gamma : \mathcal{F} \xrightarrow{\sim} \gamma_*\mathcal{F} \tag{B.3}$$

satisfying $b_{\gamma_1\gamma_2} = \gamma_{1*}(b_{\gamma_2}) \circ b_{\gamma_1}$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Let $\mathcal{F}$ be a Γ-equivariant $\mathcal{R}_X$-module. By passing to global sections in (B.2) or (B.3), we see that $H^0(X, \mathcal{F})$ acquires the structure of a right Γ-$\mathcal{R}_X(X)$-module.

Remark B.2. Let us make a few remarks on how the above definition relates to definitions in other works.

(i) If X is a scheme and $\mathcal{R}_X = \mathcal{O}_X$, the above definition is immediately seen to be equivalent to the definition given in [KS19], Definition A.1.

(ii) Let $(X, \mathcal{R}_X)$ be a ringed space with a left Γ-action. The inverses of the maps (B.1) equip $\mathcal{R}_X$ with the structure of a Γ-sheaf of rings in the sense of [Gro57], 5.1. One easily checks that our notion of Γ-equivariant $\mathcal{R}_X$-module is equivalent to the notion of a Γ-$\mathcal{R}_X$-module in the sense of loc. cit. The need to pass to the inverses of the structure maps reflects in the fact that in our setting the global sections are right Γ-modules.

Proposition B.3. *Let $(X, \mathcal{R}_X)$ be a ringed space with a left Γ-action.*

(i) *The category $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$ is an abelian category with enough injective objects.*

(ii) *If $\mathcal{F} \in \mathrm{Mod}_\Gamma(\mathcal{R}_X)$ is an injective equivariant sheaf, then $H^0(X, \mathcal{F})$ is $H^0(\Gamma, -)$-acyclic as a Γ-module (and as a Γ-$\mathcal{R}_X(X)$-module).*

(iii) *The canonical forgetful functor $\mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}(\mathcal{R}_X)$ preserves injective objects.*

Proof. In view of Remark B.2, (i) follows from [Gro57], Proposition 5.1.1 and Proposition 5.1.2 (or Théorème 1.10.1). Assertion (ii) follows from loc. cit., Lemma 5.6.1. Also note that group cohomology of Γ-modules and Γ-$\mathcal{R}_X(X)$-modules coincides, cf. [SP], Tag 0DVD. Assertion (iii) is [Gro57], Lemme 5.6.2. ☐

Definition B.4 (Equivariant cohomology)**.** *The pth derived functor of the left exact composite functor*

$$H^0(X, \Gamma; -) : \mathrm{Mod}_\Gamma(\mathcal{R}_X) \xrightarrow{H^0(X,-)} \mathrm{Mod}_{\Gamma-\mathcal{R}_X(X)} \xrightarrow{H^0(\Gamma,-)} \mathrm{Mod}_{\mathcal{R}_X(X)^\Gamma}$$

is denoted by $H^p(X, \Gamma; -)$ and called the ith equivariant cohomology of X.

Proposition B.5 (Equivariant cohomology spectral sequence)**.** *Let $\mathcal{F}$ be a Γ-equivariant $\mathcal{R}_X$-module. There is a convergent spectral sequence*

$$E_2^{pq} = H^p(\Gamma, H^q(X, \mathcal{F})) \Rightarrow H^{p+q}(X, \Gamma; \mathcal{F}) \tag{B.4}$$

of $\mathcal{R}_X(X)^\Gamma$-modules.

Proof. This follows immediately from the definition and Proposition B.3 (ii). ☐

Let $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a Γ-equivariant morphism of ringed spaces and let $\mathcal{G}$ be a Γ-equivariant $\mathcal{R}_Y$-module. Then $f^*\mathcal{G}$ can be canonically equipped with the structure of a Γ-equivariant $\mathcal{R}_X$-module: For every $\gamma \in \Gamma$, we have isomorphisms

$$\gamma^*(f^*\mathcal{G}) \cong f^*(\gamma^*\mathcal{G}) \xrightarrow{f^*(a_{\mathcal{G},\gamma})} f^*\mathcal{G},$$

which are easily checked to satisfy the necessary compatibilities. Similarly, if $\mathcal{F}$ is a Γ-equivariant $\mathcal{R}_X$-module, there is a canonical way to equip $f_*\mathcal{F}$ with the structure of Γ-equivariant $\mathcal{R}_Y$-module: For every $\gamma \in \Gamma$, there are canonical isomorphisms

$$f_*\mathcal{F} \xrightarrow{f_*(b_{\mathcal{F},\gamma})} f_*(\gamma_*\mathcal{F}) \cong \gamma_*(f_*\mathcal{F}),$$

and again one easily checks the obvious compatibilities. It is straightforward to check that the functor $f^* : \mathrm{Mod}_\Gamma(\mathcal{R}_Y) \to \mathrm{Mod}_\Gamma(\mathcal{R}_X)$ is left-adjoint to $f_* : \mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}_\Gamma(\mathcal{R}_Y)$.

Lemma B.6. *Let $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a Γ-equivariant morphism between ringed spaces with left Γ-action. Then the left exact functor*

$$f_* : \mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}_\Gamma(\mathcal{R}_Y)$$

sends injective objects to $H^0(Y, \Gamma; -)$-acyclic objects.

Proof. Let $\mathcal{F}$ be an injective Γ-equivariant $\mathcal{R}_X$-module. We consider the equivariant cohomology spectral sequence

$$E_2^{pq} = H^p(\Gamma, H^q(Y, f_*\mathcal{F})) \Rightarrow H^{p+q}(Y, \Gamma; f_*\mathcal{F}).$$

By Proposition B.3 (iii), $\mathcal{F}$ is injective as a $\mathcal{R}_X$-module. Hence $f_*\mathcal{F}$ is a flasque sheaf and $H^q(Y, f_*\mathcal{F}) = 0$ for $q > 0$. Moreover, $H^0(Y, f_*\mathcal{F}) = H^0(X, \mathcal{F})$ is $H^0(\Gamma, -)$-acyclic by Proposition B.3 (ii). It follows that $E_2^{pq} = 0$ for $p > 0$ or $q > 0$. From this the claim follows. □

We simply denote the qth right derived functor of $f_* : \mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}_\Gamma(\mathcal{R}_Y)$ by $R^q f_*$. That this is justified, follows from taking the derived functors in the commutative diagram

$$\begin{tikzcd} \mathrm{Mod}_\Gamma(\mathcal{R}_X) \arrow[r, "f_*"] \arrow[d] & \mathrm{Mod}_\Gamma(\mathcal{R}_Y) \arrow[d] \\ \mathrm{Mod}(\mathcal{R}_X) \arrow[r, "f_*"] & \mathrm{Mod}(\mathcal{R}_Y) \end{tikzcd}$$

and observing that the forgetful functor $\mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}(\mathcal{R}_X)$ is exact and preserves injective objects by Proposition B.3 (iii).

Remark B.7. Let $\mathcal{F}$ be a Γ-equivariant $\mathcal{R}_X$-module. Lemma B.6 also implies that the equivariant cohomology groups $H^p(X, \Gamma; \mathcal{F})$ do not depend on the $\mathcal{R}_X$-module structure: Let $\mathbb{Z}_X$ denote the constant sheaf. The Γ-action on $(X, \mathcal{R}_X)$ induces a Γ-action on $(X, \mathbb{Z}_X)$. The canonical morphism of ringed spaces $\iota : (X, \mathcal{R}_X) \to (X, \mathbb{Z}_X)$ is Γ-equivariant and $\iota_* : \mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}_\Gamma(\mathbb{Z}_X)$ is the functor which forgets the $\mathcal{R}_X$-module structure. The functor ι_* is exact and by Lemma B.6 it maps injective objects to acyclic objects for the functor $H^0(X, \Gamma; -) : \mathrm{Mod}_\Gamma(\mathbb{Z}_X) \to \mathrm{Ab}$. Taking derived functors of both composites in the commutative diagram

$$\begin{tikzcd} \mathrm{Mod}_\Gamma(\mathcal{R}_X) \arrow[r, "H^0(X,\Gamma;-)"] \arrow[d, "\iota_*"'] & \mathrm{Mod}_{\mathcal{R}_X(X)^\Gamma} \arrow[d] \\ \mathrm{Mod}_\Gamma(\mathbb{Z}_X) \arrow[r, "H^0(X,\Gamma;-)"] & \mathrm{Ab}, \end{tikzcd}$$

where the right vertical arrow is the forgetful functor, gives the claim.

Proposition B.8 (Leray spectral sequence)**.** *Let $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a Γ-equivariant morphism between ringed spaces with Γ-action and let $\mathcal{F}$ be a Γ-equivariant $\mathcal{R}_X$-module. Then there is a convergent spectral sequence*

$$E_2^{pq} = H^p(Y, \Gamma; R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \Gamma; \mathcal{F}).$$

of $\mathcal{R}_Y(Y)^\Gamma$-modules.

Proof. This follows immediately from $H^0(Y, \Gamma; f_*\mathcal{F}) = H^0(X, \Gamma; \mathcal{F})$ and Lemma B.6. □

Let $f : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a Γ-equivariant morphism and n an integer. For every $\mathcal{F} \in \mathrm{Mod}_\Gamma(\mathcal{R}_X)$ and $\mathcal{G} \in \mathrm{Mod}_\Gamma(\mathcal{R}_Y)$ there are canonical homomorphisms

$$H^n(Y, \Gamma; f_*\mathcal{F}) \to H^n(X, \Gamma; \mathcal{F}) \tag{B.5}$$

$$H^n(Y, \Gamma; \mathcal{G}) \to H^n(X, \Gamma; f^*\mathcal{G}). \tag{B.6}$$

of abelian groups. Indeed, let us first assume that $\mathcal{R}_X = \mathbb{Z}_X$ and $\mathcal{R}_Y = \mathbb{Z}_Y$, i.e. we restrict to the case equivariant abelian sheaves. In this case, $f^{-1} = f^*$ is an exact functor and f_* preserves injective objects. For $n = 0$, the maps in question are just given by $H^0(Y, \Gamma; f_*\mathcal{F}) = H^0(X, \Gamma; \mathcal{F})$ and the composite $H^0(Y, \Gamma; \mathcal{G}) \to H^0(Y, \Gamma; f_* f^{-1}\mathcal{G}) = H^0(X, \Gamma; f^{-1}\mathcal{G})$. Let us now construct (B.6) for arbitrary $n \geq 0$; the map (B.5) is then easily obtained via adjunction. Choose an injective resolution $\mathcal{G} \to \mathcal{I}^\bullet$ in $\mathrm{Mod}_\Gamma(\mathbb{Z}_Y)$ and an injective resolution $f^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$. Then $\mathcal{J}^\bullet$ is an injective resolution of $f^{-1}\mathcal{G}$ since f^{-1} is exact. We obtain (B.6) as the nth cohomology of

$$H^0(Y, \Gamma; \mathcal{I}^\bullet) \to H^0(X, \Gamma; f^{-1}(\mathcal{I}^\bullet)) \to H^0(X, \Gamma; \mathcal{J}^\bullet)$$

and one checks that this construction is independent of the choice of $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$. Now, for general $\mathcal{R}_X$ and $\mathcal{R}_Y$, observe that there is a canonical homomorphism $f^{-1}\mathcal{G} \to f^*\mathcal{G}$ of Γ-equivariant abelian sheaves. By virtue of Remark B.7, we can define (B.6) as the composite

$$H^n(Y, \Gamma; \mathcal{G}) \to H^n(X, \Gamma, f^{-1}\mathcal{G}) \to H^n(X, \Gamma; f^*\mathcal{G}).$$

Proposition B.9 (Functoriality)**.** *The canonical homomorphism (B.6) and its variant in usual cohomology induce a morphism of spectral sequences*

$$\begin{tikzcd} H^p(\Gamma, H^q(Y,\mathcal{G})) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}(Y,\Gamma;\mathcal{G}) \arrow[d] \\ H^p(\Gamma, H^q(X,f^*\mathcal{G})) \arrow[r, Rightarrow] & H^{p+q}(X,\Gamma;f^*\mathcal{G}). \end{tikzcd}$$

Similarly, there is a corresponding morphism for (B.5).

Proof. Let us sketch a proof based on results of [Del71], 1.4. It suffices to show the compatibility for (B.6). As above, choose injective resolutions $\mathcal{G} \to \mathcal{I}^\bullet$ in $\mathrm{Mod}_\Gamma(\mathbb{Z}_Y)$, $f^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ and $f^*\mathcal{G} \to \mathcal{K}^\bullet$ in $\mathrm{Mod}_\Gamma(\mathbb{Z}_X)$. Choose a map $\mathcal{J}^\bullet \to \mathcal{K}^\bullet$ which lifts $f^{-1}\mathcal{G} \to f^*\mathcal{G}$. Then there is an induced map $H^0(Y,\mathcal{I}^\bullet) \to H^0(X, f^{-1}I^\bullet) \to H^0(X,\mathcal{K}^\bullet)$ which induces (B.6) after applying $H^0(\Gamma,-)$ and taking cohomology. By Proposition B.3 (iii), it also induces the canonical map $H^n(Y,\mathcal{G}) \to H^n(X,f^*\mathcal{G})$ on cohomology. Equipping both sides with the canonical filtrations $\tau_{\leq -\bullet}$ (cf. [Del71], 1.4.6), we obtain a morphism

$$H^0(Y,\mathcal{I}^\bullet) \to H^0(X,\mathcal{K}^\bullet)$$

of biregularly filtered complexes. By loc. cit., 1.4.5, this induces a morphism between the associated hypercohomology spectral sequences with respect to the functor $T = H^0(\Gamma,-)$. As in loc. cit., Exemple 1.4.8, one shows that, after reindexing by $E_1^{pq} \mapsto E_2^{2p+q,-p}$, these hypercohomology spectral sequences coincide with the equivariant cohomology spectral sequences (B.4) for $\mathcal{G}$ and $f^*\mathcal{G}$, respectively. □

B.1. **Localization sequence.** Let $(X,\mathcal{R}_X)$ be ringed space equipped with a left Γ-action. Let $i : Z \hookrightarrow X$ denote the inclusion of a Γ-stable closed subspace and let $j : U = X \setminus Z \hookrightarrow X$ denote the inclusion of the complementary open subspace. It follows that $U \subset X$ is also Γ-stable. We write $i : (Z, i^{-1}\mathcal{R}_X) \to (X,\mathcal{R}_X)$ and $j : (U, j^{-1}\mathcal{R}_X) \to (X,\mathcal{R}_X)$ for the induced morphisms of ringed spaces and we denote the pullback functors by i^{-1} and j^{-1}. The ringed spaces $(Z, i^{-1}\mathcal{R}_X)$ and $(U, j^{-1}\mathcal{R}_X)$ acquire induced Γ-actions and the morphisms i and j are Γ-equivariant. If $j : U \hookrightarrow X$ is the inclusion of a Γ-stable open subset, there are canonical isomorphisms $\gamma^{-1}j_! \simeq j_!\gamma^{-1}$. These allow to promote the usual $j_!$ functor to a functor

$$j_! : \mathrm{Mod}_\Gamma(j^{-1}\mathcal{R}_U) \to \mathrm{Mod}_\Gamma(i^{-1}\mathcal{R}_X)$$

on the level of equivariant sheaves and one checks that it is left adjoint to the functor j^{-1} : $\mathrm{Mod}_\Gamma(\mathcal{R}_X) \to \mathrm{Mod}_\Gamma(j^{-1}\mathcal{R}_X)$. As it can be checked on the underlying sheaves, the counit of adjunction

$$j_!j^{-1}\mathcal{F} \to \mathcal{F}$$

is injective. For every $\mathcal{R}_X$-module, we define as usual

$$H^0_Z(X,\mathcal{F}) = \ker(H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F})).$$

If $\mathcal{F}$ is a Γ-equivariant $\mathcal{R}_X$-module, the restriction map $H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F})$ is Γ-equivariant and hence $H^0_Z(X,\mathcal{F})$ is a Γ-module. In this case, we define

$$H^0_Z(X,\Gamma;\mathcal{F}) = H^0(\Gamma, H^0_Z(X,\mathcal{F})).$$

Observe that the canonical isomorphism $\mathrm{Hom}_{\mathcal{R}_X}(\mathcal{R}_X,\mathcal{F}) = H^0(X,\mathcal{F})$ for $\mathcal{R}_X$-modules $\mathcal{F}$ induces a canonical isomorphism

$$\mathrm{Hom}_\Gamma(\mathcal{R}_X,\mathcal{F}) \cong H^0(X,\Gamma,\mathcal{F}). \tag{B.7}$$

Here Hom_Γ denotes the morphisms in the category $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$. Let $\mathcal{I}$ be an injective object in $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$. Applying $\mathrm{Hom}(-,\mathcal{I})$ to the injective Γ-equivariant homomorphism $j_!j^{-1}\mathcal{R}_X \hookrightarrow \mathcal{R}_X$, we obtain a surjective homomorphism

$$\mathrm{Hom}_\Gamma(\mathcal{R}_X,\mathcal{I}) \to \mathrm{Hom}_\Gamma(j_!j^{-1}\mathcal{R}_X,\mathcal{I}).$$

By (B.7) and adjunction, this homomorphism identifies with the restriction map

$$H^0(X,\Gamma;\mathcal{I}) \to H^0(U,\Gamma,\mathcal{I}).$$

Hence, for any injective object $\mathcal{I}$ in $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$, we have the short exact sequence

$$0 \to H^0_Z(X,\Gamma;\mathcal{I}) \to H^0(X,\Gamma;\mathcal{I}) \to H^0(U,\Gamma;\mathcal{I}) \to 0. \tag{B.8}$$

Proposition B.10 (Localization sequence)**.** *Let $(X,\mathcal{R}_X)$ be a ringed space with left Γ-action. Let Z be a Γ-invariant closed subspace and let $U = X \setminus Z$ be its open complement. For every Γ-equivariant $\mathcal{R}_X$-module, there is a functorial long exact sequence*

$$\ldots \to H^n_Z(X,\mathcal{F}) \to H^n(X,\mathcal{F}) \to H^n(U,\mathcal{F}) \to H^{n+1}_Z(X,\mathcal{F}) \to \ldots.$$

Proof. Choose an injective resolution $\mathcal{F} \hookrightarrow \mathcal{I}^\bullet$ in $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$. By (B.8), we obtain the short exact sequence of complexes

$$0 \to H^0_Z(X,\Gamma;\mathcal{I}^\bullet) \to H^0(X,\Gamma;\mathcal{I}^\bullet) \to H^0(U,\Gamma;\mathcal{I}^\bullet) \to 0.$$

Passing to cohomology groups then yields the desired exact sequence. The functoriality is clear. □

Another consequence of the above is that there is also an equivariant cohomology spectral sequence computing $H^n_Z(X,\Gamma;-)$:

Proposition B.11. *If $\mathcal{I}$ is an injective object in $\mathrm{Mod}_\Gamma(\mathcal{R}_X)$, then $H^0_Z(X,\mathcal{I})$ is $H^0(\Gamma,-)$-acyclic. In particular, for every $\mathcal{F} \in \mathrm{Mod}_\Gamma(\mathcal{R}_X)$, there is a convergent spectral sequence*

$$E_2^{pq} = H^p(\Gamma, H^q_Z(X,\mathcal{F})) \Rightarrow H^{p+q}_Z(X,\Gamma;\mathcal{F}),$$

which is functorial in $\mathcal{F}$.

Proof. As $\mathcal{I}$ is in particular an injective $\mathcal{R}_X$-module, one checks as above that

$$0 \to H^0_Z(X,\mathcal{I}) \to H^0(X,\mathcal{I}) \to H^0(U,\mathcal{I}) \to 0 \tag{B.9}$$

is an exact sequence of Γ-modules. By Proposition B.3 (ii), $H^0(X,\mathcal{I})$ and $H^0(U,\mathcal{I})$ are $H^0(\Gamma,-)$-acyclic. We have seen above in (B.8) that the map $H^0(X,\Gamma;\mathcal{I}) \to H^0(U,\Gamma;\mathcal{I})$ is surjective. Passing to the long exact sequence associated to (B.9) shows that $H^n(\Gamma, H^0_Z(X,\mathcal{I})) = 0$ for all $n > 0$, as desired. □

Lemma B.12. *Let $f : (X',\mathcal{R}_{X'}) \to (X,\mathcal{R}_X)$ be a Γ-equivariant morphism of ringed spaces. Let $\mathcal{I}$ be an injective object in $\mathrm{Mod}_\Gamma(\mathcal{R}_{X'})$. Then $f_*\mathcal{I}$ is $H^0_Z(X,\Gamma;-)$-acyclic.*

Proof. Let $U' = f^{-1}(U)$ and $Z' = f^{-1}(Z)$. Note that U' and Z' are Γ-invariant since f is Γ-equivariant. In the usual localization sequence

$$0 \to H^0_Z(X,f_*\mathcal{I}) \to H^0(X,f_*\mathcal{I}) \to H^0(U,f_*\mathcal{I}) \to H^1_Z(X,f_*\mathcal{I}) \to H^1(X,f_*\mathcal{I}) \to \ldots$$

one has $H^n(X,f_*\mathcal{I}) = H^n(U,f_*\mathcal{I}) = 0$ for all $n > 0$ since $f_*\mathcal{I}$ is flasque by Proposition B.3 (ii). Moreover, for the same reason, the map $H^0(X,f_*\mathcal{I}) = H^0(X',\mathcal{I}) \to H^0(U,f_*\mathcal{I}) = H^0(U',\mathcal{I})$ is surjective. Hence $H^n_Z(X,f_*\mathcal{I}) = 0$ for all $n > 0$. Therefore the spectral sequence

$$E_2^{pq} = H^p(\Gamma, H^q_Z(X,f_*\mathcal{I})) \Rightarrow H^{p+q}_Z(X,\Gamma;f_*\mathcal{I}).$$

from Proposition B.11 degenerates and

$$H^n(X,\Gamma;f_*\mathcal{I}) \cong H^n(\Gamma, H^0_Z(X,f_*\mathcal{I})).$$

Now, the Γ-module $H^0_Z(X,f_*\mathcal{I}) = H^0_{Z'}(X',\mathcal{I})$ is $H^0(\Gamma,-)$-acylic by Proposition B.11 and we conclude. □

In particular, just as in Remark B.7, the abelian groups $H^n_Z(X,\Gamma;\mathcal{F})$ can be calculated by regarding $\mathcal{F}$ as an equivariant abelian sheaf.

Lemma B.13. *Let $f : (X,\mathcal{R}_X) \to (X',\mathcal{R}_{X'})$ be a Γ-equivariant morphism of ringed spaces. Let $Z' \subset Y$ be a Γ-invariant closed subset with open complement $U' \subset X'$. Let $Z = f^{-1}(Z')$ and $U = f^{-1}(U')$. Let $\mathcal{G}$ be a Γ-equivariant $\mathcal{R}_{X'}$-module. Then the homomorphisms (B.6) induce a morphism*

$$\begin{tikzcd}
H^n_{Z'}(X',\Gamma;\mathcal{G}) \arrow[r] \arrow[d] & H^n(X',\Gamma;\mathcal{G}) \arrow[r] \arrow[d] & H^n(U',\Gamma;\mathcal{G}) \arrow[r] \arrow[d] & H^{n+1}_{Z'}(X',\Gamma;\mathcal{G}) \arrow[d] \\
H^n_Z(X,\Gamma;f^*\mathcal{G}) \arrow[r] & H^n(X,\Gamma;f^*\mathcal{G}) \arrow[r] & H^n(U,\Gamma;f^*\mathcal{G}) \arrow[r] & H^{n+1}_Z(X,\Gamma;f^*\mathcal{G})
\end{tikzcd}$$

of exact sequences of abelian groups.

Proof. By the above, Remark B.7 and the construction of (B.6), we may assume that $\mathcal{R}_X = \mathbb{Z}_X$ and $\mathcal{R}_Y = \mathbb{Z}_Y$. Choose injective resolutions $\mathcal{G} \to \mathcal{I}^\bullet$ and $f^{-1}\mathcal{I}^\bullet \to \mathcal{J}^\bullet$. One immediately checks that one has a commutative diagram with exact rows

$$\begin{CD} 0 @>>> H^0_{Z'}(X',\Gamma;\mathcal{I}^\bullet) @>>> H^0(X',\Gamma;\mathcal{I}^\bullet) @>>> H^0(U',\Gamma;\mathcal{I}^\bullet) @>>> 0 \\ @. @VVV @VVV @VVV \\ 0 @>>> H^0_{Z}(X,\Gamma;\mathcal{J}^\bullet) @>>> H^0(X,\Gamma;\mathcal{J}^\bullet) @>>> H^0(U,\Gamma;\mathcal{J}^\bullet) @>>> 0, \end{CD}$$

and the claim follows by passing to long exact sequences. □

Appendix C. Eisenstein-Kronecker classes and base change

In this section we review the construction of the Eisenstein-Kronecker classes of Kings-Sprang (cf. [KS19]).

C.1. **Infinitesimal neighbourhoods.** Let us begin with a remark on sheaves which are supported on a closed subscheme.

Remark C.1. In the following, let all schemes and morphisms be defined over a fixed base scheme S. Let $Z \subset X$ be a closed subscheme corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}$ and denote the associated closed immersion by $i : Z \hookrightarrow X$.

(i) Let $\mathrm{Sh}_Z(X)$ denote the full subcategory of $\mathrm{Sh}(X)$ consisting of the abelian sheaves $\mathcal{F}$ on X whose support is contained in Z. There is an equivalence of categories
$$i^{-1} : \mathrm{Sh}_Z(X) \xrightarrow{\sim} \mathrm{Sh}(Z)$$
with quasi-inverse i_*.

(ii) If $\mathcal{F} \in \mathrm{Sh}_Z(X)$ had the structure of an $\mathcal{O}_X$-module (resp. algebra), then $i^{-1}\mathcal{F}$ admits the structure of a $i^{-1}\mathcal{O}_X$-module (resp. algebra). Let $\pi : X \to S$ and $\pi_Z : Z \to S$ denote the structural morphisms. The restriction $i^{-1}\mathcal{F}$ acquires the structure of an $\pi_Z^{-1}\mathcal{O}_S$-module (resp. algebra) via the canonical ring homomorphism
$$\pi_Z^{-1}\mathcal{O}_S \to i^{-1}\mathcal{O}_X$$
obtained by applying the functor i^{-1} to the canonical homomorphism $\pi^\# : \pi^{-1}\mathcal{O}_S \to \mathcal{O}_X$. In particular, in the case where $i : S \to X$ is a section of π, there is a canonical ring homomorphism $\mathcal{O}_S \to i^{-1}\mathcal{O}_X$ and $i^{-1}\mathcal{F}$ is equipped with the structure of a $\mathcal{O}_S$-module (resp. algebra). Moreover, one has $\pi_* = i^{-1} : \mathrm{Sh}_Z(X) \xrightarrow{\sim} \mathrm{Sh}(Z)$ in this case.

(iii) Let $\mathrm{Mod}_Z(\mathcal{O}_X)$ denote the full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$ consisting of those $\mathcal{O}_X$-modules such that $\mathcal{I} \cdot \mathcal{F} = 0$. There is an equivalence of categories
$$i^{-1} = i^* : \mathrm{Mod}_Z(\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{O}_Z)$$
with quasi-inverse i_*.

Let X be a scheme. Let $i : Z \hookrightarrow X$ be a closed immersion with corresponding quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For each integer $n \geq 0$, the quasi-coherent $\mathcal{O}_X$-module
$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^{n+1}$$
has support in Z. We usually identify it with its restriction to Z and regard it as an $i^{-1}\mathcal{O}_X$-module via the equivalence of Remark C.1. The *nth infinitesimal thickening of Z in X* is the ringed space
$$X_n := (Z, \mathcal{O}_X/\mathcal{I}^{n+1}).$$
It is a scheme which canonically identifies with the relative spectrum $\underline{\mathrm{Spec}}(\mathcal{O}_X/\mathcal{I}^{n+1})$; we denote the associated closed immersion by $i_n : X_n \hookrightarrow X$. Let $f : X \to Y$ be a morphism of schemes and $Z \subset X$ and $W \subset Y$ closed subschemes fitting into a commutative diagram

$$\begin{CD} Z @>{i}>> X \\ @V{g}VV @VV{f}V \\ W @>{j}>> Y. \end{CD} \tag{C.1}$$

Let $\mathcal{I} \subset \mathcal{O}_X$ (resp. $\mathcal{K} \subset \mathcal{O}_Y$) be the quasi-coherent sheaves of ideals corresponding to $Z \subset X$ and $W \subset Y$. Then $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{I}$ and hence $f^*(\mathcal{K}^{n+1})\mathcal{O}_X \subset \mathcal{I}^{n+1}$ for all integers $n \geq 0$. This induces a homomorphism

$$f^*(\mathcal{O}_Y/\mathcal{K}^{n+1}) \to \mathcal{O}_X/\mathcal{I}^{n+1} \tag{C.2}$$

Hence, for any $n \geq 0$, we obtain an induced morphism

$$f_n : X_n \to Y_n$$

and by construction, the diagram

$$\begin{tikzcd} X_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ Y_n \arrow[r, "j_n"] & Y \end{tikzcd} \tag{C.3}$$

is commutative. If the square (C.1) is Cartesian, the induced map (C.2) is an isomorphism and hence also the square (C.3) is Cartesian.

Lastly, let us discuss the infinitesimal behaviour of the mutliplication map of a group scheme. Let S be a scheme and let $\pi : G \to S$ be a quasi-compact separated group scheme with unit section $e : S \to G$ and multiplication map $\mu : G \times_S G \to G$. Then e is a closed immersion and, for every integer $n \geq 0$, we let G_n denote the nth infinitesimal thickening of e in G. The unit section factors as $e : S \to G_n$. For all integers $n, m, k \geq 0$, the composite $G_n \times_S G_m \hookrightarrow G \times_S G \xrightarrow{\mu} G$ preserves the unit sections and we get an induced map

$$(G_n \times_S G_m)_k \to G_k,$$

where $(G_n \times_S G_m)_k$ denotes the kth infinitesimal thickening of $e : S \to G_n \times_S G_m$. We claim that

$$(G_n \times_S G_m)_k = G_n \times_S G_m$$

for all $k \geq n + m$. Since everything is local on the base S, we may assume that $S = \mathrm{Spec}(R)$ is an affine scheme. Let $\mathcal{I} \subset \mathcal{O}_G$ denote the sheaf of ideals associated to e. Then $e^{-1}\mathcal{O}_G/\mathcal{I}^{n+1} = \pi_*(\mathcal{O}_G/\mathcal{I}^{n+1})$ is a quasi-coherent $\mathcal{O}_S$-algebra and of the form $\widetilde{A_n}$ for an R-algebra A_n. Moreover, one has $G_n = \mathrm{Spec}(A_n)$. The closed immersion $e : S \to G_n$ corresponds to an ideal $I_n \subset A_n$ such that $\widetilde{I_n} = e^{-1}\mathcal{I}/\mathcal{I}^{n+1}$. In particular, one has $\mathcal{I}_n^{n+1} = 0$. With this notation, one has $G_n \times_S G_m = \mathrm{Spec}(A_n \otimes_R A_m)$ and the closed immersion $e : S \hookrightarrow G_n \times_S G_m$ corresponds to the ideal $I_n \otimes A_m + A_n \otimes I_m$. Since $I_n^{n+1} = I_m^{m+1} = 0$, it follows that

$$(I_n \otimes A_m + A_n \otimes I_m)^{n+m+1} = 0,$$

and the claim follows. In particular, for all integers $n, m \geq 0$, the multiplication map $\mu : G \times_S G \to G$ induces an S-morphism

$$G_n \times_S G_m \to G_{n+m}$$

on infinitesimal neighbourhoods.

C.2. **Formal completion of abelian schemes.** In the following let S denote a Noetherian base scheme. Let $\pi : A \to S$ be an abelian scheme and let $\pi^\vee : A^\vee \to S$ denote the dual abelian scheme. Let $\pi^\natural : A^\natural \to S$ be the universal vectorial extension of $A^\vee/S$. It sits in a short exact sequence of commutative S-group schemes

$$0 \to \omega_{A/S} \to A^\natural \to A^\vee \to 0.$$

There is a canonical isomorphism $\omega_{A^\natural/S} \cong H^1_{\mathrm{dR}}(A^\vee/S)$. Let $\mathcal{P}$ denote the Poincaré bundle on $A \times_S A^\vee$ and let $(\mathcal{P}^\natural, \nabla)$ denote the universal rigidified line bundle on $A \times_S A^\natural$, equipped with an integrable connection $\nabla : \mathcal{P}^\natural \to \Omega^1_{A\times A^\natural/A^\natural} \otimes_{\mathcal{O}_{A\times A^\natural}} \mathcal{P}^\natural$, which satisfies the theorem of the square, cf. [Lau96], section 2.

Notation C.2. *For any abelian scheme A/S, we introduce some notations which will be used throughout the following sections.*

(1) *In the following, all statements remain true if $A^\vee$ and $\mathcal{P}$ are replaced by $A^\natural$ and $\mathcal{P}^\natural$. For this reason, we restrict our treatment to $A^\vee$ and $\mathcal{P}$. Only in definitions, we briefly discuss the case of $A^\natural$ and $\mathcal{P}^\natural$ to fix notation.*

(2) *Let* A_n, $A_n^\vee$ *and* $A_n^\natural$ *denote the* n*th infinitesimal thickening of the unit sections* e, $e^\vee$ *and* $e^\natural$ *of* A, $A^\vee$ *and respectively* $A^\natural$. *Let* $i_n : A_n \to A$, $i_n^\vee : A_n^\vee \to A^\vee$ *and* $i_n^\natural : A_n^\natural \to A^\natural$ *denote the canonical closed immersions.*
(3) *We let* $\pi_n : A_n \to S$ *denote the composite* $A_n \xrightarrow{i_n} A \xrightarrow{\pi} S$. *Similarly, we define* $\pi_n^\vee$ *and* $\pi_n^\natural$.
(4) *If abelian schemes other than* A *make an appearance, we use subscripts to indicate which abelian scheme an object is associted to. E.g. if* A *and* B *are abelian schemes, then* $\mathcal{P}_A$ *and* $\mathcal{P}_B$ *denote the Poincaré bundles on* $A \times A^\vee$ *and* $B \times B^\vee$, *respectively.*
(5) *Throughout this section, we prove easy compatibility results with respect to base change. For this, let us fix the following setup: Let* $\sigma : S^\sigma \to S$ *denote a morphism of noetherian schemes. And let* $A^\sigma = A \times_{S,\sigma} S^\sigma$ *denote the abelian scheme obtained by base change of* A *along* σ. *By the universal properties of* $A^\vee$ *and* $A^\natural$, *there are canonical identifications*
$$A^{\vee,\sigma} \cong A^{\sigma,\vee} \quad \text{and} \quad A^{\natural,\sigma} \cong A^{\sigma,\natural},$$
which we will use freely. In particular, we have Cartesian squares

(C.4)
$$\begin{array}{ccc} A^\sigma & \xrightarrow{\sigma_A} & A \\ {\scriptstyle \pi_{A^\sigma}}\downarrow & & \downarrow{\scriptstyle \pi_A} \\ S^\sigma & \xrightarrow{\sigma} & S \end{array} \qquad \begin{array}{ccc} A^{\sigma,\vee} & \xrightarrow{\sigma_A^\vee} & A^\vee \\ {\scriptstyle \pi_{A^\sigma}^\vee}\downarrow & & \downarrow{\scriptstyle \pi_A^\vee} \\ S^\sigma & \xrightarrow{\sigma} & S \end{array} \qquad \begin{array}{ccc} A^{\sigma,\natural} & \xrightarrow{\sigma_A^\natural} & A^\natural \\ {\scriptstyle \pi_{A^\sigma}^\natural}\downarrow & & \downarrow{\scriptstyle \pi_A^\natural} \\ S^\sigma & \xrightarrow{\sigma} & S, \end{array}$$

Let $\mathcal{K} \subset \mathcal{O}_{A\times A^\vee}$ denote the sheaf of ideals associated with the closed immersion
$$\mathrm{id}_A \times e^\vee : A \hookrightarrow A \times_S A^\vee.$$
Similarly, one defines $\mathcal{K}^\natural$ in terms of $\mathrm{id} \times e^\natural$. Regarding $\mathcal{K}/\mathcal{K}^2$ as a quasi-coherent $\mathcal{O}_A$-module (cf. Remark C.1 (iii)), we have an isomorphism
$$\mathcal{K}/\mathcal{K}^2 \cong (\mathrm{id}_A \times e^\vee)^*\Omega^1_{A\times A^\vee/A} \cong \pi^*\omega_{A^\vee/S}$$
since $\mathrm{id}_A \times e^\vee$ is a regular closed immersion (being a section of a smooth separated morphism). Moreover, for any integer $n \geq 0$, there is a canonical isomorphism of $\mathcal{O}_A$-modules

(C.5)
$$\mathcal{K}^n/\mathcal{K}^{n+1} \cong \pi^* \operatorname{Sym}^n(\omega_{A^\vee/S}).$$

Definition C.3. *For any integer* $n \geq 0$, *we define*
$$\mathcal{O}_n := \mathcal{O}_{A\times A^\vee}/\mathcal{K}^{n+1},$$
regarded as a sheaf on A *(cf. Remark C.1 (i)). Similarly, we define* $\mathcal{O}_n^\natural := \mathcal{O}_{A\times A^\natural}/\mathcal{K}^{\natural,n+1}$

Note that $A \times_S A_n^\vee = \underline{\mathrm{Spec}}_A(\mathcal{O}_n)$ is the nth infinitesimal thickening of $\mathrm{id}_A \times e^\vee$. Note that $\mathrm{id} \times \pi_n^\vee : A \times A_n^\vee \to A$ is the identity on underlying topological spaces. In particular, it is an affine morphism, cf. [GD71], Lemme 4.2.2.1 and one has

(C.6)
$$\mathcal{O}_n = (\mathrm{id}_A \times \pi_n^\vee)_*\mathcal{O}_{A\times A_n^\vee}.$$

By Remark C.1 (ii), each $\mathcal{O}_n$ admits the structure of an $\mathcal{O}_A$-algebra. It follows that the short exact sequence
$$0 \to \mathcal{K}/\mathcal{K}^2 \to \mathcal{O}_1 \to \mathcal{O}_A \to 0$$
splits. In particular, there is a decomposition

(C.7)
$$\mathcal{O}_1 \cong \mathcal{O}_A \oplus \pi^*\omega_{A^\vee/S}$$

of $\mathcal{O}_A$-modules. If one equips the right-hand side with the "square-zero" ring structure, this becomes an isomorphism of $\mathcal{O}_A$-algebras.

Lemma C.4 (Base change)**.** *Assume the setup of Notation C.2 (5).*
(i) *For every integer* $n \geq 0$, *there is a canonical isomorphism*
$$\sigma_A^*\mathcal{O}_{A,n} \cong \mathcal{O}_{A^\sigma,n}$$
of $\mathcal{O}_{A^\sigma}$*-algebras.*
(ii) *For every* $n \geq 0$, *there is canonical isomorphism*
$$\sigma_A^*(\mathcal{K}_A^n/\mathcal{K}_A^{n+1}) \cong \mathcal{K}_{A^\sigma}^n/\mathcal{K}_{A^\sigma}^{n+1}$$
of $\mathcal{O}_{A^\sigma}$*-modules.*

(iii) *Using (i) and the canonical isomorphism* $\sigma_A^*\pi_A^*\omega_{A^\vee/S} \cong \pi_{A^\sigma}^*\omega_{A^{\sigma,\vee}/S^\sigma}$*, the splitting*

$$\mathcal{O}_{A^\sigma,1} \cong \mathcal{O}_{A^\sigma} \oplus \pi_{A^\sigma}^*(\omega_{A^{\sigma,\vee}/S^\sigma})$$

is obtained by applying σ_A^* *to the splitting (C.7).*

Proof. (i) By passing to infinitesimal thickenings, the Cartesian square

$$\begin{tikzcd} A^\sigma \arrow[r, "\sigma_A"] \arrow[d, "\mathrm{id}_{A^\sigma}\times e^\vee_{A^\sigma}"'] & A \arrow[d, "\mathrm{id}_A\times e^\vee_A"] \\ A^\sigma \times A^{\sigma,\vee} \arrow[r, "\sigma_A\times\sigma_{A^\vee}"] & A\times A^\vee \end{tikzcd}$$

gives rise to a Cartesian square

$$\begin{tikzcd} A^\sigma \times A_n^{\sigma,\vee} \arrow[r, "\sigma_A\times\sigma^\vee_{A,n}"] \arrow[d, "\mathrm{id}_{A^\sigma}\times\pi^\vee_{A^\sigma,n}"'] & A\times A_n^\vee \arrow[d, "\mathrm{id}_A\times\pi^\vee_{A,n}"] \\ A^\sigma \arrow[r, "\sigma_A"] & A, \end{tikzcd}$$

in which the vertical morphisms are affine. By (C.6) and affine base change, it follows that

$$\begin{aligned}\sigma_A^*\mathcal{O}_{A,n} &\cong \sigma_A^*(\mathrm{id}\times\pi^\vee_{A,n})_*\mathcal{O}_{A\times A_n^\vee}\\ &\cong (\mathrm{id}_{A^\sigma}\times\pi^\vee_{A^\sigma,n})_*\mathcal{O}_{A^\sigma\times A_n^{\sigma,\vee}}\\ &\cong \mathcal{O}_{A^\sigma,n}.\end{aligned}$$

Assertion (ii) follows from the chain of canonical isomorphisms

$$\begin{aligned}\sigma_A^*(\mathcal{K}_A^n/\mathcal{K}_A^{n+1}) &\cong \sigma_A^*\pi_A^*\,\mathrm{Sym}^n(\omega_{A^\vee/S})\\ &\cong \pi_{A^\sigma}^*\sigma^*\,\mathrm{Sym}^n(\omega_{A^\vee/S})\\ &\cong \pi_{A^\sigma}^*\,\mathrm{Sym}^n(\omega_{A^{\sigma,\vee}/S^\sigma})\\ &\cong \mathcal{K}^n_{A^\sigma}/\mathcal{K}^{n+1}_{A^\sigma}.\end{aligned}$$

Assertion (iii) is an immediate consequence of (i) and (ii). □

As explained at the end of Section C.2, the group scheme structure on $A^\vee$ induces for every $n, m \geq 0$ cocommutative comultiplication maps

$$\mathcal{O}_{n+m} \to \mathcal{O}_n \otimes_{\mathcal{O}_A} \mathcal{O}_m. \tag{C.8}$$

Definition C.5 (Moment map)**.** *For every integer* $n \geq 0$*, the maps (C.8) and the splitting (C.7) induce a* $\mathcal{O}_A$*-linear map*

$$\mathrm{mom} : \mathcal{O}_n \to \mathrm{TSym}^n(\mathcal{O}_1) \cong \bigoplus_{b=0}^n \pi^*\,\mathrm{TSym}^b(\omega_{A^\vee/S}),$$

called the moment map*. It is an* $\mathcal{O}_A$*-algebra homomorphism if the right-hand side regarded as a quotient of the tensor symmetric algebra. Pulling back along the unit section gives an* $\mathcal{O}_S$*-algebra homomorphism*

$$\mathrm{mom}_e : e^*\mathcal{O}_n \to \bigoplus_{b=0}^n \mathrm{TSym}^b(\omega_{A^\vee/S}).$$

Analogously, one defines mom *and* mom_e *for* $A^\natural$*.*

Lemma C.6 (Base change)**.** *Assume the setup of Notation C.2 (5). Then, under the identification* $\sigma_A^*\pi_A^*\,\mathrm{TSym}^b(\omega_{A^\vee/S}) \cong \pi_{A^\sigma}^*\,\mathrm{TSym}^b(\omega_{A^{\sigma,\vee}/S^\sigma})$ *and Lemma C.4 (i), one has*

$$\sigma_A^*(\mathrm{mom}_A) = \mathrm{mom}_{A^\sigma}$$

and $\sigma^*(\mathrm{mom}_{A,e}) = \mathrm{mom}_{A^\sigma,e}$

Proof. Since the group scheme structure $A^{\sigma,\vee} \times_{S^\sigma} A^{\sigma,\vee} \to A^{\sigma,\vee}$ canonically identifies with the base change of $A^\vee \times_S A^\vee \to A^\vee$ along σ, one deduces that, for each $n, m \geq 0$, the comultiplication map $\mathcal{O}_{A^\sigma,n+m} \to \mathcal{O}_{A^\sigma,n} \otimes_{\mathcal{O}_{A^\sigma}} \mathcal{O}_{A^\sigma,m}$ is obtained from (C.8) by applying σ_A^* and using the identifications of Lemma C.4 (i). Moreover, by Lemma C.4 (iii), the splittings (C.7) for A and A^σ are compatible under σ_A^* after identifying $\sigma_A^*\pi_A^*\omega_{A^\vee/S} \cong \pi_{A^\sigma}^*\omega_{A^{\sigma,\vee}/S^\sigma}$. The assertion about mom_e is an immediate consequence. □

C.3. **The Poincaré bundle and its completion.** Let $\mathcal{P}$ denote the Poincaré bundle on $A \times_S A^\vee$. It comes equipped with a rigidification

$$\mathrm{rig} : \mathcal{O}_{A^\vee} \cong (e \times \mathrm{id}_{A^\vee})^*\mathcal{P}. \tag{C.9}$$

Let $(\mathcal{P}^\natural, \nabla_{\mathcal{P}^\natural})$ be the universal line bundle with integrable connection on $A \times_S A^\natural$. The line bundle $\mathcal{P}^\natural$ is the pullback of $\mathcal{P}$ along the canonical homomorphism $A \times A^\natural \to A \times A^\vee$ and its rigidification $\mathcal{O}_{A^\natural} \cong (e \times \mathrm{id}_{A^\natural})^*\mathcal{P}^\natural$ is induced by (C.9). For an integer $n \geq 0$, we define $\mathcal{P}_n := \mathcal{P} \otimes_{\mathcal{O}_{A\times A^\vee}} \mathcal{O}_{A\times A^\vee}/\mathcal{K}^{n+1}$, regarded as an $\mathcal{O}_A$-module. Note that

$$\mathcal{P}_n = (\mathrm{id}_A \times \pi_n^\vee)_*(\mathcal{P}|_{A\times A_n^\vee}). \tag{C.10}$$

Similarly, we define $\mathcal{P}_n^\natural$ and there is a canonical isomorphism $\mathcal{P}_n^\natural = (\mathrm{id}_A \times \pi_n^\natural)_*(\mathcal{P}^\natural|_{A\times A_n^\natural})$. For every $n \geq 0$, the connection ∇ on $\mathcal{P}^\natural$ restricts to a connection $\nabla|_{A\times A_n^\natural} : \mathcal{P}^\natural|_{A\times A_n^\natural} \to \Omega^1_{A\times A_n^\natural/A_n^\natural} \otimes \mathcal{P}^\natural|_{A\times A_n^\natural}$. Since $\Omega^1_{A\times A_n^\natural/A_n^\natural} \cong (\mathrm{id} \times \pi_n^\natural)^*\Omega^1_{A/S}$, one obtains connections

$$\nabla_n : \mathcal{P}_n^\natural \to \Omega^1_{A/S} \otimes_{\mathcal{O}_A} \mathcal{P}_n \tag{C.11}$$

via the projection formula. For every $n \geq 0$, the rigidification (C.9) gives rise to a canonical isomorphism

$$r_n : e^*\mathcal{O}_n \xrightarrow{\sim} e^*\mathcal{P}_n. \tag{C.12}$$

This follows from the descriptions (C.6), (C.10) and affine base change, cf. the proof of Lemma C.4. Via (C.12), the unit $1 \in \Gamma(S, e^*\mathcal{O}_n)$ induces a canonical section $\mathbf{1}_n \in \Gamma(S, e^*\mathcal{P}_n)$. Similarly, there are canonical isomorphisms $r_n^\natural : e^*\mathcal{O}_n^\natural \xrightarrow{\sim} e^*\mathcal{P}_n^\natural$ giving rise to canonical sections $\mathbf{1}_n^\natural \in \Gamma(S, e^*\mathcal{P}_n^\natural)$.

Lemma C.7 (Base change)**.** *Assume the setup of Notation C.2 (5) and let $n \geq 0$ be an integer.*

(i) *There is a canonical isomorphism $\sigma_A^*(\mathcal{P}_{A,n}) \cong \mathcal{P}_{A^\sigma,n}$.*
(ii) *Under $\sigma^* e_A^* \simeq e_{A^\sigma}^*\sigma_A^*$ and the identification of (i), one has $\sigma^*(r_{A,n}) = r_{A^\sigma,n}$.*
(iii) *The canonical map $\sigma^* : \Gamma(S, e_A^*\mathcal{P}_{A,n}) \to \Gamma(S^\sigma, e_{A^\sigma}^*\mathcal{P}_{A^\sigma,n})$ maps the canonical section $\mathbf{1}_{A,n}$ to $\mathbf{1}_{A^\sigma,n}$.*
(iv) *Via the $\natural$-version of (i), one has $\sigma_A^*(\nabla_{A,n}) = \nabla_{A^\sigma,n}$.*

Proof. The canonical isomorphism

$$(\sigma_A \times \sigma_A^\vee)^*\mathcal{P}_A \cong \mathcal{P}_{A^\sigma}, \tag{C.13}$$

obtained from the universal property of $\mathcal{P}_A$, is an isomorphism of rigidified line bundles. By using (C.10), assertion (i) then follows exactly as Proposition C.4 (i). Since (C.13) respects the rigidifications, we have $(\sigma_A^\vee)^*(\mathrm{rig}_A) = \mathrm{rig}_{A^\sigma}$, where we suppress the natural isomorphism $(\sigma_A^\vee)^*(e_A \times \mathrm{id}_{A^\vee})^* \simeq (e_{A^\sigma} \times \mathrm{id}_{A^{\sigma,\vee}})^*(\sigma_A \times \sigma_A^\vee)^*$ from the notation. By passing to infinitesimal neighbourhoods, we obtain

$$(\sigma_{A,n}^\vee)^*(\mathrm{rig}_A|_{A_n^\vee}) = \mathrm{rig}_{A^\sigma}|_{A_n^{\sigma,\vee}}. \tag{C.14}$$

We now freely use the identifications (C.6) and (C.10). Consider the diagram

$$\begin{tikzcd}
\sigma^* e_A^*(\mathrm{id}_A \times \pi_{A,n}^\vee)_*\mathcal{O}_{A\times A_n^\vee} \arrow[r, "\sigma^*(r_{A,n})"] \arrow[d, "\mathrm{BC}", "\sim"'] & \sigma^* e_A^*(\mathrm{id}_A \times \pi_{A,n}^\vee)_*\mathcal{P}_A|_{A\times A_n^\vee} \arrow[d, "\mathrm{BC}", "\sim"'] \\
\sigma^*(\pi_{A,n}^\vee)_*\mathcal{O}_{A_n^\vee} \arrow[r, "\sigma^*(\pi_{A,n}^\vee)_*(\mathrm{rig}_A|_{A_n^\vee})"] \arrow[d, "\mathrm{BC}", "\sim"'] & \sigma^*(\pi_{A,n}^\vee)_*(e_A \times \mathrm{id}_{A_n^\vee})^*\mathcal{P}_A|_{A\times A_n^\vee} \arrow[d, "\mathrm{BC}", "\sim"'] \\
(\pi_{A^\sigma,n}^\vee)_*(\sigma_{A,n}^\vee)^*\mathcal{O}_{A_n^\vee} \arrow[r, "(\pi_{A^\sigma,n}^\vee)_*(\sigma_{A,n}^\vee)^*(\mathrm{rig}_A|_{A_n^\vee})"] \arrow[d, "\sim"'] & (\pi_{A^\sigma,n}^\vee)_*(\sigma_{A,n}^\vee)^*(e_A \times \mathrm{id}_{A_n^\vee})^*\mathcal{P}_A|_{A\times A_n^\vee} \arrow[d, "\sim"'] \\
(\pi_{A^\sigma,n}^\vee)_*\mathcal{O}_{A_n^{\sigma,\vee}} \arrow[r, "(\pi_{A^\sigma,n}^\vee)_*(\mathrm{rig}_{A^\sigma}|_{A_n^{\sigma,\vee}})"] \arrow[d, "\mathrm{BC}", "\sim"'] & (\pi_{A^\sigma,n}^\vee)_*(e_{A^\sigma} \times \mathrm{id}_{A_n^{\sigma,\vee}})^*\mathcal{P}_{A^\sigma}|_{A^\sigma\times A_n^{\sigma,\vee}} \arrow[d, "\mathrm{BC}", "\sim"'] \\
e_{A^\sigma}^*(\mathrm{id}_{A^\sigma} \times \pi_{A^\sigma,n}^\vee)_*\mathcal{O}_{A^\sigma\times A_n^{\sigma,\vee}} \arrow[r, "r_{A^\sigma,n}"] & e_{A^\sigma}^*(\mathrm{id}_{A^\sigma} \times \pi_{A^\sigma,n}^\vee)_*\mathcal{P}_{A^\sigma}|_{A^\sigma\times A_n^{\sigma,\vee}},
\end{tikzcd}$$

in which we indicate by BC that the arrow is given by a base change transformation. The upper square commutes by definition of $r_{A,n}$, the upper middle square commutes by naturality of the base change transformation $\sigma^*(\pi^\vee_{A,n})_* \simeq (\pi^\vee_{A^\sigma,n})_*(\sigma^\vee_{A,n})^*$, the lower middle square commutes by (C.14) and the lower square commutes by definition of $r_{A^\sigma,n}$. Therefore the outer diagram commutes and one checks that the vertical composites coincides with the corresponding isomorphisms obtained by using $\sigma^* e_A^* \simeq e_{A^\sigma}^*(\sigma_A^\vee)^*$ followed by $(\sigma_A^\vee)^*(\mathrm{id}_A \times \pi^\vee_{A,n})_* \simeq (\mathrm{id}_{A^\sigma} \times \pi^\vee_{A^\sigma,n})_*(\sigma_A \times \sigma^\vee_{A,n})^*$. This shows assertion (ii). From this, it follows that the diagram

$$\begin{tikzcd}
\Gamma(S, e_A^*\mathcal{O}_{A,n}) \arrow[r, "\sigma^*"] \arrow[d, "r_{A,n}"'] & \Gamma(S^\sigma, e_{A^\sigma}^*\mathcal{O}_{A^\sigma,n}) \arrow[d, "r_{A^\sigma,n}"] \\
\Gamma(S, e_A^*\mathcal{P}_{A,n}) \arrow[r, "\sigma^*"] & \Gamma(S^\sigma, e_{A^\sigma}^*\mathcal{P}_{A^\sigma,n})
\end{tikzcd}$$

is commutative. The upper map is a ring homomorphism and hence, by construction, the lower arrow maps $\mathbf{1}_{A,n}$ to $\mathbf{1}_{A^\sigma,n}$. Analogously to the proof of (ii), assertion (iv) follows from the fact that the canonical isomorphism $(\sigma_A \times \sigma_A^\natural)^*\mathcal{P}_A^\natural \cong \mathcal{P}_{A^\sigma}^\natural$ is an isomorphism of line bundles with connection. □

C.4. **Isogenies.** Let S be a noetherian scheme. Let A/S and B/S be abelian schemes and let $\varphi : A \to B$ be an isogeny defined over S. There are induced S-isogenies $\varphi^\vee : B^\vee \to A^\vee$ and $\varphi^\natural : B^\natural \to A^\natural$.

Lemma C.8. *Let $\varphi : A \to B$ be an S-isogeny. Let $n \geq 0$ and $\varphi_n : A_n \to B_n$ denote the induced morphism between the infinitesimal neighbourhoods of the unit sections. If φ is étale, then φ_n is an isomorphism.*

Proof. Let $\mathcal{I}_A$ (resp. $\mathcal{I}_B$) denote the sheaf of ideals defining the unit section of A (resp. B). For any $n \geq 0$, the isogeny φ induces a $\mathcal{O}_A$-linear map $\varphi^*(\mathcal{O}_B/\mathcal{I}_B^{n+1}) \to \mathcal{O}_A/\mathcal{I}_A^{n+1}$. It suffices to show that, for any $n \geq 0$, the induced $\mathcal{O}_B$-linear homomorphism

$$\mathcal{O}_B/\mathcal{I}_B^{n+1} \to \varphi_*(\mathcal{O}_A/\mathcal{I}_A^{n+1})$$

is an isomorphism. Indeed, applying π_{B*} then gives the assertion. We proceed by induction. For $n = 0$, the above map identifies with the canonical isomorphism $e_{B*}\mathcal{O}_S \cong \varphi_* e_{A*}\mathcal{O}_S$. Let $n \geq 1$ and assume that the claim is proven for $n-1$. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_B^n/\mathcal{I}_B^{n+1} \arrow[r] \arrow[d] & \mathcal{O}_B/\mathcal{I}_B^{n+1} \arrow[r] \arrow[d] & \mathcal{O}_B/\mathcal{I}_B^n \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \varphi_*(\mathcal{I}_A^n/\mathcal{I}_A^{n+1}) \arrow[r] & \varphi_*(\mathcal{O}_A/\mathcal{I}_A^{n+1}) \arrow[r] & \varphi_*(\mathcal{O}_A/\mathcal{I}_A^n) \arrow[r] & 0
\end{tikzcd}$$

Under the isomorphisms $\mathcal{I}_B^n/\mathcal{I}_B^{n+1} \cong e_{B*}\operatorname{Sym}^n(\omega_{B/S})$ and $\varphi_*(\mathcal{I}_A^n/\mathcal{I}_A^{n+1}) \cong e_{B*}\operatorname{Sym}^n(\omega_{A/S})$, the right vertical map identifies with $e_{B*}\operatorname{Sym}^n(\varphi^*)$ where $\varphi^* : \omega_{B/S} \to \omega_{A/S}$ is the morphism induced by φ. The claim follows since, for étale φ, the map φ^* is an isomorphism. □

Proposition C.9. *For every integer $n \geq 0$, the isogeny φ induces a canonical homomorphism*

$$\varphi_{\#,n} : \mathcal{P}_{A,n} \to \varphi^*\mathcal{P}_{B,n}$$

If $\varphi^\vee$ is étale, the map $\varphi_{\#,n}$ is an isomorphism.

Proof. This is [KS19], Theorem 2.6. Let us recall the construction: By the universal property of the Poincaré bundle, there is a canonical isomorphism $(\mathrm{id}_A \times \varphi^\vee)^*\mathcal{P}_A \cong (\varphi \times \mathrm{id}_{B^\vee})^*\mathcal{P}_B$ and hence an isomorphism

$$(\mathrm{id}_A \times \varphi_n^\vee)^*\mathcal{P}_A|_{A\times A_n^\vee} \cong (\varphi \times \mathrm{id}_{B_n^\vee})^*\mathcal{P}_B|_{B\times B_n^\vee}. \tag{C.15}$$

Now $\varphi_{\#,n}$ is given by the composite

$$\begin{aligned}
\mathcal{P}_{A,n} &= (\mathrm{id}_A \times \pi^\vee_{A,n})_* \mathcal{P}_A|_{A\times A^\vee_n} \\
&\xrightarrow{\mathrm{adj}} (\mathrm{id}_A \times \pi^\vee_{A,n})_*(\mathrm{id}_A \times \varphi^\vee_n)_*(\mathrm{id}_A \times \varphi^\vee_n)^*\mathcal{P}_A|_{A\times A^\vee_n} \\
&\overset{(C.15)}{\cong} (\mathrm{id}_A \times \pi^\vee_{A,n})_*(\mathrm{id}_A \times \varphi^\vee_n)_*(\varphi \times \mathrm{id}_{B^\vee_n})^*\mathcal{P}_B|_{B\times B^\vee_n} \\
&\cong (\mathrm{id}_A \times \pi^\vee_{B,n})_*(\varphi \times \mathrm{id}_{B^\vee_n})^*\mathcal{P}_B|_{B\times B^\vee_n} \\
&\cong \varphi^*(\mathrm{id}_B \times \pi^\vee_{B,n})_*\mathcal{P}_B|_{B\times B^\vee_n} = \varphi^*\mathcal{P}_{B,n},
\end{aligned}$$

where the last isomorphism comes from affine base change. If $\varphi^\vee$ is étale, it follows from Lemma C.8 that φ_n is an isomorphism and hence the unit of adjunction adj is an isomorphism as well. $\square$

Let $n \geq 0$ be an integer and let $\varphi : A \to B$ be an isogeny over S with étale dual. By the above, we have the isomorphism $\varphi_{\#,n} : \mathcal{P}_{A,n} \xrightarrow{\sim} \varphi^*\mathcal{P}_{B,n}$. Together with (C.12) this induces an isomorphism

$$\mathcal{P}_{A,n}|_{\ker(\varphi)} \xrightarrow{\sim} \pi^*_{\ker(\varphi)} e^*_B \mathcal{O}_{B,n}, \tag{C.16}$$

where $\pi_{\ker(\varphi)}$ denotes the structure morphism of $\ker(\varphi)/S$.

Definition C.10. *Assumptions as above. Let $x : S \to A$ be a φ-torsion section.*

(i) *We define ϱ_x to be the composite isomorphism*

$$\varrho_x : x^*\mathcal{P}_{A,n} \xrightarrow{x^*(\varphi_{\#,n})} x^*\varphi^*\mathcal{P}_{B,n} \cong e^*_A\varphi^*\mathcal{P}_{B,n} \xrightarrow{e^*_A(\varphi^{-1}_{\#,n})} e^*_A\mathcal{P}_{A,n} \xrightarrow{r^{-1}_{A,n}} e^*_A\mathcal{O}_{A,n}.$$

(ii) *We define*

$$_{\varrho}\mathrm{mom}_x := \mathrm{mom}_e \circ \varrho_x : x^*\mathcal{P}_{A,n} \to \bigoplus_{i=0}^{n} \mathrm{TSym}^i(\omega_{A^\vee/S})$$

and for an integer $b \geq 0$ we define

$$_{\varrho}\mathrm{mom}^b_x : x^*\mathcal{P}_{A,n} \to \mathrm{TSym}^b(\omega_{A^\vee/S})$$

as the projection of $_{\varrho}\mathrm{mom}_x$ onto the b-component.

Lemma C.11 (Base change)**.** *Assume the setup of Notation C.2 (5).*

(i) *Under the canonical equivalence $\sigma^*_A\varphi^* \simeq \varphi^{\sigma,*}$ and Lemma C.7 (i), one has*

$$\sigma^*_A(\varphi_{\#,n}) = \varphi^\sigma_{\#,n}.$$

(ii) *Additionally using $\sigma^*x^* \simeq (x^\sigma)^*$, one has*

$$\sigma^*(\varrho_{A,x}) = \varrho_{A^\sigma,x^\sigma}.$$

(iii) *Under the above identifications and $\sigma^*\omega_{A^\vee/S} \cong \omega_{A^{\sigma,\vee}/S}$, one has*

$$\sigma^*(_{\varrho}\mathrm{mom}_{A,x}) = {_{\varrho}\mathrm{mom}_{A^\sigma,x^\sigma}},$$

and similarly for $_{\varrho}\mathrm{mom}^b_{A,x}$.

Proof. A proof of (i) is tedious to write out but essentially clear. The proof follows the usual pattern: By the universal property of the Poincaré bundle, one checks that the canonical isomorphism $(\mathrm{id}_A \times \varphi^\vee)^*\mathcal{P}_A \cong (\varphi \times \mathrm{id}_{B^\vee})^*\mathcal{P}_B$ is carried under $(\sigma_A \times \sigma^\vee_B)^*$ to the corresponding isomorphism for $\varphi^\sigma : A^\sigma \to B^\sigma$. Subsequently, one verifies all necessary compatibilities of e.g. the unit of the adjunction for $(-)^* \dashv (-)_*$ and base change transformations with base change along σ. Assertion (ii) follows from (i) and Lemma C.7 (ii). Finally, (iii) is a consequence of (ii) and Lemma C.6.

$\square$

C.5. **Vanishing of cohomology.** As before, let S be a noetherian scheme and $\pi : A \to S$ an abelian scheme with zero section e. Let Γ be a discrete group which acts from the left on A/S via automorphisms of group schemes. We denote the automorphism associated to an element $\gamma \in \Gamma$ by γ as well. Note that both π and e are Γ-equivariant morphisms if we equip S with the trivial Γ-action. As usual, let

$$\omega_{A/S} := e^*\Omega^1_{A/S}$$

denote the relative co-Lie algebra of A/S. For every $\gamma \in \Gamma$, we have a canonical $\mathcal{O}_A$-linear isomorphism $\gamma^*\Omega^1_{A/S} \to \Omega^1_{A/S}$ by basic functoriality properties of differentials. One immediately checks that this equips $\Omega^1_{A/S}$ with the structure of a Γ-equivariant $\mathcal{O}_A$-module. Moreover, we get an induced Γ-equivariant structure on $\omega_{A/S}$. We equip the dual abelian scheme $A^\vee/S$ with the left Γ-action given by

$$\Gamma \to \mathrm{Aut}(A^\vee/S), \gamma \mapsto (\gamma^{-1})^\vee.$$

As above, this canonically equips $\Omega^1_{A^\vee/S}$ with the structure of a Γ-equivariant $\mathcal{O}_{A^\vee}$-module and makes $\omega_{A^\vee/S}$ into a Γ-equivariant $\mathcal{O}_S$-module. Let $n \geq 0$ be an integer and let $\mathcal{P}_n$ be as in the previous section. For every $\gamma \in \Gamma$ Proposition C.9 then provides an isomorphism

$$(\gamma_{\#,n})^{-1} : \gamma^*\mathcal{P}_n \to \mathcal{P}_n$$

and one immediately checks that this equips $\mathcal{P}_n$ with the structure of a Γ-equivariant $\mathcal{O}_A$-module. In the following, we will freely make use of the Γ-equivariant identifications $\pi^*\omega_{A/S} \cong \Omega^1_{A/S}$ and

$$\omega^g_{A^\vee/S} \cong \Omega^g_{A/S} \cong \omega^g_{A/S},$$

cf. [Lau96], Lemme 1.1.3.

Before we state the next theorem, let us recall some facts about inverse systems: An $\mathbf{N}$-indexed inverse system (A_n) satisfies the *Mittag-Leffler condition*, or simply that it *is Mittag-Leffler*, if for every $n \in \mathbf{N}$ there exists $m \geq n$ such that $\mathrm{im}(A_k \to A_n) = \mathrm{im}(A_m \to A_n)$ for all $k \geq m$. Recall that an inverse system (A_n) is *ML-zero* if for any $n \in \mathbf{N}$ there is a $m > 0$ such that $A_{n+m} \to A_n$ is the zero map. The ML-zero systems form a Serre subcategory in the abelian category of all inverse systems. In particular, ML-zero systems are stable under extensions.

Lemma C.12. *Let*

$$(A_n) \to (B_n) \to (C_n) \to 0$$

be a short exact sequence of $\mathbf{N}$*-indexed inverse systems of abelian groups. If* (A_n) *is ML-zero and* (C_n) *satisfies the Mittag-Leffler condition, then* (B_n) *satisfies the Mittag-Leffler condition.*

Proof. Let $n \in \mathbf{N}$. By assumption, we may fix an integer $m > 0$ such that $A_{n+m} \to A_n$ is the zero map. Then we may fix an integer $k \geq 0$ such that $\mathrm{im}(C_N \to C_{n+m}) = \mathrm{im}(C_{n+m+k} \to C_{n+m})$ for all $N > n+m+k$. Fix $N > n+m+k$ and let $b_n \in \mathrm{im}(B_{n+m+k} \to B_n)$. We claim that $b_n \in \mathrm{im}(B_N \to B_n)$. We have the commutative diagram

$$\begin{tikzcd}
 & B_N \arrow[r, two heads] \arrow[d] & C_N \arrow[d] \\
 & B_{n+m+k} \arrow[r] \arrow[d] & C_{n+m+k} \arrow[d] \\
A_{n+m} \arrow[r] \arrow[d, "0"'] & B_{n+m} \arrow[r] \arrow[d, "f_n"'] & C_{n+m} \\
A_n \arrow[r] & B_n &
\end{tikzcd}$$

with $B_N \to C_N$ surjective. By assumption, there exists $b_{n+m+k} \in B_{n+m+k}$, which maps to $b_n \in B_n$. Let b_{n+m} denote the image of b_{n+m+k} in B_{n+m}. The image c_{n+m} of b_{n+m} in C_{n+m} lies in $\mathrm{im}(C_{n+m+k} \to C_{n+m})$. Hence, by choice of k, there is an element $c_N \in C_N$ which maps to c_{n+m}. Choose a lift $b_N \in B_N$. Let b'_{n+m} denote the image of b_N in B_{n+m}. Then, $b'_{n+m} - b_{n+m}$ maps to zero in C_{n+m}. Hence, it comes from an element $a_{n+m} \in A_{n+m}$. The commutativity of the lower left square implies that $f_n(b'_{n+m}) = f_n(b_n)$. But this shows that b_N maps to b_n. $\square$

The the construction of the equivariant Eisenstein-Kronecker classes relies on the following computation of the cohomology of the Poincaré bundle:

Theorem C.13. *Let S be an affine Noetherian scheme. Let g denote the relative dimension of the abelian scheme A/S. Let $n \geq 0$ be an integer.*

(i) *The homomorphism*
$$R^i\pi_*(\mathcal{P}_n \otimes \Omega^g_{A/S}) \to R^i\pi_*(\mathcal{P}_{n-1} \otimes \Omega^g_{A/S})$$
induced by the canonical map $\mathcal{P}_n \to \mathcal{P}_{n-1}$ is the zero map for all $i \neq g$ and an isomorphism for $i = g$. Moreover, $R^g\pi_*(\mathcal{P}_n \otimes \Omega^g_{A/S}) \cong \mathcal{O}_S$ via the canonical map $\mathcal{P}_n \to \mathcal{P}_0 = \mathcal{O}_A$ and the trace isomorphism $\mathrm{tr} : R^g\pi_*(\Omega^g_{A/S}) \xrightarrow{\sim} \mathcal{O}_S$.

(ii) *The transition maps in the inverse system $\{H^i(A, \mathcal{P}_n \otimes \Omega^g_{A/S})\}$ are all zero for $i < g$ and are all isomorphisms for $i = g$. One has*
$$\varprojlim_n H^i(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong \begin{cases} H^0(S, \mathcal{O}_S) & \text{if } i = g \\ 0 & \text{if } i < g. \end{cases}$$

(iii) *The inverse system $\{H^i(A, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S})\}$ is ML-zero for $i < g$. For $i = g$ it satisfies the Mittag-Leffler condition. One has*
$$\varprojlim_n H^i(A, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong \begin{cases} H^0(S, \mathcal{O}_S) & \text{if } i = g \\ 0 & \text{if } i < g. \end{cases}$$

Proof. Assertion (i) is proven exactly as Theorem 1.2.1 in [Sch14]. Let us briefly summarize the argument: By definition we have the exact sequence $0 \to \pi^* \mathrm{Sym}^n(\omega_{A^\vee/S}) \to \mathcal{P}_n \to \mathcal{P}_{n-1} \to 0$. We define the finite decreasing filtration $F^i\mathcal{P}_n := \ker(\mathcal{P}_n \to \mathcal{P}_{i-1})$. The graded pieces are given by $\mathrm{gr}^i_F \mathcal{P}_n = \pi^* \mathrm{Sym}^i(\omega_{A^\vee/S})$ for $0 \leq i \leq n$ and $\mathrm{gr}^i_F \mathcal{P}_n = 0$ otherwise. This induces a filtration on $\mathcal{P}_n \otimes \pi^*\omega^g_{A^\vee/S}$; we simply denote it by F^i and its graded pieces by gr^i_F. We obtain an associated spectral sequence

$$E_1^{pq} = R^{p+q}\pi_*(\mathrm{gr}^p_F) \Rightarrow R^{p+q}\pi_*(\mathcal{P}_n \otimes \pi^*\omega_{A^\vee/S}) \tag{C.17}$$

and we have $E_1^{pq} \cong R^{p+q}\pi_*(\pi^* \mathrm{Sym}^p(\omega_{A^\vee/S}) \otimes \pi^*\omega^g_{A^\vee/S}) \cong R^{p+q}\pi_*(\mathcal{O}_A) \otimes \mathrm{Sym}^p(\omega_{A^\vee/S}) \otimes \omega^g_{A^\vee/S}$ by the projection formula. Using $\pi^*\omega^g_{A^\vee/S} \cong \Omega^g_{A/S}$, Grothendieck duality and $R^i\pi_*(\mathcal{O}_A) \cong \Lambda^i R^1\pi_*\mathcal{O}_A \cong \Lambda^i \mathrm{Lie}(A^\vee/S)$, we calculate

$$\begin{aligned} R^{p+q}\pi_*(\mathcal{O}_A) \otimes \mathrm{Sym}^p(\omega_{A^\vee/S}) \otimes \omega^g_{A^\vee/S} &\cong R^{p+q}\pi_*(\Omega^g_{A/S}) \otimes \mathrm{Sym}^p(\omega_{A^\vee/S}) \\ &\cong R^{g-p-q}\pi_*(\mathcal{O}_A)^\vee \otimes \mathrm{Sym}^p(\omega_{A^\vee/S}) \\ &\cong \Lambda^{g-p-q}(\omega_{A^\vee/S}) \otimes \mathrm{Sym}^p(\omega_{A^\vee/S}). \end{aligned}$$

One can show that, under the above identification, the differentials $d_1^{pq} : E_1^{pq} \to E_1^{p+1,q}$ agree with the corresponding differentials in the Koszul complex

$$0 \to \Lambda^{g-q}(\omega_{A^\vee/S}) \to \Lambda^{g-q-1}(\omega_{A^\vee/S}) \otimes \mathrm{Sym}^1(\omega_{A^\vee/S}) \to \ldots \to \mathrm{Sym}^{g-q}(\omega_{A^\vee}) \to 0, \tag{C.18}$$

which locally is given as in [Lan02], chapter XXI, Corollary 4.14. By loc. cit., the Koszul complex is exact. In total, we have now seen that the qth line in the spectral sequence (C.17) is given by the naive cut-off $\sigma^{\leq n}$ of the Koszul complex (C.18) since $\mathrm{gr}^p_F = 0$ for $p > n$. From this it follows that the spectral sequence degenerates at the E_2-page and that the only non-zero terms are given by $E_2^{0,g} \cong \mathcal{O}_S$ and potentially the terms $E_2^{n,q} = (R^{n+q}\pi_*(\mathcal{O}_A) \otimes \mathrm{Sym}^n(\omega_{A^\vee/S}) \otimes \omega^g_{A^\vee/S}) / \mathrm{im}(d_1^{n-1,q})$ for $-n \leq q < g - n$. From this, one deduces (cf. [Sch14], Theorem 1.2.1):

- For all $i \neq g$, the inclusion $\pi^* \mathrm{Sym}^n(\omega_{A^\vee/S}) \hookrightarrow \mathcal{P}_n$ induces an isomorphism $R^i\pi_*(\mathcal{P}_n \otimes \pi^*\omega^g_{A^\vee/S}) \cong (R^i\pi_*(\mathcal{O}_A) \otimes \mathrm{Sym}^n(\omega_{A^\vee/S} \otimes \omega^g_{A^\vee/S}) / \mathrm{im}(d_1^{i-1,0})$.
- There is an isomorphism $R^g\pi_*(\mathcal{P}_n \otimes \pi^*\omega^g_{A^\vee/S}) \cong \mathcal{O}_S$ which is induced by the composite $\mathcal{P}_n \to \mathcal{P}_{n-1} \to \ldots \to \mathcal{P}_0 = \mathcal{O}_A$ and the trace isomorphism $\mathrm{tr} : R^g\pi_*(\pi^*\omega^g_{A^\vee/S}) \xrightarrow{\sim} \mathcal{O}_S$.

In particular, this implies assertion (i) since the composite $\mathrm{Sym}^n(\omega_{A^\vee/S}) \to \mathcal{P}_n \to \mathcal{P}_{n-1}$ is zero. By considering the Leray spectral sequence, assertion (ii) follows immediately from (i). Let us now explain how to obtain (iii): For every $n \geq 0$ we have the equivariant cohomology spectral sequence

$${}_nE_2^{pq} = H^p(\Gamma, H^q(A, \mathcal{P}_n \otimes \pi^*\omega^g_{A^\vee/S})) \Rightarrow H^{p+q}(A, \Gamma; \mathcal{P}_n \otimes \pi^*\omega^g_{A^\vee/S}). \tag{C.19}$$

Since the spectral sequence in functorial, we obtain a inverse system of spectral sequences. The property for an inverse system to have only the zero map or only isomorphisms as transition maps is passed on after applying any additive functor. Hence the transitions maps in $\{{}_nE_2^{pq}\}_n$ are zero if $q \neq g$ and isomorphisms if $q = g$. We claim that the same is true for the inverse system $\{{}_nE_r^{pq}\}_n$ for all $2 \leq r \leq \infty$. Assume that the validity of the claim is established for a fixed $r \geq 2$. One has

$$ {}_nE_{r+1}^{pq} = \frac{\ker({}_nd_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1})}{\operatorname{im}({}_nd_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{pq})}. $$

Assume first that $q \neq g$. The map ${}_nE_{r+1}^{pq} \to {}_{n-1}E_{r+1}^{pq}$ is zero since it is induced by the corresponding map on the E_r-page, which is the zero map by assumption. Let $q = g$. By assumption the system $\{{}_nE_r^{pq}\}_n$ is constant up to isomorphism and hence the differential ${}_nd_r^{pq}$ factors as ${}_nE_r^{pg} \to \varprojlim_m {}_mE_r^{p+r,g-r+1} \xrightarrow{\text{can}} {}_nE_r^{p+r,g-r+1}$. As $g - r + 1 < g$, the limit vanishes and hence $\ker({}_nd_r^{pg}) = {}_nE_r^{pg}$. Furthermore, for all $n \geq 0$, one even has ${}_nE_r^{p-r,g+r-1} = 0$ by the vanishing of the coherent cohomology of A in the degree $g+r-1 > g$. Hence $\operatorname{im}({}_nd_r^{p-r,g+r-1}) = 0$. It follows that

$$ {}_nE_{r+1}^{pg} = {}_nE_r^{pg} \tag{C.20} $$

for all $n \geq 0$ which, in particular, implies the claim for $2 < r < \infty$. As we are dealing with first quadrant spectral sequences, we have $E_\infty^{pq} = E_r^{pq}$ for sufficiently large r and the claim follows.

Let $i \geq 0$. For every $n \geq 0$, the spectral sequence (C.19) provides a filtration ${}_nF^p$ on $H^i(A, \Gamma; \mathcal{P}_n \otimes \pi^*\omega_{A^\vee/S}^g)$ whose graded pieces are given by

$$ {}_n\mathrm{gr}_F^p \cong {}_nE_\infty^{p,i-p}. $$

In particular, one has ${}_nF^0 = H^i(A, \Gamma; \mathcal{P}_n \otimes \pi^*\omega_{A^\vee/S}^g)$ and ${}_nF^{i+1} = 0$. For $i < g$, it follows by induction that the systems $\{{}_nF^p\}$ are ML-zero for all p. In particular $\{H^i(A, \Gamma; \mathcal{P}_n \otimes \Omega_{A/S}^g)\}$ is ML-zero for all $i < g$. For $i = g$, it follows by induction and Lemma C.12, that the inverse systems $\{{}_nF^p\}$ are ML-zero for $p > 0$ and that $\{{}_nF^0\} = \{H^g(A, \Gamma; \mathcal{P}_n \otimes \Omega_{A/S}^g)\}$ is Mittag-Leffler. Let us consider the filtration

$$ \widehat{F}^p := \varprojlim_n {}_nF^p $$

on $\varprojlim_n H^g(A, \Gamma; \mathcal{P}_n \otimes \pi^*\omega_{A^\vee/S}^g)$. Note that $\widehat{F}^\bullet$ is a finite filtration. Since $\{{}_nF^{p+1}\}$ is Mittag-Leffler, it follows that $\mathrm{gr}_{\widehat{F}}^p \cong \varprojlim_n {}_n\mathrm{gr}_F^p$. As $\{{}_nF^p\}$ is ML-zero for all $p > 0$, we have $\widehat{F}^p = \varprojlim_n {}_n\mathrm{gr}_F^p = 0$ for all $p > 0$. For $p = 0$, (C.20) implies that

$$ \varprojlim_n {}_n\mathrm{gr}_F^0 \cong \varprojlim_n {}_nE_\infty^{0g} = \varprojlim_n {}_nE_2^{0g} $$

Part (ii) of the Theorem implies

$$ \varprojlim_n {}_nE_2^{0g} = H^0(\Gamma, \varprojlim_n H^g(A, \mathcal{P}_n \otimes \Omega_{A/S}^g)) = H^0(S, \mathcal{O}_S). $$

For the latter isomorphism, we note that $\mathrm{tr} : R^g\pi_*(\Omega_{A/S}^g) \xrightarrow{\sim} \mathcal{O}_S$ (with $\mathcal{O}_S$ carrying the trivial action) is Γ-equivariant by basic functoriality properties of the trace. $\square$

C.6. **The local cohomology groups.** Now let $D \subset A$ be a Γ-stable closed subscheme which is finite étale over S. Let $U_D := A \setminus D$ denote the open complement. We define

$$ \begin{aligned} \mathcal{O}_S[D] &:= H^0(D, \mathcal{O}_D), \\ \mathcal{O}_S[D]^0 &:= \ker(H^0(D, \mathcal{O}_D) \xrightarrow{\mathrm{tr}} H^0(S, \mathcal{O}_S)), \end{aligned} $$

where tr denotes the trace homomorphism. Note that tr is Γ-equivariant is $H^0(S, \mathcal{O}_S)$ is equipped with the trivial action. Let $\mathcal{O}_S[D]^{0,\Gamma}$ denote the Γ-invariants of $\mathcal{O}_S[D]^0$. In the following, we will always assume that D is of the form $D = \ker \delta$ for an étale S-isogeny $\delta : A \to B$ with étale dual $\delta^\vee$. Consider the localization sequence

$$ \begin{aligned} &H^{g-1}(A, \Gamma, \mathcal{P}_n \otimes \Omega_{A/S}^g) \to H^{g-1}(U_D, \Gamma; \mathcal{P}_n \otimes \Omega_{A/S}^g) \to \\ &H_D^g(A, \Gamma; \mathcal{P}_n \otimes \Omega_{A/S}^g) \to H^g(A, \Gamma, \mathcal{P}_n \otimes \Omega_{A/S}^g) \end{aligned} $$

in equivariant cohomology, cf. Proposition B.10. By Theorem C.13 (iii), the inverse system $\{H^{g-1}(A,\Gamma,\mathcal{P}_n\otimes\Omega^g_{A/S})\}$ is ML-zero and it follows that the sequence

$$0\to\varprojlim_n H^{g-1}(U_D,\Gamma;\mathcal{P}_n\otimes\Omega^g_{A/S})\to\varprojlim_n H^g_D(A,\Gamma;\mathcal{P}_n\otimes\Omega^g_{A/S})\to\varprojlim_n H^g(A,\Gamma,\mathcal{P}_n\otimes\Omega^g_{A/S})\tag{C.21}$$

is exact. Fix an integer $n\geq 0$. Recall that D is of the form $\ker(\delta)$ for an isogeny $\delta:A\to B$ with étale dual. Let $\iota_D:D\hookrightarrow A$ denote the canonical inclusion and $\pi_D:D\to S$ the structure morphism. Let $\mathcal{J}=\mathcal{J}_D\subset\mathcal{O}_A$ denote the quasi-coherent sheaf of ideals associated to ι_D.

Proposition C.14. *For every $k\geq 0$, there is a canonical isomorphism*

$$\mathrm{Ext}^q_{\mathcal{O}_A}(\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})\cong\begin{cases}0 & \text{if } q<g\\ H^{q-g}(D,\underline{\mathrm{Hom}}_{\mathcal{O}_D}(\mathcal{J}^k/\mathcal{J}^{k+1},\iota_D^*\mathcal{P}_n) & \text{if } q\geq g.\end{cases}$$

In particular, there is canonical isomorphism

$$H^0(D,\iota_D^*\mathcal{P}_n)\cong\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J},\mathcal{P}_n\otimes\Omega^g_{A/S}).\tag{C.22}$$

Proof. For the details we refer the reader to [KS19], Proposition 2.29. There is a canonical isomorphism

$$\underline{\mathrm{Ext}}^q(\mathcal{J}^{k+1}/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})\cong\underline{\mathrm{Ext}}^q_{\mathcal{O}_A}(\iota_{D*}\mathcal{O}_D,\mathcal{P}_n\otimes\Omega^g_{A/S})\otimes(\mathcal{J}^k/\mathcal{J}^{k+1})^\vee.$$

The dualizing sheaf of the local complete intersection $\iota_D:D\hookrightarrow A$ (of relative codimension g) is isomorphic to $\omega_{D/A}\cong\iota_D^*(\Omega^g_{A/S})^\vee$ and by the fundamental local isomorphism in Grothendieck duality theory (cf. [Har66], III, Proposition 7.2), one obtains

$$\underline{\mathrm{Ext}}^q_{\mathcal{O}_A}(\mathcal{J}^{k+1}/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})\cong\begin{cases}0 & \text{if } q\neq g\\ \iota_{D*}\underline{\mathrm{Hom}}_{\mathcal{O}_D}(\mathcal{J}^k/\mathcal{J}^{k+1},\iota_D^*\mathcal{P}_n) & \text{if } q=g.\end{cases}$$

where here $\mathcal{J}^k/\mathcal{J}^{k+1}$ is regarded as a $\mathcal{O}_D$-module, cf. Remark C.1 (iii). The assertion now follows from the local-to-global spectral sequence

$$E_2^{pq}=H^p(A,\underline{\mathrm{Ext}}^q_{\mathcal{O}_A}(\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S}))\Rightarrow\mathrm{Ext}^{p+q}_{\mathcal{O}_A}(\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S}).$$

∎

Note that the quasi-coherent sheaf of ideals $\mathcal{J}$ canonically is a Γ-equivariant $\mathcal{O}_A$-module since ι_D is Γ-equivariant by assumption.

Corollary C.15. *For every $k\geq 0$, there are canonical isomorphisms*

$$\mathrm{Ext}^q_{\mathcal{O}_A}(\Gamma;\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})\cong\begin{cases}0 & \text{if } q<g\\ H^0(\Gamma,\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})) & \text{if } q=g.\end{cases}$$

Proof. This immediately follows from Proposition C.14 and the spectral sequence

$$E_2^{pq}=H^p(\Gamma,\mathrm{Ext}^q_{\mathcal{O}_A}(\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S}))\Rightarrow\mathrm{Ext}^{p+q}_{\mathcal{O}_A}(\Gamma;\mathcal{J}^k/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S}),$$

cf. [Gro57], Théorème 5.6.3. ∎

Corollary C.16.

(i) *For every $k\geq 0$ and $i<g$, one has*

$$\mathrm{Ext}^i_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}^k,\mathcal{P}_n\otimes\Omega^g_{A/S})=0.$$

(ii) *For every $k\geq 0$, the canonical map*

$$\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}^k,\mathcal{P}_n\otimes\Omega^g_{A/S})\to\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})$$

is injective. In particular, the canonical homomorphism

$$\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J},\mathcal{P}_n\otimes\Omega^g_{A/S})\to\varinjlim_k\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}^{k+1},\mathcal{P}_n\otimes\Omega^g_{A/S})\tag{C.23}$$

is injective.

Proof. This follows immediately by induction from Proposition C.14 and the long exact sequence for Ext associated to the short exact sequence
$$0 \to \mathcal{J}^k/\mathcal{J}^{k+1} \to \mathcal{O}_A/\mathcal{J}^{k+1} \to \mathcal{O}_A/\mathcal{J}^k \to 0.$$
□

Now, by [Gro68], Exp. II, Théorème 6, there is a canonical isomorphism
$$\varinjlim_k \operatorname{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}^{k+1}, \mathcal{P}_n \otimes \Omega^g_{A/S}) \xrightarrow{\sim} H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}). \tag{C.24}$$
It is constructed as follows: There is the canonical isomorphism (of ∂-functors)
$$\operatorname{Ext}^g_{\mathcal{O}_A}(\iota_{D*}\iota_D^{-1}\mathcal{O}_A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \xrightarrow{\sim} H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S})$$
and for every $k \geq 0$, there are canonical maps $\iota_{D*}\iota_D^{-1}\mathcal{O}_A \to \mathcal{O}_A/\mathcal{J}^{k+1}$, obtained by applying $\iota_{D*}\iota_D^{-1}$ to the canonical map $\mathcal{O}_A \to \mathcal{O}_A/\mathcal{J}^{k+1}$. Thus, we obtain maps $\operatorname{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}^{k+1}, \mathcal{P}_n \otimes \Omega^g_{A/S}) \to H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S})$ which are clearly compatible when varying k and hence assemble to a map as in (C.24).

Corollary C.17. *For every $i < g$, one has*
$$H^i_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) = 0.$$
It follows that
$$H^i_D(A, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong \begin{cases} H^0(\Gamma, H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S})) & \text{if } i = g \\ 0 & \text{if } i < g. \end{cases}$$

Proof. The first part follows from Corollary C.16 and the isomorphism (C.24). The second part then follows immediately from the equivariant cohomology spectral sequence
$$E_2^{pq} = H^p(\Gamma, H^q_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S})) \Rightarrow H^{p+q}_D(A, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}),$$
cf. Proposition B.11. □

Remark C.18. In Theorem C.13 (iii) it was shown that
$$\varprojlim_n H^g(A, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong \varprojlim_n H^0(\Gamma, H^g(A, \mathcal{P}_n \otimes \Omega^g_{A/S})).$$
By Corollary C.17, we also obtain
$$\varprojlim_n H^{g-1}(U_D, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong \varprojlim_n H^0(\Gamma, H^{g-1}(U_D, \mathcal{P}_n \otimes \Omega^g_{A/S}))$$
and the localization sequence (C.21) is obtained by applying $\varprojlim_n H^0(\Gamma, -)$ to the localization sequences
$$H^{g-1}(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \to H^{g-1}(U_D, \mathcal{P}_n \otimes \Omega^g_{A/S}) \to H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \to H^g(A, \mathcal{P}_n \otimes \Omega^g_{A/S}).$$

Remark C.19. A slightly different approach is to use continuous cohomology as defined in [Jan88]: For an inverse system of equivariant sheaves $\{\mathcal{F}_n\}$, define the groups $H^i(A, \Gamma; \{\mathcal{F}_n\})$ via the ith derived functor of the composite $\operatorname{Mod}_\Gamma(\mathcal{O}_A)^{\mathbf{N}} \xrightarrow{H^0(A,\Gamma;-)^{\mathbf{N}}} \mathrm{Ab}^{\mathbf{N}} \xrightarrow{\varprojlim} \mathrm{Ab}$, where $\operatorname{Mod}_\Gamma(\mathcal{O}_A)^{\mathbf{N}}$ denotes the abelian category of $\mathbf{N}$-indexed inverse system of Γ-equivariant $\mathcal{O}_A$-modules. Now, instead of considering the limit of cohomology groups, one may consider the cohomology in the respective inverse systems instead. Provided by the Grothendieck spectral sequence, there are various spectral sequence in this formalism, e.g. $H^p(\Gamma, \{H^q(A, \mathcal{F}_n)\}) \Rightarrow H^{p+q}(A, \Gamma; \{\mathcal{F}_n\})$ and $H^p(\Gamma, H^q(A, \{\mathcal{F}_n\})) \Rightarrow H^{p+q}(A, \Gamma; \{\mathcal{F}_n\})$, and one has the usual $\varprojlim^1$-sequences. The benefit of this approach is that one unconditionally has a localization sequence. Also, the proof of the result analogous to Theorem C.13 (iii) becomes way more clean. The vanishing results Theorem C.13 (ii) and Corollary C.17 imply that
$$H^g(A, \Gamma; \{\mathcal{P}_n \otimes \Omega^g_{A/S}\}) \cong \varprojlim_n H^0(\Gamma, H^g(A, \mathcal{P}_n \otimes \Omega^g_{A/S}))$$
$$H^g_D(A, \Gamma; \{\mathcal{P}_n \otimes \Omega^g_{A/S}\}) \cong \varprojlim_n H^0(\Gamma, H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}))$$
$$H^g(U_D, \Gamma; \{\mathcal{P}_n \otimes \Omega^g_{A/S}\}) \cong \varprojlim_n H^0(\Gamma, H^g(U_D, \mathcal{P}_n \otimes \Omega^g_{A/S})),$$

and this allows to reformulate the results of this Chapter in this language. On the other hand, it also suggests that there is nothing new gained from this perspective. In order to avoid introducing more terminology, we do not use this approach here, even though it seems to be slightly more natural to us.

C.7. **The equivariant coherent Eisenstein-Kronecker class.** As before, let $S = \mathrm{Spec}(R)$ be an affine Noetherian scheme and A/S an abelian scheme with a left Γ-action.

Corollary C.20. *There is a canonical injection*

$$H^0(D, \iota_D^*\mathcal{P}_n) \hookrightarrow H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}).$$

Proof. This follows from (C.22), Corollary C.16 (ii) and the isomorphism (C.24). □

By (C.16), we have an isomorphism

$$\iota_D^*\mathcal{P}_n \cong \pi_D^* e_B^* \mathcal{O}_{B,n}.$$

As $\mathcal{O}_B \subset \mathcal{O}_{B,n}$ is a direct summand, we obtain an $\mathcal{O}_D$-linear inclusion

$$\mathcal{O}_D = \pi_D^* e_B^* \mathcal{O}_B \subset \pi_D^* e_B^* \mathcal{O}_{B,n} \cong \iota_D^*\mathcal{P}_n, \tag{C.25}$$

which gives rise to an injection $\mathcal{O}_S[D] = H^0(D, \mathcal{O}_D) \subset H^0(D, \iota_D^*\mathcal{P}_n)$. In particular, we obtain an injection

$$H^0(D, \mathcal{O}_D) \hookrightarrow H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \tag{C.26}$$

for every $n \geq 0$.

Lemma C.21. *The digram*

$$\begin{tikzcd} H^0(D, \mathcal{O}_D) \arrow[r, "(C.26)"] \arrow[dr, "\mathrm{tr}"'] & H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \arrow[d] \\ & H^0(S, \mathcal{O}_S), \end{tikzcd}$$

where the vertical arrow is given by the composite $H^g_D(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \to H^g(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong H^0(S, \mathcal{O}_S)$*, is commutative.*

Proof. Let us only give a sketch: Since the isomorphism $H^g(A, \mathcal{P}_n \otimes \Omega^g_{A/S}) \cong H^0(S, \mathcal{O}_S)$ factors through $H^g(A, \mathcal{P}_n \otimes \Omega^g_{A/S})$, we readily reduce the statement to $n = 0$. Unwinding the construction of (C.26), one checks that the composite

$$H^0(D, \mathcal{O}_D) \to H^g_D(A, \Omega^g_{A/S}) \to H^g(A, \Omega^g_{A/S})$$

is induced by the trace morphism $\mathrm{Tr}_{\iota_D} : R\iota_{D*}\iota_D^\flat \mathcal{O}_A \to \mathcal{O}_A$ as defined in [Har66] III, §6 (note that the fundametal local isomorphism computes $\iota_D^\flat$ in the case of a local complete intersection). The lemma then follows by the compatibility of trace morphisms under composition. □

In view of Remark C.18, applying $\varprojlim_n H^0(\Gamma, -)$ to the injections (C.26) defines an injective homomorphism

$$H^0(D, \mathcal{O}_D)^\Gamma \hookrightarrow \varprojlim_n H^g_D(A, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}).$$

It follows from Lemma C.21, that we obtain an injective homomorphism

$$\mathcal{O}_S[D]^{0,\Gamma} \to \varprojlim_n \ker(H^g_D(A, \mathcal{P}_A \otimes \Omega^g_{A/S}) \to H^0(S, \mathcal{O}_S)) \cong \varprojlim_n H^{g-1}(U_D, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}).$$

Definition C.22 (Equivariant coherent Eisenstein-Kronecker class)**.** *We denote the above map by*

$$\mathrm{EK}_{\Gamma,A} : \mathcal{O}_S[D]^{0,\Gamma} \to \varprojlim_n H^{g-1}(U_D, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S}).$$

For $f \in \mathcal{O}_S[D]^{0,\Gamma}$, the element

$$\mathrm{EK}_{\Gamma,A}(f) \in \varprojlim_n H^{g-1}(U_D, \Gamma; \mathcal{P}_n \otimes \Omega^g_{A/S})$$

is called the equivariant coherent Eisenstein-Kronecker class associated to f. *Moreover, we define*

$$\mathrm{EK}^{\natural}_{A,\Gamma} : \mathcal{O}_S[D]^{0,\Gamma} \to \varprojlim_n H^{g-1}(U_D, \Gamma; \mathcal{P}^{\natural}_n \otimes \Omega^g_{A/S})$$

as the composite of $\mathrm{EK}_{A,\Gamma}$ *with the canonical homomorphism induced by* $\mathcal{P}_n \to \mathcal{P}^{\natural}_n$.

Let $a \geq 0$ be an integer. By iterating the integrable connection $\nabla = \nabla_{\mathcal{P}^{\natural}_n}$ on $\mathcal{P}^{\natural}_n$, one obtains a map

$$\nabla^a : \mathcal{P}^{\natural}_n \to \mathrm{TSym}^a(\Omega^1_{A/S}) \otimes_{\mathcal{O}_A} \mathcal{P}^{\natural}_n. \tag{C.27}$$

Remark C.23. The map (C.27) is obtained as follows: Let $\mathcal{T}_{A/S}$ denote the sheaf of $\mathcal{O}_S$-linear derivations on $\mathcal{O}_A$ and let $\mathcal{T}_{A/S} \to \mathrm{End}_{\pi^{-1}\mathcal{O}_S}(\mathcal{P}^{\natural}_n)$, $X \mapsto \nabla_X$ denote the usual covariant derivative associated to the connection ∇. One obtains an induced map $\mathrm{Sym}^a \mathcal{T}_{A/S} \to \mathrm{End}_{\pi^{-1}\mathcal{O}_S}(\mathcal{P}^{\natural}_n)$ by sending an element $X_1 \cdots X_a$ to $\nabla_{X_1} \circ \ldots \circ \nabla_{X_a}$. This is well-defined since the Lie algebra of A is abelian, so that $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} = 0$ for any $X, Y \in \mathcal{T}_{A/S}$. This induces the desired map (C.27).

Let $f \in \mathcal{O}_S[D]^{0,\Gamma}$. Via the map (C.27), we obtain an element

$$\nabla^a \mathrm{EK}^{\natural}_{\Gamma}(f) \in \varprojlim_n H^{g-1}(U_D, \Gamma; \mathrm{TSym}^a(\Omega^1_{A/S}) \otimes \mathcal{P}^{\natural}_n \otimes \Omega^g_{A/S}). \tag{C.28}$$

Finally, we specialize these classes along torsion sections. Let $\varphi : A \to B$ be an isogeny with étale dual and let $x \in U_D(S)$ a Γ-invariant φ-torsion section. Pulling back the class (C.28) along the equivariant map $x : S \to U_D$ and using the canonical isomorphism $\omega^g_{A/S} \cong \omega^g_{A^\vee/S}$ gives rise to a class

$$x^*\nabla^a \mathrm{EK}^{\natural}_{\Gamma}(f) \in \varprojlim_n H^{g-1}(S, \Gamma; \mathrm{TSym}^a(\omega_{A/S}) \otimes x^*\mathcal{P}^{\natural}_n \otimes \omega^g_{A/S})$$

Let $b \geq 0$ be an integer. There is a canonical isomorphism $\omega_{A^\natural/S} \cong \mathcal{H}_A$ where $\mathcal{H}_A := H^1_{\mathrm{dR}}(A^\vee/S)$. Recall the moment map $\mathrm{mom}^b_x : x^*\mathcal{P}^{\natural}_n \to \mathrm{TSym}^b(\mathcal{H}_A)$ from Definition C.5. This allows to define the class

$$\mathrm{mom}^b_x(x^*\nabla^a \mathrm{EK}^{\natural}_{\Gamma}(f)) \in H^{g-1}(S, \Gamma; \mathrm{TSym}^a(\omega_{A/S}) \otimes \mathrm{TSym}^b(\mathcal{H}_A) \otimes \omega^g_{A/S}).$$

Since S is affine, the equivariant cohomology spectral sequence degenerates and we obtain

$$\begin{aligned} &H^{g-1}(S, \Gamma; \mathrm{TSym}^a(\omega_{A/S}) \otimes \mathrm{TSym}^b(\mathcal{H}_A) \otimes \omega^g_{A/S}) \\ &\cong H^{g-1}(\Gamma, \mathrm{TSym}^a(\omega_{A/S}) \otimes \mathrm{TSym}^b(\mathcal{H}_A) \otimes \omega^g_{A/S}). \end{aligned} \tag{C.29}$$

Definition C.24 (Eisenstein-Kronecker classes)**.** *Let $x \in U_D(S)$ be a Γ-invariant φ-torsion section of an isogeny $\varphi : A \to B$ with étale dual. For every $f \in \mathcal{O}_S[D]^{0,\Gamma}$, we let*

$$\mathrm{EK}^{b,a}_{\Gamma}(f, x) \in H^{g-1}(\Gamma, \mathrm{TSym}^a(\omega_{A/S}) \otimes \mathrm{TSym}^b(\mathcal{H}_A) \otimes \omega^g_{A/S})$$

denote the image of $\mathrm{mom}^b_x(x^*\nabla^a \mathrm{EK}^{\natural}_{\Gamma}(f))$ *under the map (C.29).*

C.8. **Conjugation.** Let $\sigma : S^\sigma \to S$ be a morphism between affine noetherian schemes as in Notation C.2 (5). Equip the conjugated abelian variety A^σ/S^σ with the obvious left Γ-action $\Gamma \to \mathrm{Aut}(A^\sigma/S^\sigma)$, $\gamma \mapsto \gamma^\sigma$. With this action, the canonical morphism $\sigma_A : A^\sigma \to A$ is Γ-equivariant. As before, let $D = \ker(\delta)$ be a Γ-invariant closed subscheme of A. We have a commutative diagram

$$\begin{array}{ccc} H^0(D, \mathcal{O}_D) & \xrightarrow{\sigma_D^*} & H^0(D^\sigma, \mathcal{O}_{D^\sigma}) \\ {\scriptstyle \mathrm{tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\ H^0(S, \mathcal{O}_S) & \xrightarrow{\sigma^*} & H^0(S^\sigma, \mathcal{O}_{S^\sigma}) \end{array}$$

of Γ-equivariant homomoprhisms. Hence, we obtain an induced map

$$\sigma^* : \mathcal{O}_S[D]^{0,\Gamma} \to \mathcal{O}_{S^\sigma}[D^\sigma]^{0,\Gamma}.$$

Lemma C.25 (Conjugation)**.** *Assume the setup of C.2 (5). The diagram*

$$\begin{tikzcd} \mathcal{O}_S[D]^{0,\Gamma} \arrow[r, "\sigma_D^*"] \arrow[d, "\mathrm{EK}^{(\natural)}_{A,\Gamma}"'] & \mathcal{O}_{S^\sigma}[D^\sigma]^{0,\Gamma} \arrow[d, "\mathrm{EK}^{(\natural)}_{A^\sigma,\Gamma}"] \\ \varprojlim_n H^{g-1}(U_D,\Gamma;\mathcal{P}^{(\natural)}_{A,n}\otimes\Omega^g_{A/S}) \arrow[r, "\sigma_A^*"] & \varprojlim_n H^{g-1}(U_D^\sigma,\Gamma;\mathcal{P}^{(\natural)}_{A^\sigma,n}\otimes\Omega^g_{A^\sigma/S^\sigma}), \end{tikzcd}$$

where the lower horizontal map is induced by the equivariant homomorphism σ_A and Lemma C.7 (i), is commutative.

Proof. As the isomorphism $H^g(A,\Gamma;\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})\cong H^0(S,\mathcal{O}_S)$ is compatible with σ_A^* (which follows since the trace is compatible under base change) and since the localization sequence is functorial in the sense of Lemma B.13, one reduces the claim to the commutativity of the square

$$\begin{tikzcd} H^0(D,\mathcal{O}_D) \arrow[r, "\sigma_D^*"] \arrow[d, hook] & H^0(D^\sigma,\mathcal{O}_{D^\sigma}) \arrow[d, hook] \\ H^g_D(A,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S}) \arrow[r, "\sigma_A^*"] & H^g_{D^\sigma}(A^\sigma,\mathcal{P}_{A^\sigma,n}\otimes\Omega^g_{A^\sigma/S^\sigma}). \end{tikzcd}$$

for every $n\geq 0$. Recall the construction of the map $H^0(D,\mathcal{O}_D)\hookrightarrow H^g_D(A,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})$. In the first step, one uses the isomorphism $\iota_D^*\mathcal{P}_{A,n}\xrightarrow{\iota_D^*(\delta_\#)}\iota_D^*\delta^*\mathcal{P}_{B,n}\cong\pi_D^*e_B^*\mathcal{P}_{B,n}\xrightarrow{\pi_D^*(r_{B,n})^{-1}}\pi_D^*e_B^*\mathcal{O}_{B,n}$. It easily follows from Lemma C.7 (ii) and Lemma C.11 that this isomorphism is compatible with σ_D^* in the obvious sense. Hence, also the composite $\mathcal{O}_D=\pi_D^*e_B^*\mathcal{O}_B\subset\pi_D^*e_B^*\mathcal{O}_{B,n}\cong\iota_D^*\mathcal{P}_{A,n}$ and therefore $H^0(D,\mathcal{O}_D)\subset H^0(D,\iota_D^*\mathcal{P}_{A,n})$ is compatible with σ_D^*. The second step is the isomorphism $H^0(D,\iota_D^*\mathcal{P}_{A,n})\cong\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}_D,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})$. For this, one checks that the edge isomorphism

$$\mathrm{Ext}^g(\mathcal{O}_A/\mathcal{J}_D,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})\cong H^0(A,\underline{\mathrm{Ext}}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}_D,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S}))$$

is compatible with base change; in fact, one shows, similarly as in Proposition B.9, that the entire local-to-global Ext spectral sequence is functorial. Next, in our case, the fundamental local isomorphism

$$\underline{\mathrm{Ext}}^g(\mathcal{O}_A/\mathcal{J}_D,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})\cong(\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})\otimes\iota_{D*}\omega_{D/A}$$

is compatible under arbitrary base change, cf. the end of page 52 in [Con00]. One checks without problems that the canonical identification $\omega_{D/A}\cong\iota_D^*(\Omega^g_{A/S})^\vee$ given in the proof of [KS19], Proposition 2.29 is compatible with σ_D^* and from this the compatibility of the second step follows. It remains to show the compatibility of the inclusion

$$\mathrm{Ext}^g_{\mathcal{O}_A}(\mathcal{O}_A/\mathcal{J}_D,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})\hookrightarrow\mathrm{Ext}^g_{\mathcal{O}_A}(\iota_{D^*}\iota_D^{-1}\mathcal{O}_A,\mathcal{P}_n\otimes\Omega^g_{A/S})=H^g_D(A,\mathcal{P}_{A,n}\otimes\Omega^g_{A/S})$$

with the obvious maps induced by σ_A. As the map is simply induced by the canonical map $\iota_{D*}\iota_D^{-1}\mathcal{O}_A\to\mathcal{O}_A/\mathcal{J}_D$ by functoriality, this is an easy verification. □

Let us denote the σ-semilinear homomorphism induced by the Γ-equivariant σ-semilinear homomorphism $\sigma:\mathrm{TSym}^a(\omega_{A/S})\otimes\mathrm{TSym}^b(\mathcal{H}_A)\otimes\omega^g_{A/S}\to\mathrm{TSym}^a(\omega_{A^\sigma/S^\sigma})\otimes\mathrm{TSym}^b(\mathcal{H}_{A^\sigma})\otimes\omega^g_{A^\sigma/S^\sigma}$ on group cohomology by $(-)^\sigma$.

Proposition C.26. *Let $a,b\geq 0$ be integers. Let $x\in U_D(S)$ be a Γ-invariant φ-torsion section of an isogeny φ whose dual is étale. Let $f\in\mathcal{O}_S[D]^{0,\Gamma}$. Then one has*

$$\mathrm{EK}^{b,a}_\Gamma(f,x)^\sigma=\mathrm{EK}^{b,a}_\Gamma(f^\sigma,x^\sigma).$$

Proof. This is straightforward in view of Lemma C.25 and the compatibility of the moment map under base change, cf. Lemma C.11 (iii). □

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