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1 Introduction

The aim of this work is to study the sharp interface limit of the Cahn-Hilliard equation in situations in which elastic stresses appear. The Cahn-Hilliard equation is a phase field model in the sense that interfaces are diffuse, i.e. across an interface an order parameter representing the phases changes its state rapidly, but in a smooth way. If elastic stresses are present, the Cahn-Hilliard equation has to be coupled to an elasticity system and this extended set of equations is called the Cahn-Larché system. For the Cahn-Hilliard equation it is well known that if the interfacial thickness $\varepsilon > 0$ tends to zero, the Mullins-Sekerka model is recovered. The Mullins-Sekerka model is a sharp interface model and can be formulated as a classical free boundary model. Also the sharp interface model can be extended to include elastic effects and it is the goal of this paper to discuss recent attempts to relate the Cahn-Larché system and the elastically modified Mullins-Sekerka model. We refer to the article by Garcke et al. [GLNRW] for more information on phase separation and Ostwald ripening which are both phenomena that can be modelled with the help of the Cahn-Larché system and the extended Mullins-Sekerka model. We also refer to [GLNRW] for a discussion of situations where the two models can be reasonably used to recover the above phenomena.

Some work has been done already to study the sharp interface limit of the Cahn-Larché system. Fried and Gurtin [FrGu94] and Leo, Lowengrub and Jou [LLJ98] used the method of formally matched asymptotic expansions to relate the two models. Using this technique one has to assume that a smooth solution of the sharp interface model exists and fulfills certain smoothness properties, but to our knowledge there are no rigorous results known so far for the asymptotic limit of the Cahn-Larché system.

We present three results which relate the Cahn-Larché model to the sharp interface model. In Section 4 we will first show that the Cahn-Larché free energy, which is a Ginzburg-Landau type energy supplemented by contributions from elasticity, has a Γ-limit for ε tending to zero. The Γ-limit contains the classical surface energy together with elastic terms. Furthermore we will show in Section 4 that one can pass to the limit in the Euler-Lagrange equation for minimizers of the Cahn-Larché energy in order to obtain an elastically modified Gibbs-Thomson equation. This result generalizes a result of Luckhaus and Modica [LuMo89] to the Cahn-Larché system.

For general solutions we are going to use arguments and techniques from geometric measure theory to get rigorous results. Here one uses a priori estimates and compactness arguments to show convergence of the concentration, the chemical potential and the deformation vector. The main part is then to derive the Gibbs-Thomson law from the Cahn-Larché system. We are going to use methods introduced by Ilmanen [Ilm93], Soner [Son95] and Chen [Chen96] in order to perform the limiting process in the context of the theory of varifolds. The analysis for the Cahn-Larché system is more complicated due to the fact that elastic terms appear in the Gibbs-Thomson equation through the so-called Eshelby tensor.

The outline of this work is as follows. After introducing the governing models in Section 2 we review basic knowledge on geometric measure theory and present related work in Section 3. In Section 4 we consider the stationary case and in Section 5 we discuss the general case in the context of geometric measure theory. Section 5 is part of the ongoing PhD thesis of the second author and we refer to the thesis [Kwak] for more details.

2 The models

We start reviewing elasticity theory and the models which our analysis is based on.

2.1 Introduction to mechanics

We shortly introduce the basic concepts of linear elasticity, for a detailed introduction we refer to [Gur72], [Cia88] and [Brae91]. Denoting by a bounded region $\Omega \subset \mathbb{R}^n$ the reference state, we introduce the *deformation vector* $\mathbf{u}\colon \Omega \to \mathbb{R}^n$. Since in the applications we have in mind only small deformations appear, we consider a theory which is based on the *linearized strain tensor*

$$\mathcal{E}(\mathbf{u}) = 1/2(\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

The *elastic energy density* W is typically of quadratic form

$$W(c, \mathcal{E}) = \tfrac{1}{2}\big(\mathcal{E} - \mathcal{E}^*(c)\big) : \mathcal{C}(c)\big(\mathcal{E} - \mathcal{E}^*(c)\big) \tag{1}$$

with a symmetric and positive definite *elasticity tensor* $\mathcal{C}(c)$. We call $\mathcal{E}^*(c) = \mathcal{E}^* c$ the *eigenstrain* corresponding to c which describes the energetically favorable strain at concentration c. If $\mathcal{C}(c) = \mathcal{C}$ does not depend on the concentration, we speak of *homogeneous elasticity*, otherwise we use the term *inhomogeneous elasticity*. For the theory we are going to present in this work we will make the assumption that for a suitable constant $C > 0$ the following properties of W hold

$$
\begin{aligned}
W \in C^1(\mathbb{R}\times\mathbb{R}^{n\times n}, \mathbb{R}) \quad \text{such that} \\
|W(c,\mathcal{E})| \leq C(1+|c|^2+|\mathcal{E}|^2), \\
|W_{,\mathcal{E}}(c,\mathcal{E})| \leq C(1+|c|^2+|\mathcal{E}|), \\
|W_{,c}(c,\mathcal{E})| \leq C(1+|c|+|\mathcal{E}|).
\end{aligned} \tag{2}
$$

We assume in addition that $W(c,\mathcal{E})$ only depends on the symmetric part of $\mathcal{E} \in \mathbb{R}^{n\times n}$ and $W_{,\mathcal{E}}$ is strongly monotone, i.e. there exists a constant $c_1 > 0$ such that

$$
(W_{,\mathcal{E}}(c,\mathcal{E}_1) - W_{,\mathcal{E}}(c,\mathcal{E}_2)) : (\mathcal{E}_2 - \mathcal{E}_1) \geq c_1|\mathcal{E}_2 - \mathcal{E}_1|^2. \tag{3}
$$

We remark that an elasticity energy W according to equation (1) with $\mathcal{E}^*(c) = \mathcal{E}^* c$ does not fulfill (2), if the elasticity tensor $\mathcal{C}$ depends on the concentration c.

The mechanical equilibrium is attained on a much faster time scale compared to concentration changing by diffusion. This is why we assume that the mechanical equilibrium is attained instantaneously, so that the equation for the mechanics (4) does not involve any time derivatives and we hence consider at each time $t > 0$ the *quasi-stationary* system:

$$
\operatorname{div} S = \operatorname{div} W_{,\mathcal{E}}(c,\mathcal{E}(\mathbf{u})) = 0 \tag{4}
$$

where $S = S(c,\mathcal{E}) = W_{,\mathcal{E}}(c,\mathcal{E})$ is the *stress tensor*.

For definiteness we demand the deformation vector $\mathbf{u}$ to be in $\mathrm{X}_{\mathrm{ird}}^{\perp}$ with

$$
\begin{aligned}
\mathrm{X}_{\mathrm{ird}} := \{\mathbf{u} \in H^{1,2}(\Omega,\mathbb{R}^n) \mid & \text{ there exist } b \in \mathbb{R}^n \text{ and a skew symmetric} \\
& A \in R^{n\times n} \text{ such that } \mathbf{u}(x) = b + Ax\}
\end{aligned}
$$

and $\mathrm{X}_{\mathrm{ird}}^{\perp}$ is the space perpendicular to $\mathrm{X}_{\mathrm{ird}}$ where perpendicular is meant with respect to the $H^{1,2}$-inner product. We remark that the energies of both the phase field and sharp interface models depend on $\mathbf{u}$ only through $\mathcal{E}(\mathbf{u})$ and hence the infinitesimal rigid part of $\mathbf{u}$ has no influence on the evolution of c. We have the *Korn inequality*

$$
\|\mathbf{u}\|_{H^{1,2}(\Omega)} \leq \tilde{C}\|\mathcal{E}(\mathbf{u})\|_{L^2(\Omega)}
$$

for all $\mathbf{u} \in \mathrm{X}_{\mathrm{ird}}^{\perp}$ for some constant $\tilde{C}$ (see Zeidler [Zei88]). In particular we obtain using (3) and an energy argument that $\mathbf{u} \in \mathrm{X}_{\mathrm{ird}}^{\perp}$ is uniquely determined by (4) and a stress-free boundary condition.

2.2 Phase field model

The Cahn-Larché model is based on the Ginzburg-Landau energy

$$\mathbf{E}_{\mathrm{pf}}^{\varepsilon}(c,\mathbf{u}) = \int_{\Omega} \left(\frac{\varepsilon}{2}|\nabla c|^2 + \frac{1}{\varepsilon}\Psi(c) + W(c,\mathcal{E}(\mathbf{u})) \right) \tag{5}$$

where $\varepsilon > 0$ is a small parameter related to the thickness of the diffuse interface, c is a scaled concentration difference, Ψ is a polynomial double well potential which we take to be

$$\Psi(c) = \frac{1}{4}(c^2-1)^2. \tag{6}$$

In the diffuse interface model the evolution problem related to (5) is the Cahn-Larché system

$$\partial_t c = \Delta w, \tag{7}$$

$$w = \frac{\delta \mathbf{E}_{\mathrm{pf}}^{\varepsilon}}{\delta c} = -\varepsilon \Delta c + \tfrac{1}{\varepsilon}\Psi'(c) + W_{,c}(c,\mathcal{E}(\mathbf{u})), \tag{8}$$

$$\operatorname{div} S = \operatorname{div} \frac{\delta \mathbf{E}_{\mathrm{pf}}^{\varepsilon}}{\delta \mathbf{u}} = 0 \tag{9}$$

where w is the chemical potential. We can view this system as the H^{-1} gradient flow of the energy functional (5), see [GLNRW]. This structure will lead to crucial energy estimates of the Cahn-Larché system. The existence of solutions to this phase field system has been shown in [Gar00] and [Gar03]. The results are cited in Subsection 3.2.

2.3 Sharp interface model

The energy for the sharp interface limit is given by

$$\mathbf{E}_{\mathrm{si}} = \int_{\Gamma} 2\sigma \, d\mathcal{H}^{n-1} + \sum_{k=+,-} \int_{\Omega_k} W_k(\mathcal{E}(\mathbf{u}))\, dx \tag{10}$$

where $\sigma > 0$ is a surface energy constant and Γ is the interface (a hypersurface). The notation $\int_{\Gamma} .\, d\mathcal{H}^{n-1}$ denotes the integration with respect to the $(n-1)$-dimensional surface measure (the Hausdorff measure) and Ω_-, Ω_+ are the distinct regions occupied by the two phases with the corresponding elastic energy densities

$$W_-(\mathcal{E}) := W(-1,\mathcal{E}), \quad W_+ := W(+1,\mathcal{E}).$$

To simplify notation we set $W_+ = 0$ in Ω_- and vice versa, since then we can write $\sum_k \int_{\Omega_k} W_k = \int_{\Omega} \sum_k W_k$. Furthermore the surface energy density is

$$\sigma = \tfrac{1}{2}\int_{-\infty}^{\infty}(\tfrac{1}{2}(z'(y))^2 + \Psi(z(y)))dy,$$

where z is the solution of

$$-z'' + \Psi'(z) = 0 \qquad \text{with } z(-\infty) = -1 \text{ and } z(\infty) = 1.$$

One can easily compute

$$\sigma = \int_{-\infty}^{\infty} z'(y)\sqrt{\Psi(z(y))/2}\,dy = \int_{-1}^{1}\sqrt{\Psi(y)/2}\,dy.$$

The evolution problem related to the sharp interface energy is a modified Mullins-Sekerka problem

$$\begin{aligned}
\Delta w &= 0 && \text{in } \Omega_-(t) \text{ and } \Omega_+(t), && (11)\\
V &= -\tfrac{1}{2}[\nabla w]_-^+ \cdot \nu && \text{on } \Gamma(t), && (12)\\
w &= \sigma\kappa + \tfrac{1}{2}\nu^T[W\mathrm{Id} - (\nabla\mathbf{u})^T S]_-^+\nu && \text{on } \Gamma(t), && (13)\\
\operatorname{div} S &= 0 && \text{in } \Omega_-(t) \text{ and } \Omega_+(t), && (14)\\
[S\nu]_-^+ = [\mathbf{u}]_-^+ &= 0,\ [w]_-^+ = 0 && \text{on } \Gamma(t)
\end{aligned}$$

where $\Omega_-(t)$ and $\Omega_+(t)$ are the regions occupied by the phases at time t, $\Gamma(t)$ is the interface separating these regions, ν is the unit normal along the interface pointing towards Ω_+, V is the normal velocity of the interface and $[\,.\,]_-^+$ denotes the jump of the quantity in the brackets across the interface, e.g. $[w]_-^+ = w^+ - w^-$. κ is the mean curvature of $\Gamma(t)$ with the sign convention that κ is positive, if $\Gamma(t)$ is curved in the direction of ν. In contrast to its standard definition the mean curvature is taken here to be the sum of the principle curvatures. The first two equations are classical laws describing quasi-static diffusion driven by a chemical potential w. The third equation is the modified Gibbs-Thomson equation stating that the system is in local thermodynamical equilibrium.

Since we want to restrict our analysis to closed systems, we take homogeneous Neumann boundary conditions. In the phase field model this means

$$\nabla c \cdot \nu_\Omega = \nabla w \cdot \nu_\Omega = 0, \quad S\nu_\Omega = 0,$$

where ν_Ω denotes the outer unit normal of Ω. In the sharp interface model the condition for the concentration changes to an angle condition for the interface, so altogether the boundary conditions for the sharp interface model are

$$\angle(\Gamma(t), \partial\Omega) = 90^\circ, \quad \nabla w \cdot \nu_\Omega = 0, \quad S\nu_\Omega = 0.$$

3 Preliminaries

We introduce notations and recall some known facts about measures and varifolds (see also [EvGar92], [Fed69] and [Sim83]). We end this section by precisely stating the problems we want to analyze in this paper and with a discussion of related work.

3.1 Geometric measure theory

First we recall the definition of a Radon measure μ on an open set $\Omega \subset \mathbb{R}^n$ as a Borel regular measure that is finite on compact sets. To a measure μ we introduce the notion of densities on Ω for $x \in \Omega$

$$\theta^{*n-1}(\mu, x) = \limsup_{\rho \to 0} \frac{\mu(\Omega \cap B_\rho(x))}{\omega_{n-1}\rho^{n-1}},$$
$$\theta_*^{n-1}(\mu, x) = \liminf_{\rho \to 0} \frac{\mu(\Omega \cap B_\rho(x))}{\omega_{n-1}\rho^{n-1}}.$$

Here ω_{n-1} is the volume of the $(n-1)$-dimensional unit ball. If $\theta^{*n-1}(\mu, x)$ and $\theta_*^{n-1}(\mu, x)$ coincide, this common value will be denoted by $\theta^{n-1}(\mu, x)$.

Now we look on the set of $(n-1)$-dimensional subspaces

$$\mathbb{P}^{n-1} := \{P \mid P \text{ is a } (n-1)\text{-dimensional subspace in } \mathbb{R}^n\} = \mathbb{S}^{n-1}/\{\pm 1\}.$$

We will use the same notation P for the orthogonal projection onto the subspace P. On $\mathbb{P}^{n-1}$ we use the metric induced by endomorphisms:

$$d(P, Q) := \|P - Q\|_{End}.$$

This enables us to define a varifold:

Definition 1. *A* varifold V *is a Radon measure on the* Grassmanian

$$G(\Omega) := \Omega \times \mathbb{P}^{n-1}.$$

Remark 2.

- Such varifolds are in fact $(n-1)$-varifolds. We use such varifolds, since we want to describe interfaces. One can see them to give spatial and tangential information independently of each other.
  Defining varifolds simply as Radon measures on $\Omega \times \mathbb{P}^{n-1}$, we have weakened the usual view that the tangential information is solely given by the spatial information (of a neighborhood).
- For a C^1-hypersurface $\mathcal{M}$, we can introduce a corresponding varifold V by setting
$$dV(x, P) = d\mathcal{H}^{n-1}\lfloor_{\mathcal{M}}(x)\, \delta_{T_x\mathcal{M}}(P).$$

Finally we introduce the mass measure of a varifold.

Definition 3. *The* mass measure *of a varifold is defined by*

$$\mu_V(A) := \int_{A \times \mathbb{P}^{n-1}} dV(x, P) \quad \text{for } A \subset \Omega.$$

The motivation to use varifolds is that the limiting interface will not provide sufficient smoothness to fulfill some kind of Gibbs-Thomson law in the classical sharp interface sense. In fact Schätzle has shown in [Sch97] that even the BV-formulation of the Gibbs-Thomson law breaks down when two interfaces touch each other. Introducing the notion of varifolds enable us to come up with a formulation which extends the model beyond the time of topological changes.

Remark 4. Bronsard and Stoth studied the related Allen-Cahn equation and proved that in the limit there exist interfaces with arbitrary high multiplicity, also called *phantom interfaces*, see [BroSto96]. Figure 1 gives an illustration of a time-independent example. Assume that the two regions of approximations χ^ε merge to one when letting $\varepsilon \to 0$. Then the dashed line is a phantom interface. Such phantom interfaces are not captured when using only characteristic functions.

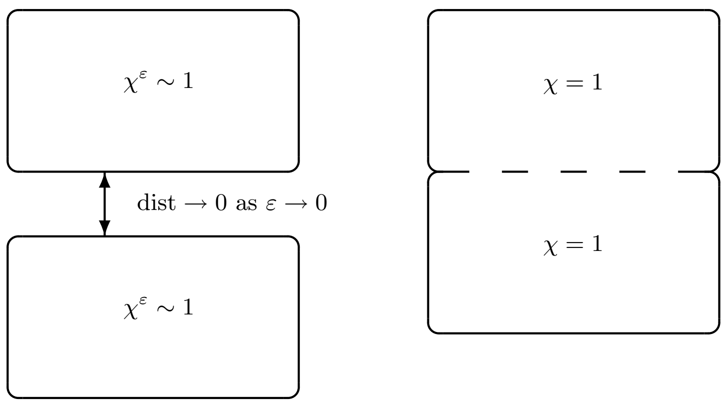

Fig. 1. An example where phase field interfaces lead to a varifold in the sharp interface limit.

First Variation of a varifold

In the smooth classical sense the Gibbs-Thomson law incorporates the mean curvature κ. Actually the curvature term occurs through the first variation of the area. For varifolds one has to use the first variation formula derived in Allard [All72] and Simon [Sim83].

As it can be found in the aforementioned works of Allard and Simon, the first variation of a varifold is given by

$$\delta V(X) = \int_{G(\Omega)} DX(x) \cdot P \, dV(x,P) \quad \text{for } X \in C_0^1(\Omega, \mathbb{R}^n) \tag{15}$$

where $DX(x) \cdot P$ is defined to be the inner product between linear mappings and $DX(x) \cdot P$ turns out to be the divergence of X with respect to the linear subspace P.

In fact, this coincides with the mean curvature in the smooth case. Using the Gauss theorem on a C^2-hypersurface $\mathcal{M}$

$$\int_{\mathcal{M}} X \cdot \nu_{\mathcal{M}} \kappa_{\mathcal{M}} \, d\mathcal{H}^{n-1} = \int_{\mathcal{M}} \operatorname{div}_{\mathcal{M}} X \, d\mathcal{H}^{n-1}$$

with $\nu_{\mathcal{M}}$ an arbitrary unit normal to $\mathcal{M}$ and $\kappa_{\mathcal{M}}$ the mean curvature of $\mathcal{M}$ with the sign according to $\nu_{\mathcal{M}}$. One notices that for $X \in C_0^1(\Omega, \mathbb{R}^n)$ the variation of the area can thus be read as the surface divergence of the vector field, i.e. the full divergence minus the normal part of DX.

In the case that the varifold is less smooth, but still has locally bounded first variation, one gets the following decomposition:

If $\|\delta V\|$ is a Radon measure, i.e.

$$\forall K \subset\subset \Omega \ \exists c_K > 0\colon \quad |\delta V(X)| < c_K \|X\|_\infty \quad \forall X \in C_0^1(K, \mathbb{R}^n),$$

the first variation of V can be seen as a bounded operator on $C^0(\Omega, \mathbb{R}^n)$ and one has a $\|\delta V\|$-measurable function $\nu\colon \Omega \to \mathbb{P}^{n-1}$ such that

$$\delta V(X) = -\int_\Omega X \cdot \nu \, d\|\delta V\|.$$

We now take the Lebesgue decomposition of $\|\delta V\|$ with respect to μ_V:

$$\delta V(X) = \int_\Omega X \cdot \nu \, d\|\delta V\| = \int_\Omega X \cdot \mathbf{H}_V \, d\mu_V + \int_Z X \cdot \nu \, d\sigma \tag{16}$$

where $\mathbf{H}_V$ is the Radon-Nikodym derivative of $\|\delta V\|$ with respect to μ_V multiplied with the normal function ν:

$$\mathbf{H}_V(x) = \nu(x) D_{\mu_V} \|\delta V\|(x).$$

$\mathbf{H}_V$ is called *generalized mean curvature vector*. The set of singularities $Z := \{x \in \mathbb{R}^n \mid D_{\mu_V} \|\delta V\|(x) = \infty\}$ is the *generalized boundary* of V with *generalized boundary measure* σ, *generalized unit co-normal* $\nu|_Z$ and $\mu_V(Z) = 0$.

Rectifiability

For a $(n-1)$-rectifiable set $M \subset \Omega$ there exists for $\mathcal{H}^{n-1}$-a.e. $x \in M$ the approximate tangent plane to M, denoted by $T_x^{\mathrm{app}} M$ (see [Sim83] for details). To such a set M one can associate a varifold V_M by setting

$$V_M(A) := \mathcal{H}^{n-1}\big(\{x \in \Omega \mid (x, T_x^{\mathrm{app}} M) \in A\}\big) \quad \text{for } A \subset G(\Omega).$$

Definition 5. *A varifold V is* rectifiable*, if there exist $\theta_i > 0$ and $(n-1)$-rectifiable sets $M_i \subset \Omega$ for $i \in \mathbb{N}$ such that*

$$V = \sum_{i \in \mathbb{N}} \theta_i \, V_{M_i}.$$

Since varifolds represent an abstract concept, one goal is to confirm rectifiability, if not even *integrality*, which is the case when all θ_i are integers in the above identity.

The relation between rectifiability and the first variation is stated in the following theorem by Allard (see [All72]).

Theorem 6 (Allard). *Suppose a varifold V has locally bounded first variation in Ω and $\theta^{n-1}(\mu_V, x) > 0$ for μ_V-a.e. $x \in \Omega$, then V is already a rectifiable varifold.*

Remark 7. Especially for a varifold V with locally bounded first variation in Ω the restriction of V onto $\{x \mid \theta^{*n-1}(\mu_V, x) > 0\} \times \mathbb{P}^{n-1}$ is rectifiable.

The next theorem by Schätzle shows that the structure of the first variation can lead to the desired rectifiability (see [Sch01]).

Theorem 8 (Schätzle). *Let W be a varifold in $\Omega \subset \mathbb{R}^n, w \in H^{1,p}(\Omega), n/2 < p < n, F \subset \Omega$ such that the characteristic function χ_F lies in $BV(\Omega)$. Furthermore we suppose*

1. $\delta W(\eta) = \int_\Omega \operatorname{div}(w\eta)\chi_F \quad \forall \eta \in C_0^1(\Omega, \mathbb{R}^n)$,
2. $|\nabla \chi_F| \le \mu_W$ *and*
3. $\|w\|_{H^{1,p}(\Omega)} + \mu_W(\Omega) \le \Lambda$ *for some* $\Lambda \in \mathbb{R}$.

Then W is rectifiable and has locally bounded first variation satisfying

$$\|\mathbf{H}_W\|_{L^s(\mu_W \lfloor_{B(x_0,r)})} \le C_{n,p}(r)\Lambda^{1+1/s} \qquad \forall B(x_0, 2r) \subset \Omega,$$

where $s \in \mathbb{R}$ such that $\frac{n-1}{s} = \frac{n}{p} - 1$.

The main part of the proof is to show a particular monotonicity formula for the density of the mass measure:

Lemma 9 (Monotonicity Formula). *For a varifold W which fulfills the assumptions of Theorem 8 the function*

$$\rho \mapsto \rho^{-(n-1)}\mu_W(B_\rho(x_0)) + C_{n,p}\min(1,d)^{-1}\Lambda\rho^\alpha \quad \forall x_0 \in \Omega, 0 < \rho < d$$

is non-decreasing for $\alpha = 1 - \frac{n-1}{s} \in (0,1)$ with $d = \operatorname{dist}(x_0, \partial\Omega)$.

Once this monotonicity formula is verified, one can use the following theorem by Ziemer:

Theorem 10 (Ziemer). *Let μ be a Radon measure on $\mathbb{R}^n$. Then the following statements are equivalent:*

1. *$\mathcal{H}^{n-1}(A) = 0$ implies that $\mu(A) = 0$ for all Borel sets $A \subset \mathbb{R}^n$ and there is a constant $\bar{C}$ such that $\left|\int \phi d\mu\right| \leq \bar{C}\|\phi\|_{BV(\mathbb{R}^n)}$ for all $\phi \in BV(\mathbb{R}^n)$.*
2. *There is a constant $\bar{C}$ such that $\mu(B(x,r)) \leq \bar{C}r^{n-1}$.*

By the theorem of Ziemer we obtain from Lemma 9 local bounds for the measure μ_W, i.e. for all $\phi \in BV(\Omega)$ and $B_\rho(x_0) \subset \Omega$

$$\left|\int_\Omega \phi\, \chi_{B_\rho(x_0)} d\mu_W\right| \leq \bar{C}\|\phi\|_{BV(\mathbb{R}^n)}.$$

Now, we choose $\phi = |w|^s$, which is in $H^{1,1}(\Omega)$ by imbedding theorems, and the first variation of the varifold W can be therefore estimated by

$$|\delta W(\eta)| \leq \left|\int (w\eta) d\mu_W\right| \leq \|w\|_{L^s(\mu_W)}\|\eta\|_{L^{s^*}(\mu_W)}.$$

By this estimate the first variation can be interpreted as a Radon measure and the above inequality leads to rectifiability of the varifold through the theorem of Allard.

3.2 Assumptions and notations

We start with solutions of the Cahn-Larché systems fulfilling the following assumptions (see also [Gar00] and [Gar03]) for $\Omega \subset \mathbb{R}^n$ open and bounded with smooth boundary. We consider for all $\varepsilon > 0$

$$\begin{aligned} c^\varepsilon &\in L^2_{\text{loc}}(0,\infty; H^{2,2}(\Omega)) \cap H^{1,2}_{\text{loc}}(0,\infty; H^{-1,2}(\Omega)),\\ w^\varepsilon &\in L^2_{\text{loc}}(0,\infty; H^{1,2}(\Omega)),\\ \mathbf{u}^\varepsilon &\in L^2_{\text{loc}}(0,\infty; H^{2,2}(\Omega)^n) \end{aligned}$$

such that the following weak formulation is fulfilled for all $T > 0$

$$\int_0^T \langle \partial_t c^\varepsilon, \zeta_1\rangle dt = \int_0^T\int_\Omega \nabla w^\varepsilon \cdot \nabla\zeta_1\, dxdt, \tag{17}$$

$$\int_0^T\int_\Omega w^\varepsilon\zeta_2\, dxdt = \int_0^T\int_\Omega \varepsilon\nabla c^\varepsilon \cdot \nabla\zeta_2 + \tfrac{1}{\varepsilon}\Psi'(c^\varepsilon)\zeta_2 + W_{,c}(c^\varepsilon, \mathbf{u}^\varepsilon)\zeta_2\, dxdt, \tag{18}$$

$$0 = \int_0^T\int_\Omega S : D\zeta_3\, dxdt \tag{19}$$

for all $\zeta_1 \in L^2(0,T;H^{1,2}(\Omega))$, $\zeta_2 \in L^2(0,T;H^{1,2}(\Omega)) \cap L^\infty(\Omega\times[0,T])$ and $\zeta_3 \in L^2(0,T;H^{1,2}(\Omega)^n)$. Here, the notation of $c^\varepsilon \in L^2_{\text{loc}}(0,\infty;H^{1,2}(\Omega))$ means that for all times $T > 0$ one has $c^\varepsilon \in L^2(0,T;H^{1,2}(\Omega))$ and $\langle .,. \rangle$ is the duality pairing between $H^{-1,2}(\Omega)$ and $H^{1,2}(\Omega)$. In contrast to other notations we define $H^{-1,2}(\Omega)$ as the dual of $\{c \in H^{1,2}(\Omega) \mid \int_\Omega c = 0\}$.

As initial conditions we assume that for all $\varepsilon > 0$

1. the initial energy is bounded: $\mathbf{E}^{\varepsilon}_{\mathrm{pf}}(0) \le \mathbf{E}_0$ and
2. the integral of the initial concentration does not depend on ε, i.e. there exists a constant $m_0 \in (-1,1)$ such that $\int_\Omega c^\varepsilon_0 = m_0|\Omega|$.

Remark 11. The existence of weak solutions of the Cahn-Larché system has been shown in [Gar00] and [Gar03]. But so far it has not been verified in general, if the concentration and deformation vector are indeed in $H^{2,2}(\Omega)$ for almost all t.

In the case that W is of the quadratic form (1) with constant elasticity tensor $\mathcal{C}$, i.e. in the homogeneous case, the equation determining $\mathbf{u}$ can be read as an elliptic system with constant coefficients, where only the right-hand side depends on the concentration:

$$\operatorname{div} S = 0 \quad \Longleftrightarrow \quad \operatorname{div}[\mathcal{C}\mathcal{E}(\mathbf{u}^\varepsilon)] = \operatorname{div}[\mathcal{C}\mathcal{E}^*(c^\varepsilon)].$$

Since c^ε is in $H^{1,2}(\Omega)$, the right-hand side is in $L^2(\Omega)$, which leads $\mathbf{u}^\varepsilon$ to be in $H^{2,2}(\Omega)$ by elliptic regularity theory.

On the other side, $W_{,c} = \mathcal{E}^* : \mathcal{C}[\mathcal{E}(\mathbf{u}^\varepsilon) - \mathcal{E}^* c^\varepsilon]$ is in $L^2(\Omega)$. Now, equation (18) can be read as an elliptic equation for c^ε and again elliptic regularity theory can be used.

If one considers inhomogeneous elasticity, the elasticity system (4) contains possibly non-continuous coefficients $\mathcal{C}(c^\varepsilon)$ so that one cannot argue as in the homogeneous case. Though, in low dimensions due to Sobolev imbedding theorems the concentration functions c^ε are continuous and therefore elliptic regularity theory for smooth coefficients can be used.

Nevertheless, for the general case we are presenting in this work we have to include this assumptions in order to get the correct Gibbs-Thomson law, see Subsection 5.4.

One important first observation for the limiting process $\varepsilon \to 0$ is to identify

$$e^\varepsilon(c^\varepsilon) := \tfrac{\varepsilon}{2}|\nabla c^\varepsilon|^2 + \tfrac{1}{\varepsilon}\Psi(c^\varepsilon)$$

as the *interfacial energy density* in the phase field model. Heuristically, this is exactly the quantity one observes to carry the interfacial energy of the phase field model, and the goal is to show convergence to a quantity that will be understood up to a factor as the $\mathcal{H}^{n-1}$-measure of the interface.

The second important function is the so-called *discrepancy measure*

$$\xi^\varepsilon(c^\varepsilon) := \tfrac{\varepsilon}{2}|\nabla c^\varepsilon|^2 - \tfrac{1}{\varepsilon}\Psi(c^\varepsilon). \tag{20}$$

As it is stated in Theorem 21, in the limit $\varepsilon \to 0$ the discrepancy measure will be non-positive, which means that the Ψ-part is larger than the $|\nabla c^\varepsilon|^2$-part.

3.3 Related works

One important source of this work is the paper by Chen [Chen96]. He has studied the asymptotic limit of the Cahn-Hilliard model. Chen showed for arbitrary spatial dimensions that solutions of the Cahn-Hilliard system converge globally in time to some generalized sharp-interface solution. He did not show that the limit varifold is rectifiable, but in the case $p = 2, n = 3$ one can use the Theorem 8 by Schätzle to deduce rectifiability for the limit varifold for the Cahn-Hilliard systems without elasticity, see Remark 17.

There is one significant difference to results for the related Allen-Cahn models which are proposed to describe motion of phase boundaries driven by surface tension:

$$\varepsilon\frac{\partial c}{\partial t} = \varepsilon\Delta c - \varepsilon^{-1}\Psi'(c).$$

Ilmanen [Ilm93] has studied the limiting behavior of the Allen-Cahn equation towards the mean curvature flow in the sense of Brakke [Bra78] and confirmed that one gets in the limit

$$\xi = 0.$$

This is also known as *equipartition of energy*. It is quite interesting to note that the interface energy is asymptotically equally distributed between the $|\nabla c^\varepsilon|^2$-and the $\Psi(c^\varepsilon)$-part. Moreover this result can be used for further results, namely it is easier to deduce the fact that the resulting interface varifold is rectifiable.

After Ilmanen [Ilm93] first used geometric measure theory to prove such convergence in $\Omega = \mathbb{R}^n$, Soner [Son95] improved the result for more general settings. Hutchinson and Tonegawa studied in [HutTon00] the asymptotic behavior of critical, not necessarily minimal points of the Cahn-Hilliard energy functional. In their work they also used geometric measure theory and derived local estimates for the discrepancy measure (20). By that, they gained convergence results for bounded domains. In their (time-independent) setting the limit varifold turns out to be integral, i.e. the interface has indeed integer multiplicity modulo a surface constant almost everywhere. Moreover local minimizers of the Cahn-Hilliard energy functional converge to a local area minimizer subject to a volume constraint. Later Tonegawa extended with similar estimates the results by Ilmanen and showed that time-dependent solutions of the Allen-Cahn equation converge to an integral varifold, cf. [Ton03].

4 The stationary case

Before we study the evolution problem, we consider the sharp interface limit of the Ginzburg-Landau energy $\mathbf{E}_{\text{pf}}^{\varepsilon}(c, \mathbf{u})$ in the limit ε tending to zero. As the Cahn-Larché system conserves the integral of the concentration c, we will consider $\mathbf{E}_{\text{pf}}^{\varepsilon}$ subject to an integral constraint on c. In fact in this case one can show that $\mathbf{E}_{\text{si}}$ is the Γ-limit of $\mathbf{E}_{\text{pf}}^{\varepsilon}$, even if we take the constraint into account. Furthermore, we present a result stating that the Lagrange multipliers related to minimizers of $\mathbf{E}_{\text{pf}}^{\varepsilon}$ will converge to a Lagrange multiplier related to a minimizer of $\mathbf{E}_{\text{si}}$ subject to a volume constraint. The results we present will generalize results of Modica [Mod87] and Luckhaus and Modica [LuMo89] to the case including elastic effects.

4.1 The Γ-limit of the Cahn-Larché energy

In this subsection we study solutions of the variational problems:

($\mathbf{P}^\varepsilon$) Find a minimizer $(c, \mathbf{u}) \in H^{1,2}(\Omega) \times X_{ird}^\perp$ of $\mathbf{E}_{\mathrm{pf}}^\varepsilon$ subject to the constraint $\frac{1}{|\Omega|}\int_\Omega c = m_0$, where $m_0 \in (-1, 1)$ is a given constant.

We will now present a result stating that solutions to ($\mathbf{P}^\varepsilon$) converge along subsequences to a minimizer of the functional

$$\mathbf{E}^0 \colon L^1(\Omega) \times \mathrm{X}_{\mathrm{ird}}^\perp \to \mathbb{R} \cup \{\infty\}$$

where

$$\mathbf{E}^0(c, \mathbf{u}) = \begin{cases} 2\sigma\mathcal{H}^{n-1}(\partial\{c = 1\} \cap \Omega) + \int_\Omega W(c, \mathcal{E}(\mathbf{u})) & \text{if } c \in BV(\Omega, \{-1, 1\}) \\ & \text{and } \frac{1}{|\Omega|}\int_\Omega c = m_0, \\ \infty & \text{otherwise.} \end{cases}$$

The following theorem now states that $\mathbf{E}^0$ is the Γ-limit of $\mathbf{E}_{\mathrm{pf}}^\varepsilon$. We also obtain that minimizers of $\mathbf{E}_{\mathrm{pf}}^\varepsilon$ approximate minimizers of $\mathbf{E}_{\mathrm{si}}$, if we take a volume constraint into account. The limiting variational problem is a partitioning problem taking interfacial energy and elastic effects into account.

The following theorem has been shown in [Gar00].

Theorem 12. *Assume that the assumptions of Ψ and W as stated above hold and let Ω be a bounded domain with Lipschitz boundary. Then it holds:*

1. *For all $(c^{\varepsilon_k}, \mathbf{u}^{\varepsilon_k})_{k\in\mathbb{N}} \in H^{1,2}(\Omega) \times \mathrm{X}_{ird}^\perp$ with $c^{\varepsilon_k} \to c$ in $L^1(\Omega)$ and $\mathbf{u}^{\varepsilon_k} \to \mathbf{u}$ in $L^2(\Omega, \mathbb{R}^n)$ as ε_k tends to zero, it holds*
$$\mathbf{E}^0(c, \mathbf{u}) \le \liminf_{k\to\infty} \mathbf{E}_{pf}^{\varepsilon_k}(c^{\varepsilon_k}, \mathbf{u}^{\varepsilon_k}).$$
2. *For any $(c, \mathbf{u}) \in L^1(\Omega) \times \mathrm{X}_{ird}^\perp$ and any sequence $\varepsilon_k \to 0, k \in \mathbb{N}$, there exists a sequence $(c^{\varepsilon_k}, \mathbf{u}^{\varepsilon_k})_{k\in\mathbb{N}} \in H^{1,2}(\Omega) \times \mathrm{X}_{ird}^\perp$ with $c^{\varepsilon_k} \to c$ in $L^1(\Omega)$ and $\mathbf{u}^{\varepsilon_k} \to \mathbf{u}$ in $L^2(\Omega, \mathbb{R}^n)$ as $\varepsilon_k \to 0$ such that*
$$\mathbf{E}^0(c, \mathbf{u}) \ge \limsup_{k\to\infty} \mathbf{E}_{pf}^{\varepsilon_k}(c^{\varepsilon_k}, \mathbf{u}^{\varepsilon_k}).$$
3. *Let $(c^\varepsilon, \mathbf{u}^\varepsilon)$ be solutions of problem ($\mathbf{P}^\varepsilon$). Then there exists a sequence $\varepsilon_k \to 0, k \in \mathbb{N}$ and $(c, \mathbf{u}) \in L^1(\Omega) \times \mathrm{X}_{ird}^\perp$ such that*
$$\begin{aligned} c^{\varepsilon_k} &\to c \quad \text{in } L^1(\Omega), \\ \mathbf{u}^{\varepsilon_k} &\to \mathbf{u} \quad \text{in } H^{1,2}(\Omega, \mathbb{R}^n) \end{aligned}$$
and $(c, \mathbf{u})$ is a global minimizer of $\mathbf{E}^0$.

For the proof and for a generalization to the situation of more than two phases we refer to [Gar00].

4.2 Convergence of the Lagrange multipliers

For a minimizer $(c, \mathbf{u})$ of $\mathbf{E}^0$ it can be shown that a constant Lagrange multiplier ω exists such that

$$2\sigma\kappa + \nu \cdot \left[W\mathrm{Id} - (\nabla\mathbf{u})^T W_{,\mathcal{E}}\right] \nu = 2\omega. \tag{21}$$

A minimizer of $\mathbf{E}^0$ minimizes $\mathbf{E}_{\mathrm{si}}$ subject to a volume constraint and ω is the Lagrange multiplier related to this constraint.

In the case that no elastic effects are present, we obtain that the mean curvature is constant and the term $\nu \cdot \left[W\mathrm{Id} - (\nabla\mathbf{u})^T W_{,\mathcal{E}}\right] \nu$ modifies this law. In particular the mean curvature can be inhomogeneous along the interface. The identity (21) and its non-equilibrium analogue (13) can be interpreted as a generalized Gibbs-Thomson equation.

Absolute minimizers $(c^\varepsilon, \mathbf{u}^\varepsilon)$ of $\mathbf{E}^\varepsilon_{\mathrm{pf}}$ have a constant Lagrange ω^ε which fulfills in a distributional sense (see [Gar00])

$$-\varepsilon\Delta c^\varepsilon + \tfrac{1}{\varepsilon}\Psi'(c^\varepsilon) + W_{,c}(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon)) = \omega^\varepsilon. \tag{22}$$

In [Gar00] it is shown that the Lagrange multipliers of $(c^\varepsilon, \mathbf{u}^\varepsilon)$ converge (along subsequences) to a Lagrange multiplier ω of the sharp interface variational problem. Here we state the result in detail.

Theorem 13. *Let Ω be a domain with a C^1-boundary and assume that Ψ and W fulfill the conditions stated above. Furthermore let $(c^\varepsilon, \mathbf{u}^\varepsilon) \in H^{1,2}(\Omega) \times \mathrm{X}^{\perp}_{ird}$ be a solution of the variational problem $(\mathbf{P}^\varepsilon)$ with Lagrange multipliers ω^ε. Then for each sequence $(\varepsilon_k)_{k\in\mathbb{N}} \to 0$ such that*

$$\begin{aligned} c^{\varepsilon_k} &\to c \quad \text{in } L^1(\Omega), \\ \mathbf{u}^{\varepsilon_k} &\to \mathbf{u} \quad \text{in } H^{1,2}(\Omega, \mathbb{R}^n) \end{aligned}$$

it holds

$$\omega^{\varepsilon_k} \to \omega,$$

where ω is the Lagrange multiplier for the absolute minimizer $(c, \mathbf{u})$ of $\mathbf{E}^0$, compare (21).

For a proof we refer to [Gar00]. We remark that although the method of Luckhaus and Modica [LuMo89] for the case without elasticity is used in the proof, one cannot follow their arguments in a straightforward way. This is due to the fact that not enough regularity is known for the minimizer $(c^\varepsilon, \mathbf{u}^\varepsilon)$ of $\mathbf{E}^\varepsilon_{\mathrm{pf}}$. In the proof of Theorem 13 one uses variations of $\mathbf{E}^\varepsilon_{\mathrm{pf}}$ with respect to the independent variables and shows that the resulting Lagrange multiplier is related to the Lagrange multiplier from (22), which is the first variation with respect to the dependent variables.

Remark 14. We also note that a minimizer $(c, \mathbf{u})$ of $\mathbf{E}^0$ also fulfills

$$\int_\Omega 2\sigma(\nabla\cdot\xi - \nu\cdot\nabla\xi\nu)|\nabla\chi_{\{c=-1\}}| + \int_\Omega (W\mathrm{Id} - (\nabla\mathbf{u})^T W_{,\mathcal{E}}) : \nabla\xi = \int_\Omega \lambda c \nabla\cdot\xi$$

for all $\xi \in C^\infty(\bar{\Omega}, \mathbb{R}^n)$ with $\xi\cdot\nu_\Omega = 0$ on $\partial\Omega$. Here, $\nu = -\frac{\nabla\chi_{\{c=-1\}}}{|\nabla\chi_{\{c=-1\}}|}$ is the generalized outer unit normal to $\{c=-1\}$ which is a $|\nabla\chi_{\{c=-1\}}|$-measurable function. The above identity is a weak formulation of the modified Gibbs-Thomson equation (21) (see [Gar00]).

5 The time-dependent case

For the evolutionary system we start with a suitable weak formulation of the sharp interface problem. Through the limiting process one cannot expect that the resulting limit objects are smooth enough such that equations (12)–(14) can be verified in a classical way. Besides concentration, the chemical potential and deformation vector which converge quite straightforward in the limiting process, to formulate a Gibbs-Thomson law we need both a characteristic function and a varifold, which represents the interface as motivated in Subsection 3.1 including possible phantom interfaces.

After stating the theorem we give an overview on the proof, not giving all the details due to the limited space and refer to [Kwak] for a full treatment.

5.1 Statement of the main theorem

First we specify the notion of a generalized solution of the sharp interface model.

Definition 15 (Generalized solution). $(M, V, w, \mathbf{u})$ *is said to be a* generalized solution of the modified Mullins-Sekerka problem*, if*

$$M \subset \Omega\times[0,\infty),\ w \in L^2_{loc}(0,\infty; H^{1,2}(\Omega)), \mathbf{u} \in L^2_{loc}(0,\infty; H^{1,2}(\Omega)^n)$$
$$V \text{ is a Radon measure on } \Omega\times\mathbb{P}^{n-1}\times(0,\infty).$$
$$\text{Moreover } \chi_M \in C^0([0,\infty); L^1(\Omega)) \cap L^\infty(0,\infty; BV(\Omega)) \text{ and}$$
$$V^t \text{ is a varifold on } \Omega \text{ for all } t>0$$

such that for all $T>0$, for almost every $0<\tau<t<T$ and for all test functions $\zeta \in C^1_0(\bar{\Omega}\times[0,T))$, $\mathbf{Y} \in C^1_0(\Omega, \mathbb{R}^n)$ and $\mathbf{X} \in L^2_0(0,T; H^{1,2}(\Omega,\mathbb{R}^n))$ the following holds:

1. $\int_0^T \int_\Omega [-2\chi_{M^t}\partial_t\zeta + \nabla w\nabla\zeta] = \int_\Omega 2\chi_{M^0}\zeta(.,0)$,
2. $2\int_\Omega \chi_{M^t}\,\mathrm{div}(w\mathbf{Y}) = \langle\partial V^t, \mathbf{Y}\rangle + \sum_{k=+,-}\int_\Omega (W^t_k Id - (\nabla\mathbf{u})^T S^t_k) : D\mathbf{Y}$,
3. $dV^t(x,P) = \sum_i \rho^t_i(x)\delta_{\nu^t_i(x)}(P)d\mu^t(x)dP$,
4. $d\mu^t(x) \geq 2\sigma|D\chi_{M^t}|(x)dx$,

5. $\mu^t(\Omega) + \sum_k \int_\Omega W_k^t + \int_\tau^t \int_\Omega |\nabla w|^2 \leq \mu^\tau(\Omega) + \sum_k \int_\Omega W_k^\tau$,
6. $\int_0^T \int_\Omega S : D\mathbf{X}\, dxdt = 0$

where $\rho_i^t \in [0,1], \sum_i \rho_i^t \geq 1, \sum_i \nu_i^t \otimes \nu_i^t = Id$ *and* μ^t *is a Radon measure on* $\bar{\Omega}$. *An upper index* $\{.\}^t$ *denotes the time.*

Remark 16. Let us discuss the definition in more detail. The first equation is the weak formulation of the diffusion equations (11) and (12). In the bulk the chemical potential will be harmonic. Equation 2 is the Gibbs-Thomson law (13) in a weak formulation (cf. Remark 14 and [Gar00]). Equations 3 and 4 describe properties of the varifold. Inequality 4 allows that the varifold can possibly see phantom interfaces. Equation 5 states the dissipation of the free energy and equation 6 states in a weak form that the stress is divergence free in the bulk, cf. (14), and at the same time one obtains that the normal jump of the stress is zero across the interface.

One should notice that the Gibbs-Thomson law has two terms which represent the interface and vanish in the bulk, but the elastic term stays a volume integral. The reason for this is that the elastic energy is a non-local volume energy. So, one has to be aware in the limiting process that both $\frac{\varepsilon}{2}|\nabla c^\varepsilon|^2$ and $\Psi(c^\varepsilon)$ converge to a $(n-1)$-dimensional measure while W stays n-dimensional.

Remark 17. In the case of Cahn-Hilliard systems, i.e. without any elastic terms, equation 2 in Definition 15 becomes of the same form as in the Theorem 8 by Schätzle. This means that one can deduce rectifiability of the varifold in the case without elasticity, at least for the case $p = 2, n = 3$.

Theorem 18 (Main). *Let the assumptions mentioned in Subsection 3.2 hold. Then there is a sequence* $\varepsilon_i \to 0$ *and a generalized solution* $(M, V, w, \mathbf{u})$ *as in Definition 15 such that for all* $T > 0$

1. $c^{\varepsilon_i} \to -1 + 2\chi_M$ *in* $C^{1/9}([0,T]; L^2(\Omega))$ *and almost everywhere,*
2. $w^{\varepsilon_i} \to w$ *weakly in* $L^2(0,T; H^{1,2}(\Omega))$,
3. $\mathbf{u}^{\varepsilon_i} \to \mathbf{u}$ *in* $L^2(0,T; H^{1,2}(\Omega)^n)$.

More precisely the varifold is obtained in the following way.

Proposition 19. *For the sequence of Theorem 18 it further holds:*

1. *There exist Radon measures* μ, μ_{kl} *on* $\bar{\Omega} \times [0,\infty)$ *such that*

$$e^{\varepsilon_i}(c^{\varepsilon_i})dxdt \to d\mu(x,t), \tag{23}$$
$$\varepsilon_i c^{\varepsilon_i}_{x_k} c^{\varepsilon_i}_{x_l} dxdt \to d\mu_{kl}(x,t) \tag{24}$$

both as Radon measures on $\bar{\Omega} \times [0,T]$ *for all* $T > 0$.

2. *For all* $\mathbf{Y} \in C_0^1(\Omega \times [0,T], \mathbb{R}^n)$ *it holds:*

$$\int_0^T \langle \partial V^t, \mathbf{Y} \rangle = \int_0^T \int_\Omega \nabla \mathbf{Y} : [d\mu(x,t) Id - (d\mu_{ij}(x,t))_{ij}].$$

Remark 20. The first part of the proposition follows easily by the energy estimates and using compactness properties of Radon measures. So it is left to show that the measures μ and μ_{ij} can be indeed identified as a varifold. This is essentially done by proving Theorem 21.

We define for $\varepsilon > 0$ the set

$$\mathcal{K}_\varepsilon := \{(c,v) \in H^{2,2}(\Omega) \times L^2(\Omega) \mid -\varepsilon\Delta c + \varepsilon^{-1}\Psi'(c) = v \text{ in } \Omega \text{ and } \partial_\nu c = 0 \text{ on } \partial\Omega\}.$$

Theorem 21. *There exist a constant $\eta_0 \in (0,1]$ and continuous and non-increasing functions $M_1(\eta), M_2(\eta)\colon (0,\eta_0] \to (0,\infty)$ such that for every $\eta \in (0,\eta_0]$, every $\varepsilon \in (0, M_1(\eta)^{-1}]$ and every $(c,v) \in \mathcal{K}_\varepsilon$ it holds*

$$\int_\Omega (\xi^\varepsilon(c))^+ \leq \eta \int_\Omega e^\varepsilon(c) + \varepsilon M_2(\eta) \int_\Omega v^2. \tag{25}$$

Remark 22. In the application of Theorem 21, v will be the sum

$$v = w^\varepsilon - W_{,c}(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon)).$$

5.2 Convergence of concentration and chemical potential

One crucial a priori estimate is due to the H^{-1} gradient flow property of the Cahn-Larché system with respect to the energy functional (5) which we already mentioned in Subsection 2.2. For more details see [Gar03].

Lemma 23. *For all $\varepsilon > 0$ and $0 < \tau < t$ it holds*

$$\mathbf{E}^\varepsilon_{pf}(t) + \int_\tau^t \int_\Omega |\nabla w^\varepsilon|^2 \leq \mathbf{E}^\varepsilon_{pf}(\tau).$$

From equations (17) and (18) one easily gets the following a priori estimates:

Lemma 24. *For all $\varepsilon > 0$ and almost all $t > 0$ it holds*

1. $\frac{1}{|\Omega|}\int_\Omega c^\varepsilon(.,t) = m_0$,
2. $\int_\Omega |c^\varepsilon|^4 \leq C(1+\mathbf{E}_0)$,
3. $\int_\Omega (|c^\varepsilon| - 1)^2 \leq C\varepsilon\mathbf{E}_0$.

Remark 25. The first equation describes one feature of the phase field model: *conservation of mass* over time. This is essentially due to the diffusion which is driven by a potential and the Neumann boundary conditions.

We introduce the auxiliary function

$$\tilde{c}^\varepsilon(x,t) := \int_{-1}^{c^\varepsilon(x,t)} \sqrt{\tilde{\Psi}(s)/2}\,ds,$$

which is also known as the *Modica ansatz*. Here $\tilde{\Psi}(s) := \min(\Psi(s), 1 + |s|^2)$ is used, so one has

$$C_1|s_1 - s_2|^2 \leq |\tilde{c}(s_1) - \tilde{c}(s_2)| \leq C_2|s_1 - s_2|(1 + |s_1| + |s_2|)$$

for some $C_1, C_2 > 0$. Using this auxiliary function it is possible to obtain bounds in $BV(\Omega)$.

Lemma 26. *For solutions to the Cahn-Larché system the Modica ansatz leads to*

$$\|\tilde{c}^\varepsilon\|_{L^\infty(0,\infty;H^{1,1}(\Omega))} + \|\tilde{c}^\varepsilon\|_{C^{1/8}([0,\infty);L^1(\Omega))} + \|c^\varepsilon\|_{C^{1/8}([0,\infty);L^2(\Omega))} \leq C. \quad (26)$$

With these uniform bounds one can pass to the limit $\varepsilon \to 0$ along a sequence and together with Lemma 24 identify a set $M \subset \Omega \times [0,\infty)$ such that we have the following lemma:

Lemma 27. *For solutions of the Cahn-Larché system there exists a sequence* $\varepsilon_j \to 0$ *such that*

- $\tilde{c}^{\varepsilon_j}(x,t) \to 2\sigma\chi_M$ *in* $C^{1/9}([0,T]; L^1(\Omega))$,
- $c^{\varepsilon_j}(x,t) \to -1 + 2\chi_M$ *in* $C^{1/9}([0,T]; L^2(\Omega))$ *and almost everywhere*

for all $T > 0$.

This set M then defines $\Omega_-(t)$ for all $t > 0$.

This proves the first convergence statement of the main theorem. For the chemical potential we observe that the following Poincaré type inequality holds:

Lemma 28. *For the solutions of the Cahn-Larché system we obtain*

$$\|w^\varepsilon(.,t)\|_{H^{1,2}(\Omega)} \leq C(\mathbf{E}^\varepsilon_{pf}(t) + \|\nabla w^\varepsilon(.,t)\|_{L^2(\Omega)}). \quad (27)$$

To prove this lemma we test equation (18) with $\mathbf{X} \cdot \nabla c^\varepsilon$ for $\mathbf{X} \in C^1_0(\Omega, \mathbb{R}^n)$ to get

$$\begin{aligned}\int w^\varepsilon \mathbf{X} \cdot \nabla c^\varepsilon &= \int \varepsilon \nabla c^\varepsilon \cdot \nabla(\mathbf{X} \cdot \nabla c^\varepsilon) + \tfrac{1}{\varepsilon}\Psi'(c^\varepsilon)\mathbf{X} \cdot \nabla c^\varepsilon + W_{,c}(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon))\mathbf{X} \cdot \nabla c^\varepsilon \\ &= \int \varepsilon \left(\nabla c^\varepsilon \cdot D\mathbf{X} \nabla c^\varepsilon - \tfrac{1}{2} \operatorname{div} \mathbf{X}\, |\nabla c^\varepsilon|^2\right) + (\tfrac{1}{\varepsilon}\Psi' + W_{,c})\mathbf{X} \cdot \nabla c^\varepsilon.\end{aligned}$$

Now we see that via partial integration

$$\int D\mathbf{X} : (\Psi \mathrm{Id}) = \int \operatorname{div} \mathbf{X}\, \Psi = -\int \mathbf{X} \cdot \nabla c^\varepsilon\, \Psi', \quad (28)$$

$$\int D\mathbf{X} : (W \mathrm{Id}) = \int \operatorname{div} \mathbf{X}\, W = -\int \mathbf{X} \cdot \nabla c^\varepsilon\, W_{,c} + \mathbf{X}_i W_{,\mathcal{E}_{kl}} \partial_i \partial_k \mathbf{u}^\varepsilon_l \quad (29)$$

$$= -\int \mathbf{X} \cdot \nabla c^\varepsilon W_{,c} - (\partial_k \mathbf{X}_i) W_{,\mathcal{E}_{kl}} \partial_i \mathbf{u}^\varepsilon_l, \quad (30)$$

where we used equation (19). With $W_{,\mathcal{E}_{kl}} = S_{kl}$ we obtain

$$\int \operatorname{div}(w^\varepsilon \mathbf{X})c^\varepsilon = \int D\mathbf{X} : \left[e^\varepsilon(c^\varepsilon)\mathrm{Id} - \varepsilon\nabla c^\varepsilon \otimes \nabla c^\varepsilon + W\mathrm{Id} - (\nabla \mathbf{u}^\varepsilon)^T S\right]. \quad (31)$$

We now introduce the mean value of w^ε as $\bar{w}^\varepsilon$ and use integration by parts to obtain

$$\int_\Omega w^\varepsilon \mathbf{X}\cdot\nabla c^\varepsilon = -\int_\Omega \nabla w^\varepsilon \cdot \mathbf{X} c^\varepsilon - \int_\Omega (w^\varepsilon - \bar{w}^\varepsilon) c^\varepsilon \operatorname{div}\mathbf{X} - \bar{w}^\varepsilon \int_\Omega c^\varepsilon \operatorname{div}\mathbf{X}. \quad (32)$$

Combining equation (31) and (32), one arrives at

$$\bar{w}^\varepsilon = \frac{1}{\int_\Omega c^\varepsilon \operatorname{div}\mathbf{X}} \int_\Omega D\mathbf{X} : \left[(e^\varepsilon(c^\varepsilon) + W(c^\varepsilon, \mathbf{u}^\varepsilon))\mathrm{Id} - \varepsilon\nabla c^\varepsilon \otimes \nabla c^\varepsilon - (\nabla \mathbf{u}^\varepsilon)^T S\right]$$
$$- \nabla w^\varepsilon \cdot \mathbf{X} c^\varepsilon - (w^\varepsilon - \bar{w}^\varepsilon) c^\varepsilon \operatorname{div}\mathbf{X}\, dx,$$

where we choose a smooth $\mathbf{X}$ such that $\int_\Omega c^\varepsilon \operatorname{div}\mathbf{X} \neq 0$. Using elliptic regularity theory, the lemma can be verified.

With this bound we conclude to weak convergence of the chemical potential.

Corollary 29. *There exist constants $C, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all $T > 0$ it holds*

$$\int_T^{T+1} \|w^\varepsilon(.,t)\|_{H^{1,2}(\Omega)} \le C. \quad (33)$$

Therefore, for a sequence $\varepsilon_j \to 0$ there exists a function $w \in L^2(0,T;H^{1,2}(\Omega))$ such that

$$w^{\varepsilon_j} \to w \qquad \textit{weakly in } L^2(0,T;H^{1,2}(\Omega)).$$

5.3 Convergence of deformation

Using the monotonicity of $W_{,\mathcal{E}}$, see (3), we obtain that the elastic energy density fulfills

$$W(c,\mathcal{E}) \ge C_0|\mathcal{E}|^2 - C_1(|c|^2+1)$$

for some constants $C_0, C_1 > 0$. Therefore we have for solutions $(c^\varepsilon, \mathbf{u}^\varepsilon)$

$$\int_\Omega |\mathcal{E}(\mathbf{u}^\varepsilon)|^2\, dx \le C\left(1 + \int_\Omega W(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon))\, dx + \int_\Omega |c^\varepsilon|^2\, dx\right).$$

Since the W-term is bounded by the total energy $\mathbf{E}^\varepsilon_{\mathrm{pf}}$ and the c^ε-term by the a priori estimate in Lemma 24, we have that $\|\mathcal{E}(\mathbf{u}^\varepsilon)\|_{L^2(\Omega)}$ is bounded uniformly in t and ε. By Korn's inequality we can also control the deformation vector $\mathbf{u}^\varepsilon$ in $H^{1,2}(\Omega)^n$.

Since $L^2(0,T;H^{1,2}(\Omega)^n)$ is a reflexive space, we have weak compactness of the deformation vector, i.e. for all sequences $(\varepsilon_j)_{j\in\mathbb{N}}$ there exists a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ such that

$$\mathbf{u}^{\varepsilon_{j_k}} \to \mathbf{u} \quad \text{weakly in } L^2_{\text{loc}}(0,\infty;H^{1,2}(\Omega)^n)$$

for some $\mathbf{u} \in L^2_{\text{loc}}(0,\infty;H^{1,2}(\Omega)^n)$.

Now we use again the monotonicity of $W_{,\mathcal{E}}$ to get

$$\begin{aligned} c_1\|\mathcal{E}(\mathbf{u}^\varepsilon - \mathbf{u})\|^2_{L^2(\Omega\times(0,T))} & \\ &\leq \int_{\Omega\times(0,T)} \big(W_{,\mathcal{E}}(c^\varepsilon,\mathcal{E}(\mathbf{u}^\varepsilon)) - W_{,\mathcal{E}}(c^\varepsilon,\mathcal{E}(\mathbf{u}))\big) : \mathcal{E}(\mathbf{u}^\varepsilon - \mathbf{u}) \\ &= -\int_{\Omega\times(0,T)} W_{,\mathcal{E}}(c^\varepsilon,\mathcal{E}(\mathbf{u})) : \mathcal{E}(\mathbf{u}^\varepsilon - \mathbf{u}). \end{aligned} \tag{34}$$

The last equality is due to the divergence free stress tensor, cf. (4). One should notice that only $W_{,\mathcal{E}}(c^\varepsilon,\mathcal{E}(\mathbf{u}^\varepsilon))$, but not $W_{,\mathcal{E}}(c^\varepsilon,\mathcal{E}(\mathbf{u}))$ is divergence free, since only in the former term the respective deformation function $\mathbf{u}^\varepsilon$ meets the condition (4).

By the strong convergence of the concentration function and the weak convergence of the deformation field, the right hand side of equation (34) goes to zero, i.e. we obtain strong convergence of the strain tensor for the sequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$. By Korn's inequality we have that the deformation vector converges strongly in $L^2_{\text{loc}}(0,\infty;H^{1,2}(\Omega)^n)$. Then for almost all t we have that $\nabla\mathbf{u}^{\varepsilon_{j_k}}(t)$ converges strongly to $\nabla\mathbf{u}(t)$ in $L^2(\Omega)$.

This verifies the third convergence statement of the main theorem. So far, we have shown the convergences as stated in the theorem, but we still have to verify, if the limit functions do represent a generalized solution according to Definition 15. Indeed the first diffusion equation immediately follows from equation (17) and the convergences of the concentration function and potential. The identity 6 in Definition 15 follows from (19) in the limit $\varepsilon \to 0$, as $\nabla\mathbf{u}^\varepsilon$ and c^ε converge strongly. The other conditions require the specification of the varifold.

5.4 The limit varifold and the Gibbs-Thomson law

This part deals with the limit varifold V. It is mainly derived from the convergence mentioned in Proposition 19 and we show that using Theorem 21 we verify the remaining conditions of Definition 15.

The energy density $e^\varepsilon(c^\varepsilon) := \frac{\varepsilon}{2}|\nabla c^\varepsilon|^2 + \frac{1}{\varepsilon}\Psi(c^\varepsilon)$ and $\varepsilon\nabla c^\varepsilon \otimes \nabla c^\varepsilon$ are bounded by the initial energy:

$$\int_0^T\int_\Omega e^\varepsilon(c^\varepsilon)\,dx\,dt \leq \int_0^T \mathbf{E}^\varepsilon_{pf}(t)\,dt \leq T\mathbf{E}_0$$

$$\int_0^T\int_\Omega \varepsilon\,|(\nabla c^\varepsilon)_i(\nabla c^\varepsilon)_j|\,dx\,dt \leq \int_0^T\int_\Omega e^\varepsilon(c^\varepsilon)\,dx\,dt \leq T\mathbf{E}_0.$$

By compactness there exist Radon measures μ and μ_{ij} according to (23) and (24). But since we also have energy estimates for all times t, we can split the measures $d\mu(x,t)$ and $d\mu_{ij}(x,t)$ into a spatial and time component, both being still Radon measures:

$$d\mu(x,t) = d\mu^t(x)dt, \qquad d\mu_{ij}(x,t) = d\mu^t_{ij}(x)dt.$$

The energy estimates in Lemma 23 show that the energies of the phase field solutions are non-increasing. This feature carries through the limit ε going to zero:

Lemma 30. *For a sequence $\varepsilon_k \to 0$ there exists a non-increasing function $\mathbf{E}\colon [0,\infty) \to [0,\infty)$ such that for all $t > 0$*

$$\mathbf{E}^{\varepsilon_k}_{pf}(t) \to \mathbf{E}(t).$$

One has to verify that this function $\mathbf{E}$ is indeed the energy of the sharp interface model. As mentioned above the interfacial energy converges to a Radon measure μ. Together with the strong convergence of the deformation vector $\mathbf{u}$, the function $\mathbf{E}$ can be identified as the energy of the limiting system:

$$\mathbf{E}(t) = \mu^t(\Omega) + \int_\Omega \sum_{k=+,-} W_k(\mathcal{E}(\mathbf{u}^t)).$$

This shows that part 5 of Definition 15 is fulfilled in the limit $\varepsilon \to 0$.

Equation (31) gives in the limit $\varepsilon \to 0$

$$\int 2\chi_{\Omega_-} \operatorname{div}(w\mathbf{X}) = \int D\mathbf{X} : [d\mu \mathrm{Id} - (d\mu_{ij})_{ij}] + \int D\mathbf{X} : (W\mathrm{Id} - (\nabla \mathbf{u})^T S).$$

Remark 31. The claim is now that $\int D\mathbf{X} : [d\mu\mathrm{Id} - (d\mu_{ij})_{ij}]$ can be seen as the first variation of a varifold. This will prove Proposition 19. Hence, part 2 of Definition 15 will be verified.

Proof (Proposition 19). For $\mathbf{Y}, \mathbf{Z} \in C^0(\bar{\Omega} \times [0,T], \mathbb{R}^n)$ one gets

$$\int_0^T \int_\Omega \mathbf{Y}^T \cdot (\varepsilon_k \nabla c^k \otimes \nabla c^k)\mathbf{Z} \le \int_0^T \int_\Omega |\mathbf{Y}||\mathbf{Z}| e^{\varepsilon_k}(c^{\varepsilon_k}) + \int_0^T \int_\Omega |\mathbf{Y}||\mathbf{Z}| \xi^{\varepsilon_k}(c^{\varepsilon_k}).$$

This means that in the limit $\varepsilon \to 0$, the last integral is non-positive and one gets the inequality

$$\int_0^T \int_\Omega \mathbf{Y}^T \cdot (d\mu_{ij})_{ij}\mathbf{Z} \le \int_0^T \int_\Omega |\mathbf{Y}||\mathbf{Z}| d\mu$$

which means that the measures μ_{ij} are absolutely continuous with respect to μ. Then there exist μ-measurable functions ν_{ij} such that $d\mu_{ij}(x,t) = \nu_{ij}(x,t)d\mu(x,t)$ and we get

$$0 \leq \int_0^T \int_\Omega \mathbf{Y} \cdot \big(\mathrm{Id} - (\nu_{ij})_{ij}\big)\mathbf{Z}\, d\mu(x,t). \tag{35}$$

Since the matrix $(\nu_{ij})_{ij}$ inherits the symmetry from (24), the matrix is positive semi-definite and by (35) is further holds

$$0 \leq (\nu_{ij})_{ij} \leq \mathrm{Id}.$$

This means one can write this matrix as $(\nu_{ij})_{ij} = \sum_{i=1}^n \tilde{\rho}_i \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i$ where $\tilde{\rho}_i \in [0,1], \sum_i \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i = \mathrm{Id}$. Moreover $\sum_i \tilde{\rho}_i \leq 1$, since actually for $y \in C^0(\Omega \times [0,T])$

$$\int_0^T \int_\Omega y\, \varepsilon_k \underbrace{\mathrm{tr}(\nabla c^k \otimes \nabla c^k)}_{=|\nabla c^{\varepsilon_k}|^2} = \int_0^T \int_\Omega y\, (e^{\varepsilon_k}(c^{\varepsilon_k}) + \xi^{\varepsilon_k}(c^{\varepsilon_k}))$$

and $\lim_{k\to\infty} \varepsilon_k\, \mathrm{tr}(\nabla c^k \otimes \nabla c^k) = \sum_i (\nu_{ii}) d\mu$. Recall that the trace of a matrix is the sum of its eigenvalues.

Setting $\rho_i := \tilde{\rho}_i + \frac{1}{n-1}\left(1 - \sum_{j=1}^n \tilde{\rho}_j\right) \in [0,1]$ we get

$$\mathrm{Id} - (\nu_{ij})_{ij} = \mathrm{Id} - \sum_i \tilde{\rho}_i \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i = \sum_i \rho_i \left(\mathrm{Id} - \boldsymbol{\nu}_i \otimes \boldsymbol{\nu}_i\right).$$

Thus we can see the limiting varifold as

$$dV(x,P) = \sum_i \rho_i(x) d\mu(x) \delta_{\boldsymbol{\nu}_i}(P)$$

where $\delta_{\boldsymbol{\nu}_i}$ is the projection onto the hyperplane normal to $\boldsymbol{\nu}_i$.

5.5 Control of discrepancy measure

In the case of homogeneous elasticity we have

$$|W_{,c}(c, \mathcal{E}(\mathbf{u}))| \leq C(1 + |c| + |\mathcal{E}(\mathbf{u})|)$$

which leads $W_{,c}(c^\varepsilon, \mathcal{E}(\mathbf{u}^\varepsilon))$ to be in $L^2(\Omega)$ for almost all times $t > 0$. So we can follow the proof of Chen in [Chen96] for the estimation of the discrepancy measure.

The proof is based on a blow-up technique for which we need some preparatory lemmas.

Lemma 32. *For each* $\eta > 0$ *there is a constant* $R(\eta) > 2$ *such that for all* $R > R(\eta)$ *the following holds:*

If

$$\hat{\Omega} = \{(x', x_n) \in B_R \mid x_n > Y(x')\}$$

is a domain in $\mathbb{R}^n$, $Y : \mathbb{R}^{n-1} \to \mathbb{R}$ *satisfying*

$$Y(0') \leq 0, \quad \nabla_{x'} Y(0') = 0', \quad \|D^2_{x'} Y\|_{C^0(B'_R)} \leq R^{-3} \tag{36}$$

and if $(\mathbf{c}, \mathbf{v}) \in H^{1,2}(\hat{\Omega}) \times L^2(\hat{\Omega})$ *with*

$$-\Delta \mathbf{c} + \Psi'(\mathbf{c}) = \mathbf{v} \qquad \text{in } \hat{\Omega}, \tag{37}$$

$$\frac{\partial}{\partial \nu} \mathbf{c} = 0 \qquad \text{on } \{(x', x_n) \in B_R \mid x_n = Y(x')\}, \tag{38}$$

$$\|\mathbf{v}\|_{L^2(B_R \cap \hat{\Omega})} \leq R^{-1} \tag{39}$$

then the following inequality holds

$$\int_{B_1 \cap \hat{\Omega}} \left(|\nabla \mathbf{c}|^2 - 2\Psi(\mathbf{c})\right)^+ \leq \eta \int_{B_2 \cap \hat{\Omega}} \left(|\nabla \mathbf{c}|^2 + \Psi'(\mathbf{c})^2 + \Psi(\mathbf{c}) + \mathbf{v}^2\right) + \int_{\{x \in B_1 \cap \hat{\Omega} \mid |\mathbf{c}| \geq 1 - \eta\}} |\nabla \mathbf{c}|^2. \tag{40}$$

Proof. For the proof, which we roughly sketch, one studies the interfacial region:

$$\hat{\Omega}_1 := \{x \in B_1 \cap \hat{\Omega} \mid |\mathbf{c}| \leq 1 - \eta\}.$$

For the case that $|\hat{\Omega}_1|$ is sufficiently small, one gets by Hölder inequality

$$\|\nabla \mathbf{c}\|_{L^2(\hat{\Omega}_1)} \leq |\hat{\Omega}_1|^{m^*} \|\nabla \mathbf{c}\|_{L^{2^*}(\hat{\Omega}_1)} \leq C\eta \|\nabla \mathbf{c}\|_{H^{1,2}(B_1 \cap \hat{\Omega})}$$

where $m^* = \frac{2\, 2^*}{2 - 2^*}$ with $2^* = \frac{2n}{n-2}$ for $n > 2$ and $2^* = 7$ otherwise. One can notice that one η appears on the right hand side, which finally leads to the statement. For the other case $|\hat{\Omega}_1|$ being large one can use a contradiction argument. Through this assumption the homogeneous equation $\Delta \mathbf{c} = \Psi'(\mathbf{c})$ is recovered. Here one can use elliptic regularity theory to get smoothness of the function $\mathbf{c}$. Comparison with viscosity functions yields that $\mathbf{c}$ would be in fact bounded in $[-1, 1]$. Results by Modica [Mod85] then finally finish the proof.

Now we need a control on the bulk energy of the interface. This is shown in the following lemma.

Lemma 33. *There exist positive constants* C_0 *and* η_0 *such that for every* $\eta \in (0, \eta_0]$*, every* $\varepsilon \in (0, 1]$ *and every* $(c, v) \in \mathcal{K}_\varepsilon$ *the following holds*

$$\int_{\{x \in \Omega \mid |c| \geq 1 - \eta\}} \left(e^\varepsilon(c) + \varepsilon^{-1} \Psi'(c)^2\right) \leq C_0 \eta \int_{\{x \in \Omega \mid |c| \leq 1 - \eta\}} \varepsilon |\nabla c|^2 + C_0 \varepsilon \int_\Omega v^2. \tag{41}$$

The proof of this lemma is based on the convexity property of Ψ'' for values $|c| \geq 1 - \eta$. One combines both

$$\int_\Omega v\psi = \int_\Omega \varepsilon\psi'(c)|\nabla c|^2 + \varepsilon^{-1}\Psi'(c)\psi$$

from the equation (18) and the Young inequality

$$\int_\Omega v\psi \le \int_\Omega \left(\tfrac{\varepsilon}{2}v^2 + \tfrac{1}{2\varepsilon}\psi^2\right)$$

where $\psi = \Psi'$ except in the bulk, so that one has bounds for ψ' in $\{|c| \ge 1-\eta\}$.

Proof (Theorem 21). We give a simple sketch of the proof for Theorem 21. As already mentioned above we use a blow-up technique. The set Ω is covered by balls $B_{R\varepsilon}(x_j)$ where R is as in Lemma 32. Changing variables to $y \to x_j + \varepsilon y$ and rescaling $\mathbf{v}^j(y) = \varepsilon v^\varepsilon(x_j + \varepsilon y)$, one gets the equation

$$-\Delta_y \mathbf{c}^j + \Psi'(\mathbf{c}^j) = \mathbf{v}.$$

By this blow-up process the right-hand side $\mathbf{v}$ can be decreased so much that Lemma 32 is applicable. Using Lemma 33 this ends the proof. If $\mathbf{v}$ does not fulfill the assumptions of Lemma 32, other elliptic estimates can be used. The careful choice of the covering then ensures that by assembling the covering the desired estimate is attained.

References

[All72] W. K. Allard. *On the first variation of a varifold.* Ann. of Math., **95** (1972), pp. 417–491.

[Brae91] D. Braess. *Finite Elemente.* Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[Bra78] K. A. Brakke. *The motion of a surface by its mean curvature.* Mathematical Notes, Princeton University Press, 1978.

[BroSto96] L. Bronsard and B. Stoth. *On the existence of high multiplicity interfaces.* Math. Res. Let., **3** (1996), pp. 41–50.

[Chen96] X. Chen. *Global asymptotic limit of solutions of the Cahn-Hilliard equation.* J. Differential Geom., **44** (1996), pp. 262–311.

[Cia88] P. G. Ciarlet. *Elasticity Theory, Volume I: Three-dimensional Elasticity.* North-Holland, Amsterdam, 1988.

[EvGar92] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, 1992.

[Fed69] H. Federer. *Geometric Measure Theory.* Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[FrPeLe95] P. Fratzl, O. Penrose, and J. L. Lebowitz. *Modelling of phase separation in alloys with coherent elastic misfit.* J. Stat. Physics, **95** (1999), nos. 5/6, pp. 1429–1503.

[FrGu94] E. Fried and M. E. Gurtin. *Dynamic solid-solid transitions with phase characterized by an order parameter.* Physica D, **72** (1994), pp. 287–308.

[Gar00] H. Garcke. *On mathematical models for phase separation in elastically stressed solids.* Habilitation thesis, University Bonn, 2000.

[Gar03] H. Garcke. *On Cahn-Hilliard systems with elasticity.* Proc. Roy. Soc. Edinburgh Sect. A, **133** (2003), no. 2, pp. 307–331.

[GLNRW] H. Garcke, M. Lenz, B. Niethammer, M. Rumpf, U. Weikard. *Multiple scales in phase separating systems with elastic misfit.* Contribution in *Analysis, Modeling and Simulation of Multiscale Problems*, A. Mielke (ed.), Springer-Verlag, Berlin, to appear Nov. 2006.

[Gur72] M. E. Gurtin. *The Linear Theory of Elasticity.* Handbuch der Physik, Vol. VIa/2, Springer, S. Flügge and C. Truesdell (eds.), Berlin, 1972.

[HutTon00] J. Hutchinson and Y. Tonegawa. *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory.* Calc. Var. and Part. Diff. Equat., **10** (2000), no. 1, pp. 49–84.

[Ilm93] T. Ilmanen. *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature.* J. Differential Geom., **38** (1993), no. 2, pp. 417–461.

[Kwak] D. J. C. Kwak. PhD thesis, University of Regensburg, in preparation.

[LLJ98] P. H. Leo, J. S. Lowengrub and H. J. Jou. *A diffuse interface model for microstructural evolution in elastically stressed solids.* Acta Mater., **46** (1998), pp. 2113–2130.

[LuMo89] S. Luckhaus, L. Modica. *The Gibbs-Thomson relation within the gradient theory of phase transitions.* Arch. Rat. Mech. Anal., **107** (1989), pp. 71–83.

[Mod85] L. Modica. *A gradient bound and a Liouville theorem for nonlinear Poisson equations.* Comm. Pure Appl. Math., **38** (1985), pp. 679–684.

[Mod87] L. Modica. *The gradient theory of phase transitions and the minimal interface criterion.* Arch. Rat. Mech. Anal., **98** (1987), pp. 123–142.

[Sim83] L. Simon. *Lectures on Geometric Measure Theory.* Proceedings of the Centre for Mathematical Analysis, Australian National University, Vol. 3, 1983.

[Sch97] R. Schätzle. *A counterexample for an approximation of the Gibbs-Thomson law.* Adv. Math. Sci. Appl., **7** (1997), no. 1, pp. 25–36.

[Sch01] R. Schätzle. *Hypersurfaces with mean curvature given by an ambient Sobolev function.* J. Differential Geom., **58** (2001), pp. 371–420.

[Son95] H. M. Soner. *Convergence of the phase field equations to the Mullins-Sekerka problem with a kinetic undercooling.* Arch. Rat. Mech. Anal., **131** (1995), pp. 139–197.

[Ton03] Y. Tonegawa. *Integrality of varifolds in the singular limit of reaction-diffusion equations.* Hiroshima Math. J., **33** (2003), no. 3, pp. 323–341.

[Zei88] E. Zeidler. *Nonlinear Functional Analysis and its Applications IV.* Applications to Mathematical Physics, Springer, Berlin-Heidelberg-New York, 1988.