Some Investigations on the Ephemerides of the Babylonian Moon Texts, System A

by

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In the Babylonian lunar theory, system A, a series of variable time intervals were recorded: time intervals as 1, 6 and 12 synodic months. These time intervals depend on the velocities of both sun and moon. In the Babylonian approach they were calculated as a sum of two terms: one depending only on the lunar velocity and the other only depending on the solar anomaly.

In the present work we demonstrate how using the modern ephemerides one can separate these time differences in a good approximation into two such terms. By comparing those with the ones in the Babylonian tables, we can then check the Babylonian approach. In this approach the terms stemming from the moon anomaly are all calculated from column Φ. This column is normally interpreted as the lunar contribution to the duration of a Saros:

$$1 \text{ Saros} = 6585 \text{ days} + \Phi^{H};$$

the velocity of the sun is assumed to be 30°/month. A supposed second term taking the solar anomaly into account has not been found in the Babylonian texts.

Our analysis of the time intervals gives the following results: In case of 1, 6 and 12 synodic months, the lunar term is the dominating one, the solar term just being a smaller correction. But in case of the Saros, the solar term is important while the lunar term plays a minor role. This throws doubt on the previous interpretation of column Φ.

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Centaurus 1980: vol. 24: pp. 36–50

The Babylonians were among others concerned with some astronomical quantities $q(v_☾, v_⊙)$ which depend on the irregularities in the movements of moon and sun. The duration of the synodic month can be mentioned as an example. According to system A (we are here using the terminology of A.C.T. [1]), this variable time interval was calculated as the sum of two independent contributions: one depending only on the variable moon velocity and the other only depending on the variable sun velocity. This method of separating the influence of the sun and moon anomalies into two additive terms was used by the Babylonians also in calculations of other interesting time intervals.

In the present work we will demonstrate a method to find out how quantities of the above mentioned type, $q(v_☾, v_⊙)$, depend on the lunar and solar anomalies. In this connection, we will shortly recall some of the properties calculated in the ephemerides of the moon, system A, mentioning only the columns of interest for the present investigation and stating their traditional interpretations (i.e. the interpretations that are offered in A.C.T. [1] and Neugebauer [2]).

Column T: Dates (i.e. year and month) of successive new moons (or full moons; we will here concentrate on the new moon texts).

Column Φ: Under the assumption of a constant velocity of the sun of 30°/month, the length of a Saros is [3]:

$$223 \text{ synodic months} = 1 \text{ Saros} = 6585^d + \Phi^H. \qquad (1)$$

(Φ is measured in units of large hours, where $1^d = 6^H$ (large hours) = 6,0° (time degrees).)

Column B: Longitude of the moon at conjunction with the sun. (Hence the difference column to column B can be understood as the velocity function of the sun: $v_⊙$ given in °/synodic months. Its period, equal to 12;22,8 synodic months, we will call $P_⊙$.)

Column F: The moon velocity in units of °/day. But since $v_☾$ is only tabulated once each synodic month (namely at mean conjunction), the period P_F of this function equals $P_F = p_a/(1-p_a) = 13;56,40$ synodic months, see ref. [2], p. 476. Here p_a is the anomalistic month measured in units of synodic months.

Column G: $29^d + G^H$ is the length of the synodic month in the first approximation, where only the variation of the moon velocity is taken into account.

Column J: Correction of column G stemming from the variation of the solar velocity.

We recall that the periods P_Φ, P_G, and P_F of the three functions Φ, G and F are identical, and that function G is derived from Φ and not, as one could suspect, from function F (which represents the lunar velocity).

In the recent years, some other functions of the same type as G (with corrections of type J) have been found and identified in Babylonian sources. (The understanding of these functions is mainly due to the work of Aaboe [4–6].) Starting out with function Φ, the three functions G, W and Λ are derived such that

$$\begin{aligned} 1 \text{ synodic month} &= 29^d + G^H \\ 6 \text{ synodic months} &= 177^d + W^H \\ 12 \text{ synodic months} &= 354^d + \Lambda^H. \end{aligned} \tag{2}$$

As in the case with Φ, the functions G, W and Λ take only the anomaly of the moon into account. The relations (2) are thus only valid on the "fast arc" of the ecliptic where the velocity of the sun is constant and equal to 30°/synodic month. Parallel to the correction J, corrections Z and Y are applied on the "slow arc" of the ecliptic such that $177^d + W^H + Z^H = 6$ synodic months, while $354^d + \Lambda^H + Y^H = 12$ synodic months [5,6]. We see how these time intervals are calculated as the sum of two terms, one depending on the variable moon velocity and the other depending on the variable sun velocity only. We will now try to demonstrate how one can separate the influence of the sun and moon velocity variations on such functions $q(v_☾, v_\odot)$. As to the duration of the synodic month, Δt, and the length of the synodic arc, $\Delta\lambda$ (i.e. the difference in longitude between consecutive new moons), an old well-known method can give us an idea of their dependence upon $v_☾$ and $v_\odot$ [7].

We consider two geometric models in which (for the sake of simplicity) the sun and the moon are assumed to move in circles, the centre of which is the earth:

Model ☾: The velocity of the moon is variable whereas the velocity of the sun is constant.
Model ⊙: The velocity of the sun is variable whereas the velocity of the moon is constant.

Knowing the maximal, minimal and mean velocities of moon and sun, we can now with model ☾ calculate the variation of Δt and $\Delta\lambda$ due to the variation of $v_☾$. Results:

$$\delta_☾(\Delta t) = \Delta t\ (\max) - \Delta t\ (\min) = 0.41 \text{ days}$$
$$\delta_☾(\Delta\lambda) = \Delta\lambda\ (\max) - \Delta\lambda\ (\min) = 0.4°.$$

Similarly using model ☉, we get the variation of Δt and $\Delta\lambda$ due to the variable sun velocity $v_\odot$:

$$\delta_\odot(\Delta t) = \Delta t\ (\max) - \Delta t\ (\min) = 0.17 \text{ days}$$
$$\delta_\odot(\Delta\lambda) = \Delta\lambda(\max) - \Delta\lambda(\min) = 2.2°.$$

From these results we can conclude that the irregularity of the sun movement is mainly determining the variation of $\Delta\lambda$. As to Δt, both $v_☾$ and $v_\odot$ have quite an influence on its variation. Still, $v_☾$ has the largest influence on the function Δt.

This method gives only a rough qualitative estimate of the dependence of Δt and $\Delta\lambda$ on $v_☾$ and $v_\odot$. It can only be used for these particular functions and not e.g. for the duration of 12 synodic months or a Saros. Therefore, we will now demonstrate another method to determine the dependence of a function of the type $q(v_☾, v_\odot)$ on $v_☾$ and $v_\odot$. To this purpose, we will introduce a new terminology. Let the times of consecutive new moons (i.e. conjunctions of sun and moon) be

$$t_0, t_1, t_2, \ldots \tag{3}$$

and the longitude of consecutive new moons be

$$\lambda_0, \lambda_1, \lambda_2, \ldots \tag{4}$$

These quantities can be found by means of Goldstine's tables for new and full moons [8].

From the t_i and λ_i we define

$$\Delta^n t_i = t_i - t_{i-n} \qquad i \in \{0, 1, \ldots\} \tag{5}$$

$$\Delta^n \lambda_i = \lambda_i - \lambda_{i-n} \qquad n \in \{1, 2, \ldots\} \tag{6}$$

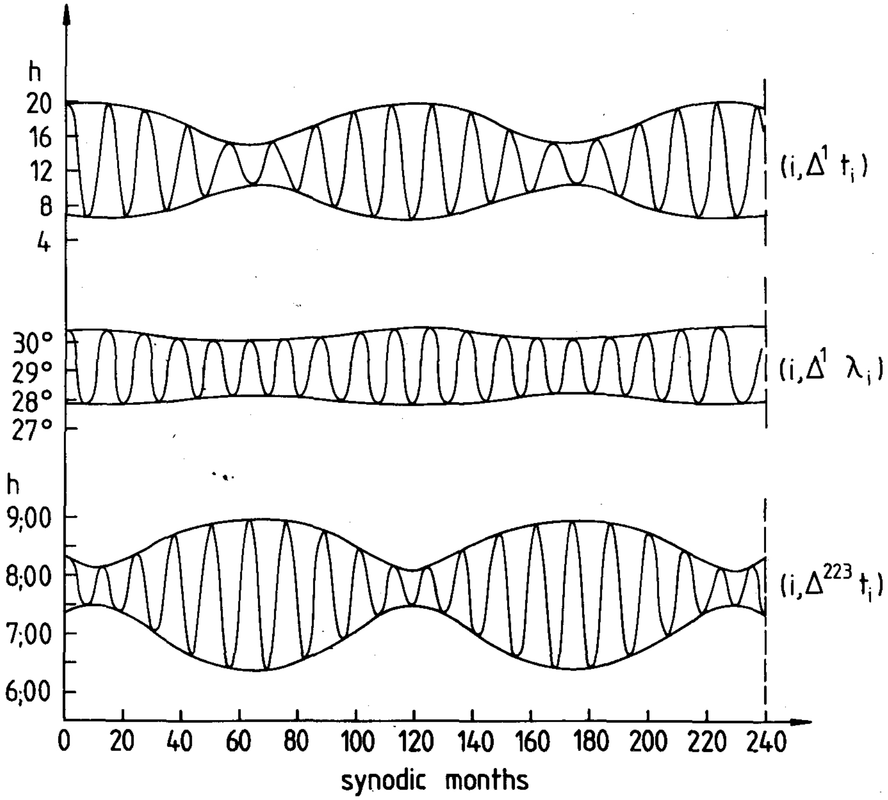

Figure 1: $\Delta^1 t$, $\Delta^1\lambda$ and $\Delta^{223} t$ plotted as functions of the lunation number.

$\Delta^n t_i$ is thus the total duration of n consecutive synodic months and $\Delta^n\lambda_i$ the length of the ecliptic arc covered by the sun during this time. For a fixed n, $\{\Delta^n t\}$ shall denote the sequence $\{\Delta^n t_1, \Delta^n t_2, \ldots\}$.

If we now plot $\Delta^1 t_i$ and $\Delta^1\lambda_i$ as functions of the number i of the lunation, we find an interesting result which is shown in figure 1. The curves $\Delta^1 t$ and $\Delta^1\lambda$ show a typical "beating" pattern, which results from a superposition of two periodic functions with slightly different periods. The curves oscillate rapidly with slowly varying amplitudes. Let us call these amplitudes (i.e. the differences between the two envelopes of the curves $\Delta^1 t$ and $\Delta^1\lambda$) $F^1(t)$ and $F^1(\lambda)$, respectively. We remark that they both seem to have the same period $D \approx 109.5$ synodic months. This we suspect to be the time of one revolution of the apside line in the ecliptic. (This is indeed the case, as shall be shown later.) Furthermore we see how $\Delta^1 t$ and $\Delta^1\lambda$ oscillate with variable periods. If we, however, abstract from this variation of the periods and calculate the mean period (of the rapid oscillations), taken over one large period D, we find an interesting result: The mean period of $\Delta^1 t$ is equal to 13.94 synodic months ($\approx P_F$), while the mean period of $\Delta^1\lambda$ is equal to 12.37 synodic months ($\approx P_\odot$).

We shall now try to understand why this is so and extract some more information from figure 1. The curves $\Delta^1 t$ and $\Delta^1\lambda$ reminded us of the sum of two periodic functions, say sine functions. Let us therefore briefly examine the behaviour of such a sum:

$$g(t) = \frac{A}{2} \sin \alpha t + \frac{B}{2} \sin \beta t. \tag{7}$$

In the simplest case, where $A = B$, we get:

$$g_0(t) = A \sin\left(\frac{\alpha + \beta}{2}\right)t \cos\left(\frac{\alpha - \beta}{2}\right)t. \tag{8}$$

The graph of such a function $g_0(t)$ is shown in the central part of figure 2. The typical pattern of the curve g_0 is immediately recognized from eq. (8) if α is not too different from β: It is a product of a sine function with a small period P_0 and a cosine function with a long period P_1 given by

$$\frac{1}{P_0} = \frac{\frac{1}{2}(\alpha + \beta)}{2\pi} = 1/2\left(\frac{1}{P_\alpha} + \frac{1}{P_\beta}\right) \tag{9}$$
$$\frac{1}{P_1} = \frac{\frac{1}{2}(\alpha - \beta)}{2\pi} = 1/2\left|\frac{1}{P_\alpha} - \frac{1}{P_\beta}\right|$$

The section of the enveloping cosine from one zero to the next we will call one period (or one section) of the envelope. Its length, D, is of course equal to $1/2\, P_1$, i.e. half the period of the cosine in eq. (8):

$$\frac{1}{D} = \left|\frac{1}{P_\alpha} - \frac{1}{P_\beta}\right| \tag{10}$$

The period of the rapid oscillations is constant and equal to P_0. In the

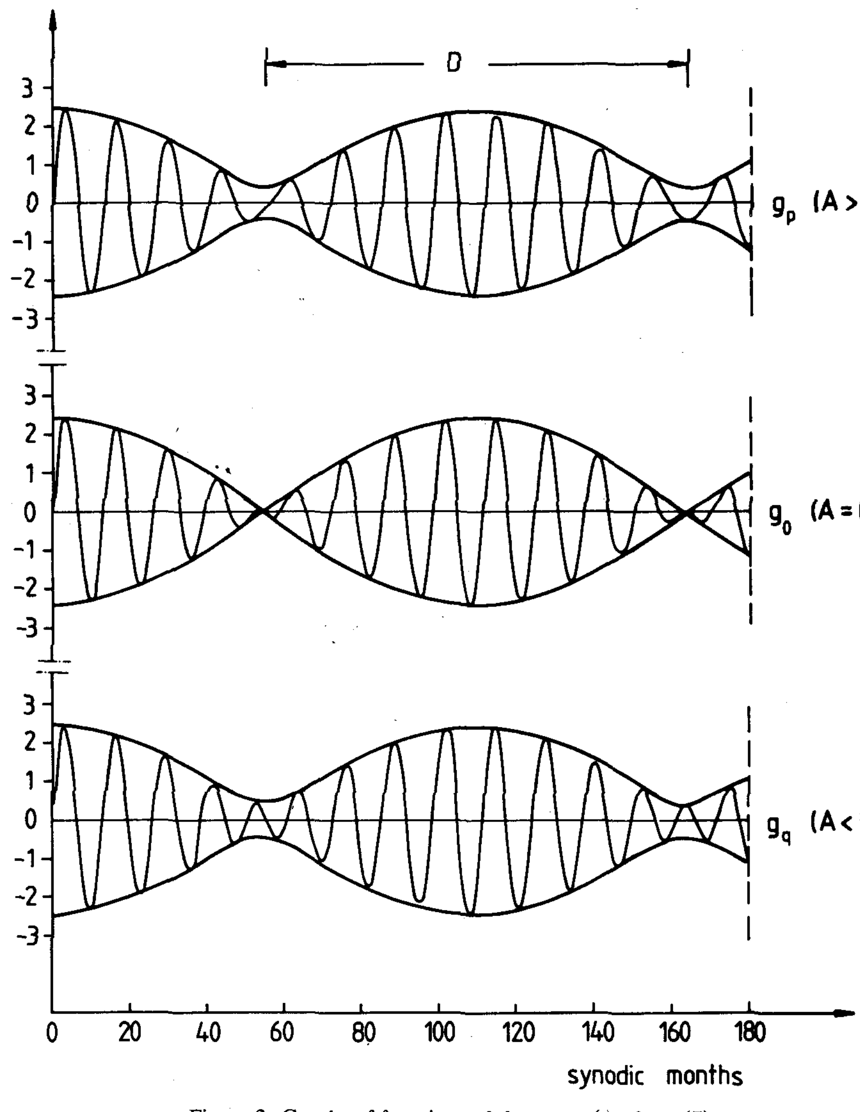

Figure 2: Graphs of functions of the type $g(t)$ of eq. (7).
The parameters in (7) are chosen as:
$P_\alpha = 2\pi/\alpha = P_\odot =$ 12;22,8 synodic months,
$P_\beta = 2\pi/\beta = P_☾ =$ 13;56,40 synodic months.
In the middle graph, g_o: $A/2 = B/2 = 1.2$.
For g_p: $A>B$, here $A/2 = 1.4$, $B/2 = 1$.
For g_q: $A<B$, here $A/2 = 1$, $B/2 = 1.4$.

case of $A \neq B$, $g(t)$ shows a slightly different pattern, see figure 2 ($g_p(t)$ for $A>B$ and $g_q1(t)$ for $A<B$.).

We notice that in these cases, the function $g(t)$ oscillates rapidly with a variable period while the period of the envelope is the same as in $g_0(t)$, namely D.

We will now point at three characteristics of the function $g(t)$ eq. (7), which are important for our understanding of $\Delta^1 t$ and $\Delta^1\lambda$, and which will be proved in detail in the appendix.

I) If $A>B$, the rapid and irregular oscillations of $g_p(t)$ will be such that their mean period, taken over one period D of the envelope, equals exactly P_α, i.e. the period of sin αt. (And, of course, correspondingly for the case $A<B$.)

II) The amplitudes of $g(t)$ vary between the limits of $V = A + B$ and $v = |A - B|$.

III) If the periods $P_\alpha = 2\pi/\alpha$ and $P_\beta = 2\pi/\beta$ of the sine functions in eq. (7) are chosen as $P_\odot \cong 12.37$ synodic months and $P_☾ \cong 13.94$ synodic months, the period D of the envelopes will be exactly equal to T, the time of revolution of the moon's apside line in the ecliptic.

We now can, of course, turn the argument around: Suppose a function $f(t)$ is known by its graph which shows a typical beating pattern. One can then, in a reasonably good approximation, write $f(t)$ as a sum of two sine functions as in eq. (7). The four parameters A, B, α and β can then be uniquely determined from the graph: The length D of one section of the envelopes can be directly measured. The mean period of the oscillation of $f(t)$, taken over one period D, can be deduced from the graph, too, by simple counting. It equals the period of the dominating term (corresponding to the larger of the amplitudes A and B), say P_α. From (10) one then calculates P_β. Finally, from the amplitude of the envelopes (measuring V and v from the graph), the amplitudes A and B themselves are determined from the rule II above.

We will now apply this method to the curves $\Delta^1 t$ and $\Delta^1\lambda$ in fig. 1. By comparing them with the curves on fig. 2, one can conclude that $\Delta^1 t$ and $\Delta^1\lambda$ in a good approximation can be written as functions of the type (7) (apart from an additive constant given by their mean values). We have already remarked that the mean rapid period of $\Delta^1 t$ equals $P_☾$, while that of $\Delta^1\lambda$ is $P_\odot$. We also stated above that the period D of their envelopes equals the revolution time of the apside line in the ecliptic. We now understand that this is so because both $\Delta^1 t$ and $\Delta^1\lambda$ are results of a superposition of two periodic functions with periods $P_☾$ and $P_\odot$, respectively. In the case of $\Delta^1 t$, the term with the period $P_☾$ is the dominating one, while for $\Delta^1\lambda$, the term with the period $P_\odot$ dominates. The amplitudes A and B of the oscillations in each curve we interpret as the variations of $\Delta^1 t$ and $\Delta^1\lambda$, caused by the sun and the moon. We therefore call them $\delta_\odot(\Delta^1 t)$ and $\delta_☾(\Delta^1 t)$, $\delta_\odot(\Delta^1\lambda)$, and $\delta_☾(\Delta^1\lambda)$ respectively. As we have demonstrated, they can be found from figure 1. The results are:

$\delta_☾(\Delta^1 t) = 9^h\!.00$ $\quad$ $\delta_\odot(\Delta^1 t) = 4^h\!.15$
$\delta_☾(\Delta^1\lambda) = 0^\circ\!.38$ $\quad$ $\delta_\odot(\Delta^1\lambda) = 2^\circ\!.18$

It is appropriate to compare these values with the corresponding values calculated with the geometric models ☾ and ⊙. We repeat those results:

$\delta_☾(\Delta t) = 0^d\!.41 = 9^h\!.8$ $\quad$ $\delta_\odot(\Delta t) = 0^d\!.17 = 4^h\!.1$
$\delta_☾(\Delta\lambda) = 0^\circ\!.4$ $\quad$ $\delta_\odot(\Delta\lambda) = 2^\circ\!.2$

Knowing that the geometric models only give a rough estimate, we can say that the agreement is excellent. The graphic analysis gives much more information and in addition, it has the advantage of being more precise and generally applicable.

Encouraged by these results, we will now go on to investigate other functions using the same graphic method. Inspired by the Babylonian columns W $(+Z)$, Λ $(+Y)$ and Φ, we have, using Goldstine's tables [8], calculated the following sequences: $\{\Delta^6 t\}$, $\{\Delta^{12} t\}$ and $\{\Delta^{223} t\}$. In figure 1, we show the graph of $\Delta^{223} t$; the others are similar to $\Delta^1 t$ in fig. 1. *A remarkable result for all these sequences is, that the* amplitudes vary with the same period of 109.5 synodic months, which is the revolution time of the moon's apside line in the ecliptic. Furthermore, in all cases the mean period of the rapid oscillations equals within an uncertainty of $\pm$ 0.02 to either $P_\odot$ = 12.37 synodic months or $P_☾$ = 13.94 synodic months. This justifies the approach of writing each of these functions as a sum of two contributions, one with the period $P_\odot$ of $v_\odot$, and the other with the period $P_☾$. We further notice that $\Delta^1\lambda$ and $\Delta^{223} t$ [*sic!*] have the same mean period, namely $P_\odot$, while

$\Delta^1 t$, $\Delta^6 t$ and $\Delta^{12} t$ all have the longer mean period $P_☾$. Hence, we can conclude that $\Delta^{223} t$ as well as $\Delta^1 \lambda$ mainly depend on the sun, the moon playing only a minor role, while $\Delta^6 t$ and $\Delta^{12} t$ as $\Delta^1 t$ mostly depend on $v_☾$. This means that our function $\Delta^{223} t$ and the Babylonian Φ do *not* alternate with the same (mean) period (column Φ being in phase with column F).

From what we have seen so far, it follows that 1 Saros approximately can be written as a constant plus two terms:

$$1 \text{ Saros} = \Delta^{223} t = \text{const.} + A_\odot + B_☾,$$

where $A_\odot$ is the dominating term. According to the traditional interpretation, however, column Φ then states the less important term, i.e. the correction $B_☾$. This raises an important question: Which kind of observation can have led the Babylonians to the function Φ? – It is hard to imagine. This may throw a little doubt on the previous interpretations of column Φ. A new interpretation, connecting column Φ more directly to the anomalistic month, would be much preferable. Remembering, that column F (i.e. the lunar velocity) as well as column G both are derived from column Φ, one should try to explain Φ as an astronomical quantity which the Babylonians could observe directly – and which contains information on the lunar velocity.

We have justified the approach of writing all the functions $\Delta^i t$, mentioned above, as sums of constants plus two periodic terms:

$$\Delta^i t = C_i + \frac{A_i}{2} \sin\left(\frac{2\pi}{P_\odot} \cdot t\right) + \frac{B_i}{2} \sin\left(\frac{2\pi}{P_☾} \cdot t\right)$$

The amplitudes A_i and B_i can, as we have demonstrated above, be found from the graphs of $\Delta^i t$. The constant C_i is of course easily determined as the mean value of $\Delta^i t$. In the following we state the results of this investigation:

	B The variation due to the anomaly of the moon	A The variation due to the anomaly of the sun	C Constant: mean value of $\Delta^i t$
$\Delta^1 t$	$9^h;00$	$4^h;10$	$29^d 12^h;44$
$\Delta^6 t$	$37^h;51$	$16^h;28$	$177^d\ 4^h;24$
$\Delta^{12} t$	$16^h;31$	$1^h;45$	$354^d\ 8^h;49$
$\Delta^{223} t$	$0^h;57$	$1^h;37$	$6585^d 7^h;42$

The Babylonian approach was also to calculate these time intervals as a constant plus the sum of two independent contributions, one depending on $v_☾$ and the other depending on $v_\odot$. For comparison, we have below stated the amplitudes of those terms:

amp. G:	$8^h;64$	amp. J:	$3^h;48$
amp. W:	$37^h;48$	amp. Z:	$22^h;22$
amp. Λ:	17^h	amp. Y:	$1^h;24$

We see that the agreement is quite good.

In case of Φ we get an interesting result: The amplitude of the zig-zag function Φ is $0^H;19,17 \approx 1^h;17$ which is far too much compared to $B = 0^h;57$. If we, however, look at the truncated Φ, the function which was first postulated by van der Waerden [3] (p. 148 ff.) and later indeed found by Aaboe [4] (p. 6 ff.), the agreement is much better: Φ was truncated by $2^H;13,20$ and $1^H;58,31,6,40$ which results in an amplitude of $\approx 0^H;14,49 \approx 0^h;59$. (This seems to support the common interpretation of column Φ.)

At this point we mention a difference between our and the Babylonian approach. In our splitting up $\Delta^i t$ we use a constant C_i plus two oscillating terms, the mean value of which is zero (i.e. the mean value of $\Delta^i t$ equals C_i). The Babylonians, however, split up these time intervals into two oscillating terms plus a constant which always equals an integer number of days, and therefore is slightly different from the mean value of $\Delta^i t$. To compensate for this, they must use oscillating terms with a mean value different from zero. Indeed, they are chosen such that the sum of their mean values plus the constant (the number of whole days) approximate the correct mean value quite well. An example:

$$\langle \Delta^1 t \rangle = 29^d 53059 = 29^d + 12^h_{;}44$$
$$29^d + \mu G + \mu J = 29^d + 14^h_{;}32 - 1^h_{;}45,16 = 29^d + 12^h_{;}47.$$

In the case of 1 Saros = $\Delta^{223}t$ one has in the texts only found the term Φ which corrects for the variable moon velocity, and no second term which could take $v_\odot$ into account. Aaboe [5] (pp. 11–15) remarked that 6585 days + μ_Φ does not approximate $\langle \Delta^{223}t \rangle$ well. He therefore constructed (parallel to the terms J, Z and Y) such a second oscillating term S, so that $6585^d + \mu_\Phi + \mu_s$ equals $\langle \Delta^{223}t \rangle$. This S is then the correction to Φ taking the variable sun velocity into account, and its amplitude is bigger than that of Φ. This is in complete agreement with our conclusions.

Appendix.

Proof of I, II and III.

We are concerned with functions of type

$$g(t) = \frac{A}{2} \sin \alpha t + \frac{B}{2} \sin \beta t \tag{7}$$

In the simplest case where $A = B$, the function $g(t)$ can be written as

$$g_0(t) = A \sin \left(\frac{\alpha+\beta}{2}\right) t \cos \left(\frac{\alpha-\beta}{2}\right) t \tag{8}$$

It has already been remarked that this function oscillates with the period

$$P_0 = \frac{2\pi}{\frac{\alpha+\beta}{2}}$$
$$\frac{1}{P_0} = \frac{1}{2}\left(\frac{1}{P_\alpha} + \frac{1}{P_\beta}\right) \tag{9}$$

while the amplitudes of g_0 vary periodically with the period D given by

$$\frac{1}{D} = \left|\frac{1}{P_\alpha} - \frac{1}{P_\beta}\right| \tag{10}$$

If $A \neq B$ we use the following identity

$$\alpha = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}$$

$$\beta = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}$$

to transform (7) into:

$$g(t) = \left(\frac{A+B}{2}\right) \sin \left(\frac{\alpha+\beta}{2}\right)t \cdot \cos\left(\frac{\alpha-\beta}{2}\right)t$$
$$+ \left(\frac{A-B}{2}\right) \cos \left(\frac{\alpha+\beta}{2}\right) t \cdot \sin \left(\frac{\alpha-\beta}{2}\right) t \qquad (11)$$

Hence we have written g as the sum of two functions g_1 and g_2 of the type g_0 discussed above. The rapid oscillation of g_1 and g_2

$$\left[\text{i.e.} \left(\frac{A+B}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)t \text{ and } \left(\frac{A-B}{2}\right) \cos \left(\frac{\alpha+\beta}{2}\right) t\right]$$

have the same period, namely P_0; but they are out of phase by + or – $\pi/2$ (depending upon $A>B$ or $A<B$). The envelopes are also out of phase: at the times $t = n{\cdot}D$, $n = 0,1,2\ldots$ the difference of the envelopes of g_1 is maximal, namely $A+B$ while that of g_2 is zero and to the times $t = \frac{1}{2} D+nD$, the difference of the envelopes of g_1 is zero and the difference of the envelopes of g_2 is maximal, namely $|A-B|$.

Knowing this, it is easy to convince oneself of II: The amplitude of $g(t)$ will vary with the period D between the limits $v = |A-B|$ and $V = A + B$.

Also, we see that if $A>B$, then $g_p(t)$ will over the period D have fulfilled exactly 1/2 oscillation more than the function

$$\sin \left(\frac{\alpha+\beta}{2}\right) t,$$

which on the other hand will have fulfilled exactly 1/2 oscillation more than a function $g_q(t)$ with $A<B$. From (9) and (11) we get:

$$D = \frac{P_\alpha \cdot P_\beta}{P_\beta - P_\alpha} = P_\alpha\left(R+\frac{1}{2}\right) = P_o \cdot R = P_\beta\left(R-\frac{1}{2}\right) \qquad (12)$$

where

$$R = \frac{P_\alpha + P_\beta}{2(P_\beta - P_\alpha)} = \frac{P_\beta}{P_\beta - P_\alpha} - \frac{1}{2}$$

(R is the number of oscillations fulfilled by sin $\left(\frac{\alpha+\beta}{2}\right)t$

during the time D. It is, of course, an irrational number in most cases.) We see now from (12) that $g_p(t)$ has fulfilled exactly as many oscillations as sin αt during the time D, namely $R + 1/2$. Or with other words: If $A>B$, the rapid and irregular oscillations of $g_p(t)$ will be such that its mean period (taken over one period D of the envelopes) exactly equals P_α. Similarly: If $A<B$, the mean period of $g_q(t)$ is exactly equal to P_β. This proves I.

Proof of III: Let p_a be the duration of the anomalistic month and p_t the duration of the tropic month. A simple reasoning will show that the revolution time T of the apside line in the ecliptic is

$$T = \frac{p_t \cdot p_a}{p_a - p_t}$$

which also can be expressed as

$$\frac{1}{T} = \frac{1}{p_t} - \frac{1}{p_a} \qquad (13)$$

However, the time periods with which we are concerned, are $P_\odot$, the tropical year measured in synodic months, and $P_☾ = P_F$ which is the period of the function tabulating the lunar velocity each mean conjunction (of ☾ and ⊙). Let us assume p_t and p_a to be given in units of synodic months. We then have the following relation (see, e.g. [2], Vol. I, pp. 476 and 375):

$$\frac{1}{P_☾} = \frac{1}{P_F} = \frac{1}{p_a} - 1 \qquad (14)$$

Furthermore,

$$\frac{1}{P_\odot} = \frac{1}{p_t} - 1 \tag{15}$$

which can be demonstrated by use of period relations: Let A, B and C be integers so that (in a good approximation)

A tropic months = B synodic months = C tropic years.

Then $A = B + C$. But $P_\odot$, the length of the tropic year measured in units of synodic months, is B/C, while $p_t = B/A$. Using this, one easily gets (15).

Combining (13), (14) and (15) we get

$$\frac{1}{T} = \frac{1}{P_\odot} - \frac{1}{P_☾} \tag{16}$$

This formula reminds us of formula (10): If in (10) we replace P_α by $P_\odot$ and P_β by $P_☾$, we get: $1/D = 1/T$.

But this means that for a function $g(t)$ where $P_\alpha = P_\odot$ and $P_\beta = P_☾$, the period D of the envelopes will be exactly T, the revolution time of the apside line. *q.e.d.*

REFERENCES

1. A.C.T.: O. Neugebauer, *Astronomical Cuneiform Texts,* 3 Vols, London, 1955.
2. O. Neugebauer, *A History of Ancient Mathematical Astronomy,* Vols. I–III, New York, 1975.
3. B. L. van der Waerden, *Anfänge der Astronomie,* Groningen, 1966, see p. 150.
4. A. Aaboe, "Some Lunar Auxiliary Tables and Related Texts from the Late Babylonian Period", *Mat. Fys. Medd. Dan. Vid. Selsk.,* 36, Nr. 12 (1968).
5. A. Aaboe, "A Computer List of New Moons for 319 B.C. to 316 B.C. from Babylon", *Mat. Fys. Medd. Dan. Vid. Selsk.,* 37, Nr. 3 (1969).
6. A. Aaboe, "Lunar and Solar Velocities and the Length of Lunation Intervals in Babylonian Astronomy", *Mat. Fys. Medd. Dan. Vid. Selsk.,* 38, Nr. 6 (1971).
7. L. Bernsen, "On the Construction of Column B in System A of the Astronomical Cuneiform Texts", *Centaurus* 14, No. 1 (1969), 23–28.
8. H. H. Goldstine, *New and Full Moons 1001 B.C. to A.D. 1651,* (Memoirs of the American Philosophical Society) Vol. 94 (1973).