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Bisectable Trapezia in Babylonian Mathematics

by

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1. Introduction

The starting point of the present investigation is the text AO 17264, published in Neugebauer, MKT, vol I, pp. 126–134. In this text a given field, shaped like a trapezium, is to be divided among six brothers in such a way that brother no 1 and brother no 2 get equal parts, brother no 3 and brother no 4 get equal parts and brother no 5 and brother no 6 get equal parts. At a first glance the problem looks like one of the many simple "brother problems" in Babylonian mathematics, but as we are going to show, the problem in AO 17264 is a rather difficult one.

Before we deal in detail with the text AO 17264 we define in section 2 a bisectable trapezium, and in section 3 we give all details about the text AO 17264. The following sections 4–11 contain our contribution to the understanding of the text. The reader who just wants to skim our paper is referred to a brief survey of these sections, given at the end of section 3.

2. Bisectable Trapezium

Let *ABCD* (fig 1) be a trapezium. The parallel sides *AD* and *BC* have the lengths x og y, respectively. Let the line *MN* (length w) be

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drawn parallel to the parallel sides of the trapezium and such that it divides the trapezium into two equal parts. Such a line of division is determined by (see below)

$$x^2 + y^2 = 2w^2. \tag{1}$$

We note that this bisector, w, is uniquely determined by the parallel sides x and y, and that it is independent of the lengths l and m of the non-parallel sides DC and AB. The reader may convince himself (e.g. by first constructing a triangle with rational sides $x-y$, l and m) that if the parallel sides of a trapezium have given rational values x and y, $x > y$, then there exist infinitely many rational values of l and m such that l and m are non-parallel sides of trapezia having parallel sides x and y. Formula (1) shows that all different trapezia with given parallel sides x and y, but with different lengths l and m of the non-parallel sides, have bisectors of the same length w.

By $[y,x]$ we denote a trapezium with parallel sides y and x. Occasionally we shall call y the bottom line and x the topline. If x and y are rational, and if also w is rational, we shall call the trapezium bisectable. As an example of a bisectable trapezium we mention one with parallel sides $y = 1$ and $x = 7$. In fact

$$1^2 + 7^2 = 2 \cdot 5^2,$$

and thus the length, w, of the bisector is equal to 5.

Formula (1), that determines the length of the bisector w, is not proved anywhere in the transmitted Babylonian text material. One

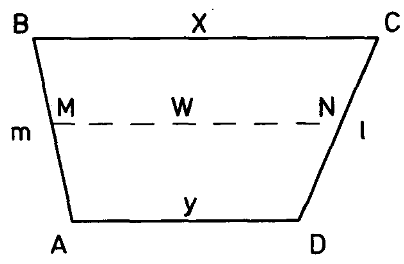

Figure 1.

proof which the Babylonians might have been able to carry out is as follows: Let (fig 2)

the area of triangle OAD be T_y,
and the area of triangle OBC be T_x,
and the area of triangle OMN be T_w.

Let the equal areas of the trapezia $AMND$ and $MBCN$ be T. Then

$$T_x = T_w + T,$$
$$T_y = T_w - T,$$

and hence

$$T_x + T_y = 2T_w.$$

Now

$$T_x, T_y, T_w$$

are proportional to

$$x^2, y^2, w^2,$$

and hence

$$x^2 + y^2 = 2w^2.$$

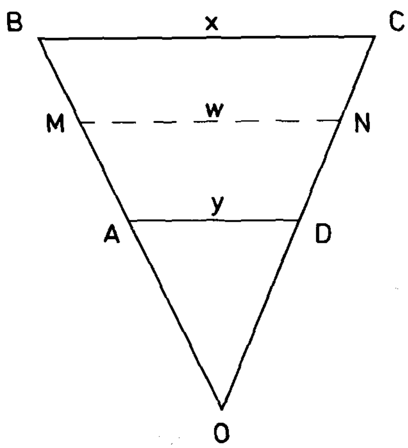

Figure 2.

3. The Text AO 17264

The text deals with a trapezium $ABCD$ (fig 3). The lengths of the parallel sides AD and BC are $x_1 = 51$ and $x_4 = 3{,}33$ (written sexagesimally), and the problem in the text consists in dividing this trapezium in the manner described in the introduction. The text solves this problem in two steps. In the first step two lines of division EF and GH, having the lengths x_2 and x_3, and being parallel to the parallel sides AD and BC, are found such that each of the three trapezia

$$[x_1, x_2]\ ,\ [x_2, x_3]\ ,\ [x_3, x_4]$$

is bisectable, and in the second step each of these trapezia is bisected. This then solves the problem.

The two lines of division x_2 and x_3 are found by the text by the formulas

$$x_3 = \frac{x_1 + x_4 + \frac{l}{m}}{\frac{1}{2}(l+m)} \tag{2}$$

and

$$x_3 - x_2 = l - m, \tag{3}$$

where l and m are the lengths of DC and AB respectively. Before we consider these formulas more closely, we notice that since x_2 and x_3 should be drawn such that each of the trapezia $[x_1, x_2]$, $[x_2, x_3]$, and $[x_3, x_4]$ is bisectable then x_2 and x_3 must satisfy the following

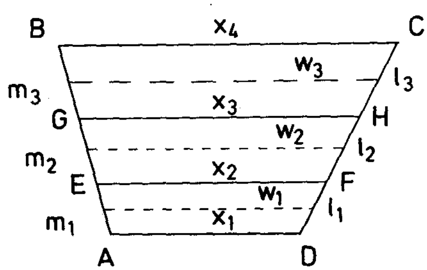

Figure 3.

system of equations

$$\begin{aligned} x_1^2+x_2^2 &= 2\ \square_1 \\ x_2^2+x_3^2 &= 2\ \square_2 \\ x_3^2+x_4^2 &= 2\ \square_3, \end{aligned} \qquad (4)$$

where $\square_i$, $i = 1,2,3$ is the square of a rational number, and the problem to be solved may now by formulated as follows: for given values of $x_1 = a$ and $x_4 = b$, $a<b$, rational solutions (x_2, x_3), $a<x_2<x_3<b$ of system (4) should be found. This formulation of the problem shows that it is by no means easy to solve the problem, considering the fact that only methods at the disposal of the Babylonian mathematicians are to be used. It therefore seems rather strange that the text uses the two simple formulas (2) and (3). Using the values $x_1 = 51$, $x_4 = 3,33$, $l = 2,15$ and $m = 1,21$ the text finds

$$x_2 = 1,33 \quad \text{and} \quad x_3 = 2,27$$

and, strange to say, these values are correct since each of the trapezia

$$\begin{aligned} [x_1, x_2] &= [51 \quad 1,33] \\ [x_2, x_3] &= [1,33 \quad 2,27] \\ [x_3, x_4] &= [2,27 \quad 3,33], \end{aligned} \qquad (5)$$

is bisectable. In fact (apart from the factor 3) the numbers in (5), written decimally, are

$$\begin{aligned} [x_1, x_2] &= [17, 31] \\ [x_2, x_3] &= [31, 49] \\ [x_3, x_4] &= [49, 71], \end{aligned}$$

and

$$\begin{aligned} 17^2+31^2 &= 2\cdot 25^2 \\ 31^2+49^2 &= 2\cdot 41^2 \\ 49^2+71^2 &= 2\cdot 61^2, \end{aligned}$$

which shows that the three trapezia are bisectable.

The formulas (2) and (3), however, cannot be correct inspite of the fact that they lead to correct values of x_2 and x_3. Neugebauer has pointed out (Neugebauer, MKT, vol I, p. 133) that formula (2) must be false, since the right-hand side of (2) does not have the dimension of a linesegment. As for formula (3) it contains the lengths l and m of the non-parallel sides, and as mentioned above these lengths may be chosen in many different ways for given values of x_1 and x_4. One might easily choose values of l and m such that (2) and (3) lead to incorrect values of x_2 and x_3, i.e. lead to non-bisectable trapezia $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_4]$.

It is evident, however, that the text AO 17264 knew of a correct solution

$$(x_1, x_2, x_3, x_4) = (17, 31, 49, 71) \tag{6}$$

of system (4), and we now conjecture that the person who made the problem in our text started from this solution and made a trapezium $ABCD$ (fig 3) where $x_1 = AD = 3 \cdot 17$ and $x_4 = BC = 3 \cdot 71$, and then asked the reader to divide this trapezium among 6 brothers such that they two and two get equal parts. The problem is solved by first dividing the trapezium $ABCD$ into three bisectable trapezia by the lines of division $x_2 = 3 \cdot 31$ and $x_3 = 3 \cdot 49$, and next by bisecting each of the bisectable trapezia $[x_1, x_2]$, $[x_2, x_3]$, $[x_3, x_4]$. But the crux of the problem is finding x_2 and x_3. The author of the text has apparently not been able to solve this problem, and he has then (and this is almost unbelievable) given some meaningless computations in formulas (2) and (3) that end up with the correct result. In no other text do we find a similar deceit.

The fact, however, that the text knew of one solution (6) of system (4) is rather surprising and calls for our unstinted praise. In section 8 we mention that, hidden in the text VAT 8512, we have found a method for computing solutions of (1) and also of (4), and we believe that this method is the one used by our text for finding the non-trivial solutions (6) of (4).

We have further examined whether other Babylonian texts deal with bisectable trapezia and we have found that besides the three bisectable trapezia mentioned above two other bisectable trapezia, [1, 7] and [7, 17] occur in some texts (Neugebauer MKT, vol 1, 289, 303, 340), (Neugebauer and Sachs, MCT, 44) and no other bisectable trapezia occur in the transmitted material. Thus the Babylonians have known 5 bisectable trapezia altogether, namely

$$[1,7],\quad [7,17],\quad [17,31],\quad [31,49],\quad [49,71],$$

and we observe that they are adjoining ones, i.e. the topline of one trapezium is equal to the bottom line of the following trapezium.

O. Neugebauer in his treatment of our text AO 17264 has computed the values m_1, m_2, and m_3 (fig 4) and he has noticed that these numbers (which by the way are not computed by the text) have constant first differences. Neugebauer suggests that this may be aimed at, since the problem in AO 17264 is a brother problem, and in such a problem a quantity is often divided into parts with constant first differences. The constancy of the first differences of m_1, m_2 and m_3 is however hardly aimed at. In section 10 it is shown that the numbers in the five bisectable trapezia above, i.e.

$$1,\quad 7,\quad 17,\quad 31,\quad 49,\quad 71$$

are values of a polynomium

$$y_i = 2i^2 - 1$$

of second degree for the consecutive values 1,2,...6 of i, and therefore the second differences are constant. In fact the first differences are

$$6,\quad 10,\quad 14,\quad 18,\quad 22$$

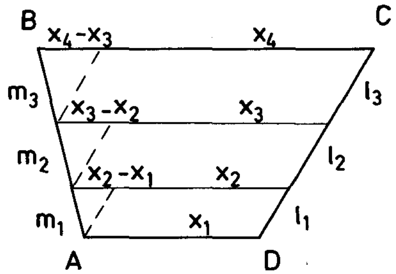

Figure 4.

and the second differences are

$$4 \quad 4 \quad 4 \quad 4.$$

Now

$$(x_1, x_2, x_3, x_4) = (17, 31, 49, 71)$$

and therefore the first differences

$$x_2 - x_1, \quad x_3 - x_2, \quad x_4 - x_3$$

are constant, and fig 4 shows that these differences are proportional to

$$m_1, \quad m_2, \quad m_3$$

and hence these numbers have constant first differences.

This ends our introductory account of the text AO 17264. We have learned that a seemingly simple problem of dividing a field among 6 brothers such that they two and two get equal parts is equivalent to the by no means simple problem of solving the system of equations (4) for given values a and b of x_1 and x_4 respectively.

In addition we have learnt that the text actually gives a correct solution of this problem, but this correct solution has been found by the wrong formulas (2) and (3). This then raises the question: what is a correct solution of the problem? Our answer is given in section 7, theorems 4 and 5. Theorem 4 gives a necessary and sufficient condition on $x_1 = a$ and $x_4 = b$ for rational solutions (x_2, x_3) of system (4), and solutions (x_2, x_3) are given in case the sufficient condition is fulfilled. Theorem 5 gives a simple necessary condition on $x_1 = a$ and $x_4 = b$ for rational solutions (x_2, x_3) of system (4). Sections 4, 5, and 6 contain the auxiliary investigation used in proving theorems 4 and 5.

Theorems 4 and 5 enable us to find all values of $x_1 = a$ and $x_4 = b$ for which (4) has rational solutions (x_2, x_3), and we can also find such solutions; but if we change the question slightly and ask if system (4), for given values of $x_1 = a$ and $x_4 = b$, has rational solutions (x_2, x_3), then it is often not possible to decide whether the sufficient condition in theorem 4 is fulfilled, and then it is not possible to decide whether system (4) has solutions, still less is it possible to find such solutions. An example of such a case is mentioned in section 7.

The last part of our paper is written in a language more congenial to Babylonian mathematics. This is particularly true of section 8. Here we give a rather simple method, hidden in a Babylonian text, for constructing bisectable trapezia. This is a remarkably simple method, which has remained unknown so far, and which we want to call to the attention of our reader. This method is, in our opinion, the method by which the Babylonians found all the bisectable trapezia that occur in our textual material.

Section 9 shows, what is well-known, (Vogel, Kurt, 1959, p. 72) that there is a connection between bisectable trapezia and Pythagorean triangles.

In section 10 we construct a sequence of adjoining bisectable trapezia by putting (in theorem 1, section 4).

$$(s,t) = (i, i+1), \quad i = 1,2,...$$

Section 11 discusses the occurrence of bisectable trapezia in later mathematical literature, and it also discusses generalizations of our problem into one of the following two directions: (1) we consider the problem of dividing a trapezium into three (or more) equal trapezia, and (2) we consider the problem of dividing a triangle into two equal parts by a line parallel to one side of the triangle.

4. *Integral Solutions of the Equation* $x^2+y^2 = 2w^2$

In this section we find in theorem 1 all integral solutions of equation (1). The method used for finding these integral solutions is well-known, and thus there is in principle nothing new in this section. On the other hand, in the literature we have not found our theorem 1, and this theorem is crucial for the proofs of some of the following theorems that are relevant for our solution of the problem in AO 17264.

Let (x, y, w) be an integral solution of (1). Here and below we assume without special mention that numbers like x, y, or w are positive (or zero). The greatest common divisor of x and y is equal to the greatest common divisor of x and w, and of y and w. If this greatest common divisor is equal to 1 the solution (x, y, w) is called a primitive solution. Along with equation (1) we consider the equation

$$x^2+y^2 = 2, \tag{7}$$

and notice that if (x, y, w) is an integral solution of (1), then $(\frac{x}{w}, \frac{y}{w})$ is a rational solution of (7). To find all rational points on the circle (7) (i.e. points with rational coordinates) it is sufficient to find all rational points on arc BC (see fig 5). One rational point on the circle is the point $A\ (-1, -1)$. Let $P\ (x, y)$ be a rational point on arc BC. The line l through A and P has a rational slope $\alpha = \frac{y+1}{x+1}$, and since the slopes of the lines AB and AC are

$$\frac{1}{\sqrt{2}+1} = \sqrt{2}-1 \quad \text{and} \quad 1$$

respectively, it follows that

$$\sqrt{2}-1<\alpha\leqq 1\,. \tag{8}$$

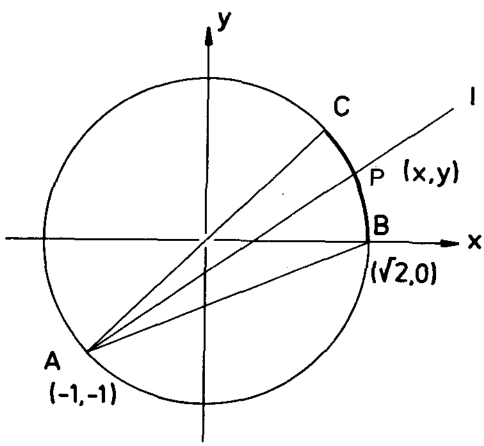

Figure 5.

The converse is also true, that is, a line l through A with rational slope α, $\sqrt{2}-1<\alpha\leqq 1$, intersects arc BC in a point with rational coordinates. To prove this we notice that the line l has the equation

$$y+1=\alpha(x+1),$$

and this line intersects the circle (7) in a point (x, y), located on arc BC, and having the rational coordinates

$$x=\frac{-\alpha^2+2\alpha+1}{1+\alpha^2} \qquad (9)$$

$$y=\frac{\alpha^2+2\alpha-1}{1+\alpha^2}.$$

Thus for α rational and $\sqrt{2}-1<\alpha\leqq 1$, (9) determines all rational points on arc BC, or all rational solutions of (7).

We are now going to find all *integral primitive solutions* (x, y, w), $y<x$, of (1). Let (x, y, w) be such a solution, then

$$\left(\frac{x}{w}, \frac{y}{w}\right)$$

is a rational solution of (7), or a rational point (different from C because $y<x$) on arc BC, and hence there exists a rational α, $\sqrt{2}-1<\alpha<1$ such that

$$\frac{x}{w}=\frac{-\alpha^2+2\alpha+1}{1+\alpha^2} \qquad (10)$$

$$\frac{y}{w}=\frac{\alpha^2+2\alpha-1}{1+\alpha^2}.$$

α being rational we can write

$$\alpha = \frac{s}{t}, \tag{11}$$

where s and t are relatively prime integers and

$$\sqrt{2}-1<\frac{s}{t}<1. \tag{12}$$

From (11) and (10) we get

$$\frac{x}{w} = \frac{-s^2+2st+t^2}{s^2+t^2}$$
$$\frac{y}{w} = \frac{s^2+2st-t^2}{s^2+t^2}. \tag{13}$$

If s *and* t *are of opposite parity* then the fractions on the right hand side of (13) are irreducible (the proof is left for the reader) and hence

$$\begin{aligned} x &= -s^2+2st+t^2 \\ y &= s^2+2st-t^2 \\ w &= s^2+t^2. \end{aligned} \tag{14}$$

If s *and* t *are of same parity* (then both of them are odd) and then the numerator and denominator in the fractions on the right-hand side of (13) are divisible by 2, and not by 4, nor by any prime $p \neq 2$ (the proof is left for the reader) and hence

$$\begin{aligned} x &= \tfrac{1}{2}(-s^2+2st+t^2) \\ y &= \tfrac{1}{2}(s^2+2st-t^2) \\ w &= \tfrac{1}{2}(s^2+t^2). \end{aligned} \tag{15}$$

The result of the preceding analysis may be expressed as follows:

To any integral primitive solution (x, y, w), $y<x$, of (1) there exist relatively prime integers s and t such that

$$\sqrt{2}-1 < \frac{s}{t} < 1, \tag{12}$$

and such that $(x\,y, w)$ is determined by (14) if s and t are of opposite parity, and (x, y, w) is determined by (15) if s and t are of the same parity.

Reversing the steps in the preceding argument we see that the converse is also true, i.e.

If s and t are relatively prime integers satisfying (12), and if (x, y, w) is determined by (14) or (15) according as s and t are of opposite or same parity, then (x, y, w) is a primitive solution of (1), and $y<x$.

Thus we have proved the following

THEOREM 1: *Let s and t be relatively prime integers such that*

$$\sqrt{2}-1 < \frac{s}{t} < 1 \tag{12}$$

all integral, primitive solutions

$$(x, y, w),\ y<x$$

of

$$x^2+y^2 = 2w^2 \tag{1}$$

are uniquely determined by (14) or (15) according as s and t are of opposite parity or of same parity.

5. Rational Solutions of the System of Equations (4)

We shall now show how all rational solutions (x_1, x_2, x_3, x_4), $x_1<x_2<x_3<x_4$, of the system

$$\begin{aligned} x_1^2+x_2^2 &= 2\ \square_1 \\ x_2^2+x_3^2 &= 2\ \square_2 \\ x_3^2+x_4^2 &= 2\ \square_3, \end{aligned} \tag{4}$$

can be constructed by means of all integral, primitive solutions of the equation

$$x^2+y^2 = 2w^2. \tag{1}$$

Let (x_1, x_2, x_3, x_4), $(x_1 < x_2 < x_3 < x_4)$ be a rational solution of system (4). Hence

$$[x_1, x_2], \quad [x_2, x_3], \quad [x_3, x_4]$$

are three bisectable trapezia. By multiplying these trapezia by $\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}$, respectively, we get three new bisectable trapezia

$$\left[1, \frac{x_2}{x_1}\right], \quad \left[1, \frac{x_3}{x_2}\right], \quad \left[1, \frac{x_4}{x_3}\right]$$

or

$$\left[1, \frac{f_1}{e_1}\right], \quad \left[1, \frac{f_2}{e_2}\right], \quad \left[1, \frac{f_3}{e_3}\right],$$

where the fractions $\frac{x_{i+1}}{x_i}$ $(i = 1,2,3)$ are reduced to fractions in their lowest terms $\frac{f_i}{e_i}$ $(i = 1,2,3)$. Hence

$$[e_1, f_1], \quad [e_2, f_2], \quad [e_3, f_3]$$

are three integral, primitive bisectable trapezia. Now

$$[x_1, x_2] = x_1\left[1, \frac{f_1}{e_1}\right]$$

$$[x_2, x_3] = x_2\left[1, \frac{f_2}{e_2}\right]$$

$$[x_3, x_4] = x_3\left[1, \frac{f_3}{e_3}\right],$$

and hence

$$x_2 = x_1 \cdot \frac{f_1}{e_1}$$

$$x_3 = x_2 \cdot \frac{f_2}{e_2} = x_1 \cdot \frac{f_1}{e_1} \cdot \frac{f_2}{e_2}$$

$$x_4 = x_3 \cdot \frac{f_3}{e_3} = x_1 \cdot \frac{f_1}{e_1} \cdot \frac{f_2}{e_2} \cdot \frac{f_3}{e_3}.$$

The result of this analysis is that to any rational solution (x_1, x_2, x_3, x_4), $x_1 < x_2 < x_3 < x_4$, of system (4) there exist three integral, primitive bisectable trapezia

$$[e_1, f_1] \quad [e_2, f_2] \quad [e_3, f_3], \quad e_i < f_i, \; i = 1, 2, 3$$

such that

$$(x_1, x_2, x_3, x_4) = x_1\left(1, \frac{f_1}{e_1}, \frac{f_1 f_2}{e_1 e_2}, \frac{f_1 f_2 f_3}{e_1 e_2 e_3}\right). \tag{16}$$

Conversely: let $[e_i, f_i]$, $e_i < f_i$, $i = 1,2,3$, be three integral, primitive bisectable trapezia (found by theorem 1) and let (x_1, x_2, x_3, x_4) be determined by (16), then (x_1, x_2, x_3, x_4) is a solution of (4). This is seen by substituting (16) into (4), and it is readily seen that $x_1 < x_2 < x_3 < x_4$. Thus we have proved the following

THEOREM 2: *All rational solutions (x_1, x_2, x_3, x_4), $x_1 < x_2 < x_3 < x_4$, of (4) are determined by (16), where x_1 is any rational number and $[e_i, f_i]$, $e_i < f_i$, $i = 1,2,3$, are any three integral primitive solutions of (1), found by theorem 1.*

On account of a later use of theorem 2, we append the following remark. Let

$$[e_1, f_1], \quad [e_2, f_2], \quad [e_3, f_3], \quad e_i < f_i, \; i = 1,2,3$$

be three integral primitive bisectable trapezia. On the basis of these three trapezia and an arbitrary rational number a, we can, by arranging the three trapezia in six different ways, form altogether six solutions of (4), namely the following ones:

$$
\begin{aligned}
&a\left(1, \frac{f_1}{e_1}, \frac{f_1 f_2}{e_1 e_2}, \frac{f_1 f_2 f_3}{e_1 e_2 e_3}\right) \\
&a\left(1, \frac{f_2}{e_2}, \frac{f_2 f_3}{e_2 e_3}, \frac{f_2 f_3 f_1}{e_2 e_3 e_1}\right) \\
&a\left(1, \frac{f_3}{e_3}, \frac{f_3 f_1}{e_3 e_1}, \frac{f_3 f_1 f_2}{e_3 e_1 e_2}\right) \\
&a\left(1, \frac{f_1}{e_1}, \frac{f_1 f_3}{e_1 e_3}, \frac{f_1 f_3 f_2}{e_1 e_3 e_2}\right) \\
&a\left(1, \frac{f_3}{e_3}, \frac{f_3 f_2}{e_3 e_2}, \frac{f_3 f_2 f_1}{e_3 e_2 e_1}\right) \\
&a\left(1, \frac{f_2}{e_2}, \frac{f_2 f_1}{e_2 e_1}, \frac{f_2 f_1 f_3}{e_2 e_1 e_3}\right)
\end{aligned} \tag{17}
$$

We notice that all six solutions, in the first "place", have the value a, and in the last "place", have the value $[a \frac{f_1 f_2 f_3}{e_1 e_2 e_3}]$.

Here we tacitly assumed that the three trapezia $[e_i, f_i]$, $i = 1,2,3$ are different two by two. If e.g. the three trapezia are all alike then (17) reduces to just one solution.

6. Number Theoretical Discussion

We now return to the equation

$$x^2 + y^2 = 2w^2 \tag{1}$$

and want to study properties of primitive integral solutions of this equation by means of elementary number theory. In this manner we get some information about the solutions of (1), information which cannot readily be obtained by looking at the solutions (14) and (15) given in theorem 1.

We arrived at the next theorem 3 in the following manner. In order to get an impression of the number of bisectable trapezia, $[y,x]$, where y and x are integers, and $y<x, x<$ some constant, e.g. $x<120$, we computed by means of theorem 1 such trapezia and found the following ones:

$$[1,7], [1,41], [7,17], [7,23], [7,103], [17,31]$$
$$[17,73], [23,47], [23,89], [31,49], [41,113]$$
$$[41,119], [47,79], [49,71], [79,119]$$

The side lenghts of these trapezia are:

$$1, 7, 17, 23, 31, 41, 47, 49, 71, 73, 79, 89, 97, 103, 113, 119$$

It is conspicuous that none of these trapezia have values of x or y in the interval from 49 to 71. We showed this result to our colleague Th. Bang and he immediately observed, first, that any integer in the interval from 49 to 71 is divisible by a prime not of the form $8n\pm 1$, and second, that the number 2 is a quadratic residue of primes of the form $8n\pm 1$ and non-residue of primes of the form $8n\pm 3$. His observations then led to the following

THEOREM 3: *Let* (x_0, y_0, w_0), $1<y_0<x_0$, *be a primitive solution of (1), and let p be a prime that divides* y_0, *then p must be of the form* $8n\pm 1$.

PROOF: Let (x_0, y_0, w_0) be a primitive solution of (1), then

$$x_0^2+y_0^2 = 2w_0^2\,.$$

Let p be a prime that divides y_0, then $p \neq 2$, for let us assume that $p = 2$, then p divides y_0^2 and $2w_0$ and hence p divides x_0^2, and so p divides x_0 contrary to the fact that x_0 and y_0 are relatively prime.

Since p divides y_0 it follows that

$$x_0^2 \equiv 2w_0^2 \pmod p$$

Since p does not divide x_0, it follows that x_0^2 is a quadratic residue modulo p. Hence $2w_0^2$ is a quadratic residue modulo p. Now p does not divide w_0, and hence w_0^2 is a quadratic residue modulo p, and finally we conclude that 2 is a quadratic residue modulo p (because residue times residue is residue, and residue times non-residue is non-residue), and hence p is of the form $8n \pm 1$.

REMARK. If p is a prime that divides x_0 then we conclude as above that p is of the form $8n \pm 1$.

ADDITIONAL REMARK. In theorem 3 we assumed $y_0 > 1$. If $y_0 = 1$, then $x_0 > 1$, and if p is a prime that divides x_0, then we conclude as above (since the number 1 is a quadratic residue modulo p) that p must be of the form $8n \pm 1$.

7. Our Solution of the Problem in AO 17264

We are now prepared to solve the problem in AO 17264, to divide a given trapezium into 3 bisectable trapezia. As mentioned in section 3 the problem consists in solving the system of equations

$$\begin{aligned} x_1^2 + x_2^2 &= 2 \quad \square_1 \\ x_2^2 + x_3^2 &= 2 \quad \square_2 \\ x_3^2 + x_4^2 &= 2 \quad \square_3 \end{aligned} \tag{4}$$

for given rational values of x_1 and x_4, $x_1 = a$ and $x_4 = b$. The problem actually consists of two parts. First: necessary and sufficient conditions on a and b should be found such that the system of equations

$$\begin{aligned} a^2 + x_2^2 &= 2 \quad \square_1 \\ x_2^2 + x_3^2 &= 2 \quad \square_2 \\ x_3^2 + b^2 &= 2 \quad \square_3 \end{aligned} \tag{18}$$

has rational solutions (x_2, x_3), $a < x_2 < x_3 < b$, and next: for values of a and b for which (18) does have solutions, such solutions should be found. These two parts of the problem are taken care of by the following

THEOREM 4: *A necessary and sufficient condition on a and b for rational solutions* (x_2, x_3) *of the system of equations (18) is that*

$$\begin{array}{ll} a & \textit{is any rational number and} \\ b & \textit{may be written in the form} \\ b = a \cdot \dfrac{f_1 f_2 f_3}{e_1 e_2 e_3} & \end{array} \tag{19}$$

where $[e_i, f_i]$, $1 \leqq e_i < f_i$, $i = 1,2,3$, *are three primitive bisectable trapezia. If (19) is fulfilled then solutions* (x_2, x_3) *of (18) are:*

$$\begin{gathered} \left(a\frac{f_1}{e_1}, a\frac{f_1 f_2}{e_1 e_2}\right) \\ \left(a\frac{f_2}{e_2}, a\frac{f_2 f_3}{e_2 e_3}\right) \\ \left(a\frac{f_3}{e_3}, a\frac{f_3 f_1}{e_3 e_1}\right) \\ \left(a\frac{f_1}{e_1}, a\frac{f_1 f_3}{e_1 e_3}\right) \\ \left(a\frac{f_3}{e_3}, a\frac{f_3 f_2}{e_3 e_2}\right) \\ \left(a\frac{f_2}{e_2}, a\frac{f_2 f_1}{e_2 e_1}\right). \end{gathered} \tag{20}$$

The "necessary part" of theorem 4 is proved as follows. Let (x_2, x_3), $a < x_2 < x_3 < b$, be a solution of (18). Hence (a, x_2, x_3, b) is a solution of (4), and by theorem 2 it then follows that there exist 3 primitive bisectable trapezia, $[e_i, f_i]$, $e_i < f_i$, $i = 1,2,3$ such that

$$(a, x_2, x_3, b) = a\left(1, \frac{f_1}{e_1}, \frac{f_1 f_2}{e_1 e_2}, \frac{f_1 f_2 f_3}{e_1 e_2 e_3}\right)$$

hence

$$b = a\frac{f_1 f_2 f_3}{e_1 e_2 e_3}$$

and thus (19) is fulfilled. In other words (19) is a necessary condition for solutions. The "sufficient part" of theorem 4 is proved as follows: We assume that a is rational and that

$$b = a \cdot \frac{f_1 f_2 f_3}{e_1 e_2 e_3}$$

where $[e_i, f_i]$, $e_i < f_i$, $i = 1,2,3$ are three bisectable trapezia. We want to prove that the system (18), for such values of a and b, has rational solutions (x_2, x_3). Now, using the remark at the end of section 5, and using the three bisectable trapezia that determine b we find that system (4) has the six solutions (17), and hence system (18) has the six solutions (20). This ends the proof of theorem 4.

A REMARK TO THEOREM 4. If (19), for given values of a and b, is satisfied by just one triplet of bisectable trapezia $[e_i, f_i]$, $e_i < f_i$, $i = 1,2,3$, then all solutions of (18) are given by (20). This follows from the analysis carried out in the proof of theorem 2. If (19), for given values of a and b is satisfied by e.g. two triplets $[e_i, f_i]$ and $[e'_i, f'_i]$, $i = 1,2,3$ then (18) has $2 \times 6 = 12$ solutions that is to say the 6 solutions (20) and the 6 solutions obtained from (20) by replacing e_i, f_i by e'_i, f'_i. If (19) is satisfied by several triplets of bisectable trapezia we proceed similarly. If some of the bisectable trapezia $[e_i, f_i]$, $i = 1,2,3$ that "enter into" b are alike, then also some of the solutions (20) are alike.

The necessary condition in theorem 4 is somewhat unhandy when applied to specific values of a and b. In a specific problem we may often with advantage use a somewhat weaker necessary condition expressed in

THEOREM 5: *Let* $\frac{b}{a} = \frac{b_1}{a_1}$ *where* a_1 *and* b_1 *are relatively prime integers. A necessary condition on a and b for rational solutions* (x_2, x_3) *of the system of equations (18) is that* $a_1 = 1$ *and* b_1 *be divisible by primes of the form* $8n \pm 1$ *and not by any other prime or* $a_1 > 1$ *and* a_1 *and* b_1 *be divisible by primes of the form* $8n \pm 1$ *and not by any other primes.*

PROOF: We assume that a and b have such values that the system of equations (18) has a solution. Hence (theorem 4) there exist 3 primitive bisectable trapezia $[e_i, f_i]$, $1 \leqq e_i < f_i$, $i = 1,2,3$ such that

$$\frac{b}{a} = \frac{b_1}{a_1} = \frac{f_1 f_2 f_3}{e_1 e_2 e_3}. \tag{21}$$

If the denominator in the fraction on the right-hand side of (21) is equal to 1, then

$$b_1 = f_1\, f_2\, f_3$$

and

$$a_1 = 1$$

and hence (theorem 3) b_1 is divisible by primes of the form $8n \pm 1$ and not by any other primes.

If the denominator in the fraction on the right-hand side of (21) is greater than 1, then (theorem 3) the numerator and the denominator of the fraction on the right-hand side of (21) are divisible by primes of the form $8n \pm 1$ and not by any other primes. The same holds good for the numerator and denominator of the fraction $\frac{a_1}{b_1}$. This ends the proof.

By means of theorem 5 we can immediately specify a great many values of a and b for which (18) does not have rational solutions, i.e. we can specify trapezia with bottom line a and topline b that cannot be divided into three bisectable trapezia. As an example we consider the system of equations (18) for $a = \frac{1}{3}$ and $b = \frac{5}{3}$, and we ask if this system of equations does have rational solutions. We find $a_1 = 1$ and $b_1 = 5$, and since b_1 is divisible by a prime not of the form $8n \pm 1$ it follows that this system (18) does not have any rational solutions (x_2, x_3). We next consider a case where a and b have such values that both a_1 and b_1 satisfy the necessary condition in theorem 5, and the question is whether the sufficient condition in theorem 4 is satisfied, i.e. whether there exist three bisectable trapezia $[e_i, f_i]$, $1 \leqq e_i < f_i$, $i = 1,2,3$ such that

$$\frac{b}{a} = \frac{f_1 f_2 f_3}{e_1 e_2 e_3}.$$

This question may be difficult to anwer. Let us e.g. consider the case where $a = 1$ and $b = 17$. We notice that the necessary condition in theorem 5 is satisfied, but we have not been able to decide whether the sufficient condition in theorem 4 is satisfied, that is we have not been able to decide where there exist three bisectable trapezia $[e_i, f_i]$ such that

$$\frac{17}{1} = \frac{f_1 f_2 f_3}{e_1 e_2 e_3}.$$

Let us next consider the case where $a = 1$ and $b = 7$. On the face of it, one would think that this case is just as difficult as the previous one, but somehow we found that

$$\frac{7}{1} = \frac{17}{7} \cdot \frac{31}{17} \cdot \frac{49}{31},$$

where each of the trapezia $[7, 17]$, $[17, 31]$, $[31, 49]$ is bisectable. Thus, for $a = 1$ and $b = 7$ the system of equations (18) has rational solutions determined by (20).

Finally we consider the specific problem in AO 17264, where

$a = 3.17$and $b = 3 \cdot 71$. Hence $a_1 = 17$ and $b_1 = 71$, and we notice that the necessary condition on a and b in theorem 5 is satisfied. Next we ask if the sufficient condition in theorem 4 is satisfied, i.e. we ask if there exist three bisectable trapezia $[e_i, f_i]$ such that

$$\frac{71}{17} = \frac{f_1}{e_1} \cdot \frac{f_2}{e_2} \cdot \frac{f_3}{e_3}.$$

It is by no means obvious whether such trapezia exist, but with an eye to the problem in AO 17264 we find

$$\frac{71}{17} = \frac{31}{17} \cdot \frac{49}{31} \cdot \frac{71}{49}$$

where $[17, 31]$, $[31, 49]$, and $[49, 71]$ are three bisectable trapezia, and hence the system of equations (18) has a solution for $a = 17$ and $b = 71$. Using the first solution in (20) we find

$$\begin{aligned}(x_2, x_3) &= \left(a\frac{f_1}{e_1}, a\frac{f_1 f_2}{e_1 e_2}\right)\\ &= \left(17 \cdot \frac{31}{17}, 17 \cdot \frac{31 \cdot 49}{17 \cdot 31}\right) = (31, 49),\end{aligned}$$

and thus we find the solution in the text. We have, in other words, not learned anything new. Our discussion of the problem does however, give us some new information. In fact, if we use the second solution in (20) we find

$$\begin{aligned}(x_2, x_3) &= \left(a\frac{f_2}{e_2}, a\frac{f_2 f_3}{e_2 e_3}\right)\\ &= \left(\frac{833}{31}, \frac{1207}{31}\right).\end{aligned}$$

This solution of the problem in AO 17264 is new. It is not mentioned by the text, and it is by no means obvious. Nor do we in the text find the remaining four solutions of (20). In the solution mentioned above we find fractions like $\frac{833}{31}$, i.e. fractions having a denominator divisible by primes of the form $8n \pm 1$, and such a fraction is not equal to a finite sexagesimal fraction. The fraction is in other words not expressible in the Babylonian number system, and so, if only for that reason, the solution is not found in the text.

According to theorem 4 solutions of the system of equations (18) (for given values of a and b) are given by (20). It is worth noticing that the solutions (20) presuppose the finding of 3 bisectable trapezia that satisfy (19), and as previously mentioned the finding of such trapezia is by no means an easy problem, and therefore the reader might get the impression that theorem 4 is of little use in connection with the problem in AO 17264. This is not true. Theorem 4 enables us to construct all problems of the type found in AO 17264, i.e. all solvable systems of equations (18). First we construct three bisectable trapezia $[e_i, f_i]$ by means of theorem 1, and next a and b are chosen such that a is arbitrary and $b = a \cdot \frac{f_1 f_2 f_3}{e_1 e_2 e_3}$. The system of equations (18) so constructed has solutions (x_2, x_3) given by (20).

Theorem 4 and 5 finish our main theoretical discussion of the problem in AO 17264. (In section 8 we shall give a second brief discussion of the problem in AO 17264). The wrong formulae (2) and (3) for the finding of x_2 and x_3 were the starting point of our investigation, and now we quite naturally ask whether we have found formulae to replace (2) and (3). The answer is only partly in the affirmative. We have not, as the text tries to do, been able to express x_2 and x_3 explicitly by a and b. (It is true that the text actually expresses x_2 and x_3 by a, b, ℓ, and m, but as mentioned in section 3 the values of l and m may be chosen in infinitely many ways for given values of a and b, and therefore the values of l and m should not enter into a formula for x_2 and x_3.) Our solutions (20) express x_2 and x_3 by a, e_i, f_i, $i = 1,2,3$, and not by a and b exclusively. It is hardly to be expected that e_i and f_i can be expressed explicitly by a and b, (in fact, as mentioned previously, we have not even been able to decide whether or not there exists e_i and f_i for $a = 1$ and $b = 17$) and therefore it is hardly to be expected that x_2 and x_3 can be expressed explicitly by a and b. In other words: simple formulae for x_2 and x_3, like formulae (2) and (3), hardly exist.

8. A Method Hidden in the Text VAT 8512 for Constructing Bisectable Trapezia

We are now going to consider a text, VAT 8512 (see Neugebauer, MKT, vol I pp 340–346), a text that at first sight does not have anything to do with bisectable trapezia, but it turns out, rather surprisingly, that the idea behind the text may be used for giving a construction of bisectable trapezia, that is a construction that lies well within the range of a Babylonian mathematician.

In the text VAT 8512 a right triangle ABK (fig 6) is divided into a triangle AML and a trapezium $MBKL$ by a line of division ML parallel to BK. Let F_1 be the area of the triangle AML and let F_2 be the area of the trapezium $BMKL$. Let $AL = b_1$ and $KL = b_2$.

It is assumed that $F_2 > F_1$ and $b_1 > b_2$. Without further explanation the text computes ML by the formula:

$$ML = \sqrt{\frac{1}{2}\left[\left(\frac{F_2 - F_1}{b_1 - b_2} + BK\right)^2 + \left(\frac{F_2 - F_1}{b_1 - b_2}\right)^2\right]} - \frac{F_2 - F_1}{b_1 - b_2}. \quad (22)$$

As shown by S. Gandz 1948, p 36 and P. Huber 1955, p 104 this formula (22) may be derived from the formula

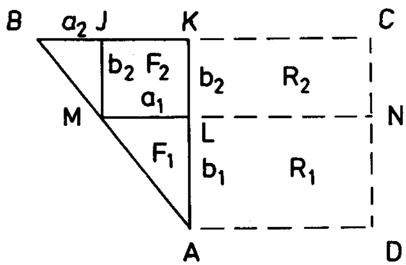

Figure 6.

$$x^2+y^2 = 2w^2 \tag{1}$$

where w divides the trapezium $[y, x]$ into two equal parts. This is seen in details as follows.

A rectangle $AKCD$, where the side AD is not yet fixed, is added to the given triangle ABK. This rectangle is divided into two rectangles R_1 and R_2 by the line ML produced to meet the line DC in N. Now:

$$R_1-R_2 = AD(b_1-b_2)$$

or

$$AD = \frac{R_1-R_2}{b_1-b_2}.$$

This holds for any value of AD. We now put

$$AD = \frac{F_2-F_1}{b_1-b_2}. \tag{23}$$

For this value of AD

$$R_1-R_2 = F_2-F_1$$

or

$$R_1+F_1 = R_2+F_2.$$

This shows that, for the chosen value (23) of AD, the large trapezium $ABCD$ is divided into two equal trapezia by the dividing line MN and hence, according to (1),

$$2MN^2 = BC^2+AD^2$$

or

$$MN = \sqrt{\tfrac{1}{2}(BC^2+AD^2)}$$

and hence

$$ML= \sqrt{\tfrac{1}{2}[(AD+BK)^2+AD^2]}-AD \tag{24}$$

and by substituting (23) into (24) we get the formula (22).

All this is well-known. But the rather few historians of mathematics who have dealt with our text have not observed that the reasoning that led to formula (22) also can be used to find bisectable trapezia. This is seen as follows.

We consider again fig 6, and assume that the sides of the similar triangles AML and MBJ, i.e. the sides

$$a_1 = ML \qquad b_1 = AL$$
$$a_2 = BJ \qquad b_2 = MJ$$

are given rational numbers. We now want to show that if these sides satisfy certain simple conditions, mentioned below in (25), then they determine a bisectable trapezium. Let the areas of triangle AML and of the trapezium $MBKL$ be F_1 and F_2 respectively. We find

$$F_1 = \tfrac{1}{2} a_1 b_1$$
$$F_2 = \tfrac{1}{2} a_2 b_2 + a_1 b_2 .$$

We now assume that the two similar triangles AML and MBJ are chosen such that

$$F_2 > F_1 \quad \text{and} \qquad (25)$$
$$b_1 > b_2 .$$

The side AD of the two rectangles R_1 and R_2 is not yet determined, but for any value of AD it holds that

$$AD = \frac{R_1 - R_2}{b_1 - b_2} .$$

If we now put

$$AD = \frac{F_2 - F_1}{b_1 - b_2} \qquad (26)$$

then

$$R_1 - R_2 = F_2 - F_1 \quad \text{or}$$
$$R_1 + F_1 = R_2 + F_2 .$$

This shows that if AD is determined by (26) then the trapezium $ABCD$, where the parallel sides AD and BC are rational, is bisected by MN, which is also rational, in other words: we have now found a simple method for constructing bisectable trapezia, a method which we shall call *method VAT 8512*. This method can be formulated as follows.

Let AML (fig 6) be a right triangle having the rational sides a_1 and b_1. Let MBJ be another right triangle having the rational sides a_2 and b_2 and being similar to the first one. The areas F_1 and F_2 of the triangle AML and the trapezium $MBKL$ are:

$$F_1 = \tfrac{1}{2} a_1 b_1$$
$$F_2 = \tfrac{1}{2} a_2 b_2 + a_1 b_2 .$$

If

$$F_1 < F_2 \quad \text{and} \quad b_1 > b_2 \tag{25}$$

then the trapezium $[y, x]$, where

$$y = \frac{F_2 - F_1}{b_1 - b_2}$$

and

$$x = a_1 + a_2 + y$$

is bisectable, and it is bisected by

$$w = a_1 + y .$$

The simplicity of method VAT 8512 appears from the following numerical example. Let the sides ML and AL of the triangle AML be

$$a_1 = 2 \quad \text{and} \quad b_1 = 4 .$$

Let the sides BJ and MJ of the triangle MBJ be

$$a_2 = 1 \quad \text{and} \quad b_2 = 2 .$$

We notice that

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

and hence the triangles are similar. We now find

$$F_1 = \tfrac{1}{2}a_1b_1 = 4$$
$$F_2 = \tfrac{1}{2}a_2b_2 + a_1b_2 = 1+4 = 5.$$

The inequalities (25) are satisfied, and hence the trapezium $[y, x]$, where

$$y = \frac{F_2 - F_1}{b_1 - b_2} = \frac{1}{2}$$
$$x = a_1 + a_2 + y = 2+1+\tfrac{1}{2} = \tfrac{7}{2}$$

is a bisectable trapezium and

$$w = a_1 + y = 2 + \tfrac{1}{2} = \tfrac{5}{2}$$

is the bisector. Multiplying this trapezium by 2 we obtain the bisectable trapezium

$$[y, x] = [1, 7]$$

and the bisector

$$w = 5\,.$$

Thus we have finally found a very simple method for determining the sides of bisectable trapezia. It is an open question whether the Babylonians actually knew of this method. In favour of the assumption that the Babylonians knew of method VAT 8512 speaks the fact that the method follows almost immediately from fig 6. It is true that this figure does not occur in the text, but the Babylonians must somehow have used this figure, otherwise formula (22) is completely incomprehensible.

However that may be, we now assign to a_1, b_1, a_2, b_2 the values given in table 1, and we find the bisectable trapezia $[y, x]$ tabulated in the penultimate row and then also the bisectable trapezia in the last row.

We notice that the trapezia in the last row are precisely the five trapezia that occur in our text material (see section 3), and now we have found these trapezia by an exceedingly simple method. In fact we believe that method VAT 8512 is the method used by the Babylonians to find bisectable trapezia.

Finally we want to prove that method VAT 8512 may be formulated in such a way that it supplies us with solutions of (1) written in the same form as in (10) section 4. To this end we first notice that method VAT 8512 may be formulated without mentioning the similar triangles AML and MBJ explicitly. We need only mention the proportional set of numbers (a_1, b_1) and (a_2, b_2), and these numbers may be fixed by fixing $(a_1; b_1)$ and a rational α. (then $a_2 = \alpha a_1$ and $b_2 = \alpha b_1$).

We now find

$$\begin{aligned} F_1 &= \tfrac{1}{2} a_1 b_1 \\ F_2 &= \tfrac{1}{2} a_2 b_2 + a_1 b_2 = \tfrac{1}{2} \alpha^2 a_1 b_1 + \alpha a_1 b_1 \\ F_2 - F_1 &= \tfrac{1}{2} a_1 b_1 (\alpha^2 + 2\alpha - 1) \\ &= \tfrac{1}{2} a_1 b_1 (\alpha + 1 - \sqrt{2})(\alpha + 1 + \sqrt{2}). \end{aligned}$$

Hence the inequalities (25) are satisfied if and only if

$$\sqrt{2} - 1 < \alpha < 1 .$$

a_1	2	3	4	5	6
b_1	4	6	8	10	12
a_2	1	2	3	4	5
b_2	2	4	6	8	10
$[y, x]$	$[\frac{1}{2}, \frac{7}{2}]$	$[\frac{7}{2}, \frac{17}{2}]$	$[\frac{17}{2}, \frac{31}{2}]$	$[\frac{31}{2}, \frac{49}{2}]$	$[\frac{49}{2}, \frac{71}{2}]$
$[y, x]$	[1, 7]	[7, 17]	[17, 31]	[31, 49]	[49, 71]

Table 1

We now assume that these inequalities are satisfied. By method VAT 8512 it follows that the trapezium $[y, x]$, where

$$y = \frac{F_2 - F_1}{b_1 - b_2} = \frac{\alpha^2 + 2\alpha - 1}{2(1-\alpha)} a_1$$

$$x = a_1 + \alpha a_1 + y = \frac{-\alpha^2 + 2\alpha + 1}{2(1-\alpha)} a_1$$

is bisectable, and the bisector is

$$w = a_1 + y = \frac{\alpha^2 + 1}{2(1-\alpha)} a_1$$

and hence

$$\frac{y}{w} = \frac{\alpha^2 + 2\alpha - 1}{\alpha^2 + 1}$$

$$\frac{x}{w} = \frac{-\alpha^2 + 2\alpha + 1}{\alpha^2 + 1}.$$

And so we find again the formulae (10) in section 4, and this time the formulae are found by means of method VAT 8512. In particular we have in this way shown that method VAT 8512 gives all rational solutions of (1).

9. *Bisectable Trapezia and Pythagorean Triangles*

The following theorem 6 reveals, what is well-known, (Vogel, Kurt, 1959, p 72) that there is a connection between bisectable trapezia and Pythagorean triangles.

THEOREM 6: *If (x, y, w), $x > y$, is a set of Pythagorean numbers, then the trapezium $[x+y, x-y]$ is bisectable and* w *is the bisector.*

The theorem may be proved as follows: Let (x, y, w), $x > y$ be a set of Pythagorean numbers. Then

$$x^2+y^2 = w^2$$

and

$$(x+y)^2+(x-y)^2 = 2x^2+2y^2 = 2w^2$$

and hence the trapezium $[x+y, x-y]$ is bisectable, and w is the bisector.

Another proof, that perhaps is new, is as follows: Let (x, y, w), $x > y$, be a set of Pythagorean numbers, and let the side of the larger square in fig 7 be $x+y$, and let the side of the inner square be $x+y-y-y = x-y$. The region between the two squares consists of 8 Pythagorean triangles. The square having the side w divides this region into two equal parts. Let w_1 (fig 8) divide the trapezium $[x+y, x-y]$ into two equal parts. Hence also the square having the side w_1 divides the region between the two squares having the sides $x+y$ and $x-y$ into two equal parts and hence $w_1 = w$.

It is however questionable whether the Babylonians knew of this connection between bisectable trapezia and Pythagorean triangles. As mentioned previously we find in the text material the following bisectable trapezia

[1,7] [7,17] [17,31] [31,49] [49,71]

and the Pythagorean triangles that lead to these trapezia are

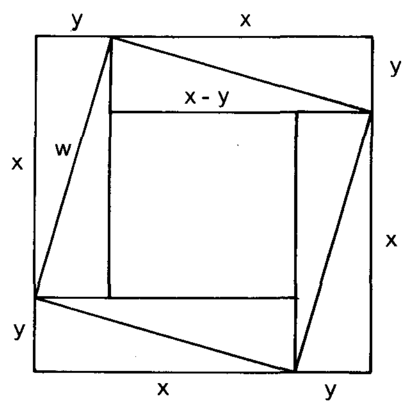

Figure 7.

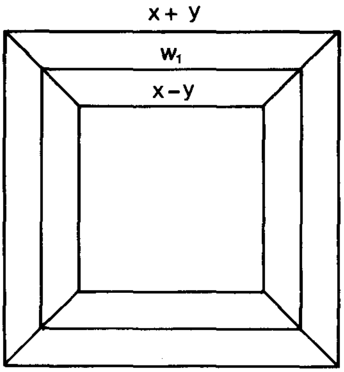

Figure 8.

$$(4,3,5) \quad (12,5,13) \quad (24,7,25) \quad (40,9,41) \quad (60,11,61)$$

but the Pythagorean triangle (60, 11, 61) does not (as far as we know) occur in our text material.

10. Adjoining Bisectable Trapezia

In this section we are going to construct a sequence of adjoining bisectable trapezia, i.e. a sequence in which the topline of one bisectable trapezium is equal to the bottom line of the next bisectable trapezium.

In section 4, theorem 1, we found all primitive bisectable trapezia expressed by parameters s and t. We now put

$$(s, t) = (i, i+1), \quad i = 1,2,3\ldots$$

Since s and t are of opposite parity formula (14) must be used, and the solution thus found is called (x_i, y_i). In this way we obtain a sequence of bisectable trapezia

$$[y_1, x_1], [y_2, x_2], \ldots, [y_i, x_i], \ldots \tag{27}$$

where

$$y_i = i^2 + 2i(i+1) - (i+1)^2 = 2i^2 - 1$$

and

$$x_i = -i^2+2i(i+1)+(i+1)^2 = 2i^2+4i+1\,.$$

We also compute y_{i+1} and find

$$y_{i+1} = 2(i+1)^2-1 = 2i^2+4i+1$$

and quite unexpectedly we find

$$x_i = y_{i+1}.$$

This shows that the two neighbouring trapezia

$$[y_i, x_i] \quad \text{and} \quad [y_{i+1}, x_{i+1}]$$

are adjoining trapezia. The sequence (27) may be written in the form

$$[y_1, y_2]\,, [y_2, y_3]\,, \dots\,, [y_i, y_{i+1}]\,, \dots$$

where

$$y_i = 2i^2-1\,.$$

For $i = 1, 2, \dots, 6$, we find

$$[1,7] \quad [7,17] \quad [17,31] \quad [31,49] \quad [49,71].$$

These are the very same trapezia that occur in our text material (see section 3), and now we have found a simple formula that leads to these trapezia. It is quite clear that the text could not reason as we did above, but since the sequence

1, 7, 17, 31, 49, 71

consists of values of a polynomium of second degree for consecutive values of i, the second differences of this sequence are constant. (We mentioned this fact in section 3.) The text might also have noticed this fact and thus been able to continue the sequence indefinitely.

ADDITIONAL REMARK. The reader may convince himself that the result found above may be generalized as follows:

Let $(s, t) = (b + ia, b + ia + a)$, $i = 1,2,...$ and let the corresponding solutions found by theorem 1 be (x_i, y_i), then

$$[y_i, x_i] \quad \text{and} \quad [y_{i+1}, x_{i+1}]$$

and adjoining bisectable trapezia, i.e.

$$x_i = y_{i+1}$$

11. Bisectable Trapezia in Later Mathematical Literature

In this last section we want to make a few remarks about the occurrence of bisectable trapezia in later mathematical literature. The problem of dividing a given trapezium into two equal trapezia by a line of division parallel to the parallel sides of the trapezium is mentioned in Euclid, *Division of Figures*, theorem 4, which states that two times the square of the line of division is equal to the sum of the squares of the parallel sides. This theorem is also mentioned by Leonardo Pisano in his *Practica Geometrica* II, p 135. Leonardo proves the theorem and considers a trapezium with parallel sides 3 and 12. He finds the square of the line of division is equal to

$$\tfrac{1}{2}(3^2 + 12^2) = \tfrac{9}{2} \cdot 17 .$$

We notice that this number is different from a square, and thus the trapezium considered by Leonardo is not bisectable (according to our definition of the word bisectable).

One might have expected to find bisectable trapezia in elementary geometries of the "Heron-type", but our search for bisectable trapezia in such works has been fruitless.

Diophant, theorem II, 19 gives rational solutions of the equation

$$x^2 - w^2 = m(w^2 - y^2)$$

where m is rational. For $m = 1$ this equation is

$$x^2 - w^2 = w^2 - y^2$$

or

$$2w^2 = x^2 + y^2$$

and this is precisely the equation (1) which determines the line of division w in a trapezium with parallel sides x and y. It is however doubtful whether this Diophant problem has its origin in bisectable trapezia. On the other hand it is rather conspicuous that the system of equations (4) (section 3) looks like many of the problems dealt with by Diophant.

One might also expect that the Babylonian mathematician has attempted to generalize the problem of bisecting a trapezium into one of the following two directions. First, he might have considered the problem of dividing a trapezium into three (or more) equal trapezia by two (or more) lines of division parallel to the parallel sides of the trapezium, and next, he might have attempted to replace the given trapezium by a triangle and thus have considered the problem of dividing a given triangle into two equal parts, one part being a triangle and the other part being a trapezium.

As to the first problem no trace of such a problem is to be found in our text material. This may find its explanation in the fact that the problem of dividing a trapezium with rational parallel sides y_1 and y_4 into three equal trapezia by lines of division y_2 and y_3 parallel to the lines y_1 and y_4 is equivalent to solving the system of equations

$$\begin{aligned} y_1^2 + y_3^2 &= 2y_2^2 \\ y_2^2 + y_4^2 &= 2y_3^2 \end{aligned} \qquad (28)$$

for y_2 and y_3, given y_1 and y_4. System (28) is equivalent to

$$y_2^2 - y_1^2 = y_3^2 - y_2^2 = y_4^2 - y_3^2 . \qquad (29)$$

It is however known, e.g. from Euler (see Dickson, vol II, p 440) that there do not exist four rational numbers whose squares form an arithemetical progression. Hence (29) has no rational solution and therefore the problem of dividing a given trapezium into three (or more) equal trapezia has no rational solution.

As to the second problem, to divide a triangle with rational base y into two equal parts, one part being a triangle, the other part being a trapezium, we can also give good reasons for not finding such a problem in our text material. In fact, the line of division w would have to satisfy the equation

$$2w^2 = y^2$$

and this equation does not have a rational solution w. This is a fact which the Babylonians most likely knew, though their knowledge was hardly based on precise reasoning.

Since it is impossible to divide a triangle into two equal parts by a line parallel to the base, it looks as if the Babylonians changed the problem slightly and considered a problem of dividing a triangle into two unequal parts and so they ended up with the rather queer problem in VAT 8512 (see section 8).

Concluding Remarks

The text Plimpton 322 (Neugebauer, O. and Sachs, A., MCT, pp 38–41) deals with integral solutions of the Pythagorean equation

$$x^2 + y^2 = z^2$$

while our text AO 17264 deals with integral solutions of the equation

$$x^2 + y^2 = 2w^2. \tag{1}$$

This means that we have two texts in Babylonian mathematics that deal with integral solutions of indeterminate equations of second degree.

The text AO 17264 actually deals with a system consisting of three equations of type (1), i.e. the system

$$\begin{aligned} x_1^2+x_2^2 &= 2 \quad \square_1 \\ x_2^2+x_3^2 &= 2 \quad \square_2 \\ x_3^2+x_4^2 &= 2 \quad \square_3 \end{aligned} \tag{4}$$

We have found all rational solutions of this system (theorem 2, section 5). The text, however, does not aim at solving this system (4). It aims at finding solutions (x_2, x_3) of (4) for given values of x_1 and x_4. We have shown in section 7 that this problem is beyond the capability of Babylonian mathematicians, and it looks as if they have given up in despair in their attempt at solving this problem and just given some meaningless computations that lead to a correct result. If this interpretation is right we here meet a procedure that is on the verge of good scholarship, but, and this must be added, a procedure that is very rare.

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