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On the Foundations of the Babylonian Column Φ: Astronomical Significance of Partial Sums of the Lunar Four

by

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Abstract:

A characteristic feature of the Babylonian mathematical astronomy is the use of periodically varying functions in the form of sequences of numbers (e.g. arithmetic progressions, zig-zag functions, or piecewise constant step functions) to describe periodically occurring astronomical phenomena. One major achievement of the Babylonian astronomers consists in a very precise determination of the periods of the number sequences used in their ephemeris texts. Any reconstruction of the Babylonian calculation schemes must explain how the fundamental periods or period relations can be determined empirically by such astronomical observations as were compiled in the Babylonian Diaries.

This paper is concerned with the Babylonian moon ephemerides. The fundamental periods used here are the length $P_{\odot}$ of the

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solar year, the period $P_☾$ of the lunar velocity, and the period P_Ω of the moon's movement in latitude. $P_☾$ and P_Ω are the periods of the moon velocity $v_☾$ and of the moon latitude $\beta_☾$, respectively, when these are measured once each synodic month, such as the Babylonians did it in their ephemeris texts.

We have recently shown that horizontal phenomena observed by the Babylonians, the so-called Lunar Four, contain information on these periods. E.g., the period $P_☾$ can be determined empirically by the sum of all Lunar Four, whereas partial sums oscillate with the period $P_\odot$. In the present paper we will explain why and how this works, through offering an astronomical interpretation and analysis of (partial) sums of the Lunar Four. In so doing, we will use the modern theory and ancient ideas on the oblique ascension of ecliptic arcs (i.e., the time it takes these arcs to pass the horizon). We also discuss the implication of this knowledge on our understanding of the development of the Babylonian astronomy.

1. Introduction on the Lunar Six

The 'Lunar Six' are some characteristic time intervals between sunrise or sunset and moonset or moonrise. These time intervals are very easy to observe: the Babylonians, indeed, recorded them regularly during the last six centuries B.C. as can be seen from the 'Diaries', the compilations of their observed data. In Figure 1 the phenomenon *KUR* is illustrated in detail as follows. The horizontal (thin) great circle is the horizon, the (thick) oblique circle is the celestial equator (as seen from Babylon), and the dotted great circle is the ecliptic. We consider a morning shortly before new moon. The sun and the moon are placed somewhere on the ecliptic near the eastern horizon; thus we have neglected the latitude of the moon. The arc of the ecliptic between moon and sun may be around 20°; the moon has thus risen visibly about $1\frac{1}{2}$ hours before sunrise. On the next morning, however, the moon will be so close to the sun that the moonrise is invisible. The time difference between this last visible moonrise (before conjunction) and the sunrise is called *KUR*. One might think that this time difference is measured by the arc of the ecliptic between moon and sun; but

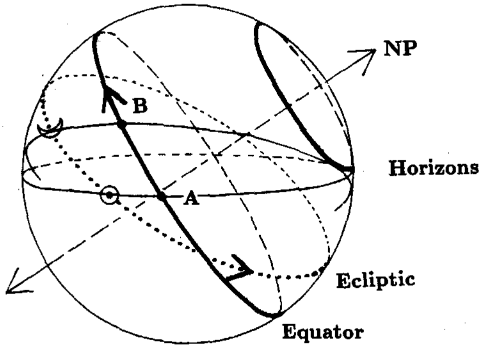

Figure 1. The celestial sphere for Babylon.

actually, times are measured on the equator and therefore we have to find the points *A* and *B* on the equator that rise at the same time as the sun and the moon, respectively. These points can be found by drawing great circles passing through the moon and the sun and being tangent to the greatest of the always visible small circles. (For drawing the celestial sphere in this way, see Olaf Schmidt [1994].) The observable *KUR* is the rising time of the elongation arc ☽☉. It is given by the length of the arc *AB* and depends on where the alongation arc ☽☉ is placed on the ecliptic and also upon the length of arc ☽☉.

KUR is one of the Lunar Six. The others of the Lunar Six, called *NA, ŠÚ, ME, NA,* and *GE* by the Babylonians,[1] are defined similarly. For completeness, let us repeat here their definitions.

Around conjunction, two characteristic time differences can be observed:

KUR=time interval between moonrise and sunrise measured on the last morning where the moon is still visible before conjunction.

NA_N=time interval between sunset and setting of the new moon crescent on the first evening where the moon is visible again after conjunction[1].

Around opposition, four characteristic time differences were regularly observed by the Babylonians. We call them in the following the 'Lunar Four':

$\check{S}\acute{U}$ = time interval from last moonset to sunrise before opposition,
NA = time interval from first sunrise after opposition to moonset[1],
ME = time interval from last moonrise to sunset before opposition,
GE = time interval from first sunset after opposition to moonrise.

Although spectacular and easy to observe, these time intervals are very complicated quantities from a theoretical point of view. Neugebauer [1957, pp. 107–109] describes in detail the factors which determine whether the new moon crescent can be seen after sunset on the evening after new moon (conjunction). The same factors also decide for how long time the new moon can be seen in the evening of its first visibility. This time interval is NA_N. The others of the Lunar Six are correspondingly determined by similar factors.

In Sections 5 and 6, the Lunar Four will be studied in more detail. Here we just mention that each of the Lunar Six strongly depends upon four variables:

1) the time interval Δt between syzygy (conjunction or opposition) and sunrise or sunset respectively:
2) the momentaneous moon velocity $v_☾$;
3) the moon's position $\lambda_☾$ in the ecliptic at the moment of the syzygy; and
4) the momentaneous latitude $\beta_☾$ of the moon.

In the case of the Lunar Four, Δt is the positive time interval between opposition (t_{op}) and sunrise (t_{sr}) or sunset (t_{ss}), respectively, so that we have

$$\begin{aligned} \Delta t &= t_{op} - t_{sr} \quad \text{for} \quad \check{S}\acute{U}, \qquad & \Delta t &= t_{sr} - t_{op} \quad \text{for} \quad NA, \\ \Delta t &= t_{op} - t_{ss} \quad \text{for} \quad ME, \qquad & \Delta t &= t_{ss} - t_{op} \quad \text{for} \quad GE\,. \end{aligned} \tag{1}$$

Taking $\check{S}\acute{U}$ as an example, we can thus write it as a function of four variables:

$$\check{S}\acute{U} = f(\Delta t, v_☾, \lambda_☾, \beta_☾) \tag{2}$$

and similarly for *NA, GE,* and *ME*, using the corresponding definitions (1) of Δt. In Figure 1 we saw that *KUR* is the rising time of a small arc of the ecliptic, say **e**, and that the magnitude of *KUR* is determined by **e**'s position in the ecliptic and by the length of **e**. Similarly for the other Lunar Six. In all cases, the length of **e**, the relevant little ecliptic arc is determined by Δt and $v_☾$, whereas the position of **e** is given by our variable $\lambda_☾$.

Due to the complexity of these dependencies, it was silently assumed for a long time that the Lunar Six were of no practical or theoretical use – at least, nobody proposed how they could have been used by the Babylonians to develop their astronomical calculation schemes. The ephemeris and procedure texts stemming from the Seleucid era, however, demonstrate that during the last three centuries B.C., the Babylonians were able to calculate and predict the magnitude of the Lunar Six. We strongly agree with O. Neugebauer [1957] who writes: 'It is one of the most brilliant achievements in the exact sciences of antiquity to have recognized the independence of these influences and to develop a theory which permits the prediction of their combined effects.'

The fact that the Babylonian astronomers were able to calculate the Lunar Four[2] by a skillful combination of these influences gives us a hint that they might as well have been able to do the reverse, i.e. to separate out the different influences from the observations. We are convinced that they really did do so, namely by simple combination of the Lunar Four data, and that their column Φ was constructed from observations of the Lunar Four. In our search for genuine Babylonian observations which possibly could have been used for constructing the column Φ, we succeeded in showing that Φ can, indeed, be derived from the Lunar Four [Brack-Bernsen 1990]. This implies that the Babylonian function Φ was found empirically, and not deduced from theoretical considerations. It also provides for the first time a proposal how the Babylonians might have used their abundant observation material of the Lunar Four.

In this paper we will explain how this is possible. We will show that by simple addition of the Lunar Four, the influence of some of the variables can be eliminated and that of others strongly reduced. By taking the sum *ŠÚ+NA* or *ME+GE,* the dependence on $\triangle t$ is eliminated and the dependence on $\beta_☾$ is strongly reduced: *ŠÚ+NA* as well as *ME+GE* are independent of $\triangle t$ and almost independent of $\beta_☾$; they depend heavily upon $\lambda_☾$ and less upon $v_☾$. By adding *ŠÚ+NA* and *ME+GE*, the dependence on $\lambda_☾$ is drastically reduced: Σ=*ŠÚ+NA+ME+GE* depends mostly on $v_☾$ and less on $\lambda_☾$. This explains why the sum Σ of the Lunar Four as a function of the lunation number varies periodically concurrently with $v_☾$.

2. Empirical information contained in the Lunar Four

The uppermost curve of Figure 2 demonstrates graphically the complexity of one of the Lunar Four. We have chosen *ŠÚ* as an example and, inspired by the Babylonians who calculated or observed characteristic moon phenomena once each synodic month, we have calculated the observable *ŠÚ* during 60 consecutive synodic months. These values of *ŠÚ* are marked in Figure 2 by crosses ($\times$), connected by straight lines, and shown as function of the lunation number *L*. We remark how *ŠÚ* varies rather chaotically and seemingly without any regularity. It is practically impossible to extract any information from *ŠÚ* by simple inspection of this curve or, as the Babylonians might have done it, by inspection of the correspondingly tabulated values of *ŠÚ*. This is not surprising since we know that *ŠÚ* is a complicated function of four variables. Similar pictures would be obtained by plotting the other Lunar Four in the same way.

It turns out, however, that the influence of the different variables on which the Lunar Four depend can be effectively reduced if some of them are combined. In Figure 2 we also show the sums *ŠÚ+NA*, *ME+GE* and the sum of all of the Lunar Four: Σ= *ŠÚ+NA+ME+GE*. They all show a much more regular behaviour than *ŠÚ*, indicating that some of the dependencies on the variables $\triangle t$, $v_☾$, $\lambda_☾$ and $\beta_☾$ have been partially eliminated by simple addition of the Lunar Four values.

Figure 3 presents a compilation of the same four quantities *ŠÚ*, *ŠÚ+NA*, *ME+GE* and Σ, now taken over a period of 450 lunations. The curves of the sums *ŠÚ+NA* and *ME+GE* are very

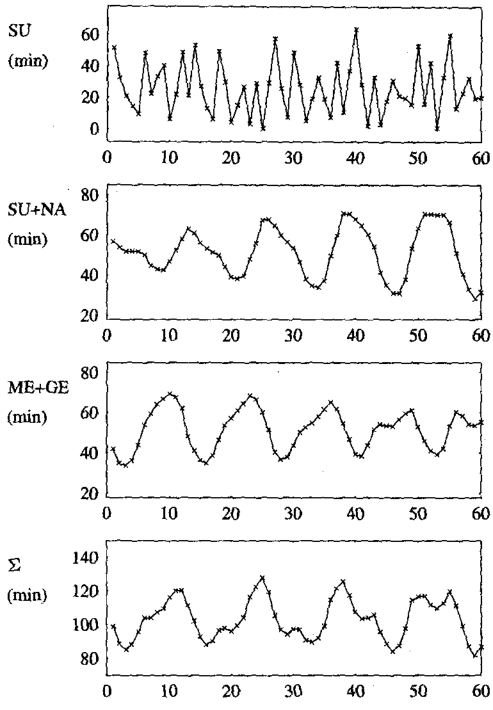

Figure 2. Different horizontal phenomena derived from the 'Lunar Four' as functions of the lunation number L, calculated for Babylon over a period of 60 months (starting on JD 1444424=Aug. 14, 759 BC). From top to bottom: *ŠÚ*, *ŠÚ+NA*, *ME+GE* and Σ.

similar. They oscillate rapidly with a mean period of ca. 12; 22 synodic months $\approx P_{\odot}$, the length of the solar year, and their amplitudes vary slowly with a period $D \approx 109.5$ synodic months, which is the revolution time of the moon's apside line in the ecliptic. Oscillating functions of this type have been discussed in an older publication [Brack-Bernsen 1980]. They result from a superposition of two oscillating functions with the periods $P_{☾}$ and $P_{\odot}$, respectively. In the case of *ŠÚ+NA* and *ME+GE*, the dominating term has the period $P_{\odot}$, reflecting the variable $\lambda_{☾}$ with this period; the influence of the variable $v_{☾}$ is being seen through the variation of the amplitudes.[3] The curve Σ, on the other hand, oscillates with

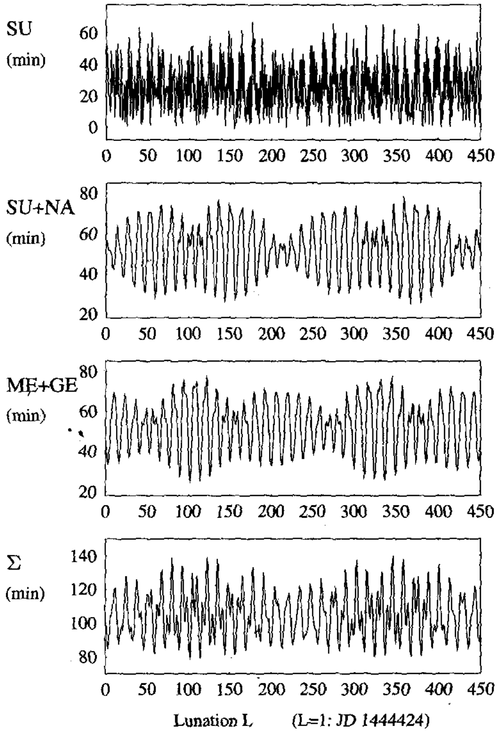

Figure 3. The same phenomena as in Figure 2, now calculated over a period of 450 synodic months.

a mean period of about 14 synodic months which is the period $P_☾$. This shows us that Σ depends most strongly on $v_☾$, while only a minor influence of $\lambda_☾$ is seen by the variation of its amplitude. The smaller irregularities in these three sum curves show that they are still influenced by other factors than those with the periods $P_\odot$ and $P_☾$.

3. Reconstruction of column Φ

The fundamental quality of column Φ consists of its period P_Φ which, with a surprisingly high accuracy, equals the period $P_☾$ of the moon velocity $v_☾$ as observed once each synodic month. Within system A of the Babylonian moon ephemerides, all calculated quantities depending on the moon velocity $v_☾$ are derived from column Φ, taking over the exact period P_Φ from column Φ. The fundamental question therefore was: from where did the Babylonians know the exact period relation which is built into their column Φ?

In searching for the origin of Φ, we had to look for observations which eventually contain information about $v_☾$ and concentrated on the Lunar Four. As already stated above, these are very complicated quantities. One of the disturbing factors stems from the fact that sun and moon both move along the ecliptic, while time and hence also the Lunar Four time intervals are measured along the equator. This fact is called the 'oblique ascension'. At the latitude of Babylon, the angle between the equator and the horizon equals 57.5°, whereas the angle between the ecliptic and the horizon varies between 34° and 81°. Therefore, depending upon its position in the ecliptic, the rising time of a 10° arc of the ecliptic varies between 6°;45 and 13°;15.

The basic idea behind the proposed reconstruction of column Φ was to reduce the influence of the oblique ascension (i.e., the dependence on $\lambda_☾$) by combining observations on the eastern with observations on the western horizon. This turned out to be possible: as we have seen in Figure 3, the sum Σ of the Lunar Four does indeed oscillate with the very period $P_☾$ we are looking for. (This indicates that Σ depends most strongly on $v_☾$ and less than each of the Lunar Four upon $\lambda_☾$, and also much less upon Δt and $\beta_☾$.) Indeed, it was shown [Brack-Bernsen, 1990] that the sum Σ of the Lunar Four:

$$\Sigma = \check{S}\acute{U}+NA+ME+GE \tag{3}$$

varies with the same period and amplitude as Φ. We therefore proposed the hypothesis: *Column Φ is derived from the sum Σ of the Lunar Four.*

Recently, using a more accurate computer code enabling us to calculate lunar phenomena at ancient times [Moshier 1992], it was shown [Brack-Bernsen 1994] that also the phases of the calculated Σ and the Babylonian Column Φ were exactly the same in the

Seleucid time. This is illustrated in Figure 4 which shows Σ over a period of 260 synodic months starting January 23, 146 B.C., compared with the Babylonian zig-zag function Φ (dashed line) for the same time period. The accordance between the 'theoretical' curve and the Babylonian function Φ is optimal. Note, in particular, that the phase between Σ and Φ, i.e. their position along the time (L) axis, had *not* been adjusted. We take this as a further convincing support of our hypothesis that Φ is derived from Σ.

We have also demonstrated [Brack-Bernsen 1994] that old Babylonian observations, as found in the text Cambyses (523 B.C.) and in the Goal-Year texts (300–50 B.C.), show the right structure and accuracy which is necessary and sufficient for the construction of Σ and thus of Φ. The Goal-Year texts also explicitly list the sums *ŠÚ+NA* and *ME+GE:* a sign that the Babylonians, indeed, were interested in partial sums of the Lunar Four. We have found no tablets with the sum Σ of all the Lunar Four, but we think it is very probable that the old astronomers went one step further and also added *ŠÚ+NA* and *ME+GE*.

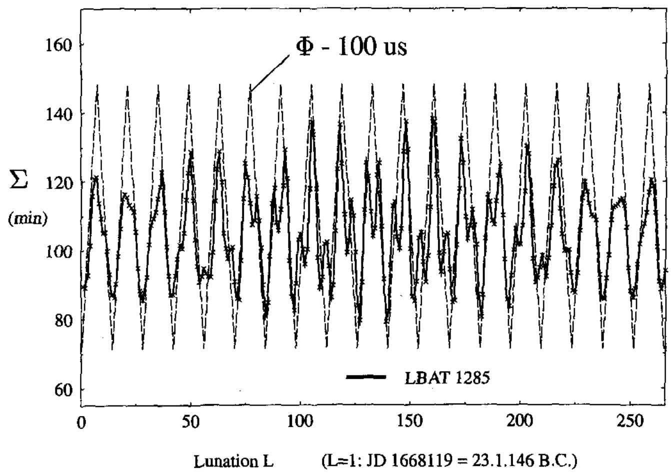

Figure 4. The sum Σ=*ŠÚ+NA+ME+GE* of the Lunar Four as function of the lunation number L (×connected by thick lines), calculated for Babylon over a period of 260 synodic months (starting on JD=1668119=Jan. 23, 146 BC). The thin dashed line shows the Babylonian zig-zag function $\Phi - 100$ *uš* determined for this time interval.

In the remainder of this paper we shall examine the partial sums *ŠÚ+NA* and *ME+GE* and the sum *Σ* in more detail, interpret them astronomically and explain why the influence of some of the variables affecting the Lunar Four can be reduced by their addition.

4. Observation of the Lunar Four

In order to get a better understanding of the Lunar Four, let us reflect upon when, how and where the Lunar Four were observed (see also Neugebauer, ACT I, pp. 229–239). We are for a moment neglecting the latitude of the moon and assume it to move along the ecliptic. We consider one full moon and assume the opposition to take place some time after sunset of a day *N*.[4]

We then have the situation indicated along the time axis in Figure 5. In the morning of day *N*, *ŠÚ* can be observed: the moon sets at the western horizon and shortly afterwards the sun rises at the eastern horizon. *ŠÚ* is the time difference between these two events, measured in time degrees[5] *uš*. Similarly, *ME* can be observed in the evening of the day *N*. Towards the end of day *N*, the opposition takes place. Thereafter, *NA* and *GE* can be observed in the morning *N*+1 and in the evening *N*+1, respectively. We see that the observations of the Lunar Four take place within a time span of about one and a half day around full moon.

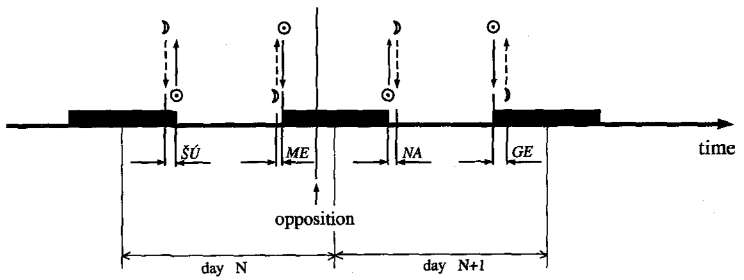

Figure 5. Times of opposition, sunrise, sunset, moonrise and moonset on the two days around opposition, marked along the time axis. Also indicated are the 'Lunar Four' time intervals *ŠÚ, ME, NA* and *GE.* (In this figure we have assumed the opposition to take place some time before midnight. In a case where the opposition takes place during daytime, the Lunar Four will occur in a different order, e.g. *ME, ŠÚ, GE, NA.*)

The Lunar Four are concerned with the sun at the horizon in one direction (east or west) and the moon in the opposite direction (west or east, respectively). In the following figures, we introduce for the sake of simplicity the symbol $\bar{\odot}$ for the 'antisun', which we define as the point on the ecliptic situated directly opposite the sun. At the very moment when the sun rises, $\bar{\odot}$ sets, and vice versa. With this definition, *ŠÚ* is the time difference between the setting of the moon ☽ and the antisun $\bar{\odot}$. Similarly, *ME* is the time difference between moonrise and antisun rise, *NA* is the time difference between antisun set and moonset, whereas *GE* is the time difference between antisun rise and moonrise.

Figure 6 illustrates the position of the moon relatively to the antisun $\bar{\odot}$ at these times (i.e., at moonsets and moonrises on the days N and $N+1$, respectively): it shows the distance (measured along the ecliptic) between the moon and $\bar{\odot}$ at the times when the Lunar Four are observed. (Remember that the opposition takes place at the moment when the moon passes the point $\bar{\odot}$.)

These distances or elongations are important for our understanding of the Lunar Four; we shall name them $\mathbf{e}_{SU}$, $\mathbf{e}_{ME}$, $\mathbf{e}_{NA}$ and $\mathbf{e}_{GE}$. They correspond to arc ☽⊙ which defines *KUR* in Figure 1 and tell us how far the moon has moved relatively to the sun in the time between opposition and the observation of the particular Lunar Four. Evidently, they depend upon the relative moon velocity $(v_{☾} - v_{\odot})$ and upon the time t_{op} at which the opposition takes place relatively to sunset t_{ss} or sunrise t_{sr}. We can take *NA* as an

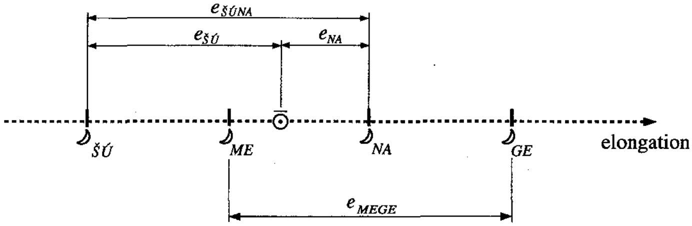

Figure 6. Positions of the moon ☽ relatively to the 'antisun' $\bar{\odot}$ at the times where the lunar four *ŠÚ, ME, NA* and *GE* are measured. ($\mathbf{e}_{SU}$, $\mathbf{e}_{ME}$, etc. denote the corresponding elongations of the moon.) The moon moves along the ecliptic from left to right; at opposition it passes the point $\bar{\odot}$.

example for all: knowing that *NA* is observed at the time of the sunrise, t_{sr}, we have:

$$\mathbf{e}_{NA} = (v_{☾} - v_{\odot}) \times (t_{sr} - t_{op}) \ .$$

The first term $(v_{☾} - v_{\odot})$ can vary between 10° per day and 14° per day; the variation of the second term Δt is much larger, namely from 0^h to 24^h, depending on whether the opposition takes place just after or before the sunrise. We can thus estimate $\mathbf{e}_{NA}$ to vary between 0° and 14°. Similarly, the other elongations are defined as

$$\mathbf{e}_{SU} = (v_{☾} - v_{\odot}) \times (t_{op} - t_{sr}) \ ,$$
$$\mathbf{e}_{ME} = (v_{☾} - v_{\odot}) \times (t_{op} - t_{ss}) \ ,$$
$$\mathbf{e}_{GE} = (v_{☾} - v_{\odot}) \times (t_{ss} - t_{op}) \ .$$

Let us now imagine what is going on at the western horizon when *ŠÚ* and *NA* are observed. The left half of Figure 7 shows us the western horizon (by the horizontal line) at sunrise on the last morning *N* before opposition (full moon), i.e. at the time when *ŠÚ* is measured. The dotted oblique line indicates the ecliptic at the western horizon on this morning; the arrow indicates the directions in which sun and moon are moving. In this figure, the ecliptic passes the horizon at a low angle of ~34°; this happens when the full moon occurs near the spring equinox. Had it taken place near the autumn equinox, the angle would have been steep, about 81°.

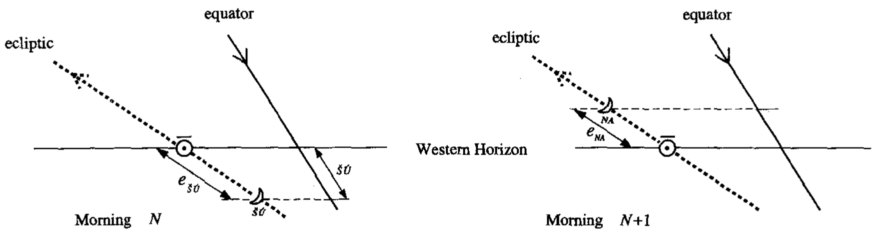

Figure 7. *Left half:* Position of moon and antisun on the western horizon at the moment of sunrise on the morning *N*. The moon has already set and the antisun is about to set. *Right half:* The situation on the next morning *N*+1: here the moon will set after the antisun.

The imaginary point $\overline{\odot}$ sets at the moment of sunrise, and through that very point on the ecliptic the moon passes at opposition. The solid oblique line indicates the equator at the western horizon, the arrow showing us the direction of the daily revolution of the sky. It is along this line that the times are measured.

The Babylonian observable *ŠÚ* is the time it takes the ecliptic arc $\mathbf{e}_{SU}$ between $☽_{SU}$ and $\overline{\odot}$ to set. This setting time is visualized in the figure, it depends on the length of $\mathbf{e}_{SU}$ and upon how steep the ecliptic stands at the western horizon. The angle between ecliptic and horizon at the moment when $\mathbf{e}_{SU}$ is setting depends on where in the ecliptic this arc is situated. Its position in the ecliptic is determined by the position $\lambda_{☾}$ in the ecliptic where the opposition takes place.

We have thus learned: *ŠÚ* is determined by $\Delta t = t_{op} - t_{sr}$, $v_{☾}$, and $\lambda_{☾}$ (and on the latitude $\beta_{☾}$ of the moon which we have neglected so far). Hereby Δt is the dominating variable.

The right part of Figure 7 shows the situation on the next morning $N+1$ when *NA* is observed. The moon now has passed the antisun $\overline{\odot}$, the opposition has taken place. *NA* is the time it takes the ecliptic arc $\overline{\odot}☽_{NA} = \mathbf{e}_{NA}$ to pass the western horizon. Analogously to the case of *ŠÚ*, we therefore get: *NA* is determined by $\Delta t = t_{sr} - t_{op}$, $v_{☾}$, and $\lambda_{☾}$ (and on the neglected lunar latitude $\beta_{☾}$).

Similar considerations of the rising full moon on the evenings before and after opposition will lead to an analogous understanding of *ME* and *GE* as the rising times of the ecliptic arcs $\mathbf{e}_{ME}$ and $\mathbf{e}_{GE}$ (again measured in time degrees).

We shall next examine the sums *ŠÚ+NA* and *ME+GE* in order to understand why they show a much more regular behaviour than each of the Lunar Four. Later, Figure 10 will show us how the Lunar Four depend heavily on $\beta_{☾}$, whereas its influence on the sums *ŠÚ+NA* and *ME+GE* is quite small.

5. *The astronomical significance of ŠÚ+NA and ME+GE*

We examine *ŠÚ+NA* as an example, since the structure of *ŠÚ+NA* and *ME+GE* in principle must be the same. Figure 8 summarizes, combining the left and the right halves of Figure 7, the situation on the western horizon on the two mornings *N* and $N+1$. Since we are concerned with the relative positions of sun

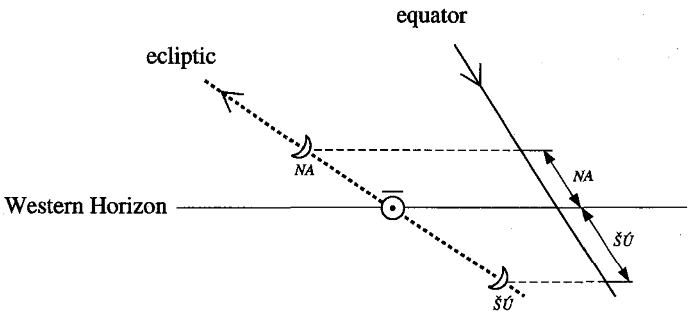

Figure 8. Position of moon and antisun at the western horizon on the two mornings N and $N+1$ around full moon at which *ŠÚ* and *NA* are measured. The opposition is assumed to take place before midnight of night N.

and moon, we have neglected the motion of the sun (and hence also of the antisun) during the time from morning N to morning $N+1$. The sum $ŠÚ+NA$ is the setting time of the ecliptic arc between $☽_{SU}$ and $☽_{NA}$. This arc is, of course, the sum of $\mathbf{e}_{SU}$ and $\mathbf{e}_{NA}$, which we shall call $\mathbf{e}_{SUNA}=\mathbf{e}_{SU}+\mathbf{e}_{NA}$. Its length equals the elongation movement of the moon in the time between morning N and morning $N+1$. (By elongation movement we understand the movement of the moon relatively to the sun.) But this elongation movement does not depend on the time at which the opposition took place. We have thus seen that by the addition of *ŠÚ* and *NA*, the variable Δt is eliminated. The elongation movement of the moon, $\mathbf{e}_{SUNA}$, only depends on the relative moon velocity $(v_{☾}-v_{\odot})$ at the day of opposition. This relative velocity is felt over a time period of slightly more than a day. The observation of $ŠÚ+NA$ starts *ŠÚ* time degrees before sunrise on day N and ends *NA* time degrees after sunrise on day $N+1$. This time interval equals thus $1+(ŠÚ+NA)/360$ days. Hence we obtain:

$$\mathbf{e}_{SUNA} = (v_{☾}-v_{\odot})\,[1+(ŠÚ+NA)/360] \;,$$

where the velocity of moon and sun are measured in degrees per day. Since $v_{\odot}=1°/\text{day}$, we get:

$$\mathbf{e}_{SUNA} = (v_{☾}-1°/\text{day})\,[1+(ŠÚ+NA)/360] \;.$$

The time which it takes $\mathbf{e}_{SUNA}$ to set only depends on its position along the ecliptic which is given by the longitude $\lambda_☾$ of the moon at opposition. We have thus seen that *ŠÚ+NA*, contrarily to *ŠÚ* or *NA*, is independent of the time of day when the opposition takes place.

Figure 9 shows the same phenomena as Figure 8, but for a situation where the opposition takes place in the morning shortly after sunrise whereas in Figure 8, it occurred before midnight. The comparison between Figures 8 and 9 clearly demonstrates that, although the single intervals *ŠÚ* and *NA* depend on the time of opposition, their sum *ŠÚ+NA* remains the same.

We now have to investigate the influence of the lunar latitude $\beta_☾$ which we have neglected so far. This is illustrated in Figure 10, where we show two situations. The first, drawn with thin lines, repeats the case of Figure 8 where the moon moves on the ecliptic (dotted line) with the latitude $\beta_☾=0$. The second case, drawn by the parallel solid line, shows the trajectory of the moon for a latitude of $\beta_☽=+3.3°$. We note that *ŠÚ* and *NA* (measured along the equator!) both change appreciably. However, their sum *ŠÚ+NA* remains the same for elementary geometrical reasons. Thus we have learned: Each single of the Lunar four depends strongly on the latitude $\beta_☾$ of the moon. This latitude varies between $+5°$ and $-5°$, so that each of the Lunar Four reflects a total latitude variation of 10°. By taking the sums *ŠÚ+NA* or *ME+GE*, the influ-

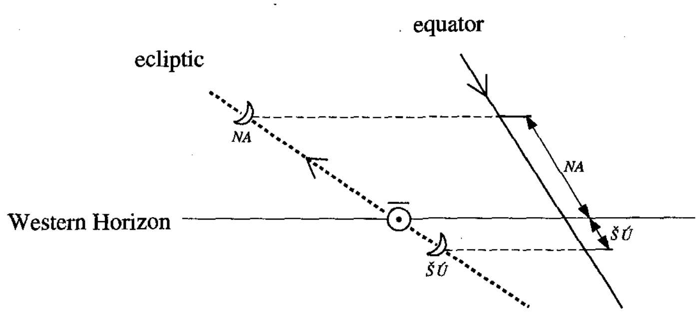

Figure 9. Position of moon and antisun at the western horizon on the two mornings *N* and *N*+1 around full moon at which *ŠÚ* and *NA* are measured. The opposition is assumed to take place shortly after sunrise on day *N*.

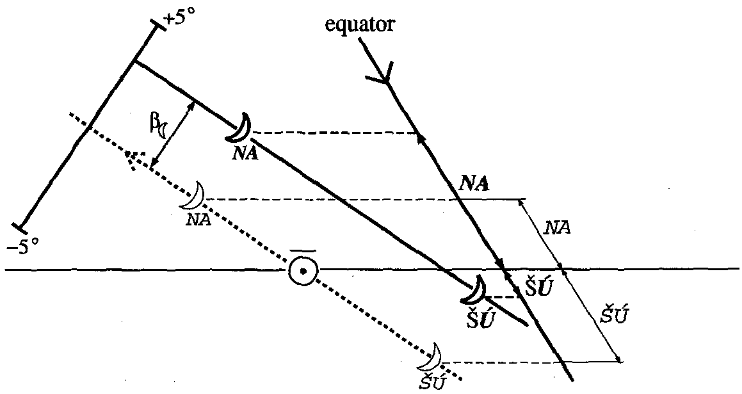

Figure 10. The same situation as in Figure 8 ($\beta_{☾}=0$) with a dotted line for the ecliptic, here in comparison to the situation where the moon has a positive latitude of $\beta_{☾}=3.3°$ and moves on the trajectory parallel to the ecliptic which is drawn by a solid line.

ence of the varying $\beta_{☾}$ is strongly reduced. In the time between the observations of *SU* and *NA* (or of *ME* and *GE*), the latitude of the moon cannot have changed by more than ~1°; only this variation of ~1° in latitude will contribute to the variation of *ŠÚ*+*NA* or *ME*+*GE*.

We summarize: *ŠÚ* and *NA* are strongly depending on Δt, $\beta_{☾}$, $\lambda_{☾}$ and $v_{☾}$. By addition of *ŠÚ* and *NA*, the variable Δt is eliminated and the influence of the variable $\beta_{☾}$ is strongly reduced. This explains the fact that the sum *ŠÚ*+*NA* in Figures 2 and 3 forms a nice and regular curve, whereas *ŠÚ* alone shows a very irregular and unpredictable behaviour.

The intervals *ME, GE* and their sum *ME*+*GE* can be treated completely analogously. We saw that *ŠÚ*+*NA* is the setting time of the ecliptic arc $\mathbf{e}_{SUNA}$. An investigation of the moon risings on the eastern horizon on the days *N* and *N*+1 will show that *ME*+*GE* is the rising time of $\mathbf{e}_{MEGE}$, an ecliptic arc situated around $\lambda_{☾}$, the length of which equals the elongation movement of the moon during the time between moonrise on day *N* and moonrise on day *N*+1.

6. *The sum Σ of ŠÚ+NA and ME+GE*

We have just seen: *ŠÚ+NA* is the setting time of $\mathbf{e}_{SUNA}$, an ecliptic arc situated around $\lambda_☾$ with a length equal to the elongation movement of the moon during the time between moonset on morning N and moonset on morning $N+1$. Analogously, *ME+GE* is the rising time of $\mathbf{e}_{MEGE}$, an ecliptic arc around $\lambda_☾$ corresponding to the elongation movement of the moon during the time between moonrise on evening N and moonrise on evening $N+1$. These two ecliptic arcs will, in a good approximation, have the same length and be situated almost at the same place of the ecliptic. Without committing an error worth mentioning, we can identify these two ecliptic arcs and replace them by $\mathbf{e}_☾$:

$$\mathbf{e}_☾ \simeq \mathbf{e}_{SUNA} \simeq \mathbf{e}_{MEGE} .$$

We have thus defined $\mathbf{e}_☾$ as the ecliptic arc situated symmetrically around $\lambda_☾$ with the length of the elongation movement of the moon on the day of opposition or, to be more precise, of the elongation movement of the moon during the time of one day plus (*ŠÚ+NA*)/360 ≃ one day plus (*ME+GE*)/360. In a good approximation, *ŠÚ+NA* and *ME+GE* are thus the setting and rising times of one and the same ecliptic arc $\mathbf{e}_☾$. Consequently, Σ is the sum of these times, i.e. the rising time of $\mathbf{e}_☾$ plus its setting time. This fact is crucial in our understanding of Σ in combination with our knowledge of the oblique ascension.

7. *The problem of oblique ascension*

Olaf Schmidt [1994, chapter III] treats the problem of the oblique ascension with modern methods. The interested reader who may want further detailed information is referred to this publication. For our present purpose we need the two important results mentioned in the following.

The first result is summarized in Figure 11, where the arc *AB* is part of the ecliptic and arc (*ADCE*) is part of the celestial equator. The yearly revolution in the ecliptic is indicated by an arrow, and

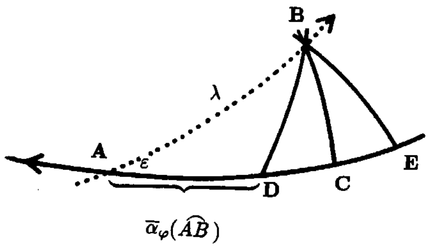

Figure 11. Enlarged part of the celestial sphere for a place whose geographical latitude equals $\varphi=32.5°$. It shows the rising and setting arcs AD and AE, respectively, of the ecliptic arc AB.

the daily revolution in the equator is similarly indicated by an arrow. The angle between the equator and the ecliptic is $\varepsilon=24°$. The geographical latitude of Babylon is $\varphi=32.5°$ and therefore the angle between the equator and the horizon at Babylon is equal to $90°-\varphi=57.5°$. In ancient astronomy, the geographic latitude of a place not on the terrestrial equator is called *sphaera obliqua.* We have drawn two positions of the horizon, namely the eastern and western horizon. Arc BD is part of the eastern horizon showing the situation when the ecliptic point B is about to rise. Arc BE is part of the western horizon. The arc AB of the eclitpic thus rises during the same time as the arc AD of the equator. We call arc AD the rising arc of arc AB and write:

$$AD = \bar{a}_\varphi(AB) \ ,$$

where φ is the geographic latitude of Babylon (see Figure 11). Similarly, we call arc AE the setting arc of arc AB and write

$$AE = \underline{a}_\varphi(AB) \ .$$

At the terrestrial equator, where the geographic latitude $\varphi=0°$, the angle between the horizon and the celestial equator is 90°, and this place is called *sphaera recta.* The horizon at *sphaera recta* is arc BC (see Figure 11). At *sphaera recta* the ecliptic arc AB rises at the same time as arc AC, and we write

$$AC = \overline{a}_0(AB) \ .$$

It also holds that AC is the setting arc of AB:

$$AC = \underline{a}_0(AB) \ .$$

We therefore write

$$AC = a(AB) \ .$$

In Figure 11 the spherical triangle DBE is equilateral because angle (CDB)=angle$(BEC)=57.5°$. Therefore

$$\text{arc } DC = \text{arc } CE \ .$$

and hence

$$\text{arc } AD + \text{arc } AE = 2\text{arc } AC$$

or

$$\overline{a}_\varphi(AB) + \underline{a}_\varphi(AB) = 2a(AB) \ .$$

Expressed in words: for any arc AB of the ecliptic with fixed end point in the vernal equinox A, the sum of the rising and setting times at *sphaera obliqua* equals twice its rising (or setting) time at *sphaera recta.* This is also true for any other arc B_1B_2 of the ecliptic. This can bee seen in the following way: the arbitrary arc B_1B_2 can by found as the difference between the two arcs of the ecliptic, AB_1 and AB_2, both having the vernal equinox A as end point.

$$B_1B_2 = AB_2 - AB_1$$

For these two arcs we know that:

$$\overline{a}_\varphi(AB_2) + \underline{a}_\varphi(AB_2) = 2a(AB_2) \ ,$$

$$\overline{a}_\varphi(AB_1)+\underline{a}_\varphi(AB_1) = 2a(AB_1) \ . \tag{4}$$

Now the rising time of arc B_1B_2 – let us call it $\overline{a}_\varphi(B_1B_2)$ – equals the rising time of AB_2 minus the rising time of arc AB_1. For the setting times, the analogous equation is true:

$$\begin{aligned}\overline{a}_\varphi(B_1B_2) &= \overline{a}_\varphi(AB_2)-\overline{a}_\varphi(AB_1)\\ \underline{a}_\varphi(B_1B_2) &= \underline{a}_\varphi(AB_2)-\underline{a}_\varphi(AB_1) \ .\end{aligned} \tag{5}$$

By addition of the two equations (5) we get:

$$\begin{aligned}\overline{a}_\varphi(B_1B_2)+\underline{a}_\varphi(B_1B_2) = {}& [\overline{a}_\varphi(AB_2)+\underline{a}_\varphi(AB_2)]\\ &- [\overline{a}_\varphi(AB_1)+\underline{a}_\varphi(AB_1)] \ .\end{aligned}$$

The combined use of Eq. (4) and Eq. (5) gives us the desired result:

$$\begin{aligned}\overline{a}_\varphi(B_1B_2)+\underline{a}_\varphi(B_1B_2) &= 2a(AB_2)-2a(AB_1)\\ &= 2a(B_1B_2) \ .\end{aligned} \tag{6}$$

In words: for any arc B_1B_2 of the ecliptic, the sum of rising and setting times at *sphaera obliqua* equals twice its rising (or setting) time at *sphaera recta.* In particular this is true for arc $\mathbf{e}_☾$, the elongation arc of the moon on the day of opposition. We therefore now know that Σ equals twice the rising time at *sphaera recta* of this arc $\mathbf{e}_☾$.

The second result in O. Schmidt [1994, chapter III] gives a handy graphic method, shown in Figure 12, for finding the rising time of a given arc of the ecliptic at *sphaera recta* (curve A) and at *sphaera obliqua* (curve C). The Curve C deals with the geographic latitude φ=32.5° of Babylon: We find the rising time of an arbitrary arc of the ecliptic B_1B_2 as the area bounded by the line segment B_1B_2, the vertical lines through B_1 and B_2 and the curve C. Thus the rising time of a minor arc of the ecliptic, say of e°, placed in ♈ is much smaller than the rising time of an ecliptic arc of e° located e.g. in ♋. In the same way, curve A gives the rising time of an arc of the ecliptic at *sphaera recta.* We notice that the variation in the rising time of an arc of the ecliptic is "smaller" at *sphaera recta* and "larger" at *sphaera obliqua* (cf. the amplitudes of A and C in Figure 12).

8. *The astronomical significance of Σ*

We have seen in Sect. 6 that Σ is the sum of the setting and rising times of $\mathbf{e}_{☾}$. (We remind the reader that $\mathbf{e}_{☾}$ is an ecliptic arc situ-

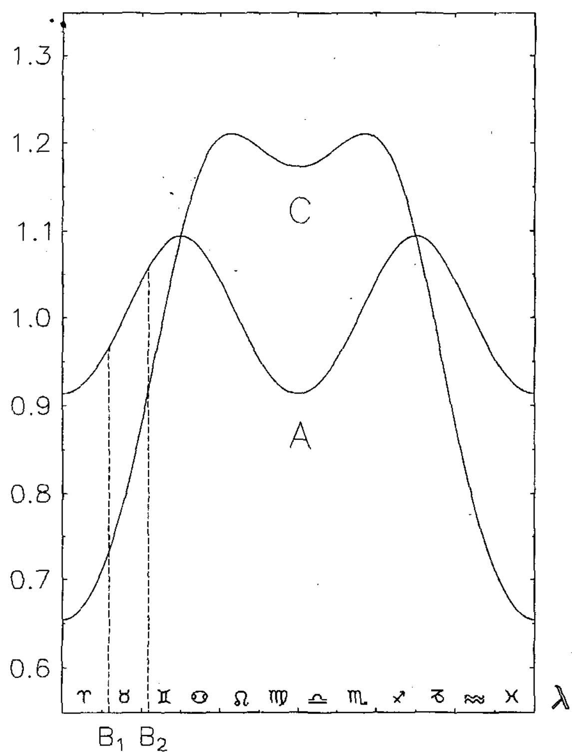

Figure 12. Graphical method of finding the rising time of a given arc of the ecliptic at *sphaera recta* (curve A) and at *sphaera obliqua* with φ=32.5° (curve C). The rising time of an arc B_1B_2 is equal to the area bounded by the line segment B_1B_2, the vertical lines through B_1 and B_2 and the curve A or C, respectively.

ated around the position $\lambda_☾$ in the ecliptic at which the opposition takes place, with a length that equals the elongation motion of the moon during the day of opposition). Let us briefly call this ecliptic arc $\mathbf{e}_☾$ the daily elongation arc of the moon. In Sect. 7 we have seen that the sum of the rising and setting times of an ecliptic arc equals twice its setting time at *sphaera recta.*

Combining these two findings, we now have found the astronomical significance of Σ: it equals twice the setting time at *sphaera recta* of the daily elongation arc of the moon on the day of opposition:

$$\Sigma = \overline{a}_\varphi(\mathbf{e}_☾)+\underline{a}_\varphi(\mathbf{e}_☾) = 2a(\mathbf{e}_☾) \ . \tag{7}$$

Σ is twice the time it takes $\mathbf{e}_☾$ to rise (or to set) when measured from the earth's equator

Knowing the astronomical significance of Σ, we can show numerically that the lunar velocity $v_☾$ is the dominating variable in the sum Σ. The influence of $\lambda_☾$ is much smaller than the influence of $v_☾$. This explains why Σ oscillates with the same period as the lunar velocity. And we can show that in case of *ME+GE*, which is the time it takes $\mathbf{e}_☾$ to rise when measured from Babylon, $\lambda_☾$ is the dominating variable and $v_☾$ has much less influence. Of course, the same is true for *ŠÚ+NA*. Both *ME+GE* and *ŠÚ+NA* oscillate with the period $P_\odot$.

The lunar velocity $v_☾$ determines the length e of the ecliptic arc $\mathbf{e}_☾$, whereas its position in the ecliptic is determined by $\lambda_☾$.

The variation due to $v_☾$: The lunar velocity varies between 11.8°/day and 15.3°/day[6]; therefore the length e of $\mathbf{e}_☾$ will vary between 10.8° and 14.3°; a variation of 14.3°−10.8°=3.5°. Therefore, the variation of Σ due to $v_☾$ is 2×3.5°=7°, and the variation of *ME+GE* due to $v_☾$ is 3.5°.

The variation due to $\lambda_☾$: Let us assume $\mathbf{e}_☾$ to be of constant length: $\mathbf{e}_☾=12°$; this is the same as assuming the lunar velocity to be constant and equal its mean value. The curves A and C give us an estimate of the variation in rising time of $\mathbf{e}_☾$ depending on its position $\lambda_☾$ in the ecliptic. Curve A varies between 0.92 and 1.08; this means that at *sphaera recta* the shortest rising time of $\mathbf{e}_☾$ (of length 12°) is 0.92×12° and the longest rising time of $\mathbf{e}_☾$ is 1.08×12°.

The variation of Σ due to $\lambda_☾$ equals twice the variation in rising time of $\mathbf{e}_☾$ at *sphaera recta:* 2×(1.08−0.92)×12°=3.84°. This variation is much smaller than 7°, the variation of Σ due to $v_☾$. The function Σ therefore oscillates with the mean period $P_☾$.

Curve C varies between 0.65 and 1.22. We can thus find the variation in rising time of $\mathbf{e}_☾$ at Babylon *sphaera obliqua:* namely (1.22−0.65)×12°=6.84°. The variation of $ME+GE$ due to $\lambda_☾$ equals 6.84°. This variation is much larger than 3.5°, the variation of $ME+GE$ due to $v_☾$. The function $ME+GE$ therefore oscillates with the mean period $P_☾$.

We now understand why it is possible to find, purely empirically, a function varying with the period $P_☾$ from the observed Lunar Four, simply by calculating their sum.

9. *Concluding remarks*

We now know the astronomical significance of Σ. It is, however, so abstract and complicated that we must assume the Babylonians did not know it. If therefore our hypothesis is right, that Φ is directly derived from Σ, we must conclude: The Babylonians succeeded in finding a purely empirical function, which contained information on the elongation movement of the moon (on the day of opposition) and hence also on its momentaneous velocity on this day. Without knowing its astronomical significance, they derived all other quantities depending on $v_☾$ and hence of the period $P_☾$ from Φ.

Through systematic treatment of their observed data, so we think, the Babylonians observed the periodicity of different astronomical quantities. Quantities of the same period were coupled: Based on one known quantity, others of the same period were derived. Scientific research always uses procedures of this kind: The discovery of regularities leads to connections which can be used for predictions – also in cases where the fundamental natural laws are not known or only partially understood.

Many other scholars (O. Neugebauer, A. Aaboe, Y. Maeyama and B. L. van der Waerden) have pointed at the prevailing role of periodic functions in the Babylonian astronomy; however, not as radically as we do it in this paper. Therefore we mention two details from the cuneiform texts which clearly support this understanding of the development of the Babylonian astronomy:

The Goal-Year texts contain collections of characteristic phenomena for the moon and the five known planets. They were used for predicting astronomical events from known phenomena of the same kind which occurred some characteristic time period earlier. In case of the moon, the characteristic time interval was 223 synodic months=1 Saros. Among the moon phenomena recorded on the Goal-Year tablets, we find the sums *ŠÚ+NA* and *ME+GE*. This shows us 1) that the Babylonians themselves, indeed, did calculate sums of the Lunar Four, and 2) that they probably used the sums *ŠÚ+NA* and *ME+GE* for prediction of Lunar Four to come one Saros later. But by so doing they have used empirically found periodic oscillations for lunar predictions.

In Figs. 2 and 3 we have seen that *ŠÚ+NA* and *ME+GE* as functions of the lunation number form nice curves, indicating that they might be easy to predict. In [Lis Brack-Bernsen 1994] it was demonstrated that the curve *ŠÚ+NA* as well as *ME+GE* was repeated almost exactly after one Saros. In the same paper, a short proposal was made about how some known values of the Lunar Four and their partial sums *ŠÚ+NA* and *ME+GE* might have been used in order to predict the Lunar Four one Saros later.

In the mean time this proposal has been confirmed by textual evidence: namely by the lines 35–38 on the back side of the text TU 11.[7] These lines show us that the Babylonians did, indeed, utilize the sum *ŠÚ+NA* for predicting *ŠÚ*. But more than that: in order to predict NA_N (new moon), they even used the sum *ŠÚ+NA* (full moon) as observed $5\frac{1}{2}$ months earlier. To us, this procedure can only be understood as an empirical utilization of periodic oscillations.[8]

We see this as a support for our reconstruction of *Φ* as a purely empirical function, derived from the sum of the Lunar Four. We therefore think that the Babylonian mathematical astronomy is more empirically founded and less theoretically than believed until now.

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NOTES

1. No difference is made in the Babylonian texts between the symbol for *NA* (new moon) observed around conjunction and that for *NA* (full moon) observed around opposition. In order to avoid confusion, we shall use the symbol NA_N for the time interval observed around conjunction.
2. Neugebauer [ACT I, pp. 229–239] has discussed the Lunar Four in detail and explained how the Babylonians calculated them.
3. Aaboe and Henderson [1975, p. 195] were the first to remark that the influence of $v_☾$ on a quantity $\Delta\lambda$ can be seen in the amplitude variation of $\Delta\lambda$ when plotted as a function of the lunation number. Here $\Delta\lambda$ denotes the ecliptic arc between the positions $\lambda_☾$ of consecutive full moons.
4. The days *N*, *N*+1 etc. here refer to the astronomical days starting at midnight – and not to the civil days of the Babylonian calendar in which a new day started in the evening at the moment of sunset.
5. The Lunar Four and all their combinations are measured in *uš*=time degrees: 1 *uš*=4 minutes, so that 360 *uš*=1 day (i.e. the time of a whole revolution of the sky about 360°).
6. The exact values $v_☾$ (max)=15.301°/day and $v_☾$ (min)=11.799°/day have been derived by Y. Maeyama. We thank him for this private communication.
7. H. Hunger, who has kindly given us a translation of this very difficult text, shall be warmly thanked at this place.
8. These lines of text TU 11 will be treated in more detail in a volume on "Ancient Astronomy and Celestial Divination", to be published under the auspices of the Dibner Institute of MIT.