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Quasi-isogenies and Morava stabilizer groups

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Abstract

For every prime p and integer $n \geqslant 3$ we explicitly construct an abelian variety $A/\mathbb{F}_{p^n}$ of dimension n such that for a suitable prime l the group of quasi-isogenies of $A/\mathbb{F}_{p^n}$ of l-power degree is canonically a dense subgroup of the n-th Morava stabilizer group at p. We also give a variant of this result taking into account a polarization. This is motivated by the recent construction of topological automorphic forms which generalizes topological modular forms [BL1].

For this, we prove some results about approximation of local units in maximal orders which is of independent interest. For example, it gives a precise solution to the problem of extending automorphisms of the p-divisible group of a simple abelian variety over a finite field to quasi-isogenies of the abelian variety of degree divisible by as few primes as possible.

1. Introduction

One of the most fruitful ways of studying the stable homotopy category is the chromatic approach: After localizing, in the sense of Bousfield, at a prime p, one is left with an infinite hierarchy of primes corresponding to the Morava K-theories $K(n)$, $n \geqslant 0$, see [R2]. The successive layers in the resulting filtration are the $K(n)$-local categories [HS] the structure of which is governed to a large extend by (the continuous cohomology of) the n-th Morava stabilizer group $\mathbb{S}_n$, i.e. the automorphism group of the one-dimensional commutative formal group of height n over $\overline{\mathbb{F}_p}$. A fundamental problem in this context is to generalize the fibration

$$L_{K(1)}S^0 \longrightarrow E_1^{hF} \longrightarrow E_1^{hF},$$

c.f. the introduction of [GHMR], to a resolution of the $K(n)$-local sphere for $n \geqslant 2$. Substantial progress on this problem for $n = 2$ and in many other cases as well has been achieved by clever use of homological algebra for $\mathbb{S}_n$-modules [GHMR],[H]. Recently, pursuing a question of M. Mahowald and C. Rezk, M. Behrens was able to give a modular interpretation of one such resolution in the case $n = 2$ [B]:

A basic observation is that $\mathbb{S}_2$ is the automorphism group of the p-divisible group of a super-singular elliptic curve E over a finite field k. Hence it seemed plausible, and was established in *loc. cit.*, that the morphisms in a resolution of a spectrum closely related to $L_{K(2)}S^0$ should have a description in terms of suitable endomorphisms of E. A key result for seeing this was to observe that for suitable primes l

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$$\left(\mathrm{End}_k(E)\left[\frac{1}{l}\right]\right)^* \subseteq \mathbb{S}_2 \qquad (1)$$

is a dense subgroup [BL2, Theorem 0.1].
One of our main results, Theorem 13, is the direct generalization of (1) to arbitrary chromatic level $n \geqslant 3$ in which E is replaced by an abelian variety of dimension n which is known to be the minimal dimension possible.
In Corollary 10 we give a variant of the arithmetic result underlying Theorem 13 in which on the left-hand-side of (1) we only allow endomorphisms which are unitary with respect to a given Rosati-involution. The motivation for this stems from recent work of M. Behrens and T. Lawson [BL1] bringing the arithmetic of suitable Shimura varieties to bear on homotopy theoretic problems of arbitrary chromatic level, generalizing the role of topological modular forms for problems of chromatic level at most two.
In deriving these results we take some intermediate steps as we now explain reviewing the individual sections in more detail:
In subsection 2.1 (resp. 2.2) we reduce the problem of approximating a local unit of a maximal order in a skew-field D, finite-dimensional over $\mathbb{Q}$, (and carrying an involution of the second kind) to a similar approximation problem for numberfields, Theorem 1 (resp. Theorem 6). The basic tool here is strong approximation for an inner (resp. outer) form of Sl_d.
The resulting problem in the numberfield case is solved in subsection 2.3 using class field theory, Theorem 7.
In subsection 3.1 we explain the application of the results obtained so far to the following problem: Given a simple abelian variety A over a finite field k one would like to extend an automorphism of the p-divisible group $A[p^\infty]$ of A to a quasi-isogeny of A the degree of which should be divisible by as few primes as possible. The additional ingredient needed here is the classical result of J. Tate that $\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathrm{End}_k(A[p^\infty])$.
Finally, subsection 3.2 contains the proof of Theorem 13 reviewed above.

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2. Approximation of local units in maximal orders

2.1 Forms of type A_{d-1}

In this subsection we consider the problem of p-adically approximating local units of a maximal order $\mathcal{O} \subseteq D$ where D is a finite dimensional skew-field over $\mathbb{Q}$. The center of D, denoted k, is a number field and we denote by d the reduced dimension of D, i.e. $\dim_k D = d^2$. The title of this subsection refers to the fact that the relevant algebraic group will turn out to be $\mathrm{Sl}_1(D)$, i.e. an inner form of type A_{d-1}. We denote by $\mathcal{O}_k \subseteq k$ the ring of integers and note that $k \cap \mathcal{O} = \mathcal{O}_k$ as an immediate consequence of [D, Kapitel VI,§11, Satz 7].
Recall that D is determined by its local invariants as follows [PR, 1.5.1]. For every place v of k there is a local invariant $\mathrm{inv}_v(D) \in \frac{1}{d}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$ and $\mathrm{inv}_v(D) = 0$ for almost all v. For a given place v, we denote by k_v the completion of k with respect to v. Then $D_v := D \otimes_k k_v$ is a central simple

k_v-algebra which determines a class $[D_v] \in \mathrm{Br}(k_v)$ in the Brauer group of k_v. There are specific isomorphisms

$$\tau_v : \mathrm{Br}(k_v) \xrightarrow{\sim} \begin{cases} \mathbb{Q}/\mathbb{Z} & , \quad v \text{ finite} \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & , \quad v \text{ real} \\ 0 & , \quad v \text{ complex} \end{cases}$$

such that $\mathrm{inv}_v(D) = \tau_v([D_v])$. In particular, D_v is a skew-field if and only if the order of $\mathrm{inv}_v(D)$ is exactly d.

We now fix a prime $0 \neq \mathfrak{p} \subseteq \mathcal{O}_k$ at which we wish to approximate. There is a unique prime $\mathfrak{P} \subseteq \mathcal{O}$ lying above $\mathfrak{p}$ [D, VI, §12, Satz 1] and we denote by $\mathcal{O}_{\mathfrak{P}}$ the $\mathfrak{P}$-adic completion of $\mathcal{O}$, c.f. [D, Kapitel VI, §11].

To describe the denominators we allow the approximating global units to have, we fix a finite set of places S of k such that

$$\mathfrak{p} \notin \mathrm{S} \text{ and there exists a place } v_0 \in \mathrm{S} \text{ such that } D_{v_0} \text{ is not a skew-field.}$$

We write S^{fin} for the set of finite places contained in S and consider the ring $\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}$ of S^{fin}-integers

$$\mathcal{O}_k \subseteq \mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}} := \{x \in k \,|\, v(x) \geqslant 0 \text{ for all finite } v \notin \mathrm{S}\} \subseteq k.$$

Since $\mathfrak{p} \notin \mathrm{S}$ we have $\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}} \subseteq \mathcal{O}_{k,\mathfrak{p}}$, the completion of $\mathcal{O}_k$ with respect to $\mathfrak{p}$. Thus

$$X := \{x \in \mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}} \,|\, v \text{ infinite and } \mathrm{inv}_v(D) = \frac{1}{2} \text{ imply } v(x) > 0\} \subseteq \mathcal{O}^*_{k,\mathfrak{p}} \text{ and} \tag{2}$$

$$(\mathcal{O} \otimes_{\mathcal{O}_k} \mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^* \subseteq \mathcal{O}^*_{\mathfrak{P}}.$$

Denoting by N the reduced norm of $k \subseteq D$ [PR, 1.4.1] we can state our first result as follows.

Theorem 1. *The closure of* $(\mathcal{O} \otimes_{\mathcal{O}_k} \mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^*$ *inside* $\mathcal{O}^*_{\mathfrak{P}}$ *equals*

$$\{x \in \mathcal{O}^*_{\mathfrak{P}} \,|\, \mathrm{N}(x) \in \mathcal{O}^*_{k,\mathfrak{p}} \text{ lies in the closure of } X\}.$$

Example 2. *1) For $k = \mathbb{Q}$ and D a definite quaternion algebra, i.e. $d = 2$ and $\mathrm{inv}_v(D) = \frac{1}{2}$ for the unique infinite place v of $\mathbb{Q}$, we can choose $\mathrm{S} = \{l\}$ for any prime $l \neq p$ at which D splits, i.e. $\mathrm{inv}_l(D) = 0$. Then $\mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}} = \{\pm 1\} \times l^{\mathbb{Z}}$ and $X = l^{\mathbb{Z}} \subseteq \mathcal{O}^*_{k,\mathfrak{p}} = \mathbb{Z}^*_p$. For $p \neq 2$ we can choose l as above such that in addition $X \subseteq \mathbb{Z}^*_p$ is dense and conclude that in this case $\mathcal{O}[\frac{1}{l}]^* \subseteq \mathcal{O}^*_{\mathfrak{P}}$ is dense. For $p = 2$ we can choose l such that the closure of X equals $1 + 4\mathbb{Z}_2$ and conclude that the closure of $\mathcal{O}[\frac{1}{l}]^*$ inside $\mathcal{O}^*_{\mathfrak{P}}$ equals*

$$\ker(\mathcal{O}^*_{\mathfrak{P}} \xrightarrow{\mathrm{N}} \mathbb{Z}^*_2 \longrightarrow \mathbb{Z}^*_2/(1 + 4\mathbb{Z}_2) \simeq \{\pm 1\}),$$

c.f. Remark 12. In the special case in which D is the endomorphism algebra of a super-singular elliptic curve in characteristic p, i.e. $\mathrm{inv}_v(D) = 0$ for all $v \neq p, \infty$, this result has been established by different means in [BL2, Theorem 0.1].

2) See Theorem 7 in subsection 2.3 for a further discussion of the closure of $X \subseteq \mathcal{O}^*_{k,\mathfrak{p}}$.

The rest of this subsection is devoted to the proof of Theorem 1 which is an application of strong approximation for algebraic groups.
The group-valued functor G on $\mathcal{O}_k$-algebras R

$$G(R) := (\mathcal{O} \otimes_{\mathcal{O}_k} R)^*$$

is representable by an affine algebraic group scheme $G/\mathrm{Spec}\,(\mathcal{O}_k)$. The reduced norm N gives an exact sequence of representable $fppf$-sheaves on $\mathrm{Spec}\,(\mathcal{O}_k)$

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\mathrm{N}} \mathbb{G}_m \longrightarrow 1. \tag{3}$$

Proposition 3. *The subgroup $G'(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) \subseteq G'(\mathcal{O}_{k,\mathfrak{p}})$ is dense.*

Proof. First note that $G'/\mathrm{Spec}\,(\mathcal{O}_k)$ is representable by an affine algebraic group scheme, hence the injectivity of the homomorphism $G'(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) \longrightarrow G'(\mathcal{O}_{k,\mathfrak{p}})$ follows from the injectivity of $\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}} \hookrightarrow \mathcal{O}_{k,\mathfrak{p}}$. Secondly, $G'(\mathcal{O}_{k,\mathfrak{p}})$ is canonically a topological group [We, Chapter I] and we claim density with respect to this topology. We have that $G'_k := G' \otimes_{\mathcal{O}_k} k = \mathrm{Sl}_1(D)$ [PR, 2.3] is an inner form of Sl_d and thus is semi-simple and simply connected. Furthermore, $G'_k \otimes_k k_{v_0} = \mathrm{Sl}_n(\tilde{D})$ for some central skew-field $\tilde{D}$ over k_{v_0} and some $n \geqslant 1$. Since D_{v_0} is not a skew-field by assumption, we have $n \geqslant 2$ and $\mathrm{rk}_{k_{v_0}} G'_k \otimes_k k_{v_0} = n-1 \geqslant 1$ [PR, Proposition 2.12], i.e. G'_k is isotropic at v_0. From strong approximation [S, Theorem 5.1.8] we conclude that

$$G'(k) \cdot G'(k_{v_0}) \subseteq G'(\mathbb{A}_k) \text{ is dense,} \tag{4}$$

where $\mathbb{A}_k$ denotes the adèle-ring of k. Fix $x \in G'(\mathcal{O}_{k,\mathfrak{p}})$ and an open subgroup $U_\mathfrak{p} \subseteq G'(\mathcal{O}_{k,\mathfrak{p}})$. Denote by $\tilde{x} \in G'(\mathbb{A}_k)$ the adèle having $\mathfrak{p}$-component x and all other components equal to 1. Then

$$U := U_\mathfrak{p} \times \prod_{v \neq \mathfrak{p} \text{ finite}} G'(\mathcal{O}_{k,v}) \times \prod_{v \text{ infinite}} G'(k_v) \subseteq G'(\mathbb{A}_k)$$

is an open subgroup and by (4) there exist $\gamma \in G'(k)$ and $\delta \in G'(k_{v_0})$ such that $\gamma\delta \in \tilde{x}U$. Since $\mathfrak{p} \neq v_0$ this implies that $\gamma_\mathfrak{p} \in \tilde{x}_\mathfrak{p} U_\mathfrak{p} = xU_\mathfrak{p}$, where $\gamma_\mathfrak{p}$ is the $\mathfrak{p}$-component of the principal adèle γ, equivalently, the image of γ under the inclusion $G'(k) \subseteq G'(k_\mathfrak{p})$. Since x and $U_\mathfrak{p}$ are arbitrary, we will be done if we can show that $\gamma \in G'(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) \subseteq G'(k)$, i.e. that for every finite place $v \notin \mathrm{S}$ we have $\gamma_v \in G'(\mathcal{O}_{k,v})$. For $v = \mathfrak{p}$ this is clear since $xU_\mathfrak{p} \subseteq G'(\mathcal{O}_{k,\mathfrak{p}})$ whereas for $v \neq \mathfrak{p}$ we have, using that $\delta_v = 1$ since $v \neq v_0 \in \mathrm{S}$,

$$(\gamma\delta)_v = \gamma_v \in (\tilde{x}U)_v = \tilde{x}_v \cdot G'(\mathcal{O}_{k,v}) = G'(\mathcal{O}_{k,v}).$$

□

To proceed, we apply (3) to the inclusion $\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}} \hookrightarrow \mathcal{O}^*_{k,\mathfrak{p}}$ to obtain a commutative diagram

(5)
$$\begin{array}{ccccccc}
1 & \longrightarrow & G'(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) & \longrightarrow & G(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) & \xrightarrow{\mathrm{N}} & \mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}} \\
 & & \downarrow & & \downarrow \iota & & \downarrow \\
1 & \longrightarrow & G'(\mathcal{O}_{k,\mathfrak{p}}) & \longrightarrow & G(\mathcal{O}_{k,\mathfrak{p}}) & \xrightarrow{\mathrm{N}_{\mathfrak{p}}} & \mathcal{O}^*_{k,\mathfrak{p}}.
\end{array}$$

By definition, $G(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) = (\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^*$ and $G(\mathcal{O}_{k,\mathfrak{p}}) = (\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathfrak{p}})^* = \mathcal{O}^*_{\mathfrak{P}}$ [D, Kapitel VI, §11, Satz 6], so Theorem 1 is concerned with the closure of the image of ι. Recall the subgroup $X\subseteq \mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}}$ from (2).

Proposition 4. *In (5) we have* $\mathrm{im}(\mathrm{N}) = X \subseteq \mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}}$.

Proof. Firstly, the commutative diagram of groups

(6)
$$\begin{array}{ccc}
D^* & \xrightarrow{\mathrm{N}} & k^* \\
\uparrow & & \uparrow \\
(\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^* & \xrightarrow{\mathrm{N}} & \mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}}
\end{array}$$

is cartesian, i.e. $x\in D^*$ and $\mathrm{N}(x)\in\mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}}$ imply $x\in(\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^*$. This follows from the similar statement, applied to both x and x^{-1}, that $x\in D^*$ and $\mathrm{N}(x)\in\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}$ imply $x\in(\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})$: By [PR, Theorem 1.15] we have

$$\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}} = \bigcap_{\mathfrak{Q}\subseteq\mathcal{O}\ \text{prime and}\ \mathfrak{Q}\cap\mathcal{O}_k\notin\mathrm{S}} \mathcal{O}_{\mathfrak{Q}}$$

and are reduced to seeing that $x\in D^*_{\mathfrak{Q}}$ and $\mathrm{N}(x)\in\mathcal{O}_{k,\mathfrak{Q}\cap\mathcal{O}_k}$ imply $x\in\mathcal{O}_{\mathfrak{Q}}$, which is true [PR, 1.4.2].
Since (6) is cartesian we see that

$$\mathrm{im}(N) = \mathrm{N}((\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^*) = \mathrm{N}(D^*)\cap\mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}} =$$
$$\{x\in k^*\,|\,v(x)>0 \text{ for } v \text{ infinite with } \mathrm{inv}_v(D)=\frac{1}{2}\}\cap\mathcal{O}^*_{k,\mathrm{S}^{\mathrm{fin}}} = X.$$

Here, the third equality is Eichler's norm Theorem [PR, Theorem 1.13].

□

Since in (5) $N_{\mathfrak{p}}$ is surjective [PR, 1.4.3] we can, using Proposition 4, rewrite (5) as

(7)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & G'(\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}}) & \longrightarrow & (\mathcal{O}\otimes_{\mathcal{O}_k}\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}})^* & \xrightarrow{\mathrm{N}} & X & \longrightarrow & 1 \\
 & & \downarrow \alpha & & \downarrow \iota & & \downarrow & & \\
1 & \longrightarrow & G'(\mathcal{O}_{k,\mathfrak{p}}) & \longrightarrow & \mathcal{O}^*_{\mathfrak{P}} & \xrightarrow{\mathrm{N}_{\mathfrak{p}}} & \mathcal{O}^*_{k,\mathfrak{p}} & \longrightarrow & 1.
\end{array}$$

Since the image of α is dense by Proposition 3 and $\mathcal{O}_{\mathfrak{P}}^*$ is compact, all that remains to be done to conclude the proof of Theorem 1 is to apply Proposition 5 below to (7).
For a subset Y of a topological space X we denote by $\overline{Y}^X$ the closure of Y in X.

Proposition 5. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H' & \longrightarrow & H & \xrightarrow{\rho} & H'' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G' & \longrightarrow & G & \xrightarrow{\pi} & G'' & \longrightarrow & 1 \end{array}$$

be a commutative diagram of first countable topological groups with exact rows, G compact, and such that $H' \subseteq G'$ is dense. Then

$$\overline{H}^G = \pi^{-1}(\overline{H''}^{G''}).$$

Proof. Assume that $g \in \overline{H}^G$. Then $g = \lim_n h_n$ for suitable $h_n \in H$ and $\pi(g) = \lim_n \pi(h_n) \in \overline{H''}^{G''}$. Conversely, given $g \in G$ with $\pi(g) = \lim_n h_n''$ for suitable $h_n'' \in H''$, choose $h_n \in H$ with $\rho(h_n) = h_n''$. The sequence $(h_n g^{-1})_n$ in G has a convergent subsequence, $\tilde{g} := \lim_i h_{n_i} g^{-1} \in G$. Then $\pi(\tilde{g}) = 1$, i.e. $\tilde{g} \in G'$ and we have $\tilde{g} = \lim_i h_i'$ for suitable $h_i' \in H'$. The sequence $((h_i')^{-1} h_{n_i})_i$ in H satisfies $\lim_i (h_i')^{-1} h_{n_i} = \tilde{g}^{-1}\tilde{g}g = g$, hence $g \in \overline{H}^G$.

□

2.2 Forms of type ${}^2A_{d-1}$

Here we consider the variant of the problem addressed in subsection 2.1 in which one seeks to approximate local units by global ones which are unitary with respect to a given involution.
Let D be a finite- dimensional skew-field of reduced dimension $d > 1$ over $\mathbb{Q}$ carrying a positive involution $*$ of the second kind, i.e. for all $x \in D^*$ we have $\mathrm{tr}_{\mathbb{Q}}^D({}^*xx) > 0$ (positivity) and $*$ restricted to the center L of D is non-trivial. Then L is a CM-field with $k := \{x \in L \,|\, x =^* x\} \subseteq L$ as its maximal real subfield [Mu, page 194]. Note that $*$ is k-linear. We assume that $\mathcal{O} \subseteq D$ is a maximal order which is invariant under $*$. Then $\mathcal{O} \cap L = \mathcal{O}_L$ and $\mathcal{O} \cap k = \mathcal{O}_k$ are the rings of integers of L and k. We consider the affine algebraic group-schemes G and T over $\mathrm{Spec}\,(\mathcal{O}_k)$ whose groups of points are given for any $\mathcal{O}_k$-algebra R by

$$G(R) = \{g \in (\mathcal{O} \otimes_{\mathcal{O}_k} R)^* \,|\, {}^*gg = 1\} \text{ and}$$

$$T(R) = \{g \in (\mathcal{O}_L \otimes_{\mathcal{O}_k} R)^* \,|\, \mathrm{N}_k^L(g) = 1\}.$$

There is a homomorphism $\mathrm{N} : G \longrightarrow T$ over $\mathrm{Spec}\,(\mathcal{O}_k)$ given on points by $\mathrm{N}(g) = \mathrm{N}_L^D(g)$ (the reduced norm of D) and we put $SG := \ker(\mathrm{N})$ to obtain an exact sequence over $\mathrm{Spec}\,(\mathcal{O}_k)$

$$1 \longrightarrow SG \longrightarrow G \xrightarrow{\mathrm{N}} T \longrightarrow 1. \tag{8}$$

Over $\mathrm{Spec}\,(k)$, this is the sequence

$$1 \longrightarrow \mathrm{SU}_1(D,1) \longrightarrow \mathrm{U}_1(D,1) \xrightarrow{\mathrm{N}} \mathrm{Res}_k^L(\mathbb{G}_m)^{(1)} \longrightarrow 1,$$

where "1" denotes the standard rank one Hermitian form on D and

$$\mathrm{Res}_k^L(\mathbb{G}_m)^{(1)} := \ker(\mathrm{Res}_k^L(\mathbb{G}_m) \xrightarrow{\mathrm{N}_k^L} \mathbb{G}_{m,k})\,;$$

c.f. [PR, 2.3] for notation and general background on the unitary groups SU and U.
We fix a prime $0 \neq \mathfrak{p} \subseteq \mathcal{O}_k$ and a finite set of *finite* places S of k with $\mathfrak{p} \notin \mathrm{S}$ and denote by $\mathcal{O}_{k,\mathrm{S}} \subseteq k$ the ring of S-integers.

Theorem 6. *The closure of $G(\mathcal{O}_{k,\mathrm{S}}) \subseteq G(\mathcal{O}_{k,\mathfrak{p}})$ equals*

$$\{g \in G(\mathcal{O}_{k,\mathfrak{p}}) \,|\, \mathrm{N}(g) \text{ lies in the closure of } T(\mathcal{O}_{k,\mathrm{S}}) \subseteq T(\mathcal{O}_{k,\mathfrak{p}})\}.$$

See Theorem 9 for the computation of the closure of $T(\mathcal{O}_{k,\mathrm{S}}) \subseteq T(\mathcal{O}_{k,\mathfrak{p}})$ in a special case.
Note that

$$G(\mathcal{O}_{k,\mathrm{S}}) = \{g \in (\mathcal{O} \otimes_{\mathcal{O}_k} \mathcal{O}_{k,\mathrm{S}})^* \,|\, {}^*gg = 1\}$$

by definition but the structure of $G(\mathcal{O}_{k,\mathfrak{p}})$ depends on the splitting behavior of $\mathfrak{p}$ in L:
If there is a unique prime $\mathfrak{q} \subseteq \mathcal{O}_L$ over $\mathfrak{p}$, then $\mathrm{inv}_{\mathfrak{q}}(D) = 0$ [Mu, page 199, (B)] and

$$G(\mathcal{O}_{k,\mathfrak{p}}) \simeq \{(x_{i,j}) \in \mathrm{Gl}_d(\mathcal{O}_{L,\mathfrak{q}}) \,|\, (\overline{x_{ji}})(x_{ij}) = 1\},$$

where $-$ denotes the non-trivial automorphism of $L_{\mathfrak{q}}$ over $k_{\mathfrak{p}}$. If $\mathfrak{p}\mathcal{O}_L = \mathfrak{q}\mathfrak{q}'$ with $\mathfrak{q} \neq \mathfrak{q}'$ then $\mathrm{inv}_{\mathfrak{q}}(D)+\mathrm{inv}_{\mathfrak{q}'}(D) = 0$ [Mu, page 197, (A)], $D_{\mathfrak{q}} \simeq D_{\mathfrak{q}'}^{opp}$ as $k_{\mathfrak{p}} = L_{\mathfrak{q}} = L_{\mathfrak{q}'}$-algebras and the involution on $D \otimes_k k_{\mathfrak{p}} = D_{\mathfrak{q}} \times D_{\mathfrak{q}'}$ exchanges the factors. Then $\mathrm{U}_1(D,1) \otimes_k k_{\mathfrak{p}} \simeq \mathrm{Gl}_1(D_{\mathfrak{q}}) \simeq \mathrm{Gl}_1(D_{\mathfrak{q}'})$, the latter isomorphism since for any group X, $x \mapsto x^{-1} : X \longrightarrow X^{opp}$ is an isomorphism. From this we get

$$G(\mathcal{O}_{k,\mathfrak{p}}) \simeq \mathcal{O}_{D_{\mathfrak{q}}}^* \simeq \mathcal{O}_{D_{\mathfrak{q}'}}^*$$

in this case.
In the rest of this subsection we give the proof of Theorem 6 which is similar to the proof of Theorem 1 and we will limit ourselves to indicating the relevant modifications.
Firstly, in analogy with Proposition 3, we have that $SG(\mathcal{O}_{k,\mathrm{S}}) \subseteq SG(\mathcal{O}_{k,\mathfrak{p}})$ is a dense subgroup: Since $\mathrm{SU}_1(D,1)$ is an outer form of Sl_d, it is semi-simple and simply connected. For any infinite place v of k, necessarily real, we have $\mathrm{SU}_1(D,1) \otimes_k k_v \simeq \mathrm{U}_d$, the standard compact form of Gl_d over $k_v \simeq \mathbb{R}$. Since $\mathrm{rk}_{\mathbb{R}}\mathrm{U}_d = d-1 \geqslant 1$, $\mathrm{SU}_1(D,1)$ is anisotropic at v and one proceeds as in the proof of Proposition 3 using $v_0 = v$ there.
Next, we explain why $\mathrm{N} : G(\mathcal{O}_{k,\mathfrak{p}}) \longrightarrow T(\mathcal{O}_{k,\mathfrak{p}})$ is surjective:
One reduces to seeing that $\mathrm{N} : G(k_{\mathfrak{p}}) \longrightarrow T(k_{\mathfrak{p}})$ is surjective as at the beginning of the proof of Proposition 4 and, for later reference, we will prove the surjectivity of $\mathrm{N} : G(k_v) \longrightarrow T(k_v)$ for every, not necessarily finite, place v of k. If v splits into w and w' in L, then v is finite (since k is totally real and L is totally imaginary, no infinite place of k splits in L) and we have a commutative diagram

$$\begin{array}{ccc} G(k_v) & \xrightarrow{\mathrm{N}} & T(k_v) \\ \downarrow \simeq & & \downarrow \simeq \\ D_v^* & \xrightarrow{\mathrm{N}_{k_v}^{D_v}} & k_v^* \end{array}$$

in which the lower horizontal arrow is surjective by [PR, 1.4.3]. If there is a unique place w of L over v then we get

$$\begin{array}{ccc} G(k_v) & \xrightarrow{\mathrm{N}} & T(k_v) \\ \downarrow \simeq & & \downarrow \simeq \\ \{(x_{ij}) \in \mathrm{Gl}_d(L_w) \,|\, (\overline{x_{ji}})(x_{ij}) = 1\} & \xrightarrow{\det} & \{x \in L_w^* \,|\, \overline{x}x = 1\} \end{array}$$

and the lower horizontal arrow is surjective since it is split by $x \mapsto \mathrm{diag}(x, 1, \ldots, 1)$.
Finally, we show that $\mathrm{N} : G(\mathcal{O}_{k,\mathrm{S}}) \longrightarrow T(\mathcal{O}_{k,\mathrm{S}})$ is surjective:
Again, it is enough to see that $\mathrm{N} : G(k) \longrightarrow T(k)$ is surjective and to this end we contemplate the following diagram:

$$\begin{array}{ccccc} G(k) & \xrightarrow{\mathrm{N}} & T(k) & \longrightarrow & \mathrm{H}^1(k, SG) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \prod_{v \in \Sigma_k^\infty} G(k_v) & \xrightarrow{\prod \mathrm{N}_v} & \prod_{v \in \Sigma_k^\infty} T(k_v) & \longrightarrow & \prod_{v \in \Sigma_k^\infty} \mathrm{H}^1(k_v, SG). \end{array}$$

Here, Σ_k^∞ denotes the set of infinite places of k, the horizontal lines are part of the cohomology sequence associated with (8) and the right-most vertical arrow is an isomorphism by the Hasse-principle for $SG \otimes_{\mathcal{O}_k} k = \mathrm{SU}_1(D, 1)$ [PR, Theorem 6.6]. Hence the surjectivity of N follows from the surjectivity of N_v for all $v \in \Sigma_k^\infty$ which has already been established.

To sum up, we have shown the existence of a diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SG(\mathcal{O}_{k,\mathrm{S}}) & \longrightarrow & G(\mathcal{O}_{k,\mathrm{S}}) & \longrightarrow & T(\mathcal{O}_{k,\mathrm{S}}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & SG(\mathcal{O}_{k,\mathfrak{p}}) & \longrightarrow & G(\mathcal{O}_{k,\mathfrak{p}}) & \longrightarrow & T(\mathcal{O}_{k,\mathfrak{p}}) & \longrightarrow & 1 \end{array}$$

fulfilling the assumptions of Proposition 5 an application of which concludes the proof of Theorem 6.

2.3 The commutative case

In subsection 2.1 the problem of approximating a local unit in a maximal order was reduced to a similar problem involving solely numberfields:
Let k be a numberfield, $0 \neq \mathfrak{p} \subseteq \mathcal{O}_k$ a prime dividing the rational prime p and Σ a possibly empty set of real places of k. For a finite set of finite places S of k not containing $\mathfrak{p}$ we consider

$$X_{\mathrm{S}} := \{x \in \mathcal{O}_{k,\mathrm{S}}^* \,|\, v(x) > 0 \text{ for all } v \in \Sigma\} \subseteq \mathcal{O}_{k,\mathrm{S}}^*$$

and wish to understand when $X_{\mathrm{S}} \subseteq \mathcal{O}_{k,\mathfrak{p}}^* =: U_{\mathfrak{p}}$ is a dense subgroup. The principal units

$$U_{\mathfrak{p}}^{(1)} := 1 + \mathfrak{p}\mathcal{O}_{k,\mathfrak{p}} \subseteq U_{\mathfrak{p}}$$

are canonically a finitely generated $\mathbb{Z}_p$-module and $U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p}$ is a finite abelian group.
It follows from Nakayama's lemma that a subgroup $Y \subseteq U_{\mathfrak{p}}$ is dense if and only if the composition $Y \hookrightarrow U_{\mathfrak{p}} \longrightarrow U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p}$ is surjective: Since $U_{\mathfrak{p}}$ is pro-finite, $Y \subseteq U_{\mathfrak{p}}$ is dense if and only if it surjects onto every finite quotient of $U_{\mathfrak{p}}$. Assume that Y does surject onto $U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p}$ and $V \subseteq U_{\mathfrak{p}}$ is arbitrary of finite index. In order to see that Y surjects onto $U_{\mathfrak{p}}/V$ we can assume that $V \subseteq U_{\mathfrak{p}}^{(1)^p}$. Then the image of Y in $U_{\mathfrak{p}}/V = \mu_{q-1} \times U_{\mathfrak{p}}^{(1)}/V$ surjects onto μ_{q-1} and $U_{\mathfrak{p}}^{(1)}/V$ is a finitely generated $\mathbb{Z}_p$-module which modulo p is generated by the image of Y. By Nakayama's lemma, Y surjects onto $U_{\mathfrak{p}}/V$.
We denote by

$$E^+ := \ker(\mathcal{O}_k^* \overset{\text{diag}}{\hookrightarrow} \bigoplus_{v\in\Sigma} k_v^* \longrightarrow \bigoplus_{v\in\Sigma} k_v^*/k_v^{*,+})$$

the group of global units of k which are positive at all places in Σ. For an infinite place v of k we write $k_v^{*,+}$ for the connected component of 1 inside k_v^*, i.e. $k_v^{*,+} \simeq \mathbb{R}^+$ (resp. $k_v^{*,+} \simeq \mathbb{C}^*$) if v is real (resp. complex). We write

$$\psi : E^+ \subseteq \mathcal{O}_k^* \hookrightarrow U_{\mathfrak{p}}$$

for the inclusion. Then $U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p}$ is a finite abelian group the minimal number of generators of which we denote by $g(\mathfrak{p}, \Sigma)$.

Theorem 7. *In the above situation:*
i) If $X_{\mathrm{S}} \subseteq U_{\mathfrak{p}}$ *is dense then* $|\mathrm{S}| \geqslant g(\mathfrak{p}, \Sigma)$.
ii) Given a set T *of places of* k *of density 1, there exists* S *as above such that* $X_{\mathrm{S}} \subseteq U_{\mathfrak{p}}$ *is dense,* $|\mathrm{S}| = g(\mathfrak{p}, \Sigma)$ *and* $\mathrm{S} \subseteq \mathrm{T}$.
iii)

$$g(\mathfrak{p}, \Sigma) \leqslant \begin{cases} [k_{\mathfrak{p}} : \mathbb{Q}_p] & , \quad \text{if } \mu_{p^\infty}(k_{\mathfrak{p}}) = \{1\}, \\ 1 + [k_{\mathfrak{p}} : \mathbb{Q}_p] & , \quad \text{if } \mu_{p^\infty}(k_{\mathfrak{p}}) \neq \{1\}. \end{cases}$$

Remark 8. *1) In general, the inequalities in iii) are strict: For* $k = \mathbb{Q}(\sqrt{2})$, $\mathfrak{p}$ *dividing* 7 *and* $\Sigma = \emptyset$ *one can check that* $g(\mathfrak{p}, \Sigma) = 0$, *i.e.* $\mathcal{O}_k^* \subseteq U_{\mathfrak{p}}$ *is dense.*
2) The proof of Theorem 7,ii) is rather constructive: One has to find principal prime ideals (λ) *of* $\mathcal{O}_k$ *with* λ *positive at all places in* Σ *(this corresponds to being trivial in* $\mathrm{Gal}(M/k)$ *in the notation of the proof) and determine the image of* λ *in* $U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p}$.

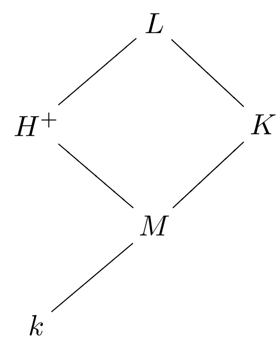
Proof. We consider the following subgroups of I_k, the idèles of k:

$$U_K := \prod_{v\nmid\infty, v\neq\mathfrak{p}} U_v \times U_{\mathfrak{p}}^{(1)^p} \times \prod_{v\in\Sigma} k_v^{*,+} \times \prod_{v\mid\infty, v\notin\Sigma} k_v^*,$$

$$U_M := \prod_{v\nmid\infty} U_v \times \prod_{v\in\Sigma} k_v^{*,+} \times \prod_{v\mid\infty, v\notin\Sigma} k_v^* \text{ and}$$

$$U_+ := \prod_{v\nmid\infty} U_v \times \prod_{v\mid\infty} k_v^{*,+}.$$

Then $U_K \subseteq U_M$ and $k^*U_K \subseteq I_k$ is of finite index. Class field theory, e.g. [N, Chapter VI], yields finite abelian extensions $k \subseteq M \subseteq K$ and the upper part of diagram (9) below. The field corresponding to k^*U_+ is the big Hilbert class field of k which we denote by H^+. Since $k^*U_K \cdot k^*U_+ = k^*U_M$ we have $H^+ \cap K = M$ and we put $L := H^+K$. We have the following diagram of fields



and some of the occurring Galois groups are identified as follows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(K/M) & \xrightarrow{\iota} & \mathrm{Gal}(K/k) & \xrightarrow{\pi} & \mathrm{Gal}(M/k) & \longrightarrow & 1 \\
 & & \beta \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq & & \\
1 & \longrightarrow & k^*U_M/k^*U_K & \longrightarrow & I_k/k^*U_K & \longrightarrow & I_k/k^*U_M & \longrightarrow & 1 \\
 & & \alpha \uparrow \simeq & & & & & & \\
 & & U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p} & & & & & &
\end{array}
\tag{9}
$$

The isomorphism α is induced by the inclusion $U_{\mathfrak{p}} \hookrightarrow k^*U_M$: One has $k^*U_M = k^*U_{\mathfrak{p}}U_K$, hence

$$k^*U_M/k^*U_K = k^*U_{\mathfrak{p}}U_K/k^*U_K \xleftarrow{\simeq} U_{\mathfrak{p}}/(U_{\mathfrak{p}} \cap k^*U_K)$$

and $U_{\mathfrak{p}} \cap k^*U_K = k^*U_{\mathfrak{p}} \cap U_K = \psi(E^+)U_{\mathfrak{p}}^{(1)^p}$.

To prove *i)*, assume that $X_{\mathrm{S}} \subseteq U_{\mathfrak{p}}$ is dense. Then $X_{\mathrm{S}} \subseteq U_{\mathfrak{p}} \longrightarrow U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p}$ is surjective, hence so is $X_{\mathrm{S}}/E^+ \longrightarrow U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p}$. The group X_{S}/E^+ is easily seen to be torsion-free and Dirichlet's unit Theorem determines its rank, hence $X_{\mathrm{S}}/E^+ \simeq \mathbb{Z}^{|\mathrm{S}|}$ and $|\mathrm{S}| \geqslant g(\mathfrak{p}, \Sigma)$.

To prove *ii)*, fix generators $x_i \in U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p}$ $(1 \leqslant i \leqslant g(\mathfrak{p}, \Sigma))$. Let $\sigma_i \in \mathrm{Gal}(L/M) \subseteq \mathrm{Gal}(L/k)$ be the unique element such that $\sigma_i|_{H^+} = \mathrm{id}$ and $\sigma_i|_K = (\iota\beta\alpha)(x_i)$. Note that $(\iota\beta\alpha)(x_i)|_M = (\pi\iota\beta\alpha)(x_i) = \mathrm{id}$ by (9). By Chebotarev's density Theorem [N, Chapter VII, Theorem 13.4], there is a finite place $v_i \in \mathrm{T}$, unramified in L/k such that $\sigma_i = \mathrm{Frob}_{v_i}^{-1}$, where Frob_{v_i} denotes the Frobenius at the place v_i, in $\mathrm{Gal}(L/k)$. Then $(\iota\beta\alpha)(x_i) = \mathrm{Frob}_{v_i}^{-1}$ in $\mathrm{Gal}(K/k)$. Since $\mathrm{Frob}_{v_i}|_{H^+} = \sigma_i^{-1}|_{H^+} = \mathrm{id}$, the prime ideal $\mathfrak{p}_i \subseteq \mathcal{O}_k$ corresponding to v_i is principal, generated by a totally positive element $\pi_i \in \mathcal{O}_k$ [N, Chapter VI, Theorem 7.3]. We claim that the image of π_i in $U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p}$ equals x_i: To see this, we apply the Artin-map $(-, K/k) : I_k \longrightarrow \mathrm{Gal}(K/k)$ to the identity $\pi_i = \pi_{i,\mathfrak{p}} \cdot (\frac{\pi_i}{\pi_{i,\mathfrak{p}}})$ in I_k, where $\pi_{i,\mathfrak{p}}$ denotes the idèle having π_i as its $\mathfrak{p}$-component and all other components equal to 1. By Artin-reciprocity we obtain $1 = (\pi_{i,\mathfrak{p}}, K/k)(\frac{\pi_i}{\pi_{i,\mathfrak{p}}}, K/k)$. Denoting $y := \frac{\pi_i}{\pi_{i,\mathfrak{p}}}$ we have $(y, K/k) = \prod_v (y_v, K_v/k_v)$ [N, Chapter VI, Theorem 5.6] and evaluate the local terms $(y_v, K_v/k_v)$ as follows:

For $v = \mathfrak{p}$ we obtain 1 since $y_{\mathfrak{p}} = 1$; for $v \neq \mathfrak{p}, v_i$ finite we obtain 1 since $y_v \in \mathcal{O}_{k,v}^*$ and v is unramified in K/k; for $v = v_i$ we obtain Frob_{v_i} since K/k is unramified at v_i and $y_{v_i} \in \mathcal{O}_{k,v_i}$ is a local uniformizer; finally, for $v|\infty$ we obtain 1 since $y_v > 0$ because π_i is totally positive.

Hence $(\pi_{i,\mathfrak{p}}, K/k) = \mathrm{Frob}_{v_i}^{-1} = (\iota\beta\alpha)(x_i)$ in $\mathrm{Gal}(K/k)$. Denoting by $\tau : U_{\mathfrak{p}} \longrightarrow U_{\mathfrak{p}}/\psi(E^+)U_{\mathfrak{p}}^{(1)^p}$ the projection we have $(\pi_{i,\mathfrak{p}}, K/k) = (\iota\beta\alpha\tau)(\pi_{i,\mathfrak{p}})$ by construction, hence $x_i = \tau(\pi_{i,\mathfrak{p}})$ by the injectivity of $\iota\beta\alpha$. This establishes the above claim saying that the global elements $\pi_i \in \mathcal{O}_k$ have the prescribed

image x_i in $U_\mathfrak{p}/\psi(E^+)U_\mathfrak{p}^{(1)^p}$. To conclude the proof of *ii)*, put $\mathrm{S} := \{v_i \mid 1 \leqslant i \leqslant g(\mathfrak{p}, \Sigma)\}$ and note that $\pi_i \in X_\mathrm{S}$ with this choice of S, hence $X_\mathrm{S} \longrightarrow U_\mathfrak{p}/\psi(E^+)U_\mathfrak{p}^{(1)^p}$ is surjective and since $E^+ \subseteq X_\mathrm{S}$, so is $X_\mathrm{S} \longrightarrow U_\mathfrak{p}/U_\mathfrak{p}^{(1)^p}$, i.e. $X_\mathrm{S} \subseteq U_\mathfrak{p}$ is dense and by construction we have $\mathrm{S} \subseteq \mathrm{T}$ and $|\mathrm{S}| = g(\mathfrak{p}, \Sigma)$.
To see *iii)* we use

$$U_\mathfrak{p} = \mu_{q-1} \times U_\mathfrak{p}^{(1)} \simeq \mu_{q-1} \times \mu_{p^\infty}(k_\mathfrak{p}) \times \mathbb{Z}_p^{[k_\mathfrak{p}:\mathbb{Q}_p]},$$

where $q = |\mathcal{O}_{k,\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{k,\mathfrak{p}}|$ [N, Chapter II, Theorem 5.7, i)] which implies that the upper bound claimed in *iii)* is in fact the minimal number of generators of $U_\mathfrak{p}/U_\mathfrak{p}^{(1)^p}$ which obviously is greater than or equal to the minimal number of generators of $U_\mathfrak{p}/\psi(E^+)U_\mathfrak{p}^{(1)^p}$, i.e. $g(\mathfrak{p}, \Sigma)$. □

In Theorem 6 we reduced the problem of approximating a local unit of a maximal order by global *unitary* units to an approximation problem for a one-dimensional torus. This seems to be substantially harder than the problem settled in Theorem 7 and we only treat the following special case here:
Let k be an imaginary quadratic field in which the rational prime p splits, $p\mathcal{O}_k = \mathfrak{p}\overline{\mathfrak{p}}$, and put

$$T := \ker(\mathrm{Res}_\mathbb{Z}^{\mathcal{O}_k}(\mathbb{G}_m) \overset{\mathrm{N}_\mathbb{Q}^k}{\longrightarrow} \mathbb{G}_m).$$

Theorem 9. *In the above situation, there is a rational prime $l \neq p$ such that $T(\mathbb{Z}[1/l]) \subseteq T(\mathbb{Z}_p)$ is a dense subgroup.*

By combining Theorems 6 and 9 we obtain the following.

Corollary 10. *Let D be a finite-dimensional skew-field over $\mathbb{Q}$ with a positive involution $*$ of the second kind and $\mathcal{O} \subseteq D$ a maximal order, stable under $*$. Assume that the center of D is an imaginary quadratic field k and let p be a rational prime which splits in k and $\mathfrak{P} \subseteq \mathcal{O}$ a prime lying above p. Then there exists a rational prime $l \neq p$ such that*

$$\left\{\alpha \in \mathcal{O}[\frac{1}{l}] \mid {}^*\alpha\alpha = 1\right\} \subseteq \mathcal{O}_\mathfrak{P}^*$$

is a dense subgroup.

Proof of Theorem 9. Put $\mathfrak{p} := \mathfrak{P} \cap \mathcal{O}_k$ and note that for every rational prime $l \neq p$

$$T(\mathbb{Z}[1/l]) = \{\alpha \in \mathcal{O}_k[1/l]^* \mid \alpha\overline{\alpha} = 1\} \subseteq T(\mathbb{Z}_p) = U_\mathfrak{p} \simeq \mathbb{Z}_p^*, \tag{10}$$

the local units of k at $\mathfrak{p}$, the final equalities following from the fact that p splits in k. Here, $-$ denotes complex conjugation. The following proof is similar to the argument of Theorem 7,*ii)* but extra care is needed to deal with the norm condition $\alpha\overline{\alpha} = 1$.
Consider the following subgroups of the idèles of k:

$$U_K := \prod_{v \neq \mathfrak{p}, \overline{\mathfrak{p}} \text{ finite}} U_v \times U_\mathfrak{p}^{(1)^p} \times U_{\overline{\mathfrak{p}}}^{(1)^p} \times \prod_{v|\infty} k_v^* \text{ and}$$

$$U_H := \prod_{v \text{ finite}} U_v \times \prod_{v|\infty} k_v^*.$$

We have a corresponding tower of abelian extensions $k \subseteq H \subseteq K$ and since U_K is stable under $\mathrm{Gal}(k/\mathbb{Q})$, the extension $K/\mathbb{Q}$ is Galois, though rarely abelian. We have an isomorphism

$$\phi : U_{\mathfrak{p}} U_{\overline{\mathfrak{p}}} / U_{\mathfrak{p}}^{(1)^p} U_{\overline{\mathfrak{p}}}^{(1)^p} \mathcal{O}_k^* \simeq \frac{U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p} \times U_{\overline{\mathfrak{p}}}/U_{\overline{\mathfrak{p}}}^{(1)^p}}{\mathcal{O}_k^*} \xrightarrow{\simeq} \mathrm{Gal}(K/H)$$

induced by the Artin-map, where $\mathcal{O}_k^*$ is embedded diagonally. Since p splits in k we have $U_{\mathfrak{p}} \simeq \mathbb{Z}_p^*$, $U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p}\mathcal{O}_k^*$ is cyclic and we fix a generator x of this group. By Chebotarev's Theorem applied to $K/\mathbb{Q}$ there exists a rational prime $l \neq p$, unramified in $K/\mathbb{Q}$ and such that for a suitable prime Λ of K lying above l we have

$$\mathrm{Frob}_{\Lambda|l}^{-1} = \phi([(x,1)]) \text{ in } \mathrm{Gal}(K/H) \subseteq \mathrm{Gal}(K/\mathbb{Q}).$$

We claim that l satisfies the conclusion of Theorem 9:
Put $\lambda := \Lambda|_k$. Since $(\mathrm{Frob}_{\Lambda|l})|_H = \mathrm{id}$, λ is a principal ideal of $\mathcal{O}_k$ a generator of which we denote by π. Then

$$\beta := \frac{\pi}{\overline{\pi}} \in \{\alpha \in \mathcal{O}_k[1/l]^* \,|\, \alpha\overline{\alpha} = 1\} = T(\mathbb{Z}[1/l]),$$

and we claim that β goes to x under the map induced by (10): As in the proof of Theorem 7,ii) one sees that

$$(\pi_{\mathfrak{p}}, \pi_{\overline{\mathfrak{p}}}) = [(x,1)] \text{ and similarly}$$
$$((\overline{\pi})_{\mathfrak{p}}, (\overline{\pi})_{\overline{\mathfrak{p}}}) = [(1,x)] \text{ in } \frac{U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p} \times U_{\overline{\mathfrak{p}}}/U_{\overline{\mathfrak{p}}}^{(1)^p}}{\mathcal{O}_k^*},$$

hence indeed

$$(\beta_{\mathfrak{p}}, \beta_{\overline{\mathfrak{p}}}) = [(x, x^{-1})]$$

and a fortiori $\beta_{\mathfrak{p}} = x$ in $U_{\mathfrak{p}}/U_{\mathfrak{p}}^{(1)^p}\mathcal{O}_k^*$. Since we have $\mathcal{O}_k^* \subseteq T(\mathbb{Z}[1/l])$ because $\mathcal{O}_k^*$ consists of roots of unity which have norm 1, we are done.

□

3. Applications

3.1 Extending automorphisms of p-divisible groups

Here we explain the application of the results from subsections 2.1 and 2.3 to the following problem: Let k be a finite field of characteristic p and A/k a simple abelian variety such that $\mathrm{End}_k(A)$ is

a maximal order in the skew-field $D := \mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The center K of D is a numberfield and $K \cap \mathrm{End}_k(A) = \mathcal{O}_K$ is its ring of integers.
The p-divisible group of A/k [T] splits as

$$A[p^\infty] = \prod_{\mathfrak{p}|p} A[\mathfrak{p}^\infty], \tag{11}$$

the product extending over all primes $\mathfrak{p}$ of $\mathcal{O}_K$ dividing p. According to J. Tate, c.f. [MW, Theorem 6], the canonical homomorphism

$$\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\simeq} \mathrm{End}_k(A[p^\infty]) \tag{12}$$

is an isomorphism. We have

$$\mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \prod_{\mathfrak{p}|p} \mathrm{End}_k(A) \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{p}} \simeq \prod_{\mathfrak{p}|p} \mathrm{End}_k(A)_{\mathfrak{P}}$$

with $\mathfrak{P}$ the unique prime of $\mathrm{End}_k(A)$ lying above $\mathfrak{p}$. Similarly, (11) implies that

$$\mathrm{End}_k(A[p^\infty]) \simeq \prod_{\mathfrak{p}|p} \mathrm{End}_k(A[\mathfrak{p}^\infty]).$$

These decompositions are compatible with (12) in that the canonical homomorphism

$$\mathrm{End}_k(A) \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\mathfrak{p}} \xrightarrow{\simeq} \mathrm{End}_k(A[\mathfrak{p}^\infty])$$

is an isomorphism for every $\mathfrak{p}|p$. We fix some $\mathfrak{p}|p$ and ask for a finite set S of finite primes of K such that $\mathfrak{p} \notin$ S and

$$(\mathrm{End}_k(A) \otimes_{\mathcal{O}_K} \mathcal{O}_{K,S})^* \hookrightarrow \mathrm{Aut}_k(A[\mathfrak{p}^\infty]) \tag{13}$$

is a dense subgroup. Note that this density is equivalent to the following assertion:
For every $\alpha \in \mathrm{Aut}_k(A[\mathfrak{p}^\infty])$ and integer $n \geqslant 1$ there is an isogeny $\phi \in \mathrm{End}_k(A)$ of degree divisible by primes in S only and some $x \in \mathcal{O}_{K,S}^*$ such that

$$\phi x|_{A[\mathfrak{p}^n]} = \alpha|_{A[\mathfrak{p}^n]},$$

i.e. the quasi-isogeny ϕx of A extends the truncation at arbitrary finite level n of α.
By Theorem 1, the inclusion (13) is dense if and only if $X \subseteq U_{\mathfrak{p}}$ is dense where $X \subseteq \mathcal{O}_K^*$ is the subgroup of global units which are positive at all real places of K at which D does not split and $U_{\mathfrak{p}} := \mathcal{O}_{K,\mathfrak{p}}^*$ are the local units of K at $\mathfrak{p}$. The density of $X \subseteq U_{\mathfrak{p}}$ in turn is firmly controlled by Theorem 7. We would like to illustrate all of this with some examples:
According to the Albert-classification [Mu, Theorem 2, p. 201], note that types I and II do not occur over finite fields, there are two possibilities:

Type III: Here, K is a totally real numberfield and D/K is a totally definite quaternion algebra. The simplest such case occurs if A/k is a super-singular elliptic curve with $\mathrm{End}_k(A) = \mathrm{End}_{\overline{k}}(A)$. In this case, it follows from Example 2,2) that, in case the characteristic of k is different from 2, for a suitable prime l

$$\left(\mathrm{End}_k(A)\left[\frac{1}{l}\right]\right)^* \hookrightarrow \mathrm{Aut}_k(A[p^\infty])$$

is dense.
To see another example of this type, let $A/\mathbb{F}_p$ correspond to a p-Weil number π with $\pi^2 = p$. Then $\dim(A) = 2$ and $A \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$ is isogeneous to the square of a super-singular elliptic curve. We have $K = \mathbb{Q}(\sqrt{p})$ and $\mathfrak{p} = (\sqrt{p})\mathcal{O}_K$, hence $A[\mathfrak{p}^\infty] = A[p^\infty]$. Furthermore, $\mathcal{O}_K^* = \{\pm 1\} \times \epsilon^{\mathbb{Z}}$ for a fundamental unit ϵ and $X \subseteq \mathcal{O}_K^*$ is of index 4. To find a small set S such that (13) is dense one first needs to compute the minimal number of generators of $U_\mathfrak{p}/XU_\mathfrak{p}^{(1)^p}$, denoted $g(\mathfrak{p}, \Sigma)$ in Theorem 7 where, in the present situation, Σ consists of both the infinite places of K. For $p = 2$ one can choose $\epsilon = 1 + \sqrt{2}$, then $X = \epsilon^{2\mathbb{Z}}$. Since $U_\mathfrak{p}/U_\mathfrak{p}^{(1)^2} \simeq \mathbb{F}_2^3$ and $\epsilon^2 \notin U_\mathfrak{p}^{(1)^2}$ one gets $g(\mathfrak{p}, \Sigma) = 2$.
For $p = 3$ we may take $\epsilon = 2 + \sqrt{3}$, then $X = \epsilon^{2\mathbb{Z}}$ again. Since now $U_\mathfrak{p}/U_\mathfrak{p}^{(1)^3} = \mu_2 \times \mathbb{F}_3^2 \simeq \mathbb{Z}/6 \times \mathbb{Z}/3$ the fact that $\epsilon^2 \notin U_\mathfrak{p}^{(1)^3}$ is not enough to conclude that $g(\mathfrak{p}, \Sigma) = 1$. However, one checks in addition that $\epsilon^2 \in U_\mathfrak{p}^{(1)}$, and concludes that $U_\mathfrak{p}/XU_\mathfrak{p}^{(1)^3} \simeq \mathbb{Z}/6$ and hence indeed $g(\mathfrak{p}, \Sigma) = 1$.
For $p \geqslant 5$ one has $U_\mathfrak{p}/U_\mathfrak{p}^{(1)^p} = \mu_{p-1} \times \mathbb{F}_p^2$ and since $\mu_{p-1} \nsubseteq K$ the image of a generator of X in $U_\mathfrak{p}/U_\mathfrak{p}^{(1)^p}$ will have non-trivial projection to $\mathbb{F}_p^2$ and one concludes that $g(\mathfrak{p}, \Sigma) = 1$.

Type IV: In this case, K is a CM-field and $X = \mathcal{O}_K^*$. The easiest such example occurs for an ordinary elliptic curve and we give two examples:
A solution of $\pi^2 + 5 = 0$ is a 5-Weil number to which there corresponds an elliptic curve $E/\mathbb{F}_5$ with $K = D = \mathbb{Q}(\sqrt{5})$. For $\mathfrak{p} = (\sqrt{5})\mathcal{O}_K$ one has $U_\mathfrak{p}/U_\mathfrak{p}^{(1)^p} = \mu_4 \times \mathbb{F}_5^2$ and since $\mathcal{O}_K^* = \{\pm 1\}$ one gets $U_\mathfrak{p}/XU_\mathfrak{p}^{(1)^p} \simeq \mathbb{Z}/10 \times \mathbb{Z}/5$, hence $g(\mathfrak{p}, \Sigma) = 2$.
Similarly, a solution of $\pi^2 - 4\pi + 5 = 0$ gives an elliptic curve over $\mathbb{F}_5$ with $D = K = \mathbb{Q}(i)$ and since 5 splits in K one has $U_\mathfrak{p}/XU_\mathfrak{p}^{(1)^p} \simeq \mathbb{Z}/10$, hence $g(\mathfrak{p}, \Sigma) = 1$ in this case.

Finally, we leave it as an easy exercise to an interested reader to check that for any prime p and integer $N \geqslant 1$ there exists a simple abelian variety $A/\mathbb{F}_p$ such that every set S for which (13) is dense necessarily satisfies $|S| \geqslant N$.

3.2 A dense subgroup of quasi-isogenies in the Morava stabilizer group

Let p be a prime and $n \geqslant 1$ an integer. The n-th Morava-stabilizer group $\mathbb{S}_n$ is the group of units of the maximal order of the central skew-field over $\mathbb{Q}_p$ of Hasse-invariant $\frac{1}{n}$.
In this section we will construct an abelian variety A/k over a finite field k of characteristic p such that for a suitable prime l the group $(\mathrm{End}_k(A)[\frac{1}{l}])^*$ is canonically a dense subgroup of $\mathbb{S}_n$. We will completely ignore the case $n = 1$ as it is very well understood. In case $n = 2$ one can take for A a super-singular elliptic curve [BL2] and the resulting dense subgroup of $\mathbb{S}_2$ has been used to great advantage in the construction of a "modular" resolution of the $K(2)$-local sphere [B].
For general n we remark that, since $\mathrm{End}_k(A) \otimes_\mathbb{Z} \mathbb{Z}_p \simeq \mathrm{End}_k(A[p^\infty])$, in order that $\mathrm{End}_k(A)$ have a

relation with $\mathbb{S}_n$ one needs $A[p^\infty]\otimes_k \overline{k}$ to have an isogeny factor of type $G_{1,n-1}$ [Ma, IV,§2,2.]. By the symmetry of p-divisible groups of abelian varieties [Ma, IV, §3, Theorem 4.1], there must then also be a factor of type $G_{n-1,1}$ which shows that $n = 2$ is somewhat special since $(1, n-1) = (n-1, 1)$ in this case. For $n \geqslant 3$ the above considerations imply that the sought for abelian variety must be of dimension at least n, as already remarked by D. Ravenel [R1, Corollary 2.4 (ii)]. Following suggestions of M. Behrens and T. Lawson we will be able to construct A having this minimal possible dimension. We start by constructing a suitable isogeny-class as follows.

Proposition 11. *Let p be a prime and $n \geqslant 3$ an integer. Then there is a simple abelian variety $A/\mathbb{F}_{p^n}$ such that the center of $\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is an imaginary quadratic field in which p splits into, say, $\mathfrak{p}$ and $\mathfrak{p}'$ such that $\mathrm{inv}_{\mathfrak{p}}(\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}) = \frac{1}{n}$, $\mathrm{inv}_{\mathfrak{p}'}(\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}) = \frac{-1}{n}$ and $\dim(A) = n$. Furthermore, A is geometrically simple with $\mathrm{End}_{\overline{\mathbb{F}_{p^n}}}(A)\otimes_{\mathbb{Z}}\mathbb{Q} = \mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$.*

Proof. We use Honda-Tate theory, see [MW] for an exposition. Let $\pi \in \overline{\mathbb{Q}}$ be a root of $f := x^2 - px + p^n \in \mathbb{Z}[x]$. Since the discriminant of f is negative, π is a p^n-Weil number and we choose $A/\mathbb{F}_{p^n}$ simple associated with the conjugacy class of π. Then $\mathbb{Q}(\pi)$ is an imaginary quadratic field and is the center of $\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$. Since $n \geqslant 3$ the Newton polygon of f at p has different slopes 1 and $n-1$ which shows that f is reducible over $\mathbb{Q}_p$ [N, Chapter II, Theorem 6.4], hence p splits in $\mathbb{Q}(\pi)$ into $\mathfrak{p}$ and $\mathfrak{p}'$ and, exchanging π and $\overline{\pi}$ if necessary, we can assume that $v_{\mathfrak{p}}(\pi) = 1$ and $v_{\mathfrak{p}}(\overline{\pi}) = n-1$. Then [MW, Theorem 8, 4.]

$$\mathrm{inv}_{\mathfrak{p}}(\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}) = \frac{v_{\mathfrak{p}}(\pi)}{v_{\mathfrak{p}}(p^n)}[\mathbb{Q}(\pi)_{\mathfrak{p}} : \mathbb{Q}_p] = \frac{1}{n} \text{ and similarly}$$

$$\mathrm{inv}_{\mathfrak{p}'}(\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}) = \frac{n-1}{n} = \frac{-1}{n}.$$

Furthermore [MW, Theorem 8, 3.], $2\cdot\dim(A) = [\mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q} : \mathbb{Q}(\pi)]^{1/2}\cdot[\mathbb{Q}(\pi):\mathbb{Q}] = 2n$. The final statement follows easily from the fact that $\pi^k \notin \mathbb{Q}$ for all $k \geqslant 1$, c.f. [HZ, Proposition 3(2)], which in turn is evident since $v_{\mathfrak{p}}(\pi) \neq v_{\mathfrak{p}}(\overline{\pi})$. □

Since the properties of $A/\mathbb{F}_{p^n}$ in Proposition 11 are invariant under $\mathbb{F}_{p^n}$-isogenies, we can, and do, choose $A/\mathbb{F}_{p^n}$ having these properties such that in addition $\mathrm{End}_{\mathbb{F}_{p^n}}(A) \subseteq \mathrm{End}_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a maximal order [Wa, proof of Theorem 3.13]. Denoting by $\mathfrak{P} \subseteq \mathrm{End}_{\mathbb{F}_{p^n}}(A)$ the unique prime lying above the prime $\mathfrak{p}$ constructed in Proposition 11, we have $(\mathrm{End}_{\mathbb{F}_{p^n}}(A))^*_{\mathfrak{P}} = \mathbb{S}_n$ since $\mathrm{inv}_{\mathfrak{p}}(\mathrm{End}^0_{\mathbb{F}_{p^n}}(A)\otimes_{\mathbb{Z}}\mathbb{Q}) = 1/n$. We choose a prime l as follows: If $p \neq 2$ we take l to be a topological generator of $\mathbb{Z}_p^*$. For $p = 2$ we take $l = 5$.

Remark 12. *Note that for $p \neq 2$ a prime $l \neq p$ topologically generates $\mathbb{Z}_p^*$ if and only if $(l \bmod p^2)$ generates $(\mathbb{Z}/p^2)^*$. Hence, by Dirichlet's Theorem on primes in arithmetic progressions, the set of all such l has a density equal to $((p-1)\phi(p-1))^{-1} > 0$ and is thus infinite. Such an l can be found rather effectively: Given $l \neq p$, compute $\alpha_k := (l^{p(p-1)/k} \bmod p^2)$ for all primes k dividing $p(p-1)$. If for all k, $\alpha_k \not\equiv 1\,(p^2)$, then l is suitable.*

Theorem 13. *In the above situation*

$$(\mathrm{End}_{\mathbb{F}_{p^n}}(A)[\frac{1}{l}])^* \hookrightarrow (\mathrm{End}_{\mathbb{F}_{p^n}}(A))^*_{\mathfrak{P}} = \mathbb{S}_n$$

is a dense subgroup.

Proof. We apply Theorem 1 with $\mathcal{O} := \mathrm{End}_{\mathbb{F}_{p^n}}(A)$, $k := \mathbb{Q}(\pi)$, $\mathfrak{p}$ the prime of $\mathcal{O}_k$ constructed in Proposition 11 and $\mathrm{S} := \{\infty, l\}$ the set consisting of the unique infinite place ∞ of k and all places dividing l. Clearly, $\mathfrak{p} \notin \mathrm{S}$ and $D := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ is not a skew-field at ∞ since $k_\infty \simeq \mathbb{C}$ and $n > 1$. Using the notation of Theorem 1 we have $\mathcal{O}_{k,\mathrm{S}^{\mathrm{fin}}} = \mathcal{O}_k[1/l]$ and $X = (\mathcal{O}_k[1/l])^*$ since k has no real place. Theorem 1 shows that the claim of Theorem 13 is equivalent to the density of $(\mathcal{O}_k[1/l])^* \subseteq \mathcal{O}_{k,\mathfrak{p}}^* \simeq \mathbb{Z}_p^*$. Since $l \in (\mathcal{O}_k[1/l])^*$, this density is clear for $p \neq 2$ by our choice of l whereas for $p = 2$ we have that $\{\pm 1\} \times 5^{\mathbb{Z}} \subseteq \mathbb{Z}_2^*$ is dense and $-1, 5 \in (\mathcal{O}_k[1/5])^*$. $\square$

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