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Strong nonlocal-to-local convergence of the Cahn-Hilliard equation and its operator

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Abstract

We prove convergence of a sequence of weak solutions of the nonlocal Cahn-Hilliard equation to the strong solution of the corresponding local Cahn-Hilliard equation. The analysis is done in the case of sufficiently smooth bounded domains with Neumann boundary condition and a $W^{1,1}$-kernel. The proof is based on the relative entropy method. Additionally, we prove the strong L^2-convergence of the nonlocal operator to the negative Laplacian together with a rate of convergence.

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Keywords: Cahn-Hilliard equation; Nonlocal Cahn-Hilliard equation; Nonlocal operators; Nonlocal-to-local convergence; Singular limit

1. Introduction

The Cahn-Hilliard equation was originally introduced in [6] to model the phenomena of spinodal decomposition in binary alloys. Since then, it has been frequently used in a variety of different mathematical models describing phenomena such as population dynamics, image processing, two-phase flows and tumor growth, cf. [8,9,11,15,18].

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The (local) Cahn-Hilliard equation as introduced in [6] reads as follows:

$$\partial_t c = m\Delta\mu \qquad \text{in } \Omega_T, \tag{1}$$

$$\mu = -\Delta c + f'(c) \quad \text{in } \Omega_T, \tag{2}$$

where $\Omega_T := (0,T)\times\Omega$, $T>0$ and $\Omega\subset\mathbb{R}^n$, $n\le 3$, is a bounded domain with C^2-boundary. Further, we require the following initial condition

$$c|_{t=0} = c_0 \quad \text{in } \Omega \tag{3}$$

and boundary conditions

$$\partial_{\mathbf{n}} c = 0, \;\; \partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \partial\Omega\times(0,T). \tag{4}$$

Here, c is a concentration parameter and μ the chemical potential associated to c. Furthermore, $m>0$ is the mobility coefficient and f is the free energy density. Typical choices for the potential in the free energy density are a smooth double-well potential, e.g. $f(c) := K(1-c^2)^2$ for some $K>0$, or the logarithmic potential $f(c) := \frac{\theta}{2}\big((1-c)\ln(1-c) + (1+c)\ln(1+c)\big) - \frac{\theta_c}{2}c^2$ for $c\in[-1,1]$, where we assume $0<\theta<\theta_c$. We note that the Cahn-Hilliard equation describes the H^{-1}-gradient flow of the free energy functional

$$\mathcal{E}^{CH}(c) := \int_\Omega \frac{1}{2}|\nabla c|^2 + f(c)\,\mathrm{d}x. \tag{5}$$

The Cahn-Hilliard equation has already been studied very intensively and there exists an extensive literature (see [8,9,11,15,18] and the references therein).

The nonlocal counterpart of the Cahn-Hilliard equation has first been presented by Giacomin and Lebowitz in [14], where the authors considered the hydrodynamic limit of a microscopic model describing an n-dimensional lattice gas evolving via the (Poisson) nearest neighbor exchange process. The nonlocal Cahn-Hilliard equation can be interpreted as the H^{-1}-gradient flow of the non-local free energy functional

$$\mathcal{E}^{NL}_\varepsilon(c) := \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)\big|c(x)-c(y)\big|^2\,\mathrm{d}y\mathrm{d}x + \int_\Omega f(c(x))\,\mathrm{d}x. \tag{6}$$

This leads to the following system:

$$\partial_t c = m\Delta\mu \qquad \text{in } \Omega_T, \tag{7}$$

$$\mu = \mathcal{L}_\varepsilon c + f'(c) \quad \text{in } \Omega_T, \tag{8}$$

where we define

$$\mathcal{L}_\varepsilon c(x) := -\int_\Omega J_\varepsilon(|x-y|)c(y)\,\mathrm{d}y + \int_\Omega J_\varepsilon(|x-y|)c(x)\,\mathrm{d}y \quad \text{for all } x\in\Omega.$$

Further, we require the initial condition

$$c|_{t=0} = c_0 \quad \text{in } \Omega \tag{9}$$

and the boundary condition

$$\partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \partial\Omega \times (0,T). \tag{10}$$

Here, $J_\varepsilon : \mathbb{R}^n \to [0,\infty)$ is a non-negative function. More precisely, we assume that $J_\varepsilon(x) = \frac{\rho_\varepsilon(x)}{|x|^2}$ for all $x \in \mathbb{R}^n$ and $J_\varepsilon \in W^{1,1}(\mathbb{R}^n)$, where $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers satisfying

$$\rho_\varepsilon : \mathbb{R} \to [0,\infty), \quad \rho_\varepsilon \in L^1(\mathbb{R}), \quad \rho_\varepsilon(r) = \rho_\varepsilon(-r) \ \text{ for all } r \in \mathbb{R},\ \varepsilon > 0,$$

$$\int_0^\infty \rho_\varepsilon(r)\, r^{n-1}\,\mathrm{d}r = \frac{2}{C_n} \ \text{ for all } \varepsilon > 0,$$

$$\lim_{\varepsilon \searrow 0} \int_\delta^\infty \rho_\varepsilon(r)\, r^{n-1}\,\mathrm{d}r = 0 \ \text{ for all } \delta > 0,$$

where $C_n := \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 \,\mathrm{d}\mathcal{H}^{n-1}(\sigma)$. Moreover, the singular potential f obeys the same assumptions as in [13], i.e.

$$f \in C^0([-1,1]) \cap C^1(-1,1),$$

$$\lim_{s\to -1} f'(s) = -\infty, \quad \lim_{s\to 1} f'(s) = +\infty, \quad f''(s) \geq -\alpha > 0.$$

Observe that the logarithmic potential mentioned above fulfills these assumptions. In our analysis, it is also possible to consider regular potentials $f : \mathbb{R} \to \mathbb{R}$, which satisfy $f''(s) \geq -\alpha$ for all $s \in \mathbb{R}$ together with a growth condition $|f'(s)| \leq C(|s|^3 + 1)$ for all $s \in \mathbb{R}$, since we only need that f'' is bounded from below. A typical choice for f is the double-well potential $f(c) := K(1-c^2)^2$ for some constant $K > 0$.

The nonlocal Cahn-Hilliard equation has already been subject to an intense research activity in the recent years. For instance, in the case of singular potentials, the authors in [13] proved well-posedness and regularity of weak solutions. Moreover, they established the validity of the strict separation property in two spatial dimensions. For further results on the nonlocal Cahn-Hilliard equation, we refer the reader to [5,8,9,11,13,15,18] and the references therein.

It is the goal of this contribution to show strong convergence of the nonlocal operator $\mathcal{L}_\varepsilon$ to $-\Delta$ with respect to the L^2-topology and to show convergence of weak solutions of (7) - (8) together with (9) - (10) to the strong solution of (1) - (2) together with (3) - (4) under suitable conditions on the initial values.

Important first results on nonlocal-to-local convergence for functionals of the form (6) were obtained by J. Bourgain, H. Brezis and P. Mironescu [3,4]. These results were extended by A.C. Ponce in [19,20], where also results on Γ-convergence were shown. The following results are based on these works to a large extent. Recent results on convergence of nonlocal quadratic forms to local quadratic forms of gradient type and further references can be found in Foghem Gounoue et al. in [12]. Nonlocal-to-local asymptotics have already been studied in [23], where the author proved the convergence of weak solutions of the fractional heat equation to the fundamental solution as $t \to \infty$. In [7], the authors studied the limits $s \to 0^+$ and $s \to 1^-$ for s-fractional heat flows in a cylindrical domain with homogeneous Dirichlet boundary conditions.

Convergence of solutions of the nonlocal Cahn-Hilliard equation, i.e., (7) - (8), to the local Cahn-Hilliard equation, i.e., (1) - (2), has already been proved by Melchionna et al. in [18] in the case of periodic boundary conditions and a regular free energy density and by Davoli et al. in [8] in the case of periodic boundary conditions and a singular free energy density. In the case of Neumann boundary conditions, convergence has been proved by Davoli et al. in [10] with an additional viscosity term in the nonlocal Cahn-Hilliard equation and in [9] for $W^{1,1}$-kernels. A corresponding result for a singular phase field system was proved by Kurima [16]. The authors in [11] proved convergence of the nonlocal to the local degenerate Cahn-Hilliard equation. In [1] and [17], the authors proved the nonlocal-to-local limit for a coupled Navier–Stokes/Cahn–Hilliard system.

In this contribution, however, we use Fourier transforms, which then guarantee a rate of convergence for the nonlocal operator in the case of $\Omega = \mathbb{R}^n$. Using a reflection argument together with perturbation and a localization argument, we can even show a rate of convergence in sufficiently smooth bounded domains $\Omega \subset \mathbb{R}^n$.

The structure of this paper is the following: In Section 2, we recall some definitions and preliminary results. In Section 3, we prove first convergence results of the nonlocal operator $\mathcal{L}_\varepsilon$. In Section 4, we then state and prove the main theorem about the strong convergence of the nonlocal operator. This will be done by localization. Finally, in Section 5, we apply our results from Section 4 to prove nonlocal-to-local convergence of the Cahn-Hilliard equation using the relative entropy method.

2. Preliminaries

In this section, we recollect some preliminary results, which we need throughout the paper. First, we briefly recall the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x.$$

We observe that this map is well-defined and defines an isometric automorphism on $L^2(\mathbb{R}^n)$, cf. Plancherel‘s Theorem. Further, we recall that the inverse of the negative Laplacian $-\Delta$ with Neumann boundary condition is a well-defined isomorphism

$$(-\Delta)^{-1} : \{c \in (H^1(\Omega))' : c_\Omega = 0\} \to \{c \in H^1(\Omega) : c_\Omega = 0\}.$$

For $c \in (H^1(\Omega))'$, we define $c_\Omega = \frac{1}{|\Omega|}\langle c, 1\rangle$. Next, we state some important inequalities.

Lemma 2.1. *For every $\delta > 0$ there exist constants $C_\delta > 0$ and $\varepsilon_\delta > 0$ with the following properties:*

1. *For every sequence* $(f_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega)$*, there holds*

$$\|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{H^1(\Omega)} \le \delta \int_\Omega \int_\Omega J_{\varepsilon_1}(x,y)\big|\nabla f_{\varepsilon_1}(x) - \nabla f_{\varepsilon_2}(y)\big|^2 \,\mathrm{d}y\mathrm{d}x + \delta \int_\Omega \int_\Omega J_{\varepsilon_2}(x,y)\big|\nabla f_{\varepsilon_1}(x) - \nabla f_{\varepsilon_2}(y)\big|^2 \,\mathrm{d}y\mathrm{d}x + C_\delta \|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{L^2(\Omega)}. \tag{11}$$

2. *For every sequence* $(f_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$*, there holds*

$$\|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{L^2(\Omega)} \le \delta \mathcal{E}_{\varepsilon_1}(f_{\varepsilon_1}) + \delta \mathcal{E}_{\varepsilon_2}(f_{\varepsilon_2}) + C_\delta \|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{(H^1(\Omega))^*}. \tag{12}$$

Proof. For a proof, we refer to [8, Lemma 4(2)]. □

Here, the term $\mathcal{E}_\varepsilon$ is defined by

$$\mathcal{E}_\varepsilon(c) := \frac{1}{4} \int_\Omega \int_\Omega J_\varepsilon(x-y)\big|c(x) - c(y)\big|^2 \,\mathrm{d}y\mathrm{d}x$$

for all $c \in H^1(\Omega)$, i.e., the first part of the nonlocal energy functional $\mathcal{E}^{NL}_\varepsilon$ in (6). In the limit $\varepsilon \searrow 0$, this term behaves as

$$\lim_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon(c) = \frac{1}{2} \int_\Omega \big|\nabla c(x)\big|^2 \,\mathrm{d}x \tag{13}$$

for all $c \in H^1(\Omega)$. For details, we refer to [3,8].

Lemma 2.2. *Let* $a, b \in L^1(\mathbb{R}^{n-1})$*,* $n \ge 2$*. Then, it holds*

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} a(x-y)\, b((1-t)y + tx) \,\mathrm{d}x\mathrm{d}y = \|a\|_{L^1(\mathbb{R}^{n-1})} \|b\|_{L^1(\mathbb{R}^{n-1})}$$

for all $t \in [0,1]$.

Proof. We define the mapping

$$\Phi_t : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \quad (x,y) \mapsto (x-y, (1-t)y + tx).$$

Then, we have

$$\det \mathrm{D}\Phi_t(x,y) = \det \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ t\mathrm{Id} & (1-t)\mathrm{Id} \end{pmatrix} = 1$$

for all $t \in [0,1]$. Using change of variables and Fubini's Theorem, we obtain

$$\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} a(x-y)\, b((1-t)y+tx)\, \mathrm{d}x\mathrm{d}y = \left(\int_{\mathbb{R}^{n-1}} a(x)\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^{n-1}} b(y)\,\mathrm{d}y\right)$$
$$= \|a\|_{L^1(\mathbb{R}^{n-1})}\|b\|_{L^1(\mathbb{R}^{n-1})}$$

for all $t \in [0,1]$. □

Remark 2.3. The space $\{c \in C_c^k(\overline{\mathbb{R}^n_+}) : \partial_{\mathbf{n}} c|_{\partial\mathbb{R}^n_+} = 0\}$ is dense in $\{c \in H^k(\mathbb{R}^n_+) : \partial_{\mathbf{n}} c|_{\partial\mathbb{R}^n_+} = 0\}$. If $k=3$, the proof is based on the following idea:
Since $C_0^\infty(\overline{\mathbb{R}^n_+})$ is dense in $H^3(\mathbb{R}^n_+)$, for any $c \in H^3(\mathbb{R}^n_+)$ there exists a sequence $(\tilde{c}_j)_{j\in\mathbb{N}} \subset C_0^\infty(\overline{\mathbb{R}^n_+})$ such that $\tilde{c}_j \to c$ in $H^3(\mathbb{R}^n_+)$ as $j \to \infty$. Next, we consider the auxiliary problem

$$\begin{aligned} (1-\Delta)w_j &= 0 && \text{in } \mathbb{R}^n_+, \\ \mathbf{n}\cdot\nabla w_j &= \mathbf{n}\cdot\nabla\tilde{c}_j && \text{on } \partial\mathbb{R}^n_+ \end{aligned}$$

for all $j \in \mathbb{N}$. Then, by linear elliptic theory, there exists a solution $w_j \in C^3(\overline{\mathbb{R}^n_+})$ for all $j \in \mathbb{N}$. Finally, the sequence $c_j := \tilde{c}_j - w_j$, $j \in \mathbb{N}$, has the desired properties.

Next, we consider the case of the bent half-space. Let $\gamma \in C_b^k(\mathbb{R}^{n-1})$ be given. Then, the space $\{c \in C_c^{k-1,1}(\overline{\mathbb{R}^n_\gamma}) : \partial_{\mathbf{n}} c|_{\partial\mathbb{R}^n_\gamma} = 0\}$ is a dense subset of $\{c \in H^k(\mathbb{R}^n_\gamma) : \partial_{\mathbf{n}} c|_{\partial\mathbb{R}^n_\gamma} = 0\}$. The proof is based on the following idea:
Since $\gamma \in C_b^k(\mathbb{R}^{n-1})$, there exists a $C^{k-1,1}$-diffeomorphism $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ with $F_\gamma(\mathbb{R}^n_+) = \mathbb{R}^n_\gamma$, $F_\gamma(x',0) = (x',\gamma(x'))$ and $-\partial_{x_n} F_\gamma(x)|_{x_n=0} = \mathbf{n}(x',\gamma(x'))$, where $\mathbf{n}$ denotes the exterior unit normal on $\partial\mathbb{R}^n_\gamma$, cf. [22, Lemma 2.1]. Let $c \in \{c \in H^k(\mathbb{R}^n_\gamma) : \partial_{\mathbf{n}} c|_{\partial\mathbb{R}^n_\gamma} = 0\}$. Moreover, we define $\tilde{c} := c \circ F_\gamma \in H^k(\mathbb{R}^n_+)$. Thanks to the first part, there exists a sequence $(\tilde{c}_j)_{j\in\mathbb{N}}$ contained in $\{c \in C_c^k(\overline{\mathbb{R}^n_+}) : \partial_{\mathbf{n}} c|_{\partial\mathbb{R}^n_+} = 0\}$ such that $\tilde{c}_j \to \tilde{c}$ in $H^k(\mathbb{R}^n_+)$ as $j \to \infty$. Finally, the sequence $c_j := \tilde{c}_j \circ F_\gamma^{-1} \in C_c^{k-1,1}(\overline{\mathbb{R}^n_\gamma})$, $j \in \mathbb{N}$, has the desired properties.

3. Convergence of the nonlocal to the local operator

In this section, we prove the strong L^2–convergence of the nonlocal operator $\mathcal{L}_\varepsilon$ to $-\Delta$. In the first case, we study the convergence on $\mathbb{R}^n$.

Lemma 3.1. *Let $c \in H^2(\mathbb{R}^n)$. Then, it holds*

$$\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} c + \Delta c\right\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as } \varepsilon \searrow 0.$$

In addition, if $c \in H^3(\mathbb{R}^n)$, we even have

$$\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} c + \Delta c\right\|_{L^2(\mathbb{R}^n)} \le K\varepsilon\|c\|_{H^3(\mathbb{R}^n)}.$$

Proof. Thanks to Plancherel's Theorem, it suffices to prove

$$\left\| \widehat{\mathcal{L}_\varepsilon^{\mathbb{R}^n} c} + \widehat{\Delta c} \right\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as } \varepsilon \searrow 0.$$

By definition, we have

$$\left\| \widehat{\mathcal{L}_\varepsilon^{\mathbb{R}^n} c} + \widehat{\Delta c} \right\|_{L^2(\mathbb{R}^n)}^2 = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \left| \big(-\mathcal{F}(J_\varepsilon)(\xi) + \mathcal{F}(J_\varepsilon)(0) - |\xi|^2 \big) \mathcal{F}(c)(\xi) \right|^2 \mathrm{d}\xi,$$

where we used that

$$(J_\varepsilon * 1)(x) = \mathcal{F}(J_\varepsilon)(0)$$

for all $x \in \mathbb{R}^n$. Next, we prove the pointwise convergence

$$\mathcal{F}(J_\varepsilon)(0) - \mathcal{F}(J_\varepsilon)(\xi) \to |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n$$

as $\varepsilon \searrow 0$. Defining the functions $f_\xi(x) := -e^{-ix\cdot\xi}$, we have

$$\begin{aligned} \mathcal{F}(J_\varepsilon)(0) - \mathcal{F}(J_\varepsilon)(\xi) &= \int\limits_{\mathbb{R}^n} J_\varepsilon(|x|)(f_\xi(x) - f_\xi(0)) \,\mathrm{d}x \\ &= \int\limits_{\mathbb{R}^n} J_\varepsilon(|x|)\big(f_\xi(x) - f_\xi(0) - \frac{1}{2}x^T D^2 f_\xi(0)x\big) \,\mathrm{d}x \\ &\quad + \int\limits_{\mathbb{R}^n} J_\varepsilon(|x|)\frac{1}{2}x^T D^2 f_\xi(0)x \,\mathrm{d}x \\ &=: I_\varepsilon^1 + I_\varepsilon^2. \end{aligned}$$

Now, we analyze these terms separately.

Ad I_ε^1: Since the function $J_\varepsilon(|x|)x_i$ is odd for all $i = 1, \dots, n$, it holds

$$\int\limits_{\mathbb{R}^n} J_\varepsilon(|x|)x_i \,\mathrm{d}x = 0 \tag{14}$$

for all $i = 1, \dots, n$. Therefore, multiplying (14) by $\partial_i f_\xi(0)$ and summing over all $i = 1, \dots, n$, gives

$$\left| I_\varepsilon^1 \right| = \left| \int\limits_{\mathbb{R}^n} J_\varepsilon(|x|)\big(f_\xi(x) - f_\xi(0) - \nabla f_\xi(0)\cdot x - \frac{1}{2}x^T D^2 f_\xi(0)x\big) \,\mathrm{d}x \right|.$$

Using Taylor's Theorem, we have $|f_\xi(x) - f_\xi(0) - \nabla f_\xi(0)\cdot x - \frac{1}{2}x^T D^2 f_\xi(0)x| \le \tilde{\varepsilon} C_{d,f}|x|^2$ for all $|x| < \delta$. Therefore, it holds

$$\begin{aligned}\left|I_\varepsilon^1\right| &\le \int_{\mathbb{R}^n} J_\varepsilon(|x|)\left|f_\xi(x) - f_\xi(0) - \nabla f_\xi(0)\cdot x - \frac{1}{2}x^T D^2 f_\xi(0)x\right| \mathrm{d}x \\ &\le \int_{B_\delta(0)} \tilde{\varepsilon} C_{d,f}\rho_\varepsilon(|x|)\,\mathrm{d}x + \int_{B_\delta(0)^c} C\rho_\varepsilon(|x|)\,\mathrm{d}x.\end{aligned}$$

Since we can choose $\tilde{\varepsilon} > 0$ arbitrarily small and since $\int_{B_\delta(0)} \rho_\varepsilon(|x|)\,\mathrm{d}x \le K$, the first term in the last line is arbitrarily small. In the second integral, the properties of ρ_ε imply that this integral vanishes as $\varepsilon \searrow 0$. Altogether, this shows $I_\varepsilon^1 \to 0$ as $\varepsilon \searrow 0$.

Ad I_ε^2: We compute

$$\begin{aligned}I_\varepsilon^2 &= \int_{\mathbb{R}^n} J_\varepsilon(|x|)\frac{1}{2}\sum_{l,m=1}^n x_l \partial_l \partial_m f_\xi(0) x_m\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{l,m=1}^n \xi_l \xi_m \int_{\mathbb{R}^n} J_\varepsilon(|x|)x_l x_m\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{m=1}^n \xi_m^2 \int_{\mathbb{R}^n} J_\varepsilon(|x|)x_m^2\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{m=1}^n \xi_m^2 \frac{1}{n}\int_{\mathbb{R}^n} J_\varepsilon(|x|)\sum_{j=1}^n x_j^2\,\mathrm{d}x = |\xi|^2.\end{aligned}$$

Here, we used the facts

$$\int_{\mathbb{R}^n} J_\varepsilon(|x|)x_l x_m\,\mathrm{d}x = 0$$

for all $m \neq l$, and

$$\int_{\mathbb{R}^n} J_\varepsilon(|x|)x_m^2\,\mathrm{d}x = \int_{\mathbb{R}^n} J_\varepsilon(|x|)x_1^2\,\mathrm{d}x$$

for all $m = 1, \dots, n$. In the last step, we used our assumptions on ρ_ε to compute

$$\frac{1}{2n}\int_{\mathbb{R}^n} \rho_\varepsilon(|x|)\,\mathrm{d}x = \frac{1}{2n}\,\omega_n \int_0^\infty \rho(r)r^{n-1}\,\mathrm{d}r = \frac{\omega_n}{n}\frac{1}{C_n} = 1,$$

where we used $\omega_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ and that (for any $j = 1, \dots, n$) it holds

$$C_n = \int_{\mathbb{S}^{n-1}} |e_1\cdot\sigma|^2\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = \int_{\mathbb{S}^{n-1}} |e_j\cdot\sigma|^2\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} |\sigma|^2\,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = \frac{\omega_n}{n}.$$

Altogether, this shows the pointwise convergence

$$\mathcal{F}(J_\varepsilon)(\xi) - \mathcal{F}(J_\varepsilon)(0) \to |\xi|^2 \quad \text{as } \varepsilon \searrow 0$$

for all $\xi \in \mathbb{R}^n$. Again, using Taylor‘s Theorem, we observe that

$$\begin{aligned} \left|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(\xi) - |\xi|^2\right| &= \left|\int_{\mathbb{R}^n} J_\varepsilon(|x|)\big(f_\xi(x) - f_\xi(0) - \nabla f_\xi(0)\cdot x - \frac{1}{2}x^T D^2 f_\xi(0)x\big)\,\mathrm{d}x\right| \\ &\le C(1+|\xi|^2)\int_{\mathbb{R}^n} \rho_\varepsilon(|x|)\,\mathrm{d}x \le C(1+|\xi|^2), \end{aligned} \tag{15}$$

which implies

$$\left|\big(\mathcal{F}(J_\varepsilon)(0) - \mathcal{F}(J_\varepsilon)(\xi) - |\xi|^2\big)\mathcal{F}(c)(\xi)\right|^2 \le C\big(|\mathcal{F}(c)(\xi)|^2 + |\xi|^2|\mathcal{F}(c)(\xi)|^2\big)$$

for all $\xi \in \mathbb{R}^n$. As $c \in H^2(\mathbb{R}^n)$, the right-hand side is integrable. Therefore, we use Lebesgue‘s dominated convergence theorem to conclude the proof for $c \in H^2(\mathbb{R}^n)$.

Now, let $c \in H^3(\mathbb{R}^n)$. Then, we even have

$$\begin{aligned} \left|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(\xi) - |\xi|^2\right| &\le C\int_{\mathbb{R}^n} J_\varepsilon(|x|)\sup_{y\in\mathbb{R}^n}|D^3 f_\xi(y)||x|^3\,\mathrm{d}x \\ &\le C|\xi|^3\int_{\mathbb{R}^n}\rho_\varepsilon(|x|)|x|\,\mathrm{d}x \le C\varepsilon|\xi|^3 \end{aligned}$$

for all $\xi \in \mathbb{R}^n$ using a third-order Taylor expansion. Consequently, it holds

$$\begin{aligned} \left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} c + \Delta c\right\|_{L^2(\mathbb{R}^2)}^2 &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |(\mathcal{F}(J_\varepsilon)(\xi) - \mathcal{F}(J_\varepsilon)(0) - |\xi|^2|)\mathcal{F}(c)(\xi)|^2\,\mathrm{d}\xi \\ &\le \frac{C\varepsilon^2}{(2\pi)^n}\int_{\mathbb{R}^n}\left||\xi|^3\mathcal{F}(c)(\xi)\right|^2\mathrm{d}\xi = C\varepsilon^2\|c\|_{H^3(\mathbb{R}^n)}^2. \end{aligned}$$

Therefore, we obtain

$$\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} c + \Delta c\right\|_{L^2(\mathbb{R}^n)} \le C\varepsilon\|c\|_{H^3(\mathbb{R}^n)},$$

which concludes the proof. □

In the next lemma, we study the situation, where x and y have positive distance, i.e., no singularity appears. In fact, this will play an important role for the following proofs.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\Omega' \subseteq \Omega$ such that $\operatorname{dist}(\overline{\Omega'}, \partial\Omega) > 0$. Then, for every $c \in L^2(\Omega)$, it holds*

$$\left\|\mathcal{R}_\varepsilon c\right\|_{L^2(\Omega')} \leq K\, \varepsilon \|c\|_{L^2(\Omega)}$$

where $K > 0$ only depends on $\operatorname{dist}(\Omega^c, \Omega')$. Here, we defined

$$\mathcal{R}_\varepsilon c(x) := \int_{\Omega^c} J_\varepsilon(|x-y|)\big(c(x) - \tilde{c}(y)\big)\, dy$$

for all $c \in L^2(\Omega)$ and almost all $x \in \Omega$. Here, $\tilde{c} \in L^2(\mathbb{R}^n)$ is an extension of c to the whole of $\mathbb{R}^n$ such that $\|\tilde{c}\|_{L^2(\mathbb{R}^n)} \leq K\|c\|_{L^2(\Omega)}$.

Proof. First of all, we observe

$$\begin{aligned}\left\|\mathcal{R}_\varepsilon c\right\|^2_{L^2(\Omega')} &\leq K \int_{\Omega'} \left| \int_{\Omega^c} J_\varepsilon(|x-y|) c(x)\, \mathrm{d}y \right|^2 \mathrm{d}x + K \int_{\Omega'} \left| \int_{\Omega^c} J_\varepsilon(|x-y|)\tilde{c}(y)\, \mathrm{d}y \right|^2 \mathrm{d}x \\ &=: K\big(I^1_\varepsilon + I^2_\varepsilon\big).\end{aligned}$$

Now, we estimate these terms separately.
Ad I^1_ε: Since $\operatorname{dist}(\overline{\Omega'}, \partial\Omega) > 0$, it holds $|x-y| \geq \delta := \operatorname{dist}(\Omega^c, \Omega') > 0$ for all $x \in \Omega^c$, $y \in \Omega'$. This yields

$$I^1_\varepsilon \leq K_\delta \int_{\Omega'} |c(x)|^2 \left(\int_{\Omega^c} \rho_\varepsilon(|x-y|)|x-y|\, \mathrm{d}y \right)^2 \mathrm{d}x \leq K_\delta\, \varepsilon^2 \|c\|^2_{L^2(\Omega)}.$$

Ad I^2_ε: Here, we estimate

$$\begin{aligned} I^2_\varepsilon &\leq \int_{\Omega'} \left(\int_{\Omega^c} J_\varepsilon(|x-y|)|\tilde{c}(y)|\, \mathrm{d}y \right)^2 \mathrm{d}x \\ &\leq \int_{\Omega'} \left(\int_{\Omega^c} J_\varepsilon(|x-y|)\, \mathrm{d}y \right) \left(\int_{\Omega^c} J_\varepsilon(|x-y|)|\tilde{c}(y)|^2\, \mathrm{d}y \right) \mathrm{d}x \\ &\leq K_\delta\, \varepsilon \int_{\Omega^c} |\tilde{c}(y)|^2 \left(\int_{\Omega'} \rho_\varepsilon(|x-y|)|x-y|\, \mathrm{d}x \right) \mathrm{d}y \\ &\leq K_\delta\, \varepsilon^2 \|c\|^2_{L^2(\Omega)}, \end{aligned}$$

where we used $|x-y| \geq \delta := \operatorname{dist}(\Omega^c, \Omega') > 0$ and Fubini's Theorem. Altogether, we obtain

$$\left\|\mathcal{R}_\varepsilon c\right\|_{L^2(\Omega')} \leq K_\delta\, \varepsilon \|c\|_{L^2(\Omega)}.$$

This concludes the proof. □

In the proof of Theorem 4.1, we want to use a localization argument. To this end, we now consider the upper half-space $\mathbb{R}^n_+$.

Lemma 3.3. *Let $c \in H^3(\mathbb{R}^n_+)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$. Then, it holds*

$$\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c + \Delta c\right\|_{L^2(\mathbb{R}^n_+)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\mathbb{R}^n_+)}.$$

Proof. Let $\tilde{c} \in H^3(\mathbb{R}^n)$ be an extension of c to $\mathbb{R}^n$ such that $\|\tilde{c}\|_{H^3(\mathbb{R}^n)} \leq K\|c\|_{H^3(\mathbb{R}^n_+)}$. Then, we observe

$$\begin{aligned}\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c + \Delta c\right\|_{L^2(\mathbb{R}^n_+)} &\leq \left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} \tilde{c} + \Delta \tilde{c}\right\|_{L^2(\mathbb{R}^n)} + \|\mathcal{R}_\varepsilon \tilde{c}\|_{L^2(\mathbb{R}^n_+)} \\ &\leq K\varepsilon\|c\|_{H^3(\mathbb{R}^n_+)} + \|\mathcal{R}_\varepsilon \tilde{c}\|_{L^2(\mathbb{R}^n_+)},\end{aligned}$$

where the error term $\mathcal{R}_\varepsilon$ is given by

$$\mathcal{R}_\varepsilon c(x) := \int_{\mathbb{R}^n_-} J_\varepsilon(|x-y|)(c(x) - \tilde{c}(y))\,\mathrm{d}y \tag{16}$$

for a.e. $x \in \mathbb{R}^n_+$. We want to prove that $\|\mathcal{R}_\varepsilon \tilde{c}\|_{L^2(\mathbb{R}^n_+)} \to 0$ as $\varepsilon \searrow 0$. To this end, we first prove the statement for functions $c \in C_0^3(\overline{\mathbb{R}^n_+})$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$ and use a density argument afterwards to conclude the proof.

Using the transformation $y = (y_1, \ldots, y_{n-1}, y_n) \mapsto (y_1, \ldots, y_{n-1}, -y_n) =: \hat{y}$, we have

$$\mathcal{R}_\varepsilon c(x) = \int_{\mathbb{R}^n_-} J_\varepsilon(|x-y|)(c(x) - \tilde{c}(y))\mathrm{d}y = \int_{\mathbb{R}^n_+} J_\varepsilon(|x-\hat{y}|)(c(x) - c(\hat{y}))\mathrm{d}y$$

for a.e. $x \in \mathbb{R}^n_+$. For $\delta > 0$, we obtain

$$\begin{aligned}|\mathcal{R}_\varepsilon c(x)| \leq &\Big|\int_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)(c(x) - c(\hat{y}))\mathrm{d}y\Big| \\ &+ \Big|\int_{\mathbb{R}^n_+\setminus(B_\delta(x')\times(0,\delta))} J_\varepsilon(|x-\hat{y}|)(c(x) - c(\hat{y}))\mathrm{d}y\Big|\end{aligned} \tag{17}$$

for a.e. $x \in \mathbb{R}^n_+$. In the second integral on the right-hand side, we use the calculations in the proof of Lemma 3.2 to obtain

$$\int_{\mathbb{R}^n_+} \Big| \int_{\mathbb{R}^n_+ \setminus (B_\delta(x') \times (0,\delta))} J_\varepsilon(|x-\hat{y}|)(c(x)-c(\hat{y}))\mathrm{d}y \Big|^2 \mathrm{d}x \le K_\delta\, \varepsilon^2 \|c\|^2_{H^3(\mathbb{R}^n_+)},$$

since it holds $\mathrm{dist}(x,\hat{y}) \ge \delta > 0$ for all $\hat{y} \in \mathbb{R}^n_+ \setminus \big(B_\delta(x') \times (0,\delta)\big)$. Thus, it suffices to consider the first integral on the right-hand side. Here, a first order Taylor expansion yields

$$\begin{aligned} c(x) - c(\hat{y}) &= \nabla c(x) \cdot (x - \hat{y}) + R_2(x,\hat{y}) \\ &= \nabla_{x'} c(x) \cdot (x' - y') + \partial_{x_n} c(x)(x_n + y_n) + R_2(x,\hat{y}), \end{aligned}$$

where $x := (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ and

$$R_2(x,\hat{y}) := \sum_{|\beta|=2} \frac{2}{\beta!} \left(\int_0^1 (1-t)\mathrm{D}^\beta c(\hat{y} + t(x-\hat{y}))\mathrm{d}t \right) (x-\hat{y})^\beta .$$

Inserting this, we end up with

$$\begin{aligned} \Big| \int_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)(c(x)-c(\hat{y}))\mathrm{d}y \Big| &\le \Big| \int_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)\nabla_{x'} c(x) \cdot (x'-y')\,\mathrm{d}y \Big| \\ &\quad + \Big| \int_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)\partial_{x_n} c(x)(x_n + y_n)\,\mathrm{d}y \Big| \\ &\quad + \Big| \int_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|) R_2(x,\hat{y})\,\mathrm{d}y \Big| \\ &=: I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta} + I^3_{\varepsilon,\delta}. \end{aligned} \tag{18}$$

Now, we estimate these integrals separately.

Ad $I^1_{\varepsilon,\delta}$: We observe that the integrand $J_\varepsilon(|x-\hat{y}|)\nabla_{x'} c(x) \cdot (x'-y')$ is odd with respect to $x'-y'$. Therefore, it holds

$$\int_{B_\delta(x')} J_\varepsilon(|x-\hat{y}|)\nabla_{x'} c(x) \cdot (x'-y')\mathrm{d}y' = 0,$$

which then implies $I^1_{\varepsilon,\delta} = 0$.

Ad $I^2_{\varepsilon,\delta}$: First of all, the properties of c and the fundamental theorem of calculus imply

$$\partial_{x_n} c(x', x_n) = \partial_{x_n} c(x', x_n) - \partial_{x_n} c(x', 0) = \int_0^1 \partial^2_{x_n} c(x', t x_n) x_n \,\mathrm{d}t.$$

This yields

$$\begin{aligned}
&\Big| \int\limits_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)\partial_{x_n} c(x)(x_n+y_n)\mathrm{d}y\Big| \\
&= \Big| \int\limits_{B_\delta(x')\times(0,\delta)} \int\limits_0^1 J_\varepsilon(|x-\hat{y}|)\partial_{x_n}^2 c(x', tx_n)x_n(x_n+y_n)\,\mathrm{d}t\,\mathrm{d}y\Big| \\
&\le \int\limits_0^1 \big|\partial_{x_n}^2 c(x', tx_n)\big| \Big(\int\limits_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)(|x_n|+|y_n|)|x_n+y_n|\mathrm{d}y\Big)\mathrm{d}t \\
&\le \int\limits_0^1 \big|\partial_{x_n}^2 c(x', tx_n)\big| a_\varepsilon(x_n)\,\mathrm{d}t,
\end{aligned}$$

where we defined

$$a_\varepsilon(x_n) := \int\limits_{B_\delta(x')\times(0,\delta)} \rho_\varepsilon(|x-\hat{y}|)\mathrm{d}y.$$

By construction and due to the properties of ρ_ε, the function a_ε is in fact independent of x'. Computing the L^2-norm of a_ε, we get

$$\begin{aligned}
\|a_\varepsilon\|^2_{L^2(\mathbb{R}_+)} &= \int\limits_0^\infty \Bigg(\int\limits_{B_\delta(x')\times(0,\delta)} \rho_\varepsilon(|x-\hat{y}|)\,\mathrm{d}y \Bigg)^2 \mathrm{d}x_n \\
&= \int\limits_0^\infty \Bigg(\int\limits_{B_\delta(x')\times(0,\delta)} \varepsilon^{-n}\rho\Big(\Big|\frac{x-\hat{y}}{\varepsilon}\Big|\Big)\,\mathrm{d}y \Bigg)^2 \mathrm{d}x_n \\
&\le \varepsilon \int\limits_0^\infty \Bigg(\int\limits_0^\infty \int\limits_{\mathbb{R}^{n-1}} \rho(|x-\hat{y}|)\,\mathrm{d}y'\mathrm{d}y_n \Bigg)^2 \mathrm{d}x_n \\
&\le \varepsilon \int\limits_0^R \Bigg(\int\limits_0^\infty \int\limits_{\mathbb{R}^{n-1}} \rho(|x-\hat{y}|)\,\mathrm{d}y'\mathrm{d}y_n \Bigg)^2 \mathrm{d}x_n \le R\varepsilon.
\end{aligned} \tag{19}$$

Here, we applied the transformations $y \mapsto \varepsilon y$ and $x_n \mapsto \varepsilon x_n$ and we used that $\operatorname{supp}\rho \subset B_R(0)$. This then implies

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)\partial_{x_n} c(x)(x_n+y_n)\mathrm{d}y\Big|^2 \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^n_+} |a_\varepsilon(x_n)|^2 \left(\frac{1}{x_n}\int_0^{x_n} \|\partial^2_{x_n} c(x',.)\|_{L^\infty(\mathbb{R}_+)}\mathrm{d}z_n\right)^2 \mathrm{d}x$$
$$\leq K \int_{\mathbb{R}^{n-1}}\int_0^\infty |a_\varepsilon(x_n)|^2 \|\partial^2_{x_n} c(x',.)\|^2_{L^\infty(\mathbb{R}_+)}\mathrm{d}x_n \mathrm{d}x'$$
$$\leq K \int_{\mathbb{R}^{n-1}} \|a_\varepsilon\|^2_{L^2(\mathbb{R}_+)} \|\partial^2_{x_n} c(x',.)\|^2_{H^1(\mathbb{R}_+)}\mathrm{d}x' \leq K\varepsilon \|c\|^2_{H^3(\mathbb{R}^n_+)},$$

where we used the embedding $H^1(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$.

Ad $I^3_{\varepsilon,\delta}$: Here, it holds

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)R_2(x,\hat{y})\mathrm{d}y\Big|^2 \mathrm{d}x$$
$$\leq K \int_{\mathbb{R}^n_+} \left(\int_{B_\delta(x')\times(0,\delta)} \int_0^1 \rho_\varepsilon(|x-\hat{y}|)|D^2c(\hat{y}+t(x-\hat{y}))|\,\mathrm{d}t\mathrm{d}y\right)^2 \mathrm{d}x$$
$$\leq K \int_{\mathbb{R}^n_+} \left(\int_{B_\delta(x')\times(0,\delta)} \rho_\varepsilon(|x-\hat{y}|)\,\mathrm{d}y\right)\left(\int_0^1 \int_{B_\delta(x')\times(0,\delta)} \rho_\varepsilon(|x-\hat{y}|)|D^2c(\hat{y}+t(x-\hat{y}))|^2\,\mathrm{d}y\mathrm{d}t\right) \mathrm{d}x$$
$$\leq K \int_{\mathbb{R}^n_+} a_\varepsilon(x_n) \left(\int_0^1 \int_{\mathbb{R}^n_+} \rho_\varepsilon(|x-\hat{y}|)\|D^2c(\hat{y}'+t(x'-\hat{y}'),.)\|^2_{L^\infty(\mathbb{R}_+)}\,\mathrm{d}y\mathrm{d}t\right) \mathrm{d}x$$
$$\leq K \int_0^R a_\varepsilon(x_n)\mathrm{d}x_n \int_0^1 \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \|\rho_\varepsilon(|x'-\hat{y}',.|)\|_{L^1(\mathbb{R}_+)}\|D^2c(\hat{y}'+t(x'-\hat{y}'),.)\|^2_{L^\infty(\mathbb{R}_+)}\,\mathrm{d}y\mathrm{d}x'\mathrm{d}t$$
$$\leq K \left(\int_0^R a_\varepsilon(x_n)\,\mathrm{d}x_n\right) \|c\|^2_{H^3(\mathbb{R}^n_+)} \leq K\varepsilon \|c\|^2_{H^3(\mathbb{R}^n_+)}, \tag{20}$$

where we first used the inequality of Cauchy-Schwarz and then Lemma 2.2 together with the embedding $H^1(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$ in the fourth step. In the last step, we used $\mathrm{supp}\rho \subset B_R(0)$ and applied the transformations $y \mapsto \varepsilon y$ and $x_n \mapsto \varepsilon x_n$. Altogether, this shows

$$\|\mathcal{R}_\varepsilon c\|_{L^2(\mathbb{R}^n_+)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\mathbb{R}^n_+)}$$

for all $c \in C_0^3(\overline{\mathbb{R}^n_+})$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$.

In the next step, we want to use a denseness argument, to conclude the proof. Let $c \in H^3(\mathbb{R}^n_+)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$ be arbitrary. Since the space $\{c \in C_0^\infty(\overline{\mathbb{R}^n_+}) : \partial_{\mathbf{n}} c = 0 \text{ on } \partial\mathbb{R}^n_+\}$ is dense in $\{c \in H^3(\mathbb{R}^n_+) : \partial_{\mathbf{n}} c = 0 \text{ on } \partial\mathbb{R}^n_+\}$, cf. Remark 2.3, there exists a sequence $(c_k)_{k\in\mathbb{N}} \subset C_0^\infty(\overline{\mathbb{R}^n_+})$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$ such that $c_k \to c$ in $H^3(\mathbb{R}^n_+)$. Thanks to our results so far, the sequence

$$\left(\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c_k + \Delta c_k\right)_{k\in\mathbb{N}} \subset L^2(\mathbb{R}^n_+)$$

is bounded. Thus, there exists a subsequence, which is again denoted by $\left(\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c_k + \Delta c_k\right)_{k\in\mathbb{N}}$, such that

$$\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c_k + \Delta c_k \rightharpoonup w \quad \text{in } L^2(\mathbb{R}^n_+) \quad \text{as } k \to \infty$$

for some $w \in L^2(\mathbb{R}^n_+)$. Since $\mathcal{L}_\varepsilon^{\mathbb{R}^n_+}$ is linear and continuous, and therefore weakly continuous, it follows that $w = \mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c + \Delta c$. Then, the weak lower semi-continuity of norms implies

$$\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c + \Delta c\right\|_{L^2(\mathbb{R}^n_+)} \le \liminf_{k\to\infty} \left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c_k + \Delta c_k\right\|_{L^2(\mathbb{R}^n_+)} \le \liminf_{k\to\infty} K\sqrt{\varepsilon}\|c_k\|_{H^3(\mathbb{R}^n_+)} \le K\sqrt{\varepsilon}$$

for all $c \in H^3(\mathbb{R}^n_+)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$ with $\|c\|_{H^3(\mathbb{R}^n_+)} \le 1$. In particular, this concludes the proof. □

Remark 3.4. The rate of convergence obtained in Lemma 3.3 is optimal. Even in the simplest case, where $n = 1$ and $c \in C_0^\infty(\overline{\mathbb{R}_+})$, we do not gain a better rate of convergence in $L^2(\mathbb{R}_+)$ unless $\partial_{\mathbf{n}}^2 c = 0$ on $\partial\mathbb{R}_+$. This shows the following calculation: Let $x > 0$. Then, we obtain for the error term

$$\begin{aligned}
\mathcal{R}_\varepsilon c(x) &= \int_0^\infty J_\varepsilon(|x-\hat{y}|)(c(x)-c(\hat{y}))\,\mathrm{d}y \\
&= \int_{B_\delta(x)\cap\mathbb{R}_+} J_\varepsilon(|x-\hat{y}|)(c(x)-c(\hat{y}))\,\mathrm{d}y + \int_{\mathbb{R}_+\setminus B_\delta(x)} J_\varepsilon(|x-\hat{y}|)(c(x)-c(\hat{y}))\,\mathrm{d}y \\
&= \int_{B_\delta(x)\cap\mathbb{R}_+} J_\varepsilon(|x-\hat{y}|)\Big(\frac{1}{2}c''(0)(x-\hat{y})^2 + R_3(x,\hat{y})\Big)\mathrm{d}y \\
&\quad + \int_{\mathbb{R}_+\setminus B_\delta(x)} J_\varepsilon(|x-\hat{y}|)(c(x)-c(\hat{y}))\,\mathrm{d}y \\
&= \frac{1}{2}c''(0) \int_{B_\delta(x)\cap\mathbb{R}_+} \rho_\varepsilon(|x-\hat{y}|)\mathrm{d}y + \mathcal{O}(\varepsilon),
\end{aligned}$$

where $\int_{B_\delta(x)\cap\mathbb{R}_+} \rho_\varepsilon(|x-\hat{y}|)\mathrm{d}y =: a_\varepsilon(x)$ and $\|a_\varepsilon(x)\|_{L^2(\mathbb{R}_+)} \geq K\sqrt{\varepsilon}$ for $\varepsilon > 0$ small enough similar as in (19). In the third step, we used a Taylor expansion. In the last step, we then applied Lemma 3.2. Here, the term $\mathcal{O}(\varepsilon)$ is measured with respect to the $L^2(\mathbb{R}_+)$-norm.

Corollary 3.5. *Let $c \in H^2(\mathbb{R}^n_+)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$. Then, it holds*

$$\left\| \mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c + \Delta c \right\|_{L^2(\mathbb{R}^n_+)} \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Proof. Thanks to Lemma 3.3, it suffices to prove that $\left\| \mathcal{L}_\varepsilon^{\mathbb{R}^n_+} c + \Delta c \right\|_{L^2(\mathbb{R}^n_+)}$ is bounded uniformly in $\varepsilon > 0$ for all $c \in H^2(\mathbb{R}^n_+)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$. Then, the Banach-Steinhaus Theorem concludes the proof.

In fact, we only need to estimate the error term (17) in a suitable way. First of all, we prove the assertion for functions $c \in C_0^2(\overline{\mathbb{R}^n_+})$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_+$ and use a density argument afterwards to conclude the proof. In the term

$$\Bigg| \int\limits_{\mathbb{R}^n_+ \setminus (B_\delta(x') \times (0,\delta))} J_\varepsilon(|x-\hat{y}|)(c(x) - c(\hat{y}))\mathrm{d}y \Bigg|,$$

we can apply Lemma 3.2, again. Thus, it suffices to consider the first part of (17). As in the proof of Lemma 3.3, it holds $I^1_{\varepsilon,\delta} = 0$. In $I^2_{\varepsilon,\delta}$, we use the properties of c and the fundamental theorem of calculus to obtain

$$\begin{aligned}
&\Bigg| \int\limits_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)\partial_{x_n} c(x)(x_n + y_n)\mathrm{d}y \Bigg| \\
&= \Bigg| \int\limits_{B_\delta(x') \times (0,\delta)} \int_0^1 J_\varepsilon(|x-\hat{y}|)\partial^2_{x_n} c(x', t x_n) x_n (x_n + y_n)\, \mathrm{d}t\mathrm{d}y \Bigg| \\
&\leq \int\limits_{B_\delta(x') \times (0,\delta)} \int_0^1 J_\varepsilon(|x-\hat{y}|)|\partial^2_{x_n} c(x', t x_n)||x_n||x_n + y_n|\, \mathrm{d}t\mathrm{d}y \\
&\leq \int_0^1 |\partial^2_{x_n} c(x', t x_n)| \Bigg(\int\limits_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)(|x_n| + |y_n|)|x_n + y_n|\mathrm{d}y \Bigg)\mathrm{d}t.
\end{aligned}$$

In the inner integral, it holds

$$\begin{aligned}
\int\limits_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)(|x_n| + |y_n|)|x_n + y_n|\mathrm{d}y &\leq \int\limits_{B_\delta(x') \times (0,\delta)} J_\varepsilon(|x-\hat{y}|)K(|x_n|^2 + |y_n|^2)\mathrm{d}y \\
&\leq \int\limits_{B_\delta(x') \times (0,\delta)} K\rho_\varepsilon(|x-\hat{y}|)\mathrm{d}y \leq K.
\end{aligned}$$

This implies

$$\Big| \int\limits_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)\partial_{x_n} c(x)(x_n+y_n)\mathrm{d}y \Big| \le \int\limits_0^1 K|\partial_{x_n}^2 c(x', tx_n)|\mathrm{d}t$$

$$= K\frac{1}{x_n}\int\limits_0^{x_n} |\partial_{x_n}^2 c(x', z_n)|\mathrm{d}z_n,$$

where we changed variables as $z_n = tx_n$ in the last step. This implies

$$\int\limits_{\mathbb{R}^n_+}\Big| \int\limits_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)\partial_{x_n} c(x)(x_n+y_n)\mathrm{d}y \Big|^2 \mathrm{d}x \le K\int\limits_{\mathbb{R}^n_+}\Big(\frac{1}{x_n}\int\limits_0^{x_n} |\partial_{x_n}^2 c(x', z_n)|\mathrm{d}z_n\Big)^2\mathrm{d}x$$

$$\le K\int\limits_{\mathbb{R}^{n-1}}\int\limits_0^\infty |\partial_{x_n}^2 c(x', x_n)|^2\mathrm{d}x_n\mathrm{d}x'$$

$$\le K\|c\|^2_{H^2(\mathbb{R}^n_+)}, \tag{21}$$

where we applied Hardy's inequality.

Ad $I^3_{\varepsilon,\delta}$: Here, we use Fubini's Theorem and Lemma 2.2 to get

$$\int\limits_{\mathbb{R}^n_+}\Big| \int\limits_{B_\delta(x')\times(0,\delta)} J_\varepsilon(|x-\hat{y}|)R_2(x,\hat{y})\mathrm{d}y \Big|^2 \mathrm{d}x$$

$$\le K\int\limits_{\mathbb{R}^n_+}\int\limits_{B_\delta(x')\times(0,\delta)}\int\limits_0^1 \rho_\varepsilon(|x-\hat{y}|)|D^2c(\hat{y}+t(x-\hat{y}))|^2\mathrm{d}t\mathrm{d}y\mathrm{d}x$$

$$\le K\int\limits_0^1\int\limits_{\mathbb{R}^{n-1}}\int\limits_{\mathbb{R}^{n-1}} \|\rho_\varepsilon(|(x'-y',.)|)\|_{L^1(\mathbb{R})}\|D^2c((1-t)y'+tx',.)\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}y'\mathrm{d}x'\mathrm{d}t$$

$$\le K\|c\|^2_{H^2(\mathbb{R}^n_+)}. \tag{22}$$

Altogether, we get the following estimate

$$\|\mathcal{R}_\varepsilon\tilde{c}\|_{L^2(\mathbb{R}^n_+)} \le K_\delta\|c\|_{H^2(\mathbb{R}^n_+)} + K\|c\|_{H^2(\mathbb{R}^n_+)}$$

for all $c \in C_0^2(\overline{\mathbb{R}^n_+})$ with $\partial_{x_n} c_{|\{x_n=0\}} = 0$. Finally, a density argument as in the proof of Lemma 3.3 finishes the proof. □

In the next step, we prove convergence on the bent half-space.

Lemma 3.6. *Let $\gamma \in C_b^3(\mathbb{R}^{n-1})$ with $\|\gamma\|_{C_b^3(\mathbb{R}^{n-1})}$ sufficiently small and $c \in H^3(\mathbb{R}^n_\gamma)$ such that $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_\gamma$. Then, it holds*

$$\left\| \mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} c + \Delta c \right\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\mathbb{R}^n_\gamma)}.$$

Proof. Let $\tilde{c} \in H^3(\mathbb{R}^n)$ denote an extension of c to $\mathbb{R}^n$ such that $\|\tilde{c}\|_{H^3(\mathbb{R}^n)} \leq K\|c\|_{H^3(\mathbb{R}^n_\gamma)}$. Then, we have

$$\begin{aligned} \left\| \mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} c + \Delta c \right\|_{L^2(\mathbb{R}^n_\gamma)} &\leq \left\| \mathcal{L}_\varepsilon^{\mathbb{R}^n} \tilde{c} + \Delta \tilde{c} \right\|_{L^2(\mathbb{R}^n)} + \|\mathcal{R}_\varepsilon \tilde{c}\|_{L^2(\mathbb{R}^n_\gamma)} \\ &\leq K\varepsilon\|c\|_{H^3(\mathbb{R}^n_\gamma)} + \|\mathcal{R}_\varepsilon \tilde{c}\|_{L^2(\mathbb{R}^n_\gamma)}, \end{aligned}$$

where we used Lemma 3.3 and defined the error term

$$\mathcal{R}_\varepsilon \tilde{c}(x) := \int_{(\mathbb{R}^n_\gamma)^c} J_\varepsilon(|x-y|)\big(c(x) - \tilde{c}(y)\big)\,\mathrm{d}y \tag{23}$$

for a.e. $x \in \mathbb{R}^n_\gamma$. Thus, it suffices to consider the error term.
Since $\gamma \in C_b^3(\mathbb{R}^{n-1})$, there exists a $C^{2,1}-$diffeomorphism $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ with $F_\gamma(\mathbb{R}^n_+) = \mathbb{R}^n_\gamma$, $F_\gamma(x', 0) = (x', \gamma(x'))$ and $-\partial_{x_n} F_\gamma(x)|_{x_n=0} = \mathbf{n}(x', \gamma(x'))$, where $\mathbf{n}$ denotes the exterior unit normal on $\partial\mathbb{R}^n_\gamma$, cf. [22, Lemma 2.1].

In the following, we assume

$$\sup_{\hat{x}\in\mathbb{R}^n} |\mathrm{D}F_\gamma(\hat{x}) - \mathrm{Id}| \leq \alpha \tag{24}$$

for some $\alpha \in (0, \frac{1}{3})$. Using the diffeomorphism F_γ and reflection afterwards, we compute

$$\begin{aligned} \mathcal{R}_\varepsilon \tilde{c}(x) &= \int_{\mathbb{R}^n_-} J_\varepsilon(|F_\gamma(\hat{x}) - F_\gamma(\hat{y})|)\big(c(F_\gamma(\hat{x})) - \tilde{c}(F_\gamma(\hat{y}))\big)\big|\det(\mathrm{D}F_\gamma(\hat{y}))\big|\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n_+} J_\varepsilon(|A_\gamma(\hat{x}, \bar{y})(\hat{x} - \bar{y})|)\big(u(\hat{x}) - u(\bar{y})\big)\big|\det(\mathrm{D}F_\gamma(\bar{y}))\big|\,\mathrm{d}y \end{aligned}$$

for almost all $x \in \mathbb{R}^n_\gamma$. Here, $u := c \circ F_\gamma : \mathbb{R}^n_+ \to \mathbb{R}$, $F_\gamma(\hat{x}) = x$, $\bar{y} = (y_1, \ldots, y_{n-1}, -y_n)$ and we used

$$F_\gamma(\hat{x}) - F_\gamma(\bar{y}) = \int_0^1 \mathrm{D}F_\gamma(\bar{y} + t(\hat{x} - \bar{y}))(\hat{x} - \bar{y})\,\mathrm{d}t =: A_\gamma(\hat{x}, \bar{y})(\hat{x} - \bar{y}).$$

Next, we rewrite the error term $\mathcal{R}_\varepsilon$ as

$$\begin{aligned}\mathcal{R}_\varepsilon c(x) = &\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\big(u(\hat{x})-u(\bar{y})\big)\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\,\mathrm{d}y\\ &+\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} K_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\big(u(\hat{x})-u(\bar{y})\big)\,\mathrm{d}y\\ &+\int\limits_{\mathbb{R}^n_+\setminus(B_\delta(\hat{x}')\times(0,\delta))} J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|)\big(u(\hat{x})-u(\bar{y})\big)\big|\det(\mathrm{D}F_\gamma(\bar{y}))\big|\,\mathrm{d}y\\ &=: I^1_\varepsilon(x)+I^2_\varepsilon(x)+I^3_\varepsilon(x), \end{aligned} \tag{25}$$

where $K_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)$ is defined by

$$\begin{aligned}K_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|) := &J_\varepsilon(|A_\gamma(\hat{x},\hat{y})(\hat{x}-\bar{y})|)\big|\det \mathrm{D}F_\gamma(\bar{y})\big|\\ &- J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\end{aligned}$$

for almost all $\hat{x},\bar{y}\in\mathbb{R}^n_+$. In $I^3_\varepsilon(x)$, we can apply Lemma 3.2, since it holds

$$|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})| \geq K|\hat{x}-\bar{y}| \geq K\delta.$$

In the next step, we estimate $I^1_\varepsilon(x)$ and $I^2_\varepsilon(x)$ separately. In the following, let $c\in C^{2,1}_0(\overline{\mathbb{R}^n_\gamma})$ with $\partial_{\mathbf{n}}c=0$ on $\partial\mathbb{R}^n_\gamma$.

Ad $I^1_\varepsilon(x)$: First, we use a Taylor expansion for u to get

$$\begin{aligned}I^1_\varepsilon(x) = &\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\nabla_{x'}u(\hat{x})\cdot(\hat{x}'-\bar{y}')\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\,\mathrm{d}y\\ &+\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\partial_{x_n}u(\hat{x})(\hat{x}_n+\bar{y}_n)\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\,\mathrm{d}y\\ &+\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)R_2(\hat{x},\bar{y})\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\,\mathrm{d}y,\end{aligned}$$

where the error term $R_2(\hat{x},\bar{y})$ is defined by

$$R_2(\hat{x},\bar{y}) := \sum_{|\beta|=2}\frac{2}{\beta!}\left(\int_0^1(1-t)\mathrm{D}^\beta c(\bar{y}+t(\hat{x}-\bar{y}))\mathrm{d}t\right)(\hat{x}-\bar{y})^\beta.$$

Observe that by our choice of F_γ, the term $A_\gamma(\hat{x},\hat{x})$ is given by

$$A_\gamma(\hat{x},\hat{x}) = \mathrm{D}F_\gamma(\hat{x}) = U(\hat{x})\left(\begin{array}{ccc|c} & & & 0\\ & A'(\hat{x}) & & \vdots\\ & & & 0\\ \hline 0 & \dots & 0 & 1\end{array}\right),$$

where $U(\hat{x}) \in \mathrm{SO}(n)$ and $A'(\hat{x}) \in \mathbb{R}^{(n-1)\times(n-1)}$, cf. [2] for details. This implies

$$|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|^2 = |A'(\hat{x})(\hat{x}'-y')|^2 + |\hat{x}_n + y_n|^2,$$

and therefore it follows that the integrand in the first term of $I^1_\varepsilon(x)$ is odd with respect to $\hat{x}'-\bar{y}'$. Consequently,

$$\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\nabla_{x'}u(\hat{x})\cdot(\hat{x}'-\bar{y}')\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\,\mathrm{d}y = 0.$$

Furthermore, F_γ satisfies

$$\partial_{x_n}u(\hat{x})_{|\{x_n=0\}} = \partial_{x_n}\big(c\circ F_\gamma\big)(\hat{x})_{|\{x_n=0\}} = \nabla c(x',\gamma(x'))\cdot(-\mathbf{n}(x',\gamma(x'))) = 0,$$

since $\partial_{\mathbf{n}}c = 0$ on $\partial\mathbb{R}^n_\gamma$. Thus, it holds

$$\partial_{x_n}u(\hat{x}) = \partial_{x_n}u(\hat{x}) - \partial_{x_n}u(\hat{x}',0) = \Big(\int_0^1 \partial^2_{x_n}u(\hat{x}',tx_n)\,\mathrm{d}t\Big)x_n.$$

Hence, we obtain in the second term of $I^1_\varepsilon(x)$

$$\begin{aligned}&\Big|\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\partial_{x_n}u(\hat{x})(\hat{x}_n+\bar{y}_n)\big|\det \mathrm{D}F_\gamma(\hat{x})\big|\,\mathrm{d}y\Big|\\ &\le \int_0^1 a_\varepsilon(x_n)|\partial^2_{x_n}u(\hat{x}',t\hat{x}_n)|\,\mathrm{d}t,\end{aligned}$$

where we used $|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})| \ge K|\hat{x}-\bar{y}|$ for all $\hat{x},\bar{y}\in\mathbb{R}^n$ as well as $\big|\det \mathrm{D}F_\gamma(\hat{x})\big| \le K$ for all $\hat{x}\in\mathbb{R}^n$. Here, we defined

$$a_\varepsilon(x_n) := \sup_{x'\in\mathbb{R}^{n-1}} \int\limits_{B_\delta(\hat{x}')\times(0,\delta)} \rho_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\,\mathrm{d}y.$$

Note that it holds $a_\varepsilon \in L^2(\mathbb{R}_+)$ and $\|a_\varepsilon\|_{L^2(\mathbb{R}_+)} \le K\sqrt{\varepsilon}$. This follows from the same calculation as in the proof of Lemma 3.3. Computing the L^2-norm, we then obtain

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\partial_{x_n} u(\hat{x})(\hat{x}_n + y_n) \big| \det \mathrm{D}F_\gamma(\hat{x}) \big| \,\mathrm{d}y \Big|^2 \mathrm{d}\hat{x}$$

$$\leq \int_{\mathbb{R}^n_+} |a_\varepsilon(\hat{x}_n)|^2 \Big(\frac{1}{\hat{x}_n} \int_0^{\hat{x}_n} |\partial_{\hat{x}_n} u(\hat{x}', z_n)| \mathrm{d}z_n \Big)^2 \mathrm{d}\hat{x}$$

$$\leq \int_{\mathbb{R}^{n-1}} \int_0^\infty |a_\varepsilon(\hat{x}_n)|^2 \|\partial_{\hat{x}_n} u(\hat{x}', .)\|^2_{L^\infty(\mathbb{R})} \mathrm{d}\hat{x}_n \mathrm{d}\hat{x}' \leq K\varepsilon \|u\|^2_{H^3(\mathbb{R}^n_+)}.$$

In the third term of $I^1_\varepsilon(x)$, we have

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|) R_2(\hat{x},\bar{y}) \big| \det \mathrm{D}F_\gamma(\hat{x}) \big| \,\mathrm{d}y \Big|^2 \mathrm{d}\hat{x}$$

$$\leq K \int_{\mathbb{R}^n_+} a_\varepsilon(\hat{x}_n) \left(\int_0^1 \int_{B_\delta(\hat{x}')\times(0,\delta)} \rho_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|) |D^2 u(\bar{y} + t(\hat{x}-\bar{y}))|^2 \,\mathrm{d}y\mathrm{d}t \right) \mathrm{d}\hat{x}$$

$$\leq K \left(\int_0^R a_\varepsilon(\hat{x}_n) \,\mathrm{d}\hat{x}_n \right) \|u\|^2_{H^3(\mathbb{R}^n_+)} \leq K\varepsilon \|u\|^2_{H^3(\mathbb{R}^n_+)}, \tag{26}$$

where we used the calculations as in (20) and Lemma 2.2. Altogether, we end up with

$$\left\| I^1_\varepsilon \right\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\sqrt{\varepsilon} \|c\|_{H^3(\mathbb{R}^n_\gamma)},$$

where we used the same estimates as in the proof of Lemma 3.3.

Ad $I^2_\varepsilon(x)$: Here, we first rewrite the term $K_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\hat{y})|)$ in the following way:

$$K_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|) = J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|)\big(\big|\det \mathrm{D}F_\gamma(\bar{y})\big| - \big|\det \mathrm{D}F_\gamma(\hat{x})\big| \big)$$
$$+ \Big(J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|) - J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|) \Big) \big|\det \mathrm{D}F_\gamma(\hat{x})\big|.$$

Using this identity, we then get

$$|I^2_\varepsilon(x)| \leq \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|)\big(\big|\det \mathrm{D}F_\gamma(\bar{y})\big| - \big|\det \mathrm{D}F_\gamma(\hat{x})\big| \big)\big(u(\hat{x}) - u(\bar{y})\big) \,\mathrm{d}y \Big|$$
$$+ \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} \Big(J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|) - J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|) \Big)$$
$$\times \big|\det \mathrm{D}F_\gamma(\hat{x})\big| \big(u(\hat{x}) - u(\bar{y})\big) \,\mathrm{d}y \Big|.$$

In the first term on the right-hand side, we use the continuity of $|\det \mathrm{D}F_\gamma|$, the fundamental theorem of calculus as well as the Cauchy-Schwarz inequality and the calculations in $I_\varepsilon^1(x)$ to estimate the $L^2(\mathbb{R}^n_+)$-norm

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|)\big(\big|\det \mathrm{D}F_\gamma(\bar{y})\big| - \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big)\big(u(\hat{x})-u(\bar{y})\big)\,\mathrm{d}y\Big|^2 \mathrm{d}\hat{x}$$

$$\leq K\varepsilon \int_{\mathbb{R}^n_+} \int_{B_\delta(\hat{x}')\times(0,\delta)} \int_0^1 \rho_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|)|Du(\bar{y}+t(\hat{x}-\bar{y}))|^2\,\mathrm{d}t\mathrm{d}y\mathrm{d}\hat{x} \leq K\varepsilon\|u\|^2_{H^3(\mathbb{R}^n_+)},$$

where we used the same arguments as in (26) in the last step. In the second term on the right-hand side, we use the mean value theorem to conclude

$$\begin{aligned}
&\big|J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|) - J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\big| \\
&= \left|\int_0^1 \nabla J_\varepsilon(|z|)_{|z=z_t} \cdot \big(A_\gamma(\hat{x},\bar{y}) - A_\gamma(\hat{x},\hat{x})\big)(\hat{x}-\bar{y})\,\mathrm{d}t\right| \\
&\leq \int_0^1 \frac{1}{\varepsilon}\frac{\rho_\varepsilon'(|z|)}{|z|^2}_{\big||z=z_t} K|\hat{x}-\bar{y}|^2\,\mathrm{d}t + \int_0^1 \frac{\rho_\varepsilon(|z|)}{|z|^3}_{\big||z=z_t} K|\hat{x}-\bar{y}|^2\,\mathrm{d}t,
\end{aligned} \tag{27}$$

where we defined

$$\begin{aligned}
z_t &:= M_t(\hat{x},\bar{y})(\hat{x}-\bar{y}), \\
M_t(\hat{x},\bar{y}) &:= A_\gamma(\hat{x},\hat{x}) + t\big(A_\gamma(\hat{x},\bar{y}) - A_\gamma(\hat{x},\hat{x})\big)
\end{aligned}$$

for all $\hat{x},\bar{y}\in\mathbb{R}^n$ and all $t\in[0,1]$. Thanks to assumption (24), it follows that $|M_t(\hat{x},\bar{y})-\mathrm{Id}|\leq 3\alpha$ for all $\hat{x},\bar{y}\in\mathbb{R}^n$ and all $t\in[0,1]$. Therefore, $M_t(\hat{x},\bar{y})^{-1}$ exists for all $\hat{x},\bar{y}\in\mathbb{R}^n$ and all $t\in[0,1]$ and satisfies the inequality

$$|M_t(\hat{x},\bar{y})^{-1}| \leq \frac{1}{1-|M_t(\hat{x},\bar{y})-\mathrm{Id}|} < \frac{1}{1-3\alpha}$$

for all $\hat{x},\bar{y}\in\mathbb{R}^n$ and all $t\in[0,1]$. Using (27), we then obtain for the second term of $I_\varepsilon^2(x)$

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} \Big(J_\varepsilon(|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|) - J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\Big) \\
&\qquad\qquad \times \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big(u(\hat{x})-u(\bar{y})\big)\,\mathrm{d}y\Big|^2\,\mathrm{d}\hat{x} \\
&\leq K\int_{\mathbb{R}^n_+}\left(\int_0^1 \int_{B_\delta(\hat{x}')\times(0,\delta)} \int_0^1 \left(\frac{1}{\varepsilon}\frac{\rho_\varepsilon'(|z_t|)}{|z_t|^2} + \frac{\rho_\varepsilon(|z_t|)}{|z_t|^3}\right)\big|Du(\bar{y}+s(\hat{x}-\bar{y}))\big||\hat{x}-\bar{y}|^3\,\mathrm{d}s\mathrm{d}y\mathrm{d}t\right)^2\mathrm{d}\hat{x}
\end{aligned}$$

$$\leq K \int_{\mathbb{R}^n_+} \left(\int_0^1 \int_{B_\delta(\hat{x}')\times(0,\delta)} \int_0^1 \Big(\frac{1}{\varepsilon}\rho'_\varepsilon(|z_t|)|\hat{x}-\bar{y}| + \rho_\varepsilon(|z_t|)\Big) \big| Du(\bar{y}+s(\hat{x}-\bar{y}))\big| \,\mathrm{d}s\mathrm{d}y\mathrm{d}t \right)^2 \mathrm{d}\hat{x}$$

$$\leq K\varepsilon \int_0^1 \int_0^1 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \Big(\frac{1}{\varepsilon}\rho'_\varepsilon(|z_t|)|\hat{x}-\bar{y}| + \rho_\varepsilon(|z_t|)\Big) \big| Du(\bar{y}+s(\hat{x}-\bar{y}))\big|^2 \,\mathrm{d}y\mathrm{d}\hat{x}\mathrm{d}s\mathrm{d}t$$

$$\leq K\varepsilon \|u\|^2_{H^3(\mathbb{R}^n_+)},$$

where we used the same arguments as above. Then, we conclude

$$\|I^2_\varepsilon\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\mathbb{R}^n_\gamma)}.$$

In particular, this implies

$$\left\|\mathcal{L}^{\mathbb{R}^n_\gamma}_\varepsilon c + \Delta c\right\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\mathbb{R}^n_\gamma)}$$

for all $c \in C^{2,1}_0(\overline{\mathbb{R}^n_\gamma})$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_\gamma$. Finally, a density argument, cf. Lemma 3.3, yields

$$\left\|\mathcal{L}^{\mathbb{R}^n_\gamma}_\varepsilon c + \Delta c\right\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\mathbb{R}^n_\gamma)}$$

for all $c \in H^3(\mathbb{R}^n_\gamma)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_\gamma$. This concludes the proof. □

Corollary 3.7. *Let $c \in H^2(\mathbb{R}^n_\gamma)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_\gamma$. Then, it holds*

$$\left\|\mathcal{L}^{\mathbb{R}^n_\gamma}_\varepsilon c + \Delta c\right\|_{L^2(\mathbb{R}^n_\gamma)} \to 0 \ \text{ as } \varepsilon \searrow 0.$$

Proof. Due to Lemma 3.6, it suffices to show that $\left\|\mathcal{L}^{\mathbb{R}^n_\gamma}_\varepsilon c + \Delta c\right\|_{L^2(\mathbb{R}^n_\gamma)}$ is bounded uniformly in $\varepsilon > 0$ for all $c \in H^2(\mathbb{R}^n_\gamma)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_\gamma$. Then, we can apply the Banach Steinhaus Theorem, which implies the assertion.
In fact, it suffices to bound the error term (23) in a suitable way. Using the same methods as in the proof of Lemma 3.6, we can rewrite $\mathcal{R}_\varepsilon$ as in (25). First of all, we prove the assertion for functions $c \in C^{1,1}_0(\overline{\mathbb{R}^n_\gamma})$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\mathbb{R}^n_\gamma$ and use a density argument afterwards to conclude the proof. Observe that it suffices to integrate in $\mathcal{R}_\varepsilon$ only over $B_\delta(\hat{x}') \times (0,\delta)$, since otherwise we can employ Lemma 3.2, cf. proof of Lemma 3.6.

Ad $I^1_\varepsilon(x)$: Using a Taylor expansion, we obtain

$$I^1_\varepsilon(x) = \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\nabla_{x'}u(\hat{x}) \cdot (\hat{x}'-\bar{y}')\big|\det \mathrm{D}F_\gamma(\hat{x})\big| \,\mathrm{d}y$$

$$+\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\partial_{x_n}u(\hat{x})(\hat{x}_n+\bar{y}_n)\left|\det \mathrm{D}F_\gamma(\hat{x})\right|\,\mathrm{d}y$$

$$+\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)R_2(\hat{x},\bar{y})\left|\det \mathrm{D}F_\gamma(\hat{x})\right|\,\mathrm{d}y.$$

As in the proof before, the first term on the right-hand side vanishes, since the integrand is odd with respect to $\hat{x}'-\bar{y}'$. In the second term, the properties of F_γ imply

$$\partial_{x_n}u(\hat{x})_{|\{x_n=0\}}=\partial_{x_n}\big(c\circ F_\gamma\big)(\hat{x})_{|\{x_n=0\}}=\nabla c(x',\gamma(x'))\cdot(-\mathbf{n}(x',\gamma(x')))=0,$$

since $\partial_{\mathbf{n}}c=0$ on $\partial\mathbb{R}^n_\gamma$, and therefore

$$\partial_{x_n}u(\hat{x})=\partial_{x_n}u(\hat{x})-\partial_{x_n}u(\hat{x}',0)=\Big(\int\limits_0^1\partial^2_{x_n}u(\hat{x}',tx_n)\,\mathrm{d}t\Big)x_n.$$

This yields

$$\Big|\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\partial_{x_n}u(\hat{x})(\hat{x}_n+\bar{y}_n)\left|\det \mathrm{D}F_\gamma(\hat{x})\right|\,\mathrm{d}y\Big|$$

$$\le K\int\limits_0^1|\partial^2_{x_n}u(\hat{x}',t\hat{x}_n)|\Big(\int\limits_{B_\delta(\hat{x}')\times(0,\delta)}\rho_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\,\mathrm{d}y\Big)\,\mathrm{d}t,$$

$$\le K\Big(\frac{1}{\hat{x}_n}\int\limits_0^{\hat{x}_n}|\partial^2_{x_n}u(\hat{x}',z_n)|\,\mathrm{d}z_n\Big)\Big(\int\limits_{B_\delta(\hat{x}')\times(0,\delta)}\rho_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\,\mathrm{d}y\Big)$$

where we used $|A_\gamma(\hat{x},\bar{y})(\hat{x}-\bar{y})|\ge C|\hat{x}-\bar{y}|$ for all $\hat{x},\bar{y}\in\mathbb{R}^n$ as well as $\left|\det \mathrm{D}F_\gamma(\hat{x})\right|\le C$ for all $\hat{x}\in\mathbb{R}^n$. Computing the L^2-norm, we then get

$$\int\limits_{\mathbb{R}^n_+}\Big|\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)\partial_{x_n}u(\hat{x})(\hat{x}_n+\bar{y}_n)\left|\det \mathrm{D}F_\gamma(\hat{x})\right|\,\mathrm{d}y\Big|^2\mathrm{d}\hat{x}\le K\|u\|_{H^2(\mathbb{R}^n_+)},$$

where we used the same ideas as in Corollary 3.5, cf. (21). In the third term, it holds

$$\int\limits_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)R_2(\hat{x},\bar{y})\left|\det \mathrm{D}F_\gamma(\hat{x})\right|\,\mathrm{d}y$$

$$\le K\int\limits_{\mathbb{R}^n_+}\int\limits_{B_\delta(\hat{x}')\times(0,\delta)}\int\limits_0^1\rho_\varepsilon(|A_\gamma(\hat{x},\hat{x})(\hat{x}-\bar{y})|)|D^2u(\bar{y}+t(\hat{x}-\bar{y}))|^2\,\mathrm{d}t\mathrm{d}y\mathrm{d}\hat{x}$$

$$\leq K \int_0^1 \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \|\rho_\varepsilon(|A_\gamma((\hat{x}', .), (\hat{x}', .))(\hat{x}' - \bar{y}', .)|)\|_{L^1(\mathbb{R})} \times \|D^2 u(\bar{y}' + t(\hat{x}' - \bar{y}'), .)\|_{L^2(\mathbb{R}_+)} \,\mathrm{d}\hat{x}'\mathrm{d}y'\mathrm{d}t$$
$$\leq K \|u\|^2_{H^2(\mathbb{R}^n_\gamma)}.$$

Here, we used similar arguments as before, cf. (22) and (26). Altogether, we have

$$\|I^1_\varepsilon\|_{L^2(\mathbb{R}^n_\gamma)} \leq K \|c\|_{H^2(\mathbb{R}^n_\gamma)}.$$

Ad $I^2_\varepsilon(x)$: Here, we have

$$|I^2_\varepsilon(x)| \leq \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x}, \bar{y})(\hat{x} - \bar{y})|)\big(\big|\det \mathrm{D}F_\gamma(\bar{y})\big| - \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big)\big(u(\hat{x}) - u(\bar{y})\big)\,\mathrm{d}y\Big|$$
$$+ \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} \Big(J_\varepsilon(|A_\gamma(\hat{x}, \bar{y})(\hat{x} - \bar{y})|) - J_\varepsilon(|A_\gamma(\hat{x}, \hat{x})(\hat{x} - \bar{y})|)\Big) \times \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big(u(\hat{x}) - u(\bar{y})\big)\,\mathrm{d}y\Big|,$$

where we used the same identity as in the proof of Lemma 3.6. In the first term on the right-hand side, we observe that

$$\Big|\big(\big|\det \mathrm{D}F_\gamma(\bar{y})\big| - \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big)\big(u(\hat{x}) - u(\bar{y})\big)\Big| \leq K\Big(\int_0^1 |Du(\bar{y} + t(\hat{x} - \bar{y}))|\mathrm{d}t\Big)|\hat{x} - \bar{y}|^2,$$

which yields

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} J_\varepsilon(|A_\gamma(\hat{x}, \bar{y})(\hat{x} - \bar{y})|)\big(\big|\det \mathrm{D}F_\gamma(\bar{y})\big| - \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big)\big(u(\hat{x}) - u(\bar{y})\big)\,\mathrm{d}y\Big|^2 \mathrm{d}\hat{x}$$
$$\leq K\|u\|^2_{H^2(\mathbb{R}^n_+)}.$$

Here, we concluded with the same arguments as in the proof before. In the second term, we again use the mean value theorem, cf. Lemma 3.6, to obtain

$$\int_{\mathbb{R}^n_+} \Big| \int_{B_\delta(\hat{x}')\times(0,\delta)} \Big(J_\varepsilon(|A_\gamma(\hat{x}, \bar{y})(\hat{x} - \bar{y})|) - J_\varepsilon(|A_\gamma(\hat{x}, \hat{x})(\hat{x} - \bar{y})|)\Big) \times \big|\det \mathrm{D}F_\gamma(\hat{x})\big|\big(u(\hat{x}) - u(\bar{y})\big)\,\mathrm{d}y\Big|^2 \mathrm{d}\hat{x}$$

$$\leq K \int_{\mathbb{R}^n_+} \left(\int_0^1 \int_{B_\delta(\hat{x}')\times(0,\delta)} \int_0^1 \Big(\frac{1}{\varepsilon}\rho_\varepsilon'(|z_t|)|\hat{x}-\bar{y}| + \rho_\varepsilon(|z_t|)\Big)\big|Du(\bar{y}+s(\hat{x}-\bar{y}))\big|\,\mathrm{d}s\mathrm{d}y\mathrm{d}t \right)^2 \mathrm{d}\hat{x}$$
$$\leq K\|u\|^2_{H^2(\mathbb{R}^n_+)},$$

where we defined

$$z_t := M_t(\hat{x},\bar{y})(\hat{x}-\bar{y}),$$
$$M_t(\hat{x},\bar{y}) := A_\gamma(\hat{x},\hat{x}) + t\big(A_\gamma(\hat{x},\bar{y}) - A_\gamma(\hat{x},\hat{x})\big)$$

for all $\hat{x},\bar{y}\in\mathbb{R}^n$ and all $t\in[0,1]$. In fact, this estimate follows by the same methods as in the proof of Lemma 3.6. Altogether, it follows that

$$\|\mathcal{R}_\varepsilon\tilde{c}\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\|c\|_{H^2(\mathbb{R}^n_\gamma)}$$

for all $c\in C_0^{1,1}(\overline{\mathbb{R}^n_\gamma})$ with $\partial_{\mathbf{n}}c=0$ on $\partial\mathbb{R}^n_\gamma$, and therefore

$$\left\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma}c + \Delta c\right\|_{L^2(\mathbb{R}^n_\gamma)} \leq K\|c\|_{H^2(\mathbb{R}^n_\gamma)}$$

for all $c\in C_0^{1,1}(\overline{\mathbb{R}^n_\gamma})$ with $\partial_{\mathbf{n}}c=0$ on $\partial\mathbb{R}^n_\gamma$. Finally, a denseness argument, cf. Lemma 3.6, yields that this estimate also holds true for all $c\in H^2(\mathbb{R}^n_\gamma)$ with $\partial_{\mathbf{n}}c=0$ on $\partial\mathbb{R}^n_\gamma$. In the last step, we employ the Banach Steinhaus Theorem. Then, the claim follows. □

4. Main result

In this section, we want to state and prove the main result on the strong convergence of the nonlocal operator $\mathcal{L}_\varepsilon$ on bounded smooth domains $\Omega\subset\mathbb{R}^n$.

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^n$, $n=2,3$, be a bounded domain with C^2-boundary. Then, for every $c\in H^2(\Omega)$ with $\partial_{\mathbf{n}}c=0$ on $\partial\Omega$, it holds*

$$\left\|\mathcal{L}_\varepsilon^{\Omega}c + \Delta c\right\|_{L^2(\Omega)} \to 0 \quad \text{as } \varepsilon\searrow 0. \tag{28}$$

Proof. Since $\partial\Omega$ is compact, there exist open sets $U_1,\dots,U_N\subset\mathbb{R}^n$ and $\gamma_1,\dots,\gamma_N\in C_0^2(\mathbb{R}^{n-1})$ such that, up to a rotation, $\Omega\cap U_j=\mathbb{R}^n_{\gamma_j}\cap U_j$ for all $j=1,\dots,N$ and $\partial\Omega\subseteq\bigcup_{j=1}^N U_j$. Since also $\Omega\setminus\big(\bigcup_{j=1}^N U_j\big)$ is compact, there exists an open, bounded set $U_0\subset\mathbb{R}^n$ such that $\Omega\setminus\big(\bigcup_{j=1}^N U_j\big)\subset U_0$ and $\overline{U_0}\subset\Omega$. Then, we choose some $\gamma_0\in C_0^2(\mathbb{R}^{n-1})$ such that $\overline{U_0}\subset\mathbb{R}^n_{\gamma_0}$. Altogether, we have

$$\overline{\Omega}\subseteq\bigcup_{j=0}^N U_j \quad\text{and}\quad \Omega\cap U_j=\mathbb{R}^n_{\gamma_j}\cap U_j \quad\text{for all } j=0,\dots,N.$$

Next, we choose a partition of unity φ_j, $j = 0, \dots, N$, on $\overline{\Omega}$. Without loss of generality, we can assume that $\partial_{\mathbf{n}_j}\varphi_j = 0$ on $\partial\mathbb{R}^n_{\gamma_j}$ for all $j = 0, \dots, N$, where $\mathbf{n}_j$ denotes the outer unit normal to $\partial\mathbb{R}^n_{\gamma_j}$. Otherwise, $\partial\mathbb{R}^n_{\gamma_j}$ admits a tubular neighborhood U_{a_j} of width $a_j > 0$, cf. [21, Section 2.3]. Then, the restriction of φ_j to $\partial\mathbb{R}^n_{\gamma_j}$ can be extended to a function $\hat{\varphi}_j$ on U_{a_j}, which is constant in the direction of $\mathbf{n}_j$ by setting $\hat{\varphi}_j(x) := \varphi_j(\pi_{\partial\mathbb{R}^n_{\gamma_j}}(x))$ for $x \in U_{a_j}$, where $\pi_{\partial\mathbb{R}^n_{\gamma_j}} : U_{a_j} \to \partial\mathbb{R}^n_{\gamma_j}$ denotes the projection on $\partial\mathbb{R}^n_{\gamma_j}$.
Finally, we choose functions $\psi_j \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi_j \subseteq U_j$, $\psi_j \geq 0$ and $\psi_j \equiv 1$ in $\operatorname{supp}\varphi_j$ for all $j = 0, \dots, N$.

Now, let $c \in H^2(\Omega)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\Omega$ be arbitrary. Using the identity $c = \sum_{j=0}^N \varphi_j c$, we get

$$\begin{aligned}
\mathcal{L}_\varepsilon^\Omega c + \Delta c &= \mathcal{L}_\varepsilon^\Omega\Big(\sum_{j=0}^N \varphi_j c\Big) + \Delta\Big(\sum_{j=0}^N \varphi_j c\Big) = \sum_{j=0}^N \Big(\mathcal{L}_\varepsilon^\Omega\big(\varphi_j c\big) + \Delta\big(\varphi_j c\big)\Big) \\
&= \sum_{j=0}^N \Big(\psi_j \mathcal{L}_\varepsilon^\Omega\big(\varphi_j c\big) + (1-\psi_j)\mathcal{L}_\varepsilon^\Omega\big(\varphi_j c\big) + \Delta\big(\varphi_j c\big)\Big) \\
&=: \sum_{j=0}^N \Big(I^1_{\varepsilon,j} + I^2_{\varepsilon,j} + \Delta\big(\varphi_j c\big)\Big).
\end{aligned}$$

In the next step, we analyze these terms separately.

Ad $I^1_{\varepsilon,j}$: For almost all $x \in \Omega$ and all $j = 0, \dots, N$ it holds:

$$\begin{aligned}
\psi_j(x)\mathcal{L}_\varepsilon^\Omega\big(\varphi_j c\big)(x) &= \psi_j(x) \int_\Omega J_\varepsilon(|x-y|)\big(\varphi_j(x)c(x) - \varphi_j(y)c(y)\big)\,\mathrm{d}y \\
&= \psi_j(x) \int_{\mathbb{R}^n_{\gamma_j}} J_\varepsilon(|x-y|)\big(\varphi_j(x)c(x) - \varphi_j(y)c(y)\big)\,\mathrm{d}y \\
&\quad + \psi_j(x) \int_{\Omega\setminus U_j} J_\varepsilon(|x-y|)\varphi_j(x)c(x)\,\mathrm{d}y \\
&\quad - \psi_j(x) \int_{\mathbb{R}^n_{\gamma_j}\setminus U_j} J_\varepsilon(|x-y|)\varphi_j(x)c(x)\,\mathrm{d}y \\
&=: \hat{I}^1_{\varepsilon,j} + \hat{I}^2_{\varepsilon,j} - \hat{I}^3_{\varepsilon,j}.
\end{aligned}$$

In $\hat{I}^2_{\varepsilon,j}$, we observe that $\delta_{2,j} := \operatorname{dist}(\operatorname{supp}\psi_j, \Omega \setminus U_j) > 0$ and therefore $|x-y| \geq \delta_{2,j}$. Hence, $\|\hat{I}^2_{\varepsilon,j}\|_{L^2(\Omega)} \to 0$ as $\varepsilon \searrow 0$ for all $j = 0, \dots, N$ by Lemma 3.2.
In $\hat{I}^3_{\varepsilon,j}$, it holds $\delta_{3,j} := \operatorname{dist}(\operatorname{supp}\psi_j, \mathbb{R}^n_{\gamma_j} \setminus U_j) > 0$. This implies $\|\hat{I}^3_{\varepsilon,j}\|_{L^2(\Omega)} \to 0$ as $\varepsilon \searrow 0$ for all $j = 0, \dots, N$ again by Lemma 3.2.
In the next step, we consider $\hat{I}^1_{\varepsilon,j} + \Delta\big(\varphi_j c\big)$. Since $\operatorname{supp}\psi_j \subseteq U_j$, we have

$$\hat{I}^1_{\varepsilon,j} + \Delta(\varphi_j c) = \psi_j \mathcal{L}_\varepsilon^{\mathbb{R}^n_{\gamma_j}}(\varphi_j c) + \Delta(\varphi_j c) = \psi_j \Big(\mathcal{L}_\varepsilon^{\mathbb{R}^n_{\gamma_j}}(\varphi_j c) + \Delta(\varphi_j c) \Big). \tag{29}$$

Due to the properties of φ_j, it holds $\varphi_j c \in H^2(\mathbb{R}^n_{\gamma_j})$ and $\partial_{\mathbf{n}}(\varphi_j c) = 0$ on $\partial\mathbb{R}^n_{\gamma_j}$. Therefore, (29) vanishes as $\varepsilon \searrow 0$ by Corollary 3.7.

Ad $I^2_{\varepsilon,j}$: Due to our choice of the functions ψ_j, it holds $\operatorname{supp}(1-\psi_j) \cap \operatorname{supp}\varphi_j = \emptyset$. Thus, we have $\delta_j := \operatorname{dist}(\operatorname{supp}(1-\psi_j), \operatorname{supp}\varphi_j) > 0$. Therefore, $\|I^2_{\varepsilon,j}\|_{L^2(\Omega)} \to 0$ as $\varepsilon \searrow 0$ for all $j = 0, \dots, N$ by Lemma 3.2. This concludes the proof. □

Corollary 4.2. *In addition, if $c \in H^3(\Omega)$ and Ω is of class C^3, it even holds*

$$\left\| \mathcal{L}_\varepsilon^\Omega c + \Delta c \right\|_{L^2(\Omega)} \le K\sqrt{\varepsilon}\|c\|_{H^3(\Omega)}.$$

Proof. This can be proven analogously as Theorem 4.1 using Lemma 3.6 instead of Corollary 3.7. □

5. Nonlocal-to-local convergence of the Cahn-Hilliard equation

Theorem 5.1. *Let the initial data $c_{0,\varepsilon} \in L^2(\Omega)$ satisfy $c_{0,\varepsilon} \to c_0$ in $L^2(\Omega)$ at rate $\mathcal{O}(\sqrt{\varepsilon})$ as $\varepsilon \searrow 0$ for $c_0 \in H^1(\Omega)$. Let the weak solution $c \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ with $f'(c) \in L^2(\Omega_T)$ of the local Cahn-Hilliard equation satisfy $c \in L^2(0,T;H^3(\Omega))$. Then, the weak solution c_ε of the nonlocal Cahn-Hilliard equation (7)-(8) converges strongly to the strong solution of the local Cahn-Hilliard equation (1)-(2) in $L^\infty(0,T;H^{-1}(\Omega)) \cap L^2(\Omega_T)$ at rate $\mathcal{O}(\sqrt{\varepsilon})$ as $\varepsilon \searrow 0$.*

Proof. Let us define the functions $u := c_\varepsilon - c$ and $w := \mu_\varepsilon - \mu$. Then, (u, w) solves the system

$$\partial_t u = \Delta w, \tag{30}$$

$$w = \mathcal{L}_\varepsilon^\Omega c_\varepsilon + \Delta c + f'(c_\varepsilon) - f'(c). \tag{31}$$

In the next step, we test (30) by $(-\Delta_N)^{-1}u$. Here, $v := (-\Delta_N)^{-1}u \in H^1(\Omega) \cap L^2_{(0)}(\Omega)$ denotes the unique solution of

$$\int_\Omega \nabla v \cdot \nabla \varphi \,\mathrm{d}x = \int_\Omega u\varphi \,\mathrm{d}x$$

for all $\varphi \in H^1(\Omega) \cap L^2_{(0)}(\Omega)$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|^2_{H^{-1}(\Omega)} = \int_\Omega \Delta w(-\Delta_N)^{-1}u \,\mathrm{d}x = -\int_\Omega wu \,\mathrm{d}x. \tag{32}$$

Here, we also used the following calculation

$$\int_\Omega \partial_t u(-\Delta_N)^{-1}u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\left|(-\Delta_N)^{-1/2}u\right|^2 \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|^2_{H^{-1}(\Omega)}.$$

Furthermore, we multiply (31) by u and integrate over Ω. Hence,

$$\int_\Omega wu \,\mathrm{d}x = \int_\Omega \mathcal{L}_\varepsilon^\Omega c_\varepsilon u \,\mathrm{d}x + \int_\Omega \Delta c u \,\mathrm{d}x + \int_\Omega \big(f'(c_\varepsilon) - f'(c)\big) u \,\mathrm{d}x.$$

Using the assumptions on f, we can estimate the last integral on the right-hand side from below by

$$\int_\Omega \big(f'(c_\varepsilon) - f'(c)\big) u \,\mathrm{d}x \geq -\int_\Omega \alpha(c_\varepsilon - c) u \,\mathrm{d}x = -\int_\Omega \alpha |u|^2 \,\mathrm{d}x.$$

Therefore, we have

$$\begin{aligned} \int_\Omega wu \,\mathrm{d}x &\geq \int_\Omega \mathcal{L}_\varepsilon^\Omega uu \,\mathrm{d}x + \int_\Omega \big(\mathcal{L}_\varepsilon^\Omega c + \Delta c\big) u \,\mathrm{d}x - \alpha \int_\Omega |u|^2 \,\mathrm{d}x \\ &= \mathcal{E}_\varepsilon(u) + \int_\Omega \big(\mathcal{L}_\varepsilon^\Omega c + \Delta c\big) u \,\mathrm{d}x - \alpha \int_\Omega |u|^2 \,\mathrm{d}x, \end{aligned} \tag{33}$$

where we used

$$\int_\Omega \mathcal{L}_\varepsilon^\Omega uu \,\mathrm{d}x = \frac{1}{4} \int_\Omega \int_\Omega J_\varepsilon(x-y) \big|u(x) - u(y)\big|^2 \,\mathrm{d}x \,\mathrm{d}y = \mathcal{E}_\varepsilon(u) \tag{34}$$

in the last step. Now, combining (32) and (33) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|u\|_{H^{-1}(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \mathcal{E}_\varepsilon(u) &\leq \Big(\alpha + \frac{1}{2}\Big) \|u\|_{L^2(\Omega)}^2 - \int_\Omega \big(\mathcal{L}_\varepsilon^\Omega c + \Delta c\big) u \,\mathrm{d}x \\ &\leq \Big(\alpha + 1\Big) \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \Big\| \mathcal{L}_\varepsilon^\Omega c + \Delta c \Big\|_{L^2(\Omega)}^2. \end{aligned}$$

Employing inequality (12) from Lemma 2.1 with $\delta = \frac{1}{2(\alpha+1)}$, we then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|u\|_{H^{-1}(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \mathcal{E}_\varepsilon(u) \leq C \|u\|_{H^{-1}(\Omega)}^2 + C \Big\| \mathcal{L}_\varepsilon^\Omega c + \Delta c \Big\|_{L^2(\Omega)}^2. \tag{35}$$

Finally, Gronwall's inequality yields

$$\|u(t)\|_{H^{-1}(\Omega)}^2 + \frac{1}{2} \|u\|_{L^2(\Omega_t)}^2 + \int_0^t \mathcal{E}_\varepsilon(u)(\tau) \,\mathrm{d}\tau \leq K\varepsilon \Big(1 + \int_0^t \|c(\tau)\|_{H^3(\Omega)}^2 \,\mathrm{d}\tau\Big) \exp\Big(\int_0^t K \,\mathrm{d}\tau\Big)$$

for almost all $t \in (0, T)$, where we used Corollary 4.2 and the assumptions on the initial data on the right-hand side. Using (13), we then conclude the proof. □

Corollary 5.2. *Under the assumptions of Corollary 5.1, it even holds*

$$c_\varepsilon \to c \quad \text{in } L^\infty(0,T;H^s(\Omega)) \quad \text{as } \varepsilon \searrow 0$$

for all $s \in (-1,0)$ *and furthermore,*

$$c_\varepsilon \to c \quad \text{in } L^2(0,T;L^p(\Omega)) \quad \text{as } \varepsilon \searrow 0$$

for all $p \in [2,6)$.

Proof. In Theorem 5.1, we have already shown that

$$c_\varepsilon \to c \quad \text{in } L^\infty(0,T;H^{-1}(\Omega))$$

as $\varepsilon \searrow 0$. Since the sequence $(c_\varepsilon)_{\varepsilon>0} \subset L^\infty(0,T;L^2(\Omega))$ is bounded, cf. [9, Theorem 2.3], we get

$$\|c_\varepsilon - c\|_{L^\infty(0,T;H^s(\Omega))} \le K\|c_\varepsilon - c\|^\theta_{L^\infty(0,T;L^2(\Omega))}\|c_\varepsilon - c\|^{1-\theta}_{L^\infty(0,T;H^{-1}(\Omega))}$$

for $s = \theta - 1$, $\theta \in (0,1)$ and thus the first assertion follows.

In the second assertion, we use that $(c_\varepsilon)_{\varepsilon>0} \subset L^2(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^6(\Omega))$ is bounded, cf. [9, Theorem 2.2]. Then, an interpolation yields

$$\|c_\varepsilon - c\|_{L^2(0,T;L^p(\Omega))} \le K\|c_\varepsilon - c\|^\theta_{L^2(\Omega_T)}\|c_\varepsilon - c\|^{1-\theta}_{L^2(0,T;L^6(\Omega))}$$

for all $p = \frac{6}{1+2\theta}$, where $\theta \in (0,1]$. This finishes the proof. □

Remark 5.3. Lastly, we want to mention that our methods from Section 5 can also be used to prove nonlocal-to-local convergence for the Allen-Cahn equation. The Allen-Cahn equation is given by

$$\partial_t c = \Delta c - f'(c) \quad \text{in } \Omega_T \tag{36}$$

together with boundary and initial conditions

$$\partial_{\mathbf{n}} c = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{37}$$

$$c|_{t=0} = c_0 \quad \text{in } \Omega. \tag{38}$$

The nonlocal version is given by

$$\partial_t c = -\mathcal{L}_\varepsilon c - f'(c) \quad \text{in } \Omega_T \tag{39}$$

with initial condition

$$c|_{t=0} = c_0 \quad \text{in } \Omega. \tag{40}$$

Observe that we do not need to impose any boundary condition for the nonlocal Allen-Cahn equation. Here, we use the same notation and the same prerequisites as before, cf. Section 1. Then, we can prove the following assertion.

Theorem 5.4. *Let $c_{0,\varepsilon} \in L^2(\Omega)$ satisfy $c_{0,\varepsilon} \to c_0$ at rate $\mathcal{O}(\sqrt{\varepsilon})$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$ for some $c_0 \in H^1(\Omega)$. Let the weak solution $c \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ with $f'(c) \in L^2(\Omega_T)$ of the local Allen-Cahn equation satisfy $c \in L^2(0,T;H^3(\Omega))$. Then, the weak solution c_ε of the nonlocal Allen-Cahn equation (39)-(40) converges strongly to the strong solution of the local Allen-Cahn equation (36)-(38) in $L^\infty(0,T;L^2(\Omega))$ at rate $\mathcal{O}(\sqrt{\varepsilon})$ as $\varepsilon \searrow 0$.*

Proof. First of all, we define $u := c_\varepsilon - c$. Then, u is a solution of

$$\partial_t u = -\mathcal{L}_\varepsilon^\Omega c_\varepsilon - f'(c_\varepsilon) - \Delta c + f'(c). \tag{41}$$

Testing (41) with u, yields

$$\begin{aligned}\int_\Omega \partial_t u u \,\mathrm{d}x &= -\int_\Omega \big(\mathcal{L}_\varepsilon^\Omega c_\varepsilon + \Delta c\big) u \,\mathrm{d}x - \int_\Omega \big(f'(c_\varepsilon) - f'(c)\big) u \,\mathrm{d}x \\ &= -\int_\Omega \big(\mathcal{L}_\varepsilon^\Omega c + \Delta c\big) u \,\mathrm{d}x - \int_\Omega \mathcal{L}_\varepsilon^\Omega u u \,\mathrm{d}x - \int_\Omega \big(f'(c_\varepsilon) - f'(c)\big) u \,\mathrm{d}x,\end{aligned}$$

where the left-hand side also satisfies

$$\int_\Omega \partial_t u u \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|_{L^2(\Omega)}^2.$$

Employing the properties of f and (34), we then end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|_{L^2(\Omega)}^2 + \mathcal{E}_\varepsilon(u) \le \Big(\alpha + \frac{1}{2}\Big)\|u\|_{L^2(\Omega)}^2 + \frac{1}{2}\Big\|\mathcal{L}_\varepsilon^\Omega c + \Delta c\Big\|_{L^2(\Omega)}^2.$$

In the last step, we use inequality (12) from Lemma 2.1 with $\delta = \frac{1}{2(\alpha + \frac{1}{2})}$, Gronwall's inequality as well as Theorem 4.1 and (13). Then, the claim follows. □

Data availability

No data was used for the research described in the article.

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