Universität Regensburg Mathematik

Periods for irregular singular connections on surfaces

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Abstract: Given an integrable connection on a smooth quasi-projective algebraic surface U over a subfield k of the complex numbers, we define *rapid decay homology* groups with respect to the associated analytic connection which pair with the algebraic de Rham cohomology in terms of period integrals. These homology groups generalize the analogous groups in the same situation over curves defined by S. Bloch and H. Esnault. In dimension two, however, new features appear in this context which we explain in detail. Assuming a conjecture of C. Sabbah on the formal classification of meromorphic connections on surfaces (known to be true if the rank is lower than or equal to 5), we prove perfectness of the period pairing in dimension two.

1   Introduction

Given a smooth n-dimensional algebraic variety U over a subfield $k \subset \mathbb{C}$, the algebraic de Rham cohomology $H^p_{dR}(U)$ is defined to be the hypercohomology of the de Rham complex $0 \to \mathcal{O}_U \to \Omega^1_{U|k} \to \ldots \to \Omega^n_{U|k} \to 0$ consisting of the sheaves of Kähler differentials on U. The set of complex valued points $U(\mathbb{C})$ of U carries a canonical structure of a smooth analytic manifold which we denote by U^{an}. There is a natural morphism from the algebraic de Rham cohomology to its transcendental counterpart $H^p_{dR}(U) \otimes_k \mathbb{C} \to H^p_{dR}(U^{\mathrm{an}})$ which is an isomorphism due to a theorem of A. Grothendieck ([9]). More generally, let $\nabla : E \to E \otimes \Omega^1_{U|k}$ denote an integrable algebraic connection on U on the vector bundle E. The integrability means that the associated sequence

$$0 \to E \xrightarrow{\nabla} E \otimes_{\mathcal{O}_U} \Omega^1_{U|k} \xrightarrow{\quad} \ldots \xrightarrow{\nabla} E \otimes_{\mathcal{O}_U} \Omega^n_{U|k} \to 0$$

is indeed a complex. Again, there is a natural morphism $H^p_{dR}(U; E, \nabla) \otimes_k \mathbb{C} \to H^p_{dR}(U^{\mathrm{an}}; E^{\mathrm{an}}, \nabla^{\mathrm{an}})$ between the algebraic and analytic de Rham cohomology, both defined as the hypercohomology of the corresponding de Rham complexes. These morphisms fail to be isomorphisms in general.

In [8], P. Deligne introduces the notion of regular singularities of the connection at infinity and proves that the comparison morphism is an isomorphism in case of a *regular* singular connection. Later Z. Mebkhout gives an alternative proof by introducing the irregularity sheaf of the connection which contrary to Deligne's proof does not use Hironaka's resolution of singularities ([16]).

Applying the analytic Poincaré lemma yields another way of stating Deligne's comparison theorem as the perfectness of the period pairing

$$(H^p_{dR}(U; E, \nabla) \otimes_k \mathbb{C}) \otimes_\mathbb{C} H_p(U^{\mathrm{an}}, \mathcal{E}^\vee) \to \mathbb{C} \tag{1.1}$$

induced from integration of differential forms over smooth topological chains. Here, $H_p(U^{\mathrm{an}}, \mathcal{E}^\vee)$ denotes the singular homology of the analytic manifold U^{an} with values in the local system $\mathcal{E}^\vee := \ker(E^{\mathrm{an},\vee} \to E^{\mathrm{an},\vee} \otimes \Omega^1_{X^{\mathrm{an}}|\mathbb{C}})$ of flat sections in the dual bundle $E^{\mathrm{an},\vee}$ which carries a natural connection $\nabla^\vee$ dual to the given connection on E.

Assume that the local system $\mathcal{E} \subset E^{\mathrm{an}}$ of analytic solutions of ∇^{an}, a locally constant sheaf of $\mathbb{C}$-vector spaces, comes equipped with the structure of an F-local system for some subfield $F \subset \mathbb{C}$. Then, the singular homology with values in $\mathcal{E}$ also carries a natural F-structure and the period pairing gives a well-defined invariant, the alternating product of the determinants of the pairing for all p, as an element in $k^{\times} \backslash \mathbb{C}^{\times} / F^{\times}$. These invariants for regular singular connections together with their ℓ-adic analogs for tamely ramified sheaves over varieties over finite fields were extensively studied by T. Saito and T. Terasoma in [23].

In their fundamental paper [3], S. Bloch and H. Esnault generalize the period pairing (1.1) to the case of *irregular* singular connections *on curves*. To this end, they define modified homology groups $H_*^{rd}(X^{\mathrm{an}}; E^{\mathrm{an}}, \nabla^{\mathrm{an}})$ on the associated Riemann surface which pairs with the algebraic de Rham cohomology in terms of period integrals and prove perfectness of the resulting pairing:

$$(H_{dR}^*(U; E, \nabla) \otimes_k \mathbb{C}) \otimes H_*^{rd}(X^{\mathrm{an}}; E^{\mathrm{an},\vee}, \nabla^{\mathrm{an},\vee}) \to \mathbb{C} \ .$$

The resulting periods are interesting objects by themselves (the integral representations of the classical Bessel-functions, Gamma-function and confluent hypergeometric functions arise in this way as periods of irregular singular connections on curves) and are mysteriously related to ramification data for wildly ramified ℓ-adic sheaves on curves over a finite field (see e.g. [25]).

In the present paper, we want to start the investigation of the higher-dimensional case by studying the period pairing for irregular singular connections on smooth algebraic surfaces U over $k \subset \mathbb{C}$. It turns out that additional features arise that do not appear in the one-dimensional case. These new phenomena affect our proceeding essentially. We will comment on these after stating the main result.

Let $\nabla^{\vee}$ be the dual connection. We define the complex $\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^{\vee})$ of sheaves of rapid decay chains on the real oriented blow-up $\widetilde{X}$ of a *good* compactification (X, D) of U^{an} (assuming a conjecture of C. Sabbah, see section 2.2 for details), the hypercohomology of which gives the **rapid decay homology**

$$H_p^{rd}(U^{\mathrm{an}}; E^{\vee}, \nabla^{\vee}) := \mathbb{H}^{-p}(\widetilde{X}, \mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^{\vee})) \ .$$

We then define a natural pairing

$$\left(H_{dR}^p(U; E, \nabla) \otimes_k \mathbb{C}\right) \otimes H_p^{rd}(U^{\mathrm{an}}; E^{\vee}, \nabla^{\vee}) \longrightarrow \mathbb{C} \qquad (1.2)$$

in terms of period integrals. Our main result, Theorem 2.5, asserts that this period pairing is a perfect duality (assuming that Sabbah's Conjecture holds for (E, ∇), which is know e.g. if rank $E \leq 5$).

We are now going to explain the new phenomena arising in the two-dimensional situation as well as Sabbah's Conjecture. For the situation on curves, the Theorem of Levelt-Turrittin (cp. [14]) asserts that the formal completion of the given flat meromorphic connection is locally isomorphic (after a cyclic covering) to a certain elementary model, namely a direct sum of irregular singular connections on line bundles times regular singular connections.

In dimension two, C. Sabbah started the investigation of the analogous questions on the formal structure of meromorphic connections (cp. [20]) indicating a subtle additional feature: If X^{an} denotes an arbitrary compactification of U^{an} with normal crossing divisor $D := X^{\mathrm{an}} \smallsetminus U^{\mathrm{an}}$ as the complement, the desired

formal decomposition can be expected after a finite sequence of point blow-ups of X^{an} only. Additionally, one asks for the elementary model to fulfill a certain technical condition, in which case it is called a *good* elementary model. In [20], C. Sabbah conjectures that such a good formal decomposition can be achieved after a finite sequence of point blow-ups and cyclic ramification along the smooth strata of the divisor. He proves the conjecture for several classes of connections, in particular it holds if the rank of the connection is lower than or equal to 5.

Assuming Sabbah's Conjecture, we prove a local duality statement in the following sense. If X is again an arbitrary compactification of U with a normal crossing divisor $D := X \smallsetminus U$, one considers the *asymptotically flat de Rham complex* $\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee})$ on the real oriented blow-up $\widetilde{X}$ of X^{an}. If $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla)$ denotes the de Rham complex with moderate growth on $\widetilde{X}$ and $\widetilde{\jmath} : U^{\mathrm{an}} \hookrightarrow \widetilde{X}$ the inclusion, there is a natural local duality pairing

$$\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) \otimes_{\mathbb{C}_{\widetilde{X}}} \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee}) \to \widetilde{\jmath}_! \mathbb{C}_U \ .$$

We prove that this is a perfect duality (in the derived sense), in Theorem 3.8. Taking global sections yields a global duality result, Theorem 4.1, by Poincaré-Verdier duality which however lacks an explicit description in terms of periods.

In case of a *good* compactification, we prove that there is a canonical isomorphism in the bounded derived category

$$\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^{\vee}) \cong \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee})[2d] \in D^b(\mathbb{C}_{\widetilde{X}}) \ , \qquad (1.3)$$

where $d := \dim_{\mathbb{C}}(X) = 2$. This isomorphism is shown to be compatible with both the period pairing and Poincaré-Verdier duality which allows to deduce our main result, Theorem 2.5, asserting the perfectness of the period pairing (1.2).

Note, that one could try to define the rapid decay sheaves literally in the same way as in the *good* case also for *non-good* compactifications. It seems likely, however, that these sheaves do not carry enough information to detect the algebraic de Rham cohomology, since the latter is dual to the hypercohomology of $\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee})$ by Theorem 4.1, whereas the proof of the isomorphism between $\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee})$ and the rapid decay complex $\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^{\vee})$ in the derived category uses the fact that we start with a good compactification in an essential way. As for the rapid decay sheaves, it is not clear how they behave under point blow-ups (in contrast to the asymptotically flat de Rham complex, see Lemma 3.7).

Finally, the explicit description of the rapid decay chains allows us to forward a given rational structure on the local system to the rapid decay homology vector spaces. To be more precise, assume that the $\mathbb{C}$-local system $\mathcal{E}$ comes equipped with a structure of a local system of F-vector spaces for some subfield $F \subset \mathbb{C}$. The rapid decay homology then naturally inherits an F-structure so that there is a well-defined determinant of the period pairing as an element in $k^{\times} \backslash \mathbb{C}^{\times} / F^{\times}$ for *irregular* singular connections on surfaces (see Definition 2.7), a generalization in dimension two of T. Saito and T. Terasoma's definition for *regular* singular connections (which works in any dimension).

Notational convention. Distinguishing between algebraic varieties and associated complex manifolds, we will decorate the latter with a superscript 'an', so that X^{an} denotes the complex manifold associated to the algebraic variety

X over $k \subset \mathbb{C}$. For readability reasons, however, we will not always pursue this notation in the case of the vector bundles and connections considered whenever it is clear, e.g. from the type of the space over which they live, in which category we are working.

2 Rapid decay homology and periods

2.1 The geometric situation

Let $k \subset \mathbb{C}$ be a subfield of the field of complex numbers and let U be a smooth quasi-projective algebraic surface defined over k. We further consider a vector bundle E, i.e. a locally free $\mathcal{O}_X$-module of rank r, together with a flat connection

$$\nabla : E \to E \otimes_{\mathcal{O}_U} \Omega^1_{U|k}$$

on E. We remark that we do not impose any condition on the behavior of ∇ at infinity in some compactification of U.

Extending scalars from k to $\mathbb{C}$, we can switch to the analytic topology. The algebraic connection ∇ induces a flat analytic connection on the vector bundle E^{an} on the analytic manifold U^{an}. We denote by

$$\mathcal{E} := \ker(E^{\mathrm{an}} \xrightarrow{\nabla} E^{\mathrm{an}} \otimes \Omega^1_{U^{\mathrm{an}}}) \subset E^{\mathrm{an}}$$

the corresponding local system of horizontal sections. By Cauchy's theorem on linear differential equations for complex variables, the subsheaf $\mathcal{E}$ of E^{an} is a locally constant sheaf of $\mathbb{C}$-vector spaces of the same rank as E^{an}.

Let $F \subset \mathbb{C}$ denote another subfield of $\mathbb{C}$. Let us assume for a moment that the local system $\mathcal{E}$ on U^{an} comes equipped with a given F-structure. More precisely, in analogy to [23] we consider the category $W_{k,F}(U)$ of triples $\mathcal{M} = ((E, \nabla), \mathcal{E}_F, \rho)$ with:

i) a vector bundle E on U with rank r together with a flat connection $\nabla : E \to E \otimes_{\mathcal{O}_U} \Omega^1_U$,

ii) a local system $\mathcal{E}_F$ of F-vector spaces on the analytic manifold U^{an},

iii) a morphism $\rho : \mathcal{E}_F \to E^{an}$ of sheaves on U^{an} inducing an isomorphism $\mathcal{E}_F \otimes_F \mathbb{C} \xrightarrow{\sim} \ker(\nabla^{an})$ of local systems of $\mathbb{C}$-vector spaces on U^{an}.

A morphism between $((E, \nabla), \mathcal{E}_F, \rho)$ and $((E', \nabla'), \mathcal{E}_{F'}, \rho')$ is given by a morphism $E \to E'$ respecting the connections together with a morphism $\mathcal{E}_F \to \mathcal{E}_F{}'$ of F-local systems with the natural compatibility condition with respect to ρ and ρ'.

In [23], T. Saito and T. Terasoma consider a similar situation, for arbitrary dimension $\dim(X)$ with the restriction that ∇ has to be regular singular at infinity (i.e. along the boundary of a suitably chosen compactification X of U). The perfectness of the resulting period pairing then follows from P. Deligne's fundamental comparison theorem already mentioned in the introduction before.

2.2 Good formal structure (after C. Sabbah)

Applying the desingularization theorem of Hironaka, we may assume that U is embedded in a smooth projective variety X such that $D := X \smallsetminus U$ is a

divisor with normal crossings. Passing to the analytic topology, we consider the meromorphic connection

$$\nabla^{\mathrm{an}} : E^{\mathrm{an}}(*D) \to E^{\mathrm{an}}(*D) \otimes \Omega^1_{X^{\mathrm{an}}|\mathbb{C}} \ .$$

According to a conjecture of C. Sabbah, any such connection admits a good formal structure after a finite sequence of point blow-ups in the following sense:

First, we recall the definition of a regular singular connection. Consider the local situation at a point $x^0 \in D$. Choosing suitable coordinates with $x^0 = 0$, we have either $D = \{x_1 x_2 = 0\}$ or $D = \{x_1 = 0\}$. Suppose, that x^0 is a crossing-point of D. Then, an integrable $\mathcal{O}_X(*D)$-connection (R, ∇) is called **regular singular**, if there is a finite dimensional $\mathbb{C}$-vector space V together with two commuting endomorphisms $\delta_i : V \to V$, $i = 1, 2$, such that R is locally isomorphic to $\mathcal{O}_{X^{\mathrm{an}}}(*D) \otimes_{\mathbb{C}} V$ with

$$\nabla(f \otimes v) = df \otimes v + f \cdot d\log x_1 \otimes \delta_1(v) + f \cdot d\log x_2 \otimes \delta_2(v) \ .$$

In the case of a smooth point of D, one asks for the endomorphism δ_1 only and requires that $\nabla(f \otimes v) = df \otimes v + f \, d\log x_1 \otimes \delta_1(v)$. In particular, a regular singular line bundle is locally isomorphic to the connection on the trivial bundle $\mathcal{O}_{X^{\mathrm{an}}}(*D)$ given by

$$\nabla_\lambda 1 = \lambda_1 \, d\log x_1 + \lambda_2 \, d\log x_2$$

with $\lambda_i \in \mathbb{C}$ and $\lambda_2 = 0$ if x^0 is a smooth point of D. Such a connection will be denoted by

$$x^\lambda := (\mathcal{O}_{X^{\mathrm{an}}}(*D), \nabla_\lambda) \ . \tag{2.1}$$

We remark, that it follows directly from the definition that any regular singular connection is locally isomorphic to a successive extension of regular singular line bundles.

Now, in dimension one, the classical Levelt-Turrittin Theorem asserts that any meromorphic connection on a curve formally decomposes into the direct sum of irregular singular line bundles times regular singular connections (cp. [14]). Sabbah's Conjecture gives an analogous formal decomposition property for surfaces with the additional difficulty that one has to allow preceding blow-ups of points on D.

If α is a section in $\mathcal{O}_{X^{\mathrm{an}}}(*D)$, we denote by e^α the meromorphic connection on the trivial line bundle $\mathcal{O}_{X^{\mathrm{an}}}(*D)$ given by $\nabla 1 = d\alpha$ (such that the horizontal sections are multiples of e^α). Its isomorphism class only depends on the class $\alpha \in \mathcal{O}_{X^{\mathrm{an}}}(*D)/\mathcal{O}_{X^{\mathrm{an}}}$. An **elementary local model** is a meromorphic connection with poles along D isomorphic to a direct sum

$$\bigoplus_{\alpha \in A} e^\alpha \otimes R_\alpha \ ,$$

where A is a finite subset of $\mathcal{O}_{X^{\mathrm{an}}}(*D)/\mathcal{O}_{X^{\mathrm{an}}}$ and the R_α are regular singular connections. The model is called **good**, if

i) the various α are pairwise different in $\mathcal{O}_{X^{\mathrm{an}}}(*D)/\mathcal{O}_{X^{\mathrm{an}}}$. In particular there is at most one trivial $\alpha = 0$.

ii) For $\alpha \neq \beta \in A$, the divisor $(\alpha - \beta)$ has support on D and is negative, and the same holds for the divisor (α) for any non-trivial $\alpha \in A$.

Now, for any stratum Y of D in the natural stratification (i.e. either Y is a connected component of the smooth part of D or a crossing point), let $\mathcal{O}_{\widehat{X|Y}}$ be the completion of $\mathcal{O}_{X^{\mathrm{an}}}$ along Y. A connection $\mathcal{M}$ meromorphic along D is said to have a **good formal decomposition along** (D, Y) **at a point** $x_0 \in Y$, if locally at x_0 there exists a good elementary model $\mathcal{M}^{el}$ and an isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{O}_{\widehat{X|Y}} \cong \mathcal{M}^{el} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{O}_{\widehat{X|Y}} \qquad \text{locally at } x_0 .$$

The connection $\mathcal{M}$ is said to have a **good formal structure along** (D, Y) at x_0, if after a bicyclic ramification along the components of D the inverse image has a good formal decomposition. Finally, $\mathcal{M}$ has a **good formal structure along** D, if it has such a structure along (D, Y) for any stratum Y at any point. The precise formulation of Sabbah's Conjecture now reads as follows.

Conjecture 2.1 (C. Sabbah, [20] I.2.5.1) *There is a finite sequence of point blow-ups* $b : X' \to X$ *over* x_0, *such that the inverse image connection* $b^* \mathcal{M}$ *has a good formal structure in a neighborhood of* x_0.

C. Sabbah proves his conjecture for several classes of connections. In particular he achieves the following result.

Theorem 2.2 (C. Sabbah, [20] I.2.5.2) *The conjecture is true for any connection* $\mathcal{M}$ *with* $\operatorname{rank} \mathcal{M} \leq 5$.

In the following, we will always assume that Sabbah's Conjecture holds.

2.3 Definition of rapid decay homology

Recall that we start with an integrable algebraic connection $\nabla : E \to E \otimes \Omega_U^1$ over the smooth quasi-projective two-dimensional variety defined over a subfield $k \subset \mathbb{C}$. Assuming Sabbah's Conjecture, we can choose an embedding of U into a smooth projective variety X, such that $D := X \smallsetminus U$ is a divisor with normal crossings and the associated meromorphic connection on the complex analytic manifold X^{an} admits a good formal structure locally at any point of D.

Since D is a normal crossing divisor, we can consider the real oriented blow-up of the irreducible components of D. We denote by $\pi : \widetilde{X} \to X^{\mathrm{an}}$ the composition of all these. Locally π reads as

$$\pi : (\mathbb{R}^+ \times S^1)^2 \to X^{\mathrm{an}} \ , \ (r_1, \vartheta_1, r_2, \vartheta_2) \mapsto (r_1 e^{i\vartheta_1}, r_2 e^{i\vartheta_2})$$

in the local case $D = \{x_1 x_2 = 0\}$ and

$$\pi : (\mathbb{R}^+ \times S^1) \times Y \to X^{\mathrm{an}} \ , \ (r_1, \vartheta_1, x_2) \mapsto (r_1 e^{i\vartheta_1}, x_2)$$

in the case $D = \{x_1 = 0\}$ where Y is an open disc in $\mathbb{C}$.

Let $\widetilde{D} := \pi^{-1}(D)$. Then π defines a diffeomorphism between $\widetilde{X} \smallsetminus \widetilde{D}$ with U^{an} which we will identify in the following and write $\widetilde{\jmath} : U^{\mathrm{an}} \hookrightarrow \widetilde{X}$ for the inclusion.

We consider singular homology of the analytic manifolds involved. Since we want to study period integrals, we will a priori work with smooth singular chains (which give the same homology as the purely topological chains by smooth approximation). In this sense, let $S_p(\widetilde{X})$ denote the $\mathbb{Q}$-vector space of smooth

singular p-chains, i.e. the free vector space over all piecewise smooth maps $c : \Delta^p \to \widetilde{X}$ from the standard p-simplex to $\widetilde{X}$. The natural boundary operator $\partial : S_p(\widetilde{X}) \to S_{p-1}(\widetilde{X})$ defines a chain complex the homology of which gives the singular homology. If $A \subset \widetilde{X}$ denotes a closed subset, the relative chain complex is defined to be the quotient $S_\bullet(\widetilde{X}, A) := S_\bullet(\widetilde{X})/S_\bullet(A)$.

These notions sheafify in the following sense (cp. [26], [24]). Let $\mathcal{C}_{\widetilde{X}}^{-p}$ denote the sheaf associated to the presheaf $V \mapsto S_p(\widetilde{X}, \widetilde{X} \smallsetminus V)$ of $\mathbb{Q}$-vector spaces. The usual boundary operator makes $\mathcal{C}_{\widetilde{X}}^{-\bullet}$ into a complex. Note that we will use the standard sign convention, i.e. if we write ∂ for the topological boundary operator on chains, the differential of $\mathcal{C}_{\widetilde{X}}^{-\bullet}$ will be given by $(-1)^r \partial$ on $\mathcal{C}_{\widetilde{X}}^{-r}$.

Let $d := \dim_{\mathbb{C}}(X) = 2$ denote the complex dimension of X. Then the sheaf $\widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]$ is the dualizing sheaf on the compact real manifold $\widetilde{X}$ with boundary $\widetilde{X} \smallsetminus U^{\mathrm{an}}$ and $\mathcal{C}_{\widetilde{X}}^{-\bullet}$ is a resolution of this sheaf (cp. [26]).

Let $\mathcal{C}_{\widetilde{X}, \widetilde{D}}^{-\bullet}$ be the complex of relative chains, i.e. the sheaf associated to $V \mapsto S_\bullet(\widetilde{X}, (\widetilde{X} \smallsetminus V) \cup \widetilde{D})$. We are going to define the complex of rapid decay chains as a subquotient of the complex $\mathcal{C}_{\widetilde{X}, \widetilde{D}}^{-\bullet} \otimes_{\mathbb{Q}} \widetilde{\jmath}_* \mathcal{E}$ of the complex of sheaves of relative chains with values in the local system $\widetilde{\jmath}_* \mathcal{E}$. Let $V \subset \widetilde{X}$ be an open subset with $V \cap \widetilde{D} \neq 0$ and $c \otimes \varepsilon$ be a local section of this complex over V. Take c to be represented by a piecewise smooth map from the standard p-simplex Δ^p to $\widetilde{X}$. We introduce the notion of rapid decay chains which works in any dimension $\dim(X) = d$.

Definition 2.3 *We say that the local section $c \otimes \varepsilon$ is a **rapid decay p-chain** if the section $\varepsilon \in \Gamma(V \smallsetminus \widetilde{D}, \mathcal{E})$ is **rapidly decaying along c inside V** in the following sense: For any $y \in c(\Delta^p) \cap \widetilde{D} \cap V$, let $e = (e_1, \ldots, e_r)$ denote a local trivialization of $E(*D)$ and $z_1, \ldots, z_d$ local coordinates such that locally $D = \{z_1 \cdots z_k = 0\}$ and that $y = 0$. With respect to the trivialization e of E, ε becomes an r-tuple of analytic functions*

$$f_i := e_i^* \varepsilon : (c(\Delta^p) \smallsetminus \widetilde{D}) \cap V \to \mathbb{C} \ , \ (z_1, \ldots, z_d) \mapsto f_i(z) \ .$$

We require that these functions have rapid decay for the argument approaching $\widetilde{D}$, i.e. that for all $N \in \mathbb{N}^k$ there is a $C_N > 0$ such that

$$|f_i(z)| \leq C_N \cdot |z_1|^{N_1} \cdots |z_k|^{N_k}$$

for all $z \in (c(\Delta^p) \smallsetminus \widetilde{D}) \cap V$ with small $|z_1|, \ldots, |z_k|$.

*The subsheaf of $\mathcal{C}_{\widetilde{X}, \widetilde{D}}^{-p} \otimes_{\mathbb{Q}} \widetilde{\jmath}_* \mathcal{E}$ generated by all rapid decay p-chains will be denoted by $\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}, -p}(\nabla)$. Together with the usual boundary operator of chains, these give the **complex of rapid decay chains** $\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla) := (\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}, -\bullet}(\nabla), \partial)$.*

We stress that we do not impose any decay condition on pairs (c, ε) with $c(\Delta^p) \subset \widetilde{D}$; nevertheless we call those pairs rapidly decaying as well. Note also that the choice of the meromorphic trivialization does not effect the notion of rapid decay.

The homology we are going to study is defined as follows. Recall that we assume Sabbah's Conjecture to hold for the given connection.

Definition 2.4 *Let (E, ∇) be an integrable connection on the smooth quasi-projective algebraic surface U over $k \subset \mathbb{C}$. The **rapid decay homology** of*

(E, ∇) *is defined to be the hypercohomology*

$$H_k^{rd}(U^{\mathrm{an}}; E, \nabla) := \mathbb{H}^{-k}(\widetilde{X}, \mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla)) \ ,$$

where X^{an} is a good compactification of U^{an} with respect to ∇ in the sense of Sabbah's Conjecture and $\pi : \widetilde{X} \to X^{\mathrm{an}}$ denotes the oriented real blow-up of the normal crossing divisor $D := X^{\mathrm{an}} \smallsetminus U^{\mathrm{an}}$.

Later, we will prove that this definition is independent of the choice of the good compactification (Proposition 3.10), justifying the notation.

Note, that the usual barycentric subdivision operator obviously can be defined on the rapid decay complex as well. Hence $\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla)$ is a homotopically fine complex of sheaves (cp. [24], p. 87) and its hypercohomology can be computed as the cohomology of the complex of global sections.

2.4   The pairing and statement of the main result

We are now going to define the pairing between de Rham cohomology of ∇ and the rapid decay homology of the dual connection $\nabla^\vee$ on the dual bundle $E^\vee$, characterized by the equality $d\langle e, \varphi \rangle = \langle \nabla e, \varphi \rangle + \langle e, \nabla^\vee \varphi \rangle$ for local sections e of E and φ of $E^\vee$. Let $\mathcal{E}^\vee \subset (E^{\mathrm{an}})^\vee$ denote the corresponding local system.

We will work on the real oriented blow-up $\pi : \widetilde{X} \to X^{\mathrm{an}}$ of D in a good compactification (X, D) of U with respect to ∇. For later purposes, we want to describe the algebraic de Rham cohomology of the connection in terms of naturally defined sheaves on $\widetilde{X}$. To this end, let $\mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D}$ denote the sheaf of functions on $\widetilde{X}$ which are holomorphic on $U^{\mathrm{an}} \subset \widetilde{X}$ and of moderate growth along $\pi^{-1}(D)$. Then $\mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D}$ is flat over $\pi^{-1}(\mathcal{O}_X)$ and if we define the **moderate de Rham complex** of (E, ∇) to be

$$\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) := \mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D} \otimes_{\pi^{-1}(\mathcal{O}_X)} \pi^{-1}(\mathrm{DR}_{X^{\mathrm{an}}}(\nabla)) \ ,$$

this complex computes the meromorphic de Rham cohomology of ∇ on X^{an} and hence the algebraic de Rham cohomology of ∇ on U (cp. with [22], Lemma 1.3 or [20], Corollaire 1.1.8):

$$H_{dR}^k(U; E, \nabla) \cong \mathbb{H}^k(\widetilde{X}, \mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla)) \ . \tag{2.2}$$

We want to define the pairing by means of integration of differential forms over smooth topological chains as a pairing of complexes of sheaves. To this end, we are going to define the sheaf of distributions with rapid decay as follows. First, let $\mathfrak{Db}_{\widetilde{X}}^{-s}$ be the usual sheaf of distributions of degree s on $\widetilde{X}$, i.e. the local sections for an open $V \subset \widetilde{X}$ are the continuous linear functionals

$$\mathfrak{Db}_{\widetilde{X}}^{-s}(V) := \mathrm{Hom}_{\mathrm{cont}}(\Gamma_c(V, \Omega_{\widetilde{X}}^{\infty, s}), \mathbb{C})$$

on the space $\Omega_{\widetilde{X}}^{\infty, s}$ of C^∞ differential forms on $\widetilde{X}$ of degree s with compact support in $\widetilde{X}$. The topology on $\Omega_{\widetilde{X}}^{\infty, s}$ is defined to be the limit topology where we write $\Omega_{\widetilde{X}}^{\infty, s}$ as the direct limit of all differential forms with support in some fixed compact $K \subset V$ with their usual Fréchet-topology, the limit taken over all these K (cp. [10], chapter II).

We now choose local coordinates x_1, x_2 in X such that locally $D = \{x = 0\}$, where $x = x_1$ in case of a smooth point of D and $x = x_1 x_2$ in case of a crossing point of D. We call a distribution $\varphi \in \mathfrak{Db}_{\widetilde{X}}^{-s}(V)$ a **rapid decay distribution** if for any compact $K \subset V$ and any element $N \in \mathbb{N}$ or $\mathbb{N}^2$ (depending on the local type of D) there exist $m \in \mathbb{N}$ and $C_{K,N} > 0$ such that for any test form η with compact support in K the estimate

$$|\varphi(\eta)| \leq C_{K,N} \sum_i \sup_{|\alpha| \leq m} \sup_K \{|x|^N |\partial^\alpha f_i|\} \tag{2.3}$$

holds, where α runs over all multi-indices of degree less than or equal to m and ∂^α denotes the α-fold partial derivative of the coefficient functions f_i of η in the chosen coordinates.

Let $\mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-s}$ denote the resulting sheaf of rapid decay distributions on $\widetilde{X}$. Varying s, we obtain a complex of sheaves where once again the sign convention applies, i.e. the differential reads as

$$\mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-s} \longrightarrow \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-s+1} \ , \ \ \varphi \mapsto \left(\eta \mapsto (-1)^s \varphi(d\eta)\right) \ .$$

Standard arguments show that the resulting complex $\mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet}$ is a fine resolution of the extension $\widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]$ of the constant sheaf by 0 along $\widetilde{\jmath} : U^{\mathrm{an}} \hookrightarrow \widetilde{X}$ shifted by the real dimension $2d$.

Now, consider a local section ω of $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}$ in degree s, as well as a rapid decay chain $c \otimes \varepsilon \in \Gamma(V, \mathcal{C}_{\widetilde{X}}^{\mathrm{rd},-r}(\nabla^\vee))$ with respect to the dual connection, c being a smooth topological r-simplex c in $\widetilde{X}$. By definition, ε is rapidly decaying along c and ω has at most moderate growth along $\widetilde{D}$, hence the local section $\langle \varepsilon, \omega \rangle \in \Gamma(V \smallsetminus \widetilde{D}, \Omega_{U^{\mathrm{an}}}^{\infty,s})$ remains rapidly decaying along c and the same holds for

$$\eta \wedge \langle \varepsilon, \omega \rangle \in \Gamma(V \cap U^{\mathrm{an}}, \Omega_{U^{\mathrm{an}}}^{\infty,r})$$

for any test form $\eta \in \Gamma_c(V, \Omega_{\widetilde{X}}^{\infty,p})$ with $p = r - s$. Therefore, the integral

$$\int_c \eta \wedge \langle \varepsilon, \omega \rangle \tag{2.4}$$

obviously converges. The rapid decay condition of ε also ensures that the distribution defined by (2.4) satisfies the estimate (2.3), since the coordinate functions of ε along the curve of integration c can be bounded from above by any power of $|x|$ by definition. For support reasons, the integral is well-defined also, i.e. independent of the choice of c in its equivalence class modulo chains in $\widetilde{X} \smallsetminus V$.

The above considerations define a morphism of sheaves

$$\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D,s}(\nabla) \otimes \mathcal{C}_{\widetilde{X}}^{\mathrm{rd},-r}(\nabla^\vee) \to \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},s-r} \ , \tag{2.5}$$

which indeed induces a morphism of complexes. More precisely, assuming that $c^{-1}(\widetilde{D}) \subset \partial \Delta^p$ (which is no restriction due to subdivision) if c_t denotes the chain one obtains by cutting off a small tubular neighborhood with radius t around the boundary $\partial \Delta^p$ from the given chain c. Then, for $c \otimes \varepsilon$ as before and ω a form with moderate growth of degree $(s-1)$ and η as before, we have the

'limit Stokes formula'

$$
\int_c \eta \wedge \langle \varepsilon, \nabla \omega \rangle + (-1)^{r+s} \int_{\partial c} \eta \wedge \langle \varepsilon, \omega \rangle = \lim_{t \to 0} \left(\int_{c_t} \eta \wedge \langle \varepsilon, \nabla \omega \rangle + (-1)^{r+s} \int_{\partial c_t} \eta \wedge \langle \varepsilon, \omega \rangle \right) =
$$

$$
\stackrel{\text{Stokes}}{=} (-1)^{r-s} \lim_{t \to 0} \int_{c_t} d\eta \wedge \langle \varepsilon, \omega \rangle = (-1)^{r-s} \int_c d\eta \wedge \langle \varepsilon, \omega \rangle \quad (2.6)
$$

where one has to keep in mind that by the given growth/decay conditions the integrals over the faces of ∂c_t 'converging' against the faces of ∂c contained in $\widetilde{D}$ vanish. Note also, that by the definition of the dual connection $d\langle \varepsilon, \omega \rangle = \langle \varepsilon, \nabla \omega \rangle$, since $\nabla^\vee \varepsilon = 0$, and that the sign conventions in defining the total complex of the distributions, the rapid decay complex and in the rule for differentiating a wedge product of forms give the appropriate sign in (2.6).

Equation (2.6), however, directly shows that (2.5) is compatible with the differentials of the complexes involved, i.e. we have obtained a pairing of complexes

$$
\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) \otimes_{\mathbb{C}} \mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^\vee) \to \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet} .
$$

Since $\mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet}$ is a resolution of $\widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]$ it follows that $\mathbb{H}^0(\widetilde{X}, \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet}) \cong \mathbb{C}$, keeping in mind that $\widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]$ is the dualizing sheaf for the compact real manifold $\widetilde{X}$ with boundary.

Taking hypercohomology of the above pairing in degree 0 thus induces a pairing

$$
H_{dR}^p(U; E, \nabla) \otimes_{\mathbb{C}} H_p^{rd}(U^{\mathrm{an}}; (E^\vee, \nabla^\vee)) \longrightarrow \mathbb{C} \quad\quad (2.7)
$$

will be called the **period pairing** of the algebraic connection (E, ∇). Our main result is the following

Theorem 2.5 (Global duality of the period pairing) *Let (E, ∇) be a flat connection on the smooth quasi-projective algebraic surface U over the subfield $k \subset \mathbb{C}$. Assume that Sabbah's Conjecture holds for (E, ∇). Then the pairing (2.7)*

$$
\left(H_{dR}^p(U; E, \nabla) \otimes_k \mathbb{C} \right) \otimes_{\mathbb{C}} H_p^{rd}(U^{\mathrm{an}}; E^\vee, \nabla^\vee) \longrightarrow \mathbb{C}
$$

is a perfect pairing of finite-dimensional $\mathbb{C}$-vector spaces.

Remark 2.6 i) For $\dim(X) = 1$ the analogous statement was proven by S. Bloch and H. Esnault in [3]. Note that in the one-dimensional situation any compactification will suffice for the definition of rapid decay homology since the given integrable connection always admits a formal structure by the classical Levelt-Turrittin Theorem and this formal structure is automatically good. In the two-dimensional situation, the suitable choice of good compactification plays an important role for the definition of the rapid decay homology. We will come back to this later (see section 3.6).
ii) We used the de Rham complex with moderate growth on $\widetilde{X}$ in order to define the period pairing referring to (2.2). However, one can obviously replace this complex by the pull-back $\pi^{-1}(\mathrm{DR}_{X^{\mathrm{an}}}(E(*D), \nabla))$ in the definition to obtain the same period pairing.
iii) In case U is affine, the period pairing admits a direct description in terms of period integrals as follows. For affine U, the algebraic de Rham cohomology

can be computed by the cohomology of global sections in U, i.e.

$$H^p_{dR}(U; E, \nabla) = H^p(\ldots \to \Gamma_U(E \otimes \Omega^q_{U|k}) \xrightarrow{\nabla} \Gamma_U(E \otimes \Omega^{q+1}_{U|k}) \to \ldots) \ .$$

Using the barycentric subdivision operator, the rapid decay complex is easily seen to be homotopically fine and thus the rapid decay homology can be computed taking global sections of the rapid decay complex also. The period pairing then obtains the following shape

$$H^p_{dR}(E, \nabla) \otimes_{\mathbb{C}} H^{rd}_p(X, \nabla^{\vee}) \to \mathbb{C} \ , \ ([\omega], [c \otimes \varepsilon]) \mapsto \int_c \langle \varepsilon, \omega \rangle$$

for a flat algebraic p-form ω on U and a rapid decay cycle $c \otimes \varepsilon$ on $\widetilde{X}$.

iv) Due to Sabbah's Theorem 2.2, the conclusion of the above theorem holds unconditionally for rank $E \leq 5$.

2.5 The determinant of periods

Consider an element $((E, \nabla), \mathcal{E}_F, \rho) \in W_{k,F}(U)$ for given subfields $k, F \subset \mathbb{C}$. Since (E, ∇) and U are defined over k, the de Rham cohomology $H_{dR}(U, \nabla)$ is a k-vector space by definition.

The F-structure $\mathcal{E}_F$ on the local system of horizontal sections given by ρ, which obviously also defines an F-structure $\mathcal{E}^{\vee}_F$ on the local system $\mathcal{E}^{\vee}$ of the dual connection, induces a canonical F-structure on the complex of rapid decay chains on a chosen good compactification X. More precisely, we define

$$\mathcal{C}^{\mathrm{rd}, -p}_{\widetilde{X}}(\nabla^{\vee})_F \subset \mathcal{C}^{-p}_{\widetilde{X}, \widetilde{D}} \otimes_{\mathbb{Q}} \widetilde{\jmath}_* \mathcal{E}^{\vee}_F$$

to be the sheaf of subvector spaces over F generated by all rapidly decaying chains in $\mathcal{C}^{-p}_{\widetilde{X}, \widetilde{D}} \otimes_{\mathbb{Q}} \widetilde{\jmath}_* \mathcal{E}^{\vee}_F$, the property of rapid decay literally being the same as in definition 2.3. Its hypercohomology gives the F-vector space $H^{rd}_p(U^{\mathrm{an}}, E, \nabla)_F$ such that

$$H^{rd}_p(U^{\mathrm{an}}, E, \nabla)_F \otimes_F \mathbb{C} \xrightarrow{\cong} H^{rd}_p(U^{\mathrm{an}}, E, \nabla) \ ,$$

the isomorphism induced by ρ. In summary, we can define the determinant of the period pairing as follows:

Definition 2.7 *For* $((E, \nabla), \mathcal{E}_F, \rho) \in W_{k,F}(U)$, *we define its* **period determinant** *to be the element*

$$\det((E, \nabla), \mathcal{E}_F, \rho) := \prod_{p \geq 0} \det(\langle \gamma^{(p)}_j, \omega^{(p)}_i \rangle)^{(-1)^p}_{i,j} \in k^{\times} \backslash \mathbb{C}^{\times} / F^{\times} \ ,$$

where $\omega^{(p)}_i$ *denotes a basis of* $H^p_{dR}(U, E, \nabla)$ *over* k *and* $\gamma^{(p)}_j$ *a basis of the F-vector space* $H^{rd}_k(U^{\mathrm{an}}, E^{\vee}, \nabla^{\vee})_F$.

Obviously, the determinant does not depend on the choices made. For regular singular connections (E, ∇) this definition coincides with the one in [23]. In case U is affine, the matrices involved carry actual period integrals as entries (see Remark 2.6 ii).

3   Local duality

We are going to study a local duality pairing on the real blow-up $\widetilde{X}$ of a good compactification X^{an} of U^{an} with respect to the given flat connection (E, ∇) on U. Our main result, Theorem 2.5, will follow from the local duality by standard globalization and the comparison of the period pairing with the local pairing.

3.1   Sheaves of functions on the real oriented blow-up

We identify the complex of rapid decay sheaves with the asymptotically flat de Rham complex on the real oriented blow-up $\widetilde{X}$ in a given good compactification X of U which we are going to define in this section. We first recall the definition of the following sheaves of functions on the real oriented blow-up. We can assume a more general situation, namely any compactification X such that $D := X \smallsetminus U$ is normal crossing. Let $\widetilde{X}$ denote the real oriented blow-up of D. For the proofs of the various properties stated in the following we refer to [20], II.1.

i) The logarithmic differential operators as well as their conjugates act on the sheaf $C^\infty_{\widetilde{X}}$ of C^∞ functions on $\widetilde{X}$ and one defines

$$\mathcal{A}_{\widetilde{X}} := \ker \overline{x}_1 \cdot \overline{\partial}_{x_1} \cap \ker \overline{x}_2 \cdot \overline{\partial}_{x_2} \subset C^\infty_{\widetilde{X}}$$

in the case $D = \{x_1 x_2 = 0\}$ and $\mathcal{A}_{\widetilde{X}} = \ker \overline{x}_1 \cdot \overline{\partial}_{x_1} \cap \ker \overline{\partial}_{x_2}$ in the case $D = \{x_1 = 0\}$. The elements of $\mathcal{A}_{\widetilde{X}}$ are holomorphic functions on $\widetilde{X}$ which admit an asymptotic development in the spirit of Majima (cp. Proposition B.2.1 in [20] and [13]).

ii) Let $\mathcal{P}^{\leq D}_{\widetilde{X}}$ denote the sheaf of C^∞-functions on $\widetilde{X}$ which are flat on $\pi^{-1}(D)$, i.e. all of whose derivations vanish on $\pi^{-1}(D)$ (cp. [15]) and let

$$\mathcal{A}^{\leq D}_{\widetilde{X}} := \mathcal{A}_{\widetilde{X}} \cap \mathcal{P}^{\leq D}_{\widetilde{X}} \ .$$

The elements of $\mathcal{A}^{\leq D}_{\widetilde{X}}$ are the holomorphic functions with vanishing asymptotic development (cp. Proposition II.1.1.11 in [20]), i.e. which are rapidly decaying on any compactum in $\widetilde{X}$. More precisely, if u is a local section of $\mathcal{A}^{\leq D}_{\widetilde{X}}$ defined on some open subset $\Omega \subset \widetilde{X}$, then for any compact $K \subset \Omega$ and any $N \in \mathbb{N}^2$, the function u satisfies an estimate of the form

$$|u(x)| \leq C_{K,N} \cdot |x_1|^{N_1} |x_2|^{N_2} \text{ for all } x \in K \smallsetminus \pi^{-1}(D) \ . \qquad (3.1)$$

iii) Let $\mathcal{A}_{\widehat{\widetilde{X}|D}}$ denote the formal completion of $\mathcal{A}_{\widetilde{X}}$ along $\pi^{-1}(D)$ and T_D the natural morphism $\mathcal{A}_{\widetilde{X}} \xrightarrow{T_D} \mathcal{A}_{\widehat{\widetilde{X}|D}} := \varprojlim_k \mathcal{A}_{\widetilde{X}}/\mathcal{I}^k_D \mathcal{A}_{\widetilde{X}}$.

According to Majima, the sequence $0 \to \mathcal{A}^{\leq D}_{\widetilde{X}} \to \mathcal{A}_{\widetilde{X}} \xrightarrow{T_D} \mathcal{A}_{\widehat{\widetilde{X}|D}} \longrightarrow 0$ is exact, generalizing the analogous theorem of Borel-Ritt in dimension one.

For $\dim(X) = 2$, both sheaves $\mathcal{A}_{\widetilde{X}}$ and $\mathcal{A}^{\leq D}_{\widetilde{X}}$ are flat as $\pi^{-1}(\mathcal{O}_X)$-algebras. Additionally, we will make use of the fact that

$$\mathcal{A}^{\leq D}_{\widetilde{X}} \hookrightarrow \left(\mathcal{P}^{\leq D}_{\widetilde{X}} \otimes_{\pi^{-1} C^\infty_{\widetilde{X}}} \pi^{-1} \Omega^{\infty,(0,\bullet)}_{X^{\mathrm{an}}}, \overline{\partial} \right) \qquad (3.2)$$

is a resolution of $\mathcal{A}_{\widetilde{X}}^{\leq D}$, where $\Omega_{X^{\mathrm{an}}}^{\infty,(p,q)}$ denotes the sheaf of C^∞-forms of degree (p,q) on X (cp. Lemme II.1.1.18 in [20]).

Recall that $\mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D}$ denotes the sheaf of functions on $\widetilde{X}$ which are holomorphic on U^{an} and of moderate growth along $\widetilde{D}$. If $\mathcal{P}^{\mathrm{mod}D}$ denotes the sheaf of C^∞-functions on $\widetilde{X}$ with moderate growth at $\pi^{-1}(D)$, the inclusion defines a resolution

$$\mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D} \hookrightarrow \left(\mathcal{P}^{\mathrm{mod}D} \otimes_{\pi^{-1}C_X^\infty} \pi^{-1}\Omega_{X^{\mathrm{an}}}^{\infty,(0,\cdot)}, \overline{\partial} \right) . \tag{3.3}$$

Both resolutions being constructed with C^∞-functions consist of fine sheaves.

3.2 Formal classification and asymptotic developments

Given a good formal structure of (E,∇) on X, the resulting formal decomposition can be lifted to an asymptotic one in the following sense. Consider the local situation at a crossing-point and assume X is a small bi-disc around the crossing point 0 with coordinates x_1, x_2 such that locally $D = \{x_1 x_2 = 0\}$. The situation at a smooth point is similar with a few obvious changes.

For an $\mathcal{O}_X(*D)$-connection $\mathcal{M}$, let $\mathcal{M}_{\widetilde{X}} := \mathcal{A}_{\widetilde{X}} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}$. In the same situation as above, one says that $\mathcal{M}$ has a **good $\mathcal{A}$-decomposition along** (D,Y) **at** x_0 if there exists a good elementary model $\mathcal{M}^{el}$ in a neighborhood of x_0 and for all $\vartheta \in \pi^{-1}(x_0)$ an isomorphism of the stalks

$$\mathcal{M}_{\widetilde{X},\vartheta} \cong \mathcal{M}^{el}_{\widetilde{X},\vartheta} ,$$

such that the induced formal isomorphism $\widehat{\mathcal{M}} \cong \widehat{\mathcal{M}^{el}}$ is independent of ϑ, where one has to keep in mind that $\mathcal{A}_{\widehat{\widetilde{X}|D}}|_{\pi^{-1}D} = \pi^{-1}\mathcal{O}_{\widehat{X|D}}|_{\pi^{-1}D}$. The notions of a **good $\mathcal{A}$-structure** is defined in an analogous manner as above. One has the following result:

Theorem 3.1 (C. Sabbah, [20] II.2.1.1) *If $\mathcal{M}$ has a good formal decomposition along (D,Y) at x_0, then it lifts to a good $\mathcal{A}$-decomposition at x_0.*

We remark, that it is essential in the proof of this theorem that the given formal decomposition is good. Hence, even if (E,∇) admits a formal decomposition at x_0 without any preceding blow-up, it may be necessary to insert point blow-ups in order to arrive at a good formal decomposition and then be able to lift it to an asymptotic decomposition.

3.3 The asymptotically flat de Rham complex

Definition 3.2 *Let $\widetilde{X}$ be the real oriented blow-up of a good compactification X^{an} of U^{an} with respect to the given flat connection (E,∇) on U. The complex*

$$\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^\vee) := \mathcal{A}_{\widetilde{X}}^{\leq D} \otimes_{\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi^{-1}(\mathrm{DR}_{X^{\mathrm{an}}}(\nabla^\vee)) \in D^b(\mathbb{C}_{\widetilde{X}})$$

*will be called the **asymptotically flat de Rham complex** of $\nabla^\vee$.*

Recall that the moderate de Rham $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla)$ complex was defined in a similar way to be

$$\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) := \mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D} \otimes_{\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi^{-1}(\mathrm{DR}_{X^{\mathrm{an}}}(\nabla)) .$$

Now, if (X,D) is good with respect to (E,∇), these complexes simplify as follows:

Proposition 3.3 *If* (E, ∇) *has a good formal structure along* D, *both complexes* $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}\,D}(E, \nabla)$ *and* $\mathrm{DR}_{\widetilde{X}}^{\leq D}(E^\vee, \nabla^\vee)$ *have cohomology in degree* 0 *only, i.e. the inclusions*

$$\mathcal{S}^{\mathrm{mod}\,D} := \mathcal{H}^0(\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}\,D}(E, \nabla)) \simeq \mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}\,D}(E, \nabla)$$

$$^\vee\mathcal{S}^{<D} := \mathcal{H}^0(\mathrm{DR}_{\widetilde{X}}^{\leq D}(E^\vee, \nabla^\vee)) \simeq \mathrm{DR}_{\widetilde{X}}^{\leq D}(E^\vee, \nabla^\vee)$$

are quasi-isomorphisms.

Proof: In [21], §7, the assertion about $\mathrm{DR}_{\widetilde{X}}^{\leq D}$ is stated assuming a *very good* formal structure (see the Remark after Théorème 7.2 and 7.3 in [21]). It follows from Théorème II.2.1.2 in [20], however, that the same proof holds if one has a good formal structure only. The proof relies on an existence theorem for flat solutions to a certain type of partial differential equations whose entries are rapidly decaying as well. In the case of $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}\,D}$ we will analogously reduce to a theorem on the existence of solutions with moderate growth, assuming the entries in the differential equations have moderate growth. This existence theorem will then be proven in the appendix.

Since (E, ∇) is assumed to have a good formal structure, there exists a bicyclic ramification $\rho : Y \to X$ such that locally on $\widetilde{Y}$ the pull-back connection $\rho^{-1}(\nabla)$ is isomorphic to its elementary model. Let $\pi_X : \widetilde{X} \to X^{\mathrm{an}}$ and $\pi_Y : \widetilde{Y} \to Y^{\mathrm{an}}$ denote the oriented real blow-up of $S := \rho^{-1}(D)$ and D respectively. Lifting ρ to $\widetilde{\rho} : \widetilde{Y} \to \widetilde{X}$, the projection formula yields

$$\boldsymbol{R}\widetilde{\rho}_* \mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}\,S}(\rho^{-1}\nabla) = \boldsymbol{R}\widetilde{\rho}_* \mathcal{A}_{\widetilde{Y}}^{\mathrm{mod}\,S} \overset{\mathbb{L}}{\otimes}_{\pi^{-1}(\mathcal{O}_X)} \pi_X^{-1} \mathrm{DR}_{X^{\mathrm{an}}}(\nabla) . \qquad (3.4)$$

Now, $\widetilde{\rho}$ being a finite map and since obviously $\boldsymbol{R}\widetilde{\rho}_* \mathcal{A}_{\widetilde{Y}}^{\mathrm{mod}\,S} = \mathcal{A}_{\widetilde{X}}^{\mathrm{mod}\,D}$ (using the resolution (3.3)), it follows that it suffices to prove the claim on $\widetilde{Y}$. Hence, we can assume that (E, ∇) itself decomposes locally on $\widetilde{X}$ as

$$\pi^{-1}(\nabla) \cong_{\mathrm{loc.}} \bigoplus_{\alpha \in A} e^\alpha \otimes R_\alpha .$$

Our claim is local in nature, so that we can assume that $\pi^{-1}(\nabla) = e^\alpha \otimes R_\alpha$.

We consider the local situation on $\widetilde{X}$ above a point $x^0 \in D$. Assume first that $x^0 = 0$ is a crossing point, i.e. locally on X^{an} we have the situation $D = \{x_1 x_2 = 0\}$. Let $\vartheta^0 \in \pi^{-1}(0)$ be the direction under consideration.

Since every regular singular connection is a successive extension of regular singular line bundles, we can further reduce to the case $R_\alpha = x^\lambda$ with a $\lambda \in \mathbb{C}^2$. Then

$$\mathrm{DR}_{\widetilde{X}, \vartheta^0}^{\mathrm{mod}\,D}(\nabla) \cong \left(0 \to \mathcal{A}_{\widetilde{X}, \vartheta^0}^{\mathrm{mod}\,D} \overset{\binom{P_1}{P_2}}{\longrightarrow} (\mathcal{A}_{\widetilde{X}, \vartheta^0}^{\mathrm{mod}\,D})^2 \overset{(P_2, -P_1)}{\longrightarrow} \mathcal{A}_{\widetilde{X}, \vartheta^0}^{\mathrm{mod}\,D} \to 0 \right) , \qquad (3.5)$$

where

$$P_i u := x_i \frac{\partial}{\partial x_i} u + x_i \frac{\partial \alpha}{\partial x_i} \cdot u + \lambda_i u .$$

The vanishing of $\mathcal{H}^2$ amounts to solving the partial differential equation

$$(\Sigma_1) : \qquad x_1 \frac{\partial}{\partial x_1} u = -x_1 \frac{\partial \alpha}{\partial x_1} \cdot u - \lambda_1 u + \rho_1 \qquad (3.6)$$

for any given $\rho_1 \in \mathcal{A}_{\widetilde{X},\vartheta^0}^{\mathrm{mod}D}$, as then $(0, u)$ gives a preimage of ρ_1. The vanishing of $\mathcal{H}^1$ additionally asks for a solution u of (3.6) which also solves

$$(\Sigma_2): \qquad x_2 \frac{\partial}{\partial x_2} u = -x_2 \frac{\partial \alpha}{\partial x_2} \cdot u - \lambda_2 u + \rho_2 ,$$

where ρ_2 is another element in $\mathcal{A}_{\widetilde{X},\vartheta^0}^{\mathrm{mod}D}$ such that the system is **integrable** in the sense that

$$P_2 \rho_1 = P_1 \rho_2 \qquad (3.7)$$

holds. By Theorem A.1, which is proven in the appendix, such solutions $u \in \mathcal{A}_{\widetilde{X},\vartheta^0}^{\mathrm{mod}D}$ always exist.

In the local case $D = \{x_1 = 0\}$ at $x^0 = 0$, we similarly may assume that $R_\alpha = x_1^{\lambda_1}$ with $\lambda_1 \in \mathbb{C}$ and then (3.5) remains valid with the same definition for P_1 and now

$$P_2' u := \frac{\partial u}{\partial x_2} + \frac{\partial \alpha}{\partial x_2} \cdot u .$$

The system of partial differential equations to be solved in $\mathcal{A}_{\widetilde{X},\vartheta^0}^{\mathrm{mod}D}$ is therefore given by the same equation (Σ_1) as above and

$$(\Sigma'): \qquad \frac{\partial}{\partial x_2} u = -\frac{\partial \alpha}{\partial x_2} \cdot u + \rho_2 ,$$

where the integrability condition now reads as $P_1 \rho_2 = P_2' \rho_1$. Again, Theorem A.1 proves the existence of such a solution.

$\square$

3.4 Local duality for good compactifications

Again, let (E, ∇) be an integrable connection on U and (X, D) a compactification of U with normal crossing divisor D. Let $\pi : \widetilde{X} \to X^{\mathrm{an}}$ be the oriented real blow-up of X along D. Multiplying an element of $\mathcal{A}_{\widetilde{X}}^{\leq D}$, i.e. a function on $\widetilde{X}$ with rapid decay at $\pi^{-1}(D)$, with a function with moderate growth, results in a rapidly decaying function again, i.e. multiplication defines a map $\mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D} \otimes_{\mathbb{C}} \mathcal{A}_{\widetilde{X}}^{\leq D} \to \mathcal{A}_{\widetilde{X}}^{\leq D}$. As a consequence, given a local section

$$\omega \in \mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D, p}(E, \nabla) = \mathcal{A}_{\widetilde{X}}^{\mathrm{mod}D} \otimes_{\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi^{-1} \Omega_{X^{\mathrm{an}}}^p \otimes_{\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi^{-1} E ,$$

i.e. a p-form with moderate growth, and a local section

$$\eta \in \mathrm{DR}_{\widetilde{X}}^{\leq D, q}(E^\vee, \nabla^\vee) = \mathcal{A}_{\widetilde{X}}^{\leq D} \otimes_{\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi^{-1} \Omega_{X^{\mathrm{an}}}^q \otimes_{\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi^{-1} E^\vee ,$$

i.e. a q-form with rapid decay, the wedge product $\omega \wedge \eta$ is rapidly decaying as well. This leads to the following definition.

Definition 3.4 *The natural pairing*

$$\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(E, \nabla) \otimes_{\mathbb{C}} \mathrm{DR}_{\widetilde{X}}^{\leq D}(E^\vee, \nabla^\vee) \to \mathrm{DR}_{\widetilde{X}}^{\leq D}(\mathcal{O}_X, d) \qquad (3.8)$$

induced by the wedge product is called the **local duality pairing** *of* (E, ∇).

Applying Proposition 3.3 to the trivial connection $(\mathcal{O}_X, d)$ (for which any compactification is good) gives

$$\widetilde{\jmath}_! \mathbb{C}_{X \smallsetminus D} = \mathcal{H}^0 \mathrm{DR}_{\widetilde{X}}^{\leq D}(\mathcal{O}_X, d) \simeq \mathrm{DR}_{\widetilde{X}}^{\leq D}(\mathcal{O}_X, d) \ ,$$

with $\widetilde{\jmath} : U^{\mathrm{an}} \hookrightarrow \widetilde{X}$ denoting the inclusion.

Now assume that (X, D) is a good compactification with respect to (E, ∇). Again by Proposition 3.3, the local duality pairing reduces to the pairing

$$\mathcal{H}^0 \mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(E, \nabla) \otimes_{\mathbb{C}} \mathcal{H}^0 \mathrm{DR}_{\widetilde{X}}^{\leq D}(E^{\vee}, \nabla^{\vee}) \longrightarrow \widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}} \ . \qquad (3.9)$$

In this situation, the local duality statement reads as follows:

Theorem 3.5 (Local Duality for good formal structure) *If the connection (E, ∇) admits a good formal structure on $X \smallsetminus D$, the pairing* (3.9)

$$\mathcal{S}^{\mathrm{mod}D} \otimes_{\mathbb{C}} {}^{\vee}\mathcal{S}^{<D} \to \widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}}$$

is a perfect duality, i.e. the induced morphisms

$$\mathcal{S}^{\mathrm{mod}D} \to \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}({}^{\vee}\mathcal{S}^{<D}, \widetilde{\jmath}_! \mathbb{C}_{X \smallsetminus D}) \ and \ {}^{\vee}\mathcal{S}^{<D} \to \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}(\mathcal{S}^{\mathrm{mod}D}, \widetilde{\jmath}_! \mathbb{C}_{X \smallsetminus D})$$

are isomorphisms in $D^b(\mathbb{C}_{\widetilde{X}})$.

Proof: Being a local problem on $\widetilde{X}$, we can assume that after a bicyclic ramification $\rho : Y \to X$ at the point $x_0 \in D$ we consider, the lift of the pull-back $\rho^{-1}\nabla$ to the oriented real-blow up $\widetilde{Y}$ of Y is locally isomorphic to its elementary model. As in the proof of Proposition 3.3, the projection formula applied to either $\mathrm{DR}_{\widetilde{X}}^{\leq D}$ or $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}$ again gives (see (3.4)) $\boldsymbol{R}\widetilde{\rho}_* \mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}S}(\rho^{-1}\nabla) \cong \mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla)$ and $\boldsymbol{R}\widetilde{\rho}_* \mathrm{DR}_{\widetilde{X}}^{\leq D}(\rho^{-1}\nabla^{\vee}) \cong \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee})$, both isomorphism obviously compatible with the duality morphism given by the wedge product. With the notation $S := \rho^{-1}(D)$ and $i : Y \smallsetminus S \hookrightarrow \widetilde{Y}$ and ${}^{\vee}\mathcal{S}_{\widetilde{Y}}^{\leq S} := \mathcal{H}^0 \mathrm{DR}_{\widetilde{Y}}^{\leq S}(\rho^{-1}\nabla^{\vee}))$, we have

$$\boldsymbol{R}\mathcal{H}om\big(\boldsymbol{R}\widetilde{\rho}_* {}^{\vee}\mathcal{S}_{\widetilde{Y}}^{\leq S}, \widetilde{\imath}_! \mathbb{C}_{Y \smallsetminus S}\big) \cong \boldsymbol{R}\widetilde{\rho}_* \boldsymbol{R}\mathcal{H}om\big({}^{\vee}\mathcal{S}^{<D}, \widetilde{\jmath}_! \mathbb{C}_{X \smallsetminus D}\big)$$

by the local Poincaré-Verdier duality, where one has to keep in mind that $\widetilde{\jmath}_! \mathbb{C}$ is up to shift by the real dimension of $\widetilde{X}$ the dualizing sheaf on the real manifold $\widetilde{X}$ with boundary $\pi^{-1}(D)$ (including corners). Therefore it remains to prove the claim on $\widetilde{Y}$, i.e. we can assume that ∇ itself decomposes locally on $\widetilde{X}$. As in Proposition 3.3, we can further reduce to the case $\pi^{-1}(\nabla) = e^{\alpha} \otimes x^{\lambda}$ for some $\alpha \in \mathcal{O}_{X^{\mathrm{an}}}(*D)/\mathcal{O}_{X^{\mathrm{an}}}$ and some $\lambda \in \mathbb{C}^2$.

We distinguish the two case, $x_0 \in D$ being a crossing-point or a smooth point, i.e. with local coordinates centered at x^0 either $D = \{x_1 x_2 = 0\}$ or $D = \{x_1 = 0\}$. Consider the case of a crossing-point first.

We define the **Stokes directions** of the elementary connection e^{α} with $\alpha(x) := x_1^{-m_1} x_2^{-m_2} u(x)$ with $u(0) \neq 0$ (which can be achieved since the decomposition is good) at the point 0 as the elements of

$$\Sigma_{\alpha}^0 := \{(\vartheta_1, \vartheta_2) \mid -m_1\vartheta_1 - m_2\vartheta_2 + \arg(u(0)) \in (\frac{\pi}{2}, \frac{3\pi}{2})\} \subset \pi^{-1}(0) \cong S^1 \times S^1 \ .$$

Now, consider a given direction $\vartheta = (\vartheta_1, \vartheta_2) \in \pi^{-1}(0)$. If $\vartheta \in \Sigma_{\alpha}^0$, then the solution $x^{-\lambda} e^{-\alpha(x)} \in \mathcal{H}^0(\pi^{-1}\mathrm{DR}_{X^{\mathrm{an}}}(\nabla^{\vee}))$ is not rapidly decaying in any small

enough open bisector $V \subset \widetilde{X}$ around ϑ whose directions $\mathcal{V} \subset \pi^{-1}(0)$ are contained in Σ^0_α (recall that u is continuous at 0). More precisely, it is even rapidly growing if $m_1 \neq 0$ or $m_2 \neq 0$ and of moderate growth for $m_1 = m_2 = 0$. Hence

$$^\vee\mathcal{S}^{<D}|_V = \widetilde{\jmath}_!\big(x^{-\lambda}e^{-\alpha(x)}\mathbb{C}_{X \smallsetminus D}\big)|_V \tag{3.10}$$

for any such V. Since $x^\lambda e^{\alpha(x)}$ is rapidly decaying in V for $m_1 \neq 0$ or $m_2 \neq 0$ and of moderate growth otherwise, one has $\mathcal{S}^{\mathrm{mod}D}|_V = x^\lambda e^{\alpha(x)} \cdot \mathbb{C}_V$ and consequently the induced morphism

$$
\begin{array}{ccc}
\boldsymbol{R}\mathcal{H}om(^\vee\mathcal{S}^{<D}, \widetilde{\jmath}_!\mathbb{C}_{X \smallsetminus D})|_V & \longrightarrow & \mathcal{S}^{\mathrm{mod}D}|_V \\[4pt]
\cong \downarrow & & \downarrow \cong \\[4pt]
\boldsymbol{R}\mathcal{H}om(\widetilde{\jmath}_!\mathbb{C}_{V \smallsetminus D}, \widetilde{\jmath}_!\mathbb{C}_{V \smallsetminus D})|_V & \xrightarrow{\;\cong\;} & \mathbb{C}_V
\end{array}
\tag{3.11}
$$

is indeed an isomorphism. Note hereby, that for the open embedding $\widetilde{\jmath}$, the functor $\widetilde{\jmath}_!$ is exact and has $\widetilde{\jmath}^*$ as its right adjoint. This easily shows that the bottom line is an isomorphism.

If $\vartheta \in \Sigma^0_{-\alpha}$ the situation is similar. If V is a subsector around ϑ, whose directions are contained in $\Sigma^0_{-\alpha}$, then, if V is small enough,

$$^\vee\mathcal{S}^{<D}|_V \cong x^{-\lambda}e^{-\alpha(x)} \cdot \mathbb{C}_V \quad \text{and} \quad \mathcal{S}^{\mathrm{mod}D}|_V \cong \widetilde{\jmath}_!\big(x^\lambda e^{\alpha(x)} \cdot \mathbb{C}_{X \smallsetminus D}\big)|_V \;,$$

and (3.11) holds again.

Finally, consider the case that ϑ separates the Stokes directions of α and $-\alpha$, i.e. $-m_1\vartheta_1 - m_2\vartheta_2 + \arg(u(0)) \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Let $V = V_1 \times V_2$ be a small open bisector. The Stokes-directions of α along D in V are defined as

$$\Sigma^D_\alpha := St^{-1}\big((\frac{\pi}{2}, \frac{3\pi}{2})\big) \subset V \cap \pi^{-1}(D) \;,$$

where $St : V \cap \pi^{-1}(D) \to \mathbb{R}/2\pi\mathbb{Z}$ is given by

$$St(r_1, \vartheta_1, r_2, \vartheta_2) := -m_1\vartheta_1 - m_2\vartheta_2 + \arg(u \circ \pi(r_1, \vartheta_1, r_2, \vartheta_2)) \;.$$

Remark that $\Sigma^0_\alpha = \Sigma^D_\alpha \cap \pi^{-1}(0)$. We further denote

$$V_\alpha := \big(V \smallsetminus \pi^{-1}(D)\big) \cup \Sigma^D_\alpha \;. \tag{3.12}$$

Hence $V_\alpha \cap \pi^{-1}(D)$ consists of those directions in $\pi^{-1}(D)$ along which $e^{\alpha(x)}$ has rapid decay for x approaching D. Let $\widetilde{\jmath}_\alpha : V_\alpha \hookrightarrow V$ denote the inclusion. Then obviously for V small enough one has

$$^\vee\mathcal{S}^{<D}|_V = \big(\widetilde{\jmath}_{-\alpha}\big)_!\big(x^{-\lambda}e^{-\alpha(x)} \cdot \mathbb{C}_{X \smallsetminus D}\big)|_V \;, \tag{3.13}$$

as well as

$$\mathcal{S}^{\mathrm{mod}D}|_V = \big(\widetilde{\jmath}_\alpha\big)_!\big(x^\lambda e^{\alpha(x)} \cdot \mathbb{C}_{X \smallsetminus D}\big)|_V \;.$$

Consequently, we deduce an analogous diagram as in (3.11) where the morphism in the bottom line now reads as

$$\boldsymbol{R}\mathcal{H}om\big((\widetilde{\jmath}_{-\alpha})_!\mathbb{C}_{X \smallsetminus D}, \widetilde{\jmath}_!\mathbb{C}_{X \smallsetminus D}\big) \longrightarrow (\widetilde{\jmath}_\alpha)_!\mathbb{C}_{V_\alpha} \;. \tag{3.14}$$

By the factorization $\widetilde{\jmath} = \widetilde{\jmath}_{-\alpha} \circ \widetilde{\iota}_{-\alpha}$ with $\widetilde{\iota}_{-\alpha} : V \smallsetminus \pi^{-1}(D) \hookrightarrow V_{-\alpha}$, we see that

$$
\begin{aligned}
\boldsymbol{R}\mathcal{H}om\big((\widetilde{\jmath}_{-\alpha})_! \mathbb{C}_{V_{-\alpha}}\, , \widetilde{\jmath}_! \mathbb{C}_{X \smallsetminus D}\big) &\cong (\widetilde{\jmath}_{-\alpha})_* \boldsymbol{R}\mathcal{H}om\big(\mathbb{C}_{V_{-\alpha}}, (\widetilde{\iota}_{-\alpha})_! \mathbb{C}_{V \smallsetminus \pi^{-1}(D)}\big) \\
&\cong (\widetilde{\jmath}_{-\alpha})_* \mathcal{H}om\big(\mathbb{C}_{V_{-\alpha}}, (\widetilde{\iota}_{-\alpha})_! \mathbb{C}_{V \smallsetminus \pi^{-1}(D)}\big) = (\widetilde{\jmath}_\alpha)_! \,\mathbb{C}_{V_\alpha}\ ,
\end{aligned}
$$

since $(V \smallsetminus V_\alpha) \cap \pi^{-1}(D)$ coincides with the closure of $V_{-\alpha} \cap \pi^{-1}(D)$ inside $\pi^{-1}(D)$. Hence, (3.14) is again an isomorphism and thus

$$
\mathcal{S}^{\mathrm{mod}D} \cong \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}\big({}^\vee\mathcal{S}^{<D}, \widetilde{\jmath}_! \mathbb{C}_{X \smallsetminus D}\big)
$$

locally on $\widetilde{X}$ over a crossing point of D. Interchanging ${}^\vee\mathcal{S}^{<D}$ and $\mathcal{S}^{\mathrm{mod}D}$ gives the analogous isomorphism.

In the local situation on $\widetilde{X}$ above a smooth point of D, i.e. above $x = 0$ where locally $D = \{x_1 = 0\}$ on X, we proceed in the same manner, where now $\alpha(x) = x^{-m_1} u(x)$ with $u(0, x_2) \neq 0$. Instead of bisectors, we consider small tubes $V := V_1 \times \Delta_2 \subset \widetilde{X}$, where V_1 is a one-dimensional sector and Δ_2 a small open disc around 0. The definition of the Stokes-directions of α along D in V are literally the same as above and the proof of the corresponding isomorphisms equally holds in this case, completing the proof of the proposition.

$\square$

3.5   Rapid decay homology and the asymptotically flat de Rham complex

Next, we want to compare the rapid decay complex $\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla)$ of a flat connection ∇ with the asymptotically flat de Rham complex in a *good* compactification. This will be a key step in the proof of the main theorem.

Theorem 3.6 *Let* (X, D) *be a good compactification of* U *with respect to the connection* (E, ∇). *We write* $d := \dim_{\mathbb{C}}(X) = 2$ *for its dimension. Then the rapid decay complex and the asymptotically flat de Rham complex are isomorphic up to a shift*

$$
\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla) \cong \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla)[2d]\ .
$$

in the derived category $D^b(\mathbb{C}_{\widetilde{X}})$.

Proof: According to Proposition 3.3, the right hand side has cohomology in degree zero only

$$
\mathcal{S}^{<D} \underset{\mathrm{def}}{=} \mathcal{H}^0(\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla)) \simeq \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla)\ ,
$$

i.e. the inclusion $\mathcal{S}^{<D} \hookrightarrow \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla)$ is a quasi-isomorphism of complexes.

Now, consider the complex $\mathcal{C}_{\widetilde{X}, \widetilde{D}}^{-\bullet}$ of sheaves of smooth topological chains on $\widetilde{X}$ relative to the boundary $\widetilde{D}$, defined to be the complex of sheaves associated to the presheaves

$$
V \mapsto S_\bullet\big(\widetilde{X}, (\widetilde{X} \smallsetminus V) \cup \widetilde{D}\big)\ . \tag{3.15}
$$

It is standard that this sheaf is a homotopically fine resolution of the constant sheaf $\mathbb{C}_{\widetilde{X}}[2d]$ shifted by the real dimension: restricted to U^{an} it is the sheaf version of the absolute singular homology of the real manifold U^{an} (cp [26]) and since we take singular homology with respect to the boundary $\widetilde{D}$, $\mathcal{C}_{\widetilde{X}, \widetilde{D}}^{-\bullet}$ gives

the constant sheaf along the boundary also: For any small enough $V \subset \widetilde{X}$, such that $V \subset \Omega$ for a contractible open Ω one has by excision

$$\mathcal{H}^{-p}(\mathcal{C}^{-\bullet}_{\widetilde{X},\widetilde{D}}) = H_p\big(\Omega, (\Omega \smallsetminus V) \cup \widetilde{D}\big) \cong \left\{ \begin{array}{ll} 0 & \text{for } p \neq 2d \\ \mathbb{C} & \text{for } p = 2d \end{array} \right. .$$

We now have a canonical morphism

$$\mathcal{C}^{-\bullet}_{\widetilde{X},\widetilde{D}} \otimes \mathcal{S}^{<D} \longrightarrow \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}(\nabla) , \qquad (3.16)$$

since for any open $V \subset \widetilde{X}$ and an asymptotically flat section $\sigma \in \Gamma_V(\mathcal{S}^{<D})$, by the characterization of the elements in $\mathcal{A}^{\leqslant D}_{\widetilde{X}}$ (see Proposition II.1.1.11 in [20] or (3.1) above), the section $\sigma \in \mathcal{S}^{<D}(V) \subset \widetilde{\jmath}_*\mathcal{E}(V)$ is rapidly decaying along any chain in $c \in \mathcal{C}^{-\bullet}_{\widetilde{X},\widetilde{D}}(V)$, hence $c \otimes \sigma \in \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}(\nabla)(V)$.

We claim that (3.16) is a quasi-isomorphism. It suffices to do so for the restriction of the sheaves to a basis of the topology of $\widetilde{X}$. If $V \subset \widetilde{X}$ is a small open contractible subset, the image of a $2d$-cell, contained in $U^{\mathrm{an}} = \widetilde{X} \smallsetminus \widetilde{D}$, then $\mathcal{S}^{<D}|_V$ is isomorphic to $\mathcal{E}|_V$, since $\mathcal{A}^{\leqslant D}_{\widetilde{X}}|_{U^{\mathrm{an}}} \cong \pi^{-1}\mathcal{O}_{U^{\mathrm{an}}}$ and on the other hand side

$$\mathcal{H}^{-p}(\mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}(\nabla))(V) \cong H_p(U^{\mathrm{an}}, U^{\mathrm{an}} \smallsetminus V; \mathcal{E})$$

coincides with the usual singular homology with values in the local system $\mathcal{E}|_{U^{\mathrm{an}}}$, since no condition of rapid decay is imposed inside U^{an} and for $V \subset U^{\mathrm{an}}$ we can use an obvious excision procedure to restrict to the situation inside U^{an}. This homology vanishes for $p \neq 2d$ and is isomorphic to $\Gamma(V, \mathcal{E})$ for $p = 2d$, the latter isomorphism being induced from the canonical one

$$\mathcal{H}^{-2d}(\mathcal{C}^{-\bullet}_{\widetilde{X},\widetilde{D}})(V) \cong H_{2d}(U^{\mathrm{an}}, U^{\mathrm{an}} \smallsetminus V) \cong \mathbb{C}$$

for the topological chains only, hence the restriction of (3.16) to U^{an} is a quasi-isomorphism.

It remains to prove the claim locally around $\widetilde{D}$. Now, since (X, D) is a good compactification, we can assume that there is a bicyclic ramification $\rho : Y \to X$ of D with lift $\widetilde{\rho} : \widetilde{Y} \to \widetilde{X}$ to the oriented real blow-ups such that locally on $\widetilde{Y}$ the pull-back $\rho^{-1}\nabla$ is isomorphic to its good formal model. Obviously, the finite map $\widetilde{\rho}$ induces isomorphisms of complexes

$$\widetilde{\rho}_*\mathcal{C}^{\mathrm{rd}}_{\widetilde{Y}}(\rho^{-1}\nabla) \cong \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}(\nabla) \quad \text{and} \quad \widetilde{\rho}_*\mathcal{C}^{-\bullet}_{\widetilde{Y},\widetilde{S}} \cong \mathcal{C}^{-\bullet}_{\widetilde{X},\widetilde{D}} ,$$

where $\widetilde{S} = \widetilde{\rho}^{-1}\widetilde{D}$. Since $\widetilde{\rho}_*\mathrm{DR}^{\leqslant S}_{\widetilde{Y}}(\rho^{-1}\nabla) = \mathrm{DR}^{\leqslant D}_{\widetilde{X}}(\nabla)$ (see (3.4) in the proof of Proposition 3.3), and the morphism (3.16) is obviously compatible with these isomorphisms, it suffices to prove the assertion on $\widetilde{Y}$, i.e. we assume $\widetilde{Y} = \widetilde{X}$, so that for $V \subset \widetilde{X}$ small enough, the connection is isomorphic on V to its good elementary model and we are reduced to the case $\nabla = e^\alpha \otimes \mathcal{R}$ with a regular singular connection $\mathcal{R}$.

Now, the solutions of a regular singular connection have moderate growth and moderate decay at most (see [8]), a local solution of $e^\alpha \otimes \mathcal{R}$ is asymptotically flat if and only if e^α has this property. Therefore

$$\mathcal{S}^{<D}(e^\alpha \otimes \mathcal{R}) = \mathcal{S}^{<D}(e^\alpha) \otimes \widetilde{\jmath}_*\mathcal{R}$$

in the obvious notation for the asymptotically flat solutions $\mathcal{S}^{<D}(\nabla)$ associated to a given connection.

We proceed as in the proof of Theorem 3.5, from which we also take the notations. First suppose that V is a small open bisector around some $\vartheta \in \pi^{-1}(0)$, where $0 \in D$ denotes a crossing point of D. Let $\Sigma_\alpha^0 \subset \pi^{-1}(0)$ denote the set of Stokes-directions of e^α.

If $\vartheta \in \Sigma_\alpha^0$ we can assume that the directions of the bisector V are all contained in Σ_α^0 and then (cp. (3.10))

$$\mathcal{S}^{<D}(e^\alpha)|_V = \widetilde{\jmath}_!\left(e^{-\alpha}\mathbb{C}_{U^{\mathrm{an}}}\right)|_V$$

and consequently, $\mathcal{S}^{<D}(\nabla)|_V = \widetilde{\jmath}_!\mathcal{E}$. For a smooth topological chain c in $\widetilde{X}$, a local section $\varepsilon = e^{-\alpha} \cdot \rho$ in $\mathcal{E}$ with a local solutions ρ of $\mathcal{R}$ will not have rapid decay along c in V as required by the definition unless the chain does not meet $\widetilde{D} \cap V$. Hence

$$\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla)|_V = \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-\bullet} \otimes \widetilde{\jmath}_!\mathcal{E} = \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-\bullet} \otimes \mathcal{S}^{<D}(\nabla)|_V \ .$$

If $\vartheta \in \Sigma_{-\alpha}^0$, we can assume that V is an open bisector such that all the arguments of points in V are contained in $\Sigma_{-\alpha}^0$. Then

$$\mathcal{S}^{<D}(e^\alpha)|_V \cong e^{-\alpha}\mathbb{C}_V \ .$$

Similarly, all twisted chains $c \otimes \varepsilon$ will have rapid decay inside V and again both complexes considered are equal to $\mathcal{C}_{\widetilde{X},\widetilde{D}}^{-\bullet} \otimes \widetilde{\jmath}_*\mathcal{E}$.

Finally, if ϑ separates the Stokes regions of α and $-\alpha$, we have (3.13):

$$\mathcal{S}^{<D}(e^\alpha)|_V \cong (\widetilde{\jmath}_{-\alpha})_!(e^{-\alpha}\mathbb{C}_{U^{\mathrm{an}}})|_V$$

where $\widetilde{\jmath}_{-\alpha} : V_{-\alpha} \hookrightarrow V$ denotes the inclusion of the subspace $V_{-\alpha}$ we defined in (3.12). The characteristic property of $V_{-\alpha}$ is that $V_{-\alpha} \cap \widetilde{D}$ consists of those directions along which $e^{-\alpha(x)}$ has rapid decay for x approaching $\widetilde{D}$. In particular, $c \otimes \varepsilon$ is a rapid decay chain on V if and only if the topological chain c in $\widetilde{X}$ approaches $\widetilde{D} \cap V$ in $V_{-\alpha}$ at most, i.e.

$$\mathrm{im}(c) \cap (\widetilde{D} \cap V) \subset V_{-\alpha} \ .$$

Again both complexes coincide:

$$\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla)|_V = \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-\bullet} \otimes (\widetilde{\jmath}_{-\alpha})_!\mathcal{E} = \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-\bullet} \otimes \mathcal{S}^{<D}(\nabla)|_V \ .$$

The situation for $\vartheta \in \widetilde{D}$ with $\pi(\vartheta)$ a smooth point of D can be handled analogously.

In summary, we have the following composition of quasi-isomorphisms

$$\mathcal{S}^{<D}[2d] \xrightarrow{\simeq} \mathbb{C}_{\widetilde{X}}[2d] \otimes \mathcal{S}^{<D} \xrightarrow{\simeq} \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-\bullet} \otimes \mathcal{S}^{<D} \xrightarrow{\simeq} \mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla)$$

and the claim of the Theorem follows.

$\square$

3.6 The general local duality theorem

Starting with the algebraic connection on the smooth quasi-projective algebraic surface U, we chose a *good* compactification (X, D) (assuming Sabbah's Conjecture holds for the given connection) in order to define the complex of sheaves $\mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}$ on the real oriented blow-up $\widetilde{X}$. By Theorem 3.6, this complex is isomorphic to the asymptotically flat de Rham complex $\mathrm{DR}^{\leq D}_{\widetilde{X}}(\nabla^\vee)$ in the derived category, the proof relying heavily on X being a *good* compactification.

Now, assume that contrary to this situation, we are given an arbitrary compactification (X, D) with a normal crossing divisor $D = X \smallsetminus U$ (but without the assumption of a good formal structure). In this situation, we will still be able to proof the local duality statement for the de Rham complexes on $\widetilde{X}$ assuming Sabbah's Conjecture.

To this end, we first study the behavior of the rapid decay and the moderate de Rham complex under blowing-up a point $z \in D$. More generally, let $b : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ be a proper morphism such that $S := b^{-1}(D)$ is again a normal crossing divisor and the restriction $b|_{Y^{\mathrm{an}} \smallsetminus S} : Y^{\mathrm{an}} \smallsetminus S \to X^{\mathrm{an}} \smallsetminus D$ is an isomorphism. Let $\pi_X : \widetilde{X} \to X^{\mathrm{an}}$ and $\pi_Y : \widetilde{Y} \to Y^{\mathrm{an}}$ denote the oriented real blow-ups. Lift b to a map $\widetilde{b} : \widetilde{Y} \to \widetilde{X}$ and consider the open embeddings $\widetilde{\jmath} : X^{\mathrm{an}} \smallsetminus D \hookrightarrow \widetilde{X}$ and $\widetilde{\imath} : Y^{\mathrm{an}} \smallsetminus S \hookrightarrow \widetilde{Y}$. The de Rham complexes on $\widetilde{X}$ behave well with respect to this situation, namely

Lemma 3.7 *There are natural isomorphism*

$$\boldsymbol{R}\widetilde{b}_*(\mathrm{DR}^{\mathrm{mod}S}_{\widetilde{Y}}(b^*\nabla)) \cong \mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}(\nabla) \ and$$
$$\boldsymbol{R}\widetilde{b}_*(\mathrm{DR}^{\leq S}_{\widetilde{Y}}(b^*\nabla)) \cong \mathrm{DR}^{\leq D}_{\widetilde{X}}(\nabla) \ .$$

Proof: Since $\mathcal{A}^{?D}_{\widetilde{X}}$ is flat over $\pi^{-1}(\mathcal{O}_{X^{\mathrm{an}}})$, where ? stand for either $<$ or mod, we conclude by the projection formula that

$$\boldsymbol{R}\widetilde{b}_*(\mathrm{DR}^{?S}_{\widetilde{Y}}) = \boldsymbol{R}\widetilde{b}_*\big(\mathcal{A}^{?S}_{\widetilde{Y}} \overset{\mathbb{L}}{\otimes}_{\pi_Y^{-1}(\mathcal{O}_{Y^{\mathrm{an}}})} \pi_Y^{-1}\mathrm{DR}_{Y^{\mathrm{an}}}(b^*\nabla)\big) \cong$$
$$\cong \boldsymbol{R}\widetilde{b}_*(\mathcal{A}^{?S}_{\widetilde{Y}}) \overset{\mathbb{L}}{\otimes}_{\pi_X^{-1}(\mathcal{O}_{X^{\mathrm{an}}})} \pi_X^{-1}\mathrm{DR}_{X^{\mathrm{an}}}(\nabla) \ ,$$

so that it remains to prove $\boldsymbol{R}\widetilde{b}_*(\mathcal{A}^{?S}_{\widetilde{Y}}) \cong \mathcal{A}^{?D}_{\widetilde{X}}$.

Consider the resolution

$$\mathcal{A}^{\leq S}_{\widetilde{Y}} \simeq \big(\mathcal{P}^{\leq S}_{\widetilde{Y}} \otimes_{\pi_Y^{-1}(C^\infty_{Y^{\mathrm{an}}})} \pi_Y^{-1}\Omega^{\infty,(0,\bullet)}_{Y^{\mathrm{an}}}, \overline{\partial}\big) \ , \tag{3.17}$$

where $\mathcal{P}^{\leq S}_{\widetilde{Y}}$ denotes the sheaf of C^∞-functions, which are flat at $\pi_Y^{-1}(S)$. Hence

$$\boldsymbol{R}\widetilde{b}_*\mathcal{A}^{\leq S}_{\widetilde{Y}} = \big(\widetilde{b}_*\mathcal{P}^{\leq S}_{\widetilde{Y}} \otimes_{\pi_X^{-1}(C^\infty_{X^{\mathrm{an}}})} \pi_X^{-1}\Omega^{\infty,(0,\bullet)}_{X^{\mathrm{an}}}, \overline{\partial}\big) \ .$$

The assertion for $\mathrm{DR}^{\leq D}_{\widetilde{X}}$ follows, since for any such $b : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ inducing an isomorphism $Y^{\mathrm{an}} \smallsetminus S \to X^{\mathrm{an}} \smallsetminus D$, one obviously has $\widetilde{b}_*\mathcal{P}^{\leq S}_{\widetilde{Y}} = \mathcal{P}^{\leq D}_{\widetilde{X}}$.

As for $\mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}$, the same arguments apply to the resolution

$$\mathcal{A}^{\mathrm{mod}S}_{\widetilde{Y}} \simeq \big(\mathcal{P}^{\mathrm{mod}S}_{\widetilde{Y}} \otimes_{\pi_Y^{-1}(C^\infty_{Y^{\mathrm{an}}})} \pi_Y^{-1}\Omega^{\infty,(0,\bullet)}_{Y^{\mathrm{an}}}, \overline{\partial}\big) \ , \tag{3.18}$$

with the sheaf $\mathcal{P}_{\widetilde{Y}}^{\mathrm{mod}S}$ of C^{∞}-functions with moderate growth along $\pi_Y^{-1}(S)$, for which $\widetilde{b}_* \mathcal{P}_{\widetilde{Y}}^{\mathrm{mod}S} = \mathcal{P}_{\widetilde{X}}^{\mathrm{mod}D}$ holds as well.

$\square$

Theorem 3.8 (Local Duality) *Let (E, ∇) be an integrable connection on U and $j : U \hookrightarrow X$ be an embedding into a smooth projective variety such that $D := X \smallsetminus U$ is a normal crossing divisor. Let $\pi : \widetilde{X} \to X^{\mathrm{an}}$ denote the real oriented blow-up of D in X^{an}. Assuming that Sabbah's Conjecture holds for (E, ∇), the local duality pairing (3.8)*

$$\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(E, \nabla) \otimes_{\mathbb{C}} \mathrm{DR}_{\widetilde{X}}^{\leq D}(E^{\vee}, \nabla^{\vee}) \to \mathrm{DR}_{\widetilde{X}}^{\leq D}(\mathcal{O}_X, d)$$

is non-degenerate.

Proof: According to Sabbah's Conjecture, there exists a sequence of point blow-ups $b : Y \to X$ such that $(b^{-1}(E, \nabla))^{\mathrm{an}}$ has a good formal decomposition along D and Proposition 3.5 can be applied.

Lifting b to a map between the real oriented blow-ups $\widetilde{b} : \widetilde{Y} \to \widetilde{X}$, we are in the situation of Lemma 3.7. We use the same notation introduced above and let $W := Y \smallsetminus S$. Then Proposition 3.5 yields an isomorphism

$$\mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}S}(b^*\nabla) \xrightarrow{\cong} \boldsymbol{R}\mathcal{H}om_{\widetilde{Y}}\big(\mathrm{DR}_{\widetilde{Y}}^{\leq S}(b^*\nabla^{\vee}), \widetilde{\iota}_! \mathbb{C}_{W^{\mathrm{an}}}\big) \ .$$

Now $\widetilde{\iota}_! \mathbb{C}_{W^{\mathrm{an}}}[2d]$ is the dualizing sheaf on the manifold $\widetilde{Y}$ with boundary $\widetilde{Y} \smallsetminus W^{\mathrm{an}}$ and similarly $\widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]$ for $\widetilde{X}$.

The resolution (3.17) again induces a fine resolution

$$\widetilde{\iota}_! \mathbb{C}_{W^{\mathrm{an}}} \simeq \big(\mathcal{P}_{\widetilde{Y}}^{\leq S} \otimes_{\pi_Y^{-1}(C_{Y^{\mathrm{an}}}^{\infty})} \pi_Y^{-1} \Omega_{Y^{\mathrm{an}}}^{\infty,(\cdot,\cdot)}, \partial, \overline{\partial}\big) \ ,$$

where the right hand is to be understood as the simple complex associated to the indicated double complex of Dolbeault-type. Applying $\widetilde{b}_*$ thus yields an isomorphism

$$\alpha : \boldsymbol{R}\widetilde{b}_* \widetilde{\iota}_! \mathbb{C}_{W^{\mathrm{an}}} = \widetilde{b}_*\big(\mathcal{P}_{\widetilde{Y}}^{\leq S} \otimes_{\pi_Y^{-1}(C_{Y^{\mathrm{an}}}^{\infty})} \pi_Y^{-1}\Omega_{Y^{\mathrm{an}}}^{\infty,(\cdot,\cdot)}\big) =$$
$$= \widetilde{b}_* \mathcal{P}_{\widetilde{Y}}^{\leq S} \otimes_{\pi_X^{-1}(C_{X^{\mathrm{an}}}^{\infty})} \pi_X^{-1}\Omega_{X^{\mathrm{an}}}^{\infty,(\cdot,\cdot)} \cong \widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}} \ .$$

This construction is the same as the one used for the isomorphisms in Lemma 3.7 above. It is therefore easy to see that these isomorphisms are compatible with the local duality pairing given by the wedge product, i.e. the diagram

$$\begin{array}{ccc}
\boldsymbol{R}\widetilde{b}_* \mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}S}(b^*\nabla) \otimes_{\mathbb{C}} \boldsymbol{R}\widetilde{b}_* \mathrm{DR}_{\widetilde{Y}}^{\leq S}(b^*\nabla^{\vee}) & \longrightarrow & \boldsymbol{R}\widetilde{b}_* \widetilde{\iota}_! \mathbb{C}_{W^{\mathrm{an}}} \\
{\scriptstyle \text{Lemma 3.7}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \alpha} \qquad (3.19) \\
\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) \otimes_{\mathbb{C}} \mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee}) & \longrightarrow & \widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}}
\end{array}$$

commutes. Hence, the morphism $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) \to \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}\big(\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee}), \widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}}\big)$ induced by the lower row of (3.19) factors as

$$\begin{array}{ccc}
\boldsymbol{R}\widetilde{b}_* \mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}S}(b^*\nabla) & \xrightarrow{\ \beta\ } & \boldsymbol{R}\widetilde{b}_* \boldsymbol{R}\mathcal{H}om_{\widetilde{Y}}\big(\mathrm{DR}_{\widetilde{Y}}^{\leq S}(b^*\nabla^{\vee}), \widetilde{\iota}_! \mathbb{C}_{W^{\mathrm{an}}}\big) \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \gamma} \qquad (3.20) \\
\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) & \longrightarrow & \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}\big(\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^{\vee}), \widetilde{j}_! \mathbb{C}_{U^{\mathrm{an}}}\big) \ ,
\end{array}$$

where γ is given by the composition of the natural morphism

$$\boldsymbol{R}\widetilde{b}_*\boldsymbol{R}\mathcal{H}om_{\widetilde{Y}}\big(\mathrm{DR}_{\widetilde{Y}}^{\leq S},\widetilde{\imath}_!\mathbb{C}_{W^{\mathrm{an}}}\big) \to \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}\big(\boldsymbol{R}\widetilde{b}_*\mathrm{DR}_{\widetilde{Y}}^{\leq S},\boldsymbol{R}\widetilde{b}_*\widetilde{\imath}_!\mathbb{C}_{W^{\mathrm{an}}}\big)$$

with the morphism α from above. By Poincaré-Verdier duality (Proposition 3.1.10 in [12]), γ is an isomorphism. In addition, β is an isomorphism due to Theorem 3.5, hence so is the bottom row of (3.20). Interchanging $\mathrm{DR}^{\mathrm{mod}D}$ and $\mathrm{DR}^{<D}$, the same arguments apply, completing the proof of the theorem.

□

Remark 3.9 The local duality theorem shows that for non-good compactifications (X, D), the complex $\mathrm{DR}_{\widetilde{X}}^{\leq D}(\nabla^\vee)$ gives the appropriate dual object to the moderate (and hence the algebraic) de Rham cohomology. The resulting pairing, however, lacks of a similar explicit description we have in case of a good compactification by period integrals.

If we assume Sabbah's Conjecture for the given connection, one can find a finite sequence of point blow-ups $b : Y \to X$, such that (Y, S) is good with respect to (E, ∇). Then $\mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}S}(b^*\nabla^\vee)$ has cohomology in degree zero only and is quasi-isomorphic to the rapid decay complex $\mathcal{C}_{\widetilde{Y}}^{\mathrm{rd}}$. On the other hand side, Lemma 3.7 yields the isomorphism $\mathrm{DR}_{\widetilde{X}}^{\mathrm{mod}D}(\nabla) \cong \boldsymbol{R}\widetilde{b}_*(\mathrm{DR}_{\widetilde{Y}}^{\mathrm{mod}S}(b^*\nabla))$, which may have non-vanishing cohomology in different degrees. In general, we will not have a topological description of this complex by rapid decay chains as in the good compactification case.

In the one-dimensional case, every compactification is automatically good and so the rapid decay complex is always isomorphic to the asymptotically flat de Rham complex in the derived sense for which the local duality theorem holds.

3.7   Independence of choice of good compactification

In the definition of the rapid decay homology (Section 2.3) we chose a *good* compactification (X, D) with respect to the given connection (E, ∇) in order to define the rapid decay complex on the real oriented blow-up $\widetilde{X}$. Theorem 3.6 and Lemma 3.7 now yield independence of the choice of the good compactification as an immediate consequence.

Proposition 3.10 *Assuming Sabbah's Conjecture, the following holds: Given two compactifications (X_1, D_1) and (X_2, D_2) of U such that $D_i = X_i \smallsetminus U$ are normal crossing divisors and (E, ∇) admits a good formal decomposition in each compactification, there is a canonical isomorphism*

$$\mathbb{H}^{-\bullet}(\widetilde{X}_1, \mathcal{C}_{\widetilde{X}_1}^{\mathrm{rd}}(\nabla)) \cong \mathbb{H}^{-\bullet}(\widetilde{X}_2, \mathcal{C}_{\widetilde{X}_2}^{\mathrm{rd}}(\nabla))$$

between the corresponding hypercohomologies.

Proof: We can pass to a common good compactification (X, D) applying the statement of Sabbah's Conjecture to the closure of U inside $X_1 \times X_2$. By Lemma 3.7, the asymptotically flat de Rham complexes on $\widetilde{X}_1$ and $\widetilde{X}_2$ both are quasi-isomorphic to the one on $\widetilde{X}$ and hence so are the rapid decay complexes by Theorem 3.6.

□

In particular, the conclusion of the proposition holds if rank $E \leq 5$ (by Sabbah's Theorem 2.2).

4   Global duality of the period pairing

We will now complete the proof of Theorem 2.5, namely the perfectness of the period pairing for the given flat connection (E, ∇) on U.

First, let (X, D) be any compactification of U with a normal crossing divisor D. Taking global sections in the local duality statement in Theorem 3.8 leads to the following global duality statement:

Theorem 4.1 *Let (E, ∇) be an integrable connection on the smooth quasi-projective surface U over $k \subset \mathbb{C}$ and let $j : U \hookrightarrow X$ be an open embedding into a smooth projective variety X such that $D := X \smallsetminus U$ is a normal crossing divisor. Assume that Sabbah's Conjecture holds for (E, ∇). Then the local duality pairing* (3.8) *induces a perfect pairing*

$$\left(H^p_{dR}(U; E, \nabla) \otimes_k \mathbb{C} \right) \otimes_{\mathbb{C}} \mathbb{H}^{2d-p}(\widetilde{X}, \mathrm{DR}^{\leq D}_{\widetilde{X}}(\nabla^\vee)) \to \mathbb{C} \ . \qquad (4.1)$$

Proof: If $\widetilde{\jmath} : U^{\mathrm{an}} \to \widetilde{X}$ denotes the inclusion of U^{an} into the real oriented blow-up of X^{an}, the local duality gives an isomorphism

$$\mathrm{DR}^{\leq D}_{\widetilde{X}}(\nabla^\vee) \xrightarrow{\cong} \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}(\mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}(\nabla), \widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}}) \ .$$

Since $\widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]$ is the dualizing sheaf on the compact real $2d$-dimensional manifold $\widetilde{X}$ with boundary $\widetilde{X} \smallsetminus U^{\mathrm{an}}$, the local Poincaré-Verdier duality ([12], Prop. 3.1.10) yields the isomorphism

$$\boldsymbol{R}\Gamma_{\widetilde{X}} \mathrm{DR}^{\leq D}_{\widetilde{X}}(\nabla^\vee)[2d] \cong$$
$$\cong \boldsymbol{R}\Gamma_{\widetilde{X}} \boldsymbol{R}\mathcal{H}om_{\widetilde{X}}(\mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}, \widetilde{\jmath}_! \mathbb{C}_{U^{\mathrm{an}}}[2d]) \cong \mathrm{Hom}^{\bullet}_{\mathbb{C}}(\boldsymbol{R}\Gamma_{\widetilde{X}} \mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}, \mathbb{C}) \ , \quad (4.2)$$

where $\mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}$ is formed with respect to ∇. Taking p-th cohomology gives the isomorphism we were looking for. Interchanging $\mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}$ and $\mathrm{DR}^{\leq D}_{\widetilde{X}}$, the same arguments prove the perfectness in the other direction.

$\square$

To prove our main result, Theorem 2.5, we are thus left to identify the global duality pairing from above with the explicitly given period pairing in the case of a *good* compactification (X, D). By Proposition 3.3, the asymptotically flat de Rham complex reduces to its 0-th cohomology sheaf ${}^\vee\mathcal{S}^{<D}$: the inclusion

$${}^\vee\mathcal{S}^{<D} \hookrightarrow \mathrm{DR}^{\leq D}_{\widetilde{X}}(\nabla^\vee)$$

is a quasi-isomorphism. Furthermore, Theorem 3.6 and the proof given above establish a quasi-isomorphism of the latter sheaf shifted by the dimension with the rapid decay complex

$${}^\vee\mathcal{S}^{<D}[2d] \xrightarrow{\cong} {}^\vee\mathcal{S}^{<D} \otimes \mathcal{C}^{-\bullet}_{\widetilde{X}, \widetilde{D}} \xrightarrow{\cong} \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}(\nabla^\vee) \ ,$$

where $\mathcal{C}^{-\bullet}_{\widetilde{X}, \widetilde{D}}$ denotes the complex of sheaves of smooth chains in $\widetilde{X}$ relative to $\widetilde{D}$ (cp. (3.15)).

Taking p-th cohomology in the duality isomorphism (4.2) therefore yields the isomorphism

$$\mathbb{H}^{-p}(\widetilde{X}, \mathcal{C}^{\mathrm{rd}}_{\widetilde{X}}(\nabla^\vee)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{H}^p(\widetilde{X}, \mathrm{DR}^{\mathrm{mod}D}_{\widetilde{X}}(\nabla)), \mathbb{C} \right) \ . \qquad (4.3)$$

We want to prove that (4.3) coincides with the morphism induced by the period pairing.

To this end, consider the resolution

$$\mathcal{A}_{\widetilde{X}}^{\le D} \otimes_{\pi^{-1}\mathcal{O}_{X^{\mathrm{an}}}} \pi^{-1}\Omega_{X^{\mathrm{an}}}^r \hookrightarrow \left(\mathcal{P}_{\widetilde{X}}^{\le D} \otimes_{\pi^{-1}C_{X^{\mathrm{an}}}^\infty} \pi^{-1}\Omega_{X^{\mathrm{an}}}^{\infty,(r,\bullet)}, \overline{\partial}\right),$$

where $\mathcal{P}_{\widetilde{X}}^{\le D}$ as before denotes the sheaf of C^∞-functions flat at $\pi^{-1}(D)$ and $\Omega_{X^{\mathrm{an}}}^{\infty,(r,s)}$ denotes the sheaf of C^∞ forms on X^{an} of degree (r,s). This resolution gives rise to the bicomplex

$$\mathcal{R}d_{\nabla^\vee}^{\bullet,\bullet} := \left(\mathcal{P}_{\widetilde{X}}^{\le D} \otimes_{\pi^{-1}C_{X^{\mathrm{an}}}^\infty} \pi^{-1}\Omega_{X^{\mathrm{an}}}^{\infty,(\bullet,\bullet)} \otimes_{\pi^{-1}\mathcal{O}_{X^{\mathrm{an}}}} \pi^{-1}E^\vee, \nabla^\vee, \overline{\partial}\right),$$

the total complex $\mathcal{R}d_{\nabla^\vee}^\bullet$ of which computes $^\vee\mathcal{S}^{<D}(\nabla^\vee) \simeq \mathrm{DR}_{\widetilde{X}}^{\le D}(\nabla^\vee)$. In particular, the complex $\mathcal{R}d_{(\mathcal{O}_X,d)}^\bullet$ associated to the trivial connection $(\mathcal{O}_X,d)$ is a fine resolution of $\widetilde{j}_!\mathbb{C}_{U^{\mathrm{an}}}$.

With these resolutions, the local duality pairing $^\vee\mathcal{S}^{<D} \otimes \mathcal{S}^{\mathrm{mod}D} \to \widetilde{j}_!\mathbb{C}_{U^{\mathrm{an}}}$ (for good compactifications) can be represented by the bottom row of the following obviously commutative diagram:

$$
\begin{array}{ccc}
^\vee\mathcal{S}^{<D}[2d] \otimes \mathcal{S}^{\mathrm{mod}D} & \longrightarrow & \widetilde{j}_!\mathbb{C}_{U^{\mathrm{an}}}[2d] \\
\simeq \downarrow & & \downarrow \simeq \\
\mathcal{C}_{\widetilde{X},\widetilde{D}}^{-,\bullet} \otimes {}^\vee\mathcal{S}^{<D} \otimes \mathcal{S}^{\mathrm{mod}D} & \xrightarrow{\;\alpha\;} & \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-,\bullet} \otimes \mathcal{R}d_{(\mathcal{O}_X,d)}^\bullet.
\end{array}
\qquad (4.4)
$$

Recall that $\mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet}$ denotes the complex of sheaves of rapid decay distributions (see section 2.4) on $\widetilde{X}$ which is a fine resolution of $\widetilde{j}_!\mathbb{C}_{U^{\mathrm{an}}}[2d]$. There is a natural quasi-isomorphism

$$\beta : \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-,\bullet} \otimes \mathcal{R}d_{(\mathcal{O}_X,d)}^\bullet \xrightarrow{\;\simeq\;} \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet}$$

of complexes mapping an element $c \otimes \rho \in \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-r} \otimes \mathcal{R}d_{(\mathcal{O}_X,d)}^s(V)$ of the left hand side over some open $V \subset \widetilde{X}$ to the distribution in $\mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},s-r}$ given by $\eta \mapsto \int_c \eta \wedge \rho$ for a test form η with compact support in V. Note hereby, that the rapid decay property of ρ ensures that the integral satisfies the estimate (2.3) necessary for the distribution to be rapidly decaying.

Similarly, we have another natural morphism

$$\gamma : \mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^\vee) \otimes \mathcal{S}^{\mathrm{mod}D} \to \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet} , \quad (c \otimes \varepsilon) \otimes \sigma \mapsto \left(\eta \mapsto \int_c \langle \varepsilon, \sigma \cdot \eta \rangle\right).$$

These morphisms fit into the following diagram

$$
\begin{array}{ccc}
\mathcal{C}_{\widetilde{X},\widetilde{D}}^{-,\bullet} \otimes {}^\vee\mathcal{S}^{<D} \otimes \mathcal{S}^{\mathrm{mod}D} & \xrightarrow{\;\alpha\;} & \mathcal{C}_{\widetilde{X},\widetilde{D}}^{-,\bullet} \otimes \mathcal{R}d_{(\mathcal{O}_X,d)}^\bullet \\
\simeq \downarrow {\scriptstyle \text{Thm 3.6}} & & \simeq \downarrow {\scriptstyle \beta} \\
\mathcal{C}_{\widetilde{X}}^{\mathrm{rd}}(\nabla^\vee) \otimes \mathcal{S}^{\mathrm{mod}D} & \xrightarrow{\;\gamma\;} & \mathfrak{Db}_{\widetilde{X}}^{\mathrm{rd},-\bullet}.
\end{array}
\qquad (4.5)
$$

Combining (4.4) and (4.5), we see that the period pairing

$$H_{dR}^p(U;E,\nabla) \otimes_{\mathbb{C}} H_p^{rd}(U^{\mathrm{an}};E^\vee,\nabla^\vee) \to \mathbb{C}$$

induced by the bottom row in (4.5) coincides with the global duality pairing (4.1) induced by the top row in (4.4) and is therefore perfect by Theorem 4.1. This completes the proof of our main result, Theorem 2.5.

A   Existence of solutions with moderate growth

In this appendix we prove the existence theorem for solutions with moderate growth for the integrable system of partial differential equations which arouse in the proof of Proposition 3.3. The proof relies heavily on methods developed by Hukuhara and further elaborated by Ramis, Sibuya ([19]) and Majima ([13]). It is a variation of the proof C. Sabbah gives for the existence of asymptotically flat solutions ([21], Appendix). However, at several places some modifications are necessary. Note, that we will work entirely in the analytic topology and therefore omit the superscript 'an' in the following.

A.1   The system of differential equations

Consider the local situation in X around a point $0 \in D$, where locally $D = \{x_1 x_2 = 0\}$ or $D = \{x_1 = 0\}$, and in either case consider the real oriented blow-up $\pi : \widetilde{X} \to X$. In order to treat both cases as simultaneously as possible, we use the notation

$$\pi : \widetilde{X} = V \times Y \to X \ ,$$

with $V \cong (\mathbb{R}_+ \times S^1)^n$ and $Y \subset \mathbb{C}^p$ an open disc around 0, where in the case $D = \{x_1 x_2 = 0\}$ we have $n = 2$, $p = 0$ whereas $n = p = 1$ in the case $D = \{x_1 = 0\}$.

In the following let ϑ_0 be a fixed element in $\pi^{-1}(0)$ and let there be given

   i) a germ $\alpha \in \mathcal{A}_{\widetilde{X}, \vartheta_0}$ on X of the form $\alpha(r, \vartheta) := x_1^{-m_1} x_2^{-m_2} \cdot \mathcal{U}(r, \vartheta)$ with $\mathcal{U}(0, \vartheta_0) \neq 0$ in the case $n = 2$ and $\alpha(r_1, \vartheta_1, y) := x_1^{-m_1} \cdot \nu(y) \cdot \mathcal{U}(r_1, \vartheta_1, y)$ with $\mathcal{U}(0, \vartheta_0, y) = 1$ for all y and $\nu(y)$ an invertible function.

   ii) germs $\rho_i \in \mathcal{A}_{\widetilde{X}, \vartheta_0}$, $i = 1, 2$.

The system of partial differential equations for functions u we have to consider is the following. For $i = 1, 2$, consider the equations

$$(\Sigma_i): \qquad x_i \frac{\partial u}{\partial x_i} = -x_i \frac{\partial \alpha}{\partial x_i} \cdot u - \lambda_i u + \rho_i =: P_i u + \rho_i \ .$$

In case $n = 2$, the system (Σ) consists of (Σ_1) and (Σ_2), whereas in the case $n = p = 1$ we replace (Σ_2) by

$$(\Sigma'): \qquad \frac{\partial u}{\partial y} = \frac{\partial \alpha}{\partial y} \cdot u + \rho_2 := P_2' u + \rho_2 \ .$$

The system is called *integrable* if $P_1 \rho_2 = P_2 \rho_1$ in the first case and $P_1 \rho_2 = P_2' \rho_1$ if $n = p = 1$.

Theorem A.1 *If the system (Σ) is integrable and $\rho_i \in \mathcal{A}_{\widetilde{X}, \vartheta_0}^{\mathrm{mod} D}$ have moderate growth, then there exists a solution $u \in \mathcal{A}_{\widetilde{X}, \vartheta_0}^{\mathrm{mod} D}$ with moderate growth. A similar statement holds, if one considers the equation (Σ_1) alone.*

The idea of the proof is to transform the system (Σ) into a fixed point problem for corresponding integral operators. The technical difficulty, solved by Hukuhara's method, is to choose the paths of integration in a suitable manner, such that the exponential part $\exp(\alpha(x))$ arising from the differential equation is rapidly decaying whenever the path meets $\pi^{-1}(D)$.

We will now explicitly describe the method in the local case over a crossing-point of D, i.e. $n = 2$ in the above notation. The case of a smooth point with $n = p = 1$ is the same with some minor modifications on which we will comment afterwards.

We will have to distinguish two cases, the purely regular and the irregular case:

i) The case (IRR): at least one exponent m_1, m_2 is positive, say $m_1 > 0$.

ii) The case (pREG): $m_1 = m_2 = 0$.

A.2   The case (IRR)

A.2.1   Paths of integration and the fundamental estimate

We start with introducing some notations. For fixed directions $\vartheta = (\vartheta_1, \vartheta_2), \eta = (\eta_1, \eta_2) \in \pi^{-1}(0)$ and a biradius $r = (r_1, r_2)$, which may be ∞ also, we write

$$W(r) := [\vartheta, \eta]_r := \{(\rho_1 e^{i\theta_1}, \rho_2 e^{i\theta_2}) \in \widetilde{X} \mid 0 < \rho_i \leq r_i \text{ and } \vartheta_i \leq \theta_i \leq \eta_i\}$$

for the corresponding closed bisector and $\mathcal{W}$ for its directions. Given a direction $\vartheta = (\vartheta_1, \vartheta_2) \in \pi^{-1}(0)$, we say that

i) $\vartheta > 0$, if $\cos(m_1\vartheta_1 + m_2\vartheta_2 - \arg\mathcal{U}(0)) > 0$; similarly for < 0,

ii) ϑ is of type $(+-)$, if $M(\vartheta) := m_1\vartheta_1 + m_2\vartheta_2 - \arg\mathcal{U}(0) \equiv \frac{\pi}{2}\mathrm{mod}2\pi$,

iii) ϑ is of type $(-+)$, if $M(\vartheta) \equiv \frac{3\pi}{2}\mathrm{mod}2\pi$.

For a bisector $W(\infty)$ with directions $\mathcal{W} \subset S^1 \times S^1$, we say that $W(\infty) > 0$ or $W(\infty) < 0$ if all $\vartheta \in \mathcal{W}$ are of this type. We further call $W(\infty)$ is of type $(+-)$ if $\{M(\vartheta) \mid \vartheta \in \mathcal{W}\} \cap \{\frac{\pi}{2}, \frac{3\pi}{2}\} = \frac{\pi}{2}$, and similar for the type $(-+)$. A bisector is called *proper* if it is of one of these types.

We consider a fixed direction $\vartheta^0 \in \pi^{-1}(0)$ as in the theorem. Let $W(\infty) = W_1(\infty) \times W_2(\infty)$ be a proper bisector around ϑ^0, where

$$W_1(\infty) = [\vartheta_1^0 - \delta, \vartheta_1^0 + \varepsilon] \text{ with } \delta, \varepsilon \in \mathbb{R}_+ \ .$$

Define

$$W_1(\infty)_+ := \begin{cases} [\vartheta_1^0, \vartheta_1^0 + \varepsilon] & \text{if } \vartheta_1^0 > 0 \\ W_1(\infty) & \text{if } \vartheta_1^0 \text{ is of type } (-+) \\ \emptyset & \text{otherwise,} \end{cases} \qquad (A.1)$$

$$W_1(\infty)_- := \begin{cases} [\vartheta_1^0 - \delta, \vartheta_1^0] & \text{if } \vartheta_1^0 > 0 \\ W_1(\infty) & \text{if } \vartheta_1^0 \text{ is of type } (+-) \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition A.2 *A* **Hukuhara-function** *in direction x_i is a piecewise affine, decreasing function $a_i : W_i(\infty) \to (0, \pi)$, such that*

$$\cos(m_1\vartheta_1 + m_2\vartheta_2 - \arg\mathcal{U}(0) + a_i(\vartheta_i)) \begin{cases} > 0 & \text{if } \vartheta_i \in W_i(\infty)_+ \\ < 0 & \text{if } \vartheta_i \in W_i(\infty)_- \ . \end{cases}$$

We cite the following lemma from [21], Lemme (C.6):

Lemma A.3 *Any $\vartheta^0 \in \pi^{-1}(0)$ possesses a basis of neighborhoods consisting of proper bisectors admitting Hukuhara-functions in both directions.*

Given such a proper bisector $W(\infty) = [\mu, \eta]_\infty$ with direction $\mathcal{W}$ together with a Hukuhara-function a_i, one defines

$$A_i : \mathcal{W}_i \to \mathbb{R}_+ \ , \ A_i(\vartheta_i) := \begin{cases} 1 & \text{if } W_i(\infty) < 0 \\ \exp(\int_{\vartheta_i}^{\eta_i} \cot(a_i(\tau))\, d\tau) & \text{in all other cases.} \end{cases}$$

Definition A.4 *The* **Hukuhara-domain** *inside $W_i(\infty)$* **with radius** r_i *is defined as $S_i(r_i) := \{x_i \mid \arg(x_i) \in \mathcal{W} \text{ and } 0 < |x_i| \le r_i \cdot A_i(\arg(x_i))\}$.*

The domain for our solutions will be

$$S(r) := \begin{cases} S_1(r_1) \times S_2(r_2) & \text{if } m_2 > 0 \\ S_1(r_1) \times W_2(r_2) & \text{if } m_2 = 0 \ , \end{cases}$$

with the standard sector $W_2(r_2)$ with radius r_2.

Remark, that the given fixed ϑ^0 endows $S(r)$ with a canonical base-point $x^0(r) = (x_1^0(r_1), x_2^0(r_2))$ on the boundary of $S(r)$ with $\arg(x_i^0(r_i)) = \vartheta_i^0$.

Now, let there be given $r \in \mathbb{R}_+^2$, $N \in \mathbb{N}^2$ and $K > 0$. Consider the space $\mathcal{F}^{\mathrm{mod}}(N, r, K)$ of all continuous functions $\varphi : S(r) \to \mathbb{C}$ which are holomorphic in the interior of $S(r)$ and satisfy the estimate

$$|\varphi(x)| \le K \cdot |x|^{-N} := K \cdot |x_1|^{-N_1}|x_2|^{-N_2} \ .$$

Consider the vector space $\mathcal{F}^{\mathrm{mod}}(N, r) := \bigcup_{K>0} \mathcal{F}^{\mathrm{mod}}(N, r, K)$, which we endow with the norm

$$||\varphi|| := \sup |\varphi(x)| \cdot |x|^N \ .$$

For any $x \in S(r)$, we choose the path $\gamma(t)$ of integration ending at x depending on the type of the bisector:
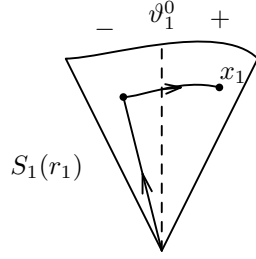
1. Case: $W(\infty) < 0$. Then $\gamma(t) = (\gamma_1(t), x_2)$, where γ_1 is the linear connection between 0 and x_1
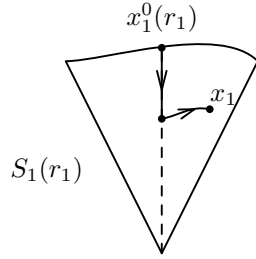
$$\gamma_1(t) := tx_1/|x_1| \ .$$

2. Case: $W(\infty)$ is of type $(+-)$ or $(-+)$. If $\cos(m_1 \arg(x_1) + m_2 \arg(x_2) - \arg(\mathcal{U}(0))) < 0$, we choose the same path as in the previous case. Otherwise, choose an auxiliary $\vartheta_1 \in \mathcal{W}_1$ such that $(\vartheta, \arg(x_2))$ fulfills this estimate. The path will then consist of two parts, the first one being the radial line between 0 and $(|x_1|A_1(\vartheta_1), x_2)$, the second part, which we will call the *Hukuhara-path*, is given by

$$[\vartheta_1, \arg x_1] \to S_1(r_1) \ , \ \vartheta \mapsto (|x_1| \exp \int_\vartheta^{\arg x_1} \cot(a_1(\tau))\, dt, x_2) \ ,$$

which we reparameterize by arclength (see the figure below for the case $(+-)$).

3. Case: $W(\infty) > 0$. Assume first that $m_2 > 0$. For $i = 1, 2$ consider the curve γ_i again consisting of two parts, one radial linear path from the base-point $x_i^0(r_i)$ of $S_i(r_i)$ to the point $|x_i|A_i(\vartheta_i^0)$, and another *Hukuhara-path* from this point a path running into x_i parallel to the boundary of $S_i(r_i)$. The paths over which we will integrate will be of the form $\gamma(t) = (\gamma_1(t), x_2)$ or $\gamma(t) = (x_1, \gamma_2(t))$.

In case $m_2 = 0$, let γ_1 be the same path in $S_1(r_1)$ as above and let γ_2 be the linear path between x_2^0 and x_2.

We emphasize, that all paths are parameterized by arclength. Then one has the following fundamental estimate (cp. with Lemma 1.7.1 in [19]):

Lemma A.5 *Let $\gamma(t)$ be any of the paths above of the form $\gamma(t) = (\gamma_1(t), x_2)$ for some x_2. Then, for given $N \in \mathbb{N}^2$ there exists a pair $r(N) \in \mathbb{R}_+^2$ such that for all $r \leq r(N)$ and all $x \in S(r)$:*

$$\frac{d}{dt}\big(|\gamma(t)|^{-N} \cdot e^{\operatorname{Re}\alpha(\gamma(t))}\big) \geq N_1|\gamma_1(t)|^{-N_1-1}|x_2|^{-N_2} \cdot e^{\operatorname{Re}\alpha(\gamma(t))} . \qquad (A.2)$$

Assuming $m_2 > 0$, a similar statement holds for the case $\gamma(t) = (x_1, \gamma_2(t))$ interchanging the indices.

Proof: We have to distinguish the different cases as above.

In case $W(\infty) < 0$, i.e. $e^{\alpha(x)}$ is rapidly decaying in $W(\infty)$ if x approaches 0, γ_1 is the radial line from 0 to x_1 and the left hand side of the inequality equals

$$t^{-N_1-1}|x_2|^{-N_2} \cdot e^{\operatorname{Re}\alpha} \cdot \Big(-N_1 + t\frac{d}{dt}\operatorname{Re}\alpha(\gamma(t))\Big) . \qquad (A.3)$$

Now $\alpha\gamma(t) = \gamma_1(t)^{-m_1}x_2^{-m_2}\mathcal{U}(\alpha\gamma(t))$ and hence $t\frac{d}{dt}\operatorname{Re}(\alpha\gamma(t)) =$

$$= t^{-m_1}|x_2|^{-m_2}(-m_1 C_1 \cos(m_1 \arg x_1 + m_2 \arg x_2 - \arg\mathcal{U}(\alpha\gamma(t))) + C_2 t) \quad (A.4)$$

where $C_1 := |\mathcal{U}(\alpha\gamma(t))| > 0$ is positive and $C_2 := |\frac{d}{dt}\mathcal{U}(\alpha\gamma(t))| \cdot \cos(m_1 \arg x_1 + m_2 \arg x_2 - \arg\mathcal{U}(\alpha\gamma(t)))$ is bounded. Since $W(\infty) < 0$, the cosine in (A.4) is negative for small r, i.e. small t (recall that the paths are parameterized by

arclength). It follows that $t\frac{d}{dt}\operatorname{Re}\alpha\gamma(t)$ can be made arbitrarily large by choosing r small enough.

The same argument applies to the radial part of γ_1 in the case $W(\infty)$ is of type $(+-)$ or $(-+)$, since this radial part also runs into the rapidly decaying sector inside $S_1(r_1)$. We are left to consider the Hukuhara-path, which parameterized by the angle ϑ reads as $\gamma(\vartheta) := (\eta(\vartheta), x_2)$ with

$$\eta(\vartheta) = |x_1| \exp\Big(\int_{\vartheta}^{\arg(x_1)} \cot(a_1(\tau))d\tau + i\vartheta \Big)$$

with the Hukuhara-function a_1 (cp. Definition A.2). We thus have

$$\frac{d}{d\vartheta}|\eta(\vartheta)| = -|\eta(\vartheta)|\cot(a_1(\vartheta)) . \tag{A.5}$$

Reparametrization by arclength t gives (recall that $0 < a_1(\vartheta) < \pi$)

$$\frac{dt}{d\vartheta} = |\frac{d\eta(\vartheta)}{d\vartheta}| = |\eta(\vartheta)(\cot(a_1(\vartheta)) + i\vartheta)| = |\eta(\vartheta)|(\sin(a_1(\vartheta)))^{-1} .$$

It follows that the left hand side of (A.2) equals

$$\pm\Big(N_1\cos(a_1(\vartheta)) + \sin(a_1(\vartheta))\frac{d}{d\vartheta}\operatorname{Re}\alpha(\gamma(\vartheta))\Big) \cdot |\gamma(t)|^{-N-1}e^{\operatorname{Re}\alpha(\gamma(t))} , \tag{A.6}$$

with the plus sign whenever $\arg(x_1) \in W_1(\infty)_+$ and the minus sign otherwise (cp. Definition A.1), since in the first case the angle ϑ increases along the Hukuhara-path whereas in the last case it decreases. Now $\operatorname{Re}\alpha(\gamma(\vartheta)) = |\eta(\vartheta)|^{-m_1}|x_2|^{-m_2}\cos(\arg(\alpha\gamma(\vartheta)))$ and using (A.5) one achieves

$$\frac{d}{dt}\Big(|\gamma(t)|^{-N}e^{\operatorname{Re}\alpha\gamma(t)}\Big) = \pm|\eta(t)|^{-N_1-1}|x_2|^{-N_2}e^{\operatorname{Re}\alpha\gamma(t)}\cdot$$
$$\cdot\big(- N_1\cos(a_1(\vartheta)) + m_1|\eta(\vartheta)|^{-m_1}|x_2|^{-m_2}\cdot R\cdot\cos(\rho(\vartheta))\big) , \tag{A.7}$$

with $\rho(\vartheta)$ of the form

$$\rho(\vartheta) = m_1\vartheta + m_2\arg(x_2) - \arg\mathcal{U}(\alpha\gamma(\vartheta)) + a_1(\vartheta) .$$

Choosing r small enough, the continuity of $\mathcal{U}$ and the definition of the Hukuhara-function $a_1(\vartheta)$ ensure that $\cos(\rho(\vartheta))$ has the appropriate sign, namely the same as the sign on the right hand side of (A.7). Hence, for r small enough, the term

$$m_1|\eta(\vartheta)|^{-m_1}|x_2|^{-m_2}\cdot R\cdot\cos(\rho(\vartheta))$$

dominates and can be made arbitrarily large.

The case $W(\infty) > 0$ is similar. The radial part of the path can be handled in the same way as the radial path in the case $W(\infty) < 0$, where one has to keep in mind that the direction is reversed, the path running inwards, and hence the sign in (A.3) changes:

$$\frac{d}{dt}\big(|\gamma(t)|^{-N}\cdot e^{\operatorname{Re}\alpha(\gamma(t))}\big) = (-1)\cdot(A.3) .$$

But since $W(\infty) > 0$, the behavior of $\operatorname{Re}\alpha(\gamma(t))$ changes in the same way, and the same argument as above applies. For the Hukuhara part, which is contained in either $W_1(\infty)_+$ or $W_1(\infty)_-$, the arguments about the sign in (A.6) applies and the proof is literally the same as above.

$\square$

A.2.2   The integral operators

We now define the integral operators whose fixed points will be the solutions to the system (Σ). We will not go too much into detail, since – once the fundamental estimate is established – this part of the proof is literally the same as in [21], C.13 – 16. Again, the integral operators will differ according to the type of $W(\infty)$:

If $W(\infty) \not> 0$, define

$$Tu(x) := e^{-\alpha(x_1, x_2)} \cdot \int_\gamma (-\lambda_1 u(\xi_1, x_2) + \rho_1(\xi_1, x_2)) \cdot e^{\alpha(\xi_1, x_2)} \frac{d\xi_1}{\xi_1} \ .$$

For $u \in \mathcal{F}^{\mathrm{mod}}(N, r, K)$, the fundamental estimate yields $|Tu(x)| \leq$

$$e^{-\mathrm{Re}\,\alpha(x)} \cdot \frac{K(1 + |\lambda_1|)}{N_1} |x_2|^{-N_2} \cdot \int_\gamma \frac{d}{dt}(|\xi_1|^{-N_1} e^{\mathrm{Re}\,\alpha(\xi_1, x_2)}) dt = \frac{K(1 + |\lambda_1|)}{N_1} \cdot |x|^{-N} \ ,$$

since $e^{\mathrm{Re}\,\alpha(\xi_1, x_2)}$ decreases rapidly for $\xi_1 \to 0$. In order to apply the fundamental estimate, we have to choose a $r \in \mathbb{R}_+^2$ small enough. Choosing N big enough, such that $1 + |\lambda_1| < N_1$, we see that T maps $\mathcal{F}^{\mathrm{mod}}(N, r, K)$ into itself and is contracting. The resulting fixed point $u \in \mathcal{F}^{\mathrm{mod}}(N, r, K)$ obviously solves (Σ_1). Using the integrability of the system (Σ), it follows that u solves the whole system. We omit the arguments here and refer to [21], (B.4) for the proof.

If $W(\infty) > 0$, we define $Tu(x) := e^{-\alpha(x)} \cdot (S_1 u + S_2 u)$, where

$$S_1 u := \int_{\gamma_1} (-\lambda_1 u(\xi_1, x_2) + \rho_1(\xi_1, x_2)) e^{\alpha(\xi_1, x_2)} \frac{d\xi_1}{\xi_1} \quad \text{and} \qquad \text{(A.8)}$$

$$S_2 u := \int_{\gamma_2} (-\lambda_2 u(x_1^0(r_1), \xi_2) + \rho_2(x_1^0(r_1), \xi_2)) e^{\alpha(x_1^0(r_1), \xi_2)} \frac{d\xi_2}{\xi_2} \ . \qquad \text{(A.9)}$$

Recall, that $x^0(r) =: x^0$ denotes the base-point in $S(r)$. Assume first that $m_2 > 0$. Let $N_1, N_2 \geq 2(1 + \max\{|\lambda_1|, |\lambda_2|\})$. Then the fundamental estimate of Lemma A.5 yields $|e^{-\alpha(x_1^0, x_2)} S_2 u| \leq$

$$\leq e^{-\mathrm{Re}\,\alpha(x_0^1, x_2)} \frac{K(1 + |\lambda_2|)}{N_2} |x_1^0|^{-N_1} \int_{\gamma_2} \frac{\partial}{\partial \xi_2}(|\xi_2|^{-N_2} e^{\mathrm{Re}\,\alpha(x_1^0, \xi_2)}) d\xi_2 \leq$$

$$\leq \frac{K}{2} |x_1^0|^{-N_1} |x_2|^{-N_2} \ . \quad \text{(A.10)}$$

But then

$$|e^{-\alpha(x)} S_2 u| \leq \frac{K}{2} |x_2|^{-N_2} \left(|x_1|^{-N_1} - e^{\mathrm{Re}\,\alpha(x)} \int_{\gamma_1} \frac{\partial}{\partial \xi_1}(|\xi_1|^{-N_1} e^{\mathrm{Re}\,\alpha(\xi_1, x_2)}) d\xi_1 \right) \ .$$

$$\text{(A.11)}$$

Using the fundamental estimate once again gives

$$|e^{-\alpha(x)} S_1 u| \leq e^{-\mathrm{Re}\,\alpha(x)} \frac{K(1 + |\lambda_1|)}{N_1} |x_2|^{-N_2} \int_{\gamma_1} \frac{\partial}{\partial \xi_1}(|\xi_1|^{-N_1} e^{\mathrm{Re}\,\alpha(\xi_1, x_2)}) d\xi_1 \ ,$$

$$\text{(A.12)}$$

and hence $|Tu| \leq \frac{K}{2} |x_1|^{-N_1} |x_2|^{-N_2}$. It follows that T is a contracting self-map of $\mathcal{F}^{\mathrm{mod}}(N, r, K)$. It is easy to see that the resulting fixed point u solves the

system (Σ), where for the verification of (Σ_2) one has to keep in mind that (Σ) is assumed to be integrable.

In the case $W(\infty) > 0$ and $m_2 = 0$, the proceeding is similar. Again, let $Tu = \exp(\alpha(x))(S_1 u + S_2 u)$ as above. Choose r small enough, such that

$$|e^{\alpha(x_1^0, x_2)}| \sup_{\xi_2 \in \operatorname{im}(\gamma_2)} |e^{-\alpha(x_1^0, \xi_2)}| < 2 \ ,$$

which is possible since $m_2 = 0$. Then, if $N_2 \geq 4(1 + |\lambda_2|)$, we have

$$|e^{-a(x_1^0, x_2)} S_2 u| \leq 2K \frac{1 + |\lambda_2|}{N_2} |x_1^0|^{-N_1} (|x_2|^{-N_2} - |x_2^0|^{-N_2}) \leq \frac{K}{2} |x_1^0|^{-N_1} |x_2|^{-N_2} \ ,$$

hence (A.10) remains valid. Since $m_1 > 0$, the fundamental estimate on γ_1 and then (A.12) are again true, which proves the claim.

A.3   The case (pREG) and the case at a smooth point

The case (pREG) bears no more difficulty, since then $\alpha(x) = \mathcal{U}(x)$ has no singularity along $\pi^{-1}(D)$. Replacing the Hukuhara-domain $S(r)$ in the case (IRR) by the standard bisector $W(r)$ and choosing γ_i to be the linear path between x_i^0 and x_i, the integral operator

$$Tu := e^{-\alpha(x)}(S_1 u + S_2 u)$$

with S_1 and S_2 as in (A.8) and (A.9) is a contracting self-map of $\mathcal{F}^{\mathrm{mod}}(N, r, K)$. The necessary estimates are easily deduced from the boundedness of $|e^{-\alpha(x)}|$ in that case.

This completes the proof of Theorem A.1 in the local case over a crossing-point of D. For a smooth point $0 \in D = \{x_1 = 0\}$, the same arguments apply after the following replacements in section A.2.2. In the case $W(\infty) \not> 0$, the second integral operator (A.9) has to be replaced by

$$S_2 u := \int_{\gamma_2} \rho_2(x_1^0(r_1), \xi_2) e^{\alpha(x_1^0(r_1), \xi_2)} \, d\xi_2 \ .$$

The estimate (A.10) can be achieved again as in the case (pREG) before, since $|e^{\alpha(x_1^0, x_2)}|$ is bounded for fixed x_1^0. The estimate (A.12) again holds because of the fundamental estimate in the irregular case $m_1 > 0$, or because of the boundedness of the exponential factor in the purely regular case.

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