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Low Mach number limit of a diffuse interface model for two-phase flows of compressible viscous fluids

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Abstract

In this paper, we consider a singular limit problem for a diffuse interface model for two immiscible compressible viscous fluids. Via a relative entropy method, we obtain a convergence result for the low Mach number limit to a corresponding system for incompressible fluids in the case of well-prepared initial data and same densities in the limit.

KEYWORDS

Cahn–Hilliard, diffuse interface model, low Mach number limit, Navier–Stokes, two-phase flow

MOS SUBJECT CLASSIFICATION

35B25; 76N10; 35Q30; 35Q35; 76T99

1 | INTRODUCTION AND MAIN RESULT

Diffuse interface models are an important modeling approach to describe two- or multi-phase flows in fluid mechanics. In comparison with classical sharp interface models they have the theoretical and practical advantage that surfaces separating the fluids do not need to be resolved explicitely. In the case of two fluids the (diffuse) interface is described as the region, where an order parameter, which will be the concentration difference of the two fluids in the following, is not close to one of two values, which describe the presence of only one fluid (± 1 in the following).

In this contribution we consider the relation between two diffuse interface models for a two-phase flow of viscous Newtonian fluids. The first one is for the case of compressible fluids and leads to the Navier-Stokes/Cahn-Hilliard system for compressible fluids:

$$\varrho\partial_t \mathbf{v} + \varrho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbb{S} + \frac{1}{M}\nabla p = -\operatorname{div}\left(\nabla c\otimes\nabla c - \frac{|\nabla c|^2}{2}\mathbb{I}\right), \tag{1.1}$$

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{v}) = 0, \tag{1.2}$$

$$\varrho\partial_t c + \varrho\mathbf{v}\cdot\nabla c = \Delta\mu, \tag{1.3}$$

$$\varrho\mu = \varrho\frac{1}{M}\frac{\partial f}{\partial c} - \Delta c, \tag{1.4}$$

in $\Omega\times(0,T)$, where $\Omega\subseteq\mathbb{R}^3$ is a bounded C^2-domain and $p = \varrho^2\frac{\partial f}{\partial\varrho}(\varrho, c)$ and

$$\mathbb{S} = 2\nu(c)D\mathbf{v} + \eta(c)\operatorname{div}\mathbf{v}\,\mathbb{I}, \tag{1.5}$$

$$D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) - \frac{1}{3}\operatorname{div}\mathbf{v}\mathbb{I}.$$

Here $c : \Omega \times (0, T) \to \mathbb{R}$ describes the concentration difference of two partly miscible compressible fluids, $\rho : \Omega \times (0, T) \to [0, \infty)$ is the density of the fluid mixture and $\mathbf{v} : \Omega \times (0, T) \to \mathbb{R}^3$ its (barycentric) mean velocity. Moreover, $\lambda, \eta : \mathbb{R} \to [0, \infty)$ are functions describing the shear and bulk viscosity of the mixture, $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a homogeneous free energy density of the mixture and $M > 0$ is an analogue of a Mach number. Precise assumptions will be given below. This system is a variant of the model derived by Lowengrub and Truskinovsky [34] in a non-dimensionalized form, compare of also [4]. Here we have set the Reynolds and Peclet number to one for simplifity and the Cahn number proportional to M, compare of [34, eq. (3.35)] for the details. We note that in the present variant the total free energy is given by

$$E_{\text{free}}(\varrho, c) = \int_\Omega \left(\varrho f(\varrho, c) + \frac{1}{2} |\nabla c|^2 \right) \mathrm{d}x, \tag{1.6}$$

while in [34] there is an additional factor ρ in front of $|\nabla c|^2$. The system is closed by the initial and boundary conditions

$$\mathbf{v}|_{\partial\Omega} = \nabla c \cdot \mathbf{n}|_{\partial\Omega} = \nabla \mu \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.7}$$

$$(\mathbf{v}, c)|_{t=0} = (\mathbf{v}_0, c_0), \tag{1.8}$$

where $\mathbf{n}$ is the exterior normal of Ω. Existence of weak solutions for this system was proved by Feireisl and the first author in [2]. This result was extended to the case of certain dynamic boundary conditions by Cherfils et al. [6]. Existence and uniqueness of strong solutions for this system was shown by Kotschote and Zacher [30], see also [29]. Existence of dissipative martingal solutions of a stochastically perturbed version of this system was shown by Feireisl and Petcu [18]. In the time-independent, stationary situation existence of weak solutions was shown by Liang and Wang [31, 32]. Existence of weak solutions for a similar Navier-Stokes/Allen-Cahn system for compressible fluids was shown by Feireisl et al. [21], while an entropy stable finite volume method for this instationary system was proposed by Feireisl, Petcu and She [20], where also existence of weak solutions of the discretized system was shown. For this system Feireisl, Petcu, and Pražák [19] studied a relative entropy and obtained results on weak-strong uniqueness and on a low Mach number limit similar to our result in the following.

It is the goal of this contribution to study the low Mach number limit $M \to 0$ for (1.1)–(1.4) and show convergence to solutions of the system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \operatorname{div}(2\nu(c) D\mathbf{v}) + \nabla \pi = -\operatorname{div}(\nabla c \otimes \nabla c), \tag{1.9a}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{1.9b}$$

$$\partial_t c + \mathbf{v} \cdot \nabla c = \Delta \mu, \tag{1.9c}$$

$$\mu = -\Delta c + G'(c). \tag{1.9d}$$

under suitable assumptions and well-prepared initial data. We note that we consider a situation, where the two fluids in limit $M \to 0$ have the same density (or the density difference is neglected). The latter system is known as “model H” and is one of the basic diffuse interface models for the two-phase flow of incompressible fluids. It first appeared in Hohenberg and Halperin [26] and was later derived in the framework of rational continuum mechanics by Gurtin et al. [24]. A first analytic result on existence of strong solutions, if $\Omega = \mathbb{R}^2$ and G is a suitably smooth double well potential was obtained by Starovoitov [36]. More complete results were presented by Boyer [5] in the case that $\Omega \subset \mathbb{R}^d$ is a periodical channel and a smooth double well potential G and the first author in [1] in the case of a bounded smooth domain and singular double well potential. We refer to Abels, Giorgini, and Garcke [3] for recent analytic results for an extension of this model to different densities and further references.

We note that, using (1.9d), one observes that (1.9a) is equivalent to

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} - \operatorname{div}(2\nu(c) D\mathbf{v}) + \nabla \pi = \mu \nabla c - \nabla G(c).$$

The mathematical study of the low Mach number limit for systems of equations describing a motion of fluids gets back to the seminal work of Klainerman and Majda [27]. Studying various types of singular limits allows us to eliminate unimportant or unwanted modes of the motion as a consequence of scaling and asymptotic analysis. The aim of the mathematical analysis of low Mach number limits is to fill up the gap between compressible fluids and their "idealized" incompressible models. There are two ways to introduce the Mach number into the system, which are different from the physical point of view, but from the mathematical one–completely equivalent. The first approach considers a varying equation of state as well as the transport coefficients see the works of Ebin [12], Schochet [35]. The second way is to evaluate qualitatively the incompressibility using the dimensional analysis. We rewrite our system in the dimensionless form by scaling each variable by its characteristic value, see Klein [28]. The mathematical analysis of singular limits in the frame of strong solutions can be referred to works of Gallagher [22], Schochet [35], Danchin [8] or Hoff [25]. The seminal works of Lions [33] and its extension by Feireisl et al. [17] on the existence of global weak solutions in the barotropic case gave a new possibility of a rigorous study of singular limits in the frame of weak solutions, see the works of Desjardins and Grenier [9], Desjardins, Grenier, Lions and Masmoudi [10].

The relative energy inequality was introduced by Dafermos [7] and in the fluid dynamic context was introduced by Germain [23]. Deriving the relative energy inequality for sufficiently smooth test functions and proving the weak-strong uniqueness it gives us very powerful and elegant tool for the purpose of measuring the stability of a solution compared to another solution with better regularity. This method was developed by Feireisl, Novotný and co-workers in the framework of singular limits problems (see e.g., [13–16] and references therein).

The structure of this contribution is as follows: In Section 2 we summarize our assumptions, basic definitions and state our main result on the low Mach number limit. Then in Section 3 we prove the main result with the aid of a relative entropy method.

Notations

In the manuscript, we denote the usual Lebesgue and Sobolev spaces by L^p and $W^{k,p}$ respectively for $1 \leq p \leq \infty$, $k \geq 0$. The corresponding norms are $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$. In particular, we define $H^k := W^{k,2}$. Throughout the paper, the letter C will indicate a generic positive constant that may change its value from line to line, or even in the same line.

2 | ASSUMPTIONS AND MAIN RESULT

We assume that f is given in the form

$$f(\varrho, c) = f_{\mathrm{e}}(\varrho) + MG(c). \tag{2.1}$$

This choice coincides with the assumptions in [2] with $H \equiv 0$ therein. We only added the factor M in front of G, which can be incorporated in G. This yields

$$p(\varrho, c) = \varrho^2 \frac{\partial f(\varrho, c)}{\partial \varrho} = p_{\mathrm{e}}(\varrho), \; f_{\mathrm{e}}(\varrho) = \int_1^{\varrho} \frac{p_{\mathrm{e}}(z)}{z^2}\, \mathrm{d}z \tag{2.2}$$

where $p_{\mathrm{e}} \in C([0,\infty)) \cap C^1(0,\infty)$. Moreover, it was assumed that

$$p_{\mathrm{e}}(0) = 0, \; \underline{p}_1 \varrho^{\gamma-1} - \underline{p}_2 \leq p_{\mathrm{e}}'(\varrho) \leq \overline{p}(1 + \varrho^{\gamma-1}) \tag{2.3}$$

for a certain $\gamma > \frac{3}{2}$ and

$$\begin{aligned} G''(c) \geq -\kappa \text{ for some } \kappa \in \mathbb{R}, \; \underline{G}_1 c - \underline{G}_2 \leq G'(c) \leq \overline{G}(1+c), \\ \left|G'(c_1) - G'(c_2)\right| \leq \overline{G}|c_1 - c_2|, \; \left|G''(c_1) - G''(c_2)\right| \leq \overline{G}|c_1 - c_2| \end{aligned} \tag{2.4}$$

for all $c, c_1, c_2 \in \mathbb{R}$. Hence (1.1)–(1.4) reduce to

$$\varrho_\varepsilon \partial_t \mathbf{v}_\varepsilon + \varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla \mathbf{v}_\varepsilon - \operatorname{div} \mathbb{S}_\varepsilon + \frac{1}{\varepsilon^2} \nabla (p_e(\varrho_\varepsilon) - p_e(1)) = \varrho_\varepsilon \mu_\varepsilon \nabla c_\varepsilon - \varrho_\varepsilon G'(c_\varepsilon) \nabla c_\varepsilon, \tag{2.5a}$$

$$\partial_t \varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon \mathbf{v}_\varepsilon) = 0, \tag{2.5b}$$

$$\varrho_\varepsilon \partial_t c_\varepsilon + \varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon = \Delta \mu_\varepsilon, \tag{2.5c}$$

$$\varrho_\varepsilon \mu_\varepsilon = \varrho_\varepsilon G'(c_\varepsilon) - \Delta c_\varepsilon, \tag{2.5d}$$

subject to the boundary conditions

$$\mathbf{v}_\varepsilon|_{\partial\Omega} = \nabla c_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = \nabla \mu_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.5e}$$

Let us recall the definition of weak solutions in the sense of [2, theorem 1.2] (with $H \equiv 0$ there):

Definition 2.1. Let $T > 0$, $Q_T = \Omega \times (0,T)$, $\varrho_{0,\varepsilon} \in L^\gamma(\Omega)$ with $\varrho_{0,\varepsilon} \geq 0$ almost everywhere, and $\mathbf{m}_{0,\varepsilon} : \Omega \to \mathbb{R}^3$ be measurable such that $\varrho_{0,\varepsilon}^{-1}|\mathbf{m}_{0,\varepsilon}|^2 \in L^1(\Omega)$. Then $\varrho_\varepsilon \in L^\infty(0,T;L^\gamma(\Omega))$ with $\varrho_\varepsilon \geq 0$, $\mathbf{v}_\varepsilon \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$, $c_\varepsilon \in L^\infty(0,T;H^1(\Omega))$ are a weak solution of (2.5) if the following holds true:

1. For every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times (0,T);\mathbb{R}^3)$

$$-\int_{Q_T} \left(\varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \left(\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon + \frac{1}{M} p_e(\varrho_\varepsilon)\, \mathbb{I} - \mathbb{S}_\varepsilon\right) : \nabla \boldsymbol{\varphi}\right) \mathrm{d}x\mathrm{d}t$$
$$= \int_{Q_T} \left((\nabla c_\varepsilon \otimes \nabla c_\varepsilon) : \nabla \boldsymbol{\varphi} - \frac{|\nabla c_\varepsilon|^2}{2} \operatorname{div} \boldsymbol{\varphi}\right) \mathrm{d}x\mathrm{d}t, \tag{2.6}$$

   where $\mathbb{S}_\varepsilon = 2\nu(c_\varepsilon) D\mathbf{v}_\varepsilon + \eta(c_\varepsilon) \operatorname{div} \mathbf{v}_\varepsilon\, \mathbb{I}$.
2. ϱ_ε is a renormalized solution of (2.5b) in the sense of DiPerna and Lions [11], that is,

$$\int_{Q_T} (\varrho_\varepsilon B(\varrho_\varepsilon) \partial_t \varphi + \varrho_\varepsilon B(\varrho_\varepsilon) \mathbf{v}_\varepsilon \cdot \nabla \varphi - b(\varrho_\varepsilon) \operatorname{div} \mathbf{v}_\varepsilon\, \varphi)\, \mathrm{d}x\mathrm{d}t = 0 \tag{2.7}$$

   for any test function $\varphi \in \mathcal{D}(\overline{\Omega} \times (0,T))$, and any

$$B(\varrho_\varepsilon) = B(1) + \int_1^{\varrho_\varepsilon} \frac{b(z)}{z^2}\, \mathrm{d}z, \tag{2.8}$$

   where $b \in C^0([0,\infty))$ is a bounded function.
3. For every $\varphi \in \mathcal{D}(\Omega \times (0,T))$

$$\int_{Q_T} (\varrho_\varepsilon c_\varepsilon\ \partial_t \varphi + \varrho_\varepsilon c_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla \varphi)\, \mathrm{d}x\mathrm{d}t = \int_{Q_T} \nabla \mu_\varepsilon \cdot \nabla \varphi\, \mathrm{d}x\mathrm{d}t \tag{2.9}$$

   and

$$\int_{Q_T} \varrho_\varepsilon \mu_\varepsilon \varphi\, \mathrm{d}x\mathrm{d}t = \int_{Q_T} \left(\varrho_\varepsilon G'(c_\varepsilon)\varphi + \nabla c_\varepsilon \cdot \nabla \varphi\right) \mathrm{d}x\mathrm{d}t. \tag{2.10}$$

4. The energy inequality

$$E(t) + \int_{Q_\tau} \left(\mathbb{S}_\varepsilon : \nabla \mathbf{v}_\varepsilon + |\nabla \mu_\varepsilon|^2\right) \mathrm{d}x\mathrm{d}\tau \leq E(s) \tag{2.11}$$

   holds for almost every $0 \leq s \leq T$ including $s = 0$ and all $t \in [s,T]$, where

$$E(t) = \int_\Omega \varrho_\varepsilon(t) \frac{|\mathbf{v}_\varepsilon(t)|^2}{2}\, \mathrm{d}x + E_{\text{free}}(\varrho_\varepsilon(t), c_\varepsilon(t)), \tag{2.12}$$

$$E(0) = E_0 = \int_\Omega \varrho_{0,\varepsilon}^{-1} \frac{|\mathbf{m}_{0,\varepsilon}|^2}{2}\, \mathrm{d}x + E_{\text{free}}(\varrho_{0,\varepsilon}, c_{0,\varepsilon}). \tag{2.13}$$

5. $\varrho_\varepsilon, \varrho_\varepsilon \mathbf{v}_\varepsilon, c_\varepsilon$ are weakly continuous with respect to $t \in [0,T]$ with values in $L^1(\Omega)$ and $\varrho_\varepsilon|_{t=0} = \varrho_{0,\varepsilon}$, $\varrho_\varepsilon \mathbf{v}_\varepsilon|_{t=0} = \mathbf{m}_{0,\varepsilon}$, $c_\varepsilon|_{t=0} = c_{0,\varepsilon}$.

We note that existence of weak solutions follows from [2, theorem 1.2] and the well-posedness result of the limit system (1.9) is referred to [1].

Our main result in this contribution is:

Theorem 2.2. Let $\gamma \geq \frac{12}{5}$, $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with C^2-boundary, $T > 0$, $M = \varepsilon^2$, $\varepsilon > 0$, and let $(\mathbf{v}, c, \mu)$ be a (sufficiently) smooth solution of (1.9). Moreover, we assume that $\varrho_{0,\varepsilon} \in L^\gamma(\Omega)$, $\mathbf{v}_{0,\varepsilon} \in L^2(\Omega)^3$, $c_{0,\varepsilon} \in H^1(\Omega)$ are given such that $\varrho_{0,\varepsilon} := 1 + \varepsilon \varrho^{(1)}_{0,\varepsilon}$ and

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty(\Omega)} + \|\mathbf{v}_{0,\varepsilon}\|_{L^2(\Omega)} + \|\nabla c_{0,\varepsilon}\|_{L^2(\Omega)} \leq C. \tag{2.14}$$

and $\varrho^{(1)}_{0,\varepsilon} \to_{\varepsilon\to 0} 0$ in $L^\infty(\Omega)$, $\mathbf{v}_{0,\varepsilon} \to_{\varepsilon\to 0} \mathbf{v}_0$ in $L^2(\Omega)^3$, $c_{0,\varepsilon} \to_{\varepsilon\to 0} c_0$ in $H^1(\Omega)$. Then

$$\varrho_\varepsilon(t,\cdot) \to_{\varepsilon\to 0} 1 \text{ in } L^1(\Omega), \quad \mathbf{v}_\varepsilon(t,\cdot) \to_{\varepsilon\to 0} \mathbf{v} \text{ in } L^2(\Omega)^3, \quad c_\varepsilon(t,\cdot) \to_{\varepsilon\to 0} c(t,\cdot) \text{ in } H^1(\Omega)$$

uniformly in $t \in [0, T]$.

Remark 2.3. Here the restriction of $\gamma \geq \frac{12}{5}$ comes essentially from addressing the nonconvex part of the potential G, cf. (3.11). If one considers a convex potential G, we may relax it to $\gamma \geq 2$, which is due to the convection $\varrho_\varepsilon \mu_\varepsilon \mathbf{v} \cdot \nabla c_\varepsilon$, compare of (3.27).

3 | LOW MACH LIMIT

3.1 | Relative energy inequality

To compare (2.5) and (1.9), we proceed with the so-called *relative energy* (*entropy*):

$$\begin{aligned}\mathcal{E}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c\right) := &\int_\Omega \left[\frac{1}{2}\varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{v}|^2 + \frac{1}{\varepsilon^2}\left(F_e(\varrho_\varepsilon) - F_e'(1)(\varrho_\varepsilon - 1) - F_e(1)\right)\right] \mathrm{d}x \\ &+ \int_\Omega \left[\frac{1}{2}|\nabla c_\varepsilon - \nabla c|^2 + \varrho_\varepsilon(G(c_\varepsilon) - G'(c)(c_\varepsilon - c) - G(c))\right] \mathrm{d}x,\end{aligned}$$

where $F_e(\varrho_\varepsilon) = \varrho_\varepsilon f_e(\varrho_\varepsilon)$. Here $(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon)$ is a weak solution to (2.5) depending on ε in the sense of Definition 2.1, while $(1, \mathbf{v}, c)$ is a pair of smooth test functions which is then chosen as the solution to (1.9).

By the weak formulation of (2.5a) for $\mathbf{v}_\varepsilon$, that is, (2.6), with $\mathbf{v}$ as the test function we obtain for every $\tau \in [0, T]$

$$\begin{aligned}-\left[\int_\Omega \varrho_\varepsilon \mathbf{v}_\varepsilon \mathbf{v} \,\mathrm{d}x\right]_{t=0}^{t=\tau} = &-\int_{Q_\tau} \varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \mathbf{v} \,\mathrm{d}x\mathrm{d}t \\ &-\int_{Q_\tau} (\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon - 2\nu(c_\varepsilon) D\mathbf{v}_\varepsilon) : \nabla \mathbf{v} \,\mathrm{d}x\mathrm{d}t \\ &-\int_{Q_\tau} \varrho_\varepsilon \mu_\varepsilon \nabla c_\varepsilon \cdot \mathbf{v} \,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} \varrho_\varepsilon \mathbf{v} \cdot \nabla c_\varepsilon G'(c_\varepsilon) \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.1}$$

Let $\frac{1}{2}|\mathbf{v}|^2$ be the test function in the weak formulation of continuity Equation (2.5b). This yields

$$\left[\int_\Omega \frac{\varrho_\varepsilon}{2}|\mathbf{v}|^2 \,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_{Q_\tau} (\varrho_\varepsilon \mathbf{v} \cdot \partial_t \mathbf{v} + \varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla \mathbf{v} \cdot \mathbf{v}) \,\mathrm{d}x\mathrm{d}t \tag{3.2}$$

for every $\tau \in [0, T]$. Summing (3.1), (3.2) and the energy inequality of weak solutions, one ends up with

$$\begin{aligned}&\left[\int_\Omega \frac{\varrho_\varepsilon}{2}|\mathbf{v}_\varepsilon - \mathbf{v}|^2 \,\mathrm{d}x\right]_{t=0}^{t=\tau} \\ &+ \left[\int_\Omega \left(\varrho_\varepsilon G(c_\varepsilon) + \frac{1}{2}|\nabla c_\varepsilon|^2\right) \mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega \frac{1}{\varepsilon^2} F_e(\varrho_\varepsilon) \,\mathrm{d}x\right]_{t=0}^{t=\tau}\end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_\tau} 2\nu(c_\varepsilon)|D\mathbf{v}_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t+\int_{Q_\tau}|\nabla\mu_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t \\
&\le -\int_{Q_\tau}\varrho_\varepsilon(\mathbf{v}_\varepsilon-\mathbf{v})\cdot(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v})\,\mathrm{d}x\mathrm{d}t-\int_{Q_\tau}\varrho_\varepsilon(\mathbf{v}_\varepsilon-\mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{v}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_\tau}2\nu(c_\varepsilon)D\mathbf{v}_\varepsilon:\nabla\mathbf{v}\,\mathrm{d}x\mathrm{d}t-\int_{Q_\tau}\varrho_\varepsilon\mu_\varepsilon\nabla c_\varepsilon\cdot\mathbf{v}\,\mathrm{d}x\mathrm{d}t+\int_{Q_\tau}\varrho_\varepsilon\mathbf{v}\cdot\nabla c_\varepsilon G'(c_\varepsilon)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.3}
$$

In view of the weak formulation of (2.5c), we have

$$
\left[\int_\Omega \varrho_\varepsilon c_\varepsilon\mu\,\mathrm{d}x\right]_{t=0}^{t=\tau}-\int_{Q_\tau}(\varrho_\varepsilon c_\varepsilon\partial_t\mu+\varrho_\varepsilon c_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mu)\,\mathrm{d}x\mathrm{d}t=-\int_{Q_\tau}\nabla\mu_\varepsilon\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t.
\tag{3.4}
$$

Moreover, with $\mu=G'(c)-\Delta c$ one gets

$$
\int_{Q_\tau}\varrho_\varepsilon c_\varepsilon\partial_t\mu\,\mathrm{d}x\mathrm{d}t=\int_{Q_\tau}\varrho_\varepsilon c_\varepsilon\partial_t(G'(c)-\Delta c)\,\mathrm{d}x\mathrm{d}t.
\tag{3.5}
$$

Adding (3.4) and (3.5) and integration by parts gives

$$
\begin{aligned}
&\left[-\int_\Omega\varrho_\varepsilon c_\varepsilon G'(c)-\nabla c_\varepsilon\cdot\nabla c\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&=\left[\int_\Omega(\varrho_\varepsilon-1)c_\varepsilon\Delta c\,\mathrm{d}x\right]_{t=0}^{t=\tau}-\int_{Q_\tau}\varrho_\varepsilon c_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_\tau}\nabla\mu_\varepsilon\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t-\int_{Q_\tau}\varrho_\varepsilon c_\varepsilon(G''(c)\partial_t c-\Delta\partial_t c)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.6}
$$

Direct calculations yield

$$
\begin{aligned}
\left[\int_\Omega\varrho_\varepsilon cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau}&=\left[\int_\Omega(\varrho_\varepsilon-1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau}+\int_0^\tau\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega cG'(c)\,\mathrm{d}x\mathrm{d}t \\
&=\left[\int_\Omega(\varrho_\varepsilon-1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau}+\int_{Q_\tau}\partial_t cG'(c)\,\mathrm{d}x\mathrm{d}t+\int_{Q_\tau}cG''(c)\partial_t c\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\left[\int_\Omega\varrho_\varepsilon cG'(c)-\varrho_\varepsilon G(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau}&=\left[\int_\Omega(\varrho_\varepsilon-1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&+\left[\int_\Omega(1-\varrho_\varepsilon)G(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau}+\int_{Q_\tau}cG''(c)\partial_t c\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.7}
$$

It follows from the strong formulation of (1.9) that

$$
\left[\int_\Omega\frac{1}{2}|\nabla c|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau}=-\int_{Q_\tau}G'(c)\partial_t c\,\mathrm{d}x\mathrm{d}t-\int_{Q_\tau}|\nabla\mu|^2\,\mathrm{d}x\mathrm{d}t-\int_{Q_\tau}\mathbf{v}\cdot\nabla c\mu\,\mathrm{d}x\mathrm{d}t.
\tag{3.8}
$$

Now we summarize from (3.3), (3.6), (3.7), (3.8) that

$$
\begin{aligned}
&\left[\int_\Omega\frac{\varrho_\varepsilon}{2}|\mathbf{v}_\varepsilon-\mathbf{v}|^2\,\mathrm{d}x\right]_{t=0}^{t=\tau}+\left[\int_\Omega\frac{1}{\varepsilon^2}F_e(\varrho_\varepsilon)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&+\left[\int_\Omega\left(\varrho_\varepsilon(G(c_\varepsilon)-G'(c)(c_\varepsilon-c)-G(c))+\frac{1}{2}|\nabla c_\varepsilon-\nabla c|^2\right)\,\mathrm{d}x\right]_{t=0}^{t=\tau}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{Q_\tau} 2\nu(c_\varepsilon)D\mathbf{v}_\varepsilon : (D\mathbf{v}_\varepsilon - D\mathbf{v})\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} |\nabla\mu_\varepsilon - \nabla\mu|^2\,\mathrm{d}x\mathrm{d}t \\
&\le -\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v})\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{v} - \mathbf{v}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&+\left[\int_\Omega (\varrho_\varepsilon - 1)c_\varepsilon\Delta c\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (\varrho_\varepsilon - 1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (1-\varrho_\varepsilon)G(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&-\int_{Q_\tau} \varrho_\varepsilon\mu_\varepsilon\nabla c_\varepsilon\cdot\mathbf{v}\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} \varrho_\varepsilon\mathbf{v}\cdot\nabla c_\varepsilon G'(c_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&-\int_{Q_\tau} \varrho_\varepsilon c_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \varrho_\varepsilon c_\varepsilon(G''(c)\partial_t c - \Delta\partial_t c)\,\mathrm{d}x\mathrm{d}t \\
&+\int_{Q_\tau} cG''(c)\partial_t c\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} G'(c)\partial_t c\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \mathbf{v}\cdot\nabla c\mu\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

In view of the strong formulation of (1.9a),

$$
\begin{aligned}
&-\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v})\,\mathrm{d}x\mathrm{d}t \\
&\quad = -\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot(\operatorname{div}(2\nu(c)D\mathbf{v}) - \nabla\pi + \mu\nabla c - \nabla G(c))\,\mathrm{d}x\mathrm{d}t \\
&\quad = -\int_{Q_\tau} (\varrho_\varepsilon - 1)(\mathbf{v}_\varepsilon - \mathbf{v})\cdot(\operatorname{div}(2\nu(c)D\mathbf{v}) + \mu\nabla c)\,\mathrm{d}x\mathrm{d}t \\
&\qquad -\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(\pi + G(c))\,\mathrm{d}x\mathrm{d}t \\
&\qquad -\int_{Q_\tau} (\mathbf{v}_\varepsilon - \mathbf{v})\cdot(\operatorname{div}(2\nu(c)D\mathbf{v}))\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} (\mathbf{v}_\varepsilon - \mathbf{v})\cdot\mu\nabla c\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

By integration by parts, one obtains

$$
\begin{aligned}
&-\int_{Q_\tau} (\mathbf{v}_\varepsilon - \mathbf{v})\cdot(\operatorname{div}(2\nu(c)D\mathbf{v}))\,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_\tau} (D\mathbf{v}_\varepsilon - D\mathbf{v}) : (2\nu(c)D\mathbf{v})\,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_\tau} (D\mathbf{v}_\varepsilon - D\mathbf{v}) : (2\nu(c_\varepsilon)D\mathbf{v})\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} (D\mathbf{v}_\varepsilon - D\mathbf{v}) : (2(\nu(c) - \nu(c_\varepsilon))D\mathbf{v})\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Then adding all together with (2.5b) multiplied by $F_\varepsilon'(\varrho_\varepsilon)$ entails that

$$
\begin{aligned}
&[\mathcal{E}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c)]_{t=0}^{t=\tau} \\
&\quad + \int_{Q_\tau} 2\nu(c_\varepsilon)|D\mathbf{v}_\varepsilon - D\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} |\nabla\mu_\varepsilon - \nabla\mu|^2\,\mathrm{d}x\mathrm{d}t \\
&\le \left[\int_\Omega (\varrho_\varepsilon - 1)c_\varepsilon\Delta c\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (\varrho_\varepsilon - 1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (1-\varrho_\varepsilon)G(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\quad - \int_{Q_\tau} (\varrho_\varepsilon - 1)(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\operatorname{div}(2\nu(c)D\mathbf{v})\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{v} - \mathbf{v}_\varepsilon)\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(\pi + G(c))\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(c\mu)\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mu(c_\varepsilon - c)\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{Q_\tau} (\varrho_\varepsilon - 1)\mathbf{v} \cdot \nabla \mu c_\varepsilon \,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \mathbf{v} \cdot \nabla (c_\varepsilon - c)(\mu_\varepsilon - \mu) \,\mathrm{d}x\mathrm{d}t \\
&- \int_{Q_\tau} (\varrho_\varepsilon - 1)\mu_\varepsilon \nabla c_\varepsilon \cdot \mathbf{v} \,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} (\varrho_\varepsilon - 1)\mathbf{v} \cdot \nabla c_\varepsilon G'(c_\varepsilon) \,\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} (D\mathbf{v}_\varepsilon - D\mathbf{v}) : (2(\nu(c) - \nu(c_\varepsilon))D\mathbf{v}) \,\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_\tau} \varrho_\varepsilon \partial_t c \left(G'(c_\varepsilon) - (c_\varepsilon - c)G''(c) - G'(c) \right) \,\mathrm{d}x\mathrm{d}t \\
&- \int_{Q_\tau} (\varrho_\varepsilon - 1) \left(G''(c) c \partial_t c - G'(c)\partial_t c \right) \,\mathrm{d}x\mathrm{d}t \\
&- \int_{Q_\tau} (\varrho_\varepsilon - 1) c_\varepsilon \Delta \partial_t c \,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} (\varrho_\varepsilon - 1)\mu_\varepsilon \partial_t c \,\mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{3.9}
$$

where we used

$$
\begin{aligned}
\int_{Q_\tau} \varrho_\varepsilon \mathbf{v} \cdot \nabla c_\varepsilon G'(c_\varepsilon) \,\mathrm{d}x\mathrm{d}t &= \int_{Q_\tau} (\varrho_\varepsilon - 1)\mathbf{v} \cdot \nabla c_\varepsilon G'(c_\varepsilon) \,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} \mathbf{v} \cdot \nabla G(c_\varepsilon) \,\mathrm{d}x\mathrm{d}t \\
&= \int_{Q_\tau} (\varrho_\varepsilon - 1)\mathbf{v} \cdot \nabla c_\varepsilon G'(c_\varepsilon) \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Concerning the potential part of G, here due to the assumption (2.4) we employ the decomposition of G such that $G(c) = G_0(c) + G_1(c)$ with $G_1(c) = -\kappa \frac{c^2}{2}$ for $\kappa > 0$ where $G_0(c)$ is convex. Then we have

$$
\int_\Omega \varrho_\varepsilon (G_0(c_\varepsilon) - G_0'(c)(c_\varepsilon - c) - G_0(c)) \,\mathrm{d}x \geq 0,
$$

and

$$
- \left[\int_\Omega \varrho_\varepsilon (G_1(c_\varepsilon) - G_1'(c)(c_\varepsilon - c) - G_1(c)) \,\mathrm{d}x \right]_{t=0}^{t=\tau} = \left[\int_\Omega \varrho_\varepsilon \frac{\kappa}{2} (c_\varepsilon - c)^2 \,\mathrm{d}x \right]_{t=0}^{t=\tau}. \tag{3.10}
$$

As $G_1(c)$ is a nonconvex part, in the following we would like to justify the following identity to ensure a suitable relative energy inequality:

$$
\begin{aligned}
&\left[\int_\Omega \frac{\varrho_\varepsilon}{2} (c_\varepsilon - c)^2 \,\mathrm{d}x \right]_{t=0}^{t=\tau} \\
&\quad = - \int_{Q_\tau} (\nabla \mu_\varepsilon - \nabla \mu) \cdot (\nabla c_\varepsilon - \nabla c) \,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} (\varrho_\varepsilon - 1)(c_\varepsilon - c)\partial_t c \,\mathrm{d}x\mathrm{d}t \\
&\quad\quad - \int_{Q_\tau} (c_\varepsilon - c)(\mathbf{v}_\varepsilon - \mathbf{v}) \cdot \nabla c \,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} (\varrho_\varepsilon - 1)(c_\varepsilon - c)\mathbf{v}_\varepsilon \cdot \nabla c \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.11}
$$

We give an essential claim for the justification.
Claim: It holds that

$$
\left[\int_\Omega \varrho_\varepsilon \frac{c_\varepsilon^2}{2} \,\mathrm{d}x \right]_{t=0}^{t=\tau} = - \int_{Q_\tau} \nabla \mu_\varepsilon \cdot \nabla c_\varepsilon \,\mathrm{d}x\mathrm{d}t. \tag{3.12}
$$

Proof of the claim. Let $0 \leq t \leq t + h \leq T$. Integrating (2.5b) over $[t, t+h]$ in its weak formulation (using a standard approximation argument) and testing with $\frac{1}{2} c_\varepsilon(t+h) c_\varepsilon(t)$ yields

$$
\int_\Omega \frac{(\varrho_\varepsilon(t+h) - \varrho_\varepsilon(t)) c_\varepsilon(t+h) c_\varepsilon(t)}{2h} \,\mathrm{d}x = \int_\Omega \frac{1}{2h} \int_t^{t+h} \varrho_\varepsilon(\tau)\mathbf{v}_\varepsilon(\tau) \,\mathrm{d}\tau \cdot \nabla (c_\varepsilon(t+h) c_\varepsilon(t)) \,\mathrm{d}x.
$$

Similarly, integrating (2.5c) on $[t, t+h]$ in its weak formulation (using a standard approximation argument) and testing with $\frac{1}{2}(c_\varepsilon(t+h) + c_\varepsilon(t))$ yields

$$\begin{aligned}&\int_\Omega \frac{(\varrho_\varepsilon(t+h)c_\varepsilon(t+h) - \varrho_\varepsilon(t)c_\varepsilon(t))(c_\varepsilon(t+h)+c_\varepsilon(t))}{2h}\,\mathrm{d}x\\&\quad = \int_\Omega \frac{1}{2h}\int_t^{t+h} (\varrho_\varepsilon(\tau)c_\varepsilon(\tau)\mathbf{v}_\varepsilon(\tau) - \nabla\mu_\varepsilon(\tau))\,\mathrm{d}\tau \cdot \nabla(c_\varepsilon(t+h)+c_\varepsilon(t))\,\mathrm{d}x.\end{aligned}$$

Now subtracting the first from the second identity gives

$$\begin{aligned}&\int_\Omega \frac{\varrho_\varepsilon(t+h)c_\varepsilon^2(t+h) - \varrho_\varepsilon(t)c_\varepsilon^2(t)}{2h}\,\mathrm{d}x\\&\quad = \int_\Omega \frac{1}{2h}\int_t^{t+h} (\varrho_\varepsilon(\tau)c_\varepsilon(\tau)\mathbf{v}_\varepsilon(\tau) - \nabla\mu_\varepsilon(\tau))\,\mathrm{d}\tau \cdot \nabla(c_\varepsilon(t+h)+c_\varepsilon(t))\,\mathrm{d}x\\&\qquad - \int_\Omega \frac{1}{2h}\int_t^{t+h} \varrho_\varepsilon(\tau)\mathbf{v}_\varepsilon(\tau)\,\mathrm{d}\tau \cdot \nabla(c_\varepsilon(t+h)c_\varepsilon(t))\,\mathrm{d}x.\end{aligned}$$

To pass to the limit $h \to 0+$ in the first term on the right-hand side it is essential that $\nabla(c_\varepsilon(.+h)+c_\varepsilon) \in L^2(0,T;L^s(\Omega))$ and $\frac{1}{2h}\int_t^{t+h}\varrho_\varepsilon(\tau)c_\varepsilon(\tau)\mathbf{v}_\varepsilon(\tau))\,\mathrm{d}\tau \in L^2(0,T;L^{s'}(\Omega))$ are bounded, where $\frac{1}{s} = \frac{1}{\gamma} - \frac{1}{6}$, compare of (3.26) below. This is the case if $\frac{1}{\gamma} + \frac{1}{6} + \frac{1}{6} \leq 1 - \frac{1}{\gamma} + \frac{1}{6}$, which is equivalent to $\gamma \geq \frac{12}{5}$. This is where we need an extra restriction on γ. The same estimates can be applied for the second term on the right-hand side. Hence we can pass to the limit $h \to 0+$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_\varepsilon \frac{c_\varepsilon^2}{2}\,\mathrm{d}x = -\int_\Omega \nabla\mu_\varepsilon \cdot \nabla c_\varepsilon\,\mathrm{d}x \quad \text{in } \mathcal{D}'(0,T),$$

which yields the claim by the fundamental theorem for Sobolev functions and integrating over $(0,T)$. □

Taking c as the test function in the weak formulation of (2.5c) and employing the strong formulation of (1.9c), we find

$$\begin{aligned}\left[\int_\Omega \varrho_\varepsilon c_\varepsilon c\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_{Q_\tau} (-\nabla\mu_\varepsilon + \varrho_\varepsilon c_\varepsilon \mathbf{v}_\varepsilon)\cdot\nabla c\,\mathrm{d}x\mathrm{d}t + \int_\Omega \varrho_\varepsilon c_\varepsilon \partial_t c\,\mathrm{d}x\\&= \int_{Q_\tau} (\varrho_\varepsilon - 1)c_\varepsilon \mathbf{v}_\varepsilon \cdot \nabla c\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} c_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla c\,\mathrm{d}x\mathrm{d}t\\&\quad - \int_{Q_\tau}\nabla\mu_\varepsilon\cdot\nabla c\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau}\nabla\mu\cdot\nabla c_\varepsilon\,\mathrm{d}x\mathrm{d}t + \int_\Omega (\varrho_\varepsilon - 1)c_\varepsilon\partial_t c\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.13}$$

Similarly, taking $\frac{c^2}{2}$ as the test function in the weak formulation of the continuity Equation (2.5b), together with (1.9c), yields

$$\begin{aligned}\left[\int_\Omega \varrho_\varepsilon \frac{c^2}{2}\,\mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_{Q_\tau} \varrho_\varepsilon c\mathbf{v}_\varepsilon\cdot\nabla c\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau}\varrho_\varepsilon c\partial_t c\,\mathrm{d}x\mathrm{d}t\\&= \int_{Q_\tau} (\varrho_\varepsilon - 1)c\mathbf{v}_\varepsilon\cdot\nabla c\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} c(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla c\,\mathrm{d}x\mathrm{d}t\\&\quad - \int_{Q_\tau}\nabla\mu\cdot\nabla c\,\mathrm{d}x\mathrm{d}t + \int_\Omega(\varrho_\varepsilon - 1)c\partial_t c\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.14}$$

Summing (3.12), (3.14), and subtracting (3.13) from the resulting equation entail the desired identity (3.11).

Now we define a modified relative energy $\widetilde{\mathcal{E}}$ by eliminating the nonconvex part of the chemical potential

$$\widetilde{\mathcal{E}}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c) \;:= \mathcal{E}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c) - \int_\Omega \varrho_\varepsilon(G_1(c_\varepsilon) - G_1'(c)(c_\varepsilon - c) - G_1(c))\,\mathrm{d}x.$$

Adding (3.9) and (3.11), one obtains *relative energy inequality*

$$
\begin{aligned}
&\left[\widetilde{\mathcal{E}}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon|1, \mathbf{v}, c)\right]_{t=0}^{t=\tau} \\
&\quad + \int_{Q_\tau} 2\nu(c_\varepsilon)|D\mathbf{v}_\varepsilon - D\mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} |\nabla\mu_\varepsilon - \nabla\mu|^2 \,\mathrm{d}x\mathrm{d}t \\
&\leq \left[\int_\Omega (\varrho_\varepsilon - 1)c_\varepsilon \Delta c \,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (\varrho_\varepsilon - 1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (1-\varrho_\varepsilon)G(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \\
&\quad - \kappa\int_{Q_\tau} (\nabla\mu_\varepsilon - \nabla\mu)\cdot(\nabla c_\varepsilon - \nabla c)\,\mathrm{d}x\mathrm{d}t - \kappa\int_{Q_\tau} (\varrho_\varepsilon - 1)(c_\varepsilon - c)\partial_t c\,\mathrm{d}x\mathrm{d}t \\
&\quad - \kappa\int_{Q_\tau} (c_\varepsilon - c)(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla c\,\mathrm{d}x\mathrm{d}t - \kappa\int_{Q_\tau} (\varrho_\varepsilon - 1)(c_\varepsilon - c)\mathbf{v}_\varepsilon\cdot\nabla c\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} (\varrho_\varepsilon - 1)(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\operatorname{div}(2\nu(c)D\mathbf{v})\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{v} - \mathbf{v}_\varepsilon)\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(\pi + G(c))\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(c\mu)\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mu(c_\varepsilon - c)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} (\varrho_\varepsilon - 1)\mathbf{v}\cdot\nabla\mu c_\varepsilon\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} \mathbf{v}\cdot\nabla(c_\varepsilon - c)(\mu_\varepsilon - \mu)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} (\varrho_\varepsilon - 1)\mu_\varepsilon\nabla c_\varepsilon\cdot\mathbf{v}\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} (\varrho_\varepsilon - 1)\mathbf{v}\cdot\nabla c_\varepsilon G'(c_\varepsilon)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{Q_\tau} (D\mathbf{v}_\varepsilon - D\mathbf{v}) : (2(\nu(c) - \nu(c_\varepsilon))D\mathbf{v})\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{Q_\tau} \varrho_\varepsilon\partial_t c\left(G'(c_\varepsilon) - (c_\varepsilon - c)G''(c) - G'(c)\right)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} (\varrho_\varepsilon - 1)\left(G''(c)c\partial_t c - G'(c)\partial_t c\right)\,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_{Q_\tau} (\varrho_\varepsilon - 1)c_\varepsilon\Delta\partial_t c\,\mathrm{d}x\mathrm{d}t - \int_{Q_\tau} (\varrho_\varepsilon - 1)\mu_\varepsilon\partial_t c\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.15}
$$

3.2 | Uniform estimates

Let $\mathbf{v} = 0$ and $c = 1$ in (3.15). Then one obtains

$$
\left[\widetilde{\mathcal{E}}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon|1, 0, 1)\right]_{t=0}^{t=\tau} + \int_{Q_\tau} 2\nu(c_\varepsilon)|D\mathbf{v}_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau} |\nabla\mu_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t \leq C
$$

for every $\tau \in [0, T]$. In a similar way as in [13, 19], we obtain the uniform estimates

$$
\operatorname*{esssup}_{t\in(0,T)} \left\|\sqrt{\varrho_\varepsilon}\mathbf{v}_\varepsilon\right\|_{L^2} \leq C, \tag{3.16}
$$

$$
\operatorname*{esssup}_{t\in(0,T)} \int_{\Omega\cap\{1/2\leq\varrho_\varepsilon\leq 2\}} \left|\frac{\varrho_\varepsilon - 1}{\varepsilon}\right|^2 \mathrm{d}x \leq C, \tag{3.17}
$$

$$
\operatorname*{esssup}_{t\in(0,T)} \int_{\Omega\setminus\{1/2\leq\varrho_\varepsilon\leq 2\}} (1 + |\varrho_\varepsilon|^\gamma)\,\mathrm{d}x \leq \varepsilon^2 C, \tag{3.18}
$$

$$
\operatorname*{esssup}_{t\in(0,T)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \leq C, \tag{3.19}
$$

$$
\int_0^T \|\nabla\mu_\varepsilon\|_{L^2(\Omega)}^2\,\mathrm{d}t \leq C, \tag{3.20}
$$

where $C > 0$ depends on the bounds for the initial data. Moreover, via Korn's inequality (cf. [16, theorem 11.21]) and $\nu_* \le \nu(c_\varepsilon) \le \nu^*$, one has

$$\int_0^T \|\nabla \mathbf{v}_\varepsilon\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le C.$$

In view of a generalized Korn–Poincaré inequality (cf. [16, theorem 11.23]), we obtain

$$\int_0^T \|\mathbf{v}_\varepsilon\|_{W^{1,2}(\Omega)}^2 \,\mathrm{d}t \le C. \tag{3.21}$$

Incorporating with (3.19) and the conservation of $\varrho_\varepsilon c_\varepsilon$, that is, for a.e. $\tau \in (0, T)$,

$$\int_\Omega \varrho_\varepsilon c_\varepsilon(\tau)\,\mathrm{d}x = \int_\Omega \varrho_0 c_0 \,\mathrm{d}x,$$

proceeding in a similar way as in [2, lemma 2.1] yields

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|c_\varepsilon(t)\|_{W^{1,2}(\Omega)} \le C. \tag{3.22}$$

Moreover, it follows from (2.4), (2.5d), (3.22) that

$$\left|\int_\Omega \varrho_\varepsilon \mu_\varepsilon \,\mathrm{d}x\right| = \left|\int_\Omega \varrho_\varepsilon G'(c_\varepsilon)\,\mathrm{d}x\right| \le C\|\varrho_\varepsilon\|_{L^\gamma(\Omega)}\left(1 + \|c_\varepsilon\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)}\right) \le C,$$

which, in accordance with (3.20), implies

$$\int_0^T \|\mu_\varepsilon(t)\|_{W^{1,2}(\Omega)}^2 \,dt \le C. \tag{3.23}$$

With Sobolev embedding in 3D, we have $\mu_\varepsilon \in L^2(0,T;L^6(\Omega))$. Combining with the fact $\varrho_\varepsilon \in L^\infty(0,T;L^\gamma(\Omega))$, one obtains $\varrho_\varepsilon\mu_\varepsilon \in L^2(0,T;L^q(\Omega))$ uniformly, with $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{6}$. Then by means of the elliptic estimates of c_ε in (2.5d), namely,

$$-\Delta c_\varepsilon = \varrho_\varepsilon \mu_\varepsilon - \varrho_\varepsilon G'(c_\varepsilon),$$

we get

$$c_\varepsilon \in L^2(0,T;W^{2,q}(\Omega)) \tag{3.24}$$

for all $1 < q < 6$ satisfying $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{6}$.

3.3 | Incompressible limit

Now we are in the position to control the right-hand side terms of (3.15) and derive the desired limit passage. First,

$$\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mathbf{v}\cdot(\mathbf{v}-\mathbf{v}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \le \int_0^\tau \|\nabla\mathbf{v}\|_{L^\infty(\Omega)}\left\|\sqrt{\varrho_\varepsilon}(\mathbf{v}_\varepsilon-\mathbf{v})\right\|_{L^2(\Omega)}^2 \mathrm{d}t \le C\int_0^\tau \widetilde{\mathcal{E}}(t)\,\mathrm{d}t.$$

By $\nu_* \le \nu \le \nu^*$, the Lipschitz continuity of $\nu(c)$, and Young's inequality, we have

$$\int_{Q_\tau} (D\mathbf{v}_\varepsilon - D\mathbf{v}) : (2(\nu(c) - \nu(c_\varepsilon))D\mathbf{v})\,\mathrm{d}x\mathrm{d}t$$

$$\leq \frac{1}{2}\int_{Q_\tau} \nu(c_\varepsilon)|D\mathbf{v}_\varepsilon - D\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t + C(\nu_*^{-1})\int_0^\tau \|D\mathbf{v}\|^2_{L^\infty(\Omega)}\|c_\varepsilon - c\|^2_{L^2(\Omega)}\,\mathrm{d}t$$
$$\leq \frac{1}{2}\int_{Q_\tau} \nu(c_\varepsilon)|D\mathbf{v}_\varepsilon - D\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t + C(\nu_*^{-1})\int_0^\tau \widetilde{\mathcal{E}}(t)\,\mathrm{d}t.$$

Moreover,

$$\begin{aligned}
&\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla\mu(c_\varepsilon - c)\,\mathrm{d}x\mathrm{d}t \\
&\quad\leq \int_0^\tau \|\varrho_\varepsilon\|_{L^\gamma(\Omega)}\|\nabla\mu\|_{L^\infty(\Omega)}\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{L^6(\Omega)}\|c_\varepsilon - c\|_{L^6(\Omega)}\,\mathrm{d}t \\
&\quad\leq \frac{1}{2}\int_{Q_\tau} \nu(c_\varepsilon)|D\mathbf{v}_\varepsilon - D\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t + C(\nu_*^{-1})\int_0^\tau \widetilde{\mathcal{E}}(t)\,\mathrm{d}t,
\end{aligned}$$

for all $\gamma > 3/2$, where we used the energy boundedness of ϱ_ε, the Sobolev embedding of $W^{1,2} \hookrightarrow L^6$ in three dimensions and the Poincaré inequality. Analogously, it follows

$$\begin{aligned}
-\int_{Q_\tau} \mathbf{v}\cdot\nabla(c_\varepsilon - c)(\mu_\varepsilon - \mu)\,\mathrm{d}x\mathrm{d}t &= \int_{Q_\tau} \mathbf{v}\cdot\nabla(\mu_\varepsilon - \mu)(c_\varepsilon - c)\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^\tau \|\mathbf{v}\|^2_{L^\infty(\Omega)}\|c_\varepsilon - c\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{4}\int_{Q_\tau} |\nabla\mu_\varepsilon - \nabla\mu|^2\,\mathrm{d}x\mathrm{d}t \\
&\leq \frac{1}{4}\int_{Q_\tau} |\nabla\mu_\varepsilon - \nabla\mu|^2\,\mathrm{d}x\mathrm{d}t + C\int_0^\tau \widetilde{\mathcal{E}}(t)\,\mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
&-\kappa\int_{Q_\tau} (\nabla\mu_\varepsilon - \nabla\mu)\cdot(\nabla c_\varepsilon - \nabla c)\,\mathrm{d}x\mathrm{d}t - \kappa\int_{Q_\tau} (c_\varepsilon - c)(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla c\,\mathrm{d}x\mathrm{d}t \\
&\quad\leq \frac{1}{4}\int_{Q_\tau} |\nabla\mu_\varepsilon - \nabla\mu|^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_{Q_\tau} \nu(c_\varepsilon)|D\mathbf{v}_\varepsilon - D\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}t + C(\nu_*^{-1})\int_0^\tau \widetilde{\mathcal{E}}(t)\,\mathrm{d}t.
\end{aligned}$$

By direct calculations and weak formulation of continuity equation for ϱ_ε,

$$\begin{aligned}
&\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(\pi + c\mu + G(c))\,\mathrm{d}x\mathrm{d}t \\
&\quad= -\varepsilon\int_{Q_\tau} \frac{\varrho_\varepsilon - 1}{\varepsilon}\partial_t(\pi + c\mu + G(c))\,\mathrm{d}x\mathrm{d}t \\
&\qquad+ \varepsilon\left[\int_\Omega \frac{\varrho_\varepsilon - 1}{\varepsilon}(\pi + c\mu + G(c))\,\mathrm{d}x\right]_{t=0}^{t=\tau} - \varepsilon\int_{Q_\tau} \frac{\varrho_\varepsilon - 1}{\varepsilon}\mathbf{v}\cdot\nabla(\pi + c\mu + G(c))\,\mathrm{d}x\mathrm{d}t. \qquad (3.25)
\end{aligned}$$

For sufficiently smooth $(\mathbf{v}, \pi, c, \mu)$, it follows from (3.17) and the Hölder inequality that (3.25) is controlled by

$$\int_{Q_\tau} \varrho_\varepsilon(\mathbf{v}_\varepsilon - \mathbf{v})\cdot\nabla(\pi + c\mu + G(c))\,\mathrm{d}x\mathrm{d}t \leq \varepsilon C,$$

where C depends on the initial data and $(\mathbf{v}, p, c, \mu)$, but is independent of $\varepsilon > 0$.

Concerning the terms associated with $(\varrho_\varepsilon - 1)$, it follows from (3.17) and (3.18) that

$$\int_{Q_\tau} (\varrho_\varepsilon - 1)f\,\mathrm{d}x\mathrm{d}t \leq \varepsilon C,$$

for all $f \in L^1(0,T;L^2(\Omega)\cap L^{\frac{\gamma}{\gamma-1}}(\Omega))$ with $\gamma > \frac{3}{2}$. Similarly,

$$\left[\int_\Omega (\varrho_\varepsilon - 1)c_\varepsilon \Delta c\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (\varrho_\varepsilon - 1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$+\left[\int_\Omega (\varrho_\varepsilon - 1)cG'(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} + \left[\int_\Omega (1-\varrho_\varepsilon)G(c)\,\mathrm{d}x\right]_{t=0}^{t=\tau} \le \varepsilon C.$$

However, for the term $\int_{Q_\tau}(\varrho_\varepsilon - 1)\mu_\varepsilon \nabla c_\varepsilon \cdot \mathbf{v}\,\mathrm{d}x\mathrm{d}t$, we know from (3.22) and (3.23) that $\mu_\varepsilon \nabla c_\varepsilon \in L^2(0,T;L^{\frac{3}{2}}(\Omega))$, which is not sufficient for all $\gamma > \frac{3}{2}$. By (3.24) and Sobolev embedding $W^{2,q} \hookrightarrow W^{1,s}$ with $\frac{1}{s} = \frac{1}{q} - \frac{1}{3} = \frac{1}{\gamma} - \frac{1}{6}$, one obtains

$$\nabla c_\varepsilon \in L^2(0,T;L^s(\Omega)) \tag{3.26}$$

for all s satisfying $\frac{1}{s} = \frac{1}{\gamma} - \frac{1}{6}$. Then

$$\int_{Q_\tau} (\varrho_\varepsilon - 1)\mu_\varepsilon \nabla c_\varepsilon \cdot \mathbf{v}\,\mathrm{d}x\mathrm{d}t \le \int_0^\tau \|\varrho_\varepsilon - 1\|_{L^\gamma(\Omega)}\|\mu_\varepsilon\|_{L^6(\Omega)}\|\nabla c_\varepsilon\|_{L^p(\Omega)}\|\mathbf{v}\|_{L^\infty(\Omega)}\,\mathrm{d}t \le \varepsilon C \tag{3.27}$$

for $1 \ge \frac{1}{\gamma} + \frac{1}{6} + \frac{1}{s} = \frac{2}{\gamma}$, which holds for all $\gamma \ge 2$.

Furthermore,

$$\begin{aligned}&\int_{Q_\tau} \varrho_\varepsilon \partial_t c\left(G'(c_\varepsilon) - (c_\varepsilon - c)G''(c) - G'(c)\right)\,\mathrm{d}x\mathrm{d}t\\&\le C\int_{Q_\tau} |c_\varepsilon - c|^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_\tau}(\varrho_\varepsilon - 1)\partial_t c\left(G'(c_\varepsilon) - (c_\varepsilon - c)G''(c) - G'(c)\right)\,\mathrm{d}x\mathrm{d}t\\&\le C\int_0^\tau \widetilde{\mathcal{E}}(t)\,\mathrm{d}t + \varepsilon C.\end{aligned}$$

Collecting all the estimates above, we then obtain a Gronwall's type inequality:

$$\left[\widetilde{\mathcal{E}}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c\right)\right]_{t=0}^{t=\tau} \le C\int_0^\tau \widetilde{\mathcal{E}}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c\right)(t)\,\mathrm{d}t + \varepsilon C$$

for all $\tau \in (0,T)$, which yields

$$\widetilde{\mathcal{E}}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon, c_\varepsilon | 1, \mathbf{v}, c\right)(\tau) \le C\left(\widetilde{\mathcal{E}}\left(\varrho_{0,\varepsilon}, \mathbf{v}_{0,\varepsilon}, c_{0,\varepsilon} | 1, \mathbf{v}_0, c_0\right) + \varepsilon\right)e^\tau$$

for all $\tau \in (0,T)$. If additionally one has for the initial data

$$\begin{aligned}\mathbf{v}_{0,\varepsilon} &\to \mathbf{v}_0, && \text{in } L^2(\Omega),\\ \varrho^{(1)}_{0,\varepsilon} &\to 0, && \text{in } L^\infty(\Omega),\\ \nabla c_{0,\varepsilon} &\to \nabla c_0, && \text{in } L^2(\Omega),\end{aligned}$$

as $\varepsilon \to 0$, one concludes the low Mach number limit immediately, which finishes the proof of Theorem 2.2.

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CONFLICT OF INTEREST STATEMENT

The authors declare that there are no conflicts of interest.

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