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Lie 2-groups from loop group extensions

Matthias Ludewig[1] · Konrad Waldorf[2]

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Abstract
We give a very simple construction of the string 2-group as a strict Fréchet Lie 2-group. The corresponding crossed module is defined using the conjugation action of the loop group on its central extension, which drastically simplifies several constructions previously given in the literature. More generally, we construct strict 2-group extensions for a Lie group from a central extension of its based loop group, under the assumption that this central extension is disjoint commutative. We show in particular that this condition is automatic in the case that the Lie group is semisimple and simply connected.

Contents

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✉ Matthias Ludewig
matthias.ludewig@mathematik.uni-regensburg.de

1 Universität Regensburg, Regensburg, Germany

2 Universität Greifswald, Greifswald, Germany

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1 Introduction

In the seminal paper [5] by Baez, Crans, Stevenson and Schreiber, a certain Fréchet Lie 2-group extension of a Lie group G of Cartan type (i.e., compact, connected, simple, simply connected) was constructed, using a particular presentation of the universal central extension of the loop group LG. For $G = \mathrm{Spin}(n)$, their construction realizes a model for the string 2-group.

In an attempt to generalize this construction, the second-named author described in [18–20] another, diffeological 2-group extension of an arbitrary Lie group G, using an arbitrary central extension of LG equipped with a certain additional structure—a multiplicative fusion product. If G is of Cartan type, such a central extension can be provided canonically, and one can prove abstractly that the corresponding 2-group is weakly equivalent to the one of Baez et al.

The purpose of the present paper is to (drastically) simplify and to unify both constructions. For this purpose, we study in the first part of this paper, Sect. 2, central extensions of loop groups and of groups of paths, in the category of Fréchet Lie groups. We identify a property of central extensions of a loop group, *disjoint commutativity*, as crucial for the construction of 2-groups. A central extension

$$1 \to \mathrm{U}(1) \to \widetilde{LG} \to LG \to 1$$

is disjoint commutative if elements $\Phi, \Phi' \in \widetilde{LG}$ commute if they project to loops $\gamma, \gamma' \in LG$ with disjoint supports. Disjoint commutativity has been introduced in [20] as a property of transgressive central extension, and it is relevant for the theory of nets of operator algebras [6]. Our first result is the following (see Corollary 2.4.4 and, for a more general statement, Theorem 2.4.10).

Theorem 1.1 *If G is semisimple and simply connected, then all central extensions of LG are disjoint commutative.*

The relevance of disjoint commutativity for Lie 2-groups lies in the construction of crossed module actions. We denote by P_eG the Fréchet Lie group of paths in G that start at the identity element e, and all whose derivatives at both end points vanish. We denote by $\widetilde{\Omega_{(0,\pi)}G}$ the restriction of $\widetilde{LG}$ to the group $\Omega_{(0,\pi)}G$ of those loops whose support is in their first half $(0, \pi) \subset S^1$. Then, we consider the Lie group homomorphism

$$t : \widetilde{\Omega_{(0,\pi)}G} \to P_eG$$

thats projects to the first half of the base loop, considered as a (closed) path. In order to turn the homomorphism t into a crossed module, it remains to provide a crossed module action α of P_eG on $\widetilde{\Omega_{(0,\pi)}G}$. In the above-mentioned paper [5] by Baez et al., such an action is constructed (in a slightly different setting) using Lie-algebraic methods and particularities of a specific model of $\widetilde{\Omega G}$. In the second above-mentioned approach [18, 20], a crossed module action is constructed using the given fusion product.

In our setting, the required action α is both simple and canonical: a path $\gamma \in P_eG$ is first „doubled“ to a thin loop in ΩG, lifted to $\widetilde{\Omega G}$, and then acts by conjugation on $\Phi \in \widetilde{\Omega_{(0,\pi)}G}$, see Sect. 3.2. In general, this canonical action α will not be a crossed module action, as it does not satisfy the so-called Peiffer identity. One of our main insights is that this problem is resolved when $\widetilde{\Omega G}$ is disjoint commutative, see Theorem 3.2.4.

Theorem 1.2 *If $\widetilde{\Omega G}$ is a disjoint commutative central extension of ΩG, then the canonical action α turns $t : \widetilde{\Omega_{(0,\pi)}G} \to P_eG$ into a central crossed module. Moreover, if G is semisimple, then α is the only such action.*

We emphasize that Theorem 1.2 provides a drastic simplification of the construction of 2-group extensions; in particular, for the construction of string 2-group models. Neither additional structure on the central extension is needed, nor any other special knowledge about its concrete model.

We denote by $X(\widetilde{\Omega G})$ the crossed module of Theorem 1.2, and now consider the special case where G is of Cartan type, and $\widetilde{\Omega G}$ has level $k \in \mathbb{Z}$. We denote by $X^{\mathrm{BCSS}}(G, k)$ the crossed module constructed by Baez et al. at the same level. In Sect. 4.1 we construct a canonical, strict homomorphism

$$X(\widetilde{\Omega G}) \to X^{\mathrm{BCSS}}(G, k), \tag{1.1}$$

of crossed modules of Fréchet Lie groups. On the other hand, we consider a disjoint commutative central extension $\widetilde{LG}$ with a fusion product λ, and denote by $X^{\mathrm{W}}(\widetilde{\Omega G}, \lambda)$ the diffeological crossed module corresponding to the diffeological 2-group of [18–20]. Under the canonical inclusion of Fréchet manifolds into diffeological spaces, we construct in Sect. 4.2 another, strict homomorphism

$$X^{\mathrm{W}}(\widetilde{\Omega G}, \lambda) \to X(\widetilde{\Omega G}).$$

Theorems 4.1.2 and 4.2.6 prove the following.

Theorem 1.3 *The homomorphisms* (1.1) *and* (1.1) *establish weak equivalences*

$$X^{\mathrm{W}}(\widetilde{\Omega G}, \lambda) \cong X(\widetilde{\Omega G}) \quad \text{and} \quad X(\widetilde{\Omega G}) \cong X^{\mathrm{BCSS}}(G, k).$$

In particular, this shows that the two earlier constructions $X^{\mathrm{W}}(\widetilde{\Omega G}, \lambda)$ and $X^{\mathrm{BCSS}}(G, k)$ are canonically and strictly isomorphic, a fact that is very difficult to observe when only looking at these two 2-groups.

Another aspect we investigate in this paper concerns the 2-groups associated to the crossed modules discussed above. As these two structures (2-groups and crossed modules) are canonically equivalent, our crossed module $X(\widetilde{\Omega G})$ determines a Fréchet Lie 2-group $\mathcal{G}(X(\widetilde{\Omega G}))$, whose group of objects is P_eG, and whose group of morphisms is the semi-direct product

$$\widetilde{\Omega_{(0,\pi)}G} \rtimes_\alpha P_eG. \tag{1.2}$$

The Fréchet Lie 2-group of Baez et al. has a similar structure. However, the diffeological construction of the second-named author results into a Lie 2-group whose group of morphisms is a subgroup of $\widetilde{\Omega G}$, and is hence „nicer“. It turns out that the missing ingredient to identify the semi-direct product (1.2) with a subgroup of $\widetilde{\Omega G}$ is a homomorphism

$$i : P_e G \longrightarrow \widetilde{\Omega G}$$

such that $i(\gamma) \in \widetilde{\Omega G}$ lies over the thin loop corresponding to the path γ. Such a map was called *fusion factorization* in [9]. Here we have the following result, see Theorem 3.3.5.

Theorem 1.4 *If G is semisimple, fusion factorizations are unique. If G is additionally simply connected, then fusion factorizations exist.*

In particular, if G is simply connected and semisimple, then every central extension of ΩG carries a unique fusion factorization. The whole situation can be summarized as follows.

Corollary 1.5 *If G is simply connected and semisimple, then for every central extension $\widetilde{\Omega G}$ of ΩG, there exists a unique central crossed module $X(\widetilde{\Omega G})$ of Fréchet Lie groups with underlying homomorphism*

$$t : \widetilde{\Omega_{(0,\pi)} G} \to P_e G.$$

Moreover, there exists a unique Lie 2-group $\mathcal{G}(\widetilde{\Omega G}, i)$, with objects and morphisms

$$\widetilde{P_e G^{[2]}} \overset{s}{\underset{t}{\rightrightarrows}} P_e G,$$

where $\widetilde{P_e G^{[2]}} \subset \widetilde{\Omega G}$ is the subgroup over those loops that are flat at 0 and π. Finally, $X(\widetilde{\Omega G})$ and $\mathcal{G}(\widetilde{\Omega G}, i)$ correspond to each other under the adjunction between crossed modules and 2-groups.

Finally, we come back to the main motivation of the whole topic, the construction of models for the string 2-group. In [5] it was proved that the geometric realization of the Lie 2-group corresponding to the crossed module $X^{\mathrm{BCSS}}(G, k)$ is a 3-connected cover of G, which—for $G = \mathrm{Spin}(d)$—is the defining property of a string 2-group. In Sect. 3.4 we generalize this result slightly from Lie groups of Cartan type to arbitrary simple and simply connected Lie groups. We have the following result, see Theorem 3.4.1.

Theorem 1.6 *If G is simple and simply connected, and $\widetilde{\Omega G}$ is a basic central extension, then the geometric realization of the (canonically isomorphic) Lie 2-groups $\mathcal{G}(X(\widetilde{\Omega G}))$ and $\mathcal{G}(\widetilde{\Omega G}, i)$ are 3-connected covers of G. In particular, if $G = \mathrm{Spin}(d)$, both are models for the string 2-group.*

2 Loop groups and their central extensions

In this section we recall some relevant results about central extensions of loop groups and path groups, and also add a couple of new results which we will use later. In particular, in Sect. 2.4, we discuss and investigate the relatively new notion of disjoint commutative central extensions.

2.1 Path groups and loop groups

Throughout, let G be a connected (finite-dimensional) Lie group. We denote the identity element by e, and we denote by $LG = C^\infty(S^1, G)$ the smooth loop group of G. We always identify $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. For $I \subset S^1$, we write

$$L_I G = \{\gamma \in LG \mid \gamma(t) = e \text{ whenever } t \notin I\}. \tag{2.1.1}$$

We say that a map $f : M \to N$ between manifolds is *flat* at a point $p \in M$, if all directional derivatives of f vanish at all orders at the point p. We observe, in particular, that all elements of $L_{(a,b)}G$ are flat at $t = a, b$ (unless $(a, b) = (0, 2\pi)$). We also denote by $\Omega G \subset LG$ the subset of loops γ that are flat at $t = 0$ and satisfy $\gamma(0) = e$. Analogously to the notation above, we also write

$$\Omega_I G = L_I G \cap \Omega G.$$

We denote by PG the space of all smooth maps $\gamma : [0, \pi] \to G$ that are flat at their endpoints, and by $P_e G \subset PG$ the subset of paths γ with $\gamma(0) = e$. We then have a short exact sequence

$$\Omega_{(0,\pi)} G \longrightarrow P_e G \xrightarrow{\text{ev}} G,$$

where the first map is the restriction of $\gamma \in \Omega_{(0,\pi)}G$ to the interval $[0, \pi]$, and the second map is the endpoint evaluation. For two paths γ_1, γ_2 with a common initial point and a common end point, we define a loop $\gamma_1 \cup \gamma_2 \in LG$ by

$$(\gamma_1 \cup \gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [0, \pi] \\ \gamma_2(2\pi - t) & t \in [\pi, 2\pi] \end{cases}. \tag{2.1.2}$$

We identify the fibre product $P_e G^{[2]} = P_e G \times_G P_e G$ with its image in LG under this map.

For non-trivial G, all loop groups and path groups discussed above are infinite-dimensional Lie groups, which are modeled on nuclear Fréchet spaces. Their Lie algebras are obtained by taking the appropriate path space inside the Lie algebra $\mathfrak{g}$ of G.

Remark 2.1.1 The Fréchet Lie groups $L_I G$, ΩG, and $P_e G$ are *regular* in the sense of [14, Def. 3.12], which means that every smooth curve in their Lie algebra can be integrated to a smooth curve in the group. This follows from the fact that such an integral can be calculated pointwise in the loop parameter, which gives a smooth curve in G. Then, as solutions to ordinary differential equations depend smoothly on the initial data, these curves yield a smooth curve in the appropriate path group.

It is a corollary of [17, Prop. 3.4.1] that if G is semisimple, there are no non-trivial Lie group homomorphisms from LG to any abelian Lie group A, i.e., every Lie group homomorphism $\varphi : LG \to A$ is $\varphi = 1$. The following generalization will be key to the present paper.

Theorem 2.1.2 *If G is a semisimple Lie group, then the Fréchet Lie group P_eG does not admit non-trivial Lie group homomorphisms to any abelian Lie group A. The same is true for the identity components of ΩG and $L_I G$, for any $I \subseteq S^1$.*

We need the following lemma.

Lemma 2.1.3 *For every smooth function $f : [0, a] \to \mathbb{R}$ that is flat at zero, there are smooth functions $g_1, g_2 : [0, a] \to \mathbb{R}$ that are also flat at zero and satisfy $f(t) = g_1(t)g_2(t)$ for all $t \in [0, a]$.*

For the proof of Lemma 2.1.3, we need the following observation: consider the following property for a map $f : [0, a] \to \mathbb{R}$ with $f(0) = 0$.

($\star$) f is smooth on $(0, a]$ and for each $n \in \mathbb{N}$, there exists $\varepsilon > 0$ such that $|f(t)| \le t^n$ for each $t \in [0, \varepsilon]$.

An easy exercise shows that f satisfies ($\star$) if and only if f is smooth on $[0, a]$ and flat at zero.

Proof of Lemma 2.1.3 By ($\star$) we may choose, for each $n \in \mathbb{N}$, an $\varepsilon_n > 0$ such that $|f(t)| \le t^n$ for each $t \in [0, \varepsilon_n]$. We choose these numbers such that the sequence $\varepsilon_1, \varepsilon_2, \dots$ is strictly decreasing and converges to zero. For each $n \in \mathbb{N}$, we choose smooth functions $h_n : [\varepsilon_{n+1}, \varepsilon_n] \to \mathbb{R}$ such that $\frac{1}{2}t^n \le h_n(t) \le t^n$ for all $t \in [\varepsilon_{n+1}, \varepsilon_n]$, in such a way that the functions h_{n+1} and h_n fit smoothly together. As $(\varepsilon_n)_{n\in\mathbb{N}}$ forms a null sequence, there is a smooth function h on $(0, \varepsilon_1]$ such that h agrees with h_n when restricted to $[\varepsilon_{n+1}, \varepsilon_n]$. Setting $h(0) = 0$, we obtain a function h which by construction satisfies $h(t) \ge \frac{1}{2}|f(t)|$ for each $t \in (0, \varepsilon_1]$, and which satisfies ($\star$), hence is flat at zero. We smoothly extend h to a function defined on all of $[0, a]$.

We now set $g_1(t) = f(t)/h(t)^{1/2}$, $g_2(t) = h(t)^{1/2}$. It is clear that the function g_2 satisfies ($\star$), and so does g_1, as $|g_1(t)| \le h(t)/h(t)^{1/2} = h(t)^{1/2}$. Hence both g_1 and g_2 are flat at zero, and we have $f(t) = g_1(t)g_2(t)$, as required. □

Proof of Theorem 2.1.2 We prove the result for P_eG, the proof for L_IG is similar. As G is semisimple, we have $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, that is, every element of $\mathfrak{g}$ is linear combination of commutators. We first show that the same is true for the Lie algebra $P_0\mathfrak{g}$ of P_eG. Let $x_1, \dots, x_n$ be a vector space basis for $\mathfrak{g}$ and choose numbers $a_k^{ij} \in \mathbb{R}$ with

$$x_k = \sum_{ij=1}^{n} a_k^{ij}[x_i, x_j].$$

Write

$$X(t) = \sum_{k=1}^{n} f_k(t)x_k$$

By Lemma 2.1.3, there exist $g_k, h_k \in P_0\mathbb{R}$ such that $f_k(t) = g_k(t)h_k(t), k = 1, \dots, n$. Then

$$X(t) = \sum_{ijk=1}^{n} f_k(t)a_k^{ij}[x_i, x_j] = \sum_{ijk=1}^{n} a_k^{ij}[g_k(t)x_i, h_k(t)x_j].$$

As commutators in $P_0\mathfrak{g}$ are taken pointwise, this witnesses X as a sum of commutators in the Lie algebra $P_0\mathfrak{g}$.

Let now $\varphi : P_eG \to A$ be a Lie group homomorphism with induced Lie algebra homomorphism $\varphi_* : P_0\mathfrak{g} \to \mathfrak{a}$, where $\mathfrak{a}$ is the Lie algebra of A. As φ_* sends commutators to commutators, it must send the commutator subspace of $P_0\mathfrak{g}$ to the commutator subspace of $\mathfrak{a}$, which is zero as A (and consequently $\mathfrak{a}$) is abelian. However, as $[P_0\mathfrak{g}, P_0\mathfrak{g}] = P_0\mathfrak{g}$, this implies that φ_* is identically zero. Since P_eG is regular (see Remark 2.1.1), this implies, together with the fact that P_eG is connected that φ itself is trivial; see [12, Lemma 7.1]. □

2.2 Classification of central extensions of loop groups

We recall that a *central extension* of a (possibly infinite-dimensional, Fréchet) Lie group H (by the group $\mathrm{U}(1)$) is a sequence

$$1 \to \mathrm{U}(1) \longrightarrow \widetilde{H} \xrightarrow{\pi} H \to 1$$

of Lie groups and Lie group homomorphisms such that it is exact as a sequence of groups, and $\widetilde{H}$ is a principal $\mathrm{U}(1)$-bundle over H. For such a central extension, we always identify $\mathrm{U}(1)$ with its image in $\widetilde{H}$. A Lie group isomorphism $f : \widetilde{H} \to \widetilde{H}'$ is an *isomorphism of central extensions* if it is base point-preserving and trivial on $\mathrm{U}(1) \subset \widetilde{H}$. We denote by $\mathrm{c}\mathcal{E}\mathrm{xt}(H)$ the groupoid of central extensions of H.

Given two central extension $\widetilde{H}$ and $\widetilde{H}'$, their tensor product $\widetilde{H} \otimes \widetilde{H}'$ (as $\mathrm{U}(1)$-principal bundles) has a group structure turning it into another central extension. This defines a symmetric monoidal structure on $\mathrm{c}\mathcal{E}\mathrm{xt}(H)$. Given a central extension $\widetilde{H}$, the dual circle bundle $\widetilde{H}^*$ has an obvious group structure turning it into a central extension that is inverse to $\widetilde{H}$ with respect to the tensor product. Hence, the set $h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(H))$ of isomorphism classes in $\mathrm{c}\mathcal{E}\mathrm{xt}(H)$ is a group.

We discuss the classification of central extensions $\widetilde{H}$ for a given (Fréchet) Lie group H. Choosing a linear section of the Lie algebra homomorphism $\widetilde{\mathfrak{h}} \to \mathfrak{h}$ induced by the projection $\widetilde{H} \to H$ gives an identification $\widetilde{h} \cong \mathfrak{h} \oplus \mathbb{R}$, under which the bracket attains the form

$$[(X, \lambda), (Y, \mu)] = ([X, Y], \omega(X, Y)),$$

for a continuous Lie algebra 2-cocycle ω on $\mathfrak{h}$. The cocycles corresponding to two different choices of splittings differ by a coboundary; hence, there is a well-defined class in the continuous Lie algebra cohomology group $H^2_c(\mathfrak{h}, \mathbb{R})$ defined by the central extension $\widetilde{H}$. This establishes a group homomorphism

$$h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(H)) \longrightarrow H^2_c(\mathfrak{h}, \mathbb{R}). \tag{2.2.1}$$

This homomorphism is neither injective or surjective in general. However, if H is simply connected, then (2.2.1) is injective [14, Thm. 7.12] and its image is the subgroup represented by cocycles ω whose group of periods

$$\mathrm{Per}\omega := \left\{ \int_Z \overline{\omega} \,\middle|\, Z \text{ a smooth 2-cycle on } H \right\} \subseteq \mathbb{R}$$

is contained in $2\pi\mathbb{Z}$. Here $\overline{\omega}$ denotes the left invariant 2-form on H determined by ω [13].

In general, if H satisfies $\pi_1(H) = 0$ but is not necessarily connected, we obtain a functor

$$\mathrm{c}\mathcal{E}\mathrm{xt}(\pi_0(H)) \to \mathrm{c}\mathcal{E}\mathrm{xt}(H),$$

given by pullback of central extensions along the group homomorphism $H \to \pi_0(H)$. On isomorphism classes, this gives a sequence

$$h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(\pi_0(H))) \longrightarrow h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(H)) \longrightarrow H^2_c(\mathfrak{h}, \mathbb{R}), \tag{2.2.2}$$

which is exact in the middle if $\pi_1(H) = 0$. Indeed, if $\widetilde{H}$ is a central extension with vanishing cohomology class, then its restriction to the identity component $H_0 \subset H$ still has vanishing cohomology class. But over H_0 the map (2.2.1) is injective, showing the restriction of $\widetilde{H}$ to H_0 must be trivial. But this implies that $\widetilde{H}$ comes from a central extension of $\pi_0(H)$.

Example 2.2.1 For connected and simply connected groups H (where the map (2.2.1) is injective), there is an explicit description of the central extension corresponding to a 2-cocycle ω on $\mathfrak{h}$ with $\mathrm{Per}\omega \subset 2\pi\mathbb{Z}$, see [17, §4.4]. Let $\overline{\omega}$ be the left invariant 2-form on H determined by ω. For a loop $\gamma \in \Omega H$, we define

$$C(\gamma) := \exp\left(i \int_{\hat{h}} \overline{\omega} \right),$$

where $h : [0, 1] \to \Omega H$ is a smooth null homotopy of γ and $\hat{h} : [0, 1] \times S^1 \to H$ is the corresponding surface in H (such a null homotopy exists as H is simply connected). By the assumption on ω, the integral of $\overline{\omega}$ over any *closed* surface lies in $2\pi\mathbb{Z}$, which implies that $C(\gamma)$ is independent of the choice of h. One then defines

$$\widetilde{H} = P_e H \times \mathrm{U}(1) \big/ \sim,$$

where $(\gamma_1, z_1) \sim (\gamma_2, z_2)$ if $\gamma_1(\pi) = \gamma_2(\pi)$ and $C(\gamma_1 \cup \gamma_2) = z_2/z_1$. The bundle projection $\pi : \widetilde{H} \to H$ is given by $(\gamma, z) \mapsto \gamma(\pi)$ and the group product is

$$[\gamma_1, z_1] \cdot [\gamma_2, z_2] = [(\mathtt{const}_{\gamma_1(\pi)} \cdot \gamma_2) * \gamma_1, z_1 z_2],$$

where $*$ denotes concatenation of paths. This gives a central extension whose image under the map (2.2.1) is the cocycle ω (Propositions 4.4.2 & 4.5.6 of [17]).

In the following we consider central extensions of the loop groups LG and ΩG of a connected Lie group G. As $\pi_k(\Omega G) = \pi_{k+1}(G)$ and

$$\pi_k(LG) = \pi_k(\Omega G) \oplus \pi_k(G) = \pi_{k+1}(G) \oplus \pi_k(G), \tag{2.2.3}$$

it follows that LG and ΩG are connected if and only if G is simply connected. Moreover, since $\pi_2(G) = 0$ for any (finite-dimensional) Lie group G, it follows that always $\pi_1(\Omega G) = 0$ while $\pi_1(LG) = \pi_1(G)$. Thus, if G is simply connected, then the map (2.2.1) is injective, hence any central extension of $H = LG$ or ΩG is determined by its corresponding Lie algebra cocycle ω.

Lemma 2.2.2 *If G is semisimple, then every 2-cocycle on $L\mathfrak{g}$ and $\Omega\mathfrak{g}$ is cohomologous to a cocycle of the form*

$$\omega(X, Y) = \int_{S^1} b(X(t), Y'(t))\mathrm{d}t \tag{2.2.4}$$

for a G-invariant symmetric bilinear form b on $\mathfrak{g}$.

Proof It is well known that every G-invariant 2-cocycle is of the form (2.2.4), see e.g., [17, Prop. 4.2.4]. For not necessarily G-invariant 2-cocycles, the result follows from the general results of [16]; see in particular Example 7.2. There, cocycles are decomposed as $f_1 + f_2$, which are necessarily *uncoupled* in the authors terminology, as $\mathfrak{g}$ is semisimple. It is not hard to figure out that f_2 is necessarily a coboundary and f_1 gives a cocycle of the form (2.2.4). □

Remark 2.2.3 If G is compact, simply connected and simple, then there is an isomorphism $H^2_c(L\mathfrak{g}, \mathbb{R}) \cong H^2_c(\Omega\mathfrak{g}, \mathbb{R}) \cong H^3(G, \mathbb{R})$ that sends the subgroup of classes defining a central extension of LG, i.e., the image of (2.2.1), onto the subgroup $H^3(G, \mathbb{Z})$.

Remark 2.2.4 Consider the central extension $\widetilde{LG}$ constructed in Example 2.2.1 from a 2-cocycle ω on $L\mathfrak{g}$. If ω is of the form (2.2.4) for a bilinear form b on $\mathfrak{g}$, then $\widetilde{LG}$ can be equivalently described as follows. The elements of $\widetilde{LG}$ can be represented by pairs (σ, z), where $\sigma : D^2 \to G$ is a smooth map and where $(\sigma_1, z_1) \sim (\sigma_2, z_2)$ if and only if

$$\frac{z_2}{z_1} = \exp\left(2\pi i \int_\Sigma \overline{\nu}\right).$$

Here $\Sigma : D^3 \to G$ is a map whose restriction $\Sigma_{|\partial D^3}$ is given by σ_1 and σ_2 on its two hemispheres, and $\overline{\nu}$ is the left invariant 3-form on G associated to the Lie algebra cocycle $\nu(x, y, z) = b([x, y], z)$ on $\mathfrak{g}$. The group structure is realized with the Mickelsson product, see Theorem 6.4.1 of [2] and [11]. The projection $\widetilde{LG} \to LG$ is given by sending $(\sigma, z) \mapsto \sigma|_{\partial D^2}$, identifying $\partial D^2 = S^1$. This description is equivalent to the one of (2.2.4) as the transgression of $\overline{\nu}$ is cohomologous to $\overline{\omega}/2\pi$, see [17, Prop. 4.4.4].

2.3 Restrictions of central extensions

In the following, we assume—as before—that G is a connected (finite-dimensional) Lie group.

Lemma 2.3.1 *If G is semisimple, then the automorphism group of a central extension of LG, ΩG, $L_I G$ or $\Omega_I G$, for $I \subseteq S^1$, is canonically isomorphic to* $\mathrm{Hom}(\pi_1(G), \mathrm{U}(1))$. *In particular, if G is semisimple and simply connected, then the categories* $\mathrm{c}\mathcal{E}\mathrm{xt}(LG)$, $\mathrm{c}\mathcal{E}\mathrm{xt}(\Omega G)$ *and* $\mathrm{c}\mathcal{E}\mathrm{xt}(L_I G)$ *have only trivial automorphism groups.*

Proof We prove the result for LG, the proof for ΩG and $L_I G$ is similar. By (2.2.3), we have $\pi_0(LG) = \pi_1(G)$, hence any group homomorphism $\varphi : \pi_1(G) \to \mathrm{U}(1)$ gives rise to an automorphism f of $\widetilde{LG}$ by setting $f(\Phi) = \varphi([\pi(\Phi)])\Phi$.

Conversely, let f be an automorphism of a central extension $\widetilde{LG}$. We define a map $\varphi : LG \to \widetilde{LG}$ by

$$\varphi(\gamma) = f(\tilde{\gamma})\tilde{\gamma}^{-1},$$

where $\tilde{\gamma} \in \widetilde{LG}$ is any lift of γ. We observe that φ is well-defined and satisfies $f(\Phi) = \varphi(\pi(\Phi))\Phi$ for all $\Phi \in \widetilde{LG}$. φ is smooth, since $\pi : \widetilde{LG} \to LG$ has smooth local sections. Then, since f is base-point preserving, we have

$$\pi(\varphi(\gamma)) = \pi(f(\tilde{\gamma}))\pi(\tilde{\gamma})^{-1} = \gamma\gamma^{-1} = \mathtt{const}_e,$$

hence $\varphi(\gamma) \in \mathrm{U}(1) \subset \widetilde{LG}$. The resulting map $\varphi : LG \to \mathrm{U}(1)$ is a group homomorphism, as

$$\varphi(\gamma\eta) = f(\tilde{\gamma})f(\tilde{\eta})\tilde{\eta}^{-1}\tilde{\gamma}^{-1} = f(\tilde{\gamma})\varphi(\eta)\tilde{\gamma}^{-1} = f(\tilde{\gamma})\tilde{\gamma}^{-1}\varphi(\eta) = \varphi(\gamma)\varphi(\eta),$$

where we used that $\mathrm{U}(1) \subset \widetilde{LG}$ is central. If G is semisimple, Theorem 2.1.2 shows that φ is trivial on the identity component $(LG)_0$. This implies that φ factors through $\pi_0(LG) = \pi_1(G)$. □

For our construction of 2-group extensions, central extensions of ΩG will be relevant. On the other hand, central extensions of LG frequently occur in practice. Therefore, we shall study the relation between the two types of central extensions. Clearly, restriction from LG to ΩG provides a functor

$$\mathrm{c}\mathcal{E}\mathrm{xt}(LG) \longrightarrow \mathrm{c}\mathcal{E}\mathrm{xt}(\Omega G). \tag{2.3.1}$$

Lemma 2.3.2 *If G is semisimple and simply connected, then the functor (2.3.1) is an equivalence.*

Proof By Lemma 2.3.1, we only have to check that the functor is a bijection on isomorphism classes. To this end, consider the commutative diagram

$$\begin{array}{ccc} h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(LG)) & \longrightarrow & h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(\Omega G)) \\ \downarrow & & \downarrow \\ H^2_c(L\mathfrak{g}, \mathbb{R}) & \longrightarrow & H^2_c(\Omega\mathfrak{g}, \mathbb{R}). \end{array} \tag{2.3.2}$$

where the top horizontal map is induced by the functor (2.3.1), the bottom horizontal map is pullback along the Lie algebra homomorphism $\Omega\mathfrak{g} \to L\mathfrak{g}$ and the vertical maps are the canonical map (2.2.2) for $H = \Omega G$ and $H = LG$, respectively. That G is simply connected implies that both vertical maps are injective. On the other hand, as G is semisimple, Lemma 2.2.2 implies that the bottom map is an isomorphism (as both consist of classes determined by cocycles of the specific form (2.2.4), which gives the same classification). We conclude that the top horizontal map must be injective.

On the other hand, given a central extension $\widetilde{\Omega G}$ of ΩG, one can construct a central extension $\widetilde{LG}$ of LG such that $\widetilde{LG}|_{\Omega G} = \widetilde{\Omega G}$ in the following way. As G is semisimple, we may assume that $\widetilde{\Omega G}$ is classified by a cocycle ω of the specific form (2.2.4). Because G is simply-connected, the Ad_G-invariance of ω integrates to a G-action on $\widetilde{\Omega G}$ lifting the conjugation action on ΩG. Identifying $LG = \Omega G \rtimes G$, defining $\widetilde{LG} := \widetilde{\Omega G} \rtimes G$ gives the claimed central extension. □

Example 2.3.3 If G is not simply connected, the conclusion of Lemma 2.3.2 is generally false. Namely, if $\widetilde{G}$ is a finite cover of G (these are defined by elements of $\mathrm{Hom}(\pi_1(G), \mathrm{U}(1))$), then pullback of $\widetilde{G}$ along the evaluation homomorphism $LG \to G$ yields a central extension of LG which is trivial when restricted to ΩG.

If $I \subsetneq (0, 2\pi)$, we can further restrict a central extension of ΩG along the inclusion $\Omega_I G \subset \Omega G$, which gives functors

$$\mathrm{c}\mathcal{E}\mathrm{xt}(\Omega G) \longrightarrow \mathrm{c}\mathcal{E}\mathrm{xt}(\Omega_I G). \tag{2.3.3}$$

Lemma 2.3.4 *If G is semisimple, then the functor (2.3.3) is an equivalence whenever $I \subsetneq (0, 2\pi)$ is connected and non-empty.*

Proof We show that the functor is fully faithful. To this end, since we are dealing with groupoids, it suffices to show that (2.3.3) induces an isomorphism of automorphism groups. Let f be an automorphism of a central extension $\widetilde{\Omega G}$ of ΩG, inducing an isomorphism f_I of the restricted central extension $\widetilde{\Omega_I G}$. As G is semisimple, we obtain from (the proof of) Lemma 2.3.1 that $f(\Phi) = \varphi([\pi(\Phi)])\Phi$ for some group homomorphism $\varphi : \pi_1(G) \to \mathrm{U}(1)$. Now if f_I is trivial, we have $\Phi = f_I(\Phi) = \varphi([\pi(\Phi)])\Phi$ for all $\Phi \in \widetilde{\Omega_I G}$, so $\varphi([\pi(\Phi)]) = 1$. But this implies that φ (hence f) is trivial, as any element of $\pi_1(G)$ can be represented by a loop in $\Omega_I G$. This shows that the induced map on automorphism groups is injective.

Similarly, if f_I is any automorphism of $\widetilde{\Omega_I G}$, then $f_I(\Phi) = \varphi([\pi(\Phi)])\Phi$ for some $\varphi : \pi_0(\Omega_I G) \to \mathrm{U}(1)$. But $\pi_0(\Omega_I G) = \pi_0(\Omega G) = \pi_1(G)$, so $f(\Phi) = \varphi([\pi(\Phi)])\Phi$

is an extension of f_I to an automorphism of $\widetilde{\Omega G}$. This shows that the induced map on automorphism groups is surjective, so the functor is fully faithful.

It remains to show that the functor is essentially surjective. Observe that $\Omega_I G = \Omega_{I^\circ} G$, where I° is the interior of I, hence we may assume that I is open. Since I is connected, there exists a diffeomorphism $\varphi : I \to (0, 2\pi)$, which we may choose φ to be affine-linear. Pre-composition with φ induces a group isomorphism $\varphi^* : \Omega G \to \Omega_I G$, which gives rise to an equivalence $\mathrm{c}\mathcal{E}\mathrm{xt}(\Omega_I G) \to \mathrm{c}\mathcal{E}\mathrm{xt}(\Omega G)$. Since G is semisimple, any central extension $\widetilde{\Omega G}$ of ΩG can be represented by a cocycle of the form (2.2.4). It follows that any central extension of $\Omega_I G$ is represented by a cocycle of the form

$$\begin{aligned}\varphi^*\omega(X, Y) &= \int_0^{2\pi} b\big((\varphi_* X)(t), (\varphi_* Y)'(t)\big)dt \\ &= \int_0^{2\pi} b\big(X(\varphi(t)), Y'(\varphi(t))\varphi'(t)\big)dt \\ &= \int_I b\big(X(t), Y'(t)\big)dt.\end{aligned}$$

But this is just the restriction of the cocycle ω.

This shows that if $\widetilde{\Omega_I G}$ is a central extension of $\Omega_I G$, then there exists a central extension $\widetilde{\Omega G}'$ of ΩG whose restriction $\widetilde{\Omega_I G}'$ to $\Omega_I G$ is classified by the same Lie algebra cocycle. Since $\pi_1(\Omega G) = \pi_2(G) = 0$, the sequence (2.2.2) is exact in the middle, and so $\widetilde{\Omega_I G}'$ and $\widetilde{\Omega_I G}$ differ by a central extension of $\pi_0(\Omega G) = \pi_1(G)$. But since the inclusion $\Omega_I G \to \Omega G$ induces an isomorphism on π_0, we can modify $\widetilde{\Omega G}'$ to achieve $\widetilde{\Omega_I G}' \cong \widetilde{\Omega_I G}$. Hence, the functor (2.3.3) is essentially surjective. □

Example 2.3.5 Without the assumption of semisimplicity, Lemma 2.3.4 is false in general: Example 2.4.7 provides an example of a non-trivial central extension of ΩG such that the restriction to a suitable $\Omega_I G$ is trivial.

2.4 Disjoint commutativity

Let G be a finite-dimensional, connected Lie group. It turns out that to construct a 2-group from central extensions of the loop group LG, it is important that these central extensions satisfy a certain extra property, *disjoint commutativity*, which was first studied systematically in [20, §3.3].

Definition 2.4.1 (Disjoint commutativity) A central extension $\widetilde{LG}$ of LG is called *disjoint commutative* if for all $I, J \subset S^1$ with $I \cap J = \emptyset$ the subgroups $\widetilde{L_I G}$ and $\widetilde{L_J G}$ of $\widetilde{LG}$ commute.

The following lemma is crucial. Recall that a bihomomorphism b on a group K is called *skew* if $b(g, h) = b(h, g)^{-1}$. Moreover, by an *interval*, we mean an open, nonempty and connected proper subset $I \subset S^1$.

Lemma 2.4.2 *Let $\widetilde{LG}$ be a central extension of LG, and suppose that G is semisimple. Then, there exists a unique bihomomorphism*

$$b : \pi_1(G) \times \pi_1(G) \longrightarrow \mathrm{U}(1) \tag{2.4.1}$$

such that, for disjoint intervals $I, J \subset S^1$ and all $\gamma \in L_I G$, $\eta \in L_J G$, we have

$$b([\gamma], [\eta]) = \tilde{\gamma}\tilde{\eta}\tilde{\gamma}^{-1}\tilde{\eta}^{-1}$$

where $\tilde{\gamma}$ and $\tilde{\eta}$ are arbitrary lifts of γ, η to $\widetilde{LG}$. Moreover, b is skew.

Proof For $\Phi \in \widetilde{L_I G}$, $\Psi \in \widetilde{L_J G}$, we have $\pi(\Phi^{-1}\Psi\Phi\Psi^{-1}) = \mathtt{const}_e$, hence the commutator $\Phi^{-1}\Psi\Phi\Psi^{-1}$ is contained in $\mathrm{U}(1) \subset \widetilde{LG}$. Observe that this commutator only depends on $\pi(\Phi)$ and $\pi(\Psi)$, as replacing $\Phi = z\Phi$ and $\Psi = w\Psi$, $z, w \in \mathrm{U}(1)$, leads to the same result. Hence we obtain a map

$$B_{IJ} : L_I G \times L_J G \longrightarrow \mathrm{U}(1), \qquad (\gamma, \eta) \longmapsto \tilde{\gamma}\tilde{\eta}\tilde{\gamma}^{-1}\tilde{\eta}^{-1}. \tag{2.4.2}$$

where $\tilde{\gamma}$ and $\tilde{\eta}$ are arbitrary lifts of γ, η to the central extension. B_{IJ} is smooth as $\widetilde{LG}$ admits smooth local sections. We calculate

$$\begin{aligned} B_{IJ}(\gamma_1, \eta) B_{IJ}(\gamma_2, \eta) &= (\tilde{\gamma}_1\tilde{\eta}\tilde{\gamma}_1^{-1}\tilde{\eta}^{-1})(\tilde{\gamma}_2\tilde{\eta}\tilde{\gamma}_2^{-1}\tilde{\eta}^{-1}) \\ &= \tilde{\gamma}_1(\tilde{\gamma}_2\tilde{\eta}\tilde{\gamma}_2^{-1}\tilde{\eta}^{-1})\tilde{\eta}\tilde{\gamma}_1^{-1}\tilde{\eta}^{-1} \\ &= (\tilde{\gamma}_1\tilde{\gamma}_2)\tilde{\eta}(\tilde{\gamma}_1\tilde{\gamma}_2)^{-1}\tilde{\eta}^{-1} \\ &= B_{IJ}(\gamma_1\gamma_2, \eta) \end{aligned}$$

and

$$\begin{aligned} B_{IJ}(\gamma, \eta_1) B_{IJ}(\gamma, \eta_2) &= (\tilde{\gamma}\tilde{\eta}_1\tilde{\gamma}^{-1}\tilde{\eta}_1^{-1})(\tilde{\gamma}\tilde{\eta}_2\tilde{\gamma}^{-1}\tilde{\eta}_2^{-1}) \\ &= \tilde{\gamma}\tilde{\eta}_1\tilde{\gamma}^{-1}(\tilde{\gamma}\tilde{\eta}_2\tilde{\gamma}^{-1}\tilde{\eta}_2^{-1})\tilde{\eta}_1^{-1} \\ &= \tilde{\gamma}(\tilde{\eta}_1\tilde{\eta}_2)\tilde{\gamma}^{-1}(\tilde{\eta}_1\tilde{\eta}_2)^{-1} \\ &= B_{IJ}(\gamma, \eta_1\eta_2), \end{aligned}$$

using that B_{IJ} takes values in the center of $\widetilde{LG}$. Hence, B_{IJ} is a bihomomorphism.

Since G is semisimple, Theorem 2.1.2 implies that B_{IJ} must be constant on the connected components of $L_I G$ and $L_J G$. It follows that there exists a unique bihomomorphism

$$B^0_{IJ} : \pi_0(L_I G) \times \pi_0(L_J G) \longrightarrow \mathrm{U}(1) \qquad \text{with} \qquad B^0_{IJ}([\gamma], [\eta]) = B_{IJ}(\gamma, \eta).$$

Since I and J are intervals, then the inclusion $L_I G \to \Omega G$ is a homotopy equivalence, hence induces an isomorphism $\pi_0(L_I G) \cong \pi_0(\Omega G) = \pi_1(G)$. We conclude that in this case, there exists a unique bihomomorphism b_{IJ} on $\pi_1(G)$ such that

$$B^0_{IJ}([\gamma], [\eta]) = b_{IJ}([\gamma], [\eta]).$$

whenever $\gamma \in L_I G$ and $\eta \in L_J G$. Another way to say this is that, given an interval $I \subset S^1$, each element of $\pi_1(G)$ has a representative γ supported in $L_I G$, and two such representatives of the same element of $\pi_1(G)$ are already homotopic in $L_I G$. We now show that all these bihomomorphism b_{IJ} is independent of the choice of I and J.

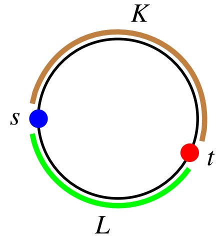
(a). First observe that if $I' \subseteq I$ and $J' \subseteq J$, then $B_{I'J'}$ is the restriction of B_{IJ} to $L_{I'}G \times L_{J'}G$.

(b). Suppose now that I, J and I', J' are two pairs of disjoint intervals of S^1 with the property that $I \cap I'$ and $J \cap J'$ are non-empty. Then, using (a), we see that for $\gamma \in L_{I\cap I'}G$, $\eta \in L_{J\cap J'}G$, we have

$$b_{I'J'}([\gamma],[\eta]) = B_{I'J'}(\gamma,\eta) = B_{I\cap I', J\cap J'}(\gamma,\eta) = B_{IJ}(\gamma,\eta) = b_{IJ}([\gamma],[\eta]).$$

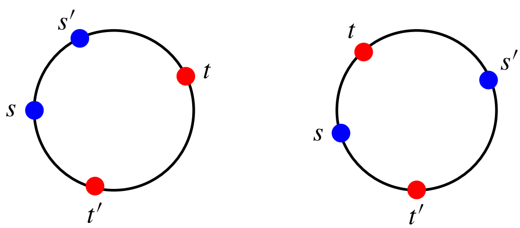
Hence $b_{IJ} = b_{I'J'}$.

(c). Next we show that $b_{IJ} = b_{JI}$. Choose $s \in I$ and $t \in J$ and let $K, L \subset S^1$ be the two disjoint intervals such that $\partial K = \partial L = \{s, t\}$.



By construction, all the intersections $I \cap K$, $K \cap J$, $J \cap L$, $L \cap I$ are non-empty. Therefore, by (b), $b_{IJ} = b_{KL} = b_{JI}$.

(d). Now let I, J and I', J' be two arbitrary pairs of disjoint intervals of S^1. Choose pairwise distinct points $s \in I$, $t \in J$, $s' \in I'$, $t' \in J'$. There are two different possible basic configurations for s, s', t, t', as depicted below.



In the first configuration, we can choose disjoint intervals $K, L \subset S^1$ such that $s, s' \in K$ and $t, t' \in L$. Then, by construction, K has non-empty intersection with both I and I' and L has non-empty intersection with both J and J'. Consequently, by (b), we obtain $b_{IJ} = b_{KL} = b_{I'J'}$. In the second configuration, we can choose disjoint intervals $K, L \subset S^1$ such that $s, t' \in K$ and $t, s' \in L$. Then K has non-empty intersection with both I and J' and L has non-empty intersection with both J and I'. Therefore, by (b) and (c), $b_{IJ} = b_{KL} = b_{J'I'} = b_{I'J'}$.

We conclude that the bihomomorphism b_{IJ} is independent of the choice of disjoint intervals $I, J \subset S^1$. We write b for this bihomomorphism on $\pi_1(G)$.

It follows from the definition of the bihomomorphisms B_{IJ} that

$$b_{IJ}(g, h) = b_{JI}(h, g)^{-1}, \qquad g, h \in \pi_1(G)$$

for any pair of disjoint intervals $I, J \subset S^1$. Since $b = b_{IJ} = b_{JI}$, this shows that the bihomomorphism b is skew. □

Theorem 2.4.3 *A central extension $\widetilde{LG}$ of the loop group LG of a semisimple Lie group G is disjoint commutative if and only if the bihomomorphism b of Lemma 2.4.2 vanishes.*

Corollary 2.4.4 *If G is simply connected and semisimple, then all central extensions of LG are disjoint commutative.*

Proof of Theorem 2.4.3 It is clear that b is trivial when $\widetilde{LG}$ is disjoint commutative. If b is trivial, then the bihomomorphisms B^0_{IJ} (and consequently the B_{IJ}) for disjoint intervals I, J are trivial as well, so that $\widetilde{LG}$ is disjoint commutative for intervals.

It remains to treat the case of general disjoint subsets $I, J \subset S^1$. Observe that $L_I G = L_{I^\circ} G$, where I° is the interior of I, hence we can assume throughout that $I, J \subseteq S^1$ are open. Suppose now that $I = I_1 \sqcup I_2 \sqcup \cdots$ is a disjoint union of possibly infinitely many intervals. Then $L_{I_1 \sqcup \cdots \sqcup I_n} G \cong L_{I_1} G \times \cdots \times L_{I_n} G$ for each $n \in \mathbb{N}$. Moreover, the union

$$\bigcup_{n=1}^{\infty} L_{I_1 \sqcup \cdots \sqcup I_n} G \subset L_I G$$

is dense. This implies that the group of connected components of $L_I G$ is the direct *sum*

$$\pi_0(L_I G) = \bigoplus_{k=1}^{\infty} \pi_0(L_{I_k} G).$$

We therefore obtain that if $J = J_1 \sqcup J_2 \sqcup \cdots \subset S^1$ is another such subset, then the bihomomorphism B^0_{IJ} is determined by the bihomomorphisms $B^0_{I_k J_l}$, $j, k \in \mathbb{N}$, which vanish by assumption. □

Let $\widetilde{LG}$ be a central extension of LG. Any $\mathrm{U}(1)$-valued group 2-cocycle κ on $\pi_1(G)$ can be used to modify the group product of $\widetilde{LG}$ according to the formula

$$\Phi \star \Psi = \kappa([\pi(\Phi)], [\pi(\Psi)]) \cdot \Phi\Psi. \tag{2.4.3}$$

We assume throughout that κ is normalized, in the sense that $\kappa(g, e) = \kappa(e, g) = \kappa(e, e) = 1$. This is equivalent to requiring that the unit elements for the two products coincide. Normalization is no serious restriction as every cocycle is cohomologous to a normalized one. Moreover, if $\kappa = d\rho$ for some U(1)-valued 1-cocycle, then the central extension $\widetilde{LG}$ with the modified product (2.4.3) is isomorphic to the original central extension.

Lemma 2.4.5 *Let G be a semisimple Lie group and let b be the obstruction bihomomorphism of Lemma 2.4.2 for a central extension $\widetilde{LG}$. Then, the obstruction bihomomorphism b' for the central extension with the modified product (2.4.3) is given by*

$$b'(g, h) = b(g, h) \cdot skew\kappa(g, h)^{-1},$$

where

$$skew\kappa(g, h) := \kappa(g, h)\kappa(h, g)^{-1}$$

is the skew *of κ.*

It is well-known that the skew of a 2-cocycle on an *abelian* group is always a bihomomorphism; notice here that $\pi_1(G)$ is abelian as G is a Lie group.

Proof Let $\Phi \in L_I G$ and $\Psi \in L_J G$ for $I = (0, \pi)$ and $J = (\pi, 2\pi)$, and let $g = [\pi(\Phi)], h = [\pi(\Psi)] \in \pi_1(G)$. The inverses of Φ and Ψ with respect to the modified product (2.4.3) are

$$\Phi^{\star-1} = \kappa(g, g^{-1})^{-1}\Phi^{-1}, \qquad \Psi^{\star-1} = \kappa(h, h^{-1})^{-1}\Psi^{-1}.$$

Then, using that $\pi_1(G)$ is abelian,

$$\begin{aligned} b'(g, h) &= B'_{IJ}(\pi(\Phi), \pi(\Psi)) \\ &= \Phi\star\Psi\star\Phi^{\star-1}\star\Psi^{\star-1} \\ &= \kappa(g, g^{-1})^{-1}\kappa(h, h^{-1})^{-1}\Phi\star\Psi\star\Phi^{-1}\star\Psi^{-1} \\ &= \kappa(g, g^{-1})^{-1}\kappa(h, h^{-1})^{-1}\kappa(g, h)\kappa(gh, g^{-1})\underbrace{\kappa(ghg^{-1}, h^{-1})}_{=\kappa(h,h^{-1})}\Phi\Psi\Phi^{-1}\Psi^{-1} \\ &= \kappa(g, g^{-1})^{-1}\kappa(g, h)\kappa(gh, g^{-1})b(g, h) \end{aligned}$$

Since κ is a group cocycle, we have

$$\kappa(gh, g^{-1}) = \kappa(hg, g^{-1}) = \kappa(h, g)^{-1}\kappa(h, 1)\kappa(g, g^{-1}) = \kappa(h, g)^{-1}\kappa(g, g^{-1}),$$

as κ is assumed to be normalized. Plugging this into the previous formula yields the desired result. ◻

Example 2.4.6 The above results provide many examples of central extensions of LG for non-simply connected Lie groups G that are not disjoint commutative. For example, suppose we have $\xi \in \mathrm{U}(1)$ and $p, q \in \mathbb{Z}$ such that $\xi^p = \xi^q = 1$. Then, the group 2-cocycle κ on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ given by

$$\kappa((k_1, k_2), (l_1, l_2)) = \xi^{k_1 l_2},$$

has non-trivial skew. For example, with the choices $\xi = -1$ and $p = q = 2$, the trivial central extension of $L(\mathrm{SO}(m) \times \mathrm{SO}(n))$ $(m, n \geq 3)$ modified by the cocycle κ provides a central extension that is not disjoint commutative.

Example 2.4.7 Things change completely upon leaving the realm of semisimple Lie groups. An example of a non-disjoint commutative central extension in the case that G has trivial fundamental group is the following. Consider $G = \mathbb{R}^+$ and let $\widetilde{LG}$ be the central extension corresponding to the group cocycle

$$\kappa(\gamma, \eta) = \exp(i \log \gamma(s) \cdot \log \eta(t)),$$

for $s, t \in S^1$ fixed. Since $L\mathbb{R}^+$ is abelian, the bihomomorphism B_{IJ} from (2.4.2) is the restriction of a bihomomorphism B defined on all of LG, which is just the skew of κ. This is non-zero whenever $s \neq t$.

Example 2.4.8 A further example of a central extension of $L\mathrm{U}(1)$ that is not disjoint commutative is given as Example 4.12 in [20].

Example 2.4.9 Consider $G = \mathrm{SO}(d)$ for $d \geq 5$. Then the group of isomorphism classes of central extensions of $L\mathrm{SO}(d)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$, where the first factor is called the *level* and the second factor comes from central extensions of $\mathrm{SO}(d)$ (compare Lemma 4.8 of [10]). Since $\pi_1(\mathrm{SO}(d)) = \mathbb{Z}_2$ and $H^2(\mathbb{Z}_2, \mathrm{U}(1)) = 0$, there are no non-trivial product modifications by group cocycles. It turns out that the obstruction bihomomorphism of the generator of $h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(L\mathrm{SO}(d)))$ is the $\mathrm{U}(1)$-valued skew bihomomorphism

$$b(k_1, k_2) = (-1)^{k_1 k_2} \tag{2.4.4}$$

on $\mathbb{Z}_2$. Hence, a central extension of $L\mathrm{SO}(d)$ is disjoint commutative if and only if it is of even level. Of course, by Corollary 2.4.4, all central extensions of $L\mathrm{SO}(d)$ become disjoint commutative when pulled back along $L\mathrm{Spin}(d) \to L\mathrm{SO}(d)$.

Recall that a bihomomorphism b on an abelian group K is *alternating* if $b(g, g) = 1$. Any alternating bihomomorphism is skew, but the converse is not always true in the presence of 2-torsion in the target. By definition, the skew of a group cocycle is always alternating. Moreover, an easy calculation shows that the skew of a coboundary is zero. Hence we obtain a well-defined group homomorphism

$$H^2(K, \mathrm{U}(1)) \longrightarrow \mathrm{Alt}^2(K, \mathrm{U}(1)) \tag{2.4.5}$$

from the second group cohomology of K to the group of alternating bihomomorphisms on K. It is a fact that this group homomorphism is always surjective [15, Proposition 3.3].

Given a non-disjoint commutative central extension $\widetilde{LG}$, one may ask whether we can modify the product by a group cocycle κ such that $\widetilde{LG}$ becomes disjoint commutative. To investigate this question, consider the map that assigns to a central extension the obstruction bihomomorphism from Lemma 2.4.2. By Lemma 2.4.5 and the surjectivity of (2.4.5), this map descends to a group homomorphism

$$\frac{h_0\big(\mathrm{c}\mathcal{E}\mathrm{xt}(LG)\big)}{H^2(\pi_1(G), \mathrm{U}(1))} \longrightarrow \frac{\mathrm{Skew}^2(\pi_1(G), \mathrm{U}(1))}{\mathrm{Alt}^2(\pi_1(G), \mathrm{U}(1))}, \tag{2.4.6}$$

where $\mathrm{Skew}^2(\pi_1(G), \mathrm{U}(1))$ denotes the group of skew bihomomorphisms on $\pi_1(G)$ and $H^2(\pi_1(G), \mathrm{U}(1))$ acts on the set of isomorphism classes of central extensions by modifying the product according to (2.4.3). Combining Theorem 2.4.3 with the surjectivity of (2.4.5), we obtain the following result.

Theorem 2.4.10 *Let G be a semisimple Lie group. A central extension $\widetilde{LG}$ of LG can be modified by a group 2-cocycle $\kappa \in H^2(\pi_1(G), \mathrm{U}(1))$ to become disjoint commutative if and only if the image of $\widetilde{LG}$ under* (2.4.6) *is zero.*

It is easy to see that the quotient on the right hand side of (2.4.6) is isomorphic to $\pi_1(G) \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Thus, we obtain the following result.

Corollary 2.4.11 *Let G be a semisimple Lie group such that $\pi_1(G)$ has no 2-torsion. Then, any central extension of LG can be modified by a group 2-cocycle to become disjoint commutative.*

In case of Example 2.4.9 the image of the basic central extension of $L\mathrm{SO}(d)$ under (2.4.6) is the non-alternating, non-zero skew bihomomorphism (2.4.4). Hence, this central extension cannot be modified by a group cocycle to become disjoint commutative.

Remark 2.4.12 In the case of $G = \mathrm{SO}(d)$ any element on the right hand side of (2.4.6) is realized by a central extension of $L\mathrm{SO}(d)$. In other words, the homomorphism (2.4.6) is surjective. We do not know whether it is surjective for *any* semisimple Lie group G.

3 Lie 2-groups from loop group extensions

This section contains the main result of the present article, namely, the construction of Lie 2-groups from loop group extensions. In Sect. 3.1 we recall the relevant facts about crossed modules and Lie 2-groups, and Sect. 3.2 contains the main construction. Section 3.3 concerns the notion of a fusion factorization that allows one to give our Lie 2-groups a more convenient form. In Sect. 3.4 we show that our Lie 2-groups deliver 3-connected covering groups, in particular, models for the string 2-group.

3.1 Strict Lie 2-groups and crossed modules

We recall that a *strict Lie 2-group* is a groupoid $\Gamma = (\Gamma_0, \Gamma_1, s, t, i, \circ, \mathrm{inv})$ whose set Γ_0 of objects and whose set Γ_1 of morphisms are (possibly Fréchet) Lie groups, whose source and target map $s, t : \Gamma_1 \to \Gamma_0$, composition $\circ : \Gamma_1 \times_{t,s} \Gamma_1 \to \Gamma_1$, identity map $i : \Gamma_0 \to \Gamma_1$, and inversion (with respect to composition) $\mathrm{inv} : \Gamma_1 \to \Gamma_1$ are all smooth group homomorphisms. We note that if Γ_1 and Γ_0 are finite-dimensional, then the fibre product $\Gamma_1 \times_{t,s} \Gamma_1$ exists since s and t are surjective Lie group homomorphisms, hence submersions; in the infinite-dimensional setting, the existence of the fibre product is a further assumption that we need to impose. We also note that the group

$$\pi_1(\Gamma) := \ker(s) \cap \ker(t) \subseteq \Gamma_1$$

is abelian.

When constructing strict Lie 2-groups it is worthwhile to notice that composition and inversion are already determined by the remaining structure. Indeed, it is straightforward to see that

$$x \circ y = x\, i(s(x))^{-1} y = x\, i(t(y))^{-1} y, \tag{3.1.1}$$

for composable morphisms $x, y \in \Gamma_1$, i.e., morphisms such that $s(x) = t(y)$. It follows from this that the inverse of a morphism $x \in \Gamma_1$ with respect to composition satisfies

$$\mathrm{inv}(x) = i(s(x)) x^{-1} i(t(x)). \tag{3.1.2}$$

Moreover, in a strict Lie 2-group the subgroups $\ker(s)$ and $\ker(t)$ of Γ_1 commute: let $x \in \ker(s)$, $y \in \ker(t)$, and let $e \in \Gamma_0$ be the unit element. Then

$$yx = (e \circ y)(x \circ e) = (e \cdot x) \circ (y \cdot e) = x \circ y = x\, i(s(x))^{-1} y = xy. \tag{3.1.3}$$

We have the following converse of these three observations.

Lemma 3.1.1 *Suppose Γ_0 and Γ_1 are Lie groups and $s, t : \Gamma_1 \to \Gamma_0$ and $i : \Gamma_0 \to \Gamma_1$ are smooth group homomorphisms such that:*

(a). $s \circ i = \mathrm{id}_{\Gamma_0} = t \circ i$.
(b). $\ker(s)$ *and* $\ker(t)$ *are commuting Lie subgroups.*

Then, together with the composition defined by (3.1.1) *and the inversion defined in* (3.1.2)*, this structure constitutes a strict Lie 2-group.*

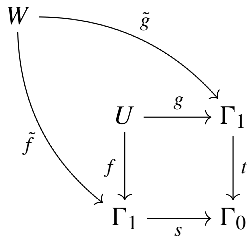
Proof First of all we prove that the fibre product $\Gamma_1 \times_{t,s} \Gamma_1$ exists in the category of Fréchet Lie groups. We consider $U := \ker(s) \times \ker(s) \times \Gamma_0$ equipped with the maps $f, g : U \to \Gamma_1$ defined by $f(x, y, z) := xi(z)$ and $g(x, y, z) := yi(t(x)z)$. Then we have

$$(s \circ g)(x, y, z) = s(yi(t(x)z)) = t(x)z = t(xi(z)) = (t \circ f)(x, y, z).$$

By (b) we see that U is a Fréchet manifold, and the maps f and g are clearly smooth. Moreover, we turn U into a Fréchet Lie group, and f and g into group homomorphisms, by declaring

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) := (x_1 i(z_1) x_2 i(z_1)^{-1}, y_1 x_1 i(z_1) y_2 i(z_1)^{-1} x_1^{-1}, z_1 z_2).$$

Now we assume that



is a commutative diagram in the category of Fréchet Lie groups. We define

$$h : W \to U; \quad h(w) := (w_1 i(s(w_1))^{-1}, w_2 i(s(w_2))^{-1}, s(w_1))$$

where $w_1 := \tilde{f}(w)$ and $w_2 := \tilde{g}(w)$. This is a smooth group homomorphism such that $g \circ h = \tilde{g}$ and $f \circ h = \tilde{f}$. It is straightforward to check that this map is unique with this property. This shows that U is the required fibre product; in particular, it exists. It is then easy to see that the composition defined by (3.1.1) is smooth and that—using (a)—turns Γ into a Fréchet Lie groupoid.

The commutativity in condition (b) is used in order to show that composition is a group homomorphism: Let $x_1, x_2, y_1, y_2 \in \Gamma_1$ with $s(x_1) = t(y_1)$, $s(x_2) = t(y_2)$. Observe that $x_2\, i(s(x_2)) \in \ker(s)$ and $i(t(y_1))^{-1} y_1 \in \ker(t)$. Therefore, we can calculate

$$\begin{aligned} x_1 x_2 \circ y_1 y_2 &= x_1 x_2\, i(t(y_1 y_2))^{-1} y_1 y_2 \\ &= x_1 x_2\, i(t(y_2))^{-1} i(t(y_1))^{-1} y_1 y_2 \\ &= x_1 x_2\, i(s(x_2))^{-1} i(t(y_1))^{-1} y_1 y_2 \\ &= x_1 i(t(y_1))^{-1} y_1 x_2\, i(s(x_2))^{-1} y_2 \\ &= (x_1 \circ y_1) \cdot (x_2 \circ y_2), \end{aligned}$$

where in the second last step, we used (b). □

Another way to present (Fréchet) Lie 2-groups is in terms of crossed modules of (Fréchet) Lie groups. Recall that a *crossed module* X of Fréchet Lie groups consists of a pair of Fréchet Lie groups G and H together with a smooth group homomorphism $t : H \to G$ and *crossed module action* of G on H, i.e., a smooth map $\alpha : G \times H \to H$, such that α is an action of G on H by group homomorphisms, and

$$t(\alpha_g(h)) = g t(h) g^{-1} \qquad \text{and} \qquad \alpha_{t(h)}(k) = hkh^{-1} \tag{3.1.4}$$

hold for all $g \in G$ and $h, k \in H$, where $\alpha_g(h) := \alpha(g, h)$. The first property means that t is G-equivariant for the G-action α on H and the conjugation action of G on itself. The second property is called the *Peiffer identity*. A nice review of the history of the notion of a crossed module can be found in [7].

Observe that for a crossed module X, the Peiffer identity imply that $A := \ker(t)$ lies in the center of H and, in particular, is abelian. By G-equivariance of t, the G-action α restricts to an action on A. The crossed module is called *central* if this action of G on A is trivial.

There is an adjoint equivalence

$$\mathcal{X} : \mathcal{L}\text{ie-2-}\mathcal{G}\text{rp} \rightleftarrows \mathcal{X}\text{-}\mathcal{M}\text{od} : \mathcal{G} \tag{3.1.5}$$

between the category $\mathcal{L}$ie-2-$\mathcal{G}$rp of Fréchet Lie 2-groups and the category $\mathcal{X}$-$\mathcal{M}$od of crossed modules of Fréchet Lie groups, when both are equipped with the obvious notion of strict morphisms. For plain crossed modules of sets, this is the Brown-Spencer theorem [3], which has been generalized to crossed modules ambient to another category by Janelidze [8]; here we use it in the Fréchet Lie group setting. Explicitly, the equivalence 3.1.5 is given by:

$$\mathcal{G}(H, G, t, \alpha) := \left(\begin{array}{ll} \Gamma_1 := H \rtimes_\alpha G & s(h, g) := g \\ \Gamma_0 := G & t(h, g) := t(h)g \\ & i(g) := (e, g) \end{array} \right)$$

$$\mathcal{X}(\Gamma, s, t, i) := \left(\begin{array}{c} H := \ker(s) \subseteq \Gamma_1 \\ G := \Gamma_0 \\ t := (t : \Gamma_1 \to \Gamma_0)_{|\ker(s)} \\ \alpha_g(h) := i(g)h\, i(g)^{-1} \end{array} \right)$$

The above description of $\mathcal{G}$ uses Lemma 3.1.1, which applies here since the Lie subgroups $\ker(s) = \{(h, 1) \mid h \in H\} \cong H$ and $\ker(t) = \{(h^{-1}, t(h)) \mid h \in H\} \cong H$ commute. It is worthwhile to look at the unit and counit maps

$$\epsilon : \mathcal{X}\mathcal{G} \Rightarrow \mathrm{id}_{\mathcal{X}\text{-}\mathcal{M}\text{od}} \qquad \eta : \mathrm{id}_{\mathcal{L}\text{ie-2-}\mathcal{G}\text{rp}} \Rightarrow \mathcal{G}\mathcal{X}$$

of the adjunction (3.1.5). While the formula for the unit ϵ is obvious, the counit η is given at a Lie 2-group Γ by the strict Lie 2-group isomorphism

$$\eta_\Gamma = \begin{pmatrix} \Gamma_1 \to \ker(s) \rtimes_\alpha \Gamma_0, & h \mapsto (h\, i(s(h))^{-1}, s(h)) \\ \Gamma_0 \to \Gamma_0, & g \mapsto g \end{pmatrix}.$$

Example 3.1.2 Given any abelian Lie group A, setting $\Gamma_0 = \{e\}$, $\Gamma_1 = A$ (and trivial s, t, i) give a strict Lie 2-group denoted by BA. The corresponding crossed module is $A \to \{e\}$, with the (necessarily trivial) action. Observe that A is forced to be abelian by the requirement of Lemma 3.1.1 (b).

Example 3.1.3 Any Lie group G can be viewed as a strict Lie 2-group, denoted G_{dis}, by setting $\Gamma_0 = \Gamma_1 = G$ and $s = t = i = \mathrm{id}_G$. The corresponding crossed module is $\{e\} \to G$.

3.2 Crossed modules from loop group extensions

Let again G be a connected (finite-dimensional) Lie group and let

$$1 \to \mathrm{U}(1) \to \widetilde{\Omega G} \xrightarrow{\pi} \Omega G \to 1$$

be a Fréchet central extension of the based loop group ΩG. We will now describe how to use this central extension to produce a crossed module of Fréchet Lie groups. For a Lie subgroup $H \subset \Omega G$, we write

$$\widetilde{H} := H \times_{\Omega G} \widetilde{\Omega G}$$

for the pullback of $\widetilde{\Omega G}$ to H, and address an element $(h, \Phi) \in \widetilde{H}$ by just Φ.

We identify $P_eG^{[2]}$ with a subgroup of ΩG using the injective map $\cup : P_eG^{[2]} \to \Omega G$, and hence consider, in the above notation, the pullback

$$\widetilde{P_eG^{[2]}} = P_eG^{[2]} \times_{\Omega G} \widetilde{\Omega G}.$$

To begin with, we have canonical maps

$$\widetilde{P_eG^{[2]}} \underset{t}{\overset{s}{\rightrightarrows}} P_eG\ , \qquad \begin{matrix} s(\Phi) = \gamma_2, \\ t(\Phi) = \gamma_1, \end{matrix} \qquad \text{whenever} \quad \pi(\Phi) = \gamma_1 \cup \gamma_2. \tag{3.2.1}$$

We note that

$$\ker(s) = \widetilde{\Omega_{(0,\pi)}}G. \tag{3.2.2}$$

As for any central extension, the conjugation action of $\widetilde{\Omega G}$ on itself descends to a smooth action of ΩG. This action is trivial on $\mathrm{U}(1) \subset \widetilde{\Omega G}$ and restricts to the subgroups $\widetilde{\Omega_I G}$, for any subset $I \subseteq S^1$. Pulling back along the "diagonal" group homomorphism

$$P_eG \to P_eG^{[2]} \xrightarrow{\cup} \Omega G,$$

where $\cup$ is defined in (2.1.2), we obtain an action of P_eG on $\widetilde{\Omega G}$. The restriction of this action to $\widetilde{\Omega_{(0,\pi)}}G$ will be denoted by α, and will be called the *canonical action* associated to $\widetilde{\Omega G}$. Explicitly, it is given by

$$\alpha : P_eG \times \widetilde{\Omega_{(0,\pi)}}G \longrightarrow \widetilde{\Omega_{(0,\pi)}}G, \qquad \alpha_\gamma(\Phi) = \widetilde{\gamma \cup \gamma} \cdot \Phi \cdot (\widetilde{\gamma \cup \gamma})^{-1}, \tag{3.2.3}$$

where $\widetilde{\gamma \cup \gamma}$ is any lift of $\gamma \cup \gamma$ to $\widetilde{\Omega G}$. As the choice of lift is unique up to an element in the center of $\widetilde{\Omega G}$, the right hand side of (3.2.3) is independent of the choice of lift.

Remark 3.2.1 We emphasize that the construction of the canonical action α is *much* simpler than the construction in [5, Lemma 24] and, in particular, that it does not depend on any additional data or a particular model for the central extension (compare also [17, Prop. 4.3.2]). That the canonical action coincides with the action from [5, Lemma 24] will be discussed in detail in Sect. 4.1.

The map t intertwines the canonical action α with the conjugation action of P_eG on itself,

$$t(\alpha_{\gamma}(\Phi)) = \gamma \cdot t(\Phi) \cdot \gamma^{-1}.$$

However, the canonical action α does *not* generally satisfy the Peiffer identity

$$\alpha_{t(\Psi)}(\Phi) = \Psi \cdot \Phi \cdot \Psi^{-1}. \tag{3.2.4}$$

Instead, we have the following lemma.

Lemma 3.2.2 *If the central extension $\widetilde{\Omega G}$ is disjoint commutative, then the canonical action α of* (3.2.3) *satisfies the Peiffer identity.*

Proof Let $\Psi, \Phi \in \ker(s) = \widetilde{\Omega_{(0,\pi)}}G$ and write $\gamma = t(\Psi)$. Then

$$\begin{aligned}\alpha_{t(\Psi)}(\Phi) &= \widetilde{\gamma \cup \gamma} \cdot \Phi \cdot \widetilde{\gamma \cup \gamma}^{-1} \\ &= (\widetilde{\gamma \cup \gamma} \cdot \Psi^{-1}) \cdot (\Psi\Phi\Psi^{-1}) \cdot (\Psi \cdot \widetilde{\gamma \cup \gamma}^{-1}).\end{aligned}$$

The middle term is contained in $\ker(s) = \widetilde{\Omega_{(0,\pi)}}G$, while the outer terms are contained in $\ker(t) = \widetilde{\Omega_{(\pi,2\pi)}}G$. Hence, by disjoint commutativity, these terms commute, leading to the desired result. □

Finally, we observe that the canonical action α is trivial on the central subgroup $\mathrm{U}(1) \subset \widetilde{\Omega_{(0,\pi)}}G$. Thus, we obtain the following result.

Theorem 3.2.3 *If $\widetilde{\Omega G}$ is a disjoint commutative central extension of ΩG, then the Lie group homomorphism $t : \widetilde{\Omega_{(0,\pi)}}G \to P_eG$ and the canonical action α of* (3.2.3) *form a central crossed module of Fréchet Lie groups, denoted by $X(\widetilde{\Omega G})$.*

Next we study the question of whether there are other options for the crossed module action α.

Theorem 3.2.4 *Let G be a semisimple Lie group and let $\widetilde{\Omega G}$ be a disjoint commutative central extension of ΩG. Let moreover α' be an action of P_eG on $\widetilde{\Omega_{(0,\pi)}}G$ turning*

$$t : \widetilde{\Omega_{(0,\pi)}}G \to P_eG$$

into a central crossed module. Then α' coincides with the canonical action α of (3.2.3).

Proof For $\gamma \in P_eG$ and $\eta \in \Omega_{(0,\pi)}G$, we define a map $\kappa_\gamma : \Omega_{(0,\pi)}G \to \widetilde{\Omega_{(0,\pi)}}G$ by

$$\kappa_\gamma(\eta) := \alpha'_\gamma(\tilde{\eta})\alpha_\gamma(\tilde{\eta})^{-1}, \tag{3.2.5}$$

where $\tilde{\eta}$ is any lift of η. This is well-defined, as any two lifts of η differ only by an element $z \in \mathrm{U}(1)$ and both actions are central, so $\alpha'_\gamma(z) = z = \alpha_\gamma(z)$. It is moreover smooth as $\widetilde{\Omega G}$ possesses smooth local sections. As both α' and α intertwine t with the conjugation action of P_eG on $\Omega_{(0,\pi)}G$, we have $t(\kappa_\gamma(\eta)) = \mathtt{const}_e$ for all $\gamma \in P_eG$, $\eta \in \Omega_{(0,\pi)}G$, hence κ_γ takes values in $\mathrm{U}(1)$. Moreover, κ_γ is a group homomorphism:

$$\begin{aligned}\kappa_\gamma(t(\Phi)t(\Psi)) &= \alpha'_\gamma(\Phi\Psi)\alpha_\gamma(\Phi\Psi)^{-1}\\ &= \alpha'_\gamma(\Phi)\underbrace{\alpha'_\gamma(\Psi)\alpha_\gamma(\Psi)^{-1}}_{\in \mathrm{U}(1)}\alpha_\gamma(\Phi)^{-1}\\ &= \alpha'_\gamma(\Phi)\alpha_\gamma(\Phi)^{-1}\alpha'_\gamma(\Psi)\alpha_\gamma(\Psi)^{-1}\\ &= \kappa_\gamma(t(\Phi))\kappa_\gamma(t(\Psi)).\end{aligned}$$

By Theorem 2.1.2, $\kappa_\gamma : \Omega_{(0,\pi)}G \to \mathrm{U}(1)$ must be the trivial group homomorphism for each $\gamma \in P_eG$. Hence α' coincides with α. □

Let $\mathcal{X}\text{-c}\mathcal{E}\mathrm{xt}(G)$ be the subcategory of $\mathcal{X}\text{-}\mathcal{M}\mathrm{od}$ consisting of those central crossed modules $(\widetilde{\Omega_{(0,\pi)}}G, P_eG, t, \alpha)$ in which $\widetilde{\Omega_{(0,\pi)}}G$ is a disjoint commutative central extension of $\Omega_{(0,\pi)}G$, and $t : \widetilde{\Omega_{(0,\pi)}}G \to P_eG$ is given as before; i.e., if $\Phi \in \widetilde{\Omega_{(0,\pi)}}G$ projects to $\gamma \cup \mathtt{const}_e$, then $t(\Phi) = \gamma$. The morphisms are crossed module morphisms whose map $P_eG \to P_eG$ is the identity, and whose map $\widetilde{\Omega_{(0,\pi)}}G \to \widetilde{\Omega_{(0,\pi)}}G'$ is a morphism of central extensions of $\Omega_{(0,\pi)}G$. On the other side, we let dc-c$\mathcal{E}$xt(ΩG) denote the full subcategory of c$\mathcal{E}$xt(ΩG) over all disjoint commutative central extensions of ΩG. Theorem 3.2.3 establishes a functor

$$X : \text{dc-c}\mathcal{E}\mathrm{xt}(\Omega G) \longrightarrow \mathcal{X}\text{-c}\mathcal{E}\mathrm{xt}(G). \tag{3.2.6}$$

In order to see this, it suffices to observe that any automorphism of a central extension $\widetilde{\Omega G}$ provides an automorphism of the restricted central extension $\widetilde{\Omega_{(0,\pi)}}G$ that intertwines the action α.

Corollary 3.2.5 *If G is simply connected and semisimple, the functor X is an equivalence of categories,* dc-c$\mathcal{E}$xt(ΩG) $\cong$ $\mathcal{X}$-c$\mathcal{E}$xt(G).

Proof By Lemma 2.3.1, the assumptions on G imply that both dc-c$\mathcal{E}$xt(ΩG) and $\mathcal{X}$-c$\mathcal{E}$xt(G) are groupoids with trivial automorphism groups. Therefore, we only have to show that the functor X is a bijection on isomorphism classes of objects.

If two crossed modules $X(\widetilde{\Omega G})$ and $X(\widetilde{\Omega G}')$ are isomorphic via an isomorphism in $\mathcal{X}$-c$\mathcal{E}$xt(G), then this in particular implies that the restricted central extensions $\widetilde{\Omega_{(0,\pi)}}G$ and $\widetilde{\Omega_{(0,\pi)}}G'$ are isomorphic. But, by Lemma 2.3.4, this implies that $\widetilde{\Omega G}$ and $\widetilde{\Omega G}'$ are themselves isomorphic. Hence the functor X is injective.

Conversely, by the same Lemma 2.3.4, any central extension $\widetilde{\Omega_{(0,\pi)}}G$ of $\Omega_{(0,\pi)}G$ is the restriction of a central extension $\widetilde{\Omega G}$ of ΩG. From the proof of that lemma it is clear that $\widetilde{\Omega G}$ is disjoint commutative if $\widetilde{\Omega_{(0,\pi)}}G$ is. □

Remark 3.2.6 The group homomorphism κ_γ from the proof of Theorem 3.2.4, defined in (3.2.5), can be defined for any two central crossed module actions α and α' for the homomorphism $t : \widetilde{\Omega_{(0,\pi)}}G \to P_eG$, for any central extension $\widetilde{\Omega G}$ and without assuming that G is semisimple. As both α and α' satisfy the Peiffer identity, κ_γ depends on γ only through the endpoint $g = \gamma(\pi)$.

Varying g, we obtain a map

$$\kappa : G \to \mathrm{Hom}(\Omega_{(0,\pi)}G, \mathrm{U}(1)).$$

The group $\mathrm{Hom}(\Omega_{(0,\pi)}G, \mathrm{U}(1))$ carries a right action of P_eG given by precomposition with the conjugation action on $\Omega_{(0,\pi)}G$, which descends to an action of G as $\Omega_{(0,\pi)}G$ acts trivially. One can then show that κ is a diffeological group 1-cocycle with values in the right G-module $\mathrm{Hom}(\Omega_{(0,\pi)}G, \mathrm{U}(1))$, equipped with the functional diffeology.

Conversely, modifying α by a general $\mathrm{Hom}(\Omega_{(0,\pi)}G, \mathrm{U}(1))$-valued diffeological group cocycle κ on G according to formula (3.2.5) gives another crossed module action of P_eG on $\Omega_{(0,\pi)}G$, and the resulting crossed module is isomorphic in $\mathcal{X}$-c$\mathcal{E}$xt(G) to the previous one if and only if κ is a coboundary.

3.3 Fusion factorizations

Let $\widetilde{\Omega G}$ be a disjoint commutative central extension of ΩG. In Theorem 3.2.3 we have constructed a canonical crossed module $X(\widetilde{\Omega G})$ associated to $\widetilde{\Omega G}$. The functor $\mathcal{G}$ from the adjunction 3.1.5 turns it into a strict Lie 2-group. Explicitly, this Lie 2-group, $\mathcal{G}(X(\widetilde{\Omega G}))$, has the underlying groupoid

$$\widetilde{\Omega_{(0,\pi)}}G \rtimes_\alpha P_eG \underset{t}{\overset{s}{\underset{\longleftarrow i \longrightarrow}{\rightrightarrows}}} P_eG, \tag{3.3.1}$$

where $i(\gamma) = (1,\gamma)$, $s(\Phi,\gamma) = \gamma$ and $t(\Phi,\gamma) = t(\Phi)\gamma$.

However, a more natural form for a strict Lie 2-group constructed from a central extension $\widetilde{\Omega G}$ would be

$$\widetilde{P_eG^{[2]}} \underset{t}{\overset{s}{\underset{\longleftarrow i \longrightarrow}{\rightrightarrows}}} P_eG, \tag{3.3.2}$$

i.e., its Lie group of morphisms is $\widetilde{P_eG^{[2]}} \subset \widetilde{\Omega G}$, and the maps s and t are as in 3.2.1. We claim that the missing ingredient to obtain such a form is the identity map i. It can be provided by a so-called fusion factorization, see [9, Definition 5.5]. A *fusion factorization* for a central extension $\widetilde{\Omega G}$ is a Lie group homomorphism

$$i : P_eG \to \widetilde{P_eG^{[2]}} \qquad \text{such that} \qquad \pi(i(\gamma)) = \gamma \cup \gamma. \tag{3.3.3}$$

Lemma 3.3.1 *Let $\widetilde{\Omega G}$ be a disjoint commutative central extension of ΩG. Then, any fusion factorization i for $\widetilde{\Omega G}$ provides an identity map completing 3.3.2 to a strict Lie 2-group $\mathcal{G}(\widetilde{\Omega G}, i)$, together with a canonical isomorphism $\mathcal{G}(\widetilde{\Omega G}, i) \cong \mathcal{G}(X(\widetilde{\Omega G}))$.*

Proof In order to show that i turns 3.3.2 into a Lie 2-group, we use Lemma 3.1.1: The requirement that $\ker(s)$ and $\ker(t)$ commute is the assumption that $\widetilde{\Omega G}$ is disjoint commutative, and the property (3.3.3) implies that both $t \circ i$ and $s \circ i$ are the identity on $P_e G$.

In order to construct the isomorphism $\mathcal{G}(\widetilde{\Omega G}, i) \cong \mathcal{G}(X(\widetilde{\Omega G}))$ we observe that

$$X(\widetilde{\Omega G}) = \mathcal{X}(\mathcal{G}(\widetilde{\Omega G}, i)) \tag{3.3.4}$$

where on the left is the crossed module of Theorem 3.2.3 and $\mathcal{X}$ is the functor from the adjunction 3.1.5. Indeed, the crossed module on the left is $\widetilde{\Omega_{(0,\pi)} G} \to P_e G$ with the canonical action given by (3.2.3), and the crossed module on the right is $\ker(s) \xrightarrow{t} P_e G$ with the action given by

$$\alpha_\gamma(\Phi) = i(\gamma)\Phi\, i(\gamma)^{-1}. \tag{3.3.5}$$

First, we recall from (3.2.2) that $\ker(s) = \widetilde{\Omega_{(0,\pi)} G}$, and observe that the Lie group homomorphisms to $P_e G$ coincide. Second, for $\gamma \in P_e G$, the fusion factorization $i(\gamma)$ provides a concrete choice for a lift of $\gamma \cup \gamma$, which means that the formulas (3.2.3) and (3.3.5) coincide. This shows the equality in (3.3.4). Now, applying the functor $\mathcal{G}$ to (3.3.4) and using the counit

$$\eta_{\mathcal{G}(\widetilde{\Omega G}, i)} : \mathcal{G}(\widetilde{\Omega G}, i) \to \mathcal{G}\mathcal{X}(\mathcal{G}(\widetilde{\Omega G}, i))$$

establishes the claimed isomorphism. □

Next we study existence and uniqueness of fusion factorizations.

Lemma 3.3.2 *Let $\widetilde{\Omega G}$ be a central extension of ΩG. If G is semisimple, there exists at most one fusion factorization for $\widetilde{\Omega G}$.*

Proof Let i and i' be two fusion factorizations. We define a map $\varphi : P_e G \to \widetilde{\Omega G}$ by

$$\varphi(\gamma) = i(\gamma) i'(\gamma)^{-1}.$$

As both $i(\gamma)$ and $i'(\gamma)$ lie over $\gamma \cup \gamma$, φ takes values in $\mathrm{U}(1) \subset \widetilde{\Omega G}$. φ is a group homomorphism, because

$$\varphi(\gamma_1)\varphi(\gamma_2) = i(\gamma_1) i'(\gamma_1)^{-1} \varphi(\gamma_2) = i(\gamma_1)\varphi(\gamma_2) i'(\gamma_1)^{-1} = \varphi(\gamma_1\gamma_2).$$

By Theorem 2.1.2, φ is trivial. Hence $i = i'$. □

Remark 3.3.3 The proof above shows that in the general (not necessarily semisimple) case, if a fusion factorization exists, then the Poincaré dual $(P_e G)^* = \mathrm{Hom}(P_e G, \mathrm{U}(1))$ acts freely and transitively on the set of fusion factorizations.

We denote by $\sigma : \Omega G \to \Omega G$ the group homomorphism obtained by pullback with the "flip" diffeomorphism $t \mapsto -t$.

Lemma 3.3.4 *Let $\widetilde{\Omega G}$ be a central extension of ΩG. Suppose there exists a group homomorphism $\tilde{\sigma} : \widetilde{\Omega G} \to \widetilde{\Omega G}$ covering σ which is $\mathrm{U}(1)$-anti-equivariant in the sense that $\tilde{\sigma}(z\Phi) = \bar{z}\tilde{\sigma}(\Phi)$ for all $z \in \mathrm{U}(1)$ and $\Phi \in \widetilde{\Omega G}$. Then there exists a unique fusion factorization i such that $\tilde{\sigma} \circ i = i$.*

Proof Consider the map

$$w : P_eG \times_{\Omega G} \widetilde{\Omega G} \longrightarrow \mathrm{U}(1), \qquad w(\gamma, \Phi) = \Phi^{-1}\tilde{\sigma}(\Phi),$$

where the fibre product is taken over the diagonal map $P_eG \to \Omega G$, $\gamma \mapsto \gamma \cup \gamma$. Since $\pi(\Phi) = \pi(\tilde{\sigma}(\Phi)) = \gamma \cup \gamma$, we have $\pi(w(\gamma, \Phi)) = \mathtt{const}_e$; hence, $w(\gamma, \Phi)$ takes values in $\mathrm{U}(1)$. Moreover, w is a group homomorphism:

$$\begin{aligned} w(\gamma, \Phi)w(\eta, \Psi) &= w(\gamma, \Phi)\Psi^{-1}\tilde{\sigma}(\Psi) \\ &= \Psi^{-1}w(\gamma, \Phi)\tilde{\sigma}(\Psi) \\ &= \Psi^{-1}\Phi^{-1}\tilde{\sigma}(\Phi)\tilde{\sigma}(\Psi) \\ &= w(\gamma\eta, \Phi\Psi). \end{aligned}$$

For $z \in \mathrm{U}(1)$, we have

$$w(\gamma, z\Phi) = (z\Phi)^{-1}\tilde{\sigma}(z\Phi) = \bar{z}^2\Phi^{-1}\tilde{\sigma}(z\Phi) = \bar{z}^2 w(\gamma, \Phi).$$

Hence, if $(\gamma, \Phi) \in \ker(w)$, then we have $(\gamma, z\Phi) \in \ker(w)$ if and only if $\bar{z}^2 = 1$, that is, $z = \pm 1$. We obtain that $\mathrm{pr}_1 : \ker(w) \to P_eG$ is a double cover. Since P_eG is contractible, this double cover is necessarily trivial. Therefore, its restriction to the identity component $\ker(w)_0$ is an isomorphism of Lie groups $\mathrm{pr}_1|_{\ker(w)_0} : \ker(w)_0 \to P_eG$. Then, $i := \mathrm{pr}_2 \circ (\mathrm{pr}_1|_{\ker(w)_0})^{-1}$ is a fusion factorization.

Conversely, any fusion factorization i such that $\tilde{\sigma} \circ i = i$ gives a section of $\mathrm{pr}_1 : \ker(w) \to P_eG$ with $i(\mathtt{const}_e) = 1$. But since the fibres of $\ker(w)$ are discrete, there is at most one such section. □

Theorem 3.3.5 *Let $\widetilde{\Omega G}$ be a central extension of ΩG, where G is simply connected and semisimple. Then, there exists a unique fusion factorization for $\widetilde{\Omega G}$.*

Proof Uniqueness was shown in Lemma 3.3.2, so it remains to show existence. We claim that our assumptions on G imply the conditions of Lemma 3.3.4. To see this, consider the dual (inverse) central extension $\widetilde{\Omega G}^*$. Then $\sigma^*\widetilde{\Omega G}^*$ is another central extension, which comes with a canonical Lie group homomorphism

$$\tilde{\sigma}' : \widetilde{\Omega G} \to \sigma^*\widetilde{\Omega G}^*$$

that covers σ and is $\mathrm{U}(1)$-anti-equivariant. By our assumptions, the homomorphism $h_0(\mathrm{c}\mathcal{E}\mathrm{xt}(\Omega G)) \longrightarrow H^2_c(L\mathfrak{g}, \mathbb{R})$ of (2.2.1) is injective, so that central extensions are determined their 2-cocycles. Now, if ω is the 2-cocycle classifying $\widetilde{\Omega G}$, then the dual extension $\widetilde{\Omega G}^*$ is classified by $-\omega$. By Lemma 2.2.2 we may assume that ω is G-equivariant, hence of the form (2.2.4). For such a cocycle ω the action of σ on $H_c^2(L\mathfrak{g}, \mathbb{R})$ replaces ω by $-\omega$, so that $\sigma^*\widetilde{\Omega G}^*$ is again classified by ω. By Lemma 2.3.1, $\sigma^*\widetilde{\Omega G}^*$ is, as a central extension, isomorphic to $\widetilde{\Omega G}$. The post-composition of this isomorphism with $\tilde{\sigma}'$ provides an anti-linear bundle map $\tilde{\sigma}$ covering σ, and Lemma 3.3.4 completes the proof. □

Remark 3.3.6 Observe that the proof of Theorem 3.3.5 actually shows that under the assumptions of Theorem 3.3.5, there exists a map $\tilde{\sigma}$ as in Lemma 3.3.4, and the unique fusion factorization i satisfies additionally $\tilde{\sigma} \circ i = i$.

3.4 Classification of the Lie 2-groups

In this section we prove that—in the case of a simple and simply connected Lie group G and for a "basic" central extension—our canonical Lie 2-group $\mathcal{G} := \mathcal{G}(X(\widetilde{\Omega G}))$ of Sect. 3.2 becomes under geometric realization a 3-connected cover of G. For this purpose we will use the methods developed in [4, 5].

We start by recalling some notions and basic facts about Lie 2-groups (as used, e.g., in [5, §4.2]). A *strict homomorphism* between strict Lie 2-groups consists of two Lie group homomorphisms (one between the morphism groups and one between the object groups), which intertwine all structure maps. The *strict kernel* of such a strict homomorphism is the 2-group obtained by taking the level-wise kernels. It is a Lie 2-group if both kernels are submanifolds (which is automatic in the finite-dimensional case). A sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of strict Lie 2-groups and strict homomorphisms is called *strictly exact* if it is exact on both object and morphism level.

Taking the nerve of a strict Lie 2-group $\mathcal{G}$ and forgetting the smooth structure, we obtain a simplicial space $N\mathcal{G}$, where $(N\mathcal{G})_0 = \mathrm{Ob}(\mathcal{G})$ and whose n-th space, $n \geq 1$, is the space of n-strings of composable morphisms,

$$(N\mathcal{G})_n = \{(x_1, \ldots, x_n) \in \mathrm{Mor}(\mathcal{G})^n \mid s(x_j) = t(x_{j-1}),\, j = 2, \ldots, n\}.$$

Applying the geometric realization functor, we obtain a topological space $|\mathcal{G}|$, the *geometric realization* of $\mathcal{G}$. Pointwise multiplication in $\mathrm{Mor}(\mathcal{G})$ endows each of the spaces $(N\mathcal{G})_n$ with the structure of a topological group (in fact, a Lie group) for which the simplicial structure maps are homomorphisms. Put differently, we have a group object in the category of simplicial topological spaces, and since the geometric realization functor preserves finite products, it sends group objects to group objects, so that $|\mathcal{G}|$ acquires the structure of a topological group (see also Lemma 1 in [4]). Any strict homomorphism $f : \mathcal{G} \to \mathcal{H}$ between two strict 2-groups induces a continuous group homomorphism $|f|$ between the corresponding geometric realizations, and if f was a weak equivalence of 2-groups, then $|f|$ is a weak homotopy equivalence. It is moreover a fact that geometric realization takes a short strictly exact sequence of Lie 2-groups to an exact sequence of topological groups [5, §4.2].

We now come to the main topic of this section. Let G be a finite-dimensional, connected, and semisimple Lie group and let $\widetilde{\Omega G}$ be a disjoint commutative central extension of its loop group ΩG. Let

$$\mathcal{G} := \mathcal{G}(X(\widetilde{\Omega G}))$$

be the Lie 2-group corresponding to the crossed module constructed in Sect. 3.2 and let $|\mathcal{G}|$ be its geometric realization. Since two objects $\beta_1, \beta_2 \in P_e G$ are isomorphic in $\mathcal{G}$ if and only if they have the same end point, $\mathcal{G}$ comes with a canonical strict homomorphism

$$\mathcal{G} \longrightarrow G_{\mathrm{dis}},$$

to the strict Lie 2-group G_{dis} (see Example 3.1.3), given by end point evaluation. Applying geometric realization and post-composing with the canonical group homomorphism $|G_{\mathrm{dis}}| \to G$ (which is a homotopy equivalence), we obtain a homomorphism of topological groups

$$|\mathcal{G}| \longrightarrow G. \tag{3.4.1}$$

We then have the following result.

Theorem 3.4.1 *If G is simple, connected, and simply connected and $\widetilde{\Omega G}$ is a basic central extension, then $|\mathcal{G}|$ is the 3-connected cover of G, via the group homomorphism (3.4.1).*

In fact, we prove the following more general statement, which immediately implies Theorem 3.4.1. To formulate it, let $\omega \in H^2_c(\Omega\mathfrak{g}, \mathbb{R})$ be the classifying 2-cocycle for the central extension $\widetilde{\Omega G}$ and let $\overline{\omega}$ be the corresponding left-invariant 2-form on ΩG. For a smooth map $f : S^2 \to \Omega G$, we define

$$\varphi(f) := \frac{1}{2\pi} \int_{S^2} f^* \overline{\omega}.$$

Since ω classifies a central extension of ΩG, it has integral periods, so φ takes values in $\mathbb{Z}$ (see the discussion in Sect. 2.2). It is moreover easy to see that $\varphi(f)$ only depends on the homotopy class of f and yields a group homomorphism $\varphi : \pi_2(\Omega G) \to \mathbb{Z}$.

Theorem 3.4.2 *Let G be a connected semisimple Lie group. For $k = 1$ and $k \geq 4$, the group homomorphism (3.4.1) induces an isomorphism $\pi_k(\mathcal{G}) \cong \pi_k(G)$. Moreover, we have an exact sequence*

$$0 \longrightarrow \pi_3(|\mathcal{G}|) \longrightarrow \pi_3(G) \cong \pi_2(\Omega G) \xrightarrow{\varphi} \mathbb{Z} \longrightarrow \pi_2(|\mathcal{G}|) \longrightarrow 0. \tag{3.4.2}$$

Proof of Theorem 3.4.1 The assumptions on G imply that $H^2(\Omega G, \mathbb{Z}) \cong \pi_2(\Omega G) = \mathbb{Z}$. That $\widetilde{\Omega G}$ is basic means that $\overline{\omega}$ is a generator of $H^2(\Omega G, \mathbb{Z}) \cong \mathbb{Z}$. Taking $f : S^2 \to \Omega G$ to be a generator of $\pi_2(\Omega G)$, we obtain that $f^*\overline{\omega}$ is a generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$, so that φ is surjective. Hence $\pi_2(|\mathcal{G}|) = \pi_3(|\mathcal{G}|) = 0$. □

Proof of Theorem 3.4.2 Let G'_{dis} be the strict 2-group with objects P_eG, morphisms $P_eG^{[2]}$, and structure maps $s := \text{pr}_2$, $t := \text{pr}_1$, and i the diagonal map (Lemma 3.1.1). Then, we have a factorization,

$$\mathcal{G} \longrightarrow G'_{\text{dis}} \longrightarrow G_{\text{dis}}, \tag{3.4.3}$$

where the first map is the identity on objects, and thereby uniquely determined on morphisms, while the morphism $G'_{\text{dis}} \to G_{\text{dis}}$ is the end point evaluation, both on objects and morphisms. It is straightforward to show that the second arrow in (3.4.3) is a weak equivalence, so the induced group homomorphism between the geometric realizations is a weak homotopy equivalence. By construction, the strict kernel of the first homomorphism in (3.4.3) is the trivial group on objects and $\mathrm{U}(1) \subset \widetilde{\Omega G}$ on morphisms; in other words, it is the strict 2-group $B\mathrm{U}(1)$ (Example 3.1.2). We therefore get a strict short exact sequence of Lie 2-groups

$$B\mathrm{U}(1) \longrightarrow \mathcal{G} \longrightarrow G'_{\text{dis}}. \tag{3.4.4}$$

We now use the explicit construction of the geometric realization $|\mathcal{G}|$, described in [4, Lemma 1 and §5.3]: Let $\mathcal{G}$ be a strict Lie 2-group and let $(H, \mathcal{G}_0, \alpha, t)$ be the corresponding crossed module. Then, there exists a weakly contractible topological group EH containing H as a normal subgroup, together with an action of $\mathcal{G}_0$ on EH extending the action of $\mathcal{G}_0$ on H. Moreover, H is embedded as a normal subgroup of the semidirect product $EH \rtimes \mathcal{G}_0$, and we have a short exact sequence of topological groups

$$H \longrightarrow EH \rtimes \mathcal{G}_0 \longrightarrow |\mathcal{G}|,$$

witnessing $|\mathcal{G}|$ as the quotient

$$|\mathcal{G}| \cong (EH \rtimes \mathcal{G}_0)/H.$$

The above construction is functorial in $\mathcal{G}$; hence, we may apply it to the strict short exact sequence (3.4.4). The object group $\mathcal{G}_0$ is contractible in each case (being either trivial or the path group P_eG), hence the geometric realization has the homotopy type of BH in each case. Identifying $\Omega_{(0,\pi)}G \cong \Omega G$ and $\widetilde{\Omega_{(0,\pi)}G} \cong \widetilde{\Omega G}$ (see Lemma 2.3.4), we obtain that under geometric realization, the strict short exact sequence (3.4.4) corresponds to the homotopy fiber sequence

$$B\mathrm{U}(1) \longrightarrow B\widetilde{\Omega G} \longrightarrow B\Omega G.$$

An inspection of the construction in [4, §5.3] reveals that, as expected, this sequence is just the one obtained from applying the classifying space functor B to the short exact sequence of the central extension $\widetilde{\Omega G}$. Since $BU(1) \simeq K(\mathbb{Z}, 2)$, this shows (using the long exact sequence of homotopy groups) that the map $\pi_k(B\widetilde{\Omega G}) \to \pi_k(B\Omega G)$ is an isomorphism for $k = 1$ and $k \geq 4$.

It is now a general fact that for a principal U(1)-bundle $U(1) \to P \to B$, the boundary map $\pi_2(B) \to \pi_1(U(1)) \cong \mathbb{Z}$ of the corresponding long exact sequence of homotopy groups is the map that sends $[f] \in \pi_2(B)$ to the first Chern number $\langle c_1(f^*P), [S^2] \rangle$ of the bundle $f^*P \to S^2$. In our case, the first Chern class of $\widetilde{\Omega G}$ is represented by the left-invariant 2-form $\overline{\omega}$, and so the result follows. ◻

4 Comparison with other constructions

In this section we carry out the comparison between our constructions of Sect. 3 and the constructions of Baez et al. and the second-named author.

4.1 The BCSS string 2-group

We start by reviewing the main construction of Baez et al. [5, Prop. 25]. We remark that their construction is presented as if it results into as that of a Fréchet Lie 2-group, but in fact it results into a crossed module of Fréchet Lie groups, to which then the functor $\mathcal{G}$ from 3.1.5 is applied without mention. So we better describe that crossed module directly.

Let $P_eG^{\text{BCSS}} \subset C^\infty([0, 2\pi], G)$ be the Fréchet submanifold of paths starting at $e \in G$. Note that—in contrast to our setting—there is no flatness assumption; moreover, paths are parameterized by $[0, 2\pi]$ instead of $[0, \pi]$. We denote by $\Omega G^{\text{BCSS}} \subset P_eG^{\text{BCSS}}$ the Fréchet manifold of closed paths, and assume that

$$1 \to U(1) \to \widetilde{\Omega G}^{\text{BCSS}} \to \Omega G^{\text{BCSS}} \to 1$$

is a central extension. A Lie group homomorphism

$$t^{\text{BCSS}} : \widetilde{\Omega G}^{\text{BCSS}} \to P_eG^{\text{BCSS}}$$

is defined by projection and inclusion. Under certain assumptions on the central extension, including the condition that G is of Cartan type and classified by a level $k \in \mathbb{Z}$, a central crossed module action

$$\alpha^{\text{BCSS}} : P_eG^{\text{BCSS}} \times \widetilde{\Omega G}^{\text{BCSS}} \to \widetilde{\Omega G}^{\text{BCSS}}$$

can be defined (in a difficult way, using Lie-algebraic methods). It will not be necessary to review this construction here, as we will prove below that it restricts to our canonical action. We denote the crossed module defined this way by $X^{\text{BCSS}}(G, k)$; it is precisely the one described in [5, Prop. 25].

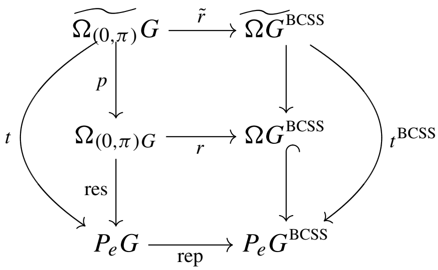
In the following we will show that $X^{\text{BCSS}}(G, k)$ is weakly equivalent to our canonical crossed module $X(\widetilde{\Omega G})$ from Theorem 3.2.3. In order to do so, we first have to specify the disjoint commutative central extension $\widetilde{\Omega G}$ required there. We consider the maps

$$\Omega_{(0,\pi)}G \xrightarrow{\text{res}} P_eG \xrightarrow{\text{rep}} P_eG^{\text{BCSS}}$$

defined by $\text{rep}(\gamma)(x) := \gamma(\frac{1}{2}x)$, for $\gamma \in P_eG$, $x \in [0, 2\pi]$, and $\text{res}(\eta)(x) := \eta(x)$, for $\eta \in \Omega_{(0,\pi)}G$ and $x \in [0, \pi]$. Note that rep and res are Lie group homomorphisms. Their composition will be denoted by $r := \text{rep} \circ \text{res}$. We let

$$\widetilde{\Omega_{(0,\pi)}}G := r^*\widetilde{\Omega G}^{\text{BCSS}}$$

be the pullback central extension. By Lemma 2.3.4, this is the restriction of a central extension $\widetilde{\Omega G}$, as required. Note that $\widetilde{\Omega G}$ is disjoint commutative since G is semisimple and simply connected, due to Corollary 2.4.4. We obtain—by construction—a commutative diagram:



Lemma 4.1.1 *The maps* $\tilde{r}$ *and* rep *constitute a strict homomorphism*

$$R : X(\widetilde{\Omega G}) \longrightarrow X^{\text{BCSS}}(G, k)$$

of crossed modules.

Proof Since the diagram is commutative, it remains to prove that the crossed module actions are exchanged, i.e., that

$$\alpha^{\text{BCSS}}_{\text{rep}(\gamma)}(\tilde{r}(\Phi)) = \tilde{r}(\alpha_\gamma(\Phi)) \tag{4.1.1}$$

for all $\gamma \in P_eG$ and $\Phi \in \widetilde{\Omega_{(0,\pi)}}G$. We note that

$$\begin{aligned} t^{\text{BCSS}}(\alpha^{\text{BCSS}}_{\text{rep}(\gamma)}(\tilde{r}(\Phi))) &= \text{rep}(\gamma) \cdot t^{\text{BCSS}}(\tilde{r}(\Phi))) \cdot \text{rep}(\gamma)^{-1} \\ &= \text{rep}(\gamma) \cdot \text{rep}(t(\Phi)) \cdot \text{rep}(\gamma)^{-1} \\ &= \text{rep}(\gamma \cdot t(\Phi) \cdot \gamma^{-1}) \\ &= r(\eta(\gamma, \Phi)), \end{aligned}$$

where $\eta(\gamma, \Phi) := (\gamma \cdot t(\Phi) \cdot \gamma^{-1}) \cup \mathtt{const}_e \in \Omega_{(0,\pi)}G$. This shows that we obtain a well-defined element

$$\alpha_\gamma(\Phi) := (\eta(\gamma, \Phi), \alpha^{\mathrm{BCSS}}_{\mathrm{rep}(\gamma)}(\tilde{r}(\Phi))) \in \widetilde{\Omega_{(0,\pi)}}G.$$

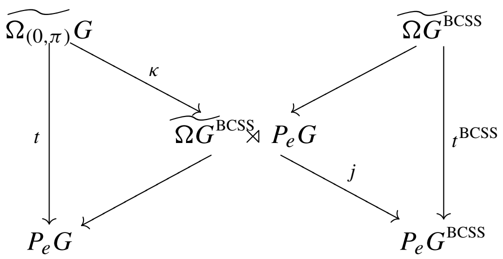
The map α_γ defined like this is a smooth, central crossed module action for $t : \widetilde{\Omega G} \to P_eG$; moreover, by construction, it satisfies (4.1.1). Since G is semisimple, it coincides with our canonical action by Theorem 3.2.4. □

We may thus say that our canonical action α is the restriction of the action α^{BCSS} along the homomorphism R.

Theorem 4.1.2 *The homomorphism R of Lemma 4.1.1 establishes a weak equivalence of crossed modules of Fréchet Lie groups,*

$$X(\widetilde{\Omega G}) \cong X^{\mathrm{BCSS}}(G, k).$$

Proof Every strict homomorphism between crossed modules determines a weak homomorphism, a.k.a. a butterfly, see [1, §4.5]. In the case of R, this butterfly is



where the group in the middle is the semi-direct product w.r.t. the action α^{BCSS} induced along rep : $P_eG \to P_eG^{\mathrm{BCSS}}$, and the NE-SW-sequence is the corresponding split extension. Moreover,

$$\kappa(\Phi) := (\tilde{r}(\Phi)^{-1}, t(\Phi)) \quad \text{and}$$
$$j(\Phi, \gamma) := \mathrm{rep}(\gamma) \cdot t^{\mathrm{BCSS}}(\Phi).$$

By [1, §5.2], a butterfly establishes a weak equivalence if it is reversible, meaning that its NW-SE-sequence

$$\widetilde{\Omega_{(0,\pi)}}G \xrightarrow{\kappa} \widetilde{\Omega G}^{\mathrm{BCSS}} \rtimes P_eG \xrightarrow{j} P_eG^{\mathrm{BCSS}}$$

is also short exact. Since that sequence is always a complex (for any butterfly), it remains to prove that it is an exact sequence of groups and a locally trivial principal bundle.

Since r is injective, the covering map $\tilde{r}$ is also injective, and hence κ is injective. In order to show the surjectivity of j, we consider $\gamma \in P_eG^{\mathrm{BCSS}}$ and choose a smooth map $\varphi : [0, 2\pi] \to [0, \pi]$ with $\varphi(0) = 0$ and $\varphi(2\pi) = \pi$ that is flat at its end points. Then, for any lift $\Phi \in \widetilde{\Omega G}^{\mathrm{BCSS}}$ of $\gamma \cdot \mathrm{rep}(\gamma \circ \varphi)^{-1} \in \Omega G^{\mathrm{BCSS}}$, we have $j(\Phi, \gamma \circ \varphi) = \gamma$, hence j is surjective. The fact that φ can be chosen to be the same for all $\gamma \in P_e G^{\mathrm{BCSS}}$ and the fact that Φ can be chosen in a locally smooth way shows that j has local sections, and hence is a principal bundle.

It remains to show exactness in the middle. Let $(\Phi, \gamma) \in \widetilde{\Omega G}^{\mathrm{BCSS}} \rtimes P_e G$ be in the kernel of j, i.e., $\mathrm{rep}(\gamma) \cdot t^{\mathrm{BCSS}}(\Phi) = \mathtt{const}_e$. Then

$$(\gamma \cup \mathtt{const}_e, \Phi^{-1}) \in \widetilde{\Omega_{(0,\pi)} G}$$

is sent to (Φ, γ) under κ. □

4.2 The diffeological string 2-group

The following construction of a diffeological 2-group is implicit in [18–20], but has not been described explicitly. It takes as input data a *fusion extension*, i.e. central extension

$$1 \to \mathrm{U}(1) \to \widetilde{LG} \to LG \to 1 \tag{4.2.1}$$

of Fréchet Lie groups that is equipped with a *multiplicative fusion product*.

In the following we use without further notice the fully faithful functor from Fréchet manifolds to diffeological spaces in order to embed everything into that setting. We let $P_e G_{\mathrm{si}}$ be the diffeological space of paths in G with sitting instants (constant in neighborhoods of its end points) starting at $e \in G$, and by $P_e G_{\mathrm{si}}^{[k]}$ its k-fold fibre products along the endpoint evaluation $\mathrm{ev} : P_e G_{\mathrm{si}} \to G$. As before, we have a smooth map $\cup : P_e G_{\mathrm{si}}^{[2]} \to LG$. A *fusion product* is a bundle morphism

$$\lambda : \mathrm{pr}_{12}^* \cup^* \widetilde{LG} \otimes \mathrm{pr}_{23}^* \cup^* \widetilde{LG} \longrightarrow \mathrm{pr}_{13}^* \cup^* \widetilde{LG}$$

over $P_e G_{\mathrm{si}}^{[2]}$ that satisfies the evident associativity condition over $P_e G_{\mathrm{si}}^{[4]}$. Moreover, it is called *multiplicative* if it is a group homomorphism, see [18–20] for more details.

Remark 4.2.1 Fusion extensions may—on first view—look odd and involved, but in fact appear very naturally. Indeed, there are at least the following three ways to obtain a fusion extension of the loop group LG of a Lie group G:

(1) Transgression of any multiplicative bundle gerbe over G results in a fusion extension of LG; this is explained in [19, §2].
(2) The *Mickelsson model* produces a canonical fusion extension for any simply connected Lie group G; this is explained in [20, Example 2.6].
(3) The operator-algebraic *implementer model* [9] produces a canonical fusion extension for $L\mathrm{Spin}(d)$.

We note that every fusion extension comes equipped with a *fusion factorization*, uniquely characterized by the property that is neutral with respect to fusion [20, Prop.

3.1.1]. The following result, which is nothing but a reformulation of the given conditions, constructs from a fusion extension a strict diffeological 2-group.

Proposition 4.2.2 *Given a fusion extension as above, the following structure yields a central strict diffeological 2-group $\mathcal{S}(\widetilde{LG}, \lambda)$:*

- *The diffeological group of objects is P_eG_{si}.*
- *The diffeological group of morphisms is*

$$\widetilde{\Omega G}^{\mathrm{dflg}} := P_eG_{\mathrm{si}}^{[2]} \times_{LG} \widetilde{LG},$$

where the fibre product is taken along the map $\cup : P_eG_{\mathrm{si}}^{[2]} \to LG$.
- *Source and target maps are $s(\gamma_1, \gamma_2, \Phi) := \gamma_2$ and $t(\gamma_1, \gamma_2, \Phi) := \gamma_1$.*
- *Composition is the fusion product λ of $\widetilde{LG}$:*

$$(\gamma_0, \gamma_1, \Phi') \circ (\gamma_1, \gamma_2, \Phi) := (\gamma_0, \gamma_2, \lambda(\Phi' \otimes \Phi)).$$

- *The identity morphism of $\gamma \in P_eG_{\mathrm{si}}$ is $(\gamma, \gamma, i(\gamma))$, where i is the fusion factorization associated to λ.*

Remark 4.2.3 It is easy to check that $\pi_1\mathcal{S}(\widetilde{LG}, \lambda) = \mathrm{U}(1)$ and $\pi_0\mathcal{S}(\widetilde{LG}, \lambda) \cong G$, so that $\mathcal{S}(\widetilde{LG}, \lambda)$ is a diffeological Lie 2-group extension

$$B\mathrm{U}(1) \longrightarrow \mathcal{S}(\widetilde{LG}, \lambda) \longrightarrow G_{\mathrm{dis}}.$$

Remark 4.2.4 As noticed in [9, §5.2] and deduced in general in Sect. 3.1, the fusion product λ is already determined by its fusion factorization i; moreover, the subgroups

$$\widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}} = \ker(s) \subset \widetilde{\Omega G}^{\mathrm{dflg}} \quad \text{and} \quad \widetilde{\Omega_{(\pi,2\pi)}G}^{\mathrm{dflg}} = \ker(t) \subset \widetilde{\Omega G}^{\mathrm{dflg}}$$

commute with each other.

The goal of this section is to compare the diffeological Lie 2-group $\mathcal{S}(\widetilde{LG}, \lambda)$ with our constructions from Sect. 3, and it is best to do this on the level of crossed modules. The diffeological crossed module $\mathcal{X}(\mathcal{S}(\widetilde{LG}, \lambda))$ is

$$t : \widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}} \to P_eG_{\mathrm{si}},$$

with the central crossed module action

$$\alpha^{\mathrm{dflg}} : P_eG_{\mathrm{si}} \times \widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}} \to \widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}}$$

given by $\alpha^{\mathrm{dflg}}(\gamma, \Phi) := i(\gamma) \cdot \Phi \cdot i(\gamma)^{-1}$.

Remark 4.2.5 As in Sect. 3.2, we can observe here immediately that this action does not even depend on the fusion factorization, and hence, that the crossed module $\mathcal{X}(\mathcal{S}(\widetilde{LG}, \lambda))$ is completely independent of the fusion product λ. However, the condition that the subgroups $\widetilde{L_{(0,\pi)}G}$ and $\widetilde{L_{(\pi,2\pi)}G}$ commute has to be imposed (it is slightly weaker than disjoint commutativity).

In order to explore the relation between the diffeological crossed module $\mathcal{X}(\mathcal{S}(\widetilde{LG}, \lambda))$ and our crossed module $X(\widetilde{\Omega G})$ from Sect. 3.2, we assume that $\widetilde{LG}$ is a disjoint commutative central extension of a Lie group G; then, both crossed modules are defined. We obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}} & \hookrightarrow & \widetilde{\Omega_{(0,\pi)}G} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ P_eG_{\mathrm{si}} & \hookrightarrow & P_eG \end{array}$$

whose horizontal arrows are inclusions (paths with sitting instants are flat). Moreover, we observe that the action α^{dflg} and or canonical action α are defined in exactly the same way. Hence, above diagram constitutes a strict homomorphism of diffeological crossed modules

$$\mathcal{X}(\mathcal{S}(\widetilde{LG}, \lambda)) \to X(\widetilde{\Omega G}). \tag{4.2.2}$$

Theorem 4.2.6 *The homomorphism* (4.2.2) *is a weak equivalence,*

$$\mathcal{X}(\mathcal{S}(\widetilde{LG}, \lambda)) \cong X(\widetilde{\Omega G}).$$

In particular, there is a canonical weak equivalences of diffeological 2-groups

$$\mathcal{S}(\widetilde{LG}, \lambda) \cong \mathcal{G}(X(\widetilde{\Omega G})) \cong \mathcal{G}(\widetilde{\Omega G}, i).$$

Proof We proceed as in the proof of Theorem 4.1.2 and consider the butterfly

$$\begin{array}{ccccc} \widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}} & & & & \widetilde{\Omega_{(0,\pi)}G} \\ & \searrow^{\kappa} & & \swarrow & \\ {\scriptstyle t}\downarrow & & \widetilde{\Omega_{(0,\pi)}G} \rtimes P_eG_{\mathrm{si}} & & \downarrow{\scriptstyle t} \\ & \swarrow & & \searrow^{j} & \\ P_eG_{\mathrm{si}} & & & & P_eG, \end{array}$$

where now $\kappa(\Phi) := (\Phi^{-1}, t(\Phi))$ and $j(\Phi, \gamma) := \gamma t(\Phi)$. We use again [1, §5.2] and have to prove that the NW-SE-sequence is short exact. The proofs that κ is injective and that j is surjective and has local sections go as for Theorem 4.1.2. For exactness in the middle, we observe that an equality $\gamma t(\Phi) = \mathtt{const}_e$ implies that $t(\Phi)$ has sitting instants, and hence $\Phi \in \widetilde{\Omega_{(0,\pi)}G}^{\mathrm{dflg}}$. □

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