Quasimorphisms and bounded Cohomology

(Quasimorphismen und beschränkte Kohomologie)

Master thesis

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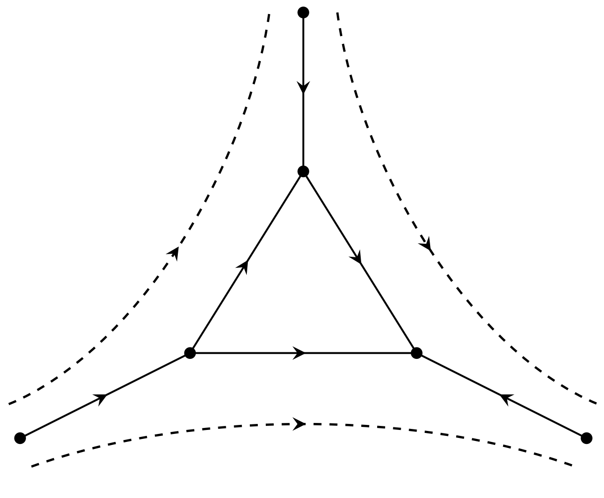

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Introduction

This thesis aims to introduce quasimorphisms and explain their connection to bounded and equivariant bounded cohomology. In particular, we extend vanishing results for cup products and Massey triple products in (equivariant) bounded cohomology of groups acting on graphs.

Bounded cohomology of groups arises from a complex of bounded invariant functions (Chapter 2). In contrast to ordinary group cohomology, it is difficult to compute in general. For example, the group cohomology of non-abelian free groups is known in every degree. On the other hand, although bounded cohomology of non-abelian free groups can be computed in degree up to 3, it is completely unknown in higher degrees. It is not even known if the bounded cohomology of non-abelian free groups in degree 4 has a finite dimension or not.

A possible approach to obtain control over the cohomology groups in higher dimensions is to look at cup products or Massey triple products of lower degree coclasses. To that effect, we study quasimorphisms as their understanding gives rise to great knowledge of bounded cohomology in degree 2 (Chapter 3) and consider the products with coclasses induced by quasimorphisms.

The centrepiece of this text is Chapter 4. In this part, we give four examples of quasimorphisms, namely Brooks, Rolli, Δ-decomposable and median quasimorphisms. The first three of them are quasimorphisms of non-abelian free groups and the last one is an example for quasimorphisms of groups acting on median graphs or CAT(0) cube complexes. Furthermore, we recall existing results of triviality for cup products and Massey triple products with coclasses induced by one of these quasimorphisms.

In the literature, one uses aligned chains in order to prove the existing vanishing results for Brooks, Rolli and Δ-decomposable quasimorphisms [1, 13]. It is not possible to adapt this approach in an easy way to prove the results for median quasimorphisms [3]. The goal of Chapter 4 is the construction of a general statement about quasimorphisms and their products that can be used not only to deduce all the existing vanishing results but also to extend them. In particular, we obtain the following vanishing results for the bounded cohomology of a non-abelian free group.

Corollary (Corollaries 4.3.7, 4.3.1, 4.3.2)**.** *Let F be a non-abelian free group and let $\phi\colon F \to \mathbb{R}$ be a Brooks, Rolli, or Δ-decomposable quasimorphism. Then ϕ is a quasimorphism and for all $n, m \in \mathbb{N}$ and $\alpha_1 \in H_b^n(F; R)$ and $\alpha_2 \in H_b^m(F; R)$,*

- *the cup products $[\delta^1\widehat{\phi}] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1\widehat{\phi}]$ are trivial in $H_b^{n+2}(F; \mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1\widehat{\phi}], \alpha_2 \rangle \subset H_b^{n+m+1}(F; \mathbb{R})$ is trivial.*

On top of that, we obtain a similar result for groups acting on median graphs that can be applied in particular to finite dimensional CAT(0) cube complexes and to right-angled Artin groups.

Corollary (Corollary 4.3.4)**.** *Let* Γ *be a discrete group acting on a restricted median graph* X*. Let* s *be an* $\mathcal{H}$*-segment in* X *and let* f_s *be the corresponding median quasimorphism of* $\Gamma \curvearrowright X$*. Then* f_s *is a symmetric quasimorphism and for all* $n, m \in \mathbb{N}$ *and* $\alpha_1 \in H^n_{\Gamma,b}(X; R)$ *and* $\alpha_2 \in H^m_{\Gamma,b}(X; R)$ *that are non-transverse to the orbit* Γs*,*

- *the cup products* $[\delta^1 f_s] \cup \alpha_1$ *and* $\alpha_1 \cup [\delta^1 f_s]$ *are trivial in* $H^{n+2}_{\Gamma,b}(X; \mathbb{R})$*, and*
- *the Massey triple product* $\langle \alpha_1, [\delta^1 f_s], \alpha_2 \rangle \subset H^{n+m+1}_{\Gamma,b}(X; \mathbb{R})$ *is trivial.*

Eventually, Chapter 5 deals with the connection between median quasimorphisms of groups acting on trees and the second bounded cohomology in degree 2. In particular, we define a characterization for actions on a tree that yields an infinite family of median quasimorphisms that gives rise to an infinite linearly independent family of coclasses in the second bounded cohomology. Moreover, we produce lattices of products of automorphism groups of trees for which the triviality of the median quasimorphisms is equivalent to the triviality of the second bounded cohomology. This characterization is related to local ∞-transitivity.

1 Preliminaries

This chapter is a quick overview over the basic tools that will be used in the thesis. In order to keep this overview short and clear, there will not be given many proofs. However, we always refer to literature where the omitted proof can be found.

We assume the knowledge of basic group theory as it can be found for example in [11, Chapter 2].

1.1 Graph theory

In this section, we recall the basics of graph theory that are needed in this thesis.

Definition 1.1.1 ((undirected) graph)**.** An *(undirected) graph* is a pair $X = (V, E)$ of disjoint sets with

$$E \subset \{e \subset V \mid |e| = 2\};$$

the elements of V are the *vertices*, the elements of E are the *edges* of X.

Definition 1.1.2 (directed graph)**.** A *directed graph* is a pair $X = (V, E)$ of disjoint sets with

$$E \subset \{(v, w) \in V \times V \mid v \neq w\};$$

the elements of V are the *vertices*, the elements of E are the *edges* of X.

When we talk about graphs, we always mean undirected graphs.

Definition 1.1.3 (neighbour, degree)**.** Let (V, E) be a graph. We say that two vertices $v, w \in V$ are *neighbours* if $\{v, w\} \in E$. The number of neighbours of a vertex v is the *degree* of this vertex, denoted by $\deg(v)$.

Definition 1.1.4 (locally finite, regular)**.** Let $n \in \mathbb{N}$.

- A graph is called *locally finite* if every vertex has finite degree.
- A graph is called *regular of degree n* if every vertex has degree n.

Definition 1.1.5 (graph isomorphism)**.** Let $X = (V, E)$ and $X' = (V', E')$ be graphs. A *graph isomorphism* between X and X' is a bijective map

$$f \colon V \to V'$$

such that for all $v, w \in V$ it is

$$\{v, w\} \in E \Leftrightarrow \{f(v), f(w)\} \in E'.$$

If such a graph isomorphism exists, then we call X and X' *isomorphic.*

Definition 1.1.6 (path, cycle, connected)**.** Let $X = (V, E)$ be a graph.

- Let $n \in \mathbb{N} \cup \{\infty\}$. A *path in X of length n* is a sequence of different vertices $v_0, \ldots, v_n \in V$ such that for all $i \in \{0, \ldots, n-1\}$ we have $\{v_i, v_{i+1}\} \in E$. If $n \in \mathbb{N}$ we say that this path connects v_0 and v_n.
- Let $n \in \mathbb{N}_{>2}$. A *cycle* of length n in X is a path $v_0, \ldots, v_{n-1}$ in X such that $\{v_{n-1}, v_0\} \in E$.
- The graph X is called *connected* if every two vertices are connected by a path.

Definition 1.1.7 (inverse path, concatenation)**.** Let $X = (V, E)$ be a graph. For a finite path $p = v_0, \ldots, v_n$ in X we define the *inverse path* $\overline{p}$ of p by the sequence $v_n, \ldots, v_0$. Let $q = x_0, \ldots, x_m$ be another path in X with $m \in \mathbb{N} \cup \{\infty\}$ such that $v_n = x_0$ and p, q don't share any other vertex. Then, we define the *concatenation* $p * q$ of p and q by the sequence

$$p * q = v_0, \ldots, v_n, x_1 \ldots, x_m.$$

Definition 1.1.8 (tree)**.** A graph is a called a *tree* if it is connected and does not contain any cycle.

Lemma 1.1.9 ([11, Proposition 3.1.10])**.** *A graph is a tree if and only if every two vertices are connected by exactly one path.*

Theorem 1.1.10 (graph metric)**.** *Let $X = (V, E)$ be a connected graph. Then the map*

$$\begin{aligned} d\colon V \times V &\to \mathbb{R}_{\geq 0} \\ (v, w) &\mapsto \min\{n \in \mathbb{N} \mid \exists \text{ a path in } X \text{ of length } n \text{ connecting } v \text{ and } w\} \end{aligned}$$

defines a metric on V, the so called graph metric on X.

From now on, we consider graphs as metric spaces equipped with the graph metric.

Definition 1.1.11 (geodesic)**.** Let $X = (V, E)$ be a graph and $v, w \in V$. A *geodesic* in X from v to w is a path of length $d(v, w)$ connecting v and w. We denote by $[v, w]$ the set of geodesics from v to w.

Remark 1.1.12. If $T = (V, E)$ is a tree and $v, w \in V$ then $[v, w]$ consists exactly of one geodesic, namely the unique path connecting v and w. In this case, we denote this unique geodesic also by $[v, w]$ with a slight abuse of notation.

Cayley graph

Definition 1.1.13 (Cayley graph)**.** Let G be a group with generating set $S \subset G$. Then the *Cayley graph of* G *with respect to* S is the graph $\mathrm{Cay}(G, S)$ having

- G as its set of vertices, and
- the set $\{\{x, xs\} \mid x \in G \text{ and } s \in (S \cup S^{-1}) \setminus \{e\}\}$ as its set of edges.

Example 1.1.14. Let $n \in \mathbb{N}$ and let F be the non-abelian group freely generated by a set $S \subset F$ of n elements. Then, $\mathrm{Cay}(F, S)$ is a regular tree of degree $2n$.

Median graphs

Definition 1.1.15 (median graph)**.** A graph $X = (V, E)$ is called a *median graph* if for every $x, y, z \in V$ there exists a unique vertex $m \in V$ with

$$\begin{aligned} d(x, y) &= d(x, m) + d(m, y), \\ d(x, z) &= d(x, m) + d(m, z), \\ d(y, z) &= d(y, m) + d(m, z). \end{aligned}$$

In other words, the vertex m is the unique vertex that lies on a geodesic in $[x, y]$, $[x, z]$, and $[y, z]$, simultaneously.

From now on, we fix a median graph $X = (V, E)$.

Definition 1.1.16 (halfspace, $\mathcal{H}$-intervals)**.** For an edge $e \in E$ that consists of two distinct vertices α and ω we define the map

$$\begin{aligned} g_e \colon V &\to \{\alpha, \omega\} \\ x &\mapsto \begin{cases} \alpha, & \text{if } d(x, \alpha) < d(x, \omega), \\ \omega, & \text{if } d(x, \omega) < d(x, \alpha), \end{cases} \end{aligned}$$

called *gate map* of closest-point projection for e. (This map is well-defined since X is a median graph.) Each of the sets $g_e^{-1}(\alpha)$ and $g_e^{-1}(\omega)$ is called a *halfspace*. We say that e is dual to the halfspaces $g_e^{-1}(\alpha)$ and $g_e^{-1}(\omega)$. Note that two edges can be dual to the same halfspace, see Figure 1.1.

We define $\mathcal{H}$ to be the set of halfspaces. For a halfspace $h \in \mathcal{H}$ we denote by $\overline{h}$ its complement $\overline{h} \coloneqq V \setminus h$. It is clear by definition that $\overline{h}$ is also a halfspace. For two vertices $x, y \in V$ we say a halfspace h separates y from x if $y \in h$ and $x \in \overline{h}$ and define the *$\mathcal{H}$-interval* $[x, y]_{\mathcal{H}}$ to be the set consisting of all halfspaces separating y from x. A geodesic $x_0, \ldots, x_n$ is said to *cross* a halfspace at time i if h separates x_{i+1} from x_i.

Theorem 1.1.17. *Let $x, y \in V$ be vertices. Then $d(x, y)$ is equal to the number of halfspaces separating y from x. More precisely, for every halfspace h separating y from x, every geodesic from x to y crosses h exactly once.*

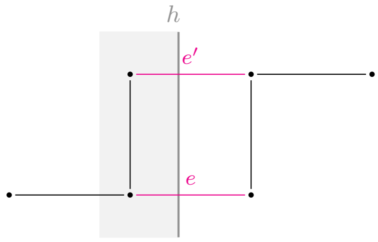

Figure 1.1: The edges e and e' are both dual to the halfspace h.

This can be seen by [8, Remark 1.7] using the connection between median graphs and structures called CAT(0) cube complexes that is developed in [8, Theorem 1.17].

Corollary 1.1.18. *Let $x, y \in V$ and let $m \in V$ be a vertex contained in a geodesic from x to y. Then*

$$[x, y]_{\mathcal{H}} = [x, m]_{\mathcal{H}} \sqcup [m, y]_{\mathcal{H}}.$$

Corollary 1.1.19. *Halfspaces are convex.*

Proof. Let $x, y \in V$ be vertices both contained in a halfspace h and let $\gamma = x_0, \ldots, x_n$ be a geodesic from $x_0 = x$ to $x_n = y$. Since h and $\overline{h}$ both don't separate y from x, we know that the the geodesic γ does cross neither h nor $\overline{h}$ by Theorem 1.1.17. Hence, all vertices of γ must lie in h. □

Definition 1.1.20 (tightly nested, segment)**.** Two halfspaces h_1 and h_2 are *tightly nested* if they are distinct, if $h_1 \supset h_2$, and if there is no other halfspace $h \in \mathcal{H}$ such that $h_1 \supset h \supset h_2$.

An *$\mathcal{H}$-segment* of length $l \in \mathbb{N}$ is a sequence $(h_1 \supset \ldots \supset h_l)$ of tightly nested halfspaces. The *reverse* $\overline{s}$ of an $\mathcal{H}$-segment $s = (h_1 \supset \ldots \supset h_l)$ is defined by $\overline{s} := (\overline{h_l} \supset \ldots \supset \overline{h_1})$.

We denote by $X_{\mathcal{H}}^{(l)}$ the set of all $\mathcal{H}$-segments of length l. For $x, y \in V$ we write $[x, y]_{\mathcal{H}}^{(l)}$ for the set of all segments of length l whose halfspaces are contained in $[x, y]_{\mathcal{H}}$.

Example 1.1.21. Let $T = (V, E)$ be a tree. In this special case of a median graph, we have a one-to-one correspondence between halfspaces and oriented edges. Furthermore, the uniqueness of paths in a tree gives rise to the fact that for $l \in \mathbb{N}$ and $x, y \in V$, there is a one-to-one correspondence between geodesics segments of length l of the (unique) geodesic connecting x to y and $\mathcal{H}$-segments that are contained in $[x, y]_{\mathcal{H}}^{(l)}$.

Definition 1.1.22 (interior)**.** A vertex $x \in V$ is said to lie in the *interior* of a segment $(h_1 \supset \ldots \supset h_l)$ if $x \in h_1 \cup \overline{h_l}$.

1.2 Group theory

In this section we recall group actions [11, Section 4.1], the connection between free groups and reduced words [11, Section 3.3.1], and amenability [15, Chapters 1-2].

Group actions

Definition 1.2.1 (group action)**.** Let G be a group and X be an object in a category $\mathcal{C}$. An *action of* G *on* X in the category $\mathcal{C}$ is a group homomorphism

$$G \to \mathrm{Aut}_{\mathcal{C}}(X).$$

In this thesis, we will mostly consider group actions on sets or on graphs.

Example 1.2.2. In the case that a group G acts on a set S via $\phi\colon G \to \mathrm{Aut}_{Set}(S)$, we write

$$g \cdot s \coloneqq \phi(g)(s)$$

for $g \in G$ and $s \in S$. Then, we can describe the group action by the map

$$\begin{aligned} G \times S &\to S \\ (g, s) &\mapsto g \cdot s. \end{aligned}$$

Recall that for a graph $X = (V, E)$ the set of automorphisms $\mathrm{Aut}_{Graph}(X)$ consists of bijections $V \to V$ that fulfils certain properties on the level of edges. Hence, a group action $\phi\colon G \to \mathrm{Aut}_{Graph}(X)$ can also be described by a map

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto g \cdot v \coloneqq \phi(g)(v). \end{aligned}$$

Example 1.2.3. Let G be a group generated by a set S. Then G acts by graph automorphisms on $\mathrm{Cay}(G, S)$ via left-translation, i.e.

$$\begin{aligned} G &\to \mathrm{Aut}_{Graph}(\mathrm{Cay}(G, S)) \\ g &\mapsto (v \mapsto gv). \end{aligned}$$

Note that this map is well-defined, as for every $g \in G$ the map $G \to G$, $h \mapsto gh$ is a bijection on the vertices of $\mathrm{Cay}(G, S)$ and preserves edges.

Free groups and reduced words

Definition 1.2.4 (reduced word)**.** Let S be a set, and let $(S \cup \widehat{S})^*$ be the set of words over S and formal inverses of elements of S. Let $n \in \mathbb{N}$ and let $s_1, \ldots, s_n \in S \cup \widehat{S}$. The word $s_1 \cdots s_n$ is *reduced* if $s_{j+1} \neq \widehat{s_j}$ and $\widehat{s_{j+1}} \neq s_j$ holds for all $j \in \{1, \ldots, n-1\}$. We write $F_{red}(S)$ for the set of all reduced words in $(S \cup \widehat{S})^*$.

We can define a group structure on $F_{red}(S)$ by the concatenation of words and possibly reducing the word at the point of concatenation [11, Proposition 3.3.5]. Then, we see that the free group F generated by S is canonically isomorphic as a group to $F_{red}(S)$.

Definition 1.2.5 (self-overlapping word)**.** Let F be a free group and $w \in F$ a non-trivial reduced word. We call w *self-overlapping* if there are reduced words $s, v \in F$ such that s is non-trivial and

$$w = svs$$

is a reduced word.

Amenability

Definition 1.2.6 (topological group)**.** A *topological group* is a group G equipped with a topology such that the inversion map

$$\begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

and the composition map

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned}$$

are continuous.

Definition 1.2.7 (locally compact)**.** A topological space X is called *locally compact* if for every $x \in X$ and every open neighbourhood U of x there exists a compact neighbourhood K of x that is contained in U, i.e. $x \in K \subset U$.

Let G be a locally compact group. We denote by $\mathcal{B}(G)$ the Borel algebra, i.e. the smallest σ-algebra on G that contains all open sets. Locally compact groups have the following important property [15, Page 2].

Theorem 1.2.8. *Let G be a locally compact group. Then there exists a non-zero, positive, regular Borel measure λ on G that is G-left-invariant, i.e.*

$$\lambda(gA) = \lambda(A)$$

for all $g \in G$ and $A \in \mathcal{B}(G)$.

In the following, every locally compact group G is considered as a measure space $(G, \mathcal{B}(G), \lambda)$.

Definition 1.2.9. For G a locally compact group, we denote by $L^\infty(G, \mathbb{C})$ the set of equivalence classes of λ-measurable, complex-valued functions ϕ that are essentially bounded, i.e.

$$\operatorname{ess\,sup}_{g\in G}|\phi(g)| \coloneqq \inf\{C \geq 0 \mid |\phi(g)| \leq C \text{ for almost all } g \in G\} < \infty.$$

We call two functions ϕ and ψ equivalent if they are equal almost everywhere.

In particular, $L^\infty(G,\mathbb{C})$ is a normed $\mathbb{C}$-vector space with the norm of $\phi \in L^\infty(G,\mathbb{C})$ given by

$$\|\phi\|_\infty \coloneqq \operatorname{ess\,sup}_{g\in G}|\phi(g)|.$$

A locally compact group G admits an action on $L^\infty(G,\mathbb{C})$ defined by

$$\begin{aligned} G \times L^\infty(G,\mathbb{C}) &\to L^\infty(G,\mathbb{C}) \\ (g,f) &\mapsto (x \mapsto f(g^{-1}x)). \end{aligned}$$

Definition 1.2.10 (operator norm)**.** Let $f\colon V \to W$ be a linear map between normed vector spaces. Then we define the *operator norm* $\|f\|$ of f by

$$\|f\| \coloneqq \inf\{M \geq 0 \mid \|f(v)\| \leq M\|v\| \text{ for all } v \in V\}.$$

There is an important theorem about linear maps with finite operator norm. A proof for this can be found in [17, Satz II.1.2].

Theorem 1.2.11. *Let $f\colon V \to W$ be a linear map between normed vector spaces. Then f is continuous if and only if $\|f\|$ is finite.*

Definition 1.2.12 (mean)**.** Let G be a locally compact group.

- A *mean* on $L^\infty(G,\mathbb{C})$ is a linear map $m\colon L^\infty(G,\mathbb{C}) \to \mathbb{C}$ such that $\|m\| = 1$ and $m(\underline{1}) = 1$, where $\underline{1}$ denotes the constant map with value 1.
- A mean $m \in L^\infty(G,\mathbb{C})$ is called *G-invariant* if for all $g \in G$ and $f \in L^\infty(G,\mathbb{C})$ it is
$$m(g \cdot f) = m(f).$$
- The group G is called *amenable* if there exists a G-invariant mean on $L^\infty(G,\mathbb{C})$.

Amenability fulfils the following inheritance properties. The corresponding proofs can be found in [15, Proposition 1.12, Proposition 1,13].

Theorem 1.2.13. *Let G be a locally compact group.*

- *If G is amenable then every closed subgroup of G is amenable.*
- *If H is a closed normal subgroup of G, then G is amenable if and only if both H and G/H are amenable.*

The second inheritance property can be generalized to all closed subgroups as follows.

Theorem 1.2.14. *Let G be a locally compact group and $H \subset G$ a closed subgroup. If H is amenable and there is a finite G-invariant measure on G/H, then G is amenable.*

Proof. Let μ be a G-invariant measure on G/H with $\mu(G/H) = 1$. This can be obtained by rescaling the finite measure on G/H. Since H is amenable, there exists an H-invariant mean m_H on $L^\infty(H, \mathbb{C})$. We define for $f \in L^\infty(G, \mathbb{C})$ the map

$$\begin{aligned} \phi_f \colon G/H &\to \mathbb{R} \\ gH &\mapsto m_H(x \mapsto f(gx)). \end{aligned}$$

First, note that this map is well-defined as we compute for $g \in G$, $h \in H$ and $x \in H$

$$f(gh \cdot x) = (g^{-1} \cdot f)(h \cdot x) = (h^{-1} \cdot (g^{-1} \cdot f))(x).$$

Hence, we have

$$m_H(x \mapsto f(ghx)) = m_H(h^{-1} \cdot (x \mapsto f(gx))) = m_H(x \mapsto f(gx))$$

by H-invariance of m_H.

As we consider Borel algebras, continuous maps are measurable. Hence, the following three maps are measurable:

$$\begin{aligned} G &\longrightarrow L^\infty(G, \mathbb{C}) \\ g &\longmapsto g^{-1} \cdot f, \end{aligned}$$

$$\begin{aligned} L^\infty(G, \mathbb{C}) &\xrightarrow{|H} L^\infty(H, \mathbb{C}) \\ \varphi &\longmapsto \varphi_{|H}, \end{aligned}$$

and

$$\begin{aligned} L^\infty(H, \mathbb{C}) &\xrightarrow{m_H} \mathbb{C} \\ \psi &\longmapsto m_H(\psi). \end{aligned}$$

Their concatenation is a measurable map from G to $\mathbb{C}$ sending an element $g \in G$ to $\phi_f(gH)$. As the projection $G \to G/H$ is an open map, we can deduce that ϕ_f is measurable. Knowing this, we are able to define a G-invariant mean on $L^\infty(G, \mathbb{C})$ via

$$\begin{aligned} m \colon L^\infty(G, \mathbb{C}) &\to \mathbb{C} \\ f &\mapsto \int_{G/H} \phi_f(gH) d\mu(gH). \end{aligned}$$

At first, we observe that m_G is G-invariant, as we compute for $g' \in G$ and $f \in L^\infty(G, \mathbb{C})$

$$\begin{aligned} m(g' \cdot f) &= \int_{G/H} \phi_{g' \cdot f}(gH) d\mu(gH) \\ &= \int_{G/H} m_H(x \mapsto (g' \cdot f)(gx)) d\mu(gH) \\ &= \int_{G/H} m_H(x \mapsto f(g'^{-1} gx)) d\mu(gH) \\ &= \int_{G/H} \phi_f(g'^{-1} gH) d\mu(gH) \\ &= \int_{G/H} \phi_f(gH) d\mu(gH), \end{aligned}$$

where the last equation follows from the G-invariance of μ. Furthermore, m is linear as m_H and the integral are linear and

$$\begin{aligned} m(\underline{1}) &= \int_{G/H} \phi_{\underline{1}}(gH) d\mu(gH) \\ &= \int_{G/H} m_H(\underline{1}) d\mu(gH) \\ &= \int_{G/H} 1 d\mu(gH) \\ &= \mu(G/H) = 1. \end{aligned}$$

Finally, we check that the operator norm of m equals 1. For this, let $f \in L^\infty(G, \mathbb{C})$. We compute

$$\begin{aligned} |m(f)| &= \left| \int_{G/H} \phi_f(gH) d\mu(gH) \right| \leq \int_{G/H} |\phi_f(gH)| d\mu(gH) \\ &= \int_{G/H} |m_H(x \mapsto f(gx))| d\mu(gH) \\ &\leq \int_{G/H} \|(x \mapsto f(gx))\|_\infty d\mu(gG) \\ &\leq \int_{G/H} \|f\|_\infty d\mu(gH) \\ &= \|f\|_\infty. \end{aligned}$$

This equation shows $\|m\| \leq 1$. Together with $m(\underline{1}) = 1$ we conclude $\|m\| = 1$. Hence, G is amenable. □

Example 1.2.15.
- Abelian locally compact groups are amenable [15, Proposition 0.15].

- An example for a group that is not amenable is the (discrete) non-abelian free group of rank 2 elements ([15, Example 0.6]).

1.3 Cohomology

In the following, let Γ be a discrete group.

Definition 1.3.1 (group ring)**.** We denote by $\mathbb{R}[\Gamma]$ the *group ring* associated to $\mathbb{R}$ and Γ, i.e. the abelian group $\bigoplus_\Gamma \mathbb{R}$ together with a multiplication defined by

$$\left(\sum_{g\in\Gamma} a_g g\right)\left(\sum_{g\in\Gamma} b_g g\right) = \sum_{g\in\Gamma}\left(\sum_{h\in\Gamma} a_{gh} b_{h^{-1}}\right) g$$

Note that $\mathbb{R}[\Gamma]$ forms a ring as its name suggests.

Definition 1.3.2 (invariant)**.** Let V be an $\mathbb{R}[\Gamma]$-module. Then we denote by V^Γ the submodule of Γ-*invariant elements* of V, i.e.

$$V^\Gamma := \{v \in V \mid g \cdot v = v \text{ for all } g \in \Gamma\}.$$

Example 1.3.3. For $n \in \mathbb{N}_{>0}$ we consider the abelian group $\operatorname{Map}(\Gamma^n, \mathbb{R})$ where the group structure is given by pointwise addition. We have a group action of Γ on $\operatorname{Map}(\Gamma^n, \mathbb{R})$ by defining $g \cdot f$ for $g \in \Gamma$ and $f\colon \Gamma^n \to \mathbb{R}$ by

$$\begin{aligned} g \cdot f\colon \Gamma^n &\to \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto f(g^{-1}x_1, \dots, g^{-1}x_n). \end{aligned}$$

This gives rise to an $\mathbb{R}[\Gamma]$-module structure on $\operatorname{Map}(\Gamma^n, \mathbb{R})$. For $f \in \operatorname{Map}(\Gamma^n, \mathbb{R})$ we define

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in \Gamma\}$$

and say that f is bounded if $\|f\|_\infty < \infty$. We consider the set

$$\ell^\infty(\Gamma^n, \mathbb{R}) := \{f\colon \Gamma^n \to \mathbb{R} \mid \|f\|_\infty < \infty\}$$

Since the Γ action on $\operatorname{Map}(\Gamma^n, \mathbb{R})$ preserves boundedness, we obtain that $\ell^\infty(\Gamma^n, \mathbb{R})$ is an $\mathbb{R}[\Gamma]$-submodule of $\operatorname{Map}(\Gamma^n, \mathbb{R})$.

Definition 1.3.4 (cochain complex, cochain, cocycle, coboundary)**.** A *cochain complex* is a sequence

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

where C^n is an $\mathbb{R}[\Gamma]$-module and $\delta^n\colon C^n \to C^{n+1}$ is an $\mathbb{R}[\Gamma]$-linear map with $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{N}$. We define δ^{-1} to be the trivial map. We call $\delta^* = (\delta^n)_{n\in\mathbb{N}}$ the *simplicial* *coboundary operator* and denote this cochain complex also by (C^*, δ^*) or just by C^* when it is clear which coboundary operator is used.

Let $\zeta \in C^n$ for some $n \in \mathbb{N}$. Then we call ζ an *n-cochain* or *cochain*. If $\zeta \in \ker(\delta^n)$ we call it *n-cocycle* or *cocycle*. If $\zeta \in \operatorname{im}(\delta^{n-1})$ we call it *n-coboundary* or *coboundary*.

Definition 1.3.5 (homomorphism of cochains)**.** Let (C^*, δ^*_C) and (D^*, δ^*_D) be two cochain complexes. A *homomorphism of cochain complexes* from C^* to D^* is a family of maps $\phi^* = (\phi^n)_{n\in\mathbb{N}}$ where for all $n \in \mathbb{N}$

- $\phi^n \colon C^n \to D^n$ is an $\mathbb{R}[\Gamma]$-linear map, and
- $\phi^{n+1} \circ \delta^n_C = \delta^n_D \circ \phi^n$.

Definition 1.3.6 (cohomology)**.** Let (C^*, δ^*) be a cochain complex. Then we define the *n-th cohomology* for $n \in \mathbb{N}$ of this cochain complex by

$$H^n(C^*) \coloneqq \frac{\ker(\delta^n \colon C^n \to C^{n+1})}{\operatorname{im}(\delta^{n-1} \colon C^{n-1} \to C^n)}$$

and call the elements of $H^n(C^*)$ *coclasses*. We write $H^*(C^*) \coloneqq (H^n(C^*))_{n\in\mathbb{N}}$.

2 Bounded Cohomology

In this chapter we define bounded cohomology of groups and equivariant bounded cohomology of group actions. This chapter is meant as a recollection of the definitions and important statements. We do not prove everything in detail but refer to the corresponding literature for the missing details.

2.1 Bounded cohomology of groups

At first, we define (bounded) cohomology of discrete groups. We follow Frigerio [7, Sections 1.1 and 1.4].

Let Γ be a discrete group. We construct a cochain complex of $\mathbb{R}[\Gamma]$-modules as follows. For $n \in \mathbb{N}$ we consider the $\mathbb{R}[\Gamma]$-module $\mathrm{Map}(\Gamma^{n+1}, \mathbb{R})$ and define the coboundary operator $\delta^n \colon \mathrm{Map}(\Gamma^{n+1}, \mathbb{R}) \to \mathrm{Map}(\Gamma^{n+2}, \mathbb{R})$ by

$$\delta^n f(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, \widehat{x}_i, \ldots, x_{n+1})$$

for $f \in \mathrm{Map}(\Gamma^{n+1}, \mathbb{R})$. Then, $(\mathrm{Map}(\Gamma^{*+1}, \mathbb{R}), \delta^*)$ is a cochain complex of $\mathbb{R}[\Gamma]$-modules called the *homogeneous complex for* Γ *and* $\mathbb{R}$. One can check via a direct computation that δ^* fulfils the properties of a coboundary operator.

In particular, δ^n is a map of $\mathbb{R}[\Gamma]$ modules for all $n \in \mathbb{N}$ and hence commutes with the Γ-action. This means it sends Γ-invariant cochains to Γ-invariant cochains.

For this reason, it is possible to consider the subcomplex $C^*(\Gamma; \mathbb{R}) = \mathrm{Map}(\Gamma^{*+1}, \mathbb{R})^\Gamma$ of the homogeneous complex consisting only of Γ-invariant cochains equipped with the restricted coboundary operator, also denoted by δ^*.

Then, the group cohomology of Γ with coefficients in $\mathbb{R}$ is then defined by

$$H^*(\Gamma; \mathbb{R}) \coloneqq H^*(C^*(\Gamma; \mathbb{R})).$$

An example for groups, where it is possible to compute the group cohomology in every degree are non-abelian free groups. In particular, the following statement holds true due to [12, Corollary 1.6.23].

Example 2.1.1. Let F be a non-abelian free group. Then, for all $k \geq 2$, we have $H^k(F; \mathbb{R}) = 0$.

In order to define the bounded cohomology of Γ, we consider the bounded subcomplex of $C^*(\Gamma; \mathbb{R})$. More precisely, as δ^* preserves boundedness, we obtain a complex consisting of $\mathbb{R}[\Gamma]$-modules $C_b^n(\Gamma;\mathbb{R}) := \ell^\infty(\Gamma^{n+1},\mathbb{R})^\Gamma \subset C^n(\Gamma;\mathbb{R})$ for $n \in \mathbb{N}$ and the coboundary operator given by δ^*.

We define the bounded cohomology of Γ with real coefficients by

$$H_b^*(\Gamma;\mathbb{R}) := H^*(C_b^*(\Gamma;\mathbb{R})).$$

Theorem 2.1.2 ([7, Corollary 6.7])**.** *The map*

$$\begin{aligned} H_b^2(\Gamma;\mathbb{R}) &\to \mathbb{R} \\ \alpha &\mapsto \inf\{\|\zeta\|_\infty \mid \zeta \in C_b^2(\Gamma;\mathbb{R}) \textit{ is a cocycle with } [\zeta]=\alpha\} \end{aligned}$$

defines a norm on $H_b^2(\Gamma;\mathbb{R})$. In particular, $H_b^2(\Gamma;\mathbb{R})$ is a Banach space, i.e. a complete normed $\mathbb{R}$-vector space.

Since every cocycle in $C_b^n(\Gamma;\mathbb{R})$ is also a cocycle in $C^n(\Gamma;\mathbb{R})$ we obtain a canonical comparison map between bounded cohomology and ordinary group cohomology.

Definition 2.1.3 (comparison map)**.** The inclusion $\iota\colon \ell^\infty(\Gamma^{*+1},\mathbb{R}) \to \mathrm{Map}((\Gamma^{*+1},\mathbb{R})$ from the bounded complex to the ordinary homogeneous complex induces a map

$$\mathrm{comp}_\Gamma^*\colon H_b^*(\Gamma;\mathbb{R}) \to H^*(\Gamma;\mathbb{R})$$

from bounded cohomology to ordinary cohomology called the *comparison map.* For $n \in \mathbb{N}$, we denote the kernel of comp_Γ^n by $EH_b^n(\Gamma;\mathbb{R})$.

The study of the kernel of the comparison map is important for the understanding of bounded cohomology. There is a relation between the kernel $EH_b^2(\Gamma;\mathbb{R})$ of the comparison map in degree two and quasimorphisms. This relation is discussed in Chapter 3 and gives an important insight in the bounded cohomology of a non-abelian group of degree 2 as we can see in the following example.

Example 2.1.4. Let F be a non-abelian free group. For $k \geq 2$ we have

$$H_b^k(F;\mathbb{R}) = EH_b^k(\Gamma;\mathbb{R})$$

since $H_b^k(F;\mathbb{R}) = 0$ by Example 2.1.1.

Next, we introduce the cup product and the Massey triple product in bounded cohomology. It is a way of generating higher dimensional coclasses out of lower dimensional ones.

We define the *cup product* on cochains of dimension $p, q \in \mathbb{N}$ by

$$\begin{aligned} \cup\colon C_b^p(\Gamma;\mathbb{R}) \otimes_\mathbb{R} C_b^q(\Gamma;\mathbb{R}) &\to C_b^{p+q}(\Gamma;\mathbb{R}) \\ f \otimes g &\mapsto f \cup g : ((\gamma_0,\dots,\gamma_{p+q}) \mapsto f(\gamma_0,\dots,\gamma_p)\cdot g(\gamma_p,\dots,\gamma_{p+q})) \end{aligned}$$

The cup product is associative and for $p, q \in \mathbb{N}$ and cochains $f \in C_b^p(\Gamma;\mathbb{R})$ and

$g \in C_b^q(\Gamma;\mathbb{R})$ we have

$$\delta^{p+q}(f \cup g) = \delta^p f \cup g + (-1)^p f \cup \delta^q g.$$

Hence, the following definition of the cup product on bounded cohomology is well-defined.

Definition 2.1.5 (cup product)**.** We define the *cup product* on coclasses of dimension $p, q \in \mathbb{N}$ by

$$\begin{aligned}\cup\colon H_b^p(\Gamma;\mathbb{R}) \otimes_{\mathbb{R}} H_b^q(\Gamma;\mathbb{R}) &\to H_b^{p+q}(\Gamma;\mathbb{R})\\ [f] \otimes [g] &\mapsto [f \cup g].\end{aligned}$$

Another operation in the bounded cohomology is the Massey triple product, first defined as an operation in singular cohomology. The definition can be adapted to bounded cohomology as follows.

Definition 2.1.6 (Massey triple product)**.** Let $p, q, k \in \mathbb{N}$ be integers and $\alpha_1 \in H_b^p(\Gamma;\mathbb{R})$, $\alpha_2 \in H_b^q(\Gamma;\mathbb{R})$, $\alpha \in H_b^k(\Gamma;\mathbb{R})$. If the cup products $\alpha_1 \cup \alpha$ and $\alpha \cup \alpha_2$ are trivial then the *Massey triple product* $\langle \alpha_1, \alpha, \alpha_2 \rangle$ exists and it is a subset of $H_b^{p+q+k-1}(\Gamma;\mathbb{R})$ whose elements are determined as follows.

A coclass $\zeta \in H_b^{p+q+k-1}(\Gamma;\mathbb{R})$ is an element of the Massey triple product $\langle \alpha_1, \alpha, \alpha_2 \rangle$ if and only if there exists $\omega_1 \in C_b^p(\Gamma;\mathbb{R})$, $\omega_2 \in C_b^q(\Gamma;\mathbb{R})$, and $\omega \in C_b^k(\Gamma;\mathbb{R})$ that are representatives of α_1, α_2, and α, respectively, and $\beta_1 \in C_b^{p+k-1}(\Gamma;\mathbb{R})$, $\beta_2 \in C_b^{q+k-1}(\Gamma;\mathbb{R})$ with $\delta\beta_1 = \omega_1 \cup \omega$ and $\delta\beta_2 = \omega \cup \omega_2$ such that

$$\zeta = (-1)^k[(-1)^p \omega_1 \cup \beta_2 - \beta_1 \cup \omega_2].$$

The Massey triple product is said to be *trivial* if it contains the trivial coclass.

Remark 2.1.7. The cochain $(-1)^p\omega_1 \cup \beta_2 - \beta_1 \cup \omega_2$ of the above definition is indeed a cocycle, as we compute

$$\begin{aligned}\delta((-1)^p\omega_1 \cup \beta_2 - \beta_1 \cup \omega_2) &= (-1)^p \delta(\omega_1 \cup \beta_2) + \delta(\beta_1 \cup \omega_2)\\ &= \omega_1 \cup \delta\beta_2 - \delta\beta_1 \cup \omega_2\\ &= \omega_1 \cup \omega \cup \omega_2 - \omega_1 \cup \omega \cup \omega_2 = 0.\end{aligned}$$

2.2 Equivariant bounded cohomology

Let Γ be a discrete group acting on a set S.

We define a group action of Γ on the set

$$\ell^\infty(S^{n+1}, \mathbb{R}) \coloneqq \{f\colon S^{n+1} \to \mathbb{R} \mid \|f\|_\infty < \infty\}$$

via

$$(g \cdot f)(s_0, \ldots, s_n) = f(g^{-1} \cdot s_0, \ldots, g^{-1} \cdot s_n)$$

for $g \in \Gamma$ and $f \in \ell^\infty(S^{n+1}, \mathbb{R})$.

As in the definition of (bounded) group cohomology, we consider only Γ-invariant bounded maps. This means, we consider the cochain complex

$$C^*_{\Gamma,b}(S; \mathbb{R}) \coloneqq \ell^\infty(S^{*+1}, \mathbb{R})^\Gamma$$

equipped with the coboundary operator δ^* that is given in dimension $n \in \mathbb{N}$ by

$$\delta^n f(s_0, \ldots, s_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(s_0, \ldots, \widehat{s}_i, \ldots, s_{n+1})$$

for $f \in C^n_{\Gamma,b}(S; \mathbb{R})$ and $s_0, \ldots, s_{n+1} \in S$.

We define the Γ-equivariant bounded cohomology of S (with real coefficients) by

$$H^*_{\Gamma,b}(S; \mathbb{R}) \coloneqq H^*(C^*_{\Gamma,b}(S; \mathbb{R})).$$

Equivariant bounded cohomology of $\Gamma \curvearrowright S$ allows conclusions on the bounded cohomology of Γ via the following map.

Let $s \in S$ and

$$\begin{aligned} o_s \colon \Gamma &\to S \\ g &\mapsto gs \end{aligned}$$

be the orbit map of s. This map induces a Γ-equivariant homomorphism of cochain complexes

$$o_s^* \colon C^*_{\Gamma,b}(S; \mathbb{R}) \to C^*_b(\Gamma; \mathbb{R})$$

that is defined by

$$o_s^n(f)(x_0, \ldots, x_n) = f(x_0 s, \ldots, x_n s)$$

for $f \in C^n_{\Gamma,b}(S; \mathbb{R})$ and $x_0, \ldots, x_n \in \Gamma$.

Hence, we obtain an induced map $o_s^* \colon H^*_{\Gamma,b}(S; \mathbb{R}) \to H^*_b(\Gamma; \mathbb{R})$ in bounded cohomology. This map is an isomorphism in the following case.

Theorem 2.2.1 ([7, Theorem 4.23]). *Suppose the stabilizer of every element of S is amenable. Then,*

$$o_s^* \colon H^*_{\Gamma,b}(S; \mathbb{R}) \to H^*_b(\Gamma; \mathbb{R})$$

is an isomorphism of cochain complexes for all $s \in S$.

An important application of this theorem will be the case of a non-abelian free group acting on one of its Cayley graphs.

Corollary 2.2.2. *Let F be a non-abelian free group with (not necessarily free) generating set $\mathcal{P} \subset F$. Then, for all $x \in F$ the orbit map*

$$o_x^* \colon H_{F,b}^*(\mathrm{Cay}(F, \mathcal{P}); \mathbb{R}) \to H_b^*(F; \mathbb{R})$$

is an isomorphism.

Proof. By Theorem 2.2.1 it suffices to show that the stabilizer of every vertex is amenable. But, for vertices s of $\mathrm{Cay}(F, \mathcal{P})$ and $g \in F$ we have $gs = s$ if and only if g is the neutral element. Hence, the stabilizer of every vertex is amenable, since it is the trivial group. □

In the same way as in bounded group cohomology, we can define the cup product and the Massey triple product in equivariant bounded cohomology.

Definition 2.2.3 (cup product)**.** Let $\Gamma \curvearrowright S$ be a group action. We define the *cup product* in equivariant bounded cohomology of dimension p and $q \in \mathbb{N}$ by

$$\begin{aligned} \cup \colon C_{\Gamma,b}^p(S; \mathbb{R}) \otimes_{\mathbb{R}} C_{\Gamma,b}^q(S; \mathbb{R}) &\to C_{\Gamma,b}^{p+q}(S; \mathbb{R}) \\ f \otimes g &\mapsto f \cup g : ((s_0, \ldots, s_{p+q}) \mapsto f(s_0, \ldots, s_p) \cdot g(s_p, \ldots, s_{p+q})) \\ \cup \colon H_{\Gamma,b}^p(S; \mathbb{R}) \otimes_{\mathbb{R}} H_{\Gamma,b}^q(S; \mathbb{R}) &\to H_{\Gamma,b}^{p+q}(S; \mathbb{R}) \\ [f] \otimes [g] &\mapsto [f \cup g]. \end{aligned}$$

The cup product on coclasses is well-defined due to similar arguments as for coclasses in bounded group cohomology. Likewise, we obtain the following equation regarding the coboundary operator: For $f \in C_{\Gamma,b}^p(\Gamma; \mathbb{R})$ and $g \in C_{\Gamma,b}^q(\Gamma; \mathbb{R})$ we have

$$\delta^{p+q}(f \cup g) = \delta^p f \cup g + (-1)^p f \cup \delta^q g.$$

Remark 2.2.4. An interesting connection between the cup product in bounded and equivariant bounded cohomology is the compatibility with the orbit map o_s^* for $s \in S$, i.e. for $f \in H_{\Gamma,b}^p(S; \mathbb{R})$, $g \in H_{\Gamma,b}^q(S; \mathbb{R})$ we have

$$o_s^{p+q}(f \cup g) = o_s^p(f) \cup o_s^q(g).$$

Definition 2.2.5 (Massey triple product)**.** Let $\Gamma \curvearrowright S$ be a group action. Let $p, q, k \in \mathbb{N}$ and $\alpha_1 \in H_{\Gamma,b}^p(S; \mathbb{R})$, $\alpha_2 \in H_{\Gamma,b}^q(S; \mathbb{R})$, $\alpha \in H_{\Gamma,b}^k(S; \mathbb{R})$. If the cup products $\alpha_1 \cup \alpha$ and $\alpha \cup \alpha_2$ are trivial then the *Massey triple product* $\langle \alpha_1, \alpha, \alpha_2 \rangle$ exists and it is a subset of $H_{\Gamma,b}^{p+q+k-1}(S; \mathbb{R})$ whose elements are determined as follows.

A coclass $\zeta \in H_{\Gamma,b}^{p+q+k-1}(S; \mathbb{R})$ is an element of the Massey triple product $\langle \alpha_1, \alpha, \alpha_2 \rangle$ if and only if there exists $\omega_1 \in C_{\Gamma,b}^p(S; \mathbb{R})$, $\omega_2 \in C_{\Gamma,b}^q(S; \mathbb{R})$, and $\omega \in C_{\Gamma,b}^k(S; \mathbb{R})$ that are representatives of α_1, α_2, and α, respectively, and $\beta_1 \in C^{p+k-1}_{\Gamma,b}(S;\mathbb{R})$, $\beta_2 \in C^{q+k-1}_{\Gamma,b}(S;\mathbb{R})$ with $\delta\beta_1 = \omega_1 \cup \omega$ and $\delta\beta_2 = \omega \cup \omega_2$ such that

$$\zeta = (-1)^k[(-1)^p\omega_1 \cup \beta_2 - \beta_1 \cup \omega_2].$$

The Massey triple product is said to be *trivial* if it contains the trivial element.

Theorem 2.2.6. *Let $\Gamma \curvearrowright S$ be a group action. The orbit map o_s^* for $s \in S$ is compatible with the Massey triple product in bounded and equivariant bounded cohomology. More precisely, for $p, q, k \in \mathbb{N}$ and $\alpha_1 \in H^p_{\Gamma,b}(S;\mathbb{R})$, $\alpha_2 \in H^q_{\Gamma,b}(S;\mathbb{R})$, $\alpha \in H^k_{\Gamma,b}(S;\mathbb{R})$, we have*

$$o_s^{p+q+k}(\langle\alpha_1, \alpha, \alpha_2\rangle) \subset \langle o_s^p(\alpha_1), o_s^k(\alpha), o_s^q(\alpha_2)\rangle.$$

Proof. Let $\omega_1 \in C^p_{\Gamma,b}(S;\mathbb{R})$, $\omega_2 \in C^q_{\Gamma,b}(S;\mathbb{R})$, and $\omega \in C^k_{\Gamma,b}(S;\mathbb{R})$ be representatives of α_1, α_2, and α, respectively, and $\beta_1 \in C^{p+k-1}_{\Gamma,b}(S;\mathbb{R})$, $\beta_2 \in C^{q+k+1}_{\Gamma,b}(S;\mathbb{R})$ with

$$\delta\beta_1 = \omega_1 \cup \omega \quad \text{and} \quad \delta\beta_2 = \omega \cup \omega_2.$$

We consider the element

$$\zeta := (-1)^k[(-1)^p\omega_1 \cup \beta_2 - \beta_1 \cup \omega_2]$$

of the Massey triple product $\langle\alpha_1, \alpha, \alpha_2\rangle$. By the compatibility of the cup product with the orbit map o_s, we have

$$o_s^{p+q+k-1}(\zeta) = (-1)^k[(-1)^p o_s^p(\omega_1) \cup o_s^{q+k-1}(\beta_2) - o_s^{p+k-1}\beta_1 \cup o_s^q\omega_2].$$

Taking a look at the appearing coclasses, we see that $o_s^p(\omega_1)$, $o_s^q(\omega_2)$, and $o_s^p(\omega)$ are representatives for the coclasses $o_s^p(\alpha_1)$, $o_s^q(\alpha_2)$, and $o_s^k(\alpha)$, respectively, and it is

$$\delta o_s^{p+k-1}(\beta_1) = o_s^{p+k}(\delta\beta_1) = o_s^{p+k}(\omega_1 \cup \omega) = o_s^p(\omega_1) \cup o_s^k(\omega).$$

As we obtain a similar equation for β_2 we conclude

$$o_s^{p+q+k-1}(\zeta) \in \langle o_s^p(\alpha_1), o_s^k(\alpha), o_s^q(\alpha_2)\rangle.$$

. □

3 Quasimorphisms

This chapter is dedicated to quasimorphisms and their connection to bounded cohomology. We first introduce quasimorphisms of groups and take a closer look at homogeneous quasimorphisms and their induced coclasses. Moreover, we also define quasimorphisms of group actions and describe their connection to quasimorphisms of groups.

3.1 Quasimorphisms of groups

In this section we introduce quasimorphisms of groups and their potential for the computation of bounded cohomology in degree 2. We follow Frigerio [7, Sections 2.3, 2.4]. For this, let Γ be a discrete group.

Definition 3.1.1 (quasimorphism)**.** A map $f\colon \Gamma \to \mathbb{R}$ is a *quasimorphism* (*of* Γ) if

$$D(f) \coloneqq \sup_{g,h\in\Gamma} |f(g) + f(h) - f(gh)| < \infty.$$

We call $D(f)$ the *defect* of f. The space of quasimorphisms has a canonical structure of an $\mathbb{R}$-module, and it is denoted by $\mathrm{QM}(\Gamma)$.

Example 3.1.2 (trivial quasimorphisms)**.** Trivial examples for quasimorphisms are group homomorphisms, whose defect is 0, and bounded functions, whose defect is bounded by triple the bound of the function. A quasimorphism is called trivial if it is a sum of a homomorphism and a bounded function. Note that a non-zero group homomorphism cannot be bounded. Hence, $\mathrm{Hom}(\Gamma;\mathbb{R}) \cap \ell^\infty(\Gamma;\mathbb{R}) = \{0\}$. We denote by

$$\mathrm{QM}_0(\Gamma) \coloneqq \mathrm{Hom}(\Gamma;\mathbb{R}) \oplus \ell^\infty(\Gamma;\mathbb{R})$$

the space of trivial quasimorphisms.

More examples for quasimorphisms will be given in Section 4.1.

3.1.1 Homogeneous quasimorphisms

An interesting class of quasimorphisms of groups are homogeneous quasimorphisms. We show that every quasimorphism has bounded distance to a unique homogeneous quasimorphism.

Definition 3.1.3. A quasimorphism $f\colon \Gamma \to \mathbb{R}$ is *homogeneous* if $f(g^n) = n \cdot f(g)$ for every $g \in \Gamma$, $n \in \mathbb{Z}$. The space of homogeneous quasimorphisms is a submodule of $\mathrm{QM}(\Gamma,\mathbb{R})$ and is denoted by $\mathrm{QM}_h(\Gamma,\mathbb{R})$.

Theorem 3.1.4. *Let $f \in \mathrm{QM}(\Gamma, \mathbb{R})$ be a quasimorphism. Then, there exists a unique element $\overline{f} \in \mathrm{QM}_h(\Gamma, \mathbb{R})$ that has finite distance from f. Moreover, we have*

$$\|f - \overline{f}\|_\infty \leq D(f) \text{ and } D(\overline{f}) \leq 4D(f).$$

Proof. For the uniqueness, let $\overline{f}, \widetilde{f} \in \mathrm{QM}_h(\Gamma, \mathbb{R})$ be homogeneous quasimorphisms with $\|f - \overline{f}\|_\infty \leq D(f)$ and $\|f - \widetilde{f}\|_\infty \leq D(f)$. By the triangle inequality, we have

$$\begin{aligned} \|\widetilde{f} - \overline{f}\|_\infty &= \|\widetilde{f} - f + f - \overline{f}\|_\infty \\ &\leq \|\widetilde{f} - f\|_\infty + \|f - \overline{f}\|_\infty \\ &\leq 2D(f). \end{aligned}$$

Hence, for all $g \in \Gamma$ and all $n \in \mathbb{N}_{>0}$, we compute

$$\begin{aligned} |\widetilde{f}(g) - \overline{f}(g)| &= |\frac{1}{n} \cdot (n\widetilde{f}(g) - n\overline{f}(g))| \\ &= \frac{1}{n} \cdot |(\widetilde{f}(g^n) - \overline{f}(g^n))| \\ &\leq \frac{1}{n} \cdot \|\widetilde{f} - \overline{f}\|_\infty \\ &\leq \frac{2}{n} D(f). \end{aligned}$$

As $\lim\limits_{n\to\infty} \frac{2}{n} D(f) = 0$, this shows $\widetilde{f}(g) - \overline{f}(g) = 0$. Finally, we have $\widetilde{f} = \overline{f}$.

For the existence, we show that for $g \in \Gamma$ the sequence $(\frac{f(g^n)}{n})_{n>0}$ is a Cauchy sequence in $\mathbb{R}$. Knowing this, we can define

$$\begin{aligned} \overline{f} \colon F &\to \mathbb{R} \\ g &\mapsto \lim_{n\to\infty} \frac{f(g^n)}{n} \end{aligned}$$

and show that this map fulfils the required properties of a homogeneous quasimorphism. First, we prove that for all $n, m \in \mathbb{N}_{>0}$ the inequality

$$|f(g^{mn}) - nf(g^m)| \leq (n-1)D(f)$$

holds. For this, we choose $m \in \mathbb{N}_{>0}$ arbitrarily and use induction to show the statement for all $n \in \mathbb{N}_{>0}$. In the base case for $n = 1$, we compute

$$|f(g^{mn}) - nf(g^m)| = |f(g^m) - f(g^m)| = 0 = (n-1)D(f).$$

Now assume that for $n \in \mathbb{N}_{>0}$ the inequality $|f(g^{mn}) - nf(g^m)| \leq (n-1)D(f)$ holds.

Then

$$\begin{aligned}|f(g^{m(n+1)}) - (n+1)f(g^m)| &= |f(g^{nm}g^m) - f(g^n m) - f(g^m) + f(g^n m) - nf(g^m)| \\ &\leq |f(g^{nm}g^m) - f(g^n m) - f(g^m)| + |f(g^n m) - nf(g^m)| \\ &\overset{(*)}{\leq} D(f) + (n-1)D(f) = nD(f)\end{aligned}$$

using the definition of the defect of a quasimorphism and the induction hypothesis in step $(*)$. With these inequalities, we are able to prove that the sequence $(\frac{f(g^n)}{n})_{n>0}$ is Cauchy as we choose for $\epsilon \in \mathbb{R}_{>0}$ an integer $N \in \mathbb{N}_{>0}$ large enough such that

$$\epsilon > \frac{2}{N}D(f).$$

We then obtain for all $n, m \geq N$ the inequality

$$\begin{aligned}\left|\frac{f(g^n)}{n} - \frac{f(g^m)}{m}\right| &= \frac{1}{mn}\left|mf(g^n) - nf(g^m)\right| \\ &\leq \frac{1}{mn}\big(|mf(g^n) - f(g^{nm})| + |f(g^{nm}) - nf(g^m)|\big) \\ &\leq \frac{1}{mn}\big((m-1)D(f) + (n-1)D(f)\big) \leq \left(\frac{1}{n} + \frac{1}{m}\right)D(f) \\ &\leq \frac{2}{N}D(f) < \epsilon\end{aligned}$$

and conclude that the sequence is a Cauchy sequence. Thus, the map

$$\begin{aligned}\overline{f}\colon F &\to \mathbb{R} \\ g &\mapsto \frac{f(g^n)}{n}\end{aligned}$$

is well defined. Its distance from f is at most $D(f)$ since for all $g \in \Gamma$ we have

$$\begin{aligned}\left|\overline{f}(g) - f(g)\right| &= \left|\lim_{n\to\infty}\left(\frac{f(g^n)}{n} - f(g)\right)\right| \\ &= \lim_{n\to\infty}\frac{1}{n}\big|(f(g^n) - nf(g))\big| \leq \lim_{n\to\infty}\frac{n-1}{n}D(f) \\ &= D(f).\end{aligned}$$

It is a quasimorphism with defect smaller or equal to $4D(f)$, since for all $g, h \in \Gamma$, we compute

$$\begin{aligned}&\left|\overline{f}(g) + \overline{f}(h) - \overline{f}(gh)\right| \\ &\quad\leq \left|\overline{f}(g) - f(g)\right| + \left|\overline{f}(h) - f(h)\right| + \left|-\overline{f}(gh) + f(gh)\right| + \left|f(g) + f(h) - f(gh)\right| \\ &\quad\leq 4D(f).\end{aligned}$$

As the last step of this proof we need to show that the constructed quasimorphism $\overline{f}$ is homogeneous. For this, let $g \in \Gamma$ and $m \in \mathbb{N}_{>0}$. Then

$$\begin{aligned}\overline{f}(g^m) &= \lim_{n\to\infty} \frac{f((g^m)^n)}{n} = \lim_{n\to\infty} \frac{f(g^{mn})}{n} \\ &= m \cdot \lim_{n\to\infty} \frac{f(g^{mn})}{mn} = m\overline{f}(g).\end{aligned}$$

□

Definition 3.1.5 (homogenization)**.** For $f \in QM(\Gamma, \mathbb{R})$ we call the unique homogeneous quasimorphism $\overline{f} \in QM_h(\Gamma, \mathbb{R})$ that has bounded distance from f the *homogenization of f*.

Corollary 3.1.6. *We have* $\mathrm{QM}(\Gamma) = \mathrm{QM}_h(\Gamma) \oplus \ell^\infty(\Gamma; \mathbb{R})$.

Proof. The existence of the homogenization shows $\mathrm{QM}(\Gamma) = \mathrm{QM}_h(\Gamma) + \ell^\infty(\Gamma; \mathbb{R})$. The uniqueness of the homogenization concludes the directness of the sum. □

3.1.2 Quasimorphisms of groups and bounded cohomology

Next we want to study the connection between quasimorphisms of groups and bounded group cohomology. We will see that quasimorphisms and in particular homogeneous quasimorphisms allow us to understand better the bounded group cohomology in degree 2.

Every quasimorphism defines a Γ-invariant (not necessarily bounded) cochain via the map

$$\begin{aligned}\widehat{\cdot} \colon \mathrm{QM}(\Gamma, \mathbb{R}) &\to C^1(\Gamma, \mathbb{R}) \\ f &\mapsto \left(\widehat{f} \colon (x, y) \mapsto f\left(x^{-1}y\right)\right).\end{aligned}$$

In particular, this map is $\mathbb{R}$-linear and it is injective, because one can reconstruct f via $f(x) = \widehat{f}(e, x)$.

We take a closer look at $\delta^1 \widehat{f}$ for an arbitrary quasimorphism $f \in QM(\Gamma)$. This map is bounded by $D(f)$ as we can estimate for $x, y, z \in \Gamma$

$$\begin{aligned}|\delta^1 \widehat{f}(x, y, z)| &= |\widehat{f}(y, z) - \widehat{f}(x, z) + \widehat{f}(x, y)| \\ &= |f(y^{-1}z) - f(x^{-1}z) + f(x^{-1}y)| \leq D(f)\end{aligned}$$

since $x^{-1}z = x^{-1}y \cdot y^{-1}z$. This shows $\delta^1 \widehat{f} \in C_b^2(\Gamma, \mathbb{R})$. Further, it is a cocycle in $C_b^2(\Gamma, \mathbb{R})$, as $\delta^2 \delta^1 \widehat{f} = 0$ by the definition of the homogeneous complex. We then obtain a well-defined $\mathbb{R}$-linear map

$$\begin{aligned}\Phi \colon \mathrm{QM}(\Gamma) &\to H_b^2(\Gamma, \mathbb{R}) \\ f &\mapsto [\delta^1 \widehat{f}].\end{aligned}$$

On the other hand, we define the $\mathbb{R}$-linear map

$$\widetilde{\cdot}\colon C^1(\Gamma;\mathbb{R}) \to \operatorname{Map}(\Gamma;\mathbb{R})$$
$$\varphi \mapsto \widetilde{\varphi}\colon (x \mapsto \varphi(e,x)).$$

By definition, this map restricts to a map $C_b^1(\Gamma;\mathbb{R}) \to \ell^\infty(\Gamma;\mathbb{R})$. Furthermore, it is injective, since we can reconstruct $\varphi \in C^1(\Gamma;\mathbb{R})$ via $\varphi(x,y) = \widetilde{\varphi}(x^{-1}y)$ for all $x,y \in \Gamma$ using the Γ-invariance of φ.

The concatenation $\widetilde{\cdot} \circ \widehat{\cdot}$ is the identity on $QM(\Gamma)$, which can be seen by a short comparison of the definitions.

An interesting observation for $\varphi \in C^1(\Gamma;\mathbb{R})$ is the equation $D(\widetilde{\varphi}) = \|\delta^1\varphi\|_\infty$, which also follows from the Γ-invariance of φ. This will be very useful in the proof of the following theorem.

Theorem 3.1.7. *Let Γ be a group. Then the sequence of $\mathbb{R}[\Gamma]$-modules*

$$0 \to \operatorname{QM}_0(\Gamma) \xhookrightarrow{\text{incl}} \operatorname{QM}(\Gamma) \xrightarrow{\Phi} H_b^2(\Gamma;\mathbb{R}) \xrightarrow{\operatorname{comp}_\Gamma^2} H^2(\Gamma;\mathbb{R})$$

is exact. In particular, it is

$$EH_b^2(\Gamma;\mathbb{R}) \cong \operatorname{QM}(\Gamma)/\operatorname{QM}_0(\Gamma) \cong \operatorname{QM}_h(\Gamma)/\operatorname{Hom}(\Gamma;\mathbb{R}).$$

Proof. By definition, the inclusion $\operatorname{QM}_0(\Gamma) \xhookrightarrow{\text{incl}} \operatorname{QM}(\Gamma)$ is injective. We want to show $\operatorname{im}(\text{incl}) = \ker(\Phi)$. For $f \in \operatorname{Hom}(\Gamma;\mathbb{R})$ we know that $\delta^1\widehat{f}$ is bounded by $D(f) = 0$. Hence, $\delta^1\widehat{f} = 0$. Also, for $f \in \ell^\infty(\Gamma;\mathbb{R})$, the map $\widehat{f}$ is bounded and hence $\delta^1\widehat{f}$ is a coboundary in $C_b^2(\Gamma;\mathbb{R})$. These two observations show that for all $f \in \operatorname{QM}_0(\Gamma)$, the coclass $[\delta^1\widehat{f}] \in H_b^2(\Gamma;\mathbb{R})$ is trivial and $\operatorname{im}(\text{incl}) \subset \ker(\Phi)$. On the other hand, let $f \in \operatorname{QM}(\Gamma)$ such that $[\delta^1\widehat{f}] = 0$ in $H_b^2(\Gamma;\mathbb{R})$. Then there exists $\varphi \in C_b^1(\Gamma;\mathbb{R})$ such that $\delta^1\widehat{f} = \delta^1\varphi$. The map $\widetilde{\varphi}$ is bounded, since φ is bounded, and $\widehat{f} - \varphi$ is a group homomorphism, as $D(\widehat{f}-\varphi) = \|\delta^1(\widehat{f}-\varphi)\|_\infty = 0$. This shows

$$f = \widetilde{\widehat{f}} - \widetilde{\varphi} + \widetilde{\varphi} = \widetilde{\widehat{f}-\varphi} + \widetilde{\varphi} \in \operatorname{QM}_0(\Gamma).$$

The last thing missing for the sequence to be exact is the equation $\operatorname{im}(\Phi) = \ker(\operatorname{comp}_\Gamma^2)$. The relation $\operatorname{im}(\Phi) \subset \ker(\operatorname{comp}_\Gamma^2)$ follows directly from the fact that $\widehat{f} \in C^1(\Gamma,\mathbb{R})$ for $f \in \operatorname{QM}(\Gamma,\mathbb{R})$. On the other hand, let $\varphi \in C_b^2(\Gamma;\mathbb{R})$ be a cocycle such that $[\varphi] = 0$ in $H^2(\Gamma;\mathbb{R})$. Then there exists $\psi \in C^1(\Gamma;\mathbb{R})$ with $\varphi = \delta^1\psi$. Since the defect $D(\widetilde{\psi}) = \|\delta^1\psi\|_\infty < \infty$ is finite, we know that $\widetilde{\psi}$ is a quasimorphism with $[\varphi] = [\delta^1\psi] = \Phi(\widetilde{\psi})$ because we have the equation $\delta^1\psi = \delta^1\widehat{\widetilde{\psi}}$ by comparing the definitions of $\widetilde{\cdot}$ and $\widehat{\cdot}$.

Using the exactness of the sequence, we are now able to describe $EH_b^2(\Gamma;\mathbb{R})$ as

$$EH_b^2(\Gamma;\mathbb{R}) = \ker(\operatorname{comp}_\Gamma^2) \cong \operatorname{QM}(\Gamma)/\operatorname{QM}_0(\Gamma)$$

where the isometry is induced by Φ. By Example 3.1.2 and Corollary 3.1.6 we know

$$EH_b^2(\Gamma;\mathbb{R}) \cong \mathrm{QM}(\Gamma)/\,\mathrm{QM}_0(\Gamma) \cong \mathrm{QM}_h(\Gamma)/\,\mathrm{Hom}(\Gamma;\mathbb{R}).$$

□

Corollary 3.1.8. *Let $f \in \mathrm{QM}_h(\Gamma)$ be a homogeneous quasimorphism. Then $\delta^1\widehat{f}$ represents the trivial coclass in $H_b^2(\Gamma;\mathbb{R})$ if and only if f is a group homomorphism.*

Proof. We know

$$\ker(\Phi) = \mathrm{QM}_0(\Gamma) = \mathrm{Hom}(\Gamma;\mathbb{R}) \oplus \ell^\infty(\Gamma;\mathbb{R}).$$

This means any unbounded map in $\ker(\Phi)$ is at finite distance to a group homomorphism.

Let $f \in \mathrm{QM}_h(\Gamma)$ be a homogeneous quasimorphisms with $\Phi(f) = [\delta^1\widehat{f}] = 0$. If f is zero-map, then f is a group homomorphism. So assume f is non-zero. Then there exists $g \in \Gamma$ such that $f(g) \neq 0$. This implies that f is unbounded as we have $f(g^n) = nf(g)$ for all $n \in \mathbb{N}$. In particular, we have that f is at finite distance to a group homomorphism. But since group homomorphisms are homogeneous quasimorphisms, we conclude with the uniqueness of the homogenization, that f has to be a group homomorphism.

On the other hand, if a homogeneous quasimorphism f is a group homomorphism, then $f \in \ker(\Phi)$ which means $[\delta^1\widehat{f}] = 0$. □

3.2 Quasimorphisms of group actions

In this section we introduce quasimorphisms of group actions and how they interact - similar to quasimorphisms of groups - with equivariant bounded cohomology of degree 2. Furthermore, we examine how to construct quasimorphisms of groups out of quasimorphisms of group actions using orbit maps.

In the following, let Γ be a discrete group and let $\Gamma \curvearrowright S$ be a group action of Γ on a set S.

Definition 3.2.1 (quasimorphism of group action)**.** A *quasimorphism of* $\Gamma \curvearrowright S$ is a map $f\colon S \times S \to \mathbb{R}$ that is Γ-invariant with respect to the diagonal action of Γ on $S \times S$ and has finite *defect*

$$D(f) := \|\delta^1 f\|_\infty = \sup_{x,y,z\in S} |f(y,z) - f(x,z) + f(x,y)|.$$

We denote by $\mathrm{QM}(\Gamma \curvearrowright S)$ the set of all quasimorphisms of $\Gamma \curvearrowright S$.

Since the defect $D(f) = \|\delta^1 f\|_\infty$ of a quasimorphism of $\Gamma \curvearrowright S$ is finite, we know that $\delta^1 f$ is a cocycle in $C^2_{\Gamma,b}(S;\mathbb{R})$. Furthermore, the orbit map gives a connection between the coclasses defined by quasimorphisms of $\Gamma \curvearrowright S$ and of Γ, see the following theorem.

Theorem 3.2.2. *Let $f\colon S \times S \to \mathbb{R}$ be a quasimorphism of $\Gamma \curvearrowright S$ and let $s \in S$. Then, the pullback of f under the orbit map o_s, i.e. the map*

$$\begin{aligned} f_s \colon \Gamma &\to \mathbb{R} \\ g &\mapsto f(s, gs) \end{aligned}$$

is a quasimorphism of Γ with $\delta^1 \widehat{f_s} = o_s^2(\delta^1 f)$.

Proof. Let $g, h \in \Gamma$. Then

$$\begin{aligned} |f_s(g) + f_s(h) - f_s(gh)| &= |f(s, gs) + f(s, hs) - f(s, ghs)| \\ &= |f(s, gs) + f(gs, ghs) - f(s, ghs)| \\ &= |\delta^1 f(s, gs, ghs)| \leq D(f). \end{aligned}$$

This allows us to estimate $D(f_s) \leq D(f)$ and hence f_s is a quasimorphism.

Furthermore, we compute for $x_0, x_1, x_2 \in \Gamma$

$$\begin{aligned} \delta^1 \widehat{f_s}(x_0, x_1, x_2) &= \widehat{f_s}(x_1, x_2) - \widehat{f_s}(x_0, x_2) + \widehat{f_s}(x_0, x_1) \\ &= f_s(x_1^{-1} x_2) - f_s(x_0^{-1} x_2) + f_s(x_0^{-1} x_1) \\ &= f(s, x_1^{-1} x_2 s) - f(s, x_0^{-1} x_2 s) + f(s, x_0^{-1} x_1 s) \\ &= f(x_1 s, x_2 s) - f(x_0 s, x_2 s) + f(x_0 s, x_1 s) \\ &= o_s^2(\delta^1 f)(x_0, x_1, x_2). \end{aligned}$$

□

4 Examples of quasimorphisms and triviality results

In the following we give three examples for quasimorphisms of non-abelian free groups, namely Brooks, Rolli, and Δ-decomposable quasimorphisms and one example for quasimorphisms of groups acting on median graphs. Furthermore, we collect results of triviality for cup products and Massey triple products with coclasses induced by one of these quasimorphisms.

The existing triviality results for Brooks, Rolli, and Δ-decomposable quasimorphisms were proven by Amontova and Bucher [1] and Marasco [13] using aligned chains. It is currently unknown how aligned chains can be adapted to the setting of group actions on median graphs. The goal of this chapter is to obtain a general statement of triviality for products with certain quasimorphisms that is adaptable simultaneously to each of the four examples we give. This chapter is ordered in the following way. We first formulate the statement in Section 4.2. Then, we show in Section 4.3 how we can use it to prove vanishing results for examples of quasi-morphisms. Finally, in Section 4.4 we focus on the proof of the general statement.

4.1 Examples

In this section we give examples for quasimorphisms both of groups and of group actions. Furthermore, we give an overview of vanishing results for cup products and Massey triple products with coclasses induced by those quasimorphisms.

4.1.1 Brooks quasimorphisms

As a first point we introduce Brooks quasimorphisms. They are named after the mathematician Robert Brooks, who mentioned them first in a paper about bounded cohomology [2]. They are the first example of an infinite family of quasimorphisms that give rise to infinitely many linearly independent coclasses in the bounded cohomology of degree 2 of a non-abelian free group.

Definition 4.1.1 (Brooks quasimorphism)**.** Let F be a non-abelian free group and let S be a free generating set of F. Let w be a non-empty reduced word over $S \cup S^{-1}$. Note that $w \neq w^{-1}$, since w is a non-trivial element of F. We define

$$\begin{aligned} \chi_w \colon F &\to \{-1, 0, +1\} \\ g &\mapsto \begin{cases} 1, & \text{if } g = w, \\ -1, & \text{if } g = w^{-1}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let l be the length of the word w. The *Brooks quasimorphism* on w is defined as the function

$$\begin{aligned} \phi_w \colon F &\to \mathbb{R} \\ g &\mapsto \sum_{j=1}^{m-l+1} \chi_w(x_j \cdots x_{j+l-1}) \end{aligned}$$

where $g = x_1 \cdots x_m$ as a reduced word over $S \cup S^{-1}$. In other words, the Brooks quasimorphism $\phi_w(g)$ counts the number of occurrences of the word w in g minus the number of occurrences of the word w^{-1} in g.

We use the convention that the sum over the empty set is equal to 0. This occurs when the length of g is smaller than the length of w.

The following is known about Brooks quasimorphisms.

Theorem 4.1.2 ([1, Theorem A])**.** *Let F be a non-abelian free group freely generated by S and let $w \in F$ be non-trivial. Then the Brooks quasimorphism f_w is a quasimorphism and for all $\alpha \in H^n_b(F; R)$ the cup product*

$$[\delta^1 \widehat{f_w}] \cup \alpha \in H_b^{n+2}(F; \mathbb{R})$$

is trivial.

4.1.2 Rolli quasimorphisms

Another family of quasimorphisms that proves that the second bounded cohomology of a non-abelian group is infinite dimensional is the family of Rolli quasimorphisms. They were first introduced by the mathematician Pascal Rolli.

In the following, let F be a non-abelian free group freely generated by the finite set $S = \{s_1, \ldots, s_n\}$. We define the set of bounded alternating maps from $\mathbb{Z}$ to $\mathbb{R}$ by

$$\ell^\infty_{alt}(\mathbb{Z}, \mathbb{R}) \coloneqq \{f \colon \mathbb{Z} \to \mathbb{R} \mid \|f\|_\infty < \infty \text{ and } f(m) = f(-m) \text{ for all } m \in \mathbb{Z}\}.$$

Definition 4.1.3 (Rolli quasimorphism)**.** Let $\lambda_1, \ldots, \lambda_n \in \ell^\infty_{alt}(\mathbb{Z}, \mathbb{R})$. Similarly to the fact that each element of F can be uniquely written as a reduced word, every $x \in F$ can be uniquely written as a factorization $x = s_{n_0}^{m_0} \cdots s_{n_k}^{m_k}$, where all exponents $m_j \in \mathbb{Z}\backslash\{0\}$

are non-zero integers and for all $j \in \{0, \dots, k-1\}$ we have $n_j \neq n_{j+1}$. We call the map

$$\begin{aligned} \phi\colon F &\to \mathbb{R} \\ x &\mapsto \sum_{j=0}^{k} \lambda_{n_j}(m_j), \end{aligned}$$

a *Rolli quasimorphism.*

In the following, we give the idea how one can prove that $H_b^2(F;\mathbb{R})$ is infinite dimensional using Rolli quasimorphisms. For this, one considers a simplified version of the Rolli quasimorphisms, namely the ones where $\lambda_1, \dots, \lambda_n \in \ell^\infty_{alt}(\mathbb{Z},\mathbb{R})$ are chosen to be the same map denoted by λ. We call the associated Rolli quasimorphism ϕ_λ and obtain a linear map

$$\begin{aligned} \ell^\infty_{alt}(\mathbb{Z},\mathbb{R}) &\to H_b^2(F;\mathbb{R}) \\ \lambda &\mapsto [\delta^1\widehat{\phi_\lambda}]. \end{aligned}$$

Rolli [16, Proposition 2.2] proves that this map is injective, and hence, the family of Rolli quasimorphisms induces an infinite linearly independent family of coclasses in $H_b^2(F;\mathbb{R})$.

The following vanishing result is known for cup products with coclasses given by Rolli quasimorphisms.

Theorem 4.1.4 ([1, Theorem A])**.** *Let $\phi\colon F \to \mathbb{R}$ be a Rolli quasimorphism. Then ϕ is a quasimorphism and for all $\alpha \in H_b^n(F;R)$ the cup product*

$$[\delta^1\widehat{\phi}] \cup \alpha \in H_b^{n+2}(F;\mathbb{R})$$

is trivial.

4.1.3 Δ-decomposable quasimorphisms

The mathematician Heuer introduced the notion of Δ-decomposable quasimorphisms. For this, we first define Δ-decompositions and then Δ-decomposable quasimorphism as it is done by Heuer [9] and by Amontova and Bucher [1].

In the following, let F be a non-abelian free group with S a free generating set of F.

Notation 4.1.5 (Sequences)**.** Let $\mathcal{A} \subset F$ be a symmetric subset, i.e. if $a \in \mathcal{A}$, then also $a^{-1} \in \mathcal{A}$. We denote by $\mathcal{A}^*$ the set of finite sequences in $\mathcal{A}$ including the empty sequence,

$$\mathcal{A}^* = \{(a_1, \dots, a_n) \mid n \in \mathbb{N} \text{ and } \forall i \in \{1, \dots, n\}\colon\ a_i \in \mathcal{A}\}$$

For $s = (a_1, \dots, a_n) \in \mathcal{A}^*$ we define n to be the length of s and we denote by s^{-1} the sequence $(a_n^{-1}, \dots, a_1^{-1})$. Note that s^{-1} is again an element of $\mathcal{A}^*$ because of the symmetry assumption on $\mathcal{A}$. For $t = (b_1, \dots, b_m) \in \mathcal{A}^*$ we define the common sequence of s and t to be the empty sequence if $a_1 \neq b_1$ and otherwise to be the sequence $(a_1, \dots, a_r)$ where r is the largest integer with $r \leq \min\{m, n\}$ such that $a_j = b_j$ for all $j \leq r$. Moreover, we define $s \cdot t := (a_1, \dots, a_n, b_1, \dots, b_m)$ to be the concatenation of s and t.

Definition 4.1.6 (Δ-decomposition)**.** Let $\mathcal{P} \subset F$ be a symmetric subset, called *pieces of* F, such that the neutral element of F does not lie in $\mathcal{P}$. A Δ-*decomposition* of F into the pieces $\mathcal{P}$ is a map $\Delta\colon F \to \mathcal{P}^*$ assigning to every $g \in F$ a sequence $(g_1, \dots, g_n) \in \mathcal{P}^*$ such that for all $g \in F$ and $\Delta(g) = (g_1, \dots, g_k)$

(i) the reduced expression of g in the letters $S \cup S^{-1}$ is given by the concatenation of the words $g_1, \dots, g_k$, i.e. the word $g_1 \cdots g_k$ is already a reduced word,

(ii) the sequence $\Delta(g^{-1})$ is given by $\Delta(g)^{-1}$ in the sense of Notation 4.1.5, and

(iii) for all $1 \leq i \leq j \leq n$ we have $\Delta(g_i \cdots g_j) = (g_i, \dots, g_j)$.

Furthermore, we demand the existence of a constant $R \in \mathbb{N}$ with the following property:

(iv) For all $g, h \in F$ let
- $c_1 \in \mathcal{P}^*$ be such that c_1^{-1} is the common sequence of $\Delta(g)$ and $\Delta(gh)$,
- $c_2 \in \mathcal{P}^*$ be such that c_2^{-1} is the common sequence of $\Delta(g^{-1})$ and $\Delta(h)$,
- $c_3 \in \mathcal{P}^*$ be such that c_3^{-1} is the common sequence of $\Delta(h^{-1})$ and $\Delta((gh)^{-1})$.

Let $r_1, r_2, r_3 \in \mathcal{P}^*$ be the unique sequences such that

$$\begin{aligned} \Delta(g) &= c_1^{-1} \cdot r_1 \cdot c_2, \\ \Delta(h) &= c_2^{-1} \cdot r_2 \cdot c_3, \\ \Delta((gh)^{-1}) &= c_3^{-1} \cdot r_3 \cdot c_1. \end{aligned}$$

Then the length or r_1, r_2, r_3 is bounded by R.

For the pair (g, h), we call c_1, c_2, c_3 the *c-part of the* Δ-*triangle* of (g, h) and r_1, r_2, r_3 the *r-part of the* Δ-*triangle* of (g, h).

Definition 4.1.7 (Δ-decomposable quasimorphism)**.** Let $\mathcal{P} \subset F$ be a symmetric subset not containing the neutral element. Let $\Delta\colon F \to \mathcal{P}^*$ be a Δ-decomposition of F into pieces $\mathcal{P}$ and let $\lambda \in \ell^\infty_{alt}(\mathcal{P}, \mathbb{R})$ be a alternating bounded map on $\mathcal{P}$, i.e. $\lambda(p^{-1}) = -\lambda(p)$ for all $p \in \mathcal{P}$. Then the map

$$\begin{aligned} \phi_{\lambda,\Delta}\colon F &\to \mathbb{R} \\ g &\mapsto \sum_{i=0}^{n} \lambda(g_i), \end{aligned}$$

with $\Delta(g) = (g_1, \dots, g_n)$ is called a Δ-*decomposable quasimorphism.*

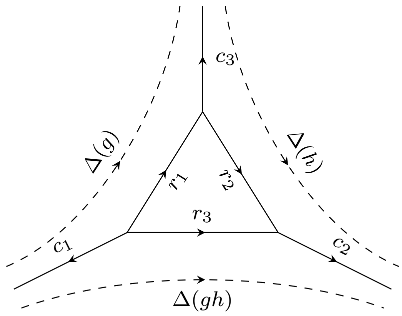

Figure 4.1: Δ-decomposition

Next, we give some examples of Δ-decomposable quasimorphisms.

Example 4.1.8 (trivial decomposition)**.** Let F be freely generated by a finite set S and let $\mathcal{P}_{triv} = S \cup S^{-1}$. Every element $x \in F$ can be uniquely written as a reduced word $s_1 \cdots s_n$ with letters $s_i \in S \cup S^{-1}$. We define $\Delta_{triv} : F \to \mathcal{P}_{triv}$ by sending an element $x \in F$ to the sequence $(s_1, \ldots, s_n) \in \mathcal{P}^*_{triv}$ determined by its reduced expression. This is indeed a Δ-decomposition and in this case every Δ_{triv}-decomposable quasimorphism is a group homomorphism.

Example 4.1.9 (Brooks quasimorphism)**.** Let $w \in F$ be a non self-overlapping word. Then every word $x \in F$ can be written as a reduced word of the form

$$x = u_1 w^{\epsilon_1} u_2 w^{\epsilon_2} \cdots u_k w^{\epsilon_k} u_{k+1},$$

where $u_i \in F$ are possibly empty such that w and w^{-1} are no subwords of u_i for all $i \in \{1, \ldots, k+1\}$ and $\epsilon_i = \pm 1$ for all $i \in \{1, \ldots, k\}$. One can show that this expression is unique due to the assumption that w is non self-overlapping. We consider the symmetric subset

$$\mathcal{P}_w := \{x \in F \mid x \text{ non-trivial and contains neither } w \text{ nor } w^{-1} \text{ as subword}\} \cup \{w, w^{-1}\}$$

of F and define the Δ-decomposition

$$\begin{aligned} \Delta_w \colon F &\to \mathcal{P}^*_w \\ x &\mapsto (u_1, w^{\epsilon_1}, u_2, w^{\epsilon_2}, \ldots, u_k, w^{\epsilon_k}, u_{k+1}) \end{aligned}$$

for x, u_i and ϵ_i described as above. The Δ-decomposable quasimorphism associated to this decomposition and to the alternating map

$$\begin{aligned}\lambda\colon \mathcal{P}_w &\to \mathbb{R}\\ x &\mapsto \begin{cases} 1, & \text{if } x = w,\\ -1, & \text{if } x = w^{-1},\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

coincides with the Brooks quasimorphism ϕ_w.

Example 4.1.10 (Rolli quasimorphism)**.** Every Rolli quasimorphism is Δ-decomposable. In order to see this, let F be a free group freely generated by the set $S = \{s_1, \ldots, s_n\}$ and let $\lambda_1, \ldots, \lambda_n \in \ell^\infty_{alt}(\mathbb{Z}, \mathbb{R})$. We consider the Rolli quasimorphism ϕ_{Rolli} that sends a factorization $s_{n_0}^{m_0} \cdots s_{n_k}^{m_k}$ to $\sum_{i=0}^{k} \lambda_{n_i}(m_i)$.

To write this as a Δ-decomposable quasimorphism, we choose the pieces

$$\mathcal{P}_R := \big\{ s^m \mid s \in S \text{ and } m \in \mathbb{Z} \setminus \{0\} \big\}$$

and consider the decomposition

$$\begin{aligned}\Delta_R\colon F &\to \mathcal{P}_R\\ x &\mapsto (s_{n_0}^{m_0}, \ldots, s_{n_k}^{m_k}),\end{aligned}$$

where x has the factorization $s_{n_0}^{m_0} \cdots s_{n_k}^{m_k}$. We define $\lambda\colon \mathcal{P}_R \to \mathbb{R}$ by

$$\begin{aligned}\lambda\colon \mathcal{P}_R &\to \mathbb{R}\\ s_i^m &\mapsto \lambda_i(m)\end{aligned}$$

for $i \in \{1, \ldots, n\}$ and $m \in \mathbb{N}_{>0}$. By construction, we obtain $\phi_{Rolli} = \phi_{\lambda_R, \Delta_R}$.

The following is known for Δ-decomposable quasimorphisms ([1],[13]).

Theorem 4.1.11. *Let $\phi\colon F \to \mathbb{R}$ be a Δ-decomposable quasimorphism. Then ϕ is a quasimorphism and for all $\alpha_1 \in H_b^n(F; R)$ and $\alpha_2 \in H_b^m(F; R)$,*

- *the cup products $[\delta^1\widehat{\phi}] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1\widehat{\phi}]$ are trivial in $H_b^{n+2}(F; \mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1\widehat{\phi}], \alpha_2 \rangle \subset H_b^{n+m+1}(F; \mathbb{R})$ is trivial.*

4.1.4 Median quasimorphisms

In the following part, we introduce median quasimorphisms. Initially, they were defined by Monod and Shalom [14] as a quasimorphism of a group Γ that acts on a tree. This definition was generalized by Brück, Fournier-Facio and Löh [3]. They defined median quasimorphisms of a group acting on a CAT(0) cube complex. A CAT(0) cube complex is shortly said a glueing of unit cubes that is simply connected and fulfils a certain condition on the links of its vertices. For more details, see [8]. The most important property of CAT(0) cube complexes in our context is that its vertices and edges form a median graph and vice versa, every median graph defines a CAT(0) cube complex ([8, Theorem 1.17]).

In this thesis, the median graph viewpoint comes more natural as we prove a general vanishing result for groups acting on graphs that can be adapted to all the given examples of quasimorphisms (Section 4.4). For this reason, we slightly modify the definition of median quasimorphisms that is given by Brück, Fournier-Facio and Löh [3].

We fix the following setup for this section: Let $X = (V, E)$ be a median graph. We equip V with the graph metric denoted by d.

Let Γ be a discrete group acting on X by graph automorphisms. Since graph automorphisms preserve the metric, every element $g \in \Gamma$ induces a bijection on the halfspaces $\mathcal{H}$ by defining $g \cdot h = \{g \cdot x \mid x \in h\}$ for $h \in \mathcal{H}$. Moreover, for two halfspaces h_1, h_2 the following holds: It is $h_1 \supset h_2$ if and only if $gh_1 \supset gh_2$ for $g \in \Gamma$. Hence, the action of Γ on X induces an action of Γ on the set $X_{\mathcal{H}}^{(l)}$ of all $\mathcal{H}$-segments of length l. For $s \in X_{\mathcal{H}}^{(l)}$ we call the elements of the orbit Γs translates of s.

Definition 4.1.12 (median quasimorphism)**.** Let $l \in \mathbb{N}$ and $s \in X_{\mathcal{H}}^{(l)}$ be an $\mathcal{H}$-segment. We define the *median quasimorphism f_s for s* by

$$\begin{aligned} f_s \colon V \times V &\to \mathbb{R} \\ (x, y) &\mapsto \Big(\text{number of translates of } s \text{ in } [x, y]_{\mathcal{H}}^{(l)}\Big) \\ &\qquad - \Big(\text{ number of translates of } s \text{ in } [y, x]_{\mathcal{H}}^{(l)}\Big) . \end{aligned}$$

It follows from the definition that f_s is Γ-invariant. Furthermore, $f_s = 0$ if $\Gamma s = \Gamma \overline{s}$.

Remark 4.1.13. Let $s \in X_{\mathcal{H}}^{(l)}$ be an $\mathcal{H}$-segment of length $l \in \mathbb{N}$ with $\Gamma s \neq \Gamma \overline{s}$. Then f_s is given by the map

$$\begin{aligned} f_s \colon V \times V &\to \mathbb{R} \\ (x, y) &\mapsto \sum_{t \in [x,y]_{\mathcal{H}}^{(l)}} \epsilon_s(t), \end{aligned}$$

where ϵ_s is defined as

$$\begin{aligned} \epsilon_s \colon X_{\mathcal{H}}^{(l)} &\to \{-1, 0, 1\} \\ t &\mapsto \begin{cases} 1, & \text{if } \Gamma t = \Gamma s, \\ -1, & \text{if } \Gamma t = \Gamma \overline{s}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It is not true that every median quasimorphism is indeed a quasimorphism as Brück, Fournier-Facio and Löh constructed a counterexample, see [3, Example 3.7].

However, if the median graph fulfils the following finiteness condition it is known that every median quasimorphism has indeed finite defect.

Definition 4.1.14 (restricted)**.** We say the median graph X is *restricted* if for every $l \in \mathbb{N}$ there exists a constant $c \in \mathbb{N}$ such that for all $x, y \in V$ and $m \in V$ that lies on a geodesic from x to y it is

$$\left| \{t \in [x, y]_{\mathcal{H}}^{(l)} \mid t \text{ contains } m \text{ in the interior}\} \right| \leq c.$$

Theorem 4.1.15. *Let X be a restricted median graph and $s \in X_{\mathcal{H}}^{(l)}$ be an $\mathcal{H}$-segment of X. Then the median quasimorphism f_s has bounded defect $\left|\delta^1 f_s\right|$ and is a quasimorphism of $\Gamma \curvearrowright X$.*

Remark 4.1.16. The definition of restricted median quasimorphism is quite unhandy.

Taking a look at the version of Brück, Fournier-Facio and Löh [3, Proposition 3.10], one notices that they formulate the statement for finite dimensional CAT(0) cube complexes. This can be translated to the viewpoint of median graphs by the condition that there is a uniform bound c such that for two vertices x and y there are at most c edges adjacent to y that belong to a geodesic from x to y, see [3, Lemma 3.3].

Although this condition seems more practical, we cannot use it in this thesis as there is a gap in the arguments of Brück, Fournier-Facio and Löh. In particular, they concluded in [3, Lemma 3.9] that the median graph corresponding to a finite dimensional cube complex is restricted. Unfortunately, the arguments in the proof of this lemma are not correct. They state for a finite-dimensional CAT(0) cube complex that if m lies in the interior of tightly nested halfspaces $h_1 \supset h_2$ then m is part of an edge that is dual to h_1. But this is not true, see Figure 4.2 for a counterexample.

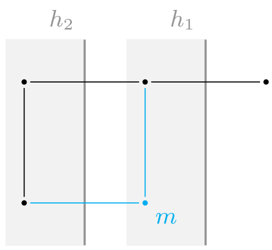

Figure 4.2: The vertex m is not contained in an edge dual to h_1.

It remains as an open problem whether [3, Lemma 3.9] is true and consequently it is unclear if median quasimorphisms of groups acting on finite-dimensional CAT(0) cube complexes have finite defect.

Trees are examples of restricted median graphs as they are uniquely geodesic and $\mathcal{H}$-segments correspond to geodesics. In this case, we define median quasimorphisms associated to geodesics.

Definition 4.1.17 (median quasimorphisms, tree case)**.** Let Γ be a group acting on a tree $T = (V, E)$ and let γ be a geodesic in T. Then the *median quasimorphism f_γ for γ*

is defined by

$$\begin{aligned} f_\gamma \colon V \times V &\to \mathbb{R} \\ (x,y) &\mapsto \big(\text{number of translates of } \gamma \text{ contained in } [x,y]\big) \\ &\qquad - \big(\text{ number of translates of } \gamma \text{ contained in } [y,x]\big). \end{aligned}$$

Note that the median quasimorphism f_γ coincides with the median quasimorphism f_s, where s denotes the $\mathcal{H}$-segment that corresponds to γ.

Remark 4.1.18. Let Γ act on a tree $T = (V, E)$ and let $v, w \in V$. For $l \in \mathbb{N}$ we denote by $[\![v, w]\!]^{(l)}$ the set of subgeodesics of length l of $[v, w]$ and we denote by $\mathcal{E}^{(l)}$ the set of geodesics of length l in T.

Now let γ be a geodesic in T of length $l \in \mathbb{N}$. If $\Gamma\gamma = \Gamma\overline{\gamma}$, then $f_\gamma = 0$. If not, we define

$$\begin{aligned} \epsilon_\gamma \colon \mathcal{E}^{(l)} &\to \{-1, 0, 1\} \\ \beta &\mapsto \begin{cases} 1, & \text{if } \Gamma\beta = \Gamma\gamma, \\ -1, & \text{if } \Gamma\beta = \Gamma\overline{\gamma}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We can compute f_γ by

$$f_\gamma(x,y) = \sum_{\beta \in [\![x,y]\!]^{(l)}} \epsilon_\gamma(\beta)$$

for $x, y \in V$.

The pullbacks of median quasimorphisms of Γ acting on a tree correspond to the median quasimorphism as considered by Monod and Shalom [14]. They will play an important part in the last Chapter 5 as we will develop connections between the triviality of the second bounded cohomology of Γ and the triviality of these quasimorphisms of Γ.

In the following, we present the triviality result for cup products with classes represented by median quasimorphisms that was developed in [3].

Assume that X is a restricted median quasimorphism. At first, we want to point out that the cup product with a coclass induced by a median quasimorphism is not always trivial, as Brück, Fournier-Facio and Löh constructed a counterexample [11, Example 3.13]. Besides the construction of this counterexample, they proved that the cup product is trivial when combining a coclass induced by a median quasimorphism with so called non-transverse coclasses. First, we introduce non-transverse coclasses and then we revise the triviality result.

Definition 4.1.19 (heads, tails)**.** Let $x = (h_1 \supset \ldots \supset h_l) \in X^{(l)}_{\mathcal{H}}$. We say that $\alpha \in V$ is a *head* of s if $\alpha \in \overline{h_1}$ and there exists an edge dual to h_1 that has α as one of its endpoints. We say that $\omega \in V$ is a *tail* of s if $\omega \in h_l$ and there exists an edge dual to h_l that has ω as one of its endpoints.

We let $\alpha(s)$ denote the set of heads of s and we let $\omega(s)$ denote the set of tails of s.

Definition 4.1.20 (non-transverse)**.** Let $s \in X^{(l)}_{\mathcal{H}}$ and $\zeta \in C^n_{\Gamma,b}(X;\mathbb{R})$. We say that ζ and s are *non-transverse* if for all $x_1,\dots,x_n \in V$ the value of $\zeta(\alpha, x_1,\dots,x_n)$ is independent of the choice of head $\alpha \in \alpha(s)$ and the value of $\zeta(\omega, x_1,\dots,x_n)$ is independent of the choice of tail $\omega \in \omega(s)$.
We say that ζ and a set $S \subset X^{(l)}_{\mathcal{H}}$ are *non-transverse* if ζ and s are non-transverse for all $s \in S$.

We define a coclass $\alpha \in H^n_{\Gamma,b}(X;\mathbb{R})$ to be *non-transverse* to Γs if it admits a representative ζ that is non-transverse to Γs.

Remark 4.1.21. Let $s \in X^{(l)}_{\mathcal{H}}$ and $\zeta \in C^n_{\Gamma,b}(X;\mathbb{R})$. By definition, we have $\alpha(\overline{s}) = \omega(s)$ and $\omega(\overline{s}) = \alpha(s)$. Hence, ζ and Γs are non-transverse if and only if ζ and $\Gamma\overline{s}$ are non-transverse.

Theorem 4.1.22 ([3, Theorem 3.17])**.** *Let $s \in X^{(l)}_{\mathcal{H}}$ be an $\mathcal{H}$-segment in X, and let f_s be the corresponding median quasimorphism of $\Gamma \curvearrowright X$. Then, for every $n \in \mathbb{N}$ and coclass $\alpha \in H^n_{\Gamma,b}(X;\mathbb{R})$ that is non-transverse to the orbit Γs, the cup product*

$$[\delta^1 f_s] \cup \alpha \in H^{n+2}_{\Gamma,b}(X;\mathbb{R})$$

is trivial.

One can deduce a triviality result for the cup product of two coclasses induced by median quasimorphisms. For this, we introduce the notion of non-transverse halfspaces.

Definition 4.1.23 ((non)-transverse halfspaces)**.** Two halfspaces h_1 and h_2 are called *transverse* if each of the four intersections

$$\begin{aligned} &h_1 \cap h_2, \quad \overline{h_1} \cap h_2, \\ &h_1 \cap \overline{h_2}, \quad \overline{h_1} \cap \overline{h_2} \end{aligned}$$

is non-empty. We then write $h_1 \pitchfork h_2$. We say that two sets H_1, H_2 of halfspaces are *non-transverse* if h_1 and h_2 are non-transverse for each $h_1 \in H_1$ and each $h_2 \in H_2$.

Corollary 4.1.24 ([3, Theorem 3.19])**.** *Let $s = (h_1 \supset \ldots \supset h_l)$ and $r = (k_1 \supset \ldots \supset k_p)$ be $\mathcal{H}$-segments in X. Suppose that each of the four pairs*

$$\begin{aligned} &\Gamma h_1, \Gamma k_1; \quad \Gamma h_1, \Gamma k_p; \\ &\Gamma h_l, \Gamma k_1; \quad \Gamma h_l, \Gamma k_p; \end{aligned}$$

is non-transverse. Then $\delta^1 f_r$ and Γs are non-transverse. In particular,

$$[\delta^1 f_s] \cup [\delta^1 f_r] = 0.$$

Right-angled Artin groups

Right-angled Artin groups are highly connected with CAT(0) cube complexes and thus with median graphs. This section is meant as a short outlook how we can apply the results for median graphs to right-angled Artin group. For this reason, we only focus on the ideas rather than the precise details. However, the details can be found in [3, Section 5]. Note that we assume here that the underlying median graphs of CAT(0) cube complexes are restricted. We refer to Remark 4.1.16 to see the doubts on this assumption.

Definition 4.1.25 (right-angled Artin group)**.** Let $G = (V, E)$ be a graph. The corresponding *right-angled Artin group* (RAAG) $\Gamma \coloneqq A(G)$ is the group defined by the presentation

$$\langle V \mid \{v^{-1}w^{-1}vw = 1 \mid \{v, w\} \in E\}\rangle.$$

In the following, let G be a graph and $\Gamma = A(G)$ be the corresponding RAAG. There is a cube complex $S(G)$ associated to Γ, called the Salvetti complex. Furthermore, the universal covering $\tilde{S}(G)$ of $S(G)$ is a finite dimensional CAT(0) cube complex on which Γ acts freely and cocompactly with an induced action on the underlying median graph denoted by $X = (V, E)$. For the details, we refer to Charney [5, Section 3.6].

Each edge $e \in E$ has a label $\lambda(e)$ which is a vertex of G. That label is preserved by the Γ-action on X and induces a well-defined label $\lambda(h) = \lambda(e)$ for h a halfspace dual to e.

The following result is known about non-transversality in the case of RAAG's.

Lemma 4.1.26 ([3, Corollary 5.2])**.** *Let $l, n \in \mathbb{N}$ and let $s = (h_1 \supset \ldots \supset h_l) \in X^{(l)}_{\mathcal{H}}$. If $\lambda(h_1)$ and $\lambda(h_l)$ are isolated vertices of G, i.e.* $\deg(\lambda(h_1)) = 0 = \deg(\lambda(h_l))$*, then every $\zeta \in C^n_{\Gamma,b}(X; \mathbb{R})$ is non-transverse to Γs.*

A direct consequence of this lemma and Theorem 4.1.22 is the following triviality result for the cup product.

Corollary 4.1.27. *Let $l, n \in \mathbb{N}$ and let $s = (h_1 \supset \ldots \supset h_l) \in X^{(l)}_{\mathcal{H}}$. If $\lambda(h_1)$ and $\lambda(h_l)$ are isolated vertices of G, i.e.* $\deg(\lambda(h_1)) = \deg(\lambda(h_l)) = 0$*, then for all $\alpha \in C^n_{\Gamma,b}(X; \mathbb{R})$ it is*

$$[\delta^1 f_s] \cup \alpha = 0.$$

We also can adapt the triviality result for two coclasses induced by median quasimorphism to this setup.

Corollary 4.1.28 ([3, Corollary 5.4])**.** *Let $s = (h_1 \supset \ldots \supset h_l)$ and $r = (k_1 \supset \ldots \supset k_p)$ be two $\mathcal{H}$-segments of X. Suppose that each of the four pairs of vertices*

$$\lambda(h_1), \lambda(k_1); \quad \lambda(h_1), \lambda(k_p);$$
$$\lambda(h_l), \lambda(k_1); \quad \lambda(h_l), \lambda(k_p);$$

is not connected by an edge in G*. Then* $[\delta^1 f_s] \cup [\delta^1 f_r] = 0$.

The idea of the proof is to show that r and s fulfil the assumptions of Corollary 4.1.24 from which we obtain the triviality result for two coclasses given by median quasimorphisms.

4.2 Main Theorem on triviality

The goal of this section is to construct certain quasimorphisms of groups acting on graphs and achieve triviality results for the cup product and the Massey triple product such that we obtain a general proof for the results for Brooks, Rolli, Δ-decomposable and median quasimorphisms.

Preliminary definitions

Let Γ be a discrete group acting on a graph $X = (V, E)$. Furthermore, let $l \in \mathbb{N}_{>0}$. We denote by E^{or} the set

$$E^{or} := \left\{ (\alpha, \omega) \in V^2 \mid \{\alpha, \omega\} \in E \right\}.$$

For $e = (\alpha, \omega) \in E^{or}$ an oriented edge, we denote by $\overline{e}$ the corresponding oriented edge of opposite orientation, i.e. $\overline{e} := (\omega, \alpha)$. Moreover, for $a = (e_1, \ldots, e_l) \in (E^{or})^l$, we define $\overline{a} := (\overline{e_l}, \ldots, \overline{e_1})$.

Definition 4.2.1. For $l \in \mathbb{N}$, $x, y \in V$ and $p = x_0, \ldots, x_n$ a path connecting x and y we denote for $i \in \{0, \ldots, n-1\}$ by $e_i = (x_i, x_{i+1})$ the oriented edge given by the vertices x_i and x_{i+1}. We then define

$$p^{(l)} := \left\{ (e_{i_1}, \ldots, e_{i_l}) \in (E^{or})^l \mid 0 \leq i_1 < i_2 < \ldots < i_l \leq n-1 \right\}.$$

In other words, $p^{(l)}$ consists of all l-tuples of oriented edges that appear on p in this given order.

For an element $a = (e_{i_1}, \ldots, e_{i_l}) \in p^{(l)}$ we define

$$\alpha(a) := x_{i_1} \quad \text{and} \quad \omega(a) := x_{i_l+1}.$$

We say a vertex m of the path p *is contained in* a if $\alpha(a) \neq m \neq \omega(a)$ and if the vertices $\alpha(a)$, m, $\omega(a)$ are passed by p in this order.

Definition 4.2.2 (quasi-median property)**.** For $x, y \in V$, let $P(x, y) \neq \emptyset$ be a set of finite paths from x to y. The family $(P(x, y))_{x,y \in V}$ is said to fulfil the *quasi-median property*, if the following holds:

There exist $R \in \mathbb{N}$ such that for all $x, y, z \in V$ there exist

- a triple $(m_x, m_y, m_z) \in V^3$, and
- paths $p_{xy} \in P(x, y)$, $p_{yz} \in P(y, z)$, and $p_{xz} \in P(x, z)$

with

$$p_{xy} = s_x * r_1 * \overline{s_y},$$
$$p_{yz} = s_y * r_2 * \overline{s_z},$$
$$p_{xz} = s_x * r_3 * \overline{s_z}$$

for suitable paths $s_x \in P(x, m_1)$, $s_y \in P(y, m_y)$, $s_z \in P(z, m_z)$ and $r_1 \in P(m_1, m_2)$, $r_2 \in P(m_2, m_3)$, $r_3 \in P(m_1, m_3)$ such that r_1, r_2, and r_3 have length at most R.

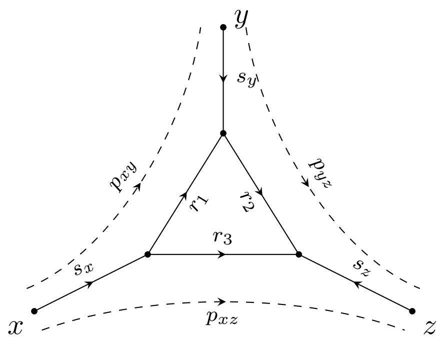

Figure 4.3: Quasi-median property

Example 4.2.3. By definition of median graphs, the sets of geodesics $P(x, y) = [x, y]$ for vertices x, y form a family that fulfils the quasi-median property. In particular, the constant R can be chosen to be 0. This holds true in particular for trees and will be helpful for the triviality results concerning Brooks quasimorphisms and median quasimorphisms.

Example 4.2.4. For Δ-decomposable quasimorphisms we will not consider actions on trees or median graphs. In this case, it is useful to look at the action of a non-abelian free group F on its Cayley graph $\mathrm{Cay}(F, \mathcal{P})$ for $\mathcal{P}$ the pieces associated to the Δ-decomposition. This is a good choice for a generating set of F because the Δ-decomposition for $x^{-1}y$ with $x, y \in F$ gives rise to a path in $\mathrm{Cay}(F, \mathcal{P})$, namely for $\Delta(x^{-1}y) = (g_1, \ldots, g_k)$ we obtain the path

$$x,\ xg_1,\ xg_1g_2,\ \ldots,\ xg_1 \cdots g_k.$$

in $\mathrm{Cay}(F, \mathcal{P})$ from x to y. If we choose $P(x, y)$ to contain only this path, then the family of these sets fulfils the quasi-median property by definition of Δ-decompositions. The constant R can be chosen to be the same as the constant bounding the r-part of the Δ-triangles.

Setup and statement

Let Γ be a discrete group acting on a graph $X = (V, E)$. For $x, y \in V$, let $P(x, y) \neq \emptyset$ be a set of finite paths from x to y such that the family $P = (P(x, y))_{x,y \in V}$ fulfils the quasi-median property and

- is compatible with the Γ-action, i.e. for all $x, y \in V$ and $g \in \Gamma$ it is
$$g \cdot P(x, y) = P(gx, gy),$$
- is compatible with inversion, i.e. for all $x, y \in V$ it is $P(x, y) = \overline{P(y, x)}$,
- is compatible with taking subpaths, i.e. for all $x, y \in V$ and all $p \in P(x, y)$, every subpath p' of p is contained in $P(x', y')$ where x' is the starting point of p' and y' its endpoint.

Furthermore, we fix $l \in \mathbb{N}_{>0}$ and a map $\lambda \colon (E^{or})^l \to \mathbb{R}$ with the following properties:

1. It is Γ-*invariant* with respect to the diagonal action of Γ on $(E^{or})^l$.
2. It is *inversely*, i.e. for all $a \in (E^{or})^l$, we have $\lambda(a) = -\lambda(\overline{a})$.
3. It is *bounded*, i.e. $\|\lambda\|_\infty < \infty$.
4. It admits for all $x, y \in V$ and for all $p, q \in P(x, y)$ a bijection $\varphi \colon p^{(l)} \to q^{(l)}$ with $\lambda_{|p^{(l)}} = \lambda_{|q^{(l)}} \circ \varphi$.
$$\begin{array}{ccc} p^{(l)} & \xrightarrow{\lambda_{|p^{(l)}}} & \mathbb{R} \\ {\scriptstyle\varphi}\downarrow & \nearrow_{\lambda_{|q^{(l)}}} & \\ q^{(l)} & & \end{array}$$
For all vertices $x, y \in V$ and paths $p, q \in P(x, y)$ we fix such a bijection $\varphi_{p,q}$.
5. It fulfils the following finiteness condition: There exists a constant $c \in \mathbb{N}$ such that for all $x, y \in V$, for all $p \in P(x, y)$, and for all vertices m on p it is
$$\left| \left\{ a \in p^{(l)} \mid m \text{ is contained in } a \right\} \cap \operatorname{supp}(\lambda) \right| \leq c.$$

Recalling the vanishing results for cup products with coclasses induced by median quasimorphisms, we see that we need to reduce the second factor of the cup products to a certain family of coclasses. Concretely, we need to adapt the notion of non-transversal cochains from the bounded cohomology of groups acting on median graphs to our setup. The following definition will do the job.

Definition 4.2.5 (well-behaved)**.** A cochain $\zeta \in C^n_{\Gamma,b}(X;\mathbb{R})$ is said to be *well-behaved* if for all vertices $x, y \in V$ and for all paths $p, q \in P(x,y)$ with $a \in p^{(l)} \cap \operatorname{supp}(\lambda)$ it is

$$\begin{aligned}\zeta(\alpha(a), x_1, \ldots, x_n) &= \zeta(\alpha(\varphi_{p,q}(a)), x_1, \ldots, x_n), \text{ and}\\ \zeta(\omega(a), x_1, \ldots, x_n) &= \zeta(\omega(\varphi_{p,q}(a)), x_1, \ldots, x_n)\end{aligned}$$

for any choice of $x_1, \ldots, x_n \in V$.

A coclass $\alpha \in H^n_{\Gamma,b}(X;\mathbb{R})$ is said to be *well-behaved* if it admits a well-behaved representative.

Main Theorem 4.2.6. *Having this setup for X, P, and λ the following holds:*

1. *The map*

$$\begin{aligned} f\colon V \times V &\to \mathbb{R}\\ (x,y) &\mapsto \sum_{a \in p^{(l)}} \lambda(a),\end{aligned}$$

   with $p \in P(x,y)$, is a well-defined, antisymmetric quasimorphism of $\Gamma \curvearrowright X$.

2. *Let $\alpha \in H^n_{\Gamma,b}(X;\mathbb{R})$ be well-behaved. Then the cup products $[\delta^1 f] \cup \alpha$ and $\alpha \cup [\delta^1 f]$ are trivial in $H^{n+2}_{\Gamma,b}(X;\mathbb{R})$.*

3. *Let $\alpha_1 \in H^n_{\Gamma,b}(X;\mathbb{R})$ and $\alpha_2 \in H^m_{\Gamma,b}(X;\mathbb{R})$ both be well-behaved. Then, the Massey triple product $\langle \alpha_1, [\delta^1 f], \alpha_2\rangle$ is trivial in $H^{n+m+1}_{\Gamma,b}(X;\mathbb{R})$.*

4.3 Application

In this chapter, we observe how the Main Theorem 4.2.6 can be adapted to the four examples of quasimorphisms we gave in Section 4.1. We distinguish two cases, one for Rolli and Δ-decomposable quasimorphisms and one for Brooks and median quasimorphisms.

4.3.1 Δ-decomposable and Rolli quasimorphisms

We obtain the following result for Rolli-quasimorphisms.

Corollary 4.3.1. *Let F be a non-abelian free group and $\phi\colon F \to \mathbb{R}$ be a Rolli quasimorphism. Then ϕ is a quasimorphism and for all $n, m \in \mathbb{N}$ and $\alpha_1 \in H^n_b(F;R)$ and $\alpha_2 \in H^m_b(F;R)$,*

- *the cup products $[\delta^1 \widehat{\phi}] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1 \widehat{\phi}]$ are trivial in $H^{n+2}_b(F;\mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1 \widehat{\phi}], \alpha_2\rangle \subset H^{n+m+1}_b(F;\mathbb{R})$ is trivial.*

As every Rolli quasimorphism is Δ-decomposable (Example 4.1.10), this corollary is a special case of the following.

Corollary 4.3.2. *Let F be a non-abelian free group and $\phi\colon F \to \mathbb{R}$ be a Δ-decomposable quasimorphism Then ϕ is a quasimorphism and for all $n, m \in \mathbb{N}$ and $\alpha_1 \in H^n_b(F;R)$ and $\alpha_2 \in H^m_b(F;R)$,*

- *the cup products $[\delta^1\widehat{\phi}] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1\widehat{\phi}]$ are trivial in $H^{n+2}_b(F;\mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1\widehat{\phi}], \alpha_2\rangle \subset H^{n+m+1}_b(F;\mathbb{R})$ is trivial.*

In order to apply Main Theorem 4.2.6 to Δ-decomposable quasimorphisms, we need the following lemma.

Lemma 4.3.3. *Let F be a non-abelian free group, $\mathcal{P} \subset F$ a symmetric subset not containing the neutral element and $\Delta\colon F \to \mathcal{P}^*$ a Δ-decomposition. Let $\lambda \in \ell^\infty_{alt}(\mathcal{P},\mathbb{R})$. The map*

$$\begin{aligned} f_{\lambda,\Delta}\colon F \times F &\to \mathbb{R} \\ (x,y) &\mapsto \phi_{\lambda,\Delta}(x^{-1}y) \end{aligned}$$

is a symmetric quasimorphism of $F \curvearrowright \operatorname{Cay}(F,\mathcal{P})$ and for all integers $n, m \in \mathbb{N}$ and coclasses $\alpha_1 \in H^n_{F,b}(\operatorname{Cay}(F,\mathcal{P});\mathbb{R})$ and $\alpha_2 \in H^m_{F,b}(\operatorname{Cay}(F,\mathcal{P});\mathbb{R})$,

- *the cup products $[\delta^1 f_{\lambda,\Delta}] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1 f_{\lambda,\Delta}]$ are trivial in $H^{n+2}_{F,b}(\operatorname{Cay}(F,\mathcal{P});\mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1 f_{\lambda,\Delta}], \alpha_2\rangle \subset H^{n+m+1}_{F,b}(\operatorname{Cay}(F,\mathcal{P});\mathbb{R})$ is trivial.*

Proof. At first, we consider in $\operatorname{Cay}(F,\mathcal{P})$ the family $P = (P(x,y))_{x,y\in F}$ of sets of paths, where for $x, y \in F$ the set $P(x,y)$ contains only the path from x to y determined by the Δ-decomposition of $x^{-1}y$. By Example 4.2.4 the family P fulfils the quasi-median property and by the definition of a Δ-decomposition it is clear that P is compatible with inversion and with taking subpaths. Furthermore, the family P is compatible with the F-action, since we have for $x, y \in F$ the equality $x^{-1}y = (gx)^{-1}(gy)$ and therefore, $\Delta(gx^{-1}gy) = \Delta(x^{-1}y)$.

The next step of the proof is to choose $l \in \mathbb{N}$ and a map $\lambda'\colon (E^{or})^l \to \mathbb{R}$ such that $\operatorname{Cay}(F,\mathcal{P})$, P and λ' are as in the setup of Subsection 4.2.

For this, let $l = 1$ and consider

$$\begin{aligned} \lambda'\colon E^{or} &\to \mathbb{R} \\ (x,y) &\mapsto \lambda(x^{-1}y). \end{aligned}$$

This map is well-defined as for $(x,y) \in E^{or}$ we have $x^{-1}y \in \mathcal{P}$ and thus, the term $\lambda(x^{-1}y)$ is defined. Furthermore, the map is F-invariant, inversely and bounded, since λ is alternating and bounded.

Since for $x, y \in F$ the set $P(x,y)$ consists only of one element, we choose for the unique path $p \in P(x,y)$ the bijection $\varphi_{p,p} = \mathrm{id}_{|p^{(1)}}$ and have $\lambda'_{|p^{(1)}} = \lambda'_{|p^{(1)}} \circ \mathrm{id}_{|p^{(1)}}$.

It remains to prove the finiteness condition, i.e. we need to show there exists a constant $c \in \mathbb{N}$ such that for all $x, y \in V$, for all $p \in P(x, y)$ and for all vertices m of p

$$\left|\left\{a \in p^{(1)} \mid m \text{ is contained in } a\right\} \cap \operatorname{supp}(\lambda)\right| \leq c.$$

For this, we choose $c = 0$. Then, for $x, y \in V$ and the unique map $p \in P(x, y)$ every element of $p^{(1)}$ is just an oriented edge of p which has no interior. Therefore, we count for all vertices m of p

$$\left|\left\{a \in p^{(1)} \mid m \text{ is contained in } a\right\} \cap \operatorname{supp}(\lambda)\right| = 0.$$

For $x, y \in F$ with Δ-decomposition $\Delta(x^{-1}y) = (g_1, \ldots, g_k)$ we compute

$$f_{\lambda,\Delta}(x, y) = \phi_{\lambda,\Delta}\left(x^{-1}y\right) = \sum_{i=0}^{k} \lambda(g_i) = \sum_{a \in p^{(1)}} \lambda'(a)$$

for p the unique path in $P(x, y)$. As we are in the setup of Subsection 4.2, we conclude with Main Theorem 4.2.6 that $f_{\lambda,\Delta}$ is a symmetric quasimorphism.

In order to obtain the statements for the cup product and the Massey triple product, we first observe that in our case every coclass $\zeta \in H^n_{\Gamma,b}(X; \mathbb{R})$ with $n \in \mathbb{N}$ is well-behaved since for all $x, y \in F$ the set $P(x, y)$ consists only of one path. Now, Main Theorem 4.2.6 does the job. □

Eventually, we are able to prove the vanishing results for Δ-decomposable quasimorphisms.

Proof of Corollary 4.3.2. Let $\mathcal{P} \subset F$ be a symmetric subset, $\Delta\colon F \to \mathcal{P}^*$ a decomposition and $\lambda \in \ell^\infty_{alt}(\mathcal{P}, \mathbb{R})$. We want to prove the corollary for the Δ-decomposable quasimorphism $\phi_{\lambda,\Delta}$. At first, we observe for $x \in F$

$$\phi_{\lambda,\Delta}(x) = f_{\lambda,\Delta}(e, x) = f_{\lambda,\Delta,e}(x).$$

Using Theorem 3.2.2 and Lemma 4.3.3, we deduce that $\phi_{\lambda,\Delta} = f_{\lambda,\Delta,e}$ is a quasimorphism as it is the pullback of the quasimorphism $f_{\lambda,\Delta}$ under the orbit map o_e.

Furthermore, since the orbit map $o_e^2\colon H^2_{F,b}(\operatorname{Cay}(F, \mathcal{P}); \mathbb{R}) \to H^2_b(F; \mathbb{R})$ is an isomorphism by Corollary 2.2.2 with

$$o_e^2\left(\left[\delta^1 f_{\lambda,\Delta,e}\right]\right) = \left[\delta^1 \widehat{\phi_{\lambda,\Delta}}\right]$$

and the cup product and the Massey triple product are compatible with o_e^* (Remark 2.2.4 and Theorem 2.2.6) we obtain the vanishing results for the cup product and the Massey triple product using the triviality results we proved in Lemma 4.3.3. □

4.3.2 Median quasimorphisms and Brooks quasimorphisms

The following results for median quasimorphisms are consequences of Main Theorem 4.2.6.

Corollary 4.3.4. *Let Γ be a discrete group acting on a restricted median graph X. Let s be an $\mathcal{H}$-segment in X, and let f_s be the corresponding median quasimorphism of $\Gamma \curvearrowright X$. Then f_s is a symmetric quasimorphism and for all $n, m \in \mathbb{N}$ and coclasses $\alpha_1 \in H^n_{\Gamma,b}(X;R)$ and $\alpha_2 \in H^m_{\Gamma,b}(X;R)$ that are non-transverse to the orbit Γs,*

- *the cup products $[\delta^1 f_s] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1 f_s]$ are trivial in $H^{n+2}_{\Gamma,b}(X;R)$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1 f_s], \alpha_2\rangle \subset H^{n+m+1}_{\Gamma,b}(X;R)$ is trivial.*

The following two statements can be deduced directly from the results for median quasimorphisms, the first of them generalizing Corollary 4.1.24 about cup products of two coclasses induced by median quasimorphisms. Afterwards we prove Corollary 4.3.4 about median quasimorphisms.

Corollary 4.3.5. *Let Γ be a discrete group acting on a restricted median graph X. Let $s = (h_1 \supset \ldots \supset h_l)$, $r = (k_1 \supset \ldots \supset k_p)$ and $t = (b_1 \supset \ldots \supset b_q)$ be $\mathcal{H}$-segments in X. Suppose each of the four pairs*

$$\begin{aligned} &\Gamma h_1, \Gamma k_1; \quad \Gamma h_1, \Gamma k_p; \\ &\Gamma h_l, \Gamma k_1; \quad \Gamma h_l, \Gamma k_p; \end{aligned}$$

and each of the four pairs

$$\begin{aligned} &\Gamma h_1, \Gamma b_1; \quad \Gamma h_1, \Gamma b_q; \\ &\Gamma h_l, \Gamma b_1; \quad \Gamma h_l, \Gamma b_q; \end{aligned}$$

is non-transverse. Then,

- *the cup product $[\delta^1 f_r] \cup [\delta^1 f_s] \in H^4_{\Gamma,b}(X;\mathbb{R})$ is trivial, and*
- *the Massey triple product $\langle [\delta^1 f_r], [\delta^1 f_s], [\delta^1 f_t]\rangle \subset H^5_{\Gamma,b}(X;\mathbb{R})$ is trivial.*

Proof. Brück, Fournier-Facio and Löh [3, Theorem 3.19] proved that in this case f_r and Γs as well as f_t and Γs are non-transverse. We conclude with Corollary 4.3.4. □

Corollary 4.3.6. *Let Γ be a discrete group acting on a tree T and let γ be a geodesic in T. Then, for all $n, m \in \mathbb{N}$ and $\alpha_1 \in H^n_{\Gamma,b}(T;\mathbb{R})$ and $\alpha_2 \in H^m_{\Gamma,b}(T;\mathbb{R})$,*

- *the cup products $[\delta^1 f_\gamma] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1 f_\gamma]$ are trivial in $H^{n+2}_{\Gamma,b}(T;\mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1 f_\gamma], \alpha_2\rangle \subset H^{n+m+1}_{\Gamma,b}(T;\mathbb{R})$ is trivial.*

Proof. This follows from the fact that for all $n \geq 1$ and $\zeta \in C^n_{\Gamma,b}(T;\mathbb{R})$ and for all $\mathcal{H}$-segments s of T we have that ζ non-transverse to Γs [3, Section 4]. Hence, we may apply Corollary 4.3.4. □

Furthermore, we can deduce triviality of cup products and Massey triple products with coclasses induced by Brooks quasimorphisms.

Corollary 4.3.7. *Let F be a non-abelian free group and $\phi\colon F \to \mathbb{R}$ be a Brooks quasimorphism. Then ϕ is a quasimorphism and for all $n, m \in \mathbb{N}$ and $\alpha_1 \in H_b^n(F;\mathbb{R})$ and $\alpha_2 \in H_b^m(F;\mathbb{R})$,*

- *the cup products $[\delta^1\widehat{\phi}] \cup \alpha_1$ and $\alpha_1 \cup [\delta^1\widehat{\phi}]$ are trivial in $H_b^{n+2}(F;\mathbb{R})$, and*
- *the Massey triple product $\langle \alpha_1, [\delta^1\widehat{\phi}], \alpha_2\rangle \subset H_b^{n+m-1}(F;\mathbb{R})$ is trivial.*

Proof. Let $S \subset F$ be a free generating set and $w = x_1 \cdots x_l \in F$ a non-trivial word. Then, F acts on the tree $T = \mathrm{Cay}(F, S)$ by left-translation. The word w gives rise to a geodesic

$$\gamma = e,\ x_1,\ x_1x_2,\ \ldots,\ x_1\cdots x_l$$

in $\mathrm{Cay}(F,S)$. We consider the quasimorphism $f_{\gamma,e}$, the pullback of the median quasimorphism f_γ for γ under the orbit map o_e, and compute for $g = g_1 \cdots g_n \in F$

$$f_{\gamma,e}(g) = \sum_{\beta \in [\![e,g]\!]^{(l)}} \epsilon_\gamma(\beta) = \sum_{j=1}^{n-l+1} \chi_w(g_j \cdots g_{j+l-1}) = \phi_w(g).$$

Hence, we have $f_{\gamma,e} = \phi_w$. As f_γ is a quasimorphism of $F \curvearrowright \mathrm{Cay}(F,S)$ by Corollary 4.3.6 and as the pullback preserves quasimorphisms (Theorem 3.2.2) we deduce that ϕ_w is a quasimorphism of F.

By Corollary 4.3.6, we know that for all $\alpha_1 \in H^n_{\Gamma,b}(F;\mathbb{R})$ and $\alpha_2 \in H^m_{\Gamma,b}(F;\mathbb{R})$ the cup products

$$[\delta^1 f_\gamma] \cup \alpha_1, \quad \text{and} \quad \alpha_1 \cup [\delta^1 f_\gamma]$$

as well as the Massey triple product

$$\langle \alpha_1, [\delta^1 f_\gamma], \alpha_2 \rangle$$

are trivial. Since the orbit map $o_e^2\colon H^2_{F,b}(\mathrm{Cay}(F,\mathcal{P});\mathbb{R}) \to H^2_b(F;\mathbb{R})$ is an isomorphism by Corollary 2.2.2 with

$$o_e^2([\delta^1 f_{\gamma,e}]) = [\delta^1 \widehat{\phi_w}]$$

and the cup product and the Massey triple product are compatible with o_e^* by Remark 2.2.4 and Theorem 2.2.6, we obtain the vanishing results for the cup product and the Massey triple product. □

Finally, we prove Corollary 4.3.4 about median quasimorphisms of groups acting on restricted median graphs.

Proof of Corollary 4.3.4. Let $X = (V, E)$ be the restricted median graph of the assumptions of Corollary 4.3.4. We fix the following notation: For an oriented edge $e = (\alpha(e), \omega(e)) \in E^{or}$ we denote by h_e the halfspace defined by the (undirected) edge $\{\alpha(e), \omega(e)\}$ containing $\omega(e)$.

For $x, y \in V$ we define $P(x, y)$ to be the set of all geodesics from x to y. Since X is a median graph, the family $P = (P(x, y))_{x,y\in V}$ fulfils the quasi-median property. As Γ-translates, inverses and sub-paths of geodesics are again geodesics, it is clear that P is compatible with the Γ-action, inversion and taking sub-paths.

Let $s \in X^{(l)}_{\mathcal{H}}$ be an $\mathcal{H}$-segment. If $\Gamma s = \Gamma \overline{s}$, then $f_s = 0$ is a quasimorphism and the triviality results follow easily. On the other hand, if $\Gamma s \neq \Gamma \overline{s}$, we define

$$\lambda\colon (E^{or})^l \to \mathbb{R}$$
$$(e_1, \ldots, e_l) \mapsto \begin{cases} 1, & \text{if } \Gamma(h_{e_1}, \ldots, h_{e_l}) = \Gamma s, \\ -1, & \text{if } \Gamma(h_{e_1}, \ldots, h_{e_l}) = \Gamma \overline{s}, \\ 0, & \text{otherwise.} \end{cases}$$

Note: Since Γ acts transitive on $X^{(l)}_{\mathcal{H}}$ we know

$$\operatorname{supp}(\lambda) \subset \left\{(e_1, \ldots, e_l) \in (E^{or})^l \mid (h_{e_1} \supset \ldots \supset h_{e_l}) \in X^{(l)}_{\mathcal{H}}\right\} =: (E^{or})^l_{\mathcal{H}}.$$

We quickly check

- λ is Γ-invariant, since for $g \in \Gamma$ and $(e_1, \ldots, e_l) \in (E^{or})^l$ we have

$$\Gamma \cdot (h_{ge_1}, \ldots, h_{ge_l}) = \Gamma \cdot (gh_{e_1}, \ldots, gh_{e_l}) = \Gamma \cdot (h_{e_1}, \ldots, h_{e_l}),$$

- λ is inversely by definition,
- $\|\lambda\|_\infty \leq 1$ is finite.

In order to be in the setup of Section 4.2, we need to verify that λ admits for all $x, y \in V$ and for all $p, q \in P(x, y)$ a bijection $\varphi\colon p^{(l)} \to q^{(l)}$ with $\lambda_{|p^{(l)}} = \lambda_{|q^{(l)}} \circ \varphi$. This technical part of the proof uses the connection between $(E^{or})^l_{\mathcal{H}}$ and $X^{(l)}_{\mathcal{H}}$.

Recall we denoted by $(E^{or})^l_{\mathcal{H}}$ the set

$$(E^{or})^l_{\mathcal{H}} = \left\{(e_1, \ldots, e_l) \in (E^{or})^l \mid (h_{e_1} \supset \ldots \supset h_{e_l}) \in X^{(l)}_{\mathcal{H}}\right\}.$$

For $x, y \in V$ and $p \in P(x, y)$ a geodesic from x to y we consider the map

$$\Psi_p\colon p^{(l)} \cap (E^{or})^l_{\mathcal{H}} \to [x, y]^{(l)}_{\mathcal{H}}$$
$$(e_1, \ldots, e_l) \mapsto (h_{e_1} \supset \ldots \supset h_{e_l}).$$

It is not instantly clear that this map is well-defined. More precisely, we need to check that the image indeed lies in $[x, y]^{(l)}_{\mathcal{H}}$. At first, we have $\operatorname{im}(\Psi_p) \subset X^{(l)}_{\mathcal{H}}$ by the definition of $(E^{or})^l_{\mathcal{H}}$. On the other hand, for $(e_1,\dots,e_l)\in p^{(l)}$ we obtain for $i\in\{1,\dots,l\}$ and $e_i=(\alpha_i,\omega_i)$ the equalities

$$\begin{aligned} d(x,\omega_i) &= d(x,\alpha_i)+1, \text{ and}\\ d(y,\omega_i) &= d(y,\alpha_i)-1. \end{aligned}$$

Hence, h_{e_i} separates y from x, which yields $\Psi_p(e_1,\dots,e_l)\in[x,y]^{(l)}_{\mathcal{H}}$. On top of that, Ψ_p is injective since the geodesic p from x to y crosses a halfspace of $[x,y]^{(l)}_{\mathcal{H}}$ exactly once, see Theorem 1.1.17. We can also show that Ψ_p is surjective. For this, choose an arbitrary element $(h_1,\dots,h_l)\in[x,y]^{(l)}_{\mathcal{H}}$. We know for $i\in\{1,\dots,l\}$ that p crosses h_i exactly once. Hence, we find $e_i\in p^{(1)}$ with $h_{e_i}=h_i$. The problem now is that it is not clear that $(e_1,\dots,e_l)\in p^{(l)}$ because we do not know yet that the edges occur in this order in p. To show this, let $i\in\{1,\dots,l-1\}$ and $e_i=(\alpha_i,\omega_i)$, $e_{i+1}=(\alpha_{i+1},\omega_{i+1})$. We have

$$d(\omega_{i+1},\alpha_i)=d(\omega_{i+1},\omega_i)\pm 1.$$

If $d(\omega_{i+1},\alpha_i)=d(\omega_{i+1},\omega_i)+1$, then $\omega_{i+1}\in h_i$. Since h_{i+1} is convex and contains y this means e_i and e_{i+1} appear on the geodesic p in this order and we are done. Conversely, if $d(\omega_{i+1},\alpha_i)=d(\omega_{i+1},\omega_i)-1$, then $\omega_i\in\overline{h_i}$. This contradicts $h_{i+1}\subset h_i$ and is not possible. We conclude that $e_1,\dots,e_l$ appear in p in this order and Ψ_p is surjective.

For $p,q\in P(x,y)$ we obtain now a bijection

$$\Psi_q^{-1}\circ\Psi_p\colon p^{(l)}\cap(E^{or})^l_{\mathcal{H}}\to q^{(l)}\cap(E^{or})^l_{\mathcal{H}}.$$

Since $\left|p^{(l)}\right|=\left|q^{(l)}\right|<\infty$ as p and q geodesics from x to y of the same (finite) length, we can expand $\Psi_q^{-1}\circ\Psi_p$ to a bijection

$$\varphi_{p,q}\colon p^{(l)}\to q^{(l)}.$$

We need to check $\lambda_{|p^{(l)}}=\lambda|\, q^{(l)}\circ\varphi_{p,q}$. For this, let $a\in p^{(l)}$. If $a\notin\operatorname{supp}(\lambda)$ we know by the construction of $\varphi_{p,q}$ that $\lambda(a)=0=\lambda(\varphi_{p,q}(a))$. If $a\in\operatorname{supp}(\lambda)\subset p^{(l)}\cap(E^{or})^l_{\mathcal{H}}$ then

$$\begin{aligned} \lambda(a) &= \epsilon_s(\Psi_p(a))=\epsilon_s((\Psi_q\circ\Psi_q^{-1})\circ\Psi_p(a))\\ &= \epsilon_s(\Psi_q\circ\varphi_{p,q}(a))=\lambda(\varphi_{p,q}(a)). \end{aligned}$$

The last property of λ that we need to check in order to obtain the setup of Section 4.2 is the finiteness condition. We need to show the existence of a constant $c\in\mathbb{N}$ such that for all $x,y\in V$, for all $p\in P(x,y)$ and for all vertices m in p we have

$$\left|\left\{a\in p^{(l)}\mid m\text{ is contained in }a\right\}\cap\operatorname{supp}(\lambda)\right|\le c.$$

Here, we need the assumption that X is restricted. Note for $x,y\in V$, $p\in P(x,y)$ and vertices m on p it is m contained in some $a\in p^{(l)}\cap(E^{or})^l_{\mathcal{H}}$ if and only if m lies in the interior of $\Psi_p(a)$. Hence, λ fulfils the finiteness condition if there exists a constant $c \in \mathbb{N}$ such that for all $x, y \in V$, for all $p \in P(x,y)$ and for all vertices m of p we have

$$\left|\left\{t \in [x,y]^{(l)}_{\mathcal{H}} \mid m \text{ lies in the interior of } t\right\}\right| \leq c.$$

Such a constant c exists by the definition of restricted median graphs.

Since we are now in the setup of Main Theorem 4.2.6, we conclude that

$$\begin{aligned} f_s \colon V^2 &\to \mathbb{R} \\ (x,y) &\mapsto \sum_{t \in [x,y]^{(l)}_{\mathcal{H}}} \epsilon_s(t) = \sum_{a \in p^{(l)}} \lambda(a), \end{aligned}$$

with $p \in P(x,y)$, is a symmetric quasimorphism.

To conclude the triviality results for the cup product and Massey triple product with the median quasimorphisms f_s it suffices to show that every coclass $\alpha \in H^n_{\Gamma,b}(X;\mathbb{R})$ that is non-transverse to the orbit Γs admits a representative $\zeta \in C^n_{\Gamma,b}(X;\mathbb{R})$ that is well-behaved.

For this, let $\alpha \in H^n_{\Gamma,b}(X;\mathbb{R})$ be non-transverse to the orbit Γs. Then, it admits a representative $\zeta \in C^n_{\Gamma,b}(X;\mathbb{R})$ that is non-transverse to Γs. Let $x, y \in V$, $p, q \in P(x,y)$ and $x_1, \ldots, x_n \in V$. First, we observe that for $a \in p^{(l)} \cap \operatorname{supp}(\lambda)$ and $c \coloneqq \varphi_{p,q}(a)$ the segments $\Psi_p(a)$ and $\Psi_q(c)$ coincide and hence

- $\alpha(a)$ and $\alpha(\varphi_{p,q}(a))$ are both heads of the $\mathcal{H}$-segment $\Psi_p(a)$,
- $\omega(a)$ and $\omega(\varphi_{p,q}(a))$ are both tails of the $\mathcal{H}$-segment $\Psi_p(a)$.

Furthermore, since $a \in \operatorname{supp}(\lambda)$ we have $(h_{e_1} \supset \ldots \supset h_{e_l}) \in \Gamma s \cup \Gamma \overline{s}$ and conclude

$$\zeta(\alpha(a), x_1, \ldots, x_n) = \zeta(\alpha(\varphi_{p,q}(a)), x_1, \ldots, x_n)$$

using that ζ is non-transverse to Γs. This shows that ζ is well-behaved and thus, the triviality results follow from Main Theorem 4.2.6. ☐

Right-angled Artin groups

These generalized results for median quasimorphisms are adaptable to right-angled Artin groups as we will see in the following.

In particular, we obtain two results generalizing Corollaries 4.1.27 and 4.1.28. Let G be a graph and $\Gamma \coloneqq A(G)$ the corresponding RAAG. We denote by X the median graph that underlies the universal covering of the Salvetti complex of Γ and consider the induced action of Γ on X.

Corollary 4.3.8. *Let $l \in \mathbb{N}$ and let $s = (h_1 \supset \ldots \supset h_l) \in X^{(l)}_{\mathcal{H}}$. If $\lambda(h_1)$ and $\lambda(h_l)$ are isolated vertices of G, i.e.* $\deg(\lambda(h_1)) = 0 = \deg(\lambda(h_l))$, *then for $n, m \in \mathbb{N}$ and $\alpha_1 \in H^n_{\Gamma,b}(X;\mathbb{R})$ and $\alpha_2 \in H^m_{\Gamma,b}(X;\mathbb{R})$,*

- *the cup products* $[\delta^1 f_s] \cup \alpha_1$ *and* $\alpha_1 \cup [\delta^1 f_s]$ *are trivial in* $H^{n+2}_{\Gamma,b}(X;\mathbb{R})$*, and*
- *the Massey triple product* $\langle \alpha_1, [\delta^1 f_s], \alpha_2 \rangle \subset H^{n+m+1}_{\Gamma,b}(X;\mathbb{R})$ *is trivial.*

Proof. By Lemma 4.1.26 we know that α_1 and Γs as well as α_2 and Γs are non-transverse. We conclude with Corollary 4.3.4. □

Corollary 4.3.9. *Let* $s = (h_1 \supset \ldots \supset h_l)$*,* $r = (k_1 \supset \ldots \supset k_p)$ *and* $t = (b_1 \supset \ldots \supset b_q)$ *be* $\mathcal{H}$*-spaces in* X*. Suppose each of the four pairs of vertices*

$$\lambda(h_1), \lambda(k_1); \quad \lambda(h_1), \lambda(k_p);$$
$$\lambda(h_l), \lambda(k_1); \quad \lambda(h_l), \lambda(k_p);$$

and each of the four pairs of vertices

$$\lambda(h_1), \lambda(b_1); \quad \lambda(h_1), \lambda(b_q);$$
$$\lambda(h_l), \lambda(b_1); \quad \lambda(h_l), \lambda(b_q);$$

is not connected by an edge in G*. Then,*

- *the cup product* $[\delta^1 f_r] \cup [\delta^1 f_s] \in H^4_{\Gamma,b}(X;\mathbb{R})$ *is trivial, and*
- *the Massey triple product* $\langle [\delta^1 f_r], [\delta^1 f_s], [\delta^1 f_t] \rangle \subset H^5_{\Gamma,b}(X;\mathbb{R})$ *is trivial.*

Proof. The $\mathcal{H}$-segments fulfil the assumptions of Corollary 4.3.5, as one can see in [3, Corollary 5.4]. We then conclude the triviality results. □

4.4 Proof of the Main Theorem

In order to prove boundedness of maps in all three parts of Main Theorem 4.2.6, we will make use of the following lemma several times.

Lemma 4.4.1. *Assume we are in the Setup of Main Theorem 4.2.6 and let* $\tau\colon (E^{or})^l \to \mathbb{R}$ *be a map such that*

- $\tau(a) = \tau(\overline{a})$ *for all* $a \in (E^{or})^l$*, and*
- *for all vertices* $x, y \in V$ *and all paths* $p, q \in P(x,y)$ *it is* $\tau(a) = \tau(\varphi_{p,q}(a))$ *for* $a \in p^{(l)}$*.*

Then, for all $x, y, z \in V$ *and for all* $p_1 \in P(x,y)$*,* $p_2 \in P(y,z)$ *and* $p_3 \in P(x,z)$ *we have*

$$\left| \sum_{a \in p_1^{(l)}} \lambda(a)\tau(a) + \sum_{a \in p_2^{(l)}} \lambda(a)\tau(a) - \sum_{a \in p_3^{(l)}} \lambda(a)\tau(a) \right| \leq 3 \cdot (R+1) c \|\lambda\|_\infty \|\tau\|_\infty,$$

where R *is the constant given by the quasi-median property of* $(P(x,y))_{x,y\in V}$ *and* c *is the constant given by the finiteness condition (Property 5) of* λ*.*

Proof. Let $x, y, z \in V$. Furthermore, choose p_{xy}, p_{yz}, p_{xz} and $s_x, s_y, s_z, r_1, r_2, r_3$ as in Definition 4.2.2 of the quasi-median property, i.e.

$$\begin{aligned} p_{xy} &= s_x * r_1 * \overline{s_y}, \\ p_{yz} &= s_y * r_2 * \overline{s_z}, \\ p_{xz} &= s_x * r_3 * \overline{s_z}, \end{aligned}$$

and the length of r_1, r_2, and r_3 is bounded by R.

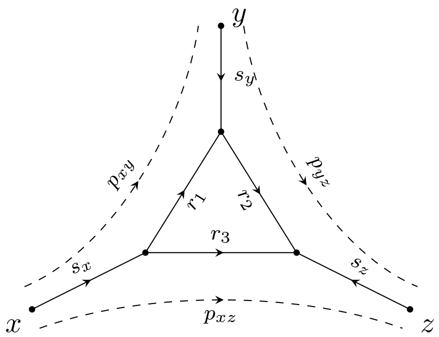

Figure 4.4: Quasi-median property

The second property of τ implies $\sum_{a \in p_1^{(l)}} \lambda(a)\tau(a) = \sum_{a \in p_{xy}^{(l)}} \lambda(a)\tau(a)$ and similar results for the other two summands. This allows us to use the partition of the paths p_{xy}, p_{yz}, and p_{xz} that is given by the quasi-median property.

In particular, both $s_x^{(l)}$ and $\overline{s_y}^{(l)}$ are distinct subsets of $p_{xy}^{(l)}$ which implies

$$\begin{aligned} \sum_{a \in p_1^{(l)}} \lambda(a)\tau(a) &= \sum_{a \in p_{xy}^{(l)}} \lambda(a)\tau(a) \\ &= \sum_{a \in s_x^{(l)}} \lambda(a)\tau(a) + \sum_{a \in \overline{s_y}^{(l)}} \lambda(a)\tau(a) + \sum_{a \in p_{xy}^{(l)} \setminus (s_x^{(l)} \cup \overline{s_y}^{(l)})} \lambda(a)\tau(a) \\ &= \sum_{a \in s_x^{(l)}} \lambda(a)\tau(a) - \sum_{a \in s_y^{(l)}} \lambda(a)\tau(a) + \sum_{a \in p_{xy}^{(l)} \setminus (s_x^{(l)} \cup \overline{s_y}^{(l)})} \lambda(a)\tau(a). \end{aligned}$$

We obtain similar equations for p_2 and p_3, namely

$$\sum_{a \in p_2^{(l)}} \lambda(a)\tau(a) = \sum_{a \in s_y^{(l)}} \lambda(a)\tau(a) - \sum_{a \in s_z^{(l)}} \lambda(a)\tau(a) + \sum_{a \in p_{yz}^{(l)} \setminus (s_y^{(l)} \cup \overline{s_z}^{(l)})} \lambda(a)\tau(a),$$

and

$$\sum_{a\in p_3^{(l)}} \lambda(a)\tau(a) = \sum_{a\in s_x^{(l)}} \lambda(a)\tau(a) - \sum_{a\in s_z^{(l)}} \lambda(a)\tau(a) + \sum_{a\in p_{xy}^{(l)}\setminus(s_x^{(l)}\cup\overline{s_z}^{(l)})} \lambda(a)\tau(a).$$

Representative for all three cases, we take a closer look at the set $p_{xy}^{(l)} \setminus (s_x^{(l)} \cup \overline{s_y}^{(l)})$. We consider two cases. The first case is $l = 1$. Then, the set $p_{xy}^{(1)} \setminus (s_x^{(1)} \cup \overline{s_y}^{(1)})$ coincides with $r_1^{(1)}$, the set of edges of r_1. Hence, it consists of less than R elements. This constant is smaller than $(R+1)c$ and the statement is true in this case. On the other side, we consider $l \geq 2$. Then,

$$\begin{aligned} p_{xy}^{(l)} \setminus (s_x^{(l)} \cup \overline{s_y}^{(l)}) &= \left\{ a \in p_{xy}^{(l)} \mid \exists m \in r_1 \cap V \text{ such that } m \text{ is contained in } a \right\} \\ &= \bigcup_{m\in r_1\cap V} \left\{ a \in p_{xy}^{(l)} \mid m \text{ is contained in } a \right\}, \end{aligned}$$

which means that

$$\left| \left(p_{xy}^{(l)} \setminus (s_x^{(l)} \cup \overline{s_y}^{(l)}) \right) \cap \operatorname{supp}(\lambda) \right| \leq |r_1 \cap V| \cdot c \leq (R+1) \cdot c \tag{4.1}$$

as r_1 contains a maximum of $(R+1)$ vertices. We obtain the same inequalities by changing the roles of x, y, and z. This allows us to compute

$$\begin{aligned} &\left| \sum_{a\in p_1^{(l)}} \lambda(a)\tau(a) + \sum_{a\in p_2^{(l)}} \lambda(a)\tau(a) - \sum_{a\in p_3^{(l)}} \lambda(a)\tau(a) \right| \\ &= \left| \sum_{a\in p_{xy}^{(l)}\setminus(s_x^{(l)}\cup\overline{s_y}^{(l)})} \lambda(a)\tau(a) + \sum_{a\in p_{yz}^{(l)}\setminus(s_y^{(l)}\cup\overline{s_z}^{(l)})} \lambda(a)\tau(a) - \sum_{a\in p_{xy}^{(l)}\setminus(s_x^{(l)}\cup\overline{s_z}^{(l)})} \lambda(a)\tau(a) \right| \\ &\leq 3(R+1) \cdot c \cdot \|\lambda\|_\infty \|\tau\|_\infty \end{aligned}$$

using the triangle inequality. □

Proof of Main Theorem 4.2.6, Part 1

The map

$$\begin{aligned} f \colon V \times V &\to \mathbb{R} \\ (x, y) &\mapsto \sum_{a\in p^{(l)}} \lambda(a), \end{aligned}$$

is well-defined by Property 4 of λ. Moreover, f is antisymmetric, since we have a bijection $p^{(l)} \to \overline{p}^{(l)}$ and λ is inversely. Now, we check that f is a quasimorphism, i.e. we show that it is Γ-invariant and has finite defect. For the Γ-invariance, let $x, y \in V$ and $g \in \Gamma$. We choose a path $p \in P(x, y)$ and observe that $g \cdot p \in P(gx, gy)$ and that we have a bijection

$$\begin{aligned} p^{(l)} &\to (g \cdot p)^{(l)} \\ (e_1, \ldots, e_l) &\mapsto (g \cdot e_1, \ldots, g \cdot e_l). \end{aligned}$$

By the Γ-invariance of λ we compute

$$f(x, y) = \sum_{a \in p^{(l)}} \lambda(a) = \sum_{a \in p^{(l)}} \lambda(g \cdot a) = \sum_{a \in (g \cdot p)^{(l)}} \lambda(a) = f(g \cdot x, g \cdot y).$$

In order to show that f has finite defect, we use Lemma 4.4.1 with $\tau := \underline{1}$ the constant map at 1 and obtain for vertices $x, y, z \in V$ and paths $p_1 \in P(x, y)$, $p_2 \in P(y, z)$, and $p_3 \in P(x, z)$ the inequality

$$\begin{aligned} |f(y, z) - f(x, z) + f(x, y)| &= \left| \sum_{a \in p_2^{(l)}} \lambda(a) - \sum_{a \in p_3^{(l)}} \lambda(a) + \sum_{a \in p_1^{(l)}} \lambda(a) \right| \\ &\leq 3(R + 1) \cdot c \cdot \|\lambda\|_\infty. \end{aligned}$$

This bound is independent of x, y, and z, which means that $D(f)$ is finite and f is a quasimorphism of $\Gamma \curvearrowright X$.

Proof of Main Theorem 4.2.6, Part 2

Let $\zeta \in C^n_{\Gamma,b}(X; \mathbb{R})$ be well-behaved. We recall this means for all vertices $x, y \in V$ and for all paths $p, q \in P(x, y)$ with a fixed bijection $\varphi_{p,q} \colon p^{(l)} \to q^{(l)}$ as in Property 4 of λ we have for all $x_1, \ldots, x_n \in V$ and $a \in p^{(l)} \cap \operatorname{supp}(\lambda)$ the equalities

$$\begin{aligned} \zeta(\alpha(a), x_1, \ldots, x_n) &= \zeta(\alpha(\varphi_{p,q}(a)), x_1, \ldots, x_n), \text{ and} \\ \zeta(\omega(a), x_1, \ldots, x_n) &= \zeta(\omega(\varphi_{p,q}(a)), x_1, \ldots, x_n). \end{aligned}$$

We want to find a Γ-invariant map $\eta \colon V^{n+1} \to \mathbb{R}$ such that

$$\beta := f \cup \zeta + \delta^n \eta$$

is bounded and therefore an element of $C^{n+1}_{\Gamma,b}(X; \mathbb{R})$. Then $\delta^{n+1}\beta = \delta^1 f \cup \zeta$ lies in the image of δ^{n+1} and the cup product $[\delta^1 f] \cup [\zeta]$ is trivial.

We define

$$\begin{aligned} \tilde{\zeta} \colon (E^{or})^l \times V^n &\to \mathbb{R} \\ (a, x_1, \ldots, x_n) &\mapsto \frac{1}{2}\Big(\zeta(\alpha(a), x_1, \ldots, x_n) + \zeta(\omega(a), x_1, \ldots, x_n)\Big). \end{aligned}$$

Because ζ is well-behaved we know that for vertices $x, y \in V$, paths $p, q \in P(x,y)$, and $a \in p^{(l)} \cap \operatorname{supp}(\lambda)$ we have for all $x_1, \dots, x_n \in V$

$$\tilde{\zeta}(a, x_1, \dots, x_n) = \tilde{\zeta}(\varphi_{p,q}(a), x_1, \dots, x_n).$$

For this reason, we obtain a well-defined map

$$\begin{aligned} \eta \colon V^{n+1} &\to \mathbb{R} \\ (x_0, \dots, x_n) &\mapsto \sum_{a \in p^{(l)}} \lambda(a)\tilde{\zeta}(a, x_1, \dots, x_n) \end{aligned}$$

with $p \in P(x_0, x_1)$. Well-defined means in this case that it does not depend on the choice of $p \in P(x_0, x_1)$.

At first, we observe that η is Γ-invariant. For this, let $x_0, \dots, x_n \in V$, $g \in \Gamma$ and $p \in P(x_0, x_1)$. We compute using the Γ-invariance of λ and ζ

$$\begin{aligned} \eta(gx_0, \dots, gx_n) &= \sum_{a \in (g \cdot p)^{(l)}} \lambda(a) \cdot \tilde{\zeta}(a, gx_1, \dots, gx_n) \\ &= \sum_{a \in p^{(l)}} \lambda(ga) \cdot \tilde{\zeta}(ga, gx_1, \dots, gx_n) \\ &= \sum_{a \in p^{(l)}} \lambda(a) \cdot \tilde{\zeta}(a, x_1, \dots, x_n). \end{aligned}$$

Now we want to show that $\beta := f \cup \zeta + \delta^n \eta$ is bounded. Let $x_0, \dots, x_{n+1} \in V$. We choose $p_{01} \in P(x_0, x_1)$, $p_{02} \in P(x_0, x_2)$ and $p_{12} \in P(x_1, x_2)$. Our goal is to apply Lemma 4.4.1 for $\tau := \widetilde{\zeta}(\,\cdot\,, x_2, \dots, x_{n+1})$. This is possible, as the equation

$$\tau(a) = \widetilde{\zeta}(a, x_2, \dots, x_{n+1}) = \widetilde{\zeta}(\overline{a}, x_2, \dots, x_{n+1}) = \tau(\overline{a})$$

holds for all $a \in (E^{or})^{(l)}$ because $\alpha(a) = \omega(\overline{a})$ and vice versa, and for vertices $x, y \in V$ and paths $p, q \in P(x,y)$ we have for $\varphi_{p,q}$ the fixed bijection $p^{(l)} \to q^{(l)}$ with $\lambda_{|p^{(l)}} = \lambda_{|q^{(l)}} \circ \varphi_{p,q}$

$$\begin{aligned} \tau(\varphi_{p,q}(a)) &= \widetilde{\zeta}(\varphi_{p,q}(a), x_2, \dots, x_{n+1}) \\ &= \widetilde{\zeta}(a, x_2, \dots, x_{n+1}) \\ &= \tau(a) \end{aligned}$$

as ζ is well-behaved. We also stress that $t = \widetilde{\zeta}$ is bounded by $\|\zeta\|_\infty < \infty$.

We recall the cocycle condition for ζ that gives for every $a \in p_{01}^{(l)}$ and $x \in \{\alpha(a), \omega(a)\}$ the equation

$$\zeta(x, x_2, \dots, x_{n+1}) = \zeta(x_1, \dots, x_{n+1}) + \sum_{i=2}^{n+1} (-1)^i \zeta(x, x_1, \dots, \widehat{x_i}, \dots, x_{n+1})$$

and hence, by definition of $\tilde{\zeta}$ we obtain

$$\tilde{\zeta}(a, x_2, \dots, x_{n+1}) = \zeta(x_1, \dots, x_{n+1}) + \sum_{i=2}^{n+1} (-1)^i \tilde{\zeta}(a, x_1, \dots, \widehat{x_i}, \dots, x_{n+1}). \tag{4.2}$$

Knowing this, we compute

$$\begin{aligned}
\beta(x_0, \dots, x_{n+1}) &= f(x_0, x_1)\zeta(x_1, \dots, x_{n+1}) + \sum_{i=0}^{n+1} (-1)^i \eta(x_0, \dots, \widehat{x_i}, \dots, x_{n+1}) \\
&= \sum_{a \in p_{01}^{(l)}} \lambda(a)\zeta(x_1, \dots, x_{n+1}) + \sum_{i=2}^{n+1} (-1)^i \eta(x_0, x_1, x_2 \dots, \widehat{x_i}, \dots, x_{n+1}) \\
&\quad + \eta(x_1, \dots, x_{n+1}) - \eta(x_0, x_2, \dots, x_{n+1}) \\
&= \sum_{a \in p_{01}^{(l)}} \lambda(a) \left(\zeta(x_1, \dots, x_{n+1}) + \sum_{i=2}^{n+1} (-1)^i \tilde{\zeta}(a, x_1, x_2 \dots, \widehat{x_i}, \dots, x_{n+1}) \right) \\
&\quad + \eta(x_1, \dots, x_{n+1}) - \eta(x_0, x_2, \dots, x_{n+1}) \\
&= \sum_{a \in p_{01}^{(l)}} \lambda(a)\tilde{\zeta}(a, x_2, \dots, x_{n+1}) \\
&\quad + \sum_{a \in p_{12}^{(l)}} \lambda(a)\tilde{\zeta}(a, x_2, \dots, x_{n+1}) - \sum_{a \in p_{02}^{(l)}} \lambda(a)\tilde{\zeta}(a, x_2, \dots, x_{n+1}) \\
&= \sum_{a \in p_{01}^{(l)}} \lambda(a)\tau(a) + \sum_{a \in p_{12}^{(l)}} \lambda(a)\tau(a) - \sum_{a \in p_{02}^{(l)}} \lambda(a)\tau(a).
\end{aligned}$$

We conclude with Lemma 4.4.1 that β is bounded by $3 \cdot (R+1) \cdot c \cdot \|\lambda\|_\infty \|\tau\|_\infty$. This bound is finite, as $\|\tau\|_\infty < \infty$ is finite.

So far, we have shown that the cup product $[\delta^1 f] \cup \alpha$ is trivial for any $\alpha \in C^n_{\Gamma,b}(X; \mathbb{R})$ that is well-behaved. Part 2 of the Main Theorem 4.2.6 also states that the cup product $\alpha \cup [\delta^1 f]$ is trivial. The computation works in a similar way.

We again want to find a Γ-equivariant map $\vartheta \colon V^{n+1} \to \mathbb{R}$ such that the cochain

$$\beta' := \zeta \cup f - \delta^n \vartheta$$

is bounded as then $\delta^{n+1}(-1)^n \beta' = \zeta \cup \delta^1 f$ lies in the image of δ^{n+1} and thus the cup product $[\zeta] \cup [\delta^1 f]$ is trivial.

We define

$$\begin{aligned} \vartheta\colon V^{n+1} &\to \mathbb{R} \\ (x_0,\dots,x_n) &\mapsto \sum_{a\in p^{(l)}} \lambda(a)\tilde{\zeta}(a,x_0,\dots,x_{n-1}) \end{aligned}$$

where $p \in P(x_{n-1}, x_n)$. Just as the map η before, this map is well-defined and Γ-invariant since ζ is well-behaved.

Let $x_0, \dots, x_{n+1} \in V$. Similar to the proof of boundedness of η we want to apply Lemma 4.4.1 to $\tau \coloneqq \widetilde{\zeta}(\,\cdot\,, x_0, \dots, x_{n-1})$. This is possible, as τ defined in this way fulfils the assumptions of Lemma 4.4.1 similar to the previous case. The finiteness is then a consequence of the fact that τ is bounded by $\|\zeta\|_\infty < \infty$.

In order to be able to apply Lemma 4.4.1, we need the cocycle condition of ζ. In particular, we use for all $x \in V$

$$(-1)^n\zeta(x,x_0,\dots,x_{n-1}) = \zeta(x_0,\dots,x_n) - \sum_{i=0}^{n-1}(-1)^i\zeta(x,x_0,\dots,\widehat{x_i},\dots x_{n-1},x_n).$$

that implies the following equation for all $a \in p^{(l)}_{n,n+1}$

$$(-1)^n\tilde{\zeta}(a,x_0,\dots,x_{n-1}) = \zeta(x_0,\dots,x_n) - \sum_{i=0}^{n-1}(-1)^i\tilde{\zeta}(a,x_0,\dots,\widehat{x_i},\dots x_{n-1},x_n). \quad (4.3)$$

We choose paths $p_{n,n+1} \in P(x_n, x_{n+1})$, $p_{x_{n-1},x_n} \in P(x_{n-1}, x_n)$, and $p_{n-1,n+1} \in P(x_{n-1}, x_{n+1})$

and compute for $\beta' = \zeta \cup f - \delta^n \vartheta$

$$\begin{aligned}
\beta'(x_0,\dots,x_n) &= \zeta(x_0,\dots,x_{n+1}) f(x_n,x_{n+1}) - \sum_{i=0}^{n+1} (-1)^i \vartheta(x_0,\dots,\widehat{x_i},\dots,x_{n+1}) \\
&= \sum_{a\in p^{(l)}_{n,n+1}} \lambda(a)\zeta(x_0,\dots,x_n) - \sum_{i=0}^{n-1}(-1)^i\vartheta(x_0,\dots,\widehat{x_i},\dots,x_{n-1},x_n,x_{n+1}) \\
&\quad - (-1)^n\vartheta(x_0,\dots,x_{n-1},x_{n+1}) + (-1)^n\vartheta(x_0,\dots,x_n) \\
&= \sum_{a\in p^{(l)}_{n,n+1}} \lambda(a)\left(\zeta(x_0,\dots,x_n) - \sum_{i=0}^{n-1}(-1)^i\tilde{\zeta}(a,x_1,x_2\dots,\widehat{x_i},\dots,x_{n+1})\right) \\
&\quad - (-1)^n\vartheta(x_0,\dots,x_{n-1},x_{n+1}) + (-1)^n\vartheta(x_0,\dots,x_n) \\
&= \sum_{a\in p^{(l)}_{n,n+1}} (-1)^n\lambda(a)\tilde{\zeta}(a,x_0,\dots,x_{n-1}) \\
&\quad - \sum_{a\in p^{(l)}_{n-1,n+1}} (-1)^n\lambda(a)\tilde{\zeta}(a,x_0,\dots,x_{n-1}) \\
&\quad + \sum_{a\in p^{(l)}_{n-1,n}} (-1)^n\lambda(a)\tilde{\zeta}(a,x_0,\dots,x_{n-1}) \\
&= (-1)^n\left(\sum_{a\in p^{(l)}_{n,n+1}} \lambda(a)\tau(a) - \sum_{a\in p^{(l)}_{n-1,n+1}} \lambda(a)\tau(a) + \sum_{a\in p^{(l)}_{n-1,n}} \lambda(a)\tau(a)\right).
\end{aligned}$$

Eventually, Lemma 4.4.1 allows us to conclude that β' is bounded.

Proof of Main Theorem 4.2.6, Part 3

Let $\zeta_1 \in C^n_{\Gamma,b}(X;\mathbb{R})$ and $\zeta_2 \in C^m_{\Gamma,b}(X;\mathbb{R})$ both be well-behaved. We want to show that the Massey triple product $\langle[\zeta_1],[\delta^1 f],[\zeta_2]\rangle$ is trivial.

Recall, for

$$\begin{aligned}
\vartheta\colon V^{n+1} &\to \mathbb{R} \\
(x_0,\dots,x_n) &\mapsto \sum_{a\in p^{(l)}} \lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})
\end{aligned}$$

with $p^{(l)} \in P(x_{n-1},x_n)$ and $\beta_1 := (-1)^n(\zeta_1\cup f - \delta^n\vartheta)$ we have

$$\delta^{n+1}\beta_1 = \zeta_1 \cup \delta^1 f.$$

On the other hand, for

$$\begin{aligned}\eta\colon V^{m+1} &\to \mathbb{R}\\ (x_0,\dots,x_m) &\mapsto \sum_{a\in p^{(l)}} \lambda(a)\tilde{\zeta_2}(a,x_1,\dots,x_m)\end{aligned}$$

with $p^{(l)} \in P(x_0, x_1)$ and $\beta_2 \coloneqq f \cup \zeta_2 + \delta^m \eta$ we have

$$\delta^{m+1}\beta_2 = \delta^1 f \cup \zeta_2.$$

This was done in the proof of Part 2 of Main Theorem 4.2.6.

Now

$$[(-1)^n \zeta_1 \cup \beta_2 - \beta_1 \cup \zeta_2] \in \langle [\zeta_1], [\delta^1 f], [\zeta_2]\rangle$$

by definition of the Massey triple product and we will show in the following that this element is trivial.

At first, we observe

$$\begin{aligned}(-1)^n\zeta_1\cup\beta_2 - \beta_1\cup\zeta_2 &= (-1)^n\zeta_1\cup(f\cup\zeta_2+\delta^m\eta) - (-1)^n(\zeta_1\cup f - \delta^n\vartheta)\cup\zeta_2\\ &= (-1)^n\zeta_1\cup\delta^m\eta + (-1)^n\delta^n\vartheta\cup\zeta_2\\ &= \delta^{n+m}(\zeta_1\cup\eta + (-1)^n\vartheta\cup\zeta_2). \qquad (4.4)\end{aligned}$$

The procedure is now as follows: We want to find a Γ-invariant map $\kappa\colon V^{n+m} \to \mathbb{R}$ such that

$$\beta \coloneqq \zeta_1 \cup \eta + (-1)^n \vartheta \cup \zeta_2 - \delta^{n+m-1}\kappa$$

is bounded. Then we have

$$\begin{aligned}\delta^{n+m}\beta &= \delta^{n+m}(\zeta_1\cup\eta + (-1)^n\vartheta\cup\zeta_2 - \delta^{n+m-1}\kappa)\\ &= \delta^{n+m}(\zeta_1\cup\eta + (-1)^n\vartheta\cup\zeta_2)\\ &= (-1)^n\zeta_1\cup\beta_2 - \beta_1\cup\zeta_2\end{aligned}$$

using that $\delta^{n+m-1}\kappa$ is a coboundary and Equation 4.4. This implies that the Massey triple product $\langle [\zeta_1], [\delta^1 f], [\zeta_2]\rangle$ as it contains the trivial coclass $0 = [(-1)^n\zeta_1\cup\beta_2 - \beta_1\cup\zeta_2]$.

We consider

$$\begin{aligned}\kappa\colon V^{n+m} &\to \mathbb{R}\\ (x_1,\dots,x_{n+m}) &\mapsto \sum_{a\in p^{(l)}} \lambda(a)\tilde{\zeta_1}(a,x_1,\dots,x_n)\tilde{\zeta_2}(a,x_{n+1},\dots,x_{n+m})\end{aligned}$$

with $p \in P(x_n, x_{n+1})$. This map does not depend on the choice of $p \in P(x_n, x_{n+1})$

since ζ_1 and ζ_2 are well-behaved. Hence, κ is a well-defined map. Furthermore, it is Γ-invariant, as λ, ζ_1 and ζ_2 are Γ-invariant. In the following, let

$$\beta \coloneqq \zeta_1 \cup \eta + (-1)^n \vartheta \cup \zeta_2 - \delta^{n+m-1}\kappa.$$

To show the boundedness of β, let $x_0, \dots, x_{n+m} \in V$ and $p_{n-1,n} \in P(x_{n-1}, x_n)$, $p_{n-1,n+1} \in P(x_{n-1}, x_{n+1})$, and $p_{n,n+1} \in P(x_n, x_{n+1})$. Our goal is to apply Lemma 4.4.1 for

$$\tau \coloneqq \widetilde{\zeta_1}(\,\cdot\,, x_0, \dots, x_{n-1}) \cdot \widetilde{\zeta_2}(\,\cdot\,, x_{n+1}, \dots, x_{n+m}).$$

We checked in the proof of Part 2 that the maps $\widetilde{\zeta_1}(\,\cdot\,, x_0, \dots, x_{n-1})$ and $\widetilde{\zeta_2}(\,\cdot\,, x_{n+1}, \dots, x_{n+m})$ fulfil the assumptions of Lemma 4.4.1. It is then an easy conclusion that τ does so, too. We also note that τ is bounded by $\|\zeta_1\|_\infty \cdot \|\zeta_2\|_\infty < \infty$.

We compute

$$\begin{aligned}
\beta(x_0, \dots, x_{n+m}) &= \zeta_1(x_0, \dots, x_n)\eta(x_n, \dots, x_{n+m}) + (-1)^n\vartheta(x_0, \dots, x_n)\zeta_2(x_n, \dots, x_{n+m}) \\
&\quad - \sum_{i=0}^{n+m} (-1)^i \kappa(x_0, \dots, \widehat{x_i}, \dots, x_{n+m}) \\
&= \zeta_1(x_0, \dots, x_n)\eta(x_n, \dots, x_{n+m}) \ + \ (-1)^n\vartheta(x_0, \dots, x_n)\zeta_2(x_n, \dots, x_{n+m}) \\
&\quad - \sum_{i=0}^{n-1} (-1)^i \sum_{a \in p^{(l)}_{n,n+1}} \lambda(a)\tilde{\zeta}_1(a, x_0, \dots, \widehat{x_i}, \dots, x_{n-1}, x_n)\tilde{\zeta}_2(a, x_{n+1}, \dots, x_{n+m}) \\
&\quad - (-1)^n \sum_{a \in p^{(l)}_{n-1,n+1}} \lambda(a)\tilde{\zeta}_1(a, x_0, \dots, x_{n-1})\tilde{\zeta}_2(a, x_{n+1}, \dots, x_{n+m}) \\
&\quad - \sum_{i=n+1}^{n+m} (-1)^i \sum_{a \in p^{(l)}_{n-1,n}} \lambda(a)\tilde{\zeta}_1(a, x_0, \dots, x_{n-1})\tilde{\zeta}_2(a, x_n, x_{n+1}, \dots, \widehat{x_i}, \dots, x_{n+m})
\end{aligned}$$

At first, we look at the blue part of the equation. To tidy up the equation, we define for $a \in p^{(l)}_{n,n+1}$

$$c_a \coloneqq \lambda(a)\tilde{\zeta}_2(a, x_{n+1}, \dots, x_{n+m}).$$

Then,

$$\begin{aligned}
&\zeta_1(x_0,\dots,x_n)\eta(x_n,\dots,x_{n+m})\\
&\qquad -\sum_{i=0}^{n-1}(-1)^i\sum_{a\in p^{(l)}_{n,n+1}}\lambda(a)\tilde{\zeta}_1(a,x_0,\dots,\widehat{x_i},\dots,x_{n-1},x_n)\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m})\\
&= \sum_{a\in p^{(l)}_{n,n+1}}\zeta_1(x_0,\dots,x_n)\lambda(a)\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m})\\
&\qquad -\sum_{a\in p^{(l)}_{n,n+1}}\sum_{i=0}^{n-1}(-1)^i c_a\tilde{\zeta}_1(a,x_0,\dots,\widehat{x_i},\dots,x_{n-1},x_n)\\
&= \sum_{a\in p^{(l)}_{n,n+1}}c_a\left(\zeta_1(x_0,\dots,x_n)-\sum_{i=0}^{n-1}(-1)^i\tilde{\zeta}_1(a,x_0,\dots,\widehat{x_i},\dots,x_{n-1},x_n)\right).
\end{aligned}$$

As we have done it in Equation 4.3, we obtain due to the cocycle condition on ζ_1

$$(-1)^n\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})=\zeta_1(x_0,\dots,x_n)-\sum_{i=0}^{n-1}(-1)^i\tilde{\zeta}_1(a,x_0,\dots,\widehat{x_i},\dots,x_{n-1},x_n).$$

Applying this, we have

$$\begin{aligned}
&\zeta_1(x_0,\dots,x_n)\eta(x_n,\dots,x_{n+m})\\
&\qquad -\sum_{i=0}^{n-1}(-1)^i\sum_{a\in p^{(l)}_{n,n+1}}\lambda(a)\tilde{\zeta}_1(a,x_0,\dots,\widehat{x_i},\dots,x_{n-1},x_n)\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m})\\
&= \sum_{a\in p^{(l)}_{n,n+1}}c_a(-1)^n\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})\\
&= (-1)^n\sum_{a\in p^{(l)}_{n,n+1}}\lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m}).
\end{aligned}$$

Similar things can be done for the pink part of the equation. For this, we define for $a\in p_{n-1,n}$ the constant

$$d_a := \lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})$$

and compute

$$\begin{aligned}
&(-1)^n\vartheta(x_0,\dots,x_n)\zeta_2(x_n,\dots,x_{n+m}) \\
&\qquad - \sum_{i=n+1}^{n+m}(-1)^i \sum_{a\in p^{(l)}_{n-1,n}} \lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})\tilde{\zeta}_2(a,x_n,x_{n+1},\dots,\widehat{x_i},\dots,x_{n+m}) \\
&= \sum_{a\in p^{(l)}_{n-1,n}} (-1)^n\lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})\zeta_2(x_n,\dots,x_{n+m}) \\
&\qquad - \sum_{a\in p^{(l)}_{n-1,n}} (-1)^n \sum_{i=1}^{m} d_a(-1)^i\tilde{\zeta}_2(a,x_n,x_{n+1},\dots,\widehat{x_{n+i}},\dots,x_{n+m}) \\
&= \sum_{a\in p^{(l)}_{n-1,n}} (-1)^n d_a\left(\zeta_2(x_n,\dots,x_{n+m}) - \sum_{i=1}^{m}(-1)^i\tilde{\zeta}_2(a,x_n,x_{n+1},\dots,\widehat{x_{n+i}},\dots,x_{n+m})\right).
\end{aligned}$$

As well as in Equation 4.2 the cocycle condition of ζ_2 allows the equation

$$\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m}) = \zeta_2(x_n,\dots,x_{n+m}) - \sum_{i=1}^{m}(-1)^i\tilde{\zeta}_2(a,x_n,x_{n+1},\dots,\widehat{x_{n+i}},\dots,x_{n+m}).$$

Hence, we conclude

$$\begin{aligned}
&(-1)^n\vartheta(x_0,\dots,x_n)\zeta_2(x_n,\dots,x_{n+m}) \\
&\qquad - \sum_{i=n+1}^{n+m}(-1)^i \sum_{a\in p^{(l)}_{n-1,n}} \lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})\tilde{\zeta}_2(a,x_n,x_{n+1},\dots,\widehat{x_i},\dots,x_{n+m}) \\
&= \sum_{a\in p^{(l)}_{n-1,n}} (-1)^n d_a\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m}) \\
&= (-1)^n \sum_{a\in p^{(l)}_{n-1,n}} \lambda(a)\tilde{\zeta}_1(a,x_0,\dots,x_{n-1})\tilde{\zeta}_2(a,x_{n+1},\dots,x_{n+m}).
\end{aligned}$$

Now we just need to put all together and obtain

$$\begin{aligned}
&\beta(x_0,\dots,x_{n+m})\\
&\quad= (-1)^n \sum_{a\in p^{(l)}_{n,n+1}} \lambda(a)\tilde{\zeta_1}(a,x_0,\dots,x_{n-1})\tilde{\zeta_2}(a,x_{n+1},\dots,x_{n+m})\\
&\qquad\quad - (-1)^n \sum_{a\in p^{(l)}_{n-1,n+1}} \lambda(a)\tilde{\zeta_1}(a,x_0,\dots,x_{n-1})\tilde{\zeta_2}(a,x_{n+1},\dots,x_{n+m})\\
&\qquad\quad + (-1)^n \sum_{a\in p^{(l)}_{n-1,n}} \lambda(a)\tilde{\zeta_1}(a,x_0,\dots,x_{n-1})\tilde{\zeta_2}(a,x_{n+1},\dots,x_{n+m})\\
&\quad= (-1)^n \left(\sum_{a\in p^{(l)}_{n,n+1}} \lambda(a)\tau(a) - \sum_{a\in p^{(l)}_{n-1,n+1}} \lambda(a)\tau(a) + \sum_{a\in p^{(l)}_{n-1,n}} \lambda(a)\tau(a)\right).
\end{aligned}$$

Finally, we apply Lemma 4.4.1 and conclude that β is bounded since τ is bounded.

5 Median quasimorphisms on trees

In this section we look at pullbacks of median quasimorphisms of groups acting on trees. After presenting the preliminary definitions and properties of trees and their automorphism groups in Section 5.1, we give a characterization for actions on trees that gives rise to an infinite family of non-trivial median quasimorphisms (Section 5.2). We work out the proof of this characterization in Section 5.3. Lastly, we formulate a consequence of this characterization. It states in particular that for cocompact lattices of a products of automorphism groups of regular trees the triviality of the second bounded cohomology is equivalent to the triviality of the median quasimorphism. We follow the paper of Iozzi, Pagliantini and Sisto [10].

At first, we recall Remark 4.1.18, where we described the construction of median quasimorphism of a group Γ acting on a tree $T = (V, E)$.

For $l \in \mathbb{N}$ and s a geodesic of length l in T, the median quasimorphism f_s for s is the zero-map if $\Gamma s = \Gamma \overline{s}$. Otherwise, we have

$$f_s(x, y) = \sum_{\beta \in [\![x,y]\!]^{(n)}} \epsilon_s(\beta)$$

for $x, y \in V$, where the map ϵ_s is defined as

$$\begin{aligned} \epsilon_s \colon \mathcal{E}^{(l)} &\to \{-1, 0, 1\} \\ \beta &\mapsto \begin{cases} 1, & \text{if } \Gamma\beta = \Gamma s, \\ -1, & \text{if } \Gamma\beta = \Gamma\overline{s}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

As mentioned, we only look at pullbacks of such quasimorphisms under the orbit map for some $v \in V$. Therefore, we call these pullbacks for simplicity just median quasimorphisms and no confusion will arise.

5.1 Preliminaries on trees

Definition 5.1.1 ((bi)-infinite chains, ends)**.** Let $T = (V, E)$ be a tree. We call a geodesic $\gamma \colon \mathbb{N} \to V$ or $\gamma \colon \mathbb{Z} \to V$ an *infinite* or *bi-infinite chain*, respectively. We define an equivalence relation on the set of infinite chains by

$$\gamma_1 \sim \gamma_2 \colon \Leftrightarrow \gamma_1(\mathbb{N}) \cap \gamma_2(\mathbb{N}) \text{ is infinite.}$$

We define the *set of ends* δT *of* T, also called the *boundary of* T, to be the set of equivalence classes of infinite chains.

Remark 5.1.2. The relation on the set of infinite chains is indeed an equivalence relation since T is a tree and hence two infinite chains γ_1 and γ_2 have an infinite intersection if and only if there exist $n, m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ it is $\gamma_1(n+k) = \gamma_2(m+k)$.

Remark 5.1.3 (group action on ends)**.** Let T be a tree with a non-empty set of ends. Let Γ be a group acting on T. Then Γ also acts on the boundary of T via

$$\begin{aligned} \Gamma \times \delta T &\to \delta T \\ (g, [\gamma]) &\mapsto [n \mapsto g \cdot \gamma(n)]. \end{aligned}$$

Lemma 5.1.4. *Every non-empty locally finite tree with infinitely many vertices has a non-empty set of ends.*

Proof. Let $T = (V, E)$ be a non-empty locally finite tree with infinitely many vertices. Let $v \in V$. We want to construct an infinite chain $\gamma\colon \mathbb{N} \to V$. Note that a map $\gamma\colon \mathbb{N} \to V$ is a geodesic if for all $n \in \mathbb{N}$ we have $\gamma(n) \neq \gamma(n+2)$ and $\{\gamma(n), \gamma(n+1)\} \in E$.

We define for $x \in V$ with $d(v, x) = 1$ the number

$$\mu_v(x) := |\{y \in V \mid \text{the unique path from } v \text{ to } y \text{ crosses } x\}|$$

and obtain the equality

$$\infty = |V| = |\{v\}| + \sum_{x \in V,\ d(v,x)=1} \mu_v(x) = 1 + \sum_{x \in V,\ d(v,x)=1} \mu_v(x).$$

Hence, there is at least one neighbour x of v with $\mu_v(x) = \infty$ as v has only finitely many neighbours by the local finiteness of T.

We define now an infinite chain γ inductively as follows. For $n = 0$, we set $\gamma(0) := v$. For $n = 1$ choose a neighbour x of v with $\mu_v(x) = \infty$ and define $\gamma(1) = x$. Assume for some $n > 0$ we have defined $\gamma(i)$ for all $i \leq n$ such that $\mu_{\gamma(n-1)}(\gamma(n)) = \infty$. Since $\gamma(n)$ has only finitely many neighbours, this means we can find a neighbour y with $y \neq \gamma(n-1)$ and $\mu_{\gamma(n)}(y) = \infty$. We define $\gamma(n+1) := y$. This inductively defines an infinite chain $\gamma\colon \mathbb{N} \to V$ in T. Hence, we have $\delta T \neq \emptyset$. ☐

Example 5.1.5. Let $T = (V, E)$ be a tree such that every vertex has degree at least 2. Then T has at most 2 ends as one can fix a vertex v and construct two different infinite chains by the two neighbours of v. Furthermore, if every vertex has degree at least 3 then T has infinitely many ends.

Definition 5.1.6 (topology on trees with ends)**.** Let $T = (V, E)$ be a tree. For $e \in E$ and $x \in e$ a vertex of e we define by $T_{e,x} = (V_{e,x}, E_{e,x})$ the subtree of T whose vertex set $V_{e,x}$ is the halfspace defined by e containing x and the set of edges $E_{e,x} := \{e \in E \mid e \subset V_{e,x}\}$ being all edges in E that consists of two vertices of $V_{e,x}$.

We equip $V \cup \delta T$ with the topology given as follows. Every vertex $v \in V$ is open and for $\zeta \in \delta T$ we have a basis of open neighbourhoods given by sets of the form $V_{e,x} \cup \delta T_{e,x}$, where $e \in E$ is an edge and $x \in e$ such that $\zeta \in \delta T_{e,x}$.

When we speak of δT as a topological space we always refer the subspace topology on $\delta T \subset V \cup \delta T$.

Definition 5.1.7 (convex hull)**.** Let $T = (V, E)$ be a locally finite tree and $F \subset \delta T$ be a closed subset containing at least two elements. Then, we call the subgraph of T consisting exactly of all bi-infinite chains connecting two elements of F the *convex hull* of F.

Lemma 5.1.8. *Let $T = (V, E)$ be a locally finite tree and $F \subset \delta T$ be a closed subset containing at least 2 elements. Then, the convex hull of F is a subtree of T whose boundary with F.*

Proof. Let $T' = (V', E')$ denote the convex hull of F, i.e. the graph consisting exactly of all bi-infinite chains connecting elements of F. Since T' is a subgraph of T, it is clear that it contains no cycles. Hence, for T' being a tree we only need to show that T' is connected. To do this, let $x, y \in T'$. We want to find a bi-infinite chain γ connecting two points of F that crosses both x and y. For this, we choose two bi-infinite chains γ_1, γ_2 connecting two points in F with $\gamma_1(0) = x$ and $\gamma_2(0) = y$. Let $s := [x, y]$ be the geodesic in T from x to y. Then at least one of the two branches $(\gamma_1)_{|\mathbb{Z}_{\geq 0}}$ and $(\gamma_1)_{|\mathbb{Z}_{\leq 0}}$ of γ_1 intersects with s only in the point x, say without loss of generality this holds for $(\gamma_1)_{|\mathbb{Z}_{\leq 0}}$. The same holds for $(\gamma_2)_{|\mathbb{Z}_{\geq 0}}$ and $(\gamma_2)_{|\mathbb{Z}_{\leq 0}}$, say without loss of generality that $(\gamma_2)_{|\mathbb{Z}_{\geq 0}}$ intersects with s only in y. We consider the map

$$\begin{aligned} \gamma \colon \mathbb{Z} &\to T \\ z &\mapsto \begin{cases} \gamma_1(z), & \text{if } z \leq 0, \\ s(z), & \text{if } 0 < z < n, \\ \gamma_2(z - n), & \text{if } z \geq n. \end{cases} \end{aligned}$$

Since the tree paths that are concatenated to obtain γ do not overlap, γ is indeed a geodesic and a bi-infinite chain connecting two points in F. Hence, the geodesic connecting x and y is contained in T'.

In order to compute the set of ends of T' we see $F \subset \delta T'$ by definition of T'. On the other hand, let $\zeta \notin F$. Since F is closed in δT we find an edge $e \in E$ and a vertex $x \in e$ such that $\zeta \in \delta T_{e,x}$ and $\delta T_{e,x} \cap F = \emptyset$. Hence, a bi-infinite path connecting elements of F cannot cross vertices of $T_{e,x}$ and $\delta T'$ does not contain an infinite chain representing ζ. This means $\delta T' \subset F$. All in all, we obtain $\delta T' = F$. □

Definition 5.1.9 (semiregular tree)**.** A tree $T = (V, E)$ is called *semiregular* if the automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on the set of edges E but not necessarily transitively on the set of vertices V.

Definition 5.1.10 (minimal action)**.** The action of a group Γ on a tree T is called minimal if there is no non-empty Γ-invariant proper subtree.

In the following, we define a topology on the automorphism group of regular trees in order to talk about closed subsets.

Definition 5.1.11 (topology on $\mathrm{Aut}(T)$)**.** Let T be a regular tree of degree at least 2. For $g \in \mathrm{Aut}(T)$ and a finite subset $F \subset V$, we define

$$U_F(g) \coloneqq \{h \in \mathrm{Aut}(T) \mid \ h(x) = g(x) \text{ for all } x \in F\}.$$

We equip the set of graph automorphisms $\mathrm{Aut}(T)$ on T with the topology that is defined by having sets of the form $U_F(g)$ as a basis of open neighbourhoods of $g \in \mathrm{Aut}(T)$.

We use the following theorems without proof.

Theorem 5.1.12 ([6, Lemma 8.2])**.** *Let T be a regular tree of degree greater than 2. Then T contains a free group of rank 2 as a discrete subset.*

Theorem 5.1.13 ([6, Chapter 1, Part 8])**.** *Let T be a regular tree of degree greater than 2 and $\zeta \in \delta T$. Then the stabilizer of ζ, i.e. the set*

$$\mathrm{Stab}_{\mathrm{Aut}(T)}(\zeta) = \{f \in \mathrm{Aut}(T) \mid f \cdot \zeta = \zeta\}$$

is an amenable subgroup of $\mathrm{Aut}(T)$*.*

Next, we introduce local ∞-transitivity for subgroups of $\mathrm{Aut}(T)$.

Definition 5.1.14 (local ∞-transitivity)**.** A subgroup of the automorphism group Γ of a locally finite tree is *locally ∞-transitive* if for all vertices $v \in V$ and for all integers $n \in \mathbb{N}$ the stabiliser $\mathrm{Stab}_\Gamma(v)$ acts transitively on the n-sphere centred at v.

Lemma 5.1.15. *Let $T = (V, E)$ be a regular tree of degree at least 2 and let $H \subset \mathrm{Aut}(T)$ be a closed subgroup. Then, for a vertex $v \in V$ the following are equivalent:*

1. $\mathrm{Stab}_H(v)$ *acts transitively on all spheres of finite radius centred at v.*
2. $\mathrm{Stab}_H(v)$ *acts transitively on δT.*

In particular, H is locally ∞-transitive if and only if for all $v \in V$ the stabilizer $\mathrm{Stab}_H(v)$ acts transitively on δT.

Proof. The last part of the lemma follows directly from the definition of local ∞-transitivity.

For the main part, suppose for $v \in V$ that $\mathrm{Stab}_H(v)$ acts transitively on δT. Let $n \in \mathbb{N}$ and $w_1, w_2 \in V$ that both have distance n to v. As every vertex in T has degree at least 2, we can find two infinite chains γ_1 and γ_2 both starting at v with $\gamma_1(n) = w_1$ and $\gamma_2(n) = w_2$. As $\mathrm{Stab}_H(v)$ acts transitively on δT, there exists $g \in \mathrm{Stab}_H(v)$ such that $g \cdot \gamma_1$ and γ_2 define the same end. As these two infinite chains have the same starting point, this means $g \cdot \gamma_1 = \gamma_2$ and hence, $g \cdot w_1 = w_2$.

On the other hand, assume that $\mathrm{Stab}_H(v)$ acts transitively on all spheres of finite radius centred at v for some $v \in V$. Let $\zeta_1, \zeta_2 \in \delta T$ be two ends of T represented by the infinite chains γ_1, γ_2 respectively, both starting at v.

For $n \in \mathbb{N}$, choose $h_n \in \mathrm{Stab}_H(v)$ such that $h_n \cdot \gamma_1(n) = \gamma_2(n)$. This exists, as $\mathrm{Stab}_H(v)$ acts transitively on the n-sphere around v. By the uniqueness of geodesics in T this means in particular

$$(h_n \cdot \gamma_1)_{|\{0,\dots,n\}} = (\gamma_2)_{|\{0,\dots,n\}}.$$

If $\{h_n \mid n \in \mathbb{N}\}$ is finite, then for one $n_0 \in \mathbb{N}$ we know that $h_{n_0} \cdot \gamma_1 = \gamma_2$. If the set is infinite, then we constructed an infinite subset of the compact set $\mathrm{Stab}_{\mathrm{Aut}(T)}(v)$, [6, Chapter 1, Part 4]. Hence, there exists a limit point h of this infinite set in $\mathrm{Stab}_{\mathrm{Aut}(T)}(v)$. For this limit point h, we also know that it lies in H as H is closed. This means $h \in \mathrm{Stab}_H(v)$. Furthermore, as h is a limit point of the automorphisms h_n, we know that $h \cdot \gamma_1 = \gamma_2$. This shows $h \cdot \alpha = \beta$ and we conclude that $\mathrm{Stab}_H(v)$ acts transitively on δT. □

We end this section with an additional definition.

Definition 5.1.16. Let G be a locally compact topological group. A *cocompact lattice* in G is a discrete subgroup Γ of G such that the quotient space G/X is compact with respect to the quotient topology.

5.2 Main statement

In the following, we consider a discrete group Γ and a group action of Γ on a tree $T = (V, E)$. Furthermore, we fix a base vertex $v \in V$.

We then have the following theorem, which yields meaningful consequences on the second bounded cohomology of Γ.

Theorem 5.2.1 ([10, Theorem 1])**.** *Suppose that Γ acts minimally on the tree T and that every vertex of T has degree greater than 2. Then, one of the following holds*

1. *Γ fixes a point in the boundary δT of T,*
2. *T is semiregular and for every vertex $x \in V$ and for every $n \in \mathbb{N}$ the group Γ acts transitively on the set of geodesics of length n starting at a vertex in Γx,*
3. *There exists an infinite family $(s_n)_{n\in\mathbb{N}}$ of geodesics such that $([\delta^1 \widehat{f_{s_n,v}}])_{n\in\mathbb{N}}$ is an infinite family of linearly independent coclasses in $H^2_b(\Gamma;\mathbb{R})$.*

We take a closer look at the different cases. In particular, we obtain the following.

Lemma 5.2.2. *Suppose we are in the setup of Theorem 5.2.1. If Case 3 holds, then $\dim(H^2_b(\Gamma;\mathbb{R})) = \infty$.*

Furthermore, Case 2 has a big influence on median quasimorphisms.

Lemma 5.2.3. *Suppose we are in the setup of Theorem 5.2.1. If Case 2 holds, then every median quasimorphism is trivial.*

Proof. To show this, let s be a geodesic in $T = (V, E)$ and $x \in V$. We consider the median quasimorphism $f_{s,x}$. Since for every $n \in \mathbb{N}$ the action of Γ on the set of geodesics of length n starting at a vertex of Γx is transitive, there is for every $g \in \Gamma$ an element $h \in \Gamma$ with $[x, gx] = h \cdot [gx, x]$. This induces a bijection

$$\begin{aligned} [\![x, gx]\!]^{(n)} &\to [\![gx, x]\!]^{(n)}, \\ \beta &\mapsto h \cdot \beta. \end{aligned}$$

Hence, we compute for $g \in \Gamma$

$$\begin{aligned} f_s(x, gx) &= \sum_{\beta \in [\![x,gx]\!]^{(n)}} \epsilon_s(\beta) \\ &= \sum_{\beta \in [\![gx,x]\!]^{(n)}} \epsilon_s(h^{-1}\beta) \\ &= \sum_{\beta \in [\![gx,x]\!]^{(n)}} \epsilon_s(\beta) \\ &= f_s(gx, x), \end{aligned}$$

where the second to last equality holds true due to the definition of ϵ_s. This shows that f_s is symmetric. As we know by Corollary 4.3.4 that f_s is also anti-symmetric, we conclude $f_s = 0$ and thus, $f_{s,x} = 0$. □

About mutual exclusiveness

Despite the version of Iozzi, Pagliantini and Sisto [10], we stated the theorem without mutual exclusiveness. This is due to the problem that there is an inaccuracy in the corresponding abstract [10, Section 2.2]. What holds true is that Cases 1 and 2 cannot simultaneously occur. To see this, suppose we are in Case 1 . Choose $x \in V$ and let γ be the infinite chain starting at x representing the fixed point in the boundary. We consider the geodesic $s = [x, \gamma(1)]$ of length 2. Then, every translate of s lies on the infinite chain representing the fixed point. In particular, for all $g \in \mathrm{Stab}_\Gamma(x)$ it is $g \cdot \gamma(1) = \gamma(1)$. This contradicts Case 2 as x has more than one neighbour.

Furthermore, by Lemma 5.2.3 we know that Cases 2 and 3 cannot simultaneously occur.

However, it is not clear whether Cases 1 and 3 exclude themselves or not. In the corresponding abstract by Iozzi, Pagliantini and Sisto [10, Section 2.2] it is stated that in Case 1 every median quasimorphism is bounded. If this was true, then every coclass corresponding to a median quasimorphism would be trivial and hence, Case 3 cannot hold.

But there is an example of a group acting minimally on a tree that has a fixed point in the boundary and an unbounded median quasimorphisms that is also not a group homomorphism. This would produce a non-trivial coclass in the second bounded cohomology.

Having this counterexample, we see that the arguments of Iozzi, Pagliantini and Sisto cannot hold true. Nevertheless, this example does not give an evidence that Cases 1 and 3 can simultaneously occur. The connection between the two Cases remains as an open problem.

In the last part of this passage we present the example that contradicts the arguments given in [10, Section 2.2].

We consider the free group F freely generated by the set $S = \{a, b\}$ and the corresponding Cayley graph $\mathrm{Cay}(F, S) = (V, E)$ that is a regular tree of degree 4, see [11, Theorem 3.3.1].

As F is freely generated by S a group action of F on $\mathrm{Cay}(F, S)$ is determined by defining the action of a and b on $\mathrm{Cay}(F, S)$. Visually explained, multiplication with a moves everything one level up and multiplication with b rotates the left, lower and right branches of the origin e, see Figure 5.1.

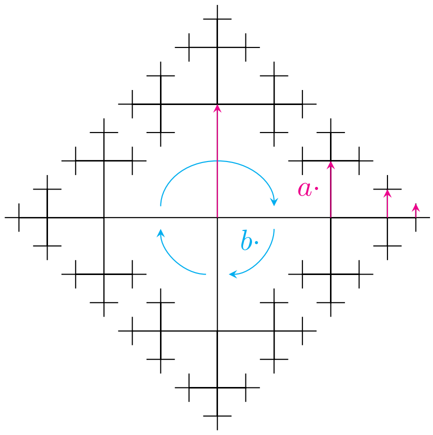

Figure 5.1: Group action of F on $\mathrm{Cay}(F, S)$

Formally, this can be described as follows. We consider the group isomorphism

$$\begin{aligned} \varphi\colon F &\to F \\ a &\mapsto b \\ b &\mapsto a^{-1}. \end{aligned}$$

Note: This isomorphism defines a clockwise rotation of the branches of the origin. We define the group action of F on $\mathrm{Cay}(F, S)$ by

$$a \cdot w = aw$$

$$b \cdot w = \begin{cases} w, & \text{if } w \text{ starts with } a, \\ \varphi(w), & \text{if } w \text{ starts with } b \text{ or } a^{-1}, \\ \varphi^2(w), & \text{otherwise.} \end{cases}$$

for $w \in F$ a reduced word.

One can check that this is indeed a minimal group action. Furthermore, it fixes the point in the boundary defined by the infinite chain

$$\begin{aligned}\gamma\colon \mathbb{N} &\to V\\ n &\mapsto a^n.\end{aligned}$$

To obtain an unbounded median quasimorphism that is not a group homomorphism, consider the geodesic $s = [e, a^2]$. Since s is part of the infinite chain γ defining the fixed point in the boundary, we know that any translate of s lies on an infinite chain defining this fixed point. This means, $F \cdot s \neq F \cdot \overline{s}$ and $f_{s,e}(a^n) = n - 1$ for all $n \in \mathbb{N}$. In particular, $f_{s,e}$ is unbounded. Furthermore, the median quasimorphism $f_{s,e}$ is not a group homomorphism, as $f_{s,e}(a) = 0$ but $f_{s,e}(a^2) = 1$.

5.3 Proof of the statement

We assume that the assumptions of Theorem 5.2.1 are fulfilled and Cases 1 and 2 do not hold. In order to find a family of geodesics $(s_n)_{n\in\mathbb{N}}$ such that $([\delta^1 \widehat{f_{s_n,v}}])_{n\in\mathbb{N}}$ is a linearly independent family in $H^2_b(\Gamma;\mathbb{R})$, we define a Γ-equivariant labelling on geodesics connecting two points of Γv, i.e. a word in the alphabet $\{a, b, c\}$ with some additional properties.

We then show that we have a family of geodesics with complicated enough labellings such that two of them overlap only in short subgeodesics in relation to their lengths. This family of geodesics allows us to construct geodesics $(s_n)_{n\in\mathbb{N}}$ such that $([\delta^1 \widehat{f_{s_n,v}}])_{n\in\mathbb{N}}$ is linearly independent.

5.3.1 The labelling

In the following, we will be mostly interested in geodesics connecting vertices of Γv and the number of translates of v the geodesic crosses. For this, we introduce the following definition.

Definition 5.3.1 (orbit length, $o(n)$-geodesic, o-edge, o-vertex)**.** The *orbit-length* or shorter *o-length* of a geodesic γ is the number of vertices in the orbit Γv crossed by γ minus 1. We denote the o-length of γ by $|\gamma|_o$.

An *$o(n)$-geodesic* is a geodesic of orbit length n connecting two vertices of Γv. If we do not need to emphasize the length we use the notation *o-geodesic.* An $o(1)$-geodesic is also called *o-edge* and an *o-vertex* is a vertex in the orbit Γv.

In order to define a Γ-invariant labelling on the o-geodesics we need a positive integer k and a Γ-invariant assignment of a letter from $\{a, b, c\}$ to each $o(k)$-geodesics, i.e. two $o(k)$-geodesics that lie in the same orbit are labelled by the same letter.

Then, we define a labelling on the o-geodesics via the following: Every o-geodesic of orbit length smaller than k has the empty word as label and for an o-geodesic of o-length greater than k we assign a word in $\{a, b, c\}$ via the following procedure. Let γ be an

$o(L)$-geodesic with $L > k$. For $i \in \{1, \dots, L-k+1\}$ we denote by γ_i the $o(k)$-subgeodesic of γ starting at the i-th o-vertex of γ. Then, the label of γ is defined to be the word having the label of γ_i as i-th letter. Note that in this case, the label consists of $L-k+1$ letters.

As the labelling of the $o(k)$-geodesics is Γ-invariant, the induced labelling on all o-geodesics is also Γ-invariant.

Definition 5.3.2 (good word)**.** A word in the letters $\{a, b, c\}$ is called *good*, if it is the concatenation of words of the form ab^N for $N \geq 2$.

We use that if we are not in Case 1 or 2 we can find a positive integer k and an assignment of a letter from $\{a, b, c\}$ to each orbit of $o(k)$-geodesics such that every good word can be realized as the label of some o-geodesic, and

- either we assign the label a, b respectively, to a unique orbit of o-geodesics, or
- the inverse of any element labelled a is not in an orbit labelled b.

We skip the technical construction of such a labelling and proceed with the proof of Theorem 5.2.1 using the existence of such a label. The details for the construction can be found in [10, Section 2.6].

5.3.2 The choice of geodesics

From now on, we suppose that we are not in Case 1 or 2 of Theorem 5.2.1. Furthermore, we fix an integer k and a Γ-invariant assignment of a letter of $\{a, b, c\}$ to each $o(k)$-geodesic such that the induced labelling on the o-geodesic fulfils the properties described above.

In the following, we take a closer look at good words and look at some lemmas that help us to prove Theorem 5.2.1.

We use the following notation: If w is a word in $\{a, b, c\}$, then we denote by $\overline{w}$ the word we obtain by reading w from right to left.

The following two lemmas are quite technical. We will construct good words and geodesics labelled with these words such that their overlap is bounded. This will play an important role in the definition of the geodesics s_n that give rise to a linearly independent family $(\delta^1 \widehat{f_{s_n,v}})$.

Lemma 5.3.3. *There exists a family of words $(w_{n,i})_{n\in\mathbb{N}, i\in\{1,2,3\}}$ in the alphabet $\{a, b\}$ such that*

- *for every $n \in \mathbb{N}$ and every $i \in \{1, 2, 3\}$ the word $w_{n,i}$ has length at least $20n$ and is good, i.e. it is the concatenation of words of the form ab^N for $N \geq 2$.*
- *for each integer l there exists $n_0(l)$ such that for all $n, m \geq n_0(l)$ and for all $i, j \in \{1, 2, 3\}$*

– *The words* $w_{n,i}$ *and* $w_{m,j}$ *share a subword of length at least*

$$\frac{\min\{|w_{n,i}|+l-1,|w_{m,j}|+l-1\}}{10}-l$$

if and only if $n=m$ *and* $i=j$.

– $w_{n,i}$ *doesn't share a common subword of length at least*

$$\frac{\min\{|w_{n,i}|+l-1,|w_{m,j}|+l-1\}}{10}-l$$

with $\overline{w_{m,j}}$.

Proof. In order to construct such words, we define for $n \in \mathbb{N}$ and $i \in \{1,2,3\}$ the positive integers $v_{n,i} \coloneqq 100\cdot(3n+i)$ and set

$$w_{n,i} \coloneqq ab^{v_{n,i}}ab^{v_{n,i}+1}\cdots ab^{v_{n,i}+99}.$$

By definition of the $v_{n,i}$, if there are $n,m \in \mathbb{N}$ and $i,j \in \{1,2,3\}$ such that in both words $w_{n,i}$ and $w_{m,j}$ the same exponent of b appears, then $n=m$ and $i=j$.

For $l \in \mathbb{N}$ we choose $n_0(l)=l$. Then, for all $n \geq n_0(l)$ and $i \in \{1,2,3\}$ we know

$$|w_{n,i}| = 100+100v_{n,i}+\sum_{k=1}^{99}k > 5000+100v_{n,i}$$

and we estimate

$$\frac{|w_{n,i}|+l-1}{10}-l > 500+10v_{n,i}-\frac{9l+1}{10} > 500+10v_{n,i}-l > 500+9v_{n,i}.$$

A subword of $w_{n,i}$ with length at least $\frac{|w_{n,i}|+l-1}{10}-l$ has to contain a subword of the form $ab^{v_{n,i}+z}ab^{v_{n,i}+z+1}$ for some $z \in \{1,\dots,99\}$. Together with the uniqueness of the exponent, this concludes the proof. □

We fix such a family $(w_{n,i})_{n\in\mathbb{N},i\in\{1,2,3\}}$ of good words up to the end of this subsection and remind us that we chose the labelling in such a way that every good word can be realized as the label of some o-geodesic. This allows us to bound the length of common subgeodesics of two geodesics labelled as such good words, as we see in the following.

Definition 5.3.4. We say a geodesic γ is labelled w^{-1} for w a word in $\{a,b,c\}$ if γ^{-1} is labelled by w.

Lemma 5.3.5. *Let* $n,m \geq n_0(k)$ *and* $i,j \in \{1,2,3\}$. *Furthermore, let* γ_1,γ_2 *be* o-*geodesics labelled by* $w_{n,i}^{\epsilon_1}$ *and* $w_{m,j}^{\epsilon_2}$, *respectively, for* $\epsilon_1,\epsilon_2 \in \{1,-1\}$. *If* γ_1 *and* γ_2 *share a common* o-*subgeodesic of* o-*length at least* $\frac{\min\{|\gamma_1|_o,|\gamma_2|_o\}}{10}-1$, *then* $\epsilon_1=\epsilon_2$ *and* $i=j, n=m$.

Proof. In order to make clear the connection to Lemma 5.3.3 above, we make use of the equation

$$|\gamma_1|_o = |w_{n,i}| + k - 1$$

for the orbit length of γ_1. The same holds for the orbit length of γ_2.

Suppose that γ_1 and γ_2 share a common subgeodesic of o-length

$$L \geq \frac{\min\{|\gamma_1|_o, |\gamma_2|_o\}}{10} - 1.$$

In order to obtain conclusions on subwords of $w_{n,i}$ and $w_{m,j}$ we need to check that the o-length L is greater than k as in this case the two words then need to share a common subword of length $L-k+1$. This follows from the assumption on $w_{n,i}$ and $w_{m,j}$ to have length at least $20n$, $20m$, respectively. Then

$$L \geq \frac{\min\{|w_{n,i}| + k - 1, |w_{m,j}| + k - 1\}}{10} - 1 \geq \frac{\min\{20n, 20m\}}{10} - 1 \geq 2k - 1 \geq k.$$

If $\epsilon_1 = \epsilon_2$, then the existence of such a long common subgeodesic implies that $w_{n,i}$ and $w_{m,j}$ share a subword of length at least

$$\left(\frac{\min\{|\gamma_1|_o, |\gamma_2|_o\}}{10} - 1\right) - k + 1 = \frac{\min\{|w_{n,i}| + k - 1, |w_{m,j}| + k - 1\}}{10} - k.$$

This allows us to conclude $m = n$ and $i = j$ by Theorem 5.2.1.

To finish the proof, we need to show if $\epsilon_1 \neq \epsilon_2$ then γ_1 and γ_2 cannot share such a long subgeodesic. To do this, we suppose $\epsilon_1 = 1$ and $\epsilon_2 = -1$. This means γ_2^{-1} is labelled $w_{m,j}$. We want to show that the label of γ_2 cannot share a subword with $w_{n,i}$ of length at least $\frac{\min\{|w_{n,i}|+k-1, |w_{m,j}|+k-1\}}{10} - k$ as this means they cannot share such a long subgeodesic.

Firstly, we consider the case where γ_2 is labelled $\overline{w_{m,j}}$. Then, by Lemma 5.3.3 the words $w_{n,i}$ and $\overline{w_{m,j}}$ do not share a long enough subword.

If γ_2 is not labelled $\overline{w_{m,j}}$, then there exists an $o(k)$-subgeodesic of γ_2^{-1} labelled a or b such that its inverse is labelled b or a, respectively. By the assumption on our labelling, we know that in this case the label a and the label b are assigned to a unique orbit of $o(k)$-geodesics. Hence, the label of γ_2 is obtained by reading $w_{m,j}$ from right to left and replacing a with b and b with a, which means it is the concatenation of words $a^N b$ with $N \geq 2$. It is clear that $w_{n,i}$ and the label of γ_2 cannot share such a long subword. □

Definition 5.3.6 ((long) $w_{n,i}$-subgeodesic)**.** Let $n \in \mathbb{N}$, $i \in \{1, 2, 3\}$ and γ be an o-geodesic labelled $w_{n,i}$. An o-subgeodesic of γ is called *$w_{n,i}$-subgeodesic.* It is called *long,* if its o-length is at least $(|w_{n,i}| + k - 1)/2$.

Definition 5.3.7 (almost concatenation)**.** Let γ_1 and γ_2 be two o-geodesics. If the endpoint of γ_1 and the starting point of γ_2 coincide or can be connected by an o-edge, then the concatenation of γ_1, the o-edge (if necessary), and γ_2 is an o-geodesic. It is called the *almost concatenation* of γ_1 and γ_2.

In the next lemma, we show the existence of a family $(s_n)_{n\in\mathbb{N}}$ of o-geodesics with some interesting properties. This family will be the one we use for the proof of Theorem 5.2.1.

Lemma 5.3.8. *Let $n \in \mathbb{N}$. Then there exist $g_{n,1},\ g_{n,2},\ g_{n,3} \in \Gamma$ and an o-geodesic s_n obtained by almost concatenating a long $w_{n,3}$-subgeodesic and a long $w_{n,1}$-subgeodesic such that the following holds for each positive integer N.*

1. *The o-geodesic from v to $(g_{n,1}, g_{n,3})^N v$ is the almost concatenation of a long $w_{n,1}$-subgeodesic, $N-1$ translates of s_n and a long $w_{n,3}$-subgeodesic.*
2. *The o-geodesic from v to $(g_{n,1}g_{n,2})^N v$, respectively $(g_{n,2}^{-1}g_{n,3})^N v$, is obtained alternately almost concatenating long $w_{n,1}$-subgeodesics and long $w_{n,2}$-subgeodesics, respectively long $w_{n,2}^{-1}$-subgeodesics and long $w_{n,3}$-subgeodesics.*

Proof. We choose for $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$ the elements $g_{n,i}$ in such a way that the o-geodesic from v to $g_{n,i}v$ is labelled $w_{n,i}$. Let $N \in \mathbb{N}$. We only prove Statement 1 of the lemma, the second one works in a similar way.

Let γ_1 be the longest o-subgeodesic of $[v,\ g_{n,1}v]$ that does not overlap with $[g_{n,3}^{-1}v,\ v]$ and $[g_{n,1}v,\ g_{n,1}g_{n,3}v]$. Note, that γ_1 is a long $w_{n,1}$-subgeodesic since we can bound the overlap by Lemma 5.3.5. In the same way we choose γ_3 to be the longest o-subgeodesic of $[v,\ g_{n,3}v]$ that does not overlap with $[g_{n,1}^{-1}v,\ v]$ and $[g_{n,3}v,\ g_{n,3}g_{n,1}v]$. Similar to γ_1, we see that γ_3 is a long $w_{n,3}$-subgeodesic. Then, we define s_n to be the almost concatenation of $g_{n,1}\gamma_3$ and $g_{n,1}g_{n,3}\gamma_1$ that has to exist by the construction of the two o-geodesics.

Furthermore, let γ_3' be the o-subgeodesic of $[(g_{n,1}g_{n,3})^{N-1}g_{n,1}v, g_{n,1}g_{n,3}v]$ that almost concatenates with $(g_{n,1}g_{n,3})^{N-1}s_n$ and let γ_1' be the longest o-subgeodesic of $[v, g_{n,1}v]$ that almost concatenates with s_n. The almost concatenations exist by definition of s_n. Then, one can show that the almost concatenation of

$$\gamma_1',\ (g_{n,1}g_{n,3})^1 s_n,\ \ldots,\ (g_{n,1}g_{n,3})^{N-1}s_n,\ \gamma_3'$$

is the geodesic $[v, (g_{n,1}, g_{n,3})^N v]$ see Figure 5.2. □

In order to prove Theorem 5.2.1 we use the following lemma about linearly independence of coclasses in the second bounded cohomology.

Lemma 5.3.9. *Let $(f_n)_{n\in\mathbb{N}}$ be a family of non-trivial quasimorphisms on a discrete group G. Suppose that there exists a family of elements $(\eta_n, h_n)_{n\in\mathbb{N}}$ of G such that for all positive integers z, m, n we have*

$$f_n(\eta_m^z) = f_n(h_m^z) = 0$$

and

$$f_n((\eta_m h_m)^z) \begin{cases} = 0, & \text{if } n \neq m, \\ \geq z-1, & \text{if } n = m. \end{cases}$$

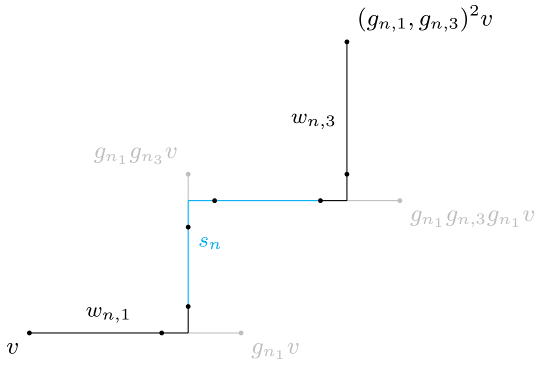

Figure 5.2: The geodesic $[v, (g_{n,1}, g_{n,3})^2 v]$

Then $(\delta^1 \widehat{f_n})_{n\in\mathbb{N}}$ *is a linearly independent family in* $H^2_b(G;\mathbb{R})$*.*

Proof. For every $n \in \mathbb{N}$ we consider the homogenization $\overline{f_n}$ of f_n, which is in particular of finite distance of f_n and hence, defines the same coclass in $H^2_b(\Gamma;\mathbb{R})$ (Theorem 3.1.4). By looking at the construction of the homogenization in the proof of Theorem 3.1.4, which precisely constructs it pointwise by $\overline{f_n}(x) = \lim_{k\to\infty} \frac{f_n(x^k)}{k}$ for all $n \in \mathbb{N}$ and $x \in G$, we obtain for positive integers m, n the equations

$$\overline{f_n}(\eta_m) = \overline{f_n}(h_m) = 0$$

and

$$\overline{f_n}(\eta_m h_m) \begin{cases} = 0, & \text{if } n \neq m, \\ \geq \lim_{k\to\infty} \frac{k-1}{k} = 1, & \text{if } n = m. \end{cases}$$

With help of these equations, we prove the linear independence of the coclasses associated to the quasimorphisms f_n. For this, let $I \subset \mathbb{N}$ be a finite subset and let $(\lambda_i)_{i\in I}$ be a family of scalars in $\mathbb{R}$ such that

$$0 = \sum_{i\in I} \lambda_i \left[\delta^1 \widehat{f_i}\right] = \sum_{i\in I} \lambda_i \left[\delta^1 \widehat{\overline{f_i}}\right]$$

in $H^2_b(G;\mathbb{R})$. Since δ^1 and $\widehat{\cdot}$ are $\mathbb{R}$-linear maps, this is equivalent to

$$0 = \left[\delta^1 \left(\widehat{\sum_{i\in I} \lambda_i \overline{f_i}}\right)\right].$$

Thus, the homogeneous quasimorphism $\sum_{i\in I} \lambda_i \overline{f_i}$ needs to be a group homomorphism by Theorem 3.1.7. This condition has a great influence on the λ_i's, as we can compute for an arbitrary $j \in I$

$$\lambda_j \overline{f_j}(\eta_j h_j) = \sum_{i\in I} \lambda_i \overline{f_i}(\eta_j h_j) = \sum_{i\in I} \lambda_i \overline{f_i}(\eta_j) + \sum_{i\in I} \lambda_i \overline{f_i}(h_j) = 0.$$

As $\overline{f_j}(\eta_j h_j) \neq 0$ we conclude $\lambda_j = 0$ which proves the linear independence of the finite family $(f_n)_{n\in I}$. As the finite set $I \subset \mathbb{N}$ was chosen arbitrarily, we can conclude that the family $(f_n)_{n\in\mathbb{N}}$ is linearly independent. □

Proof of Theorem 5.2.1. We are still supposing that Cases 1 and 2 of Theorem 5.2.1 do not hold. Let the family $(g_{n,i})_{n\in\mathbb{N}, i\in\{1,2,3\}}$ of Γ and the family $(s_n)_{n\in\mathbb{N}}$ be as in Lemma 5.3.8. We show that we can apply Lemma 5.3.9 to

$$\begin{aligned} f_n &= f_{s_{n+n_0(k)},v}, \\ \eta_n &= g_{n+n_0(k),1} \cdot g_{n+n_0(k),2}, \text{ and} \\ h_n &= g^{-1}_{n+n_0(k),2} \cdot g_{n+n_0(k),3} \end{aligned}$$

for $n \in \mathbb{N}$. Let $n \geq n_0(k)$ and z be positive integers. Let $m \geq n_0(k)$, $i \neq j \in \{1,2,3\}$ and $\epsilon_1, \epsilon_2 \in \{1,-1\}$ such that a translate of s_n is contained in the geodesic joining v and $(g^{\epsilon_1}_{m,i} g^{\epsilon_2}_{m,j})^z v$. We show this is only possible for $n = m$, $\{i,j\} = \{1,3\}$ and $\epsilon_1 = \epsilon_2 = 1$. By construction of s_n as the almost concatenation of a long $w_{n,1}$-subgeodesic γ_1 and a long $w_{n,3}$-subgeodesic γ_3 we know that γ_1 and γ_3 are contained in the geodesic from v to $(g^{\epsilon_1}_{m,i} g^{\epsilon_2}_{m,j})^z v$. We now distinguish different cases for γ_1. In case 1, the geodesic γ_1 contains a long $w^{\epsilon_1}_{m,i}$-subgeodesic. By Lemma 5.3.5 this implies $m = n$, $i = 1$ and $\epsilon_1 = 1$. In Case 2, the geodesic γ_1 contains a long $w^{\epsilon_2}_{m,j}$-subgeodesic. Similarly to Case 1 we obtain $m = n$, $i = 1$ and $\epsilon_1 = 1$. If we are in neither of the previous cases, then

- γ_1 is contained in an almost concatenation of a long $w^{\epsilon_1}_{m,i}$-subgeodesic and a long $w^{\epsilon_2}_{m,j}$-subgeodesic, or
- γ_1 is contained in an almost concatenation of a long $w^{\epsilon_2}_{m,j}$-subgeodesic and a long $w^{\epsilon_1}_{m,i}$-subgeodesic.

In both cases, one half of γ_1, or more precisely a $w_{n,1}$-subgeodesic of γ_1 of o-length at least $(|w_{n,1}| + k - 1)/4 - 1$ is contained in a long $w^{\epsilon_1}_{m,i}$-subgeodesic or a long $w^{\epsilon_2}_{m,j}$-subgeodesic. By Lemma 5.3.5 this can only be true if $m = n$ and $i = 1, \epsilon_1 = 1$ or $j = 1, \epsilon_2 = 1$, respectively. We obtain similar results for γ_3, which proves the claim.

By Lemma 5.3.8 we know that at least $z - 1$ translates of s_n are contained in the geodesic joining v and $(g_{n,1} g_{n,3})^z v$. Thus, we conclude for all $m, z \in \mathbb{N}$

$$\begin{aligned} &f_{s_n,v}((g_{m,1} g_{m,2})^z) = f_{s_n,v}((g^{-1}_{m,2} g_{m,3})^z) = 0, \text{ and} \\ &f_{s_n,v}((g_{m,1} g_{m_3})^z) \begin{cases} = 0, & \text{if } n \neq m, \\ \geq z - 1, & \text{if } n = m. \end{cases} \end{aligned}$$

We apply Lemma 5.3.9 and obtain that the family $([\delta^1 f_{\widehat{s_{n+n_0(k)},v}}])_{n\in\mathbb{N}}$ is linearly independent in $H_b^2(\Gamma;\mathbb{R})$. $\square$

5.4 Consequences

In this section, we want to prove the following corollary of Theorem 5.2.1.

Corollary 5.4.1. *For $i \in \{1,\ldots,k\}$, let T_i be a regular tree of finite degree greater than 2. Let $\Gamma \subset \prod_{i=1}^k \operatorname{Aut}(T_i)$ be a co-compact lattice. We denote by $pr_j\colon \prod_{i=1}^k \operatorname{Aut}(T_i) \to \operatorname{Aut}(T_j)$ the projection maps for $j \in \{1,\ldots,k\}$. Then, the following are equivalent:*

1. *For every $i \in \{1,\ldots,k\}$ the closure $H_i := \overline{pr_i(\Gamma)}$ is locally ∞-transitive.*
2. *$H_b^2(\Gamma;\mathbb{R}) = 0$.*
3. *Any coclass induced by a median quasimorphism $\Gamma \to \mathbb{R}$ is trivial.*

We need some preliminary lemmas to be able to prove Corollary 5.4.1.

Lemma 5.4.2. *Let $T = (V,E)$ be a locally finite tree such that each vertex has degree at least 2. Let $\Lambda \subset \operatorname{Aut}(T)$ be a subgroup. If there exists a Λ-invariant subset $S \subset V$ and for all $n \in \mathbb{N}$ a vertex $v_n \in V$ with $d(S,v_n) \geq n$, then the set of orbits $\Lambda \setminus V$ is infinite.*

Proof. Let $S \subset V$ be a Λ-invariant subset and $(v_n)_{n\in\mathbb{N}}$ a family of vertices such that for all $n \in \mathbb{N}$ it is $d(S,v_n) \geq n$. By the Λ-invariance of S we deduce for all $\lambda \in \Lambda$

$$d(S,\lambda v_n) = d(\lambda S,\lambda v_n) = d(S,v_n) \geq n.$$

This means, we can find an increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers and a sequence $(w_n)_{n\in\mathbb{N}}$ of vertices of T such that for all $\lambda \in \Lambda$ we have

$$d(S,w_n) = d(S,\lambda w_n) = k_n.$$

By the Λ-invariance of the distance of the vertices w_n to S, we see that $(\Lambda \cdot w_n)_{n\in\mathbb{N}}$ is an infinite family of disjoint Λ-orbits in V. $\square$

Lemma 5.4.3. *Let $T = (V,E)$ be a locally finite tree such that each vertex has degree at least 2. Let $\Lambda \subset \operatorname{Aut}(T)$ be a subgroup such that the set of orbits $\Lambda \setminus V$ is finite and there is no fixed point in the boundary δT. Then Λ acts minimally on T.*

Proof. We show at first that Λ does not fix a proper closed subset of δT. Assume for a contradiction that $F \subset \delta T$ is a non-empty proper closed subset that is fixed by Λ. By assumption, F contains at least 2 ends. We denote by $T' = (V',E')$ the convex hull of T. Since T' is the subtree of T consisting exactly of the bi-infinite chains connecting two points in F, we deduce that T' is Λ-invariant. Recall from Lemma 5.1.8 that we have $\delta T' = F$. This means in particular $T \neq T'$ and we can find an edge $e \in E$ that is not contained in T'. Let $x \in E$ such that $T' \cap T_{e,x} = \emptyset$. As up to x all vertices of

$T_{e,x}$ have degree at least 2 by assumption, we find for all $n \in \mathbb{N}$ a vertex $v_n \in T_{e,x}$ with $d(x, v_n) = n$. Since T' does not lie in the connected component $T_{e,x}$ we compute for all $n \in \mathbb{N}$

$$d(T', v_n) \geq d(x, v_n) = n.$$

By Lemma 5.4.2, this means that $\Lambda \setminus V$ is infinite and we obtain a contradiction to the assumption.

The considerations above show, that there is no non-empty proper closed subset of δT that is fixed by Λ. We want to use this to prove the minimality of $\Lambda \curvearrowright T$. For this, let $T' = (V', E')$ be a Λ-invariant non-empty proper subtree of T. At first we show that T' has infinitely many vertices. As T' is non-empty we can choose a vertex v of T. By the Λ-invariance of T', we have $\Lambda v \subset V'$. Now it suffices to prove that Λv consists of infinitely many vertices. In order to do this, suppose for a contradiction that Λv is a finite set. Then, we can find for all $n \in \mathbb{N}$ a vertex $v_n \in V$ with $d(\Lambda v, v_n) \geq n$ by the local finiteness of T. As Λv clearly is a Λ-invariant subset of T we can apply Lemma 5.4.2 and obtain that $\Lambda \setminus V$ is infinite. This contradicts the assumption. Thus, Λv and thereby also V' is infinite. By Lemma 5.1.4, the boundary $\delta T'$ is non-empty. Furthermore, $\delta T'$ is a Λ-invariant subset of δT by the Λ-invariance of T'. We show that $\delta T' \subset \delta T$ is a proper closed subset. For the properness, let $e = \{\alpha(e), \omega(e)\} \in E$ be a vertex of T that is not contained in T'. This exists, since $T' \neq T$. Without loss of generality, we assume $T' \subset T_{e,\alpha(e)}$. Then, $\delta T_{e,\omega(e)} \subset \delta T \setminus \delta T'$ is non-empty. For $\delta T' \subset \delta T$ to be closed, we show that $\delta T \setminus \delta T'$ is open in δT. Let $\xi \in \delta T \setminus \delta T'$ and $\gamma \colon \mathbb{N} \to V$ be an infinite chain connecting a vertex $v \in V'$ with ξ. Then, there exists an $n \in \mathbb{N}_{>0}$ such that $\gamma(n) \notin V'$ and therefore $e := \{\gamma(n-1), \gamma(n)\} \notin E'$. This means $T' \subset T_{e,\gamma(n-1)}$ and hence, ξ is contained in the open subset $\delta T_{e,\gamma(n)} \subset \delta T \setminus \delta T'$. Now $\delta T'$ is a non-empty proper closed subset of δT that is Λ-invariant. But this is not possible by the first part of the proof and we conclude that no proper subtree of T is fixed by Λ. □

Lemma 5.4.4. *Let G be a locally compact group and H be a subgroup. Then G/H is compact if and only if there exists a compact subset $K \subset G$ with $G = K \cdot H$.*

Proof. We denote by $\pi \colon G \to G/H$ the canonical quotient map. If there exists a compact subset $K \subset G$ with $G = K \cdot H$, then $G/H = \pi(K)$ is compact as π is continuous and $\pi_{|K}$ is surjective.

On the other hand, suppose that G/H is compact. For every $x \in G/H$ we choose a representative $y_x \in G$. Since G is locally compact, there is an open neighbourhood U_x of y_x and a compact subset $K_x \subset G$ with

$$y_x \in U_x \subset K_x.$$

Then, $\pi(U_x)$ is an open neighbourhood of x in G/H as $\pi^{-1}(\pi(U_x)) = \bigcup_{h \in H} h \cdot U_x$ is a union of open subsets of G. As G/H is compact, we find finitely many $x_1, \ldots, x_n \in G/H$

such that

$$G/H = \bigcup_{x \in G/H} \pi(U_x) = \pi(\bigcup_{i=1}^{n} U_{x_i}).$$

Then $K := \bigcup_{i=1}^{n} K_{x_i}$ is compact as a finite union of compact subsets with

$$G = \left(\bigcup_{i=1}^{n} U_{x_i}\right) \cdot H \subset \left(\bigcup_{i=1}^{n} K_{x_i}\right) \cdot H = K \cdot H.$$

□

Now we are able to prove Corollary 5.4.1.

Proof of Corollary 5.4.1. For $i \in \{1, \dots, k\}$, let T_i be a regular tree of finite degree greater than 2. Let $G := \prod_{i=1}^{k} \mathrm{Aut}(T_i)$ and $\Gamma \subset G$ be a cocompact lattice. We denote by $pr_j \colon G \to \mathrm{Aut}(T_j)$ the projection maps.

Suppose we are in Case 1, i.e. for every $i \in \{1, \dots, k\}$ the closure $H_i = \overline{pr_i(\Gamma)}$ is locally ∞-transitive. By Lemma 5.1.15 this means that H_i acts transitively on δT_i. Using [4, Corollary 26] we conclude $H_b^2(\Gamma; \mathbb{R}) = 0$.

The implication from Case 2 to Case 3 is clear.

For the last implication from Case 3 to Case 1, we make use of Theorem 5.2.1. At first, we observe for $i \in \{1, \dots, k\}$ that we obtain a group action of Γ on $\mathrm{Aut}(T_i)$ induced by the left-translation action of Γ on G. Furthermore, for $H_i = \overline{pr_i(\Gamma)}$ we obtain a well-defined continuous and surjective map

$$G/\Gamma \to \mathrm{Aut}(T_i)/H_i,$$

hence, $\mathrm{Aut}(T_i)/H_i$ is compact.

Now we consider a non-zero, positive, regular, $\mathrm{Aut}(T_i)$-left invariant Borel measure λ on G that exists according to Theorem 1.2.8. For this measure, we have $0 < \lambda(K) < \infty$ for $K \subset \mathrm{Aut}(T_i)$ compact. As $\mathrm{Aut}(T_i)/H_i$ is compact, there exists by Lemma 5.4.4 a compact subset K of $\mathrm{Aut}(T_i)$, whose projection $\pi(K)$ to the quotient space coincides with $\mathrm{Aut}(T_i)/H_i$. The finiteness of $\lambda(K)$ then induces a finite $\mathrm{Aut}(T_i)$-invariant measure on $\mathrm{Aut}(T_i)/H_i$.

As $\mathrm{Aut}(T_i)$ contains a free group of rank 2 as discrete subgroup (Theorem 5.1.12) and $\mathrm{Aut}(T_i)$ is a Hausdorff topological group, this means that this free group is a closed subgroup of $\mathrm{Aut}(T_i)$. In particular, it is a closed subgroup that is not amenable. By the inheritance properties of amenability (Theorem 1.2.13) we conclude that $\mathrm{Aut}(T_i)$ is not amenable.

If the closed subgroup H_i of $\mathrm{Aut}(T_i)$ would fix a point ζ in the boundary δT_i, then $H_i \subset \mathrm{Stab}_{\mathrm{Aut}(T)}(\zeta)$ would be a closed subgroup of the stabilizer of ζ, which is amenable by Theorem 5.1.13. By the inheritance properties of amenability, this implies that H_i is amenable, too. In this case, we obtain a closed amenable subgroup H_i of $\mathrm{Aut}(T_i)$ such that $\mathrm{Aut}(T_i)/H_i$ admits a finite $\mathrm{Aut}(T_i)$-invariant measure. By Theorem 1.2.14, $\mathrm{Aut}(T_i)$ has to be amenable, which is a contradiction to the previous paragraph. We conclude that $\mathrm{Aut}(T_i)$ does not fix a point in the boundary δT_i.

Let V denote the vertex set of $\mathrm{Aut}(T_i)$. The next step of the proof is to show that the group action of H_i on V has only finitely many orbits. To see this, we choose at first a compact subset K of $\mathrm{Aut}(T_i)$ such that $\mathrm{Aut}(T_i) = K \cdot H_i$. This exists by Lemma 5.4.4 as $\mathrm{Aut}(T_i)/H_i$ is compact. We consider

$$K' \coloneqq K^{-1} \subset \mathrm{Aut}(T_i),$$

which is also a compact subset of the topological group $\mathrm{Aut}(T_i)$. We then have

$$\mathrm{Aut}(T_i) = H_i \cdot K'.$$

We choose $v \in V$ and take a look at the canonical map

$$\begin{aligned} \varphi\colon K' &\to K' \cdot v \\ f &\mapsto f(v). \end{aligned}$$

This map is continuous, as $K' \cdot v \subset V$ is discrete and for every $x \in K' \cdot v$ we can find $f_x \in K'$ with $f_x(v) = x$ and hence,

$$\varphi^{-1}(x) = \{f \in K' \mid f(v) = x = f_x(v)\} = U_{\{v\}}(f_x) \cap K'$$

is open in K' by the definition of the topology on $\mathrm{Aut}\, T_i$. As K' is compact and φ surjective, we obtain that $K' \cdot v$ is a compact subset of the discrete set V and consequently it is finite. Furthermore, $K' \cdot v$ contains at least one representative for each orbit of $H_i \curvearrowright V$ because it is

$$H_i \cdot (K' \cdot v) = (H_i \cdot K') \cdot v = \mathrm{Aut}(T_i) \cdot v = V.$$

Hence, $H_i \setminus V$ is finite and we are able to apply Lemma 5.4.3 and obtain that the action $H_i \curvearrowright T_i$ is minimal. With this, we verified that the assumptions of Theorem 5.2.1 are satisfied. Furthermore, as H_i does not fix a point in the boundary of T_i and as every co-class represented by a median quasimorphism is zero, we are in Case 2 of Theorem 5.2.1, which means that for every vertex $x \in V$ and every $n \in \mathbb{N}$ the group H_i acts transitively on the set of geodesics of length n starting at a vertex in Γx. In particular, this means for every $x \in V$ that H_i acts transitively on all finite spheres centred at x. We conclude that H_i is locally ∞-transitive. □

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