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The Linearised Einstein Equations on Lorentzian Manifolds

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Zusammenfassung

In der Allgemeinen Relativitätstheorie wird die Raumzeit des Universums als vierdimensionale Lorentz-Mannigfaltigkeit beschrieben, deren metrischer Tensor die Einstein-Gleichungen erfüllt. Diese sind nichtlinear und somit schwer lösbar. Kleine Störungen der Metrik können aber gut auch mithilfe der linearisierten Einsteingleichungen modelliert werden. Dies ermöglicht zum Beispiel die Beschreibung der Ausbreitung von Gravitationswellen im Vakuum.

In dieser Arbeit sollen die Einsteingleichungen linearisiert und dann Existenz und Eindeutigkeit ihrer Lösungen für das Cauchy-Problem studiert werden. Es handelt sich dabei im mathematischen Sinn nicht um Wellengleichungen. Aufgrund einer linearisierten Diffeomorphismus-Invarianz besitzen sie viele "Eichlösungen", die nicht messbar sind und zu physikalisch ununterscheidbaren Raumzeiten führen. Das Eindeutigkeitsresultat gilt deswegen nur "bis auf Eichung", und um das Existenzresultat zu zeigen, muss eine Hilfs-Wellengleichung gelöst werden.

Es ist physikalisch nicht zu erwarten, dass die Anfangsdaten oder die Lösung der Gleichungen glatt sind. Aus diesem Grund werden die Resultate für Tensoren mit beliebigem reellen Sobolev-Grad gezeigt.

Summary

In General Relativity, the space-time of the universe is described as a four-dimensional Lorentzian manifold whose metric tensor satisfies the Einstein equations. These are nonlinear and therefore hard to solve. However, small perturbations of the metric can also be modelled using the linearised Einstein equations. This allows, for example, for the description of the propagation of gravitational waves in the vacuum.

In this paper, the linearised Einstein equations shall be derived, and afterwards the existence and uniqueness of their solutions for the Cauchy problem shall be studied. Mathematically, these are not wave equations. Due to a linearised diffeomorphism invariance, they admit many "gauge solutions", which are not measurable and yield physically indistinguishable space-times. Therefore, the uniqueness result only holds "up to gauge", and to prove the existence result, an auxiliary wave equation must be solved.

On physical grounds, it need not be expected that the initial data or the solutions of the equations are smooth. For this reason, the results are shown for tensors with arbitrary real Sobolev degree.

1. Preliminaries

As stated in the introduction, we want to derive and study the linearised Einstein equations, following the treatment given in the paper [1] by Lindblad Petersen. Due to the "gauge invariance", the resulting Cauchy problem is ill-posed for Sobolev sections. As shown in [1], one can however obtain a well-posed Cauchy problem in suitable quotient spaces (the spaces of "solutions modulo gauge solutions" and "initial data modulo gauge-producing initial data" are isomorphic). The next step would then be to study the initial data ("modulo gauge-producing initial data") to obtain a classification of the possible solutions, see [1], theorem 6.2. However, as stated above, here we restrict ourselves to the derivation of the equations and the proofs of existence and uniqueness "up to gauge". In the last section, we will study a simple example motivated by physics.

In the following, some notation is defined that shall be assumed throughout the rest of the text unless otherwise stated. (Some standard notation, for the sake of completeness, is also defined in Appendix A.)

Setting 1.1. We denote by (M, g) the underlying smooth Lorentzian Einstein manifold, where we assume M to have vanishing Einstein constant.[1] The dimension of M shall be $n+1$, where $n \geq 2$. Thus the Einstein equations read as

$$\mathrm{ric}_g = 0.$$

We use subscripts or superscripts to distinguish the various objects associated with the metric, e.g. like above we write ric_g for the Ricci tensor of g, but these are dropped if they should be clear from the context.

We assume M to be globally hyperbolic.[2]

Definition 1.2.

1. A hypersurface $\Sigma \subset M$ is called a *Cauchy surface*, if every inextensible timelike curve in M meets Σ exactly once. Here a vector $V \in \mathrm{T}M$ is called *timelike* resp. *spacelike* if $g(V, V) < 0$ resp. $g(V, V) > 0$, and a submanifold $N \subset M$ is called *timelike* resp. *spacelike* if all vectors tangent to it are timelike resp. spacelike.

2. By [2], there exists (due to the global hyperbolicity of M) a smooth *Cauchy temporal function* $t : M \to \mathbb{R}$, i.e. for all $\tau \in t(M)$, $\Sigma_\tau := t^{-1}(\tau)$ is a smooth spacelike Cauchy hypersurface and $\mathrm{grad}(t) := (\mathrm{d}t)^\sharp$ is timelike and past directed. Thus the metric can be written as
$$g = -\alpha^2(\mathrm{d}t \otimes \mathrm{d}t) + \tilde{g}_\tau,$$
where $\alpha : M \to \mathbb{R}$ is a positive function and $\tilde{g}_\tau$ is the positive definite metric on Σ_τ, depending smoothly on $\tau \in t(M)$. This may also be regarded as a tensor on M when composing it (implicitly) with the projection onto $((\mathrm{d}t)^\sharp)^\perp$. We abbreviate $\nabla_t := \nabla_{\mathrm{grad}(t)}$.

3. The future pointing unit normal ν to Σ_τ is then given by $\nu = -\frac{1}{\alpha}\mathrm{grad}(t)|_{\Sigma_\tau}$.

4. For a subset $K \subset M$, we denote by $J(K)$ the union of its causal past and future, i.e. the set of all points in M that can be reached from K by timelike or spacelike curves.

In the following definitions, we choose one of the Cauchy surfaces Σ_τ and denote it by Σ.

[1] The word "smooth" is here used for "as often differentiable as needed".

[2] We do not define this term here, as we will only need the properties of M stated in the following.

Definition 1.3. The first fundamental form on Σ shall be denoted by $\tilde{g}$ and the second fundamental form by $\tilde{k}$.[3] For the second fundamental form we use the definition

$$\tilde{k}(X,Y) = g(\nabla_X \nu, Y)$$

for $X, Y \in \mathrm{T}\Sigma$.[4]

Definition and Remark 1.4. 1. Let $\mathcal{T}^{i,j}M := (\mathrm{T}M)^{\otimes i} \otimes (\mathrm{T}^*M)^{\otimes j}$ be the bundle of (i,j)-tensors on M (for nonnegative integers i,j). We will need to consider different spaces of sections in this bundle (also called tensor fields resp. by abuse of notation simply tensors). Precise definitions will be given below, together with suitable topologies.

2. We note that $C^\infty(M, \mathcal{T}^{0,0}M) = C^\infty(M)$ (the space of smooth $\mathbb{R}$-valued functions on M), $C^\infty(M, \mathcal{T}^{1,0}M) = \mathcal{X}(M)$ (the space of smooth vector fields on M), and $C^\infty(M, \mathcal{T}^{0,1}M) = \Omega^1(M)$ (the space of smooth 1-forms on M).

3. We denote the bundle of symmetric $(0,2)$-tensors on M by $\mathrm{S}^2M \subset \mathcal{T}^{0,2}M$ (i.e. for $S \in \mathcal{T}^{0,2}M$, we have $S \in \mathrm{S}^2M \Leftrightarrow \forall_{X,Y\in \mathrm{T}M} : S(X,Y) = S(Y,X)$). We define the *symmetrization* $\mathrm{sym} : \mathcal{T}^{0,2}M \to \mathrm{S}^2M$ by

$$\mathrm{sym}(S)(X,Y) = \frac{1}{2}(S(X,Y) + S(Y,X))$$

for $S \in \mathcal{T}^{0,2}M$, $X, Y \in \mathrm{T}M$.

4. We introduce a short-hand notation which will be useful later: For $S \in \mathrm{S}^2M$, let

$$\overline{S} := S - \frac{1}{2}\mathrm{tr}_g(S)g$$

(which can also be applied pointwise to sections of S^2M).

Definition 1.5. We define the *divergence* δ^g: For $S \in C^\infty(M, \mathcal{T}^{0,j}M)$, where $j \geq 1$, let

$$\delta^g(S) := -(\mathrm{tr}_g)_{11}(\nabla_{\cdot} S(\cdot, \cdots)),$$

where the first tensor slot is here the one in ∇. If $\{e_i\}_{0\leq i\leq n}$ is a local orthonormal frame, this means that for $X_1, \ldots, X_{j-1} \in \mathrm{T}M$, we have

$$\delta^g(S)(X_1, \ldots, X_{j-1}) = -\sum_{i=0}^{n} \epsilon_i (\nabla_{e_i} S)(e_i, X_1, \ldots, X_{j-1}),$$

where $\epsilon_i := g(e_i, e_i)$.

Definition 1.6. The *Lichnerowicz operator* $\Box_L^g$ on smooth symmetric $(0,2)$-tensor fields shall be defined as follows: For $S \in C^\infty(M, \mathrm{S}^2M)$, let

$$\Box_L^g S := \nabla^*\nabla S + \mathrm{ric}_g \circ S + S \circ \mathrm{ric}_g - 2\overset{\circ}{\mathrm{R}}_g S,$$

where

$$\nabla^*\nabla S := -(\mathrm{tr}_g)_{12}(\nabla^2_{\cdot,\cdot} S(\cdot,\cdot)))$$

[3] In general, we decorate tensors on Σ by tildes.

[4] The first fundamental form is the restriction of g to $\mathrm{T}\Sigma$, as usual.

is the *connection-Laplace operator* and

$$\mathring{\mathrm{R}}_g S := (\mathrm{tr}_g)_{14}(S(\mathrm{R}_g(\cdot,\cdot)\cdot,\cdot))$$

is the action of R on symmetric $(0,2)$-tensor fields. We note that $\Box_L^g$ is a wave operator (defined below). Sometimes we will also need the *Laplace-Beltrami operator* $\Box^g$, which is defined by

$$\Box^g f = -\mathrm{tr}_g(\nabla \mathrm{d} f).$$

This is a wave operator as well.

2. Topological vector spaces of sections

We will need to work with distributional sections of vector bundles, which shall be defined in this section. The definitions are mostly quoted from [3], Sections 2.3 and 2.4, and [1], Section 3.

Setting 2.1. We remain in Setting 1.1. Furthermore, we choose an auxiliary Riemannian metric $\overline{g}$ on M, with associated Levi-Civita connection $\overline{\nabla}$.[5] For simplicity we choose $\overline{g}$ such that the volume elements dvol_g and $\mathrm{dvol}_{\overline{g}}$ coincide.

Additionally, in the following, let $E \to M$ be a real vector bundle over M which is equipped with an (auxiliary) fiberwise scalar product g_E that depends smoothly on the basepoint. One chooses a connection ∇^E on E and combines it with the Levi-Civita connection $\overline{\nabla}$ on T^*M to obtain connections on $(\mathrm{T}^*M)^{\otimes j} \otimes E$ for all integers $j \geq 0$, still denoted ∇^E. Furthermore the scalar products $\overline{g}$ and g_E induce norms $|\cdot|$ on $(\mathrm{T}^*M)^{\otimes j} \otimes E$ for all integers $j \geq 0$. Let $I = \{1, \dots, m\}$ for $m \in \mathbb{N}$ or $I = \mathbb{N}$ be an index set, and for $i \in I$, let $K_i \subset M$ be compact subsets, such that $K_i \subset \mathrm{int}(K_{i+1})$ for $i \in I$ (in the finite case, for $m \neq i \in I$) and $M = \bigcup_{i \in I} K_i$.

Definition 2.2 ([3], Section 2.3)**.** The *space of smooth sections* in E is denoted by

$$C^\infty(M, E).$$

For $f \in C^\infty(M,E)$ and $j \in \mathbb{N}_0$, the j'th covariant derivative $(\nabla^E)^j f := \underbrace{\nabla^E \cdots \nabla^E}_{j \text{ times}} f$ is a smooth section of $(\mathrm{T}^*M)^{\otimes j} \otimes E$. For $K \subset M$ compact, we define the seminorm

$$\|f\|_{K,m} := \max_{j=0,\dots,m} \max_{x \in K} |(\nabla^E)^j f(x)|.$$

These seminorms shall define the topology of $C^\infty(M,E)$, which is called the *C^∞-topology.*

We have:

Lemma 2.3. *The topology defined on $C^\infty(M,E)$ defined above turns this space into a Fréchet space, i.e. a complete topological vector space which is Hausdorff and whose topology can also be defined using a countable family of seminorms. This topology does neither depend on the choice of the connection ∇^E, nor on the choice of the scalar products g_E and $\overline{g}$.*

The statement can be proven analogously to Theorem 1.1.5 in [4]. The countable family of seminorms which induces the same topology is given by $\{\|\cdot\|_{K_i,i}\}_{i \in I}$.

[5]The purpose of this metric is to obtain e.g. a positive operator $(\nabla^E)^*\nabla^E + \mathrm{id}$, see below. We only need it in this section.

Definition 2.4 ([3], Section 2.3)**.** 1. For $K \subset M$ compact, we define

$$C_K^\infty(M,E) := \{f \in C^\infty(M,E) \,|\, \mathrm{supp}(f) \subset K\},$$

the *space of smooth sections in E supported in K.*

2. The *space of compactly supported smooth sections* in E is defined as

$$C_\mathrm{c}^\infty(M,E) := \bigcup_{\substack{K\subset M\\ \text{compact}}} C_K^\infty(M,E).$$

This is the same as $\bigcup_{i\in I} C_{K_i}^\infty(M,E)$. The topology on $C_\mathrm{c}^\infty(M,E)$ is defined to be the strict inductive limit topology induced by the inclusions $C_{K_i}^\infty(M,E) \hookrightarrow C_\mathrm{c}^\infty(M,E)$, where $i \in I$.

Remark 2.5. 1. By definition, the strict inductive limit topology on $C_\mathrm{c}^\infty(M,E)$, induced by the inclusions $C_{K_i}^\infty(M,E) \hookrightarrow C_\mathrm{c}^\infty(M,E)$, is the finest locally convex vector space topology that makes these inclusions continuous. As $C_{K_i}^\infty(M,E)$ are Fréchet spaces for all $i \in I$, this turns $C_\mathrm{c}^\infty(M,E)$ into a so-called LF-space (see e.g. [5], Chapter 13). This topology on $C_\mathrm{c}^\infty(M,E)$ can also be proven to be independent of the choice of the K_i.

2. There is a continuous inclusion $C_\mathrm{c}^\infty(M,E) \hookrightarrow C^\infty(M,E)$, but the topology on $C_\mathrm{c}^\infty(M,E)$ defined above is finer than the subspace topology induced by this inclusion.

Definition 2.6 ([3], Section 2.4)**.** The *space of distributional sections* in E is denoted

$$\mathcal{D}'(M,E).$$

As a set, this is the set of continuous linear functionals on $C_\mathrm{c}^\infty(M,E^*)$, where E^* is the dual bundle of E. It is topologized by the weak* topology. The evaluation of $u \in \mathcal{D}'(M,E)$ on $\phi \in C_\mathrm{c}^\infty(M,E^*)$ is denoted $u[\phi]$.

Definition 2.7 ([3], Section 2.5)**.** For $f_1, f_2 \in C_\mathrm{c}^\infty(M,E)$, we define the *$L^2$-scalar product* of f_1 and f_2 by

$$(f_1,f_2)_{L^2(M,E)} := \int_M g_E(f_1,f_2)\,\mathrm{dvol}_g,$$

where dvol_g is the volume element of the semi-Riemannian metric g (which coincides with $\mathrm{dvol}_{\overline{g}}$). We define the *$L^2$-norm* of $f \in C_\mathrm{c}^\infty(M,E)$ by

$$\|f\|_{L^2(M,E)} := \sqrt{(f,f)_{L^2(M,E)}} = \sqrt{\int_M g_E(f,f)\,\mathrm{dvol}_g}.$$

We define $L^2(M,E)$ to be the completion of $C_\mathrm{c}^\infty(M,E)$ with respect to this norm, the *space of square integrable sections* in E.

Remark 2.8. 1. We have a continuous inclusion $L^2(M,E) \hookrightarrow \mathcal{D}'(M,E)$, defined by

$$f \mapsto \left(\phi \mapsto (f,\phi)_{L^2(M,E)}\right).$$

We consider elements in $L^2(M,E)$ as functions (which are only defined up to the L^2-equivalence relation, identifying all functions which coincide outside of a set of measure zero). Then this can also be written as

$$f \mapsto \left(\phi \mapsto \int_M g_E(f,\phi)\,\mathrm{dvol}_g\right).$$

2. Analogously to the smooth case, we could also consider for fixed $K \subset M$ compact a space $L^2_K(M,E)$ of square integrable sections in E supported in K, and a space $L^2_c(M,E)$ of square integrable compactly supported sections.

Definition and Remark 2.9 ([3], Section 2.6.2)**.** 1. Let $K \subset M$ be compact and let $s \in \mathbb{R}$. Let $K_1 \subset M$ be compact such that $K \subset \operatorname{int}(K_1)$ and the boundary of K_1 is smooth. Let $K' := K_1 \cup_{\partial K_1} K_2$ be the double of K_1 (as a smooth manifold). This means that we let K_2 to be simply another copy of K_1 and glue the two copies along their boundary $\partial K_1 = \partial K_2$. Analogously, we double $E|_{K_1}$ to a bundle $E' \to K'$, and we extend the given metrics and connections on K to smooth metrics and connections on K'. Then we can consider elements of $C^\infty_K(M,E)$ also as elements of $C^\infty(K',E')$ (extending them by zero to K_2).

The operator $(\nabla^E)^*\nabla^E + \mathrm{id} : C^\infty(K',E') \to C^\infty(K',E')$ is positive and essentially self-adjoint with respect to the $L^2(K',E')$-scalar product, hence its closure is a positive self-adjoint extension. (Here $(\nabla^E)^*$ is the formal adjoint of $\nabla^E : L^2(K',E') \supset C^\infty(K',E') \to C^\infty(K',\mathrm{T}^*K' \otimes E') \subset L^2(K',\mathrm{T}^*K' \otimes E')$, where the scalar product on $L^2(K',\mathrm{T}^*K' \otimes E')$ is induced from the Riemannian metric on $\mathrm{T}K'$ and the scalar product on E'. See the next section for the definition of the formal adjoint.)

The square root of this extension shall be denoted D. Then we define the s'th *Sobolev norm* of $f \in C^\infty_K(M,E)$ by

$$\|f\|_{H^s_K(M,E)} := \|\mathrm{D}^s f\|_{L^2(K',E')}.$$

The *space of sections of Sobolev regularity s supported in K* shall be defined to be the completion of $C^\infty_K(M,E)$ with respect to this norm and denoted by $H^s_K(M,E)$. It does not depend on the choices of K_1 and the scalar products and connections on K' (i.e. neither on the choice of the extensions from K_1 to K' nor on the scalar products and connections on K_1 themselves). We note $H^0_K(M,E) = L^2_K(M,E)$.

2. We define for $s \in \mathbb{R}$ the *space of sections of Sobolev regularity s with compact support* in E by

$$H^s_c(M,E) := \bigcup_{\substack{K \subset M \\ \text{compact}}} H^s_K(M,E),$$

where for $K \subset L$ compact subsets of M and $s \in \mathbb{R}$, the inclusion $C^\infty_K(M,E) \hookrightarrow C^\infty_L(M,E)$ induces an inclusion $H^s_K(M,E) \hookrightarrow H^s_L(M,E)$. Analogously to the smooth case, this is the same as $\bigcup_{i \in I} H^s_{K_i}(M,E)$ and shall carry the strict inductive limit topology induced by the inclusions $H^s_{K_i}(M,E) \hookrightarrow H^s_c(M,E)$. The resulting topology is again independent of the choice of the K_i.

Remark 2.10. 1. For $r > s \in \mathbb{R}$, there exist continuous inclusions $H^r_K(M,E) \hookrightarrow H^s_K(M,E)$ and $H^r_c(M,E) \hookrightarrow H^s_c(M,E)$.

2. Here we only considered the compactly supported case, analogously we could define spaces of possibly non-compactly supported Sobolev sections $H^s(M,E)$. (However, these depend on the choices of g_E and ∇^E if M is not compact. For $H^s_K(M,E)$, where $K \subset M$ is compact, and for $H^s_c(M,E)$, this is not the case, see definition and remark 2.11 below. We will only use $H^s(M,E)$ in the case that M is compact, when it equals $H^s_c(M,E)$ anyways.)

3. We note that the spaces $H^s_K(M,E)$ (for $K \subset M$ compact, for $s \in \mathbb{R}$) embed continuously into $\mathcal{D}'(M,E)$: The pairing $C^\infty_K(M,E) \times C^\infty_c(M,E^*) \to \mathbb{R}$, $(f,\phi) \mapsto \int_M \phi(f)\,\mathrm{dvol}_g$ extends uniquely to a bicontinuous pairing $H^s_K(M,E) \times C^\infty_c(M,E^*) \to \mathbb{R}$.[6] Using this, every $f \in H^s_K(M,E)$ defines a continuous functional on $C^\infty_c(M,E^*)$ similar to the L^2-case, yielding continuous embeddings $H^s_K(M,E) \hookrightarrow \mathcal{D}'(M,E)$ and $H^s_c(M,E) \hookrightarrow \mathcal{D}'(M,E)$.

The preceding remark allows for the definition of local Sobolev spaces:

Definition and Remark 2.11 ([3], Section 2.6.4)**.** 1. The *space of sections with local Sobolev regularity* $s \in \mathbb{R}$ *in* E is defined by

$$H^s_{\mathrm{loc}}(M,E) := \left\{ f \in \mathcal{D}'(M,E) \,|\, \chi f \in H^s_c(M,E) \,\text{for all}\, \chi \in C^\infty_c(M,\mathbb{R}) \right\}.$$

For $\chi \in C^\infty_c(M,\mathbb{R})$, the map

$$f \mapsto \|\chi f\|_{H^s(\mathrm{supp}(\chi), E|_{\mathrm{supp}(\chi)})}$$

defines a seminorm on $H^s_{\mathrm{loc}}(M,E)$. The family of such seminorms is used to define a topology on $H^s_{\mathrm{loc}}(M,E)$. By considering cutoff functions $(\chi_j)_{j\in\mathbb{N}}$ such that the sets $\{\chi_j \equiv 1\}$ exhaust M, we note that the same topology can also be defined using a countable family of seminorms. This turns $H^s_{\mathrm{loc}}(M,E)$ into a Fréchet space.

2. We define

$$H^\infty_{\mathrm{loc}}(M,E) := \bigcap_{s\in\mathbb{R}} H^s_{\mathrm{loc}}(M,E) = C^\infty(M,E)$$

(the last equality by the Sobolev embedding theorem).

3. Similarly to lemma 2.3, one can show that the topologies on $H^s_K(M,E)$ (where $K \subset M$ is compact), $H^s_c(M,E)$, H^s_{loc} and $H^\infty_{\mathrm{loc}}(M,E)$ are independent of the auxiliary metrics $\overline{g}$ and g_E and the connection ∇^E (since on compact subsets of M, one can

[6] To prove this, it needs to be verified that if $(f_n)_{n\in\mathbb{N}} \subset C^\infty_K(M,E)$ is a Cauchy sequence with respect to the $H^s_K(M,E)$-norm and $\phi \in C^\infty_c(M,E^*)$, then $\left(\int_M \phi(f_n)\,\mathrm{dvol}_g\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. This is done as follows: There exists $u \in C^\infty_c(M,E)$ such that $\phi(p)(v(p)) = g_E(u(p),v(p))$ for all $p \in M$, $v \in C^\infty_K(M,E)$. Let $\tilde{K} := K \cup \mathrm{supp}(\phi)$. We calculate, where $\tilde{\mathrm{D}}, \tilde{K}', \tilde{E}'$ are constructed as D, K', E' in definition and remark 2.9, but with $\tilde{K}$ instead of K,

$$\begin{aligned}\left|\int_M \phi(f_n)\,\mathrm{dvol}_g - \int_M \phi(f_m)\,\mathrm{dvol}_g\right| &= \left|\int_{\tilde{K}} g_E(u, \tilde{\mathrm{D}}^{-s}\tilde{\mathrm{D}}^s f_n - \tilde{\mathrm{D}}^{-s}\tilde{\mathrm{D}}^s f_m)\,\mathrm{dvol}_g\right| \\ &= |(u, \tilde{\mathrm{D}}^{-s}\tilde{\mathrm{D}}^s f_n - \tilde{\mathrm{D}}^{-s}\tilde{\mathrm{D}}^s f_m)_{L^2(\tilde{K}',\tilde{E}')}| \\ &= |((\tilde{\mathrm{D}}^{-s})^* u, \tilde{\mathrm{D}}^s f_n - \tilde{\mathrm{D}}^s f_m)_{L^2(\tilde{K}',\tilde{E}')}| \\ &\le \left\|(\tilde{\mathrm{D}}^{-s})^* u\right\|_{L^2(\tilde{K}',\tilde{E}')} \left\|\tilde{\mathrm{D}}^s f_n - \tilde{\mathrm{D}}^s f_m\right\|_{L^2(\tilde{K}',\tilde{E}')} \to 0\end{aligned}$$

for $n, m \to \infty$. Here in the second-to-last step, we used the Cauchy-Schwarz inequality, and the last expression converges to 0 for $n, m \to \infty$ because $(\tilde{\mathrm{D}}^s f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\tilde{K}',\tilde{E}')$ (by continuity of the embedding $H^s_K(M,E) \hookrightarrow H^s_{\tilde{K}}(M,E)$ and the definition of the norm on the latter space), and $\left\|(\tilde{\mathrm{D}}^{-s})^* u\right\|_{L^2(\tilde{K}',\tilde{E}')} < \infty$.

(Here $(\tilde{\mathrm{D}}^{-s})^*$ is the Hilbert space adjoint of the unbounded pseudodifferential operator $\tilde{\mathrm{D}}^{-s}$; the theory of such operators is e.g. described in [6], chapter III.3. Since $\tilde{\mathrm{D}}^{-s}$ is constructed using only real functions it follows from the functional calculus that it is actually the same as $(\tilde{\mathrm{D}}^{-s})^*$. Since $u \in C^\infty(\tilde{K}, E|_{\tilde{K}}) \subset H^{-s}_{\tilde{K}}(M,E)$, therefore by definition $(\tilde{\mathrm{D}}^{-s})^* u \in L^2(\tilde{K}',\tilde{E}')$, i.e. $\left\|(\tilde{\mathrm{D}}^{-s})^* u\right\|_{L^2(\tilde{K}',\tilde{E}')} < \infty$.)

estimate the Sobolev norms resulting from different choices against each other). In particular, when we later consider subbundles of tensor bundles, we will not specify this data any more. Instead we will usually work with the semi-Riemannian metric g and the Levi-Civita connection ∇.

As we want to study wave equations, it will be natural to consider sections whose differentiability properties in the time direction are different than the ones in the spatial directions. For this, we define time-dependent function spaces, adapted to the Cauchy temporal function $t : M \to \mathbb{R}$.

Definition 2.12 ([3], Section 2.7, [1], Chapter 3)**.** For $s \in \mathbb{R}$, observe that the family $\{H^s_{\mathrm{loc}}(\Sigma_\tau, E|_{\Sigma_\tau})\}_{\tau \in t(M)}$ is a bundle of Fréchet spaces over the interval $t(M) \subset \mathbb{R}$. The space of m-times continuously differentiable sections of this bundle is denoted by

$$C^m(t(M), H^s_{\mathrm{loc}}(\Sigma_\cdot, E|_{\Sigma_\cdot})),$$

which is a Fréchet space when endowed with a topology induced by the C^m-seminorms on compact intervals in $t(M)$. We define the *space of sections of finite energy of infinite order* in E by

$$CH^s_{\mathrm{loc}}(M, E, t) := \bigcap_{j=0}^{\infty} C^j\left(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_\cdot, E|_{\Sigma_\cdot})\right).$$

These intersections carry an induced Fréchet topology.

The spaces $CH^s_{\mathrm{loc}}(M, E, t)$ will be the natural spaces to work with when solving the linearised Einstein equations. We note that they depend on the time function. It can be shown (as in [3], Corollary 18) that the solutions to the linearised Einstein equations will not do so.

As final remarks of this section, it shall be noted how some standard tensor operations can be defined for distributional tensors.

Remark 2.13.

1. If $X \in \mathcal{D}'(M, \mathrm{T}M)$ and $Y \in C^\infty(M, \mathrm{T}M)$, then the distribution $g(X, Y)$ is defined as follows: For $\phi \in C^\infty_{\mathrm{c}}(M, \underline{\mathbb{R}})$ (where $\underline{\mathbb{R}}$ is the trivial bundle with fiber $\mathbb{R}$), we let
$$g(X, Y)[\phi] := X[\phi g(\cdot, Y)]$$
(which is well-defined as $\phi g(\cdot, Y)$ is compactly supported and smooth). By density of the smooth sections in the distributional sections, this extends to a bicontinuous and symmetric mapping $g : \mathcal{D}'(M, \mathrm{T}M) \times \mathcal{D}'(M, \mathrm{T}M) \to \mathcal{D}'(M, \underline{\mathbb{R}})$.

2. Similarly, we can insert vector fields in distributional tensors: If $S \in \mathcal{D}'(M, \mathcal{T}^{0,2}M)$, $X, Y \in C^\infty_{\mathrm{c}}(M, \mathrm{T}M)$, $\phi \in C^\infty_{\mathrm{c}}(M, \underline{\mathbb{R}})$, then we define
$$S(X, Y)[\phi] := S[\phi(X \otimes Y)].$$

3. Furthermore, for $S \in \mathcal{D}'(M, \mathcal{T}^{0,2}M)$, $T \in C^\infty_{\mathrm{c}}(M, \mathcal{T}^{0,2}M)$ and $\phi \in C^\infty_{\mathrm{c}}(M, \underline{\mathbb{R}})$, we let
$$g(S, T)[\phi] := S[\phi g(\cdot, T)]$$
which again extends also in the second slot to distributional sections. In particular, we can define the trace of S by
$$\mathrm{tr}_g(S) := g(S, g).$$

3. Differential operators

Setting 3.1. We remain in Setting 1.1. Furthermore, in the following, let $E \to M$ and $F \to M$ be real vector bundles over M, with nondegenerate bilinear forms g_E, g_F which are here (contrarily to Setting 2.1) not required to be scalar products. These do in general not induce a norm any more, but the nondegenerate bilinear forms $(\cdot,\cdot)_{L^2(M,E)}$ and $(\cdot,\cdot)_{L^2(M,F)}$ remain defined.

Definition 3.2 ([6], Definition 1.1, [7], Appendices E, F)**.**

1. A *(linear) differential operator* $P : C^\infty(M,E) \to C^\infty(M,F)$ of order $k \in \mathbb{N}_0$ is an $\mathbb{R}$-linear map such that for every point in M, there exists a neighbourhood U with local coordinates $(x_0,\dots,x_n)$ and local trivializations $E|_U \cong U \times \mathbb{R}^p$, $F|_U \cong U \times \mathbb{R}^q$ (for $p, q \in \mathbb{N}$) in which it can be written in the form
$$P = \sum_{|\alpha| \le k} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$
where each A^α is a $(q \times p)$-matrix and $A^\alpha \neq 0$ for some α with $|\alpha| = k$.

2. The *formal adjoint operator* $P^* : C^\infty(M,F) \to C^\infty(M,E)$ of P is the unique linear differential operator which satisfies
$$(Pf_1, f_2)_{L^2(M,F)} = (f_1, P^* f_2)_{L^2(M,E)}$$
for all $f_1 \in C^\infty(M,E)$, $f_2 \in C^\infty(M,F)$ with $\operatorname{supp}(f_1) \cap \operatorname{supp}(f_2)$ compact. (An explicit form of P^* can be derived by iterated partial integration and an application of Stokes' resp. Gauß' theorem. Note that this depends on the metrics g, g_E and g_F by definition of the L^2-scalar products.)

3. The *(principal) symbol* $\sigma_\xi(P;x) : E_x \to F_x$ of P at $x \in M$, where $\xi \in \mathrm{T}^*M$, and E_x, F_x are the fibers of E, F over x, is a linear map defined as follows: For $z \in E_x$, let $u \in C^\infty(M,E)$ with $u(x) = z$. Let furthermore $\phi \in C^\infty(M)$ with $\phi(x) = 0$, $d\phi(x) = \xi$ and then set
$$\sigma_\xi(P;x)z := \frac{1}{k!} P(\phi^k u)|_x.$$
This definition is independent of the choices of u and ϕ.

Remark 3.3.

1. Using the formal adjoint, one can define weak derivatives and extend any differential operators also to distributional sections. Namely for $f_1 \in \mathcal{D}'(M,E)$ and a differential operator P, one defines $Pf_1 \in \mathcal{D}'(M,F)$ to be the unique distribution satisfying the equation from the definition of P^* above (i.e. $(Pf_1, f_2)_{L^2(M,F)} = (f_1, P^* f_2)_{L^2(M,E)}$) for all $f_2 \in C^\infty_{\mathrm{c}}(M,F)$. (If $P = \nabla^E$ for a connection ∇^E on E, then the so-defined Pf_1 is often called the *weak derivative* of f_1.) We obtain a continuous map $P : \mathcal{D}'(M,E) \to \mathcal{D}'(M,F)$.

2. Furthermore, this construction is compatible with the Sobolev spaces, i.e. if P has order k, then for all $s \in \mathbb{R}$ and $K \subset M$ compact, the maps $P : H^s_{\mathrm{loc}}(M,E) \to H^{s-k}_{\mathrm{loc}}(M,F)$, $P : H^s_K(M,E) \to H^{s-k}_K(M,F)$ and $P : H^s_{\mathrm{c}}(M,E) \to H^{s-k}_{\mathrm{c}}(M,F)$ defined by the formula above are well-defined and continuous. Analogous statements hold for the finite energy sections.[7]

[7] Every distribution is infinitely often differentiable in the weak sense defined above. The actual "smoothness" of distributional sections is, by the Sobolev embedding theorem, expressed by the Sobolev degree. (Note that every compactly supported distribution is, by a standard result, of some Sobolev regularity $s \in \mathbb{R}$.)

Example 3.4. One calculates that $\delta : C^\infty(M, \mathcal{T}^{i,j}M) \to C^\infty(M, \mathcal{T}^{i,j-1}M)$ is the formal adjoint of $\nabla : C^\infty(M, \mathcal{T}^{i,j-1}M) \to C^\infty(M, \mathcal{T}^{i,j}M)$.[8] On the other hand, δ, when restricted to the smooth sections of S^2M, can also be seen as the formal adjoint of the symmetrized covariant derivative $\mathrm{sym} \circ \nabla : C^\infty(M, \mathcal{T}^{0,1}M) \to C^\infty(M, \mathrm{S}^2M)$.

An important class of differential operators are the wave operators.

Definition 3.5 ([1], Definition A.6)**.** A linear differential operator $P : \mathcal{D}'(M, E) \to \mathcal{D}'(M, E)$ is called a *wave operator* if in local coordinates, it takes the form

$$P = -\sum_{i,j=0}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \text{lower order terms},$$

where g^{ij} are the coefficients of the semi-Riemannian metric on M (in the considered local coordinate system). Equivalently, the principal symbol of P is given by this metric, i.e.

$$\sigma_\xi(P; x)z = ((-g^{ij}\xi_i\xi_j)z)|_x$$

for all $x \in M$, $\xi \in \mathrm{T}^*M$, $z \in E_x$.

As already stated, Dric is not a wave operator (it does indeed describe the propagation of gravitational waves, but it cannot be a wave operator due to the gauge solutions). However, in the proofs of existence and uniqueness of the solutions to the linearised Einstein equations, in many places "auxiliary" linear wave equations will be crucial. For these, an existence and uniqueness theorem is known:

Theorem 3.6 ([1], Theorem A.7)**.** *Let $s \in \mathbb{R} \cup \{\infty\}$ and let P be a wave operator. Let $(u_0, u_1, f) \in H^s_{\mathrm{loc}}(\Sigma, E|_\Sigma) \oplus H^{s-1}_{\mathrm{loc}}(\Sigma, E|_\Sigma) \oplus CH^{s-1}_{\mathrm{loc}}(M, E, t)$. Then there is a unique $u \in CH^s_{\mathrm{loc}}(M, E, t)$ such that*

$$\begin{aligned} Pu &= f, \\ u|_\Sigma &= u_0, \\ \nabla_\nu u|_\Sigma &= u_1. \end{aligned}$$

Moreover, we have finite speed of propagation*, i.e.*

$$\mathrm{supp}(u) \subset J\left(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup K\right)$$

for any subset $K \subset M$ such that $\mathrm{supp}(f) \subset J(K)$.

This theorem can be proven using standard techniques from the theory of partial differential equations. In [8], a similar one is proven using Riesz distributions in the smooth case. In [3], it is extended by approximation arguments to the distributional, but spatially compactly supported case. For the general theorem stated above, one needs to argue by finite speed of propagation to translate the proofs given there.

Furthermore, as a technical aid, we will need some elliptic differential operators.

Definition 3.7 ([7], Appendix G)**.** A linear differential operator $P : \mathcal{D}'(M, E) \to \mathcal{D}'(M, F)$ is called an *elliptic operator* if at every $x \in M$ and $\xi \in \mathrm{T}^*M$, its principal symbol $\sigma_\xi(P; x) : E_x \to F_x$ is an isomorphism. It is called an *overdetermined* resp. *underdetermined elliptic operator*, if for every $x \in M, \xi \in \mathrm{T}^*M$, the principal symbol $\sigma_\xi(P; x) : E_x \to F_x$ is injective resp. surjective.

[8] The connection-Laplace operator $\nabla^*\nabla$ can now also be seen as the composition of $\nabla^* = \delta$ and ∇, which gives the same resulting formula (as desired).

We will need elliptic operators (of first order) to show the regularity of certain functions. Here the following theorem is crucial:

Theorem 3.8 (cf. [9], Theorem 1.2.A, [6], Theorem III.5.2)**.** *Assume M to be compact. Let $P : \mathcal{D}'(M,E) \to \mathcal{D}'(M,F)$, defined locally by the formula in definition 3.2, be a linear elliptic operator of order k, and assume that $A^\alpha \in C^\infty(M, \mathbb{R}^{q\times p})$ for all α. Let $u \in H^s(M,E)$ satisfy (almost everywhere) in M in the distributional sense the equation $Pu = f$, where $f \in H^s(M,F)$ for $s \in \mathbb{R}$. Then*

$$u \in H^{s+k}(M,E).$$

Furthermore, the $H^{s+k}(M,E)$-norm of u can be estimated by

$$\|u\|_{H^{s+k}(M,E)} \le C\|f\|_{H^s(M,F)} + C'\|u\|_{H^s(M,E)},$$

where C, C' are constants independent of u and f.

The first part of the theorem (i.e. $u \in H^{s+k}(M,E)$) follows e.g. from the more general Theorem 1.2.A in [9]. A proof of the second part (the norm estimate) can e.g. be found in [6], Theorem III.5.2. Both proofs use so-called parametrices.

Remark 3.9.

1. We note as a corollary that there is a similar statement for local Sobolev spaces over possibly noncompact manifolds and the corresponding seminorms on them.
2. If P is an overdetermined elliptic operator, then P^*P is elliptic. Noting that if u solves $Pu = f$, then u solves also $P^*Pu = P^*f$, we can conclude from the elliptic regularity theorem above an analogous statement for overdetermined elliptic operators.[9]

4. Linearisation of the Einstein and Constraint equations

Setting 4.1. We remain in Settings 1.1, 2.1 and 3.1 (now with concrete bundles instead of the placeholders E and F). We start out with the (non-linear) Einstein equation of the vacuum with vanishing cosmological constant for globally hyperbolic spacetimes of dimension at least 3 ([1], Section 4.1):

$$\mathrm{ric}_g = 0. \tag{4.1}$$

These can be derived using the formulas from this section by a variational principle from the Einstein-Hilbert functional ([10], Section 5):

$$\mathcal{S}_M = \int_M \mathrm{scal}_g \,\mathrm{dvol}_g.$$

Einstein metrics turn out to be extremal points of this functional.

The induced first and second fundamental forms on Σ need to satisfy the Constraint equations ([1], Section 4.2):

$$\Phi_1(\tilde{g}, \tilde{k}) := \mathrm{scal}_{\tilde{g}} - \tilde{g}(\tilde{k}, \tilde{k}) + (\mathrm{tr}_{\tilde{g}}\tilde{k})^2 = 0, \tag{4.2}$$

$$\Phi_2(\tilde{g}, \tilde{k}) := -\,\delta^{\tilde{g}}(\tilde{k}) - d(\mathrm{tr}_{\tilde{g}}\tilde{k}) = 0. \tag{4.3}$$

The first equation here is often called the *energy constraint*, the second one the *momentum constraint*. See e.g. [11] for more elaboration.

[9] In the estimate, we get then at first the $H^{s-k}(M,E)$-norm of P^*f instead of the $H^s(M,F)$-norm of f, but the former is bounded by a multiple of the latter as $P^* : H^s(M,F) \to H^{s-k}(M,E)$ is continuous.

Definition and Remark 4.2. For $h \in C^\infty_c(M, \mathrm{S}^2M)$ and sufficiently small $t \in \mathbb{R}$, $g + th$ is again a semi-Riemannian metric on M. The differential of the Riemannian curvature tensor on (M, g) in the direction of h is then defined by

$$\mathrm{DR}_g(h) := \lim_{t\to 0} \frac{1}{t}(\mathrm{R}_{g+th} - \mathrm{R}_g).$$

This means that for all $X, Y, Z \in \mathrm{T}M$, we have

$$\mathrm{DR}_g(h)(X, Y, Z) = \lim_{t\to 0} \frac{1}{t}(\mathrm{R}_{g+th}(X, Y)Z - \mathrm{R}_g(X, Y)Z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{R}_{g+th}(X, Y)Z.$$

This is again a $(1, 3)$-tensor. (Instead of $t \mapsto g + th$, also any other curve $t \mapsto \gamma(t)$, where $\gamma(0) = g$, $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\gamma(t) = h$, could have been used. This is a Gâteaux differential, where one considers R. as a map from $H^{m+2}_{\mathrm{loc}}(M, \mathrm{S}^2M)$ to $H^m_{\mathrm{loc}}(M, \mathcal{T}^{1,3}M)$ for some $m \geq 0$. Existence of the differential follows then from the local formula for R_g, see [7], 1.173. We simply calculate the pointwise limit, recognizing that this must then also be the limit with respect to the Sobolev seminorms, since this is unique.) Analogously $\mathrm{D}(g^{-1})(h)$, $\mathrm{Dric}_g(h)$, $\mathrm{Dscal}_g(h)$, $\mathrm{D}\nabla^g(h)$ and $\mathrm{D}\delta^g(h)$ shall be defined.[10]

We have:

Proposition 4.3 (cf. [10], Proposition 5.1, and the proof of [7], Proposition 1.184)**.** *Let $h \in C^\infty_c(M, \mathrm{S}^2M)$. The differentials of* R*,* ric*,* scal*,* ∇*, and* δ *at* g*, in the direction of* h*, are determined by the formulas*

1. *Levi-Civita connection*

$$g(\mathrm{D}\nabla^g(h)(X, Y), Z) = \frac{1}{2}\left((\nabla^g{}_X h)(Y, Z) + (\nabla^g{}_Y h)(X, Z) - (\nabla^g{}_Z h)(X, Y)\right), \quad (4.4)$$

2. *Divergence on* 1*-forms*

$$\mathrm{D}\delta^g(h)(\alpha) = g(h, \nabla^g\alpha) - g\left(\alpha, \delta^g(h) + \frac{1}{2}d(\mathrm{tr}_g h)\right), \quad (4.5)$$

3. *Riemann curvature tensor*[11]

$$\mathrm{DR}_g(h)(X, Y)Z = (\nabla^g{}_X(\mathrm{D}\nabla^g(h)))(Y, Z) - (\nabla^g{}_Y(\mathrm{D}\nabla^g(h)))(X, Z), \quad (4.6)$$

4. *Ricci tensor*

$$\mathrm{Dric}_g(h) = \frac{1}{2}\Box^g_L h - (\mathrm{sym} \circ \nabla^g)(\delta^g h) - \frac{1}{2}\nabla^g \mathrm{d}(\mathrm{tr}_g h), \quad (4.7)$$

which can also be written as

$$\mathrm{Dric}_g(h) = \frac{1}{2}\left(\Box^g_L h - \mathcal{L}_{(\delta^g(\bar{h}))^\sharp} g\right), \quad (4.8)$$

[10] For $\mathrm{D}\nabla^g(h)$, we note that $\nabla^{g+th} - \nabla^g$ is the difference of two connections on M, hence a $(1, 2)$-tensor. Therefore, $\mathrm{D}\nabla^g(h)$ will be a $(1, 2)$-tensor as well, where the limit is taken with respect to a Sobolev norm just as for $\mathrm{DR}_g(h)$, $\mathrm{Dric}_g(h)$ and $\mathrm{Dscal}_g(h)$.
The notation $\mathrm{D}(g^{-1})(h)$ is actually slightly inconsistent to improve readability.

[11] Some of the brackets will be omitted if they should be clear from the context.

5. *scalar curvature*

$$\mathrm{Dscal}_g(h) = \Box^g(\mathrm{tr}_g h) + \delta^g(\delta^g h) - g(\mathrm{ric}_g, h) \tag{4.9}$$

for all $X, Y, Z \in \mathrm{T}M$, $\alpha \in \Omega^1(M)$.

A proof is given in Appendix B.

Equation (4.8) motivates the following definition.

Definition 4.4 ([1], Definition 4.3)**.** The *linearised Ricci curvature* $\mathrm{Dric} : \mathcal{D}'(M, \mathcal{T}^{0,2}M) \to \mathcal{D}'(M, \mathcal{T}^{0,2}M)$ is defined by

$$\mathrm{Dric}(h) := \frac{1}{2}\left(\Box_L h - \mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g\right) \tag{4.10}$$

for all $h \in \mathcal{D}'(M, \mathcal{T}^{0,2}M)$. If $h \in \mathcal{D}'(M, \mathcal{T}^{0,2}M)$ satisfies $\mathrm{Dric}(h) = 0$, it is said to satisfy the *linearised Einstein equations (of the vacuum with vanishing cosmological constant).*

Remark 4.5.

1. We note that the Lie derivative can be defined also for distributions, using the formula (A.1).
2. It shall be emphasized that Dric is not a wave operator. (This follows e.g. from the existence of the "gauge solutions" proven below.)
3. We omit the dependency of Dric on g in the notation for brevity.

Analogously, we have to linearise the Constraint equations.

Proposition and Definition 4.6 ([1], Definition 4.6)**.** *Let $(\tilde{g}, \tilde{k}), (\tilde{h}, \tilde{m}) \in \mathcal{D}'(\Sigma, \mathcal{T}^{0,2}\Sigma) \times \mathcal{D}'(\Sigma, \mathcal{T}^{0,2}\Sigma)$. We define*

$$\mathrm{D}\Phi(\tilde{h}, \tilde{m}) := \begin{pmatrix} \mathrm{D}\Phi_1(\tilde{h}, \tilde{m}) \\ \mathrm{D}\Phi_2(\tilde{h}, \tilde{m}) \end{pmatrix} \tag{4.11}$$

with

$$\begin{aligned} \mathrm{D}\Phi_1(\tilde{h}, \tilde{m}) :=& - \delta^{\tilde{g}}(-\delta^{\tilde{g}}\tilde{h} + \mathrm{d}\mathrm{tr}_{\tilde{g}}\tilde{h}) - \tilde{g}(\mathrm{ric}_{\tilde{g}}, \tilde{h}) \\ &+ 2\tilde{g}(\tilde{k} \circ \tilde{k} - (\mathrm{tr}_{\tilde{g}}\tilde{k})\tilde{k}, \tilde{h}) - 2\tilde{g}(\tilde{k}, \tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathrm{D}\Phi_2(\tilde{h}, \tilde{m})(X) :=& - \tilde{g}(\tilde{h}, \tilde{\nabla}.\tilde{k}(\cdot, X)) + \tilde{g}\left(\tilde{k}(\cdot, X), \delta^{\tilde{g}}(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g})\right) \\ &- \frac{1}{2}\tilde{g}(\tilde{k}, \tilde{\nabla}_X \tilde{h}) + \mathrm{d}(\tilde{g}(\tilde{k}, \tilde{h}))(X) - \delta^{\tilde{g}}(\tilde{m} - (\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g})(X) \end{aligned} \tag{4.13}$$

for all $X \in \mathrm{T}\Sigma$. Here $\tilde{\nabla} := \nabla^{\tilde{g}}$ denotes the induced Levi-Civita connection on Σ (which is compatible with $\tilde{g}$) and the metric composition $\tilde{k} \circ \tilde{k}$ is done using $\tilde{g}$. We say that the tuple $(\tilde{h}, \tilde{m})$ satisfies the linearised Constraint equations*, linearised around $(\tilde{g}, \tilde{k})$, if $\mathrm{D}\Phi(\tilde{h}, \tilde{m}) = 0$.*

If $(\tilde{g}, \tilde{k}), (\tilde{h}, \tilde{m})$ are smooth and $(\tilde{h}, \tilde{m})$ is compactly supported, then

$$\mathrm{D}\Phi_1(\tilde{h}, \tilde{m}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_1(\tilde{g} + t\tilde{h}, \tilde{k} + t\tilde{m}), \tag{4.14}$$

$$\mathrm{D}\Phi_2(\tilde{h}, \tilde{m}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Phi_2(\tilde{g} + t\tilde{h}, \tilde{k} + t\tilde{m}) \tag{4.15}$$

with Φ_1, Φ_2 defined in equations (4.2), (4.3).

Only equations (4.14), (4.15) require a proof. This can be found in Appendix B.

We consider now the first and second fundamental form $(\tilde{g}, \tilde{k})$ on Σ which are induced by the metric g. If g changes infinitesimally (but the embedding of Σ into M remains the same), these will also change infinitesimally. We recall

$$\tilde{g}(X,Y) = g(X,Y), \qquad \tilde{k}(X,Y) = g(\nabla_X \nu, Y)$$

for all $X, Y \in \mathrm{T}\Sigma$.

Proposition and Definition 4.7 ([1], Definition 4.7)**.** *Let $h \in \mathcal{D}'(M, \mathrm{S}^2 M)$. We define $(D\tilde{g}(h), D\tilde{k}(h)) \in \mathcal{D}'(\Sigma, \mathrm{S}^2\Sigma) \times \mathcal{D}'(\Sigma, \mathrm{S}^2\Sigma)$ by*

$$\mathrm{D}\tilde{g}(h)(X,Y) := h(X,Y), \tag{4.16}$$

$$\mathrm{D}\tilde{k}(h)(X,Y) := -\frac{1}{2}h(\nu,\nu)\tilde{k}(X,Y) - \frac{1}{2}\nabla_X h(\nu, Y) - \frac{1}{2}\nabla_Y h(\nu, X) + \frac{1}{2}\nabla_\nu h(X,Y) \tag{4.17}$$

for $X, Y \in \mathrm{T}\Sigma$. We call $\mathrm{D}\tilde{g}(h)$ and $\mathrm{D}\tilde{k}(h)$ the linearised first and second fundamental forms *induced by h.*

If $h \in C^\infty_{\mathrm{c}}(M, \mathrm{S}^2 M)$ and we consider $\tilde{g}$, $\tilde{k}$ as maps from $C^\infty(M, \mathrm{S}^2 M)$ to $C^\infty(\Sigma, \mathrm{S}^2\Sigma)$ which assign to g the first and second fundamental form on Σ resulting from g (and the embedding of Σ in M), we have

$$\mathrm{D}\tilde{g}(h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{g}(g + th), \tag{4.18}$$

$$\mathrm{D}\tilde{k}(h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tilde{k}(g + th). \tag{4.19}$$

Only equations (4.18), (4.19) require a proof, which is again done in Appendix B.

We recall that the first and second fundamental form $\tilde{g}$ and $\tilde{k}$ on Σ must satisfy the equations ([11], equations (16), (17))

$$\begin{aligned} 2\mathrm{ric}_g(\nu,\nu) + \mathrm{tr}_g(\mathrm{ric}_g) &= \Phi_1(\tilde{g}, \tilde{k}), \\ \mathrm{ric}_g(\nu, \cdot) &= \Phi_2(\tilde{g}, \tilde{k}). \end{aligned}$$

We can linearise this equation around g in the direction of h. Using $\mathrm{ric}_g = 0$ and the previous calculations, we obtain for $h \in C^\infty_{\mathrm{c}}(M, \mathrm{S}^2 M)$ and $\mathrm{D}\tilde{g}(h)$, $\mathrm{D}\tilde{k}(h)$ defined by equations (4.16), (4.17):[12]

$$2(\mathrm{Dric}(h))(\nu,\nu) + \mathrm{tr}_g(\mathrm{Dric}(h)) = \mathrm{D}\Phi_1(\mathrm{D}\tilde{g}(h), \mathrm{D}\tilde{k}(h)), \tag{4.20}$$

$$\mathrm{Dric}(h)(\nu, \cdot) = \mathrm{D}\Phi_2(\mathrm{D}\tilde{g}(h), \mathrm{D}\tilde{k}(h)). \tag{4.21}$$

In particular, if $\mathrm{Dric}(h) = 0$ (i.e. h satisfies the linearised Einstein equations), then the induced initial data $(\mathrm{D}\tilde{g}(h), \mathrm{D}\tilde{k}(h))$ must satisfy $\mathrm{D}\Phi(\mathrm{D}\tilde{g}(h), \mathrm{D}\tilde{k}(h)) = 0$. The existence theorem proven later may be seen as the converse statement.

We conclude this section with the "gauge invariance" of the linearised Einstein equations, which explains their large kernel and the fact that they are not uniquely solvable. We will later also only prove a uniqueness result "up to gauge".

Lemma 4.8 ([1], Lemma 4.5)**.** *Let $V \in \mathcal{D}'(M, \mathrm{T}M)$. Then $\mathcal{L}_V g$ solves the linearised Einstein equations, i.e.* $\mathrm{Dric}(\mathcal{L}_V g) = 0$.

[12] Note that we have product rules for the differentiation in the direction of h, similar to eq. B.1. Also, by definition of the differential, we have a chain rule.

Proof. As the smooth sections in TM lie dense in the distributional sections and the map $\mathrm{Dric} \circ (\mathcal{L}.g) : \mathcal{D}'(M, \mathrm{T}M) \to \mathcal{D}'(M, \mathrm{S}^2M)$ is continuous, it suffices to prove the claim for smooth V. So assume V to be smooth; let then φ_t be its flow. By definition of the differential of ric_g, we have

$$\mathrm{Dric}(\mathcal{L}_V g) = \lim_{t\to 0} \frac{1}{t}(\mathrm{ric}_{\varphi_t^*(g)} - \mathrm{ric}_g),$$

since $\varphi_0^*(g) = g$, $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi_t^*(g) = \mathcal{L}_V g$.

However, we have

$$\mathrm{ric}_{\varphi_t^*(g)} = \varphi_t^*(\mathrm{ric}_g) = \varphi_t^*(0) = 0 = \mathrm{ric}_g$$

by assumption on g. Hence $\mathrm{Dric}(\mathcal{L}_V g) = \lim_{t\to 0} \frac{0}{t} = 0$. □

5. Existence of solutions

We remain in the situation of the previous sections. Recalling again that the linearised Einstein equations are not wave equations, the existence of a solution can not directly be concluded. However, we can search for solutions of $\Box_L h = 0$. For such a linear wave equation, an existence theorem is known (theorem 3.6). We will show that for suitable initial conditions, the solution of $\Box_L h = 0$ also satisfies $\delta^g(\overline{h}) = 0$, thus it will solve $\mathrm{Dric}(h) = 0$ as well.

As a first step, we need to translate the condition on h to induce a given first and fundamental form on the Cauchy surface Σ into conditions on $h|_\Sigma$ and $\nabla_\nu h|_\Sigma$, i.e. initial conditions for a linear wave equation. Furthermore, we show that $\delta^g(\overline{h})$ vanishes at least on Σ if these initial conditions are chosen suitably.

Lemma 5.1 ([1], Lemma 5.1)**.** *For $s \in \mathbb{R}$, let $(\tilde{h}, \tilde{m}) \in H^s_{\mathrm{loc}}(\Sigma, \mathrm{S}^2\Sigma) \times H^{s-1}_{\mathrm{loc}}(\Sigma, \mathrm{S}^2\Sigma)$. Assume that for $h \in CH^s_{\mathrm{loc}}(M, \mathrm{S}^2M, t)$, its restriction to $\mathrm{S}^2M|_\Sigma$ satisfies*[13]

$$\begin{aligned} h(X,Y) &= \tilde{h}(X,Y), & \nabla_\nu h(X,Y) &= 2\tilde{m}(X,Y) - (\tilde{h}\circ\tilde{k} + \tilde{k}\circ\tilde{h})(X,Y), \\ h(\nu,X) &= 0, & \nabla_\nu h(\nu,X) &= -\delta^{\tilde{g}}\left(\tilde{h} - \tfrac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right)(X), \\ h(\nu,\nu) &= 0, & \nabla_\nu h(\nu,\nu) &= -2\mathrm{tr}_{\tilde{g}}\tilde{m}, \end{aligned}$$

for all $X, Y \in \mathrm{T}\Sigma$ (where the composition is defined using $\tilde{g}$ instead of g). Then $\tilde{h}, \tilde{m}$ are the first and second linearised fundamental forms induced by h,[14] *and*

$$\delta^g(\overline{h})|_\Sigma = 0.$$

A proof can be found in Appendix B.

We are now able to formulate and prove the existence theorem.

Theorem 5.2 ([1], Theorem 5.2)**.** *Let $s \in \mathbb{R}\cup\{\infty\}$ and assume that $(\tilde{h}, \tilde{m}) \in H^s_{\mathrm{loc}}(\Sigma, \mathrm{S}^2\Sigma) \times H^{s-1}_{\mathrm{loc}}(\Sigma, \mathrm{S}^2\Sigma)$ satisfies*

$$\mathrm{D}\Phi(\tilde{h}, \tilde{m}) = 0.$$

Then there exists a unique

$$h \in CH^s_{\mathrm{loc}}(M, \mathrm{S}^2M, t),$$

[13] Note that this restriction is an element of $H^s_{\mathrm{loc}}(\Sigma, \mathrm{S}^2M|_\Sigma)$ by definition of $CH^s_{\mathrm{loc}}(M, \mathrm{S}^2M, t)$. In particular, we do not need to define a trace operator here, as we would have to do for Sobolev sections.

[14] Spelled out, this means $\tilde{h} = D\tilde{g}(h)$, $\tilde{m} = D\tilde{k}(h)$.

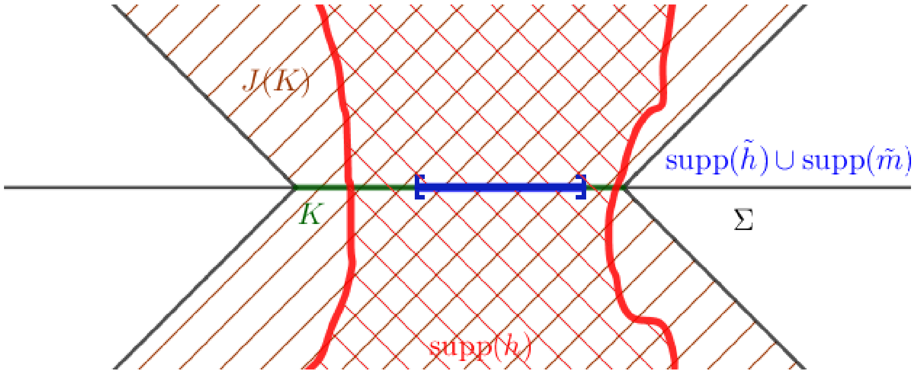

Figure 1: Situation in theorem 5.2, in flat Minkowski space

inducing $(\tilde{h}, \tilde{m})$ *as linearised first and second fundamental forms, such that* $h|_\Sigma$ *and* $\nabla_\nu h|_\Sigma$ *are as in Lemma 5.1 and*

$$\Box_L h = 0,$$
$$\delta^g(\overline{h}) = 0.$$

In particular,

$$\mathrm{Dric}(h) = 0.$$

Moreover,

$$\mathrm{supp}(h) \subset J\left(\mathrm{supp}(\tilde{h}) \cup \mathrm{supp}(\tilde{m})\right).$$

The situation of the theorem is sketched in figure 1, where M is taken to be flat, two-dimensional Minkowski space.

Remark 5.3 ([1], Remark 5.3)**.** We note that an arbitrary solution to the linearised Einstein equations inducing the given linearised first and second fundamental forms need not have finite speed of propagation, only the specific one considered here. Any gauge solution with arbitrary support could be added to it without changing the induced linearised first and second fundamental forms, while changing the support of the solution.

In the proof, we will need two lemmas.

Lemma 5.4 ([1], Lemma 5.5)**.** *If* $h \in \mathcal{D}'(M, \mathrm{S}^2M)$*, then*

$$\delta^g\left(\mathrm{Dric}(h) - \frac{1}{2}\mathrm{tr}_g(\mathrm{Dric}(h))g\right) = 0. \tag{5.1}$$

Proof. We recall the contracted second Bianchi identity: For any Lorentzian metric $\hat{g}$ on M, we have

$$\delta^{\hat{g}}\left(\mathrm{ric}_{\hat{g}} - \frac{1}{2}\mathrm{tr}_{\hat{g}}(\mathrm{ric}_{\hat{g}})\hat{g}\right) = 0.$$

A linearisation of this equation around g, using $\mathrm{ric}_{\hat{g}} = 0$, proves the lemma for smooth h. Since the smooth sections are dense in $\mathcal{D}'(M, \mathrm{S}^2M)$ and Dric is continuous, this proves the equation for general h. □

Lemma 5.5. *Let* $(N, \hat{g})$ *be a semi-Riemannian manifold with Levi-Civita connection* $\hat{\nabla}$*. Then*

$$\delta^{\hat{g}}\left(\mathcal{L}_V\hat{g} - \frac{1}{2}\mathrm{tr}_{\hat{g}}(\mathcal{L}_V\hat{g})\hat{g}\right) = \delta^{\hat{g}}(\hat{\nabla}V^\flat) - \mathrm{ric}_{\hat{g}}(V, \cdot) \tag{5.2}$$

for all $V \in \mathcal{D}'(N, \mathrm{T}N)$*.*

The proof is again given in Appendix B.

Proof of Theorem 5.2. Consider the Cauchy problem

$$\Box_L h = 0,$$

where $h|_\Sigma$ and $\nabla_\nu h|_\Sigma$ shall satisfy the conditions of Lemma 5.1. We note $(h|_\Sigma, \nabla_\nu h|_\Sigma) \in H^s_{\mathrm{loc}}(\Sigma, \mathrm{S}^2 M|_\Sigma) \times H^{s-1}_{\mathrm{loc}}(\Sigma, \mathrm{S}^2 M|_\Sigma)$. By Theorem 3.6, there exists a unique solution $h \in CH^s_{\mathrm{loc}}(M, \mathrm{S}^2 M, t)$ to this Cauchy problem. Furthermore, this solution satisfies $\mathrm{supp}(h) \subset J\left(\mathrm{supp}(\tilde{h}) \cup \mathrm{supp}(\tilde{m})\right)$ (by the same theorem).

It remains to show $\delta^g \overline{h} = 0$ to prove the theorem. This will be done by showing that $\delta^g \overline{h}$ solves a linear wave equation with vanishing initial conditions, and then apply the uniqueness part of Theorem 3.6.

We have $\delta^g \overline{h} \in CH^{s-1}_{\mathrm{loc}}(M, \mathrm{T}^* M, t)$ (since $h \in CH^s_{\mathrm{loc}}(M, \mathrm{S}^2 M, t)$). The lemmas 5.4 and 5.5 imply together with $\Box_L h = 0$ and $\mathrm{ric}_g = 0$ that

$$\begin{aligned} 0 =& \delta^g \left(\mathrm{Dric}(h) - \frac{1}{2} \mathrm{tr}_g(\mathrm{Dric}(h)) g \right) \\ =& \frac{1}{2} \delta^g \left(-\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g + \frac{1}{2} \mathrm{tr}_g \left(\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g \right) g \right) \\ =& -\frac{1}{2} \delta^g (\nabla (\delta^g(\overline{h}))). \end{aligned} \tag{5.3}$$

Now we want to show $\nabla_\nu(\delta^g(\overline{h}))|_\Sigma = 0$. In the following, we calculate on Σ (but suppress this in the notation). Because of the linearized Constraint equations and the initial conditions from Lemma 5.1, we deduce

$$\begin{aligned} 0 = \mathrm{D}\Phi_1(\tilde{h}, \tilde{m}) =& \mathrm{tr}_g(\mathrm{Dric}(h)) + 2\mathrm{Dric}(h)(\nu, \nu) \\ =& \frac{1}{2} \left(\mathrm{tr}_g(-\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g) - 2\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g(\nu, \nu) \right) \end{aligned} \tag{5.4}$$

The trace can be evaluated pointwise, using a local geodesic frame $\{e_i\}_{1 \le i \le n}$ on $\mathrm{T}\Sigma$ which is extended by ν to a local geodesic frame on $\mathrm{T}M$.[15] We note that for X, Y sections of $\mathrm{T}\Sigma$ with vanishing covariant derivative, we have

$$\begin{aligned} -\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g(X, Y) =& -g(\nabla_X (\delta^g(\overline{h}))^\sharp, Y) - g(X, \nabla_Y (\delta^g(\overline{h}))^\sharp) \\ =& -\partial_X (g((\delta^g(\overline{h}))^\sharp, Y)) - \partial_Y (X, g((\delta^g(\overline{h}))^\sharp)) \\ =& -\partial_X ((\delta^g(\overline{h}))(Y)) - \partial_Y ((\delta^g(\overline{h}))(X)). \end{aligned}$$

In particular, we recognize that this vanishes for $X = Y = e_i$, as $\delta^g(\overline{h})$ is identically zero on Σ. Thus in eq. (5.4), the trace term simplifies to $\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g(\nu, \nu)$ (where the sign vanished because of $g(\nu, \nu) = -1$). We obtain

$$\begin{aligned} 0 = -\frac{1}{2} \mathcal{L}_{(\delta_g(\overline{h}))^\sharp}(\nu, \nu) =& \frac{1}{2} \left(\partial_\nu ((\delta^g(\overline{h}))(\nu)) + \partial_\nu ((\delta^g(\overline{h}))(\nu)) \right) \\ =& (\nabla_\nu (\delta^g(\overline{h})))(\nu) \end{aligned} \tag{5.5}$$

[15] Recall that this means that the orthonormal sections $\{e_i\}_{1 \le i \le n}$ are chosen to have vanishing covariant derivative at some given point $p \in M$. Actually we need to replace ν by some vector field which coincides only at p with ν, but has vanishing covariant derivative there. The calculation below is true only at p, but the resulting tensor equations must be true regardless of the particular choices made, and at every point on Σ.

(recalling that we can assume $\nabla_\nu \nu = 0$ for the last equation, as these are tensor equations).

Furthermore we have, again using eq. (5.4), and that $\delta^g(\overline{h})$ vanishes on Σ:

$$\begin{aligned} 0 = \mathrm{D}\Phi_2(\tilde{h}, \tilde{m})(X) &= \mathrm{Dric}(h)(\nu, X) = -\frac{1}{2}\mathcal{L}_{(\delta^g(\overline{h}))^\sharp}(\nu, X) \\ &= -\frac{1}{2}\partial_\nu((\delta^g(\overline{h}))(X)) - \frac{1}{2}\partial_X((\delta^g(\overline{h}))(\nu)) = \frac{1}{2}(\nabla_\nu(\delta^g(\overline{h})))(X). \end{aligned} \tag{5.6}$$

We summarize the results of the previous calculations: Equation (5.3) implies that $\delta^g(\overline{h}) \in CH^{s-1}_{\mathrm{loc}}(M, \mathrm{T}^*M, t)$ satisfies

$$\delta^g(\nabla(\delta^g(\overline{h}))) = 0.$$

From the equations (5.5), (5.6), we deduce

$$\nabla_\nu(\delta^g(\overline{h}))|_\Sigma = 0.$$

Furthermore, in Lemma 5.1, we have shown

$$\delta^g(\overline{h})|_\Sigma = 0.$$

As $\delta^g \circ \nabla$ is a wave operator, the uniqueness part of Theorem 3.6 thus applies to yield $\delta^g(\overline{h}) = 0$ (as 0 certainly also satisfies these initial conditions). This finishes the proof of the theorem. □

6. Uniqueness up to gauge

In the situation of the previous sections, having now solved the linearised Einstein equations, we continue by showing a uniqueness result "up to gauge". This means that we will show that a solution inducing vanishing linearized first and second fundamental forms, can be written as a suitable Lie derivative of the metric.

Theorem 6.1 ([1], Theorem 5.7)**.** *Let $s \in \mathbb{R} \cup \{\infty\}$. Assume that $h \in CH^s_{\mathrm{loc}}(M, \mathrm{S}^2M, t)$ satisfies*

$$\mathrm{Dric}(h) = 0$$

and that the induced first and second linearized fundamental forms vanish. Then there exists a vector field $V \in CH^{s+1}_{\mathrm{loc}}(M, \mathrm{T}M, t)$ such that

$$h = \mathcal{L}_V g.$$

If $\mathrm{supp}(h) \subset J(K)$ *for some compact $K \subset \Sigma$, then we can choose V such that* $\mathrm{supp}(V) \subset J(K)$.

Proof. We define V as the solution of a suitable wave equation. We know that $\delta^g(\overline{h}) \in CH^{s-1}_{\mathrm{loc}}(M, \mathrm{T}^*M, t)$. Thus the Cauchy problem

$$\begin{aligned} \delta^g(\nabla V) &= (\delta^g(\overline{h}))^\sharp, \\ V|_\Sigma &= 0, \\ \nabla_\nu V|_\Sigma &= \frac{1}{2}h(\nu,\nu)\nu + h(\nu, \cdot)^\sharp \end{aligned} \tag{6.1}$$

admits a unique solution $V \in CH^s_{\mathrm{loc}}(M, \mathrm{T}M, t)$ by theorem 3.6, since $\delta^g \circ \nabla$ is a wave operator. This also satisfies $\mathrm{supp}(V) \subset J(K)$ if $\mathrm{supp}(h) \subset J(K)$ for some compact $K \subset \Sigma$.

By Lemma 5.5, using the Ricci-flatness of M and the defining properties of V we have

$$\delta^g(\overline{\mathcal{L}_V g}) = \delta^g(\nabla V^\flat) = \delta^g(\overline{h}) \tag{6.2}$$

(where we used that $\flat$ commutes with δ^g). Hence (using $\mathrm{Dric}(h) = 0$ and $\mathrm{Dric}(\mathcal{L}_V g) = 0$, the latter by Lemma 4.8), we obtain

$$\begin{aligned} 0 =& 2\mathrm{Dric}(h - \mathcal{L}_V g) \\ =& \Box_L(h - \mathcal{L}_V g) - \mathcal{L}_{(\delta^g(\overline{h} - \overline{\mathcal{L}_V g}))^\sharp} g = \Box_L(h - \mathcal{L}_V g). \end{aligned} \tag{6.3}$$

We now want to apply the uniqueness part of Theorem 3.6 to the wave operator $\Box_L$ to deduce that $h - \mathcal{L}_V g = 0$. This will be implied by (6.3) if we can show

$$(h - \mathcal{L}_V g)|_\Sigma = 0, \tag{6.4}$$

$$\nabla_\nu(h - \mathcal{L}_V g)|_\Sigma = 0. \tag{6.5}$$

Note also that we know $\mathcal{L}_V g \in CH^{s-1}_{\mathrm{loc}}(M, \mathrm{S}^2 M, t)$ since $V \in CH^s_{\mathrm{loc}}(M, \mathrm{T}M, t)$, so we have $h - \mathcal{L}_V g \in CH^{s-1}_{\mathrm{loc}}(M, \mathrm{S}^2 M, t)$ and the uniqueness theorem for linear wave equations can indeed be applied. We show first eq. (6.4). By the properties of V given in (6.1), and $\mathrm{D}\tilde{g}(h) = 0$ (by the assumption on h), we get for all $X, Y \in \mathrm{T}\Sigma$:

$$h(X, Y) = \mathrm{D}\tilde{g}(h)(X, Y) = 0 = g(\nabla_X V, Y) + g(\nabla_Y V, X) = \mathcal{L}_V g(X, Y).$$

(Here we used $\nabla_X V = \nabla_Y V = 0$, since $X, Y \in \mathrm{T}\Sigma$ and $V|_\Sigma = 0$.) Also

$$h(X, \nu) = g(\nabla_\nu V, X) = g(\nabla_\nu V, X) + g(\nabla_X V, \nu) = \mathcal{L}_V g(\nu, X),$$

and

$$h(\nu, \nu) = 2g(\nabla_\nu V, \nu) = \mathcal{L}_V g(\nu, \nu)$$

(recall $g(\nu, \nu) = -1$ for the calculation of the last equation). These three calculations show (6.4). We continue by showing eq. (6.5). Since $\mathrm{D}\tilde{k}(h) = 0$, we get for $X, Y \in \mathrm{T}\Sigma$, using its defining formula (4.17):

$$\nabla_\nu h(X, Y) = h(\nu, \nu)\tilde{k}(X, Y) + \nabla_X h(\nu, Y) + \nabla_Y h(\nu, X). \tag{6.6}$$

On the other hand, we have $\nabla_X \nu \in \mathrm{T}\Sigma$, since

$$g(\nabla_X \nu, \nu) = \frac{1}{2}\partial_X g(\nu, \nu) = 0. \tag{6.7}$$

We can extend X to a vector field on Σ, which then commutes with ν. Then $\nabla_\nu X = \nabla_X \nu - [X, \nu] = \nabla_X \nu \in \mathrm{T}\Sigma$. Therefore, the second covariant derivatives of V in the directions of ν and X can be written simply as compositions of first covariant derivatives, $\nabla_\nu(\nabla_X V) = \nabla^2_{\nu,X} V$, $\nabla_X(\nabla_\nu V) = \nabla^2_{X,\nu} V$, since $V|_\Sigma = 0$ (thus $\nabla_{\nabla_X \nu} V = 0$ as $\nabla_X \nu \in \mathrm{T}\Sigma$, etc.). Furthermore $(\nabla_X h)(\nu, Y) = \partial_X h(\nu, Y) - h(\nu, \nabla_X Y)$ by the product rule for tensors (since $h|_\Sigma(\nabla_X \nu, Y) = \mathrm{D}\tilde{g}(h)(\nabla_X \nu, Y) = 0$ as $\nabla_X \nu, Y \in \mathrm{T}\Sigma$). Analogous statements can be made for Y. Thus, using the defining properties (6.1) of V, we can calculate

$$\begin{aligned} \nabla_\nu \mathcal{L}_V g(X, Y) =& g(\nabla^2_{\nu,X} V, Y) + g(\nabla^2_{\nu,Y} V, X) \\ =& g(\nabla^2_{X,\nu} V, Y) + g(\nabla^2_{Y,\nu} V, X) + \mathrm{R}(\nu, X, V, Y) + \mathrm{R}(\nu, Y, V, X) \\ =& \partial_X g(\nabla_\nu V, Y) - g(\nabla_\nu V, \nabla_X Y) + \partial_Y g(\nabla_\nu V, X) - g(\nabla_\nu V, \nabla_Y X) \\ =& \partial_X h(\nu, Y) - h(\nu, \nabla_X Y) - \frac{1}{2}h(\nu, \nu)g(\nu, \nabla_X Y) \\ &+ \partial_Y h(\nu, X) - h(\nu, \nabla_Y X) - \frac{1}{2}h(\nu, \nu)g(\nu, \nabla_Y X) \\ =& \nabla_X h(\nu, Y) + \nabla_Y h(\nu, X) + h(\nu, \nu)\tilde{k}(X, Y). \end{aligned} \tag{6.8}$$

(In the last step, we also used $g(\nu, \nabla_X Y) = \partial_X g(\nu, Y) - g(\nabla_X \nu, Y) = -\tilde{k}(X,Y)$ by definition of $\tilde{k}$ and since $g(\nu, Y)$ identically vanishes, and analogously $g(\nu, \nabla_Y X) = -\tilde{k}(X,Y)$.) A comparision of (6.6) and (6.8) yields

$$\nabla_\nu(h - \mathcal{L}_V g)(X,Y) = 0 \tag{6.9}$$

for $X, Y \in \mathrm{T}\Sigma$. It remains to show $\nabla_\nu(h - \mathcal{L}_V g)(\nu, \cdot) = 0$. Eq. (6.2) is equivalent to[16]

$$\delta^g(\mathcal{L}_V g)(W) + \frac{1}{2}\partial_W \mathrm{tr}_g(\mathcal{L}_V g) = \delta^g(h)(W) + \frac{1}{2}\partial_W \mathrm{tr}_g(h) \tag{6.10}$$

for all $W \in \mathrm{T}M$. Now eq. (6.4) already implies that $\mathrm{tr}_g(\mathcal{L}_V g)|_\Sigma = \mathrm{tr}_g(h)|_\Sigma$. Thus for $X \in \mathrm{T}\Sigma$, we have $\partial_X \mathrm{tr}_g(\mathcal{L}_V g) = \partial_X \mathrm{tr}_g(h)$ and the above equation then yields

$$\delta^g(\mathcal{L}_V g)(X) = \delta^g(h)(X).$$

If we evaluate the divergence using a local orthonormal frame $\{e_i\}_{1\leq i\leq n}$ on $T\Sigma$, extended by ν to a local orthonormal frame on $\mathrm{T}M|_\Sigma$, we note that $\mathcal{L}_V g|_\Sigma(e_i, X) = h|_\Sigma(e_i, X) = \mathrm{D}\tilde{g}(h)(e_i, X) = 0$ (the first equation again by (6.4), the third one by assumption on h). Thus the equation above implies $-(\nabla_\nu(\mathcal{L}_V g))(\nu, X) = -(\nabla_\nu h)(\nu, X)$, so

$$\nabla_\nu(h - \mathcal{L}_V g)(\nu, X) = 0. \tag{6.11}$$

Finally, to calculate $\nabla_\nu(h - \mathcal{L}_V g)|_\Sigma(\nu,\nu)$, we use again eq. (6.10), now with $W = \nu$. Commuting covariant derivatives with metric traces (where the partial derivative on functions is just the same as the covariant derivative), and using (6.4) and (6.9) to simplify the divergence and the trace, this yields (on Σ):

$$\begin{aligned}
0 =& \delta^g(\overline{\mathcal{L}_V g})(\nu) - \delta^g(\overline{h})(\nu) \\
=& \delta^g(\mathcal{L}_V g - h)(\nu) + \frac{1}{2}\partial_\nu(\mathrm{tr}_g(\mathcal{L}_V g) - \mathrm{tr}_g(h)) \\
=& \delta^g(\mathcal{L}_V g - h)(\nu) + \frac{1}{2}\mathrm{tr}_g(\nabla_\nu(\mathcal{L}_V g - h)) \\
=& \nabla_\nu(\mathcal{L}_V g - h)(\nu,\nu) - \frac{1}{2}\nabla_\nu(\mathcal{L}_V g - h)(\nu,\nu) \\
=& \frac{1}{2}\nabla_\nu(\mathcal{L}_V g - h)(\nu,\nu).
\end{aligned}$$

Thus $\nabla_\nu(h - \mathcal{L}_V g)(\nu,\nu) = 0$. Together with eq. (6.9) and eq. (6.11), this implies eq. (6.5).

As already stated, the uniqueness part of 3.6 now shows that $h = \mathcal{L}_V g$. This concludes the proof of uniqueness "up to gauge". The regularity of V is shown in the following lemma. □

Lemma 6.2 ([1], Lemma 5.8). *Let $V \in CH^s_{\mathrm{loc}}(M, \mathrm{T}M, t)$ with $\mathcal{L}_V g \in CH^s_{\mathrm{loc}}(M, \mathrm{S}^2M, t)$. Then $V \in CH^{s+1}_{\mathrm{loc}}(M, \mathrm{T}M, t)$.*

Proof. The proof is based on elliptic regularity theory. Let $j \in \mathbb{N}_0$. We have

$$\nabla^j_{t,\ldots,t} V \in C^0(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_., \mathrm{T}M|_{\Sigma_.}))$$

and need to show that $\nabla^j_{t,\ldots,t} V \in C^0(t(M), H^{s-j+1}_{\mathrm{loc}}(\Sigma_., \mathrm{T}M|_{\Sigma_.}))$.

Let $\tau \in t(M)$. The projection of vectors onto their parallel and normal components to Σ induces a split $\mathrm{T}M|_{\Sigma_\tau} \cong \mathbb{R} \oplus \mathrm{T}\Sigma_\tau$. Write $\nabla^j_{t,\ldots,t} V|_{\Sigma_\tau} = (\nabla^j_{t,\ldots,t} V)^\perp|_{\Sigma_\tau} \nu_\tau + (\nabla^j_{t,\ldots,t} V)^\parallel|_{\Sigma_\tau}$

[16] Cf. eq. (B.33).

with $(\nabla^j_{t,\dots,t}V)^{\|}|_{\Sigma_\tau} \in H^{s-j}_{\mathrm{loc}}(\Sigma_\tau, \mathrm{T}\Sigma_\tau)$, $(\nabla^j_{t,\dots,t}V)^{\perp}|_{\Sigma_\tau} \in H^{s-j}_{\mathrm{loc}}(\Sigma_\tau, \mathbb{R})$. We show that $\tau \mapsto (\nabla^j_{t,\dots,t}V)^{\|}|_{\Sigma_\tau}$ and $\tau \mapsto (\nabla^j_{t,\dots,t}V)^{\perp}|_{\Sigma_\tau}$ are two continuous sections of the bundles $(H^{s-j+1}_{\mathrm{loc}}(\Sigma_\tau, \mathrm{T}\Sigma_\tau))_{\tau\in t(M)}$ and $(H^{s-j+1}_{\mathrm{loc}}(\Sigma_\tau, \mathbb{R}))_{\tau\in t(M)}$, which then proves the lemma.[17]

By commuting derivatives, we note that

$$\begin{aligned}\mathcal{L}_{\nabla^j_{t,\dots,t}V}g(X,Y) =& g(\nabla_X(\nabla^j_{t,\dots,t}V),Y) + g(\nabla_Y(\nabla^j_{t,\dots,t}V),X)\\ =& (\nabla_t)^j\mathcal{L}_V g(X,Y) + P_j(V)(X,Y)\end{aligned}$$

for $X, Y \in C^\infty(M, \mathrm{T}M)$, where P_j is some differential operator of order j. By assumption on V, this shows that

$$\mathcal{L}_{\nabla^j_{t,\dots,t}V}g \in CH^{s-j}_{\mathrm{loc}}(M, \mathrm{S}^2M, t). \tag{6.12}$$

The induced first and second fundamental forms on Σ_τ shall be denoted $\tilde{g}_\tau$ and $\tilde{k}_\tau$. For $X, Y \in \mathrm{T}\Sigma$, $W \in CH^{s-j}(M, \mathrm{T}M, t)$, we have by definition

$$(\mathcal{L}_W g)|_{\Sigma_\tau}(X,Y) = g|_{\Sigma_\tau}(\nabla_X W, Y) + g|_{\Sigma_\tau}(X, \nabla_Y W). \tag{6.13}$$

The first term here is equal to

$$\begin{aligned}g|_{\Sigma_\tau}(\nabla_X W, Y) =& g|_{\Sigma_\tau}\Big((\nabla_X W^{\|})|_{\Sigma_\tau} + (\partial_X W^\perp)|_{\Sigma_\tau}\nu_\tau + W^\perp|_{\Sigma_\tau}\nabla_X\nu, Y\Big)\\ =& \tilde{g}_\tau(\tilde{\nabla}_X(W^{\|}|_{\Sigma_\tau}), Y) + W^\perp|_{\Sigma_\tau} g(\nabla_X\nu, Y)\\ =& \tilde{g}_\tau(\tilde{\nabla}_X(W^{\|}|_{\Sigma_\tau}), Y) + W^\perp|_{\Sigma_\tau}\tilde{k}_\tau(X,Y),\end{aligned}$$

where in the second-to last equation, the second term vanished because $Y \in \mathrm{T}\Sigma$, and for the same reason we could replace $g|_{\Sigma_\tau}$ with $\tilde{g}_\tau$ and ∇ with $\tilde{\nabla} := \nabla^{\tilde{g}}$ in the first term. Analogously, the second term in (6.13) equals

$$g|_{\Sigma_\tau}(X, \nabla_Y W) = \tilde{g}_\tau(X, \tilde{\nabla}_Y(W^{\|}|_{\Sigma_\tau})) + W^\perp|_{\Sigma_\tau}\tilde{k}_\tau(X,Y).$$

Thus

$$\begin{aligned}(\mathcal{L}_W g)|_{\Sigma_\tau}(X,Y) =& \tilde{g}_\tau(\tilde{\nabla}_X(W^{\|}|_{\Sigma_\tau}), Y) + \tilde{g}_\tau(X, \tilde{\nabla}_Y(W^{\|}|_{\Sigma_\tau})) + 2W^\perp|_{\Sigma_\tau}\tilde{k}_\tau(X,Y)\\ =& \mathcal{L}_{W^{\|}|_{\Sigma_\tau}}\tilde{g}_\tau(X,Y) + 2W^\perp|_{\Sigma_\tau}\tilde{k}_\tau(X,Y)\end{aligned} \tag{6.14}$$

(where in the last equation, we could use the fact that for vectors tangential to Σ and tensors on Σ, we may also define the Lie derivative using $\tilde{\nabla}$).

If we now use eq. (6.12) in eq. (6.14) with $W := \nabla^j_{t,\dots,t}V$, we observe that

$$\left(\tau \mapsto \mathcal{L}_{(\nabla^j_{t,\dots,t})^{\|}|_{\Sigma_\tau}}\tilde{g}_\tau\right) \in CH^{s-j}_{\mathrm{loc}}(M, \mathrm{S}^2\Sigma_\cdot, t) \subset C^0(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_\cdot, \mathrm{S}^2\Sigma_\cdot)).$$

This holds since the left-hand side and the second term on the right-hand side of eq. (6.14) have the needed regularity, hence also the first term on the right-hand side must have this regularity.

Now

$$W \mapsto \mathcal{L}_W\tilde{g}_\tau$$

is a linear differential operator from $\mathrm{T}\Sigma_\tau$ to $\mathrm{S}^2\Sigma_\tau$ of injective principal symbol. Thus by elliptic regularity theory, we conclude (cf. Remark 3.9)

$$(\nabla^j_{t,\dots,t}V)^{\|} \in C^0(t(M), H^{s-j+1}_{\mathrm{loc}}(\Sigma_\cdot, \mathrm{T}\Sigma_\cdot)) \tag{6.15}$$

[17] Note that the assumption means that these are already continuous sections of the corresponding H^{s-j}_{loc}-bundles.

for all integers $j \geq 0$. (In passing we note that the elliptic theory does not only yield that $(\nabla^j_{t,\dots,t}V)^{\|}(\tau) \in H^{s-j+1}_{\mathrm{loc}}(\Sigma_\tau, \mathrm{T}\Sigma_\tau)$ for all $\tau \in t(M)$, but also the elliptic estimates for all seminorms defining the topology of $H^{s-j+1}_{\mathrm{loc}}(\Sigma_\tau, \mathrm{T}\Sigma_\tau)$, which depend continuously on $\tau \in t(M)$. Using this, one recognizes that $(\nabla^j_{t,\dots,t}V)^{\|}$ is even a continuous section of $(H^{s-j+1}_{\mathrm{loc}}(\Sigma_\tau, \mathrm{T}\Sigma_\tau))_{\tau\in t(M)}$.)

We recall $\nu_\tau = -\frac{1}{\alpha}\mathrm{grad}(t)$, thus $(\nabla^j_{t,\dots,t}V)^\perp = g(\nabla^j_{t,\dots,t}V, \frac{1}{\alpha}\mathrm{grad}(t))$. For $X \in \mathrm{T}\Sigma_\tau$, we have, using that X and $\nabla_X(\frac{1}{\alpha}\mathrm{grad}(t))$ are parallel to Σ_τ:[18]

$$\begin{aligned} \mathrm{d}((\nabla^j_{t,\dots,t}V)^\perp)(X) =&\partial_X(g(\nabla^j_{t,\dots,t}V, \frac{1}{\alpha}\mathrm{grad}(t))) \\ =&\frac{1}{\alpha}g(\nabla_X(\nabla^j_{t,\dots,t}V), \mathrm{grad}(t)) + g((\nabla^j_{t,\dots,t}V)^{\|}, \nabla_X(\frac{1}{\alpha}\mathrm{grad}(t))) \\ =&\frac{1}{\alpha}\mathcal{L}_{\nabla^j_{t,\dots,t}V}g(X, \mathrm{grad}(t)) - \frac{1}{\alpha}g(\nabla_t(\nabla^j_{t,\dots,t}V), X) \\ &+ g((\nabla^j_{t,\dots,t}V)^{\|}, \nabla_X(\frac{1}{\alpha}\mathrm{grad}(t))). \end{aligned} \tag{6.16}$$

The first term here lies in $C^0(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_\cdot, \underline{\mathbb{R}})$ by eq. (6.12) (as $CH^{s-j}_{\mathrm{loc}}(M, \mathrm{S}^2M, t) \subset C^0(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_\cdot, \mathrm{S}^2M|_{\Sigma_\cdot})))$. The last one does so by eq. (6.15). For the second one, we note that although V might not be $(j+1)$-times continuously differentiable in the time direction, this term actually only contains derivatives of V up to order j, as X is parallel to Σ_τ (so the highest-order derivatives drop out). Thus we obtain $\mathrm{d}((\nabla^j_{t,\dots,t}V)^\perp)(X) \in C^0(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_\cdot, \underline{\mathbb{R}})$. As $X \in \mathrm{T}\Sigma_\tau$ was arbitrary, this implies

$$\mathrm{d}|_{\mathrm{T}\Sigma_\tau}((\nabla^j_{t,\dots,t}V)^\perp) \in C^0(t(M), H^{s-j}_{\mathrm{loc}}(\Sigma_\cdot, \mathrm{T}^*\Sigma_\cdot)).$$

As $\mathrm{d}|_{\mathrm{T}\Sigma_\tau}$ is a first-order linear differential operator mapping functions on Σ_τ onto one-forms, whose principal symbol is injective, we can conclude again, by Remark 3.9, that $(\nabla^j_{t,\dots,t}V)^\perp \in C^0(t(M), H^{s+1-j}_{\mathrm{loc}}(\Sigma_\cdot, \underline{\mathbb{R}}))$ for all integers $j \geq 0$. Assembling this together with (6.15), it follows that

$$\nabla^j_{t,\dots,t} \in C^0(t(M), H^{s+1-j}_{\mathrm{loc}}(\Sigma_\cdot, \mathrm{T}M|_{\Sigma_\cdot})$$

for all integers $j \geq 0$. This is equivalent to $V \in CH^{s+1}_{\mathrm{loc}}(M, \mathrm{T}M, t)$. □

7. Example: Plane waves in Minkowski space

In this last section, we want to show a simple example motivated by physics. The treatment is based on [12], Kapitel 32.

Setting 7.1. We consider Minkowski space (M, g), where $M := \mathbb{R}^4$, $g := -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3$ in cartesian coordinates $x := \mathrm{id}$. In the following, all components are considered with respect to cartesian coordinates, and we adopt the usual physics notation and summation convention. The greek indices shall run from 0 to 3.

Let $h \in C^\infty(M, \mathrm{S}^2M)$. By adding a suitable (physically irrelevant) gauge solution, we can assume that $\delta^g\overline{h} = 0$: As in the proof of Theorem 6.1, using an auxiliary wave equation, one can show that for any $h \in C^\infty(M, \mathrm{S}^2M)$, there exists $V \in C^\infty(M, \mathrm{T}M)$ such that $\delta^g(\overline{h} - \overline{\mathcal{L}_V g}) = 0$. By replacing h with $h - \mathcal{L}_V g$, we can thus assume $\delta^g\overline{h} = 0$.

[18] For the latter one, recall eq. (6.7).

In the following, we assume h to take on the particularly simple form of a *plane wave*, i.e. there exist $0 \neq p \in \mathbb{R}^4 \cong \mathrm{T}_x M$, $0 \neq A \in \bigodot^2(\mathbb{R}^4) \cong \bigodot^2(\mathrm{T}_x M)$ (for all $x \in M$, where $\bigodot^2(V)$ denotes the symmetric $(0,2)$-tensor product of the vector space V with itself)[19] such that

$$h_{\mu\nu}(x) := \mathrm{Re}(A_{\mu\nu} e^{ip_\alpha x^\alpha})$$

(where $p_\alpha x^\alpha := g(p,x) = -p^0 x^0 + p^1 x^1 + p^2 x^2 + p^3 x^3$ under the isomorphism $\mathrm{T}_x M \cong \mathbb{R}^4 \cong M$ for $x \in M$).[20]

Abbreviate $\partial_\mu := \frac{\partial}{\partial x^\mu}$, $\partial^\mu := g^{\mu\nu}\partial_\nu$ for $\mu = 0,1,2,3$. As Minkowski space is flat, we have $\Box_L = \partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$. Then

$$\begin{aligned}(D\mathrm{ric}(h))_{\mu\nu} =& \frac{1}{2}\Box_L h_{\mu\nu} \\ =& \frac{1}{2}\mathrm{Re}(A_{\mu\nu} e^{ip_\alpha x^\alpha}(-(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2)) \\ =& \frac{1}{2}\mathrm{Re}(A_{\mu\nu} p_\beta p^\beta e^{ip_\alpha x^\alpha}).\end{aligned}$$

In particular, $D\mathrm{ric}(h) = 0$ holds if and only if $p_\beta p^\beta = 0$. Thus p must be a lightlike vector; plane gravitational waves indeed propagate with the speed of light.[21].

The condition $\delta^g \overline{h} = 0$ is (using again the flatness of Minkowski space) equivalent to

$$\begin{aligned}0 = (\delta^g \overline{h})_\mu = -\partial^\nu \overline{h}_{\nu\mu} = -\partial^\nu \mathrm{Re}\left(A_{\nu\mu} e^{ip_\alpha x^\alpha} - \frac{1}{2} A^\beta_\beta g_{\nu\mu} e^{ip_\alpha x^\alpha}\right) \\ = -\mathrm{Re}\left(ip^\nu A_{\nu\mu} e^{ip_\alpha x^\alpha} - \frac{1}{2} ip^\nu A^\beta_\beta g_{\nu\mu} e^{ip_\alpha x^\alpha}\right)\end{aligned}$$

for $\mu = 0,1,2,3$. We conclude

$$p^\nu A_{\nu\mu} - \frac{1}{2} p^\nu A^\beta_\beta g_{\nu\mu} = 0. \tag{7.1}$$

A priori, it could have been assumed that A consisted of 10 independent numbers (degrees of freedom), since a symmetric $(0,2)$-tensor on $\mathbb{R}^4$ has 10 independent components. The constraints (7.1) impose four conditions and reduce these to 6.

[19] The letter x is used for points and for the chart, but there should not be a risk of confusion.

[20] Note that the concept of a plane wave on an arbitrary manifold is not well-defined, since it depends on charts via the isomorphisms $\mathbb{R}^4 \cong \mathrm{T}_x M$ and $\bigodot^2(\mathbb{R}^4) \cong \bigodot^2(\mathrm{T}_x M)$ for $x \in M$. The definition of the plane wave above and the underlying physical concept imply that the components of p and A are constant with respect to a given chart. In Minkowski space, we can however use a preferred class of charts to define the plane waves, namely those where the metric takes on the usual diagonal form $\mathrm{diag}(-1,1,1,1)$. They can all be related by global Lorentz transformations, so in particular if a vector resp. tensor field has constant components in one of them, it has constant components in all of them.

The relevance of this example is justified by the Fourier transform; many physically realistic waves can be written as superpositions of such plane waves. However note that some, also physically imaginable, solutions to the linearised Einstein equations might be not square-integrable, such that the Fourier transform is not well-defined. Also they might be so irregular that the interpretation of the Fourier transform as a decomposition into plane waves is not possible.

Also we first had to impose the "gauge condition" $\delta^g \overline{h} = 0$ and afterwards wrote down the plane-wave ansatz. Hence we can not even claim that all plane-wave solutions to the linearised Einstein equations can by addition of a gauge solution brought into a plane-wave form with $\delta^g \overline{h} = 0$. (The addition of the gauge solution $\mathcal{L}_V g$ might spoil the plane-wave form, as there is no reason why $\mathcal{L}_V g$ should be a plane wave. On the other hand, one could try to interpret this as first imposing the gauge condition on an arbitrary solution and afterwards doing a Fourier transformation, i.e. decomposition into plane waves. The addition of the gauge solution might however make the second step impossible, if the gauge solution is not decaying at infinity fast enough or is too irregular.)

Here the plane waves just serve as an example.

[21] Until now, we only knew that they could not propagate faster

By eventually performing a Lorentz transformation, we can assume $p^\mu = p$ for $\mu = 0, 3$ and $p^\mu = 0$ for $\mu = 1, 2$, where $0 \neq p \in \mathbb{R}$. Then the equations (7.1) read explicitly

$$\begin{aligned} pA_{00} + pA_{30} - \frac{1}{2}p(A_{00} - A_{11} - A_{22} - A_{33}) &= 0, \\ pA_{01} + pA_{31} &= 0, \\ pA_{02} + pA_{32} &= 0, \\ pA_{03} + pA_{33} + \frac{1}{2}p(A_{00} - A_{11} - A_{22} - A_{33}) &= 0. \end{aligned}$$

Thus we can write all 16 components of $A_{\mu\nu}$ in terms of the six independent variables $A_{00}, A_{11}, A_{12}, A_{13}, A_{23}, A_{33}$. Namely,

$$A_{01} = -A_{31}, \qquad A_{02} = -A_{32}, \qquad A_{03} = -\frac{A_{33} + A_{00}}{2}, \qquad A_{22} = -A_{11}$$

(and the rest is determined by symmetry). Furthermore, some of the remaining degrees of freedom are unphysical, as we have some residual gauge freedom even after imposing the condition $\delta^g \overline{h} = 0$:

Let $v \in \mathbb{C}^4$. The vector field $V^\beta(x) := \mathrm{Re}(v^\beta e^{ip_\alpha x^\alpha})$ satisfies

$$(\mathcal{L}_V g)_{\mu\nu} = (\partial_\mu V^\beta) g_{\beta\nu} + (\partial_\nu V^\beta) g_{\mu\beta} = \mathrm{Re}\left(iv^\beta p_\mu g_{\beta\nu} e^{ip_\alpha x^\alpha} + iv^\beta p_\nu g_{\mu\beta} e^{ip_\alpha x^\alpha} \right). \tag{7.2}$$

Also we calculate, using Lemma 5.5, $\mathrm{ric}_g = 0$ and $p_\gamma p^\gamma = 0$:

$$(\delta^g \overline{\mathcal{L}_V g})_\beta = (\delta^g (\nabla V^\flat))_\beta = (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\mathrm{Re}(v_\beta e^{ip_\alpha x^\alpha}) = \mathrm{Re}(p_\gamma p^\gamma v_\beta e^{ip_\alpha x^\alpha}) = 0$$

for $\beta = 0, 1, 2, 3$. Thus if h satisfies $\delta^g \overline{h} = 0$, also $\delta^g(\overline{h} - \overline{\mathcal{L}_V g}) = 0$. This shows that we can replace the plane wave h with the plane wave $h - \mathcal{L}_V g$, where V is as above, without spoiling the divergence condition. They are physically equivalent.

The calculation (7.2) shows that this replacement of h by $h - \mathcal{L}_V g$ corresponds to the replacement

$$A_{\mu\nu} \to A_{\mu\nu} - iv^\beta p_\mu g_{\beta\nu} - iv^\beta p_\nu g_{\mu\beta}.$$

Since $p^0 = p^3 = p$, $p^1 = p^2 = 0$, this corresponds to the replacements

$$\begin{aligned} A_{00} &\to A_{00} - 2iv^0 p, \\ A_{11} &\to A_{11}, \\ A_{12} &\to A_{12}, \\ A_{13} &\to A_{13} - iv^1 p, \\ A_{23} &\to A_{23} - iv^2 p, \\ A_{33} &\to A_{33} - 2iv^3 p \end{aligned}$$

(note $p_0 = -p^0$, $g_{00} = -1$). Therefore, we can reduce the number of degrees of freedom from six to two physically relevant ones. Namely, choosing $v^0 := \frac{A_{00}}{2ip}$, $v^1 := \frac{A_{13}}{ip}$, $v^2 := \frac{A_{23}}{ip}$, $v^3 := \frac{A_{33}}{2ip}$, we can arrange that the amplitude (still denoted $A_{\mu\nu}$) takes on the form

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are two independent parameters, corresponding to two different polarization states. There is no gauge transformation which can reduce the number of degrees of freedom further. Furthermore, the gravitational wave is transverse (i.e. divergence-free, which follows from $\delta^g \overline{h} = 0$ and $h = \overline{h}$ here) and trace-free. Such a tensor is also called TT-tensor.

Without being able to go into depth here, it shall be noted that the structure of the plane wave amplitude found above has direct physical consequences. For example, it implies that hypothetical quanta of gravitational waves (gravitons) need to have spin 2 (due to their transformation behavior under rotation, cf. [12], Kapitel 32). For further study, the reader should consult the extensive physics literature.

Conclusion and Outlook

In the preceding work, we have been able to study infinitesimal deformations of Einstein metrics on Lorentzian manifolds. The linearised Einstein equations were derived and solved. Given initial conditions on a Cauchy surface satisfying the linearised Constraint equations, it was shown that there exists a solution inducing this initial data. The Cauchy problem for the linearised Einstein equations in the spaces of Sobolev sections is nevertheless not well-posed: The solution is only unique "up to gauge", as proven as well in this work.

A further study of the Cauchy problem can be found in [1]. There it has been shown that the uniqueness "up to gauge" can be used to derive a well-posed Cauchy problem in suitable quotient spaces. Furthermore, it is interesting to study further the spaces of the initial data (modulo "gauge producing initial data"), as this allows for a classification of the possible waves as well. This is done as well in the reference given above, but only in the case of vanishing scalar curvature. A classification result for the general case is yet to prove, but it will probably be complex.

The interest for Einstein metrics and their infinitesimal deformations can be justified by pure mathematics and they can be studied by mathematicians for its own sake. However, in particular the Lorentzian case is highly relevant for the physical reality, the world we live in. This was hinted at in the preceding example, but the actual problems to be solved in physics are far more complex.

However, in any case the linearised Einstein equations are only an approximation to reality. The real problem is non-linear and has been studied for decades. Abstract results are also known in this case, but the study of these is hard. It may be hoped that a better knowledge of the linearised Einstein equations can also help to understand the non-linear ones.

For the working physicist, the abstract results derived here are "obvious". Nevertheless, in a certain sense they deepen the understanding of our world as it is.

A. Basic definitions of differential geometry and some formulas

For the sake of completeness, we record here some definitions used in the text, to be applied in Setting 1.1. Most of them are standard, but nevertheless they differ in some of the literature.

Definition A.1. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $n \in \mathbb{N}_0$, we write $|\alpha| := \alpha_1 + \dots + \alpha_n$, and $\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \prod_{i=1}^n \left(\frac{\partial}{\partial x^i}\right)^{\alpha_i}$ (where $\left(\frac{\partial}{\partial x^i}\right)^0 := \mathrm{id}$ for $i = 1, \dots, n$).

Definition A.2. 1. The canonical volume form on M is denoted dvol_g.

2. For the curvature tensor we use the definition

$$\mathrm{R}(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z$$

for all $X, Y, Z \in \mathrm{T}M$.[22] Here

$$\nabla^2_{X,Y} Z := \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z$$

is the second covariant derivative.

Definition and Remark A.3. 1. A *(generalized) orthonormal basis* $\{e_i\}_{0 \le i \le m-1}$ of some m-dimensional vector space with a scalar product $\langle \cdot, \cdot \rangle$ shall be vectors such that $\langle e_i, e_j \rangle = \epsilon_i \delta_{ij}$ for all $0 \le i, j \le m-1$ and numbers $\epsilon_i \in \{\pm 1\}$ (for $0 \le i \le m-1$). (These equations serve as a definition of the numbers ϵ_i.)

2. For any $p \in M$, there exists a neighbourhood U of p and sections $\{e_i\}_{0 \le i \le n}$ of $\mathrm{T}U$ such that these form a orthonormal basis with respect to $g|_q$ at each $q \in U$ (a *local orthonormal frame*). Local orthonormal frames on Σ will be denoted by $\{e_i\}_{1 \le i \le n}$ instead of $\{e_i\}_{0 \le i \le n-1}$.

3. We call a vector field X *synchronous at p* if $(\nabla X)|_p = 0$. We note that for any $p \in M$, there even exists a local orthonormal frame in a neighbourhood of p which is synchronous at p; such a frame is called a *local geodesic frame*.[23]

Definition and Remark A.4. 1. Viewing elements of $C^\infty(M, \mathcal{T}^{i,j}M)$ as $C^\infty(M)$-multilinear maps from $(\mathrm{T}M)^j$ to $(\mathrm{T}M)^{\otimes i}$, we define their trace (resp. contraction): For $S \in C^\infty(M, \mathcal{T}^{i,j}M)$ with $i, j \ge 1$ and $0 \le k \le i, 0 \le l \le j$ we define with respect to local coordinates $(x^0, \dots, x^n)$, for all vectors $X_1, \dots, X_{j-1} \in \mathrm{T}M$:

$$(\mathrm{tr}^l_k S)(X_1, \dots, X_{j-1}) = \sum_{a=0}^n dx^a(S(X_1, \dots, \frac{\partial}{\partial x^a}, \dots, X_{j-1})),$$

where the vector fields $\frac{\partial}{\partial x^a}$ are inserted at the k'th position and the dual basis elements dx^a are paired with the l'th tensor factor. This defines an element of $C^\infty(M, \mathcal{T}^{i-1,j-1}M)$ which does not depend on the choice of coordinates. It could just as well have been defined using a local orthonormal frame $\{e_a\}_{0 \le a \le n}$ and its (algebraic) dual frame $\{e_a^*\}_{0 \le a \le n}$.[24] Furthermore the trace commutes with covariant differentiation.

[22] Here and often else, $X, Y, Z \in \mathrm{T}M$ are interpreted as vectors over some given point; as R is a tensor, $\mathrm{R}(X,Y)Z$ is well-defined if X, Y, Z are all vectors corresponding to the same basepoint. This shall be implied in such a statement. Analogously, in the next formula we implicitly assume Z to be a vector field on M.

[23] These can be defined by parallel transport of an orthonormal basis of $\mathrm{T}_p M$ along geodesics in a normal neighbourhood of p.

[24] The dual frame is characterised by $e_a^*(e_b) = \delta_{ab}$ for $0 \le a, b \le n$.

2. We can also define a *metric trace* (resp. contraction) for (i,j)-tensors, where we contract two covector slots: For $S \in C^\infty(M, \mathcal{T}^{i,j}M)$ with $j \geq 2$ and $0 \leq k < l \leq j$, we define, using an orthonormal frame, for vectors $X_1, \ldots, X_{j-2} \in \mathrm{T}M$:

$$((\mathrm{tr}_g)_{kl} S)(X_1, \ldots, X_{j-2}) = \sum_{a=0}^{n} \epsilon_a S(X_1, \ldots, e_a, \ldots, e_a, \ldots, X_{j-2}),$$

where the vector fields e_a are inserted at the k'th and l'th position. This defines an element of $C^\infty(M, \mathcal{T}^{i,j-2}M)$ which does not depend on the choice of orthonormal frame ([13], Lemma 1.2.5); also tr_g commutes with covariant differentiation like the other trace. When they are clear from the context, the indices k, l will be omitted when writing both kinds of traces.

Definition and Remark A.5. 1. The metric g can be extended to 1-forms (i.e. $(0,1)$-tensor fields) and $(0,2)$-tensor fields: If $\alpha, \beta \in \Omega^1(M)$, we define, using a local orthonormal frame $\{e_i\}_{0 \leq i \leq n}$:

$$g(\alpha, \beta) := \sum_{i=0}^{n} \epsilon_i \alpha(e_i) \beta(e_i).$$

If $S, T \in C^\infty(M, \mathcal{T}^{0,2}M)$, we define

$$g(S,T) := \sum_{i,j=0}^{n} \epsilon_i \epsilon_j S(e_i, e_j) T(e_i, e_j).$$

(These definitions are independent of the choice of frame.)

2. We define the *"musical isomorphisms"*

$$\flat_k : \mathcal{T}^{i,j} \to \mathcal{T}^{i-1,j+1}, \qquad \flat_k := \mathrm{tr}^k_{j+1} \circ (\cdot \otimes g)$$

(for $1 \leq k \leq i$, where $i \geq 1$ is assumed) and

$$\sharp_k : \mathcal{T}^{i,j} \to \mathcal{T}^{i+1,j-1}, \qquad \sharp_k := \mathrm{tr}^{i+1}_k \circ (\cdot \otimes g^{-1})$$

(for $1 \leq k \leq j$, where $j \geq 1$ is assumed). When applied point-wise, these commute with covariant differentiation as the traces do, and g and g^{-1} are parallel. By definition, $\flat_i$ is the inverse of $\sharp_j$, wherever this statement is well-defined. We also write $\omega^\sharp := \sharp_1(\omega)$ for one-forms ω and $V^\flat := \flat_1(V)$ for vector fields V.

3. For $S \in C^\infty(M, \mathcal{T}^{i,j}M)$, $T \in C^\infty(M, \mathcal{T}^{k,l}M)$ with $j, k \geq 1$, we define the *composition*

$$C^\infty(M, \mathcal{T}^{i+k-1,j+l-1}M) \ni S \circ T := \mathrm{tr}^{i+1}_j (S \otimes T),$$

i.e. one contracts the last covector slot of S with the first vector slot of T. Sometimes also two $(0,2)$-tensors will have to be composed: For $S, T \in C^\infty(M, \mathcal{T}^{0,2}M)$, we define $S \circ T \in C^\infty(M, \mathcal{T}^{0,2}M)$ by

$$(S \circ T)(X,Y) := g(S(X,\cdot), T(Y,\cdot)) = \sum_{i=0}^{n} \epsilon_i S(X, e_i) T(Y, e_i) = \mathrm{tr}_g(S(X,\cdot), T(Y,\cdot))$$

for $X, Y \in \mathrm{T}M$.[25]

[25] Consequently, a more appropriate symbol would actually be $\circ^g$ as it depends on g via the orthonormal frame.

4. We note that
$$(\mathrm{tr}_g)_{kl} S = \mathrm{tr}^1_k \mathrm{tr}^2_l (g^{-1} \otimes S).$$
Here the traces shall be applied from the right to the left, and g^{-1} is the inverse of g, where g is interpreted as a map from $\mathrm{T}M$ to T^*M.[26] If S is a $(2,0)$-tensor field, one can also write $\mathrm{tr}_g S = \mathrm{tr}(g^{-1} \circ S)$.

5. Also we have for smooth $(0,2)$-tensor fields S and T:
$$g(S,T) = \mathrm{tr}(g^{-1} \circ S \circ g^{-1} \circ T)$$
and
$$S \circ T = S \circ g^{-1} \circ T$$
(where the composition on the left-hand side is the metric-dependent one while the one on the right-hand side is metric-independent).

Definition and Remark A.6. The Lie derivative is denoted $\mathcal{L}$, as usual. We recall that $\mathcal{L}$ commutes with contractions as well, and the definitions of $\mathcal{L}$ and ∇ on functions (simply the partial derivative), and that both derivatives satisfy a product rule for tensor fields. Noting that inserting a vector field X on the k'th position into a (i,j)-tensor S (where $1 \le k \le j$) is equal to the contraction $c^{i+1}_k(S \otimes X)$, we obtain the product rules for the covariant differentiation and Lie derivatives of $(0,j)$-tensors: For $X_1, \ldots, X_j$ vector fields on M, we have

$$\partial_X(S(X_1,\ldots,X_j)) = (\mathcal{L}_X S)(X_1,\ldots,X_j) + \sum_{l=1}^{j} S(X_1,\ldots,\mathcal{L}_X X_l,\ldots,X_j),$$
$$\partial_X(S(X_1,\ldots,X_j)) = (\nabla_X S)(X_1,\ldots,X_j) + \sum_{l=1}^{j} S(X_1,\ldots,\nabla_X X_l,\ldots,X_j)$$

(where $X \in \mathcal{X}(M)$ in the former and $X \in \mathrm{T}M$ in the latter equation). In particular, using the torsion-freeness and a product rule (for g) for the Levi-Civita connection, we obtain for vector fields X, Y, Z on M:

$$\begin{aligned}(\mathcal{L}_X g)(Y,Z) &= \partial_X(g(Y,Z)) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) = \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_X Z).\end{aligned} \tag{A.1}$$

Note that this formula defines $\mathcal{L}_X g$ also for distributional X.

These definitions here were stated for M only, but if needed, analogous definitions for other manifolds will be used.

B. Calculations

This appendix is devoted to the proof of the various computations used in the main body of the text. Much of the treatment follows [13] and [10], which are based on [7].

[26] Thus g^{-1} is a $(2,0)$-tensor field such that $g \circ g^{-1} = \mathrm{id}_{\mathrm{T}^*M}$ and $g^{-1} \circ g = \mathrm{id}_{\mathrm{T}M}$; using a local orthonormal frame $\{e_i\}_{0 \le i \le n}$ one can write $g^{-1} = \sum_{i=0}^{n} \epsilon_i e_i \otimes e_i$.

Proof of Proposition 4.3, 1. We abbreviate g_t for $g+th$ and ∇^t for ∇^{g+th} (wherever this is well-defined); then $\nabla^g = \nabla^0$. In the following we choose some $p \in M$ and only calculate at this point, but this is often suppressed in the notation. Recalling that $\mathrm{D}\nabla^g(h)$ is a tensor[27], we note that for vector fields X, Y, Z on M, the value of $g(\mathrm{D}\nabla^g(h)(X,Y),Z)|_p$ only depends on the values of X, Y, Z at p. Thus we can assume that X, Y, Z are synchronous at p (with respect to ∇^0). By torsion-freeness of ∇^0 this implies that the commutators of X, Y and Z also vanish at p.

The Koszul formula yields

$$\begin{aligned}2g_t(\nabla^t{}_X Y, Z) =& \partial_X(g_t(Y,Z)) + \partial_Y(g_t(X,Z)) - \partial_Z(g_t(X,Y)) \\ &- g_t(X,[Y,Z]) + g_t(Y,[Z,X]) + g_t(Z,[X,Y])\end{aligned}$$

for all t where this is a well-defined statement. The second line vanishes as the commutators of X, Y, Z are zero. Thus we obtain, inserting $g_t = g + th$, $g_0 = g$ and subtracting the two resulting formulas,

$$\begin{aligned}&2(g+th)(\nabla^t{}_X Y, Z) - 2g(\nabla^0{}_X Y, Z) \\ &\quad = \partial_X((g+th)(Y,Z)) - \partial_X(g(Y,Z)) + \partial_Y((g+th)(X,Z)) \\ &\qquad - \partial_Y(g(X,Z)) - \partial_Z((g+th)(X,Y)) + \partial_Z(g(X,Y)) \\ &\quad = t\left(\partial_X(h(Y,Z)) + \partial_Y(h(X,Z)) - \partial_Z(h(X,Y))\right).\end{aligned}$$

However on the left-hand side of this equation, the second term actually is zero due to synchronousness. Therefore,

$$(g+th)(\nabla^t{}_X Y, Z) = \frac{1}{2} t\left(\partial_X(h(Y,Z)) + \partial_Y(h(X,Z)) - \partial_Z(h(X,Y))\right)$$

and upon differentiating, we obtain

$$\begin{aligned}&\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(\nabla^t{}_X Y, Z)\right) + \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} th(\nabla^t{}_X Y, Z)\right) \\ &\quad = \frac{1}{2}\left(\partial_X(h(Y,Z)) + \partial_Y(h(X,Z)) - \partial_Z(h(X,Y))\right).\end{aligned}$$

Here on the left-hand side, the first term equals $g(\mathrm{D}\nabla^g(h)(X,Y),Z)$ by definition of ∇^t and since differentiation with respect to t commutes with g. We note that, since h is a tensor and the vector fields are all synchronous at p, the right-hand side can be rewritten as

$$\begin{aligned}&\frac{1}{2}\left(\partial_X(h(Y,Z)) + \partial_Y(h(X,Z)) - \partial_Z(h(X,Y))\right) \\ &\quad = \frac{1}{2}\left((\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) - (\nabla_Z h)(X,Y)\right).\end{aligned}$$

[28] So

$$\begin{aligned}&g(\mathrm{D}\nabla^g(h)(X,Y),Z) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} th(\nabla^t{}_X Y, Z) \\ &\quad = \frac{1}{2}\left((\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) - (\nabla_Z h)(X,Y)\right).\end{aligned}$$

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[27] See the notes above the statement of Proposition 4.3.

[28] This follows since $h(\nabla_X Y, Z) = h(Y, \nabla_X Z) = h(\nabla_Y X, Z) = h(X, \nabla_Y Z) = h(\nabla_Z X, Y) = h(X, \nabla_Z Y) = 0$. In the end we need to have a tensor equation to make the formula valid; otherwise the assumption of synchronous vector fields would have been invalid.

To prove the claimed equation (4.4), it only remains to show that the second term on the left-hand side of the above equation vanishes. This follows from

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} th({\nabla^t}_X Y, Z) = \lim_{t\to 0}\frac{1}{t} th({\nabla^t}_X Y, Z) = \lim_{t\to 0} h({\nabla^t}_X Y, Z) = h({\nabla^0}_X Y, Z) = 0,$$

as taking the limit with respect to t commutes with h, and Y is synchronous at p (with respect to $\nabla^0 = \nabla$). □

For the proof of part 2., we first show an auxiliary lemma.

Lemma B.1 (cf. [10], Lemma 5.4)**.** *For $h \in C^\infty(M, \mathrm{S}^2 M)$ and $r \in C^\infty(M, \mathcal{T}^{0,2}M)$, we have*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{tr}_{g+th}(r) = -g(h, r).$$

Proof. At first we note that $\phi : I \to C^\infty(M, \mathcal{T}^{1,1}M), t \mapsto (g + th)^{-1} \circ (g + th)$ (where $I \subset \mathbb{R}$ is a sufficiently small interval) is the composition of the map $\psi : I \times I \to C^\infty(M, \mathcal{T}^{1,1}M), (t_1, t_2) \mapsto (g + t_1 h)^{-1} \circ (g + t_2 h)$ with the diagonal map $\Delta : I \to I \times I$, $t \mapsto (t, t)$. Thus the chain rule yields

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi(t) &= \left(\mathrm{d}_{\Delta(0)}\psi\right)\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Delta\right)\\ &= \left(\left.\tfrac{\mathrm{d}}{\mathrm{d}t_1}\right|_{t_1=0}(g + t_1 h)^{-1}\circ g,\quad \left.\tfrac{\mathrm{d}}{\mathrm{d}t_2}\right|_{t_2=0} g^{-1}\circ(g + t_2 h)\right)\begin{pmatrix}1\\1\end{pmatrix}\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t_1}(g + t_1 h)^{-1}\right|_{t_1=0}\circ g + \left.\frac{\mathrm{d}}{\mathrm{d}t_2}\right|_{t_2=0} g^{-1}\circ(g + t_2 h)\\ &=(\mathrm{D}(g^{-1})(h))\circ g + g^{-1}\circ h. \end{aligned} \tag{B.1}$$

This can be seen as a product rule for differentiation with respect to t. However, by definition ϕ is a constant function (taking as value the $(1,1)$-tensor corresponding to the identity map on $\mathrm{T}M$). Hence the left-hand side of eq. (B.1) is zero, implying that

$$\mathrm{D}(g^{-1})(h) = -g^{-1}\circ h\circ g^{-1} \tag{B.2}$$

and therefore

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{tr}_{g+th}(r) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} &\mathrm{tr}((g + th)^{-1}\circ r)\\ =&\mathrm{tr}((\mathrm{D}(g^{-1})(h))\circ r) = -\mathrm{tr}(g^{-1}\circ h\circ g^{-1}\circ r)\\ =& -g(h, r),\end{aligned}$$

where it was used that the limit $t \to 0$ commutes with the metric-independent trace. □

Proof of Proposition 4.3, 2. We recall $\delta^g(\alpha) = -\mathrm{tr}_g(\nabla.\alpha(\cdot))$ for $\alpha \in \Omega^1(M)$. In Lemma B.1 we found

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{tr}_{g+th}(\nabla.\alpha(\cdot)) = -g(h, \nabla.\alpha(\cdot)).$$

Thus we get

$$\mathrm{D}\delta^g(h)(\alpha) = -\left(-g(h, \nabla.\alpha(\cdot)) + \mathrm{tr}_g(\mathrm{D}\nabla^g(h)(\alpha))\right) \tag{B.3}$$

by a product rule which can be proven as eq. (B.1).

We use now a generalized orthonormal frame $\{e_i\}_{0\leq i\leq n}$ to evaluate the metric trace. By eq. (4.4), we have for all $X, Y \in \mathrm{T}M$ (as $\mathrm{D}\nabla^g(h)(X,Y) = \sum_{j=0}^n \epsilon_j g(\mathrm{D}\nabla^g(h)(X,Y), e_j)e_j$):

$$\mathrm{D}\nabla^g(h)(X,Y) = \sum_{j=0}^n \frac{1}{2}\epsilon_j \left((\nabla_X h)(Y,e_j) + (\nabla_Y h)(X,e_j) - (\nabla_{e_j} h)(X,Y)\right) e_j. \tag{B.4}$$

By the product rule and since ∇ commutes with traces, we have

$$\begin{aligned}\partial_X(\alpha(Y)) =&\nabla_X(\mathrm{tr}_1^1(\alpha\otimes Y))\\ =&\mathrm{tr}_1^1((\nabla_X\alpha)\otimes Y) + \mathrm{tr}_1^1(\alpha\otimes(\nabla_X Y))\\ =&(\nabla_X\alpha)(Y) + \alpha(\nabla_X Y).\end{aligned}$$

As the partial derivatives are unchanged when changing the metric, we deduce from this

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\nabla^{g+th}{}_Y\alpha)(X) + \alpha(\nabla^{g+th}{}_X Y) = (\mathrm{D}\nabla^g(h)(\alpha))(X,Y) + \alpha(\mathrm{D}\nabla^g(h)(X,Y)).$$

Thus

$$\begin{aligned}(\mathrm{D}\nabla^g(h)(\alpha))(X,Y) =& -\alpha(\mathrm{D}\nabla^g(h)(X,Y))\\ =& -\alpha\left(\sum_{j=0}^n \frac{1}{2}\epsilon_j\left((\nabla_X h)(Y,e_j) + (\nabla_Y h)(X,e_j) - (\nabla_{e_j}h)(X,Y)\right)e_j\right).\end{aligned}$$

If we use the same generalized orthonormal frame $\{e_i\}_{0\leq i\leq n}$ to evaluate the metric trace, we get

$$\begin{aligned}-\mathrm{tr}_g(\mathrm{D}\nabla^g(h)(\alpha)) =&\alpha\left(\sum_{i,j=0}^n \frac{1}{2}\epsilon_i\epsilon_j\left((\nabla_{e_i}h)(e_i,e_j) + (\nabla_{e_i}h)(e_i,e_j) - (\nabla_{e_j}h)(e_i,e_i)\right)e_j\right)\\ =&\sum_{i,j=0}^n \frac{1}{2}\epsilon_i\epsilon_j\alpha(e_j)\left(2(\nabla_{e_i}h)(e_i,e_j) - (\nabla_{e_j}h)(e_i,e_i)\right)\\ =& -\sum_{j=0}^n \frac{1}{2}\epsilon_j\alpha(e_j)(2(\delta^g(h))(e_j) + \nabla_{e_j}(\mathrm{tr}_g h))\\ =& -g\left(\alpha,\delta^g(h)\right) - \frac{1}{2}g\left(\alpha, d(\mathrm{tr}_g h)\right).\end{aligned} \tag{B.5}$$

(We note $\sum_{i=0}^n \epsilon_i(\nabla_{e_j}h)(e_i,e_i) = \nabla_{e_j}(\mathrm{tr}_g h) = \partial_{e_j}(\mathrm{tr}_g h) = \mathrm{d}(\mathrm{tr}_g h)(e_i)$, since ∇_{e_j} commutes with metric traces, and the covariant derivative for a function equals the partial derivative.) Equations (B.3) and (B.5) together yield (4.5). □

Proof of Proposition 4.3, 3. As in 1. we choose $p \in M$ and assume X, Y, Z to be vector fields synchronous at p (which is valid as $\mathrm{DR}_g(h)$ is a tensor). We have

$$(\mathrm{DR}_g(h))(X,Y,Z) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nabla^t{}_X(\nabla^t{}_Y Z) - \nabla^t{}_Y(\nabla^t{}_X Z) - \nabla^t{}_{[X,Y]}Z. \tag{B.6}$$

Arguments as in the proof of eq. (B.1) yield product rules for the first and the second term in eq. (B.6). We obtain

$$\begin{aligned}(\mathrm{DR}_g(h))(X,Y,Z) =&(\mathrm{D}\nabla^g(h))(X,\nabla_Y Z) + \nabla_X((\mathrm{D}\nabla^g(h))(Y,Z))\\ &- (\mathrm{D}\nabla^g(h))(Y,\nabla_X Z) - \nabla_Y((\mathrm{D}\nabla^g(h))(X,Z)) - (\mathrm{D}\nabla^g(h))([X,Y],Z)\\ =&\nabla_X((\mathrm{D}\nabla^g(h))(Y,Z)) - \nabla_Y((\mathrm{D}\nabla^g(h))(X,Z))\end{aligned}$$

by synchronousness of X and Y at p. As X, Y, Z are synchronous at p, similar arguments to the ones made in the proof of 1., using the product rule for ∇, show that the right-hand side coincides (at p) with

$$(\nabla_X(\mathrm{D}\nabla^g(h)))(Y,Z) - (\nabla_Y(\mathrm{D}\nabla^g(h)))(X,Z).$$

This proves equation (4.6) (since both sides are tensorial). □

To continue with the proof of Proposition 4.3, we need three auxiliary calculations, which are done in the following lemmata.

Lemma B.2 ([10], Lemma 5.2). *Let h be a symmetric $(0,2)$-tensor field, and let $\{e_i\}_{0\le i\le n}$ be a locally defined generalized orthonormal frame. Then for any $X \in \mathrm{T}M$,*

$$\sum_{i=0}^{n} \epsilon_i h(\nabla_X e_i, e_i) = 0.$$

Proof. We write $\nabla_X e_i = \sum_{j=1}^{n} \alpha_{ji} e_j$. Differentiation of the orthogonality relation yields

$$\begin{aligned} 0 = \partial_X(g(e_i,e_k)) =& g(\nabla_X e_i, e_k) + g(e_i, \nabla_X e_k) \\ =& \sum_{j=0}^{n} (\alpha_{ji}\epsilon_j\delta_{jk} + \alpha_{jk}\epsilon_j\delta_{ji}) \\ =& \alpha_{ki}\epsilon_k + \alpha_{ik}\epsilon_i. \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{i=0}^{n} \epsilon_i h(\nabla_X e_i, e_i) =& \sum_{i,k=0}^{n} \epsilon_i h(\alpha_{ki} e_k, e_i) \\ =& \frac{1}{2} \sum_{i,k=0}^{n} (\epsilon_i \alpha_{ki} + \epsilon_k \alpha_{ik}) h(e_k, e_i) \\ =& \frac{1}{2} \sum_{i,k=0}^{n} \epsilon_i\epsilon_k \underbrace{(\alpha_{ki}\epsilon_k + \alpha_{ik}\epsilon_i)}_{=0} h(e_k, e_i) \\ =& 0. \end{aligned}$$

In the second equality, it has been used that the sum is symmetric under exchange of the indices i and k. □

Corollary B.3 ([10], Corollary 5.3). *Let h be a symmetric $(0,2)$-tensor field. Then $\overset{\circ}{\mathrm{R}}h$ is also a symmetric $(0,2)$-tensor.*

Proof. Let $\{e_i\}_{0\le i\le n}$ be a local orthonormal frame and let $X, Y \in \mathrm{T}M$. By definition,

$$(\overset{\circ}{\mathrm{R}}h)(X,Y) = \sum_{i=0}^{n} \epsilon_i h(\mathrm{R}(e_i, X)Y, e_i).$$

Because of the Bianchi identity, this is equal to

$$-\sum_{i=0}^{n} \epsilon_i h(\mathrm{R}(Y, e_i)X, e_i) - \sum_{i=0}^{n} \epsilon_i h(\mathrm{R}(X, Y)e_i, e_i).$$

Due to antisymmetry of R in its first two arguments, the first sum equals $(\overset{\circ}{\mathrm{R}}h)(Y,X)$. Thus to prove the corollary, we need to show that the second sum is zero. We calculate

$$\begin{aligned}\sum_{i=0}^{n}\epsilon_i h(\mathrm{R}(X,Y)e_i,e_i) =&\sum_{i=0}^{n}\epsilon_i \big(h(\nabla_X(\nabla_Y e_i),e_i)-h(\nabla_Y(\nabla_X e_i),e_i)-h(\nabla_{[X,Y]}e_i,e_i)\big)\\ =&\sum_{i=0}^{n}\epsilon_i \left(\partial_X(h(\nabla_Y e_i,e_i))-(\nabla_X h)(\nabla_Y e_i,e_i)-h(\nabla_X e_i,\nabla_Y e_i)\right.\\ &-\partial_Y(h(\nabla_X e_i,e_i))+(\nabla_Y h)(\nabla_X e_i,e_i)\\ &\left.+h(\nabla_Y e_i,\nabla_X e_i)-h(\nabla_{[X,Y]}e_i,e_i)\right).\end{aligned}$$

The third and the sixth term in the bracket add up to zero. For the other terms, we apply the previous lemma to the symmetric $(0,2)$-tensors h, $\nabla_X h$ and $\nabla_Y h$ to show that the whole sum becomes zero. This concludes the proof. □

Lemma B.4. *We define the Riemannian curvature tensor on* $(0,2)$*-tensor fields by*

$$\mathrm{R}(X,Y)h=\nabla^2_{X,Y}h-\nabla^2_{Y,X}h$$

for all $h\in C^\infty(M,\mathcal{T}^{0,2}M)$*,* $X,Y\in \mathrm{T}M$*. Then for all* $A,B,V,W\in\mathrm{T}M$*, we have*

$$(\mathrm{R}(A,B)h)(V,W)=-h(\mathrm{R}(A,B)V,W)-h(V,\mathrm{R}(A,B)W). \tag{B.7}$$

Proof. The map $\mathrm{R}(\cdot,\cdot)h(\cdot,\cdot)$ is tensorial in all of its slots since already the tensor derivatives are tensorial in all of their slots. The same holds for the right-hand side of the claimed equation. Thus it suffices to prove it at a point $p\in M$, where it can be assumed that A,B,V,W are synchronous at p. We have (where the traces are applied from the right to the left)

$$\begin{aligned}(\mathrm{R}(A,B)h)(V,W)=&\mathrm{tr}_1^1\mathrm{tr}_2^2((\mathrm{R}(A,B)h(\cdot,\cdot))\otimes V\otimes W)\\ =&\mathrm{tr}_1^1\mathrm{tr}_2^2((\nabla^2_{A,B}h)\otimes V\otimes W)-\mathrm{tr}_1^1\mathrm{tr}_2^2((\nabla^2_{B,A}h)\otimes V\otimes W).\end{aligned} \tag{B.8}$$

A twofold application of the product rule yields

$$\begin{aligned}\nabla^2_{A,B}(h\otimes V\otimes W)=&(\nabla^2_{A,B}h)\otimes V\otimes W+(\nabla_A h)\otimes(\nabla_B(V\otimes W))\\ &+(\nabla_B h)\otimes(\nabla_A(V\otimes W))+h\otimes(\nabla^2_{A,B}(V\otimes W)).\end{aligned}$$

After again applying the product rule in the second and the third term, they are recognized to be zero at p by synchronousness of V,W there. Using this, and an analogous equation with the roles of A and B interchanged, in eq. (B.8), we obtain

$$\begin{aligned}(\mathrm{R}(A,B)h)(V,W)=&\mathrm{tr}_1^1\mathrm{tr}_2^2(\nabla^2_{A,B}(h\otimes V\otimes W))-\mathrm{tr}_1^1\mathrm{tr}_2^2(h\otimes(\nabla^2_{A,B}(V\otimes W)))\\ &-\mathrm{tr}_1^1\mathrm{tr}_2^2(\nabla^2_{B,A}(h\otimes V\otimes W))+\mathrm{tr}_1^1\mathrm{tr}_2^2(h\otimes(\nabla^2_{B,A}(V\otimes W))).\end{aligned} \tag{B.9}$$

As $\nabla^2_{A,B}$ commutes with contractions, the first term here equals

$$\nabla^2_{A,B}(\mathrm{tr}_1^1\mathrm{tr}_2^2(h\otimes V\otimes W))=\nabla^2_{A,B}(h(V,W)).$$

However, we have $h(V,W)\in C^\infty(M)$, hence these covariant derivatives are partial derivatives and this term equals $\partial_A(\partial_B(h(V,W)))$. Analogously, the third term in eq. (B.9) is equal to $-\partial_B(\partial_A(h(V,W)))$. Thus the first and the third term in eq. (B.9) add up to

$$\partial_A(\partial_B(h(V,W)))-\partial_B(\partial_A(h(V,W)))=\partial_{[A,B]}(h(V,W))=0$$

at p, since A, B are synchronous there.

We note $\nabla^2_{A,B}(V \otimes W) = (\nabla^2_{A,B}V) \otimes W + (\nabla_A V) \otimes (\nabla_B W) + (\nabla_B V) \otimes (\nabla_A W) + V \otimes (\nabla^2_{A,B}W)$, where the second and the third summand are zero at p. Therefore, eq. (B.9) can at p also be rewritten as

$$\begin{aligned}(\mathrm{R}(A,B)h)(V,W) =& - \mathrm{tr}^1_1\mathrm{tr}^2_2(h \otimes (\nabla^2_{A,B}V) \otimes W) - \mathrm{tr}^1_1\mathrm{tr}^2_2(h \otimes V \otimes (\nabla^2_{A,B}W)) \\ &+ \mathrm{tr}^1_1\mathrm{tr}^2_2(h \otimes (\nabla^2_{B,A}V) \otimes W) + \mathrm{tr}^1_1\mathrm{tr}^2_2(h \otimes V \otimes (\nabla^2_{B,A}W)) \\ =& - h(\nabla^2_{A,B}V, W) - h(V, \nabla^2_{A,B}W) + h(\nabla^2_{B,A}V, W) + h(V, \nabla^2_{B,A}W) \\ =& - h(\mathrm{R}(A,B)V, W) - h(V, \mathrm{R}(A,B)W).\end{aligned}$$

□

Proof of Proposition 4.3, 4. We recall that taking the trace of a linear map does not depend on the metric, hence it commutes with differentiation with respect to t. For $X, Y \in \mathrm{T}M$, we obtain, using eq. (4.6) (where we note that taking the trace tr^1_1 of $(X,Y,Z) \mapsto (\nabla_Y \mathrm{D}\nabla^g(h))(X,Z)$ is the same as taking the trace tr^1_2 of $(Y,X,Z) \mapsto (\nabla_Y \mathrm{D}\nabla^g(h))(X,Z)$, which is $(\nabla.\mathrm{D}\nabla^g(h))(\cdot,\cdot)$ by definition):

$$\begin{aligned}(\mathrm{Dric}_g(h))(X,Y) =&(\mathrm{D}(\mathrm{tr}^1_1\mathrm{R}_g)(h))(X,Y) \\ =&(\mathrm{tr}^1_1(\mathrm{DR}_g(h)))(X,Y) \\ =&\left(\mathrm{tr}^1_1(\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot)) - \mathrm{tr}^1_2(\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot))\right)(X,Y).\end{aligned}$$

We use now a local generalized orthonormal frame $\{e_i\}_{0 \le i \le n}$ to evaluate the traces, where we note $e^*_i = \epsilon_i g(\cdot, e_i)$. Thus

$$\begin{aligned}(\mathrm{Dric}_g(h))(X,Y) =&\sum_{i=0}^n e^*_i((\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot))(e_i,X,Y)) - e^*_i((\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot))(X,e_i,Y)) \\ =&\sum_{i=0}^n \epsilon_i \,(g((\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot)),\cdot)(e_i,X,Y,e_i) \\ &-g((\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot)),\cdot)(X,e_i,Y,e_i))\,.\end{aligned}$$

Now by the product rule for ∇, using that ∇ commutes with contractions (up to index shifts) and g is parallel, we get

$$\begin{aligned}g((\nabla.\mathrm{D}\nabla^g(h)(\cdot,\cdot)),\cdot) =&\mathrm{tr}^1_4((\nabla\mathrm{D}\nabla^g(h)) \otimes g) \\ =&\nabla(\mathrm{tr}^1_3(\mathrm{D}\nabla^g(h) \otimes g)) - \mathrm{tr}^1_4((\mathrm{D}\nabla^g(h)) \otimes (\nabla g)) \\ =&\nabla(\mathrm{tr}^1_3(\mathrm{D}\nabla^g(h) \otimes g)) = \nabla(C_g h)\end{aligned}$$

where $C_g h := \mathrm{tr}^1_3(\mathrm{D}\nabla^g(h) \otimes g)$, or in other words $C_g h(X,Y,Z) = g(\mathrm{D}\nabla^g(h)(X,Y),Z)$ for all $X,Y,Z \in \mathrm{T}M$ where this is is a syntactically valid statement. Thus we can rewrite

$$\begin{aligned}(\mathrm{Dric}_g(h))(X,Y) =&\sum_{i=0}^n \epsilon_i \,((\nabla(C_g h))(e_i,X,Y,e_i) - (\nabla(C_g h))(X,e_i,Y,e_i)) \\ =&\sum_{i=0}^n \epsilon_i \,((\nabla_{e_i}(C_g h))(X,Y,e_i) - (\nabla_X(C_g h))(e_i,Y,e_i))\,. \qquad \text{(B.10)}\end{aligned}$$

In the following, as the equation to be proved is a tensor equation, we can choose $p \in M$, assume X, Y and the e_i to be synchronous at p and calculate only at p. Using part 1., i.e. eq. (4.4), the second summand in eq. (B.10) is then readily calculated to be

$$\begin{aligned}
-\sum_{i=0}^{n} \epsilon_i (\nabla_X (C_g h))(e_i, Y, e_i) &= -\sum_{i=0}^{n} \epsilon_i \partial_X ((C_g h)(e_i, Y, e_i)) \\
&= -\frac{1}{2} \sum_{i=0}^{n} \epsilon_i \partial_X \left((\nabla_{e_i} h)(Y, e_i) + (\nabla_Y h)(e_i, e_i) - (\nabla_{e_i} h)(e_i, Y) \right) \\
&= -\frac{1}{2} \partial_X \left(\sum_{i=0}^{n} \epsilon_i \partial_Y (h(e_i, e_i)) \right) \\
&= -\frac{1}{2} \partial_X \partial_Y (\mathrm{tr}_g h) \\
&= -\frac{1}{2} (\nabla \mathrm{d}(\mathrm{tr}_g h))(X, Y). \qquad \text{(B.11)}
\end{aligned}$$

(We used the synchronousness in the first and in the last step; in the third step we used that $\nabla_{e_i} h$ is a symmetric tensor to cancel the first with the last term in the bracket.)

In the first term of (B.10), we again apply eq. (4.4) and obtain

$$\begin{aligned}
&\sum_{i=0}^{n} \epsilon_i (\nabla_{e_i} (C_g h))(X, Y, e_i) \\
&\quad = \frac{1}{2} \sum_{i=0}^{n} \epsilon_i \left((\nabla^2_{e_i, X} h)(Y, e_i) + (\nabla^2_{e_i, Y} h)(X, e_i) - (\nabla^2_{e_i, e_i} h)(X, Y) \right). \qquad \text{(B.12)}
\end{aligned}$$

In the first summands in eq. (B.12), we commute the covariant derivatives using the Riemannian curvature tensor on $(0,2)$-tensor fields. This yields

$$\begin{aligned}
\frac{1}{2} \sum_{i=0}^{n} \epsilon_i (\nabla^2_{e_i, X} h)(Y, e_i) =& \frac{1}{2} \sum_{i=0}^{n} \epsilon_i \left((\nabla^2_{X, e_i} h)(Y, e_i) + (\mathrm{R}(e_i, X) h)(Y, e_i) \right) \\
=& \frac{1}{2} \sum_{i=0}^{n} \epsilon_i \left((\nabla^2_{X, e_i} h)(Y, e_i) - h(\mathrm{R}(e_i, X) Y, e_i) - h(Y, \mathrm{R}(e_i, X) e_i) \right)
\end{aligned} \qquad \text{(B.13)}$$

by Lemma B.4.

On the other hand we calculate

$$\begin{aligned}
\frac{1}{2} (\mathrm{ric}_g \circ h)(X, Y) =& \frac{1}{2} \sum_{i,j=1}^{n} \epsilon_i \epsilon_j g(\mathrm{R}(e_j, e_i) X, e_j) h(Y, e_i) \\
=& \frac{1}{2} \sum_{i,j=1}^{n} \epsilon_i \epsilon_j g(\mathrm{R}(X, e_j) e_j, e_i) h(Y, e_i) = \frac{1}{2} \sum_{j=1}^{n} \epsilon_j h(Y, \mathrm{R}(X, e_j) e_j), \qquad \text{(B.14)}
\end{aligned}$$

which we use, together with the antisymmetry of R in its first two arguments, in eq. (B.13) to deduce

$$\frac{1}{2} \sum_{i=0}^{n} \epsilon_i (\nabla^2_{e_i, X} h)(Y, e_i) = -\frac{1}{2} (\overset{\circ}{\mathrm{R}} h)(X, Y) + \frac{1}{2} (\mathrm{ric}_g \circ h)(X, Y) + \frac{1}{2} \sum_{i=0}^{n} \epsilon_i (\nabla^2_{X, e_i} h)(Y, e_i). \qquad \text{(B.15)}$$

Similarly we deal with the second summands in eq. (B.12) and obtain

$$\frac{1}{2}\sum_{i=0}^{n}\epsilon_i(\nabla^2_{e_i,Y}h)(X,e_i) = -\frac{1}{2}(\mathring{\mathrm{R}}h)(Y,X) + \frac{1}{2}(\mathrm{ric}_g \circ h)(Y,X) + \frac{1}{2}\sum_{i=0}^{n}\epsilon_i(\nabla^2_{Y,e_i}h)(X,e_i).$$

By Corollary B.3, $\mathring{\mathrm{R}}h$ is a symmetric tensor. Also we note $(\mathrm{ric}_g \circ h)(Y,X) = (h \circ \mathrm{ric}_g)(X,Y)$. Thus this can be rewritten as

$$\frac{1}{2}\sum_{i=0}^{n}\epsilon_i(\nabla^2_{e_i,Y}h)(X,e_i) = -\frac{1}{2}(\mathring{\mathrm{R}}h)(Y,X) + \frac{1}{2}(h \circ \mathrm{ric}_g)(X,Y) + \frac{1}{2}\sum_{i=0}^{n}\epsilon_i(\nabla^2_{Y,e_i}h)(X,e_i). \tag{B.16}$$

The third summands in eq. (B.12) are recognized to add up to half of the connection Laplacian $\nabla^*\nabla h$ of h, applied to X and Y. Using this and equations (B.11), (B.12), (B.15), (B.16) in (B.10), we obtain

$$\begin{aligned}\mathrm{Dric}_g(h)(X,Y) =& -\frac{1}{2}(\nabla \mathrm{d}(\mathrm{tr}_g h))(X,Y) \\ &+ \frac{1}{2}\left(\nabla^*\nabla h + \mathrm{ric}_g \circ h + h \circ \mathrm{ric}_g - 2\mathring{\mathrm{R}}h\right)(X,Y) \\ &+ \frac{1}{2}\sum_{i=0}^{n}\epsilon_i\left((\nabla^2_{X,e_i}h)(Y,e_i) + (\nabla^2_{Y,e_i}h)(X,e_i)\right).\end{aligned} \tag{B.17}$$

By definition

$$\delta^g h = -\sum_{i=0}^{n}\epsilon_i(\nabla_{e_i}h)(e_i,\cdot),$$

which leads to

$$((\mathrm{sym}\circ\nabla)(\delta^g h))(X,Y) = -\frac{1}{2}\sum_{i=0}^{n}\epsilon_i\left((\nabla^2_{X,e_i}h)(e_i,Y) + (\nabla^2_{Y,e_i}h)(e_i,X)\right).$$

That is, by symmetry of h, up to a sign the last term in eq. (B.17). As the second term of eq. (B.17) is one half of the Lichnerowicz operator, we obtain in total

$$\mathrm{Dric}_g(h) = \frac{1}{2}\square_L h - ((\mathrm{sym}\circ\nabla)(\delta^g h)) - \frac{1}{2}(\nabla \mathrm{d}(\mathrm{tr}_g h)).$$

This proves equation 4.7. □

The proof of equation 4.8 requires another auxiliary lemma.

Lemma B.5. *Let* $\omega \in \Omega^1(M)$. *Then*

$$\mathcal{L}_{\omega^\sharp}g = 2(\mathrm{sym}\circ\nabla)(\omega). \tag{B.18}$$

Proof. Let $V, W \in \mathrm{T}M$. Then

$$\begin{aligned}(\mathcal{L}_{\omega^\sharp}g)(V,W) =& g(\nabla_V\omega^\sharp, W) + g(V, \nabla_W\omega^\sharp) \\ =& \partial_V(g(\omega^\sharp, W)) - g(\omega^\sharp, \nabla_V W) + \partial_W(g(V,\omega^\sharp)) - g(\nabla_W V, \omega^\sharp) \\ =& \partial_V(\omega(W)) - \omega(\nabla_V W) + \partial_W(\omega(V)) - \omega(\nabla_W V) \\ =& (\nabla_V\omega)(W) + (\nabla_W\omega)(V) \\ =& 2(\mathrm{sym}\circ\nabla)(\omega)(V,W).\end{aligned}$$

□

Proof of Proposition 4.3, 4., continued. We recall $\overline{h} := h - \frac{1}{2}\mathrm{tr}_g(h)g$. By Lemma B.5, we have

$$\begin{aligned}\frac{1}{2}\mathcal{L}_{(\delta^g(\overline{h}))^\sharp} g =&(\mathrm{sym}\circ\nabla)(\delta^g(\overline{h}))\\ =&(\mathrm{sym}\circ\nabla)(\delta^g(h)) - \frac{1}{2}(\mathrm{sym}\circ\nabla)(-(\mathrm{tr}_g)_{11}\nabla.(\mathrm{tr}_g(h)g(\cdot,\cdot))\\ =&(\mathrm{sym}\circ\nabla)(\delta^g(h)) + \frac{1}{2}(\mathrm{sym}\circ\nabla)((\mathrm{tr}_g)_{11}\nabla.(\mathrm{tr}_g(h))\otimes g(\cdot,\cdot))\\ &+\frac{1}{2}(\mathrm{sym}\circ\nabla)((\mathrm{tr}_g)_{11}(\mathrm{tr}_g(h))\otimes(\nabla.g)(\cdot,\cdot))\\ =&(\mathrm{sym}\circ\nabla)(\delta^g(h)) + \frac{1}{2}(\mathrm{sym}\circ\nabla)((\mathrm{tr}_g)_{11}\partial.(\mathrm{tr}_g(h))\otimes g(\cdot,\cdot))\\ =&(\mathrm{sym}\circ\nabla)(\delta^g(h)) + \frac{1}{2}(\mathrm{sym}\circ\nabla)(\partial.(\mathrm{tr}_g(h)))\\ =&(\mathrm{sym}\circ\nabla)(\delta^g(h)) + \frac{1}{2}(\mathrm{sym}\circ\nabla)(\mathrm{d}(\mathrm{tr}_g(h))).\end{aligned} \tag{B.19}$$

(We also used $\nabla g = 0$ and $\nabla = \partial$ on $C^\infty(M)$.) We note that for vector fields V, W synchronous at some $p \in M$, we have at p:

$$\begin{aligned}\frac{1}{2}(\mathrm{sym}\circ\nabla)(\mathrm{d}(\mathrm{tr}_g h))(V,W) =&\frac{1}{4}\left(\partial_V(\partial_W(\mathrm{tr}_g h)) + \partial_W(\partial_V(\mathrm{tr}_g h))\right)\\ =&\frac{1}{2}\partial_V(\partial_W(\mathrm{tr}_g h)) - \frac{1}{4}\partial_{[V,W]}(\mathrm{tr}_g h)\\ =&\frac{1}{2}\partial_V(\partial_W(\mathrm{tr}_g h))\\ =&\frac{1}{2}(\nabla d(\mathrm{tr}_g h))(V,W).\end{aligned}$$

As both sides are tensorial in V, W, this shows $(\mathrm{sym}\circ\nabla)(\mathrm{d}(\mathrm{tr}_g h)) = \nabla\mathrm{d}(\mathrm{tr}_g h)$. Inserting this in eq. (B.19) and comparing with eq. (4.7), we obtain eq. (4.8). This concludes the proof of part 4. of Proposition 4.3. □

Proof of Proposition 4.3, 5. We have by Lemma B.1, part 4. of the proposition and a product rule that is proven similarly to eq. (B.1):

$$\begin{aligned}\mathrm{Dscal}_g(h) =& \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{tr}_{g+th}(\mathrm{ric}_{g+th})\\ =&\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{tr}_{g+th}(\mathrm{ric}_g)\right) + \mathrm{tr}_{g+th}(\mathrm{Dric}_g(h))\\ =&-g(h,\mathrm{ric}_g) + \mathrm{tr}_g\left(\frac{1}{2}\Box_L h - (\mathrm{sym}\circ\nabla)(\delta^g h) - \frac{1}{2}\nabla\mathrm{d}(\mathrm{tr}_g h)\right).\end{aligned} \tag{B.20}$$

We recall $\Box^g f = -\mathrm{tr}_g(\nabla\mathrm{d}f)$ for $f \in C^\infty(M)$, so the last term here is $\frac{1}{2}\Box^g(\mathrm{tr}_g h)$. Furthermore for any 1-form ω and $X, Y \in \mathrm{T}M$, we have

$$((\mathrm{sym}\circ\nabla)\omega)(X,Y) = \frac{1}{2}\left((\nabla\omega)(X,Y) + (\nabla\omega)(Y,X)\right),$$

so, using a local orthonormal frame $\{e_i\}_{0\le i\le n}$, we get

$$\mathrm{tr}_g((\mathrm{sym}\circ\nabla)\omega) = \sum_{i=0}^n \epsilon_i(\nabla\omega)(e_i,e_i) = -\delta^g\omega.$$

Inserting this in eq. (B.20), we get

$$\mathrm{Dscal}_g(h) = -g(h, \mathrm{ric}_g) + \frac{1}{2}\mathrm{tr}_g(\Box_L h) + \delta^g(\delta^g h) + \frac{1}{2}\Box^g(\mathrm{tr}_g h). \tag{B.21}$$

Now we want to compute $\mathrm{tr}_g(\Box_L h)$. We first note that

$$\begin{aligned}\mathrm{tr}_g(\overset{\circ}{\mathrm{R}}h) &= \sum_{i=0}^{n} \epsilon_i (\overset{\circ}{\mathrm{R}}h)(e_i, e_i) \\ &= \sum_{i,j=0}^{n} \epsilon_i \epsilon_j h(\mathrm{R}(e_j, e_i)e_i, e_i),\end{aligned}$$

while we also have

$$\begin{aligned}\mathrm{tr}_g(\mathrm{ric}_g \circ h) &= \sum_{i=0}^{n} \epsilon_i (\mathrm{ric}_g \circ h)(e_i, e_i) \\ &= \sum_{i,k=0}^{n} \epsilon_i \epsilon_k \mathrm{ric}_g(e_i, e_k) h(e_k, e_i) \\ &= \sum_{i,j,k=0}^{n} \epsilon_i \epsilon_j \epsilon_k g(\mathrm{R}(e_i, e_j)e_j, e_k)) h(e_k, e_i) \\ &= \sum_{i,j=0}^{n} \epsilon_i \epsilon_j h\left(\sum_{k=1}^{n} \epsilon_k g(\mathrm{R}(e_i, e_j)e_j, e_k)e_k, e_i\right) \\ &= \sum_{i,j=0}^{n} \epsilon_i \epsilon_j h(\mathrm{R}(e_i, e_j)e_j, e_i).\end{aligned}$$

Interchanging i and j in the latter calculation and comparing it with the one above, we obtain $\mathrm{tr}_g(\overset{\circ}{\mathrm{R}}h) = \mathrm{tr}_g(\mathrm{ric}_g \circ h)$. Using this and cyclic invariance of the trace, we also get

$$\begin{aligned}\mathrm{tr}_g(h \circ \mathrm{ric}_g) &= \mathrm{tr}_g(h \circ g^{-1} \circ \mathrm{ric}_g) \\ &= \mathrm{tr}(g^{-1} \circ h \circ g^{-1} \circ \mathrm{ric}_g) \\ &= \mathrm{tr}(g^{-1} \circ \mathrm{ric}_g \circ g^{-1} \circ h) \\ &= \mathrm{tr}_g(\mathrm{ric}_g \circ h) = \mathrm{tr}_g(\overset{\circ}{\mathrm{R}}h).\end{aligned}$$

Together with the definition of $\Box_L$, these calculations show

$$\mathrm{tr}_g(\Box_L h) = \mathrm{tr}_g(\nabla^* \nabla h). \tag{B.22}$$

On the other hand, g^{-1} is parallel, so

$$\begin{aligned}\Box^g(\mathrm{tr}_g h) &= \Box^g(\mathrm{tr}(g^{-1} \circ h)) \\ &= -\sum_{i=0}^{n} \epsilon_i \nabla^2_{e_i, e_i} \mathrm{tr}(g^{-1} \circ h) \\ &= -\mathrm{tr}\left(g^{-1} \circ \left(\sum_{i=0}^{n} \epsilon_i \nabla^2_{e_i, e_i} h\right)\right) \\ &= \mathrm{tr}_g(\nabla^* \nabla h).\end{aligned}$$

Inserting eq. (B.22) in this equation shows

$$\Box^g(\mathrm{tr}_g h) = \mathrm{tr}_g(\Box_L h). \tag{B.23}$$

We insert eq. (B.23) in eq. (B.21) and obtain

$$\mathrm{Dscal}_g(h) = \Box^g(\mathrm{tr}_g h) + \delta^g(\delta^g h) - g(h, \mathrm{ric}_g).$$

Using symmetry of g on the bundle of $(0,2)$-tensors, this proves eq. (4.9). $\Box$

Proof of Proposition 4.6. We note that by a product rule which is proven similarly to eq. (B.1), we have for $i = 1, 2$:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_i(\tilde{g} + t\tilde{h}, \tilde{k} + t\tilde{m}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_i(\tilde{g} + t\tilde{h}, \tilde{k}) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_i(\tilde{g}, \tilde{k} + t\tilde{m}).$$

Thus to prove the equations (4.14), (4.15) we need to show that the sum of the partial derivatives of Φ_i with respect to $\tilde{g}$ in the direction of $\tilde{h}$ and with respect to $\tilde{k}$ in the direction of $\tilde{m}$ is equal to $\mathrm{D}\Phi_i$ (for $i = 1, 2$). We write D_1 for the differentiation with respect to the $\tilde{g}$-variable (so e.g. $\mathrm{D}_1\mathrm{scal}_{\tilde{g}}(\tilde{h}) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{scal}_{\tilde{g}+t\tilde{h}}$, etc.) and D_2 for differentiation with respect to the $\tilde{k}$-variable (so e.g. $\mathrm{D}_2(\mathrm{tr}_{\tilde{g}}\tilde{k})(\tilde{m}) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{tr}_{\tilde{g}}(\tilde{k} + t\tilde{m}) = \mathrm{tr}_{\tilde{g}}\tilde{m}$). With this notation, the above formula reads $\mathrm{D}\Phi_i(\tilde{h}, \tilde{m}) = \mathrm{D}_1\Phi_i(\tilde{h}) + \mathrm{D}_2\Phi_i(\tilde{m})$.

We start with the proof of eq. (4.14), where we recall the energy constraint equation (4.2):

$$\Phi_1(\tilde{g}, \tilde{k}) = \mathrm{scal}_{\tilde{g}} - \tilde{g}(\tilde{k}, \tilde{k}) + (\mathrm{tr}_{\tilde{g}}\tilde{k})^2 = 0. \tag{4.2}$$

We found in Proposition 4.3 (now decorating everything with tildes, and with D_1 instead of D):

$$\mathrm{D}_1\mathrm{scal}_{\tilde{g}}(\tilde{h}) = \Box^{\tilde{g}}(\mathrm{tr}_{\tilde{g}}\tilde{h}) + \delta^{\tilde{g}}(\delta^{\tilde{g}}\tilde{h}) - \tilde{g}(\mathrm{ric}_{\tilde{g}}, \tilde{h}). \tag{B.24}$$

Here we note that

$$\Box^{\tilde{g}}(\mathrm{tr}_{\tilde{g}}\tilde{h}) = -\mathrm{tr}_{\tilde{g}}(\tilde{\nabla}\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{h})) = -\delta^{\tilde{g}}(\mathrm{d}\mathrm{tr}_{\tilde{g}}\tilde{h}). \tag{B.25}$$

Furthermore, we calculate, using the product rule, cyclic invariance of the trace and eq. (B.2):

$$\begin{aligned}\mathrm{D}_1\left(-\tilde{g}(\tilde{k}, \tilde{k})\right)(\tilde{h}) =& \mathrm{D}_1\left(-\mathrm{tr}(\tilde{g}^{-1} \circ \tilde{k} \circ \tilde{g}^{-1} \circ \tilde{k})\right)(\tilde{h}) \\ =& 2\mathrm{tr}(\tilde{g}^{-1} \circ \tilde{h} \circ \tilde{g}^{-1} \circ \tilde{k} \circ \tilde{g}^{-1} \circ \tilde{k}) \\ =& 2\tilde{g}(\tilde{k} \circ g^{-1} \circ \tilde{k}, \tilde{h}) = 2\tilde{g}(\tilde{k} \circ \tilde{k}, \tilde{h})\end{aligned} \tag{B.26}$$

(where in the last formula, we used a metric-dependent composition instead of the metric-independent composition in the formula before).

From Lemma B.1 and by the product rule we obtain

$$\mathrm{D}_1\left((\mathrm{tr}_{\tilde{g}}\tilde{k})^2\right)(\tilde{h}) = -2(\mathrm{tr}_{\tilde{g}}\tilde{k})\tilde{g}(\tilde{h}, \tilde{k}) \tag{B.27}$$

Thus, using equations (B.24), (B.25), (B.26), (B.27), we get

$$\mathrm{D}_1\Phi_1(\tilde{h}) = -\delta^{\tilde{g}}(\mathrm{d}\mathrm{tr}_{\tilde{g}}\tilde{h}) + \delta^{\tilde{g}}(\delta^{\tilde{g}}\tilde{h}) - \tilde{g}(\mathrm{ric}_{\tilde{g}}, \tilde{h}) + 2\tilde{g}(\tilde{k} \circ \tilde{k}, \tilde{h}) - 2(\mathrm{tr}_{\tilde{g}}\tilde{k})\tilde{g}(\tilde{h}, \tilde{k}). \tag{B.28}$$

We now compute $\mathrm{D}_2\Phi_1(\tilde{m})$. The first term in eq. (4.2) is independent of $\tilde{k}$, hence its derivative with respect to $\tilde{k}$ is zero. For the derivative of the second term, we obtain by the product rule and cyclic invariance of the trace:

$$\begin{aligned}\mathrm{D}_2\left(-\tilde{g}(\tilde{k},\tilde{k})\right)(\tilde{m}) =& \mathrm{D}_2\left(-\mathrm{tr}(\tilde{g}^{-1}\circ\tilde{k}\circ\tilde{g}^{-1}\circ\tilde{k})\right)(\tilde{m})\\ =& -2\mathrm{tr}(\tilde{g}^{-1}\circ\tilde{k}\circ\tilde{g}^{-1}\circ\tilde{m}) = -2\tilde{g}(\tilde{k},\tilde{m}).\end{aligned}$$

Also we have, by the product rule and since the trace is linear:

$$\mathrm{D}_2\left((\mathrm{tr}_{\tilde{g}}(\tilde{k}))^2\right)(\tilde{m}) = 2(\mathrm{tr}_{\tilde{g}}(\tilde{k}))(\mathrm{tr}_{\tilde{g}}(\tilde{m})) = 2\tilde{g}(\tilde{k},(\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}).$$

Altogether

$$\mathrm{D}_2\Phi_2(\tilde{m}) = -2\tilde{g}(\tilde{k},\tilde{m}) + 2\tilde{g}(\tilde{k},(\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g}),$$

which, together with (B.28) and $\mathrm{D}\Phi_1(\tilde{h},\tilde{m}) = \mathrm{D}_1\Phi_1(\tilde{h}) + \mathrm{D}_2\Phi_1(\tilde{m})$, yields (4.14).

We continue with the proof of eq. (4.15), recalling the momentum constraint equation (4.3).

$$\Phi_2(\tilde{g},\tilde{k}) := -\delta^{\tilde{g}}(\tilde{k}) - \mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{k}) = 0. \tag{4.3}$$

Let $X \in \mathrm{T}\Sigma$; without loss of generality X is extended to some vector field on Σ (still denoted X). It suffices to compute only at one specific point and assume X to be synchronous there. In proposition 4.3 it was found (again decorating everything with tildes now and writing D_1 instead of D) that

$$\mathrm{D}_1\delta^{\tilde{g}}(\tilde{h})(\alpha) = \tilde{g}(\tilde{h},\tilde{\nabla}\alpha) - \tilde{g}(\alpha,\delta^{\tilde{g}}\tilde{h} + \frac{1}{2}\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{h}))$$

for $\alpha \in \Omega^1(\Sigma)$. In particular, this also holds for $\alpha := \tilde{k}(\cdot,X) = \mathrm{tr}^1_2(\tilde{k}\otimes X)$.

We now use an orthonormal frame $\{e_i\}_{1\leq i\leq n}$ to evaluate the divergence of α (where $\tilde{g}(e_i,e_j) = +\delta_{ij}$ as $\tilde{g}$ is Riemannian, hence we drop the signs ϵ_i). Using the product rule and that $\tilde{\nabla}$ commutes with contractions, we get[29]

$$\begin{aligned}\delta^{\tilde{g}}\alpha =& -\sum_{i=1}^{n}\mathrm{tr}^1_2(\tilde{\nabla}_{e_i}(\tilde{k}\otimes X))(e_i)\\ =& -\sum_{i=1}^{n}\left((\tilde{\nabla}_{e_i}\tilde{k})(e_i,X) + \tilde{k}(e_i,\tilde{\nabla}_{e_i}X)\right)\\ =& (\delta^{\tilde{g}}\tilde{k})(X) - \sum_{i=1}^{n}\tilde{k}(e_i,\tilde{\nabla}_{e_i}X).\end{aligned}$$

Rearranging terms, we get

$$(\delta^{\tilde{g}}\tilde{k})(X) = \delta^{\tilde{g}}\alpha + \sum_{i=1}^{n}\tilde{k}(e_i,\tilde{\nabla}_{e_i}X). \tag{B.29}$$

Differentiating the first term here with respect to $\tilde{g}$ yields by the above formula and the definition of α:

$$\mathrm{D}_1(\delta^{\tilde{g}}\alpha)(X) = \tilde{g}(\tilde{h},\tilde{\nabla}.\tilde{k}(\cdot,X)) - \tilde{g}\left(\tilde{k}(\cdot,X),\delta^{\tilde{g}}\tilde{h} + \frac{1}{2}\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{h})\right). \tag{B.30}$$

[29] Here on the left-hand side δ is the divergence of a 1-form, while on the right-hand side it is the divergence of a $(0,2)$-tensor.

For the second term in eq. (B.29), we note that by eq. (B.4), we have

$$(\mathrm{D}_1\tilde{\nabla}(\tilde{h}))(e_i, X) = \sum_{j=1}^{n} \frac{1}{2}\left(\tilde{\nabla}_{e_i}\tilde{h}(X, e_j) + \tilde{\nabla}_X\tilde{h}(e_i, e_j) - \tilde{\nabla}_{e_j}\tilde{h}(X, e_i)\right) e_j.$$

Thus the second term in (B.29), differentiated with respect to $\tilde{g}$, becomes

$$\begin{aligned}\mathrm{D}_1\left(\sum_{i=1}^{n}\tilde{k}(e_i, \tilde{\nabla}_{e_i}X)\right) &= \sum_{i,j=1}^{n}\frac{1}{2}\left(\tilde{\nabla}_{e_i}\tilde{h}(X, e_j) + \tilde{\nabla}_X\tilde{h}(e_i, e_j) - \tilde{\nabla}_{e_j}\tilde{h}(X, e_i)\right)\tilde{k}(e_i, e_j) \\ &= \sum_{i,j=1}^{n}\frac{1}{2}\tilde{\nabla}_X\tilde{h}(e_i, e_j)\tilde{k}(e_i, e_j) = \frac{1}{2}\tilde{g}(\tilde{\nabla}_X\tilde{h}, \tilde{k}). \qquad \text{(B.31)}\end{aligned}$$

[30] We use (B.30) and (B.31) in (B.29) to deduce

$$(\mathrm{D}_1(\delta^{\tilde{g}}\tilde{k})(\tilde{h}))(X) = \tilde{g}(\tilde{h}, \tilde{\nabla}.\tilde{k}(\cdot, X)) - \tilde{g}\left(\tilde{k}(\cdot, X), \delta^{\tilde{g}}\tilde{h} + \frac{1}{2}\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{h})\right) + \frac{1}{2}\tilde{g}(\tilde{\nabla}_X\tilde{h}, \tilde{k}). \quad \text{(B.32)}$$

Let $\{e_i\}_{1\leq i\leq n}$ be a local orthonormal frame. Let $Y \in \mathrm{T}\Sigma$. We calculate, using $\nabla g = 0$:

$$\begin{aligned}\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{h})(Y) = \partial_Y(\mathrm{tr}_{\tilde{g}}\tilde{h}) &= \sum_{i=1}^{n}\epsilon_i\partial_{e_i}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}(e_i, Y) \\ &= \sum_{i=1}^{n}(\nabla_{e_i}((\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}))(e_i, Y) = -\delta^{\tilde{g}}((\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g})(Y).\end{aligned}$$

Thus

$$\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{h}) = -\delta^{\tilde{g}}((\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}). \qquad \text{(B.33)}$$

We now want to differentiate the second term in (4.3) with respect to $\tilde{g}$. We note that d commutes with differentiation with respect to $\tilde{g}$ (as d does not depend on the metric). Thus we find

$$\mathrm{D}_1(\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{k}))(\tilde{h}) = -\mathrm{d}(\tilde{g}(\tilde{h}, \tilde{k})) \qquad \text{(B.34)}$$

by Lemma B.1. We apply this to X and assemble this, the definition of Φ_2 and equations (B.33), (B.32) to get

$$\begin{aligned}(\mathrm{D}_1\Phi_2(\tilde{h}))(X) = &-\tilde{g}(\tilde{h}, \tilde{\nabla}.\tilde{k}(\cdot, X)) + \tilde{g}\left(\tilde{k}(\cdot, X), \delta^{\tilde{g}}\tilde{h} - \frac{1}{2}\delta^{\tilde{g}}((\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g})\right) \\ &-\frac{1}{2}\tilde{g}(\tilde{\nabla}_X\tilde{h}, \tilde{k}) + \mathrm{d}(\tilde{g}(\tilde{h}, \tilde{k}))(X). \qquad \text{(B.35)}\end{aligned}$$

To calculate the derivative with respect to $\tilde{k}$, we note first that due to linearity of $\delta^{\tilde{g}}$ and since $\delta^{\tilde{g}}$ does not depend on $\tilde{k}$, we have

$$(\mathrm{D}_2(\delta^{\tilde{g}}(\tilde{k}))(\tilde{m}))(X) = (\delta^{\tilde{g}}(\tilde{m}))(X).$$

[30] Note that also the orthonormal basis vectors depend on $\tilde{g}$. One can rewrite $\sum_{i=1}^{n}\tilde{k}(e_i, \tilde{\nabla}_{e_i}X) = \mathrm{tr}(\tilde{g}^{-1}\otimes \tilde{k}(\cdot, \tilde{\nabla}.X))$ and then use eq. (B.2) to deduce that on the right-hand side of the first equality one needs to add a term of the form $-\mathrm{tr}(\tilde{g}^{-1}\circ\tilde{h}\circ\tilde{g}^{-1}\otimes\tilde{k}(\cdot, \tilde{\nabla}.X)) = -\sum_{i,j=1}^{n}\tilde{h}(e_i, e_j)\tilde{k}(e_i, \tilde{\nabla}_{e_j}X)$. However this vanishes since X is assumed to be synchronous at the considered point.

In the third equality, we used symmetry of $\tilde{k}$, such that upon an index renaming in the negative summands they cancel with the first positive summands.

Also

$$(\mathrm{D}_2(\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{k}))(\tilde{m}))(X) = (\mathrm{d}(\mathrm{tr}_{\tilde{g}}\tilde{m}))(X) = -\delta^{\tilde{g}}((\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g})(X)$$

(using (B.33) with $\tilde{m}$ instead of $\tilde{h}$) and we obtain, summing up and applying the signs:

$$(\mathrm{D}_2\Phi_2(\tilde{m}))(X) = -(\delta^{\tilde{g}}(\tilde{m}))(X) + \delta^{\tilde{g}}((\mathrm{tr}_{\tilde{g}}\tilde{m})\tilde{g})(X). \tag{B.36}$$

We now use equations (B.35) and (B.36) in $\mathrm{D}\Phi_2(\tilde{h}, \tilde{m}) = \mathrm{D}_1\Phi_2(\tilde{h}) + \mathrm{D}_2\Phi_2(\tilde{m})$ to deduce (4.15). □

Lemma B.6. *Regarding the unit normal vector field ν to the hypersurface $\Sigma \subset M$ as a function of g, its derivative in the direction of h is given by*

$$\mathrm{D}\nu(h) = -h(\nu, \cdot)^\sharp - \frac{1}{2}h(\nu,\nu)\nu.$$

Proof. Let $X \in \mathrm{T}\Sigma$, then we have $1 = g(\nu, \nu)$ and $0 = g(\nu, X)$. These equations are readily differentiated to yield (together with the product rule and symmetry of g)

$$\begin{aligned} 0 =& h(\nu,\nu) + 2g(\mathrm{D}\nu(h), \nu), \\ 0 =& h(\nu, X) + g(\mathrm{D}\nu(h), X). \end{aligned}$$

Let $\{e_i\}_{1\le i\le n}$ be an orthonormal frame on $\mathrm{T}\Sigma$ (which satisfies $g(e_i, e_j) = +\delta_{ij}$ by the sign convention for g). Using this and $g(\nu, \nu) = -1$, we calculate (expanding $\mathrm{D}\nu(h)$ in terms of the basis vector fields $\{\nu, e_1, \ldots, e_n\}$ and inserting the above equations, where we insert the vectors e_i in place of X):

$$\begin{aligned} \mathrm{D}\nu(h) =& -g(\mathrm{D}\nu(h), \nu)\nu + \sum_{i=1}^{n} g(\mathrm{D}\nu(h), e_i)e_i \\ =& \frac{1}{2}h(\nu,\nu)\nu - \sum_{i=1}^{n} h(\nu, e_i)e_i \\ =& h(\nu,\nu)\nu - \sum_{i=1}^{n} h(\nu, e_i)e_i - \frac{1}{2}h(\nu,\nu)\nu \\ =& -h(\nu,\cdot)^\sharp - \frac{1}{2}h(\nu,\nu)\nu. \end{aligned}$$

□

Proof of Proposition 4.7. Equation (4.18) follows directly from the definition. We prove equation (4.19).

By the product rule for the Levi-Civita connection and $g(\nu, Y) = 0$ for all $Y \in \mathrm{T}\Sigma$, we have $\tilde{k}(X, Y) = g(\nabla_X \nu, Y) = -g(\nu, \nabla_X Y)$. We differentiate this in the direction of h and apply a product rule to obtain

$$\mathrm{D}\tilde{k}(h)(X, Y) = -h(\nu, \nabla_X Y) - g(\nu, \mathrm{D}\nabla(h)(X, Y)) - g(\mathrm{D}\nu(h), \nabla_X Y). \tag{B.37}$$

We decompose $\mathrm{T}M|_\Sigma \cong \mathbb{R} \oplus \mathrm{T}\Sigma$ into vector components perpendicular and tangential to Σ. Using this decomposition, we can write $Z = Z^\parallel - g(\nu, Z)\nu$ for $Z \in C^\infty(\Sigma, \mathrm{T}M|_\Sigma)$, where $Z^\parallel \in C^\infty(\Sigma, \mathrm{T}\Sigma)$.[31] Then the third term in eq. (B.37) is equal to

$$\begin{aligned} -g(\mathrm{D}\nu(h), (\nabla_X Y)^\parallel - g(\nabla_X Y, \nu)\nu) =& h(\nu, (\nabla_X Y)^\parallel) - \frac{1}{2}h(\nu,\nu)g(\nabla_X Y, \nu) \\ =& h(\nu, (\nabla_X Y)^\parallel) + \frac{1}{2}h(\nu,\nu)\tilde{k}(X, Y) \end{aligned}$$

[31] Note the sign in front of the second term due to the metric.

by again the product rule for g and (the proof of) Lemma B.6.

For the first term in eq. (B.37) we find

$$\begin{aligned}-h(\nu, \nabla_X Y) =& - h(\nu, (\nabla_X Y)^{\|}) + h(\nu,\nu) g(\nu, \nabla_X Y) \\ =& - h(\nu, (\nabla_X Y)^{\|}) - h(\nu,\nu)\tilde{k}(X,Y)\end{aligned}$$

(again using the product rule for g). Thus the first and the third term in eq. (B.37) combine to

$$-\frac{1}{2} h(\nu,\nu)\tilde{k}(X,Y). \tag{B.38}$$

For the second term in eq. (B.37), we found in Proposition 4.3 that it is equal to $-\frac{1}{2}((\nabla_X h)(\nu, Y) + (\nabla_Y h)(\nu, Y) - (\nabla_\nu h)(X,Y))$. Thus, inserting this and (B.38) in (B.37), we get

$$\mathrm{D}\tilde{k}(h)(X,Y) = -\frac{1}{2} h(\nu,\nu)\tilde{k}(X,Y) - \frac{1}{2}(\nabla_X h)(\nu, Y) - \frac{1}{2}(\nabla_Y h)(X,\nu) + \frac{1}{2}(\nabla_\nu h)(X,Y),$$

which is eq. (4.19). □

Proof of Lemma 5.1. We calculate the formulae for smooth sections of the tensor bundle. As these lie dense in the Sobolev sections, the lemma then follows also for Sobolev sections. In the following, we denote by $\{e_i\}_{1\le i\le n}$ a local orthonormal frame on $\mathrm{T}\Sigma$ (defined where necessary). The signs ϵ_i are here all equal to $+1$. Let $X, Y \in \mathrm{T}\Sigma$.

Clearly $\tilde{h}$ is the linearised first fundamental form induced by h. Furthermore we have

$$(\nabla_Y h)(\nu, X) = \partial_Y h(\nu, X) - h(\nabla_Y \nu, X) - h(\nu, \nabla_Y X) = -h(\nabla_Y \nu, X)$$

(because $h(\nu,\cdot)$ identically vanishes on Σ). On the other hand, we calculate:

$$\begin{aligned}(\tilde{h}\circ\tilde{k})(X,Y) =& \tilde{g}(\tilde{h}(X,\cdot), \tilde{k}(\cdot, Y)) \\ =& \sum_{i=1}^{n} \tilde{g}(\tilde{h}(X, e_i), \tilde{k}(e_i, Y)) = \sum_{i=1}^{n} \tilde{h}(X, \tilde{k}(Y, e_i) e_i) \\ =& \sum_{i=1}^{n} \tilde{h}(X, g(\nabla_Y \nu, e_i) e_i) = \sum_{i=1}^{n} h(X, g(\nabla_Y \nu, e_i) e_i) = h(X, \nabla_Y \nu),\end{aligned}$$

where it has been used that X is tangential to replace $\tilde{h}$ with h. Thus

$$(\nabla_Y h)(\nu, X) = -(\tilde{h}\circ\tilde{k})(X,Y). \tag{B.39}$$

Analogously we obtain $(\nabla_X h)(\nu, Y) = -(\tilde{k}\circ\tilde{h})(X,Y)$. Therefore,

$$\begin{aligned}\mathrm{D}\tilde{k}(h) =& -\frac{1}{2} h(\nu,\nu)\tilde{k}(X,Y) - \frac{1}{2}(\nabla_X h)(\nu, Y) - \frac{1}{2}(\nabla_Y h)(\nu, X) + \frac{1}{2}(\nabla_\nu h)(X,Y) \\ =& 0 - \frac{1}{2}(\nabla_X h)(\nu, Y) - \frac{1}{2}(\nabla_Y h)(\nu, X) + \tilde{m}(X,Y) - \frac{1}{2}(\tilde{h}\circ\tilde{k} + \tilde{k}\circ\tilde{h})(X,Y) \\ =& \tilde{m}(X,Y)\end{aligned}$$

by the properties of h, so $\tilde{m}$ is indeed the linearised second fundamental form induced by h.

We continue by showing $\delta^g(\overline{h})|_\Sigma = 0$, which we do by showing that $\delta^g(\overline{h})|_\Sigma(X) = 0$ for all $X \in \mathrm{T}\Sigma$, and then $\delta^g(\overline{h})|_\Sigma(\nu) = 0$.

We note that

$$h(\nabla_Y Y, X) = h(\tilde{\nabla}_Y Y, X) + h(-g(\nabla_Y Y, \nu)\nu, X) = \tilde{h}(\tilde{\nabla}_Y Y, X)$$

and

$$h(Y, \nabla_Y X) = h(Y, \tilde{\nabla}_Y X) + h(Y, -g(\nabla_Y X)\nu) = \tilde{h}(Y, \tilde{\nabla}_Y X)$$

(since by assumption, $h(\nu, \cdot)$ is zero on Σ, and for tangential vectors, h equals $\tilde{h}$). Therefore,

$$\begin{aligned}(\nabla_Y h)(Y, X) =& \partial_Y(h(Y, X)) - h(\nabla_Y Y, X) - h(Y, \nabla_Y X)\\ =& \partial_Y(\tilde{h}(Y, X)) - \tilde{h}(\tilde{\nabla}_Y Y, X) - \tilde{h}(Y, \tilde{\nabla}_Y X) = (\tilde{\nabla}_Y \tilde{h})(Y, X).\end{aligned}$$

The same argument shows $(\nabla_Y((\mathrm{tr}_g h)g))(Y, X) = (\tilde{\nabla}_Y((\mathrm{tr}_g h)\tilde{g}))(Y, X)$. Furthermore, extending $\{e_i\}_{1\le i\le n}$ by ν to a generalized orthonormal frame on $\mathrm{T}M$, and using this to calculate the metric traces, one recognizes $(\mathrm{tr}_g h)|_\Sigma = \mathrm{tr}_{\tilde{g}}\tilde{h}$, because $(h(\nu, \nu))|_\Sigma = 0$. Also

$$(\nabla_\nu((\mathrm{tr}_g h)g))(\nu, X) = ((\partial_\nu(\mathrm{tr}_g h))g + (\mathrm{tr}_g h)(\nabla_\nu g))(\nu, X) = 0,$$

because $g(\nu, X) = 0$ and g is parallel (so both terms vanish).

These preliminaries can then be used to calculate $\delta^g(\overline{h})|_\Sigma(X)$: We get

$$\begin{aligned}\delta^g(\overline{h})|_\Sigma(X) =& \left(\nabla_\nu(h - \frac{1}{2}(\mathrm{tr}_g h)g)\right)(\nu, X) - \sum_{i=1}^n \left(\nabla_{e_i}(h - \frac{1}{2}(\mathrm{tr}_g h)g)\right)(e_i, X)\\ =& (\nabla_\nu h)(\nu, X) - \sum_{i=1}^n \left(\tilde{\nabla}_{e_i}(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g})\right)(e_i, X)\\ =& -\delta^{\tilde{g}}\left(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right)(X) + \delta^{\tilde{g}}\left(\tilde{h} - \frac{1}{2}(\mathrm{tr}_{\tilde{g}}\tilde{h})\tilde{g}\right)(X) = 0 \end{aligned}\tag{B.40}$$

by the properties of h. So $\delta^g(\overline{h})|_\Sigma$, applied to any vector tangential to Σ, is indeed zero.

Now we calculate $\delta^g(\overline{h})|_\Sigma(\nu)$. We obtain

$$\delta^g(\overline{h})|_\Sigma(\nu) = (\nabla_\nu(h - \frac{1}{2}(\mathrm{tr}_g h)g))(\nu, \nu) - \sum_{i=1}^n \left(\nabla_{e_i}(h - \frac{1}{2}(\mathrm{tr}_g h)g)\right)(\nu, e_i). \tag{B.41}$$

The terms not involving traces of h add up to

$$(\nabla_\nu h)(\nu, \nu) - \sum_{i=1}^n (\nabla_{e_i} h)(\nu, e_i) = -2\mathrm{tr}_{\tilde{g}}\tilde{m} + \sum_{i=1}^n \frac{1}{2}(\tilde{h}\circ\tilde{k} + \tilde{k}\circ\tilde{h})(e_i, e_i), \tag{B.42}$$

where we recall $(\nabla_Y h)(\nu, X) = -(\tilde{h}\circ\tilde{k})(X, Y)$ and $(\nabla_X h)(\nu, Y) = -(\tilde{k}\circ\tilde{h})(X, Y)$ for tangential X and Y.[32]

For the terms involving the traces, we calculate, noting that g is parallel, hence its

[32] Cf. eq. (B.39). Using this and the fact that we plugged in two times the same tangential vector, we obtain the "symmetric" formula above.

covariant derivative vanishes, and $g(\nu, e_i) = 0$ for all $1 \leq i \leq n$:

$$
\begin{aligned}
-\frac{1}{2}&(\nabla_\nu((\mathrm{tr}_g h)g))(\nu,\nu) + \frac{1}{2}\sum_{i=1}^n (\nabla_{e_i}((\mathrm{tr}_g h)g))(\nu, e_i) \\
&= \frac{1}{2}(\partial_\nu(h(\nu,\nu)))g(\nu,\nu) - \frac{1}{2}\sum_{j=1}^n(\partial_\nu(h(e_j,e_j)))g(\nu,\nu) \\
&\quad + \frac{1}{2}\sum_{i=1}^n(\partial_{e_i}(-h(\nu,\nu)))g(\nu,e_i) + \frac{1}{2}\sum_{i,j=1}^n \epsilon_i(\partial_{e_i}(h(e_j,e_j)))g(\nu,e_i) \\
&= -\frac{1}{2}(\partial_\nu(h(\nu,\nu))) + \frac{1}{2}\sum_{j=1}^n(\partial_\nu(h(e_j,e_j))).
\end{aligned}
$$

As the result must be a tensor, we may assume that $\{e_j\}_{1\leq j\leq n}$ and ν all have vanishing covariant derivative at p. (We may replace them with vector fields which coincide with $\{e_j\}_{1\leq j\leq n}$ and ν at p, but are synchronous there.) Then $\partial_\nu(h(\nu,\nu)) = (\nabla_\nu h)(\nu,\nu)$ and $\partial_\nu(h(e_j,e_j)) = (\nabla_\nu h)(e_j,e_j)$.

Thus the terms involving the traces become, by the properties of h,

$$
\begin{aligned}
-\frac{1}{2}&(-2\mathrm{tr}_{\tilde g}\tilde m) + \frac{1}{2}\sum_{j=1}^n(2\tilde m - (\tilde h \circ \tilde k + \tilde k \circ \tilde h))(e_j,e_j) \\
&= \mathrm{tr}_{\tilde g}\tilde m + \mathrm{tr}_{\tilde g}\tilde m - \frac{1}{2}\sum_{j=1}^n(\tilde h \circ \tilde k + \tilde k \circ \tilde h)(e_j,e_j). \qquad \text{(B.43)}
\end{aligned}
$$

Inserting equations (B.42), (B.43) into eq. (B.41), we obtain that the terms cancel and $\delta^g(\overline{h})|_\Sigma(\nu) = 0$. Together with eq. (B.40), we deduce from this $\delta^g(\overline{h})|_\Sigma = 0$, and this finishes the proof of the lemma. □

Proof of Lemma 5.5. Let $X \in \mathcal{D}'(N, \mathrm{T}N)$. Let $\{e_i\}_{1\leq i\leq n}$ be a local orthonormal frame with respect to $\hat g$. We may assume $\hat\nabla_{e_i}e_i = 0$ throughout the calculation below, as the result must be a tensor equation. We calculate, using eq. (B.33) with $(\mathrm{tr}_{\hat g}\mathcal{L}_V\hat g)\hat g$ in the first equation, $\mathrm{tr}_{\hat g}\mathcal{L}_V\hat g = 2\sum_{j=1}^n \epsilon_j \hat g(\hat\nabla_{e_j}V, e_j)$ in the second equation, and that $\hat\nabla$ commutes with $\flat$ in the last equation:

$$
\begin{aligned}
&\delta^{\hat{g}}\left(\mathcal{L}_V\hat{g}-\frac{1}{2}\mathrm{tr}_{\hat{g}}(\mathcal{L}_V\hat{g})\hat{g}\right)(X)\\
&\quad=-\left(\sum_{i=1}^{n}\epsilon_i(\hat{\nabla}_{e_i}\mathcal{L}_V\hat{g})(e_i,X)\right)+\frac{1}{2}d(\mathrm{tr}_{\hat{g}}\mathcal{L}_V\hat{g})(X)\\
&\quad=-\left(\sum_{i=1}^{n}\epsilon_i(\hat{\nabla}_{e_i}\mathcal{L}_V\hat{g})(e_i,X)\right)+\partial_X\left(\sum_{j=1}^{n}\epsilon_j\hat{g}(\hat{\nabla}_{e_j}V,e_j)\right)\\
&\quad=\sum_{i=1}^{n}\epsilon_i\left(-\hat{\nabla}_{e_i}\mathcal{L}_V\hat{g}(e_i,X)+\partial_X\hat{g}(\hat{\nabla}_{e_i}V,e_i)\right)\\
&\quad=\sum_{i=1}^{n}\epsilon_i\left(-\hat{g}(\hat{\nabla}^2_{e_i,e_i}V,X)-\hat{g}(\hat{\nabla}^2_{e_i,X}V,e_i)-\hat{g}(\hat{\nabla}_{e_i}V,\hat{\nabla}_{e_i}X)-\hat{g}(\hat{\nabla}_XV,\hat{\nabla}_{e_i}e_i)\right.\\
&\qquad\left.+\hat{g}(\hat{\nabla}^2_{X,e_i}V,e_i)+\hat{g}(\hat{\nabla}_{e_i}V,\hat{\nabla}_{e_i}X)\right)\\
&\quad=\sum_{i=1}^{n}\epsilon_i\left(-\hat{g}(\hat{\nabla}^2_{e_i,e_i}V,X)-\hat{g}(\hat{\nabla}^2_{e_i,X}V,e_i)+\hat{g}(\hat{\nabla}^2_{X,e_i}V,e_i)\right)\\
&\quad=\sum_{i=1}^{n}\epsilon_i\left(-(\hat{\nabla}_{e_i}(\hat{\nabla}_{e_i}V))^\flat(X))-\hat{g}(\mathrm{R}_{\hat{g}}(e_i,X)V,e_i)\right)\\
&\quad=\delta^{\hat{g}}(\hat{\nabla}V^\flat)(X)-\mathrm{ric}_{\hat{g}}(V,X).
\end{aligned}
$$

This is eq. (5.2). □

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Erklärung

Dies ist eine korrigierte und überarbeitete Version der Arbeit. Ich habe keine anderen als die angegebenen Quellen sowie Rückmeldungen von Prof. Dr. Bernd Ammann benutzt. Die Arbeit ist aber nicht mehr identisch mit der, die ich zur Erlangung des Bachelorgrades eingereicht habe.

Linus Götzfried

Regensburg, den 1.1.2024