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A remark on crystalline cohomology

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Abstract
We propose a new approach to crystalline cohomology based on the observation that one can lift smooth algebras uniquely "up to coherent homotopy."

Keywords Crystalline cohomology · PD de Rham cohomology

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1 Introduction

In this note, we explore an approach to crystalline cohomology using certain homotopies congruent to the identity. In the introduction we present the idea in its simplest setting. Let k be a perfect field of characteristic $p > 0$ and let R be its ring of Witt vectors. For a smooth k-algebra $\overline{A}$, a classical approach to defining a good cohomology theory, see, for example, [8, 12], is to lift $\overline{A}$ to a p-adic complete smooth R-algebra A and to consider the p-complete de Rham cohomology of A over R.

The issue with this naive approach is the difficulty of gluing this cohomology theory for more global objects over k. Indeed, the lift A over R for a given $\overline{A}$ is unique only up to non-unique isomorphism. This non-uniqueness causes significant problems. To address this, consider two lifts A_1 and A_2 of $\overline{A}$ and two isomorphisms

$$\phi_1, \phi_2 \colon A_1 \xrightarrow{\sim} A_2$$

which are congruent to the identity modulo p. Using the smoothness of A_1, one can construct a formal homotopy

$$h \colon A_1 \to A_2[\![T]\!]$$

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such that the composition of h with the map $T \mapsto 0$ is ϕ_1, and the composition of h with the map $T \mapsto p$ is ϕ_2. Using a Poincaré lemma, one then shows that ϕ_1 and ϕ_2 induce the same map on de Rham cohomology in the derived category of p-complete R-modules $\mathrm{D}(R)^\wedge$.

However, the next challenge in globalizing this construction is that h is again not unique. Nevertheless, it is unique up to a certain formal homotopy, and so on. The aim of this note is to make this idea precise using the language of higher category theory, thereby providing a new approach to crystalline cohomology.

In this note, we do not discuss the notion of crystals within this framework but focus solely on crystalline cohomology with trivial coefficients.

Let us remark that there is a methodologically related approach to derived crystalline cohomology in Mao's paper [11] using free simplicial resolutions.

Notation. We denote by $\Delta[m] \in \mathbf{sSet}$ the standard, simplicial m-simplex, i.e. the functor represented by $[m]$. We denote by $\partial\Delta[m]$ its boundary.

2 A simplicial ring

Let R be a ring and let $\pi \in R$ be a non-zero divisor. Set $\overline{R} = R/(\pi)$. We assume that $\overline{R} \neq 0$. Consider the simplicial ring $R\Delta_\bullet$ with $R\Delta_m = R[\![T_0, \dots, T_m]\!]$ and with

$$R\Delta_m(\sigma)\colon R\Delta_m \to R\Delta_n, \quad R\Delta_m(\sigma)(T_i) = \sum_{j\in\sigma^{-1}(i)} T_j$$

for a morphism $\sigma : [n] \to [m]$.

Lemma 2.1 *The canonical map $R\Delta_\bullet \xrightarrow{\sim} R$ induced by $T_i \mapsto 0$ is a trivial Kan fibration.*

Proof Recall that any surjection of simplicial groups is a Kan fibration. Consider the short exact sequence of simplicial groups

$$0 \to I_\bullet \to R\Delta_\bullet \to R \to 0.$$

We have to show that $\pi_i(I_\bullet) = 0$ for all $i \geq 0$. Let $S_\bullet$ be the free simplicial R-module with $S_m = R^{[m]}$. It is well known that $S_\bullet$ (often denoted ER) is contractible. One way to see this is to observe that $S_\bullet$ is the nerve of the category with set of objects given by R and with a unique morphism between any two objects. Since this category has an initial object, its nerve is contractible. It follows that $I_\bullet = \prod_{n>0} \mathrm{Sym}^n_R(S_\bullet)$ is also contractible as a simplicial R-module. □

Consider the simplicial ring $R\Delta^\pi_\bullet$ with

$$R\Delta^\pi_m = R\Delta_m/(T_0 + \cdots + T_m - \pi).$$

Lemma 2.2 *The canonical map $R\Delta^\pi_\bullet \xrightarrow{\sim} \overline{R}$ induced by $T_i \mapsto 0$ is a trivial Kan fibration.*

Proof Consider the following commutative diagram of simplicial R-modules with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & R\Delta_\bullet \arrow[r, "\sum_i T_i - \pi"] \arrow[d, "\wr"] & R\Delta_\bullet \arrow[r] \arrow[d, "\wr"] & R\Delta^\pi_\bullet \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & R \arrow[r, "-\pi"] & R \arrow[r] & \overline{R} \arrow[r] & 0
\end{tikzcd}$$

The two vertical arrows on the left are equivalences by Lemma 2.1, so the right vertical map is an equivalence too. □

Set

$$R\partial\Delta_m^\pi = R\Delta_m^\pi/(T_0\cdots T_m)$$

for $m \geq 0$. Recall that for a simplicial set $S_\bullet$ and a simplicial commutative ring $A_\bullet$ the set of morphisms of simplicial sets $\mathrm{Hom}(S_\bullet, A_\bullet)$ forms a commutative ring itself.

Lemma 2.3 *For any $m \geq 1$ the canonical map*

$$R\partial\Delta_m^\pi \xrightarrow{\sim} \mathrm{Hom}(\partial\Delta[m], R\Delta_\bullet^\pi) \times_{\mathrm{Hom}(\partial\Delta[m],\overline{R})} \mathrm{Hom}(\Delta[m], \overline{R}) \tag{2.1}$$

is an isomorphism. The canonical map $R\partial\Delta_0^\pi \xrightarrow{\sim} \overline{R}$ is an isomorphism.

Proof The second assertion is immediate. For the first one, note that we have a canonical isomorphism $R\Delta_m^\pi = \mathrm{Hom}(\Delta[m], R\Delta_\bullet^\pi)$. As $R\Delta_\bullet^\pi \to \overline{R}$ is a trivial Kan fibration by Lemma 2.2, the restriction map

$$R\Delta_m^\pi = \mathrm{Hom}(\Delta[m], R\Delta_\bullet^\pi) \to \mathrm{Hom}(\partial\Delta[m], R\Delta_\bullet^\pi) \times_{\mathrm{Hom}(\partial\Delta[m],\overline{R})} \mathrm{Hom}(\Delta[m], \overline{R})$$

is surjective for $m \geq 1$. An element $f \in R\Delta_m^\pi$ lies in its kernel if and only if $\partial_i(f) = 0$ for $i = 0, \ldots, m$.

Observe that, as π is a non-zero divisor in R, so is $\sum_i T_i - \pi$ in $R\Delta_m$. Using this, one sees that any subsequence of $T_0, \ldots, T_m, \sum_i T_i - \pi$ is a regular sequence in $R\Delta_m$. By Claim 2.4 below this implies that any permutation of $T_0, \ldots, T_m$ forms a regular sequence in $R\Delta_m^\pi$.

For $f \in R\Delta_m^\pi$ we get that $\partial_i(f) = 0$ if and only if f is divisible by T_i. By Claim 2.4 this finally implies that $\partial_i(f)$ vanishes for all $i = 0, \ldots, m$ if and only if f is divisible by the product $T_0 \cdots T_m$. This finishes the proof of the lemma. □

Claim 2.4 *Let A be a ring and $\mathbf{a} = (a_1, \ldots, a_r) \in A^r$ be a sequence of elements which do not generate the unit ideal. Then we have the implications* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *among the properties:*

(1) *any permutation of* $\mathbf{a}$ *is a regular sequence,*
(2) *any subsequence of* $\mathbf{a}$ *is a regular sequence,*
(3) *any element of A divisible by $a_1, \ldots, a_r$ is divisible by $a_1 \cdots a_r$.*

For the equivalence of (1) and (2) in the claim see [14, Lemma 07DW]. It is an easy exercise to check that (1) implies (3).

3 Lifting smooth algebras

Let the notation be as in Sect. 2 and assume additionally that R is π-adically complete. Let $\mathrm{SmAlg}_{\overline{R}}$ be the category of smooth $\overline{R}$-algebras and let FSmAlg_R be the category of π-adically complete topological R-algebras of finite type which are formally smooth over R.

We consider the simplicial enrichment FSmAlg_R^Δ of the category FSmAlg_R by defining the mapping simplicial sets by

$$\mathrm{Hom}_R^\Delta(A, B) := \mathrm{Hom}_R(A, B\Delta_\bullet^\pi) = \mathrm{Hom}_{R\Delta_\bullet^\pi}(A\Delta_\bullet^\pi, B\Delta_\bullet^\pi)$$

for $A, B \in \mathrm{FSmAlg}_R$. Then we obtain simplicially enriched functors

$$\mathrm{FSmAlg}_R \to \mathrm{FSmAlg}_R^\Delta \xrightarrow{\mathrm{BC}} \mathrm{SmAlg}_{\overline{R}}$$

where the categories on the left and on the right are endowed with the trivial simplicial enrichment, the first functor is the inclusion of the underlying category of zero simplices, and BC is the base change functor given by $A \mapsto \overline{A} = A/\pi$ on objects and the obvious map on mapping simplicial sets.

Proposition 3.1 *The functor* BC *above is an equivalence of simplicial categories. Moreover, every mapping simplicial set in* FSmAlg_R^Δ *is Kan.*

We remark that the second assertion means that the simplicial category FSmAlg_R^Δ is fibrant in the Bergner model structure on simplicial categories [4], though we will not make use of this model structure.

Proof The essential surjectivity follows from the fact that the obstruction to the deformation of a smooth algebra vanishes. In order to prove the proposition we will show that for $A, B \in \mathrm{FSmAlg}_R$ the map of simplicial sets

$$S_\bullet := \mathrm{Hom}_R(A, B\Delta_\bullet^\pi) \to \mathrm{Hom}_{\overline{R}}(\overline{A}, \overline{B}) =: T_\bullet$$

is a trivial Kan fibration. This means we have to show that the canonical map

$$S_m \to \mathrm{Hom}(\partial\Delta[m], S_\bullet) \times_{\mathrm{Hom}(\partial\Delta[m], T_\bullet)} T_m$$

is surjective for all $m \geq 0$. For $m \geq 1$ Lemma 2.3 tells us that this map is isomorphic to the map

$$\mathrm{Hom}_R(A, B\Delta_m^\pi) \to \mathrm{Hom}_R(A, B\partial\Delta_m^\pi).$$

This map is surjective as $B\Delta_m^\pi \to B\partial\Delta_m^\pi$ is a surjection whose domain is complete with respect to the kernel ideal generated by $T_0 \cdots T_m$ and as A is formally smooth over R.

For $m = 0$, we have to check that $S_0 = \mathrm{Hom}_R(A, B) \to T_0 \cong \mathrm{Hom}_R(A, \overline{B})$ is surjective. This follows from the formal smoothness of A over R and the fact that B is π-adically complete. □

Remark 3.2 We remark that the base change functor $\mathrm{FSmAlg}_R \to \mathrm{SmAlg}_{\overline{R}}$ is not a localisation. Indeed, assume that $A \to B$ is a map in FSmAlg_R such that its base change $\overline{A} \to \overline{B}$ is an isomorphism. As A and B are π-torsion free and π-adically complete, it follows inductively that $A/\pi^n A \to B/\pi^n B$ is an isomorphism for all n and hence so is $A \to B$. In other words, the above functor is conservative, i.e. it does not invert any non-isomorphism. So if it was a localisation, it would be an equivalence, which is clearly not the case. We don't expect the induced functor on presheaves of spaces $\mathrm{PSh}(\mathrm{FSmAlg}_R) \to \mathrm{PSh}(\mathrm{SmAlg}_{\overline{R}})$ to be a localisation either.

4 Simplicially enriched formal schemes

Let R be as in Sect. 3. Let $\mathrm{Sm}_{\overline{R}}$ be the category of smooth $\overline{R}$-schemes and let $\mathrm{SmAff}_{\overline{R}}$ be the subcategory of smooth, affine schemes. Let $\mathrm{FSmAff}_R = \mathrm{FSmAlg}_R^{\mathrm{op}}$ be the category of affine, π-adic formal schemes over R which are topologically of finite type and formally smooth. The latter inherits a simplicial enrichment from FSmAlg_R^Δ, which we denote by FSmAff_R^Δ.

We write $\mathrm{N}(-)$ for the (simplicial) nerve functor (see [9, Def. 1.1.5.5]). As $\mathrm{FSmAff}_R^{\Delta}$ is fibrant by Proposition 3.1, its nerve is an ∞-category [9, Prop. 1.1.5.10]. From Proposition 3.1 we obtain a canonical equivalence

$$\mathrm{N}(\mathrm{FSmAff}_R^{\Delta}) \xrightarrow{\sim} \mathrm{SmAff}_{\overline{R}}$$

Using this equivalence, we equip the left-hand side with the Grothendieck topology corresponding to the Zariski or étale topology on $\mathrm{SmAff}_{\overline{R}}$. In the following, $\mathrm{Sh}(-)$ denotes the ∞-category of sheaves of spaces.

Proposition 4.1 *There are canonical equivalences*

$$\mathrm{Sh}(\mathrm{Sm}_{\overline{R}}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{SmAff}_{\overline{R}}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{N}(\mathrm{FSmAff}_R^{\Delta})).$$

Proof The functors are induced by pullback along $\mathrm{N}(\mathrm{FSmAff}_R^{\Delta}) \xrightarrow{\sim} \mathrm{SmAff}_{\overline{R}} \to \mathrm{Sm}_{\overline{R}}$. Clearly, the second functor in the statement is an equivalence. As the Zariski/étale topology on $\mathrm{Sm}_{\overline{R}}$ has a basis consisting of affines, the first functor is an equivalence, too. $\square$

Remark 4.2 Proposition 4.1 is an analog of the constrution of the "motivic Monsky-Washnitzer functor" $\mathrm{DA}(k) \to \mathrm{RigDA}(K)$ due to Ayoub [2, Cor. 1.4.24, Rem. 1.4.25] and Vezzani [15], where K is a complete discretely valued field with residue field k.

5 PD-de Rham cohomology

Fix a prime number p. Let R be a ring, $\pi \in R$ an element, and assume that the ideal (π) is endowed with a PD-structure. Assume further that p is nilpotent in $\overline{R} = R/(\pi)$ and that R is π-adically complete.

Definition 5.1 We denote by $\mathrm{PFSmAlg}_R$ the following category. Its objects are pairs (B, I) consisting of an R-algebra B and a finitely generated ideal $I \subseteq B$, called ideal of definition, such that B is I-adically complete and formally smooth over R, I contains the image of π, and B/I is smooth over $\overline{R}$. The morphisms $\phi\colon (B, I) \to (B', I')$ are the R-algebra morphisms $\phi\colon B \to B'$ with $\phi(I) \subset I'$.

Note that the category $\mathrm{PFSmAlg}_R$ has finite coproducts.

Example 5.2 Our basic example of an object in $\mathrm{PFSmAlg}_R$ is

$$(B, I) = (A[\![T_0, \ldots, T_m]\!], (\pi, T_0, \ldots, T_m))$$

where A is a π-adic complete R-algebra which is formally smooth and topologically of finite type over R.

Let $\mathrm{Ch}(R)$ be the 1-category of complexes of R-modules and $\mathrm{Ch}(R)^{\wedge}$ the subcategory of derived π-complete complexes of R-modules.

Consider the PD-de Rham complex as the functor $\mathrm{dR}\colon \mathrm{PFSmAlg}_R \to \mathrm{Ch}(R)^{\wedge}$ defined as the (derived) limit[1]

$$\mathrm{dR}(B, I) = \Omega^{*,\mathrm{PD}}_{(B,I)/(R,\pi)} = \lim_n \Omega^*_{\mathrm{D}_I(B \otimes_R R_n)/R_n}$$

where $R_n = R/(\pi^n)$ and where we take the PD-envelope of I relative to the PD-structure of (π). The differential forms are the ones compatible with the PD-structure [14, Sec. 07HQ]. If the ideal of definition I is clear from the context we simply write $\mathrm{dR}(B)$ for $\mathrm{dR}(B, I)$.

[1] As the transition maps are surjective, the naive limit in fact agrees with the derived limit.

Lemma 5.3 *Assume that π is a non-zero divisor in R. All modules in the complex* $\mathrm{dR}(B, I)$ *are π-torsion free and there is an isomorphism of complexes* $\mathrm{dR}(B, I) \otimes_R \overline{R} \cong \Omega^*_{\mathrm{D}_I(B \otimes_R \overline{R})/\overline{R}}$.

Proof Write $B_n = B \otimes_R R_n$, $\overline{B} = B_1 = B/(\pi)$. We first fix n and i and show that $\Omega^i_{\mathrm{D}_I(B_n)/R_n}$ is a flat R_n-module which satisfies

$$\Omega^i_{\mathrm{D}_I(B_n)/R_n} \otimes_{R_n} \overline{R} = \Omega^i_{\mathrm{D}_I(\overline{B})/\overline{R}}.$$

Lift $\overline{C} = B/I$ to a π-adically complete smooth R-algebra C, and let $C_n = C/(\pi^n)$. By the formal smoothness of B we can lift the surjection $B \to B/I = \overline{C}$ to a surjection $B \to C$. We get induced surjections $B_n \to C_n$ whose kernel we denote by J_n. Note that $J_1 = I\overline{B}$ is finitely generated by assumption. As π is nilpotent in B_n and $J_{n-1} \cong J_n/\pi^{n-1}J_n$ it follows that J_n is finitely generated, too. As B_n and C_n are formally smooth over R_n, [7, Cor. 0_{IV}(19.5.4)] implies that J_n/J_n^2 is a projective C_n-module and that there is an isomorphism $\mathrm{Sym}_{C_n}(J_n/J_n^2)^\wedge \cong B_n$, where $^\wedge$ indicates the completion for the augmentation ideal. By Zariski descent (noting that the divided power envelope commutes with localisation and is insensitive to the I-adic completion) we may assume that J_n/J_n^2 is free, and hence B_n is isomorphic to the power series ring $C_n[\![T_1, \ldots, T_r]\!]$. It then follows from [6, Rem. 3.20 1), 6)] that $\mathrm{D}_I(B_n) = \mathrm{D}_{J_n}(B_n)$ and that one can drop the compatibility condition with the PD structure on (π), and hence the PD envelope is isomorphic to the PD polynomial algebra $C_n\langle T_1, \ldots, T_n\rangle$. In particular, it is flat over C_n and hence over R_n, and $\mathrm{D}_I(B_n) \otimes_{R_n} \overline{R} = \mathrm{D}_I(\overline{B})/\overline{R}$. As B_n is formally smooth over R_n (and p is nilpotent in R_n), $\Omega^i_{B_n/R_n}$ is a projective B_n-module. Thus

$$\Omega^i_{\mathrm{D}_I(B_n)/R_n} \cong \Omega^i_{B_n/R_n} \otimes_{B_n} \mathrm{D}_I(B_n)$$

is flat over R_n and obviously satisfies the base change formula asserted above.

Write $M_n = \Omega^i_{\mathrm{D}_I(B_n)/R_n}$. As this is a flat R_n-module and $R_n[\pi] = \pi^{n-1}R_n$ (as π is a non-zero divisor in R), it follows that the π-torsion submodule $M_n[\pi]$ is $\pi^{n-1}M_n$. Hence the pro-module $\{M_n[\pi]\}$ is zero, and hence multiplication by π is injective on $\lim_n M_n$ with cokernel $\lim_n M_n/\pi M_n = M_1$. This is precisely the claim of the lemma. □

Berthelot proved a base change formula, the Poincaré lemma, and étale descent for PD-de Rham cohomology. We recall these in the form we need in the following proposition.

Proposition 5.4 *Assume that π is a non-zero divisor in R.*

(1) *The canonical map* $\mathrm{dR}(B, I) \otimes^{\mathrm{L}}_R \overline{R} \xrightarrow{\sim} \mathrm{dR}(B/I)$ *is an equivalence for (B, I) in* $\mathrm{PFSmAlg}_R$. *Here the de Rham cohomology on the right is over $\overline{R}$ with the element $\overline{\pi} = 0$.*
(2) *For $(B, I) \to (B', I')$ a morphism in* $\mathrm{PFSmAlg}_R$ *such that $B/I \to B'/I'$ is an isomorphism the induced map* $\mathrm{dR}(B, I) \xrightarrow{\sim} \mathrm{dR}(B', I')$ *is an equivalence.*
(3) *Let $(B, I) \to (C, K)$ be a map in* $\mathrm{PFSmAlg}_R$ *such that $B \to C$ is faithfully flat and étale, and K is generated by the image of I. For the associated Cech nerve $B \to C^\bullet$ (equipped with the obvious simplicial ideal of definition $K^\bullet$) we get an equivalence* $\mathrm{dR}(B, I) \xrightarrow{\sim} \lim_\bullet \mathrm{dR}(C^\bullet, K^\bullet)$.

Proof By Lemma 5.3 we have $\mathrm{dR}(B, I) \otimes^{\mathrm{L}}_R \overline{R} \simeq \Omega^*_{\mathrm{D}_I(\overline{B})/\overline{R}}$. We saw in the proof of the lemma that $\mathrm{D}_I(\overline{B})$ is locally a PD polynomial algebra over B/I. As in [14, Lemma 07LD], the elementary Poincaré lemma [5, Lemma V.2.1.2] then implies that the canonical map $\Omega^*_{\mathrm{D}_I(\overline{B})/\overline{R}} \to \Omega^*_{(B/I)/\overline{R}}$ is a quasi-isomorphism.

(2) As both complexes are derived π-complete, this follows immediately from (1).

(3) This follows from (1) and the fact that de Rham cohomology has étale descent. $\square$

6 A new approach to crystalline cohomology

In this section we construct a simplicially enriched version of the PD-de Rham complex functor and use it together with Proposition 4.1 to construct crystalline cohomology (Definition 6.3).

We start with some purely categorical preliminaries. Let $\mathcal{C}$ be a category. We denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$. Then $s\mathcal{C}$ is canonically simplicially enriched; see [13, discussion before Prop. 2.1.2]. For $X_\bullet, Y_\bullet \in s\mathcal{C}$, we denote the mapping simplicial set in $s\mathcal{C}$ by $\mathrm{Map}_{s\mathcal{C}}(X_\bullet, Y_\bullet)$. Concretely, an n-simplex in $\mathrm{Map}_{s\mathcal{C}}(X_\bullet, Y_\bullet)$ is given by a collection of maps $f(\sigma)\colon X_q \to Y_q$ for every $q \geq 0$ and $\sigma \in \Delta[n]_q$, satisfying an obvious compatibility condition. If $\mathcal{D}$ is a second category, any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a simplicially enriched (simplicial, for short) functor $s\mathcal{C} \to s\mathcal{D}$.

Now assume that $\mathcal{C}$ has finite coproducts. Fix a simplicial object $\Delta_\bullet \in s\mathcal{C}$. We define a simplicial enrichment of $\mathcal{C}$, denoted $\mathcal{C}^\Delta$, via the formula

$$\mathrm{Map}_{\mathcal{C}^\Delta}(X, Y)_n := \mathrm{Hom}_{\mathcal{C}}(X, Y \sqcup \Delta_n) = \mathrm{Hom}_{\mathcal{C}_{\Delta_n/}}(X \sqcup \Delta_n, Y \sqcup \Delta_n) \qquad ([n] \in \Delta^{\mathrm{op}}).$$

We denote by $s\mathcal{C}_{\Delta_\bullet/}$ the simplicial slice category consisting of objects of $s\mathcal{C}$ together with a map from $\Delta_\bullet$.

Lemma 6.1 *The functor $\mathcal{C} \to s\mathcal{C}_{\Delta_\bullet/}$, $X \mapsto X \sqcup \Delta_\bullet$, extends to a simplicial functor $\mathcal{C}^\Delta \to s\mathcal{C}_{\Delta_\bullet/}$ and this simplicial functor is fully faithful, i.e. it induces isomorphisms on mapping simplicial sets.*

Proof Fix $[n] \in \Delta^{\mathrm{op}}$. We have canonical isomorphisms

$$\begin{aligned}\mathrm{Map}_{s\mathcal{C}_{\Delta_\bullet/}}(X \sqcup \Delta_\bullet, Y \sqcup \Delta_\bullet)_n &\cong \mathrm{Map}_{s\mathcal{C}}(X, Y \sqcup \Delta_\bullet)_n \\ &\cong \mathrm{Hom}_{\mathcal{C}}(X, Y \sqcup \Delta_n) \\ &= \mathrm{Map}_{\mathcal{C}^\Delta}(X, Y)_n.\end{aligned}$$

The first isomorphisms follows from the definition of the slice category and the universal property of the coproduct. For the second, note that, as X is a constant simplicial object, an n-simplex f in $\mathrm{Map}_{s\mathcal{C}}(X, Y \sqcup \Delta_\bullet)$ is uniquely determined by $f(\mathrm{id}_{[n]})\colon X \to Y \sqcup \Delta_n$. These isomorphisms are functorial in $[n] \in \Delta^{\mathrm{op}}$ and hence give the desired simplicial enrichment. $\square$

We now return to the setting of the previous sections. Let R be a π-adically complete ring where $\pi \in R$ is non zero-divisor, the ideal (π) is equipped with a PD-structure, and p is nilpotent in $R/(\pi)$.

Let $\mathrm{D}(R)$ denote the derived ∞-category of R, and $\mathrm{D}(R)^\wedge$ its subcategory formed by the derived π-complete complexes.

For $A \in \mathrm{FSmAlg}_R$, abusing notation slightly, we write $A\Delta^\pi_\bullet$ for the simplicial object

$$(A \widehat{\otimes}_R R\Delta^\pi_\bullet, (\pi, T_0, \ldots, T_\bullet)) \in s\mathrm{PFSmAlg}.$$

We will use as the abstract simplicial object $\Delta_\bullet$ above the simplicial object $R\Delta^\pi_\bullet$.

Lemma 6.2 *There is a functor*

$$\mathrm{dR}^{\Delta}\colon \mathrm{N}(\mathrm{FSmAlg}_R^{\Delta}) \to \mathrm{D}(R)^{\wedge}$$

which on objects is given by

$$\mathrm{dR}^{\Delta}(A) = \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{dR}(A\Delta_{\bullet}^{\pi}).$$

Proof The PD-de Rham complex as studied in the previous section defines a functor $\mathrm{dR}\colon \mathrm{PFSmAlg}_R \to \mathrm{Ch}(R)$ where $\mathrm{Ch}(R)$ denotes the category of chain complexes of R-modules. We now consider the composite of simplicial functors

$$\mathrm{FSmAlg}_R^{\Delta} \to \mathrm{PFSmAlg}_R^{\Delta} \to s\mathrm{PFSmAlg}_{R,\Delta_{\bullet}^{\pi}/} \to s\mathrm{PFSmAlg}_R \to s\mathrm{Ch}(R).$$

Here the first functor is given by $A \mapsto (A, (\pi))$, the second is the one from Lemma 6.1, the third one is the canonical one, and the last one is induced by dR. Applying the simplicial nerve, we obtain a functor of ∞-categories $\mathrm{N}(\mathrm{FSmAlg}_R^{\Delta}) \to \mathrm{N}(s\mathrm{Ch}(R))$. According to [10, Prop. 1.3.4.7], the latter category is the Dwyer-Kan-localization of the 1-category $s\mathrm{Ch}(R)$ at the class of simplicial homotopy equivalences. As the total complex functor $\mathrm{Tot}\colon s\mathrm{Ch}(R) \to \mathrm{Ch}(R)$ sends simplicial homotopy equivalences to quasi-isomorphisms, the composite $s\mathrm{Ch}(R) \to \mathrm{Ch}(R) \to \mathrm{D}(R)$ factors uniquely through a functor $t\colon \mathrm{N}(s\mathrm{Ch}(R)) \to D(R)$. We can thus construct the desired functor dR^{Δ} as the composite

$$\mathrm{N}(\mathrm{FSmAlg}_R^{\Delta}) \to \mathrm{N}(s\mathrm{Ch}(R)) \xrightarrow{t} \mathrm{D}(R) \to \mathrm{D}(R)^{\wedge}$$

where the last functor is the π-adic completion. As the total complex is a model for the (homotopy) colimit over Δ^{op} (see e.g. [1, Cor. 3.13]), and as the π-adic completion preserves colimits, we get the claimed description of the functor on objects. $\square$

It follows from Proposition 5.4(3) that the functor dR^{Δ} satisfies descent with respect to the étale topology. By the universal property of the ∞-category of sheaves, the Yoneda embedding induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Sh}(\mathrm{N}(\mathrm{FSmAff}_R^{\Delta}))^{\mathrm{op}}, \mathrm{D}(R)^{\wedge}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{desc}}(\mathrm{N}(\mathrm{FSmAlg}_R^{\Delta}), \mathrm{D}(R)^{\wedge})$$

where on the left-hand side we consider left adjoint functors and on the right-hand side functors satisfying étale descent. As the left-hand ∞-category identifies with $\mathrm{Fun}^{\mathrm{L}}\big(\mathrm{Sh}(\mathrm{Sm}_{\overline{R}})^{\mathrm{op}}, \mathrm{D}(R)^{\wedge}\big)$ by Proposition 4.1, the functor dR^{Δ} uniquely extends to a functor $\mathrm{dR}^{\Delta}\colon \mathrm{Sh}(\mathrm{Sm}_{\overline{R}})^{\mathrm{op}} \to \mathrm{D}(R)^{\wedge}$.

Definition 6.3 We define crystalline cohomology as the functor

$$\mathrm{Cris}\colon \mathrm{Sm}_{\overline{R}}^{\mathrm{op}} \to \mathrm{D}(R)^{\wedge}$$

which is the composite of dR^{Δ} with the Yoneda embedding $\mathrm{Sm}_{\overline{R}} \to \mathrm{Sh}(\mathrm{Sm}_{\overline{R}})$.

7 Comparison

Let the notation be as above, i.e. R be a π-adically complete ring, where $\pi \in R$ is a non-zero divisor, (π) is endowed with a PD-structure, and p is nilpotent in $R/(\pi)$.

In this section we compare our functor Cris from Sect. 6 to the de Rham functor $\mathrm{dR}\colon \mathrm{FSm}_R^{\mathrm{op}} \to \mathrm{D}(R)^{\wedge}$ on the category FSm_R of π-adic formal schemes over R which are topologically of finite type and formally smooth over R and to Berthelot's crystalline cohomology [5]

$$\mathrm{BCris}\colon \mathrm{Sm}_{\overline{R}}^{\mathrm{op}} \to \mathrm{D}(R)^{\wedge}, \quad \overline{X} \mapsto \lim_n R\Gamma_{\mathrm{cris}}(\overline{X}/R_n, \mathcal{O}^{\mathrm{cris}}).$$

Proposition 7.1 *There is a canonical isomorphism of functors*

$$\mathrm{dR} \xrightarrow{\sim} \mathrm{Cris} \circ (- \otimes_R \overline{R})\colon \mathrm{FSm}_R^{\mathrm{op}} \to \mathrm{D}(R)^{\wedge}.$$

Proof By Zariski descent it is sufficient to construct this isomorphism after restriction to FSmAff_R. For A in FSmAlg_R the functor on the left gives $\mathrm{dR}(A)$ and the functor on the right gives $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{dR}(A\Delta^{\pi}_{\bullet})$ by Lemma 6.2. The canonical morphism of simplicial rings $A \to A\Delta^{\pi}_{\bullet}$ induces the requested natural transformation $\mathrm{dR} \to \mathrm{Cris} \circ (- \otimes_R \overline{R})$. Proposition 5.4 implies that it is an equivalence. □

Proposition 7.2 *For* $\overline{X} \in \mathrm{Sm}_{\overline{R}}$ *there is a canonical isomorphism*

$$\mathrm{Cris}(\overline{X}) \cong \mathrm{BCris}(\overline{X}).$$

Proof By Zariski descent it is sufficient to construct this isomorphism after restriction to $\mathrm{SmAff}_{\overline{R}}$. Berthelot constructs a natural comparison isomorphism

$$\mathrm{BCris}(B/I) \cong \mathrm{dR}(B, I) \text{ in } \mathrm{D}(R)^{\wedge}$$

of his crystalline cohomology to PD-de Rham cohomology for (B, I) in $\mathrm{PFSmAlg}_R$, see [5, Thm. V.2.3.2], [3], [14, Prop. 07LG]. For any simplical object $(B_{\bullet}, I_{\bullet})$ in $\mathrm{PFSmAlg}_R$ we get an induced isomorphism

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{dR}(B_{\bullet}, I_{\bullet}) \cong \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{BCris}(B_{\bullet}/I_{\bullet}).$$

Taking $B_{\bullet} = A\Delta^{\pi}_{\bullet}$ for A in FSmAlg_R one gets the required equivalence. □

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