Internal Higher Categories and Applications

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Abstract

The goal of this thesis is to develop the theory of $(\infty, 1)$-categories internal to an arbitrary ∞-topos $\mathcal{B}$. These objects can be defined as complete Segal objects in $\mathcal{B}$ or equivalently as sheaves of ∞-categories on $\mathcal{B}$, and we will simply refer to them as $\mathcal{B}$-categories for short. Here we develop an extensive amount of tools that are needed to efficiently work with these objects in practice.

Beginning with the theory of adjunctions, we define notions of limits and colimits in $\mathcal{B}$-categories and prove that presheaf $\mathcal{B}$-categories are free cocompletions. We continue by developing a theory of accessible and presentable $\mathcal{B}$-categories and finally we study $\mathcal{B}$-topoi. Here one of our main results is that $\mathcal{B}$-topoi are equivalent to relative topoi over $\mathcal{B}$, so geometric morphism with target $\mathcal{B}$. We then apply this result to study smooth and proper geometric morphisms of ∞-topoi from an internal point of view. We conclude with an application in étale homotopy theory where we use some of our machinery to construct and understand a condensed refinement of the usual étale homotopy type of schemes.

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Contents

CHAPTER 1

Introduction

1.1. Motivation

Ever since the fundamental work of Lurie [57], the language of ∞-categories has become an indispensable tool in many areas of mathematics that employ homotopical thinking. The main goal of this thesis is to contribute to the further development of this language by systematically studying ∞-categories *parametrized* by an ∞-topos $\mathcal{B}$. Here an ∞-category parametrized by $\mathcal{B}$ is defined to be a sheaf of ∞-categories on $\mathcal{B}$, so a limit preserving functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. We will call these objects $\mathcal{B}$*-categories* for short. Let us write $\mathrm{Cat}(\mathcal{B}) = \mathrm{Fun}^{\mathrm{lim}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ for the ∞-category of $\mathcal{B}$-categories. We now continue by giving some examples where $\mathcal{B}$-categories show up in practice.

One can think of parametrized ∞-categories as a way of encoding additional algebraic structure on an ∞-category. As a concrete example let G be a compact Lie-group and Orb_G its orbit category. Then a $\mathrm{PSh}(\mathrm{Orb}_G)$-category C is simply a functor $\mathsf{C}\colon \mathrm{Orb}_G \to \mathrm{Cat}_\infty$. We think of a $\mathrm{PSh}(\mathrm{Orb}_G)$-category as an ∞-category with an action of the group G and for an orbit G/H we think of the value $\mathsf{C}(G/H)$ as the additional information of the "topological" H-fixed points of the action on C. In fact the idea of systematically studying $\mathrm{PSh}(\mathrm{Orb}_G)$-categories originated in [11], [12], [13], [71] etc. and these methods are by now a standard tool in equivariant homotopy theory, see [22], [21] etc.

Let us now give an example in which parametrized ∞-categories arise in more geometric situations. For this let $f\colon X \to Y$ be a map of topological spaces. Then the functor

$$\mathrm{Open}(Y) \to \mathrm{Cat}_\infty, \;\; U \mapsto \mathrm{Sh}(f^{-1}(U))$$

defines a sheaf of ∞-categories on Y that we denote by $\mathrm{Sh}^Y(X)$. It turns out that $\mathrm{Sh}^Y(X)$ suffices to completely reconstruct the map f and provides a useful way of encoding the topological information of f. In fact we will see in Chapter 6 that certain categorical properties of the $\mathrm{Sh}(Y)$-category $\mathrm{Sh}^Y(X)$ give necessary and sufficient conditions for the functors $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and f^* to be compatible with base change along some other map $g\colon Z \to Y$.

Lastly $\mathcal{B}$-categories can be used to encode a topological structure on an ∞-category by letting $\mathcal{B}$ be the ∞-topos of pyknotic spaces or condensed anima, introduced by Barwick-Haine [10] or Clausen-Scholze [20]. As a concrete example, in [9, §13] the authors construct a $\mathrm{Pyk}(S)$-category $\mathrm{Gal}(X)$ for any qcqs scheme X that serves as a refinement of the category of points of the étale topos of $X_{\text{ét}}$ and captures the additional information of a pro-finite topology on the category of points. Remembering this additional structure on the category $\mathrm{Gal}(X)$ then allows to fully recover $X_{\text{ét}}$ from X, see [9, Corollary 0.5.2] and [59, Theorem 2.3.1].

The above examples show that parametrized ∞-categories naturally appear in many different contexts. However, just like in ordinary higher category theory, it is often a non-trivial task to construct functors between $\mathcal{B}$-categories, since these encode many higher coherences, let alone prove properties of those. The main way to overcome this issue is to construct functors, natural transformations etc. using universal properties. But in order to be able to even formulate these universal properties one needs a suitably developed framework for the higher category theory of parametrized ∞-categories. Providing such a framework is the main goal of this thesis. In fact we will develop most of the contents of Lurie's Higher topos theory [57] in the context of $\mathcal{B}$-categories.

Let us now take a conceptually slightly different point of view on $\mathcal{B}$-categories. For this recall that via the nerve functor an ∞-category can be equivalently thought of as a functor $\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathcal{S}$ satisfying the following conditions: For any $n \geq 2$ the canonical map

$$\mathcal{C}_n \xrightarrow{(d_{\{0,1\}}, d_{\{1,2\}}, \dots, d_{\{n-1,n\}})} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \dots \times_{\mathcal{C}_0} \mathcal{C}_1$$

is an equivalence and furthermore the square

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "\sigma_0"] \arrow[d, "(\mathrm{id},\mathrm{id})"'] & \mathcal{C}_3 \arrow[d, "d_{\{0,2\}} \times d_{\{1,3\}}"] \\ \mathcal{C}_0 \times \mathcal{C}_0 \arrow[r, "\sigma_0"'] & \mathcal{C}_1 \times \mathcal{C}_1 \end{tikzcd}$$

is cartesian. Here we think of $\mathcal{C}_1$ as the space of morphisms in $\mathcal{C}$, so that informally speaking, the first condition says that $\mathcal{C}_n$ should be equivalent to the space of composable sequences of n morphisms. We refer to these conditions for all $n \geq 2$ as the *Segal conditions*. The second condition enforces that $\mathcal{C}_0$ is equivalent to the space of equivalence in $\mathcal{C}$, i.e. arrows that have a left and a right inverse. If a simplicial object satisfies this condition, we say that it satisfies *univalence*. It turns out that simplicial objects in $\mathcal{S}$ satisfying these two conditions model the ∞-category of ∞-categories. More precisely, the nerve functor defines an equivalence between the ∞-category Cat_∞ and the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$ spanned by the simplicial objects satisfying the Segal conditions and univalence, see [51] and [40].

One may now be inclined to consider a more general version of the above definition by replacing $\mathcal{S}$ with a general ∞-topos $\mathcal{B}$ in order to define a notion of an ∞-*category internal to* $\mathcal{B}$. It turns out that the above equivalence generalizes and $\mathcal{B}$-categories are equivalent to ∞-categories internal to $\mathcal{B}$. Indeed, if we denote by $\mathrm{CSS}(\mathcal{B})$ the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B})$ spanned by those simplicial objects satisfying the Segal conditions and univalence, it is not hard to see that there is an equivalence of ∞-categories

$$\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{CSS}(\mathcal{B}).$$

The above equivalence shows that *parametrized* and *internal* higher category theory are equivalent theories. In practice both of these perspectives are useful for different reasons. The internal point of view often allows to give conceptually very clean definitions and proofs by "internalizing" the respective concepts from usual ∞-category theory, effectively hiding the complexity of the base ∞-topos $\mathcal{B}$. On the other hand the parametrized perspective is crucial, because in practice almost all interesting examples arise as some explicit sheaves of ∞-categories. Therefore we try get the best from both worlds in this thesis: We give conceptually clean constructions and proofs using the internal perspective, but also unwind their explicit meaning for the corresponding sheaves of ∞-categories.

1.2. Outline and Main results

We continue by giving a rough overview of the contents of each chapter of this thesis.

1.2.1. Fundamentals of $\mathcal{B}$-category theory. Since we will directly build on the foundations of $\mathcal{B}$-category theory laid out in [62], we begin by recalling the basic definitions and results from loc. cit. in § 2.1. After that we study two fundamental kinds of construction in $\mathcal{B}$-category theory in § 2.2, namely subcategories and localisations. These constructions are both direct adaptions of their analogues in usual ∞-category theory. Furthermore we give a quick proof of the Straightening-Unstraightening equivalence in the context of $\mathcal{B}$-categories in § 2.3.

In § 2.4 we study adjunctions of $\mathcal{B}$-categories. Defining an adjunction of $\mathcal{B}$-categories is fairly straightforward. The ∞-category $\mathrm{Cat}(\mathcal{B})$ is enriched over Cat_∞ and therefore yields an $(\infty, 2)$-category. Thus we may simply take as a notion of adjunction the usual notion of adjunction in an $(\infty, 2)$-category. We prove a number of expected properties of adjunctions between $\mathcal{B}$-categories by internalizing some of the arguments from [17, §6.1]. We also give an explicit section-wise criterion for the existence of an adjoint functor that is very convenient in practice and which we will constantly use in the later chapters of this thesis. Except for § 2.3, the content of this chapter first appeared in [63].

1.2.2. Colimits and Cocompletions. In this chapter we study (co)limits and (co)completions in the context of $\mathcal{B}$-category theory. In § 3.1 we begin by defining a notion of (co)limits in $\mathcal{B}$-categories by adapting Joyal's definition [49, Definition 4.5] to our context and provide a number of explicit examples. We then define what it means for a $\mathcal{B}$-category to be cocomplete in § 3.2. While this notion is defined in a purely internal way, it admits the following very concrete reformulation:

PROPOSITION 1.2.2.1 (Corollary 3.2.4.7). *A $\mathcal{B}$-category C is cocomplete if and only if the following conditions are satisfied:*

(1) For every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ is cocomplete and for any $s\colon B \to A$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves colimits.

(2) For every map $p\colon P \to A$ in $\mathcal{B}$ the functor p^* has a left adjoint $p_!$ such that for every pullback square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

in $\mathcal{B}$ the natural map $q_!t^* \to s^*p_!$ is an equivalence.

The dual statement for completeness holds as well.

To state the main theorem of this section, we need to introduce the following example of a $\mathcal{B}$-category. Unstraightening the codomain fibration $\mathrm{ev}_1\colon \mathcal{B}^{\Delta^1} \to \mathcal{B}$ defines a functor

$$\Omega_{\mathcal{B}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad A \mapsto \mathcal{B}_{/A}.$$

Because $\mathcal{B}$ is an ∞-topos, $\Omega_{\mathcal{B}}$ is a sheaf and thus defines a $\mathcal{B}$-category. This example is fundamental for the theory of $\mathcal{B}$-categories because it plays the role that the ∞-category of spaces $\mathcal{S}$ plays in ordinary higher category theory. Some crucial evidence for this claim is already given in [62], where it is shown that $\Omega_{\mathcal{B}}$ is the base of the universal left fibration. Using Proposition 1.2.2.1 we can now easily verify that $\Omega_{\mathcal{B}}$ has the expected property of being cocomplete. In fact the following much stronger statement is true, which in particular characterizes $\Omega_{\mathcal{B}}$ as the free cocompletion of the point:

THEOREM 1.2.2.2 (Theorem 3.4.1.1). *For any $\mathcal{B}$-category C and any cocomplete $\mathcal{B}$-category E, restriction along the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ induces an equivalence*

$$(h_{\mathsf{C}})^*\colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}).$$

In other words, the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ exhibits the $\mathcal{B}$-category of presheaves on C as the free cocompletion of C.

More generally we construct a free U-cocompletion for any *internal class* U and prove that it has the expected universal property, see Theorem 3.4.1.13.

The contents of this chapter first appeared as a part of [63].

1.2.3. Presentable $\mathcal{B}$-categories. Ever since Lurie's seminal work [57], the theory of presentable ∞-categories has taken an important role, both within the development of the general theory of ∞-categories and in applications. This is due to the many favourable properties of presentable ∞-categories, such as the adjoint functor theorems or the existence of a well-behaved and explicit tensor product. Thus the goal of this chapter is to generalize this notion to the context of $\mathcal{B}$-categories.

As in usual ∞-category theory we would like to call a $\mathcal{B}$-category presentable if it is accessible and cocomplete. Therefore we begin by introducing a suitable notion of accessible $\mathcal{B}$-categories. For this we define the notion of a U-filtered $\mathcal{B}$-category for some internal class U. Then a $\mathcal{B}$-category C is called

U-*accessible* if there is a small $\mathcal{B}$-category C_0 and a sufficiently nice[1] internal class U such that C is obtained by freely adjoining U-filtered colimits. We will say that C is *accessible* if it is U-accessible for some (sufficiently nice) U. Thus we can now define C to be presentable if it is accessible and cocomplete. Our first main result is the following characterization of presentable $\mathcal{B}$-categories:

THEOREM 1.2.3.1 (Theorem 4.4.2.4). *For a large $\mathcal{B}$-category D, the following are equivalent:*

(1) D is presentable;
(2) there is a $\mathcal{B}$-category C and a sound doctrine U such that D is a U-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$;
(3) D is accessible and cocomplete;
(4) there is a sound doctrine U such that D is U-accessible and $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is $\mathrm{op}(\mathsf{U})$-cocomplete;
(5) there is a doctrine U and a small $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category C such that $\mathsf{D} \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})$ (see Definition 4.4.6.1);
(6) The following two conditions are satisfied:
   (a) the associated sheaf $\mathsf{D}\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ takes values in the ∞-category $\mathrm{Pr}^{\mathrm{L}}_\infty$ of presentable ∞-categories and colimit-preserving functors;
   (b) for every map $s\colon B \to A$ in $\mathcal{B}$ the associated transition functor $s^*\colon \mathsf{D}(A) \to \mathsf{D}(B)$ admits a left adjoint $s_!$, and for every pullback square in $\mathcal{B}$ the induced commutative square in $\mathrm{Pr}^{\mathrm{L}}_\infty$ is left adjointable.

Note that the equivalence of conditions (1)-(5) is an analogue of the characterization of presentable ∞-categories due to Lurie and Simpson [57, Theorem 5.5.1.1]. Condition (6), on the other hand, gives a very explicit and simple way of checking whether a $\mathcal{B}$-category is presentable in practice. Having such a plethora of equivalent characterisations of the notion of presentability at hand, it is straightforward to prove the expected theorems that revolve around these objects: Among other things, we prove adjoint functor theorems and discuss how one can construct (internal) limits and colimits of such $\mathcal{B}$-categories. Furthermore, we construct the *tensor product* of presentable $\mathcal{B}$-categories and we show:

THEOREM 1.2.3.2 (Propositions 4.6.3.2, 4.6.3.7 and 4.6.3.10). *There is a strong monoidal and fully faithful left adjoint $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \hookrightarrow \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$, where $\mathcal{B}$ is viewed as a presentably symmetric monoidal ∞-category via its cartesian monoidal structure and where $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ is the ∞-category of presentable $\mathcal{B}$-categories and cocontinuous functors. Furthermore this functor is an equivalence whenever $\mathcal{B}$ is generated under colimits by (-1)-truncated objects.*

The contents of this chapter first appeared in [65].

1.2.4. $\mathcal{B}$-topoi. Many of the explicit examples of presentable $\mathcal{B}$-categories that we have in mind arise in the following way: To any geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ of ∞-topoi with left adjoint f^*, one can associate a presentable $\mathcal{B}$-category X given by the sheaf of ∞-categories $\mathcal{X}_{/f^*(-)}$ on $\mathcal{B}$. In fact, $\mathcal{B}$-categories that arise via the above construction have even more favourable properties than just presentability. They give examples of *$\mathcal{B}$-topoi*, the topic of this chapter.

Recall that a presentable ∞-category $\mathcal{X}$ is an ∞-topos if the functor $\mathcal{X}_{/-}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that is classified by unstraightening the codomain fibration is a limit preserving functor. In this case we also say that $\mathcal{X}$ satisfies *descent*. Now if X is a $\mathcal{B}$-category with finite limits we also have a cartesian fibration $\mathrm{ev}_1\colon \mathsf{X}^{\Delta^1} \to \mathsf{X}$ of $\mathcal{B}$-categories, that via unstraightening for $\mathcal{B}$-categories classifies a functor $\mathsf{X}_{/-}\colon \mathsf{X}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$[2]. We will say that X satisfies *descent* if this functor is limit preserving and that X is a *$\mathcal{B}$-topos* if it is presentable and satisfies descent. The first main result of this chapter is the following classification of $\mathcal{B}$-topoi:

THEOREM 1.2.4.1 (Theorem 5.2.1.5). *For a large $\mathcal{B}$-category X, the following are equivalent:*

[1]The precise condition is that we require U to be a *sound doctrine*, see Definition 4.1.2.7.

[2]Here $\mathsf{Cat}_{\mathcal{B}}$ denotes the $\mathcal{B}$-category of $\mathcal{B}$-categories.

(1) X *is a* $\mathcal{B}$*-topos;*
(2) X *satisfies the* internal Giraud axioms*:*
  (a) X *is presentable;*
  (b) X *has universal colimits;*
  (c) groupoid objects in X *are effective;*
  (d) Ω*-colimits in* X *are disjoint;*
(3) there is a $\mathcal{B}$*-category* C *such that* X *arises as a left exact and accessible Bousfield localisation of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*;*
(4) X *is* Ω*-cocomplete and takes values in the* ∞*-category* $\mathrm{Top}_\infty^{\mathrm{L,\acute{e}t}}$ *of* ∞*-topoi and étale algebraic morphisms;*
(5) X *is an* $\mathrm{Top}_\infty^{\mathrm{L,\acute{e}t}}$*-valued sheaf that preserves pushouts.*

The first three items in Theorem 1.2.4.1 can be understood as the $\mathcal{B}$-categorical analogue of Lurie's characterisation of ∞-topoi in [57, Theorem 6.1.0.1]. By contrast, the last two items of the theorem provide an *external* characterisation of $\mathcal{B}$-topoi, i.e. one in terms of the underlying sheaves of ∞-categories.

As usual we will say that a *geometric morphism* of $\mathcal{B}$-topoi is a right adjoint functor $f_*\colon \mathsf{X} \to \mathsf{Y}$ between $\mathcal{B}$-topoi whose left adjoint is left exact. We denote the ∞-category of $\mathcal{B}$-topoi and geometric morphisms between them by $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$. As a consequence of the fact that $\Omega_{\mathcal{B}}$ is the free cocompletion of the point, we deduce that $\Omega_{\mathcal{B}}$ is the terminal object of $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$. It follows that the global sections functor $\mathrm{Top}^{\mathrm{R}}(\mathcal{B}) \to \mathrm{Top}_\infty^{\mathrm{R}}$ factors through $(\mathrm{Top}_\infty^{\mathrm{R}})_{/\mathcal{B}}$. Using the explicit characterisations of Theorem 1.2.4.1 we can prove the following result:

THEOREM 1.2.4.2 (Theorem 5.2.5.1). *The global sections functor induces an equivalence* $\Gamma\colon \mathrm{Top}^{\mathrm{R}}(\mathcal{B}) \simeq (\mathrm{Top}_\infty^{\mathrm{R}})_{/\mathcal{B}}$ *between the* ∞*-category of* $\mathcal{B}$*-topoi and the* ∞*-category of* ∞*-topoi over* $\mathcal{B}$*.*

The inverse to the equivalence in Theorem 1.2.4.2 can also be described very explicitly: It is precisely the construction already described above, that sends a geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ to the $\mathcal{B}$-category given by the sheaf of ∞-categories $f_*\Omega_{\mathcal{X}} = \mathcal{X}_{/f^*(-)}$ on $\mathcal{B}$.

Theorem 1.2.4.2 provides a useful way for understanding relative ∞-topoi, because it allows to translate properties of geometric morphisms to properties of the corresponding $\mathcal{B}$-topoi. Since, conceptually speaking, $\mathcal{B}$-topoi behave in essentially the same way as ordinary ∞-topoi, the latter is often quite approachable. This way of thinking about relative topoi is well-known in the 1-categorical case since the work of Moens [67]. In fact many parts of Johnstone's book [46] are using this strategy in a crucial way.

We also give an example how to apply Theorem 1.2.4.2 in ∞-topos theory. We internalize the proof that for two ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$ their product in $\mathrm{Top}_\infty^{\mathrm{R}}$ is given by the tensor product $\mathcal{X} \otimes \mathcal{Y}$. Using the equivalence of Theorem 1.2.4.2, it follows that for a span of ∞-topoi $\mathcal{X} \xrightarrow{f_*} \mathcal{B} \xleftarrow{g_*} \mathcal{Y}$ the fibre product $\mathcal{X} \times_{\mathcal{B}} \mathcal{Y}$ in $\mathrm{Top}_\infty^{\mathrm{R}}$ is given by the tensor product of presentable $\mathcal{B}$-categories $f_*\Omega_{\mathcal{X}} \otimes^{\mathcal{B}} g_*\Omega_{\mathcal{Y}}$. In combination with Theorem 1.2.3.2 this shows that the fibre product $\mathcal{X} \times_{\mathcal{B}} \mathcal{Y}$ in $\mathrm{Top}_\infty^{\mathrm{R}}$ and the relative tensor product $\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y}$ in Pr^{L} agree whenever $\mathcal{B}$ is generated under colimits by (-1)-truncated objects, generalizing [7, Corollary 1.10].

Finally, we conclude this chapter by laying the foundations for the theory of *localic* $\mathcal{B}$-topoi.

The contents of this chapter first appeared as part of [64] and [66].

1.2.5. Application: Smooth and proper morphisms of ∞-topoi. We now continue by giving a further application of the results of this thesis in ∞-topos theory. Recall from [57, Definition 7.3.1.4] that a geometric morphism $p_*\colon \mathcal{X} \to \mathcal{B}$ of ∞-topoi is called *proper* if it satisfies *proper base change*, so if for any rectangle in $\mathrm{Top}_\infty^{\mathrm{R}}$

$$\begin{tikzcd} \mathcal{X}'' \arrow[r, "t_*"] \arrow[d, "r_*"] & \mathcal{X}' \arrow[r, "s_*"] \arrow[d, "q_*"] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{B}'' \arrow[r, "l_*"'] & \mathcal{B}' \arrow[r, "h_*"'] & \mathcal{B} \end{tikzcd}$$

in which both squares are cartesian, the mate transformation:

$$l^* q_* \to r_* t^*$$

is invertible. Dually we say that p_* is *smooth* if it satisfies *smooth base change*, so if the dual mate transformation

$$q^* l_* \to t_* r^*$$

is invertible. Knowing that a specific geometric morphism is smooth or proper can be very useful in applications, but the above definitions are often rather hard to check. Therefore the main goal of this chapter is to provide more explicit criteria for smoothness and properness.

Let us start by explaining the characterization for smooth geometric morphisms. Recall that a geometric morphism of ∞-topoi $p_* \colon \mathcal{X} \to \mathcal{B}$ is *locally contractible* if p^* admits a further left adjoint $p_\sharp \colon \mathcal{X} \to \mathcal{B}$ with the property that for any span $p^* A \to p^* B \leftarrow X$ in $\mathcal{X}$ the canonical map

$$p_\sharp(p^* A \times_{p^* B} X) \to A \times_B p_\sharp X$$

is an equivalence. Translating p_* to be the $\mathcal{B}$-topos $p_* \Omega_{\mathcal{X}}$ via the equivalence of Theorem 1.2.4.2, it is not hard to check that p_* is locally contractible if and only if the unique algebraic morphism of $\mathcal{B}$-topoi $\Omega_{\mathcal{B}} \to p_* \Omega_{\mathcal{X}}$ admits a left adjoint. In this case we also say that $p_* \Omega_{\mathcal{X}}$ is a *locally contractible* $\mathcal{B}$-topos. We then characterize locally contractible $\mathcal{B}$-topoi as those $\mathcal{B}$-topoi that are generated by their *contractible* objects, see Proposition 6.1.2.5. As a consequence we deduce the following:

Theorem 1.2.5.1 (Theorem 6.1.3.2). *A geometric morphism $p_* \colon \mathcal{X} \to \mathcal{B}$ is smooth if and only if it is locally contractible.*

We now continue by giving a characterization of proper morphisms of ∞-topoi. In order to motivate our criterion, recall from [57, Corollary 7.3.4.12] that a Hausdorff space is compact if and only if the global sections functor $\Gamma_* \colon \mathrm{Sh}(X) \to \mathcal{S}$ preserves filtered colimits. Thus one may be tempted to more generally call an ∞-topos $\mathcal{X}$ *compact* if the global sections functor $\Gamma_* \colon \mathcal{X} \to \mathcal{S}$ preserves filtered colimits. Given the theory that we have developed so far we can now relativize this condition. For this recall that we call a $\mathcal{B}$-category I *filtered* if the colimit functor $\mathrm{colim}_{\mathsf{I}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) \to \Omega_{\mathcal{B}}$ is left exact. We then say that a geometric morphism $p_* \colon \mathcal{X} \to \mathcal{B}$ is *compact* if the associated terminal geometric morphism of $\mathcal{B}$-topoi $(\Gamma_{\mathcal{B}})_* \colon p_* \Omega_{\mathcal{X}} \to \Omega_{\mathcal{B}}$ commutes with colimits indexed by filtered $\mathcal{B}$-categories.

Since the notion of properness of a geometric morphism also gives a sensible notion of relative compactness one might ask if these two notions agree. We give an affirmative answer:

Theorem 1.2.5.2 (Theorem 6.2.1.12). *A geometric morphism $p_* \colon \mathcal{X} \to \mathcal{B}$ is proper if and only if it is compact.*

That such a characterization of proper geometric morphisms should hold is already mentioned in [57, Remark 7.3.1.5], but the necessary language to make it precise was not yet available. However, note that even in the case $\mathcal{B} = \mathcal{S}$, where the statement makes no reference to internal higher category theory, Theorem 1.2.5.2 was previously unknown. In fact, our proof crucially uses techniques from internal higher category theory even in this case.

Finally we apply the above theorem to show that a proper and separated map $p \colon X \to Y$ of arbitrary topological spaces induces a proper geometric morphism $p_* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. This generalizes [57, Theorem 7.3.1.16], where X was assumed to be completely regular.

The contents of this chapter first appeared as parts of [64] and [66].

1.2.6. Applications in étale homotopy theory. In the final chapter we will present applications of the previous results in étale homotopy theory. These applications have their origin in the work of Barwick-Glasman-Haine on *Exodromy* results in algebraic geometry [9] and Lurie's work on *Ultracategories* [59]. The ∞-topos internally to which we are going to work is the ∞-topos of *condensed* or *pyknotic* spaces. Our starting point is the question to what extend a coherent ∞-topos $\mathcal{X}$ can be recovered from its category of points.

In general the answer to this question is that it cannot[3], but we can fix this issue by remembering the natural pyknotic structure that the category of points $\mathrm{Pt}(\mathcal{X})$ acquires. Indeed, for any ∞-topos $\mathcal{X}$ we define $\underline{\mathrm{Pt}}(\mathcal{X})$ to be the $\mathrm{Pyk}(\mathcal{S})$-category given by the functor

$$\underline{\mathrm{Pt}}(\mathcal{X})\colon \mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}}) \to \mathrm{Cat}_\infty, \quad S \mapsto \mathrm{Fun}_*(\mathrm{Sh}(S), \mathcal{X}).$$

It is expected that one can recover the full ∞-topos $\mathcal{X}$ from $\underline{\mathrm{Pt}}(\mathcal{X})$ whenever $\mathcal{X}$ is bounded and coherent. Lurie proves this for coherent 1-topoi in [59].

For our applications we will focus on the case where $\mathcal{X} = X_{\text{ét}}$ is the ∞-topos of étale sheaves on a qcqs scheme X. In this case it turns out that the ∞-topos $X_{\text{ét}}$ can already be recovered from a smaller variant of $\underline{\mathrm{Pt}}(\mathcal{X})$. Following [9], we write $\mathrm{Gal}(X)$ for the $\mathrm{Pyk}(\mathcal{S})$-category given by the functor

$$\mathrm{Gal}(X)\colon \mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}}) \to \mathrm{Cat}_\infty, \quad S \mapsto \mathrm{Fun}_*^{\mathrm{coh}}(\mathrm{Sh}(S), X_{\text{ét}}),$$

where $\mathrm{Fun}_*^{\mathrm{coh}}(\mathrm{Sh}(S), X_{\text{ét}})$ denotes the ∞-category of coherent geometric morphisms. Considering the ∞-category $\mathcal{S}_\pi$ as a constant $\mathrm{Pyk}(\mathcal{S})$-category, Barwick-Glasman-Haine show that there is an equivalence of ∞-categories

$$X_{\text{ét}}^{\mathrm{cons}} \simeq \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathcal{S}_\pi),$$

where the left hand side denotes the full subcategory spanned by the constructible étale sheaves, see [9, Corollary 0.5.2, Theorem 13.2.11]. The first main result of this chapter extends the Exodromy equivalence to a much larger class of sheaves. To simplify notation we write $\mathbf{Pyk}(\mathcal{S}) = \Omega_{\mathrm{Pyk}(\mathcal{S})}$.

THEOREM 1.2.6.1 (Theorem 7.2.0.1). *Let X be a qcqs scheme. Then the exodromy equivalence induces an equivalence of ∞-topoi*

$$X_{\text{proét}}^{\mathrm{hyp}} \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S})).$$

Here the source category is the ∞-topos of hypercomplete pro-étale sheaves on X, as introduced in [14, §4]. The above theorem shows that when working internally to pyknotic spaces the structure of the a priori complicated pro-étale ∞-topos greatly simplifies because it is “just” a presheaf category. It also gives a conceptual explanation why the topos of pro-étale sheaves has so many convenient properties. For example it immediately implies that for an arbitrary map of schemes $f\colon X \to Y$ the pullback functor

$$f^*\colon Y_{\text{proét}}^{\mathrm{hyp}} \to X_{\text{proét}}^{\mathrm{hyp}}$$

commutes with *all* limits, see Corollary 7.2.3.12.

We will then apply Theorem 1.2.6.1 to understand the *étale homotopy theory* of schemes. Recall that the *étale homotopy type* of a scheme X, introduced by Artin-Mazur-Friedlander in [8],[24], can from a modern point of view be defined as the shape of the ∞-topos of hypercomplete étale sheaves $X_{\text{ét}}^{\mathrm{hyp}}$, see [44, § 5]. Since $X_{\text{ét}}^{\mathrm{hyp}}$ is typically not locally contractible, the étale homotopy type is often not a space but a pro-space, which can be hard to work with. However, Theorem 1.2.6.1 shows that the situation improves if we work in the pro-étale setting instead. The $\mathrm{Pyk}(\mathcal{S})$-topos $X_{\text{proét}}^{\mathrm{hyp}}$ is locally contractible, it is even a presheaf topos. Therefore we may also easily compute its shape relative to $\mathrm{Pyk}(\mathcal{S})$, which we denote by $\Pi_\infty^{\text{proét}}(X)$, as the groupoidification $\mathrm{Gal}(X)^{\mathrm{gpd}}$.

We call the pyknotic space $\Pi_\infty^{\text{proét}}(X)$ the *pro-étale homotopy type of X*. It is related to the usual étale homotopy type via the fact that $\Pi_\infty^{\text{ét}}(X)$ is equivalent to the pro-truncated homotopy type of $\Pi_\infty^{\text{proét}}(X)$ (see Proposition 7.3.1.7). Having this explicit description of $\Pi_\infty^{\text{proét}}(X)$ as the groupoidification of a $\mathrm{Pyk}(\mathcal{S})$-category allows us to apply methods from the theory of $\mathrm{Pyk}(\mathcal{S})$-categories. Concretely we will prove a variant of Quillen's Theorem B [74, Theorem B], internally to any ∞-topos $\mathcal{B}$ and also up to

[3]The sheaf topoi associated with any two pro-finite sets of the same cardinality have the same category of points

localisation at a class of morphisms.[4] We then apply this result to deduce that for a smooth and proper morphism $X \to Y$ of qcqs schemes and a geometric point $y \to Y$ the fibre of the induced map

$$\Pi_\infty^{\text{proét}}(X) \to \Pi_\infty^{\text{proét}}(Y)$$

at y is equivalent to $\Pi_\infty^{\text{proét}}(X_y)$ after pro-finitely completing at the primes that are invertible on Y, see Theorem 7.3.3.10.

The results of § 7.2 have first appeared in [91] and the results of § 7.3 are going to appear in joint work with Barwick, Haine, Holzschuh, Lara, Mair and Martini.

1.3. Related work

The series of papers on *parametrized higher category theory and parametrized higher algebra* by Barwick-Dotto-Glasman-Nardin-Shah [11], [13], [82], [71], [83] already develops a substantial amount of the theory of internal higher categories in the special case where $\mathcal{B} = \mathrm{PSh}(\mathcal{C})$ is a presheaf category on some orbital ∞-category. More concretely, in the case of an arbitrary presheaf ∞-topos, many of the results in Chapter 3 and some results from Chapter 4 on filteredness have also been proven by Shah in [82] and [83]. If $\mathcal{C}$ is also assumed to be orbital, a theory of presentability has been developed by Hilman in [43].

The theory of $\mathcal{B}$-categories is also closely related to various flavours of *synthetic ∞-category theory*. One instance of such a theory is Riehl-Shulman's *simplicial homotopy type theory* [79]. By [84], simplicial homotopy type theory can be interpreted in simplicial objects in any ∞-topos and therefore provides a purely synthetic way of arguing about $\mathcal{B}$-categories. Many concepts of internal higher category theory have been developed from this point of view by Buchholtz-Weinberger in [16] and Weinberger in [90], [89], [88]. Our theory is also related to the ongoing work of Cisinski-Cnossen-Nguyen-Walde on the formalization of higher category theory [19]. The theory of $\mathcal{B}$-categories will give rise to an example of a "synthetic category theory" and then many of our results can be proven in this more general setup.

Let us also mention that in the 1-categorical context, the notion of an internal category has been first formulated by Grothendieck in [31] and since then has been developed by many mathematicians such as Bunge, Bénabou, Celeyrette, Joyal and Tierney. In particular many results of this thesis are well-known for a long time in the 1-categorical context. Also many of the 1-categorical analogues of our topos-theoretic results are well-known. See [46] for an extensive account. More specifically let us mention that the equivalence between relative and internal topoi is due to Moens [67] and the characterization of proper morphisms is due Lindgren [54] and also appears in [69].

Finally let us again mention that the contents of Chapters 2-6 are all part of joint work with Louis Martini [63], [65], [64], [66]. The contents of § 7.2 first appeared in [91], but some of the arguments have been simplified and some results have been added. The contents of § 7.3 are going to appear in joint work with Barwick, Haine, Holzschuh, Lara, Martini and Mair.

[4] Such a result was also proven in [68]. We will give our own proof here, that is better adapted to the language of this thesis.

CHAPTER 2

Fundamentals of $\mathcal{B}$-category theory

The goal of this chapter is to set the stage and prove some preliminary results for the later chapters of this thesis. We will begin with an extensive recollection of the main framework and results from [62] in § 2.1. After that we continue by studying subcategories and localisations of $\mathcal{B}$-categories in § 2.2. Both of these notions are rather straightforward adoptions of the corresponding notions in usual higher category theory. Then we will give a proof of the straightening-unstraightening equivalence for cocartesian fibrations of $\mathcal{B}$-categories in § 2.3.2. This result was first proven in [61], but we decided to give our own proof here because it is completely different from the on given in [61].

In § 2.4 we study adjunctions between $\mathcal{B}$-categories. They are defined via the standard 2-categorical definition in terms of units, counits and triangle identities, but we also provide a number of equivalent characterizations, as in [17, § 6.1]. We also prove a useful section-wise characterization for the existence of an adjoint: A functor of $\mathcal{B}$-categories $f\colon \mathsf{C} \to \mathsf{D}$ has a left adjoint if and only if for every $A \in \mathcal{B}$ the functor of $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ admits a left adjoint and for any $s\colon A \to B$ in $\mathcal{B}$ the induced mate transformation is invertible, see Proposition 2.4.2.9.

2.1. Background and recollections on $\mathcal{B}$-categories

In this section we recall the basic framework of higher category theory internal to an ∞-topos from [62]. We give quick arguments for some of the easier statements if possible, but mostly we just refer to loc. cit. for proofs.

2.1.1. General conventions and notation. We generally follow the conventions and notation from [62]. For the convenience of the reader, we will briefly recall the main setup.

Throughout this paper we freely make use of the language of higher category theory. We will generally follow a model-independent approach to higher categories. This means that as a general rule, all statements and constructions that are considered herein will be invariant under equivalences in the ambient ∞-category, and we will always be working within such an ambient ∞-category.

We denote by Δ the simplex category, i.e. the category of non-empty totally ordered finite sets with order-preserving maps. Every natural number $n \in \mathbb{N}$ can be considered as an object in Δ by identifying n with the totally ordered set $\langle n \rangle = \{0, \dots n\}$. For $i = 0, \dots, n$ we denote by $\delta^i\colon \langle n-1 \rangle \to \langle n \rangle$ the unique injective map in Δ whose image does not contain i. Dually, for $i = 0, \dots n$ we denote by $\sigma^i\colon \langle n+1 \rangle \to \langle n \rangle$ the unique surjective map in Δ such that the preimage of i contains two elements. Furthermore, if $S \subset n$ is an arbitrary subset of k elements, we denote by $\delta^S\colon \langle k \rangle \to \langle n \rangle$ the unique injective map in Δ whose image is precisely S. In the case that S is an interval, we will denote by $\sigma^S\colon \langle n \rangle \to \langle n-k \rangle$ the unique surjective map that sends S to a single object. If $\mathcal{C}$ is an ∞-category, we refer to a functor $C\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ as a simplicial object in $\mathcal{C}$. We write C_n for the image of $n \in \Delta$ under this functor, and we write d_i, s_i, d_S and s_S for the image of the maps δ^i, σ^i, δ^S and σ^S under this functor. Dually, a functor $C^\bullet\colon \Delta \to \mathcal{C}$ is referred to as a cosimplicial object in $\mathcal{C}$. In this case we denote the image of δ^i, σ^i, δ^S and σ^S by d^i, s^i, d^S and σ^S.

The 1-category Δ embeds fully faithfully into the ∞-category of ∞-categories by means of identifying posets with 0-categories and order-preserving maps between posets with functors between such 0-categories. We denote by Δ^n the image of $\langle n \rangle \in \Delta$ under this embedding.

2.1.2. Set-theoretical foundations. Once and for all we will fix three Grothendieck universes $\mathbf{U} \in \mathbf{V} \in \mathbf{W}$ that contain the first infinite ordinal ω. A set is *small* if it is contained in $\mathbf{U}$, *large* if it is contained in $\mathbf{V}$ and *very large* if it is contained in $\mathbf{W}$. An analogous naming convention will be adopted for ∞-categories and ∞-groupoids. The large ∞-category of small ∞-groupoids is denoted by $\mathcal{S}$, and the very large ∞-category of large ∞-groupoids by $\widehat{\mathcal{S}}$. The (even larger) ∞-category of very large ∞-groupoids will be denoted by $\widehat{\widehat{\mathcal{S}}}$. Similarly, we denote the large ∞-category of small ∞-categories by Cat_∞ and the very large ∞-category of large ∞-categories by $\widehat{\mathrm{Cat}}_\infty$. We shall not need the ∞-category of very large ∞-categories in this thesis.

2.1.3. ∞-topoi. For ∞-topoi $\mathcal{A}$ and $\mathcal{B}$, a *geometric morphism* is a functor $f_*\colon \mathcal{B} \to \mathcal{A}$ that admits a left exact left adjoint, and an *algebraic morphism* is a left exact functor $f^*\colon \mathcal{A} \to \mathcal{B}$ that admits a right adjoint. The *global sections* functor is the unique geometric morphism $\Gamma_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{S}$ into the ∞-topos of ∞-groupoids $\mathcal{S}$. Dually, the unique algebraic morphism originating from $\mathcal{S}$ is denoted by $\mathrm{const}_{\mathcal{B}}\colon \mathcal{S} \to \mathcal{B}$ and referred to as the *constant sheaf* functor. We will often omit the subscripts if they can be inferred from the context. For an object $A \in \mathcal{B}$, we denote the induced étale geometric morphism by $(\pi_A)_*\colon \mathcal{B}_{/A} \to \mathcal{B}$.

2.1.4. Universe enlargement. If $\mathcal{B}$ is an ∞-topos, we define its *universe enlargement* $\widehat{\mathcal{B}} = \mathrm{Sh}_{\widehat{\mathcal{S}}}(\mathcal{B})$, where the right-hand side denotes the ∞-category of presheaves $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ which preserve small limits; this is an ∞-topos relative to the larger universe $\mathbf{V}$ [57, Remark 6.3.5.17]. Moreover, the Yoneda embedding gives rise to an inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ that commutes with small limits and colimits and with the internal hom [62, Proposition 2.4.4]. The operation of enlarging universes is transitive: when defining the ∞-topos $\widehat{\widehat{\mathcal{B}}}$ relative to $\mathbf{W}$ as the universe enlargement of $\widehat{\mathcal{B}}$ with respect to the inclusion $\mathbf{V} \in \mathbf{W}$, the ∞-category $\widehat{\widehat{\mathcal{B}}}$ is equivalent to the universe enlargement of $\mathcal{B}$ with respect to $\mathbf{U} \in \mathbf{W}$ [62, Remark 2.4.1].

2.1.5. Factorisation systems. Recall from [57, § 5.2.8], that if $f\colon a \to b$ and $g\colon x \to y$ are morphisms in an ∞-category $\mathcal{C}$, we say that f is left orthogonal to g (or g is right orthogonal to f) if the commutative square

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(b,x) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{map}_{\mathcal{C}}(b,y) \arrow[d, "f^*"] \\ \mathrm{map}_{\mathcal{C}}(a,x) \arrow[r, "g_*"'] & \mathrm{map}_{\mathcal{C}}(a,y) \end{tikzcd}$$

is cartesian. If $\mathcal{C}$ is a presentable ∞-category and if S is a small set of maps in $\mathcal{C}$, there is a unique factorisation system $(\mathcal{L}, \mathcal{R})$ in which a map is contained in $\mathcal{R}$ if and only if it is *right orthogonal* to the maps in S, and where $\mathcal{L}$ is dually defined as the set of maps that are left orthogonal to the maps in $\mathcal{R}$, see [57, Proposition 5.5.5.7]. We refer to $\mathcal{L}$ as the *saturation* of S; this is the smallest set of maps containing S that is stable under pushouts, contains all equivalences and is stable under small colimits in $\mathrm{Fun}(\Delta^1, \mathcal{C})$, see e.g. [62, Proposition 2.5.6]. An object $c \in \mathcal{C}$ is said to be *S-local* if the unique morphism $c \to 1$ is contained in $\mathcal{R}$.

If $\mathcal{C}$ is cartesian closed with internal hom $[-,-]$ and has pullbacks, we say that $f\colon a \to b$ is *internally left orthogonal* to $g\colon x \to y$ (or g is *internally right orthogonal* to f) if the commutative square

$$\begin{tikzcd} {[b,x]} \arrow[r, "g_*"] \arrow[d, "f^*"'] & {[b,y]} \arrow[d, "f^*"] \\ {[a,x]} \arrow[r, "g_*"'] & {[a,y]} \end{tikzcd}$$

in $\mathcal{C}$ is cartesian. If $\mathcal{C}$ is again presentable and S a small set of morphisms in $\mathcal{C}$, we can analogously construct a factorisation system $(\mathcal{L}', \mathcal{R}')$ in which $\mathcal{R}'$ is the set of maps in $\mathcal{B}$ that are internally right orthogonal to the maps in S [5]. Explicitly, a map is contained in $\mathcal{R}'$ if and only if it is right orthogonal to maps of the form $s \times \mathrm{id}_c$ for any $s \in S$ and any $c \in \mathcal{C}$. The left complement $\mathcal{L}'$ is comprised of the maps in $\mathcal{C}$ that are left orthogonal to the maps in $\mathcal{R}'$ and is referred to as the *internal* saturation of S.

Equivalently, $\mathcal{L}'$ is the saturation of the set of maps $s \times \mathrm{id}_c$ for $s \in S$ and $c \in \mathcal{C}$. An object $c \in \mathcal{C}$ is said to be *internally S-local* if the unique morphism $c \to 1$ is contained in $\mathcal{R}'$.

Given any factorisation system $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ in which $\mathcal{L}$ is the saturation of a small set of maps in $\mathcal{C}$, the inclusion $\mathcal{R} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ admits a left adjoint that carries a map $f \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ to the map $r \in \mathcal{R}$ that arises from the unique factorisation $f \simeq rl$ into maps $l \in \mathcal{L}$ and $r \in \mathcal{R}$ [57, Lemma 5.2.8.19]. By taking fibres over an object $c \in \mathcal{C}$, one furthermore obtains a Bousfield localisation $\mathcal{C}_{/c} \leftrightarrows \mathcal{R}_{/c}$ such that if $f\colon d \to c$ is an object in $\mathcal{C}_{/c}$ and if $f \simeq rl$ is its unique factorisation into maps $l \in \mathcal{L}$ and $r \in \mathcal{R}$, the adjunction unit is given by l.

2.1.6. Simplicial objects, $\mathcal{B}$-categories and $\mathcal{B}$-groupoids. We will now give the definition of a $\mathcal{B}$-category, our main subject of study. However we at first introduce some notation:

Notation 2.1.6.1. If $\mathcal{B}$ is an arbitrary ∞-topos, we denote by $\mathcal{B}_\Delta = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{B})$ the ∞-topos of simplicial objects in $\mathcal{B}$. Note that the adjunction $(\mathrm{const} \dashv \Gamma)\colon \mathcal{S} \leftrightarrows \mathcal{B}$ yields via postcomposition an induced adjunction $(\mathrm{const} \dashv \Gamma)\colon \mathcal{S}_\Delta \leftrightarrows \mathcal{B}_\Delta$ on the level of simplicial objects. We will often implicitly identify a simplicial ∞-groupoid K with its image in $\mathcal{B}_\Delta$ along $\mathrm{const}_\mathcal{B}$.

For every $n \geq 1$, we denote by $I^n = \Delta^1 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^0} \Delta^1 \hookrightarrow \Delta^n$ the n-spine, viewed as a simplicial ∞-groupoid. Furthermore, we denote by $E^1 = (\Delta^0 \sqcup \Delta^0) \sqcup_{(\Delta^1 \sqcup \Delta^1)} \Delta^3$ the walking equivalence.

Definition 2.1.6.2 ([62, Definitions 3.1.5 and 3.2.1]). A $\mathcal{B}$-*category* is a simplicial object $\mathsf{C} \in \mathcal{B}_\Delta$ that is internally local with respect to $I^2 \hookrightarrow \Delta^2$ (Segal conditions) and $E^1 \to \Delta^0$ (univalence). We denote by $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ the full subcategory spanned by the $\mathcal{B}$-categories. A $\mathcal{B}$-*groupoid* is a simplicial object $G \in \mathcal{B}_\Delta$ which is internally local with respect to $\Delta^1 \to \Delta^0$. We denote by $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ the full subcategory spanned by the $\mathcal{B}$-groupoids.

Remark 2.1.6.3 ([62, Proposition 3.2.7]). A combinatorial argument, see [62, Lemma 3.2.5 and 3.2.6], shows that a simplicial object C is a $\mathcal{B}$-category if and only if it satisfies the following two conditions:

(1) (*Segal conditions*) For all $n \geq 2$ the map

$$\mathsf{C}_n \xrightarrow{(d_{\{0,1\}}, d_{\{1,2\}}, \ldots, d_{\{n-1,n\}})} \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \times_{\mathsf{C}_0} \ldots \times_{\mathsf{C}_0} \mathsf{C}_1$$

in $\mathcal{B}$ is an equivalence.

(2) (*Univalence*) The commutative square

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, "\sigma_0"] \arrow[d, "(\mathrm{id},\mathrm{id})"'] & \mathsf{C}_3 \arrow[d, "d_{\{0,2\}} \times d_{\{1,3\}}"] \\ \mathsf{C}_0 \times \mathsf{C}_0 \arrow[r, "\sigma_0"'] & \mathsf{C}_1 \times \mathsf{C}_1 \end{tikzcd}$$

in $\mathcal{B}$ is a pullback.

Remark 2.1.6.4. There are several non-equivalent definitions of the walking equivalence. For example, Charles Rezk [77, § 6] defines the walking equivalence as the simplicial set J that arises as the nerve of the category with two objects and a unique isomorphism between them. Our model E^1 (that we adopted from [55, Notation 1.1.12]), on the other hand, is comprised of a map together with *separate* left and right inverses. Nevertheless, either choice gives rise to the same notion of $\mathcal{B}$-categories: there is a natural map $E^1 \to J$ which is contained in the internal saturation of $I^2 \hookrightarrow \Delta^2$, i.e. which becomes an equivalence when imposing the Segal conditions. This can be extracted from the discussion in [77, § 6], see also [76, § 2.4].

Proposition 2.1.6.5 ([62, Proposition 3.2.9, Remark 3.2.10 and Proposition 3.2.11]). *The inclusion* $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathcal{B}_\Delta$ *preserves filtered colimits and admits a left adjoint which preserves finite products. Therefore,* $\mathrm{Cat}(\mathcal{B})$ *is presentable and an exponential ideal in* $\mathcal{B}_\Delta$*, so in particular cartesian closed.*

PROOF. The statement about preservation of filtered colimits follows because the limits involved in the explicit description of Remark 2.1.6.3 are all finite and filtered colimits commute with finite limits in $\mathcal{B}_\Delta$. Furthermore the fact that the left adjoint $\mathcal{B}_\Delta \to \mathrm{Cat}(\mathcal{B})$ preserves finite products is easily seen to be equivalent to $\mathrm{Cat}(\mathcal{B})$ being an exponential ideal in $\mathcal{B}_\Delta$. Thus we need to see that for any $K \in \mathcal{B}_\Delta$ and $\mathsf{C} \in \mathrm{Cat}(B)$, the internal hom $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta}(K, \mathsf{C})$ is a $\mathcal{B}$-category. But this is clear from Definition 2.1.6.2, because $\mathcal{B}$-categories are defined to be *internally* local with respect to $I^2 \to \Delta^2$ and $E^1 \to \Delta^0$. □

NOTATION 2.1.6.6. We will denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ the internal hom in $\mathrm{Cat}(\mathcal{B})$ and refer to it as the *functor $\mathcal{B}$-category* bifunctor.

PROPOSITION 2.1.6.7 ([62, after Corollary 3.2.12]). *A simplicial object in $\mathcal{B}$ is a $\mathcal{B}$-groupoid if and only if it is constant (i.e. contained in the essential image of the diagonal embedding $\iota\colon \mathcal{B} \hookrightarrow \mathcal{B}_\Delta$), and every $\mathcal{B}$-groupoid is a $\mathcal{B}$-category. Moreover, the resulting embedding $\mathcal{B} \simeq \mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ admits both a left adjoint $(-)^{\mathrm{gpd}}$ (the* groupoidification functor*) and a right adjoint $(-)^{\simeq}$ (the* core $\mathcal{B}$-groupoid functor*). Explicitly, if C is a $\mathcal{B}$-category, one has $\mathsf{C}^{\mathrm{gpd}} \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathsf{C}$ and $\mathsf{C}^{\simeq} \simeq \mathsf{C}_0$.*

PROOF. If a simplicial object $K \in \mathcal{B}_\Delta$ is internally local with respect to $\Delta^1 \to \Delta^0$, an easy combinatorial argument (see [62, Lemma 3.1.3]) shows that it is local with respect to all maps of the form $\Delta^n \times A \to \Delta^0 \times A$. This implies that K is constant. The converse follows because the left adjoint $\mathrm{colim}_{\Delta^{\mathrm{op}}}\colon \mathcal{B}_\Delta \to \mathcal{B}$ sends maps of the form $K \times \Delta^1 \to K \times \Delta^0$ for $K \in \mathcal{B}_\Delta$ to equivalences. Indeed, as a sifted colimit, it commutes with products and clearly inverts the map $\Delta^1 \to \Delta^0$. Since a constant simplicial object satisfies the conditions of Remark 2.1.6.3, it also follows that any $\mathcal{B}$-groupoid is a $\mathcal{B}$-category. The remaining part of the Proposition is clear. □

DEFINITION 2.1.6.8. If C is a $\mathcal{B}$-category, we denote by C^{op} the simplicial object that is obtained by precomposing $\mathsf{C}\colon \Delta^{\mathrm{op}} \to \mathcal{B}$ with the involution $(-)^{\mathrm{op}}\colon \Delta \simeq \Delta$ that carries $\langle n \rangle$ (viewed as a 0-category) to its opposite $\langle n \rangle^{\mathrm{op}}$. The simplicial object C^{op} is again a $\mathcal{B}$-category that we refer to as the *opposite $\mathcal{B}$-category* of C.

REMARK 2.1.6.9. The equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Cat}(\mathcal{B})$ from Definition 2.1.6.8 restricts to the identity on $\mathrm{Grpd}(\mathcal{B})$. In fact, this follows immediately from the observation that $\mathcal{B}$-groupoids are constant simplicial objects (see Proposition 2.1.6.7).

REMARK 2.1.6.10 ([62, § 3.3]). If $f_*\colon \mathcal{B} \to \mathcal{A}$ is a geometric morphism and if f^* is the associated algebraic morphism, postcomposition induces an adjunction $f^* \dashv f_*\colon \mathrm{Cat}(\mathcal{A}) \leftrightarrows \mathrm{Cat}(\mathcal{B})$. In particular, one obtains an adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}\colon \mathrm{Cat}_\infty \leftrightarrows \mathrm{Cat}(\mathcal{B})$. We will often implicitly identify an ∞-category $\mathcal{C}$ with the associated *constant $\mathcal{B}$-category* $\mathrm{const}_{\mathcal{B}}(\mathcal{C}) \in \mathrm{Cat}(\mathcal{B})$. Furthermore, if the geometric morphism f_* is *étale*, the further left adjoint $f_!$ of f^* also induces a functor $f_!\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ that identifies $\mathrm{Cat}(\mathcal{B})$ with $\mathrm{Cat}(\mathcal{A})_{/f_!1}$.

CONSTRUCTION 2.1.6.11. By making use of the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}\colon \mathrm{Cat}_\infty \leftrightarrows \mathrm{Cat}(\mathcal{B})$ and the internal hom $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ as well as the product $- \times -$ in $\mathrm{Cat}(\mathcal{B})$, one can define bifunctors

(Functor ∞-category) $\quad \mathrm{Fun}_{\mathcal{B}}(-,-) = \Gamma_{\mathcal{B}} \circ \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$

(Powering) $\quad (-)^{(-)} = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{const}_{\mathcal{B}}(-), -)\colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$

(Tensoring) $\quad - \otimes - = \mathrm{const}_{\mathcal{B}}(-) \times -\colon \mathrm{Cat}_\infty \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$

which fit into equivalences

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(- \otimes -, -) \simeq \mathrm{map}_{\mathrm{Cat}_\infty}(-, \mathrm{Fun}_{\mathcal{B}}(-,-)) \simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-, (-)^{(-)})$$

(see [62, § 3.4]). In particular, we have $\mathrm{Fun}_{\mathcal{B}}(-,-)^{\simeq} \simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-,-)$, so that $\mathrm{Fun}_{\mathcal{B}}(-,-)$ gives rise to a Cat_∞-enrichment of $\mathrm{Cat}(\mathcal{B})$ and therefore an $(\infty, 2)$-categorical enhancement of $\mathrm{Cat}(\mathcal{B})$ [62, Remark 3.4.3].

Remark 2.1.6.12 ([62, Proposition 3.1.2]). There is an equivalence of functors $\mathrm{id}_{\mathrm{Cat}(\mathcal{B})} \simeq ((-)^{\Delta^\bullet})^{\simeq}$. In other words, for any $\mathcal{B}$-category C and any integer $n \geq 0$ one may canonically identify $\mathsf{C}_n \simeq (\mathsf{C}^{\Delta^n})_0$.

2.1.6.13. We conclude this section with a remark on *large* $\mathcal{B}$-categories: observe that postcomposition with the universe enlargement $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ from § 2.1.4 determines an inclusion $\mathrm{Cat}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})$ that is natural in $\mathcal{B}$ both with respect to geometric and algebraic morphisms of ∞-topoi [62, § 3.3]. Furthermore, the inclusion commutes with small limits and the internal hom [62, Proposition 3.4.1] and therefore also the tensoring, powering and functor ∞-category bifunctors [62, Corollary 3.4.2]. We refer to the objects in $\mathrm{Cat}(\widehat{\mathcal{B}})$ as *large* $\mathcal{B}$-categories (or as $\widehat{\mathcal{B}}$-categories) and to the objects in $\mathrm{Cat}(\mathcal{B})$ as *small* $\mathcal{B}$-categories. If not specified otherwise, every $\mathcal{B}$-category is small. Note, however, that by replacing the universe $\mathbf{U}$ with the larger universe $\mathbf{V}$ (i.e. by working internally to $\widehat{\mathcal{B}}$), every statement about $\mathcal{B}$-categories carries over to one about large $\mathcal{B}$-categories as well. Also, we will often omit specifying the relative size of a $\mathcal{B}$-category if it is evident from the context, and we will continue writing $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ for the internal hom even if C and D are large.

2.1.7. $\mathcal{B}$-categories as sheaves of ∞-categories. One may equivalently regard a $\mathcal{B}$-category as a *sheaf* of ∞-categories on $\mathcal{B}$, by which we mean a functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that preserves small limits. Let us first introduce the following notation:

Notation 2.1.7.1. If $\mathcal{C}$ is an ∞-category with all small colimits and $\mathcal{D}$ an ∞-category with all small limits we write $\mathrm{Sh}_{\mathcal{D}}(\mathcal{C}) = \mathrm{Fun}^{\lim}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.

Proposition 2.1.7.2 ([62, Proposition 3.5.1 and Remark 3.5.6]). *Let $\iota\colon \mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ denote the diagonal embedding. There is a canonical equivalence of $\mathcal{B}$-categories*

$$\Phi\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}).$$

that restricts to the equivalence $\mathcal{B} \simeq \mathrm{Sh}_{\mathcal{S}}(\mathcal{B})$ along the diagonal $\iota\colon \mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$.

Proof. If $\mathcal{C}$ is an arbitrary ∞-category with finite limits, we write $\mathrm{Fun}^{\mathrm{CSS}}(\Delta^{\mathrm{op}}, \mathcal{C})$ for the full subcategory spanned the functors that satisfy the conditions appearing in Remark 2.1.6.3. Recall from [51, 40] that the nerve functor

$$\mathcal{N}\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}), \quad \mathcal{C} \mapsto \mathrm{map}_{\mathrm{Cat}_\infty}(\Delta^\bullet, \mathcal{C})$$

restricts to an equivalence $\mathrm{Cat}_\infty \simeq \mathrm{Fun}^{\mathrm{CSS}}(\Delta^{\mathrm{op}}, \mathcal{S})$. We therefore obtain a chain of natural equivalences

$$\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Fun}^{\mathrm{CSS}}(\Delta^{\mathrm{op}}, \mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{CSS}}(\Delta^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$$

because limits in functor ∞-categories are computed pointwise. The second claim is clear from the construction of the above equivalence. □

2.1.7.3. Explicitly the functor Φ sends a $\mathcal{B}$-category C to the sheaf of complete Segal-spaces given by $\mathrm{map}_{\mathcal{B}}(-, \mathsf{C}_\bullet)$. Using Remark 2.1.6.12 we obtain a chain of natural equivalences

$$\mathrm{map}_{\mathcal{B}}(-, \mathsf{C}_\bullet) \simeq \mathrm{map}_{\mathcal{B}_\Delta}(\iota(-), \mathsf{C}^{\Delta^\bullet}) \simeq \mathrm{map}_{\mathrm{Cat}_\infty}(\Delta^\bullet, \mathrm{Fun}_{\mathcal{B}}(\iota(-), \mathsf{C})),$$

where $\iota\colon \mathcal{B} \to \mathcal{B}_\Delta$ is the diagonal functor. It follows that we may identify $\Phi(\mathsf{C})$ with the sheaf $\mathrm{Fun}_{\mathcal{B}}(\iota(-), \mathsf{C})$. Hereafter, we will often implicitly identify a $\mathcal{B}$-category C with the associated sheaf $\Phi(A)$. That is, we usually write $\mathsf{C}(A) = \Phi(\mathsf{C})(A) \simeq \mathrm{Fun}_{\mathcal{B}}(\iota(A), \mathsf{C})$ for the ∞-category of *local sections* over $A \in \mathcal{B}$, and we write $s^*\colon \mathsf{C}(B) \to \mathsf{C}(A)$ for the restriction functor along a map $s\colon B \to A$ in $\mathcal{B}$.

Remark 2.1.7.4 (cf. [62, Remark 3.1.1]). More explicitly, the ∞-category $\mathsf{C}(A) = \mathrm{Fun}_{\mathcal{B}}(\iota(A), \mathsf{C})$ is given by the complete Segal space whose space of n-morphisms is given by the ∞-groupoid $\mathrm{map}_{\mathcal{B}}(A, \mathsf{C}_n)$. In particular, the equivalence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ from Proposition 2.1.7.2 commutes both with taking core $\mathcal{B}$-groupoids and opposite $\mathcal{B}$-categories, in the sense that we have equivalences of sheaves $\mathsf{C}^{\simeq}(-) \simeq \mathsf{C}(-)^{\simeq}$ and $\mathsf{C}^{\mathrm{op}}(-) \simeq \mathsf{C}(-)^{\mathrm{op}}$.

Remark 2.1.7.5. One may interpret Proposition 2.1.7.2 as a correspondence between *internal* and *parametrised* higher category theory. Both approaches have their specific advantages: the upshot of the internal approach is that one can often use a statement about ∞-categories and simply interpret it internally in $\mathcal{B}$ in order to obtain the corresponding statement for $\mathcal{B}$-categories. On the other hand, it is usually easier to construct a particular $\mathcal{B}$-category via its associated sheaf of ∞-categories. In fact, most examples that are of practical interest arise in this way.

Remark 2.1.7.6 ([62, § 3.5]). The equivalence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ is natural in $\mathcal{B}$: if $f_* \colon \mathcal{B} \to \mathcal{A}$ is a geometric morphism and f^* denotes its left adjoint, one obtains commutative squares

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] \arrow[d, "f_*"] & \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \arrow[d, "f_*"] \\ \mathrm{Cat}(\mathcal{A}) \arrow[r, "\simeq"] & \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{A}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] & \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \\ \mathrm{Cat}(\mathcal{A}) \arrow[r, "\simeq"] \arrow[u, "f^*"] & \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{A}). \arrow[u, "f^*"'] \end{tikzcd}$$

Explicitly, $f_* \colon \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \to \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{A})$ is given by restriction along $f^* \colon \mathcal{A} \to \mathcal{B}$. In particular, we may identify $\mathsf{C}(1) \simeq \Gamma_{\mathcal{B}}(\mathsf{C})$ for every $\mathcal{B}$-category C. Furthermore, $f^* \colon \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{A}) \to \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ is given by left Kan extension along $f^* \colon \mathcal{A} \to \mathcal{B}$. Thus, if the latter functor admits an additional left adjoint $f_!$, then $f^* \colon \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{A}) \to \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B})$ is simply given by precomposition with $f_!$.

Remark 2.1.7.7 ([62, Proposition 3.5.1]). The equivalence between $\mathcal{B}$-categories and sheaves of ∞-categories respects universe enlargement in the following sense: there is a commutative square

$$\begin{tikzcd} \mathrm{Cat}(\mathcal{B}) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Sh}_{\mathrm{Cat}_\infty}(\mathcal{B}) \arrow[d, hook] \\ \mathrm{Cat}(\widehat{\mathcal{B}}) \arrow[r, "\simeq"] & \mathrm{Sh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}) \end{tikzcd}$$

in which the lower horizontal equivalence is obtained by sending a large $\mathcal{B}$-category C to $\mathrm{Fun}_{\widehat{\mathcal{B}}}(\iota(-), \mathsf{C})$, where $\iota \colon \mathcal{B} \hookrightarrow \widehat{\mathcal{B}} \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})$ is the inclusion.

2.1.7.8. We conclude this section by noting that the sheaf-theoretic perspective on $\mathcal{B}$-categories also gives rise to a *fibrational* point of view: on account of the inclusion $\mathrm{Sh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$ and by making use of the straightening/unstraightening equivalence $\mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}) \simeq \mathrm{Cart}(\mathcal{B})$ between $\widehat{\mathrm{Cat}}_\infty$-valued presheaves on $\mathcal{B}$ and *cartesian fibrations* over $\mathcal{B}$ (see [57, § 3.2]), we obtain a full embedding $\mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{Cart}(\mathcal{B})$ which sends a (large) $\mathcal{B}$-category C to its underlying cartesian fibration $\int \mathsf{C} \to \mathcal{B}$.

2.1.8. Objects and morphisms in $\mathcal{B}$-categories. Next we introduce notions of objects and morphisms in a $\mathcal{B}$-category:

2.1.8.1. Observe that by combining Proposition 2.1.7.2 and 2.1.7.3 with the two-variable adjunctions between the bifunctors $\mathrm{Fun}_{\mathcal{B}}(-,-)$, $- \otimes -$ and $(-)^{(-)}$, one obtains equivalences

$$\mathsf{C}^{\Delta^n}(A)^{\simeq} \simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(A, \mathsf{C}^{\Delta^n}) \simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\Delta^n \otimes A, \mathsf{C}) \simeq \mathrm{map}_{\mathrm{Cat}_\infty}(\Delta^n, \mathsf{C}(A))$$

for every $A \in \mathcal{B}$, every $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$ and each $n \in \mathbb{N}$ (where we leave the diagonal embedding $\mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ implicit). Moreover, by combining Proposition 2.1.6.7 with Remark 2.1.6.12, we may furthermore compute

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(A, \mathsf{C}^{\Delta^n}) \simeq \mathrm{map}_{\mathcal{B}}(A, \mathsf{C}_n).$$

In other words, the datum of a map $A \to \mathsf{C}^{\Delta^n}$ in $\mathrm{Cat}(\mathcal{B})$ is equivalent to that of a map $\Delta^n \otimes A \to \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})$, a map $A \to \mathsf{C}_n$ in $\mathcal{B}$ as well as a functor $\Delta^n \to \mathsf{C}(A)$ of ∞-categories.

Definition 2.1.8.2. Let C be a $\mathcal{B}$-category and let $A \in \mathcal{B}$ be an object. For a given integer $n \geq 0$, an *n-morphism in* C *in context* A is a map $A \to \mathsf{C}^{\Delta^n}$ in $\mathrm{Cat}(\mathcal{B})$. If $n = 0$, we simply speak of an *object* in C in context A, and for $n = 1$ we refer to such a map as a *morphism* in C in context A. Given objects $c, d\colon A \rightrightarrows \mathsf{C}$, one defines the *mapping $\mathcal{B}_{/A}$-groupoid* $\mathrm{map}_{\mathsf{C}}(c, d)$ as the pullback

$$\begin{tikzcd} \mathrm{map}_{\mathsf{C}}(c,d) \arrow[r] \arrow[d] & \mathsf{C}_1 \arrow[d, "(d_1, d_0)"] \\ A \arrow[r, "(c,d)"] & \mathsf{C}_0 \times \mathsf{C}_0. \end{tikzcd}$$

We denote a section $f\colon A \to \mathrm{map}_{\mathsf{C}}(c, d)$ by $f\colon c \to d$.

Remark 2.1.8.3 ([62, § 3.6]). Equivalently, the mapping $\mathcal{B}_{/A}$-groupoid $\mathrm{map}_{\mathsf{C}}(c, d)$ can be defined as the pullback of $(d_1, d_0)\colon \mathsf{C}^{\Delta^1} \to \mathsf{C} \times \mathsf{C}$ along $(c, d)\colon A \to \mathsf{C} \times \mathsf{C}$.

Remark 2.1.8.4. Viewed as an $\mathcal{S}$-valued sheaf on $\mathcal{B}_{/A}$, the object $\mathrm{map}_{\mathsf{C}}(c, d)$ from Definition 2.1.8.2 is given by the assignment

$$\mathcal{B}_{/A} \ni (s\colon B \to A) \mapsto \mathrm{map}_{\mathsf{C}(B)}(s^*c, s^*d)$$

where $s^*c = cs$ and likewise for d.

2.1.8.5. More generally, if $c_0, \dots, c_n$ are objects in context A in C, one writes $\mathrm{map}_{\mathsf{C}}(c_0, \dots, c_n)$ for the pullback of $(d_n, \dots, d_0)\colon \mathsf{C}_n \to \mathsf{C}_0^{n+1}$ along the map $(c_0, \dots, c_n)\colon A \to \mathsf{C}_0^{n+1}$. Using the Segal conditions, one obtains an equivalence

$$\mathrm{map}_{\mathsf{C}}(c_0, \dots, c_n) \simeq \mathrm{map}_{\mathsf{C}}(c_0, c_1) \times_A \cdots \times_A \mathrm{map}_{\mathsf{C}}(c_{n-1}, c_n).$$

By combining this identification with the map $\mathrm{map}_{\mathsf{C}}(c_0, \dots, c_n) \to \mathrm{map}_{\mathsf{C}}(c_0, c_n)$ that is induced by the map $d_{\{0,n\}}\colon \mathsf{C}_n \to \mathsf{C}_1$, one obtains a composition map

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \times_A \cdots \times_A \mathrm{map}_{\mathsf{C}}(c_{n-1}, c_n) \to \mathrm{map}_{\mathsf{C}}(c_0, c_n).$$

Given maps $f_i\colon c_{i-1} \to c_i$ in C for $i = 1, \dots, n$, we write $f_1 \cdots f_n$ for their composition. By making use of the simplicial identities, it is straightforward to verify that composition is associative and unital, i.e. that the relations $f(gh) \simeq (fg)h$ and $f\,\mathrm{id} \simeq f \simeq \mathrm{id}\, f$ as well as their higher analogues hold whenever they make sense, see [77, Proposition 5.4] for a proof.

Remark 2.1.8.6. As a $\mathcal{B}$-category C is determined by the associated sheaf of ∞-categories on $\mathcal{B}$ but not just by the underlying ∞-category $\Gamma_{\mathcal{B}}(\mathsf{C})$ of global sections, it is crucial that we allow objects and morphisms in C to have arbitrary context $A \in \mathcal{B}$. In other words, we need to allow objects and morphisms to be only *locally* defined, where by the term *local* we allude to the point of view that the base ∞-topos $\mathcal{B}$ can be thought of as a spatial object. Alternatively, this phenomenon can be viewed as a shadow of the notion of contexts in type theory (hence the name), where they are needed to keep track of the types of the variables that occur in a formula. More precisely, when regarding the theory of $\mathcal{B}$-categories as a model of simplicial homotopy type theory [79], the type-theoretic notion of contexts exactly translates into our notion of contexts.

Remark 2.1.8.7. At first, the fact that objects and morphisms of a $\mathcal{B}$-category C have non-global context A might appear to complicate things, but in practice this is usually not the case: in fact, by making use of the adjunction $(\pi_A)_! \dashv \pi_A^*\colon \mathcal{B}_{/A} \leftrightarrows \mathcal{B}$ and by the observations made in Remark 2.1.6.10, the datum of an object $c\colon A \to \mathsf{C}$ precisely corresponds to that of an object $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$, where $\pi_A^*\mathsf{C} \in \mathrm{Cat}(\mathcal{B}_{/A})$ is the image of C along the base change functor $\pi_A^*\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}_{/A}) \simeq \mathrm{Cat}(\mathcal{B})_{/A}$. In other words, upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and C with $\pi_A^*\mathsf{C}$, object in context A are turned into objects in global context. Very often, we will make use of this correspondence in order to be able to restrict our attention to objects and morphisms in global context (see § 2.1.14 below for more details on this strategy).

REMARK 2.1.8.8. Observe that for every $\mathcal{B}$-category C there is a distinguished object $\tau\colon \mathsf{C}_0 \to \mathsf{C}$ that is determined by the counit of the adjunction $\iota \dashv (-)_0\colon \mathrm{Cat}(\mathcal{B}) \leftrightarrows \mathcal{B}$ from Proposition 2.1.6.7. We refer to τ as the *tautological* object of C. By definition, *every* object $c\colon A \to \mathsf{C}$ arises as a pullback of τ, in the sense that we have $c \simeq c^*\tau$ (where $c^*\colon \mathsf{C}(\mathsf{C}_0) \to \mathsf{C}(A)$ is the restriction functor). In that way, many questions about an arbitrary object in a $\mathcal{B}$-category can be reduced to questions about the tautological object.

We conclude this section with a discussion of *equivalences* in $\mathcal{B}$-categories. To that end, given any object $c\colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C, let us denote by $\mathrm{id}_c\colon c \to c$ the morphism that is determined by the lift $s_0 c\colon A \to \mathsf{C}_0 \to \mathsf{C}_1$ of $(c, c)\colon A \to \mathsf{C}_0 \times \mathsf{C}_0$.

DEFINITION 2.1.8.9. A morphism $f\colon c \to d$ in C is an *equivalence* if there are maps $g\colon c \to d$ and $h\colon c \to d$ (all in context A) such that $gf \simeq \mathrm{id}_c$ and $fh \simeq \mathrm{id}_d$.

As a consequence of univalence, one finds:

PROPOSITION 2.1.8.10 ([62, Corollary 3.6.3]). *A map $f\colon A \to \mathsf{C}^{\Delta^1}$ in a $\mathcal{B}$-category C is an* equivalence *if it factors through $s_0\colon \mathsf{C} \hookrightarrow \mathsf{C}^{\Delta^1}$.*

In other words, every equivalence $f\colon A \to \mathsf{C}_1$ is equivalent (in the ∞-groupoid $\mathsf{C}_1(A)$) to an identity.

2.1.9. Fully faithful functors and full subcategories. We now discuss fully faithful functors and full subcategories in the setting of $\mathcal{B}$-categories:

DEFINITION 2.1.9.1. A functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is said to be *fully faithful* if it is internally right orthogonal to the map $\Delta^0 \sqcup \Delta^0 \to \Delta^1$. Dually, a functor is *essentially surjective* if is (internally) left orthogonal to the class of fully faithful functors.

2.1.9.2. It follows formally that fully faithful functors are stable under small limits in $\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B}))$ and are preserved by the endofunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, -)$ for every $\mathcal{B}$-category C [62, Proposition 3.8.4]. Moreover, it immediately follows that functor of $\mathcal{B}$-categories is an equivalence if and only if it is fully faithful and essentially surjective [62, Proposition 3.8.3], and every functor can be uniquely factored into an essentially surjective and a fully faithful functor. In other words, the *essential image* of a functor between $\mathcal{B}$-categories is well-defined.

Fully faithful and essentially surjective functors can be characterised as follows. For the proofs we refer to [62].

PROPOSITION 2.1.9.3 ([62, Proposition 3.8.6 and 3.8.7]). *For a functor $f\colon \mathsf{C} \to \mathsf{D}$ of $\mathcal{B}$-categories, the following are equivalent:*

(1) The functor f is fully faithful;
(2) the square

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "f_1"] \arrow[d] & \mathsf{D}_1 \arrow[d] \\ \mathsf{C}_0 \times \mathsf{C}_0 \arrow[r, "f_0 \times f_0"] & \mathsf{D}_0 \times \mathsf{D}_0 \end{tikzcd}$$

is a pullback;

(3) for every $A \in \mathcal{B}$ and any two objects $c_0, c_1\colon A \to \mathsf{C}$ in context A, the morphism

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

that is induced by f is an equivalence in $\mathcal{B}_{/A}$;

(4) for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ of ∞-categories is fully faithful.

PROPOSITION 2.1.9.4 ([62, Corollary 3.8.12]). *A functor $f\colon \mathsf{C} \to \mathsf{D}$ is essentially surjective if and only if $f_0\colon \mathsf{C}_0 \to \mathsf{D}_0$ is a cover (i.e. an effective epimorphism) in $\mathcal{B}$.*

Fully faithful functors are in particular monomorphisms, hence the full subcategory $\mathrm{Sub}^{\mathrm{full}}(\mathsf{D}) \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ that is spanned by the fully faithful functors into D is a poset whose objects we call *full subcategories* of D.

PROPOSITION 2.1.9.5 ([62, Proposition 3.9.3]). *Taking core $\mathcal{B}$-groupoids determines an equivalence of posets* $\mathrm{Sub}^{\mathrm{full}}(\mathsf{D}) \simeq \mathrm{Sub}(\mathsf{D}_0)$ *between the poset of full subcategories of* D *and the poset of subobjects of* $\mathsf{D}_0 \in \mathcal{B}$.

2.1.9.6. In particular, Proposition 2.1.9.5 implies that specifying a full subcategory of D is equivalent to specifying a subobject of D_0. Therefore, if $(d_i \colon A_i \to \mathsf{D})_{i\in I}$ is a family of objects in D, we may define the full subcategory of D that is *spanned* by these objects as the unique full subcategory of D whose core $\mathcal{B}$-groupoid is given by the image of the induced morphism $(d_i) \colon \bigsqcup_i A_i \to \mathsf{D}$ in $\mathcal{B}$ [62, Definition 3.9.7]. Note that this is possible even if the family is large [62, Remark 3.9.8].

2.1.10. The universe for $\mathcal{B}$-groupoids. In this section we will introduce a large $\mathcal{B}$-category that we call . It will play the role in $\mathcal{B}$-category theory that the ∞-category of spaces $\mathcal{S}$ plays in usual ∞-category theory.

DEFINITION 2.1.10.1. By straightening the codomain fibration $\mathrm{Fun}(\Delta^1, \mathcal{B}) \to \mathcal{B}$, one obtains a functor $\mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ that preserves small limits since $\mathcal{B}$ is an ∞-topos [57, Proposition 6.1.3.9]. In other words, $\mathcal{B}_{/-}$ is a sheaf of (large) ∞-categories and therefore (by Remark 2.1.7.7) determined by a large $\mathcal{B}$-category $\Omega_{\mathcal{B}}$ that we refer to as the *universe for $\mathcal{B}$-groupoids* [62, § 3.7].

2.1.10.2. We will often omit the subscript if it is clear from the context. By definition, we have equivalences $\Omega(A) \simeq \mathcal{B}_{/A} \simeq \mathrm{Grpd}(\mathcal{B}_{/A})$. In other words, the objects in Ω in context A are precisely given by the $\mathcal{B}_{/A}$-groupoids, an observation which justifies its name.

Moreover, we have:

PROPOSITION 2.1.10.3 ([62, Proposition 3.7.3]). *For any two objects g, h in Ω in context $A \in \mathcal{B}$ that correspond to $\mathcal{B}_{/A}$-groupoids $\mathsf{G}, \mathsf{H} \in \mathrm{Grpd}(\mathcal{B}_{/A}) \simeq \mathcal{B}_{/A}$, there is an equivalence*

$$\mathrm{map}_{\Omega}(g, h) \simeq \underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\mathsf{G}, \mathsf{H})$$

in $\mathcal{B}_{/A}$, where $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(-,-)$ denotes the internal hom in $\mathcal{B}_{/A}$.

REMARK 2.1.10.4. The universe Ω is to be regarded as the $\mathcal{B}$-categorical analogue of the ∞-category $\mathcal{S}$ of ∞-groupoids. In fact, the first main result of this thesis (Theorem 3.4.1.1) implies in particular that Ω is characterised among $\mathcal{B}$-categories by the same universal property that characterises $\mathcal{S}$ among ∞-categories (namely as the free cocompletion of the point).

DEFINITION 2.1.10.5. We refer to a full subcategory of Ω as a *subuniverse*.

2.1.10.6. It follows from item (4) of Proposition 2.1.9.3 and the definition of Ω that every such subuniverse corresponds precisely to a *local class* of morphisms in $\mathcal{B}$, i.e. a class S that satisfies the condition that a morphism $p \colon P \to A$ in $\mathcal{B}$ is contained in S if and only if it is *locally* contained in S, i.e. if and only if for every cover $(s_i) \colon \sqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, the maps $s_i^*(p) \colon A_i \times_A P \to A_i$ are contained in S (see [57, § 6.1.3 and Proposition 6.2.3.14]). In other words. we have:

PROPOSITION 2.1.10.7 ([62, Proposition 3.9.12]). *There is an equivalence between the partially ordered set of local classes in $\mathcal{B}$ and* $\mathrm{Sub}^{\mathrm{full}}(\Omega)$.

NOTATION 2.1.10.8. For a given local class S, we denote the associated subuniverse by Ω_S.

EXAMPLE 2.1.10.9 (see the discussion towards the end of [62, § 4.5]). Let us say that a map $p \colon P \to A$ in $\widehat{\mathcal{B}}$ is *small* if for every map $A' \to A$ in which $A' \in \mathcal{B}$, the pullback $A' \times_A P$ is contained in $\mathcal{B}$ as well. This determines a local class of morphisms in $\widehat{\mathcal{B}}$ and therefore by Proposition 2.1.10.7 a subuniverse of $\Omega_{\widehat{\mathcal{B}}} \in \mathrm{Cat}(\widehat{\widehat{\mathcal{B}}})$ which can be identified with $\Omega_{\mathcal{B}} \in \mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{Cat}(\widehat{\widehat{\mathcal{B}}})$. This exhibits $\Omega_{\mathcal{B}}$ as a full subcategory of $\Omega_{\widehat{\mathcal{B}}}$.

2.1.11. Left fibrations and the Grothendieck construction. In this subsection we will recall one of the main theorems from [62], which provides an equivalence between the $\mathcal{B}$-category of functors to the universe $\Omega_{\mathcal{B}}$ and a suitably defined category of left fibrations. This is the $\mathcal{B}$-categorical analogue of the usual straightening-unstraightening equivalence for left fibrations [57, Theorem 2.2.1.2].

Definition 2.1.11.1. A functor $p\colon \mathsf{P} \to \mathsf{C}$ between $\mathcal{B}$-categories is called a *left fibration* if it is internally right orthogonal to the map $d^1\colon \Delta^0 \hookrightarrow \Delta^1$. A functor that is contained in the internal saturation of this map is said to be *initial*.

2.1.11.2. As an immediate consequence of the definition one obtains a factorisation system between initial maps and left fibrations.

We continue with some easy characterizations of left fibrations:

Lemma 2.1.11.3. *A functor $p\colon \mathsf{P} \to \mathsf{C}$ of $\mathcal{B}$-categories is a left fibration if and only for any $A \in \mathcal{B}$ the induced functor $\mathsf{P}(A) \to \mathsf{C}(A)$ of ∞-categories is a left fibration.*

Proof. By definition, $p\colon \mathsf{P} \to \mathsf{C}$ is internally right orthogonal to d^1 if and only if the square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{P}) \arrow[r] \arrow[d, "(d^1)^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{C}) \arrow[d, "(d^1)^*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^0, \mathsf{P}) \arrow[r] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^0, \mathsf{C})
\end{tikzcd}$$

is cartesian. Since a square of sheaves of ∞-categories is cartesian if and only if it is so section-wise, the claim follows from the natural equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^n, \mathsf{D})(A) \simeq \mathrm{Fun}_{\mathcal{B}}(A, \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^n, \mathsf{D})) \simeq \mathrm{Fun}(\Delta^n, \mathrm{Fun}_{\mathcal{B}}(A, \mathsf{D})) \simeq \mathrm{Fun}(\Delta^n, \mathsf{D}(A))$$

and the analogous characterization for left fibrations between ordinary ∞-categories, see [17, Proposition 3.4.5]. □

Construction 2.1.11.4. The restriction of the codomain fibration $d_0\colon \mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B})) \to \mathrm{Cat}(\mathcal{B})$ to the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}(\mathcal{B}))$ that is spanned by the left fibrations is a cartesian fibration (as left fibrations are stable under pullback) and therefore determines via straightening a functor $\mathrm{LFib}\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$. By precomposing this functor with the product bifunctor $-\times-\colon \mathcal{B}\times\mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$ (where we leave the diagonal embedding $\mathcal{B} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ implicit), we therefore end up with a functor

$$\mathrm{LFib}(-\times-)\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}), \quad \mathsf{C} \mapsto \mathsf{LFib}_{\mathsf{C}} = \mathrm{LFib}(-\times \mathsf{C}).$$

Theorem 2.1.11.5 ([62, Theorem 4.5.1]). *For every $\mathcal{B}$-category C, the presheaf $\mathsf{LFib}_{\mathsf{C}}$ is a sheaf and therefore defines a large $\mathcal{B}$-category. Furthermore, there is an equivalence*

$$\mathsf{LFib}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$$

of large $\mathcal{B}$-categories that is natural in $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$.

The proof given in [62] is following along the lines of [17], adapting all necessary notions to the $\mathcal{B}$-categorical setting. In § 2.3.2 we will give a rather different proof, generalizing the proof for $\mathcal{B}$ being the ∞-topos of pyknotic spaces, given in [91, §3].

Remark 2.1.11.6. By means of the projection $\mathrm{pr}_0\colon A \times \mathsf{C} \to A$, every functor $p\colon \mathsf{P} \to A \times \mathsf{C}$ can be regarded as a map in $\mathrm{Cat}(\mathcal{B})_{/A} \simeq \mathrm{Cat}(\mathcal{B}_{/A})$ (cf. Remark 2.1.6.10). Now since the forgetful functor $(\pi_A)_!\colon \mathcal{B}_{/A} \to \mathcal{B}$ creates pullbacks, it follows (using Lemma 2.1.11.3) that p is a left fibration of $\mathcal{B}_{/A}$-categories if and only if it is a left fibration of $\mathcal{B}$-categories. Consequently, the functor $(\pi_A)_!$ induces an equivalence

$$\mathrm{LFib}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}) \simeq \mathrm{LFib}_{\mathcal{B}}(A \times \mathsf{C})$$

(where the subscript indicates internal to which ∞-topos we are taking left fibrations). In other words, the objects of $\mathsf{LFib}_{\mathsf{C}}$ in context A are precisely given by the left fibrations (internal to $\mathcal{B}_{/A}$) over $\pi_A^* \mathsf{C}$.

Remark 2.1.11.7. Dually, a functor $p\colon \mathsf{P} \to \mathsf{C}$ of $\mathcal{B}$-categories is a *right fibration* if it is internally right orthogonal to $d^0\colon \Delta^0 \hookrightarrow \Delta^1$, and a functor that is contained in the internal saturation of the latter map is said to be *final*. Equivalently, p is a right fibration precisely if p^{op} (see Definition 2.1.6.8) is a left fibration, and a functor j is final if and only if j^{op} is initial. Again, one obtains a factorisation system between final maps and right fibrations, and by the same construction as for left fibrations (or by simply dualising this construction in the appropriate way) one ends up with a functor

$$\mathrm{RFib}(-\times -)\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}), \quad \mathsf{C} \mapsto \mathsf{RFib}_{\mathsf{C}} = \mathrm{RFib}(\mathsf{C}\times -).$$

For every $\mathcal{B}$-category C, we have $\mathsf{RFib}_{\mathsf{C}} \simeq \mathsf{LFib}_{\mathsf{C}^{\mathrm{op}}}$, hence $\mathsf{RFib}_{\mathsf{C}}$ defines a large $\mathcal{B}$-category as well, and one furthermore obtains a natural straightening/unstraightening equivalence

$$\mathsf{RFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),$$

where $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \Omega)$ is the large $\mathcal{B}$-category of *presheaves* on C.

2.1.12. Slice $\mathcal{B}$-categories and initial objects. We now turn to the most important example of a left fibration:

Definition 2.1.12.1. For any $\mathcal{B}$-category C and any object $c\colon A \to \mathsf{C}$, one defines the *slice $\mathcal{B}$-category* $\mathsf{C}_{c/}$ via the pullback

$$\begin{tikzcd} \mathsf{C}_{c/} \arrow[r] \arrow[d, "(\pi_c)_!"] & \mathsf{C}^{\Delta^1} \arrow[d, "(d^1,d^0)"] \\ A\times \mathsf{C} \arrow[r, "c\times \mathrm{id}"] & \mathsf{C}\times\mathsf{C}. \end{tikzcd}$$

Remark 2.1.12.2 ([62, Remark 4.2.2]). In the situation of Definition 2.1.12.1, Remark 2.1.8.7 allows us to transpose $c\colon A \to \mathsf{C}$ to an object $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$. Thus, we can also define the slice $\mathcal{B}_{/A}$-category $(\pi_A^*\mathsf{C})_{\bar{c}/}$, which also comes with a projection $(\pi_{\bar{c}})_!\colon (\pi_A^*\mathsf{C})_{\bar{c}/} \to \pi_A^*\mathsf{C}$. This turns out to produce the same result, in the sense that when applying the forgetful functor $(\pi_A)_!\colon \mathrm{Cat}(\mathcal{B}_{/A}) \to \mathrm{Cat}(\mathcal{B})$ to the map $(\pi_{\bar{c}})_!\colon (\pi_A^*\mathsf{C})_{\bar{c}/} \to \pi_A^*\mathsf{C}$, we recover the map $(\pi_c)_!\colon \mathsf{C}_{c/} \to A\times\mathsf{C}$ from Definition 2.1.12.1. Thus, when regarded as a $\mathcal{B}_{/A}$-category, we may identify $\mathsf{C}_{c/}$ with $(\pi_A^*\mathsf{C})_{\bar{c}/}$.

Remark 2.1.12.3. Dually, by performing the pullback of (d^1, d^0) along $\mathrm{id}\times c\colon \mathsf{C}\times A \to \mathsf{C}\times\mathsf{C}$, one defines the slice $\mathcal{B}$-category $\mathsf{C}_{/c}$ together with its projection $(\pi_c)_!\colon \mathsf{C}_{/c} \to \mathsf{C}\times A$. Alternatively, this $\mathcal{B}$-category can be defined via the identity $\mathsf{C}_{/c} \simeq (\mathsf{C}^{\mathrm{op}}_{c/})^{\mathrm{op}}$.

Proposition 2.1.12.4 ([62, Proposition 4.2.7]). *For every object $c\colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C, the functor $(\pi_c)_!\colon \mathsf{C}_{c/} \to A\times\mathsf{C}$ is a left fibration of $\mathcal{B}$-categories.*

Proof. Combining Remark 2.1.12.2 with Remark 2.1.11.6, we may transpose c to an object $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$ and only have to check that the functor $(\pi_{\bar{c}})_!\colon (\pi_A^*\mathsf{C})_{\bar{c}/} \to \pi_A^*\mathsf{C}$ of $\mathcal{B}_{/A}$-categories is a left fibration. In other words we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ to assume that $A = 1$. In that case the claim is an easy consequence of Lemma 2.1.11.3 and the analogous statement for ordinary ∞-categories (see e.g. [17, Proposition 4.2.7]). □

Remark 2.1.12.5. By Remark 2.1.12.2, the functor $(\pi_c)_!$ in Proposition 2.1.12.4 can be regarded as a map in $\mathrm{Cat}(\mathcal{B}_{/A})$ and is as such a left fibration as well (by either applying Proposition 2.1.12.4 to the transposed object $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$ or by using Remark 2.1.11.6).

Definition 2.1.12.6. Let C be a $\mathcal{B}$-category. An object $c\colon A \to \mathsf{C}$ is said to be *initial* if the transpose map $1 \to \pi_A^*\mathsf{C}$ defines an initial functor in $\mathrm{Cat}(\mathcal{B}_{/A})$.

Remark 2.1.12.7. In the situation of Definition 2.1.12.6, one dually says that c is *final* if the transpose map $1 \to \pi_A^*\mathsf{C}$ defines a final functor in $\mathrm{Cat}(\mathcal{B}_{/A})$.

Remark 2.1.12.8 ([62, Remark 4.3.7]). For every object $A \in \mathcal{B}$, the forgetful functor

$$(\pi_A)_! \colon \operatorname{Cat}(\mathcal{B}_{/A}) \simeq \operatorname{Cat}(\mathcal{B})_{/A} \to \operatorname{Cat}(\mathcal{B})$$

creates initial maps. Therefore, if C is a $\mathcal{B}$-category, an object $c \colon A \to \mathsf{C}$ is initial if and only if the map $(c, \mathrm{id}) \colon A \to \mathsf{C} \times A$ is an initial functor in $\operatorname{Cat}(\mathcal{B})$.

2.1.12.9. Observe that if $c \colon A \to \mathsf{C}$ is an object in a $\mathcal{B}$-category C, the identity $\mathrm{id}_c \colon A \to \mathsf{C}^{\Delta^1}$ takes values in $\mathsf{C}_{c/}$. We therefore obtain a section $\mathrm{id}_c \colon A \to \mathsf{C}_{c/}$ of the structure map $\mathsf{C}_{c/} \to A$ (which coincides with the image of $\mathrm{id}_{\bar{c}} \colon 1_{\mathcal{B}_{/A}} \to (\pi_A^* \mathsf{C})_{\bar{c}/}$ along the forgetful functor $(\pi_A)_!$, see Remark 2.1.12.2).

Proposition 2.1.12.10 ([62, Proposition 4.3.9 and Remark 4.3.10]). *For any $\mathcal{B}$-category and any object $c \colon A \to \mathsf{C}$, the section $\mathrm{id}_c \colon A \to \mathsf{C}_{c/}$ is initial as a map in $\operatorname{Cat}(\mathcal{B}_{/A})$ and therefore defines an initial object of $\mathsf{C}_{c/}$.*

Corollary 2.1.12.11 ([62, Corollary 4.3.19]). *Let C be a $\mathcal{B}$-category and let $c \colon A \to \mathsf{C}$ be an object in C. The factorisation of c into an initial map and a left fibration is given by the composition $\mathrm{pr}_1 (\pi_c)_! \mathrm{id}_c \colon A \to (\mathsf{C})_{c/} \to \mathsf{C}$ where $\mathrm{pr}_1 \colon A \times \mathsf{C} \to \mathsf{C}$ is the projection.*

Proposition 2.1.12.12 ([62, Proposition 4.3.20]). *Let C be a $\mathcal{B}$-category. For any object $c \colon A \to \mathsf{C}$, the following are equivalent:*

(1) c is an initial object;
(2) the projection $(\pi_c)_! \colon \mathsf{C}_{c/} \to A \times \mathsf{C}$ is an equivalence;
(3) for any object $d \colon B \to \mathsf{C}$ the map $\mathrm{map}_{\mathsf{C}}(\mathrm{pr}_0^* c, \mathrm{pr}_1^* d) \to A \times B$ is an equivalence in $\mathcal{B}$.

Corollary 2.1.12.13 ([62, Corollary 4.3.21]). *Let C be a $\mathcal{B}$-category and let c and d be objects in C in context $A \in \mathcal{B}$ such that c is initial. Then there is a unique map $c \to d$ in C in context A that is an equivalence if and only if d is initial as well.*

Remark 2.1.12.14. Let us briefly conclude with a remark about the functoriality of the slice construction. Let C be a $\mathcal{B}$-category and $f \colon c \to d$ a morphism in context $A \in \mathcal{B}$. We have an evident commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_c"'] & \mathsf{C}_{/d} \arrow[d] \\ \mathsf{C}_{/c} \arrow[r] & A \times \mathsf{C} \end{tikzcd}$$

and since id_c is final, there is a unique functor $f_! \colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ making the above square commute. Dually we also get an induced functor $f^* \colon \mathsf{C}_{d/} \to \mathsf{C}_{c/}$.

2.1.13. Yoneda's lemma. The theory of left fibrations can be used to derive a version of Yoneda's lemma for $\mathcal{B}$-categories. First, we need a functorial version of the mapping $\mathcal{B}$-groupoid construction. To that end, let us denote by $- \star - \colon \Delta \times \Delta \to \Delta$ the ordinal sum bifunctor. We may now define:

Definition 2.1.13.1 ([62, Definition 4.2.4]). Let $\epsilon \colon \Delta \to \Delta$ denote the functor $\langle n \rangle \mapsto \langle n \rangle^{\mathrm{op}} \star \langle n \rangle$. For any $\mathcal{B}$-category C, we define the *twisted arrow $\mathcal{B}$-category* $\mathsf{Tw}(\mathsf{C})$ to be the simplicial object given by the composition

$$\Delta^{\mathrm{op}} \xrightarrow{\epsilon^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{\mathsf{C}} \mathcal{B}.$$

This defines a functor $\mathsf{Tw} \colon \operatorname{Cat}(\mathcal{B}) \to \mathcal{B}_{\Delta}$.

Note that the functor ϵ in Definition 2.1.13.1 comes along with two canonical natural transformations

$$(-)^{\mathrm{op}} \to \epsilon \leftarrow \mathrm{id}_{\Delta}$$

which induces a map of simplicial objects

$$\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$$

that is natural in C.

Combining the natural equivalences of 2.1.8.1 with Lemma 2.1.11.3, we get:

PROPOSITION 2.1.13.2 ([62, Proposition 4.2.5]). *For every $\mathcal{B}$-category C, the simplicial object $\mathsf{Tw}(\mathsf{C})$ is a $\mathcal{B}$-category, and the map $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ is a left fibration.*

By applying the straightening/unstraightening equivalence from Theorem 2.1.11.5 to the left fibration $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$, one now ends up with a bifunctor

$$\mathrm{map}_{\mathsf{C}} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \Omega$$

that sends a pair of objects $(c, d) \colon A \to \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ to the object $\mathrm{map}_{\mathsf{C}}(c, d) \in \mathcal{B}_{/A}$ from Definition 2.1.8.2. Upon transposing this bifunctor across the adjunction $\mathsf{C}^{\mathrm{op}} \times - \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, -)$, one obtains the *Yoneda embedding*

$$h_{\mathsf{C}} \colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).$$

THEOREM 2.1.13.3 ([62, Theorem 4.7.8]). *For any $\mathcal{B}$-category C, there is a commutative diagram*

$$\begin{tikzcd} \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "h \times \mathrm{id}"] \arrow[dr, "\mathrm{ev}"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-,-)"] \\ & \Omega \end{tikzcd}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$ *(where* ev *is the evaluation map).*

COROLLARY 2.1.13.4 ([62, Corollary 4.7.16]). *For every $\mathcal{B}$-category C, the Yoneda embedding h_{C} is fully faithful.*

REMARK 2.1.13.5 ([62, Proposition 4.7.20]). Explicitly, an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is contained in C if and only if the associated right fibration $p \colon \mathsf{P} \to \mathsf{C} \times A$ admits a final section $A \to \mathsf{P}$ over A (i.e. if P has a final object in global context when viewed as a $\mathcal{B}_{/A}$-category). If this is the case, one obtains an equivalence $\mathsf{C}_{/c} \simeq \mathsf{P}$ over $\mathsf{C} \times A$ where c is the image of the final section $A \to \mathsf{P}$ along the functor $\mathsf{P} \to \mathsf{C}$.

2.1.14. Context reduction techniques. As a general rule, every construction and every statement that we make in $\mathcal{B}$-category theory has to be *local* in $\mathcal{B}$ and has to be *invariant under étale transposition*, in the following sense:

(locality) For every $A \in \mathcal{B}$, the base change functor $\pi_A^* \colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}_{/A})$ preserves all of the structure that we use when reasoning about $\mathcal{B}$- (resp. $\mathcal{B}_{/A}$-)categories. Furthermore, for every cover (i.e. effective epimorphism) $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ and every object $c \colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C, a proposition is true for c if and only if it is true for each of the pullbacks $s_i^*(c) \colon A_i \to \mathsf{C}$.

(étale transposition invariance) For every object $c \colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C, a proposition holds for c if and only if the same proposition, interpreted internally in $\mathcal{B}_{/A}$, is true for the transposed object $\bar{c} \colon 1_{\mathcal{B}_{/A}} \to \pi_A^* \mathsf{C}$ (see Remark 2.1.8.7).

REMARK 2.1.14.1. More concretely, the locality rule asserts that

(1) π_A^* preserves limits and colimits;
(2) there is an equivalence $\pi_A^* \mathrm{const}_{\mathcal{B}} \simeq \mathrm{const}_{\mathcal{B}_{/A}}$;
(3) π_A^* commutes with the internal hom $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, -)$ [62, Lemma 4.2.3];
(4) π_A^* carries the universe $\Omega_{\mathcal{B}}$ to the universe $\Omega_{\mathcal{B}_{/A}}$ [62, Remark 3.7.2].

From these preservation properties, one can now infer that virtually all constructions that we carry out in $\mathrm{Cat}(\mathcal{B})$ are preserved by π_A^*, see Example 2.1.14.7 below for a few specific instances.

REMARK 2.1.14.2. In the locality rule, we need not assume that a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ is *small*. In fact, since $\mathcal{B}$ is presentable and therefore admits a small full subcategory $\mathcal{G} \subset \mathcal{B}$ that is *dense* in $\mathcal{B}$ (i.e.

which has the property that every $A \in \mathcal{B}$ is the colimit of the diagram $\mathcal{G}_{/A} \to \mathcal{B}$), every large cover can be refined by a small one.

REMARK 2.1.14.3. Very often, we simply impose invariance under étale transposition by *defining* a property of $c\colon A \to \mathsf{C}$ as a property of its transpose $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$ (see for example Definition 2.1.12.6).

REMARK 2.1.14.4. Locality and invariance under étale transposition imply that the context of an object is largely irrelevant: if we wish to study the properties of an object $c\colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C, we may simply pass to the slice ∞-topos $\mathcal{B}_{/A}$, replace C by $\pi_A^*\mathsf{C}$ and c by its transpose $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{C}$ and can thus assume that c has had global context to begin with. Note that by locality, $\pi_A^*\mathsf{C}$ arises from the very same constructions (internally in $\mathcal{B}_{/A}$) that are used to define C (internally in $\mathcal{B}$), hence every statement about the objects of C also makes sense as a statement about the objects of $\pi_A^*\mathsf{C}$. A typical example of how this procedure is used in practice is the proof of Proposition 2.1.12.4.

REMARK 2.1.14.5. If C is a $\mathcal{B}$-category and if $P(c)$ is a proposition about an object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$, then locality implies that there is a full subcategory $\mathsf{P} \hookrightarrow \mathsf{C}$ that classifies P, in the sense that an object $c\colon A \to \mathsf{C}$ is contained in P if and only if $P(c)$ is true. In fact, we may define P as the full subcategory that is spanned by the objects $c\colon A \to \mathsf{C}$ in arbitrary context A for which $P(c)$ holds. Explicitly, P is the unique full subcategory of C for which $\mathsf{P}_0 \hookrightarrow \mathsf{C}_0$ is the image of the map

$$\bigsqcup_{\substack{c\colon A \to \mathsf{C} \\ P(c) \text{ holds}}} A \to \mathsf{C}_0$$

(cf. Proposition 2.1.9.5). This means that for the tautological object $\tau\colon \mathsf{P}_0 \to \mathsf{P}$ (see Remark 2.1.8.8) there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow \mathsf{P}_0$ such that $P(s_i^*\tau)$ holds for each i. Since every object of P is a pullback of τ and since covers are stable under base change in $\mathcal{B}$, this implies that for every object $c\colon A \to \mathsf{P}$ there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $P(s_i^*c)$ holds. Using the locality rule, we thus deduce that $P(c)$ must be true. Consequently, an object $c\colon A \to \mathsf{C}$ is contained in P if and only if $P(c)$ holds, as claimed.

REMARK 2.1.14.6. By combining Remarks 2.1.14.4 and 2.1.14.5, if $P(c)$ is a proposition about an object $c\colon A \to \mathsf{C}$ in a $\mathcal{B}$-category C and if $\mathsf{P} \hookrightarrow \mathsf{C}$ is the associated classifying full subcategory, then $\pi_A^*\mathsf{P} \hookrightarrow \pi_A^*\mathsf{C}$ classifies the proposition P interpreted internally in $\mathcal{B}_{/A}$. In fact, $\pi_A^*\mathsf{P}$ is the full subcategory of $\pi_A^*\mathsf{C}$ that is spanned by those objects $\bar{c}\colon B \to \pi_A^*\mathsf{C}$ in context $B \in \mathcal{B}_{/A}$ for which the transpose $c\colon B \to \mathsf{C}$ satisfies $P(c)$ (interpreted internally in $\mathcal{B}$), which by invariance under étale transposition is equivalent to $\bar{c}$ satisfying $P(\bar{c})$ (interpreted internally in $\mathcal{B}_{/A}$).

EXAMPLE 2.1.14.7. Suppose that C is a $\mathcal{B}$-category. Then locality asserts that for every $A \in \mathcal{B}$, one obtains an equivalence $\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$ (cf. the list in Remark 2.1.14.1). In light of this equivalence, one can furthermore identify $\pi_A^*(h_{\mathsf{C}})$ with $h_{\pi_A^*\mathsf{C}}$ [62, Lemma 4.7.14] (where h_{C} is the Yoneda embedding). Hence, an object $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is representable if and only if its transpose $\bar{F}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\pi_A^*\mathsf{C})$ is representable, so that this property is indeed invariant under étale transposition. It also satisfies the second part of the locality principle, which can be seen as follows: given a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, the presheaf F being representable precisely means that the map $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ factors through the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, so clearly F being representable implies that $s_i^*(F) = Fs_i$ is representable. Conversely, if each $s_i^*(F)$ is representable, one can form the lifting problem

$$\begin{tikzcd}
\bigsqcup_i A_i \arrow[r] \arrow[d, two heads, "(s_i)"] & \mathsf{C} \arrow[d, hook, "h"] \\
A \arrow[r, "F"] \arrow[ur, dashed] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

which admits a unique solution (since covers and monomorphisms form a factorisation system in $\widehat{\mathcal{B}}$), hence the result follows.

2.2. Subcategories and Localisations

Forming subcategories or localizing at a class of morphisms are two of the most fundamental tools in (higher) category theory. In this section we will study the corresponding notions in the world of $\mathcal{B}$-categories. We will begin by proving a number of preliminary results about monomorphisms of $\mathcal{B}$-categories in § 2.2.1. In § 2.2.2 we then study subcategories of $\mathcal{B}$-categories. Our main result here is that subcategories of C uniquely correspond to subobjects of C_1 that are closed under equivalences and composition. We conclude by studying localisations of $\mathcal{B}$-categories in § 2.2.3.

2.2.1. Monomorphisms of $\mathcal{B}$-categories. Recall that a *monomorphism* in $\mathrm{Cat}(\mathcal{B})$ (i.e. a (-1)-truncated map) is a functor that is internally left orthogonal to the map $\Delta^0 \sqcup \Delta^0 \to \Delta^0$. In other words, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is a monomorphism if and only if the square

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "(\mathrm{id}_{\mathsf{C}}, \mathrm{id}_{\mathsf{C}})"] & \mathsf{D} \arrow[d, "(\mathrm{id}_{\mathsf{D}}, \mathrm{id}_{\mathsf{D}})"] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "f \times f"] & \mathsf{D} \times \mathsf{D} \end{tikzcd}$$

is a pullback, or equivalently that the diagonal map $\mathsf{C} \to \mathsf{C} \times_{\mathsf{D}} \mathsf{C}$ is an equivalence. We say that a monomorphism $f\colon \mathsf{C} \hookrightarrow \mathsf{D}$ exhibits C as a *subcategory* of D. We will study subcategories more extensively in § 2.2.2.

Proposition 2.2.1.1. *A functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is a monomorphism if and only if both f_0 and f_1 are monomorphisms in $\mathcal{B}$. In particular, both the inclusion* $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ *and the core $\mathcal{B}$-groupoid functor* $(-)^{\simeq}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Grpd}(\mathcal{B})$ *preserve monomorphisms.*

Proof. Since limits in $\mathrm{Cat}(\mathcal{B})$ are computed levelwise, the map f is a monomorphism precisely if f_n is a monomorphism in $\mathcal{B}$ for all $n \geq 0$. Owing to the Segal conditions, this is automatically satisfied whenever only f_0 and f_1 are monomorphisms. □

Proposition 2.2.1.2. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories. Then the following are equivalent:*

(1) f is a monomorphism;
(2) $f^{\simeq}$ is a monomorphism in $\widehat{\mathcal{B}}$, and for any $A \in \mathcal{B}$ and any two objects $c_0, c_1\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$, the morphism

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

that is induced by f is a monomorphism in $\widehat{\mathcal{B}}_{/A}$;
(3) for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ is a monomorphism of ∞-categories;
(4) the map of cartesian fibrations over $\mathcal{B}$ that is determined by f is a monomorphism of ∞-categories.

Proof. As monomorphisms are defined by a limit condition, one easily sees that conditions (1), (3) and (4) are equivalent, by making use of the equivalence of ∞-categories $\mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B}) \simeq \mathrm{Cart}(\mathcal{B})$ (here the latter denotes the ∞-category of cartesian fibrations over $\mathcal{B}$, see § 2.1.7) and the fact that the inclusion $\mathrm{Cat}(\widehat{\mathcal{B}}) \hookrightarrow \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{B})$ creates limits. Moreover, Proposition 2.2.1.1 implies that f is a monomorphism if and only if both f_0 and f_1 are monomorphisms in $\widehat{\mathcal{B}}$. It therefore suffices to show that f_1 is a monomorphism if and only if for every $A \in \mathcal{B}$ and any two objects $c_0, c_1\colon A \to \mathsf{C}$ in context A, the morphism

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

that is induced by f is a monomorphism in $\widehat{\mathcal{B}}_{/A}$, provided that f_0 is a monomorphism. By definition, the map that f induces on mapping $\mathcal{B}$-groupoids fits into the commutative diagram

$$\begin{tikzcd}
& \operatorname{map}_{\mathsf{C}}(c_0, c_1) \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1)) \arrow[dl] \arrow[dd] \\
\mathsf{C}_1 \arrow[rr, "f_1"] \arrow[dd] & & \mathsf{D}_1 \arrow[dd] & \\
& A \arrow[rr, "\mathrm{id}"] \arrow[dl] & & A \arrow[dl] \\
\mathsf{C}_0 \times \mathsf{C}_0 \arrow[rr, "f_0 \times f_0"] & & \mathsf{D}_0 \times \mathsf{D}_0. &
\end{tikzcd}$$

in which the two squares on the left and on the right are pullbacks. As f_0 is a monomorphism, the bottom square is a pullback, which implies that the top square is a pullback as well. Hence if f_1 is a monomorphism, then the morphism on mapping $\mathcal{B}$-groupoids must be a monomorphism as well. Conversely, suppose that f induces a monomorphism on mapping $\mathcal{B}$-groupoids. Let $P \simeq (\mathsf{C}_0 \times \mathsf{C}_0) \times_{\mathsf{D}_0 \times \mathsf{D}_0} \mathsf{D}_1$ denote the pullback of the front square in the above diagram. Then f_1 factors as $\mathsf{C}_1 \to P \to \mathsf{D}_1$ in which the second arrow is a monomorphism. It therefore suffices to show that the map $\mathsf{C}_1 \to P$ is a monomorphism as well. Note that the map $\operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1)) \to \mathsf{D}_1$ factors through the inclusion $P \hookrightarrow \mathsf{D}_1$ such that the induced map $\operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1)) \to P$ arises as the pullback of the map $P \to \mathsf{C}_0 \times \mathsf{C}_0$ along (c_0, c_1). As the object $\mathsf{C}_0 \times \mathsf{C}_0$ is obtained as the colimit of the diagram

$$\mathcal{B}_{/\mathsf{C}_0 \times \mathsf{C}_0} \to \mathcal{B} \hookrightarrow \widehat{\mathcal{B}},$$

we obtain a cover $\bigsqcup_{A \to \mathsf{C}_0 \times \mathsf{C}_0} A \twoheadrightarrow \mathsf{C}_0 \times \mathsf{C}_0$ in $\widehat{\mathcal{B}}$ and therefore a cover

$$\bigsqcup_{(c_0, c_1)} \operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1)) \twoheadrightarrow P.$$

We conclude the proof by observing that there is a pullback diagram

$$\begin{tikzcd}
\bigsqcup_{(c_0, c_1)} \operatorname{map}_{\mathsf{C}}(c_0, c_1) \arrow[r, two heads] \arrow[d, hook] & \mathsf{C}_1 \arrow[d] \\
\bigsqcup_{(c_0, c_1)} \operatorname{map}_{\mathsf{D}}(f(c_0), f(c_1)) \arrow[r, two heads] & P
\end{tikzcd}$$

in which the left vertical map is a monomorphism. Thus $\mathsf{C}_1 \to P$ is also a monomorphism by [57, Proposition 6.2.3.17]. □

Example 2.2.1.3. For any $\mathcal{B}$-category C, the canonical map $\mathsf{C}^{\simeq} \to \mathsf{C}$ is a monomorphism. In fact, using Proposition 2.2.1.2 this follows from the observation that on the level of cartesian fibrations over $\mathcal{B}$ this map is given by the inclusion of the wide subcategory of $\int \mathsf{C}$ spanned by the cartesian arrows and that this defines a monomorphism of ∞-categories.

A *strong epimorphism* in $\mathrm{Cat}(\mathcal{B})$ is a functor that is left orthogonal to the collection of monomorphisms. As a consequence of Proposition 2.2.1.1, one finds:

Proposition 2.2.1.4. *A functor between $\mathcal{B}$-groupoids is a strong epimorphism if and only if it is essentially surjective. Furthermore, both the inclusion* $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ *and the functor* $(-)^{\mathrm{gpd}} \colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Grpd}(\mathcal{B})$ *preserve strong epimorphisms.*

Proof. Let f be a functor between $\mathcal{B}$-categories. Then f^{gpd} is left orthogonal to a map g in $\mathrm{Grpd}(\mathcal{B})$ if and only if f is left orthogonal to g when viewing the latter as a map in $\mathrm{Cat}(\mathcal{B})$. Since by Proposition 2.2.1.1 g is a monomorphism in $\mathrm{Grpd}(\mathcal{B})$ if and only if g is a monomorphism in $\mathrm{Cat}(\mathcal{B})$, the map f^{gpd} is a strong epimorphism whenever f is one. Now if f is an essentially surjective map between $\mathcal{B}$-groupoids and if g is a monomorphism in $\mathrm{Cat}(\mathcal{B})$, then f is left orthogonal to g if and only if f is left orthogonal to $g^{\simeq}$, hence f is a strong epimorphism in $\mathrm{Cat}(\mathcal{B})$ since the core $\mathcal{B}$-groupoid functor preserves monomorphisms by Proposition 2.2.1.1 and since [62, Corollary 3.8.11] implies that a map between $\mathcal{B}$-groupoids is a monomorphism if and only if it is fully faithful. As every strong epimorphism is in particular essentially surjective (since fully faithful functors are always monomorphisms and since essentially surjective maps are left orthogonal to fully faithful functors), this argument also shows that the inclusion $\mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ preserves strong epimorphisms. □

REMARK 2.2.1.5. In light of Proposition 2.2.1.1 it might be tempting to expect that a map $f\colon \mathsf{C} \to \mathsf{D}$ in $\mathrm{Cat}(\mathcal{B})$ is a strong epimorphism if and only if f_0 and f_1 are covers. In fact, since the Segal conditions imply that f_0 and f_1 being a cover is equivalent to f being a cover in the ∞-topos $\mathcal{B}_\Delta$ (where covers are given by levelwise covers in $\mathcal{B}$), this is easily seen to be a sufficient condition. It is however not necessary. For example, the functor $(d_2, d_0)\colon \Delta^1 \sqcup \Delta^1 \to \Delta^2$ in Cat_∞ is a strong epimorphism since every subcategory of Δ^2 that contains the image of this functor must necessarily be Δ^2, but this map is not surjective on the level of morphisms.

2.2.2. Subcategories of $\mathcal{B}$-categories. For any ∞-category $\mathcal{C}$ with finite limits and any object $c \in \mathcal{C}$, we write $\mathrm{Sub}_{\mathcal{C}}(c)$ for the poset of *subobjects* of c, i.e. the full subcategory of $\mathcal{C}_{/c}$ that is spanned by the (-1)-truncated objects. Since a functor $f\colon \mathsf{C} \to \mathsf{D}$ is a monomorphism in $\mathrm{Cat}(\mathcal{B})$ if and only if f is a (-1)-truncated object in $\mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$, it makes sense to define:

DEFINITION 2.2.2.1. Let D be a $\mathcal{B}$-category. A *subcategory* of D is defined to be an object in $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{D})$.

WARNING 2.2.2.2. If C is a $\mathcal{B}$-category, not every subobject of C in $\mathcal{B}_\Delta$ need to be a $\mathcal{B}$-category. Therefore, the two posets $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$ and $\mathrm{Sub}_{\mathcal{B}_\Delta}(\mathsf{C})$ are in general different.

Recall from the discussion in § 2.1.6 (but see also § 2.1.8) that if C is a $\mathcal{B}$-category and A is an object in $\mathcal{B}$, the datum of a map $A \to \mathsf{C}_1$ is equivalent to that of a map $A \to \mathsf{C}^{\Delta^1}$, which is in turn equivalent to that of a map $\Delta^1 \otimes A \to \mathsf{C}$. Hence, the identity $\mathsf{C}_1 \to \mathsf{C}_1$ transposes to a functor $\Delta^1 \otimes \mathsf{C}_1 \to \mathsf{C}$.

LEMMA 2.2.2.3. *For any $\mathcal{B}$-category C, the functor $\Delta^1 \otimes \mathsf{C}_1 \to \mathsf{C}$ is a strong epimorphism in $\mathrm{Cat}(\mathcal{B})$.*

PROOF. In light of Remark 2.2.1.5, it suffices to show that the functor $\Delta^1 \otimes \mathsf{C}_1 \to \mathsf{C}$ induces a cover on level 0 and level 1. On level 0, the map is given by

$$(d_1, d_0)\colon \mathsf{C}_1 \sqcup \mathsf{C}_1 \to \mathsf{C}_0$$

which is clearly a cover since precomposition with $s_0 \sqcup s_0\colon \mathsf{C}_0 \sqcup \mathsf{C}_0 \to \mathsf{C}_1 \sqcup \mathsf{C}_1$ recovers the diagonal $\mathsf{C}_0 \sqcup \mathsf{C}_0 \to \mathsf{C}_0$ which is always a cover in $\mathcal{B}$. On level 1, one obtains the map

$$(s_0 d_1, \mathrm{id}, s_0 d_0)\colon \mathsf{C}_1 \sqcup \mathsf{C}_1 \sqcup \mathsf{C}_1 \to \mathsf{C}_1$$

which is similarly a cover in $\mathcal{B}$, as desired. □

PROPOSITION 2.2.2.4. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories and let $\mathsf{E} \hookrightarrow \mathsf{D}$ be a subcategory. The following are equivalent:*

(1) f factors through the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$;

(2) $f^{\simeq}$ factors through $\mathsf{E}^{\simeq} \hookrightarrow \mathsf{D}^{\simeq}$, and for each pair of objects $(c_0, c_1)\colon A \to \mathsf{C}_0 \times \mathsf{C}_0$ in context $A \in \mathcal{B}$, the map

$$\mathrm{map}_{\mathsf{C}}(c_0, c_1) \to \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1))$$

that is induced by f factors through the inclusion

$$\mathrm{map}_{\mathsf{E}}(f(c_0), f(c_1)) \hookrightarrow \mathrm{map}_{\mathsf{D}}(f(c_0), f(c_1));$$

(3) for each map $\Delta^1 \otimes A \to \mathsf{C}$ in context $A \in \mathcal{B}$ its image in D is contained in E.

PROOF. It is immediate that (1) implies (2) and that (2) implies (3). Suppose therefore that condition (3) holds. As in the proof of Proposition 2.2.1.2, the collection of all maps $A \to \mathsf{C}_1$ constitutes a cover

$$\bigsqcup_{A \to \mathsf{C}_1} A \twoheadrightarrow \mathsf{C}_1$$

in $\widehat{\mathcal{B}}$. By applying Proposition 2.2.1.4 and [62, Corollary 3.8.12], we may view this map as a strong epimorphism between large $\mathcal{B}$-groupoids. Since strong epimorphisms are *internally* left orthogonal to monomorphisms and therefore closed under products in $\mathrm{Cat}(\widehat{\mathcal{B}})$, we deduce that the induced map $\bigsqcup_{A\to \mathsf{C}_1} \Delta^1\otimes A \to \Delta^1\otimes \mathsf{C}_1$ is a strong epimorphism. Together with Lemma 2.2.2.3, we therefore obtain a strong epimorphism $\bigsqcup_{A\twoheadrightarrow \mathsf{C}_1} \Delta^1\otimes A \to \mathsf{C}$. Using the assumptions, we may now construct a lifting problem

$$\begin{tikzcd}
\bigsqcup_{A\to \mathsf{C}_1} \Delta^1\otimes A \arrow[r] \arrow[d, two heads] & \mathsf{E} \arrow[d, hook] \\
\mathsf{C} \arrow[r, "f"] \arrow[ur, dashed] & \mathsf{D}
\end{tikzcd}$$

which admits a unique solution, hence condition (1) follows. $\square$

Corollary 2.2.2.5. *A functor $f\colon \mathsf{C}\to\mathsf{D}$ of $\mathcal{B}$-categories factors through the inclusion $\mathsf{D}^{\simeq}\hookrightarrow \mathsf{D}$ if and only if f sends all morphisms in C to equivalences in D.* $\square$

Definition 2.2.2.6. Let $f\colon \mathsf{C}\to\mathsf{D}$ be a map in $\mathrm{Cat}(\mathcal{B})$ and let $\mathsf{C}\twoheadrightarrow\mathsf{E}\hookrightarrow\mathsf{D}$ be the factorisation of f into a strong epimorphism and a monomorphism. Then the subcategory $\mathsf{E}\hookrightarrow\mathsf{D}$ is referred to as the 1-*image* of f.

In [62, § 3.9] we have shown that full subcategories of a $\mathcal{B}$-category C can be parametrised by the subobjects of C_0 in $\mathcal{B}$ (see also Proposition 2.1.9.5). Our goal hereafter is to obtain a similar result for *all* subcategories of C. To that end, note that the functor

$$(-)^{\Delta^1}\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}} \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}}$$

admits a left adjoint that is given by the composition

$$\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}} \xrightarrow{\Delta^1\otimes -} \mathrm{Cat}(\mathcal{B})_{/\Delta^1\otimes\mathsf{C}^{\Delta^1}} \xrightarrow{\mathrm{ev}_!} \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$$

in which ev denotes the evaluation map. Similarly, the functor

$$(-)^{\simeq}\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}} \to \mathcal{B}_{/\mathsf{C}_1}$$

has a left adjoint that is given by the composition

$$\mathcal{B}_{/\mathsf{C}_1} \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}_1} \xrightarrow{i_!} \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}^{\Delta^1}}$$

where $i\colon \mathsf{C}_1\simeq (\mathsf{C}^{\Delta^1})^{\simeq}\hookrightarrow \mathsf{C}^{\Delta^1}$ denotes the canonical inclusion. By Proposition 2.2.1.1, the functor $(-)_1 = (-)^{\simeq}\circ(-)^{\Delta^1}$ sends a monomorphism $\mathsf{D}\hookrightarrow\mathsf{C}$ to the inclusion $\mathsf{D}_1\hookrightarrow\mathsf{C}_1$ and therefore restricts to a functor $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})\to\mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$. Since the inclusion $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})\hookrightarrow\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ admits a left adjoint that sends a functor $f\colon\mathsf{D}\to\mathsf{C}$ to its 1-image in C, we thus obtain an adjunction

$$(\langle-\rangle \dashv (-)_1)\colon \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)\leftrightarrows \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$$

in which the left adjoint $\langle-\rangle$ sends a monomorphism $S\hookrightarrow\mathsf{C}_1$ to the 1-image $\langle S\rangle$ of the associated map $\Delta^1\otimes S\to\mathsf{C}$. Note that for any subcategory $\mathsf{D}\hookrightarrow\mathsf{C}$, the counit $\langle\mathsf{D}_1\rangle\to\mathsf{D}$ is given by the unique solution to the lifting problem

$$\begin{tikzcd}
\Delta^1\otimes\mathsf{D}_1 \arrow[r] \arrow[d, two heads] & \mathsf{D} \arrow[d, hook] \\
\langle\mathsf{D}_1\rangle \arrow[r, hook] \arrow[ur, dashed] & \mathsf{C}
\end{tikzcd}$$

in which the upper horizontal map is the transpose of the identity $\mathsf{D}_1\to\mathsf{D}_1$. By Lemma 2.2.2.3, this is a strong epimorphism, hence we conclude that the map $\langle\mathsf{D}_1\rangle\to\mathsf{D}$ must be an equivalence. We have thus shown:

Proposition 2.2.2.7. *For any $\mathcal{B}$-category C, the functor $(-)_1\colon \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})\to\mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$ exhibits the poset $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$ as a reflective subposet of $\mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$.* $\square$

Remark 2.2.2.8. The inclusion $(-)_1\colon \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$ is in general not an equivalence. For example, consider $\mathcal{B} = \mathcal{S}$ and $\mathsf{C} = \Delta^2$: here the two maps $d^{\{0,1\}}\colon \Delta^1 \to \Delta^2$ and $d^{\{1,2\}}\colon \Delta^1 \to \Delta^2$ determine a proper subobject of Δ^2_1, but the associated subcategory of Δ^2 is nevertheless Δ^2 itself.

As Remark 2.2.2.8 exemplifies, one obstruction to $(-)_1\colon \mathrm{Sub}(\mathsf{C}) \hookrightarrow \mathrm{Sub}(\mathsf{C}_1)$ being an equivalence is that the collection of maps that determine a subobject $S \hookrightarrow \mathsf{C}_1$ need not be stable under composition. In other words, to make sure that a subobject of C_1 arises as the object of morphisms of a subcategory of C, we need to impose a composability condition on this subobject. Altogether, we obtain the following characterisation of the essential image of $(-)_1$:

Proposition 2.2.2.9. *For any $\mathcal{B}$-category C, a subobject $S \hookrightarrow \mathsf{C}_1$ lies in the essential image of the inclusion $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathcal{B}}(\mathsf{C}_1)$ if and only if*

(1) it is closed under equivalences, i.e. the map $(s_0d_1, s_0d_0)\colon S \sqcup S \to \mathsf{C}_1$ factors through $S \hookrightarrow \mathsf{C}_1$;
(2) it is closed under composition, i.e. the restriction of the composition map $d_1\colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \to \mathsf{C}_1$ along the inclusion $S \times_{\mathsf{C}_0} S \hookrightarrow \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ factors through $S \hookrightarrow \mathsf{C}_1$.

The remainder of this section is devoted to the proof of Proposition 2.2.2.9. Our strategy is to make use of the intuition that the datum of a subcategory of C should be equivalent to the datum of a collection of objects in C, together with a composable collection of maps between these objects. Our goal hereafter is turn this surmise into a formal statement.

2.2.2.10. For any integer $k \geq 0$, let $i_k\colon \Delta^{\leq k} \hookrightarrow \Delta$ denote the full subcategory spanned by $\langle n \rangle$ for $n \leq k$, and let $\mathcal{B}_\Delta^{\leq k}$ denote the ∞-category of $\mathcal{B}$-valued presheaves on $\Delta^{\leq k}$. The truncation functor $i_k^*\colon \mathcal{B}_\Delta \to \mathcal{B}_\Delta^{\leq k}$ admits both a left adjoint $(i_k)_!$ and a right adjoint $(i_k)_*$ given by left and right Kan extension. Note that both $(i_k)_!$ and $(i_k)_*$ are fully faithful. We will generally identify $\mathcal{B}_\Delta^{\leq k}$ with its essential image in $\mathcal{B}_\Delta$ along the *right* Kan extension $(i_k)_*$. We define the associated *coskeleton* functor as $\mathrm{cosk}_k = (i_k)_* i_k^*$ and the *skeleton* functor as $\mathrm{sk}_k = (i_k)_! i_k^*$. The unit of the adjunction $i_k^* \dashv (i_k)_*$ provides a map $\mathrm{id}_{\mathcal{B}_\Delta} \to \mathrm{cosk}_k$, and the counit of the adjunction $(i_k)_! \dashv i_k^*$ provides a map $\mathrm{sk}_k \to \mathrm{id}_{\mathcal{B}_\Delta}$. We say that $C \in \mathcal{B}_\Delta$ is *k-coskeletal* if the map $C \to \mathrm{cosk}_k(C)$ is an equivalence, i.e. if C is contained in $\mathcal{B}_\Delta^{\leq k} \subset \mathcal{B}_\Delta$. Dually, C is *k-skeletal* if the map $\mathrm{sk}_k(C) \to C$ is an equivalence. Note that the adjunction $\mathrm{sk}_k \dashv \mathrm{cosk}_k$ implies that a simplicial object is k-coskeletal if and only if it is local with respect to the maps $\mathrm{sk}_k(D) \to D$ for every $D \in \mathcal{B}_\Delta$.

Definition 2.2.2.11. For any integer $k \geq 0$, a map $f\colon C \to D$ in $\mathcal{B}_\Delta$ is said to be *k-coskeletal* if it is right orthogonal to $\mathrm{sk}_k(K) \to K$ for every $K \in \mathcal{B}_\Delta$.

Note that by using the adjunction $\mathrm{sk}_k \dashv \mathrm{cosk}_k$ and Yoneda's lemma, one has the following criterion for a map between simplicial objects in $\mathcal{B}$ to be k-coskeletal:

Proposition 2.2.2.12. *For any integer $k \geq 0$, a map $f\colon C \to D$ in $\mathcal{B}_\Delta$ is k-coskeletal precisely if the canonical map $C \to D \times_{\mathrm{cosk}_k(D)} \mathrm{cosk}_k(C)$ is an equivalence.* □

For any $n \geq 1$, denote by $\partial\Delta^n$ the simplicial ∞-groupoid $\mathrm{sk}_{n-1}\Delta^n$ and by $\partial\Delta^n \hookrightarrow \Delta^n$ the natural map induced by the adjunction counit.

For later use, we record the following obvious consequence of the skeletal filtration on simplical sets:

Lemma 2.2.2.13. *Let $j\colon K \hookrightarrow L$ be a monomorphism of finite simplicial sets and assume that $\mathrm{sk}_k K = \mathrm{sk}_k L$ for some $k \in \mathbb{N}$. Then j is contained in the smallest saturated class containing the maps $\partial\Delta^l \to \Delta^l$ for $k < l \leq \dim L$.* □

Lemma 2.2.2.14. *Let $k \geq 0$ be an integer. Then the following sets generate the same saturated class of morphisms in $\mathcal{B}_\Delta$:*

(1) $\{\mathrm{sk}_k D \to D \mid D \in \mathcal{B}_\Delta\}$;
(2) $\{\partial\Delta^n \otimes A \hookrightarrow \Delta^n \otimes A \mid n > k,\ A \in \mathcal{B}\}$.

(3) $\{\partial\Delta^{k+1} \otimes D \hookrightarrow \Delta^{k+1} \otimes D \mid D \in \mathcal{B}_\Delta\}$.

Proof. We start by showing that the saturations of (1) and (2) agree. Given $A \in \mathcal{B}$, note that since the truncation functor i_k^* commutes with postcomposition by both the pullback functor $\pi_A^* \colon \mathcal{B} \to \mathcal{B}_{/A}$ and its right adjoint $(\pi_A)_*$, the uniqueness of adjoints implies that the functor sk_k commutes with $- \times A \colon \mathcal{B}_\Delta \to \mathcal{B}_\Delta$. By a similar argument, the functor sk_k commutes with $\mathrm{const} \colon \mathcal{S}_\Delta \to \mathcal{B}_\Delta$. We therefore obtain an equivalence $\mathrm{sk}_k(\Delta^m \otimes A) \simeq \mathrm{sk}_k(\Delta^m) \otimes A$ with respect to which the canonical map $\mathrm{sk}_k(\Delta^m \otimes A) \to \Delta^m \otimes A$ corresponds to the map obtained by applying the functor $- \otimes A$ to the map $\mathrm{sk}_k(\Delta^m) \to \Delta^m$. This already implies that the set in (2) is contained in the set in (1), so that the saturation of (2) is contained in the saturation of (1). Conversely, as any $D \in \mathcal{B}_\Delta$ can be written as a colimit of objects of the form $\Delta^n \otimes A$ (see [62, Lemma 4.5.2]), the above argument also shows that every map in (1) is a colimit of maps of the form $\mathrm{sk}_k(\Delta^n) \otimes A \to \Delta^n \otimes A$. Since moreover $\mathrm{const}_{\mathcal{B}}$ and $- \otimes A$ are colimit-preserving functors, one finds that (1) is contained in the saturation of (2) as soon as we can show that any saturated class S of maps in $\mathcal{S}_\Delta$ which contains $\partial\Delta^n \to \Delta^n$ for all $n > k$ must also contain the maps $\mathrm{sk}_k \Delta^m \to \Delta^m$ for all m. But that is immediate from Lemma 2.2.2.13.

Next, to show that the saturation of (2) contains (3), we may again assume $D \simeq \Delta^m \otimes A$. In this case, the inclusion $\partial\Delta^{k+1} \times \Delta^m \hookrightarrow \Delta^{k+1} \times \Delta^m$ can be obtained as an iterated pushout of maps of the form $\partial\Delta^n \hookrightarrow \Delta^n$ for $n > k$ (by Lemma 2.2.2.13), hence the claim follows. For the converse inclusion, we will use induction on n, the case $n = k+1$ being satisfied by definition. Given that for a fixed $n > k$ the inclusion $\partial\Delta^n \otimes A \hookrightarrow \Delta^n \otimes A$ is contained in the saturation of (3), Lemma 2.2.2.13 allows us to build the inclusion $\partial\Delta^n \times \Delta^1 \hookrightarrow \mathrm{sk}_n(\Delta^n \times \Delta^1)$ as an iterated pushout along $\partial\Delta^n \hookrightarrow \Delta^n$. Therefore, the map $\mathrm{sk}_n(\Delta^n \times \Delta^1) \otimes A \hookrightarrow (\Delta^n \times \Delta^1) \otimes A$ is contained in the saturation of (3) by the left cancellation property. Let $\alpha \colon \Delta^{n+1} \to \Delta^n \times \Delta^1$ be defined by $\alpha(i) = (i, 0)$ for $i = 0, \dots, n$ and $\alpha(n+1) = (n+1, 1)$, and let $\beta \colon \Delta^n \times \Delta^1 \to \Delta^{n+1}$ be given by $\beta(i, 0) = i$ and $\beta(i, 1) = n+1$. We then obtain a retract diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "\alpha'"] \arrow[d, hook] & \mathrm{sk}_n(\Delta^n \times \Delta^1) \arrow[r, "\beta'"] \arrow[d, hook] & \partial\Delta^{n+1} \arrow[d, hook] \\
\Delta^{n+1} \arrow[r, "\alpha"] & \Delta^n \times \Delta^1 \arrow[r, "\beta"] & \Delta^{n+1}
\end{tikzcd}$$

in which α' and β' are given by the restriction of α and β, respectively. We therefore conclude that the map $\partial\Delta^{n+1} \otimes A \hookrightarrow \Delta^{n+1} \otimes A$ is in the saturation of (3), as desired. □

As a consequence of Lemma 2.2.2.14, one finds:

Proposition 2.2.2.15. *For any integer $k \geq 0$, a map $f \colon C \to D$ in $\mathcal{B}_\Delta$ is k-coskeletal if and only if it is internally right orthogonal to the map $\partial\Delta^{k+1} \hookrightarrow \Delta^{k+1}$.* □

We can use Proposition 2.2.2.15 to show that every monomorphism between $\mathcal{B}$-categories is 1-coskeletal. To that end, recall that we denote by $I^2 \hookrightarrow \Delta^2$ the inclusion of the 2-spine (see § 2.1.6). We now obtain:

Lemma 2.2.2.16. *Let S be the internal saturation of $\Delta^0 \sqcup \Delta^0 \to \Delta^0$ and $I^2 \hookrightarrow \Delta^2$ in $\mathcal{B}_\Delta$. Then S contains the map $\partial\Delta^2 \hookrightarrow \Delta^2$.*

Proof. Let $f \colon K \to L$ be a map in $\mathcal{B}_\Delta$ that is internally right orthogonal to the maps $\Delta^0 \sqcup \Delta^0 \to \Delta^0$ and the inclusion of the 2-spine $I^2 \hookrightarrow \Delta^2$. Then f is a monomorphism. Now consider the commutative diagram

$$\begin{tikzcd}
K^{\Delta^2} \arrow[dr] \arrow[rrr, bend left] \arrow[dd, bend right] & & & & & \\
& P \arrow[rr] \arrow[dl] \arrow[dd, hook] \arrow[dr, phantom, "\ulcorner"] & & K^{\partial\Delta^2} \arrow[rr] \arrow[dl] \arrow[dd, hook] & & K^{I^2} \arrow[dl, "\mathrm{id}"] \arrow[dd, hook] \\
Q \arrow[rr] \arrow[dr] \arrow[dd, hook] & & R \arrow[rr] \arrow[dd, hook] \arrow[dr, phantom, "\ulcorner"] & & K^{I^2} \arrow[dd, hook] & \\
& L^{\Delta^2} \arrow[rr] \arrow[dl, "\mathrm{id}"] & & L^{\partial\Delta^2} \arrow[rr] \arrow[dl, "\mathrm{id}"] & & L^{I^2} \arrow[dl, "\mathrm{id}"] \\
L^{\Delta^2} \arrow[rr] & & L^{\partial\Delta^2} \arrow[rr] & & L^{I^2} &
\end{tikzcd}$$

in which P, Q and R are defined by the condition that the respective square is a pullback diagram. We need to show that the map $K^{\Delta^2} \to P$ is an equivalence. As by assumption on f the map $K^{\Delta^2} \to Q$ is an equivalence, it suffices to show that $P \to Q$ is an equivalence as well. But this map is already a monomorphism, hence the claim follows from the observation that $P \to Q$ must be a cover as the map $K^{\Delta^2} \to Q$ is one. □

Proposition 2.2.2.17. *Every monomorphism between $\mathcal{B}$-categories is 1-coskeletal.*

Proof. Lemma 2.2.2.16 implies that every monomorphism between $\mathcal{B}$-categories is internally right orthogonal to $\partial\Delta^2 \hookrightarrow \Delta^2$ and therefore 1-coskeletal. □

Let C be a $\mathcal{B}$-category and let $\mathrm{Cat}(\mathcal{B})^{\leq 1}_{/\mathsf{C}}$ be the full subcategory of $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ that is spanned by the 1-coskeletal maps into C. By restricting the inclusion $\mathrm{Cat}(\mathcal{B})^{\leq 1}_{/\mathsf{C}} \hookrightarrow \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ to (-1)-truncated objects (i.e. to monomorphisms into D), one obtains a full embedding

$$\mathrm{Sub}^{\leq 1}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$$

of partially ordered sets. Proposition 2.2.2.17 now implies:

Corollary 2.2.2.18. *For any $\mathcal{B}$-category C, the inclusion $\mathrm{Sub}^{\leq 1}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$ is an equivalence.* □

For any $\mathcal{B}$-category C, the functor $(\mathrm{cosk}_1)_{/\mathsf{C}} \colon (\mathcal{B}_\Delta)_{/\mathsf{C}} \to (\mathcal{B}^{\leq 1}_{\Delta})_{/\mathrm{cosk}_1 \mathsf{C}}$ that is induced by the coskeleton functor on the slice ∞-categories admits a fully faithful right adjoint η^* that is given by base change along the adjunction unit $\eta \colon \mathsf{C} \to \mathrm{cosk}_1 \mathsf{C}$. Upon restricting to subobjects, we therefore obtain an adjunction

$$\mathrm{Sub}_{\mathcal{B}_\Delta}(\mathsf{C}) \underset{\eta^*}{\overset{(\mathrm{cosk}_1)_{/\mathsf{C}}}{\rightleftarrows}} \mathrm{Sub}_{\mathcal{B}^{\leq 1}_{\Delta}}(\mathrm{cosk}_1 \mathsf{C}).$$

In general, the functor η^* does not take values in $\mathrm{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$, but we may explicitly characterise those subobjects of $\mathrm{cosk}_1 \mathsf{C}$ that do give rise to a $\mathcal{B}$-category. To that end, note that given a subobject $D \hookrightarrow \mathrm{cosk}_1 \mathsf{C}$ in $\mathcal{B}^{\leq 1}_{\Delta}$, the restriction of $d_1 \colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \to \mathsf{C}_1$ along the inclusion $D_1 \times_{D_0} D_1 \hookrightarrow \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$ determines a map $d_1 \colon D_1 \times_{D_0} D_1 \to \mathsf{C}_1$.

Definition 2.2.2.19. Let C be a $\mathcal{B}$-category. A subobject $D \hookrightarrow \mathrm{cosk}_1 \mathsf{C}$ in $\mathcal{B}^{\leq 1}_{\Delta}$ is said to be *closed under composition* if the map $d_1 \colon D_1 \times_{D_0} D_1 \to \mathsf{C}_1$ factors through $D_1 \hookrightarrow \mathsf{C}_1$. We denote by $\mathrm{Sub}^{\mathrm{comp}}_{\mathcal{B}^{\leq 1}_{\Delta}}(\mathrm{cosk}_1 \mathsf{C})$ the full subcategory of $\mathrm{Sub}_{\mathcal{B}^{\leq 1}_{\Delta}}(\mathrm{cosk}_1 \mathsf{C})$ that is spanned by these subobjects.

Lemma 2.2.2.20. *Let $A \in \mathcal{B}$ be an arbitrary object and let S be a saturated set of maps in $\mathcal{B}_\Delta$ that contains the internal saturation of $\partial\Delta^2 \hookrightarrow \Delta^2$ as well as the map $I^2 \otimes A \hookrightarrow \Delta^2 \otimes A$. Then S contains $I^n \otimes A \hookrightarrow \Delta^n \otimes A$ for all $n \geq 2$.*

Proof. We may assume $n > 2$. By [50, Proposition 2.13], it suffices to show that for all $0 < i < n$ the inclusion $\Lambda^n_i \otimes A \hookrightarrow \Delta^n \otimes A$ is contained in S. On account of the factorisation $\Lambda^n_i \hookrightarrow \partial\Delta^n \hookrightarrow \Delta^n$ in which the first map is obtained as a pushout along $\partial\Delta^{n-1} \hookrightarrow \Delta^n$, this is immediate. □

Proposition 2.2.2.21. *Let $D \hookrightarrow \operatorname{cosk}_1 \mathsf{C}$ be a subobject in $\mathcal{B}_{\Delta}^{\leq 1}$. Then $\eta^* D$ is a $\mathcal{B}$-category if and only if D is closed under composition. In particular, η^* defines an equivalence* $\operatorname{Sub}^{\mathrm{comp}}_{\mathcal{B}_{\Delta}^{\leq 1}}(\operatorname{cosk}_1 \mathsf{C}) \simeq \operatorname{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C})$.

Proof. If $\eta^* D$ is a $\mathcal{B}$-category, the fact that applying cosk_1 to the inclusion $\eta^* D \hookrightarrow \mathsf{C}$ recovers the subobject $D \hookrightarrow \operatorname{cosk}_1 \mathsf{C}$ implies that D is closed under composition. Conversely, suppose that D is closed under composition. Since $E^1 \to 1$ is a cover in $\mathcal{B}_{\Delta}$ (where E^1 is the walking equivalence, see § 2.1.6), every monomorphism of simplicial objects in $\mathcal{B}$ is internally right orthogonal to $E^1 \to 1$. Therefore $\eta^* D$ is univalent. We still need to show that $\eta^* D$ satisfies the Segal conditions. Since $\eta^* D \hookrightarrow \mathsf{C}$ is 1-coskeletal, Lemma 2.2.2.20 implies that we only need to show that $(\eta^* D)_2 \to C_1 \times_{D_0} D_1$ is an equivalence. As this map is a monomorphism, it furthermore suffices to show that it is a cover in $\mathcal{B}$. Note that since the natural map $(-)^{\Delta^2} \to (-)^{\partial \Delta^2}$ induces an equivalence on 1-coskeletal objects, the identification $\partial \Delta^2 \simeq I^2 \sqcup_{\Delta^0 \sqcup \Delta^0} \Delta^1$ gives rise to a commutative square

$$\begin{tikzcd}
 & \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[rr] \arrow[ddrr, "d_1"] & & (\operatorname{cosk}_1 \mathsf{C})_2 \arrow[rr] \arrow[dd] & & \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \arrow[dd] \\
D_1 \times_{D_0} D_1 \arrow[ur, hook] \arrow[rr, dashed] \arrow[ddrr, dashed, bend right, "d_1"'] & & D_2 \arrow[ur, hook] \arrow[rr] \arrow[dd] & & D_1 \times_{D_0} D_1 \arrow[ur, hook] \arrow[dd] & \\
 & & & \mathsf{C}_1 \arrow[rr] & & \mathsf{C}_0 \times \mathsf{C}_0 \\
 & & D_1 \arrow[ur, hook] \arrow[rr] & & D_0 \times D_0 \arrow[ur, hook] &
\end{tikzcd}$$

in which the two squares in the front and in the back of the cube are pullbacks and where the dashed arrows exist as D is closed under composition. By combining this diagram with the pullback square

$$\begin{tikzcd}
(\eta^* D)_2 \arrow[r] \arrow[d, hook] & D_2 \arrow[d, hook] \\
\mathsf{C}_2 \arrow[r] & (\operatorname{cosk}_1 \mathsf{C})_2,
\end{tikzcd}$$

one concludes that the map $(\eta^* D)_2 \to D_1 \times_{D_0} D_1$ admits a section and is therefore a cover, as desired. Lastly, the claim that that η^* induces an equivalence $\operatorname{SubObj}^{\mathrm{comp}}(\operatorname{cosk}_1 \mathsf{C}) \simeq \operatorname{Sub}(\mathsf{C})$ now follows easily with Corollary 2.2.2.18. ☐

Proof of Proposition 2.2.2.9. It is clear that any subobject $S \hookrightarrow \mathsf{C}_1$ that arises as the object of morphisms of a subcategory of C must necessarily satisfy the two conditions, so it suffices to prove the converse. Let $D_0 \hookrightarrow \mathsf{C}$ be the image of $(d_1, d_0)\colon S \sqcup S \to \mathsf{C}_0$. As S is closed under equivalences in C, the restriction of $s_0\colon \mathsf{C}_0 \to \mathsf{C}_1$ to D_0 factors through $S \hookrightarrow \mathsf{C}_1$. By setting $D_1 = S$, we thus obtain a subobject $D \hookrightarrow \operatorname{cosk}_1 \mathsf{C}$ in $\mathcal{B}_{\Delta}^{\leq 1}$. By assumption, this subobject is closed under composition in the sense of Definition 2.2.2.19, hence Proposition 2.2.2.21 implies that $\eta^* D$ is a subcategory of C. Hence $S = D_1$ arises as the object of morphisms of $\eta^* D$ and is therefore contained in the essential image of $(-)_1\colon \operatorname{Sub}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}) \hookrightarrow \operatorname{Sub}_{\mathcal{B}}(\mathsf{C}_1)$. ☐

2.2.3. Localisations of $\mathcal{B}$-categories. Recall that a functor between $\mathcal{B}$-categories is said to be *conservative* if it is internally right orthogonal to the map $\Delta^1 \to \Delta^0$ (cf. [62, Definition 4.1.10]). Hereafter we discuss the left complement of the associated factorisation system, i.e. the saturated class that is internally generated by $\Delta^1 \to \Delta^0$.

Definition 2.2.3.1. A functor between $\mathcal{B}$-categories is an *iterated localisation* if it is left orthogonal to every conservative functor.

Remark 2.2.3.2. By definition, a functor $f\colon \mathsf{C} \to \mathsf{D}$ is conservative if and only if the commutative square

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "\mathrm{diag}"'] & \mathsf{D} \arrow[d, "\mathrm{diag}"] \\ \mathsf{C}^{\Delta^1} \arrow[r, "f^{\Delta^1}"'] & \mathsf{D}^{\Delta^1} \end{tikzcd}$$

is cartesian. Since this can be check after evaluating at any $A \in \mathcal{B}$, the natural equivalences $\mathsf{C}^{\Delta^n}(A) \simeq \mathsf{C}(A)^{\Delta^n}$ show that f is conservative if and only if $f(A)$ is conservative for all A.

The saturated class of iterated localisations in $\mathrm{Cat}(\mathcal{B})$ is internally generated by $\Delta^1 \to \Delta^0$. Since this map is a strong epimorphism by Remark 2.2.1.5, we deduce:

Proposition 2.2.3.3. *Every iterated localisation in* $\mathrm{Cat}(\mathcal{B})$ *is a strong epimorphism and therefore in particular essentially surjective. Dually, every monomorphism is conservative.* □

Definition 2.2.3.4. Let C be a $\mathcal{B}$-category and let $\mathsf{S} \to \mathsf{C}$ be a functor. The *localisation* of C at S is the $\mathcal{B}$-category $\mathsf{S}^{-1}\mathsf{C}$ that fits into the pushout square

$$\begin{tikzcd} \mathsf{S} \arrow[r] \arrow[d] & \mathsf{S}^{\mathrm{gpd}} \arrow[d] \\ \mathsf{C} \arrow[r] & \mathsf{S}^{-1}\mathsf{C}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We refer to the map $\mathsf{C} \to \mathsf{S}^{-1}\mathsf{C}$ as the *localisation functor* that is associated with the map $\mathsf{S} \to \mathsf{C}$. More generally, a functor $\mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories is said to be a localisation if there is a functor $\mathsf{S} \to \mathsf{C}$ and an equivalence $\mathsf{D} \simeq \mathsf{S}^{-1}\mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})_{\mathsf{C}/}$.

Remark 2.2.3.5. The above definition is a direct analogue of the construction of localisations of ∞-categories, see [17, Proposition 7.1.3].

By definition, the groupoidification functor $\mathsf{S} \to \mathsf{S}^{\mathrm{gpd}}$ in Definition 2.2.3.4 is an iterated localisation. One therefore finds:

Proposition 2.2.3.6. *For any* $\mathcal{B}$*-category* C *and any functor* $\mathsf{S} \to \mathsf{C}$*, the localisation functor* $\mathsf{C} \to \mathsf{S}^{-1}\mathsf{C}$ *is an iterated localisation.* □

Lemma 2.2.3.7. *Let* G *be a* $\mathcal{B}$*-groupoid and let* $\mathsf{G} \to \mathsf{C}$ *be a strong epimorphism in* $\mathrm{Cat}(\mathcal{B})$*. Then* C *is a* $\mathcal{B}$*-groupoid as well.*

Proof. Since G is a $\mathcal{B}$-groupoid, Corollary 2.2.2.5 implies that the functor $\mathsf{G} \to \mathsf{C}$ factors through the inclusion $\mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$. We may therefore construct a lifting problem

$$\begin{tikzcd} \mathsf{G} \arrow[r] \arrow[d] & \mathsf{C}^{\simeq} \arrow[d, hook] \\ \mathsf{C} \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \mathsf{C} \end{tikzcd}$$

which admits a unique solution. Hence the identity on C factors through $\mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$, which evidently implies that $\mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$ is already an equivalence. □

Lemma 2.2.3.8. *For any strong epimorphism* $f\colon \mathsf{C} \to \mathsf{D}$ *in* $\mathrm{Cat}(\mathcal{B})$*, the commutative square*

$$\begin{tikzcd} \mathsf{C} \arrow[r] \arrow[d, "f"] & \mathsf{C}^{\mathrm{gpd}} \arrow[d, "f^{\mathrm{gpd}}"] \\ \mathsf{D} \arrow[r] & \mathsf{D}^{\mathrm{gpd}} \end{tikzcd}$$

is cocartesian.

Proof. If $\mathsf{P} = \mathsf{D} \sqcup_{\mathsf{C}} \mathsf{C}^{\mathrm{gpd}}$ denotes the pushout, we need to show that the induced functor $g\colon \mathsf{P} \to \mathsf{D}^{\mathrm{gpd}}$ is an equivalence. Since iterated localisations are stable under pushout, the map $\mathsf{D} \to \mathsf{P}$ is an iterated localisation, which (by the left cancellation property) implies that g must be an iterated localisation as well. We therefore only need to show that g is conservative. Since $\mathsf{D}^{\mathrm{gpd}}$ is a $\mathcal{B}$-groupoid, this is equivalent to P being a $\mathcal{B}$-groupoid as well [62, Corollary 4.1.17]. But since strong epimorphisms are also preserved by pushouts, the map $\mathsf{C}^{\mathrm{gpd}} \to \mathsf{P}$ is a strong epimorphism, hence Lemma 2.2.3.7 implies the claim. □

Proposition 2.2.3.9. *Let $f\colon \mathsf{S} \to \mathsf{T}$ and $g\colon \mathsf{T} \to \mathsf{C}$ be functors in* $\mathrm{Cat}(\mathcal{B})$*, and suppose that f is a strong epimorphism. Then the induced functor $\mathsf{S}^{-1}\mathsf{C} \to \mathsf{T}^{-1}\mathsf{C}$ is an equivalence.*

Proof. This is a direct consequence of the pasting lemma for pushout squares and Lemma 2.2.3.8. □

Remark 2.2.3.10. Proposition 2.2.3.9 implies that when considering localisations of a $\mathcal{B}$-category C, we may restrict our attention to *subcategories* $\mathsf{S} \hookrightarrow \mathsf{C}$ instead of general functors, as we can always factor a functor $\mathsf{S} \to \mathsf{C}$ into a strong epimorphism followed by a monomorphism. Alternatively, by making use of the strong epimorphism $\Delta^1 \otimes \mathsf{S}_0 \to \mathsf{S}$ from Lemma 2.2.2.3, we can always assume that S is of the form $\Delta^1 \otimes A$ for some $A \in \mathcal{B}$.

Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories. Let $f^*\mathsf{D}^{\simeq} \hookrightarrow \mathsf{C}$ be the subcategory that is defined by the pullback square

$$\begin{tikzcd} f^*\mathsf{D}^{\simeq} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{D}^{\simeq} \arrow[d, hook] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D}. \end{tikzcd}$$

Since $\mathsf{D}^{\simeq}$ is a $\mathcal{B}$-groupoid, the map $f^*\mathsf{D}^{\simeq} \to \mathsf{D}^{\simeq}$ factors through $f^*\mathsf{D}^{\simeq} \to (f^*\mathsf{D}^{\simeq})^{\mathrm{gpd}}$. Consequently, one obtains a factorisation of f into the composition

$$\mathsf{C} \to (f^*\mathsf{D}^{\simeq})^{-1}\mathsf{C} \xrightarrow{f_1} \mathsf{D}.$$

Let us set $\mathsf{C}_1 = (f^*\mathsf{D}^{\simeq})^{-1}\mathsf{C}$. By replacing C by C_1 and f by f_1 and iterating this procedure, we obtain an $\mathbb{N}$-indexed diagram in $\mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$. Let $f_\infty\colon \mathsf{E} \to \mathsf{D}$ denote the colimit of this diagram. By construction, the map f factors into the composition $\mathsf{C} \to \mathsf{E} \to \mathsf{D}$ in which the first map is a countable composition of localisations and therefore an iterated localisation in the sense of Definition 2.2.3.1. We claim that the map f_∞ is conservative. To see this, consider the cartesian square

$$\begin{tikzcd} f_\infty^*\mathsf{D}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{D}^{\simeq} \arrow[d] \\ \mathsf{E} \arrow[r, "f_\infty"] & \mathsf{D}. \end{tikzcd}$$

On account of filtered colimits being universal in $\mathrm{Cat}(\mathcal{B})$ (see Proposition 2.1.6.5), we obtain an equivalence $f_\infty^*\mathsf{D}^{\simeq} \simeq \operatorname{colim}_n f_n^*\mathsf{D}^{\simeq}$. By construction, the categories $f_n^*\mathsf{D}^{\simeq}$ sit inside the $\mathbb{N}$-indexed diagram

$$\cdots \to f_{n-1}^*\mathsf{D}^{\simeq} \to (f_{n-1}^*\mathsf{D}^{\simeq})^{\mathrm{gpd}} \to f_n^*\mathsf{D}^{\simeq} \to (f_n^*\mathsf{D}^{\simeq})^{\mathrm{gpd}} \to f_{n+1}^*\mathsf{D}^{\simeq} \to (f_{n+1}^*\mathsf{D}^{\simeq})^{\mathrm{gpd}} \to \cdots$$

such that the functor $\cdot 2\colon \mathbb{N} \to \mathbb{N}$ that is given by the inclusion of all even natural numbers recovers the $\mathbb{N}$-indexed diagram $n \mapsto f_n^*\mathsf{D}^{\simeq}$ that is defined by the cartesian square above. As both the inclusion of all even natural numbers and that of all odd natural numbers define final functors $\mathbb{N} \to \mathbb{N}$, we conclude that $f_\infty^*\mathsf{D}^{\simeq}$ is obtained as the colimit of the diagram $n \mapsto (f_n^*\mathsf{D}^{\simeq})^{\mathrm{gpd}}$ and is therefore a *groupoid* in $\mathcal{B}$. Applying [62, Corollary 4.1.16], this shows that f_∞ is conservative. Therefore the factorisation of f into the composite $\mathsf{C} \to \mathsf{E} \to \mathsf{D}$ as constructed above is the unique factorisation of f into an iterated localisation and a conservative functor. Applying this construction when f is already an iterated localisation, one in particular obtains:

Proposition 2.2.3.11. *Every iterated localisation between $\mathcal{B}$-categories is obtained as a countable composition of localisation functors.* □

Our next goal is to prove the universal property of a localisation functor. To that end, given any two $\mathcal{B}$-categories C and D and any functor $\mathsf{S} \to \mathsf{C}$, note that as the base change functor $\pi_A^*\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B}_{/A})$ from Remark 2.1.6.10 preserves the internal hom $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ [62, Lemma 4.2.3], an object of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ in context $A \in \mathcal{B}$ is precisely given by a functor of $\mathcal{B}_{/A}$-categories $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$. Therefore, the collection of functors $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ in arbitrary context $A \in \mathcal{B}$ whose restriction along $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{C}$ factors through $\pi_A^*\mathsf{D}^{\simeq}$ span a full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ (see § 2.1.9) that we denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D})$.

Remark 2.2.3.12 (locality of $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D})$). Note that a functor $f\colon \pi_A^*\mathsf{S} \to \pi_A^*\mathsf{D}$ factors through $\pi_A^*\mathsf{D}^{\simeq}$ if and only if the transposed map $A \times \mathsf{S} \to \mathsf{D}$ factors through $\mathsf{D}^{\simeq}$. As the map $\mathsf{D}^{\simeq} \hookrightarrow \mathsf{D}$ is a monomorphism by Example 2.2.1.3, the same argument as in Example 2.1.14.7 shows that this condition is *local*, in the sense that for every cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, the functor f factors through $\pi_A^*\mathsf{D}^{\simeq}$ if and only if each of the functors $s_i^*(f)$ factors through $\pi_{A_i}^*\mathsf{D}^{\simeq}$. As a consequence, *every* object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D})$ encodes a functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ whose restriction along $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{C}$ factors through $\pi_A^*\mathsf{C}^{\simeq}$. In conjunction with [62, Lemma 4.2.3], this observation furthermore implies that there is a canonical equivalence $\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{S}}(\pi_A^*\mathsf{C},\pi_A^*\mathsf{D})$ for every $A \in \mathcal{B}$, cf. Remark 2.1.14.6.

Remark 2.2.3.13. By Corollary 2.2.2.5 and Remark 2.2.3.12, a functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ defines an object in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D})$ precisely if its restriction along $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{C}$ sends every map in $\pi_A^*\mathsf{S}$ to an equivalence in $\pi_A^*\mathsf{C}$.

Proposition 2.2.3.14. *Let C be a $\mathcal{B}$-category and let $\mathsf{S} \to \mathsf{C}$ be a functor. Then precomposition with the localisation functor $L\colon \mathsf{C} \to \mathsf{S}^{-1}\mathsf{C}$ induces an equivalence*

$$L^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C},\mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D})$$

for any $\mathcal{B}$-category D.

Proof. By applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,\mathsf{D})$ to the pushout square that defines the localisation of C at S, one obtains a pullback square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C},\mathsf{D}) \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D}) \arrow[d] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{\mathrm{gpd}},\mathsf{D}) \arrow[r] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S},\mathsf{D}).
\end{tikzcd}$$

We claim that the two horizontal functors are fully faithful. To see this, it suffices to consider the lower horizontal map. This is a fully faithful functor precisely if it is internally right orthogonal to the map $\Delta^0 \sqcup \Delta^0 \to \Delta^1$, and by making use of the adjunction between tensoring and powering in $\mathrm{Cat}(\mathcal{B})$, one sees that this is equivalent to the induced functor $\mathsf{D}^{\Delta^1} \to \mathsf{D} \times \mathsf{D}$ being internally right orthogonal to the map $\mathsf{S} \to \mathsf{S}^{\mathrm{gpd}}$. Hence it suffices to show that the functor $\mathsf{D}^{\Delta^1} \to \mathsf{D} \times \mathsf{D}$ is conservative. But that is immediate from Remark 2.2.3.2.

Since for any $A \in \mathcal{B}$ a functor $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{D}$ factors through $\pi_A^*\mathsf{D}^{\simeq}$ if and only if it factors through the map $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{S}^{\mathrm{gpd}}$, one obtains a commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D}) \arrow[r, hook] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D}) \arrow[d] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{\mathrm{gpd}},\mathsf{D}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S},\mathsf{D}).
\end{tikzcd}$$

and therefore a map $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C},\mathsf{D})$. Since every object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C},\mathsf{D})$ by definition gives rise to an object in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{S}}(\mathsf{C},\mathsf{D})$, this map must also be essentially surjective and is thus an equivalence. $\square$

2.3. Straightening-Unstraightening for $\mathcal{B}$-categories

One of the most fundamental results in ∞-category theory is the straightening-unstraightening equivalence relating functors with target Cat_∞ and cocartesian fibrations, see [57, Theorem 3.2.0.1]. The goal of this section is to prove the analogous result in the world of $\mathcal{B}$-categories. This was first proven in [61] but we decided to include our own proof in § 2.3.2 since it is rather different from the one given in [61]. While the proof in loc. cit. internalizes some of the ideas of [57, § 3.2], ours is bootstrapped from [57, Theorem 3.2.0.1] and the explicit description of unstraightening as a lax colimit in [27]. To be able to prove straightening-unstraightening we at first need to define the $\mathcal{B}$-category of $\mathcal{B}$-categories, which we do in § 2.3.1.

2.3.1. The large $\mathcal{B}$-category of $\mathcal{B}$-categories. The goal in this section is to define the large $\mathcal{B}$-category of $\mathcal{B}$-categories. What makes this possible is the following general construction:

CONSTRUCTION 2.3.1.1. Recall that Lurie's tensor product of presentable ∞-categories introduced in [56, § 4.8.1] defines a functor

$$- \otimes -\colon \mathrm{Pr}^{\mathrm{R}} \times \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Pr}^{\mathrm{R}}, \quad (\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Sh}_{\mathcal{C}}(\mathcal{D})$$

that preserves limits in each variable. Since the functor $\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ factors through the inclusion $\mathrm{Pr}^{\mathrm{R}} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ we may consider the composite

$$\mathrm{Pr}^{\mathrm{R}} \times \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times \mathcal{B}_{/-}} \mathrm{Pr}^{\mathrm{R}} \times \mathrm{Pr}^{\mathrm{R}} \xrightarrow{-\otimes -} \mathrm{Pr}^{\mathrm{R}} \to \widehat{\mathrm{Cat}}_\infty.$$

Its transpose defines a functor $\mathrm{Pr}^{\mathrm{R}} \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$. It follows from [57, Theorem 5.5.3.18] that this map factors through the full subcategory spanned by the limit-preserving functors and thus defines a functor

$$- \otimes \Omega\colon \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Cat}(\widehat{\mathcal{B}}).$$

By the explicit description of the tensor product of presentable ∞-categories [56, Proposition 4.8.1.17], this functor is equivalently given by $\mathrm{Sh}_{-}(\mathcal{B}_{/-})$. In other words, given any presentable ∞-category $\mathcal{E}$, the associated large $\mathcal{B}$-category $\mathcal{E} \otimes \Omega$ is given by the composition

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}_{/-}} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \xrightarrow{\mathrm{Sh}_{\mathcal{E}}(-)} \widehat{\mathrm{Cat}}_\infty.$$

Let us now consider the above construction in the special case $\mathcal{E} = \mathrm{Cat}_\infty$. By definition, $\mathrm{Cat}_\infty \otimes \Omega$ is given by the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}_{/-}} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \xrightarrow{\mathrm{Sh}_{\mathrm{Cat}_\infty}(-)} \widehat{\mathrm{Cat}}_\infty$$

and thus agrees with the presheaf of ∞-categories $\mathrm{Cat}(\mathcal{B}_{/-})$ defined in [62, § 3.3]. In particular it follows that the latter is indeed a sheaf. Therefore we feel inclined to make the following definition:

DEFINITION 2.3.1.2. We define the *large $\mathcal{B}$-category* $\mathsf{Cat}_{\mathcal{B}}$ *of (small) $\mathcal{B}$-categories* to be $\mathsf{Cat}_{\mathcal{B}} = \mathrm{Cat}_\infty \otimes \Omega$, i.e. as the large $\mathcal{B}$-category that corresponds to the sheaf $\mathrm{Cat}(\mathcal{B}_{/-})$.

REMARK 2.3.1.3 (locality of the $\mathcal{B}$-category of $\mathcal{B}$-categories). By definition of $\mathsf{Cat}_{\mathcal{B}}$, there is a canonical equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$ (where $\pi_A^*\colon \mathrm{Cat}(\widehat{\mathcal{B}}) \to \mathrm{Cat}(\widehat{\mathcal{B}}_{/A})$ denotes the base change functor induced by $\pi_A^*\colon \mathcal{B} \to \mathcal{B}_{/A}$, cf. Remark 2.1.6.10). In fact, by Remark 2.1.7.6 we may compute $\pi_A^* \mathsf{Cat}_{\mathcal{B}} \simeq \mathrm{Cat}(\mathcal{B}_{/(\pi_A)_!(-)})$, which is evidently equivalent to $\mathrm{Cat}((\mathcal{B}_{/A})_{/-})$.

REMARK 2.3.1.4. By applying $- \otimes \Omega$ to the equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat}_\infty \simeq \mathrm{Cat}_\infty$, one obtains an equivalence $(-)^{\mathrm{op}}\colon \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}}$. On global sections over $A \in \mathcal{B}$, this equivalence recovers the equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat}(\mathcal{B}_{/A}) \simeq \mathrm{Cat}(\mathcal{B}_{/A})$ that carries a $\mathcal{B}_{/A}$-category to its opposite (cf. Remark 2.1.7.4).

REMARK 2.3.1.5. By working internally to $\widehat{\mathcal{B}}$, we may define the (very large) $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ of large $\mathcal{B}$-categories. By regarding $\mathsf{Cat}_{\mathcal{B}}$ as a very large $\mathcal{B}$-category, we furthermore obtain a fully faithful functor $i\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$. In fact, by the discussion in [62, § 3.3], the inclusion $\mathrm{Cat}(\mathcal{B}_{/A}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}}_{/A})$

defines an embedding of presheaves $\mathrm{Cat}(\mathcal{B}_{/-}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}}_{/-})$ on $\mathcal{B}$. Since moreover restriction along the inclusion $\mathcal{B} \hookrightarrow \widehat{\mathcal{B}}$ defines an equivalence

$$\mathrm{Sh}_{\mathrm{Cat}(\widehat{\mathcal{S}})}(\widehat{\mathcal{B}}) \simeq \mathrm{Sh}_{\mathrm{Cat}(\widehat{\mathcal{S}})}(\mathcal{B})$$

(see the argument in [62, Remark 2.4.1]), we obtain the desired fully faithful functor $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ in $\mathrm{Cat}(\widehat{\widehat{\mathcal{B}}})$. Explicitly, an object $A \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ in context $A \in \widehat{\mathcal{B}}$ that corresponds to a $\mathcal{B}_{/A}$-category $\mathsf{C} \to A$ is contained in $\mathsf{Cat}_{\mathcal{B}}$ precisely if for any map $s\colon A' \to A$ with $A' \in \mathcal{B}$ the pullback $s^*\mathsf{C}$ is small.

2.3.2. Straightening-Unstraightening. The goal of this section is to give a quick proof of the straightening-unstraightening equivalence for $\mathcal{B}$-categories, see Theorem 2.3.2.7. This was first proven in [61]. The proof given here is a straight-forward generalization of the proof of "continuous straightening-unstraightening" given in [91] and while it might be conceptually less satisfying than the one in [61], it has the advantage that it's a bit shorter, so we decided to include it here.

Definition 2.3.2.1. A functor $p\colon \mathsf{D} \to \mathsf{C}$ of $\mathcal{B}$-categories is a *cocartesian fibration* if and only if for any $A \in \mathcal{B}$ the map $p(A)\colon \mathsf{D}(A) \to \mathsf{C}(A)$ is a cocartesian fibration and for any $s\colon B \to A$, the functor $s^*\colon \mathsf{D}(A) \to \mathsf{D}(B)$ sends $p(A)$-cocartesian edges to $p(B)$-cocartesian edges.

For two cocartesian fibrations $p\colon \mathsf{D} \to \mathsf{C}$ and $p'\colon \mathsf{D}' \to \mathsf{C}$ a *cocartesian functor* f from p to p' is a commutative triangle

$$\begin{tikzcd} \mathsf{D} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathsf{D}' \arrow[dl, "p'"] \\ & \mathsf{C} & \end{tikzcd}$$

such that for any $A \in \mathcal{B}$, the induced functor $f(A)\colon \mathsf{D}(A) \to \mathsf{D}'(A)$ sends $p(A)$-cocartesian edges to $p'(A)$-cocartesian edges. We say that a morphism $f\colon \Delta^1 \to \mathsf{D}(A)$ is *p-cocartesian* if for all $s\colon B \to A$ the morphism s^*f is $p(B)$-cocartesian.

Remark 2.3.2.2. There is also a more internal definition of cocartesian fibrations and functors, see [61, Definition 3.1.1]. That this definition agrees with the one above is proven in [61, Proposition 3.1.7].

Remark 2.3.2.3. Evidently $p\colon \mathsf{D} \to \mathsf{C}$ is a cocartesian fibration if and only if for any $d\colon A \to \mathsf{D}$ and morphism $f\colon p(d) \to y$ in C, there is some cocartesian morphism $f'\colon d \to d'$ lifting f.

Remark 2.3.2.4. Let us denote by $\mathrm{Fun}^{\mathrm{cocart}}(\Delta^1, \mathrm{Cat}_\infty)$ the full subcategory spanned by the cocartesian fibrations where the morphisms are the commutative squares

$$\begin{tikzcd} \mathcal{D} \arrow[r, "s"] \arrow[d, "p"'] & \mathcal{D}' \arrow[d, "p'"] \\ \mathcal{C} \arrow[r, "t"'] & \mathcal{C}' \end{tikzcd}$$

such that s sends p-cocartesian edges to p'-cocartesian edges. Then the inclusion $\mathrm{Fun}^{\mathrm{cocart}}(\Delta^1, \mathrm{Cat}_\infty) \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ preserves all limits. Indeed, by combining [27, Theorem 4.5] with [56, Proposition 7.3.2.6] we see that the inclusion even admits a left adjoint. It follows that if $\mathcal{B} = \mathrm{PSh}(\mathcal{C}_0)$ is a presheaf topos, it suffices to check the conditions appearing in Definition 2.3.2.1 on the full subcategory $\mathcal{C}_0 \subseteq \mathrm{PSh}(\mathcal{C}_0)$.

For later use we make the following observation:

Remark 2.3.2.5. Suppose that $\mathcal{C}$ is an ∞-category and $p\colon \mathsf{D} \to \mathrm{const}\,\mathcal{C}$ a functor. Then in order to show that p is cocartesian fibration, it suffices to check that

(1) For any $A \in \mathcal{B}$ the functor $(\eta^A)^*p(A)\colon (\eta^A)^*\mathsf{D}(A) \to \mathcal{C}$ given by pulling back $p(A)\colon \mathsf{D}(A) \to \mathrm{const}\,\mathcal{C}(A)$ along the adjunction unit $\eta^A\colon \mathcal{C} \to \mathrm{const}\,\mathcal{C}(A)$ is a cocartesian fibration.
(2) For any $s\colon B \to A$ the induced functor $s^*\colon (\eta^A)^*\mathsf{D}(A) \to (\eta^B)^*\mathsf{D}(B)$ preserves cocartesian edges.

Indeed, to see that $p(A)$ is a cocartesian fibration it suffices to check this after pulling back along any map $\alpha\colon \Delta^n \to \operatorname{const} \mathcal{C}(A)$. But by Proposition A.2, we may find some cover $s\colon B \twoheadrightarrow A$, such that $s^*\alpha$ factors through $\eta_B\colon \mathcal{C} \to \operatorname{const}\mathcal{C}(B)$. In particular we have a diagram

$$\begin{tikzcd} \Delta^n \times_{\operatorname{const}\mathcal{C}(B)} \mathsf{D}(B) \arrow[r] \arrow[d] & (\eta^B)^*\mathsf{D}(B) \arrow[r] \arrow[d] & \mathsf{D}(B) \arrow[d] \\ \Delta^n \arrow[r] \arrow[rr, bend right, "s^*\alpha"'] & \mathcal{C} \arrow[r, "\eta^B"'] & \operatorname{const}\mathcal{C}(B) \end{tikzcd}$$

in which all squares are cartesian. Since the middle arrow is a cocartesian fibration, it follows that the functor $\Delta^n \times_{\operatorname{const}\mathcal{C}(B)} \mathsf{C}(B) \to \Delta^n$ is a cocartesian fibration and the same holds for any further pullback along some $t\colon C \to B$. Since $B \to A$ is an effective epimorphism it follows using condition (2) and Remark 2.3.2.4, that also $\Delta^n \times_{\operatorname{const}\mathcal{C}(A)} \mathsf{C}(A) \to \Delta^n$ is a cocartesian fibration and thus $p(A)$ is a cocartesian fibration. A similar argument shows that for any $s\colon B \to A$, the functor s^* preserves cocartesian edges and the claim follows.

Definition 2.3.2.6. If C is a $\mathcal{B}$-category, we write $\mathrm{Cocart}_{\mathsf{C}}$ for the subcategory of $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}}$ spanned by the cocartesian fibrations and cocartesian functors between them. Since cocartesian fibrations and functors are stable under pullbacks, it follows that the functor $\mathrm{Cat}(\mathcal{B})_{/-}$ (that we get by straightening the codomain fibration $\mathrm{ev}_1\colon \mathrm{Cat}(\mathcal{B})^{\Delta^1} \to \mathrm{Cat}(\mathcal{B})$) restricts to a functor

$$\mathrm{Cocart}_{-}\colon \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty.$$

For any $\mathsf{C} \in \mathrm{Cat}(\mathcal{B})$ we may now precompose with the functor $(\mathsf{C} \times -)^{\mathrm{op}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}(\mathcal{B})^{\mathrm{op}}$ in order to obtain a functor

$$\mathsf{Cocart}_{\mathsf{C}}\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty.$$

The goal of this section is to prove the following:

Theorem 2.3.2.7. *Let C be a $\mathcal{B}$-category. Then there is an equivalence of functors $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$*

$$\mathrm{St}_{\mathsf{C}}\colon \mathsf{Cocart}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})\colon \mathrm{Un}_{\mathsf{C}}.$$

In particular $\mathsf{Cocart}_{\mathsf{C}}$ is a $\mathcal{B}$-category. Furthermore this equivalence is natural in C.

Remark 2.3.2.8. By [61, Remark 5.2.4] the functor $\mathsf{Cocart}_{\mathsf{C}}$ agrees with sheaf corresponding to the (large) $\mathcal{B}$-category of cocartesian fibrations defined there.

Remark 2.3.2.9. Dually we say that a functor $f\colon \mathsf{D} \to \mathsf{C}$ of $\mathcal{B}$-categories is a *cartesian fibration* if f^{op} is a cocartesian fibration. It follows that applying $(-)^{\mathrm{op}}\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ induces an equivalence $\mathsf{Cart}_{\mathsf{C}} \simeq \mathsf{Cocart}_{\mathsf{C}^{\mathrm{op}}}$ (here $\mathrm{Cart}_{\mathsf{C}}$ is defined in the same way as in Definition 2.3.2.6). Thus it follows from Theorem 2.3.2.7 that we also have a natural equivalence

$$\mathrm{St}_{\mathsf{C}}\colon \mathsf{Cart}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}_{\mathcal{B}})\colon \mathrm{Un}_{\mathsf{C}}.$$

Let us start with the following setup. Let $\mathcal{C}_0$ be an ∞-category and $\mathsf{C}\colon \mathcal{C}_0^{\mathrm{op}} \to \mathrm{Cat}_\infty$ a $\mathrm{PSh}(\mathcal{C}_0)$-category classified by a cocartesian fibration $p\colon \tilde{\mathsf{C}} \to \mathcal{C}_0^{\mathrm{op}}$. Then the straightening-unstraightening equivalence

$$\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathrm{Cat}_\infty) \simeq \mathrm{Cocart}_{\mathcal{C}_0^{\mathrm{op}}}$$

induces an equivalence of ∞-categories

$$\psi\colon \mathrm{Cat}(\mathrm{PSh}(\mathcal{C}_0))_{/\mathsf{C}} \to (\mathrm{Cocart}_{\mathcal{C}_0^{\mathrm{op}}})_{/\tilde{\mathsf{C}}}$$

that is natural in C. The following lemma now explicitly identifies the essential image of $\mathrm{Cocart}_{\mathsf{C}}$ under the equivalence ψ:

LEMMA 2.3.2.10. *Under the equivalence ψ, the subcategory* $\mathrm{Cocart}_{\mathsf{C}}$ *corresponds to the subcategory* $\mathrm{Cocart}_{\tilde{\mathsf{C}}} \subset (\mathrm{Cocart}_{\mathcal{C}_0^{\mathrm{op}}})_{/\tilde{\mathsf{C}}}$.

PROOF. We start by checking that the two subcategories $\psi(\mathrm{Cocart}_{\mathsf{C}})$ and $\mathrm{Cocart}_{\tilde{\mathsf{C}}}$ have the same objects. For this observe that by Remark 2.3.2.4 a functor of $\mathrm{PSh}(\mathcal{C}_0)$-categories $q\colon \mathsf{D} \to \mathsf{C}$ is a cocartesian fibration if and only if the associated morphism $\tilde{q}\colon \tilde{\mathsf{D}} \to \tilde{\mathsf{C}}$ of cocartesian fibrations over $\mathcal{C}_0^{\mathrm{op}}$ satisfies the conditions of [55, Lemma 1.4.14] and thus if only if it is a cocartesian fibration.

To conclude the proof we therefore only need to see that a morphism $f\colon \mathsf{D} \to \mathsf{D}'$ over C is a cocartesian functor if and only if the associated functor $\tilde{f}\colon \tilde{\mathsf{D}} \to \tilde{\mathsf{D}}'$ over $\tilde{\mathsf{C}}$ preserves cocartesian edges. Unwinding the definitions this follows immediately from the next Lemma. □

LEMMA 2.3.2.11. *Suppose that we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "f"] \arrow[dr, "s"] \arrow[ddr, bend right, "p"'] & & \mathcal{E} \arrow[dl, "t"'] \arrow[ddl, bend left, "r"] \\
& \mathcal{C} \arrow[d, "q"] & \\
& \mathcal{C}_0 &
\end{tikzcd}$$

where all arrows expect f are cocartesian fibrations and we assume that f sends p-cocartesian edges to r-cocartesian edges, Then f sends s-cocartesian edges to t-cocartesian edges if and only if for every $A \in \mathcal{C}_0$ the induced map on fibres

$$\begin{tikzcd}
\mathcal{D}_A \arrow[rr, "f_A"] \arrow[dr, "s_A"'] & & \mathcal{E}_A \arrow[dl, "t_A"] \\
& \mathcal{C}_A &
\end{tikzcd}$$

sends s_a cocartesian edges to t_a cocartesian edges.

PROOF. It is clear that if f preserves cocartesian edges, then so do any of the maps on fibres. For the converse note that by [57, Proposition 2.4.2.11] an edge $\alpha\colon d \to d'$ is s-cocartesian if and only if we can factor it as the composite of a p-cocartesian edge followed by an $s_{p(d')}$-cocartesian edge. Since we have an analogous description for t-cocartesian edges the claim follows from our assumption because f sends p-cocartesian edges to r-cocartesian edges. □

One of the main ingredients for our proof of Theorem 2.3.2.7 is the following:

THEOREM 2.3.2.12 ([27, Corollary 7.6]). *Let $\mathcal{C}$ be an ∞-category and let $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be a functor. There is a canonical equivalence*

$$\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/} \xrightarrow{\simeq} \mathrm{Un}(F)$$

of ∞-categories that is natural in F. Here $\mathrm{Un}(F)$ denotes the total space of the cocartesian fibration classifying F.

For our applications we will need the following slightly more precise formulation, which can be easily deduced from the results in [27]:

COROLLARY 2.3.2.13. *Let $\mathcal{C}$ be an ∞-category and let $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ be a functor. There is a canonical equivalence*

$$\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/} \xrightarrow{\simeq} \mathrm{Un}(F)$$

of cocartesian fibrations that is furthermore natural in F. In particular for any $f\colon x \to y$ in $\mathcal{C}$, the canonical functor

$$c_f\colon F(x) \times \mathcal{C}_{y/} \to \operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/} \xrightarrow{\simeq} \mathrm{Un}$$

preserves cocartesian edges.

Proof. For any cocartesian fibration $p\colon X \to \mathcal{C}$ there is a natural equivalence

$$\begin{aligned} \mathrm{map}_{\mathrm{Cocart}(\mathcal{C})}(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C})} F(-) \times \mathcal{C}_{-/}, X) &\simeq \lim_{\mathrm{Tw}(\mathcal{C})} \mathrm{map}_{\mathrm{Cocart}(\mathcal{C})}(F(-) \times \mathcal{C}_{-/}, X) \\ &\simeq \lim_{\mathrm{Tw}(\mathcal{C})} \mathrm{map}_{\mathrm{Cat}_\infty}(F(-), \mathrm{Fun}_{\mathrm{Cocart}(\mathcal{C})}(\mathcal{C}_{-/}, X)) \\ &\simeq \mathrm{map}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)}(F, \mathrm{St}(X)). \end{aligned}$$

Here the second equivalence follows by adjunction and the last equivalence follows from [27, Lemma 9.10] and [29, Proposition 2.3]. This proves the first part of the claim. The second part follows since the explicit formula in Theorem 2.3.2.12 shows that the forgetful functor $\mathrm{Cocart}(\mathcal{C}) \to \mathrm{Cat}_{\infty/\mathcal{C}}$ preserves colimits. □

Lemma 2.3.2.14. *Let $\mathcal{C}$ be an ∞-category such that $\mathcal{B}$ is a left exact accessible localisation of* $\mathrm{PSh}(\mathcal{C})$*. For any two $\mathcal{B}$-categories C and D, there is an equivalence*

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \simeq \int_c \mathrm{Fun}(\mathsf{C}(Lc), \mathsf{D}(Lc)) := \lim_{\mathrm{Tw}(\mathcal{C})} \mathrm{Fun}(\mathsf{C}(L(-)), \mathsf{D}(L(-))).$$

that is natural in $\mathsf{C}, \mathsf{D} \in \mathrm{Cat}(\mathcal{B})$.

Proof. If $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ is the inclusion, the observation that we have a canonical equivalence $\mathrm{Fun}_{\mathcal{B}}(-,-) \simeq \mathrm{Fun}_{\mathrm{PSh}(\mathcal{C})}(i(-), i(-))$ implies that we can assume that $\mathcal{B} = \mathrm{PSh}(\mathcal{C})$. In this case, we have an equivalence $\mathrm{Cat}(\mathcal{B}) \simeq \mathrm{PSh}_{\mathrm{Cat}_\infty}(\mathcal{C})$. By [29, Proposition 2.3] there is therefore an equivalence

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{C}, \mathsf{D}) \simeq \int_c \mathrm{map}_{\mathrm{Cat}_\infty}(\mathsf{C}(c), \mathsf{D}(c))$$

that is natural in C and D. Thus, for any ∞-category $\mathcal{K}$ we have a chain of natural equivalences

$$\begin{aligned} \mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{K}, \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})) &\simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\mathcal{K} \otimes \mathsf{C}, \mathsf{D}) \\ &\simeq \int_c \mathrm{map}_{\mathrm{Cat}_\infty}(\mathcal{K} \times \mathsf{C}(c), \mathsf{D}(c)) \\ &\simeq \mathrm{map}_{\mathrm{Cat}_\infty}(K, \int_c \mathrm{Fun}(\mathsf{C}(c), \mathsf{D}(c))) \end{aligned}$$

which implies that we also get an equivalence

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \simeq \int_c \mathrm{Fun}(\mathsf{C}(c), \mathsf{D}(c))$$

that is natural in C and D. □

We can now already prove the desired equivalence in the case where $\mathcal{B}$ is a presheaf topos:

Proposition 2.3.2.15. *Let C be a $\mathrm{PSh}(\mathcal{C}_0)$-category. Then there is an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{PSh}(\mathcal{C}_0)}(\mathsf{C}, \mathsf{Cat}_{\mathrm{PSh}(\mathcal{C}_0)}) \simeq \mathrm{Cocart}_{\mathsf{C}}$$

which is furthermore natural in C.

Proof. Let $p\colon \tilde{\mathsf{C}} \to \mathcal{C}_0^{\mathrm{op}}$ be the corresponding cocartesian fibration. Then Lemma 2.3.2.10 shows that there is a natural equivalence

$$\mathrm{Cocart}_{\mathsf{C}} \simeq \mathrm{Cocart}_{\tilde{\mathsf{C}}}.$$

Now by Theorem 2.3.2.12 we have a natural equivalence $\tilde{\mathsf{C}} \simeq \mathrm{colim}_{\mathrm{Tw}(\mathcal{C}_0^{\mathrm{op}})} \mathsf{C}(-) \times (\mathcal{C}_{/-})^{\mathrm{op}}$ and therefore we get a chain of natural equivalences

$$\begin{aligned} \mathrm{Cocart}_{\tilde{\mathsf{C}}} \simeq \mathrm{Fun}(\operatorname*{colim}_{\mathrm{Tw}(\mathcal{C}_0^{\mathrm{op}})} \mathsf{C}(-) \times (\mathcal{C}_{/-})^{\mathrm{op}}, \mathrm{Cat}_\infty) &\simeq \lim_{\mathrm{Tw}(\mathcal{C}_0^{\mathrm{op}})} \mathrm{Fun}(\mathsf{C}(-), \mathrm{Fun}(\mathcal{C}_{/-})^{\mathrm{op}}, \mathrm{Cat}_\infty)) \\ &= \int_c \mathrm{Fun}(\mathsf{C}(c), \mathsf{Cat}_{\mathrm{PSh}(\mathcal{C}_0)}(c)) \\ &\simeq \mathrm{Fun}_{\mathrm{PSh}(\mathcal{C}_0)}(\mathcal{C}_0, \mathsf{Cat}_{\mathrm{PSh}(\mathcal{C}_0)}) \end{aligned}$$

using straightening-unstraightening for the first equivalence and Lemma 2.3.2.14 for the last. □

We are now ready to prove the main result:

Proof of Theorem 2.3.2.7. Observe that the functor defining the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}(\mathcal{B}))$ is given by the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{-\times A} \mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \xrightarrow{\mathrm{Fun}_{\mathcal{B}}(-,\mathsf{Cat}(\mathcal{B}))} \widehat{\mathrm{Cat}}_{\infty}.$$

It follows that in order prove Theorem 2.3.2.7 it suffices to prove that we have an equivalence of ∞-categories

$$\mathrm{Cocart}_{\mathsf{C}} \simeq \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}(\mathcal{B})))$$

that is natural in C. For this we move one universe up and consider C as $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$-category so that we may apply Proposition 2.3.2.15 to deduce that we have a natural equivalence

$$\mathrm{Cocart}_{\tilde{\mathsf{C}}} \simeq \mathrm{Fun}_{\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})}(\mathsf{C}, \mathsf{Cat}_{\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})}) \simeq \int_A \mathrm{Fun}((\mathsf{C}(A), \mathrm{Fun}((\mathcal{B}_{/A})^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty})).$$

As before, we denote by $p\colon \tilde{\mathsf{C}} \to \mathcal{B}^{\mathrm{op}}$ the cocartesian fibration classifying C. Now we observe using Lemma 2.3.2.10 in the case $\mathcal{C}_0 = \mathcal{B}$, that $\mathrm{Cocart}_{\mathsf{C}}$ is naturally equivalent to the full subcategory of $\mathrm{Cocart}_{\tilde{\mathsf{C}}}$ spanned by the cocartesian fibrations $q\colon \tilde{\mathsf{D}} \to \tilde{\mathsf{C}}$ satisfying the following condition: The composite cocartesian fibration

$$\tilde{\mathsf{D}} \xrightarrow{q} \tilde{\mathsf{C}} \xrightarrow{p} \mathcal{B}^{\mathrm{op}}$$

classifies a limit preserving functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty} \subseteq \widehat{\mathrm{Cat}}_{\infty}$. On the other hand we have that

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}}) \simeq \int_A \mathrm{Fun}(\mathsf{C}(A), \mathrm{Fun}^{\lim}((\mathcal{B}_{/A})^{\mathrm{op}}, \mathrm{Cat}_{\infty})) \subseteq \int_A \mathrm{Fun}(\mathsf{C}(A), \mathrm{Fun}((\mathcal{B}_{/A})^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty})).$$

Therefore it remains to check that a cocartesian fibration $q\colon \tilde{\mathsf{D}} \to \tilde{\mathsf{C}}$ classifies a limit preserving functor to $\mathrm{Cat}_{\infty} \subseteq \widehat{\mathrm{Cat}}_{\infty}$ after composing with $p\colon \tilde{\mathsf{C}} \to \mathcal{B}^{\mathrm{op}}$ if and only if for any $f\colon A \to B \in \mathrm{Tw}(\mathcal{B}^{\mathrm{op}})$ and $c \in \mathsf{C}(B)$ the functor classified by pulling back q along

$$\varphi_{(c,f)}\colon \{c\} \times (\mathcal{B}_{/A})^{\mathrm{op}} \to \mathsf{C}(B) \times (\mathcal{B}_{/A})^{\mathrm{op}} \to \tilde{\mathsf{C}}$$

classifies a limit preserving functor $(\mathcal{B}_{/A})^{\mathrm{op}} \to \mathrm{Cat}_{\infty} \subseteq \widehat{\mathrm{Cat}}_{\infty}$. Let us begin by assuming that $p \circ q$ classifies a limit preserving functor to Cat_{∞}. In this case note that the cocartesian fibration $q_{(c,f)}\colon (\mathcal{B}_{/A})^{\mathrm{op}} \times_{\tilde{\mathsf{C}}} \tilde{\mathsf{D}} \to (\mathcal{B}_{/A})^{\mathrm{op}}$ given by pulling back q along $\varphi_{(c,f)}$ also sits inside a cartesian square

$$(+) \qquad \begin{tikzcd} (\mathcal{B}_{/A})^{\mathrm{op}} \times_{\tilde{\mathsf{C}}} \tilde{\mathsf{D}} \arrow[r] \arrow[d, "q_{(x,f)}"'] & (\mathcal{B}_{/A})^{\mathrm{op}} \times_{\mathcal{B}^{\mathrm{op}}} \tilde{\mathsf{D}} \arrow[d] \\ (\mathcal{B}_{/A})^{\mathrm{op}} \arrow[r] & (\mathcal{B}_{/A})^{\mathrm{op}} \times_{\mathcal{B}^{\mathrm{op}}} \tilde{\mathsf{C}} \end{tikzcd}$$

where the right vertical arrow is given by pulling back along the forgetful functor $(\mathcal{B}_{/A})^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$. Furthermore note that since the functor $\varphi_{(c,f)}$ sends any edge in $(\mathcal{B}_{/A})^{\mathrm{op}}$ to a cocartesian edge in $\tilde{\mathsf{C}}$ (using Corollary 2.3.2.13), the above square is also a pullback in $\mathrm{Cocart}_{(\mathcal{B}_{/A})^{\mathrm{op}}}$. Also note that the projection $(\mathcal{B}_{/A})^{\mathrm{op}} \times_{\mathcal{B}^{\mathrm{op}}} \tilde{\mathsf{C}} \to (\mathcal{B}_{/A})^{\mathrm{op}}$ classifies the composite

$$(\mathcal{B}_{/A})^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{C}(-)} \mathrm{Cat}_{\infty}$$

which preserves all limits. Since the same holds for $(\mathcal{B}_{/A})^{\mathrm{op}} \times_{\mathcal{B}^{\mathrm{op}}} \tilde{\mathsf{D}} \to (\mathcal{B}_{/A})^{\mathrm{op}}$ it follows that $q_{(x,f)}$ classifies a functor which is a pullback of limit preserving functors with values in Cat_{∞} and thus has the same properties.

For the converse let us fix diagram $d\colon K \to \mathcal{B}$. Then for any $c \in \mathsf{C}(\operatorname{colim} d)$ we consider the functor

$$\varphi_{(c,\mathrm{id}_{\operatorname{colim} d})}\colon (\mathcal{B}_{/\operatorname{colim} d})^{\mathrm{op}} \to \tilde{\mathsf{C}}$$

as above, which induces a cartesian square of the form (+) in $\mathrm{Cocart}_{\mathcal{B}_{/\operatorname{colim} d}}$ Let us denote the functor classified by $q_{(c,\mathrm{id}_{\operatorname{colim} d})}$ by F_c. Then the induced commutative diagram of functors yields a commutative cube

$$\begin{tikzcd}
& \lim_d F_c(-) \arrow[rr] \arrow[dd] & & \lim_d \mathsf{D}(-) \arrow[dd] \\
F_c(\operatorname{colim} d) \arrow[ur, "\simeq"] \arrow[rr] \arrow[dd] & & \mathsf{D}(\operatorname{colim} d) \arrow[ur] \arrow[dd] & \\
& * \arrow[rr] & & \lim_d \mathsf{C}(-) \\
* \arrow[ur] \arrow[rr, "c"] & & \mathsf{C}(\operatorname{colim} d) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

where the lower right diagonal functor is an equivalence because C is a $\mathcal{B}$-category and the upper left diagonal is an equivalence by assumption. Since we have such a cube for any $c \in \mathsf{C}(\operatorname{colim} d)$ it follows that the map $\mathsf{D}(\operatorname{colim} d) \to \lim_d \mathsf{D}(-)$ is a fibre-wise equivalence and therefore an equivalence. Furthermore D lands in $\mathrm{Cat}_\infty \subset \widehat{\mathrm{Cat}}_\infty$ because F_c and C do by assumption. This completes the proof. $\square$

Let us now remark that one can also deduce straightening-unstraightening for left fibrations from Theorem 2.3.2.7. For this observe that applying Construction 2.3.1.1 to the fully faithful functor $\mathcal{S} \to \mathrm{Cat}_\infty$ we obtain a functor $j\colon \Omega_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$. Furthermore it follows directly from Proposition 2.1.9.3 that j is fully faithful. Therefore composing with j gives a fully faithful functor $j_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$ by [62, Proposition 3.8.4]. Furthermore note that an arbitrary morphism of left fibrations of $\mathcal{B}$-categories is a cocartesian functor so that we also get an inclusion of a full subcategory $\mathsf{LFib}_{\mathsf{C}} \subseteq \mathsf{Cocart}_{\mathsf{C}}$. We can then conclude:

Corollary 2.3.2.16. *Let C be a $\mathcal{B}$-category. Then the equivalence of Theorem 2.3.2.7 restricts to an equivalence of full subcategories*

$$\mathsf{LFib}_{\mathsf{C}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}}).$$

Proof. It follows from the proof of Theorem 2.3.2.7 that a cocartesian fibration $q(A)\colon \mathsf{D} \to \mathsf{C}$ corresponds to a functor that factors through $\Omega_{\mathcal{B}} \subseteq \mathsf{Cat}_{\mathcal{B}}$ if and only if for every $A \in \mathcal{B}$ all fibres of the functors $q(A)\colon \mathsf{D}(A) \to \mathsf{C}(A)$ are ∞-groupoids. This is the case if and only if they are left fibrations by Proposition 2.1.11.3, so the claim follows. $\square$

2.4. Adjunctions

In this section we will study *adjunctions* between $\mathcal{B}$-categories. We begin in § 2.4.1 by defining such adjunctions as ordinary adjunctions in the underlying bicategory of $\mathrm{Cat}(\mathcal{B})$. In § 2.4.2 we compare our definition with *relative* adjunctions and prove a convenient section-wise criterion for when a functor admits a left or right adjoint. In § 2.4.3 we discuss an alternative approach to adjunctions based on an equivalence of mapping $\mathcal{B}$-groupoids. Finally, we discuss the special case of *reflective subcategories* in § 2.4.4.

2.4.1. Definitions and basic properties. Let C and D be $\mathcal{B}$-categories, let $f, g\colon \mathsf{C} \rightrightarrows \mathsf{D}$ be two functors and let $\alpha\colon f \to g$ be a morphism of functors, i.e. a map in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$. If $h\colon \mathsf{E} \to \mathsf{C}$ is any other functor, we denote by $\alpha h\colon fh \to gh$ the map $h^*(\alpha)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{E}, \mathsf{D})$. Dually, if $k\colon \mathsf{D} \to \mathsf{E}$ is an arbitrary functor, we denote by $k\alpha\colon kf \to kg$ the map $k_*(\alpha)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$. Having established the necessary notational conventions, we may now define:

Definition 2.4.1.1. Let C and D be $\mathcal{B}$-categories. An *adjunction* between C and D is a tuple (l, r, η, ϵ), where $l\colon \mathsf{C} \to \mathsf{D}$ and $r\colon \mathsf{D} \to \mathsf{C}$ are functors and where $\eta\colon \mathrm{id}_{\mathsf{D}} \to rl$ and $\epsilon\colon lr \to \mathrm{id}_{\mathsf{C}}$ are maps such that there are commutative triangles

$$
\begin{tikzcd}
l \arrow[r, "l\eta"] \arrow[dr, "\mathrm{id}"'] & lrl \arrow[d, "\epsilon l"] \\
& l
\end{tikzcd}
\qquad
\begin{tikzcd}
rlr \arrow[d, "r\epsilon"'] & r \arrow[l, "\eta r"'] \arrow[dl, "\mathrm{id}"] \\
r &
\end{tikzcd}
$$

in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ and in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{D}, \mathsf{C})$, respectively. We denote such an adjunction by $l \dashv r$, and we refer to η as the *unit* and to ϵ as the *counit* of the adjunction. We say that a pair $(l, r) \colon \mathsf{C} \leftrightarrows \mathsf{D}$ *defines an adjunction* if there exist transformations η and ϵ as above such that the tuple (l, r, η, ϵ) is an adjunction.

Analogous to how adjunctions between ∞-categories can be defined (see [48, §17]), Definition 2.4.1.1 is equivalent to an adjunction in the underlying homotopy bicategory of the $(\infty, 2)$-category $\mathrm{Cat}(\mathcal{B})$ (see § 2.1.6). We may therefore make use of the usual bicategorical arguments to derive results for adjunctions in $\mathrm{Cat}(\mathcal{B})$. Hereafter, we list a few of these results, we refer the reader to [30, § I.6] and [80, § 2.1] for proofs.

Proposition 2.4.1.2. *If $(l \dashv r) \colon \mathsf{C} \leftrightarrows \mathsf{D}$ and $(l' \dashv r') \colon \mathsf{D} \leftrightarrows \mathsf{E}$ are adjunctions between $\mathcal{B}$-categories, then the composite functors define an adjunction $(ll' \dashv r'r) \colon \mathsf{C} \leftrightarrows \mathsf{E}$.* ☐

Proposition 2.4.1.3. *Adjoints are unique if they exist, i.e if $(l \dashv r)$ and $(l \dashv r')$ are adjunctions between $\mathcal{B}$-categories, then $r \simeq r'$. Dually, if $(l \dashv r)$ and $(l' \dashv r)$ are adjunctions, then $l \simeq l'$.* ☐

Proposition 2.4.1.4. *In order for a pair $(l, r) \colon \mathsf{C} \leftrightarrows \mathsf{D}$ of functors between $\mathcal{B}$-categories to define an adjunction, it suffices to provide maps $\eta \colon \mathrm{id}_{\mathsf{D}} \to rl$ and $\epsilon \colon lr \to \mathrm{id}_{\mathsf{C}}$ such that the compositions $\epsilon l \circ l\eta$ and $r\epsilon \circ \eta r$ are equivalences.* ☐

Corollary 2.4.1.5. *If $f \colon \mathsf{C} \to \mathsf{D}$ is an equivalence between $\mathcal{B}$-categories, then the pair (f, f^{-1}) defines an adjunction.* ☐

Corollary 2.4.1.6. *For any adjunction $(l \dashv r) \colon \mathsf{C} \leftrightarrows \mathsf{D}$ between $\mathcal{B}$-categories and any equivalence $f \colon D \simeq D'$, the induced pair $(lf^{-1}, fr) \colon \mathsf{C} \leftrightarrows \mathsf{D}'$ defines an adjunction as well.* ☐

If $\mathcal{A}$ and $\mathcal{B}$ are ∞-topoi and $f \colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ is a functor, we will often need to know whether f carries an adjunction $l \dashv r$ in $\mathrm{Cat}(\mathcal{B})$ to an adjunction $f(l) \dashv f(r)$ in $\mathrm{Cat}(\mathcal{A})$. This is obviously the case whenever f is a functor of $(\infty, 2)$-categories. Since we do not wish to dive too deep into $(\infty, 2)$-categorical arguments, we will instead make use of the straightforward observation that f preserves adjunctions whenever there is a bifunctorial map

$$\mathrm{Fun}_{\mathcal{B}}(-, -) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))$$

that recovers the action of f on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor.

Lemma 2.4.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be ∞-topoi and let $f \colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ be a functor that preserves finite products. Suppose furthermore that there is a morphism of functors $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$, where $\mathrm{const}_{\mathcal{B}} \colon \mathrm{Cat}_{\infty} \to \mathrm{Cat}(\mathcal{B})$ and $\mathrm{const}_{\mathcal{A}} \colon \mathrm{Cat}_{\infty} \to \mathrm{Cat}(\mathcal{A})$ are the constant sheaf functors. Then f induces a bifunctorial map $\mathrm{Fun}_{\mathcal{B}}(-, -) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))$ that recovers the action of f on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor. Moreover, if f is fully faithful and if the map $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$ restricts to an equivalence on the essential image of f, then this map is an equivalence.*

Proof. Since f preserves finite products, the map $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$ induces a map

$$- \otimes f(-) \to f(- \otimes -)$$

of bifunctors $\mathrm{Cat}_{\infty} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$. This map gives rise to the first arrow in the composition

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(f(- \otimes -), f(-)) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(- \otimes f(-), f(-)) \simeq \mathrm{map}_{\mathrm{Cat}_{\infty}}(-, \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))),$$

and by precomposition with the morphism $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-\otimes -, -) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(f(-\otimes -), f(-))$ that is induced by f and Yoneda's lemma, we end up with the desired morphism of functors

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))$$

that recovers the morphism $\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(-,-) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{A})}(f(-), f(-))$ upon restriction to core ∞-groupoids. By construction, this map is an equivalence whenever f is fully faithful and the map $\mathrm{const}_{\mathcal{A}} \to f \circ \mathrm{const}_{\mathcal{B}}$ is an equivalence. □

REMARK 2.4.1.8. In the situation of Lemma 2.4.1.7, the construction in the proof shows that if C and D are $\mathcal{B}$-categories, the functor

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \to \mathrm{Fun}_{\mathcal{A}}(f(\mathsf{C}), f(\mathsf{D}))$$

that is induced by f and the morphism of functors $\varphi\colon -\otimes f(-) \to f(-\otimes -)$ is given as the transpose of the composition

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \otimes f(\mathsf{C}) \xrightarrow{\varphi} f(\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \otimes \mathsf{C}) \xrightarrow{f(\mathrm{ev})} f(\mathsf{D})$$

in which $\mathrm{ev}\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \otimes \mathsf{C} \to \mathsf{D}$ denotes the counit of the adjunction $- \otimes \mathsf{C} \dashv \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, -)$.

Using Lemma 2.4.1.7, one now finds:

COROLLARY 2.4.1.9. *Let $f_*\colon \mathcal{B} \to \mathcal{A}$ be a geometric morphism of ∞-topoi. If a pair (l, r) of functors in* $\mathrm{Cat}(\mathcal{B})$ *defines an adjunction, then the pair $(f_*(l), f_*(r))$ defines an adjunction in* $\mathrm{Cat}(\mathcal{A})$*. Moreover, the converse is true whenever f_* is fully faithful.*

Dually, for any algebraic morphism $f^*\colon \mathcal{A} \to \mathcal{B}$ of ∞-topoi, if a pair (l, r) of functors in $\mathrm{Cat}(A)$ *defines an adjunction, then the pair $(f^*(l), f^*(r))$ defines an adjunction in* $\mathrm{Cat}(\mathcal{B})$*, and the converse is true whenever f^* is fully faithful.*

PROOF. This follows immediately from Lemma 2.4.1.7 on account of the equivalence $\mathrm{const}_{\mathcal{B}} \simeq f^* \circ \mathrm{const}_{\mathcal{A}}$ and the map $\mathrm{const}_{\mathcal{A}} \to f_* \mathrm{const}_{\mathcal{B}}$ that is induced by the adjunction unit $\mathrm{id}_{\mathcal{A}} \to f_* f^*$. □

Recall from Proposition 2.1.6.7 that the inclusion $\mathcal{B} \simeq \mathrm{Grpd}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ admits a left adjoint $(-)^{\mathrm{gpd}}$. We now obtain:

COROLLARY 2.4.1.10. *The groupoidification functor $(-)^{\mathrm{gpd}}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Grpd}(\mathcal{B})$ preserves adjunctions and therefore carries any left or right adjoint functor to an equivalence in* $\mathrm{Grpd}(\mathcal{B})$.

PROOF. The first part follows by applying Lemma 2.4.1.7 to the map $\eta\, \mathrm{const}_{\mathcal{B}}\colon \mathrm{const}_{\mathcal{B}} \to (-)^{\mathrm{gpd}} \circ \mathrm{const}_{\mathcal{B}}$ in which $\eta\colon \mathrm{id}_{\mathrm{Cat}(\mathcal{B})} \to (-)^{\mathrm{gpd}}$ denotes the adjunction unit. As for the second part, it suffices to note that if $(l \dashv r)\colon \mathsf{G} \leftrightarrows \mathsf{H}$ is an adjunction between $\mathcal{B}$-groupoids, then since both $\mathrm{Fun}_{\mathcal{B}}(\mathsf{G}, \mathsf{G})$ and $\mathrm{Fun}_{\mathcal{B}}(\mathsf{H}, \mathsf{H})$ are ∞-groupoids both unit and counit must be an equivalence. □

COROLLARY 2.4.1.11. *For any simplicial object $K \in \mathcal{B}_\Delta$, the endofunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$ on* $\mathrm{Cat}(\mathcal{B})$ *preserves adjunctions in* $\mathrm{Cat}(\mathcal{B})$.

PROOF. By bifunctoriality of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$, precomposition with the terminal map $K \to 1$ in $\mathcal{B}_\Delta$ gives rise to the diagonal functor $\mathrm{id}_{\mathrm{Cat}(\mathcal{B})} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$, and combining this map with the functor $\mathrm{const}_{\mathcal{B}}$ then defines a map $\mathrm{const}_{\mathcal{B}}(-) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathrm{const}_{\mathcal{B}}(-))$, hence Lemma 2.4.1.7 applies. □

REMARK 2.4.1.12. In the situation of Corollary 2.4.1.11, Remark 2.4.1.8 shows that for any two $\mathcal{B}$-categories C and D, the induced map

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \to \mathrm{Fun}_{\mathcal{B}}(\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}))$$

is the one that is determined by the composition

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \otimes (\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \times K) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_K} \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \otimes \mathsf{C} \xrightarrow{\mathrm{ev}_{\mathsf{C}}} \mathsf{D}$$

in light of the two adjunctions $- \times K \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$ and $- \otimes \mathsf{C} \dashv \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, -)$. Here ev_K and ev_{C}, respectively, denote the counits of these adjunctions.

Combining Corollary 2.4.1.9 with Corollary 2.4.1.11, one furthermore obtains:

COROLLARY 2.4.1.13. *For any simplicial object $K \in \mathcal{B}_\Delta$, the functor* $\mathrm{Fun}_{\mathcal{B}}(K, -)\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$ *carries adjunctions in* $\mathrm{Cat}(\mathcal{B})$ *to adjunctions in* Cat_∞. □

Similarly as above, if $\mathcal{A}$ and $\mathcal{B}$ are ∞-topoi and if $f\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{A})$ is a functor such that there is a bifunctorial map

$$\mathrm{Fun}_{\mathcal{B}}(-,-) \to \mathrm{Fun}_{\mathcal{A}}(f(-), f(-))^{\mathrm{op}}$$

that recovers the action of f on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor, the functor f sends an adjunction $l \dashv r$ in $\mathrm{Cat}(\mathcal{B})$ to an adjunction $f(r) \dashv f(l)$ in $\mathrm{Cat}(\mathcal{A})$. One therefore finds:

PROPOSITION 2.4.1.14. *The equivalence* $(-)^{\mathrm{op}}\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})$ *sends an adjunction $l \dashv r$ to an adjunction $r^{\mathrm{op}} \dashv l^{\mathrm{op}}$.*

PROOF. This follows from the evident equivalence

$$(-)^{\mathrm{op}}\colon \mathrm{Fun}_{\mathcal{B}}(-,-) \simeq \mathrm{Fun}_{\mathcal{B}}((-)^{\mathrm{op}}, (-)^{\mathrm{op}})^{\mathrm{op}}$$

of bifunctors $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$, which shows that if $l \dashv r$ is an adjunction with unit η and counit ϵ, then the pair $(r^{\mathrm{op}}, l^{\mathrm{op}})$ defines an adjunction on account of the maps $\epsilon^{\mathrm{op}}\colon \mathrm{id} \to l^{\mathrm{op}} r^{\mathrm{op}}$ and $\eta^{\mathrm{op}}\colon r^{\mathrm{op}} l^{\mathrm{op}} \to \mathrm{id}$ that correspond to ϵ and η via the above equivalence. □

The contravariant versions of the functors considered in Corollary 2.4.1.11 and Corollary 2.4.1.13 preserve adjunctions as well: If C is an arbitrary $\mathcal{B}$-category, functoriality of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{C})$ defines a map

$$\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\mathsf{E}, \mathsf{D}) \to \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}, \mathsf{C}))$$

that is natural in E and D. The composition

$$\begin{aligned}\mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(- \otimes \mathsf{E}, \mathsf{D}) &\to \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(- \otimes \mathsf{E}, \mathsf{C}))\\ &\simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}) \times (- \otimes \mathsf{E}), \mathsf{C})\\ &\simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}((- \otimes \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C})) \times \mathsf{E}, \mathsf{C})\\ &\simeq \mathrm{map}_{\mathrm{Cat}(\mathcal{B})}(- \otimes \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}, \mathsf{C}))\end{aligned}$$

(in which each step is natural in D and E) and Yoneda's lemma now give rise to a map

$$\mathrm{Fun}_{\mathcal{B}}(\mathsf{E}, \mathsf{D}) \to \mathrm{Fun}_{\mathcal{B}}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{C}), \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}, \mathsf{C}))$$

that defines a morphism of functors $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \times \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}_\infty$ and that recovers the action of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{C})$ on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor. One therefore finds:

PROPOSITION 2.4.1.15. *For any $\mathcal{B}$-category C, the two functors $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{C})$ and $\mathrm{Fun}_{\mathcal{B}}(-, \mathsf{C})$ carry an adjunction $l \dashv r$ in $\mathrm{Cat}(\mathcal{B})$ to an adjunction $r^* \dashv l^*$ in $\mathrm{Cat}(\mathcal{B})$ and in Cat_∞, respectively.* □

2.4.2. Adjunctions via relative adjunctions of cartesian fibrations. Recall from the discussion in 2.1.7 that every pair $(l, r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ of functors between (large) $\mathcal{B}$-categories give rise to a pair of functors $(\int l, \int r)\colon \int \mathsf{C} \leftrightarrows \int \mathsf{D}$ between the associated cartesian fibrations over $\mathcal{B}$. In this section, our goal is to characterise those pairs $(\int l, \int r)$ that come from an adjunction $l \dashv r$.

Given any small ∞-category $\mathcal{C}$, there is a bifunctor

$$- \otimes -\colon \mathrm{Cat}_\infty \times \mathrm{Cart}(\mathcal{C}) \to \mathrm{Cart}(\mathcal{C})$$

that sends a pair $(\mathcal{X}, \mathcal{P} \to \mathcal{C})$ to the cartesian fibration $\mathcal{X} \times \mathcal{P} \to \mathcal{P} \to \mathcal{C}$ in which the first arrow is the natural projection. Explicitly, a morphism in $\mathcal{X} \times \mathcal{P}$ is cartesian precisely if its projection to $\mathcal{P}$ is cartesian in $\mathcal{P}$ and its projection to $\mathcal{X}$ is an equivalence. For an arbitrary fixed cartesian fibration $\mathcal{P} \to \mathcal{C}$, the functor $-\otimes\mathcal{P}\colon \mathrm{Cat}_\infty \to \mathrm{Cart}(\mathcal{C}) \hookrightarrow (\widehat{\mathrm{Cat}}_\infty)_{/\mathcal{C}}$ admits a right adjoint $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{P},-)$ that sends a map $\mathcal{Q}\to\mathcal{C}$ to the ∞-category that is defined by the pullback square

$$\begin{tikzcd} \mathrm{Fun}_{/\mathcal{C}}(\mathcal{P},\mathcal{Q}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{P},\mathcal{Q}) \arrow[d] \\ 1 \arrow[r] & \mathrm{Fun}(\mathcal{P},\mathcal{C}) \end{tikzcd}$$

in which the vertical map on the right is given by postcomposition with $\mathcal{Q}\to\mathcal{C}$ and in which the lower horizontal arrow picks out the cartesian fibration $\mathcal{P}\to\mathcal{C}$ [57, Proposition 5.2.5.1]. If $\mathcal{Q}\to\mathcal{C}$ is a cartesian fibration, let $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P},\mathcal{Q})\hookrightarrow \mathrm{Fun}_{/\mathcal{C}}(\mathcal{P},\mathcal{Q})$ denote the full subcategory that is spanned by those functors that preserve cartesian edges, and observe that this defines a functor

$$\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P},-)\colon \mathrm{Cart}(\mathcal{C})\to\mathrm{Cat}_\infty\,.$$

As the equivalence $\mathrm{map}_{/\mathcal{C}}(\mathcal{X}\otimes\mathcal{P},\mathcal{Q})\simeq \mathrm{map}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{X},\mathrm{Fun}_{/\mathcal{C}}(\mathcal{P},\mathcal{Q}))$ identifies functors $\mathcal{X}\otimes\mathcal{P}\to\mathcal{Q}$ that preserve cartesian arrows with functors $\mathcal{X}\to\mathrm{Fun}_{/\mathcal{C}}(\mathcal{P},\mathcal{Q})$ that take values in $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P},\mathcal{Q})$, one obtains an adjunction $(-\otimes\mathcal{P}\dashv \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P},-))\colon \widehat{\mathrm{Cat}}_\infty \leftrightarrows \mathrm{Cart}(\mathcal{C})$. By making use of the bifunctoriality of $-\otimes-$, the assignment $\mathcal{P}\mapsto \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{P},-)$ gives rise to a bifunctor $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(-,-)$ in a unique way such that there is an equivalence

$$\mathrm{map}_{\mathrm{Cart}(\mathcal{C})}(-\otimes-,-)\simeq \mathrm{map}_{\mathrm{Cat}_\infty}(-,\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(-,-)).$$

Note that there is an equivalence $\int(-\otimes-)\simeq -\otimes\int(-)$ of bifunctors $\mathrm{Cat}_\infty\times\mathrm{Cat}(\mathrm{PSh}_{\mathcal{S}}(\mathcal{C}))\to\mathrm{Cart}(\mathcal{C})$ in which the tensoring on the left-hand side is given by the canonical tensoring in $\mathrm{Cat}(\mathrm{PSh}_{\mathcal{S}}(\mathcal{C}))$ over Cat_∞, i.e. by the bifunctor $\mathrm{const}(-)\times -$. By the uniqueness of adjoints, one therefore finds:

Proposition 2.4.2.1. *For any small ∞-category $\mathcal{C}$, there is an equivalence*

$$\mathrm{Fun}_{\mathrm{PSh}_{\mathcal{S}}(\mathcal{C})}(-,-)\simeq \mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\textstyle\int(-),\int(-))$$

of bifunctors $\mathrm{Cat}(\mathrm{PSh}_{\mathcal{S}}(\mathcal{C}))^{\mathrm{op}}\times\mathrm{Cat}(\mathrm{PSh}_{\mathcal{S}}(\mathcal{C}))\to\mathrm{Cat}_\infty$ *that recovers the action of* $\int\colon \mathrm{PSh}_{\mathcal{S}}(\mathcal{C})\to\mathrm{Cart}(\mathcal{C})$ *on mapping ∞-groupoids upon postcomposition with the core ∞-groupoid functor.* □

Recall the notion of a *relative adjunction* between cartesian fibrations as defined by Lurie in [56, § 7.3]:

Definition 2.4.2.2. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{P}$ and $\mathcal{Q}$ be cartesian fibrations over $\mathcal{C}$. A *relative adjunction* between $\mathcal{P}$ and $\mathcal{Q}$ is defined to be an adjunction $(l\dashv r)\colon \mathcal{Q}\leftrightarrows\mathcal{P}$ between the underlying ∞-categories such that both l and r define maps in $\mathrm{Cart}(\mathcal{C})$ and such that the structure map $p\colon\mathcal{P}\to\mathcal{C}$ sends the adjunction counit ϵ to the identity transformation on p and the structure map $q\colon\mathcal{Q}\to\mathcal{C}$ sends the adjunction unit η to the identity transformation on q.

By construction of the bifunctor $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(-,-)$, it is immediate that a pair $(l,r)\colon\mathcal{Q}\leftrightarrows\mathcal{P}$ of maps in $\mathrm{Cart}(\mathcal{C})$ defines a relative adjunction if and only if there are maps $\eta\colon \mathrm{id}_{\mathcal{Q}}\to rl$ and $\epsilon\colon lr\to\mathrm{id}_{\mathcal{P}}$ in $\mathrm{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{Q},\mathcal{Q})$ and $\mathrm{Fun}^{\mathrm{Cart}}_{\mathcal{C}}(\mathcal{P},\mathcal{P})$, respectively, that satisfy the triangle identities from Definition 2.4.1.1. Proposition 2.4.2.1 therefore implies:

Corollary 2.4.2.3. *For any small ∞-category $\mathcal{C}$, a pair $(l,r)\colon \mathsf{C}\leftrightarrows\mathsf{D}$ of functors between $\mathrm{PSh}_{\mathcal{S}}(\mathcal{C})$-categories defines an adjunction if and only if the associated pair $(\int l,\int r)$ defines a relative adjunction in* $\mathrm{Cart}(\mathcal{C})$. □

Observe that as by [57, Lemma 6.3.5.28] the inclusion $\widehat{\mathcal{B}}\hookrightarrow\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$ defines a geometric morphism of ∞-topoi (relative to the universe $\mathbf{V}$), Corollary 2.4.1.9 implies that the pair (l,r) defines an adjunction between large $\mathcal{B}$-categories if and only if it defines an adjunction in $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$. We may therefore conclude:

Corollary 2.4.2.4. *A pair $(l,r)\colon\mathsf{C}\leftrightarrows\mathsf{D}$ of functors between large $\mathcal{B}$-categories defines an adjunction if and only if the associated pair $(\int l,\int r)$ defines a relative adjunction in* $\mathrm{Cart}(\mathcal{B})$. □

The upshot of Corollary 2.4.2.4 is that we may make use of Lurie's results on relative adjunctions [56, § 7.3.2] in order to formulate a useful criterion for when a functor between $\mathcal{B}$-categories admits a right and a left adjoint, respectively. For this we need to recall the *mate* construction:

DEFINITION 2.4.2.5. For any right lax square in $\mathrm{Cat}(\mathcal{B})$ of the form

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "r_1"] \arrow[d, "f"'] & \mathsf{D}_1 \arrow[d, "g"] \arrow[dl, phantom, "\swarrow\varphi"] \\ \mathsf{C}_2 \arrow[r, "r_2"'] & \mathsf{D}_2 \end{tikzcd}$$

such that both r_1 and r_2 admit left adjoints l_1 and l_2 exhibited by units $\eta_i\colon \mathrm{id} \to r_i l_i$ and counits $\epsilon_i\colon l_i r_i \to \mathrm{id}$, there is a left lax square

$$\begin{tikzcd} \mathsf{C}_1 \arrow[d, "f"'] & \mathsf{D}_1 \arrow[l, "l_1"'] \arrow[d, "g"] \\ \mathsf{C}_2 \arrow[ur, phantom, "\nwarrow\psi"] & \mathsf{D}_2 \arrow[l, "l_2"] \end{tikzcd}$$

in which ψ is defined as the composite map

$$l_2 g \xrightarrow{l_2 g \eta_1} l_2 g r_1 l_1 \xrightarrow{l_2 \varphi l_1} l_2 r_2 f l_1 \xrightarrow{\epsilon_2 f l_1} f l_1.$$

Conversely, when starting with the latter left lax square, the original right lax square is recovered by means of the composition

$$g r_1 \xrightarrow{\eta_2 g r_1} r_2 l_2 g r_1 \xrightarrow{r_2 \psi r_1} r_2 f l_1 r_1 \xrightarrow{r_2 f \epsilon_1} r_2 f.$$

The left lax square determined by ψ is referred to as the *mate* of the right lax square determined by ψ, and vice versa.

REMARK 2.4.2.6. In the 2-categorical context mates have been studied under the name *adjoint squares* by Gray in [30, §I.6], and under the name mate in [52, §2]. In the $(\infty, 2)$-categorical setting they have been studied by Haugseng, see the discussion following [37, Remark 4.5]. In the case where the starting 2-cell is invertible, which we will mostly use, they are also already considered in [56, Definition 4.7.4.13].

REMARK 2.4.2.7. The mate construction is functorial in the following sense: Consider the composition of right lax squares

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "r_1"] \arrow[d, "f_1"'] & \mathsf{D}_1 \arrow[d, "g_1"] \arrow[dl, phantom, "\swarrow\varphi_1"] \\ \mathsf{C}_2 \arrow[r, "r_2"] \arrow[d, "f_2"'] & \mathsf{D}_2 \arrow[d, "g_2"] \arrow[dl, phantom, "\swarrow\varphi_2"] \\ \mathsf{C}_3 \arrow[r, "r_3"'] & \mathsf{D}_3, \end{tikzcd}$$

by which we simply mean the composition $(\varphi_2 f_1) \circ (g_2 \varphi_1)$. Then the mate of the composite square is given by the composition of left lax squares

$$\begin{tikzcd} \mathsf{C}_1 \arrow[d, "f_1"'] & \mathsf{D}_1 \arrow[l, "l_1"'] \arrow[d, "g_1"] \\ \mathsf{C}_2 \arrow[d, "f_2"'] \arrow[ur, phantom, "\nwarrow\psi_1"] & \mathsf{D}_2 \arrow[l, "l_2"'] \arrow[d, "g_2"] \\ \mathsf{C}_3 \arrow[ur, phantom, "\nwarrow\psi_2"] & \mathsf{D}_3, \arrow[l, "l_3"] \end{tikzcd}$$

in which ψ_1 denotes the mate of φ_1 and ψ_2 denotes the mate of φ_2. This is easily checked using the triangle identities for adjunctions and the interchange law in bicategories.

Similarly, one can show that the mate of the *horizontal* composition of right lax squares

$$\begin{tikzcd} \mathsf{C}_1 \arrow[r, "r_1"] \arrow[d, "f"] & \mathsf{D}_1 \arrow[r, "r_1'"] \arrow[d, "g"] \arrow[dl, Rightarrow, "\varphi_1"'] & \mathsf{E}_1 \arrow[d, "h"] \arrow[dl, Rightarrow, "\varphi_2"'] \\ \mathsf{C}_2 \arrow[r, "r_2"] & \mathsf{D}_2 \arrow[r, "r_2'"] & \mathsf{E}_2 \end{tikzcd}$$

(i.e. the composite $r_2'\varphi_1 \circ \varphi_2 r_1$) is given by the horizontal composition of the associated mates.

Lemma 2.4.2.8. *Let $\mathcal{C}$ be an ∞-category and let $p\colon \mathcal{P} \to \mathcal{C}$ and $q\colon \mathcal{Q} \to \mathcal{C}$ be cartesian fibrations. A map $r\colon \mathcal{P} \to \mathcal{Q}$ in* $\mathrm{Cart}(\mathcal{C})$ *is a relative right adjoint if and only if*

(1) for all $c \in \mathcal{C}$ the functor $r|_c\colon \mathcal{P}|_c \to \mathcal{Q}|_c$ that is induced by r on the fibres over c admits a left adjoint $l_c\colon \mathcal{Q}|_c \to \mathcal{P}|_c$;

(2) For every morphism $g\colon d \to c$ in $\mathcal{C}$, the mate of the commutative square

$$\begin{tikzcd} \mathcal{P}|_c \arrow[r, "r|_c"] \arrow[d, "g^*"'] & \mathcal{Q}|_c \arrow[d, "g^*"] \arrow[dl, Rightarrow, "\simeq"'] \\ \mathcal{P}|_d \arrow[r, "r|_d"] & \mathcal{Q}|_d \end{tikzcd}$$

commutes.

If this is the case, the relative left adjoint l of r recovers the map l_c on the fibres over $c \in \mathcal{C}$.

Dually, a map $l\colon \mathcal{Q} \to \mathcal{P}$ in $\mathrm{Cart}(\mathcal{C})$ *is a relative left adjoint if and only if*

(1) for all $c \in \mathcal{C}$ the functor $l|_c\colon \mathcal{Q}|_c \to \mathcal{P}|_c$ that is induced by l on the fibres over c admits a right adjoint $r_c\colon \mathcal{P}|_c \to \mathcal{Q}|_c$;

(2) For every morphism $g\colon d \to c$ in $\mathcal{C}$, the mate of the commutative square

$$\begin{tikzcd} \mathcal{P}|_c \arrow[d, "g^*"'] & \mathcal{Q}|_c \arrow[l, "l|_c"'] \arrow[d, "g^*"] \\ \mathcal{P}|_d \arrow[ur, Rightarrow, "\simeq"'] & \mathcal{Q}|_d \arrow[l, "l|_d"] \end{tikzcd}$$

commutes.

If this is the case, the relative right adjoint r of l recovers the map r_c on the fibres over $c \in \mathcal{C}$.

Proof. The second statement is the content of (the dual of) [56, Proposition 7.3.2.11]. The first statement, on the other hand, is a formal consequence of the second: in fact, in light of the straightening equivalence, there is an equivalence $(-)^{\vee,\mathrm{op}}\colon \mathrm{Cart}(\mathcal{C}) \simeq \mathrm{Cart}(\mathcal{C})$ that is determined by the equivalence $(-)^{\mathrm{op}}_*\colon \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{C}) \simeq \mathrm{PSh}_{\widehat{\mathrm{Cat}}_\infty}(\mathcal{C})$ given by postcomposition with the involution $(-)^{\mathrm{op}}\colon \widehat{\mathrm{Cat}}_\infty \simeq \widehat{\mathrm{Cat}}_\infty$. By combining Proposition 2.4.1.14 with Corollary 2.4.2.3, the equivalence $(-)^{\vee,\mathrm{op}}$ carries a relative left adjoint to a relative right adjoint, and it is evidently true that it translates the two conditions in the second statement to the two conditions in the first one. Since we already know that the second statement is verified, the first one therefore follows as well. □

By combining Corollary 2.4.2.4 with Lemma 2.4.2.8, we conclude:

Proposition 2.4.2.9. *A functor $r\colon \mathsf{C} \to \mathsf{D}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$ is a right adjoint if and only if the following two conditions hold:*

(1) For any object $A \in \mathcal{B}$, the induced functor $r(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ is the right adjoint in an adjunction $(l_A, r(A), \eta_A, \epsilon_A)$.

(2) For any morphism $s\colon B \to A$ in $\mathcal{B}$, the mate of the commutative square

$$\begin{tikzcd} \mathsf{C}(A) \arrow[r, "r(A)"] \arrow[d, "s^*"'] & \mathsf{D}(A) \arrow[d, "s^*"] \arrow[dl, Rightarrow, "\simeq"'] \\ \mathsf{C}(B) \arrow[r, "r(B)"] & \mathsf{D}(B) \end{tikzcd}$$

commutes.

If this is the case, then the left adjoint l of r is given on objects $A \in \mathcal{B}$ by l_A and on morphisms $s\colon B \to A$ by the mate of the commutative square defined by $r(s)$.

Dually, a functor $l\colon \mathsf{D} \to \mathsf{C}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$ is a left adjoint if and only if the following two conditions hold:

(1) For any object $A \in \mathcal{B}$, the induced map $l(A)\colon \mathsf{D}(A) \to \mathsf{C}(A)$ is the left adjoint in an adjunction $(l(A), r_A, \eta_A, \epsilon_A)$.

(2) For any morphism $s\colon B \to A$ in $\mathcal{B}$, the mate of the commutative square

$$\begin{tikzcd} \mathsf{C}(A) \arrow[d, "s^*"'] & \mathsf{D}(A) \arrow[l, "l(A)"'] \arrow[d, "s^*"] \\ \mathsf{C}(B) \arrow[ur, Rightarrow, "\simeq"] & \mathsf{D}(B) \arrow[l, "l(B)"'] \end{tikzcd}$$

commutes.

If this is the case, then a right adjoint r of l is given on objects $A \in \mathcal{B}$ by r_A and on morphisms $s\colon B \to A$ by the mate of the commutative square defined by $l(s)$. □

Remark 2.4.2.10. In the situation of Proposition 2.4.2.9, suppose that the functor $r\colon \mathsf{C} \to \mathsf{D}$ is fully faithful and suppose that condition (1) is satisfied. Since the mate of the commutative square in condition (2) is given by the composition

$$l_B s^* \xrightarrow{l_B s^* \eta_A} l_B s^* r(A) l_A \xrightarrow{\simeq} l_B r(B) s^* l_A \xrightarrow{\epsilon_B s^* l_A} s^* l_A$$

in which the map ϵ_B is an equivalence, the composition is an equivalence whenever the map $l_B s^* \eta_A$ is an equivalence. Since furthermore the map $l_A \eta_A$ is an equivalence as well, we may in this case replace condition (2) by the a priori weaker condition that there exists an *arbitrary* equivalence $l_B s^* \simeq s^* l_A$.

Combining Lemma 2.4.2.8 with Corollary 2.4.2.3 furthermore implies:

Corollary 2.4.2.11. *Let $r\colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories and let $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ be a left exact localisation where $\mathcal{C}$ is some small ∞-category. Then r is a right adjoint if and only if the following two conditions hold:*

(1) For any object $c \in \mathcal{C}$, the induced functor $r(Lc)\colon \mathsf{C}(Lc) \to \mathsf{D}(Lc)$ is a right adjoint.

(2) For any morphism $s\colon d \to c$ in $\mathcal{C}$, the mate of the commutative square

$$\begin{tikzcd} \mathsf{C}(Lc) \arrow[r, "r(Lc)"] \arrow[d, "Ls^*"'] & \mathsf{D}(Lc) \arrow[d, "Ls^*"] \arrow[dl, Rightarrow, "\simeq"'] \\ \mathsf{C}(Ld) \arrow[r, "r(Ld)"] & \mathsf{D}(Ld) \end{tikzcd}$$

commutes. □

Using the criterion from Proposition 2.4.2.9, we are now able to provide a large class of examples for adjunctions between $\mathcal{B}$-categories:

Example 2.4.2.12. In Construction 2.3.1.1, we defined a functor $-\otimes\Omega\colon \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Cat}(\mathcal{B})$ that carries a presentable ∞category $\mathcal{C}$ to the sheaf of ∞-categories $\mathcal{C}\otimes\mathcal{B}_{/-}$ (where $-\otimes-$ is Lurie's tensor product of presentable ∞-categories). Therefore, if $g\colon \mathcal{C} \to \mathcal{D}$ is a right adjoint functor between presentable ∞-categories, we get an induced functor

$$g\otimes\Omega\colon \mathcal{C}\otimes\Omega \to \mathcal{D}\otimes\Omega$$

of large $\mathcal{B}$-categories. We note that for any morphism $s\colon B \to A$ in $\mathcal{B}$ the mate of the commutative square

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{B}_{/A} \arrow[r, "g\otimes \mathcal{B}_{/A}"] \arrow[d, "\mathcal{C}\otimes s^*"] & \mathcal{D} \otimes \mathcal{B}_{/A} \arrow[d, "\mathcal{D}\otimes s^*"] \\ \mathcal{C} \otimes \mathcal{B}_{/B} \arrow[r, "g\otimes \mathcal{B}_{/B}"] & \mathcal{D} \otimes \mathcal{B}_{/B} \end{tikzcd}$$

may be identified with the square induced by passing to left adjoints in the commutative diagram

$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{B}_{/A} \arrow[r, "g\otimes \mathcal{B}_{/A}"] & \mathcal{D} \otimes \mathcal{B}_{/A} \\ \mathcal{C} \otimes \mathcal{B}_{/B} \arrow[r, "g\otimes \mathcal{B}_{/B}"] \arrow[u, "\mathcal{C}\otimes s_*"] & \mathcal{D} \otimes \mathcal{B}_{/B} \arrow[u, "\mathcal{D}\otimes s_*"] \end{tikzcd}$$

Thus it follows from Proposition 2.4.2.9 that $g \otimes \Omega$ is a right adjoint.

We conclude this section by applying the above example in two concrete cases. At first we note that the large $\mathcal{B}$-category $\Omega_\Delta = \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ (where Δ is viewed as a constant $\mathcal{B}$-category) may naturally be identified with the large $\mathcal{B}$-category $\mathcal{S}_\Delta \otimes \Omega$. Therefore, by applying the functor $- \otimes \Omega$ from Construction 2.3.1.1 to the inclusion $\mathrm{Cat}_\infty \hookrightarrow \mathrm{PSh}_{\mathcal{S}}(\Delta)$, one obtains a canonical inclusion of large $\mathcal{B}$-categories

$$\iota\colon\ \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta).$$

Now Example 2.4.2.12 shows:

Proposition 2.4.2.13. *The inclusion $\iota\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ admits a left adjoint $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \to \mathsf{Cat}_{\mathcal{B}}$.* □

Similarly, the inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ induces an inclusion $\Omega \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$, so that Example 2.4.2.12 together with Proposition 2.1.6.7 yields:

Proposition 2.4.2.14. *The inclusion $\Omega \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ admits both a right adjoint $(-)^{\simeq}$ and a left adjoint $(-)^{\mathrm{gpd}}$ that recover the core $\mathcal{B}$-groupoid and the groupoidification functor on local sections.* □

2.4.3. Adjunctions in terms of mapping $\mathcal{B}$-groupoids.

The notion of an adjunction between ∞-categories can be formalised in several ways. One way is the bicategorical approach that we have chosen in Definition 2.4.1.1, but an equivalent way to define an adjunction is by means of a triple (l, r, α) in which $(l, r)\colon \mathcal{C} \leftrightarrows \mathcal{D}$ is a pair of functors and

$$\alpha\colon\ \mathrm{map}_{\mathcal{D}}(-, r(-)) \simeq \mathrm{map}_{\mathcal{C}}(l(-), -)$$

is an equivalence (see for Example [17, Theorem 6.1.23]). The aim of this section is to obtain an analogous characterisation for adjunctions between $\mathcal{B}$-categories. To that end, recall from § 2.1.11 that there is a factorisation system in $\mathrm{Cat}(\mathcal{B})$ between initial functors and left fibrations. Recall, furthermore, that there is a functor $\mathrm{Cat}(\mathcal{B})^{\mathrm{op}} \to \mathrm{Cat}(\widehat{\mathcal{B}})$ that carries a $\mathcal{B}$-category C to the large $\mathcal{B}$-category $\mathsf{LFib}_{\mathsf{C}}$ of left fibrations over C and that carries a functor $f\colon \mathsf{C} \to \mathsf{D}$ to the pullback functor $f^*\colon \mathsf{LFib}_{\mathsf{C}} \to \mathsf{LFib}_{\mathsf{D}}$ that carries a left fibration $q\colon \mathsf{Q} \to A \times \mathsf{D}$ in context $A \in \mathcal{B}$ to its pullback along $\mathrm{id} \times f\colon A \times \mathsf{C} \to A \times \mathsf{D}$. Now the key result from which we will derive our desired characterisation of adjunctions is the following statement:

Proposition 2.4.3.1. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories. Then the pullback functor*

$$f^*\colon\ \mathsf{LFib}_{\mathsf{D}} \to \mathsf{LFib}_{\mathsf{C}}$$

admits a left adjoint $f_!$ that is fully faithful whenever f is. If $p\colon \mathsf{P} \to A \times \mathsf{C}$ is an object in $\mathsf{LFib}_{\mathsf{C}}$, the left fibration $f_!(p)$ over $A \times \mathsf{D}$ is the unique functor that fits into a commutative diagram

$$\begin{tikzcd} \mathsf{P} \arrow[r, "i"] \arrow[d, "p"] & f_!\mathsf{P} \arrow[d, "f_!(p)"] \\ A \times \mathsf{C} \arrow[r, "\mathrm{id} \times f"] & A \times \mathsf{D} \end{tikzcd}$$

such that i is initial.

In order to prove Proposition 2.4.3.1, we need the following lemma:

Lemma 2.4.3.2. *If $f\colon \mathsf{C} \to \mathsf{D}$ and $g\colon \mathsf{D} \to \mathsf{E}$ are functors in $\mathrm{Cat}(\mathcal{B})$ such that g is fully faithful, then gf is initial if and only if both f and g are initial.*

Proof. As initial functors are closed under composition, gf is initial whenever both f and g are, so it suffices to show the converse direction. Since initial functors are the left complement in a factorisation system, they satisfy the left cancellability property, so that it suffices to show that f is initial given that gf is. We will make use of the $\mathcal{B}$-categorical version of Quillen's theorem A [62, Corollary 4.4.8]. Let therefore $d\colon A \to \mathsf{D}$ be an object in context $A \in \mathcal{B}$. On account of the commutative diagram

$$\begin{tikzcd} \mathsf{C}_{/d} \arrow[r] \arrow[d] & \mathsf{D}_{/d} \arrow[r] \arrow[d] & \mathsf{E}_{/g(d)} \arrow[d] \\ \mathsf{C} \times A \arrow[r] & \mathsf{D} \times A \arrow[r] & \mathsf{E} \times A \end{tikzcd}$$

in which the left square is a pullback, it suffices to show that the right square is a pullback as well, which follows immediately from g being fully faithful. $\square$

Proof of Proposition 2.4.3.1. We wish to apply Proposition 2.4.2.9. Fixing an object $A \in \mathcal{B}$, first note that the functor

$$f^*\colon \mathrm{LFib}(A \times \mathsf{D}) \to \mathrm{LFib}(A \times \mathsf{C})$$

that is given by pullback along $(\mathrm{id} \times f)\colon A \times \mathsf{C} \to A \times \mathsf{D}$ has a left adjoint $f_!$. In fact, on account of the commutative square

$$\begin{tikzcd} \mathrm{LFib}(A \times \mathsf{D}) \arrow[r, "f^*"] \arrow[d, hook, "i"] & \mathrm{LFib}(A \times \mathsf{C}) \arrow[d, hook, "i"] \\ \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{D}} \arrow[r, "f^*"] & \mathrm{Cat}(\mathcal{B})_{/A \times \mathsf{C}}, \end{tikzcd}$$

one may define the desired left adjoint $f_!$ on the level of left fibrations as the composition $L_{/A\times\mathsf{D}} \circ (\mathrm{id} \times f)_! \circ i$, where $L_{/A\times\mathsf{C}}\colon \mathrm{Cat}(\mathcal{B})_{/A\times\mathsf{D}} \to \mathrm{LFib}(A\times\mathsf{D})$ denotes the localisation functor and where $(\mathrm{id} \times f)_!$ denotes the forgetful functor. By construction, this functor sends $p\colon \mathsf{P} \to A \times \mathsf{C}$ to the left fibration $f_!(p)\colon \mathsf{Q} \to A \times \mathsf{D}$ that arises from the factorisation of $(\mathrm{id} \times f)p\colon \mathsf{P} \to A \times \mathsf{D}$ into an initial map and a left fibration. Note that the counit of this adjunction is given by the canonical map $\mathsf{P} \to \mathsf{Q} \times_{\mathsf{C}} \mathsf{D}$. If f is fully faithful, Lemma 2.4.3.2 implies that this map is initial and therefore an equivalence since it is already a left fibration. As a consequence f being fully faithful implies that $f_!$ is fully faithful as well. Therefore, by using Proposition 2.4.2.9 the proof is complete once we show that for any map $s\colon B \to A$ in $\mathcal{B}$, the lax square

$$\begin{tikzcd} \mathrm{LFib}(A \times \mathsf{D}) \arrow[d, "s^*"'] & \mathrm{LFib}(A \times \mathsf{C}) \arrow[l, "f_!"'] \arrow[d, "s^*"] \arrow[dl, phantom, "\Leftarrow\varphi"] \\ \mathrm{LFib}(B \times \mathsf{D}) & \mathrm{LFib}(B \times \mathsf{C}) \arrow[l, "f_!"] \end{tikzcd}$$

commutes. To see this, let $p\colon \mathsf{P} \to A \times \mathsf{C}$ be a left fibration, and consider the commutative diagram

$$\begin{tikzcd} & s^* f_! \mathsf{P} \arrow[rr] \arrow[dd, "s^* f_!(p)" near start] & & f_! \mathsf{P} \arrow[dd, "f_!(p)"] \\ s^* \mathsf{P} \arrow[ur, "s^* i"] \arrow[rr] \arrow[dd, "s^* p"] & & \mathsf{P} \arrow[ur, "i"] \arrow[dd, "p" near start] & \\ & B \times \mathsf{D} \arrow[rr, "s \times \mathrm{id}" near start] & & A \times \mathsf{D} \\ B \times \mathsf{C} \arrow[ur, "\mathrm{id} \times f"] \arrow[rr, "s \times \mathrm{id}"'] & & A \times \mathsf{C} \arrow[ur, "\mathrm{id} \times f"'] & \end{tikzcd}$$

in which $f_!(p)i\colon \mathsf{P} \to f_!\mathsf{P} \to A \times \mathsf{D}$ is the factorisation of $(\mathrm{id} \times f)p$ into an initial map and a left fibration. The map $\varphi\colon f_!s^*(p) \to s^*f_!(p)$ is given by the unique lift in the commutative square

$$\begin{tikzcd} s^*\mathsf{P} \arrow[r, "s^*i"] \arrow[d, "j"] & s^*f_!\mathsf{P} \arrow[d, "s^*f_!(p)"] \\ f_!s^*\mathsf{P} \arrow[r, "f_!s^*p"] \arrow[ur, dotted, "\varphi"] & B \times \mathsf{D} \end{tikzcd}$$

in which j is initial. To complete the proof, it therefore suffices to show that s^*i is initial, which follows from the fact that the map $s\colon B \to A$ is a right fibration and therefore proper, cf. [62, § 4.4]. □

Corollary 2.4.3.3. *For any functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories, the functor*

$$f^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \Omega) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$$

admits a left adjoint $f_!$ that fits into a commutative diagram

$$\begin{tikzcd} \mathsf{C}^{\mathrm{op}} \arrow[r, "f^{\mathrm{op}}"] \arrow[d, hook, "h_{\mathsf{C}^{\mathrm{op}}}"] & \mathsf{D}^{\mathrm{op}} \arrow[d, hook, "h_{\mathsf{D}^{\mathrm{op}}}"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega) \arrow[r, "f_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \Omega) \end{tikzcd}$$

in which the two vertical arrows are given by the Yoneda embedding. Moreover, f is fully faithful if and only if $f_!$ is fully faithful.

Proof. The existence of the left adjoint $f_!$ follows immediately from Proposition 2.4.3.1 on account of the straightening/unstraightening equivalence for left fibrations (Theorem 2.1.11.5). To show that the composition $\mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \Omega)$ factors through the Yoneda embedding $\mathsf{D}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \Omega)$, it suffices to show that for every representable left fibration $p\colon \mathsf{P} \to A \times \mathsf{C}$ the associated left fibration $f_!(p)\colon \mathsf{Q} \to A \times \mathsf{D}$ is representable as well. This follows immediately from the fact that there is an initial map $i\colon \mathsf{P} \to \mathsf{Q}$, which implies that Q admits an initial section $A \to \mathsf{Q}$ whenever P admits such a section (cf. Remark 2.1.13.5). □

Proposition 2.4.3.4. *A pair of functors $(l, r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ between $\mathcal{B}$-categories defines an adjunction if and only if there is an equivalence of functors*

$$\alpha\colon \operatorname{map}_{\mathsf{D}}(l(-), -) \simeq \operatorname{map}_{\mathsf{C}}(-, r(-)).$$

Proof. Suppose that $l \dashv r$ is an adjunction in $\mathrm{Cat}(\mathcal{B})$. Then Proposition 2.4.1.15 gives rise to an adjunction $l^* \dashv r^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \leftrightarrows \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. On the other hand, Corollary 2.4.3.3 provides a left adjoint $r_!$ to r^*, hence the uniqueness of adjoints implies that there is an equivalence $\beta\colon r_! \simeq l^*$. We therefore conclude that there is an equivalence $\alpha\colon h_{\mathsf{C}}r \simeq l^*h_{\mathsf{D}}$, where h_{C} and h_{D} denotes the Yoneda embedding of C and D, respectively. On account of the adjunction $- \times \mathsf{D}^{\mathrm{op}} \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, -)$, the datum of such an equivalence corresponds precisely to an equivalence

$$\alpha\colon \operatorname{map}_{\mathsf{D}}(l(-), -) \simeq \operatorname{map}_{\mathsf{C}}(-, r(-)),$$

as desired.

Conversely, suppose that the pair (l, r) comes along with an equivalence α as above. As functoriality of the twisted arrow construction (Definition 2.1.13.1) gives rise to a morphism of functors $\operatorname{map}_{\mathsf{C}}(-, -) \to \operatorname{map}_{\mathsf{D}}(l(-), l(-))$, one obtains a map

$$\operatorname{map}_{\mathsf{C}}(-, -) \to \operatorname{map}_{\mathsf{D}}(l(-), l(-)) \simeq \operatorname{map}_{\mathsf{C}}(-, rl(-)).$$

As the Yoneda embedding is fully faithful (Corollary 2.1.13.4), this map arises uniquely from a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to rl$. In fact, we may view the above map as a functor

$$\mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\Delta^1}$$

that sends an object $d\colon A\to \mathsf{C}$ to the map

$$\mathrm{map}_{\mathsf{C}}(-,d)\to \mathrm{map}_{\mathsf{D}}(l(-),l(d))\simeq \mathrm{map}_{\mathsf{C}}(-,rl(d))$$

in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As the Yoneda embedding $\mathsf{C}\hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is fully faithful, this map must arise from a map in C, hence the above functor factors through the fully faithful functor $\mathsf{C}^{\Delta^1}\hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\Delta^1}$ that is induced by the Yoneda embedding. By a similar argument, one obtains a map $\epsilon\colon lr\to \mathrm{id}_{\mathsf{D}}$. We complete the proof by showing that η and ϵ satisfy the conditions of Proposition 2.4.1.4, i.e. that the maps $r\epsilon\circ\eta r$ and $\epsilon l\circ l\eta$ are equivalences. We show this for the first case, the second case follows from an analogous argument. Since equivalences of functors can be detected objectwise by [62, Corollary 4.7.17], it suffices to show that for any object $d\colon A\to\mathsf{D}$ the map

$$r(d)\xrightarrow{\eta rd} rlr(d)\xrightarrow{r\epsilon d} r(d)$$

is an equivalence. Now bifunctoriality of the equivalence $\mathrm{map}_{\mathsf{D}}(l(-),-)\simeq \mathrm{map}_{\mathsf{C}}(-,r(-))$ implies that there is a commutative diagram

$$\begin{tikzcd} r(d) \arrow[r, "\eta rd"] \arrow[d, "\mathrm{id}_{r(d)}"] & rlr(d) \arrow[d, "r\epsilon d"] \\ r(d) \arrow[r, "\mathrm{id}_{r(d)}"] & r(d) \end{tikzcd}$$

that arises from the transposed commutative diagram

$$\begin{tikzcd} lr(d) \arrow[r, "\mathrm{id}_{lr(d)}"] \arrow[d, "\mathrm{id}_{lr(d)}"] & lr(d) \arrow[d, "\epsilon d"] \\ lr(d) \arrow[r, "\epsilon d"] & d, \end{tikzcd}$$

which proves the claim. □

Recall that if $r\colon \mathsf{D}\to\mathsf{C}$ is a functor between $\mathcal{B}$-categories and if $c\colon A\to\mathsf{D}$ is an arbitrary object, the functor $\mathrm{map}_{\mathsf{C}}(c,r(-))\colon A\times\mathsf{D}\to\Omega$ precisely classifies the left fibration $\mathsf{D}_{c/}\to A\times\mathsf{D}$ that arises as the pullback of the slice projection $(\pi_c)_!\colon \mathsf{C}_{c/}\to A\times\mathsf{C}$ along $\mathrm{id}\times r\colon A\times\mathsf{D}\to A\times\mathsf{C}$ (see [62, Definition 4.2.1]). We now obtain:

Corollary 2.4.3.5. *Let $r\colon\mathsf{D}\to\mathsf{C}$ be a functor between large $\mathcal{B}$-categories. Then r admits a left adjoint l if and only if for any object $c\colon A\to\mathsf{C}$ in context $A\in\mathcal{B}$ the copresheaf* $\mathrm{map}_{\mathsf{C}}(c,r(-))$ *(viewed as an object in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\Omega)$ in context A) is representable by an object in D, in which case the representing object is given by $l(c)$ and the associated initial object in $\mathsf{D}_{c/}$ is given by the unit map $\eta c\colon c\to rl(c)$.*

Proof. By Proposition 2.4.3.4, the functor r admits a left adjoint if and only if there is a functor $l\colon\mathsf{C}\to\mathsf{D}$ and an equivalence

$$\alpha\colon\ \mathrm{map}_{\mathsf{D}}(l(-),-)\simeq\mathrm{map}_{\mathsf{C}}(-,r(-)).$$

Therefore, if r admits a left adjoint then $\mathrm{map}_{\mathsf{C}}(c,r(-))$ is representable by $l(c)\colon A\to\mathsf{D}$, and the explicit construction of the equivalence α in Proposition 2.4.3.4 shows that the equivalence

$$\mathsf{D}_{l(c)/}\simeq\mathsf{D}_{c/}$$

over $A\times\mathsf{D}$ that arises from α sends the initial section $\mathrm{id}_{l(c)}\colon A\to\mathsf{D}_{l(c)/}$ to the unit map $\eta c\colon c\to rl(c)$.

Conversely, if $\mathrm{map}_{\mathsf{C}}(c,r(-))$ is representable for every object c in C in context $A\in\mathcal{B}$, then the functor $hr\colon\mathsf{D}\to\mathsf{C}\hookrightarrow\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})=\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\Omega_{\widehat{\mathcal{B}}})$ transposes to a functor

$$\mathsf{C}^{\mathrm{op}}\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\Omega_{\widehat{\mathcal{B}}})$$

that factors through the Yoneda embedding $\mathsf{D}^{\mathrm{op}}\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\Omega_{\widehat{\mathcal{B}}})$ by [62, Proposition 3.9.4] and therefore defines a functor $l\colon\mathsf{C}\to\mathsf{D}$. By construction, this functor comes with an equivalence $\mathrm{map}_{\mathsf{D}}(l(-),-)\simeq\mathrm{map}_{\mathsf{C}}(-,r(-))$, hence the claim follows. □

Let C and D be $\mathcal{B}$-categories and let $\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ be the full subcategory that is spanned by those functors $\pi_A^*\mathsf{D} \to \pi_A^*\mathsf{C}$ in $\mathrm{Cat}(\mathcal{B}_{/A})$ (for every $A \in \mathcal{B}$) that admit a left adjoint. Dually, let $\underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathsf{C},\mathsf{D}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ denote the full subcategory spanned by those functors that admit a right adjoint.

Remark 2.4.3.6 (locality of adjunctions). If C and D are $\mathcal{B}$-categories and $A \in \mathcal{B}$ is an arbitrary object, the property of a functor $f\colon \pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ to be a right adjoint is local in $\mathcal{B}$ (see § 2.1.14). In fact, by Corollary 2.4.3.5 this property is equivalent to the condition that for every object c in $\pi_A^*\mathsf{C}$ (in arbitrary context), the functor $\mathrm{map}_{\pi_A^*\mathsf{C}}(c, f(-))$ is representable. Hence the claim follows from the fact that the representability of such functors is a local condition (see Example 2.1.14.7). In particular, this implies that every object in $\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ in context $A \in \mathcal{B}$ encodes a right adjoint functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$, and one furthermore has a canonical equivalence $\pi_A^*\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C}) \simeq \underline{\mathsf{Fun}}^R_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{D},\pi_A^*\mathsf{C})$ for every $A \in \mathcal{B}$ (see Remarks 2.1.14.5 and 2.1.14.6).

Remark 2.4.3.7 (étale transposition invariance). By its very definition, the property of an object $f\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ to be a right adjoint (i.e. to be contained in $\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C})$) is invariant under étale transposition (see § 2.1.14).

Corollary 2.4.3.8. *For any two $\mathcal{B}$-categories C and D, there is an equivalence*

$$\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C}) \simeq \underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}$$

that sends a functor between D and C to its left adjoint, and vice versa.

Proof. By postcomposition with the Yoneda embedding $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, the $\mathcal{B}$-category $\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ embeds into $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{C}^{\mathrm{op}},\Omega)$. Likewise, the $\mathcal{B}$-category $\underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}} \simeq \underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{D}^{\mathrm{op}})$ embeds into the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{C}^{\mathrm{op}},\Omega)$. To finish the proof, we only need to show that an object $f\colon A\times\mathsf{D}\times\mathsf{C}^{\mathrm{op}} \to \Omega$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{C}^{\mathrm{op}},\Omega)$ in context $A \in \mathcal{B}$ is contained in the essential image of $\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ if and only if it is contained in the essential image of $\underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}$. By Remarks 2.4.3.6 and 2.4.3.7 (and the fact that the base change functor π_A^* preserves the internal hom, cf. Remark 2.1.14.1), we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$ (see Remark 2.1.14.4). By Corollary 2.4.3.5, the functor f is contained in $\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{D},\mathsf{C})$ if and only if $f(d,-)$ is representable for any object d in D and $f(-,c)$ is representable for any object c in C, which is in turn equivalent to f being contained in the essential image of $\underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathsf{C},\mathsf{D})^{\mathrm{op}}$. Thus the claim follows. □

2.4.4. Reflective subcategories. In this brief section we discuss the special case of an adjunction where the right adjoint is fully faithful. Again this material is quite standard for ordinary ∞-categories, see for example [57, §5.2.7].

Definition 2.4.4.1. Let $i\colon \mathsf{C} \hookrightarrow \mathsf{D}$ be a fully faithful functor between $\mathcal{B}$-categories. Then C is said to be *reflective* in D if i admits a left adjoint. Dually, C is *coreflective* if i admits a right adjoint.

Proposition 2.4.4.2. *If $(l \dashv r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ is an adjunction between $\mathcal{B}$-categories, then l is fully faithful if and only if the adjunction unit η is an equivalence, and r is fully faithful if and only if the adjunction counit ϵ is an equivalence.*

Proof. The functor l is fully faithful if and only if the map

$$\mathrm{map}_{\mathsf{C}}(-,-) \to \mathrm{map}_{\mathsf{D}}(l(-),l(-))$$

is an equivalence [62, Proposition 3.8.7]. By postcomposition with the equivalence

$$\mathrm{map}_{\mathsf{D}}(l(-),l(-)) \simeq \mathrm{map}_{\mathsf{C}}(-,rl(-))$$

that is provided by Proposition 2.4.3.4, this is in turn equivalent to the map

$$\mathrm{map}_{\mathsf{C}}(-,-) \to \mathrm{map}_{\mathsf{C}}(-,rl(-))$$

being an equivalence. But this map is obtained as the image of the adjunction unit $\eta\colon \Delta^1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{C})$ along the fully faithful functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}, \Omega)$ that is induced by postcomposition with the Yoneda embedding $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. The claim thus follows from the observation that fully faithful functors are conservative (since the map $\Delta^1 \to \Delta^0$ is essentially surjective, see [62, Lemma 3.8.8]). The dual statement about r and ϵ is proved by an analogous argument. $\square$

By combining Proposition 2.4.4.2 with Proposition 2.4.1.4, one immediately deduces:

Corollary 2.4.4.3. *Let $i\colon \mathsf{D} \hookrightarrow \mathsf{C}$ be a fully faithful functor between $\mathcal{B}$-categories. Then D is reflective in C if and only if i admits a retraction $L\colon \mathsf{C} \to \mathsf{D}$ together with a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to iL$ such that both ηi and $L\eta$ are equivalences.* $\square$

If $\mathsf{D} \hookrightarrow \mathsf{C}$ is a reflective subcategory, then the reflection functor $L\colon \mathsf{C} \to \mathsf{D}$ is a retraction and therefore in particular essentially surjective (cf. Proposition 2.1.9.4). Consequently, we may recover the subcategory D from the endofunctor $iL\colon \mathsf{C} \to \mathsf{C}$ be means of its factorisation into an essentially surjective and a fully faithful functor. Conversely, given an arbitrary endofunctor $f\colon \mathsf{C} \to \mathsf{C}$, Corollary 2.4.4.3 shows that the essential image of f defines a reflective subcategory precisely if there is a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to f$ such that both ηf and $f\eta$ are equivalences. Let us record this observation for future use in the following proposition.

Proposition 2.4.4.4. *Let C be a $\mathcal{B}$-category, let $f\colon \mathsf{C} \to \mathsf{C}$ be a functor and let $iL\colon \mathsf{C} \twoheadrightarrow \mathsf{D} \hookrightarrow \mathsf{C}$ be its factorisation into an essentially surjective and a fully faithful functor. Then $L \dashv i$ precisely if there is a map $\eta\colon \mathrm{id}_{\mathsf{C}} \to f$ such that both ηf and $f\eta$ are equivalences.* $\square$

Example 2.4.4.5. If $(\mathcal{L}, \mathcal{R})$ is a factorisation system in $\mathcal{B}$, then for any $A \in \mathcal{B}$ the full subcategory $\mathcal{R}_{/A} \hookrightarrow \mathcal{B}_{/A}$ is reflective: the associated reflection functor $L_{/A}\colon \mathcal{B}_{/A} \to \mathcal{R}_{/A}$ is induced by the unique factorisation of maps. Such a factorisation system $(\mathcal{L}, \mathcal{R})$ is called a *modality* if $\mathcal{L}$ is closed under base change in $\mathcal{B}$, which precisely means that for every map $s\colon B \to A$ in $\mathcal{B}$ the natural map $L_{/B}s^* \to s^* L_{/A}$ is an equivalence. Using Proposition 2.4.2.9, we thus conclude that the right orthogonality class $\mathcal{R}$ of any modality $(\mathcal{L}, \mathcal{R})$ defines a reflective subcategory of Ω. In Example 3.2.4.4 below, we will characterise those reflective subcategories of Ω that arise in such a way.

Reflective subcategories are examples of *localisations* in the sense of § 2.2.3:

Proposition 2.4.4.6. *Let $(l \dashv r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ be a reflective subcategory. Then l is the localisation of C at the subcategory $\mathsf{S} = l^{-1}\mathsf{D}^{\simeq} \hookrightarrow \mathsf{C}$.*

Proof. By construction of S, we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{S} \arrow[r] \arrow[d, hook] & \mathsf{S}^{\mathrm{gpd}} \arrow[r] \arrow[d] & \mathsf{D}^{\simeq} \arrow[d, hook] \\ \mathsf{C} \arrow[r, "L"] \arrow[rr, bend right, "l"'] & \mathsf{S}^{-1}\mathsf{C} \arrow[r, "g"] & \mathsf{D}, \end{tikzcd}$$

hence we only need to show that g is an equivalence. Let us define $h = Lr$. Then $gh \simeq lr \simeq \mathrm{id}$, hence h is a right inverse of g. We finish the proof by showing that h is a left inverse of g as well. Since $L^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{C}, \mathsf{S}^{-1}\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{S}^{-1}\mathsf{C})$ is fully faithful by Proposition 2.2.3.14, it suffices to produce an equivalence $hgL \simeq L$. Let $\eta\colon \mathrm{id} \to rl$ be the adjunction unit. Since $l\eta$ is an equivalence, the map $l\eta c$ factors through the core $\mathsf{D}^{\simeq} \hookrightarrow \mathsf{D}$ for every object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$. By construction of S, this means that ηc is contained in S, hence $L\eta c$ is an equivalence. Since equivalences of functors can be detected objectwise [62, Corollary 4.7.17], we conclude that $L\eta\colon L \to Lrl \simeq hgL$ is the desired equivalence. $\square$

It will be useful to have a name for the class of localisations that arise from reflective subcategories:

Definition 2.4.4.7. Let $\mathsf{S} \to \mathsf{C}$ be a functor between $\mathcal{B}$-categories. The localisation $L\colon \mathsf{C} \to S^{-1}\mathsf{C}$ is said to be a *Bousfield localisation* if L admits a fully faithful right adjoint $i\colon \mathsf{S}^{-1}\mathsf{C} \hookrightarrow \mathsf{C}$.

Remark 2.4.4.8. The extra condition on the right adjoint in Definition 2.4.4.7 to be fully faithful is superfluous: in fact, by Proposition 2.2.3.14 the functor $L^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(S^{-1}\mathsf{C}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is fully faithful and by Proposition 2.4.1.15 L^* is left adjoint to i^*. We therefore obtain an equivalence $L^* \simeq i_!$, hence Corollary 2.4.3.3 implies that i must be fully faithful as well.

CHAPTER 3

Colimits and cocompletions

We begin this chapter by studying limits and colimits in $\mathcal{B}$-category theory in § 3.1. The definition of a (co)limit will be a straightforward adaption of the usual definition for ∞-categories. After developing the general theory, we provide a number of explicit examples and show that some important $\mathcal{B}$-categories have all limits and colimits. We also provide some techniques how to split up colimits indexed by $\mathcal{B}$-categories into more manageable parts, which will be convenient in the next sections and chapters.

In § 3.2 we define what it means for a $\mathcal{B}$-category to be cocomplete. One subtle but important difference with ordinary ∞-category theory, is that a $\mathcal{B}$-category is not necessarily cocomplete even if every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in an arbitrary context A admits a colimit. Instead one has to require that the same property also holds for $\pi_A^* \mathsf{C}$, internally to $\mathcal{B}_{/A}$ for every $A \in \mathcal{B}$. More generally we define a notion of U-cocompleteness for any subsheaf $\mathsf{U} \subseteq \mathsf{Cat}_{\mathcal{B}}$ and unwind these definitions in a number of examples.

In § 3.3 we develop a theory of Kan-extensions in the world of $\mathcal{B}$-categories. We prove that Kan-extensions exist, whenever the $\mathcal{B}$-categories involved admit a sufficient amount of colimits. Finally, we prove the main results of this chapter in § 3.4. We show that for any $\mathcal{B}$-category C, the Yoneda-embedding $h_{\mathsf{C}}\colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ exhibits $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ as the free cocompletion of C. More generally we construct the free U-cocompletion of C for any internal class U and prove that is has the expected universal property. We also give some concrete examples of free U-cocompletions, in particular when C is the point.

3.1. Limits and colimits

In this section we discuss limits and colimits in a $\mathcal{B}$-category. We set up the general theory in § 3.1.1–3.1.3. All in all our treatment is quite parallel to the one in ordinary higher category theory, see for example [48, §19] or [17, §6.2]. In § 3.1.4 and § 3.1.5 we discuss limits and colimits in the universe Ω and in the $\mathcal{B}$-category of $\mathcal{B}$-categories $\mathsf{Cat}_{\mathcal{B}}$. In § 3.1.6 we discuss how to compute colimits in slice categories and in § 3.1.7 we collect some results that show under which circumstances an adjunction induces an adjunction on slice categories. In § 3.1.8 we show that initial and final functors can be characterised by their property of preserving limits and colimits. Finally, in § 3.1.9 we explain how general internal limits and colimits can be decomposed into groupoidal and constant limits and colimits.

3.1.1. Definitions and first examples. Let C be a $\mathcal{B}$-category. Recall from Proposition 2.1.6.5 that for any simplicial object I in $\mathcal{B}$ the internal hom $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in $\mathcal{B}_\Delta$ is a $\mathcal{B}$-category. We refer to the objects of this $\mathcal{B}$-category as *I-indexed diagrams in* C. Note that this $\mathcal{B}$-category is equivalent to $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$, where I is the image of the simplicial object I along the localisation functor $\mathcal{B}_\Delta \to \mathrm{Cat}(\mathcal{B})$. Thus, in what follows we can always safely assume that I is a $\mathcal{B}$-category.

Now recall from [62, Definition 4.2.1] that to any pair of maps $f\colon \mathsf{D} \to \mathsf{C}$ and $g\colon \mathsf{E} \to \mathsf{C}$ in $\mathrm{Cat}(\mathcal{B})$ we can associate the *comma $\mathcal{B}$-category* $\mathsf{D} \downarrow_{\mathsf{C}} \mathsf{E} = (\mathsf{D} \times \mathsf{E}) \times_{\mathsf{C} \times \mathsf{C}} \mathsf{C}^{\Delta^1}$. We may now define:

Definition 3.1.1.1. Let C be a $\mathcal{B}$-category and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ be an I-indexed diagram in C in context $A \in \mathcal{B}$, for some $I \in \mathcal{B}_\Delta$. The *$\mathcal{B}$-category of cones over d* is defined as the comma $\mathcal{B}$-category $\mathsf{C}_{/d} = \mathsf{C} \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})} A$ formed from $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ and the diagonal map $\mathrm{diag}\colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$. Dually, the *$\mathcal{B}$-category of cocones under d* is defined as the comma $\mathcal{B}$-category $\mathsf{C}_{d/} = A \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})} \mathsf{C}$.

In the situation of Definition 3.1.1.1, the $\mathcal{B}$-category of cones $\mathsf{C}_{/d}$ admits a structure map into $\mathsf{C} \times A$ that fits into the pullback square

$$\begin{tikzcd}
\mathsf{C}_{/d} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})_{/d} \arrow[d, "(\pi_d)_!"] \\
\mathsf{C} \times A \arrow[r, "\operatorname{diag} \times \mathrm{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \times A
\end{tikzcd}$$

in which the vertical map on the right is the forgetful functor from the slice $\mathcal{B}$-category, cf. Definition 2.1.12.1. Since this is a right fibration (Proposition 2.1.12.4), so is the map $\mathsf{C}_{/d} \to \mathsf{C} \times A$. In other words, we may regard this map as an object in $\mathsf{RFib}_{\mathsf{C}}$ in context A. Dually, the map $\mathsf{C}_{d/} \to A \times \mathsf{C}$ is a left fibration and therefore defines an object in $\mathsf{LFib}_{\mathsf{C}}$ in context A. With respect to the straightening/unstraightening equivalence $\mathsf{RFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ from Theorem 2.1.11.5, the right fibration $\mathsf{C}_{/d} \to \mathsf{C} \times A$ corresponds to the presheaf $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(\operatorname{diag}(-), d)$ on C, and the left fibration $\mathsf{C}_{d/} \to A \times \mathsf{C}$ corresponds to the copresheaf $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(d, \operatorname{diag}(-))$ on C.

Remark 3.1.1.2 (locality of cones). In the situation of Definition 3.1.1.1, if $B \in \mathcal{B}$ is an arbitrary object, it follows immediately from Remark 2.1.14.1 that one obtains a canonical equivalence of $\mathcal{B}_{/B}$-categories $\pi_B^*(\mathsf{C}_{/d}) \simeq (\pi_B^*\mathsf{C})_{/\pi_B^*(d)}$.

Remark 3.1.1.3 (étale transposition invariance for cones). In the situation of Definition 3.1.1.1, let us denote by $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*I, \pi_A^*\mathsf{C})$ the transpose of d. Since the forgetful functor $(\pi_A)_!\colon \mathrm{Cat}(\mathcal{B}_{/A}) \to \mathrm{Cat}(\mathcal{B})$ preserves pullbacks, we deduce from Remark 2.1.12.2 that the map $\mathsf{C}_{/d} \to \mathsf{C} \times A$ arises as the image of $(\pi_A^*\mathsf{C})_{/\bar{d}} \to \pi_A^*\mathsf{C}$ along $(\pi_A)_!$. In other words, when regarded as a $\mathcal{B}_{/A}$-category, we can identify $\mathsf{C}_{/d}$ with $(\pi_A^*\mathsf{C})_{/\bar{d}}$.

Remark 3.1.1.4. Let I be a simplicial object in $\mathcal{B}$ and let C be a $\mathcal{B}$-category. Recall from [62, Definition 4.3.11] the definition of the *right cone* $I^{\triangleright}$ as the pushout

$$\begin{tikzcd}
I \sqcup I \arrow[r, "(d^1, d^0)"] \arrow[d, "\mathrm{id} \times \pi_I"] & \Delta^1 \otimes I \arrow[d] \\
I \sqcup 1 \arrow[r, "(\iota, \infty)"] & I^{\triangleright}.
\end{tikzcd}$$

By applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, \mathsf{C})$ to this diagram, one obtains an equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright}, \mathsf{C}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})} \mathsf{C}$$

over $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \times \mathsf{C}$, in which the right-hand side denotes the comma $\mathcal{B}$-category that is formed from the cospan

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \xrightarrow{\mathrm{id}} \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \xleftarrow{\operatorname{diag}} \mathsf{C}.$$

By construction, if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ is an I-indexed diagram in C in context $A \in \mathcal{B}$, one obtains a pullback square

$$\begin{tikzcd}
\mathsf{C}_{d/} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright}, \mathsf{C}) \arrow[d, "(\iota^*, \infty^*)"] \\
A \times \mathsf{C} \arrow[r, "d \times \mathrm{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \times \mathsf{C}.
\end{tikzcd}$$

In other words, the pullback of $\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright}, \mathsf{C})$ along $d \times \mathrm{id}$ recovers the $\mathcal{B}$-category of cocones under d. We may therefore regard any object $\bar{d}\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleright}, \mathsf{C})$ as a cocone $d \to \operatorname{diag} c$ under the diagram $d = \iota^*\bar{d}$ with $c = \infty^*\bar{d}$.

Dually, one defines the *left cone* $I^{\triangleleft}$ as the pushout

$$\begin{tikzcd}
I \sqcup I \arrow[r, "(d^1, d^0)"] \arrow[d, "\pi_I \times \mathrm{id}"] & \Delta^1 \otimes I \arrow[d] \\
1 \times I \arrow[r] & I^{\triangleleft}
\end{tikzcd}$$

and therefore obtains an equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleleft}, \mathsf{C}) \simeq \mathsf{C} \downarrow_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})} \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$$

over $\mathsf{C} \times \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$. Consequently, the pullback of $\underline{\mathsf{Fun}}_{\mathcal{B}}(I^{\triangleleft}, \mathsf{C})$ along $\mathrm{id} \times d$ recovers the $\mathcal{B}$-category of cones $\mathsf{C}_{/d}$ over d.

DEFINITION 3.1.1.5. Let C be a $\mathcal{B}$-category and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ be an I-indexed diagram in context A in C, for some $A \in \mathcal{B}$ and some $I \in \mathcal{B}_{\Delta}$. A *limit* cone of d is a map $\mathrm{diag}(\lim d) \to d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in context A that defines a final section $A \to \mathsf{C}_{/d}$ over A. Dually, a *colimit* cocone of d is a map $d \to \mathrm{diag}(\operatorname{colim} d)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in context A that defines an initial section $A \to \mathsf{C}_{d/}$ over A.

REMARK 3.1.1.6. The above definition is a direct analogue of Joyal's original definition of limits and colimits in an ∞-category [49].

REMARK 3.1.1.7. In the situation of Definition 3.1.1.5, Remark 2.1.13.5 implies that an I-indexed diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a colimit cocone if and only if the presheaf $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(d, \mathrm{diag}(-))$ is representable, in which case the representing object is given by $\operatorname{colim} d$. In other words, if d admits a colimit cocone, one obtains an equivalence $\mathsf{C}_{\operatorname{colim} d/} \simeq \mathsf{C}_{d/}$ over $A \times \mathsf{C}$, and conversely if there is an object $c\colon A \to \mathsf{C}$ and an equivalence $\mathsf{C}_{c/} \simeq \mathsf{C}_{d/}$ over $A \times \mathsf{C}$ then the image of the object id_c in $\mathsf{C}_{c/}$ along this equivalence defines a colimit cocone of d. A similar observation can be made for limits. In particular, the colimit and limit of a diagram are unique up to equivalence if they exist.

REMARK 3.1.1.8 (locality of limits and colimits). The existence of limits and colimits is a *local* condition: in fact, by the same reasoning as in Remark 2.4.3.6, a diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a limit in C if and only if for every cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ the diagram $s_i^*(d)\colon A_i \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a limit in C. Analogous observations can be made for colimits.

REMARK 3.1.1.9 (étale transposition invariance for limits and colimits). In light of Remark 3.1.1.3, a cone $\mathrm{diag}\, c \to d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ in context A transposes to a cone $\mathrm{diag}\, \bar{c} \to \bar{d}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$ in context $1_{\mathcal{B}_{/A}}$ (where $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{B}_{/A})(\pi_A^* I, \pi_A^* \mathsf{C})$ and $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^* \mathsf{C}$ are the transpose of d and c, respectively), and the former defines an initial section $A \to \mathsf{C}_{/d}$ over A if and only if the latter defines an initial object $1_{\mathcal{B}_{/A}} \to \pi_A^* \mathsf{C}_{/\bar{d}}$. In other words, we may compute the limit of $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ as the transpose of the limit of $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^* \mathsf{C})$. Analogous observations can be made for colimits.

EXAMPLE 3.1.1.10. Let C be $\mathcal{B}$-category and let $c\colon A \to \mathsf{C}$ be an object, viewed as a 1-indexed diagram $c\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(1, \mathsf{C}) \simeq \mathsf{C}$. Then there are equivalences $\lim c \simeq c \simeq \operatorname{colim} c$, and the associated limit and colimit cocones are given by $\mathrm{id}_c\colon A \to \mathsf{C}_{/c}$ and $\mathrm{id}_c\colon A \to \mathsf{C}_{c/}$.

EXAMPLE 3.1.1.11. For any $\mathcal{B}$-category C and any object $c\colon A \to \mathsf{C}$, the object c is initial if and only if it defines a colimit of the initial diagram $d\colon \varnothing \to \mathsf{C}$, and dually c is final if and only if it defines a limit of d. In fact, since $\varnothing$ is initial in $\mathrm{Cat}(\mathcal{B})$, there is an equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\varnothing, \mathsf{C}) \simeq 1$, which implies that the left fibration $\mathsf{C}_{d/} \to A \times \mathsf{C}$ is an equivalence. Consequently, a section $A \to \mathsf{C}_{d/}$ is initial if and only if the map $A \to A \times \mathsf{C}$ is, which is in turn the case if and only if the associated map $1 \to \pi_A^* \mathsf{C}$ is initial in $\mathrm{Cat}(\mathcal{B}_{/A})$. As this is precisely the condition that c is an initial object in C, the result follows. The statement about final objects and limits follows by dualisation.

PROPOSITION 3.1.1.12. *Let C be a $\mathcal{B}$-category and let I be a simplicial object in $\mathcal{B}$. The following conditions are equivalent:*

(1) every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a colimit $\operatorname{colim} d$*;*

(2) the diagonal functor $\mathrm{diag}\colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ *admits a left adjoint* $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \to \mathsf{C}$*.*

If either of these conditions are satisfied, the functor colim *carries d to* $\operatorname{colim} d$*, and the adjunction unit* $d \to \mathrm{diag}\operatorname{colim} d$ *defines a colimit cocone of d. The dual statement for limits holds as well.*

Proof. By the dual of Corollary 2.4.3.5, the functor diag admits a left adjoint if and only if for every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ the functor $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(d, \mathrm{diag}(-))$ is representable by an object in C, in which case the left adjoint sends d to the representing object in C. By definition, this functor classifies the left fibration $\mathsf{C}_{d/} \to A \times \mathsf{C}$. Therefore, Remark 3.1.1.7 shows that diag admits a left adjoint if and only if every diagram d admits a colimit $\mathrm{colim}\, d\colon A \to \mathsf{C}$, in which case this is the representing object of the functor $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(d, \mathrm{diag}(-))$. Corollary 2.4.3.5 moreover shows that in this case the adjunction unit $d \to \mathrm{diag}\,\mathrm{colim}\, d$ defines an initial section $A \to \mathsf{C}_{d/}$. □

Example 3.1.1.13. Let C be a large $\mathcal{B}$-category and G be a $\mathcal{B}$-groupoid. By using Proposition 2.4.2.9, the following two conditions are equivalent:

(1) C admits G-indexed colimits;
(2) for every $A \in \mathcal{B}$ the functor $\pi_{\mathsf{G}}^*\colon \mathsf{C}(A) \to \mathsf{C}(\mathsf{G} \times A)$ admits a left adjoint $(\pi_{\mathsf{G}})_!$ such that for every map $s\colon B \to A$ in $\mathcal{B}$ the natural morphism $(\pi_{\mathsf{G}})_! s^* \to s^*(\pi_{\mathsf{G}})_!$ is an equivalence.

In particular, if C has G-indexed colimits, then the colimit of a diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{C})$ can be identified with the image of $d \in \mathsf{C}(\mathsf{G} \times A)$ along the functor $(\pi_{\mathsf{G}})_!$.

Dually, the following two conditions are equivalent:

(1) C admits G-indexed limits;
(2) for every $A \in \mathcal{B}$ the functor $\pi_{\mathsf{G}}^*\colon \mathsf{C}(A) \to \mathsf{C}(\mathsf{G} \times A)$ admits a right adjoint $(\pi_{\mathsf{G}})_*$ such that for every map $s\colon B \to A$ in $\mathcal{B}$ the natural morphism $s^*(\pi_{\mathsf{G}})_* \to (\pi_{\mathsf{G}})_* s^*$ is an equivalence.

In particular, if C has G-indexed limits, then the limit of a diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{C})$ can be identified with the image of $d \in \mathsf{C}(\mathsf{G} \times A)$ along the functor $(\pi_{\mathsf{G}})_*$.

Example 3.1.1.14. Let C be a large $\mathcal{B}$-category and let $\mathcal{J}$ be an ∞-category. By using Proposition 2.4.2.9, the following two conditions are equivalent:

(1) C admits $\mathcal{J}$-indexed colimits;
(2) for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits $\mathcal{J}$-indexed colimits, and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves such colimits.

Dually, the following two conditions are equivalent:

(1) C admits $\mathcal{J}$-indexed limits;
(2) for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits $\mathcal{J}$-indexed limits, and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves such limits.

Remark 3.1.1.15. Let $\mathcal{C}$ be a small ∞-category such that $\mathcal{B}$ is a left exact and accessible localisation of $\mathrm{PSh}(\mathcal{C})$. Let $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ be the localisation functor. Then Corollary 2.4.2.11 implies that in the situation of Example 3.1.1.13 and Example 3.1.1.14, it suffices to check the condition in (2) for the special case where $A = L(c)$, $B = L(d)$ and $s = L(t)$ for some objects $c, d \in \mathcal{C}$ and some map $t\colon d \to c$ in $\mathcal{C}$.

3.1.2. Preservation of limits and colimits. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories and let I be a simplicial object in $\mathcal{B}$. Let $f_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ be the functor that is given by postcomposition with f. For any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$, the functor f_* gives rise to an evident commutative square

$$\begin{tikzcd} \mathsf{C}_{/d} \arrow[r, "f_*"] \arrow[d] & \mathsf{D}_{/f_*d} \arrow[d] \\ \mathsf{C} \times A \arrow[r, "f \times \mathrm{id}"] & \mathsf{D} \times A. \end{tikzcd}$$

Suppose that d has a limit in C, i.e. there is a limit cone given by a final section $A \to \mathsf{C}_{/d}$ over A. We say that the functor f *preserves* this limit if the image of this limit cone along f_* defines a final section of $\mathsf{D}_{/f_*d}$. Dually, if d has a colimit in C then f is said to preserve this colimit if the image of the colimit cocone along f_* is an initial section of $\mathsf{D}_{f_*d/}$ over A.

Remark 3.1.2.1 (locality of preservation of limits and colimits). The property that a functor $f\colon \mathsf{C} \to \mathsf{D}$ preserves the limit (colimit) of a diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ is a *local* condition: in fact, the same reasoning as in Remark 2.4.3.6 implies that f preserves the limit of d if and only if for every cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ the limit of the induced diagram $s_i^*(d)$ is preserved by f. Analogous observations can be made for colimits.

Remark 3.1.2.2 (étale transposition invariance for the preservation of limits and colimits). Note that by means of the projections to A, the functor $f_*\colon \mathsf{C}_{/d} \to \mathsf{D}_{/f_*d}$ can be regarded as a map in $\mathrm{Cat}(\mathcal{B}_{/A})$. When viewed as such, Remark 3.1.1.3 implies that this map can be identified with the functor $(\pi_A^* f)_*\colon (\pi_A^*\mathsf{C})_{/\bar{d}} \to (\pi_A^*\mathsf{D})_{/(\pi_A^* f)_*\bar{d}}$ (where $\bar{d}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* I, \pi_A^*\mathsf{C})$ denotes the transpose of d). Together with Remark 3.1.1.9, this implies that f preserves the limit of d if and only if $\pi_A^* f$ preserves the limit of $\bar{d}$. Analogous observations hold for colimits.

Lemma 3.1.2.3. *Let $(l \dashv r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories, and let $f\colon c \to r(d)$ be a map in C in context $A \in \mathcal{B}$. Then f is an equivalence if and only if the transpose map $g\colon l(c) \to d$ defines a final section of $\mathsf{C}_{/d}$ over A.*

Proof. By Corollary 2.4.3.5, the counit $\epsilon d\colon lr(d) \to d$ defines a final section of $\mathsf{C}_{/d}$ over A, hence the dual of Corollary 2.1.12.13 implies that there is a map $g \to \epsilon d$ in $\mathsf{C}_{/d}$ that is an equivalence if and only if g is final. On account of the equivalence $\mathsf{C}_{/d} \simeq \mathsf{C}_{/r(d)}$, this map corresponds to a map $f \to \mathrm{id}_{r(d)}$ in $\mathsf{C}_{/r(d)}$. The result now follows from the straightforward observation that the latter is an equivalence if and only if f is an equivalence in C. □

Proposition 3.1.2.4. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories and let I be a simplicial object in $\mathcal{B}$ such that C and D admit all I-indexed limits, i.e the diagonal maps $\mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ and $\mathsf{D} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ admit right adjoints (cf. Proposition 3.1.1.12). Then f preserves all I-indexed limits precisely if the mate of the commutative square*

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \arrow[d, "f_*"] & \mathsf{C} \arrow[l, "\mathrm{diag}"'] \arrow[d, "f"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}) & \mathsf{D} \arrow[l, "\mathrm{diag}"'] \end{tikzcd}$$

commutes. The dual statement about colimits holds as well.

Proof. Suppose that f preserves all I-indexed limits. The mate of the commutative square in the statement of the proposition is encoded by a map $\varphi\colon f \lim \to \lim f_*$ that is given by the composite

$$f \lim \xrightarrow{\eta f \lim} \lim \mathrm{diag} f \lim \xrightarrow{\simeq} \lim f_* \mathrm{diag} \lim \xrightarrow{\lim f_* \epsilon} \lim f_*$$

in which η and ϵ are the units and counits of the two adjunctions $\mathrm{diag} \dashv \lim$. By [62, Corollary 4.7.17], this map is an equivalence if and only if for any $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ the associated map $\varphi(d)\colon f(\lim d) \to \lim f_* d$ is an equivalence in D. Now since the transpose map $\mathrm{diag} f(\lim d) \to f_* d$ is given by postcomposing the equivalence $\mathrm{diag} f(\lim d) \simeq f_* \mathrm{diag}(\lim d)$ with the map $f_* \epsilon d$ and since Proposition 3.1.1.12 implies that ϵd is precisely the limit cone over d in D, the claim follows from Lemma 3.1.2.3. □

Remark 3.1.2.5. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories, let I be an arbitrary simplicial object in $\mathcal{B}$ and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ be a diagram that has a limit in C. Suppose furthermore that $f_* d$ has a limit in D. Then the universal property of final objects (see Corollary 2.1.12.13) gives rise to a unique map

$$\begin{tikzcd} \mathrm{diag} f(\lim d) \arrow[rr] \arrow[dr] & & \mathrm{diag} \lim f_* d \arrow[dl] \\ & f_* d & \end{tikzcd}$$

in $\mathsf{D}_{/f_* d}$ that is an equivalence if and only if f preserves the limit of d. Since $\mathsf{D}_{/f_* d} \to \mathsf{D}$ is a right fibration and therefore in particular conservative (cf. [62, Definition 4.1.10]), this is in turn equivalent to the map $f(\lim d) \to \lim f_* d$ being an equivalence in D. If both C and D admit I-indexed limits, this map is nothing but the mate transformation $f \lim \to \lim f_*$ from Proposition 3.1.2.4 evaluated at the object d.

EXAMPLE 3.1.2.6. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories and let G be a $\mathcal{B}$-groupoid. Suppose that both C and D admit G-indexed colimits. By using Proposition 2.4.2.9 and Proposition 3.1.2.4, the following two conditions are equivalent:

(1) f preserves G-indexed colimits;
(2) for every $A \in \mathcal{B}$ the natural morphism $(\pi_{\mathsf{G}})_! f(\mathsf{G} \times A) \to f(A)(\pi_{\mathsf{G}})_!$ is an equivalence.

Dually, if C and D admit G-indexed limits, the following two conditions are equivalent:

(1) f preserves G-indexed limits;
(2) for every $A \in \mathcal{B}$ the natural morphism $f(A)(\pi_{\mathsf{G}})_* \to (\pi_{\mathsf{G}})_* f(A)$ is an equivalence.

EXAMPLE 3.1.2.7. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between large $\mathcal{B}$-categories, let $\mathcal{J}$ be an ∞-category and suppose that both C and D admit $\mathcal{J}$-indexed colimits. By using Proposition 2.4.2.9 and Proposition 3.1.2.4, the following two conditions are equivalent:

(1) f preserves $\mathcal{J}$-indexed colimits;
(2) for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ preserves $\mathcal{J}$-indexed colimits.

Dually, if C and D admit $\mathcal{J}$-indexed limits, the following two conditions are equivalent:

(1) f preserves $\mathcal{J}$-indexed limits;
(2) for every $A \in \mathcal{B}$ the functor $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ preserves $\mathcal{J}$-indexed limits.

Checking whether a functor between $\mathcal{B}$-categories preserves certain limits or colimits becomes simpler when the functor is fully faithful:

PROPOSITION 3.1.2.8. *Let $f\colon \mathsf{C} \hookrightarrow \mathsf{D}$ be a fully faithful functor between $\mathcal{B}$-categories, let I be a simplicial object in $\mathcal{B}$ and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ be a diagram in C. Suppose that $f_*(d)$ admits a colimit in D such that* $\operatorname{colim} f_* d$ *is contained in C. Then* $\operatorname{colim} f_* d$ *already defines a colimit of d in C. The analogous statement for limits holds as well.*

PROOF. Since f is fully faithful, the canonical square

$$\begin{tikzcd} \mathsf{C}_{d/} \arrow[r, hook, "f_*"] \arrow[d] & \mathsf{D}_{f_*d/} \arrow[d] \\ A \times \mathsf{C} \arrow[r, hook, "\mathrm{id} \times f"] & A \times \mathsf{D} \end{tikzcd}$$

is a pullback and f_* is fully faithful. Therefore, if $\operatorname{colim} f_* d\colon A \to \mathsf{D}_{f_*d/}$ is an initial section such that the underlying object $\operatorname{colim} f_* d$ in D is contained in C, then the entire colimit cocone is contained in the essential image of f_*, i.e. defines a section $A \to \mathsf{C}_{d/}$ over A. By Lemma 2.4.3.2, this section must be initial as well, hence the result follows. □

COROLLARY 3.1.2.9. *Let $f\colon \mathsf{C} \hookrightarrow \mathsf{D}$ be a fully faithful functor between $\mathcal{B}$-categories, and suppose that both C and D admit I-indexed colimits for some simplicial object I in $\mathcal{B}$. Then f preserves I-indexed colimits if and only if the restriction of* $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}) \to \mathsf{D}$ *along the inclusion $f_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ factors through the inclusion $f\colon \mathsf{C} \hookrightarrow \mathsf{D}$. The analogous statement for limits holds as well.* □

We conclude this section with a discussion of the preservation of (co)limits by adjoint functors. We will need the following lemma:

LEMMA 3.1.2.10. *Let $(l \dashv r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories and let $i\colon L \to K$ be a map between simplicial objects in $\mathcal{B}$. Then the two commutative squares*

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[r, "l_*"] \arrow[d, "i^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}) \arrow[d, "i^*"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C}) \arrow[r, "l_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{D}) \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[d, "i^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}) \arrow[l, "r_*"'] \arrow[d, "i^*"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C}) & \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{D}) \arrow[l, "r_*"'] \end{tikzcd}$$

that are obtained from the bifunctoriality of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ *are related by the mate correspondence.*

PROOF. To prove the lemma, we may argue in the homotopy bicategory of the $(\infty, 2)$-category $\mathrm{Cat}(\mathcal{B})$. Then the claim follows from the fact that the natural transformation $\underline{\mathsf{Fun}}_{\mathcal{B}}(K,-) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L,-)$ determines a pseudonatural transformation between 2-functors. See [52, Proposition 2.5] for an argument in the strict case. $\square$

PROPOSITION 3.1.2.11. *Let* $(l \dashv r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ *be an adjunction between* $\mathcal{B}$*-categories. Then* l *preserves all colimits that exist in* C*, and* r *preserves all limits that exist in* D*.*

PROOF. We will show that the right adjoint $r\colon \mathsf{D} \to \mathsf{C}$ preserves all limits that exist in D, the dual statement about l and colimits follows by taking opposite $\mathcal{B}$-categories. Let therefore I be a simplicial object in $\mathcal{B}$ and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ be a diagram that has a limit in D. We need to show that the image of the final section $\operatorname{diag}\lim d \to d$ along $r_*\colon \mathsf{D}_{/d} \to \mathsf{C}_{/r_* d}$ is final. By Corollary 2.4.1.11, the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(I,-)$ sends the adjunction $l \dashv r$ to an adjunction $l_* \dashv r_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C}) \leftrightarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})$, hence by using Proposition 2.4.3.4 one obtains a chain of equivalences

$$\begin{aligned}
\operatorname{map}_{\mathsf{C}}(-, r(\lim d)) &\simeq \operatorname{map}_{\mathsf{D}}(l(-), \lim d) \\
&\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(\operatorname{diag} l(-), d) \\
&\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(l_* \operatorname{diag}(-), d) \\
&\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(\operatorname{diag}(-), r_* d)
\end{aligned}$$

of presheaves on C. We complete the proof by showing that this equivalence sends the identity $\mathrm{id}_{r(\lim d)}$ to the map $\operatorname{diag} r(\lim d) \simeq r_* \operatorname{diag}\lim d \to r_* d$ that arises as the image of the limit cone $\operatorname{diag}\lim d \to d$ under the functor r_*. By construction, the image of the identity $\mathrm{id}_{r(\lim d)}$ under this chain of equivalences is given by the composition

$$\operatorname{diag} r(\lim d) \xrightarrow{\eta \operatorname{diag} r} r_* l_* \operatorname{diag} r(\lim d) \xrightarrow{\simeq} r_* \operatorname{diag} lr(\lim d) \xrightarrow{r_* \operatorname{diag} \epsilon} r_* \operatorname{diag} \lim d \to r_* d$$

in which the right-most map is the image of the limit cone $\operatorname{diag}\lim d \to d$ under the functor r_*, the map η denotes the unit of the adjunction $l_* \dashv r_*$ and ϵ denotes the counit of the adjunction $l \dashv r$. As the composition of the first three maps is precisely the mate of the equivalence $l_* \operatorname{diag} \simeq \operatorname{diag} l$ and therefore recovers the equivalence $\operatorname{diag} r(\lim d) \simeq r_* \operatorname{diag}(\lim d)$ by Lemma 3.1.2.10, the result follows. $\square$

PROPOSITION 3.1.2.12. *Let* $(l \dashv r)\colon \mathsf{C} \leftrightarrows \mathsf{D}$ *be an adjunction in* $\mathrm{Cat}(\mathcal{B})$ *that exhibits* D *as a reflective subcategory of* C*, let* I *be a simplicial object in* $\mathcal{B}$ *and let* $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})$ *be a diagram in context* $A \in \mathcal{B}$ *such that* $r_* d$ *admits a colimit in* C*. Then* $l(\operatorname{colim} r_* d)$ *defines a colimit of* d *in* D*. Dually, if* $r_* d$ *admits a limit in* C*, then* $l(\lim r_* d)$ *defines a limit of* d *in* D*.*

PROOF. Suppose first that $r_* d$ admits a colimit in C. Since r is fully faithful, we obtain a chain of equivalences

$$\begin{aligned}
\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(d, \operatorname{diag}(-)) &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(r_* d, \operatorname{diag} r(-)) \\
&\simeq \operatorname{map}_{\mathsf{C}}(\operatorname{colim} r_* d, r(-)) \\
&\simeq \operatorname{map}_{\mathsf{D}}(l(\operatorname{colim} r_* d), -),
\end{aligned}$$

which shows that the colimit of d in D exists and is explicitly given by $l(\operatorname{colim} r_* d)$.

Next, let us suppose that $r_* d$ admits a limit in C. By the triangle identities, the functor l sends the adjunction unit $\eta\colon \mathrm{id} \to rl$ to an equivalence. In particular, the map $\lim r_* d \to rl(\lim r_* d)$ is sent to an equivalence in D. Note that on account of the equivalence

$$\operatorname{map}_{\mathsf{C}}(-, \lim r_* d) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(\operatorname{diag} l(-), d),$$

the presheaf $\mathrm{map}_{\mathsf{C}}(-, \lim r_* d)$ sends any map in C that is inverted by l to an equivalence in Ω. Applying this observation to $\eta\colon \lim r_* d \to rl(\lim r_* d)$, we obtain a retraction $\varphi\colon rl(\lim r_* d) \to \lim r_* d$ of η that gives rise to a retract diagram

$$\begin{tikzcd}
\lim r_* d \arrow[r, "\eta"] \arrow[d, "\eta"] & rl(\lim r_* d) \arrow[r, "\varphi"] \arrow[d, "\eta"] & \lim r_* d \arrow[d, "\eta"] \\
rl(\lim r_* d) \arrow[r, "rl\eta"] & rlrl(\lim r_* d) \arrow[r, "rl\varphi"] & rl(\lim r_* d)
\end{tikzcd}$$

in which the two maps in the lower row are equivalences. By the triangle identities and the fact that since r is fully faithful the adjunction counit $\epsilon\colon lr \to \mathrm{id}$ is an equivalence (see Proposition 2.4.4.2), the vertical map in the middle must be an equivalence as well, hence we conclude that $\eta\colon \lim r_* d \to rl(\lim r_* d)$ too is an equivalence. Therefore, the computation

$$\begin{aligned}
\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{D})}(\mathrm{diag}(-), d) &\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(I,\mathsf{C})}(\mathrm{diag}\, r(-), r_* d) \\
&\simeq \mathrm{map}_{\mathsf{C}}(r(-), \lim r_* d) \\
&\simeq \mathrm{map}_{\mathsf{C}}(r(-), rl(\lim r_* d)) \\
&\simeq \mathrm{map}_{\mathsf{D}}(lr(-), l(\lim r_* d)) \\
&\simeq \mathrm{map}_{\mathsf{D}}(-, l(\lim r_* d))
\end{aligned}$$

proves the claim. □

Remark 3.1.2.13. We adopted the strategy for the proof of the second claim in Proposition 3.1.2.12 from Denis-Charles Cisinski's proof of the analogous statement for ∞-categories, see [17, Proposition 6.2.17].

3.1.3. Limits and colimits in functor categories. In this section, we discuss the familiar fact that limits and colimits in functor ∞-categories can be computed objectwise in the context of $\mathcal{B}$-categories.

Proposition 3.1.3.1. *Let I be a simplicial object in $\mathcal{B}$ and let C be a $\mathcal{B}$-category that admits all I-indexed limits. Then $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})$ admits all I-indexed limits for any simplicial object K in $\mathcal{B}$, and the precomposition functor $i^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C})$ preserves I-indexed limits for any map $i\colon L \to K$ in $\mathcal{B}_{\Delta}$. The dual statement for colimits is true as well.*

Proof. Proposition 3.1.1.12 implies that the diagonal functor $\mathrm{diag}\colon \mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ admits a right adjoint $\lim\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \to \mathsf{C}$. By Corollary 2.4.1.11, the functor $\lim_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})$ therefore defines a right adjoint to the diagonal functor $\mathrm{diag}_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}))$. As postcomposing the latter with the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}))$ recovers the diagonal functor $\mathrm{diag}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}))$, Corollary 2.4.1.11 implies that $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})$ admits all I-indexed limits. If $i\colon L \to K$ is an arbitrary map in $\mathcal{B}_{\Delta}$, the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[r, "\mathrm{diag}_*"] \arrow[d, "i^*"] \arrow[rr, bend left, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})) \arrow[r, "\simeq"] \arrow[d, "i^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})) \arrow[d, "(i^*)_*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C}) \arrow[r, "\mathrm{diag}_*"] \arrow[rr, bend right, "\mathrm{diag}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})) \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C}))
\end{tikzcd}$$

and the functoriality of the mate construction (cf. Remark 2.4.2.7) imply that in order to show that the functor $i^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C})$ preserves I-indexed limits, we only need to show that the mate of the left square in the above diagram commutes, which is an immediate consequence of Lemma 3.1.2.10. □

Proposition 3.1.3.2. *Let I be a simplicial object in $\mathcal{B}$ and let C and D be $\mathcal{B}$-categories such that D admits I-indexed limits. Let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}))$ be a diagram in context $A \in \mathcal{B}$, and let $\operatorname{diag} F \to d$ be a cone over d, where $F\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is an arbitrary object. Then $\operatorname{diag} F \to d$ is a limit cone if and only if for every map $s\colon B \to A$ in $\mathcal{B}$ and every $c\colon B \to \mathsf{C}$ the induced map $\operatorname{diag}(\overline{F})(c) \to \bar{d}(c)$ is a limit cone in $\pi_A^*\mathsf{D}$ in context B (where $\operatorname{diag} \overline{F} \to \bar{d}$ denotes the transpose of $\operatorname{diag} F \to d$ across the adjunction $(\pi_A)_! \dashv \pi_A^*$). The dual statements for colimits holds as well.*

Proof. Using that π_A^* preserves the internal hom (Remark 2.1.14.1) together with the étale transposition invariance of limits (Remark 3.1.1.9), we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume $A \simeq 1$ (see Remark 2.1.14.4). By means of the adjunction $\operatorname{diag} \dashv \lim$ and Lemma 3.1.2.3, the map $\operatorname{diag} F \to d$ defines a limit cone if and only if the transpose map $F \to \lim d$ is an equivalence in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$. Using that equivalences in functor $\mathcal{B}$-categories are detected object-wise (see [62, Corollary 4.7.17]), this is in turn the case precisely if for every $c\colon B \to \mathsf{C}$ the map $F(c) \to (\lim d)(c)$ is an equivalence in context B. Note that by Remark 3.1.1.8, this map transposes to the map $\pi_B^*(F)(\bar{c}) \to \lim \pi_B^*(d)(\bar{c})$ (where $\bar{c}\colon 1_{\mathcal{B}_{/B}} \to \pi_B^*\mathsf{C}$ is the transpose of c). Using Proposition 3.1.3.1, we can identify the latter with the map $\pi_B^*(F)(\bar{c}) \to \lim(\pi_B^*(d)(c))$, i.e. with the transpose of the morphism of diagrams $\operatorname{diag} \pi_B^*(F)(\bar{c}) \to \pi_B^*(d)(\bar{c})$. Hence, we conclude that $\operatorname{diag} F \to d$ is a limit cone if and only if $\operatorname{diag} \pi_B^*(F)(\bar{c}) \to \pi_B^*(d)(\bar{c})$ is one for each $c\colon B \to \mathsf{C}$. Now by Remark 3.1.1.3, the latter transposes to $\operatorname{diag} F(c) \to d(c)$, hence the claim follows from the invariance of limit cones under étale transposition (Remark 3.1.1.9). □

Proposition 3.1.3.3. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories, let I be a simplicial object in $\mathcal{B}$ and suppose that both C and D admits I-indexed limits and that f preserves such limits. Then for every simplicial object K in $\mathcal{B}$, the induced functor $f_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D})$ preserves I-indexed limits as well. The dual statement for colimits holds too.*

Proof. Similarly as in the proof in Proposition 3.1.3.1, we need to show that the mate of the left square in the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \arrow[r, "\operatorname{diag}_*"] \arrow[d, "f_*"] \arrow[rr, bend left, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})) \arrow[r, "\simeq"] \arrow[d, "(f_*)_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C})) \arrow[d, "(f_*)_*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}) \arrow[r, "\operatorname{diag}_*"] \arrow[rr, bend right, "\operatorname{diag}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D})) \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D}))
\end{tikzcd}$$

commutes, which follows from the observation that this mate is obtained by applying the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(K, -)$ to the mate of the commutative square

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\operatorname{diag}"] \arrow[d, "f"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C}) \arrow[d, "f_*"] \\
\mathsf{D} \arrow[r, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{D}),
\end{tikzcd}$$

which by assumption is an equivalence. Hence the claim follows. □

3.1.4. Limits and colimits in the universe Ω. Our goal of this section is to prove that the universe Ω for $\mathcal{B}$-groupoids admits small limits and colimits, and to give explicit constructions of those. We start with the case of colimits:

Proposition 3.1.4.1. *The universe Ω for small $\mathcal{B}$-groupoids admits small colimits. Moreover, if I is a $\mathcal{B}$-category and if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ is an I-indexed diagram in context $A \in \mathcal{B}$, then the colimit*

$\operatorname{colim} d\colon A \to \Omega$ *is given by the* $\mathcal{B}_{/A}$*-groupoid* $(\int d)^{\mathrm{gpd}}$ *, where* $\int d \to A \times \mathsf{I}$ *denotes the left fibration that is classified by* d.

Proof. In light of Proposition 3.1.1.12, we need to show that the diagonal functor $\operatorname{diag}\colon \Omega \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ has a left adjoint, which is a consequence of Corollary 2.4.3.3. The explicit description of this colimit furthermore follows from Proposition 2.4.3.1. □

Remark 3.1.4.2. For the special case $\mathcal{B} \simeq \mathcal{S}$, the explicit construction of colimits in Proposition 3.1.4.1 is given in [57, Corollary 3.3.4.6].

Remark 3.1.4.3. Let $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ be a left exact accessible localisation with left adjoint L, where $\mathcal{C}$ is a small ∞-category. Let I be a $\mathcal{B}$-category and let $d\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ be a diagram classified by a left fibration $\mathsf{P} \to \mathsf{I}$. By Proposition 3.1.4.1 we have that $\operatorname{colim} d \simeq \mathsf{P}^{\mathrm{gpd}} \simeq \operatorname{colim}_{\Delta^{\mathrm{op}}} \mathsf{P}$. Therefore $\operatorname{colim} d$ is given by applying L to the presheaf

$$c \mapsto (\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathsf{P})(c) \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathsf{P}(c)) \simeq \mathsf{P}(c)^{\mathrm{gpd}}.$$

Since [62, Corollary 4.6.8] implies that for every $c \in \mathcal{C}$ the left fibration $\mathsf{P}(c) \to \mathsf{I}(c)$ classifies the functor $\Gamma_{\mathcal{B}_{/L(c)}} \circ d(c)\colon \mathsf{I}(c) \to \mathcal{S}$, we conclude that $\operatorname{colim} d \in \mathcal{B}$ is given by applying L to the presheaf $c \mapsto \operatorname{colim}(\Gamma \circ d(c))$.

We now record the following easy consequence:

Corollary 3.1.4.4. *Suppose that* I *is a* $\mathcal{B}$*-category,* $d\colon \mathsf{I} \to \Omega$ *a diagram and* $f\colon \mathsf{J} \to \mathsf{I}$ *an essentially surjective functor. Then we get an induced effective epimorphism* $\operatorname{colim} d \circ f \to \operatorname{colim} d$ *in* $\mathcal{B}$.

Proof. Let $\mathsf{P} \to \mathsf{I}$ denote the left fibration classifying d. Consider the cartesian square

$$\begin{tikzcd} \mathsf{Q} \arrow[r, "f'"'] \arrow[d] & \mathsf{P} \arrow[d] \\ \mathsf{J} \arrow[r, "f"] & \mathsf{I} \end{tikzcd}$$

where $\mathsf{Q} \to \mathsf{J}$ classifies $d \circ f$. Since essentially surjective functors are stable under pullback, we get that f' is also essentially surjective. We claim that thus $(f')^{\mathrm{gpd}}\colon \mathsf{Q}^{\mathrm{gpd}} \to \mathsf{P}^{\mathrm{gpd}}$ is an effective epimorphism. Indeed by Proposition 2.1.9.4 the induced map $\mathsf{Q}_0 \to \mathsf{P}_0$ is an effective epimorphism. Since the canonical map $\mathsf{P}_0 \to \mathsf{P}^{\mathrm{gpd}}$ is also an effective epimorphism (this can be easily deduced from [57, Lemma 6.2.3.13]), the claim follows from [57, Corollary 6.2.3.12]. Hence $(f')^{\mathrm{gpd}}$ is also essentially surjective and the claim follows from Proposition 3.1.4.1. □

We will now proceed by showing that Ω also admits small limits. By Proposition 3.1.1.12, we need to show that for any $\mathcal{B}$-category I the diagonal functor $\operatorname{diag}\colon \Omega \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ admits a right adjoint. To that end, recall that since $\mathrm{Cat}(\mathcal{B})$ is cartesian closed, the pullback functor $\pi_{\mathsf{I}}^*\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{I}}$ admits a right adjoint $(\pi_{\mathsf{I}})_*$ that is given by sending a functor $p\colon \mathsf{P} \to \mathsf{I}$ to the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{P})_{/\mathsf{I}}$ that is defined by the pullback square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{P})_{/\mathsf{I}} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{P}) \arrow[d, "p_*"] \\ 1 \arrow[r, "\mathrm{id}_{\mathsf{I}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{I}). \end{tikzcd}$$

If p is a left fibration, then so is p_*, hence $(\pi_{\mathsf{I}})_*$ sends p to a $\mathcal{B}$-groupoid in this case. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (where $A \in \mathcal{B}$ is an arbitrary object) and using the locality of LFib (see Remark 2.1.11.6), this argument also shows that the pullback functor $\pi_{\mathsf{I}}^*\colon \mathcal{B}_{/A} \to \mathrm{LFib}(A \times \mathsf{I})$ admits a right adjoint $(\pi_{\mathsf{I}})_*$ for any $A \in \mathcal{B}$. Moreover, if $s\colon B \to A$ is a map in $\mathcal{B}$, the natural map $s^*(\pi_{\mathsf{I}})_* \to (\pi_{\mathsf{I}})_* s^*$ is an equivalence whenever the transpose map $s_!(\pi_{\mathsf{I}})^* \to (\pi_{\mathsf{I}})^* s_!$ is one, and as this latter condition is evidently satisfied, Proposition 2.4.2.9 and Theorem 2.1.11.5 now show:

PROPOSITION 3.1.4.5. *The universe Ω for small $\mathcal{B}$-groupoids admits small limits. More precisely, if* I *is a $\mathcal{B}$-category and if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \Omega)$ is an* I*-indexed diagram in context $A \in \mathcal{B}$, then the limit* $\lim d\colon A \to \Omega$ *is given by the $\mathcal{B}_{/A}$-groupoid $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{I}, \int \bar{d})_{/\pi_A^*\mathsf{I}}$ in $\mathcal{B}_{/A}$, where $\int \bar{d} \to \pi_A^* \mathsf{I}$ is the left fibration that is classified by the transpose $\bar{d}\colon \pi_A^* \mathsf{I} \to \Omega_{\mathcal{B}_{/A}}$ of d.* ☐

PROOF. The discussion before the proposition shows the existence of limits. The explicit description of the limit follows from the description of the right adjoint $(\pi_{\mathsf{I}})_*$ in the case $A = 1$ and the invariance of limits under étale transposition, Remark 3.1.1.9. ☐

REMARK 3.1.4.6. For the special case $\mathcal{B} \simeq \mathcal{S}$, the explicit construction of limits in Proposition 3.1.4.5 is given in [57, Corollary 3.3.3.3].

If I is an arbitrary $\mathcal{B}$-category, the fact that right adjoint functors preserve limits (Proposition 3.1.2.11) combined with the fact that the final object 1_Ω is the limit of the unique diagram $\varnothing \to \Omega$ (Example 3.1.1.11) show that $\operatorname{diag}(1_\Omega)\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ defines a final object in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$. We will denote this object by $1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}$. Proposition 3.1.4.5 now implies:

COROLLARY 3.1.4.7. *For any $\mathcal{B}$-category* I*, the limit functor* $\lim_{\mathsf{I}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \to \Omega$ *is explicitly given by the representable functor* $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}, -)$*, where* $1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ *denotes the final object in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$*.*

PROOF. Since Proposition 3.1.4.5 already implies the existence of $\lim_{\mathsf{I}}$, the claim follows from the equivalence $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}, -) \simeq \operatorname{map}_\Omega(1_\Omega, \lim_{\mathsf{I}}(-))$ and the fact that $\operatorname{map}_\Omega(1_\Omega, -)$ is equivalent to the identity functor on Ω, see [62, Proposition 4.6.3]. ☐

Recall from § 2.1.10 that there is a canonical embedding $i\colon \Omega_{\mathcal{B}} \hookrightarrow \Omega_{\widehat{\mathcal{B}}}$. For later use, we note:

PROPOSITION 3.1.4.8. *The inclusion $i\colon \Omega_{\mathcal{B}} \hookrightarrow \Omega_{\widehat{\mathcal{B}}}$ preserves small limits and colimits.*

PROOF. We begin with the case of colimits. Using Corollary 3.1.2.9, it suffices to show that the restriction of the colimit functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\widehat{\mathcal{B}}}) \to \Omega_{\widehat{\mathcal{B}}}$ along the inclusion $i_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\widehat{\mathcal{B}}})$ takes values in $\Omega_{\mathcal{B}}$ for any $\mathcal{B}$-category I. Since Proposition 3.1.4.1 implies that the colimit of any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\widehat{\mathcal{B}}})$ is given by the (large) $\mathcal{B}_{/A}$-groupoid $(\int d)^{\mathrm{gpd}}$, the claim follows from [62, Proposition 3.3.3], together with the fact that d taking values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}})$ is tantamount to $\int d$ being a small $\mathcal{B}_{/A}$-category, cf. [62, Corollary 4.5.9].

As for the case of limits, by Corollary 3.1.4.7 we need to verify that $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\widehat{\mathcal{B}}})}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\widehat{\mathcal{B}}})}, i_*(-))$ takes values in $\Omega_{\mathcal{B}}$. Since we have $1_{\Omega_{\widehat{\mathcal{B}}}} \simeq i(1_\Omega)$, we find that $1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\widehat{\mathcal{B}}})} \simeq i_*(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)})$, so that the functor $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\widehat{\mathcal{B}}})}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\widehat{\mathcal{B}}})}, i_*(-))$ can be identified with $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}(1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)}, -)$ (since i_* is fully faithful). Hence the claim follows. ☐

We have now assembled the necessary results in order to prove the following:

PROPOSITION 3.1.4.9. *For any $\mathcal{B}$-category* C*, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ of presheaves on* C *admits small limits and colimits. Moreover, for any $\mathcal{B}$-category* I *and any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$, a cone* $\operatorname{diag} c \to d$ *defines a limit of d if and only if the induced cone* $\operatorname{diag} h(c) \to h_* d$ *defines a limit in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. In particular, the Yoneda embedding h preserves small limits.*

PROOF. The fact that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ admits small limits and colimits follows immediately from combining Proposition 3.1.3.1 with Propositions 3.1.4.5 and 3.1.4.1. Now if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ is an I-indexed diagram in C and if $\operatorname{diag} c \to d$ is an arbitrary cone that is represented by a section $A \to \mathsf{C}_{/d}$ over A, we obtain a commutative diagram

$$\begin{tikzcd}
& \mathsf{C}_{/c} \arrow[rr, hook] \arrow[dl] \arrow[dd] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \arrow[dl] \arrow[dd] \\
\mathsf{C}_{/d} \arrow[rr, hook, crossing over] \arrow[dd] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_* d} & \\
& \mathsf{C} \times A \arrow[rr, hook, "h\times\mathrm{id}" near start] \arrow[dl, "\mathrm{id}"] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times A \arrow[dl, "\mathrm{id}"] \\
\mathsf{C} \times A \arrow[rr, hook, "h\times\mathrm{id}"] & & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times A \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in which the square in the front and the one in the back are cartesian as h is fully faithful. Therefore, the upper horizontal square must be cartesian as well. The cone $\operatorname{diag} c \to d$ defines a limit of d if and only if the map $\mathsf{C}_{/c} \to \mathsf{C}_{/d}$ is an equivalence. Likewise, the induced cone $\operatorname{diag} h(c) \to h_* d$ defines a limit of $h_* d$ precisely if the map $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_* d}$ is an equivalence. To complete the proof, we therefore need to show that the first map is an equivalence if and only if the second map is one. As the upper square in the previous diagram is cartesian, the second condition implies the first. Conversely, the map $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_* d}$ corresponds via Theorem 2.1.11.5 to a map between presheaves on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ which are both representable by objects in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Therefore, there is a unique map $h(c) \to \lim h_* d$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that the induced map

$$\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-, h(c)) \to \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(-, \lim h_* d)$$

recovers the morphism $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_* d}$ on the level of presheaves on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As Yoneda's lemma (Theorem 2.1.13.3) implies that restricting this map along $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ recovers the map $h(c) \to \lim h_* d$, the latter being an equivalence implies that the morphism $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h_* d}$ is an equivalence as well, as desired. □

Corollary 3.1.4.10. *For any $\mathcal{B}$-category C and any object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$, the corepresentable functor $\operatorname{map}_{\mathsf{C}}(c, -)\colon A \times \mathsf{C} \to \Omega$ transposes to a functor $\pi_A^* \mathsf{C} \to \Omega_{\mathcal{B}_{/A}}$ that preserves all limits that exist in $\pi_A^* \mathsf{C}$.*

Proof. By Example 2.1.14.7, the transpose of $\operatorname{map}_{\mathsf{C}}(c, -)$ can be identified with $\operatorname{map}_{\pi_A^* \mathsf{C}}(\bar{c}, -)$, where $\bar{c}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^* \mathsf{C}$ is the transpose of c. Therefore, by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume that $A \simeq 1$. On account of Yoneda's lemma, the functor $\operatorname{map}_{\mathsf{C}}(c, -)$ is equivalent to the composition $c^* h$, where h denotes the Yoneda embedding and $c^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \Omega$ is the evaluation functor at c. By Proposition 3.1.4.9 and Proposition 3.1.3.1, both of these functors preserve limits, hence the claim follows. □

Our next goal is to show that Ω is *cartesian closed*. To that end, denote by $- \times -\colon \Omega \times \Omega \to \Omega$ the product functor. One now finds:

Proposition 3.1.4.11. *The universe Ω for small $\mathcal{B}$-groupoids is cartesian closed, in that there is an equivalence*

$$\operatorname{map}_{\Omega}(- \times -, -) \simeq \operatorname{map}_{\Omega}(-, \operatorname{map}_{\Omega}(-, -))$$

of functors $\Omega^{\mathrm{op}} \times \Omega^{\mathrm{op}} \times \Omega \to \Omega$.

Proof. First, we claim that the transpose $\varphi\colon \Omega \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)$ of the product bifunctor $- \times -\colon \Omega \times \Omega \to \Omega$ takes values in $\underline{\mathsf{Fun}}^L_{\mathcal{B}}(\Omega, \Omega)$. To see this, we need to show that the image of every $\mathcal{B}_{/A}$-groupoid G along φ defines a left adjoint functor of $\mathcal{B}_{/A}$-categories. Note that since π_A^* preserves adjunctions (Corollary 2.4.1.9) and the internal hom (Remark 2.1.14.1), we may identify $\pi_A^*(- \times -)$ with the product bifunctor of $\pi_A^* \Omega$ and $\pi_A^*(\varphi)$ with its transpose. Together with the equivalence $\pi_A^* \Omega \simeq \Omega_{\mathcal{B}_{/A}}$ from Remark 2.1.14.1, this implies that the image $\varphi(\mathsf{G})\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)$ transposes to the product functor $\mathsf{G} \times -\colon \Omega_{\mathcal{B}_{/A}} \to \Omega_{\mathcal{B}_{/A}}$. Thus, by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume without loss of generality that

$A \simeq 1$. In this case, Example 3.1.1.14 implies that the functor $\mathsf{G} \times -\colon \Omega \to \Omega$ is given on local sections over $A \in \mathcal{B}$ by the ∞-categorical product functor

$$\mathcal{B}_{/A} \xrightarrow{\pi_A^* \mathsf{G} \times -} \mathcal{B}_{/A}$$

which admits a right adjoint $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{G}, -)$. If $s\colon B \to A$ is a map in $\mathcal{B}$, we deduce from [62, Lemma 4.2.3] that the natural map $s^* \underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{G}, -) \to \underline{\mathrm{Hom}}_{\mathcal{B}_{/B}}(\pi_B^* \mathsf{G}, s^*(-))$ is an equivalence, hence Proposition 2.4.2.9 shows that the functor $\mathsf{G} \times -\colon \Omega \to \Omega$ admits a right adjoint, as desired.

As a consequence of what we've just shown and Corollary 2.4.3.8, we now obtain a bifunctor $f\colon \Omega^{\mathrm{op}} \times \Omega \to \Omega$ that fits into an equivalence

$$\mathrm{map}_{\Omega}(- \times -, -) \simeq \mathrm{map}_{\Omega}(-, f(-,-)).$$

We complete the proof by showing that f is equivalent to the mapping bifunctor $\mathrm{map}_{\Omega}(-,-)$. Note that by [62, Proposition 4.6.3] the functor $\mathrm{map}_{\Omega}(1_{\Omega}, -)$ is equivalent to the identity on Ω. Hence the chain of equivalences

$$f(-,-) \simeq \mathrm{map}_{\Omega}(1_{\Omega}, f(-,-)) \simeq \mathrm{map}_{\Omega}(1_{\Omega} \times -, -) \simeq \mathrm{map}_{\Omega}(-,-)$$

in which the second step follows from the evident equivalence $1_{\Omega} \times - \simeq \mathrm{id}_{\Omega}$ gives rise to the desired identification. $\square$

In [62, Proposition 3.7.3], it was shown that for any two objects $g, h\colon A \rightrightarrows \Omega$ in context $A \in \mathcal{B}$ that correspond to $\mathcal{B}_{/A}$-groupoids G, H, there is an equivalence $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\mathsf{G}, \mathsf{H}) \simeq \mathrm{map}_{\Omega}(g, h)$ of $\mathcal{B}_{/A}$-groupoids (where $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(\mathsf{G}, \mathsf{H})$ denotes the internal hom in $\mathcal{B}_{/A}$). We are now able to upgrade this result to a *functorial* equivalence.

Proposition 3.1.4.12. *The mapping $\mathcal{B}$-groupoid bifunctor* $\mathrm{map}_{\Omega}(-,-)$ *recovers the internal hom bifunctor* $\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(-,-)\colon \mathcal{B}_{/A}^{\mathrm{op}} \times \mathcal{B}_{/A} \to \mathcal{B}_{/A}$ *when taking local sections over $A \in \mathcal{B}$.*

Proof. By [62, Lemma 4.7.13] and Remark 2.1.14.1, we can identify $\pi_A^*(\mathrm{map}_{\Omega}(-,-))$ with the functor $\mathrm{map}_{\Omega_{\mathcal{B}_{/A}}}(-,-)$. Therefore, by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ we may assume without loss of generality that $A \simeq 1$. Also, [62, Corollary 4.6.8] implies that one may identify the bifunctor $\mathrm{map}_{\mathcal{B}}(-,-)\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{S}$ with the composition

$$\mathcal{B}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{\Gamma_{\mathcal{B}}(\mathrm{map}_{\Omega}(-,-))} \mathcal{B} \xrightarrow{\Gamma_{\mathcal{B}}} \mathcal{S}.$$

Since applying $\Gamma_{\mathcal{B}}$ to the bifunctor $- \times -\colon \Omega \times \Omega \to \Omega$ recovers the ordinary product bifunctor on $\mathcal{B}$, Proposition 3.1.4.11 yields an equivalence

$$\mathrm{map}_{\mathcal{B}}(- \times -, -) \simeq \mathrm{map}_{\mathcal{B}}(-, \Gamma_{\mathcal{B}}(\mathrm{map}_{\Omega}(-,-))),$$

which finishes the proof. $\square$

3.1.5. Limits and colimits in $\mathsf{Cat}_{\mathcal{B}}$. Recall that by the discussion in § 2.3.1, the assignment $A \mapsto \mathrm{Cat}(\mathcal{B}_{/A})$ defines a sheaf of ∞-categories on $\mathcal{B}$ that we denote by $\mathsf{Cat}_{\mathcal{B}}$ and that we refer to as the $\mathcal{B}$-category of (small) $\mathcal{B}$-categories. By combining Proposition 3.1.2.12 with Proposition 2.4.2.13 and the fact that presheaf $\mathcal{B}$-categories admits small limits and colimits (Proposition 3.1.4.9), we find:

Proposition 3.1.5.1. *The $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ admits small limits and colimits.* $\square$

Next, our goal is to show that $\mathsf{Cat}_{\mathcal{B}}$ is *cartesian closed*. To that end, let $- \times -\colon \mathsf{Cat}_{\mathcal{B}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ be the product functor.

Proposition 3.1.5.2. *There is a functor* $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ *together with an equivalence*

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(- \times -, -) \simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(-, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)).$$

In other words, the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ is cartesian closed.

Proof. This is proved in exactly the same way as Proposition 3.1.4.11. Using Corollary 2.4.3.8, it is enough to show that the product bifunctor transposes to a functor $\mathsf{Cat}_{\mathcal{B}} \to \underline{\mathsf{Fun}}^L_{\mathcal{B}}(\mathsf{Cat}_{\mathcal{B}}, \mathsf{Cat}_{\mathcal{B}})$. Using the equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}$ from Remark 2.3.1.3, we may carry out the same reduction steps as in the proof of Proposition 3.1.4.11, so that it will be sufficient to prove that for every $\mathcal{B}$-category C the functor $\mathsf{C} \times -\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ has a right adjoint. To see this, note that this functor is given on local sections over $A \in \mathcal{B}$ by the ∞-categorical product functor

$$\mathrm{Cat}(\mathcal{B}_{/A}) \xrightarrow{\pi_A^*\mathsf{C}\times -} \mathrm{Cat}(\mathcal{B}_{/A}).$$

which admits a right adjoint $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}, -)$. Furthermore, if $s\colon B \to A$ is a map in $\mathcal{B}$, we deduce from [62, Lemma 4.2.3] that the natural map $s^*\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}, -) \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/B}}(\pi_B^*\mathsf{C}, s^*(-))$ is an equivalence. Hence, Proposition 2.4.2.9 shows that the functor $\mathsf{C} \times -\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ admits a right adjoint, as desired. $\square$

Remark 3.1.5.3. By making use of [62, Corollary 4.6.8] and the fact that the product bifunctor $- \times -$ on $\mathsf{Cat}_{\mathcal{B}}$ recovers the ∞-categorical product bifunctor on $\mathrm{Cat}(\mathcal{B}_{/A})$ upon taking local sections over $A \in \mathcal{B}$, the equivalence

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(- \times -, -) \simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(-, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-))$$

from Proposition 3.1.5.2 implies that the bifunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ recovers the internal hom of $\mathrm{Cat}(\mathcal{B}_{/A})$ when being evaluated at $A \in \mathcal{B}$, which justifies our choice of notation.

Corollary 3.1.5.4. *The mapping $\mathcal{B}$-groupoid bifunctor $\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \Omega$ is equivalent to the composition of the bifunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ with the core $\mathcal{B}$-groupoid functor $(-)^{\simeq}\colon \mathsf{Cat}_{\mathcal{B}} \to \Omega$.*

Proof. On account of Proposition 2.4.2.14 and the fact that the functor $\mathrm{map}_{\Omega}(1_{\Omega}, -)$ is equivalent to the identity on Ω (see [62, Proposition 4.6.3]), we obtain equivalences

$$\begin{aligned}
\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)^{\simeq} &\simeq \mathrm{map}_{\Omega}(1_{\Omega}, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)^{\simeq}) \\
&\simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(1_{\Omega}, \underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)) \\
&\simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(1_{\Omega} \times -, -) \\
&\simeq \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(-,-)
\end{aligned}$$

in which the last equivalence follows from the evident equivalence $1_{\Omega} \times - \simeq \mathrm{id}_{\mathsf{Cat}_{\mathcal{B}}}$. $\square$

3.1.6. Colimits in slice $\mathcal{B}$-categories. It is well-known that if $\mathcal{C}$ is an ∞-category and $c \in \mathcal{C}$ is an arbitrary object, the colimit of a diagram $d\colon \mathcal{I} \to \mathcal{C}_{/c}$ can be computed as the colimit of the underlying diagram $(\pi_c)_! d\colon \mathcal{I} \to \mathcal{C}$. In this section we will establish the analogous statement for $\mathcal{B}$-categories.

Lemma 3.1.6.1. *Let C be a $\mathcal{B}$-category and let $f\colon c \to d$ be a map in C in context $1 \in \mathcal{B}$ such that c is an initial object in C. Then f defines an initial object in $\mathsf{C}_{/d}$.*

Proof. Let $g\colon c' \to d$ be an arbitrary map in C in context $1 \in \mathcal{B}$. We have an evident commutative square

$$\begin{tikzcd}
(\mathsf{C}_{/d})_{/g} \arrow[r, "\simeq"] \arrow[d, "(\pi_g)_!"] & \mathsf{C}_{/c'} \arrow[d, "(\pi_{c'})_!"] \\
\mathsf{C}_{/d} \arrow[r, "(\pi_d)_!"] & \mathsf{C}.
\end{tikzcd}$$

in which the upper horizontal map is an equivalence as it is a right fibration that preserves final objects. Moreover, since c is initial, the diagram

$$\begin{tikzcd}
1 \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & \mathsf{C}_{/d} \arrow[d, "(\pi_d)_!"] \\
1 \arrow[r, "c"] & \mathsf{C}
\end{tikzcd}$$

is a pullback. Thus we obtain a cartesian square

Consequently, we obtain an equivalence $\mathrm{map}_{/d}(f,g) \simeq \mathrm{map}_{\mathsf{C}}(c,c')$. Since c is initial, we conclude that $\mathrm{map}_{/d}(f,g) \simeq 1$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, the same conclusion holds for every map $g\colon c' \to \pi_A^* d$ in context A. Hence, we deduce from [62, Proposition 4.3.14] that f is initial. □

Lemma 3.1.6.2. *Let* C *be a* $\mathcal{B}$*-category and let* $f\colon c \to d$ *be a map in* C *in context* $1 \in \mathcal{B}$*. Then there is an equivalence* $(\mathsf{C}_{c/})_{/f} \simeq (\mathsf{C}_{/d})_{f/}$ *that commutes with the projections to* $\mathsf{C}_{/d}$ *and* $\mathsf{C}_{c/}$*.*

Proof. Note that the projection $(\pi_c)_!\colon \mathsf{C}_{c/} \to \mathsf{C}$ induces a left fibration $(\pi_c)_!\colon (\mathsf{C}_{c/})_{/f} \to \mathsf{C}_{/d}$. By considering the commutative square

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & c \arrow[d, "f"] \\ c \arrow[r, "f"] & d \end{tikzcd}$$

as an object $\varphi\colon 1 \to (\mathsf{C}_{c/})_{/f}$, we obtain a commutative square

$$\begin{tikzcd} 1 \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_f"] & (\mathsf{C}_{c/})_{/f} \arrow[d, "(\pi_c)_!"] \\ (\mathsf{C}_{/d})_{f/} \arrow[r, "(\pi_c)_!"] \arrow[ur, dotted] & \mathsf{C}_{/d}. \end{tikzcd}$$

As the left vertical map is initial, the dotted filler exists, hence the proof is complete once we show that φ is initial too. By construction, the right fibration $(\pi_f)_!\colon (\mathsf{C}_{c/})_{/f} \to \mathsf{C}_{c/}$ carries φ to an initial object. The desired result therefore follows from Lemma 3.1.6.1. □

Proposition 3.1.6.3. *Let* I *and* C *be* $\mathcal{B}$*-categories and let* $c\colon 1 \to \mathsf{C}$ *be an object. Let* $d\colon \mathsf{I} \to \mathsf{C}_{/c}$ *be a diagram and suppose that the diagram* $(\pi_c)_! d\colon \mathsf{I} \to \mathsf{C}$ *admits a colimit in* C*. Then* d *admits a colimit in* $\mathsf{C}_{/c}$*, and* $(\pi_c)_!$ *preserves this colimit.*

Proof. On account of the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/\operatorname{diag}(c)}$, the diagram $d\colon \mathsf{K} \to \mathsf{C}_{/c}$ corresponds to an object $d' = (\pi_c)_! d \to \operatorname{diag}(c)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/\operatorname{diag}(c)}$, which can be equivalently regarded as a cocone $\overline{d'}\colon 1 \to \mathsf{C}_{d'/}$. One therefore obtains a unique map

$$\begin{tikzcd} & d' \arrow[dl] \arrow[dr, "\overline{d'}"] & \\ \operatorname{diag}(\operatorname{colim} d') \arrow[rr] & & \operatorname{diag}(c) \end{tikzcd}$$

in $\mathsf{C}_{d'/}$ (by the universal property of initial objects, see [62, Proposition 4.3.14]) which can be regarded as an object in $(\mathsf{C}_{d'/})_{/\overline{d'}}$. Now Lemma 3.1.6.2 gives rise to an equivalence

$$(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{d'/})_{/\overline{d'}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c})_{d/}$$

over $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c})$ the pullback of which along the diagonal map determines an equivalence $(\mathsf{C}_{d'/})_{/\overline{d'}} \simeq (\mathsf{C}_{/c})_{d/}$ that fits into a commutative diagram

$$\begin{tikzcd} (\mathsf{C}_{d'/})_{/\overline{d'}} \arrow[rr, "\simeq"] \arrow[dr, "(\pi_{\overline{d'}})_!"'] & & (\mathsf{C}_{/c})_{d/} \arrow[dl, "(\pi_c)_!"] \\ & \mathsf{C}_{d'/} & \end{tikzcd}$$

Consequently, the colimit cocone $d' \to \operatorname{colim} d'$ lifts along $(\pi_c)_!$ to a cocone under d. By Lemma 3.1.6.1, this lift defines an initial object and therefore a colimit cocone, hence the claim follows. □

3.1.7. Adjunctions of slice $\mathcal{B}$-categories. In this section we collect some basic facts about adjunctions between slice $\mathcal{B}$-categories. More precisely, we show how an adjunction between $\mathcal{B}$-categories induces an adjunction on slice $\mathcal{B}$-categories, and we furthermore investigate the relation between the existence of pullbacks in a $\mathcal{B}$-category and adjunctions between its slices. Everything discussed in this section is well-known for ∞-categories, and our proofs are straightforward adaptations of their ∞-categorical counterparts.

Proposition 3.1.7.1. *Let $(l \dashv r) \colon \mathsf{C} \leftrightarrows \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories, and let $c \colon A \to \mathsf{C}$ be an arbitrary object. Then the induced functor $r_{/c} \colon \mathsf{C}_{/c} \to \mathsf{D}_{/r(c)}$ of $\mathcal{B}_{/A}$-categories admits a left adjoint l_c that is explicitly given by the composition*

$$l_c \colon \mathsf{D}_{/r(c)} \xrightarrow{l_{/r(c)}} \mathsf{C}_{/lr(c)} \xrightarrow{(\epsilon c)_!} \mathsf{C}_{/c},$$

where $\epsilon c \colon lr(c) \to c$ is the counit of the adjunction $(l \dashv r)$.

Proof. Using [62, Remark 4.2.2], we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and the adjunction $l \dashv r$ by its image along π_A^* and can therefore assume without loss of generality that $A \simeq 1$. Let us fix an adjunction unit η and an adjunction counit ϵ. In light of the commutative diagram

$$\begin{tikzcd}
 & & \mathsf{C}_{/c} \arrow[rr, "r_{/c}"] \arrow[ddl, bend right, "(\pi_c)_!"'] & & \mathsf{D}_{/r(c)} \arrow[ddl, bend left, "(\pi_{r(c)})_!"] \\
\mathsf{D}_{/r(c)} \arrow[r, "l_{/r(c)}"] \arrow[urr, bend left, "l_c"] \arrow[d, "(\pi_{r(c)})_!"'] & \mathsf{C}_{/lr(c)} \arrow[rr, "r_{/lr(c)}"] \arrow[ur, "(\epsilon c)_!"] \arrow[d, "(\pi_{lr(c)})_!"'] & & \mathsf{D}_{/rlr(c)} \arrow[ur, "(\epsilon r(c))_!"] \arrow[d, "(\pi_{rlr(c)})_!"'] & \\
\mathsf{D} \arrow[r, "l"] & \mathsf{C} \arrow[rr, "r"] & & \mathsf{D} &
\end{tikzcd}$$

we obtain an equivalence $rl(\pi_{r(c)})_! \simeq (\pi_{r(c)})_! r_{/c} l_c$, which in turn yields a commutative diagram

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_{r(c)}"] \arrow[d, hook, "d^0"] \arrow[rr, bend left, "\epsilon r(c)"] & \mathsf{D}_{/r(c)} \arrow[r, "r_{/c} l_c"] \arrow[d, "d^0"] & \mathsf{D}_{/r(c)} \arrow[d, "(\pi_{r(c)})_!"] \\
\Delta^1 \arrow[r, "\mathrm{id} \otimes \mathrm{id}_{r(c)}"] \arrow[rr, bend right, "r\eta c"'] & \Delta^1 \otimes \mathsf{D}_{/r(c)} \arrow[r, "\eta(\pi_{r(c)})_!"] \arrow[ur, dashed, "\eta_c"] & \mathsf{D}.
\end{tikzcd}$$

Note that as d^0 is a final functor, the lift η_c exists. Moreover, since restricting η_c along $\mathrm{id} \otimes \mathrm{id}_{r(c)}$ produces a lift of the outer square in the above diagram, the uniqueness of such lifts and the triangle identities for the adjunction $l \dashv r$ imply that η_c carries the final object $\mathrm{id}_{r(c)}$ to the map in $\mathsf{D}_{/r(c)}$ that is encoded by the commutative triangle

$$\begin{tikzcd}
r(c) \arrow[r, "r\eta(c)"] \arrow[dr, "\mathrm{id}_{r(c)}"'] & rlr(c) \arrow[d, "\epsilon r(c)"] \\
 & r(c).
\end{tikzcd}$$

In particular, the functor $\mathsf{D}_{/r(c)} \xrightarrow{d^1} \Delta^1 \otimes \mathsf{D}_{/r(c)} \xrightarrow{\eta_c} \mathsf{D}_{/r(c)}$ preserves the final object. Since this functor by construction commutes with the projection $(\pi_{r(c)})_!$, it must therefore be equivalent to the identity on $\mathsf{D}_{/r(c)}$, so that η_c encodes a map $\mathrm{id}_{\mathsf{D}_{/r(c)}} \to r_{/c} l_c$.

Dually, we also have an equivalence $lr(\pi_c)_! \simeq (\pi_c)_! l_c r_{/c}$, so that the map $\epsilon(\pi_c)_! \colon \Delta^1 \otimes \mathsf{C}_{/c} \to \mathsf{C}$ encodes a morphism of functors $(\pi_c)_! l_c r_{/c} \to (\pi_c)_!$. By an analogous argument as above, we can now construct a lift $\epsilon_c \colon \Delta^1 \otimes \mathsf{C}_{/c} \to \mathsf{C}_{/c}$ of $\epsilon(\pi_c)_!$ along $(\pi_c)_!$ that encodes a morphism of functors $l_c r_{/c} \to \mathrm{id}_{\mathsf{C}_{/c}}$. To complete the proof, it now suffices to show that the two compositions $r_{/c} \xrightarrow{\eta_c r_{/c}} r_{/c} l_c r_{/c} \xrightarrow{r_{/c}\epsilon_c} r_{/c}$ and $l_c \xrightarrow{l_c \eta_c} l_c r_{/c} l_c \xrightarrow{\epsilon l_c} l_c$ are equivalences, by Proposition 2.4.1.4. Using that $(\pi_{r(c)})_!$ and $(\pi_c)_!$ are right fibrations and therefore in particular conservative, it suffices to show that these two morphisms become equivalences after postcomposition with $(\pi_{r(c)})_!$ and $(\pi_c)_!$, respectively. Therefore, the claim follows from the triangle identities for η and ϵ, together with the observation that by construction we may identify $(\pi_{r(c)})_!\eta_c \simeq \eta(\pi_{r(c)})_!$ and $(\pi_c)_!\epsilon_c \simeq \epsilon(\pi_c)_!$. □

LEMMA 3.1.7.2. *Let C be a $\mathcal{B}$-category and let $c\colon 1 \to \mathsf{C}$ be an arbitrary object. For any map $f\colon d \to c$, there is a pullback square*

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}_{/c}}(-, f) \arrow[r, "(\pi_c)_!"] \arrow[d] & \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), d) \arrow[d, "f_*"] \\
\mathrm{diag}(1_\Omega) \arrow[r] & \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), c)
\end{tikzcd}$$

in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/c})$.

PROOF. We have a commutative diagram

$$\begin{tikzcd}
\mathsf{C}_{/d} \arrow[r, "(f_!, \mathrm{id})"] \arrow[d, "f_!"] & \mathsf{C}_{/c} \times_{\mathsf{C}} \mathsf{C}_{/d} \arrow[r] \arrow[d] & \mathsf{C}_{/d} \arrow[d, "f_!"] \\
\mathsf{C}_{/c} \arrow[r, "(\mathrm{id}, \mathrm{id})"] & \mathsf{C}_{/c} \times_{\mathsf{C}} \mathsf{C}_{/c} \arrow[r] \arrow[d] & \mathsf{C}_{/c} \arrow[d, "(\pi_c)_!"] \\
 & \mathsf{C}_{/c} \arrow[r, "(\pi_c)_!"] & \mathsf{C}
\end{tikzcd}$$

in which all square are cartesian. It follows that the upper right square defines a cartesian square of right fibrations over C and thus the claim follows because $(\pi_c)_!$ induces an equivalence $(\mathsf{C}_{/c})_{/f} \xrightarrow{\simeq} \mathsf{C}_{/d}$ of right fibrations over $\mathsf{C}_{/c}$. □

PROPOSITION 3.1.7.3. *Let C be a $\mathcal{B}$-category with a final object $1_{\mathsf{C}}\colon 1 \to \mathsf{C}$. Then the following are equivalent:*

(1) C admits finite products;
(2) for every $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$, the projection $(\pi_c)_!\colon \mathsf{C}_{/c} \to \pi_A^*\mathsf{C}$ admits a right adjoint π_c^*.

Moreover, if either of these conditions is satisfied, the composition $(\pi_c)_!\pi_c^*$ is equivalent to the endofunctor $- \times c$ on $\pi_A^*\mathsf{C}$.

PROOF. Let us first assume that C admits finite products. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and C with $\pi_A^*\mathsf{C}$, we may assume that $A \simeq 1$. Suppose that $d\colon 1 \to \mathsf{C}$ is an arbitrary object. On account of the equivalence $1_{\mathsf{C}} \times c \simeq c$, we have a commutative square

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}"] \arrow[d, "d_0"] & 1 \arrow[r, "\mathrm{id}_c"] \arrow[d, "1_{\mathsf{C}}"'] & \mathsf{C}_{/c} \arrow[d, "(\pi_c)_!"] \\
\Delta^1 \arrow[r, "\pi_d"] & \mathsf{C} \arrow[r, "- \times c"] \arrow[ur, dashed, "\pi_c^*"] & \mathsf{C}
\end{tikzcd}$$

(in which $\pi_d\colon d \to 1_{\mathsf{C}}$ denotes the unique map), and since 1_{C} is final, the lift π_c^* exists. Note that the projection $\mathrm{pr}_0\colon - \times c \to \mathrm{id}_{\mathsf{C}}$ defines a map $\epsilon\colon (\pi_c)_!\pi_c^* \to \mathrm{id}_{\mathsf{C}}$. Now the fact that π_c^* by construction preserves final objects implies that this functor carries the unique map $\pi_d\colon d \to 1_{\mathsf{C}}$ to the unique map $\pi_{\pi_c^*(d)}\colon \pi_c^*(d) \to \mathrm{id}_c$. As this implies that the image of $\pi_{\pi_c^*(d)}$ along $(\pi_c)_!$ recovers the projection $\mathrm{pr}_1\colon d \times c \to c$, the commutative square

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}}(-, (\pi_c)_!\pi_c^*(d)) \arrow[r, "\epsilon_*"] \arrow[d, "(\pi_c)_!(\pi_{\pi_c^*(d)})_*"] & \mathrm{map}_{\mathsf{C}}(-, d) \arrow[d] \\
\mathrm{map}_{\mathsf{C}}(-, c) \arrow[r] & \mathrm{diag}(1_\Omega)
\end{tikzcd}$$

is a pullback in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Together with Lemma 3.1.7.2, this shows that the composition

$$\mathrm{map}_{\mathsf{C}_{/c}}(-, \pi_c^*(d)) \xrightarrow{(\pi_c)_!} \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), (\pi_c)_!\pi_c^*(d)) \xrightarrow{\epsilon_*} \mathrm{map}_{\mathsf{C}}((\pi_c)_!(-), d)$$

is an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and C with $\pi_A^*\mathsf{C}$, the same assertion is true for any object $d\colon A \to \mathsf{C}$. Hence π_c^* is right adjoint to $(\pi_c)_!$.

Conversely, suppose that $(\pi_c)_!$ admits a right adjoint π_c^* for all objects $c\colon A \to \mathsf{C}$, and let us show that C admits finite products. By induction, it suffices to consider binary products. Given any pair of objects $(c, d)\colon A \to \mathsf{C} \times \mathsf{C}$, we need to show that the presheaf $\mathrm{map}_{\pi_A^*\mathsf{C}}(\mathrm{diag}(-), (c, d))$ is representable. We may again assume that $A \simeq 1$. Let us show that the object $(\pi_c)_!\pi_c^*(d)$ represents this presheaf. Note that there is a pullback square

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}\times\mathsf{C}}(\mathrm{diag}(-), (c, d)) \arrow[r] \arrow[d] & \mathrm{map}_{\mathsf{C}}(-, d) \arrow[d] \\
\mathrm{map}_{\mathsf{C}}(-, d) \arrow[r] & \mathrm{diag}(1_\Omega).
\end{tikzcd}$$

To complete the proof, it therefore suffices to show that the maps $\epsilon_*\colon \mathrm{map}_{\mathsf{C}}(-, (\pi_c)_!\pi_c^*(d)) \to \mathrm{map}_{\mathsf{C}}(-, d)$ and $(\pi_c)_!(\pi_{\pi_c^*(d)})_*\colon \mathrm{map}_{\mathsf{C}}(-, (\pi_c)_!\pi_c^*(d)) \to \mathrm{map}_{\mathsf{C}}(-, c)$ exhibit $\mathrm{map}_{\mathsf{C}}(-, (\pi_c)_!\pi_c^*(d))$ as a product of $\mathrm{map}_{\mathsf{C}}(-, c)$ and $\mathrm{map}_{\mathsf{C}}(-, d)$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. By the object-wise criterion for equivalences and Corollary 2.4.1.9, this follows once we show that for every $z\colon 1 \to \mathsf{C}$ the commutative square

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{C}}(z, (\pi_c)_!\pi_c^*(d)) \arrow[r, "\epsilon_*"] \arrow[d, "(\pi_c)_!(\pi_{\pi_c^*(d)})_*"] & \mathrm{map}_{\mathsf{C}}(z, d) \arrow[d] \\
\mathrm{map}_{\mathsf{C}}(z, c) \arrow[r] & 1
\end{tikzcd}$$

is a pullback square in $\mathcal{B}$. By descent in $\mathcal{B}$ and Lemma 3.1.7.2, this follows once we show that for any map $f\colon \pi_A^*(z) \to \pi_A^*(c)$ the composition

$$\mathrm{map}_{\mathsf{C}_{/c}}(f, \pi_c^*(d)) \xrightarrow{(\pi_c)_!} \mathrm{map}_{\mathsf{C}}(\pi_A^*(z), \pi_A^*(\pi_c)_!\pi_c^*(d)) \xrightarrow{\pi_A^*(\epsilon)_*} \mathrm{map}_{\mathsf{C}}(\pi_A^*(z), \pi_A^*(d))$$

is an equivalence. Since this is just the adjunction property of $(\pi_c)_! \dashv \pi_c^*$, the claim follows. ☐

REMARK 3.1.7.4. In the situation of Proposition 3.1.7.3, the proof shows that in light of the equivalence $(\pi_c)_!\pi_c^* \simeq - \times c$, the counit of the adjunction $(\pi_c)_! \dashv \pi_c^*$ can be identified with $\mathrm{pr}_0\colon -\times c \to \mathrm{id}_{\pi_A^*\mathsf{C}}$. Similarly, if $d \to c$ is an arbitrary map in context $A \in \mathcal{B}$, the unit $d \to \pi_c^*(\pi_c)_!$ is characterised by the condition that the composition $(\pi_c)_!d \to (\pi_c)_!\pi_c^*(\pi_c)_!d \simeq ((\pi_c)_!d) \times c \to (\pi_c)_!d$ is equivalent to the identity. It is therefore determined by the map $(\pi_c)_!d \to ((\pi_c)_!d) \times c$ that is given by the identity on the first factor and the structure map $d \to c$ on the second factor.

COROLLARY 3.1.7.5. *For any $\mathcal{B}$-category C, the following are equivalent:*

(1) C admits pullbacks;
(2) for every map $f\colon c \to d$ in C in context $A \in \mathcal{B}$, the projection $f_!\colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ admits a right adjoint f^*.

Moreover, if either of these conditions are satisfied, then the composition $f_!f^*$ can be identified with the pullback functor $- \times_d c$.

PROOF. In light of Proposition 3.1.7.3, it will be enough to show that C admits pullbacks if and only if for every object $c\colon A \to \mathsf{C}$ the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/c}$ admits binary products. Using Example 3.1.1.14, this is easily reduced to the corresponding statement for ∞-categories, which appears as [17, Theorem 6.6.9]. ☐

COROLLARY 3.1.7.6. *Let $(l \dashv r)\colon \mathsf{C} \to \mathsf{D}$ be an adjunction between $\mathcal{B}$-categories, and suppose that C admits pullbacks. Then for any object $d\colon A \to \mathsf{D}$ in D in context $A \in \mathcal{B}$, the induced functor $l_{/d}\colon \mathsf{D}_{/d} \to \mathsf{C}_{/l(c)}$ admits a right adjoint r_d that is explicitly given by the composition*

$$r_d\colon \mathsf{C}_{/l(d)} \xrightarrow{r_{/l(d)}} \mathsf{D}_{/rl(d)} \xrightarrow{(\eta d)^*} \mathsf{D}_{/d}$$

in which $(\eta d)^* = d \times rl(d) -$ is the pullback functor along the adjunction unit.

Proof. By Corollary 3.1.7.5 the functor $(\eta d)^*$ indeed exists and is right adjoint to the projection $(\eta d)_!$. Now by Proposition 3.1.7.1, the functor $r_{/l(d)}\colon \mathsf{C}_{/l(d)} \to \mathsf{D}_{/rl(d)}$ admits a left adjoint $l_{r(d)}$ that is given by the composition $(\epsilon l(d))_! l_{/rl(d)}$. Therefore, the functor r_d is right adjoint to the composition

$$D_{/d} \xrightarrow{(\eta d)_!} \mathsf{D}_{/rl(d)} \xrightarrow{l_{/rl(d)}} C_{/lrl(d)} \xrightarrow{(\epsilon l(d))_!} \mathsf{C}_{/l(d)}.$$

It now suffices to notice that on account of the triangle identities, this functor is equivalent to $l_{/d}$. □

3.1.8. A characterisation of initial and final functors. In this section, we show that initial and final functors (see § 2.1.11) can be characterised as those functors along which restriction of diagrams does not change their limits and colimits, respectively. For the case $\mathcal{B} \simeq \mathcal{S}$, this characterisation is proved in [57, Proposition 4.1.1.8] or [17, Theorem 6.4.5]. For the general case, note that precomposition with a functor $i\colon \mathsf{J} \to \mathsf{I}$ of $\mathcal{B}$-categories defines a functor $i^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\mathsf{C})$ that induces a functor $i^*\colon \mathsf{C}_{d/} \to \mathsf{C}_{i^*d/}$ over $A \times \mathsf{C}$ for every I-indexed diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ in C.

Proposition 3.1.8.1. *For any functor $i\colon \mathsf{J} \to \mathsf{I}$ between $\mathcal{B}$-categories, the following are equivalent:*

(1) i is final;
(2) for every large $\mathcal{B}$-category C and every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ in context $A \in \mathcal{B}$, the functor $i^*\colon \mathsf{C}_{d/} \to \mathsf{C}_{i^*d/}$ is an equivalence;
(3) For every large $\mathcal{B}$-category C and every diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ in context $A \in \mathcal{B}$ that admits a colimit $\operatorname{colim} d$, the image of the colimit cocone $d \to \operatorname{diag}\operatorname{colim} d$ along the functor $i^*\colon \mathsf{C}_{d/} \to \mathsf{C}_{i^*d/}$ defines a colimit cocone of i^*d.
(4) The mate of the commutative square

$$\begin{tikzcd} \Omega \arrow[r, "\operatorname{diag}"] \arrow[d, "\operatorname{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega) \arrow[d, "i^*"] \\ \Omega \arrow[r, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\Omega) \end{tikzcd}$$

commutes.

The dual characterisation of initial functors holds as well.

Proof. We begin by showing that (1) implies (2). Suppose that i is final, and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ be an arbitrary diagram. By making use of Remark 3.1.1.9 and the fact that the base change functor π_A^* preserves final functors [62, Remark 4.4.9], we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$ (see Remark 2.1.14.4). On account of [62, Proposition 4.1.18], it suffices to show that the induced map $i^*|_c$ on the fibres over every $c\colon A \to \mathsf{C}$ is an equivalence. By the same argument as above, we may again assume $A \simeq 1$. Now the commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "c"] \arrow[d, "d"] & \mathsf{C} \arrow[d, "d\times\operatorname{diag}"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}) \arrow[r, "\operatorname{id}\times\operatorname{diag}(c)"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}) \end{tikzcd}$$

shows that the fibre of the left fibration $\mathsf{C}_{d/} \to \mathsf{C}$ over c is equivalent to the fibre of the right fibration $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}_{/c}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$ (that is given by postcomposition with $(\pi_c)_!\colon \mathsf{C}_{/c} \to \mathsf{C}$) over $d\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})$. Similarly, the fibre of $\mathsf{C}_{i^*d/} \to \mathsf{C}$ over c is equivalent to the fibre of the right fibration $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\mathsf{C}_{/c}) \to$

$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C})$ over i^*d such that the map $i^*|_c$ fits into the commutative diagram

$$\begin{tikzcd}
& \mathsf{C}_{i^*d/}|_c \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C}_{/c}) \arrow[dd] \\
\mathsf{C}_{d/}|_c \arrow[ur, "i^*|_c"] \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}_{/c}) \arrow[ur, "i^*"] \arrow[dd] & \\
& 1 \arrow[rr, "i^*d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C}) \\
1 \arrow[ur, "\mathrm{id}"] \arrow[rr, "d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \arrow[ur, "i^*"] &
\end{tikzcd}$$

in which the two squares in the front and in the back are cartesian. Since i is final, the right square must be cartesian as well, hence $i^*|_c$ is an equivalence, so that (2) holds. Condition (3) follows immediately from (2). For the special case $\mathsf{C} = \Omega$, the same argument as in the proof of Proposition 3.1.2.4 shows that condition (3) is equivalent to the condition that the map $\operatorname{colim}_{\mathsf{J}} i^* \to \operatorname{colim}_{\mathsf{I}}$ must be an equivalence, hence condition (3) implies condition (4). Lastly, suppose that the map $\operatorname{colim}_{\mathsf{J}} i^* \to \operatorname{colim}_{\mathsf{I}}$ is an equivalence, and let us show that i is final. It will be enough to show that i is internally left orthogonal to the universal right fibration $\widehat{\Omega}^{\mathrm{op}} \to \Omega^{\mathrm{op}}$ (see [62, § 4.6]) as every right fibration between (small) $\mathcal{B}$-categories arises as a pullback of this functor. By Proposition 3.1.4.5, the universe Ω admits small limits, hence if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega^{\mathrm{op}})$ is an arbitrary diagram both $\Omega^{\mathrm{op}}_{d/}$ and $\Omega^{\mathrm{op}}_{i^*d/}$ admits an initial section. By assumption, the functor $i^*\colon \Omega^{\mathrm{op}}_{d/} \to \Omega^{\mathrm{op}}_{i^*d/}$ sends the colimit cocone $d \to \operatorname{diag}\operatorname{colim} d$ to an initial section of $\Omega^{\mathrm{op}}_{i^*d/}$, which implies that the functor $i^*\colon \Omega^{\mathrm{op}}_{d/} \to \Omega^{\mathrm{op}}_{i^*d/}$ must be initial as well. But this map is already a left fibration since it can be regarded as a map between left fibrations over Ω^{op}, hence we conclude that this functor must be an equivalence. Similarly as above and by making use of the equivalence $\widehat{\Omega} \simeq \Omega_{1_\Omega/}$ over Ω from [62, Proposition 4.6.3], one obtains a commutative diagram

$$\begin{tikzcd}
& \Omega^{\mathrm{op}}_{i^*d/}|_{\pi_A^*(1_\Omega)} \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \widehat{\Omega}^{\mathrm{op}}) \arrow[dd] \\
\Omega^{\mathrm{op}}_{d/}|_{\pi_A^*(1_\Omega)} \arrow[ur, "i^*|_{\pi_A^*(1_\Omega)}"] \arrow[rr] \arrow[dd] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \widehat{\Omega}^{\mathrm{op}}) \arrow[ur, "i^*"] \arrow[dd] & \\
& A \arrow[rr, "i^*d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega^{\mathrm{op}}) \\
A \arrow[ur, "\mathrm{id}"] \arrow[rr, "d"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega^{\mathrm{op}}) \arrow[ur, "i^*"] &
\end{tikzcd}$$

in which the squares in the front, in the back and on the left are cartesian. As the maps $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \widehat{\Omega}^{\mathrm{op}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega^{\mathrm{op}})$ and $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \widehat{\Omega}^{\mathrm{op}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega^{\mathrm{op}})$ are right fibrations, the vertical square on the right is cartesian already when its underlying square of core $\mathcal{B}$-groupoids is. We therefore deduce that this square must be a pullback as well, which means that i is final. □

REMARK 3.1.8.2. Let C be a large $\mathcal{B}$-category, let $i\colon \mathsf{J} \to \mathsf{I}$ be a functor between $\mathcal{B}$-categories and let us fix an I-indexed diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$. Suppose that both d and i^*d admit a colimit in C. Then the universal property of initial objects (see [62, Corollary 4.3.21]) gives rise to a unique map

$$\begin{tikzcd}
& i^*d \arrow[dl] \arrow[dr] & \\
\operatorname{diag}\operatorname{colim} i^*d \arrow[rr] & & \operatorname{diag}\operatorname{colim} d
\end{tikzcd}$$

in $\mathsf{C}_{i^*d/}$ that is an equivalence if and only if the cocone $i^*d \to \operatorname{diag}\operatorname{colim} d$ (which is the image of the colimit cocone $d \to \operatorname{diag}\operatorname{colim} d$ along i^*) is a colimit cocone. Proposition 3.1.8.1 now implies that this map is always an equivalence when i is final, and conversely i must be final whenever this map is an equivalence for every $\mathcal{B}$-category C and every diagram d that has a colimit in C (in fact, Proposition 3.1.8.1 shows that it suffices to consider $\mathsf{C} = \Omega$).

3.1.9. Decomposition of colimits I. In [57, § 4.2], Lurie provides techniques for computing colimits in an ∞-category by means of decomposing diagrams into more manageable pieces. For example, he proves that an ∞-category has small colimits if and only if it has small coproducts and pushouts. In this section, we aim for similar results in the context of internal higher category theory. We are mainly interested in the decomposition of arbitrary colimits into colimits indexed by constant $\mathcal{B}$-categories (i.e. $\mathcal{B}$-categories that are in the image of the functor $\mathrm{const}_{\mathcal{B}} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathcal{B})$, see Remark 2.1.6.10) and $\mathcal{B}$-groupoids. In these two cases, colimits admit rather explicit descriptions that are often simpler to understand in practice (see Examples 3.1.1.14 and 3.1.1.13). Note that in ∞-category theory such a decomposition is not really visible since internal to the ∞-topos of spaces $\mathcal{S}$, any $\mathcal{S}$-groupoid is automatically constant. However, the technique of proof that we use is still mostly the same as in [57, § 4.2]. Our main result will be the following proposition:

Proposition 3.1.9.1. *A large $\mathcal{B}$-category C admits small colimits if and only if it admits colimits indexed by* constant *$\mathcal{B}$-categories and by $\mathcal{B}$-groupoids, and a functor $f \colon \mathsf{C} \to \mathsf{D}$ between large $\mathcal{B}$-categories that admit small colimits preserves such colimits if and only if it preserves colimits indexed by constant $\mathcal{B}$-categories and by $\mathcal{B}$-groupoids.*

The proof of Proposition 3.1.9.1 requires a few preparations.

Lemma 3.1.9.2. *Let $(\mathsf{C}_i)_{i \in I}$ be a small family of $\mathcal{B}$-categories, and let $c_i \colon 1 \to \mathsf{C}_i$ be an object in context $1 \in \mathcal{B}$ for every $i \in I$. If each c_i is initial then the induced object $c = (c_i)_{i \in I} \colon 1 \to \mathsf{C} = \prod_i \mathsf{C}_i$ is initial as well.*

Proof. By Proposition 2.1.12.12, the object $(c_i)_{i \in I}$ is initial precisely if the projection

$$(\pi_c)_! \colon \mathsf{C}_{c/} \to \mathsf{C}$$

is an equivalence. The result thus follows from the observation that $(\pi_c)_!$ is equivalent to the product

$$\prod_i (\pi_{c_i})_! \colon \prod_i (\mathsf{C}_i)_{c_i/} \to \prod_i \mathsf{C}_i$$

and is therefore an equivalence since each of the maps $(\pi_{c_i})_!$ is one. □

The key input in the proof of Proposition 3.1.9.1 is the following Proposition. The strategy of proof is the same as in [57, Proposition 4.4.2.6].

Proposition 3.1.9.3. *Let κ be a regular cardinal, let $\mathcal{K}$ be a κ-small ∞-category and let*

$$\alpha \colon \mathcal{K} \to \mathrm{Cat}(\mathcal{B}), \quad k \mapsto \mathsf{J}_k$$

be a diagram with colimit $\mathsf{J} = \mathrm{colim}_k \mathsf{J}_k$ in $\mathrm{Cat}(\mathcal{B})$. Suppose that C is a $\mathcal{B}$-category and that $d \colon \mathsf{J} \to \mathsf{C}$ is a diagram such that

(1) for every $k \in \mathcal{K}$ the restricted diagram $d_k \colon \mathsf{J}_k \to \mathsf{C}$ admits a colimit in C;
(2) C admits colimits indexed by κ-small constant $\mathcal{B}$-categories.

Then d admits a colimit in C.

Proof. We consider the full subcategory $\mathcal{C}$ of $(\mathrm{Cat}_\infty)_{/\mathcal{K}}$ spanned by all functors $\varphi \colon \mathcal{L} \to \mathcal{K}$ such that the conclusion of the proposition holds for $\alpha \circ \varphi$. We wish to show that the $\mathcal{C}$ contains $\mathrm{id}_{\mathcal{K}}$. For this it suffices to see that $\mathcal{C}$ contains all maps $\Delta^n \to \mathcal{K}$ and is closed under κ-small coproducts and pushouts (as every κ-small simplicial set can be build as an iterated pushout of κ-small coproducts of simplices). Since Δ^n has a final object, the first part is clear. Thus it remains to prove the proposition in the cases where $\mathcal{K}$ is a κ-small set and $\mathcal{K} = \Lambda^2_0$. Suppose first that $\mathcal{K}$ is a κ-small set. Then the inclusions

$i_k\colon \mathsf{J}_k \hookrightarrow \mathsf{J}$ for each $k \in \mathcal{K}$ determine a pullback square

$$\begin{tikzcd} \mathsf{C}_{d/} \arrow[r, "(i_k^*)_{k\in\mathcal{K}}"] \arrow[d] & \prod_k \mathsf{C}_{d_k/} \arrow[d] \\ \mathsf{C} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{C}). \end{tikzcd}$$

By assumption, each of the categories $\mathsf{C}_{d_k/}$ admits an initial global section, hence Lemma 3.1.9.2 implies that the induced global section $1 \to \prod_k \mathsf{C}_{d_k/}$ is initial as well. Phrased differently, the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{C}) \to \Omega$ that classifies the left fibration $\prod_k \mathsf{C}_{d_k/} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \Omega)$ is corepresented by the diagram $(\operatorname{colim} d_k)_{k\in\mathcal{K}}\colon \mathcal{K} \to \mathsf{C}$. Since diag by assumption admits a left adjoint, we thus conclude that the left fibration $\mathsf{C}_{d/} \to \mathsf{C}$ is classified by the functor corepresented by $\bigsqcup_k \operatorname{colim} d_k\colon 1 \to \mathsf{C}$, which implies that d has a colimit in C.

Let us now assume $\mathcal{K} = \Lambda^2_0$, i.e. that J is given by a pushout. Then there is an equivalence

$$\begin{tikzcd} \mathsf{C}_{d/} \arrow[rr, "\simeq"] \arrow[dr] & & \lim_k \mathsf{C}_{d_k/} \arrow[dl] \\ & \mathsf{C} & \end{tikzcd}$$

of left fibrations over C, which with Example 3.1.1.14 implies that the functor $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\mathsf{C})}(d, \operatorname{diag}(-))$ is given by the $\mathcal{K}^{\mathrm{op}}$-indexed limit of functors $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}_k,\mathsf{C})}(d_k, \operatorname{diag}(-))$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$. Since C by assumption admits $\mathcal{K}$-indexed colimits, its opposite C^{op} admits $\mathcal{K}^{\mathrm{op}}$-indexed limits. Moreover, since each of the functors $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}_k,\mathsf{C})}(d_k, \operatorname{diag}(-))$ is contained in the essential image of the Yoneda embedding $\mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$, we conclude that $\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\mathsf{C})}(d, \operatorname{diag}(-))$ is corepresentable since the Yoneda embedding commutes with limits (Proposition 3.1.4.9). Hence the diagram d admits a colimit in C. $\square$

By a similar argument as in the proof of Proposition 3.1.9.3 one shows:

PROPOSITION 3.1.9.4. *Let κ be a regular cardinal, let $\mathcal{K}$ be a κ-small ∞-category and let*

$$\alpha\colon \mathcal{K} \to \mathrm{Cat}(\mathcal{B}), \quad k \mapsto \mathsf{J}_k$$

be a diagram with colimit $\mathsf{J} = \operatorname{colim}_k \mathsf{J}_k$ in $\mathrm{Cat}(\mathcal{B})$. Let C be a $\mathcal{B}$-category that satisfies the conditions of Proposition 3.1.9.3, let $d\colon \mathsf{J} \to \mathsf{C}$ be a diagram and suppose that $f\colon \mathsf{C} \to \mathsf{D}$ is a functor in $\mathrm{Cat}(\mathcal{B})$ such that

(1) for every $k \in \mathcal{K}$ the functor f preserves the colimit of the restricted diagram $d_k\colon \mathsf{J}_k \to \mathsf{C}$;
(2) f preserves colimits indexed by κ-small constant $\mathcal{B}$-categories.

Then f preserves the colimit of d. $\square$

PROOF OF PROPOSITION 3.1.9.1. Let J be a $\mathcal{B}$-category and let $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \mathsf{C})$ be a diagram in context $A \in \mathcal{B}$. We want to show that d admits a colimit in C. By making use of Remark 3.1.1.9, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$ (see Remark 2.1.14.4). Recall from [62, Lemma 4.5.2 and the discussion following it] that we have a canonical equivalence

$$\mathsf{J} \simeq \operatorname*{colim}_{(\Delta^n \times \mathsf{G})_{/\mathsf{J}}} \Delta^n \otimes \mathsf{G}.$$

Furthermore it follows from Proposition 3.1.8.1 that a $\mathcal{B}$-category C has $\Delta^n \otimes \mathsf{G}$-indexed colimits if and only if it has G-indexed colimits since Δ^n admits a final object. So if C admits colimits indexed by constant $\mathcal{B}$-categories and $\mathcal{B}$-groupoids G, we may apply Proposition 3.1.9.3 to conclude that d has a colimit in C. The argument for the preservation of small colimits is analogous, by making use of Proposition 3.1.9.4 instead. $\square$

3.2. Cocompleteness

This section is dedicated to a more global study of (co)limits in a $\mathcal{B}$-category. More precisely, if U is an internal class of $\mathcal{B}$-categories (i.e. a full subcategory of $\mathsf{Cat}_{\mathcal{B}}$, see Definition 3.2.1.1), we define and study what it means for a $\mathcal{B}$-category C to be U-*(co)complete* and for a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories to be U-*(co)continuous*. For the special case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$, this will yield the correct internal analogue of the usual notion of cocompleteness and cocontinuity in (higher) category theory. One should note that this will be a strictly stronger notion than to simply admit all internal colimits that are indexed by small $\mathcal{B}$-categories, cf. Example 3.2.4.12 below. We begin in § 3.2.1 by defining the notion of an internal class U of $\mathcal{B}$-categories, which is the internal analogue of a collection of ∞-categories. In § 3.2.2, we give the definition of U-cocompleteness and U-cocontinuity with respect to such an internal class and we recast some of the results from § 3.1 in this language. In § 3.2.3, we define the large $\mathcal{B}$-category of U-cocomplete $\mathcal{B}$-categories, and in § 3.2.4 we study the special case where U is the internal class of *all* (small) $\mathcal{B}$-categories.

3.2.1. Internal classes. In this section we introduce the correct $\mathcal{B}$-categorical analogue of *classes* of ∞-categories:

Definition 3.2.1.1. An *internal class* of $\mathcal{B}$-categories is a full subcategory $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$.

Remark 3.2.1.2. The reason why we define an internal class to be a full subcategory $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ rather than just a subcategory $\mathcal{U} \hookrightarrow \mathrm{Cat}(\mathcal{B})$ in the usual ∞-categorical sense is that when using internal classes as indexing classes for colimits, only the former notion leads to a theory of cocompleteness that is local in $\mathcal{B}$ (cf. § 2.1.14), whereas the latter does not. For example, it is not reasonable to call a $\mathcal{B}$-category *cocomplete* even when it admits I-indexed colimits for every $\mathcal{B}$-category I (see Definition 3.1.1.5), because it could still happen that there is a $\mathcal{B}_{/A}$-category J (for some $A \in \mathcal{B}$) such that $\pi_A^*\mathsf{C}$ does not have all J-indexed colimits (see Example 3.2.4.12 below). Instead, on should ask that C admits all colimits indexed by the maximal internal class $\mathsf{Cat}_{\mathcal{B}}$ (Example 3.2.1.3), which precisely amounts to asking that every small diagram $\mathsf{I} \to \pi_A^*\mathsf{C}$ of $\mathcal{B}_{/A}$-categories admits a colimit for every $A \in \mathcal{B}$. In this way, the notion of cocompleteness is forced to be local.

Example 3.2.1.3. By Remark 2.3.1.5, the (large) $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ may be regarded as an internal class of large $\mathcal{B}$-categories, so as a subcategory of the (very large) $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}$.

Example 3.2.1.4. On account of the adjunction $\mathrm{const} \dashv \Gamma\colon \widehat{\mathrm{Cat}}_{\infty} \leftrightarrows \mathrm{Cat}(\widehat{\mathcal{B}})$, the transpose of the functor $\mathrm{const}\colon \mathrm{Cat}_{\infty} \to \mathrm{Cat}(\mathcal{B}) \simeq \Gamma(\mathsf{Cat}_{\mathcal{B}})$ defines a map $\mathrm{const}(\mathrm{Cat}_{\infty}) \to \mathsf{Cat}_{\mathcal{B}}$ in $\mathrm{Cat}(\widehat{\mathcal{B}})$. The essential image of this functor thus defines an internal class of $\mathcal{B}$-categories that we denote by $\mathsf{LConst} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ and that we refer to as the internal class of *locally constant* $\mathcal{B}$*-categories*. By construction, this is the full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the constant $\mathcal{B}$-categories, i.e. by those objects $1 \to \mathsf{Cat}_{\mathcal{B}}$ that correspond to categories of the form $\mathrm{const}(\mathcal{C})$ for some $\mathcal{C} \in \mathrm{Cat}_{\infty}$. Thus, a $\mathcal{B}_{/A}$-category C defines an object in LConst in context $A \in \mathcal{B}$ precisely if there is a cover $(s_i)_{i\in I}\colon \bigsqcup_{i\in I} A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^*\mathsf{C}$ is a constant $\mathcal{B}_{/A_i}$-category for each $i \in I$.

Example 3.2.1.5. On account of the inclusion $\Omega \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ from Proposition 2.4.2.14, the universe Ω can be viewed as an internal class of $\mathcal{B}$-categories.

3.2.2. U-cocomplete $\mathcal{B}$-categories. In this section we define and study the condition on a $\mathcal{B}$-category to admit colimits indexed by objects in an internal class U of $\mathcal{B}$-categories (see Definition 3.2.1.1).

Definition 3.2.2.1. Let U be an internal class of $\mathcal{B}$-categories. A $\mathcal{B}$-category C is said to be U-*cocomplete* if $\pi_A^*\mathsf{C}$ admits I-indexed colimits for every object $\mathsf{I} \in \mathsf{U}(A)$ and every $A \in \mathcal{B}$. Similarly, if $f\colon \mathsf{C} \to \mathsf{D}$ is a functor between $\mathcal{B}$-categories that are both U-cocomplete, we say that f is U-*cocontinuous* if $\pi_A^* f$ preserves I-indexed colimits for any $A \in \mathcal{B}$ and any $\mathsf{I} \in \mathsf{U}(A)$. We simply say that a (large) $\mathcal{B}$-category

C is *cocomplete* if it is $\mathsf{Cat}_{\mathcal{B}}$-cocomplete (when viewing $\mathsf{Cat}_{\mathcal{B}}$ as an internal class of $\widehat{\mathcal{B}}$-categories), and we call a functor between cocomplete (large) $\mathcal{B}$-categories *cocontinuous* if it is $\mathsf{Cat}_{\mathcal{B}}$-cocontinuous.

Dually, we say that a $\mathcal{B}$-category C is U-*complete* if $\pi_A^*\mathsf{C}$ admits I-indexed limits for every object $\mathsf{I} \in \mathsf{U}(A)$ and every $A \in \mathcal{B}$. If $f\colon \mathsf{C} \to \mathsf{D}$ is a functor between $\mathcal{B}$-categories that are both U-complete, we say that f is U-*continuous* if $\pi_A^* f$ preserves I-indexed limits for any $A \in \mathcal{B}$ and any $\mathsf{I} \in \mathsf{U}(A)$. We simply say that a (large) $\mathcal{B}$-category C is *complete* if it is $\mathsf{Cat}_{\mathcal{B}}$-complete, and we call a functor between complete (large) $\mathcal{B}$-categories *continuous* if it is $\mathsf{Cat}_{\mathcal{B}}$-continuous.

Remark 3.2.2.2. If U is an internal class of $\mathcal{B}$-categories, let $\mathrm{op}(\mathsf{U})$ be the internal class that arises as the image of U along the equivalence $(-)^{\mathrm{op}}\colon \mathsf{Cat}_{\mathcal{B}} \simeq \mathsf{Cat}_{\mathcal{B}}$ from Remark 2.3.1.4. Then a $\mathcal{B}$-category C is U-complete if and only if C^{op} is $\mathrm{op}(\mathsf{U})$-cocomplete, and a functor f is U-continuous if and only if f^{op} is $\mathrm{op}(\mathsf{U})$-cocontinuous. For this reason, we may dualise statements about $\mathrm{op}(\mathsf{U})$-cocompleteness and $\mathrm{op}(\mathsf{U})$-cocontinuity to obtain the corresponding statements about U-completeness and U-continuity.

Remark 3.2.2.3 (locality of U-cocompleteness and U-cocontinuity). Since both the existence of (co)limits and the preservation of such (co)limits are local conditions (Remark 3.1.1.8 and Remark 3.1.2.1), one finds that if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, a $\mathcal{B}$-category C is U-(co)complete if and only if $\pi_{A_i}^*\mathsf{C}$ is $\pi_{A_i}^*\mathsf{U}$-(co)complete, and a functor $f\colon \mathsf{C} \to \mathsf{D}$ between U-(co)complete $\mathcal{B}$-categories is U-(co)continuous if and only if $\pi_{A_i}^*(f)$ is $\pi_{A_i}^*\mathsf{U}$-(co)continuous.

Remark 3.2.2.4. Let U be an internal class of $\mathcal{B}$-categories that is spanned by a collection of objects $(\mathsf{I}_i \in \mathsf{Cat}_{\mathcal{B}}(A_i))_{i \in I}$ in $\mathsf{Cat}_{\mathcal{B}}$ (in the sense of § 2.1.9). Then Remark 3.1.1.8 implies that a $\mathcal{B}$-category C is U-cocomplete whenever $\pi_{A_i}^*\mathsf{C}$ has I_i-indexed colimits for all $i \in I$. Moreover, Remark 3.1.2.1 implies that a functor $f\colon \mathsf{C} \to \mathsf{D}$ between U-cocomplete $\mathcal{B}$-categories is U-cocontinuous whenever $\pi_{A_i}^* f$ preserves I_i-indexed colimits for all $i \in I$.

Since by Corollary 2.4.1.9 the functor π_A^* carries adjunctions in $\mathcal{B}$ to adjunctions in $\mathcal{B}_{/A}$ for every $A \in \mathcal{B}$, Proposition 3.1.2.11 implies:

Proposition 3.2.2.5. *A left adjoint functor between U-cocomplete categories is U-cocontinuous, while a right adjoint between U-complete categories is U-continuous.* □

Similarly, Proposition 3.1.2.12 shows:

Proposition 3.2.2.6. *Suppose that U is an internal class of $\mathcal{B}$-categories and let D be a U-cocomplete $\mathcal{B}$-category. Then every reflective and every coreflective subcategory of D is U-cocomplete as well.* □

As we have a natural equivalence $\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*(-), \pi_A^*(-))$ for every $A \in \mathcal{B}$ (see Remark 2.1.14.1), Propositions 3.1.3.1 and 3.1.3.3 show:

Proposition 3.2.2.7. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories. Then $f_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{D})$ is a U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories for all $K \in \mathcal{B}_\Delta$. Moreover, for all $i\colon L \to K$ in $\mathcal{B}_\Delta$, the map $i^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(K, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(L, \mathsf{C})$ is U-cocontinuous as well.* □

Example 3.2.2.8. The universe Ω for small $\mathcal{B}$-groupoids is complete and cocomplete since Ω admits small limits and colimits (Proposition 3.1.4.1 and Proposition 3.1.4.5) and since for any $A \in \mathcal{B}$ there is a natural equivalence $\pi_A^*\Omega \simeq \Omega_{\mathcal{B}_{/A}}$ (Remark 2.1.14.1). By the same argument and Proposition 3.1.4.8, the inclusion $i\colon \Omega_{\mathcal{B}} \hookrightarrow \Omega_{\widehat{\mathcal{B}}}$ is continuous and cocontinuous.

Furthermore we conclude:

Proposition 3.2.2.9. *For any $\mathcal{B}$-category C, the presheaf $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is complete and cocomplete. If C is U-complete for some internal class U, the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is U-continuous, and for every $c\colon A \to \mathsf{C}$ the corepresentable copresheaf $\mathrm{map}_{\mathsf{C}}(c,-)\colon A \times \mathsf{C} \to \Omega$ transposes to a $\pi_A^*\mathsf{U}$-continuous functor $\pi_A^*\mathsf{C} \to \Omega_{\mathcal{B}_{/A}}$.*

PROOF. The first claim is an immediate consequence of Example 3.2.2.8 and Proposition 3.2.2.7. For the second claim, we have to see that $\pi_A^* h\colon \pi_A^*\mathsf{C} \to \pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ preserves all limits indexed by the objects in $\mathsf{U}(A)$. By Example 2.1.14.7, we may identify $\pi_A^* h_{\mathsf{C}}$ with $h_{\pi_A^*\mathsf{C}}$, so that we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Now the claim follows from Proposition 3.1.4.9. Lastly, the third claim is a direct consequence of Corollary 3.1.4.10. □

EXAMPLE 3.2.2.10. By combining Proposition 3.2.2.9 with Proposition 3.2.2.6, one finds that the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ is complete and cocomplete.

Also note that Proposition 3.1.6.3 in combination with the fact that the slice construction is compatible with $\pi_A^*(-)$ by Remark 2.1.14.1 we conclude:

COROLLARY 3.2.2.11. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category. For every object* $c\colon 1 \to \mathsf{C}$*, the slice* $\mathcal{B}$*-category* $\mathsf{C}_{/c}$ *is* U*-cocomplete, and the forgetful functor* $(\pi_c)_!$ *is* U*-cocontinuous.* □

3.2.3. The large $\mathcal{B}$-category of U-cocomplete $\mathcal{B}$-categories. In Proposition 2.2.2.7, we show that in order to define a (non-full) subcategory of a $\mathcal{B}$-category C, it suffices to specify a subobject of its object of morphisms C_1, i.e. an arbitrary family of maps in C. With this in mind, we define:

DEFINITION 3.2.3.1. For any internal class U of $\mathcal{B}$-categories, the large $\mathcal{B}$-category of U-cocomplete $\mathcal{B}$-categories $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ is defined as the subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the $\pi_A^*\mathsf{U}$-cocontinuous functors between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. In the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$ (viewed as an internal class of large $\mathcal{B}$-categories), we denote the resulting very large $\mathcal{B}$-category by $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{cc}}$.

REMARK 3.2.3.2 (locality of $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$). The subobject of $(\mathsf{Cat}_{\mathcal{B}})_1$ that is spanned by the $\pi_A^*\mathsf{U}$-cocontinuous functors between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}$-categories is stable under equivalences and composition in the sense of Proposition 2.2.2.9. As moreover U-cocompleteness and U-cocontinuity are local conditions (Remark 3.2.2.3), we conclude (by the same argument as in Remark 2.1.14.5) that an object $A \to \mathsf{Cat}_{\mathcal{B}}$ is contained in $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ if and only if the associated $\mathcal{B}_{/A}$-category is $\pi_A^*\mathsf{U}$-complete, and a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}_{/A}$-categories defines a morphism in $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ in context $A \in \mathcal{B}$ precisely if it is a $\pi_A^*\mathsf{U}$-cocontinuous functor between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories. In particular, if C and D are $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories, a functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is contained in the image of the monomorphism

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}}(\mathsf{C},\mathsf{D}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\mathsf{C},\mathsf{D})$$

if and only if it is $\pi_A^*\mathsf{U}$-cocontinuous. Moreover, there is a canonical equivalence $\pi_A^*\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}\text{-cc}}$ for every $A \in \mathcal{B}$ (by the same argument as in Remark 2.1.14.6).

DEFINITION 3.2.3.3. Let U be an internal class of $\mathcal{B}$-categories. If C and D are U-cocomplete $\mathcal{B}$-categories, we will denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{D})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ that is spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ in context $A \in \mathcal{B}$ such that the corresponding functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is $\pi_A^*\mathsf{U}$-cocontinuous. In the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$, we will denote the associated large $\mathcal{B}$-category by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{C},\mathsf{D})$.

REMARK 3.2.3.4 (locality of $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{D})$). In the situation of Definition 3.2.3.3, note that by combining Remark 3.1.5.3 and Corollary 3.1.5.4 with Remark 3.2.3.2, we obtain an equivalence

$$\mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}}(\mathsf{C},\mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{D})^{\simeq}.$$

As a consequence, Remark 3.2.3.2 implies that an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{D})$ is contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{D})$ if and only if the associated functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is $\pi_A^*\mathsf{U}$-cocontinuous, and we obtain a canonical equivalence

$$\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{D}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}\text{-cc}}(\pi_A^*\mathsf{C},\pi_A^*\mathsf{D})$$

for every $A \in \mathcal{B}$ (see Remark 2.1.14.6).

The notion of U-cocompleteness and U-cocontinuity allows for some flexibility in the choice of internal class U. For example, Proposition 3.1.8.1 implies that whenever I is a $\mathcal{B}$-category that is contained in U and $f\colon \mathsf{I} \to \mathsf{J}$ is a final functor, adjoining the $\mathcal{B}$-category J to U does not affect whether a $\mathcal{B}$-category is U-cocomplete or not. As it will be convenient later to impose certain stability conditions on an internal class, we define:

Definition 3.2.3.5. A *colimit class* in $\mathcal{B}$ is an internal class U of $\mathcal{B}$-categories that contains the final $\mathcal{B}$-category 1 and that is stable under final functors, i.e. satisfies the property that whenever $\mathsf{I} \to \mathsf{J}$ is a final functor in $\mathcal{B}_{/A}$ for some $A \in \mathcal{B}$, then $\mathsf{I} \in \mathsf{U}(A)$ implies that $\mathsf{J} \in \mathsf{U}(A)$.

For every internal class U of $\mathcal{B}$-categories one can construct a colimit class $\mathsf{U}^{\mathrm{colim}}$ that is uniquely specified by the condition that $\mathsf{U}^{\mathrm{colim}}$ is the minimal colimit class that contains U. Explicitly, this class is spanned by those $\mathcal{B}_{/A}$-categories J that admit a final functor from either an object in $\mathsf{U}(A)$ or the final $\mathcal{B}_{/A}$-category $1 \in \mathrm{Cat}(\mathcal{B}_{/A})$. Thus, a $\mathcal{B}_{/A}$-category I is contained in $\mathsf{U}^{\mathrm{colim}}(A)$ if and only if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that for each i the $\mathcal{B}_{/A_i}$-category $s_i^*\mathsf{I}$ admits a final functor from either an object in $\mathsf{U}(A_i)$ or the final object $1 \in \mathrm{Cat}(\mathcal{B}_{/A_i})$. By combining Proposition 3.1.8.1 with Remark 3.2.2.4, we deduce that a $\mathcal{B}$-category C is U-cocomplete if and only if it is $\mathsf{U}^{\mathrm{colim}}$-cocomplete, and similarly a functor $f\colon \mathsf{C} \to \mathsf{D}$ is U-cocontinuous if and only if it is $\mathsf{U}^{\mathrm{colim}}$-cocontinuous. Together with the evident observation that the above description of the objects in $\mathsf{U}^{\mathrm{colim}}$ is local in $\mathcal{B}$ (so that one obtains an equivalence $\pi_A^*(\mathsf{U}^{\mathrm{colim}}) \simeq (\pi_A^*\mathsf{U})^{\mathrm{colim}}$ for all $A \in \mathcal{B}$, cf. § 2.1.14), this implies that one has $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \simeq \mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}^{\mathrm{colim}}\text{-cc}}$. Thus, for the sake of discussing colimits, we may therefore always assume that an internal class is a colimit class.

3.2.4. Cocompleteness and cocontinuity. In § 3.1.9, we saw that every small internal colimit can be decomposed into colimits indexed by $\mathcal{B}$-groupoids and by constant $\mathcal{B}$-categories. In the terminology introduced in § 3.2.2, this result can be formulated as follows:

Proposition 3.2.4.1. *A large $\mathcal{B}$-category C is cocomplete if and only if it is both Ω- and LConst-cocomplete, and a functor between cocomplete large $\mathcal{B}$-categories is cocontinuous if and only if it is both Ω- and LConst-cocontinuous.*

Proof. We show the case of cocompleteness, the case of cocontinuity is completely analogous. We need to show that for every $A \in \mathcal{B}$ the $\mathcal{B}_{/A}$-category $\pi_A^*\mathsf{C}$ admits colimits indexed by all small $\mathcal{B}_{/A}$-categories if it admits colimits indexed by all small $\mathcal{B}_{/A}$-groupoids and by the objects of $\mathsf{LConst}(A)$. Note that by construction of LConst (Example 3.2.1.4) and by the equivalence $\mathrm{const}_{\mathcal{B}_{/A}} \simeq \pi_A^* \mathrm{const}_{\mathcal{B}}$ for every $A \in \mathcal{B}$ (Remark 2.1.14.1), we may identify $\pi_A^* \mathsf{LConst}$ with the internal class of locally constant $\mathcal{B}_{/A}$-categories. Therefore, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$. In this case, the result follows immediately from Proposition 3.1.9.3 (since every constant $\mathcal{B}$-category defines an object in $\mathsf{LConst}(1)$). □

In light of Proposition 3.2.4.1, it seems reasonable to investigate Ω- and LConst-cocompleteness separately. We begin with the case of $\mathcal{B}$-groupoidal colimits. By combining Example 3.1.1.13 with Example 3.1.2.6, we find:

Proposition 3.2.4.2. *Let S be a local class of maps in $\mathcal{B}$ and let Ω_S be the associated subuniverse (see § 2.1.10), where we view Ω_S as an internal class of large $\mathcal{B}$-categories. Then a large $\mathcal{B}$-category C is Ω_S-cocomplete if and only if the following two conditions are satisfied:*

(1) for every map $p\colon P \to A$ in S, the functor $p^*\colon \mathsf{C}(A) \to \mathsf{C}(P)$ admits a left adjoint $p_!$;
(2) for every pullback square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"'] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

in $\mathcal{B}$ in which p and q are contained in S, the natural map $q_!t^* \to s^*p_!$ is an equivalence.

Furthermore, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between (large) Ω_S-cocomplete $\mathcal{B}$-categories is Ω_S-cocontinuous precisely if for every map $p\colon P \to A$ in S the natural map $p_!f(P) \to f(A)p_!$ is an equivalence. $\square$

Example 3.2.4.3. If S is a local class in $\mathcal{B}$, the associated subuniverse $\Omega_S \hookrightarrow \Omega$ is closed under Ω_S-colimits (i.e. Ω_S is Ω_S-cocomplete and the inclusion $\Omega_S \hookrightarrow \Omega$ is Ω_S-cocontinuous) if and only if S is stable under composition. For example, this is always the case when S is the right class of a factorisation system in $\mathcal{B}$.

Example 3.2.4.4. Recall from Example 2.4.4.5 that every modality $(\mathcal{L}, \mathcal{R})$ in $\mathcal{B}$ (i.e. a factorisation system in which $\mathcal{L}$ is stable under base change in $\mathcal{B}$) determines a reflective subcategory $\Omega_{\mathcal{R}}$ of Ω. Conversely, if $\Omega_{\mathcal{R}} \hookrightarrow \Omega$ is an arbitrary reflective subcategory, then [85, Theorem 4.8] shows that the associated local class $\mathcal{R}$ in $\mathcal{B}$ arises from a modality as in Example 2.4.4.5 precisely if $\mathcal{R}$ is stable under composition, i.e. if $\Omega_{\mathcal{R}} \hookrightarrow \Omega$ is closed under $\Omega_{\mathcal{R}}$-colimits. Hence modalities in $\mathcal{B}$ correspond precisely to those reflective subuniverses that are closed under self-indexed colimits in Ω.

Let $\mathcal{K}$ be a class of ∞-categories, i.e. a full subcategory of Cat_∞. As in example 3.2.1.4 we obtain a functor $\mathcal{K} \to \mathsf{Cat}_{\mathcal{B}}$ by transposing the map $\mathrm{const}_{\mathcal{B}}\colon \mathcal{K} \hookrightarrow \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathcal{B})$ across the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}$. We denote the essential image of this functor by $\mathsf{LConst}_{\mathcal{K}}$. By construction, for every $A \in \mathcal{B}$ the internal class $\pi_A^* \mathsf{LConst}_{\mathcal{K}}$ is the full subcategory of $\mathsf{Cat}_{\mathcal{B}_{/A}}$ that is spanned by $\mathrm{const}_{\mathcal{B}_{/A}}(\mathcal{I})$ for each $\mathcal{I} \in \mathcal{K}$. Hence a $\mathcal{B}_{/A}$-category C defines an object in $\mathsf{LConst}_{\mathcal{K}}(A)$ if and only if there is a cover $(s_i)_i\colon \bigsqcup A_i \twoheadrightarrow A$ such that $s_i^*\mathsf{C} \simeq \mathrm{const}_{\mathcal{B}_{/A_i}}(\mathcal{I}_i)$ for some $\mathcal{I}_i \in \mathcal{K}$. Using Remark 3.2.2.4, Examples 3.1.1.14 and 3.1.2.7 now imply:

Proposition 3.2.4.5. *If $\mathcal{K}$ is a class of ∞-categories, a $\mathcal{B}$-category C is $\mathsf{LConst}_{\mathcal{K}}$-cocomplete if and only if for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits colimits indexed by every object in $\mathcal{K}$ and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves such colimits. Furthermore, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathsf{LConst}_{\mathcal{K}}$-cocomplete $\mathcal{B}$-categories is $\mathsf{LConst}_{\mathcal{K}}$-cocontinuous if and only if for all $A \in \mathcal{B}$ the functor $f(A)$ preserves all colimits that are indexed by objects in $\mathcal{K}$.* $\square$

In Construction 2.3.1.1 we define a functor $- \otimes \Omega\colon \mathrm{Pr}^{\mathrm{R}} \to \mathrm{Cat}(\widehat{\mathcal{B}})$. Its explicit formula and Proposition 3.2.4.5 now yield:

Corollary 3.2.4.6. *For every class of ∞-categories $\mathcal{K}$ there is an equivalence $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{LConst}_{\mathcal{K}}\text{-cc}} \simeq \mathrm{Cat}_\infty^{\mathcal{K}\text{-cc}} \otimes \Omega$ with respect to which the inclusion $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{LConst}_{\mathcal{K}}\text{-cc}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is obtained by applying $- \otimes \Omega$ to the inclusion $\mathrm{Cat}_\infty^{\mathcal{K}\text{-cc}} \hookrightarrow \mathrm{Cat}_\infty$.* $\square$

By combining Propositions 3.2.4.1, 3.2.4.2 and 3.2.4.5 we now arrive at the following:

Corollary 3.2.4.7. *A $\mathcal{B}$-category C is cocomplete if and only if the following conditions are satisfied:*

(1) For every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ is cocomplete and for any $s\colon B \to A$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves colimits.

(2) For every map $p\colon P \to A$ in $\mathcal{B}$ the functor p^* has a left adjoint $p_!$ such that for every pullback square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

the natural map $q_!t^* \to s^*p_!$ is an equivalence.

Furthermore a functor $f\colon \mathsf{C} \to \mathsf{D}$ of cocomplete $\mathcal{B}$-categories is cocontinuous if and only if for every $A \in \mathcal{B}$ the functor $f(A)$ preserves colimits, and for every map $p\colon P \to A$ in $\mathcal{B}$ the natural map $p_!f(P) \to f(A)p_!$ is an equivalence. $\square$

Example 3.2.4.8. Let $\mathcal{C}$ be a presentable ∞-category. Then Corollary 3.2.4.7 and its dual show that the $\mathcal{B}$-category of Construction 2.3.1.1 is both complete and cocomplete. In fact $\mathcal{C} \otimes \Omega$ will give rise to a *presentable* $\mathcal{B}$*-category*, which are defined to be suitable localisations of presheaf $\mathcal{B}$-categories. We will pursue a detailed study of presentable $\mathcal{B}$-categories in Chapter 4.

Remark 3.2.4.9. Let $\mathcal{C}$ be a small ∞-category such that $\mathcal{B}$ is a left exact and accessible localisation of $\mathrm{PSh}(\mathcal{C})$, and let $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ be the localisation functor. Then in order to see that a $\mathcal{B}$-category C is cocomplete, it suffices to check the conditions of Corollary 3.2.4.7 for objects in $\mathcal{C}$: Indeed, as the existence of colimits is a local condition (Remark 3.1.1.8), one may assume without loss of generality that the object A appearing in condition (1) and (2) of Corollary 3.2.4.2 is of the form $L(a)$ for some $a \in \mathcal{C}$. By furthermore using Remark 3.1.1.15, one can also assume that $B = L(b)$ and $s = L(s')$ for some $d \in \mathcal{C}$ and some map $s'\colon b \to a$ in $\mathcal{C}$. Finally, provided that C is LConst-cocomplete, Proposition 3.1.9.3 allows us to further assume that $P = L(p)$ and $u = L(u')$ for some $p \in \mathcal{C}$ and some map $u'\colon p \to a$ in $\mathcal{C}$. Together with Proposition 3.2.4.1, these observations imply that C is cocomplete if and only if

(1) for every $a \in \mathcal{C}$ the ∞-category $\mathsf{C}(L(a))$ has small colimits, and for every $t\colon b \to a$ in $\mathcal{C}$ the functor $L(t)^*\colon \mathsf{C}(L(a)) \to \mathsf{C}(L(b))$ preserves small colimits;
(2) for every pullback square
$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "v"] & p \arrow[d, "u"] \\ b \arrow[r, "s"] & a \end{tikzcd}$$
in $\mathrm{PSh}(\mathcal{C})$ where $s\colon b \to a$ and $u\colon p \to a$ are maps in $\mathcal{C}$, the functors $L(u)^*\colon \mathsf{C}(L(a)) \to \mathsf{C}(L(p))$ and $L(v)^*\colon \mathsf{C}(L(d)) \to \mathsf{C}(L(Q))$ admits left adjoints $L(u)_!$ and $L(v)_!$ such that the natural map $L(v)_! L(t)^* \to L(s)^* L(u)_!$ is an equivalence.

Example 3.2.4.10. Let $\mathcal{X}$ be an ∞-topos and let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism. We may consider the limit-preserving functor
$$\mathcal{X}_{/f^*(-)}\colon \mathcal{B}^{\mathrm{op}} \xrightarrow{(f^*)^{\mathrm{op}}} \mathcal{X}^{\mathrm{op}} \xrightarrow{\mathcal{X}_{/-}} \widehat{\mathrm{Cat}}_\infty$$
which defines a large $\mathcal{B}$-category X. Clearly X is LConst-cocomplete. Furthermore, for every pullback square
$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "S"] & A \end{tikzcd}$$
in $\mathcal{B}$, the lax square
$$\begin{tikzcd} \mathcal{X}_{/f^*(Q)} \arrow[d, "f^*(q)_!"] \arrow[dr, phantom, "\searrow"] & \mathcal{X}_{/f^*(P)} \arrow[l, "f^*(t)^*"'] \arrow[d, "f^*(p)_!"] \\ \mathcal{X}_{/f^*(B)} & \mathcal{X}_{/f^*(A)} \arrow[l, "f^*(s)^*"] \end{tikzcd}$$
commutes since f^* preserves pullbacks. Thus it follows from Corollary 3.2.4.7 that X is cocomplete. Dually one shows that X is also complete. In fact X will be an example of a $\mathcal{B}$*-topos*, i.e. a left exact localisation (in a suitable sense) of a presheaf $\mathcal{B}$-category and any $\mathcal{B}$-topos arises in this way. We will make this claim precise in Chapter 5.

Example 3.2.4.11. One may also combine Proposition 3.2.4.2 and 3.2.4.5 in a more general way. Namely let S be a local class of maps in $\mathcal{B}$ and $\mathcal{K}$ a class of ∞-categories, and consider the internal class $\langle S, \mathcal{K}\rangle$ generated by Ω_S and $\mathsf{LConst}_{\mathcal{K}}$ (i.e. the essential image of the functor $\Omega_S \sqcup \mathsf{LConst}_{\mathcal{K}} \to \mathsf{Cat}_{\mathcal{B}}$). Then Remark 3.2.2.4 shows that a $\mathcal{B}$-category C is $\langle S, \mathcal{K}\rangle$-cocomplete if and only if

(1) for every $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits colimits indexed by objects in $\mathcal{K}$, and for every map $s\colon B \to A$ in $\mathcal{B}$ the transition functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserve these colimits;

(2) for every map $p\colon P \to A$ in S the functor p^* admits a left adjoint $p_!$ that is compatible with base change in the sense of Proposition 3.2.4.2.

Example 3.2.4.12. The notion of being cocomplete is strictly stronger than simply admitting small colimits. For a concrete counterexample, consider be the category of (topological) manifolds Man. There is a functor

$$\underline{\mathsf{Sh}} = \mathrm{Sh}(-)\colon \mathrm{Man} \to \mathrm{Pr}^{\mathrm{L}}$$

that takes a manifold M to the ∞-category of sheaves of spaces on M. This defines a limit-preserving functor

$$\underline{\mathsf{Sh}}\colon \mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$$

via Kan extension and thus a $\mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})$-category that in particular admits all colimits indexed by constant $\mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})$-categories. Furthermore $\underline{\mathsf{Sh}}$ has all colimits indexed by $\mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})$-groupoids: by Proposition 3.1.9.3 it suffices to see this for representable $\mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})$-groupoids. By Corollary 2.4.2.11, we have to check that for any two manifolds M and N the functor

$$\pi_M^*\colon \mathrm{Sh}(N) \to \mathrm{Sh}(M \times N)$$

admits a left adjoint and for any map $\alpha\colon N' \to N$ the mate of the commutative square

$$\begin{tikzcd} \mathrm{Sh}(N) \arrow[r, "\pi_M^*"] \arrow[d, "\alpha^*"] & \mathrm{Sh}(M \times N) \arrow[d, "\alpha_X^*"] \\ \mathrm{Sh}(N') \arrow[r, "\pi_M^*"] & \mathrm{Sh}(M \times N') \end{tikzcd}$$

is an equivalence. Since the projections $M \times N \to N$ and $M \times N' \to N'$ are topological submersions, the left adjoint exists and the mate is an equivalence by the smooth base change isomorphism, see [86, Lemma 3.25]. Therefore $\underline{\mathsf{Sh}}$ admits small colimits. However, if $\underline{\mathsf{Sh}}$ was cocomplete, it would follow that for *any* continuous map $f\colon M \to N$ of manifolds, the pullback functor

$$f^*\colon \mathrm{Sh}(N) \to \mathrm{Sh}(M)$$

would have a left adjoint. This is certainly not the case. For example if Y is a point, the pullback f^* is simply the stalk functor at the point determined by f, and in general stalk functors don't preserve infinite products. However if we let Sub denote the local class in $\mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})$ that is generated by the topological submersions in Man, the above arguments show that the $\mathrm{PSh}_{\mathcal{S}}(\mathrm{Man})$-category $\underline{\mathsf{Sh}}$ is in fact $\langle \mathrm{Sub}, \mathrm{Cat}_\infty \rangle$-cocomplete (see Example 3.2.4.11).

3.3. Kan extensions

The goal of this chapter is to develop the theory of Kan extensions of functors between $\mathcal{B}$-categories. The main theorem about the existence of Kan extensions will be discussed in § 3.3.3, but its proof requires a few preliminary steps. We begin in § 3.3.1 by discussing the *co-Yoneda lemma*, which states that every presheaf can be obtained as the colimit of its Grothendieck construction. Secondly, § 3.3.2 contains a discussion of what we call U-*small* presheaves, those that can be obtained as U-colimits of representables.

3.3.1. The co-Yoneda lemma. If C is a $\mathcal{B}$-category and if $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is a presheaf on C, Yoneda's lemma (Theorem 2.1.13.3) and the straightening/unstraightening equivalence (Theorem 2.1.11.5) allow us to identify the pullback $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ of the right fibration $(\pi_F)_!\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ along the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ with the right fibration $\int F \to \mathsf{C}$ that is classified by F. Let us denote by $h_{/F}\colon \mathsf{C}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ the induced embedding. Since $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ admits a final object $\mathrm{id}_F\colon 1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$, Proposition 3.1.8.1 implies that the functor $(\pi_F)_!$ admits a colimit that is given by F itself (cf. Example 3.1.1.10). Using Remark 3.1.8.2, the functor $h_{/F}$ therefore induces a canonical map

$$\operatorname{colim} hp \simeq \operatorname{colim}(\pi_F)_! h_{/F} \to \operatorname{colim}(\pi_F)_! \simeq F$$

of presheaves on C. Our goal in this section is to prove that this map is an equivalence:

PROPOSITION 3.3.1.1. *Let* C *be a* $\mathcal{B}$*-category, let* $F\colon 1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *be a presheaf on* C *and let* $p\colon C_{/F} \to \mathsf{C}$ *be the associated right fibration. Then the map* $\operatorname{colim} hp \to F$ *is an equivalence.*

REMARK 3.3.1.2. The analogue of Proposition 3.3.1.1 for usual ∞-categories can be found in [57, Lemma 5.1.5.3].

The proof of Proposition 3.3.1.1 requires a few preparations. We begin with the following special case:

PROPOSITION 3.3.1.3. *For any* $\mathcal{B}$*-category* C*, the colimit of the Yoneda embedding* $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *is given by the final object* $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$.

PROOF. Using Proposition 3.1.4.8 in conjunction with Proposition 3.1.2.8, it suffices to show that the colimit of $\hat{h}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \Omega_{\widehat{\mathcal{B}}})$ is given by $1_{\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})}$. On account of the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{pr}_0^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})) \arrow[d, "\simeq"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \Omega_{\widehat{\mathcal{B}}}) \arrow[r, "(\mathrm{id} \times h)^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}, \Omega_{\widehat{\mathcal{B}}})
\end{tikzcd}$$

and Corollary 2.4.3.3, the colimit of $\hat{h}$ in $\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$ is equivalent to $(\mathrm{pr}_0)_!(\mathrm{id} \times h)_!(i \operatorname{map}_{\mathsf{C}})$, where $i\colon \Omega \hookrightarrow \Omega_{\widehat{\mathcal{B}}}$ denotes the inclusion. On the other hand, Yoneda's lemma provides a commutative square

$$\begin{tikzcd}
\mathrm{Tw}(\mathsf{C}) \arrow[r, "j"] \arrow[d] & \int \mathrm{ev} \arrow[d] \\
\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "\mathrm{id} \times h"] & \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

in which j is initial (see the proof of [62, Theorem 4.7.8]), which together with Proposition 2.4.3.1 implies that $(\mathrm{id} \times h)_!(i \operatorname{map}_{\mathsf{C}})$ is given by the functor $i \circ \mathrm{ev}$. Note that by postcomposing pr_0^* with the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \Omega_{\widehat{\mathcal{B}}}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}))$, we recover the diagonal functor $\mathrm{diag}\colon \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}))$. As this equivalence furthermore transforms the composition $i \circ \mathrm{ev}$ into the inclusion $i_*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$, we conclude that the colimit of $\hat{h}$ is equivalent to the colimit of i_*. Since $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ is a final object, the result thus follows from Proposition 3.1.8.1, together with Example 3.1.1.10. ◻

REMARK 3.3.1.4. In the situation of Proposition 3.3.1.3, Proposition 2.4.3.1 implies that the colimit of the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ classifies the left fibration $q\colon \mathsf{Q} \to \mathsf{C}^{\mathrm{op}}$ that is defined by the unique commutative square

$$\begin{tikzcd}
\mathrm{Tw}(\mathsf{C}) \arrow[r, "i"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\
\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \arrow[r, "\mathrm{pr}_0"] & \mathsf{C}^{\mathrm{op}}
\end{tikzcd}$$

in which i is initial. By Proposition 3.3.1.3, the map q is an equivalence, hence we conclude that the projection $\mathrm{pr}_0\, p\colon \mathrm{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}}$ must be initial.

LEMMA 3.3.1.5. *Let* C *be a* $\mathcal{B}$*-category and let* $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ *be a presheaf on* C*. Then there is a canonical equivalence* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ *that fits into a commutative diagram*

$$\begin{tikzcd}
\mathsf{C}_{/F} \arrow[d, hook, "h_{(\mathsf{C}_{/F})}"] \arrow[dr, "(h_{\mathsf{C}})_{/F}"] & \\
\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \arrow[r, "\simeq"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}
\end{tikzcd}$$

Proof. Let $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ be the projection, and let $p_!\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the left adjoint of the precomposition functor p^*. By Corollary 2.4.3.3, there is an equivalence $p_! h_{(\mathsf{C}_{/F})} \simeq h_{\mathsf{C}} p$, hence it suffices to show that $p_!$ factors through $(\pi_F)_!\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ via an equivalence. By construction of $p_!$, this functor sends the final object $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})}$ to F, hence we obtain a lifting problem

$$\begin{tikzcd} 1 \arrow[r, "F"] \arrow[d, "1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})}"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[d, "(\pi_F)_!"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \arrow[r, "p_!"] \arrow[ur, dashed] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which F and $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ define final maps and $(\pi_F)_!$ is a right fibration. On account of the factorisation system between final maps and right fibrations, the dashed arrow exists and has to be final as well. To complete the proof, it therefore suffices to show that it is also a right fibration, which follows once we verify that $p_!$ is a right fibration. By Proposition 2.4.3.1, this map evaluates at any $A \in \mathcal{B}$ to the the functor $\mathrm{RFib}(A \times \mathsf{C}_{/F}) \to \mathrm{RFib}(A \times \mathsf{C})$ that is given by restricting the right fibration $\mathrm{Cat}(\mathcal{B})_{/A\times\mathsf{C}_{/F}} \to \mathrm{Cat}(\mathcal{B})_{/A\times\mathsf{C}}$ of ∞-categories. Since the canonical square

$$\begin{tikzcd} \mathrm{RFib}(A \times \mathsf{C}_{/F}) \arrow[r, hook] \arrow[d] & \mathrm{Cat}(\mathcal{B})_{/A\times\mathsf{C}_{/F}} \arrow[d] \\ \mathrm{RFib}(A \times \mathsf{C}) \arrow[r, hook] & \mathrm{Cat}(\mathcal{B})_{/A\times\mathsf{C}} \end{tikzcd}$$

is a pullback, it thus follows that $p_!$ is sectionwise a right fibration and must therefore be a right fibration itself. □

Proof of Proposition 3.3.1.1. The map $\operatorname{colim} hp \to F$ is determined by the cocone under $hp \simeq (\pi_F)_! h_{/F}$ that arises as the image of the colimit cocone $(\pi_F)_! \to \mathrm{diag}(F)$ along the functor

$$(h_{/F})^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{(\pi_F)_!/} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{hp/}.$$

By making use of the equivalence $\varphi\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})$ from Lemma 3.3.1.5, we now obtain a commutative square

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})_{\varphi/} \arrow[r, "(h_{/F})^*"] \arrow[d, "(p_!)_*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})_{h_{\mathsf{C}_{/F}}/} \arrow[d, "(p_!)_*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{(\pi_F)_!/} \arrow[r, "(h_{/F})^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{hp/}. \end{tikzcd}$$

As $p_!$ is a left adjoint and therefore preserves colimits, we may thus replace C by $\mathsf{C}_{/F}$ and can therefore assume without loss of generality $F = 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$, in which case the desired result follows from Proposition 3.3.1.3. □

3.3.2. U-small presheaves. In this section we study those subcategories of the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ of presheaves on a $\mathcal{B}$-category C that are spanned by U-colimits of representable presheaves for an arbitrary internal class U of $\mathcal{B}$-categories.

Definition 3.3.2.1. Let C be a $\mathcal{B}$-category and let U be an internal class of $\mathcal{B}$-categories. We say that a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context $A \in \mathcal{B}$ is U-*small* if $\mathsf{C}_{/F}$ is contained in $\mathsf{U}^{\mathrm{colim}}(A)$ (see the discussion after Definition 3.2.3.5). We denote by $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by the U-small presheaves.

Remark 3.3.2.2 (locality of U-small presheaves). The property of a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ to be U-small is local in $\mathcal{B}$. That is, for every cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$, the presheaf F is U-small if and only if $s_i^*(F)$ is U-small. This follows immediately from the fact that since $\mathsf{U}^{\mathrm{colim}}(-)$ is a subsheaf of $\mathrm{Cat}(\mathcal{B}_{/-})$, the property to be contained in $\mathsf{U}^{\mathrm{colim}}(A)$ can be checked locally. As a consequence (see Remark 2.1.14.5), a presheaf F is contained in $\mathsf{Small}_{\mathcal{B}}(\mathsf{C})$ if and only if F is U-small. From this observation and Remark 3.3.2.3 below, it furthermore follows (by the argument in Remark 2.1.14.6) that there is a natural equivalence

$$\mathsf{Small}^{\pi_A^*\mathsf{U}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}) \simeq \pi_A^* \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}).$$

for every $A \in \mathcal{B}$.

Remark 3.3.2.3 (étale transposition invariance of U-small presheaves). Since for every $A \in \mathcal{B}$ we have an equivalence $\pi_A^*(\mathsf{U}^{\mathrm{colim}}) \simeq (\pi_A^*\mathsf{U})^{\mathrm{colim}}$ (see the discussion following Definition 3.2.3.5) and on account of Remark 2.1.12.2, it follows that a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is U-small if and only if its transpose $\hat{F}\colon 1_{\mathcal{B}_{/A}} \to \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$ is $\pi_A^*\mathsf{U}$-small.

Remark 3.3.2.4. For the special case where $\mathcal{B} \simeq \mathcal{S}$ and where U is the class of κ-filtered ∞-categories for some regular cardinal κ, the ∞-category of U-small presheaves on a small ∞-category is precisely its ind-completion by κ-filtered colimits in the sense of [57, § 5.3.5]. In general, however, the $\mathcal{B}$-category $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ need not be a free cocompletion, see § 3.4.1 below.

Example 3.3.2.5. For any internal class U of $\mathcal{B}$-categories and for any $\mathcal{B}$-category C, the presheaf represented by an object c in C in context $A \in \mathcal{B}$ is U-small: the canonical section $\mathrm{id}_c\colon A \to \mathsf{C}_{/c}$ provides a final map from an object contained in $\mathsf{U}^{\mathrm{colim}}(A)$, which implies that $\mathsf{C}_{/c}$ defines an object of $\mathsf{U}^{\mathrm{colim}}$ as well. By making use of [62, Proposition 3.9.4], the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ thus factors through the inclusion $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

Proposition 3.3.2.6. *For any $\mathcal{B}$-category C and any internal class U of $\mathcal{B}$-categories, the $\mathcal{B}$-category $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is closed under U-colimits of representables in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. More precisely, for any object $A \to \mathsf{U}$ in context $A \in \mathcal{B}$ that corresponds to a $\mathcal{B}_{/A}$-category I, the colimit functor $\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \to \pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ restricts to a functor*

$$\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^*\mathsf{C}) \to \pi_A^* \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}).$$

Proof. By using Example 2.1.14.7 and Remark 3.3.2.2, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$, so that it will be enough to show that for any diagram $d\colon B \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in context $B \in \mathcal{B}$ the colimit $\mathrm{colim}\, hd\colon B \to \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})$ is a U-small presheaf on C. By the same argument and Remark 3.1.1.9, we may again replace $\mathcal{B}$ with $\mathcal{B}_{/B}$, so that we can also reduce to $B \simeq 1$. Let $pi\colon \mathsf{I} \to \mathsf{P} \to \mathsf{C}$ be the factorisation of d into a final functor and a right fibration. By Proposition 3.1.8.1 we find $\mathrm{colim}\, hd \simeq \mathrm{colim}\, hp$, hence Proposition 3.3.1.1 implies $\mathsf{P} \simeq \mathsf{C}_{/\mathrm{colim}\, hd}$. Since i is a final functor into P from the $\mathcal{B}$-category $\mathsf{I} \in \mathsf{U}(1)$, this shows that $\mathrm{colim}\, hd$ is U-small. □

We finish this section by showing that for any $\mathcal{B}$-category C, the functor $h\colon \mathsf{C} \hookrightarrow \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ that is induced by the Yoneda embedding has a left adjoint whenever C is U-cocomplete.

Proposition 3.3.2.7. *Let U be an internal class of $\mathcal{B}$-categories. If C is a U-cocomplete $\mathcal{B}$-category, the functor $h\colon \mathsf{C} \hookrightarrow \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ that is induced by the Yoneda embedding admits a left adjoint $L\colon \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{C}$.*

Proof. As C being U-cocomplete is equivalent to C being $\mathsf{U}^{\mathrm{colim}}$-cocomplete, we may assume without loss of generality that U is already a colimit class. Let F be an object in $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ in context $A \in \mathcal{B}$. On account of Proposition 2.4.3.5, it suffices to show that the copresheaf $\mathrm{map}_{\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}(F, h(-))$ is corepresentable by an object in C. Using Example 2.1.14.7 together with Remark 3.3.2.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume without loss of generality that F is a U-small presheaf in context $1 \in \mathcal{B}$ (see Remark 2.1.14.4). In this case, we have $\mathsf{C}_{/F} \in \mathsf{U}(1)$, where $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ is the right fibration that is classified by F. Now Proposition 3.3.1.1 and Proposition 3.1.8.1 give rise to an equivalence $F \simeq \operatorname{colim} hp$. Thus, one obtains a chain of equivalences

$$\begin{aligned}\operatorname{map}_{\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}(F, h(-)) &\simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\operatorname{colim} hp, h(-))\\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))}(hp, \operatorname{diag} h(-))\\ &\simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \mathsf{C})}(p, \operatorname{diag}(-))\\ &\simeq \operatorname{map}_{\mathsf{C}}(\operatorname{colim} p, -),\end{aligned}$$

which shows that the presheaf $\operatorname{map}_{\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})}(F, h(-))$ is represented by $L(F) = \operatorname{colim} p$. □

3.3.3. The functor of left Kan extension. Throughout this section, let C, D and E be $\mathcal{B}$-categories and let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor.

DEFINITION 3.3.3.1. A *left Kan extension* of a functor $F\colon \mathsf{C} \to \mathsf{E}$ along f is a functor $f_!F\colon \mathsf{D} \to \mathsf{E}$ together with an equivalence

$$\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})}(f_!F, -) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})}(F, f^*(-)).$$

Dually, a *right Kan extension* of F along f is a functor $f_*F\colon \mathsf{D} \to \mathsf{E}$ together with an equivalence

$$\operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})}(-, f_*F) \simeq \operatorname{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})}(f^*(-), F).$$

REMARK 3.3.3.2 (locality of Kan extensions). In the situation of Definition 3.3.3.1, if $A \in \mathcal{B}$ is an arbitrary object, one easily deduces from Remark 2.1.14.1 and [62, Lemma 4.7.13] that the functor $\pi_A^*(f_!F)$ is a left Kan extension of π_A^*F along π_A^*f.

REMARK 3.3.3.3. As usual, the theory of right Kan extensions can be formally obtained from the theory of left Kan extensions by taking opposite $\mathcal{B}$-categories. We will therefore only discuss the case of left Kan extensions.

REMARK 3.3.3.4. The theory of Kan extensions for the special case $\mathcal{B} \simeq \mathcal{S}$ is discussed in [48, §22], [57, § 4.3], or [17, §6.4].

The main goal of this section is to prove the following theorem about the existence of left Kan extensions:

THEOREM 3.3.3.5. *Let U be an internal class of $\mathcal{B}$-categories such that for every object $d\colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/d}$ is contained in $\mathsf{U}^{\mathrm{colim}}(A)$. Then, whenever E is U-cocomplete, the functor $f^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ has a left adjoint $f_!$ which is fully faithful whenever f is fully faithful.*

PROOF. To begin with, by replacing U with $\mathsf{U}^{\mathrm{colim}}$, we may assume without loss of generality that U is a colimit class and therefore that $\mathsf{C}_{/d}$ is contained in U for every object d in D.

By Corollary 2.4.3.3, the functor $(f \times \mathrm{id})^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D} \times \mathsf{E}^{\mathrm{op}}, \Omega) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{E}^{\mathrm{op}}, \Omega)$ admits a left adjoint $(f \times \mathrm{id})_!$. We now claim that the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \xrightarrow{h_*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{E}^{\mathrm{op}}, \Omega) \xrightarrow{(f\times\mathrm{id})_!} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D} \times \mathsf{E}^{\mathrm{op}}, \Omega) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}))$$

takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{E}))$. To see this, let $F\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ be an object in context $A \in \mathcal{B}$. Using Example 2.1.14.7 together with Remark 3.3.2.2 and the fact that as π_A^* preserves adjunctions (Corollary 2.4.1.9) we may identify $\pi_A^*(f \times \mathrm{id})_!$ with $(\pi_A^*(f) \times \mathrm{id})_!$, which allows us to replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and therefore to reduce to the case where $A \simeq 1$ (see Remark 2.1.14.4). Let $p\colon \mathsf{P} \to \mathsf{C} \times \mathsf{E}^{\mathrm{op}}$ be the left fibration that is classified by the transpose of hF, and let $qi\colon \mathsf{P} \to \mathsf{Q} \to \mathsf{D} \times \mathsf{E}^{\mathrm{op}}$ be the factorisation of $(f \times \mathrm{id})p$ into an initial functor and a left fibration. Then $q\colon \mathsf{Q} \to \mathsf{D} \times \mathsf{E}^{\mathrm{op}}$ classifies $(f \times \mathrm{id})_!(hF)$, hence we need to show that for any object $d\colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ the fibre $\mathsf{Q}|_d \to A \times \mathsf{E}^{\mathrm{op}}$ is classified by a U-small presheaf on E. By the same argument as above, we may again assume that $A \simeq 1$. Consider the commutative diagram

$$\begin{tikzcd}
 & & & & \mathsf{Q}|_d \arrow[dl] \arrow[dr, "s"] \arrow[dd] & \\
 & \mathsf{P}_{/d} \arrow[rr, "i_{/d}"] \arrow[dl] \arrow[dd] & & \mathsf{Q}_{/d} \arrow[rr, "j"] \arrow[dl] \arrow[dd] & & \mathsf{R} \arrow[dd, "r"] \\
\mathsf{P} \arrow[rr, "i"] \arrow[dd, "p"] & & \mathsf{Q} \arrow[dd, "q"] & & \mathsf{E}^{\mathrm{op}} \arrow[dl] \arrow[dr, "\mathrm{id}"] & \\
 & \mathsf{C}_{/d} \times \mathsf{E}^{\mathrm{op}} \arrow[rr] \arrow[dl] & & \mathsf{D}_{/d} \times \mathsf{E}^{\mathrm{op}} \arrow[rr] \arrow[dl] & & \mathsf{E}^{\mathrm{op}} \\
\mathsf{C} \times \mathsf{E}^{\mathrm{op}} \arrow[rr, "f \times \mathrm{id}"] & & \mathsf{D} \times \mathsf{E}^{\mathrm{op}} & & &
\end{tikzcd}$$

in which R is uniquely determined by the condition that j be initial and r be a left fibration. Since $i_{/d}$ is the pullback of i along a right fibration and since right fibrations are proper [62, Proposition 4.4.7], this map is initial. As a consequence, the composition $ji_{/d}$ is initial as well, which implies that the left fibration r is classified by the colimit of the composition $\mathsf{C}_{/d} \to \mathsf{C} \to \mathsf{E} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E})$. By Proposition 3.3.2.6 and the condition on $\mathsf{C}_{/d}$ to be contained in $\mathsf{U}(1)$, the left fibration r is classified by a U-small presheaf. To prove our claim, we therefore need only show that the map $s\colon \mathsf{Q}|_d \to \mathsf{R}$ is an equivalence. As this is a map of right fibrations over E^{op}, we may work fibrewise [62, Proposition 4.1.18]. If $e\colon A \to \mathsf{E}^{\mathrm{op}}$ is an object in context $A \in \mathcal{B}$, we obtain an induced commutative triangle

$$\begin{tikzcd}
 & (\mathsf{Q}|_d)|_e \arrow[dl] \arrow[dr, "s|_e"] & \\
(\mathsf{Q}_{/d})|_e \arrow[rr, "j|_e"] & & \mathsf{R}|_e
\end{tikzcd}$$

over A. Since the projections $\mathsf{Q}_{/d} \to \mathsf{E}^{\mathrm{op}}$ and $\mathsf{R} \to \mathsf{E}^{\mathrm{op}}$ are smooth by [62, Propositions 4.4.6 and 4.4.7] and since initial functors are a fortiori covariant equivalences (see [62, § 4.4]), we deduce from [62, Proposition 4.4.10] that $j|_e$ exhibits $\mathsf{R}|_e$ as the groupoidification of $(\mathsf{Q}_{/d})|_e$. Moreover, the map $(\mathsf{Q}|_d)|_e \to (\mathsf{Q}_{/d})|_e$ is a pullback of the final map $A \to \mathsf{D}_{/d} \times A$ along a smooth map and therefore final as well. Since final functors induce equivalences on groupoidifications, we thus conclude that $s|_e$ must be an equivalence, as desired.

By making use of the discussion thus far, we may now define $f_!$ as the composition of the two horizontal arrows in the top row of the commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \arrow[r] \arrow[d, hook, "h"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{E})) \arrow[r, "L_*"] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E}) \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{E}^{\mathrm{op}}, \Omega) \arrow[r, "(f\times \mathrm{id})_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D} \times \mathsf{E}^{\mathrm{op}}, \Omega) &
\end{tikzcd}$$

in which L denotes the left adjoint to the Yoneda embedding that is supplied by Proposition 3.3.2.7. It is now clear from the construction of $f_!$ that this functor defines a left adjoint of f^*.

Lastly, suppose that f is fully faithful. We show that in this case the adjunction counit $\mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})} \to f^* f_!$ is an equivalence. Since equivalences are computed objectwise (see [62, Corollary 4.7.17]), we only have to show that for every object F in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ the induced map $F \to f^* f_! F$ is an equivalence. Since π_A^* preserves adjunctions and the internal hom (Corollary 2.4.1.9 and Remark 2.1.14.1), we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that F is in context $1 \in \mathcal{B}$ (see Remark 2.1.14.4). By construction of the adjunction $f_! \dashv f^*$, the unit $F \to f^* f_! F$ is determined by the composition

$$h_*(F) \xrightarrow{\eta_1 h_*(F)} (f \times \mathrm{id})^*(f \times \mathrm{id})_! h_*(F) \xrightarrow{(f\times\mathrm{id})^* \eta_2 (f\times\mathrm{id})_! h_*(F)} (f \times \mathrm{id})^* h_* L_* (f \times \mathrm{id})_! h_*(F)$$

in which η_1 is the unit of the adjunction $(f \times \mathrm{id})_! \dashv (f \times \mathrm{id})^*$ and η_2 is the unit of the adjunction $L_* \dashv h_*$. By Corollary 2.4.3.3, the first map is an equivalence, hence it suffices to show that the second one is an equivalence as well. Again, it suffices to show this objectwise. Let therefore c be an object of C, as above without loss of generality in context $1 \in \mathcal{B}$. By the above argument, the object $(f \times \mathrm{id})_! h_*(F)(c)$ is given by the colimit of the diagram $hF(\pi_c)_! \colon \mathsf{C}_{/c} \to \mathsf{C} \to \mathsf{E} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E})$. By making use of the final section $\mathrm{id}_c \colon 1 \to \mathsf{C}_{/c}$, this presheaf is therefore representable by $F(c)$, which implies the claim. □

REMARK 3.3.3.6. In the situation of Theorem 3.3.3.5, the construction of $f_!$ shows that if $F \colon \mathsf{D} \to \mathsf{E}$ is a functor, the counit $f_! f^* F \to F$ is given by the composition

$$L_*(f \times \mathrm{id})_!(f \times \mathrm{id})^* h_*(F) \xrightarrow{L_* \epsilon_1 h_* F} L_* h_*(F) \xrightarrow{\epsilon_2} F$$

where ϵ_1 is the counit of the adjunction $(f \times \mathrm{id})_! \dashv (f \times \mathrm{id})^*$ and ϵ_2 is the counit of the adjunction $L_* \dashv h_*$. Since the latter is an equivalence, the functor F arises as the left Kan extension of $f^* F$ precisely if the map $L_* \epsilon_1 h_*(F)$ is an equivalence. Let $q \colon \mathsf{Q} \to \mathsf{D} \times \mathsf{E}^{\mathrm{op}}$ be the left fibration that is classified by $h_*(F)$ and let $p \colon \mathsf{P} \to \mathsf{C} \times \mathsf{E}^{\mathrm{op}}$ be the pullback of q along $f \times \mathrm{id}$. Let furthermore $q' \colon \mathsf{Q}' \to \mathsf{C} \times \mathsf{E}^{\mathrm{op}}$ be the functor that arises from factoring $(f \times \mathrm{id})p$ into an initial map and a left fibration. On the level of left fibrations over $\mathsf{D} \times \mathsf{E}^{\mathrm{op}}$, the map $\epsilon_1 h_*(F)$ is then given by the map g that arises as the unique lift in the commutative diagram

$$\begin{tikzcd} \mathsf{P} \arrow[r, "i"] \arrow[d, "i^*"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{Q}' \arrow[r, "q"] \arrow[ur, dotted, "g"] & \mathsf{D} \times \mathsf{E}^{\mathrm{op}}. \end{tikzcd}$$

Then the condition that $L_* \epsilon_1 h_* F$ is an equivalence corresponds to the condition that for any object $d \colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ the map $g|_d \colon \mathsf{Q}'|_d \to \mathsf{Q}|_d$, viewed as a map over $\pi_A^* \mathsf{E}^{\mathrm{op}}$, induces an equivalence $\mathrm{colim}(q'|_d^{\mathrm{op}}) \simeq \mathrm{colim}(q|_d^{\mathrm{op}})$ in $\pi_A^* \mathsf{E}$. Note that by a similar argument as in the proof of Theorem 3.3.3.5, the map $g|_d$ fits into a commutative square

$$\begin{tikzcd} \mathsf{Q}'_{/d} \arrow[r, "g_{/d}"] \arrow[d, "j'"] & \mathsf{Q}_{/d} \arrow[d, "j"] \\ \mathsf{Q}'|_d \arrow[r, "g|_d"] & \mathsf{Q}|_d \end{tikzcd}$$

in which j' and j are initial. As a consequence, the map $g|_d$ is determined by the factorisation of the map $j i_{/d}$ in the commutative diagram

$$\begin{tikzcd} \mathsf{P}_{/d} \arrow[r, "i_{/d}"] \arrow[d] & \mathsf{Q}_{/d} \arrow[r, "j"] \arrow[d] & \mathsf{Q}|_d \arrow[d] \\ \mathsf{C}_{/d} \times \mathsf{E}^{\mathrm{op}} \times A \arrow[r, "f_{/d} \times \mathrm{id}"] & \mathsf{D}_{/d} \times \mathsf{E}^{\mathrm{op}} \times A \arrow[r] & \mathsf{E}^{\mathrm{op}} \times A \end{tikzcd}$$

into an initial map and a right fibration. This argument shows that the map $g|_d$ classifies the canonical map

$$\mathrm{colim}\, hF(\pi_d)_! f_{/d} \to \mathrm{colim}\, hF(\pi_d)_!$$

of presheaves on $\pi_A^* \mathsf{E}$ that is induced by the functor $f_{/d} \colon \mathsf{C}_{/d} \to \mathsf{D}_{/d}$. Since L is a left inverse of h that preserves colimits, we thus conclude that F is a left Kan extension of its restriction $f^* F$ precisely if the map $f_{/d} \colon \mathsf{C}_{/d} \to \mathsf{D}_{/d}$ induces an equivalence

$$\mathrm{colim}\, F(\pi_d)_! f_{/d} \simeq \mathrm{colim}\, F(\pi_d)_! \simeq F(d)$$

in $\pi_A^* \mathsf{E}$ for every object $d \colon A \to \mathsf{D}$.

Recall from [62, § 4.7] that a large $\mathcal{B}$-category D is *locally* small if the left fibration $\mathrm{Tw}(\mathsf{D}) \to \mathsf{D}^{\mathrm{op}} \times \mathsf{D}$ is small (in the sense of [62, § 4.5]). Theorem 3.3.3.5 now implies:

COROLLARY 3.3.3.7. *If $f \colon \mathsf{C} \to \mathsf{D}$ is a functor of $\mathcal{B}$-categories such that C is small and D is locally small (but not necessarily small). If E is a cocomplete large $\mathcal{B}$-category, the functor of left Kan extension $f_!$ always exists.*

Proof. By Theorem 3.3.3.5, it suffices to show that for any object $d\colon A \to \mathsf{D}$ in context $A \in \mathcal{B}$ the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/d}$ is small, which follows immediately from the observation that the right fibration $\mathsf{C}_{/d} \to \mathsf{C} \times A$ a pullback of the small fibration $\mathrm{Tw}(\mathsf{D}) \to \mathsf{D}^{\mathrm{op}} \times \mathsf{D}$ and therefore small itself. $\square$

We conclude this section with an application of the theory of Kan extensions to a characterisation of *colimit cocones*. If I is a $\mathcal{B}$-category, recall from Remark 3.1.1.4 that the associated right cone $\mathsf{I}^{\triangleright}$ comes equipped with two functors $\iota\colon \mathsf{I} \to \mathsf{I}^{\triangleright}$ and $\infty\colon 1 \to \mathsf{I}^{\triangleright}$. Our goal is to prove:

Proposition 3.3.3.8. *Let I and C be $\mathcal{B}$-categories and suppose that C admits I-indexed colimits. Then the functor of left Kan extension*

$$\iota_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$$

along $\iota\colon \mathsf{I} \to \mathsf{I}^{\triangleright}$ exists and is fully faithful, and its essential image coincides with the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$ that is spanned by the colimit cocones.

The proof of Proposition 3.3.3.8 relies on the following two general facts:

Lemma 3.3.3.9. *Suppose that*

$$\begin{tikzcd} \mathsf{P} \arrow[r, "g"] \arrow[d, "p"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

is a cartesian square in $\mathrm{Cat}(\mathcal{B})$ such that q admits a fully faithful left adjoint. Then p admits a fully faithful left adjoint as well.

Proof. By assumption q has a section $l_1\colon \mathsf{D} \to \mathsf{Q}$ which pulls back along f to form a section $l_0\colon \mathsf{C} \hookrightarrow \mathsf{P}$ of p. Moreover, the adjunction counit $\epsilon_1\colon \Delta^1 \otimes \mathsf{Q} \to \mathsf{Q}$ fits into a commutative diagram

$$\begin{tikzcd} \Delta^1 \otimes \mathsf{D} \arrow[r, "\mathrm{id} \otimes l_1"] \arrow[d, "s^0"] & \Delta^1 \otimes \mathsf{Q} \arrow[r, "s^0"] \arrow[d, "\epsilon_1"] & \mathsf{Q} \arrow[d, "q"] \\ \mathsf{D} \arrow[r, "l_1"] & \mathsf{Q} \arrow[r, "q"] & \mathsf{D}, \end{tikzcd}$$

hence pullback along f defines a map $\epsilon_0\colon \Delta^1 \otimes \mathsf{P} \to \mathsf{P}$ that fits into a commutative square

$$\begin{tikzcd} \Delta^1 \otimes \mathsf{C} \arrow[r, "\mathrm{id} \otimes l_0"] \arrow[d, "s^0"] & \Delta^1 \otimes \mathsf{P} \arrow[r, "s^0"] \arrow[d, "\epsilon_0"] & \mathsf{P} \arrow[d, "p"] \\ \mathsf{C} \arrow[r, "l_0"] & \mathsf{P} \arrow[r, "p"] & \mathsf{C}, \end{tikzcd}$$

By construction, the map $\epsilon_0 d^0$ is equivalent to the identity on P, and the map $\epsilon_1 d^1$ recovers the functor $l_1 p$. The previous commutative diagram now precisely expresses that both $p\epsilon_0$ and $\epsilon_0 l_0$ are equivalence, hence the desired result follows from Corollary 2.4.4.3. $\square$

Lemma 3.3.3.10. *Fully faithful functors in $\mathrm{Cat}(\mathcal{B})$ are stable under pushout.*

Proof. If

$$\begin{tikzcd} \mathsf{C} \arrow[r, "h"] \arrow[d, hook, "f"] & \mathsf{E} \arrow[d, "g"] \\ \mathsf{D} \arrow[r, "k"] & \mathsf{F} \end{tikzcd}$$

is a pushout square in $\mathrm{Cat}(\mathcal{B})$ in which f is fully faithful, applying the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)$ results in a pullback square

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{F}) \arrow[r, "g^*"] \arrow[d, "k^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}) \arrow[d, "h^*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r, "f^*"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which f^* admits a fully faithful left adjoint $f_!$. By Lemma 3.3.3.9, this implies that g^* admits a fully faithful left adjoint as well, hence that the functor of left Kan extension $g_!$ is fully faithful. This in turn implies that g must be fully faithful too, see Corollary 2.4.3.3. □

PROOF OF PROPOSITION 3.3.3.8. Let U be the smallest colimit class in $\mathcal{B}$ that contains I. Then C is U-cocomplete (by Remark 3.2.2.4). Hence the existence of $\iota_!$ follows from Theorem 3.3.3.5 once we show that for every object $j\colon A \to \mathsf{I}^{\triangleright}$ the $\mathcal{B}_{/A}$-category $\mathsf{I}_{/j}$ is contained in $\mathsf{U}(A)$. By definition of the right cone, we have a cover $\mathsf{I}_0 \sqcup 1 \twoheadrightarrow (\mathsf{I}^{\triangleright})_0$ which induces a cover $A_0 \sqcup A_1 \twoheadrightarrow A$ by taking the pullback along $j\colon A \to (\mathsf{I}^{\triangleright})_0$. Let $j_0\colon A_0 \to \mathsf{I}^{\triangleright}$ and $j_1\colon A_1 \to \mathsf{I}^{\triangleright}$ be the induced objects. Since j_0 factors through the inclusion $\iota_0\colon \mathsf{I}_0 \hookrightarrow (\mathsf{I}^{\triangleright})_0$ and since ι is fully faithful by Lemma 3.3.3.10, we obtain an equivalence $\mathsf{I}_{/j_0} \simeq \mathsf{I}_{/j_0'}$ over A_0, where j_0' is the unique object in I such that $\iota(j_0') \simeq j_0$. Since j_1 factors through the inclusion of the cone point $\infty\colon 1 \to \mathsf{I}^{\triangleright}$ which defines a final object in $\mathsf{I}^{\triangleright}$, we furthermore obtain an equivalence $\mathsf{I}_{/j_1} \simeq \pi_{A_i}^*\mathsf{I}$. Therefore the $\mathcal{B}_{/A}$-category $\mathsf{I}_{/j}$ is *locally* contained in U and therefore contained in U itself, for U defines a sheaf on $\mathcal{B}$. We therefore deduce that the functor of left Kan extension $\iota_!$ exists. Since Lemma 3.3.3.10 implies that ι is fully faithful, Corollary 2.4.3.3 furthermore shows that $\iota_!$ is fully faithful as well.

We finish the proof by identifying the essential image of $\iota_!$. By combining Remark 3.1.1.4 with Lemma 3.3.3.9, if $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(I, \mathsf{C})$ is a diagram, the object $\iota_!(d)$ defines a fully faithful left adjoint $A \to \mathsf{C}_{d/}$ to the projection $\mathsf{C}_{d/} \to A$. By Example 3.1.1.11, this precisely means that $\iota_!(d)$ is an initial section over A and is therefore a colimit cocone. Conversely, if $\bar{d}\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$ is a cocone under $d = \iota^*\bar{d}$, the map $\epsilon\bar{d}\colon \iota_! d \to \bar{d}$ defines a map in $\mathsf{C}_{d/}$. By the above argument, the domain of this map is a colimit cocone, hence if $\bar{d}$ defines a colimit cocone in $\mathsf{C}_{d/}$ as well, the map $\epsilon\bar{d}$ must necessarily be an equivalence since *any* map between two initial objects in a $\mathcal{B}_{/A}$-category is an equivalence (see Corollary 2.1.12.13). □

3.4. Cocompletion

The main goal of this section is to construct and study the *free cocompletion* by U-colimits of an arbitrary $\mathcal{B}$-category, for any internal class U of $\mathcal{B}$-categories. In § 3.4.1 we give the construction of this $\mathcal{B}$-category and prove its universal property. § 3.4.2 contains a criterion to detect free cocompletions, and in § 3.4.3 we study the free U-cocompletion of the point. Finally, in § 3.4.4 we improve on the results of § 3.1.9 and explain how to decompose colimits of $\mathcal{B}$-categories in greater generality.

3.4.1. The free U-cocompletion. Let C be a $\mathcal{B}$-category and let U be an internal class of $\mathcal{B}$-categories. The goal of this section is to construct the free U-cocompletion of C, i.e. the initial U-cocomplete $\mathcal{B}$-category that is equipped with a functor from C.

We begin our discussion of the free cocompletions with the maximal case $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$:

THEOREM 3.4.1.1. *For any $\mathcal{B}$-category C and any cocomplete large $\mathcal{B}$-category E, the functor of left Kan extension $(h_{\mathsf{C}})_!$ along the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ induces an equivalence*

$$(h_{\mathsf{C}})_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}).$$

In other words, the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ exhibits the $\mathcal{B}$-category of presheaves on C as the free cocompletion of C.

REMARK 3.4.1.2. The analogue of Theorem 3.4.1.1 for ∞-categories is the content of [57, Theorem 5.1.5.6] or [17, Theorem 6.3.13].

The proof of Theorem 3.4.1.1 relies on the following lemma:

LEMMA 3.4.1.3. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories and assume that C is small. Then the left Kan extension $(h_{\mathsf{C}})_!(h_{\mathsf{D}}f)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{D}) = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \Omega_{\widehat{\mathcal{B}}})$ of $h_{\mathsf{D}}F$ along h_{C} is equivalent to the* *composition*

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \xhookrightarrow{i_*} \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \xrightarrow{f_!} \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{D}),$$

where $i\colon \Omega_{\mathcal{B}} \hookrightarrow \Omega_{\widehat{\mathcal{B}}}$ *is the inclusion from § 2.1.10.*

Proof. Since $(h_{\mathsf{C}})_!$ is fully faithful and since the restriction of $f_! i_*$ along h_{C} recovers the functor $h_{\mathsf{D}} f$, it suffices to show that $f_! i_*$ is a left Kan extension along its restriction. By Remark 3.3.3.6, this is the case precisely if for any presheaf F on C the inclusion $h_{/F}\colon \mathsf{C}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ induces an equivalence

$$\operatorname{colim} f_!(\pi_F)_! h_{/F} \simeq \operatorname{colim} f_!(\pi_F)_! \simeq f_!(F).$$

Since $f_! i_*$ commutes with small colimits (Proposition 3.1.4.8) and since $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ admits small colimits (Proposition 3.2.2.7), it suffices to show that the map

$$\operatorname{colim}(\pi_F)_! h_{/F} \to F$$

is an equivalence in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, which follows immediately from Proposition 3.3.1.1. $\square$

Proof of Theorem 3.4.1.1. Let us first show that for any object $f\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$ in context $A \in \mathcal{B}$ the object $(h_{\mathsf{C}})_!(f)$ is contained in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$. By making use of Remarks 3.2.3.4, 2.1.14.1 and 3.3.3.2 as well as Example 2.1.14.7, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$ (see Remark 2.1.14.4). Hence, we only need to show that $h_!(f)$ is cocontinuous. By again making use of Remark 3.3.3.2 and Example 2.1.14.7, it is enough to show that $h_!(f)$ preserves I-indexed colimits for every small $\mathcal{B}$-category I. By Lemma 3.4.1.3 and the explicit construction of $h_!$ in Theorem 3.3.3.5, the functor $h_!(f)$ is equivalent to the composition

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \xhookrightarrow{i_*} \underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C}) \xrightarrow{f_!} \underline{\mathsf{Small}}^{\mathsf{Cat}_{\mathcal{B}}}_{\widehat{\mathcal{B}}}(\mathsf{E}) \xrightarrow{L} \mathsf{E}$$

in which L is left adjoint to the Yoneda embedding h_{E}. Since all three functors preserve small colimits, the claim follows.

By what we have just shown, the embedding $h_!$ takes values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$ and therefore determines an inclusion $h_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$. To show that this functor is essentially surjective as well, we need only show that any object $g\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})$ in context $A \in \mathcal{B}$ whose associated functor in $\mathrm{Cat}(\widehat{\mathcal{B}}_{/A})$ is cocontinuous is a left Kan extension of its restriction along h. By the same reduction argument as above, we may again assume $A \simeq 1$. By using Remark 3.3.3.6, the functor g is a Kan extension of gh precisely if for any presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ the functor $h_{/F}\colon \mathsf{C}_{/F} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ induces an equivalence

$$\operatorname{colim} g(\pi_F)_! h_{/F} \simeq g(F)$$

in E. Since Proposition 3.3.1.1 implies that the canonical map $\operatorname{colim}(\pi_F)_! h_{/F} \to F$ is an equivalence in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ and since g is cocontinuous, this is immediate. $\square$

Remark 3.4.1.4. In the situation of Theorem 3.4.1.1, suppose that E is in addition locally small. If $f\colon \mathsf{C} \to \mathsf{E}$ is an arbitrary functor, its left Kan extension $h_!(f)$ is not only cocontinuous, but even admits a right adjoint. In fact, by the explicit construction of $h_!(f)$ in the proof of Theorem 3.4.1.1, we may compute

$$\begin{aligned}\operatorname{map}_{\mathsf{E}}(h_!(f)(-), -) &\simeq \operatorname{map}_{\mathsf{E}}(L f_! i_*(-), -)\\ &\simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\widehat{\mathcal{B}}}(\mathsf{C})}(i_*(-), f^* h_{\mathsf{E}}(-))\end{aligned}$$

and since E is locally small, the functor $f^* h_{\mathsf{E}}$ takes values in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, hence the claim follows. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and using Remark 3.3.3.2 and Example 2.1.14.7, the same argument works for arbitrary objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E})$, hence we conclude that the functor $h_!$ takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})^L$ and therefore gives rise to an equivalence

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E})^L \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}).$$

This is a special (and in a certain sense universal) case of the adjoint functor theorem for presentable $\mathcal{B}$-categories. The general case will be treated in the next chapter, see Proposition 4.4.3.1.

Our next goal is to generalise Theorem 3.4.1.1 to an arbitrary internal class U of $\mathcal{B}$-categories. For this, we need to make the following general observation:

Lemma 3.4.1.5. *Let E be a $\mathcal{B}$-category, let $\mathsf{C} \hookrightarrow \mathsf{E}$ be a full subcategory and let $\mathsf{U} \subset \mathsf{V}$ be two internal classes of $\mathcal{B}$-categories. Suppose that E is V-cocomplete. Then there exists a full subcategory $\mathsf{D} \hookrightarrow \mathsf{E}$ that is closed under U-colimits (i.e. that is U-cocomplete and the inclusion into E is U-cocontinuous), contains C and is the smallest full subcategory of E with these properties, in that whenever $\mathsf{D}' \hookrightarrow \mathsf{E}$ has the same properties there is an inclusion $\mathsf{D} \hookrightarrow \mathsf{D}'$ over E.*

Proof. Recall that the full subposet $\mathrm{Sub}^{\mathrm{full}}(\mathsf{E}) \hookrightarrow \mathrm{Sub}(\mathsf{E})$ that is spanned by the fully faithful functors is a reflective subcategory (cf. the discussion in [62, § 3.9]), which implies that this subposet is closed under limits in $\mathrm{Sub}(\mathsf{E})$, i.e. meets. To complete the proof, we therefore only need to show that the collection of full subcategories of E that contain C and that are closed under U-colimits in E is closed under limits in $\mathrm{Sub}(\mathsf{E})$. Clearly, if $(\mathsf{D}_i)_{i\in I}$ is a collection of full subcategories in E that each contain C, then so does their meet $\mathsf{D} = \bigwedge_i \mathsf{D}_i$. Similarly, suppose that each $\mathcal{B}$-category D_i is closed under U-colimits in E, and let $A \in \mathcal{B}$ be an arbitrary context. Since π_A^* commutes with limits and carries fully faithful functors to fully faithful functors, we may assume without loss of generality that $A \simeq 1$. We thus only need to show that the meet of the D_i is closed under I-indexed colimits in E for any $\mathsf{I} \in \mathsf{U}(1)$. Let $d \colon B \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{D})$ be a diagram in context $B \in \mathcal{B}$. Since by assumption the object $\operatorname{colim} d$ is contained in D_i for every $i \in I$ and thus defines an object in D, the result follows. □

In light of Lemma 3.4.1.5, we may now define:

Definition 3.4.1.6. For any $\mathcal{B}$-category C and any internal class U of $\mathcal{B}$-categories, we define the large $\mathcal{B}$-category $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ as the smallest full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that contains C and is closed under U-colimits.

Remark 3.4.1.7. Suppose that U is a *small* internal class of $\mathcal{B}$-categories and C is a $\mathcal{B}$-category. Then $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is small as well. To see this, let us first fix a small full subcategory of generators $\mathcal{G} \subset \mathcal{B}$ (i.e. a full subcategory such that every object in $\mathcal{B}$ admits a small cover by objects in $\mathcal{G}$). Since U is small, there exists a small regular cardinal κ such that for every $\mathcal{B}$-category I in U in context $G \in \mathcal{G}$ the object $\mathsf{I}_0 \in \mathcal{B}_{/G}$ is κ-compact. We construct a diagram $\mathsf{E}^\bullet \colon \kappa \to \mathrm{Sub}^{\mathrm{full}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ by transfinite recursion as follows: set $\mathsf{E}^0 = \mathsf{C}$ and $\mathsf{E}^\lambda = \bigvee_{\tau<\lambda} \mathsf{E}^\tau$ for any limit ordinal $\lambda < \kappa$, where the right-hand side denotes the join operation in the poset $\mathrm{Sub}^{\mathrm{full}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. For $\lambda < \kappa$, we furthermore define $\mathsf{E}^{\lambda+1}$ to be the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by E^λ together with those objects that arise as the colimit of a diagram of the form $d \colon \mathsf{I} \to \pi_G^* \mathsf{E}^\lambda$ for $G \in \mathcal{G}$ and $\mathsf{I} \in \mathsf{U}(G)$. Let us set $\mathsf{E} = \bigvee_{\tau<\kappa} \mathsf{E}^\tau$. Since κ is small and E^τ is a small large $\mathcal{B}$-category for every $\tau < \kappa$, the large $\mathcal{B}$-category E is small as well. We claim that E is U-cocomplete. In fact, it suffices to show that for every $G \in \mathcal{G}$ and every diagram $d \colon \mathsf{I} \to \pi_G^* \mathsf{E}$ the object $\operatorname{colim} d$ is contained in $\pi_G^* \mathsf{E}$ as well. Since I_0 is κ-compact in $\mathcal{B}_{/G}$ and since κ is κ-filtered as it is regular, the map $d_0 \colon \mathsf{I}_0 \to \mathsf{E}_0 = \bigvee_{\tau<\kappa} \mathsf{E}_0^\tau$ factors through E_0^τ for some $\tau < \kappa$. As a consequence, the colimit $\operatorname{colim} d$ is contained in $\mathsf{E}^{\tau+1}$ and therefore a fortiori in E, as claimed. Now since E is U-cocomplete and contains C, it must also contain $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$, which is therefore small.

In the situation of Definition 3.4.1.6, Proposition 3.3.1.1 implies that there are inclusions

$$\mathsf{C} \hookrightarrow \mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}).$$

In general, the middle inclusion is not an equivalence, as the following example shows.

Example 3.4.1.8. Let $\mathcal{B} = \mathcal{S}$ be ∞-topos of spaces, let $\mathsf{C} = (\Delta^1)^{\mathrm{op}}$ and let U be the smallest colimit class that contains Λ_0^2. An ∞-category is thus U-cocomplete precisely if it admits pushouts. An object in

$\mathrm{Fun}(\Delta^1, \mathcal{S})$ is representable when viewed as a presheaf on $(\Delta^1)^{\mathrm{op}}$ precisely if it is one of the two maps $0 \to 1$ and $1 \to 1$. Hence $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{S})$ that is spanned by the maps $n \to 1$ for natural numbers $n \leq 2$. But this ∞-category is not closed under pushouts in $\mathrm{Fun}(\Delta^1, \mathcal{S})$: for example, the map $S^1 \to 1$ is a pushout of objects in $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ which is not contained in $\mathsf{Small}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ itself.

Lemma 3.4.1.9. *Let $A \in \mathcal{B}$ be an arbitrary object, let U be an internal class of $\mathcal{B}$-categories and let $f\colon \mathsf{C} \to \mathsf{D}$ be a $\pi_A^*\mathsf{U}$-cocontinuous functor of $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-category. Then $(\pi_A)_*(f)$ is a U-cocontinuous functor of U-cocomplete $\mathcal{B}$-categories.*

Proof. Let $B \in \mathcal{B}$ be an arbitrary object. We need to show that for every $\mathsf{I} \in \mathsf{U}(B)$ the $\mathcal{B}_{/B}$-categories $\pi_B^*(\pi_A)_*\mathsf{C}$ and $\pi_B^*(\pi_A)_*\mathsf{D}$ admit I-indexed colimits and that $\pi_B^*(\pi_A)_*(f)$ preserves these. Note that if $\mathrm{pr}_0\colon A \times B \to A$ and $\mathrm{pr}_1\colon A \times B \to B$ are the two projections, the natural map $\pi_B^*(\pi_A)_* \to (\mathrm{pr}_1)_* \mathrm{pr}_0^*$ is an equivalence, owing to the transpose map $(\mathrm{pr}_0)_! \mathrm{pr}_1^* \to \pi_A^*(\pi_B)_!$ being one. Thus, we may identify $\pi_B^*(\pi_A)_*(f)$ with $(\mathrm{pr}_1)_* \mathrm{pr}_0^*(f)$. Now since f is a $\pi_A^*\mathsf{U}$-cocontinuous functor between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories, it follows that $\mathrm{pr}_0^*(f)$ is a $\pi_{A\times B}^*\mathsf{U}$-cocontinuous functor between $\pi_{A\times B}^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A\times B}$-categories (Remark 3.2.2.3). Therefore, by passing to $\mathcal{B}_{/B}$, we can assume that $B \simeq 1$. In other words, we only need to show that for every $\mathsf{I} \in \mathsf{U}(1)$ the two horizontal maps in the commutative square

$$\begin{tikzcd}
(\pi_A)_*\mathsf{C} \arrow[r, "\mathrm{diag}"] \arrow[d, "(\pi_A)_*(f)"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\pi_A)_*\mathsf{C}) \arrow[d, "(\pi_A)_*(f)_*"] \\
(\pi_A)_*\mathsf{C} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\pi_A)_*\mathsf{C})
\end{tikzcd}$$

have left adjoints and that the associated mate transformation is an equivalence. This is a consequence of the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(-, (\pi_A)_*(-)) \simeq (\pi_A)_*\underline{\mathsf{Fun}}_{\mathcal{B}}(\pi_A^*(-), -)$ (which follows by adjunction from the evident equivalence $\pi_A^*(- \times -) \simeq \pi_A^*(-) \times_A \pi_A^*(-)$) and the fact that by Corollary 2.4.1.9 the geometric morphism $(\pi_A)_*$ preserves adjunctions. $\square$

Lemma 3.4.1.10. *Let U be an internal class of $\mathcal{B}$-categories and let*

$$\begin{tikzcd}
\mathsf{Q} \arrow[r, hook, "j"] \arrow[d, "q"] & \mathsf{P} \arrow[d, "p"] \\
\mathsf{D} \arrow[r, hook, "i"] & \mathsf{C}
\end{tikzcd}$$

be a pullback square in $\mathrm{Cat}(\mathcal{B})$ *in which i and j are fully faithful. Assume furthermore that D, C and P are U-cocomplete and p and i are U-cocontinuous. Then Q is U-cocomplete and j is U-cocontinuous.*

Proof. We need to show that for every $A \in \mathcal{B}$ and every $\mathsf{I} \in \mathsf{U}(A)$, the $\mathcal{B}_{/A}$-category $\pi_A^*\mathsf{Q}$ admits I-indexed colimits and the functor $\pi_A^* j$ preserves them. Since π_A^* preserves pullbacks and fully faithful functors and on account of Remark 3.2.2.3, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Now we obtain a commutative diagram

$$\begin{tikzcd}
& \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Q}) \arrow[rr, hook] \arrow[dl] \arrow[dd, dashed] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{P}) \arrow[dl] \arrow[dd, "\mathrm{colim}"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{D}) \arrow[rr, hook] \arrow[dd, "\mathrm{colim}"] & & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \arrow[dd, "\mathrm{colim}"] & \\
& \mathsf{Q} \arrow[rr, hook] \arrow[dl] & & \mathsf{P} \arrow[dl] \\
\mathsf{D} \arrow[rr, hook] & & \mathsf{C} &
\end{tikzcd}$$

where the dashed arrow exists on account of the lower square being a pullback. Thus Proposition 3.1.2.8 yields that Q admits I-indexed colimits and that j preserves these, as desired. $\square$

Proposition 3.4.1.11 (locality of $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$). *For any $\mathcal{B}$-category C, any internal class U of $\mathcal{B}$-categories and any object $A \in \mathcal{B}$, there is a natural equivalence*

$$\pi_A^* \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{PSh}}^{\pi_A^*\mathsf{U}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}).$$

Proof. It follows from Example 2.1.14.7 that there is a commutative diagram

$$\begin{tikzcd}
\pi_A^*\mathsf{C} \arrow[d, hook, "\pi_A^* h"] \arrow[ddr, hook, bend left, "h_{\pi_A^*\mathsf{C}}"] & \\
\pi_A^*\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, hook] \arrow[dr, hook] & \\
\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "\simeq"] & \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}),
\end{tikzcd}$$

and it is clear that $\pi_A^*\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is closed under $\pi_A^*\mathsf{U}$-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$. It therefore suffices to show that if $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$ is a full subcategory that contains $\pi_A^*\mathsf{C}$ and that is likewise closed under $\pi_A^*\mathsf{U}$-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C})$, this subcategory must contain $\pi_A^*\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Consider the commutative diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, hook] \arrow[d, "\eta_A"] & \mathsf{D}' \arrow[r, hook] \arrow[d] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "\eta_A"] \\
(\pi_A)_*\pi_A^*\mathsf{C} \arrow[r, hook] & (\pi_A)_*\mathsf{D} \arrow[r, hook] & (\pi_A)_*\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

in which η_A denotes the adjunction unit of $\pi_A^* \dashv (\pi_A)_*$ and in which D' is defined by the condition that the right square is a pullback. Note that the triangle identities for the adjunction $\pi_A^* \dashv (\pi_A)_*$ imply that D contains $\pi_A^*\mathsf{D}'$. The proof is therefore finished once we show that D' is closed under U-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. To prove this claim, note that we may identify $(\pi_A)_*\pi_A^* \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(A, -)$. With respect to this identification, the unit η_A corresponds to precomposition with the unique map $\pi_A \colon A \to 1$. Thus, Proposition 3.2.2.7 implies that η_A is a U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories. Also, Lemma 3.4.1.9 implies that the inclusion $(\pi_A)_*\mathsf{D} \hookrightarrow (\pi_A)_*\pi_A^*\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is closed under U-colimits. Therefore, the result follows from Lemma 3.4.1.10. □

Lemma 3.4.1.12. *Let U be an internal class and let C and D be U-cocomplete $\mathcal{B}$-categories. Let $\alpha \colon f \to g$ be a map in $\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ in context $1 \in \mathcal{B}$. Then α is U-cocontinuous when viewed as a functor $\mathsf{C} \to \mathsf{D}^{\Delta^1}$ (where D^{Δ^1} is indeed U-cocomplete by Proposition 3.2.2.7).*

Proof. We need to show that for every $A \in \mathcal{B}$ and every $\mathsf{I} \in \mathsf{U}(A)$, the functor $\pi_A^*\alpha$ preserves I-indexed colimits. Since by Remark 2.1.14.1 the base change functor π_A^* commutes with cotensoring, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Now consider the commutative diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathrm{diag}"] \arrow[d, "\alpha"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \arrow[d, "\alpha_*"] \\
\mathsf{D}^{\Delta^1} \arrow[r, "\mathrm{diag}"] \arrow[d, "(d_1, d_0)"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{D}^{\Delta^1}) \arrow[d, "(d_1, d_0)_*"] \\
\mathsf{D} \times \mathsf{D} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{D} \times \mathsf{D}).
\end{tikzcd}$$

In order to show that α preserves I-indexed colimits, we need to verify that the mate transformation φ of the upper square is an equivalence. On account of Proposition 3.1.3.1, the mate of the lower square is an equivalence. We claim that the mate of the composite square is an equivalence as well, i.e. that $(f, g) \colon \mathsf{C} \to \mathsf{D} \times \mathsf{D}$ preserves I-indexed colimits. To see this, let $d \colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ be a diagram in context $A \in \mathcal{B}$. Using Remark 3.1.2.2, we may once again replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$ (see Remark 2.1.14.4). Now as $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, -)$ commutes with limits, we obtain an equivalence

$(\mathsf{D}\times\mathsf{D})_{(f,g)_*d/}\simeq \mathsf{D}_{f_*d/}\times\mathsf{D}_{g_*d/}$, so that the claim follows once we show that the image of the initial cocone $1\to\mathsf{C}_{d/}$ along the functor

$$(f_*,g_*)\colon \mathsf{C}_{d/}\to \mathsf{D}_{f_*d/}\times\mathsf{D}_{g_*d/}$$

is initial as well. This in turn follows from the assumption that both f and g preserve I-indexed colimits, together with the fact that the product of two initial maps is again initial.

As a consequence of what we've shown so far and the functoriality of mates, we conclude that postcomposing φ with the functor $(d_1,d_0)\colon \mathsf{C}^{\Delta^1}\to\mathsf{D}\times\mathsf{D}$ yields an equivalence. Therefore, φ is itself an equivalence since (d_1,d_0) is conservative by Remark 2.2.3.2. □

Theorem 3.4.1.13. *Let* C *be a* $\mathcal{B}$*-category, let* U *be an internal class of* $\mathcal{B}$*-categories and let* E *be a* U*-cocomplete large* $\mathcal{B}$*-category. Then the functor of left Kan extension along* $h_{\mathsf{C}}\colon\mathsf{C}\hookrightarrow\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *exists and determines an equivalence*

$$(h_{\mathsf{C}})_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})\simeq\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}).$$

In other words, the $\mathcal{B}$*-category* $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *is the free* U*-cocompletion of* C*.*

Proof. Let us define $\mathsf{E}'=\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\Omega_{\widehat{\mathcal{B}}})^{\mathrm{op}}$. By Proposition 3.2.2.9, the inclusion $h^{\mathrm{op}}_{\mathsf{E}}\colon\mathsf{E}\hookrightarrow\mathsf{E}'$ that is given by the Yoneda embedding is U-cocontinuous. Let $j\colon\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})\hookrightarrow\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the inclusion. By Theorem 3.3.3.5, the functors of left Kan extension along h_{C} and j exist and define inclusions

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}')\xhookrightarrow{(h_{\mathsf{C}})_!}\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')\xhookrightarrow{j_!}\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}'),$$

and by Theorem 3.4.1.1 the essential image of the composition is the full subcategory spanned by those objects in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')$ which define cocontinuous functors. Since j is by construction U-cocontinuous, the restriction functor $j^*\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')$ restricts to a functor

$$j^*\colon\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')\to\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}').$$

Consequently, we deduce that the left Kan extension functor $(h_{\mathsf{C}})_!\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}')\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')$ factors through an inclusion

$$(h_{\mathsf{C}})_!\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}')\hookrightarrow\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}').$$

We claim that this functor is essentially surjective and therefore an equivalence. On account of Remarks 3.2.3.4 and 3.3.3.2 as well as Proposition 3.4.1.11, it suffices to show (by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, see Remark 2.1.14.4) that any U-cocontinuous functor $f\colon\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})\to\mathsf{E}'$ is a left Kan extension along its restriction to C. Let $\epsilon\colon(h_{\mathsf{C}})_!h^*_{\mathsf{C}}f\to f$ be the adjunction counit, and let D be the full subcategory of $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by those objects F in $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ (in arbitrary context) for which ϵF is an equivalence. We need to show that $\mathsf{D}=\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. By construction, we have $\mathsf{C}\hookrightarrow\mathsf{D}$, so that it suffices to show that D is closed under U-colimits in $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Note that the inclusion $\mathsf{D}\hookrightarrow\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is precisely the pullback of $s_0\colon\mathsf{E}'\hookrightarrow(\mathsf{E}')^{\Delta^1}$ along $\epsilon\colon\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})\to(\mathsf{E}')^{\Delta^1}$. Since Proposition 3.2.2.7 implies that s_0 is cocontinuous and Lemma 3.4.1.12 shows that ϵ is U-cocontinuous, we deduce from Lemma 3.4.1.10 that the inclusion $\mathsf{D}\hookrightarrow\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ is indeed closed under U-colimits.

To finish the proof, we need to show that the equivalence $(h_{\mathsf{C}})_!\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}')\simeq\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')$ restricts to the desired equivalence

$$(h_{\mathsf{C}})_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})\simeq\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}).$$

As clearly h^*_{C} restricts in the desired way, it suffices to show that $(h_{\mathsf{C}})_!$ restricts as well. By the same reduction steps as above, this follows once we show that for every functor $f\colon\mathsf{C}\to\mathsf{E}$, the left Kan extension $(h_{\mathsf{C}})_!f\colon\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})\to\mathsf{E}'$ factors through E. Consider the commutative diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, hook, dotted] \arrow[dr, hook, "h"'] & \mathsf{D} \arrow[r] \arrow[d, hook] & \mathsf{E} \arrow[d, hook] \\
& \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "(h_{\mathsf{C}})_!f"] & \mathsf{E}'
\end{tikzcd}$$

in which the square is a pullback. Since both $h_{\mathsf{C}})_!f$ and $\mathsf{E} \hookrightarrow \mathsf{E}'$ are U-cocontinuous, it follows from Lemma 3.4.1.10 that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is closed under U-colimits and must therefore be an equivalence. As a consequence, the functor $(h_{\mathsf{C}})_!f$ factors through E, as needed. □

Corollary 3.4.1.14. *Let C be a $\mathcal{B}$-category and let $\mathsf{U} \subset \mathsf{V}$ be internal classes such that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is V-cocomplete. Then the inclusion $i\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{V}}(\mathsf{C})$ admits a left adjoint. In particular, if C itself is V-cocomplete, the inclusion $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{V}}(\mathsf{C})$ admits a left adjoint.*

Proof. By choosing $\mathsf{U} = \varnothing$ (i.e. the initial object in $\mathrm{Cat}(\mathcal{B})$), the second claim is an immediate consequence of the first. To prove the first statement, let $j\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ be the inclusion. Then Theorem 3.4.1.13 allows us to construct a candidate for the left adjoint $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{V}}(\mathsf{C}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ of i as the left Kan extension of j along ij. By construction, L is V-cocontinuous. As i is U-cocontinuous and since we have equivalences $j^*(Li) \simeq (ij)^*(L) \simeq j$, Theorem 3.4.1.13 moreover gives rise to an equivalence $Li \simeq \mathrm{id}_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})}$. Similarly, since $j^*(i) \simeq ij$, one obtains an equivalence $i \simeq j_!(ij)$. Therefore, transposing the identity on ij across the adjunction $(ij)_! \dashv (ij)^*$ gives rise to a map $\eta\colon \mathrm{id}_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{V}}(\mathsf{C})} \to iL$ such that ηi is an equivalence, being a map between U-cocontinuous functors that restricts to an equivalence along j. By making use of Corollary 2.4.4.3, we conclude that L is a left adjoint once we verify that $L\eta$ is an equivalence as well. As both domain and codomain of this map are V-cocontinuous functors, this is the case already if its restriction along ij is an equivalence, which follows from the construction of η. □

Corollary 3.4.1.15. *Let U be a* small *internal class of $\mathcal{B}$-categories. Then the inclusion $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ admits a left adjoint that carries a $\mathcal{B}$-category C to its free U-cocompletion. Moreover, the adjunction unit is given by the Yoneda embedding $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$.*

Proof. By Remark 3.4.1.7, the free U-cocompletion $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is indeed a small $\mathcal{B}$-category. Therefore, the Yoneda embedding $h_{\mathsf{C}}\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is a well-defined map in $\mathsf{Cat}_{\mathcal{B}}$. By Corollary 2.4.3.5, it suffices to show that the composition

$$\varphi\colon\ \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), -) \hookrightarrow \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), -) \to \mathrm{map}_{\mathsf{Cat}_{\mathcal{B}}}(\mathsf{C}, -)$$

is an equivalence of functors $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \to \Omega$. Using that equivalences of functors are detected object-wise [62, Corollary 4.7.17], this follows once we show that the evaluation of this map at any object $A \to \mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ yields an equivalence of $\mathcal{B}_{/A}$-groupoids. By combining Remark 3.2.3.2 with Proposition 3.4.1.11 and with Example 2.1.14.7, we may pass to $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$ (see Remark 2.1.14.4). In this case, the result follows from Theorem 3.4.1.13 in light of the observation that by Remark 3.2.3.4, the evaluation of φ at a U-cocomplete $\mathcal{B}$-category E is precisely the restriction of the equivalence from Theorem 3.4.1.13 to core $\mathcal{B}$-groupoids. □

3.4.2. Detecting cocompletions. In this section we give a characterisation when a functor $f\colon \mathsf{C} \to \mathsf{D}$ exhibits D as the free U-cocompletion of C. To achieve this, we need the notion of *U-cocontinuous objects*, which is in a certain way an internal analogue of the notion of a κ-compact object in an ∞-category:

Definition 3.4.2.1. Let D be a U-cocomplete $\mathcal{B}$-category. We define the full subcategory $\mathsf{D}^{\mathsf{U}\text{-cc}} \hookrightarrow \mathsf{D}$ of U-*cocontinuous objects* as the pullback

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{U}\text{-cc}} \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{D}, \Omega)^{\mathrm{op}} \arrow[d, hook] \\
\mathsf{D} \arrow[r, hook, "h_{\mathsf{D}^{\mathrm{op}}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \Omega)^{\mathrm{op}}.
\end{tikzcd}$$

Remark 3.4.2.2 (locality of U-cocontinuous objects). In the situation of Definition 3.4.2.1, we may combine Example 2.1.14.7 with Remark 3.2.3.4 to deduce that there is a canonical equivalence $\pi_A^*(\mathsf{D}^{\mathsf{U}\text{-cc}}) \simeq (\pi_A^*\mathsf{D})^{\pi_A^*\mathsf{U}\text{-cc}}$ of full subcategories of $\pi_A^*\mathsf{D}$, for every $A \in \mathcal{B}$.

Remark 3.4.2.3 (étale transposition invariance of U-cocontinuous objects). By Remark 3.4.2.2, an object $d\colon A \to \mathsf{D}$ is contained in $\mathsf{D}^{\mathsf{U}\text{-cc}}$ if and only if its transpose $\bar{d}\colon 1 \to \pi_A^*\mathsf{D}$ is $\pi_A^*\mathsf{U}$-cocontinuous.

The following proposition and its proof is an adaptation of [57, Proposition 5.1.6.10].

Proposition 3.4.2.4. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between $\mathcal{B}$-categories such that D is U-cocomplete, and let $\hat{f}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \to \mathsf{D}$ be its unique U-cocontinuous extension. Then the following are equivalent:*

(1) $\hat{f}$ is an equivalence;

(2) f is fully faithful, takes values in $\mathsf{D}^{\mathsf{U}\text{-cc}}$, and generates D under U-colimits.

Proof. We first note that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})^{\mathsf{U}\text{-cc}}$ contains C. Indeed, Yoneda's lemma implies that the composition

$$\mathsf{C} \xhookrightarrow{h_{\mathsf{C}}} \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \xhookrightarrow{h_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})^{\mathrm{op}}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}), \Omega)^{\mathrm{op}}$$

can be identified with the opposite of the transpose of the evaluation functor $\mathrm{ev}\colon \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \to \Omega$. Together with Proposition 3.4.1.11 and Remark 2.1.14.1, this implies that the image of $c\colon A \to \mathsf{C}$ along this composition transposes to the functor

$$\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}}(\pi_A^*\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}) \xrightarrow{\mathrm{ev}_c} \Omega_{\mathcal{B}_{/A}}$$

which is $\pi_A^*\mathsf{U}$-cocontinuous by Proposition 3.2.2.7. Therefore, (1) implies (2).

Conversely, suppose that condition (2) is satisfied. We first prove that $\hat{f}$ is fully faithful. Tot that end, if $c\colon A \to \mathsf{C}$ is an arbitrary object, we claim that the morphism

$$\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})}(c, -) \to \mathrm{map}_{\mathsf{D}}(\hat{f}(c), \hat{f}(-))$$

is an equivalence. By combining Remarks 3.3.3.2 and 3.3.2.2 with Proposition 3.4.1.11 and Example 2.1.14.7, we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and can thus assume that $A \simeq 1$ (see Remark 2.1.14.4). In this case, the fact that C is contained in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})^{\mathsf{U}\text{-cc}}$ and condition (2) imply that both domain and codomain of the morphism are U-cocontinuous functors. By using Lemma 3.4.1.12 and the fact that the above morphism restricts to an equivalence on C, the universal property of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ thus implies that this map is an equivalence of functors. By what we just have shown, if $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is an arbitrary object, the natural transformation

$$\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})}(-, F) \to \mathrm{map}_{\mathsf{D}}(\hat{f}(-), \hat{f}(F))$$

restricts to an equivalence on C. As this map transposes to a morphism of $\pi_A^*\mathsf{U}$-cocontinuous functors (using Proposition 3.2.2.9 and the fact that $\hat{f}$ is U-cocontinuous), the same argument as above shows that the entire natural transformation is in fact an equivalence and therefore that $\hat{f}$ is fully faithful, as desired. As therefore $\hat{f}$ exhibits $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ as a full subcategory of D that is closed under U-colimits and that contains C, the assumption that D is generated by C under U-colimits implies that $\hat{f}$ is an equivalence. □

3.4.3. Cocompletion of the point. Let U be an internal class of $\mathcal{B}$-categories. Our goal in this section is to study the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \hookrightarrow \Omega$. To that end, let us denote by $\mathrm{gpd}(\mathsf{U}) \hookrightarrow \Omega$ the image of U along the groupoidification functor $(-)^{\mathrm{gpd}}\colon \mathsf{Cat}_{\mathcal{B}} \to \Omega$ from Proposition 2.4.2.14.

Definition 3.4.3.1. We call an internal class U *closed under groupoidification*, if for any $A \in \mathcal{B}$ and $\mathsf{I} \in \mathsf{U}(A)$ the groupoidification $\mathsf{I}^{\mathrm{gpd}}$ is also contained in U. For any internal class U we can form its *closure under groupoidification*, denoted $\overline{\mathsf{U}}$, that is defined as the internal class spanned by U and $\mathrm{gpd}(\mathsf{U})$.

Remark 3.4.3.2. Since for any $\mathcal{B}$-category I, the morphism $\mathsf{I} \to \mathsf{I}^{\mathrm{gpd}}$ is final, it follows that any colimit class (in the sense of Definition 3.2.3.5) is closed under groupoidification. Furthermore, for any internal class U, we have inclusions $\mathsf{U} \subseteq \overline{\mathsf{U}} \subseteq \mathsf{U}^{\mathrm{colim}}$. In particular the discussion after Definition 3.2.3.5 shows that a $\mathcal{B}$-category is U-cocomplete if and only if it is $\overline{\mathsf{U}}$-cocomplete. The same statement holds for U-cocontinuity.

REMARK 3.4.3.3. If U is closed under groupoidification, the adjunction $(-)^{\mathrm{gpd}} \dashv \iota\colon \mathsf{Cat}_{\mathcal{B}} \leftrightarrows \Omega$ restricts to an adjunction

$$((-)^{\mathrm{gpd}} \dashv i)\colon \mathsf{U} \leftrightarrows \mathrm{gpd}(\mathsf{U}).$$

PROPOSITION 3.4.3.4. *For any internal class U of $\mathcal{B}$-categories, there is an inclusion $\mathrm{gpd}(\mathsf{U}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ which is an equivalence whenever $\overline{\mathsf{U}}$ is closed under U-colimits in $\mathsf{Cat}_{\mathcal{B}}$.*

PROOF. By construction, the canonical map $\mathsf{U} \hookrightarrow \overline{\mathsf{U}}$ induces an equivalence $\mathrm{gpd}(\mathsf{U}) \simeq \mathrm{gpd}(\overline{\mathsf{U}})$. Therefore we may assume that U is closed under groupoidification. For any $\mathcal{B}$-category I contained in $\mathsf{U}(1)$, its groupoidification $\mathsf{I}^{\mathrm{gpd}}$ is the colimit of the functor $\mathsf{I} \to 1 \hookrightarrow \Omega$ (see Proposition 3.1.4.1) and therefore by definition contained in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$. Note that by using Remarks 2.1.14.1 and 2.3.1.3 as well as Corollary 2.4.1.9, for every $A \in \mathcal{B}$ the functor π_A^* carries the adjunction $(-)^{\mathrm{gpd}} \dashv \iota\colon \mathsf{Cat}_{\mathcal{B}} \leftrightarrows \Omega$ to the adjunction $(-)^{\mathrm{gpd}} \dashv \iota\colon \mathsf{Cat}_{\mathcal{B}_{/A}} \leftrightarrows \Omega_{\mathcal{B}_{/A}}$. Together with Proposition 3.4.1.11, this observation and the above argument also yields that for every $\mathsf{I} \in \mathsf{U}(A)$ the groupoidification $\mathsf{I}^{\mathrm{gpd}}$ defines an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$. Thus, the groupoidification functor $(-)^{\mathrm{gpd}}\colon \mathsf{Cat}_{\mathcal{B}} \to \Omega$ restricts to a functor $\mathsf{U} \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ and therefore gives rise to the desired inclusion $\mathrm{gpd}(\mathsf{U}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$. Now by definition of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$, this inclusion is an equivalence if and only if $\mathrm{gpd}(\mathsf{U})$ is closed under U-colimits in Ω. But if the subcategory $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is closed under U-colimits in $\mathsf{Cat}_{\mathcal{B}}$ it follows by Remark 3.4.3.3 that $\mathrm{gpd}(\mathsf{U}) = \mathsf{U} \cap \Omega$, hence the claim follows from Lemma 3.4.1.10. $\square$

EXAMPLE 3.4.3.5. Let S be a local class of maps in $\mathcal{B}$ and let $\Omega_S \hookrightarrow \Omega$ be the associated full subcategory of Ω (cf. Proposition 2.1.10.7). Then Ω_S is clearly closed under groupoidification. Recall that Ω_S is closed under Ω_S-colimits in Ω precisely if the local class S is stable under composition (see Example 3.2.4.3). Therefore, if S is stable under composition, Proposition 3.4.3.4 provides an equivalence $\Omega_S \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_S}(1)$.

If S is not closed under composition, the free cocompletion $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_S}(1)$ still admits an explicit description. Namely, an object $c\colon A \to \Omega$ in context $A \in \mathcal{B}$ defines an object of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_S}(1)$ if and only if it is locally a composition of two morphisms in S. To be more precise, c is in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_S}(1)$ if and only if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that every $s_i^* c \in \Omega(A_i) = \mathcal{B}_{/A_i}$ can be written as a composition $g_i f_i$ of two morphisms $g_i\colon P_i \to Q_i$ and $f_i\colon Q_i \to A_i$ that are in S. This description holds since the full subcategory spanned by these objects is clearly closed under Ω_S-indexed colimits and it is easy to see that it is the smallest full subcategory of Ω with this property.

EXAMPLE 3.4.3.6. The following observation is due to Bastiaan Cnossen: Let $\mathcal{B} = \mathrm{PSh}_{\mathcal{S}}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ with pullbacks and let S be a class of morphisms in $\mathcal{C}$ that is closed under pullbacks in $\mathcal{C}$. It generates a local class in $\mathcal{B} = \mathrm{PSh}_{\mathcal{S}}(\mathcal{C})$ that we denote by W. As in Example 3.2.4.11, we obtain an internal class $\mathsf{U}_S = \langle W, \mathrm{Cat}_\infty \rangle$, so that we may now consider the free U_S-cocompletion $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}_S}(1)$ of the point. It may be explicitly described as the presheaf on $\mathcal{C}$ given by

$$\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}_S}(1)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad c \mapsto \mathrm{PSh}_{\mathcal{S}}(S_{/c})$$

where $S_{/c}$ denotes the full subcategory of $\mathcal{C}_{/c}$ spanned by the morphisms in S. In particular it agrees with the $\mathrm{PSh}(\mathcal{C})$-category underlying the *initial cocomplete pullback formalism* described in [23, § 4].

We conclude this section by showing that any U-cocomplete large $\mathcal{B}$-category E is *tensored* over $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ in the following sense:

DEFINITION 3.4.3.7. A large $\mathcal{B}$-category E is *tensored* over $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ if there is a functor $-\otimes -\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \times \mathsf{E} \to \mathsf{E}$ together with an equivalence

$$\mathrm{map}_{\mathsf{E}}(- \otimes -, -) \simeq \mathrm{map}_{\Omega_{\widehat{\mathcal{B}}}}(-, \mathrm{map}_{\mathsf{E}}(-,-)).$$

PROPOSITION 3.4.3.8. *If E is a U-cocomplete large $\mathcal{B}$-category, then E is tensored over $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$.*

Proof. Since E is U-cocomplete, Proposition 3.1.3.1 implies that the functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{E})$ is U-cocomplete as well. As a consequence, we may apply Theorem 3.4.1.13 to extend the identity $\mathrm{id}_{\mathsf{E}}\colon 1\to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{E})$ in a unique way to a U-cocontinuous functor $f\colon \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)\to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\mathsf{E})$. We define the desired bifunctor $-\otimes-$ as the transpose of f. To see that it has the desired property, note that $\mathrm{map}_{\mathsf{E}}(-\otimes-,-)$ is the transpose of the composition

$$\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}^{\mathrm{op}},\mathsf{E}^{\mathrm{op}}) \xhookrightarrow{(h_{\mathsf{E}^{\mathrm{op}}})_*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}^{\mathrm{op}}\times\mathsf{E},\Omega_{\widehat{\mathcal{B}}}),$$

whereas the functor $\mathrm{map}_{\Omega_{\widehat{\mathcal{B}}}}(-,\mathrm{map}_{\mathsf{E}}(-,-))$ transposes to the functor

$$\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(1)^{\mathrm{op}} \xhookrightarrow{i} \Omega^{\mathrm{op}}_{\widehat{\mathcal{B}}} \xhookrightarrow{h_{\Omega^{\mathrm{op}}_{\widehat{\mathcal{B}}}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega_{\widehat{\mathcal{B}}},\Omega_{\widehat{\mathcal{B}}}) \xrightarrow{\mathrm{map}^*_{\mathsf{E}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E}^{\mathrm{op}}\times\mathsf{E},\Omega_{\widehat{\mathcal{B}}}).$$

As the opposite of either of these functors is U-cocontinuous, Theorem 3.4.1.13 implies that they are both uniquely determined by their value at the point $1\colon 1\to\Omega$. Since $\mathrm{map}_{\Omega_{\widehat{\mathcal{B}}}}(1,-)$ is equivalent to the identity functor, we find that both of these functors send $1\colon 1\to\Omega$ to $\mathrm{map}_{\mathsf{E}}$ and that they are therefore equivalent, as required. □

Remark 3.4.3.9. By dualising Proposition 3.4.3.8, one obtains that a U-complete large $\mathcal{B}$-category E is *powered* over $\underline{\mathsf{PSh}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(1)$: since $\underline{\mathsf{PSh}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(1)^{\mathrm{op}}$ is the free U-completion of the final $\mathcal{B}$-category $1\in\mathrm{Cat}(\mathcal{B})$, there is a functor $(-)^{(-)}\colon \underline{\mathsf{PSh}}^{\mathrm{op}(\mathsf{U})}_{\mathcal{B}}(1)^{\mathrm{op}}\times\mathsf{E}\to\mathsf{E}$ that fits into an equivalence

$$\mathrm{map}_{\mathsf{E}}(-,(-)^{(-)})\simeq \mathrm{map}_{\Omega_{\widehat{\mathcal{B}}}}(-,\mathrm{map}_{\mathsf{E}}(-,-)).$$

3.4.4. Decomposition of colimits II. In § 3.1.9 we showed that whenever C is a $\mathcal{B}$-category that admits colimits indexed by κ-small constant $\mathcal{B}$-categories and $\mathcal{K}\to\mathrm{Cat}(\mathcal{B})$, $k\mapsto\mathsf{J}_k$ is a diagram that is indexed by a κ-small constant $\mathcal{B}$-category $\mathcal{K}$, then C admits colimits indexed by $\mathsf{J}=\mathrm{colim}_k\,\mathsf{J}_k$ as soon as it admits J_k-indexed colimits for all $k\in\mathcal{K}$. In this section, our goal is to generalise this result by allowing $\mathcal{K}$ to be an arbitrary $\mathcal{B}$-category instead of merely a constant one. More precisely, we will show:

Proposition 3.4.4.1. *Let U be an internal class, let $d\colon\mathsf{I}\to\mathsf{U}$ be a diagram such that $\mathsf{I}\in\mathsf{U}(1)$, and let $\mathsf{K}=\mathrm{colim}\,d$. Then every U-cocomplete $\mathcal{B}$-category admits K-indexed colimits, and every U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories preserves K-indexed colimits.*

Our strategy for the proof of Proposition 3.4.4.1 is to take the colimit of a K-indexed diagram in the free cocompletion of C (i.e. in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$) and to show that this colimit can be reflected back into C. We therefore need to study such K-indexed colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ first.

Lemma 3.4.4.2. *For every $\mathcal{B}$-category C, the large $\mathcal{B}$-category $\mathsf{RFib}_{\mathsf{C}}$ is a reflective subcategory of $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$.*

Proof. To begin with, we note that the sheaf associated with $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ is given by $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times-}$. In fact, the latter defines a $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$-category, and there is a right fibration of $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$-categories

$$p\colon\ \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times-}\to\mathrm{Cat}(\mathcal{B})_{/-}$$

that is section-wise given by postcomposition with the projection onto the second factor. By [62, Proposition 3.3.5], the codomain can be identified with (the underlying $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$-category of) the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$. Since $\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times-}$ has a final object (in the $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$-categorical sense, which is easily deduced from Examples 3.1.1.11 and 3.1.1.14) that is carried to C along the right fibration p, we thus obtain an equivalence $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}\simeq\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times-}$ of $\mathrm{PSh}_{\widehat{\mathcal{S}}}(\mathcal{B})$-categories. Since the domain is a (large) $\mathcal{B}$-category, so is the codomain, and this equivalence defines an identification of (large) $\mathcal{B}$-category. Now using this identification, we find that the inclusion $\mathrm{RFib}(\mathsf{C}\times-)\hookrightarrow\mathrm{Cat}(\mathcal{B})_{/\mathsf{C}\times-}$ determines a fully faithful functor $i\colon\mathsf{RFib}_{\mathsf{C}}\hookrightarrow(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ such that $i(A)$ admits a left adjoint L_A for every $A\in\mathcal{B}$. Moreover, if $s\colon B\to A$ is an arbitrary map in $\mathcal{B}$, the fact that s is smooth implies that the natural map $L_Bs^*\to s^*L_A$ is an equivalence (see the discussion in [62, § 4.4]), hence the claim follows. □

PROPOSITION 3.4.4.3. *Let* U *be an internal class of* $\mathcal{B}$*-categories and let* C *be a small* $\mathcal{B}$*-category. Then, for any diagram* $d\colon \mathsf{I} \to \mathsf{U}$ *with colimit* K *in* $\mathsf{Cat}_{\mathcal{B}}$ *and any diagram* $p\colon \mathsf{K} \to \mathsf{C}$ *with colimit* F *in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*, there is a diagram* $d'\colon \mathsf{I} \to \underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *such that* $F \simeq \operatorname{colim} d'$*.*

PROOF. The cocone $d \to \operatorname{diag}(\mathsf{C})$ implies that we may view d as a diagram $d\colon \mathsf{I} \to \mathsf{U}_{/\mathsf{C}} \hookrightarrow (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$. By Proposition 3.1.6.3, the colimit of this diagram is $p\colon \mathsf{K} \to \mathsf{C}$. Since $F \simeq \operatorname{colim} p$, there is a final functor $\mathsf{K} \to \mathsf{C}_{/F}$ over C, hence Lemma 3.4.4.2 implies that the localisation functor $L\colon (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}}$ carries $p\colon \mathsf{K} \to \mathsf{C}$ to the right fibration $\mathsf{C}_{/F} \to \mathsf{C}$. In other words, the presheaf F arises as the colimit of the diagram $d' = Ld\colon \mathsf{I} \to \mathsf{U}_{/\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. It now suffices to observe that by construction of L, this functor takes values in $\underline{\mathsf{Small}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. □

PROOF OF PROPOSITION 3.4.4.1. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a U-cocontinuous functor between U-cocomplete $\mathcal{B}$-categories. Consider the commutative diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, hook, "h_{\mathsf{C}}"] & \mathsf{D} \arrow[d, hook, "h_{\mathsf{D}}"] \\ \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "\hat{f}"] & \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

that arises from applying the universal property of $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ to the composition $\mathsf{C} \to \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{D})$. As C and D are U-cocomplete, the vertical inclusions admit left adjoints L_{C} and L_{D} by Corollary 3.4.1.14. Now if $p\colon \mathsf{K} \to \mathsf{C}$ is a diagram, Proposition 3.4.4.3 implies that there is a diagram $p'\colon \mathsf{I} \to \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ such that $\operatorname{colim} p'$ is equivalent to the colimit of $h_{\mathsf{C}}p$. In particular, the colimit of $h_{\mathsf{C}}p$ is contained in $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Consequently, $L_{\mathsf{C}} \operatorname{colim} p'$ defines a colimit of p by Proposition 3.1.2.11. By replacing C with D, this argument also shows that every diagram $\mathsf{K} \to \mathsf{D}$ admits a colimit in D. Moreover, as f and $\hat{f}$ are U-cocontinuous, the universal property of $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ implies that the canonical map $L_{\mathsf{D}}\hat{f} \to fL_{\mathsf{C}}$ is an equivalence. Consequently, as $L_{\mathsf{D}}\hat{f}$ preserves the colimit of $h_{\mathsf{C}}p$, so does fL_{C}. As the colimit cocone of p is the image of the colimit cocone of $h_{\mathsf{D}}p$ along L_{C}, we conclude that f preserves the colimit of p. Now by replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the above argumentation, one concludes that both C and D admit K-indexed colimits and that f commutes with such colimits. □

CHAPTER 4

Presentable $\mathcal{B}$-categories

Our theory of presentable $\mathcal{B}$-categories relies on the interplay between internal classes U of $\mathcal{B}$-categories and their associated internal classes $\mathsf{Filt}_{\mathsf{U}}$ of U-*filtered* $\mathcal{B}$-categories. We study the concept of U-filteredness in § 4.1. In particular, we study certain conditions on the internal class U which guarantee that every $\mathcal{B}$-category can be decomposed into a U-filtered colimit of objects in U. This is a technical condition which is crucial for the development of accessibility in the world of $\mathcal{B}$-categories. In § 4.2, we discuss how one can construct an ample amount of internal classes U which satisfy these conditions. Building upon these rather technical preparations, we then define the concept of U-accessibility in § 4.3 and prove a few basic results that we will need for our discussion of presentable $\mathcal{B}$-categories. For example, we give a characterisation of U-accessible $\mathcal{B}$-categories by making use of the notion of U-*compactness*. In § 4.4, we introduce and study presentable $\mathcal{B}$-categories. Aside from discussing multiple characterisations of these $\mathcal{B}$-categories, we prove an adjoint functor theorem and discuss limits and colimits of presentable $\mathcal{B}$-categories. In the final two sections of this chapter, we switch gears and study a few concepts of higher algebra in the world of $\mathcal{B}$-categories. In § 4.5, we set up the main framework and use it to characterise dualisable objects in the $\mathcal{B}$-category of modules over an $\mathbb{E}_\infty$-ring in $\mathcal{B}$. In § 4.6, we discuss tensor products of $\mathcal{B}$-categories and in particular a symmetric monoidal structure on the $\mathcal{B}$-category of presentable $\mathcal{B}$-categories. We then use this structure to realise $\mathcal{B}$-modules in the ∞-category of presentable ∞-categories as presentable $\mathcal{B}$-categories.

4.1. Filtered $\mathcal{B}$-categories

Classically, if κ is a (regular) cardinal, a 1-category $\mathcal{J}$ is said to be κ-*filtered* if the colimit functor $\operatorname{colim}_{\mathcal{J}}\colon \operatorname{Fun}(\mathcal{J}, \mathrm{Set}) \to \mathrm{Set}$ commutes with κ-small limits. In [57], Lurie generalised this concept to the notion of a κ-*filtered* ∞-*category* $\mathcal{J}$, which is an ∞-category for which $\operatorname{colim}_{\mathcal{J}}\colon \operatorname{Fun}(\mathcal{J}, \mathcal{S}) \to \mathcal{S}$ preserves κ-small limits. The main goal of this section is to discuss an analogous concept for $\mathcal{B}$-categories. Following ideas originally introduced in 1-category theory by Adámek-Borceux-Lack-Rosický [1] and later generalised to ∞-categories by Charles Rezk [78], we will introduce the notion of a U-*filtered* $\mathcal{B}$-*category*, where U is an arbitrary internal class, i.e. a full subcategory of the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ of $\mathcal{B}$-categories (cf. § 2.1.10). The main definitions and basic properties of such U-filtered $\mathcal{B}$-categories are discussed in § 4.1.1. In § 4.1.2, we introduce a slightly weaker notion, that of a *weakly* U-filtered $\mathcal{B}$-category. Classically, a κ-filtered (∞-)category can be equivalently described as an (∞-)category in which every κ-small diagram has a cocone. The notion of weak U-filteredness is a generalisation of this condition. However, as the terminology suggests, this notion is a priori weaker than that of U-filteredness. Following [1], we will call an internal class U a *doctrine* if both conditions happen to be equivalent. In § 4.1.3 and § 4.1.4, we will study two other important properties of internal classes: *regularity* and the *decomposition property*. Recall that a cardinal κ is said to be regular if κ-small sets are closed under κ-small sums. The notion of regularity for internal classes aims at capturing this property in the world of $\mathcal{B}$-categories. The decomposition property, on the other hand, is the condition that every $\mathcal{B}$-category can be obtained as a U-filtered colimit of objects in U. Hence, this notion can be viewed as an analogue to the fact that every (∞-)category is a κ-filtered colimit of κ-small (∞-)categories. We will make use of this property when we discuss the notion of U-compactness in § 4.1.5.

4.1.1. U-filtered $\mathcal{B}$-categories. In this section, we introduce and study the notion of U-filteredness in the world of $\mathcal{B}$-categories, where U is an arbitrary internal class. We begin with the following definition, which is an evident generalisation of the classical concept of a κ-filtered (∞-)category:

Definition 4.1.1.1. For any internal class U of $\mathcal{B}$-categories, a $\mathcal{B}$-category J is said to be U-*filtered* if the colimit functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega) \to \Omega$ is U-continuous. We define the internal class $\mathsf{Filt}_{\mathsf{U}}$ of U-filtered categories as the full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by those $\mathcal{B}_{/A}$-categories J that are $\pi_A^*\mathsf{U}$-filtered, for every $A \in \mathcal{B}$.

Remark 4.1.1.2. In the situation of Definition 4.1.1.1, the fact that U-continuity is a local condition by Remark 3.2.2.3 implies that *every* object $A \to \mathsf{Filt}_{\mathsf{U}}$ is $\pi_A^*\mathsf{U}$-filtered (which a priori has no reason to be true). In particular, the sheaf associated with $\mathsf{Filt}_{\mathsf{U}}$ is given on local sections over $A \in \mathcal{B}$ by the full subcategory of $\mathrm{Cat}(\mathcal{B}_{/A})$ that is spanned by the $\pi_A^*\mathsf{U}$-filtered categories. For any $A \in \mathcal{B}$, we therefore obtain a canonical equivalence $\pi_A^* \mathsf{Filt}_{\mathsf{U}} \simeq \mathsf{Filt}_{\pi_A^*\mathsf{U}}$.

Remark 4.1.1.3. Clearly, if $\mathsf{U} \hookrightarrow \mathsf{V}$ is an inclusion of internal classes, every V-filtered $\mathcal{B}$-category is in particular U-filtered. Therefore, one obtains an inclusion $\mathsf{Filt}_{\mathsf{V}} \hookrightarrow \mathsf{Filt}_{\mathsf{U}}$.

Remark 4.1.1.4. If I and J are $\mathcal{B}$-categories, note that the horizontal mate of the commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I} \times \mathsf{J}, \Omega) \arrow[d, "\operatorname{colim}_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega) \arrow[l, "\operatorname{diag}_*"'] \arrow[d, "\operatorname{colim}"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) & \Omega \arrow[l, "\operatorname{diag}"]
\end{tikzcd}$$

(with respect to the two adjunctions $\operatorname{diag} \dashv \lim$ and $\operatorname{diag}_* \dashv \lim_*$) is equivalent to the horizontal mate of the commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \arrow[r, "\operatorname{diag}_*"] \arrow[d, "\lim"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I} \times \mathsf{J}, \Omega) \arrow[d, "\lim_*"] \\
\Omega \arrow[r, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega).
\end{tikzcd}$$

(with respect to the two adjunctions $\operatorname{colim} \dashv \operatorname{diag}$ and $\operatorname{colim}_* \dashv \operatorname{diag}_*$). As a consequence, the functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega) \to \Omega$ commutes with I-indexed limits if and only if the functor $\lim\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \to \Omega$ commutes with J-indexed colimits. Thus, if U is an internal class of $\mathcal{B}$-categories, a $\mathcal{B}$-category J is U-filtered precisely if for all $A \in \mathcal{B}$ and all $\mathsf{I} \in \mathsf{U}(A)$ the limit functor $\lim\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}_{/A}}) \to \Omega_{\mathcal{B}_{/A}}$ commutes with $\pi_A^*\mathsf{J}$-indexed colimits.

Recall from [62, § 4.2] the definition of the *right cone* $\mathsf{J}^{\triangleright}$ of a $\mathcal{B}$-category J. It comes with an inclusion $\iota\colon \mathsf{J} \hookrightarrow \mathsf{J}^{\triangleright}$ such that for every $\mathcal{B}$-category C that admits J-indexed colimits, the functor of left Kan extension $\iota_!$ exists and carries an I-indexed diagram in C to its colimit cocone, see Proposition 3.3.3.8. We now obtain:

Proposition 4.1.1.5. *A $\mathcal{B}$-category J is U-filtered with respect to some internal class U if and only if the inclusion $\iota_!\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \Omega)$ is U-continuous.*

Proof. By definition, the functor $\iota_!$ is U-continuous if and only if for all $A \in \mathcal{B}$ the functor $\pi_A^*(\iota_!) \simeq (\pi_A^*\iota)_!$ (cf. [62, Lemma 4.2.3] and Corollary 2.4.1.9) preserves limits of I-indexed diagrams for all $\mathsf{I} \in \mathsf{U}(A)$, and J is U-filtered if and only if the colimit functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{J}, \Omega_{\mathcal{B}_{/A}}) \to \Omega_{\mathcal{B}_{/A}}$ commutes with I-indexed limits for all $\mathsf{I} \in \mathsf{U}(A)$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, it therefore suffices to show that for any $\mathsf{I} \in \mathsf{U}(1)$, the functor $\iota_!$ commutes with I-indexed limits if and only if $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega) \to \Omega$ preserves I-limits. Note that by combining Proposition 3.1.8.1 with the fact that the cone point $\infty\colon 1 \to \mathsf{I}^{\triangleright}$ is final, one finds that the colimit functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright}, \Omega) \to \Omega$ is given by evaluation at ∞. As a consequence, Proposition 3.1.3.1 implies that the colimit functor $\operatorname{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright},\Omega)\to\Omega$ preserves I-indexed limits. Owing to the commutative diagram

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\Omega) \arrow[d, hook, "\iota_!"] \arrow[dr, "\operatorname{colim}_{\mathsf{J}}"] & \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright},\Omega) \arrow[r, "\operatorname{colim}_{\mathsf{J}^{\triangleright}}"'] & \Omega, \end{tikzcd}$$

the functoriality of mates thus implies that $i_!$ preserving I-indexed limits implies that $\operatorname{colim}_{\mathsf{J}}$ commutes with I-indexed limits as well. The converse direction, on the other hand, follows from combining the functoriality of the mate construction with the straightforward observation that $(\iota^*,\infty^*)\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}^{\triangleright},\Omega)\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\Omega)\times\Omega$ is a conservative functor. □

By an analogous argument as in the proof of Proposition 4.1.1.5 and by furthermore using Remark 4.1.1.4, one obtains:

PROPOSITION 4.1.1.6. *A category* J *in* $\mathcal{B}$ *is* U*-filtered with respect to some internal class* U *if and only if for all* $A\in\mathcal{B}$ *and all* $\mathsf{I}\in\mathsf{U}(A)$ *the functor of right Kan extension*

$$\iota_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\mathcal{B}_{/A}})\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft},\Omega_{\mathcal{B}_{/A}})$$

preserves $\pi_A^*\mathsf{J}$*-indexed colimits.* □

REMARK 4.1.1.7. By Proposition 4.1.1.6 and Proposition 3.1.8.1, if $\mathsf{J}\to\mathsf{K}$ is a final functor such that J is U-filtered, then K must be U-filtered as well. Since the final $\mathcal{B}$-category 1 is trivially U-filtered for every choice of internal class U, this means that $\mathsf{Filt}_{\mathsf{U}}$ is a *colimit class* in the sense of Definition 3.2.3.5.

For later use, let us note the following closure property of U-filtered $\mathcal{B}$-categories:

PROPOSITION 4.1.1.8. *For any internal class* U*, the internal class* $\mathsf{Filt}_{\mathsf{U}}$ *is closed under* $\mathsf{Filt}_{\mathsf{U}}$*-colimits in* $\mathsf{Cat}_{\mathcal{B}}$*.*

PROOF. By Remark 4.1.1.2, it suffices to show that if J is a U-filtered $\mathcal{B}$-category and $d\colon\mathsf{J}\to\mathsf{Filt}_{\mathsf{U}}$ is a diagram, its colimit K in $\mathsf{Cat}_{\mathcal{B}}$ is also U-filtered. Given any $\mathsf{I}\in\mathsf{U}(1)$, Proposition 3.4.4.1 shows that the functor of right Kan extension $\iota_*\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega)\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft},\Omega)$ commutes with K-indexed colimits. As for any $A\in\mathcal{B}$ the $\mathcal{B}_{/A}$-category $\pi_A^*\mathsf{K}$ is the colimit of π_A^*d, the same argument also shows that for all $\mathsf{I}\in\mathsf{U}(A)$ the functor $\iota_*\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft},\Omega_{\mathcal{B}_{/A}})\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\Omega_{\mathcal{B}_{/A}})$ commutes with $\pi_A^*\mathsf{K}$-indexed colimits. Hence Proposition 4.1.1.6 implies that K is U-filtered. □

4.1.2. Weakly U-filtered $\mathcal{B}$-categories. Recall from Remark 3.2.2.2 that if U is an internal class, we denote by $\operatorname{op}(\mathsf{U})$ the internal class that arises as the image of U along the equivalence $(-)^{\mathrm{op}}\colon\mathsf{Cat}_{\mathcal{B}}\simeq\mathsf{Cat}_{\mathcal{B}}$. In practice, we will often require that every $\operatorname{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category is U-filtered. However, this is not true for every internal class U, not even in the case $\mathcal{B}=\mathcal{S}$ [78, §6]. In this section, we will therefore study a slightly weaker notion than that of a filtered U-category, which will encompass the class of $\operatorname{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-categories. We adopted the idea of weak U-filteredness from Charles Rezk [78], who in turn generalised ideas from [1] to ∞-categories.

DEFINITION 4.1.2.1. If U is an internal class of $\mathcal{B}$-categories, a $\mathcal{B}$-category J is *weakly* U*-filtered* if for every $A\in\mathcal{B}$ and every $\mathsf{I}\in\mathsf{U}(A)$ the diagonal functor $\pi_A^*\mathsf{J}\to\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\mathrm{op}},\pi_A^*\mathsf{J})$ is final. We define the internal class $\mathsf{wFilt}_{\mathsf{U}}$ as the full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the weakly $\pi_A^*\mathsf{U}$-filtered $\mathcal{B}_{/A}$-categories, for every $A\in\mathcal{B}$.

REMARK 4.1.2.2. In the situation of Definition 4.1.2.1, as the condition of a functor of $\mathcal{B}$-categories being final is *local* in $\mathcal{B}$ [62, Remark 4.4.9], every object $A\to\mathsf{wFilt}_{\mathsf{U}}$ is weakly $\pi_A^*\mathsf{U}$-filtered. In particular, there is a canonical equivalence $\pi_A^*\,\mathsf{wFilt}_{\mathsf{U}}\simeq\mathsf{wFilt}_{\pi_A^*\mathsf{U}}$ for all $A\in\mathcal{B}$.

Remark 4.1.2.3. If $\mathsf{U} \hookrightarrow \mathsf{V}$ is an inclusion of internal classes, every weakly V-filtered $\mathcal{B}$-category is in particular weakly U-filtered. One therefore obtains an inclusion $\mathsf{wFilt}_{\mathsf{V}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$.

Example 4.1.2.4. By Quillen's theorem A for $\mathcal{B}$-categories [62, Corollary 4.4.8], every functor that admits a left adjoint is final. Consequently, every $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category I is in particular weakly U-filtered.

Proposition 4.1.2.5. *A $\mathcal{B}$-category J is weakly U-filtered if and only if for every $A \in \mathcal{B}$ and every $\mathsf{I} \in \mathsf{U}(A)$ the colimit functor* $\mathrm{colim}\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{J}, \Omega_{\mathcal{B}_{/A}}) \to \Omega_{\mathcal{B}_{/A}}$ *preserves I-indexed limits of corepresentables.*

Proof. To begin with, note that since the colimit of a diagram $f\colon \mathsf{J} \to \Omega$ is given by the groupoidification of the associated left fibration $\mathsf{J}_{f/}$ by Proposition 3.1.4.1, the colimit of every corepresentable is given by the final object in Ω. In other words, there is a commutative square

$$\begin{tikzcd} \mathsf{J}^{\mathrm{op}} \arrow[r, hook, "h_{\mathsf{J}^{\mathrm{op}}}"] \arrow[d, "\pi_{\mathsf{J}^{\mathrm{op}}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega) \arrow[d, "\mathrm{colim}"] \\ 1 \arrow[r, hook, "1_\Omega"] & \Omega. \end{tikzcd}$$

As a result, for any diagram $d\colon \mathsf{I} \to \mathsf{J}^{\mathrm{op}}$, the presheaf $\mathrm{colim}\, h_{\mathsf{J}^{\mathrm{op}}} d$ is equivalent to the constant functor $1_\Omega \pi_{\mathsf{J}^{\mathrm{op}}} \colon \mathsf{J}^{\mathrm{op}} \to \Omega$. As the inclusion $1_\Omega \hookrightarrow \Omega$ admits a left adjoint (see Example 3.1.1.11) and is therefore continuous by Proposition 3.2.2.5, we conclude that the limit $\lim(\mathrm{colim}\, h_{\mathsf{J}^{\mathrm{op}}} d)$ is given by the final object in Ω. Hence the canonical map

$$\mathrm{colim}(\lim h_{\mathsf{J}^{\mathrm{op}}} d) \to \lim(\mathrm{colim}\, h_{\mathsf{J}^{\mathrm{op}}} d)$$

is an equivalence if and only if the domain of this map is the final object as well. On account of the chain of equivalences

$$\begin{aligned} \lim h_{\mathsf{J}^{\mathrm{op}}} d &\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\Omega)}(h_{\mathsf{J}^{\mathrm{op}}}(-), \lim h_{\mathsf{J}^{\mathrm{op}}} d) \\ &\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J},\Omega))}(\mathrm{diag}\, h_{\mathsf{J}^{\mathrm{op}}}(-), h_{\mathsf{J}^{\mathrm{op}}} d) \\ &\simeq \mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{J}^{\mathrm{op}})}(\mathrm{diag}(-), d), \end{aligned}$$

the functor $\lim h_{\mathsf{J}^{\mathrm{op}}} d$ classifies the left fibration $\mathsf{J}_{d^{\mathrm{op}}/} \to \mathsf{J}$. Hence $\mathrm{colim}(\lim h_{\mathsf{J}^{\mathrm{op}}} d)$ is the final object if and only if $(\mathsf{J}_{d^{\mathrm{op}}/})^{\mathrm{gpd}} \simeq 1$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, the same argumentation goes through for any $\mathsf{I} \in \mathsf{U}(A)$ and any diagram $d\colon \mathsf{I} \to \pi_A^* \mathsf{J}^{\mathrm{op}}$. By Quillen's theorem A for $\mathcal{B}$-categories [62, Corollary 4.4.8], the result thus follows. □

Corollary 4.1.2.6. *For every internal class U of $\mathcal{B}$-categories, any U-filtered $\mathcal{B}$-category is weakly U-filtered. In other words, there is an inclusion $\mathsf{Filt}_{\mathsf{U}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$ of internal classes.* □

Following the terminology introduced in [1], we may now make the following definition:

Definition 4.1.2.7. An internal class U of $\mathcal{B}$-categories is said to be *sound* if the inclusion $\mathsf{wFilt}_{\mathsf{U}} \hookrightarrow \mathsf{Filt}_{\mathsf{U}}$ is an equivalence. It is called *weakly sound* if for every $A \in \mathcal{B}$, every $\mathrm{op}(\pi_A^*\mathsf{U})$-cocomplete $\mathcal{B}_{/A}$-category is $\pi_A^*\mathsf{U}$-filtered.

Remark 4.1.2.8. On account of Remark 4.1.1.2 and Remark 4.1.2.2, the étale base change of a (weakly) sound internal class is (weakly) sound as well.

We finish this section with another characterisation of weakly U-filtered $\mathcal{B}$-categories that will be useful later. Recall from Definition 3.3.2.1 that if C is an arbitrary $\mathcal{B}$-category and V is an arbitrary internal class of $\mathcal{B}$-categories, we denote by $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{V}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ the full subcategory that is spanned by those objects $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ for which the domain of the associated right fibration $\mathsf{C}_{/F}$ is contained in $\mathsf{V}^{\mathrm{colim}}(A)$ (where $\mathsf{V}^{\mathrm{colim}}$ is the smallest colimit class containing V, see Definition 3.2.3.5). We now obtain:

PROPOSITION 4.1.2.9. *A $\mathcal{B}$-category J is weakly U-filtered if and only if the inclusion*

$$\mathsf{J} \hookrightarrow \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$$

induced by the Yoneda embedding is final.

PROOF. By Quillen's theorem A for $\mathcal{B}$-categories [62, Corollary 4.4.8], the inclusion $\mathsf{J} \hookrightarrow \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ is final if and only if for every $A \in \mathcal{B}$ and every $\mathrm{op}(\mathsf{U})$-small presheaf $F\colon A \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ the groupoidification of the $\mathcal{B}_{/A}$-category $\mathsf{J}_{F/}$ is the final object in $\mathcal{B}_{/A}$. By the same reasoning, J is weakly U-filtered if and only if for every $\mathsf{I} \in \mathsf{U}(A)$ and every diagram $d\colon \mathsf{I}^{\mathrm{op}} \to \pi_A^* \mathsf{J}$ the groupoidification of the $\mathcal{B}_{/A}$-category $\pi_A^*\mathsf{J}_{d/}$ is final in $\mathcal{B}_{/A}$. Hence it suffices to show that for every such diagram d, there is an object $F\colon A \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ such that $(\pi_A^*\mathsf{J})_{d/} \simeq \mathsf{J}_{F/}$, and vice versa. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and by using Remark 3.3.2.2, we may assume that $A \simeq 1$. Now by Proposition 3.3.2.6, the colimit of $h_{\mathsf{J}} d\colon \mathsf{I} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})$ is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ and therefore defines a U-small presheaf F. By construction, we have an equivalence $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{J})_{F/} \simeq \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{J})_{h_{\mathsf{J}} d/}$ whose pullback along the Yoneda embedding determines an equivalence $\mathsf{J}_{d/} \simeq \mathsf{J}_{F/}$. Hence, if $\mathsf{J}_{F/}^{\mathrm{gpd}}$ is final, so is $\mathsf{J}_{d/}^{\mathrm{gpd}}$. Conversely, if we are given an arbitrary U-small presheaf F, the fact that $\mathsf{J}_{F/}^{\mathrm{gpd}}$ being final is *local* in $\mathcal{B}$ implies (by definition of what it means for a presheaf to be U-small) that we may safely assume that there is a diagram $d\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{J}$ with $\mathsf{I} \in \mathsf{U}(1)$ such that $F \simeq \operatorname{colim} h_{\mathsf{J}} d$. By the same argument as above, we thus conclude that if $\mathsf{J}_{d/}^{\mathrm{gpd}}$ is final, so is $\mathsf{J}_{F/}$, which finishes the proof. $\square$

4.1.3. Regular classes. Recall that a cardinal κ is said to be *regular* if it is infinite and if any κ-small union of κ-small sets is still κ-small. In this section, we will study an analogue of this condition in the context of internal classes of $\mathcal{B}$-categories. To that end, recall from the discussion in [61, § 6.1] that the Yoneda embedding $\Delta \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$ (where Δ is implicitly regarded as a constant $\mathcal{B}$-category) factors through the embedding $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta)$, so that we may regard Δ as an internal class of $\mathcal{B}$-categories. We may now define:

DEFINITION 4.1.3.1. An internal class U is said to be *right regular* if U contains Δ and if U is closed under U-colimits in $\mathsf{Cat}_{\mathcal{B}}$. We define the *right regularisation* $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$ of U to be the smallest right regular class that contains U.

Dually, U is called *left regular* if it contains Δ and is closed under $\mathrm{op}(\mathsf{U})$-colimits in $\mathsf{Cat}_{\mathcal{B}}$, and we define the *left regularisation* $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$ as the smallest left regular class that contains U.

Finally, we say that U is *regular* if it is both left and right regular, and we define the *regularisation* $\mathsf{U}^{\mathrm{reg}}$ of U as the smallest regular class that contains U.

REMARK 4.1.3.2. An internal class U of $\mathcal{B}$-categories is left regular if and only if $\mathrm{op}(\mathsf{U})$ is right regular, and there is an evident equivalence $\mathrm{op}(\mathsf{U}^{\mathrm{reg}}_{\leftarrow}) \simeq \mathrm{op}(\mathsf{U})^{\mathrm{reg}}_{\rightarrow}$ of internal classes. In particular, if we have an equivalence $\mathsf{U} \simeq \mathrm{op}(\mathsf{U})$ of internal classes, then the notions of left and right regularity collapse to the notion of regularity, and the left/right regularisation of U is already its regularisation (cf. Corollary 4.1.3.5 below).

REMARK 4.1.3.3. By the same argument as in the proof of Proposition 3.4.1.11, there is an equivalence $\pi_A^*(\mathsf{U}^{\mathrm{reg}}_{\rightarrow}) \simeq (\pi_A^*\mathsf{U})^{\mathrm{reg}}_{\rightarrow}$ for any internal class U and any $A \in \mathcal{B}$. In particular, the étale base change of a right regular class is still right regular. Similar observations can be made for the (left) regularisation of U.

PROPOSITION 4.1.3.4. *For every internal class U of $\mathcal{B}$-categories, a $\mathcal{B}$-category is U-cocomplete if and only if it is $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$-cocomplete, and a functor between $\mathcal{B}$-categories is U-cocontinuous if and only if it is $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$-cocontinuous.*

Dually, a $\mathcal{B}$-category is U-complete if and only if it is $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-complete, and a functor between $\mathcal{B}$-categories is U-complete if and only if it is $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-continuous.

PROOF. We only prove the first statement, the second one follows by dualisation. So let C be a U-cocomplete $\mathcal{B}$-category, and let V be the largest internal class of $\mathcal{B}$-categories subject to the condition that C is V-cocomplete. Clearly V contains Δ since every $\mathcal{B}$-category is Δ-cocomplete. Moreover, Proposition 3.4.4.1 implies that for any $\mathsf{I} \in \mathsf{V}(1)$ and any diagram $d\colon \mathsf{I} \to \mathsf{V}$ with colimit K, the $\mathcal{B}$-category C admits K-indexed colimits, which implies that $\mathsf{K} \in \mathsf{V}(1)$ by maximality of V. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the same argument, one concludes that V is closed under V-colimits in $\mathsf{Cat}_{\mathcal{B}}$ and must therefore contain $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$. An analogous argument also shows that every U-cocontinuous functor is $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$-cocontinuous. □

Corollary 4.1.3.5. *The right (left) regularisation of an internal class* U *is the smallest internal class that contains* U *and* Δ *and that is closed under* U*-colimits (*$\mathrm{op}(\mathsf{U})$*-colimits) in* $\mathsf{Cat}_{\mathcal{B}}$*.*

Proof. This is an immediate consequence of the observation that by Proposition 4.1.3.4, an internal class V of $\mathcal{B}$-categories is closed under U-colimits ($\mathrm{op}(\mathsf{U})$-colimits) in $\mathsf{Cat}_{\mathcal{B}}$ if and only if it is closed under $\mathsf{U}^{\mathrm{reg}}_{\rightarrow}$-colimits ($\mathrm{op}(\mathsf{U}^{\mathrm{reg}}_{\leftarrow})$-colimits) in $\mathsf{Cat}_{\mathcal{B}}$. □

Proposition 4.1.3.6. *For every internal class* U*, the inclusion* $\mathsf{Filt}_{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}} \hookrightarrow \mathsf{Filt}_{\mathsf{U}}$ *is an equivalence.*

Proof. In light of Remark 4.1.3.3 and Remark 4.1.1.2, it suffices to show that every U-filtered $\mathcal{B}$-category J is already $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-filtered. This amounts to showing that the functor $\mathrm{colim}_{\mathsf{J}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \to \Omega$ is $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$-continuous. By Proposition 4.1.3.4, this is immediate. □

Corollary 4.1.3.7. *The left regularisation of a (weakly) sound internal class is also (weakly) sound.*

Proof. Suppose that U is sound, i.e. that $\mathsf{Filt}_{\mathsf{U}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$ is an equivalence. Since $\mathsf{U} \hookrightarrow \mathsf{U}^{\mathrm{reg}}_{\leftarrow}$ implies that we have an inclusion $\mathsf{wFilt}_{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}}$, Proposition 4.1.3.6 implies that the inclusion $\mathsf{Filt}_{\mathsf{U}^{\mathrm{reg}}_{\leftarrow}} \hookrightarrow \mathsf{wFilt}_{\mathsf{U}^{\mathrm{reg}}}$ is also an equivalence, hence $\mathsf{U}^{\mathrm{reg}}_{\leftarrow}$ is sound. The case where U is weakly sound follows from a similar argument. □

For the study of accessibility and presentability of $\mathcal{B}$-categories, we will generally need to restrict our attention to those internal classes of $\mathcal{B}$-categories that are themselves *small* $\mathcal{B}$-categories. It will therefore be useful to give such internal classes a dedicated name. Again following [1], we thus define:

Definition 4.1.3.8. An internal class U of $\mathcal{B}$-categories is a *doctrine* if U is a small $\mathcal{B}$-category.

Proposition 4.1.3.9. *The (left/right) regularisation of a doctrine is still a doctrine.*

Proof. It suffices to show that any doctrine U is contained in a regular doctrine V. We will explicitly construct such a doctrine in § 4.2.2 below, cf. Remark 4.2.2.22. □

4.1.4. The decomposition property. It is well-known that for every regular cardinal κ, any ∞-category can be written as a κ-filtered colimit of κ-small ∞-categories. In order to obtain a well-behaved notion of accessibility internal to $\mathcal{B}$, it will be crucial to have an analogue of this property for $\mathcal{B}$-categories. This leads us to the following definition:

Definition 4.1.4.1. An internal class U of $\mathcal{B}$-categories is said to have the *decomposition property* if for every $A \in \mathcal{B}$ and every $\mathcal{B}_{/A}$-category C, there is a $\pi_A^*\mathsf{U}$-filtered $\mathcal{B}_{/A}$-category J and a diagram $d\colon \mathsf{J} \to \pi_A^*\mathsf{U}$ with colimit C.

Remark 4.1.4.2. In the situation of Definition 4.1.4.1, by applying the decomposition property to $\mathsf{D} = \mathsf{C}^{\mathrm{op}}$, one deduces that C can also be obtained as a $\pi_A^*\mathsf{U}$-filtered colimit of a diagram in $\mathrm{op}(\pi_A^*\mathsf{U})$.

The main goal of this section is to show:

Proposition 4.1.4.3. *Every left regular and weakly sound internal class* U *has the decomposition property.*

Before we can prove Proposition 4.1.4.3, we need a few preparations.

Lemma 4.1.4.4. *Let C be a small $\mathcal{B}$-category and let $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be a full subcategory that contains C. Then any presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is the colimit of the diagram $\mathsf{D}_{/F} \to \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.*

Proof. By Proposition 3.1.6.3, it suffices to show that the final object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ is the colimit of the inclusion $\mathsf{D}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$. In light of the inclusions $\mathsf{C}_{/F} \hookrightarrow \mathsf{D}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ and by making use of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})$ from Lemma 3.3.1.5, we may thus assume that F is the final object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Moreover, since the inclusion $\Omega_{\mathcal{B}} \hookrightarrow \Omega_{\widehat{\mathcal{B}}}$ is cocontinuous by Example 3.2.2.8, we may enlarge our universe and thus assume without loss of generality that D is *small*. Now let $i\colon \mathsf{C} \hookrightarrow \mathsf{D}$ and $j\colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the inclusions. Since the identity on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is the left Kan extension of the Yoneda embedding h along itself by Theorem 3.4.1.1, we obtain equivalences $j \simeq j^* j_! i_!(h) \simeq i_!(h)$, where the functor $i_!$ exists by Corollary 3.3.3.7 since D is small. Therefore, the identity on $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is also the left Kan extension of j along itself. The claim now follows from the explicit description of the left Kan extension in Remark 3.3.3.6, which implies that we have equivalences $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})} \simeq j_!(j)(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}) \simeq \operatorname{colim} j$. □

Lemma 4.1.4.5. *Let U be an internal class of $\mathcal{B}$-categories and let*

$$\begin{tikzcd} \mathsf{Q} \arrow[r, "g"] \arrow[d, "q"] & \mathsf{P} \arrow[d, "p"] \\ \mathsf{D} \arrow[r, "f"] & \mathsf{C} \end{tikzcd}$$

be a pullback square in $\mathrm{Cat}(\mathcal{B})$ in which D, C and P are U-cocomplete and both f and p are U-cocontinuous. Then Q is U-cocomplete and both q and g are U-cocontinuous.

Proof. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ for $A \in \mathcal{B}$ if necessary and using Remarks 3.1.1.3 and 3.1.2.2, it will suffice to prove that any diagram $d\colon \mathsf{K} \to \mathsf{Q}$ with $\mathsf{K} \in \mathsf{U}(1)$ admits a colimit in Q and that furthermore q preserves this colimit. The pullback square in the statement of the lemma induces a commutative diagram

$$\begin{tikzcd} & 1 \arrow[rr] \arrow[dl, "\overline{d}"'] \arrow[dd] & & 1 \arrow[dl, "\overline{gd}"'] \arrow[dd] \\ \mathsf{Q}_{d/} \arrow[rr, "g_*"] \arrow[dd, "q_*"] & & \mathsf{P}_{gd/} \arrow[dd, "p_*"] & \\ & 1 \arrow[rr] \arrow[dl, "\overline{qd}"'] & & 1 \arrow[dl, "\overline{pgd}"'] \\ \mathsf{D}_{qd/} \arrow[rr, "f_*"] & & \mathsf{C}_{pgd/} & \end{tikzcd}$$

in which the front square is a pullback, the three cocones $\overline{qd}$, $\overline{gd}$ and $\overline{pgd}$ are colimit cocones and the cocone $\overline{d}$ is determined by the universal property of pullbacks. To finish the proof, it suffices to show that $\overline{d}$ is a colimit cocone, i.e. initial. Given any $\overline{d}'\colon 1 \to \mathsf{Q}_{d/}$, we obtain a pullback square

$$\begin{tikzcd} \operatorname{map}_{\mathsf{Q}_{d/}}(\overline{d}, \overline{d}') \arrow[r] \arrow[d] & \operatorname{map}_{\mathsf{P}_{gd/}}(\overline{gd}, g_*\overline{d}') \arrow[d] \\ \operatorname{map}_{\mathsf{D}_{fd/}}(\overline{qd}, q_*\overline{d}') \arrow[r] & \operatorname{map}_{\mathsf{C}_{pgd/}}(\overline{pgd}, p_*g_*\overline{d}') \end{tikzcd}$$

in $\mathcal{B}$. Since $\overline{qd}$, $\overline{gd}$ and $\overline{pgd}$ are initial, it follows that the cospan in the lower right corner is constant on the final object $1 \in \mathcal{B}$, hence $\operatorname{map}_{\mathsf{Q}_{d/}}(\overline{d}, \overline{d}') \simeq 1$. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and $\overline{d}$ with $\pi_A^*(\overline{d})$, the same is true for any object $\overline{d}'\colon A \to \mathsf{Q}_{d/}$. As a consequence, $\overline{d}$ must be initial. □

Proof of Proposition 4.1.4.3. By Remark 4.1.3.3 and Remark 4.1.2.8, it suffices to show that every $\mathcal{B}$-category C is a U-filtered colimit of a diagram in U. As U by regularity contains Δ and since the localisation functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta) \to \mathsf{Cat}_{\mathcal{B}}$ is cocontinuous, we deduce from Lemma 4.1.4.4 that C arises as the colimit of the diagram $\mathsf{U}_{/\mathsf{C}} \to \mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$. We therefore only need to show that $\mathsf{U}_{/\mathsf{C}}$ is U-filtered. Using that U is weakly sound, it will suffice to show that $\mathsf{U}_{/\mathsf{C}}$ is $\mathrm{op}(\mathsf{U})$-cocomplete. By Corollary 3.2.2.11, the

$\mathcal{B}$-category $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ is cocomplete and the projection $(\pi_{\mathsf{C}})_!$ is cocontinuous. As the inclusion $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ is closed under $\mathrm{op}(\mathsf{U})$-colimits, the desired result follows from Lemma 4.1.4.5. □

COROLLARY 4.1.4.6. *Let* U *be a weakly sound internal class of* $\mathcal{B}$*-categories. Then a (large)* $\mathcal{B}$*-category* C *is cocomplete if and only if* C *is both* $\mathrm{op}(\mathsf{U})$*- and* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete. Similarly, a functor* $f\colon \mathsf{C} \to \mathsf{D}$ *between cocomplete (large)* $\mathcal{B}$*-categories is cocontinuous if and only if it is both* $\mathrm{op}(\mathsf{U})$*- and* $\mathsf{Filt}_{\mathsf{U}}$*-cocontinuous.*

PROOF. We prove the first statement, the second one follows by a similar argument. Since the claim is clearly necessary, it suffices to prove the converse. So let us assume that C is both $\mathrm{op}(\mathsf{U})$- and $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. By Proposition 4.1.3.4 and Proposition 4.1.3.6, we may assume without loss of generality that U is left regular. Proposition 4.1.4.3 now implies that U has the decomposition property. By definition and in light of Remark 4.1.4.2, this means that $(\mathrm{op}(\mathsf{U}) \cup \mathsf{Filt}_{\mathsf{U}})^{\mathrm{reg}}_{\to} = \mathsf{Cat}_{\mathcal{B}}$. Appealing once more to Proposition 4.1.3.4, the claim follows. □

4.1.5. U-compact objects. Recall from § 3.4.2 that if V is an internal class and if C is a V-cocomplete $\mathcal{B}$-category, we say that an object $c\colon A \to \mathsf{C}$ is V*-cocontinuous* if the functor $\mathrm{map}_{\mathsf{C}}(c, -)\colon \pi_A^*\mathsf{C} \to \Omega_{\mathcal{B}_{/A}}$ is $\pi_A^*\mathsf{V}$-cocontinuous. In this section, we specialise this concept to the case where $\mathsf{V} = \mathsf{Filt}_{\mathsf{U}}$ for some internal class U. This leads us to the notion of a U*-compact object*, which is the internal analogue of the concept of a κ-compact object in an ∞-category, where κ is a cardinal.

DEFINITION 4.1.5.1. Let U be an internal class of $\mathcal{B}$-categories, and let C be a $\mathsf{Filt}_{\mathsf{U}}$-cocomplete $\mathcal{B}$-category. An object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$ is said to be U*-compact* if it is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous, i.e. if the functor $\mathrm{map}_{\mathsf{C}}(c, -)\colon \pi_A^*\mathsf{C} \to \Omega_{\mathcal{B}_{/A}}$ is $\mathsf{Filt}_{\pi_A^*\mathsf{U}}$-cocontinuous. We denote by $\mathsf{C}^{\mathsf{U}\text{-cpt}}$ the full subcategory of C that is spanned by the U-compact objects.

REMARK 4.1.5.2. In the situation of Definition 4.1.5.1, an object $c\colon A \to \mathsf{C}$ is contained in $\mathsf{C}^{\mathsf{U}\text{-cpt}}$ if and only if it is U-compact. This is a direct consequence of Remark 3.4.2.2. Together with Remark 4.1.1.2, this implies that if $A \in \mathcal{B}$ is an arbitrary object in $\mathcal{B}$, there is a natural equivalence $\pi_A^*(\mathsf{C}^{\mathsf{U}\text{-cpt}}) \simeq (\pi_A^*\mathsf{C})^{\pi_A^*\mathsf{U}\text{-cpt}}$.

LEMMA 4.1.5.3. *Let* U *be an internal class of* $\mathcal{B}$*-categories, and let* C *be a* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete* $\mathcal{B}$*-category. Then the full subcategory* $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}\text{-cc}}(\mathsf{C}, \Omega) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ *of* $\mathsf{Filt}_{\mathsf{U}}$*-cocontinuous functors is closed under* U*-limits.*

PROOF. Using Remarks 3.2.3.4 and 4.1.1.2, it will suffice to show that whenever I is a $\mathcal{B}$-category that is contained in $\mathsf{U}(1)$ and $d\colon \mathsf{I} \to \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}\text{-cc}}(\mathsf{C}, \Omega)$ is a diagram, then the limit $\lim d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous. We may compute $\lim d$ as the composition

$$\mathsf{C} \xrightarrow{d'} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \xrightarrow{\lim_{\mathsf{I}}} \Omega,$$

where d' is the transpose of d. Since $\lim_{\mathsf{I}}$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous by Remark 4.1.1.4, it thus suffices to show that d' is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous as well. This can be extracted as a special case of Lemma 4.6.1.3 below. □

PROPOSITION 4.1.5.4. *Let* U *be an internal class and let* C *be an* $\mathrm{op}(\mathsf{U})$*- and* $\mathsf{Filt}_{\mathsf{U}}$*-cocomplete* $\mathcal{B}$*-category. Then the subcategory* $\mathsf{C}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{C}$ *is closed under* $\mathrm{op}(\mathsf{U})$*-colimits in* C*.*

PROOF. By using Remark 4.1.5.2, it suffices to show that whenever I is a $\mathcal{B}$-category that is contained in $\mathsf{U}(1)$ and $d\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{C}^{\mathsf{U}\text{-cpt}}$ is a diagram, the colimit $\mathrm{colim}\, d$ in C is U-compact. As we have noted in Proposition 3.2.2.9, the Yoneda embedding $h_{\mathsf{C}^{\mathrm{op}}}\colon \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ is $\mathrm{op}(\mathsf{U})$-continuous, so that we can identify $\mathrm{map}_{\mathsf{C}}(\mathrm{colim}\, d, -)$ with the limit of the diagram $h_{\mathsf{C}^{\mathrm{op}}} d^{\mathrm{op}}$. The desired result now follows from Lemma 4.1.5.3. □

DEFINITION 4.1.5.5. If $\mathsf{C} \hookrightarrow \mathsf{D}$ is a fully faithful functor of $\mathcal{B}$-categories, the $\mathcal{B}$-category $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$ of *retracts* of C in D is the full subcategory of D that is spanned by those objects $d\colon A \to \mathsf{D}$ in context

$A \in \mathcal{B}$ for which there is an object $c\colon A \to \mathsf{C}$ and a commutative diagram

$$\begin{tikzcd} & c \arrow[dr] & \\ d \arrow[ur] \arrow[rr, "\mathrm{id}"] & & d. \end{tikzcd}$$

REMARK 4.1.5.6. In the situation of Definition 4.1.5.5, there are inclusions $\mathsf{C} \hookrightarrow \mathsf{Ret}_{\mathsf{D}}(\mathsf{C}) \hookrightarrow \mathsf{D}$. Furthermore, an object $d\colon A \to \mathsf{D}$ is contained in $\mathsf{Ret}_{\mathsf{C}}(\mathsf{D})$ precisely if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $s_i^*(d)\colon A_i \to \mathsf{D}$ is a retract of an object $c\colon A_i \to \mathsf{C}$. Therefore, if C is small and D is locally small (in the sense of [62, Definition 4.7.1]), then $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$ is small as well: in fact, by [62, Proposition 4.7.4], this follows once we verify that $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})_0$ is small. Since the latter admits a small cover

$$\bigsqcup_{G \in \mathcal{G}} \bigsqcup_{d \in \mathrm{Ret}_{\mathsf{D}(G)}(\mathsf{C}(G))} G \twoheadrightarrow \mathsf{Ret}_{\mathsf{D}}(\mathsf{C})_0$$

where $\mathcal{G} \subset \mathcal{B}$ is a small generating subcategory and where $\mathrm{Ret}_{\mathsf{D}(G)}(\mathsf{C}(G))$ denotes the full subcategory of $\mathsf{D}(G)$ that is spanned by the retracts of $\mathsf{C}(G)$, which is clearly a small ∞-category, this is immediate.

LEMMA 4.1.5.7. *Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category. Then the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C}, \Omega) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ of U-cocontinuous functors is closed under retracts.*

PROOF. By Remark 3.2.3.4, it will suffice to show that whenever a copresheaf $F\colon \mathsf{C} \to \Omega$ is a retract of a U-cocontinuous functor $G\colon \mathsf{C} \to \Omega$, then F is U-cocontinuous as well. Let $R = \Delta^2 \sqcup_{\Delta^1} \Delta^0$ be the walking retract diagram, i.e. the quotient of Δ^2 that is obtained by collapsing $d^1\colon \Delta^1 \hookrightarrow \Delta^2$ to a point. Then the datum of retract $F \to G \to F$ is tantamount to a map $r\colon \mathsf{C} \to \Omega^R$. Since the retract of an equivalence is an equivalence as well, the functor $d_{\{1\}}\colon \Omega^R \to \Omega$ that is obtained by evaluation at $\{1\} \in \Delta^2 \twoheadrightarrow R$ must be conservative. By combining this observation with the fact that $d_{\{1\}}$ is cocontinuous, the equivalence $d_{\{1\}} r \simeq G$ and the functoriality of mates, we conclude that the map $\operatorname{colim}_{\mathsf{I}} r_* \to r \operatorname{colim}_{\mathsf{I}}$ is an equivalence for every $\mathsf{I} \in \mathsf{U}(1)$. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the same argument, we thus find that r is U-cocontinuous. As we recover F by postcomposing r with the cocontinuous functor $d_{\{0\}}\colon \Omega^R \to \Omega$, the claim follows. $\square$

PROPOSITION 4.1.5.8. *Let U be an internal class of $\mathcal{B}$-categories and let C be a $\mathsf{Filt}_{\mathsf{U}}$-cocomplete $\mathcal{B}$-category. Then $\mathsf{C}^{\mathsf{U}\text{-cpt}}$ is closed under retracts in C, in the sense that the inclusion $\mathsf{C}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{Ret}_{\mathsf{C}}(\mathsf{C}^{\mathsf{U}\text{-cpt}})$ is an equivalence.*

PROOF. It suffices to show that the retract of a U-compact object in C is U-compact as well, which immediately follows from Lemma 4.1.5.7. $\square$

We conclude this section with a characterisation of U-compact objects in presheaf $\mathcal{B}$-categories. This will require the following lemma:

LEMMA 4.1.5.9. *Let U be an internal class of $\mathcal{B}$-categories and let $\mathsf{C} \hookrightarrow \mathsf{D}$ be a full inclusion of $\mathcal{B}$-categories such that D is $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. Let J be a U-filtered $\mathcal{B}$-category, let $d\colon \mathsf{J} \to \mathsf{C} \hookrightarrow \mathsf{D}$ be a diagram and suppose that $F = \operatorname{colim} d$ is a U-compact object in D. Then F is contained in $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$.*

PROOF. The object F being U-compact implies that the canonical map

$$\varphi\colon \operatorname{colim} \operatorname{map}_{\mathsf{D}}(F, d(-)) \to \operatorname{map}_{\mathsf{D}}(F, F)$$

must be an equivalence. Thus the identity on F gives rise to a global section

$$\mathrm{id}_F\colon 1 \to \operatorname{colim} \operatorname{map}_{\mathsf{D}}(F, d(-)).$$

Let $p\colon \mathsf{P} \to \mathsf{J}$ be the left fibration that is classified by the copresheaf $\operatorname{map}_{\mathsf{D}}(F, d(-))$. Since the map $\mathsf{P} \to \mathsf{P}^{\mathrm{gpd}} \simeq \operatorname{colim} \operatorname{map}_{\mathsf{D}}(F, d(-))$ is essentially surjective (by [62, Lemma 3.8.8]), the map $\mathsf{P}_0 \to \mathsf{P}^{\mathrm{gpd}}$ is a cover in $\mathcal{B}$ [62, Corollary 3.9.5], so that we can find a cover $s\colon A \twoheadrightarrow 1$ in $\mathcal{B}$ and a local section $x\colon A \to \mathsf{P}$ such that the composite with $\mathsf{P} \to \mathsf{P}^{\mathrm{gpd}}$ recovers $\pi_A^* \mathrm{id}_F$. Let $j = p(x)$. Then x defines an object

$f\colon A \to \mathsf{P}|_j \simeq \mathrm{map}_{\mathsf{D}}(\pi_A^* F, d(j))$ that is carried to $\pi_A^* \mathrm{id}_F$ by the canonical morphism $\mathrm{map}_{\mathsf{D}}(\pi_A^* F, d(j)) \to \mathrm{map}_{\mathsf{D}}(\pi_A^* F, \pi_A^* D)$. In other words, composing $f\colon \pi_A^* F \to d(j)$ with the map $d(j) \to \pi_A^* F$ into the colimit yields $\pi_A^* F$. As this precisely means that $\pi_A^* F$ is a retract of $d(j)$, the claim follows. $\square$

PROPOSITION 4.1.5.10. *Let* U *be an internal class of* $\mathcal{B}$*-categories that has the decomposition property, and let* C *be a* $\mathcal{B}$*-category. Then there is an equivalence*

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}} \simeq \mathsf{Ret}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}))$$

of full subcategories in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*. In particular,* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ *is small.*

PROOF. Yoneda's lemma implies that every representable presheaf is U-compact. By combining this observation with Proposition 4.1.5.8 and Proposition 4.1.5.4, one thus obtains an inclusion

$$\mathsf{Ret}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}.$$

As for the converse inclusion, suppose that $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is a U-compact presheaf. By Remark 4.1.4.2, there exists a U-filtered $\mathcal{B}$-category J and a diagram $d\colon \mathsf{J} \to \mathrm{op}(\mathsf{U})$ such that $\mathsf{C}_{/F} \simeq \operatorname{colim} d$ in $\mathsf{Cat}_{\mathcal{B}}$. Proposition 3.4.4.3 then shows that F is the colimit of a J-indexed diagram in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})$. As F is U-compact and J is U-filtered, Lemma 4.1.5.9 shows that F is locally a retract of an object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})$. By Remarks 4.1.5.2 and 3.3.2.2, if $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is an arbitrary object, we can replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and carry out the same argument as above, which shows that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is contained in $\mathsf{Ret}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}))$. $\square$

COROLLARY 4.1.5.11. *Let* U *be an internal class of* $\mathcal{B}$*-categories that has the decomposition property, and let* J *be a weakly* U*-filtered* $\mathcal{B}$*-category. Then the inclusion* $\mathsf{J} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ *is final.*

PROOF. Proposition 4.1.5.10 shows that any U-compact presheaf $F\colon \mathsf{J}^{\mathrm{op}} \to \Omega$ arises as a retract of some object $G\colon 1 \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ after passing to a suitable cover of $1 \in \mathcal{B}$. Thus, the right fibration $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})_{/F} \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})$ is *locally* a retract of a representable right fibration, so that we must have $(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J})_{/F})^{\mathrm{gpd}} \simeq 1$ as the latter property can be checked locally in $\mathcal{B}$. By Remarks 4.1.5.2 and 3.3.2.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ to arrive at the same conclusion for any object $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$. Using Quillen's theorem A [62, Corollary 4.4.8], this shows that the inclusion $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathrm{op}(\mathsf{U})}(\mathsf{J}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is final. Hence the claim follows from Proposition 4.1.2.9. $\square$

COROLLARY 4.1.5.12. *A left regular class* U *is sound if and only if it is weakly sound.*

PROOF. Using Remarks 4.1.1.2 and 4.1.2.8, it suffices to show that whenever U is weakly sound, every weakly U-filtered $\mathcal{B}$-category J is U-filtered. Since Proposition 4.1.4.3 implies that U has the decomposition property, Corollary 4.1.5.11 shows that the inclusion $\mathsf{J} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is final. By Proposition 4.1.5.4, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}}$ is $\mathrm{op}(\mathsf{U})$-cocomplete and therefore U-filtered since U is by assumption weakly sound. Now if $\mathsf{I} \in \mathsf{U}(1)$ is chosen arbitrarily, the fact that $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ is cocomplete allows us to extend any diagram $d\colon \mathsf{J} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$ to a diagram $d'\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}\text{-cpt}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$, using the universal property of presheaf $\mathcal{B}$-categories. As the inclusion $\mathsf{J} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{U}}$-cpt is final, the limit functor $\lim_{\mathsf{I}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \to \Omega$ preserves the colimit of d if and only if it preserves the colimit of d' (see Proposition 3.1.8.1), which is indeed the case as $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is U-filtered. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and carrying out the same argument (which is possible by Remark 4.1.5.2), this already implies that J must be U-filtered, as desired. $\square$

4.2. Cardinality in internal higher category theory

In our treatment of accessibility and presentability for $\mathcal{B}$-categories later in this paper, we will rely on the existence of an ample amount of doctrines that satisfy the decomposition property. Therefore, it will be crucial to know that there are sufficiently many *(left) regular and sound doctrines* in any ∞-topos $\mathcal{B}$. The main objective of this section is to construct such internal classes. More precisely, our approach is to first construct what we call the *canonical bifiltration* of the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$, i.e. a 2-dimensional filtration by internal classes which can be regarded as a way to order $\mathcal{B}$-categories by *size*. The first dimension of this bifiltration is parametrised by *cardinals*, and the second one by the poset of *local classes* in $\mathcal{B}$. The canonical bifiltration will be *exhaustive*, so that every $\mathcal{B}$-category can be assigned an upper bound in size, and it will be exclusively comprised of regular doctrines. We carry out the construction of this bifiltration in § 4.2.1. In § 4.2.2, we discuss how one can extract a particularly well-behaved subfiltration from the canonical bifiltration that is still exhaustive and in which each member is *sound*. The latter will be parametrised by a class of cardinals that satisfy a property which depends on the ∞-topos $\mathcal{B}$ and that we refer to as $\mathcal{B}$-*regularity*. Finally, we discuss a particular member of the canonical bifiltration in § 4.2.3, that of *finite* $\mathcal{B}$-categories.

4.2.1. The canonical bifiltration of the $\mathcal{B}$-category of $\mathcal{B}$-categories. Recall that when $\mathcal{K}$ is an arbitrary class of ∞-categories (i.e. a full subcategory of Cat_∞), we denote by $\mathsf{LConst}_{\mathcal{K}}$ the essential image of the canonical functor $\mathcal{K} \hookrightarrow \mathrm{Cat}_\infty \to \mathsf{Cat}_{\mathcal{B}}$ in which the second map is obtained as the transpose of $\mathrm{const}_{\mathcal{B}} \colon \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathcal{B}) = \Gamma\,\mathsf{Cat}_{\mathcal{B}}$. If S is a local class of morphisms in $\mathcal{B}$, we denote by $\langle \mathcal{K}, S\rangle$ the internal class of $\mathcal{B}$-categories that is generated by $\mathsf{LConst}_{\mathcal{K}}$ and Ω_S.

Definition 4.2.1.1. Let $\mathcal{K} \subset \mathrm{Cat}_\infty$ be a class of ∞-categories and let S be a local class of morphisms in $\mathcal{B}$. We define the internal class $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$ of $\langle \mathcal{K}, S\rangle$-*small* $\mathcal{B}$-*categories* as the left regularisation of $\langle \mathcal{K}, S\rangle$. We denote its underlying ∞-category of global sections by $\mathrm{Cat}(\mathcal{B})^{\langle \mathcal{K}, S\rangle}$.

Remark 4.2.1.2. In the situation of Definition 4.2.1.1, let us denote by $\pi_A^* S$ the class of those maps in $\mathcal{B}_{/A}$ whose underlying map in $\mathcal{B}$ is contained in S. Since $(\pi_A)_!$ preserves small colimits and covers, this is still a local class, and one has a natural equivalence $\pi_A^*\Omega_S \simeq \Omega_{\pi_A^* S}$ of subuniverses. With this understood, Remark 4.1.3.3 gives rise to a canonical equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^{\langle \mathcal{K}, \pi_A^* S\rangle}$ for every $A \in \mathcal{B}$.

By combining Proposition 4.1.3.4 with Example 3.2.4.11, one finds:

Proposition 4.2.1.3. *A (large) $\mathcal{B}$-category C is $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$-complete precisely if*

(1) The ∞-category $\mathsf{C}(A)$ admits limits indexed by objects in $\mathcal{K}$, and for every map $s \colon B \to A$ the transition functor $s^* \colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves these limits;

(2) For every map $p \colon P \to A$ in S, the functor s^* admits a right adjoint $s_* \colon \mathsf{C}(P) \to \mathsf{C}(A)$ and for every cartesian square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"] & P \arrow[d, "p"] \\ B \arrow[r, "s"] & A \end{tikzcd}$$

in $\mathcal{B}$ in which p (and therefore q) are contained in S, the natural map $s^*p_* \to q_*t^*$ is an equivalence.

Moreover, a functor $f \colon \mathsf{C} \to \mathsf{D}$ of $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$-complete $\mathcal{B}$-categories is $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$-continuous precisely if for all $A \in \mathcal{B}$ the functor $f(A)$ preserves limits indexed by objects in $\mathcal{K}$, and for all maps $p \colon P \to A$ in S the natural morphism $f(A)p_* \to p_* f(P)$ is an equivalence.

The dual statements about cocompleteness and cocontinuity (both understood with respect to the right regular class $\mathrm{op}(\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle})$) hold as well. $\square$

In the situation of Definition 4.2.1.1, note that whenever $\mathcal{K}$ is a doctrine (i.e. a small ∞-category) and S is bounded (i.e. the subuniverse Ω_S that corresponds to S is small), Proposition 4.1.3.9 implies that $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$ is a doctrine. Therefore, assigning to a pair $(\mathcal{K}, S)$ the regular class $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$ defines a map of posets

$$\mathrm{Sub}^{\mathrm{small}}_{\mathrm{full}}(\mathrm{Cat}_\infty) \times \mathrm{Sub}^{\mathrm{small}}_{\mathrm{full}}(\Omega) \to \mathrm{Sub}^{\mathrm{small}}_{\mathrm{full}}(\mathsf{Cat}_{\mathcal{B}})$$

that we refer to as the *canonical bifiltration of* $\mathsf{Cat}_{\mathcal{B}}$.

REMARK 4.2.1.4. The canonical bifiltration is *exhaustive*. In fact, if C is an arbitrary $\mathcal{B}_{/A}$-category, we may find a small ∞-category $\mathcal{J}$ and a diagram $d\colon \mathcal{J} \to \mathrm{Cat}(\mathcal{B}_{/A})$ with colimit C such that for all $j \in \mathcal{J}$ one has $d(j) \simeq \Delta^n \otimes B$ for some $n \geq 0$ and some $B \in \mathcal{B}_{/A}$. Note that $\Delta^n \otimes B$ can be identified with the B-indexed colimit of the constant diagram in $\mathsf{Cat}_{\mathcal{B}_{/A}}$ with value Δ^n. Therefore, by choosing $\mathcal{K}$ to be the doctrine of ∞-categories spanned by the single object $\mathcal{J} \in \mathrm{Cat}_\infty$ and choosing S to be the bounded local class that is generated by the maps $B \to A$, we find that C is $\langle \mathcal{K}, S\rangle$-small.

EXAMPLE 4.2.1.5. Since every $\mathcal{B}$-category is a small colimit of objects of the form $\Delta^n \otimes A$ with $n \geq 0$ and $A \in \mathcal{B}$, we deduce that the regularisation of $\langle \mathrm{Cat}_\infty, \mathrm{all}\rangle$ is $\mathsf{Cat}_{\mathcal{B}}$ (where the local class all is the class of all morphisms in $\mathcal{B}$).

4.2.2. $\boldsymbol{\kappa}$-small $\boldsymbol{\mathcal{B}}$-categories. Let κ be a cardinal. Recall from [57, § 6.1.6] that a map $p\colon P \to A$ in $\mathcal{B}$ is said to be *relatively κ-compact* if for every κ-compact $B \in \mathcal{B}$ and every map $s\colon B \to A$, the pullback s^*P is κ-compact as well. We denote by κ-cpt the local class of morphisms in $\mathcal{B}$ that is generated by the relatively κ-compact morphisms, and we let $\Omega_{\mathcal{B}}^{\kappa}$ be the associated subuniverse. Explicitly, a map $p\colon P \to A$ is contained in κ-cpt precisely if there is a cover $(s_i)_i\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $s_i^* p$ is relatively κ-compact for each i.

Let us denote by $\mathrm{Cat}_\infty^\kappa$ the doctrine of κ-small ∞-categories. We may now define:

DEFINITION 4.2.2.1. A $\mathcal{B}$-category is said to be *κ-small* if it is $\langle \mathrm{Cat}_\infty^\kappa, \kappa\text{-cpt}\rangle$-small. We will use the notation $\mathsf{Cat}_{\mathcal{B}}^{\kappa} = \mathsf{Cat}_{\mathcal{B}}^{\langle \mathrm{Cat}_\infty^\kappa, \kappa\text{-cpt}\rangle}$ to denote the internal class of κ-small $\mathcal{B}$-categories, and we denote its underlying ∞-category of global sections by $\mathrm{Cat}(\mathcal{B})^\kappa$.

REMARK 4.2.2.2. Note that for general $A \in \mathcal{B}$ there is no reason to expect an equivalence $\pi_A^*\Omega_{\mathcal{B}}^\kappa \simeq \Omega_{\mathcal{B}_{/A}}^\kappa$. Therefore, we can also not expect to have an equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}}^\kappa \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^\kappa$. The situation improves, however, when A is assumed to be κ-compact. In this case, the observation that an object in $\mathcal{B}_{/A}$ is κ-compact if and only if its underlying object in $\mathcal{B}$ is κ-compact implies that a map in $\mathcal{B}_{/A}$ is relatively κ-compact if and only its underlying map in $\mathcal{B}$ is relatively κ-compact, so that we obtain an equivalence $\pi_A^*\Omega_{\mathcal{B}}^\kappa \simeq \Omega_{\mathcal{B}_{/A}}^\kappa$. By using Remark 4.2.1.2, this equivalence in turn induces an equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}}^\kappa \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^\kappa$.

The internal class $\mathsf{Cat}_{\mathcal{B}}^\kappa$ is not very well-behaved for arbitrary cardinals κ. Therefore, we will restrict our attention to a certain class of cardinals that are in a sense *adapted* to the ∞-topos $\mathcal{B}$.

DEFINITION 4.2.2.3. We say that cardinal κ is *$\mathcal{B}$-regular* if

(1) κ is regular and uncountable;
(2) $\mathcal{B}$ is κ-accessible;
(3) the full subcategory $\mathcal{B}^{\kappa\text{-cpt}} \hookrightarrow \mathcal{B}$ of κ-compact objects in $\mathcal{B}$ is closed under finite limits and subobjects in $\mathcal{B}$.

REMARK 4.2.2.4. Every uncountable regular cardinal κ is $\mathcal{S}$-regular. In fact, condition (2) is immediate, and $1 \in \mathcal{S}$ is certainly κ-compact. Moreover, if $P = A \times_C B$ is a pullback of κ-compact ∞-groupoids, descent implies $P \simeq \mathrm{colim}_{a \in A} P|_a$. Since κ-compact ∞-groupoids are precisely those which are κ-small [57, Corollary 5.4.1.5] and since κ-compact objects in $\mathcal{S}$ are stable under κ-small colimits, it suffices to show that $P|_a$ is κ-compact. We may therefore reduce to the case where $A \simeq 1$. By the same reasoning, we can assume $B \simeq 1$ as well. But then P can be identified with a mapping ∞-groupoid of C, which is κ-small by again making use of [57, Corollary 5.4.1.5]. Finally, the identification of κ-compact ∞-groupoids with κ-small ∞-groupoids also shows that these are stable under subobjects. Hence condition (3) is satisfied as well.

REMARK 4.2.2.5. Note that there is an ample amount of $\mathcal{B}$-regular cardinals, in the sense that if κ' is an arbitrary cardinal one can always find a larger cardinal $\kappa \geq \kappa'$ that is $\mathcal{B}$-regular. Indeed, by enlarging κ' if necessary one can always arrange for $\mathcal{B}$ to be κ'-accessible. Then for any (uncountable) $\kappa \gg \kappa'$

(in the sense of [57, Definition A.2.6.3]) for which $\mathcal{B}^{\kappa'\text{-cpt}}$ is κ-small, an object in $\mathcal{B}$ is κ-compact if and only if the underlying presheaf on $\mathcal{B}^{\kappa'\text{-cpt}}$ takes values in the full subcategory $\mathcal{S}^{\kappa\text{-cpt}} \hookrightarrow \mathcal{S}$ of κ-compact ∞-groupoids [57, Lemma 5.4.7.5]. In combination with Remark 4.2.2.4, this shows that κ is $\mathcal{B}$-regular. In particular, this argument shows that we can always find a $\mathcal{B}$-regular κ such that $\kappa \gg \kappa'$.

Remark 4.2.2.6. If κ is a $\mathcal{B}$-regular cardinal, then κ is also $\mathcal{B}_{/A}$-regular for every *κ-compact* object $A \in \mathcal{B}$. In fact, since an object in $\mathcal{B}_{/A}$ is κ-compact if and only if its image along $(\pi_A)_!$ is κ-compact, every object in $\mathcal{B}_{/A}$ is a κ-filtered colimit of κ-compact objects, which shows that (2) is satisfied. Condition (3) follows from A being κ-compact, together with the fact that $(\pi_A)_!$ preserves pullbacks (and consequently also subobjects).

Remark 4.2.2.7. For every $\mathcal{B}$-regular cardinal κ, the ∞-topos $\mathcal{B}$ admits a presentation by the full subcategory $\mathcal{B}^{\kappa\text{-cpt}} \subset \mathcal{B}$ of κ-compact objects, in the sense that its Yoneda extension $\mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}) \to \mathcal{B}$ is a left exact and accessible Bousfield localisation [57, Proposition 6.1.5.2]. Moreover, the inclusion $\mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}})$ commutes with κ-filtered colimits. In particular, the global sections functor $\Gamma\colon \mathcal{B} \to \mathcal{S}$ commutes with κ-filtered colimits, so that $\mathrm{const}_{\mathcal{B}}$ restricts to a functor $\mathcal{S}^{\kappa\text{-cpt}} \to \mathcal{B}^{\kappa\text{-cpt}}$.

The first main result in this section will be the following characterisation of κ-small $\mathcal{B}$-categories when κ is $\mathcal{B}$-regular:

Proposition 4.2.2.8. *Let κ be a $\mathcal{B}$-regular cardinal, and let C be a $\mathcal{B}$-category. Then the following are equivalent:*

(1) C is κ-small;
(2) C is a κ-compact object in $\mathrm{Cat}(\mathcal{B})$;
(3) C is contained in the smallest full subcategory of $\mathrm{Cat}(\mathcal{B})$ that is spanned by objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and that is closed under κ-small colimits;
(4) C is a κ-compact object in $\mathcal{B}_\Delta$;
(5) C_0 and C_1 are κ-compact objects in $\mathcal{B}$.

Remark 4.2.2.9. On account of Remark 4.2.2.2, Proposition 4.2.2.8 implies that for every κ-compact object $A \in \mathcal{B}$, we can identify $\mathsf{Cat}^{\kappa}_{\mathcal{B}}(A)$ with the full subcategory of κ-compact objects in $\mathrm{Cat}(\mathcal{B}_{/A})$.

The proof of Proposition 4.2.2.8 requires a few preparations. We begin by establishing that the class of relatively κ-compact maps in $\mathcal{B}$ is already local.

Lemma 4.2.2.10. *Let κ be a $\mathcal{B}$-regular cardinal and let I be a small set. For every $i \in I$, let $P_i \to A_i$ be a relatively κ-compact map in $\mathcal{B}$. Then $\bigsqcup_i P_i \to \bigsqcup_i A_i$ is relatively κ-compact.*

Proof. Let G be κ-compact, and let $s\colon G \to \bigsqcup_i A_i$ be a map. Write $I = \mathrm{colim}_j I_j$ as a κ-filtered union of its κ-small subsets, so that one obtains equivalences $\bigsqcup_i A_i \simeq \mathrm{colim}_j \bigsqcup_{i \in I_j} A_i$ and $\bigsqcup_i P_i \simeq \mathrm{colim}_j \bigsqcup_{i\in I_j} P_i$. As G is κ-compact, there is some j such that s factors through the inclusion $\bigsqcup_{i\in I_j} A_i \hookrightarrow \bigsqcup_i A_i$. By descent, we obtain a pullback diagram

$$\begin{tikzcd}
\bigsqcup_{i\in I_j} P_i \arrow[r] \arrow[d] & \bigsqcup_i P_i \arrow[d] \\
\bigsqcup_{i\in I_j} A_i \arrow[r] & \bigsqcup_i A_i,
\end{tikzcd}$$

which implies that the pullback of $\bigsqcup_i P_i \to \bigsqcup_i A_i$ to G is equivalent to the pullback of $\bigsqcup_{i\in I_j} P_i \to \bigsqcup_{i \in I_j} A_i$ to G. By again using descent, this pullback can be identified with the coproduct $\bigsqcup_{i\in I_j} P_i \times_{A_i} G_i$, where $G_i = G \times_{\bigsqcup_i A_i} A_i$. As G_i is a subobject of G and therefore κ-compact, the fibre product $P_i \times_{A_i} G_i$ is κ-compact as well. Since I_j is κ-small, we conclude that also $\bigsqcup_{i\in I_j} P_i \times_{A_i} G_i$ is κ-compact, as desired. $\square$

Proposition 4.2.2.11. *Let κ be a $\mathcal{B}$-regular cardinal. Then every object in $\Omega^{\kappa}_{\mathcal{B}}$ is already relatively κ-compact. In other words, the class of relatively κ-compact maps in $\mathcal{B}$ is local.*

Proof. Let $P \to A$ be an object in $\Omega^{\kappa}_{\mathcal{B}}(A)$. By definition, there is a cover $(s_i)\colon \bigsqcup_{i\in I} A_i \twoheadrightarrow A$ such that $s_i^* P \to A_i$ is relatively κ-compact. By Lemma 4.2.2.10, the map $\bigsqcup_i P_i \to \bigsqcup_i A_i$ is relatively κ-compact. The result therefore follows once we show that relatively κ-compact maps are stable under Δ^{op}-indexed colimits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$. By [57, Lemma 6.1.6.6] they are stable under pushouts, so we only need to consider the case of small coproducts, which again follows from Lemma 4.2.2.10. □

Remark 4.2.2.12. In [57, Proposition 6.1.6.7], Lurie shows that the class of relatively κ-compact maps in $\mathcal{B}$ is local already when $\mathcal{B}$ is κ-accessible and $\mathcal{B}^{\kappa\text{-cpt}}$ is stable under finite limits in $\mathcal{B}$. However, we failed to understand how Lurie derives this result without the additional assumption that $\mathcal{B}^{\kappa\text{-cpt}}$ is also stable under subobjects in $\mathcal{B}$. Therefore, we decided to reiterate Lurie's proof with this added assumption.

Next, we need to establish that every $\mathcal{B}$-regular cardinal is also $\mathcal{B}_\Delta$-regular. This will be a consequence of the following characterisation of κ-compact simplicial objects in $\mathcal{B}$:

Proposition 4.2.2.13. *If κ is a $\mathcal{B}$-regular cardinal, then the ∞-topos $\mathcal{B}_\Delta$ is κ-accessible, and if C is a simplicial object in $\mathcal{B}$, the following are equivalent:*

(1) C is κ-compact;
(2) $C_n \in \mathcal{B}^{\kappa\text{-cpt}}$ for all $n \geq 0$;
(3) C is contained in the smallest subcategory of $\mathcal{B}_\Delta$ that is spanned by objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and that is closed under κ-small colimits;

Proof. Remark 4.2.2.7 implies that the inclusion $\mathcal{B}_\Delta \hookrightarrow \mathrm{PSh}(\Delta \times \mathcal{B}^{\kappa\text{-cpt}})$ commutes with κ-filtered colimits, which immediately implies that $\mathcal{B}_\Delta$ is κ-accessible. Moreover, since Δ is a κ-small ∞-category, every simplicial object in $\mathcal{B}$ that is level-wise κ-compact is also κ-compact in $\mathcal{B}_\Delta$ [57, Proposition 5.3.4.13], hence (2) implies (1). If C satisfies (3), the fact that for every $k \geq 0$ the functor $(-)_k$ commutes with small colimits implies that C_k is contained in the smallest full subcategory of $\mathcal{B}$ that contains all objects of the form $\Delta^n_k \times G$ for $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and $n \geq 0$ and that is closed under κ-small colimits. Since Δ^n_k is a finite set, this implies that C_k is κ-compact, hence (2) follows. Finally, suppose that C is κ-compact. We may write C as a small colimit of objects of the form $\Delta^n \otimes G$ for $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ and therefore by [57, Corollary 4.2.3.10] as a κ-filtered colimit $C \simeq \mathrm{colim}_i C^i$ where each C^i is a κ-small colimits of objects of the form $\Delta^n \otimes G$. As C is κ-compact, there is some i_0 such that the identity on C factors through $C^{i_0} \to \mathsf{C}$. In other words, C is a retract of C^{i_0}. As retracts are countable and therefore a fortiori κ-small colimits, (3) follows. □

Corollary 4.2.2.14. *If κ is a $\mathcal{B}$-regular cardinal, then κ is $\mathcal{B}_\Delta$-regular as well. Moreover, a map in $\mathcal{B}_\Delta$ is relatively κ-compact if and only if it is level-wise given by a relatively κ-compact morphism in $\mathcal{B}$.* □

Lemma 4.2.2.15. *If κ is a $\mathcal{B}$-regular cardinal and if C is a κ-compact simplicial object in $\mathcal{B}$, then C^K is κ-compact for every ω-compact simplicial ∞-groupoid K.*

Proof. As κ-compact objects in $\mathcal{B}_\Delta$ are stable under retracts and as every ω-compact simplicial ∞-groupoid is a retract of a finite colimit of n-simplices, we may assume without loss of generality that K is a finite colimit of n-simplices. Therefore C^K is a finite limit of objects of the form C^{Δ^n}, so that Corollary 4.2.2.14 implies that we may reduce to the case $K = \Delta^n$. Now on account of the identity $(C^{\Delta^n})_k \simeq (C^{\Delta^n \times \Delta^k})_0$ and by using the fact that $\Delta^n \times \Delta^k$ is again ω-compact, we can identify $(C^{\Delta^n})_k$ as a finite limit of objects of the form $(C^{\Delta^l})_0 \simeq C_l$, which shows that $(C^{\Delta^n})_k$ is κ-compact. By Proposition 4.2.2.13, one concludes that C^{Δ^n} is κ-compact in $\mathcal{B}_\Delta$. □

Lemma 4.2.2.16. *For every ω-compact simplicial ∞-groupoid K, the functor $(-)^K\colon \mathcal{B}_\Delta \to \mathcal{B}_\Delta$ commutes with filtered colimits.*

PROOF. As every ω-compact simplicial ∞-groupoid K is a retract of a finite colimit of n-simplices, we may assume without loss of generality $K = \Delta^n$. As it suffices to show that$(-)_k^{\Delta^n}$ commutes with filtered colimits for all $k \geq 0$, the same argumentation as in the proof of Lemma 4.2.2.15 shows that we may reduce to showing that $(-)_0^{\Delta^n}$ commutes with filtered colimits. On account of the equivalence $(-)_0^{\Delta^n} \simeq (-)_n$, this is immediate. □

LEMMA 4.2.2.17. *Let κ be a $\mathcal{B}$-regular cardinal, let C be a κ-compact simplicial object in $\mathcal{B}_\Delta$ and let $C \to L(C)$ be the unit of the adjunction $(L \dashv i)\colon \operatorname{Cat}(\mathcal{B}) \leftrightarrows \mathcal{B}_\Delta$. Then $L(C)$ is κ-compact as well.*

PROOF. We will make use of the ∞-categorical version of the small object argument as developed in [4, § 2.3]. For the convenience of the reader, we briefly explain the setup, at least in the special case that is relevant for this proof. Suppose that S is a finite set of maps in $\mathcal{S}_\Delta$ such that for every map $s\colon K \to L$ in S the functors $(-)^K$ and $(-)^L$ commute with filtered colimits in $\mathcal{B}_\Delta$. Let $(\mathcal{L}, \mathcal{R})$ be the factorisation system in $\mathcal{B}_\Delta$ that is internally generated by the set S. To any object $C \in \mathcal{B}_\Delta$, we can now assign a sequence

$$\mathbb{N} \to \mathcal{B}_\Delta, \quad k \mapsto C(k)$$

by setting $C(0) = C$ and by recursively defining a map $C(k) \to C(k+1)$ via the pushout

$$\begin{tikzcd} \bigsqcup_{s\colon K \to L} L \otimes C(k)^L \sqcup_{K \otimes C(k)^L} K \otimes C(k)^K \arrow[r] \arrow[d] & C(k) \arrow[d] \\ \bigsqcup_{s\colon K \to L} L \otimes C(k)^K \arrow[r] & C(k+1) \end{tikzcd}$$

in which the coproduct ranges over all maps $s\colon K \to L$ in S. Then [4, Theorem 2.3.4] shows that the object $\operatorname{colim}_k C(k)$ is internally local with respect to the maps in S, i.e. contained in $\mathcal{R}_{/1}$, and that furthermore the map $C \to \operatorname{colim}_k C(k)$ is contained in $\mathcal{L}$, so that it is equivalent to the unit of the adjunction $\mathcal{R}_{/1} \leftrightarrows \mathcal{B}_\Delta$ evaluated at $C \in \mathcal{B}_\Delta$.

Now if we let S be the set $\{E^1 \to 1, I^2 \hookrightarrow \Delta^2\}$, Lemma 4.2.2.16 shows that we are in the above situation. Consequently, if C is a κ-compact object in $\mathcal{B}_\Delta$, the $\mathcal{B}$-category $L(C)$ can be computed as a countable colimit of the objects $C(k)$ as constructed above. Hence it suffices to show that each $C(k)$ is κ-compact, which easily follows from κ being $\mathcal{B}_\Delta$-regular (Corollary 4.2.2.14) and Lemma 4.2.2.15. □

PROOF OF PROPOSITION 4.2.2.8. We first show that (2)–(5) are equivalent. By combining Proposition 4.2.2.13 with the Segal conditions, one finds that (4) and (5) are equivalent. Moreover, since $\operatorname{Cat}(\mathcal{B})$ is an ω-accessible localisation of $\mathcal{B}_\Delta$, the localisation functor preserves κ-compact objects, which shows that (4) implies (2). Suppose now that C is a κ-compact object in $\operatorname{Cat}(\mathcal{B})$. As in the proof of Proposition 4.2.2.13, we can find a κ-filtered ∞-category $\mathcal{J}$ such that $\mathsf{C} \simeq \operatorname{colim}_{j \in \mathcal{J}} \mathsf{C}^j$ where each C^j is a κ-small colimit of objects of the form $\Delta^n \otimes G$, where $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$. Hence C is a retract of some C^j, so that (3) holds. Lastly, since we can compute any small colimit in $\operatorname{Cat}(\mathcal{B})$ by first taking the colimit of the underlying diagram in $\mathcal{B}_\Delta$ and then applying the reflector $L\colon \mathcal{B}_\Delta \to \operatorname{Cat}(\mathcal{B})$, Lemma 4.2.2.17 implies that every κ-small colimit in $\operatorname{Cat}(\mathcal{B})$ of objects of the form $\Delta^n \otimes G$ with $n \geq 0$ and $G \in \mathcal{B}^{\kappa\text{-cpt}}$ is also κ-compact in $\mathcal{B}_\Delta$. Thus (3) implies (4).

Finally, since $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$ is closed under both $\mathsf{LConst}_{\operatorname{Cat}_\infty^\kappa}$- and $\Omega_{\mathcal{B}}^\kappa$-colimits and since $\Delta^n \otimes G$ can be regarded as the colimit of the constant G-indexed diagram with value Δ^n, it is clear that (3) implies (1). To show the converse, let V be the internal class that is spanned by those $\mathcal{B}_{/A}$-categories (for $A \in \mathcal{B}^{\kappa\text{-cpt}}$) that satisfy the $\mathcal{B}_{/A}$-categorical analogue of (3). Note that if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover by κ-compact objects and if D is a $\mathcal{B}$-category such that $\pi_{A_i}^* \mathsf{D}$ satisfies the $\mathcal{B}_{/A_i}$-categorical analogue of condition (3), then D satisfies (3): in fact, since we have already established that (3) and (4) are equivalent, this is a consequence of Proposition 4.2.2.11 and Corollary 4.2.2.14. As a consequence, for every κ-compact object $A \in \mathcal{B}$, we can identify $\mathsf{V}(A)$ with the class of $\mathcal{B}_{/A}$-categories that satisfy the $\mathcal{B}_{/A}$-categorical version of condition (3). As V clearly contains both $\mathsf{LConst}_{\operatorname{Cat}_\infty^\kappa}$ and $\Omega_{\mathcal{B}}^\kappa$, the proof will be complete once we show that V is closed under both $\mathsf{LConst}_{\operatorname{Cat}_\infty^\kappa}$- and $\Omega_{\mathcal{B}}^\kappa$-colimits. By our description of $\mathsf{V}(A)$ for every κ-compact $A \in \mathcal{B}$ and the fact that colimits can be computed locally by Remark 3.1.1.8, this is clear for the first case. To show the second case, we need to verify that for every relatively κ-compact map $p\colon P \to A$, the functor $p_!\colon \mathrm{Cat}(\mathcal{B}_{/P}) \to \mathrm{Cat}(\mathcal{B}_{/A})$ restricts to a map $\mathsf{V}(P) \to \mathsf{V}(A)$. Using again that the class of relatively κ-compact maps is local, it is enough to consider the case where A (and therefore also P) is κ-compact. To show the claim, we may again use the explicit description of $\mathsf{V}(A)$ and $\mathsf{V}(P)$ to deduce that it suffices to verify that $p_!$ carries κ-small colimits of objects in $\mathrm{Cat}(\mathcal{B}_{/P})$ of the form $\Delta^n \otimes Q$ (with $Q \to P$ relatively κ-compact) to κ-small colimits of objects in $\mathrm{Cat}(\mathcal{B}_{/A})$ of the form $\Delta^n \otimes Q$ (with $Q \to A$ κ-compact). Since $p_!$ preserves small colimits and acts by postcomposition with p, this follows from the fact that relatively κ-compact maps are closed under composition. □

Corollary 4.2.2.18. *For every $\mathcal{B}$-regular cardinal κ, the internal class $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$ is a doctrine.*

Proof. As $\mathcal{B}$ is generated by $\mathcal{B}^{\kappa}$, Remark 4.2.2.2 implies that we only need to show that the collection of κ-small $\mathcal{B}$-categories is small, which is an immediate consequence of (2) in Proposition 4.2.2.8 □

By construction, the internal class $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$ is regular for every cardinal κ. We conclude this section by proving that whenever κ is $\mathcal{B}$-regular, the doctrine $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$ is sound.

Lemma 4.2.2.19. *Let κ be a $\mathcal{B}$-regular cardinal, and let $\mathcal{J}$ be a κ-filtered ∞-category. Then $\mathcal{J}$ is $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-filtered when viewed as a constant $\mathcal{B}$-category.*

Proof. By Proposition 4.1.1.5, we need to show that the inclusion $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{J}, \Omega) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{J}^{\triangleright}, \Omega)$ is $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-continuous. Since $\mathcal{J}$ is κ-filtered, the inclusion section-wise preserves κ-small limits. It therefore suffices to show that it is Ω^{κ}-continuous. This amounts to showing that for every $A \in \mathcal{B}$ and every $G \in \Omega^{\kappa}(A)$ the geometric morphism $\mathcal{B}_{/G} \to \mathcal{B}_{/A}$ commutes with $\mathcal{J}$-indexed colimits. As the preservation of colimits is a local condition 3.1.2.1 and as $\mathcal{B}$ is generated by the κ-compact objects in $\mathcal{B}$, we may assume that A is κ-compact. In light of Remark 4.2.2.6, we may thus replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore reduce to the case $A \simeq 1$. As κ is $\mathcal{B}$-regular, the collection of κ-compact objects in $\mathcal{B}$ is stable under finite limits. Therefore, for every $H \in \mathcal{B}^{\kappa\text{-cpt}}$ the functor $\mathrm{map}_{\mathcal{B}}(G \times H, -)$ preserves $\mathcal{J}$-filtered colimits. By Yoneda's lemma, this implies that the functor $\underline{\mathrm{Hom}}_{\mathcal{B}}(G, -)$ also preserves $\mathcal{J}$-filtered colimits. On account of the pullback square

$$\begin{tikzcd}
(\pi_G)_* \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{B}}(G, (\pi_G)_!(-)) \arrow[d] \\
\mathrm{diag}(1) \arrow[r, "\mathrm{id}_G"] & \mathrm{diag}(\underline{\mathrm{Hom}}_{\mathcal{B}}(G, G))
\end{tikzcd}$$

in $\mathrm{Fun}(\mathcal{B}_{/G}, \mathcal{B})$ and the fact that the cospan in the lower right corner consists of functors which preserve $\mathcal{J}$-indexed colimits, the claim follows from the fact that $\mathcal{J}$-indexed colimits commute with finite limits. □

Lemma 4.2.2.20. *Let κ be a $\mathcal{B}$-regular cardinal, and let J be a $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-cocomplete $\mathcal{B}$-category. Then the canonical functor $\Gamma\mathsf{J} \to \mathsf{J}$ that is obtained from the counit of the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ is final.*

Proof. For every $G \in \mathcal{B}^{\kappa\text{-cpt}}$, the functor $\mathsf{J}(1) \to \mathsf{J}(G)$ admits a left adjoint and is therefore in particular final. In other words, if $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}})$ denotes the inclusion, then the functor $\epsilon\colon \Gamma_{\mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}})} i\mathsf{J} \to i\mathsf{J}$ is section-wise final. But since the local sections functor $\mathrm{ev}_G\colon \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}) \to \mathcal{S}$ defines an algebraic morphism of ∞-topoi and since every algebraic morphism preserves both final functors and right fibrations, applying ev_G to any factorisation of ϵ in $\mathrm{Cat}(\mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}))$ into a final functor and a right fibration yields a factorisation of $\epsilon(G)$ into a final functor and a right fibration in Cat_{∞}. Consequently, the map ϵ must already be final. As we recover the map $\Gamma\mathsf{J} \to \mathsf{J}$ by applying the algebraic morphism $L\colon \mathrm{PSh}(\mathcal{B}^{\kappa\text{-cpt}}) \to \mathcal{B}$ to ϵ, the claim follows. □

Proposition 4.2.2.21. *If κ is a $\mathcal{B}$-regular cardinal, then $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$ is sound.*

Proof. On account of Corollary 4.1.5.12, it suffices to show that $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$ is weakly sound. Together with the fact that $\mathcal{B}$ is generated by its κ-compact objects and Remark 4.2.2.2, it is therefore enough to prove that every $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$-cocomplete $\mathcal{B}$-category J is $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$-filtered. By Lemma 4.2.2.20 and Remark 4.1.1.7, we can furthermore assume that J is the constant $\mathcal{B}$-category associated with an ∞-category that admits κ-small colimits and that is therefore κ-filtered [57, Proposition 5.3.3.3]. As a consequence, the result follows from Lemma 4.2.2.19. $\square$

REMARK 4.2.2.22. As a consequence of Proposition 4.2.2.21, if C is an arbitrary $\mathcal{B}$-category, there is always a regular and sound doctrine U such that $\mathsf{C} \in \mathsf{U}(1)$. In fact, we only need to choose a $\mathcal{B}$-regular cardinal κ such that C is κ-compact (and therefore κ-small by Proposition 4.2.2.8) and set $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}^{\kappa}$. More generally, if V is a doctrine, we can find a $\mathcal{B}$-regular cardinal κ such that V_0 is κ-compact and such that the tautological object $\tau \colon \mathsf{V}_0 \to \mathsf{V}$ corresponds to a κ-small $\mathcal{B}_{/\mathsf{V}_0}$-category. As every object of V (in arbitrary context $A \in \mathcal{B}$) arises as a pullback of τ, this implies that V is contained in $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$.

COROLLARY 4.2.2.23. *For every $\mathcal{B}$-regular cardinal κ, there is an equivalence*

$$\Omega_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}} \simeq \Omega_{\mathcal{B}}^{\kappa}$$

of full subcategories in Ω.

PROOF. Since $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$ is a sound doctrine by Proposition 4.2.2.21, it has the decomposition property. We may therefore apply Proposition 4.1.5.10 to deduce an equivalence

$$\Omega_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}} \simeq \mathsf{Ret}_{\Omega}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}}(1)).$$

Note that if $\mathsf{G} \colon 1 \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}}(1)$ is an arbitrary object, there is a cover $\bigsqcup_i A_i \twoheadrightarrow 1$ of $\mathcal{B}$ (without loss of generality by κ-compact objects) and for each i a κ-small $\mathcal{B}_{/A_i}$-category J such that $\pi_{A_i}^* \mathsf{G} \simeq \mathsf{J}^{\mathrm{gpd}}$. Since κ is by definition uncountable, we thus find that $\pi_{A_i}^* \mathsf{G}$ arises as a κ-small colimit of κ-compact objects in $\mathcal{B}_{/A_i}$ (using the characterisation of κ-small $\mathcal{B}_{/A_i}$-categories in Proposition 4.2.2.8) and is therefore itself κ-compact. Using Proposition 4.2.2.11, this implies that G is κ-compact itself. By Remark 3.3.2.2, the same argument can be carried out for every object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}}(1)$ in context $A \in \mathcal{B}^{\kappa\text{-cpt}}$, and since the collection of κ-compact objects in $\mathcal{B}$ generate $\mathcal{B}$ under small colimits, this implies that we have an inclusion

$$\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}}(1) \hookrightarrow \Omega_{\mathcal{B}}^{\kappa}.$$

Using again that κ is uncountable, the collection of κ-compact objects in $\mathcal{B}$ is closed under retracts, and as the same is true for the class of κ-compact objects in $\mathcal{B}_{/A}$ for every $A \in \mathcal{B}^{\kappa\text{-cpt}}$, we find that $\Omega_{\mathcal{B}}^{\kappa}$ is closed under retracts in Ω, so that we obtain an inclusion

$$\Omega_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}} \hookrightarrow \Omega_{\mathcal{B}}^{\kappa}.$$

Conversely, it is clear that whenever G is a κ-small $\mathcal{B}$-groupoid, the associated object $\mathsf{G} \colon 1 \to \Omega$ is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}}(1)$. Again, the same is true for every κ-small $\mathcal{B}_{/A}$-groupoid whenever A is κ-compact. Hence we obtain an inclusion

$$\Omega_{\mathcal{B}}^{\kappa} \hookrightarrow \Omega_{\mathcal{B}}^{\mathsf{Cat}_{\mathcal{B}}^{\kappa}\text{-cpt}},$$

which finishes the proof. $\square$

4.2.3. Finite $\mathcal{B}$-categories. In this section we will discuss another important example of a regular and sound doctrine. Recall that a quasicategory $\mathcal{C}$ is called *finite* if there is a finite simplicial set and a Joyal equivalence $K \to \mathcal{C}$. This is equivalent to $\mathcal{C}$ being contained in the smallest subcategory of Cat_{∞} that contains $\varnothing$, Δ^0 and Δ^1 and is closed under pushouts (see [87, Proposition 2.5]). We denote the associated doctrine of ∞-categories by $\mathrm{Fin}_{\mathcal{S}}$. Let us denote by eq the local class of equivalences in $\mathcal{B}$. We may now define:

DEFINITION 4.2.3.1. A $\mathcal{B}$-category is said to be *finite* if it is $(\mathrm{Fin}_{\mathcal{S}}, \mathrm{eq})$-small, and we shall denote by $\mathsf{Fin}_{\mathcal{B}} = \mathsf{Cat}_{\mathcal{B}}^{\langle \mathrm{Fin}_{\mathcal{S}}, \mathrm{eq}\rangle}$ the associated regular doctrine of finite $\mathcal{B}$-categories. We will denote by $\mathrm{Fin}(\mathcal{B})$ the underlying ∞-category of global sections. We say that a $\mathcal{B}$-category I is *filtered* if it is $\mathsf{Fin}_{\mathcal{B}}$-filtered. We will say that a $\mathcal{B}$-category *has finite (co)limits* if it is $\mathsf{Fin}_{\mathcal{B}}$-(co)complete, and a functor *preserves* *finite (co)limits* if it is $\mathsf{Fin}_{\mathcal{B}}$-(co)continuous. Dually, we say that a $\mathcal{B}$-category *has filtered colimits* if it is Filt-cocomplete, and a functor *preserves filtered colimits* if it is Filt-cocontinuous. If C is a $\mathcal{B}$-category that has filtered colimits, an object $c\colon A \to \mathsf{C}$ is said to be *compact* if it is $\mathsf{Filt}_{\mathsf{Fin}_{\mathcal{B}}}$-compact, and we denote the full subcategory of compact objects in C by $\mathsf{C}^{\mathrm{cpt}}$.

Remark 4.2.3.2. By Remark 4.2.1.2 and the evident fact that $\pi_A^*\mathrm{eq} = \mathrm{eq}$ as local classes in $\mathcal{B}_{/A}$, there is a canonical equivalence $\pi_A^* \mathsf{Fin}_{\mathcal{B}} \simeq \mathsf{Fin}_{\mathcal{B}_{/A}}$ for all $A \in \mathcal{B}$.

Remark 4.2.3.3. Every filtered $\mathcal{B}$-category J satisfies $\mathsf{J}^{\mathrm{gpd}} \simeq 1$. In fact, by Corollary 4.1.2.6 it is weakly filtered, thus in particular the unique functor $I \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\varnothing, \mathsf{I}) \simeq 1$ is final.

Proposition 4.2.3.4. *There is an equivalence $\mathsf{Fin}_{\mathcal{B}} \simeq \mathsf{LConst}_{\mathrm{Fin}_{\mathcal{S}}}$ of internal classes. In other words, a finite $\mathcal{B}$-category is simply a locally constant sheaf of finite ∞-categories.*

Proof. Since $\Omega_{\mathrm{eq}} \simeq 1_{\Omega}$ as full subcategories, we can describe $\mathsf{Fin}_{\mathcal{B}}$ as the regularisation of $\mathsf{LConst}_{\mathrm{Fin}_{\mathcal{S}}}$. But since Cat_{∞} is compactly generated, we may apply Corollary A.4 and conclude that $\mathsf{LConst}_{\mathrm{Fin}_{\mathcal{S}}}$ is already closed under $\mathsf{LConst}_{\mathrm{Fin}_{\mathcal{S}}}$-colimits in $\mathsf{Cat}_{\mathcal{B}}$. Hence the claim follows. □

By Proposition 4.2.1.3, finite limits and preservation of finite limits can be checked section-wise:

Proposition 4.2.3.5. *Let C be a $\mathcal{B}$-category. Then*

(1) C has finite limits if and only if $\mathsf{C}(A)$ has finite limits for every $A \in \mathcal{B}$ and for every $s\colon B \to A$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves finite limits.

(2) A functor $f\colon \mathsf{C} \to \mathsf{D}$ between $\mathcal{B}$-categories that have finite limits preserves such limits if and only if $f(A)\colon \mathsf{C}(A) \to \mathsf{D}(A)$ preserves finite limits for every $A \in \mathcal{B}$.

The dual statements about finite colimits hold as well. □

One can construct an ample amount of filtered $\mathcal{B}$-categories from *presheaves* of filtered ∞-categories:

Proposition 4.2.3.6. *Say that $\mathcal{B}$ is given as a left exact accessible localisation $L\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ where $\mathcal{C}$ is a small ∞-category. Let J be any $\mathrm{PSh}(\mathcal{C})$-category such $\mathsf{J}(c)$ is filtered for every $c \in \mathcal{C}$. Then $L\mathsf{J}$ is a filtered $\mathcal{B}$-category.*

Proof. Let $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ be the inclusion. Since L is left exact, it induces a functor of $\mathrm{PSh}(\mathcal{C})$-categories $L\colon \Omega_{\mathrm{PSh}(\mathcal{C})} \to i\Omega_{\mathcal{B}}$ that for every $A \in \mathrm{PSh}(\mathcal{C})$ is given by

$$L_{/A}\colon \mathrm{PSh}(\mathcal{C})_{/A} \to \mathcal{B}_{/LA}.$$

By Proposition 4.2.3.5, the functor L thus preserves finite limits. Furthermore, it readily follows from Proposition 2.4.2.9 that L admits a right adjoint i that is fully faithful. Therefore, we have a commutative diagram

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, \Omega_{\mathrm{PSh}(\mathcal{C})}) \arrow[r, "\mathrm{colim}_{\mathsf{J}}"] \arrow[d, "L_*"] & \Omega_{\mathrm{PSh}(\mathcal{C})} \arrow[d, "L"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, i\Omega_{\mathcal{B}}) \arrow[r, "\mathrm{colim}_{\mathsf{J}}"'] & i\Omega_{\mathcal{B}}. \end{tikzcd}$$

Since there is an equivalence $i\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{J}, i\Omega_{\mathcal{B}}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(L\mathsf{J}, \Omega_{\mathcal{B}})$ that is natural in J, the lower colimit functor in the above diagram can be identified with the functor $i\,\mathrm{colim}_{L\mathsf{J}}\colon i\underline{\mathsf{Fun}}_{\mathcal{B}}(L\mathsf{J}, \Omega_{\mathcal{B}}) \to i\Omega_{\mathcal{B}}$. Using that i is fully faithful, we get that this map is equivalent to the composition $L\,\mathrm{colim}_{\mathsf{J}}\, i_*$. Therefore, it suffices to show that the upper colimit functor in the above diagram preserves finite limits. To see this, since $\mathrm{PSh}(\mathcal{C})_{/c} \simeq \mathrm{PSh}(\mathcal{C}_{/c})$ for every $c \in \mathcal{C}$ and since $\mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ generates $\mathrm{PSh}(\mathcal{C})$ under small colimits, it suffices to show that the functor $(-)^{\mathrm{gpd}}\colon \mathrm{LFib}(\mathsf{J}) \to \mathrm{PSh}(\mathcal{C})$ commutes with finite limits, cf. Propositions 3.1.4.1 and 4.2.3.5. Since for every $c \in \mathcal{C}$ the evaluation functor $\mathrm{ev}_c\colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{S}$ commutes with small colimits, the lax square

$$\begin{tikzcd} \mathrm{LFib}_{\mathrm{PSh}(\mathcal{C})}(\mathsf{J}) \arrow[r, "(-)^{\mathrm{gpd}}"] \arrow[d, "\mathrm{ev}_c"] & \mathrm{PSh}(\mathcal{C}) \arrow[d, "\mathrm{ev}_c"] \\ \mathrm{LFib}_{\mathcal{S}}(\mathsf{J}(c)) \arrow[r, "(-)^{\mathrm{gpd}}"] & \mathcal{S} \end{tikzcd}$$

is commutative. By assumption and the fact that ev_c preserves limits, the functor $(-)^{\mathrm{gpd}} \circ \mathrm{ev}_c$ commutes with finite limits, hence so does $\mathrm{ev}_c \circ (-)^{\mathrm{gpd}}$. The claim now follows from the fact that $(\mathrm{ev}_c)_{c\in\mathcal{C}} \colon \mathrm{PSh}(\mathcal{C}) \to \prod_{c\in\mathcal{C}} \mathcal{S}$ is a conservative functor. $\square$

This leads to the main result of this section:

PROPOSITION 4.2.3.7. *The doctrine* $\mathsf{Fin}_{\mathcal{B}}$ *is sound.*

PROOF. Since $\mathsf{Fin}_{\mathcal{B}}$ is by definition regular, Corollary 4.1.5.12 implies that suffices it to show that $\mathsf{Fin}_{\mathcal{B}}$ is weakly sound. Using Remark 4.2.3.2, we only need to show that every $\mathcal{B}$-category J that has finite colimits is already filtered. But since J in particular admits finite *constant* colimits, it is section-wise filtered, hence the result follows from Proposition 4.2.3.6. $\square$

As a result of Proposition 4.2.3.7, we can now classify the compact objects of Ω. To that end, Let us denote by $\mathsf{LConst}_{\mathcal{S}^{\mathrm{cpt}}}$ the full subcategory of Ω that arises as the essential image of the map $\mathcal{S}^{\mathrm{cpt}} \to \Omega$ (which is defined as the transpose of $\mathrm{const}_{\mathcal{B}} \colon \mathcal{S}^{\mathrm{cpt}} \to \mathcal{B}$). We now obtain:

COROLLARY 4.2.3.8. *There is an equivalence*

$$\Omega_{\mathcal{B}}^{\mathrm{cpt}} \simeq \mathsf{LConst}_{\mathcal{S}^{\mathrm{cpt}}}$$

of full subcategories in Ω.

PROOF. Since $\mathsf{Fin}_{\mathcal{B}}$ is a sound doctrine by Proposition 4.2.3.7, it has the decomposition property. We may therefore apply Proposition 4.1.5.10 to deduce an equivalence

$$\Omega_{\mathcal{B}}^{\mathrm{cpt}} \simeq \mathsf{Ret}_{\Omega}(\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Fin}_{\mathcal{B}}}(1)).$$

Hence, if $\mathsf{G} \colon A \to \Omega^{\mathrm{cpt}}$ is an arbitrary object, there is a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^*\mathsf{G}$ is a retract of an object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Fin}_{\mathcal{B}}}(1)$ in context A_i, for every i. By further refining this cover, we can furthermore assume that for each i there is a finite $\mathcal{B}_{/A_i}$-category J_i such that $\pi_{A_i}^*\mathsf{G}$ is a retract of $\mathsf{J}_i^{\mathrm{gpd}}$. Hence $s_i^*\mathsf{G}$ is a retract of an object in $\mathsf{LConst}_{\mathcal{S}^{\mathrm{cpt}}}$ in context A_i, so that Corollary A.4 implies that $s_i^*\mathsf{G}$ is itself contained in $\mathsf{LConst}_{\mathcal{S}^{\mathrm{cpt}}}$, which necessarily implies that G is contained in $\mathsf{LConst}_{\mathcal{S}^{\mathrm{cpt}}}$. Conversely, if G is an object of $\mathsf{LConst}_{\mathcal{S}^{\mathrm{cpt}}}$ in context $A \in \mathcal{B}$, we can find a cover $(s_i) \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $s_i^*\mathsf{G}$ is a constant $\mathcal{B}_{/A_i}$-groupoid coming from a compact ∞-groupoid, which in turn implies that $s_i^*\mathsf{G}$ is a retract of a constant $\mathcal{B}_{/A_i}$-groupoid coming from a *finite* ∞-groupoid. As this implies that $s_i^*\mathsf{G}$ is a retract of an object in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Fin}_{\mathcal{B}}}(1)$ in context A_i, we conclude that $s_i^*\mathsf{G}$ must be contained in Ω^{cpt}, so that G is contained in Ω^{cpt} as well. $\square$

The goal for the remainder of this section is to discuss a more explicit description of filtered $\mathcal{B}$-categories in the case where $\mathcal{B}$ is *hypercomplete*. To that end, recall that the filtered ∞-categories can be characterised as those ∞-categories $\mathcal{C}$ for which every map $\mathcal{K} \to \mathcal{C}$ from a finite ∞-category $\mathcal{K}$ can be extended to a map from the cone $\mathcal{K}^{\triangleright} \to \mathcal{C}$. In other words, the ∞-category $\mathcal{C}$ is filtered if and only if for any finite ∞-category $\mathcal{K}$ the functor $j^* \colon \mathrm{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$ induced by restricting along the inclusion $j \colon \mathcal{K} \hookrightarrow \mathcal{K}^{\triangleright}$ is essentially surjective. This characterisation admits an immediate internal analogue:

DEFINITION 4.2.3.9. A $\mathcal{B}$-category J is called *quasi-filtered* if for every finite ∞-category $\mathcal{K}$ the functor $j^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}^{\triangleright}, \mathsf{J}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})$ is essentially surjective.

As the terminology suggests, every filtered $\mathcal{B}$-category is quasi-filtered. To prove this, we require the following lemma, which gives a very explicit description of the notion of quasi-filteredness:

LEMMA 4.2.3.10. *Let J be a $\mathcal{B}$-category. Then J is quasi-filtered if and only if for any $A \in \mathcal{B}$ and any diagram $\mathcal{K} \to \mathsf{J}(A)$ where $\mathcal{K}$ is a finite ∞-category there exists a cover $(s_i)_i \colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that for every i we can find a map $\mathcal{K}^{\triangleright} \to \mathsf{J}(A_i)$ making the diagram*

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d, hook, "j"] & \mathsf{J}(A) \arrow[d, "s_i^*"] \\ \mathcal{K}^{\triangleright} \arrow[r] & \mathsf{J}(A_i) \end{tikzcd}$$

commute.

PROOF. Let us first assume that J is quasi-filtered. Choose a diagram $\mathcal{K} \to \mathsf{J}(A)$ that corresponds to a map $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})$, and let us form the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "s"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}^{\triangleright}, \mathsf{J})^{\simeq} \arrow[d, "(j^*)^{\simeq}"] \\ A \arrow[r] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}. \end{tikzcd}$$

Since j^* is essentially surjective, $(j^*)^{\simeq}$ is a cover [62, Corollary 3.8.12], hence so is the map s. Thus $s \colon P \twoheadrightarrow A$ gives the desired cover. For the converse, we may pick the diagram $\mathcal{K} \to \mathsf{J}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq})$ that is determined by the identity $\mathrm{id} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}$. By assumption we may now find a cover $(s_i)_i \colon \bigsqcup_i A_i \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq}$ such that the diagram

$$\begin{tikzcd} \bigsqcup_i A_i \arrow[r] \arrow[d, two heads] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}^{\triangleright}, \mathsf{J})^{\simeq} \arrow[d, "j^*"'] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq} \arrow[r, "\mathrm{id}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{J})^{\simeq} \end{tikzcd}$$

commutes. Thus j^* is also a cover, as desired. $\square$

PROPOSITION 4.2.3.11. *Every filtered $\mathcal{B}$-category is quasi-filtered.*

PROOF. Suppose that J is a filtered $\mathcal{B}$-category. In light of Lemma 4.2.3.10, it suffices to show that for every finite ∞-category $\mathcal{K}$, every diagram $d \colon \mathcal{K} \to \mathsf{J}$ locally extends to a map $\mathcal{K}^{\triangleright} \to \mathsf{J}$. Note that J being filtered implies that $\mathsf{J}_{d/}^{\mathrm{gpd}} \simeq 1$. Therefore there is a cover $A \twoheadrightarrow 1$ in $\mathcal{B}$ such that $\mathsf{J}_{d/}(A)$ is non-empty. Unwinding the definitions, this exactly provides the desired local extension of d. $\square$

In [15, Éxpose V, Definition 8.11] Deligne chose (a 1-categorical analogue of) Definition 4.2.3.9 to *define* filtered 1-categories internal to a 1-topos, so one might be inclined to surmise that the notions of filteredness and quasi-filteredness coincide. In light of Proposition 4.2.3.11, the second is always implied by the first, and the converse is in fact true in the case where $\mathcal{B} \simeq \mathcal{S}$ (see [57, Proposition 5.4.1.22]). For general ∞-topoi, however, this is no longer the case, the obstruction being the presence of non-trivial ∞-connected objects:

PROPOSITION 4.2.3.12. *Let $\mathsf{G} \in \mathcal{B}$ be an ∞-connective object. Then G is a quasi-filtered $\mathcal{B}$-category.*

PROOF. It is well-known (see [70]) that G is ∞-connective if and only if for an arbitrary finite ∞-category $\mathcal{K}$, the diagonal map $\mathsf{G} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{G})$ is a cover. This clearly implies the claim. $\square$

Since any filtered $\mathcal{B}$-groupoid is necessarily equivalent to the final object, Proposition 4.2.3.12 shows that any non-trivial ∞-connective object gives rise to a $\mathcal{B}$-category that is quasi-filtered but not filtered. In the remainder of this section we will show that is essentially the only obstruction. More precisely we will show that if $\mathcal{B}$ is hypercomplete, then any quasi-filtered $\mathcal{B}$-category is filtered.

LEMMA 4.2.3.13. *Let C be a quasi-filtered $\mathcal{B}$-category and assume that $\mathcal{B}$ is hypercomplete. Then $\mathsf{C}^{\mathrm{gpd}} \simeq 1$.*

PROOF. Since $\mathcal{B}$ is hypercomplete, it suffices to see that the diagonal map $\mathsf{C}^{\mathrm{gpd}} \to \mathrm{map}_{\Omega}(K, \mathsf{C}^{\mathrm{gpd}})$ is a cover for any finite ∞-groupoid K, as in this case $\mathsf{C}^{\mathrm{gpd}}$ is ∞-connective (see [70] again). So it is enough to see that for every $A \in \mathcal{B}$, every map $f\colon K \to \mathsf{C}^{\mathrm{gpd}}(A)$ from a finite ∞-groupoid K locally factors through the point. Replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$ we may assume that $A = 1$, so that f corresponds to a map $g\colon K \to \mathsf{C}^{\mathrm{gpd}}$. Now recall that since the doctrine of finite $\mathcal{B}$-categories is sound and regular, we can find a filtered $\mathcal{B}$-category J and a diagram $d\colon \mathsf{J} \to \mathsf{Fin}_{\mathcal{B}}$ with colimit C. Since $(-)^{\mathrm{gpd}}$ is cocontinuous and K is a compact object of Ω by Corollary 4.2.3.8, the canonical map

$$\operatorname*{colim}_{\mathsf{J}} \mathrm{map}_{\Omega}(K, d(-)^{\mathrm{gpd}}) \to \mathrm{map}_{\Omega}(K, \mathsf{C}^{\mathrm{gpd}})$$

is an equivalence. If we denote the left fibration classifying $\mathrm{map}_{\Omega}(K, d(-)^{\mathrm{gpd}})$ by $p\colon \mathsf{P} \to \mathsf{J}$ it follows from Proposition 3.1.4.1, that we have an essentially surjective functor

$$\mathsf{P} \to \mathsf{P}^{\mathrm{gpd}} \simeq \operatorname*{colim}_{\mathsf{J}} \mathrm{map}_{\Omega}(K, d(-)^{\mathrm{gpd}}.$$

In particular the global section $g\colon 1 \to \mathrm{map}_{\Omega}(K, \mathsf{C}^{\mathrm{gpd}})$ lifts locally to a section of P. In other words, we may find a cover $\bigsqcup_k A_k \twoheadrightarrow 1$ and objects $j_k\colon A_k \to \mathsf{J}$ for each k such that $\pi_{A_k}^* g$ factors through the canonical map $d(j_k)^{\mathrm{gpd}} \to \pi_{A_k}^* \mathsf{C}^{\mathrm{gpd}}$. Since $d(j_k)$ is a finite $\mathcal{B}_{/A}$-category we may pass to a further cover and can therefore assume that $d(j_k)$ is the constant $\mathcal{B}_{/A_k}$-category associated to a finite ∞-category. Therefore, the assumption that C is quasi-filtered implies that locally the map $d(j_k)^{\mathrm{gpd}} \to \pi_{A_k}^* \mathsf{C}^{\mathrm{gpd}}$ factors through the final object, hence the claim follows. $\square$

PROPOSITION 4.2.3.14. *Suppose that $\mathcal{B}$ is hypercomplete. Then any quasi-filtered $\mathcal{B}$-category is filtered.*

PROOF. Let C be quasi-filtered. Since $\mathsf{Fin}_{\mathcal{B}}$ is sound, we only have to verify that for any finite $\mathcal{B}$-category K the diagonal functor $\mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{C})$ is final. Since being final is a local property, we may assume that K is the constant $\mathcal{B}$-category attached to some finite ∞-category $\mathcal{K}$ (see Proposition 4.2.3.4). Now for any diagram $d\colon \mathcal{K} \to \mathsf{C}$, we will show that the slice $\mathcal{B}$-category $\mathsf{C}_{d/}$ is again quasi-filtered. To see this, let $\mathcal{K}'$ be a finite ∞-category and consider an arbitrary map $f\colon \mathcal{K}' \to \mathsf{C}_{d/}(A)$ for some $A \in \mathcal{B}$. Passing from $\mathcal{B}$ to $\mathcal{B}_{/A}$ and using that $\pi_A^*(\mathsf{C}_{d/}) \simeq (\pi_A^* \mathsf{C})_{\pi_A^* d/}$ we may assume that $A \simeq 1$. Since $\mathcal{K}$ is constant, the global sections of $\mathsf{C}_{/d}$ recover the slice ∞-category $(\Gamma \mathsf{C})_{d/}$. Therefore f is given by a map $f'\colon \mathcal{K} \diamond \mathcal{K}' \to \mathsf{C}(1)$ out of the join such that the restriction along $\mathcal{K} \hookrightarrow \mathcal{K} \diamond \mathcal{K}'$ recovers d. But since finite ∞-categories are stable under the join construction, we may find a covering $(s_i)_i\colon \bigsqcup A_i \twoheadrightarrow 1$ and for every i an extension $(\mathcal{K} \diamond \mathcal{K}')^{\triangleright} \to \mathsf{C}(A_i)$ of $\pi_{A_i}^* f'$. But these precisely correspond to maps $(\mathcal{K}')^{\triangleright} \to \mathsf{C}(A_i)$ extending $s_i^* \circ f\colon \mathcal{K}' \to \mathsf{C}(A_i)$, which shows that $\mathsf{C}_{d/}$ is quasi-filtered and that therefore $\mathsf{C}_{d/}^{\mathrm{gpd}} \simeq 1$ by Lemma 4.2.3.13. Repeating the above argument with $\mathcal{B}_{/A}$ instead of $\mathcal{B}$ we get that the same holds for a diagram d in any context A, so the claim follows from Quillen's Theorem A [62, Corollary 4.4.8]. $\square$

4.3. Accessible $\mathcal{B}$-categories

In the classical 1-categorical literature, a *κ-accessible* 1-category is one that can be obtained as the free cocompletion of a small 1-category under κ-filtered colimits [53, 60]. In [57, § 5.4], Lurie generalises this concept to ∞-categories. In this section we will introduce and study an analogous notion for $\mathcal{B}$-categories, that of a U-*accessible* $\mathcal{B}$-category for any sound doctrine U. As with our discussion of U-filteredness, we draw much of our inspiration from ideas in [1] and [78]. Our exposition is tailored to the study of presentable $\mathcal{B}$-categories in § 4.4, so we will not provide an exhaustive treatment of accessibility for $\mathcal{B}$-categories, but rather set up only the basic machinery that we will need for our discussion of presentability later on. We begin in § 4.3.1 by giving the definition of a U-accessible $\mathcal{B}$-category and proving some basic results that will be useful later. In § 4.3.2, we discuss accessible functors. In § 4.3.3, we give a characterisation of U-accessible $\mathcal{B}$-categories as those that are generated by U-compact objects under U-filtered colimits. Finally, we discuss the notion of U-*flatness* in § 4.3.4.

4.3.1. Accessibility. If U is an arbitrary internal class of $\mathcal{B}$-categories and if C is a $\mathcal{B}$-category, we will use the notation $\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) = \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$ to denote the free $\mathsf{Filt}_{\mathsf{U}}$-cocompletion of U. We write $\mathrm{Ind}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ for the underlying ∞-category of global sections. If $\mathsf{U} = \mathsf{Fin}_{\mathcal{B}}$, we will simply write $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C})$ for the associated free $\mathsf{Filt}_{\mathsf{Fin}_{\mathcal{B}}}$-cocompletion and $\mathrm{Ind}_{\mathcal{B}}(\mathsf{C})$ for its underlying ∞-category of global sections. We may now define:

Definition 4.3.1.1. Let U be a sound doctrine. A large $\mathcal{B}$-category D is U-*accessible* if there is a $\mathcal{B}$-category C and an equivalence $\mathsf{D} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. A large $\mathcal{B}$-category is called *accessible* if it is U-accessible for some sound doctrine U.

Remark 4.3.1.2. By combining Remark 4.1.1.2 with Proposition 3.4.1.11, we find that for every $A \in \mathcal{B}$ there is a canonical identification $\pi_A^* \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}_{\mathcal{B}_{/A}}^{\pi_A^* \mathsf{U}}(\pi_A^* \mathsf{C})$ for every $\mathcal{B}$-category C and every sound doctrine U.

Remark 4.3.1.3. In light of Proposition 4.1.3.6 one has $\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}_{\leftarrow}^{\mathrm{reg}}}(\mathsf{C})$ for every $\mathcal{B}$-category C and every internal class U. In particular, a large $\mathcal{B}$-category D is U-accessible if and only if it is $\mathsf{U}_{\leftarrow}^{\mathrm{reg}}$-accessible. When arguing about accessible $\mathcal{B}$-categories, we can therefore always assume that U is in addition *left regular* (cf. Corollary 4.1.3.7).

Suppose that D is a U-accessible $\mathcal{B}$-category, i.e. that we have $\mathsf{D} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ for some $\mathcal{B}$-category C. Recall from § 3.4.1 that there is an inclusion $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. The following proposition shows that this inclusion is in fact an equivalence.

Proposition 4.3.1.4. *For any internal class* U *of* $\mathcal{B}$*-categories and any* $\mathcal{B}$*-category* C*, the fully faithful functor* $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ *is an equivalence. In other words, the Yoneda embedding* $\mathsf{C} \hookrightarrow \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$ *exhibits* $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$ *as the free* $\mathsf{Filt}_{\mathsf{U}}$*-cocompletion of* C*.*

Proof. It will be enough to show that $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$ is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. By combining Remark 4.1.1.2 with Remark 3.3.2.2, this follows once we prove that for any U-filtered $\mathcal{B}$-category J, the colimit of any diagram $d\colon \mathsf{J} \to \underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is contained in $\underline{\mathsf{Small}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{U}}}(\mathsf{C})$. Let us set $F = \operatorname{colim} d$ and let $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ be the associated right fibration. We need to show that $\mathsf{C}_{/F}$ is U-filtered. On account of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \mathsf{RFib}_{\mathsf{C}}$ and in light of Lemma 3.4.4.2, we may regard d as a diagram $d\colon \mathsf{J} \to \mathsf{RFib}_{\mathsf{C}} \hookrightarrow (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$ that takes values in the full subcategory $(\mathsf{Filt}_{\mathsf{U}})_{/\mathsf{C}}$ (as $\mathsf{Filt}_{\mathsf{U}}$ is a colimit class by Remark 4.1.1.7). Let $\mathsf{K} \to \mathsf{C}$ be the colimit of d in $(\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$. As the right fibration $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ is the image of $\mathsf{K} \to \mathsf{C}$ along the localisation functor $L\colon (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}} \to \mathsf{RFib}_{\mathsf{C}}$ (see Proposition 3.1.2.11), there is a final map $\mathsf{K} \to \mathsf{C}_{/F}$ over C. It therefore suffices to show that K is U-filtered. Now Proposition 3.1.6.3 implies that K is the colimit of the diagram $(\pi_{\mathsf{C}})_! d\colon \mathsf{J} \to (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}} \to \mathsf{Cat}_{\mathcal{B}}$. By construction, this diagram takes values in $\mathsf{Filt}_{\mathsf{U}}$. Therefore, the result follows from Proposition 4.1.1.8. □

Remark 4.3.1.5. In light of Proposition 4.3.1.4, if C is a $\mathcal{B}$-category and if U is a sound doctrine, Remark 4.1.1.7 implies that a presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context $A \in \mathcal{B}$ is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ if and only if the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/F}$ is $\pi_A^* \mathsf{U}$-filtered.

For later use, let us record that our notion of accessibility is stable under the formation of slice $\mathcal{B}$-categories:

Proposition 4.3.1.6. *Let* U *be a sound doctrine and let* D *be a* U*-accessible* $\mathcal{B}$*-category. Then* $\mathsf{D}_{/d}$ *is* $\pi_A^* \mathsf{U}$*-accessible, for any choice of object* $d\colon A \to \mathsf{D}$*.*

Proof. Using Remark 4.3.1.2, we may assume that $A \simeq 1$. Choose a $\mathcal{B}$-category C such that $\mathsf{D} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$. Let $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ be the presheaf that corresponds to d under this equivalence. We then obtain a commutative diagram

$$\begin{tikzcd} \mathsf{C}_{/F} \arrow[r, hook] \arrow[d, "p"] & \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[r, hook] \arrow[d, "(\pi_F)_!"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \arrow[d, "(\pi_F)_!"] \\ \mathsf{C} \arrow[r, hook, "h_{\mathsf{C}}"] & \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \end{tikzcd}$$

in which both squares are cartesian. By Lemma 3.3.1.5, the vertical map on the right can be identified with $p_!\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that the upper row in the above diagram recovers the Yoneda embedding $h_{\mathsf{C}_{/F}}$. With respect to this identification, a presheaf on $\mathsf{C}_{/F}$ is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C})_{/F}$ precisely if the domain of the associated right fibration is U-filtered. We therefore obtain an equivalence $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}_{/F})$, hence the result follows. □

4.3.2. Accessible functors. It will be convenient to also have a notion of accessibility for functors between accessible $\mathcal{B}$-categories at our disposal:

Definition 4.3.2.1. Let U be a sound doctrine. A functor $f\colon \mathsf{C} \to \mathsf{D}$ of large $\mathcal{B}$-categories is U-*accessible* if C and D are $\mathsf{Filt}_{\mathsf{U}}$-cocomplete and f is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous. We will call f *accessible* if it is U-accessible for some sound doctrine U. We denote by $\underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ the full subcategory spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ such that the corresponding $\mathcal{B}_{/A}$-functor $\pi_A^*\mathsf{C} \to \pi_A^*\mathsf{D}$ is accessible. We will denote by $\mathrm{Fun}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ the underlying ∞-category of global sections.

Remark 4.3.2.2. Let $f\colon \mathsf{C} \to \mathsf{D}$ be U-accessible for some sound doctrine U. By Remark 4.2.2.22 we may find a $\mathcal{B}$-regular cardinal κ such that $\mathsf{U} \subset \mathrm{Cat}^{\kappa}_{\mathcal{B}}$. It follows that a functor is accessible if and only if it is $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-accessible for some $\mathcal{B}$-regular cardinal κ.

Remark 4.3.2.3. Let $f\colon A \to \underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ be an arbitrary object. By definition, this means that there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that the functors $s_i^* f\colon \pi_{A_i}^*\mathsf{C} \to \pi_{A_i}^*\mathsf{D}$ are accessible for all $i \in I$. By Remark 4.3.2.2, we may find a $\mathcal{B}_{/A}$-regular cardinal κ such that all A_i are κ-compact (in $\mathcal{B}_{/A}$) and $s_i^* f$ is $\mathrm{Cat}^{\kappa}_{\mathcal{B}/A_i}$-accessible for every κ. Hence Remarks 4.1.1.2 and 4.2.2.2 together with Remark 3.2.2.3 imply that f is $\mathsf{Filt}_{\mathrm{Cat}^{\kappa}_{\mathcal{B}/A}}$-cocontinuous, so in particular accessible. Thus, an object $f\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ is contained in $\underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D})$ if and only if f defines an accessible functor between $\mathcal{B}_{/A}$-categories. In particular, one obtains a canonical equivalence $\pi_A^*\underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{D}) \simeq \underline{\mathsf{Fun}}^{\mathrm{acc}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{C}, \pi_A^*\mathsf{D})$ for every $A \in \mathcal{B}$.

Somewhat surprisingly, provided that both domain and codomain have a sufficient amount of colimits, accessibility of a functor between $\mathcal{B}$-categories is an entirely section-wise concept:

Proposition 4.3.2.4. *Let κ be a $\mathcal{B}$-regular cardinal and let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor between cocomplete $\mathcal{B}$-categories that is section-wise κ-accessible. Then the functor f is $\mathsf{Filt}_{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}$-accessible.*

Proof. As κ is $\mathcal{B}$-regular, Remarks 4.2.2.2 and 4.1.1.2 imply that it suffices to show that f preserves the colimit of every diagram $d\colon \mathsf{J} \to \mathsf{C}$ with J a $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-filtered $\mathcal{B}$-category. As C is cocomplete, there exists an extension $d'\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}} \to \mathsf{C}$ of d. By Corollary 4.1.5.11, the inclusion $\mathsf{J} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}}$ is final, hence we may replace J by $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{J})^{\mathsf{Cat}^{\kappa}_{\mathcal{B}}\text{-cpt}}$ and d by d' and can thus assume that J is $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-cocomplete (see Proposition 4.1.5.4). Using Lemma 4.2.2.20 and Remark 4.1.1.7, we can further reduce to the case where J is the constant $\mathcal{B}$-category that is associated with an ∞-category with κ-small colimits. As by [57, Proposition 5.3.3.3] every such ∞-category is κ-filtered, the result follows. □

Corollary 4.3.2.5. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor of cocomplete large $\mathcal{B}$-categories. Then f is accessible if and only if f is section-wise accessible.*

Proof. By Remark 4.3.2.2, we can assume that f is $\mathrm{Cat}^{\kappa}_{\mathcal{B}}$-accessible for some $\mathcal{B}$-regular cardinal κ. Then it follows from Lemma 4.2.2.19 that f commutes with colimits indexed by constant $\mathcal{B}$-categories attached to κ-filtered ∞-categories. In other words, $f(A)$ commutes with κ-filtered colimits for every $A \in \mathcal{B}$ and is thus section-wise accessible. For the converse, we pick a small full subcategory $\mathcal{G} \hookrightarrow \mathcal{B}$ that generates $\mathcal{B}$ under small colimits. Then we may find a $\mathcal{B}$-regular cardinal κ such that $f(G)$ is κ-accessible for every $G \in \mathcal{G}$. Since the preservation of colimits is a local condition by Remark 3.1.2.1 and since every object $A \in \mathcal{B}$ admits a cover by objects in $\mathcal{G}$, we conclude that $f(A)$ preserves κ-filtered colimits for all $A \in \mathcal{B}$. Therefore f is accessible by Proposition 4.3.2.4. $\square$

4.3.3. U-compact objects in accessible $\mathcal{B}$-categories. In [57, Proposition 5.4.2.2], Lurie characterises κ-accessible ∞-categories as those that are generated by a small collection of κ-compact objects under κ-filtered colimits. In this section, our goal is to obtain an analogue of this statement for accessible $\mathcal{B}$-categories. We begin with the following characterisation of the U-compact objects in a U-accessible $\mathcal{B}$-category:

Proposition 4.3.3.1. *Let U be an internal class of $\mathcal{B}$-categories, let C be a $\mathcal{B}$-category and let $\mathsf{D} = \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Then there is an equivalence $\mathsf{D}^{\mathsf{U}\text{-cpt}} \simeq \mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$ of full subcategories in D. In particular, $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small.*

Proof. In light of Remark 4.1.5.6, the second claim follows immediately from the first. Now by Yoneda's lemma and the fact that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits, every representable presheaf on C defines a U-compact object in D. In other words, one obtains an inclusion $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$. By combining this observation with Proposition 4.1.5.8, one obtains an inclusion $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C}) \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$. Conversely, let $F\colon A \to \mathsf{D}^{\mathsf{U}\text{-cpt}}$ be an arbitrary object. We need to show that F is contained in $\mathsf{Ret}_{\mathsf{D}}(\mathsf{C})$. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (which is made possible by Remark 4.1.5.2 and 4.3.1.2), we can assume $A \simeq 1$. The desired result thus follows from Lemma 4.1.5.9. $\square$

We can now state and prove our characterisation of U-accessible $\mathcal{B}$-categories. To that end, if D is a $\mathsf{Filt}_{\mathsf{U}}$-cocomplete $\mathcal{B}$-category and $\mathsf{C} \hookrightarrow \mathsf{D}$ is a full subcategory, we shall say that D is *generated* under $\mathsf{Filt}_{\mathsf{U}}$-colimits by C if D is the smallest full subcategory of itself that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits and contains C. We now obtain:

Proposition 4.3.3.2. *Let U be a sound doctrine and let D be a large $\mathcal{B}$-category. Then the following are equivalent:*

1. *D is U-accessible;*
2. *D is locally small and $\mathsf{Filt}_{\mathsf{U}}$-cocomplete, the (a priori large) $\mathcal{B}$-category $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small and generates D under $\mathsf{Filt}_{\mathsf{U}}$-colimits;*
3. *D is $\mathsf{Filt}_{\mathsf{U}}$-cocomplete, and there is a small full subcategory $\mathsf{C} \hookrightarrow \mathsf{D}$ such that $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$ and such that C generates D under $\mathsf{Filt}_{\mathsf{U}}$-colimits.*

Proof. If D is U-accessible, there is a small $\mathcal{B}$-category C and an equivalence $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. In particular, D is locally small and $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. Furthermore, Proposition 4.3.3.1 implies that $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small. Since D is generated by C under $\mathsf{Filt}_{\mathsf{U}}$-colimits and therefore by $\mathsf{D}^{\mathsf{U}\text{-cpt}}$, we conclude that (1) implies (2). Moreover, (2) trivially implies (3), and the fact that (3) implies (1) immediately follows from Proposition 3.4.2.4. $\square$

Recall from Definition 2.4.4.7 that a localisation $L\colon \mathsf{D} \to \mathsf{E}$ is a *Bousfield localisation* if L admits a (necessarily fully faithful) right adjoint i. Proposition 4.3.3.2 now implies:

Corollary 4.3.3.3. *Let U be a sound doctrine and let D be a U-accessible $\mathcal{B}$-category. Suppose that E is a Bousfield localisation of D such that the inclusion $i\colon \mathsf{E} \hookrightarrow \mathsf{D}$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous. Then E is U-accessible as well.*

Proof. Let $\mathsf{C} \hookrightarrow \mathsf{E}$ be the image of $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ along the localisation functor $L\colon \mathsf{D} \to \mathsf{E}$. As E is locally small and $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small by Proposition 4.3.3.2, the $\mathcal{B}$-category C is small as well [62, Lemma 4.7.5]. In light of the adjunction $L \dashv i$, the assumption that i is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous implies that L preserves U-compact objects. In other words, we have $\mathsf{C} \hookrightarrow \mathsf{D}^{\mathsf{U}\text{-cpt}}$. By Proposition 4.3.3.2, the large $\mathcal{B}$-category D

is generated by $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ under $\mathsf{Filt}_{\mathsf{U}}$-colimits, i.e. D is the smallest full subcategory of itself that contains $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ and that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits. Let $\mathsf{E}' \hookrightarrow \mathsf{E}$ be the smallest full subcategory that contains C and that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits, and let us consider the commutative diagram

$$\begin{tikzcd} \mathsf{D}^{\mathsf{U}\text{-cpt}} \arrow[r, hook] \arrow[d, two heads] & \mathsf{D}' \arrow[r, hook] \arrow[d] & \mathsf{D} \arrow[d, two heads, "L"] \\ \mathsf{C} \arrow[r, hook] & \mathsf{E}' \arrow[r, hook] & \mathsf{E} \end{tikzcd}$$

in which the right square is a pullback. Since L is cocontinuous, the inclusion $\mathsf{D}' \hookrightarrow \mathsf{D}$ is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits (using Lemma 4.1.4.5) and must therefore be an equivalence. As the inclusion i is a section of L, this implies that the inclusion $\mathsf{E}' \hookrightarrow \mathsf{E}$ is an equivalence as well. By using Proposition 4.3.3.2, we thus conclude that D is U-accessible. $\square$

Definition 4.3.3.4. A Bousfield localisation $L\colon \mathsf{D} \to \mathsf{E}$ of a U-accessible $\mathcal{B}$-category D is said to be U-*accessible* if the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$ is U-accessible. More generally, a Bousfield localisation $L\colon \mathsf{D} \to \mathsf{E}$ is accessible if there is a sound doctrine U such that D is U-accessible and the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous.

Remark 4.3.3.5. Proposition 4.3.3.2 also shows that accessibility is a *local* condition: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$ and if U is a sound doctrine, then a large $\mathcal{B}$-category D being U-accessible is equivalent to each $\pi_{A_i}^*\mathsf{D}$ being $\pi_{A_i}^*\mathsf{U}$-accessible. In fact, Remark 4.3.1.2 shows that we have an equivalence $\pi_A^*\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}}(\pi_A^*\mathsf{C})$ for every $\mathcal{B}$-category C and every $A \in \mathcal{B}$, hence the condition is necessary. To show that it is sufficient, first recall that since $\mathsf{Filt}_{\mathsf{U}}$-cocompleteness is a local condition by Remark 3.2.2.3 we deduce that D must be $\mathsf{Filt}_{\mathsf{U}}$-cocomplete. Moreover, if $\mathsf{E} \hookrightarrow \mathsf{D}$ is the smallest full subcategory that is closed under $\mathsf{Filt}_{\mathsf{U}}$-colimits in D and that contains $\mathsf{D}^{\mathsf{U}\text{-cpt}}$, the fact that $\pi_{A_i}^*\mathsf{E}$ is closed under $\mathsf{Filt}_{\pi_{A_i}^*\mathsf{U}}$-colimits and Remark 4.1.5.2 imply that the inclusion $\mathsf{E} \hookrightarrow \mathsf{D}$ is locally an equivalence and therefore already an equivalence. To show that D is U-accessible, Proposition 4.3.3.2 thus implies that it suffices to verify that also the condition of a large $\mathcal{B}$-category to be (locally) small is local in $\mathcal{B}$, which is clear from the definitions.

Proposition 4.3.3.2 can furthermore be used to show that presheaf $\mathcal{B}$-categories are U-accessible for *every* choice of a sound doctrine U:

Proposition 4.3.3.6. *For every $\mathcal{B}$-category C and every sound doctrine U, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is U-accessible.*

Proof. In light of Remark 4.3.1.3, we can assume that U is a left regular doctrine. By Proposition 4.1.5.10, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}$ is small. Using Proposition 4.3.3.2, it therefore suffices to show that every object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ can be obtained as a U-filtered colimit of U-compact objects. If $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is an arbitrary presheaf, Lemma 4.1.4.4 shows that F is the colimit of the diagram $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}_{/F} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. By Lemma 4.1.4.5, the $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}}_{/F}$ is $\mathrm{op}(\mathsf{U})$-cocomplete and therefore in particular U-filtered (Example 4.1.2.4). Hence F is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathsf{U}\text{-cpt}})$. Finally, upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ (which is made possible by Remark 4.1.5.2), the same conclusion holds for objects in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context A, which finishes the proof. $\square$

4.3.4. Flatness. Recall from § 3.4.1 that if C is a $\mathcal{B}$-category, the functor of left Kan extension along the Yoneda embedding $h_{\mathsf{C}^{\mathrm{op}}}\colon \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ induces an equivalence

$$(h_{\mathsf{C}^{\mathrm{op}}})_!\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega), \Omega)$$

where the right-hand side denotes the large $\mathcal{B}$-category of cocontinuous functors between $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ and Ω.

Definition 4.3.4.1. Let C be a $\mathcal{B}$-category and let U be an arbitrary internal class of $\mathcal{B}$-categories. A presheaf $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is said to be U-*flat* if the functor

$$\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C}, \Omega_{\mathcal{B}_{/A}}) \to \Omega_{\mathcal{B}_{/A}}$$

that is encoded by $(h_{\mathsf{C}^{\mathrm{op}}})_!(F)$ is $\pi_A^*\mathsf{U}$-continuous. We denote the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by the U-flat presheaves by $\underline{\mathsf{Flat}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$, and we denote its underlying ∞-category of global sections by $\mathrm{Flat}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$.

Remark 4.3.4.2. In the situation of Definition 4.3.4.1, the fact that U-continuity is a local condition by Remark 3.2.2.3 together with Remark 3.3.3.2 and [62, Lemma 4.7.14] implies that the presheaf F is U-flat if and only if for every cover $(s_i)\colon \bigsqcup A_i \twoheadrightarrow A$ in $\mathcal{B}$ the presheaf $s_i^* F$ is U-flat. In particular, *every* object in $\underline{\mathsf{Flat}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C})$ is U-flat, and there is a canonical equivalence $\pi_A^* \underline{\mathsf{Flat}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{Flat}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}}(\pi_A^*\mathsf{U})$ for every $A \in \mathcal{B}$.

Lemma 4.3.4.3. *Let C be a $\mathcal{B}$-category and let $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ be a presheaf. Then the Yoneda extension $(h_{\mathsf{C}^{\mathrm{op}}})_!(F)$ is equivalent to the composition*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega) \xrightarrow{p^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \Omega) \xrightarrow{\mathrm{colim}} \Omega,$$

where $p\colon \mathsf{C}_{/F} \to \mathsf{C}$ is the right fibration that is classified by F.

Proof. As both p^* and colim are cocontinuous functors, the universal property of presheaf $\mathcal{B}$-categories implies that it suffices to find an equivalence $\mathrm{colim}\, p^* h_{\mathsf{C}^{\mathrm{op}}} \simeq F$. Let us denote by $h_{/F}\colon \mathsf{C}_{/F} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ the functor that is induced by the Yoneda embedding h_{C} by taking slice $\mathcal{B}$-categories. Now there is a commutative diagram

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \arrow[r, "h_{\mathsf{C}^{\mathrm{op}}}"] \arrow[d, "h_{\mathsf{C}}^{\mathrm{op}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega) \arrow[r, "p^*"] \arrow[d, "(h_{\mathsf{C}})_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \Omega) \arrow[d, "(h_{/F})_!"'] \arrow[dr, bend left, "\mathrm{colim}"] & \\
\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{\mathrm{op}} \arrow[r, "h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \Omega) \arrow[r, "(\pi_F)_!^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}, \Omega) \arrow[r, "\mathrm{id}_F^*"] & \Omega
\end{tikzcd}$$

in which the commutativity of the right square follows from the straightening equivalence for right fibrations (Theorem 2.1.11.5) together with p being a right fibration and therefore *proper* in the sense of [62, § 4.4], see [62, Proposition 4.4.7]. In light of Yoneda's lemma, it is now immediate that the composition of the left vertical map with the lower row in the above diagram recovers F, as desired. $\square$

Recall from Example 3.1.1.11 that if C is a $\mathcal{B}$-category that admits a final object $1_{\mathsf{C}}\colon 1 \to \mathsf{C}$, then this object is the limit of the unique diagram $\varnothing \to \mathsf{C}$. In other words, the map $1_{\mathsf{C}}\colon 1 \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\varnothing, \mathsf{C}) \to \mathsf{C}$ is right adjoint to the unique functor $\pi_{\mathsf{C}}\colon \mathsf{C} \to 1$. We will denote by $\pi\colon \mathrm{id}_{\mathsf{C}} \to 1_{\mathsf{C}}\pi_{\mathsf{C}}$ the associated adjunction unit.

Lemma 4.3.4.4. *If $p\colon \mathsf{P} \to \mathsf{C}$ is a right fibration of $\mathcal{B}$-categories, the commutative square*

$$\begin{tikzcd}
\mathrm{id}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \arrow[r, "\eta"] \arrow[d, "\pi"] & p^* p_! \arrow[d, "p^* p_! \pi"] \\
1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \pi_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \arrow[r, "\eta 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \pi_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}"] & p^* p_! 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})} \pi_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}
\end{tikzcd}$$

is a pullback square in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}), \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}))$.

Proof. By Proposition 3.1.3.2, it suffices to show that for every object $F\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ in context $A \in \mathcal{B}$ the induced diagram

$$\begin{tikzcd}
F \arrow[r, "\eta F"] \arrow[d, "\pi"] & p^* p_!(F) \arrow[d, "p^* p_! \pi"] \\
\pi_A^*(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}) \arrow[r, "\eta \pi_A^*(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})})"] & p^* p_!(\pi_A^*(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}))
\end{tikzcd}$$

is a pullback. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we can assume $A \simeq 1$. In light of the straightening equivalence for right fibrations, this diagram corresponds to the commutative square

$$\begin{tikzcd} \mathsf{P}_{/F} \arrow[r] \arrow[d] & \mathsf{P}_{/F} \times_{\mathsf{C}} \mathsf{P} \arrow[d] \\ \mathsf{P} \arrow[r] & \mathsf{P} \times_{\mathsf{C}} \mathsf{P} \end{tikzcd}$$

of right fibrations over P. As this square is clearly a pullback, the claim follows. $\square$

LEMMA 4.3.4.5. *Let I be a $\mathcal{B}$-category and let*

$$\begin{tikzcd} d \arrow[r, "\varphi"] \arrow[d] & h \arrow[d] \\ \mathrm{diag}(\mathsf{G}) \arrow[r, "\mathrm{diag}(s)"] & \mathrm{diag}(\mathsf{H}) \end{tikzcd}$$

be a pullback square in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$, where $s \colon \mathsf{G} \to \mathsf{H}$ is an arbitrary map of $\mathcal{B}$-groupoids. Then the commutative square

$$\begin{tikzcd} \mathrm{colim}(d) \arrow[r, "\mathrm{colim}(\varphi)"] \arrow[d] & \mathrm{colim}(h) \arrow[d] \\ \mathsf{G} \arrow[r, "s"] & \mathsf{H} \end{tikzcd}$$

that is obtained by transposing the first square across the adjunction $\mathrm{colim} \dashv \mathrm{diag}$ *is a pullback square as well.*

PROOF. In light of the Grothendieck construction, the above pullback square corresponds to a pullback square

$$\begin{tikzcd} \mathsf{I}_{/d} \arrow[r] \arrow[d] & \mathsf{I}_{/h} \arrow[d] \\ \mathsf{G} \arrow[r, "s"] & \mathsf{H} \end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$. By Proposition 3.1.4.1, we need to show that the groupoidification functor carries this diagram to a pullback square in $\mathcal{B}$. As s is a right fibration and therefore proper [62, Proposition 4.4.7], this is immediate. $\square$

PROPOSITION 4.3.4.6. *Let U be a sound internal class and let C be a $\mathcal{B}$-category. Then there is an equivalence $\underline{\mathsf{Flat}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ of full subcategories in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.*

PROOF. In light of Remarks 4.3.1.2 and 4.3.4.2, it will be enough to show that a presheaf $F \colon \mathsf{C}^{\mathrm{op}} \to \Omega$ defines an object of $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ if and only if it is U-flat. So suppose first that F is contained in $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. Then $\mathsf{C}_{/F}$ is U-filtered. Let $p \colon \mathsf{C}_{/F} \to \mathsf{C}$ be the projection. By Lemma 4.3.4.3, the Yoneda extension $(h_{\mathsf{C}^{\mathrm{op}}})_! F$ can be computed as the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega) \xrightarrow{p^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \Omega) \xrightarrow{\mathrm{colim}} \Omega,$$

and since both p^* and colim are U-continuous, we deduce that F is U-flat.

Conversely, suppose that F is U-flat. By Lemma 4.3.4.4, the commutative square

$$\begin{tikzcd} h_{\mathsf{C}_{/F}} \arrow[r, "\eta h_{\mathsf{C}_{/F}}"] \arrow[d, "\pi h_{\mathsf{C}_{/F}}"] & p^* p_! h_{\mathsf{C}_{/F}} \arrow[d, "p^* p_! \pi h_{\mathsf{C}_{/F}}"] \\ 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})} \pi_{\mathsf{C}_{/F}} \arrow[r, "\eta 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})} \pi_{\mathsf{C}_{/F}}"] \arrow[d, "\simeq"] & p^* p_! 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})} \pi_{\mathsf{C}_{/F}} \arrow[d, "\simeq"] \\ \mathrm{diag}_{\mathsf{C}_{/F}}(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})}) \arrow[r, "\mathrm{diag}_{\mathsf{C}_{/F}}(\eta 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})})"] & \mathrm{diag}_{\mathsf{C}_{/F}}(p^* F) \end{tikzcd}$$

is a pullback in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}))$. By Proposition 3.3.1.1, the composition of the two vertical maps on the left is a colimit cocone, hence so is the composition of the two vertical maps on the right, for $p^*p_!$ preserves all colimits. Let $d\colon \mathsf{K} \to (\mathsf{C}_{/F})^{\mathrm{op}}$ be a diagram with $\mathsf{K} \in \mathsf{U}(1)$. By postcomposition with $\lim_{\mathsf{K}} d^* \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \Omega) \to \Omega$, the above pullback square induces a cartesian square

$$\begin{tikzcd}
\lim_{\mathsf{K}} d^* h_{\mathsf{C}_{/F}} \arrow[r] \arrow[d] & \lim_{\mathsf{K}} d^* p^* p_! h_{\mathsf{C}_{/F}} \arrow[d] \\
\mathrm{diag}_{\mathsf{C}_{/F}}(1_\Omega) \arrow[r] & \mathrm{diag}_{\mathsf{C}_{/F}}(\lim_{\mathsf{K}} d^* p^* F).
\end{tikzcd}$$

We claim that the right vertical map in the this last diagram is still a colimit cocone. To see this, note that the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \Omega) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \Omega))$ carries the diagram $d^*p^*p_!h_{\mathsf{C}_{/F}}$ to the composition

$$\mathsf{K} \xrightarrow{d} (\mathsf{C}_{/F})^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} \mathsf{C}^{\mathrm{op}} \xrightarrow{h_{\mathsf{C}^{\mathrm{op}}}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega) \xrightarrow{p^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}_{/F}, \Omega).$$

Now the functor $\lim_{\mathsf{K}}$ preserving the colimit of $d^*p^*p_!h_{\mathsf{C}_{/F}}$ is equivalent to the colimit functor $\mathrm{colim}_{\mathsf{C}_{/F}}$ preserving the limit of the diagram $p^*h_{\mathsf{C}^{\mathrm{op}}}p^{\mathrm{op}}d$ (cf. the argument in Remark 4.1.1.4). As the functor p^* commutes with all limits, this in turn follows once $\mathrm{colim}_{\mathsf{C}_{/F}} p^*$ preserves the limit of $h_{\mathsf{C}^{\mathrm{op}}}p^{\mathrm{op}}d$, which follows from the equivalence $\mathrm{colim}_{\mathsf{C}_{/F}} p^* \simeq (h_{\mathsf{C}^{\mathrm{op}}})_!(F)$ from Lemma 4.3.4.3 and the assumption that F is U-flat. As a consequence, we now deduce from Lemma 4.3.4.5 that the map $\mathrm{colim}_{\mathsf{C}_{/F}} \lim_{\mathsf{K}} d^* h_{\mathsf{C}_{/F}} \to 1_\Omega$ must be an equivalence. Now observe that the proof of Proposition 4.1.2.5, shows that $(\mathsf{C}_{/F})_{d^{\mathrm{op}}/}$ classifies $\lim_{\mathsf{K}} d^* h_{\mathsf{C}_{/F}}$ and thus we get $(\mathsf{C}_{/F})^{\mathrm{gpd}}_{d^{\mathrm{op}}/} \simeq 1$ by Proposition 3.1.4.1. As d was chosen arbitrarily and as replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ allows us to derive the same conclusion for any diagram $d\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, (\mathsf{C}_{/F})^{\mathrm{op}})$ in context $A \in \mathcal{B}$, this shows that $\mathsf{C}_{/F}$ is weakly U-filtered by Quillen's Theorem A and therefore U-filtered by soundness of U. Hence F is contained in $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. $\square$

4.4. Presentable $\mathcal{B}$-categories

In this section we introduce and study *presentable* $\mathcal{B}$-categories. Classically, a (locally) presentable 1-category is one that is locally small and is generated by a small collection of κ-compact objects under small colimits [26]. In [57, § 5.5], Lurie generalised this concept to ∞-categories. In particular, his treatment contains a multitude of equivalent characterisations of presentability [57, Theorem 5.5.1.1]. One of the main goals of this section is to obtain a comparable result for $\mathcal{B}$-categories. As a starting point, we chose to *define* a presentable $\mathcal{B}$-category as a *Bousfield* localisation of a presheaf $\mathcal{B}$-category at a (small) subcategory. To make sense of this, we need to study the notion of *local objects* in a $\mathcal{B}$-category, which we do in § 4.4.1. In § 4.4.2, we formally define presentable $\mathcal{B}$-categories and prove our main result about various different characterisations of this condition (Theorem 4.4.2.4), building upon our work on accessible $\mathcal{B}$-categories. In § 4.4.3, we discuss adjoint functor theorems for presentable $\mathcal{B}$-categories, and in § 4.4.4 we construct large $\mathcal{B}$-categories of presentable $\mathcal{B}$-categories and show that these are complete and cocomplete. We apply some of the preceding results to generalize the construction of § 3.4.1 in order to define free cocompletions *with relations* in § 4.4.5. Finally, we discuss the notion of U-*sheaves* in § 4.4.6: these are U-continuous functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$, where C is an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category and D is a large complete $\mathcal{B}$-category. We show that if D is presentable, such U-sheaves form a presentable $\mathcal{B}$-category as well, and that this provides yet another equivalent characterisation of the notion of presentability.

4.4.1. Local objects. Recall from § 2.2.3 the definition of a localisation of a $\mathcal{B}$-category. If $j\colon \mathsf{S} \to \mathsf{D}$ is a functor of $\mathcal{B}$-categories, we obtain a localisation functor $L\colon \mathsf{D} \to \mathsf{S}^{-1}\mathsf{D} = \mathsf{D} \sqcup_{\mathsf{S}} \mathsf{S}^{\mathrm{gpd}}$. If E is an arbitrary $\mathcal{B}$-category, L satisfies the universal property that $L^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{D}, \mathsf{E}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$ is fully faithful and identifies the domain with the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$ that is spanned by those functors $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{D}$ whose restriction along $\pi_A^*(j)$ factors through the inclusion $\pi_A^*\mathsf{D}^{\simeq} \hookrightarrow \pi_A^*\mathsf{D}$. We may now define:

Definition 4.4.1.1. If $\mathsf{S} \to \mathsf{D}$ is a functor, we define the associated *$\mathcal{B}$-category* $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *of* S*-local objects in* D as the full subcategory of D that is defined via the pullback

$$\begin{tikzcd} \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \arrow[r, hook] \arrow[d, hook, "i"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{D}) \arrow[d, hook, "L^*"] \\ \mathsf{D} \arrow[r, hook, "h"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$. We refer to an object $d\colon A \to \mathsf{D}$ as being S-*local* if it is contained in $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$.

Remark 4.4.1.2. If $A \in \mathcal{B}$ is an arbitrary object, we deduce from [62, Lemma 4.2.3 and Lemma 4.7.14] that there is a canonical equivalence $\pi_A^* \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \simeq \mathsf{Loc}_{\pi_A^*\mathsf{S}}(\pi_A^*\mathsf{D})$ of full subcategories in $\pi_A^*\mathsf{D}$. In particular, this implies that an object $d\colon A \to \mathsf{D}$ is S-local if and only if its transpose $\overline{d}\colon 1_{\mathcal{B}_{/A}} \to \pi_A^*\mathsf{D}$ defines a $\pi_A^*\mathsf{S}$-local object.

Remark 4.4.1.3. Explicitly, an object $d\colon 1 \to \mathsf{D}$ is contained in $\mathsf{Loc}_{\mathsf{S}}$ precisely if the restriction of the presheaf $h(d)$ along j factors through the inclusion $\Omega^{\simeq} \hookrightarrow \Omega$, which is the case if and only if for every map $s\colon e \to e'$ in S in context $A \in \mathcal{B}$ the morphism $j(s)^*\colon \mathrm{map}_{\mathsf{D}}(j(e'), \pi_A^* d) \to \mathrm{map}_{\mathsf{D}}(j(e), \pi_A^* d)$ is an equivalence of $\mathcal{B}_{/A}$-groupoids. By Remark 4.4.1.2, an analogous description holds for S-local objects in arbitrary context.

Remark 4.4.1.4. In the situation of Definition 4.4.1.1, we deduce from Proposition 2.2.3.9 that if $\mathsf{T} \hookrightarrow \mathsf{D}$ is the 1-*image* of the map $\mathsf{S} \to \mathsf{D}$ (in the sense of Definition 2.2.2.6), the canonical map $\mathsf{S}^{-1}\mathsf{D} \to \mathsf{T}^{-1}\mathsf{D}$ is an equivalence. Consequently, the induced map $\mathsf{Loc}_{\mathsf{T}}(\mathsf{D}) \to \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ must be an equivalence as well. Therefore, we may always assume that S is a *subcategory* of D.

Remark 4.4.1.5. Suppose that $(f_i\colon c_i \to d_i)_{i\in I}$ is a (small) family of maps in D, with $A_i \in \mathcal{B}$ being the context of f_i. By the discussion in § 2.2.2, the subcategory $\mathsf{S} \hookrightarrow \mathsf{D}$ that is *generated* by this family is given by the 1-image of the induced map $\bigsqcup_i \Delta^1 \otimes A_i \to \mathsf{D}$. By combining Remark 4.4.1.3 and 4.4.1.4, an object $d\colon 1 \to \mathsf{D}$ is S-local if and for each $i \in I$ the map

$$f_i^*\colon \mathrm{map}_{\mathsf{D}}(d_i, \pi_{A_i}^* d) \to \mathrm{map}_{\mathsf{D}}(c_i, \pi_{A_i}^* d)$$

is an equivalence in $\mathcal{B}_{/A_i}$.

The theory of local objects is intimately connected to the notion of *Bousfield localisations*, i.e. of reflective subcategories:

Proposition 4.4.1.6. *Let* D *be a* $\mathcal{B}$*-category and let* $L\colon \mathsf{D} \to \mathsf{C}$ *be a Bousfield localisation. Let* $\mathsf{S} = L^{-1}(\mathsf{C}^{\simeq}) \hookrightarrow \mathsf{D}$. *Then the inclusion* $:\mathsf{C} \hookrightarrow \mathsf{D}$ *of* L *induces an equivalence* $\mathsf{C} \simeq \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *of full subcategories in* D. *Furthermore, if* D *is* U*-accessible and* L *is a* U*-accessible Bousfield localisation, there is a* small *subcategory* $\mathsf{T} \hookrightarrow \mathsf{S}$ *such that* $\mathsf{C} \simeq \mathsf{Loc}_{\mathsf{T}}(\mathsf{D})$.

Proof. We begin with the first statement. By Proposition 2.4.4.6, the functor $L\colon \mathsf{D} \to \mathsf{C}$ identifies C with the localisation $\mathsf{S}^{-1}\mathsf{D}$. In light of the very definition of $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$, the claim thus follows once we show that the commutative square

$$\begin{tikzcd} \mathsf{C} \arrow[r, hook, "h_{\mathsf{C}}"] \arrow[d, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, hook, "L^*"] \\ \mathsf{D} \arrow[r, hook, "h_{\mathsf{D}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

is a pullback. Using [62, Lemma 4.7.14], it will be enough to show that if $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is a presheaf such that $L^*(F)\colon \mathsf{D}^{\mathrm{op}} \to \Omega$ is representable by an object $d\colon 1 \to \Omega$, then F is representable as well. This immediately follows from the computation

$$F \simeq L_! L^* F \simeq L_! h_{\mathsf{D}}(d) \simeq h_{\mathsf{C}} L(d),$$

see Corollary 2.4.3.3. Now if D is U-accessible and L is a U-accessible Bousfield localisation, let us set $\mathsf{E} = \mathsf{D}^{\mathsf{U}\text{-cpt}}$ and $\mathsf{T} = i^{-1}(\mathsf{S}) \hookrightarrow \mathsf{E}$, where $i\colon \mathsf{E} \hookrightarrow \mathsf{D}$ is the inclusion. Since E is small by Proposition 4.3.3.1, so is T, and we obtain a full inclusion $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{Loc}_{\mathsf{T}}(\mathsf{D})$. We need to show that this is an equivalence. By Remark 4.4.1.2, it will be enough to show that every T-local object $d\colon 1 \to \mathsf{D}$ is already S-local. Let $\eta\colon \mathrm{id}_{\mathsf{D}} \to iL$ be the adjunction unit. We then obtain a map

$$\eta^*\colon \operatorname{map}_{\mathsf{D}}(iL(-), d) \to \operatorname{map}_{\mathsf{D}}(-, d),$$

and since d is T-local the restriction of η^* to E is an equivalence. But as both domain and codomain of this map are $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous when viewed as functors $\mathsf{D} \to \Omega^{\mathrm{op}}$, the fact that we have $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{E})$ immediately implies that η^* is already an equivalence, so that d is S-local. □

In the situation of Proposition 4.4.1.6, the question naturally arises whether the converse is true: namely, whether the inclusion $i\colon \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ always defines a Bousfield localisation (i.e. admits a left adjoint) for every $\mathcal{B}$-category D and every functor $\mathsf{S} \to \mathsf{D}$. In general, this is false, but there is a large class of $\mathcal{B}$-categories D and functors $\mathsf{S} \to \mathsf{D}$ for which this is nonetheless the case:

Proposition 4.4.1.7. *Let D be an Ω-cocomplete large $\mathcal{B}$-category that takes values in the ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ of presentable ∞-categories. Let furthermore $i\colon \mathsf{S} \to \mathsf{D}$ be a functor where S is small. Then the inclusion $i\colon \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ admits a left adjoint and therefore exhibits $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ as a Bousfield localisation of D. Moreover, this Bousfield localisation is accessible.*

Proof. By Remark 2.2.3.10, we may assume without loss of generality that S is a subcategory of D, i.e. that i is a monomorphism. Let us first show that $i(A)\colon \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})(A) \hookrightarrow \mathsf{D}(A)$ admits a left adjoint for every object $A \in \mathcal{B}$. Choose a small subcategory of generators $\mathcal{G} \hookrightarrow \mathcal{B}$. Then an object $d\colon A \to \mathsf{D}$ is contained in $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})(A)$ precisely if for every $g\colon G \to A$ with $G \in \mathcal{G}$ and every map $s\colon p \to q$ in $\mathsf{S}(G)$ the induced map

$$s^*\colon \operatorname{map}_{\mathsf{D}(G)}(q, g^*d) \to \operatorname{map}_{\mathsf{D}(G)}(p, g^*d)$$

is an equivalence in $\mathcal{S}$ (cf. [62, Corollary 4.6.8]). As g^* admits a left adjoint $g_!$, the object d is thus contained in $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})(A)$ if and only if d is local with respect to the set of maps

$$T_A = \bigcup_{G \to A} \{g_!(s) \mid s \in \mathsf{S}(G)^{\Delta^1}\}$$

in $\mathsf{D}(A)$. By construction, T_A is a small set, and since $\mathsf{D}(A)$ is by assumption a presentable ∞-category, we deduce from [57, Proposition 5.5.4.15] that $i(A)$ admits a left adjoint L_A and that $i(A)$ is accessible.

Next, we show that for every map $p\colon P \to A$ in $\mathcal{B}$ the natural map $L_B p^* \to p^* L_A$ is an equivalence. By Remark 2.4.2.10, we only need to show that $p^* L_G$ sends the adjunction unit of $L_A \dashv i(A)$ to an equivalence. Recall from [57, Section 5.5.4] that the set of maps in $\mathsf{D}(A)$ that is inverted by L_A coincides with the *strong saturation* of T_A, which is the smallest set of maps in $\mathsf{D}(A)$ containing T_A that is stable under pushouts, satisfies the two out of three property and is stable under small colimits in $\mathsf{D}(A)^{\Delta^1}$. Therefore the adjunction unit η is contained in the strong saturation of T_A, and since p^* commutes with colimits (being a morphism in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$) we conclude that it will suffice to show that p^* sends maps in T_A to maps in the strong saturation of T_G. Let us therefore fix a map $g\colon G \to A$ with $G \in \mathcal{G}$ as well as a map $s \in \mathsf{S}(G)^{\Delta^1}$. Since D is Ω-cocomplete, we find $p^* g_!(s) \simeq h_! q^*(s)$, where h and q are defined via the pullback square

$$\begin{tikzcd} Q \arrow[r, "h"] \arrow[d, "q"] & P \arrow[d, "b"] \\ G \arrow[r, "g"] & A. \end{tikzcd}$$

Now $q^*(s)$ is a map in $\mathsf{S}(P)$ and therefore inverted by L_P, hence $h_! q^*(s)$ is inverted by L_B whenever $h_!$ sends maps in T_P to maps in the strong saturation of T_B, which is immediate by definition of T_P.

Finally, we may employ Proposition 2.4.2.9 to deduce that i admits a left adjoint L. Furthermore, as D is by assumption both Ω- and LConst-cocomplete and therefore cocomplete by Proposition 3.2.4.1, and since every reflective subcategory of a cocomplete $\mathcal{B}$-category is cocomplete as well by Proposition 3.2.2.6, the fact that i is section-wise accessible already implies that i is accessible (see Corollary 4.3.2.5). $\square$

COROLLARY 4.4.1.8. *Let C and S be (small) $\mathcal{B}$-categories and let $j\colon \mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be a functor. Then there is a sound doctrine U such that $\mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ is a U-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Conversely, any U-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ can be identified with the full subcategory $\mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ for some small $\mathcal{B}$-category S and some functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.*

PROOF. By the straightening equivalence for right fibrations, for any $A \in \mathcal{B}$ there is a natural equivalence of ∞-categories $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A) \simeq \mathrm{RFib}(\mathsf{C} \times A)$, and since the right-hand side is a localisation of the presentable ∞-category $\mathrm{Cat}(\mathcal{B})_{/A\times \mathsf{C}}$ at a small set of objects, we find that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is section-wise given by a presentable ∞-category. Moreover, if $s\colon B \to A$ is a map in $\mathcal{B}$, the functor $s^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(B)$ admits a right adjoint s_* by the theory of Kan extensions and therefore in particular commutes with small colimits. As $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is cocomplete, we are therefore in the situation of Proposition 4.4.1.7, which implies the claim. $\square$

4.4.2. Presentability. In this section we define the concept of a presentable $\mathcal{B}$-category and discuss various characterisations of this notion.

DEFINITION 4.4.2.1. A large $\mathcal{B}$-category is said to be *presentable* if there exist $\mathcal{B}$-categories C and S as well as a functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that D is equivalent to $\mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$.

REMARK 4.4.2.2. In the situation of Definition 4.4.2.1, the fact that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is locally small implies that the 1-image S' of the functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (i.e. the subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is obtained by factoring the functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ into a strong epimorphism and a monomorphism) is small as well. In fact, by combining Proposition 2.2.3.3 with [62, Proposition 4.7.2] it is clear that S' is locally small, hence [62, Proposition 4.7.4] implies that S' is small whenever S'_0 is contained in $\mathcal{B}$, which follows in turn from the observation that S' is a subcategory of the essential image of $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, which is small by [62, Lemma 4.7.5]. As a consequence, Remark 2.2.3.10 shows that we may always assume that S is a subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.

DEFINITION 4.4.2.3. We call a large $\mathcal{B}$-category C *section-wise accessible* if the associated sheaf takes values in the subcategory $\mathrm{Acc} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ of accessible ∞-categories. Analogously, we call C *section-wise presentable* if it factors through the inclusion $\mathrm{Pr}^{\mathrm{L}}_\infty \hookrightarrow \widehat{\mathrm{Cat}}_\infty$.

We now come to the main characterisation of presentable $\mathcal{B}$-categories.

THEOREM 4.4.2.4. *For a large $\mathcal{B}$-category D, the following are equivalent:*

(1) D is presentable;
(2) there is a $\mathcal{B}$-category C and an accessible Bousfield localisation $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$;
(3) D is accessible and cocomplete;
(4) D is cocomplete, and there is a $\mathcal{B}$-regular cardinal κ such that D is $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-accessible;
(5) D is cocomplete and section-wise accessible;
(6) D is Ω-cocomplete and section-wise presentable.

PROOF. The fact that (1) and (2) are equivalent is an immediate consequence of Corollary 4.4.1.8. Now if we assume that (2) is satisfied, we may find a $\mathcal{B}$-regular cardinal κ such that the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is $\mathsf{Filt}_{\mathsf{Cat}^{\kappa}_{\mathcal{B}}}$-cocontinuous (see Remark 4.3.2.2), which by Corollary 4.3.3.3 implies that D is $\mathsf{Cat}^{\kappa}_{\mathcal{B}}$-accessible. As any reflective subcategory of a cocomplete $\mathcal{B}$-category is cocomplete as well by Proposition 3.2.2.6, we conclude that (4) is satisfied. Trivially, (4) implies (3). Lastly, if $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ for some sound doctrine U and some $\mathcal{B}$-category C and if D is furthermore cocomplete, we deduce from Corollary 3.4.1.14 that the inclusion $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ admits a left adjoint, hence (3) implies (2).

To show that (2) implies (5), since Proposition 3.2.2.6 already shows that D is cocomplete, it remains to see that D is section-wise accessible. For every $A \in \mathcal{B}$, the ∞-category $\mathsf{D}(A)$ is a Bousfield localisation of the presentable ∞-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A) \simeq \mathrm{RFib}(\mathsf{C} \times A)$. Since Corollary 4.3.2.5 implies that this Bousfield localisation is *accessible*, one concludes that $\mathsf{D}(A)$ is an accessible ∞-category. Furthermore, since D is also complete, the functor $s^* \colon \mathsf{D}(A) \to \mathsf{D}(B)$ preserves colimits for any map $s \colon B \to A$ in $\mathcal{B}$, so it is in particular accessible. Thus D is section-wise accessible. The fact that (5) implies (6) is an immediate consequence of Proposition 3.2.4.1 and Proposition 3.2.4.5.

To complete the proof, we show that (6) implies (2). For this, let $\mathcal{C}$ be a small ∞-category such that there is a left exact and accessible localisation $L \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$. Let $F \colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_\infty$ be the composition

$$F \colon \mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{PSh}(\mathcal{C})^{\mathrm{op}} \xrightarrow{L} \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{D}} \widehat{\mathrm{Cat}}_\infty.$$

Since $\mathcal{C}$ is small we may find a regular cardinal κ and a functor $F_0 \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ such that F is given by composing F_0 with the functor $\mathrm{Ind}_\kappa(-) \colon \mathrm{Cat}_\infty \to \widehat{\mathrm{Cat}}_\infty$. We let C denote the sheafification of F_0, which is a small $\mathcal{B}$-category. Let U be the internal class that is spanned by the constant $\mathcal{B}$-categories associated to κ-filtered ∞-categories. We claim that D is the free U-completion of C. Note that by assumption, D is both Ω and LConst-cocomplete, hence D must be cocomplete (Proposition 3.2.4.1) and therefore a fortiori U-cocomplete. As a consequence, it suffices to verify the universal property. Let E be an arbitrary U-cocomplete (large) $\mathcal{B}$-category. By Proposition 3.2.4.5 and Remark 3.2.4.9, a functor $f \colon \mathsf{D} \to \mathsf{E}$ is U-cocontinuous if only if for every $c \in \mathcal{C}$ the functor $f(c)$ preserves κ-filtered colimits (where we slightly abuse notation and implicitly identify $c \in \mathcal{C}$ with its image along $L \colon \mathrm{PSh}_{\mathcal{S}}(\mathcal{C}) \to \mathcal{B}$). Let us write $\mathrm{Fun}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{D}, \mathsf{E})$ for the global sections of $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{D}, \mathsf{E})$, i.e. the full subcategory of $\mathrm{Fun}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$ spanned by the U-cocontinuous functors. By Lemma 2.3.2.14 it follows that we have a chain of equivalences

$$\begin{aligned} \mathrm{Fun}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{D}, \mathsf{E}) &\simeq \int_{c \in \mathcal{C}} \mathrm{Fun}^{\kappa}(\mathsf{D}(c), \mathsf{E}(c)) \\ &\simeq \int_{c \in \mathcal{C}} \mathrm{Fun}(F_0(c), \mathsf{E}(c)) \\ &\simeq \mathrm{Fun}_{\mathrm{PSh}(\mathcal{C})}(F_0, \mathsf{E}) \\ &\simeq \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}) \end{aligned}$$

that is natural in E. Using Yoneda's lemma, this already shows that D is the free U-cocompletion of C. In particular, it follows from the explicit description of the free U-cocompletion that we have a commutative triangle of fully faithful functors

$$\begin{tikzcd} \mathsf{D} \arrow[r, hook, "j"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \\ \mathsf{C} \arrow[u, hook, "i"] \arrow[ur, hook, "h"'] & \end{tikzcd}$$

Since D is cocomplete, the inclusion j admits a left adjoint by Corollary 3.4.1.14. In particular, the inclusion $j(A) \colon \mathsf{D}(A) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A)$ is a right adjoint functor between presentable ∞-categories for each object $A \in \mathcal{B}$, so it follows from [57, Proposition 5.5.1.2] that $j(A)$ is an accessible functor. Hence Corollary 4.3.2.5 implies that j is accessible, which completes the proof. □

We end this section by recording a few consequences of Theorem 4.4.2.4. We begin by noting that as Theorem 4.4.2.4 implies that every presentable $\mathcal{B}$-category is a reflective subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ for some $\mathcal{B}$-category C, we deduce from Proposition 3.2.2.6:

COROLLARY 4.4.2.5. *Every presentable $\mathcal{B}$-category is complete and cocomplete.* □

COROLLARY 4.4.2.6. *Let D be a presentable $\mathcal{B}$-category and let K be a $\mathcal{B}$-category. Then $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{D})$ is presentable.*

PROOF. By Theorem 4.4.2.4, we may choose a $\mathcal{B}$-category C and a sound doctrine U such that D is a U-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. In light of Proposition 3.2.2.7, this implies that the large $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \mathsf{D})$ is a U-accessible Bousfield localisation of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{K}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{K}^{\mathrm{op}} \times \mathsf{C})$, hence the result follows. □

Corollary 4.4.2.7. *Let* D *be a presentable* $\mathcal{B}$*-category and let* $d\colon A \to \mathsf{D}$ *be an arbitrary object. Then* $\mathsf{D}_{/d}$ *is a presentable* $\mathcal{B}_{/A}$*-category.*

Proof. We may assume that $A \simeq 1$ (cf. Remark 4.4.2.9 below). Using Corollary 3.2.2.11, one finds that $\mathsf{D}_{/d}$ is cocomplete. By Theorem 4.4.2.4, it therefore suffices to show that $\mathsf{D}_{/d}$ is accessible, which is a consequence of Proposition 4.3.1.6. □

Corollary 4.4.2.8. *Let* D *be a presentable* $\mathcal{B}$*-category and let* $\mathsf{S} \to \mathsf{D}$ *be a functor where* S *is small. Then there is a sound doctrine* U *such that* $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *is a* U*-accessible Bousfield localisation of* D*. In particular,* $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *is presentable.*

Proof. Since D is cocomplete by Corollary 4.4.2.5 and section-wise presentable by Theorem 4.4.2.4, the claim follows from Proposition 4.4.1.7. □

Remark 4.4.2.9. As yet another consequence of Theorem 4.4.2.4, the condition of a large $\mathcal{B}$-category to be presentable is a local condition: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then a $\mathcal{B}$-category D is presentable if and only if each $\pi_{A_i}^*\mathsf{D}$ is a presentable $\mathcal{B}_{/A_i}$-category. This follows from condition (3) in Theorem 4.4.2.4, together with cocompleteness being a local condition (see Remark 3.2.2.3 and Remark 4.3.3.5).

4.4.3. The adjoint functor theorem. Recall from Proposition 3.2.2.5 that any left adjoint functor between cocomplete large $\mathcal{B}$-categories is cocontinuous. Therefore, if D and E are cocomplete large $\mathcal{B}$-categories, there is a canonical inclusion

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{L}}(\mathsf{D},\mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D},\mathsf{E}).$$

If D is presentable and E is locally small, then this inclusion is in fact an equivalence:

Proposition 4.4.3.1 (Adjoint functor theorem I). *Let* D *and* E *be large* $\mathcal{B}$*-categories such that* D *is presentable and* E *is cocomplete and locally small. Then every cocontinuous functor* $f\colon \mathsf{D} \to \mathsf{E}$ *admits a right adjoint. In particular, there is an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{L}}(\mathsf{D},\mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D},\mathsf{E})$$

of (large) $\mathcal{B}$*-categories.*

Proof. In light of Remark 4.4.2.9, it is clear that the second statement immediately follows from the first. Now choose $\mathcal{B}$-categories C and S as well as a functor $\mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that $\mathsf{D} \simeq \mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. If $f\colon \mathsf{D} \to \mathsf{E}$ is a cocontinuous functor, then $fL\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{E}$ is cocontinuous as well and therefore a left adjoint by Remark 3.4.1.4. To show that f admits a right adjoint, we therefore only need to verify that the right adjoint r of fL factors through D. Since $\mathsf{D} \simeq \mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ as full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ by Theorem 4.4.2.4, this is in turn equivalent to $h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}r$ factoring through the functor

$$L^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})),$$

which is clear on account of r being right adjoint to fL. □

Recall from Corollary 4.4.2.8 that if D is a presentable $\mathcal{B}$-category and $\mathsf{S} \to \mathsf{D}$ is a functor where S is small, the $\mathcal{B}$-category $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ is an accessible Bousfield localisation of D and therefore in particular presentable. We may now use Proposition 4.4.3.1 to derive a universal property of $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ among presentable $\mathcal{B}$-categories. To that end, recall from § 2.2.3 that if E is another presentable $\mathcal{B}$-category, we denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})_{\mathsf{S}}$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ that is spanned by those objects $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ for which the restriction of the associated functor $\pi_A^*\mathsf{D} \to \pi_A^*\mathsf{E}$ along $\pi_A^*\mathsf{S} \to \pi_A^*\mathsf{D}$ takes values in the subcategory $\pi_A^*\mathsf{E}^{\simeq}$. We will denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D},\mathsf{E})_{\mathsf{S}}$ its intersection with the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D},\mathsf{E})$. We now obtain:

COROLLARY 4.4.3.2. *Let* $\mathsf{S} \to \mathsf{D}$ *be a functor of* $\mathcal{B}$*-categories where* S *is small and* D *is presentable, and let* E *be another presentable* $\mathcal{B}$*-category. Then precomposition with the left adjoint* $L\colon \mathsf{D} \to \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ *induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}.$$

PROOF. To begin with, note that as L is in particular a localisation functor (see Proposition 2.4.4.6), the universal property of localisations Proposition 2.2.3.14 implies that

$$L^*\colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}), \mathsf{E}) \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})$$

is fully faithful. Therefore, it suffices to identify its essential image with $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$. Since the restriction of L along $\mathsf{S} \to \mathsf{D}$ takes values in $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})^{\simeq}$, it is clear that L^* takes values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{E})_{\mathsf{S}}$, so that it suffices to show that every object $A \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}, \mathsf{S})_{\mathsf{S}}$ is contained in the essential image of L^*. By combining Remark 4.4.1.2 with Remarks 3.2.3.4 and 2.2.3.12, it will be enough to verify that any cocontinuous functor $f\colon \mathsf{D} \to \mathsf{E}$ whose restriction along $\mathsf{S} \to \mathsf{D}$ takes values in $\mathsf{E}^{\simeq}$ factors through L. Note that the assumption on f precisely means that f factors through the localisation $l\colon \mathsf{D} \to \mathsf{S}^{-1}\mathsf{D}$, so that $f^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{E}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ factors through $l^*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}^{-1}\mathsf{D}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$. Since $f^* \simeq g_!$ where g is the right adjoint of f that is provided by Proposition 4.4.3.1, the very definition of $\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ implies that g factors through the inclusion $i\colon \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ via a functor $g'\colon \mathsf{E} \to \mathsf{Loc}(\mathsf{D})$. Since the composite fi defines a left adjoint of g', the claim follows by passing to left adjoints. $\square$

There is also a dual version to Proposition 4.4.3.1 that classifies right adjoint functors between presentable $\mathcal{B}$-categories.

PROPOSITION 4.4.3.3 (Adjoint functor theorem II). *Let* $f\colon \mathsf{D} \to \mathsf{E}$ *be a functor between presentable* $\mathcal{B}$*-categories. Then the following are equivalent:*

(1) f *admits a left adjoint;*
(2) f *is continuous and accessible;*
(3) f *is continuous and section-wise accessible.*

PROOF. By Corollary 4.3.2.5, (2) and (3) are equivalent. Moreover, since Theorem 4.4.2.4 implies that f is section-wise given by a functor between presentable ∞-categories, the adjoint functor theorem for presentable ∞-categories [57, Corollary 5.5.2.9] shows that (1) implies (3). For the converse, note that the same result implies that $f(A)$ admits a left adjoint l_A for every $A \in \mathcal{B}$. By Proposition 2.4.2.9, it now suffices to see that the natural map $l_B s^* \to s^* l_A$ is an equivalence for every map $s\colon B \to A$ in $\mathcal{B}$. This is equivalent to seeing that the transpose map $f(A)s_* \to s_* f(B)$ that is given by passing to right adjoints is an equivalence. But this is just another way of saying that f is Ω-continuous. $\square$

4.4.4. The large $\mathcal{B}$-category of presentable $\mathcal{B}$-categories. Recall from § 3.2.3 that we defined the (very large) $\mathcal{B}$-category $\mathsf{Cat}^{\mathrm{cc}}_{\widehat{\mathcal{B}}}$ of cocomplete large $\mathcal{B}$-categories as the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ which is determined by the subobject of $(\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ that is spanned by the cocontinuous functors between cocomplete large $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. By Remark 3.2.3.2 a functor of large $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Cat}^{\mathrm{cc}}_{\widehat{\mathcal{B}}}$ precisely if it is a cocontinuous functor between cocomplete large $\mathcal{B}$-categories. We may now define:

DEFINITION 4.4.4.1. The large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories is defined as the full subcategory of $\mathsf{Cat}^{\mathrm{cc}}_{\widehat{\mathcal{B}}}$ that is spanned by the presentable $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. We denote by $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ the ∞-category of global sections of $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$.

REMARK 4.4.4.2. As presentability is a local condition (Remark 4.4.2.9) and by Remark 3.2.3.2, a large $\mathcal{B}_{/A}$-category defines an object in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ if and only if it is presentable, and a functor between such large $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ if and only if it is cocontinuous. Consequently, the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ identifies $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ with the sheaf $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/-})$ on $\mathcal{B}$. In particular, one obtains a canonical equivalence $\pi_A^* \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$.

REMARK 4.4.4.3. A priori, $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is a very large $\mathcal{B}$-category. However, note that the set of equivalence classes of presentable $\mathcal{B}$-categories is $\mathbf{V}$-small as it admits a surjection from the $\mathbf{V}$-small union

$$\bigsqcup_{\mathsf{C}\in\mathrm{Cat}(\mathcal{B})} \mathrm{Sub}_{\mathrm{small}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$$

where $\mathrm{Sub}_{\mathrm{small}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ denotes the $\mathbf{V}$-small poset of *small* subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ is furthermore locally $\mathbf{V}$-small, this shows that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is in fact only a large $\mathcal{B}$-category.

Recall from [61, § 6.2] that we denote by $\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}$ the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is determined by the subobject $L \hookrightarrow (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ of left adjoint functors. By Proposition 4.4.3.1, the inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ factors through the inclusion $\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$. Suppose now that D and E are presentable $\mathcal{B}$-categories. By combining Remark 3.1.5.3 and Corollary 3.1.5.4 with the fact that $L \hookrightarrow (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ is closed under equivalences and composition in the sense of Proposition 2.2.2.9 and by furthermore making use of Remark 3.2.3.2, we find that the induced inclusion $\mathrm{map}_{\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{D},\mathsf{E}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}}(\mathsf{D},\mathsf{E})$ is obtained by applying the core $\mathcal{B}$-groupoid functor to the equivalence

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$$

from Proposition 4.4.3.1. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and using Remark 4.4.4.2, the same assertion holds for objects in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ in context $A \in \mathcal{B}$, so that we conclude:

PROPOSITION 4.4.4.4. *The inclusion* $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}$ *is fully faithful.* □

Dually, let us denote by $\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}}$ the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is determined by the subobject $R \hookrightarrow (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ of right adjoint functors.

DEFINITION 4.4.4.5. The $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories is defined as the full subcategory of $\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}}$ that is spanned by the presentable $\mathcal{B}_{/A}$-categories for every $A \in \mathcal{B}$. We denote by $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ the underlying ∞-category of global sections.

REMARK 4.4.4.6. As in Remark 4.4.4.2, a large $\mathcal{B}_{/A}$-category defines an object in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ if and only if it is presentable, and a functor between such large $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ if and only if it is a right adjoint. As a consequence, the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ corresponds to the sheaf $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B}_{/-})$ on $\mathcal{B}$ that is spanned by the presentable $\mathcal{B}_{/A}$-categories and right adjoint functors. In particular, one obtains a canonical equivalence $\pi_A^*\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}} \simeq \mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$.

PROPOSITION 4.4.4.7. *There is a canonical equivalence* $(\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}})^{\mathrm{op}} \simeq \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *that carries a right adjoint functor between presentable* $\mathcal{B}$*-categories to its left adjoint.*

PROOF. By [61, Proposition 6.2.1], there is such an equivalence $(\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}})^{\mathrm{op}} \simeq \mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}}$, and since this functor necessarily acts as the identity on the underlying core $\mathcal{B}$-groupoids, it restricts to the desired equivalence by virtue of Proposition 4.4.4.4. □

EXAMPLE 4.4.4.8. We are now in the position to provide a large class of examples of presentable $\mathcal{B}$-categories: recall from Construction 2.3.1.1 that there is a functor $-\otimes\Omega\colon \mathrm{Pr}^{\mathrm{R}}_{\infty} \to \mathrm{Cat}(\widehat{\mathcal{B}})$ that sends a presentable ∞-category $\mathcal{E}$ to the large $\mathcal{B}$-category $\mathcal{E}\otimes\Omega = \mathcal{E}\otimes\mathcal{B}_{/-}$ (where $-\otimes-$ is Lurie's tensor product of presentable ∞-categories). By Example 3.2.4.8, the $\mathcal{B}$-category $\mathcal{E}\otimes\Omega$ is cocomplete, so that Theorem 4.4.2.4 implies that it is presentable as it takes values in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. Moreover, we deduce from Example 2.4.2.12 that whenever $g\colon \mathcal{E}\to\mathcal{E}'$ is a map in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$, the induced functor $g\otimes\Omega$ is a right adjoint. Consequently, we conclude that the functor $-\otimes\Omega$ takes values in $\mathsf{Pr}^{\mathrm{R}}(\mathcal{B})$. In particular, by applying this observation to $\mathcal{E} = \mathrm{Cat}_{\infty}$, we find that $\mathsf{Cat}_{\mathcal{B}}$ is presentable.

Our next goal is to show that $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ is complete and cocomplete. For completeness, we first need a lemma. To that end, recall from § 3.2.3 that we denote by $\mathsf{Cat}^{\Omega\text{-cc}}_{\widehat{\mathcal{B}}}$ the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is spanned by the $\Omega_{\mathcal{B}_{/A}}$-cocontinuous functors between $\Omega_{\mathcal{B}_{/A}}$-cocomplete (large) $\mathcal{B}$-categories. We now find:

LEMMA 4.4.4.9. *The $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\Omega\text{-cc}}$ is LConst-complete, and the inclusion $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\Omega\text{-cc}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is LConst-continuous.*

PROOF. By Remark 3.2.3.2, it is enough to show that for any small ∞-category $\mathcal{K}$ and any functor $d\colon \mathcal{K} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\Omega\text{-cc}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$, the following two conditions are satisfied:

(1) $\lim d$ is Ω-cocomplete;
(2) for every Ω-cocomplete large $\mathcal{B}$-category C, a functor $f\colon \mathsf{C} \to \lim d$ is Ω-cocontinuous if and only if the compositions $\mathsf{C} \to \lim d \to d(k)$ are Ω-cocontinuous for all $k \in \mathcal{K}$.

Recall from [56, Corollary 4.7.4.18] that the subcategory $\mathrm{Fun}^{\mathrm{LAdj}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \hookrightarrow \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ that is spanned by the right adjoint functors and the left adjointable squares (i.e. those commutative squares of ∞-categories whose associated mate transformation is an equivalence) admits small limits and that the inclusion preserves small limits. Let us fix a map $p\colon P \to A$ in $\mathcal{B}$, and let us denote by $\mathrm{Cat}(\widehat{\mathcal{B}})^{\Omega\text{-cc}}$ the ∞-category of global sections of $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\Omega\text{-cc}}$. Now evaluation at p defines a functor $\mathrm{Cat}(\widehat{\mathcal{B}}) \to \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ that restricts to a map $\mathrm{Cat}(\widehat{\mathcal{B}})^{\Omega\text{-cc}} \to \mathrm{Fun}^{\mathrm{LAdj}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$. Since limits in $\mathrm{Cat}(\widehat{\mathcal{B}})$ are computed section-wise, this already shows that $p^*\colon \lim d(A) \to \lim d(P)$ admits a left adjoint. Similarly, if $s\colon B \to A$ is a map in $\mathcal{B}$ and if $q\colon Q \to B$ denotes the pullback of p along s, evaluating large $\mathcal{B}$-categories at this pullback square yields a morphism $\Delta^1 \times \mathrm{Cat}(\widehat{\mathcal{B}})^{\Omega\text{-cc}} \to \mathrm{Fun}^{\mathrm{LAdj}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$. Consequently, applying $\lim d$ to the very same pullback square must yield a left-adjointable square of ∞-categories, which implies that condition (1) is satisfied. By the same argument, if C is Ω-cocomplete and if $f\colon \mathsf{C} \to \lim d$ is a functor, evaluating f at p yields a commutative square of ∞-categories that is left-adjointable if and only if the evaluation of the composition $\mathsf{C} \to \lim d \to d(k)$ at p is left-adjointable for all $k \in \mathcal{K}$. Hence (2) follows. ☐

PROPOSITION 4.4.4.10. *The $\mathcal{B}$-category $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ is complete, and the inclusion $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is continuous.*

PROOF. By the dual of Proposition 3.2.4.1, it suffices to show that $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ is both Ω- and LConst-complete and that the inclusion $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ is both Ω- and LConst-continuous. Using Remark 4.4.4.2, this follows once we show that whenever K is either given by the constant $\mathcal{B}$-category Λ_0^2 or by a $\mathcal{B}$-groupoid, the large $\mathcal{B}$-category $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ admits K-indexed limits and the inclusion $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves K-indexed limits.

Let us first assume that $\mathsf{K} = \Lambda_0^2$, i.e. suppose that

$$\begin{tikzcd} \mathsf{Q} \arrow[r, "g"] \arrow[d, "q"] & \mathsf{P} \arrow[d, "p"] \\ \mathsf{D} \arrow[r, "f"] & \mathsf{C} \end{tikzcd}$$

is a pullback diagram in $\mathrm{Cat}(\widehat{\mathcal{B}})$ in which f and p are cocontinuous functors between presentable $\mathcal{B}$-categories. By Theorem 4.4.2.4, the cospan determined by f and p takes values in $\mathrm{Pr}^{\mathrm{L}}_\infty$. Therefore, [57, Proposition 5.5.3.13] implies that Q takes values in $\mathrm{Pr}^{\mathrm{L}}_\infty$ and that g and q are section-wise cocontinuous. Moreover, Lemma 4.4.4.9 shows that Q is Ω-cocomplete and that g and q are Ω-cocomplete. By again making use of Theorem 4.4.2.4, we thus conclude that the Q is presentable and that g and q are cocontinuous. Now if Z is another presentable $\mathcal{B}$-category, a similar argumentation shows that a functor $\mathsf{Z} \to \mathsf{Q}$ is cocontinuous if and only if its composition with both g and q are cocontinuous. In total, this shows that $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ admits pullbacks and that the inclusion $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves pullbacks.

Let us now assume $\mathsf{K} = \mathsf{G}$ for some $\mathcal{B}$-groupoid G. In order to show that $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ has G-indexed limits and that the inclusion $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves G-indexed limits, another application of Remark 4.4.4.2 allows us to reduce to showing that the adjunction $(\pi_{\mathsf{G}})_* \dashv \pi_{\mathsf{G}}^*))\colon \mathrm{Cat}(\widehat{\mathcal{B}}_{/\mathsf{G}}) \leftrightarrows \mathrm{Cat}(\widehat{\mathcal{B}})$ restricts to an adjunction between $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/\mathsf{G}})$ and $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. Recall that on the level of $\widehat{\mathrm{Cat}}_\infty$-valued sheaves, the functor $(\pi_{\mathsf{G}})_*$ is given by precomposition with π_{G}^*. By combining the characterisation of presentable $\mathcal{B}$-categories as Ω-cocomplete $\mathrm{Pr}^{\mathrm{L}}_\infty$-valued sheaves (Theorem 4.4.2.4) with the explicit description of Ω-cocompleteness from Proposition 3.2.4.2 and the section-wise characterisation of left adjoint functors (Proposition 2.4.2.9), it is therefore immediate that $(\pi_{\mathsf{G}})_*$ restricts to a functor $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}_{/\mathsf{G}}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$.

Moreover, the adjunction unit $\mathrm{id}_{\mathrm{Cat}(\widehat{\mathcal{B}})} \to (\pi_{\mathsf{G}})_* \pi_{\mathsf{G}}^*$ is given by precomposition with the adjunction counit $(\pi_{\mathsf{G}})_! \pi_{\mathsf{G}}^* \to \mathrm{id}_{\mathcal{B}}$, and the adjunction counit $\pi_{\mathsf{G}}^*(\pi_{\mathsf{G}})_* \to \mathrm{id}_{\mathrm{Cat}(\widehat{\mathcal{B}}_{/A})}$ is given by precomposition with the adjunction unit $\mathrm{id}_{\mathcal{B}_{/A}} \to \pi_{\mathsf{G}}^*(\pi_{\mathsf{G}})_!$. Thus, by the section-wise characterisation of left adjoint functors and the fact that presentable $\mathcal{B}$-categories are Ω-cocomplete, these two maps must also restrict in the desired way, hence the result follows. □

PROPOSITION 4.4.4.11. *The large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ is complete, and the inclusion $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is continuous.*

PROOF. As in the proof of Proposition 4.4.4.10, it suffices to show that for either $\mathsf{K} = \Lambda^2_0$ or $\mathsf{K} = \mathsf{G}$ for G a $\mathcal{B}$-groupoid, the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ admits K-indexed limits and the inclusion $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves K-indexed limits. The first case follows as in the proof of Proposition 4.4.4.10, by making use of the dual version of Lemma 4.4.4.9, [57, Theorem 5.5.3.18] and the fact that a continuous and section-wise accessible functor between presentable $\mathcal{B}$-categories admits a left adjoint (Proposition 4.4.3.3). The argument for the second case is carried out in a completely analogous way as the one in the proof of Proposition 4.4.4.10, the only difference being that one must use the Ω-completeness of presentable $\mathcal{B}$-categories and not their Ω-cocompleteness. □

REMARK 4.4.4.12. As a consequence of Proposition 4.4.4.11, we can furthermore deduce that $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ is generated under pullbacks by presheaf $\mathcal{B}$-categories. In fact, if D is a presentable $\mathcal{B}$-category, we may find small $\mathcal{B}$-categories C and S and a functor $j \colon \mathsf{S} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ so that $\mathsf{D} \simeq \mathsf{Loc}_{\mathsf{S}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. By definition of the right-hand side, we therefore obtain a pullback square

$$\begin{tikzcd} \mathsf{D} \arrow[r] \arrow[d, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}^{\mathrm{gpd}}) \arrow[d, "\gamma^*"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "j^* h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{S}) \end{tikzcd}$$

in $\mathrm{Cat}(\widehat{\mathcal{B}})$, where $\gamma \colon \mathsf{S} \to \mathsf{S}^{\mathrm{gpd}}$ is the natural map. By Remark 3.4.1.4, the functor $j^* h_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$ is a right adjoint: its left adjoint is the left Kan extension $(h_{\mathsf{S}})_!(j)$ of j along the Yoneda embedding h_{S}. Since γ^* is a right adjoint as well, Proposition 4.4.4.11 implies that this diagram is a pullback square in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$.

Finally, by combining Proposition 4.4.4.10 and Proposition 4.4.4.11 with Proposition 4.4.4.7, we conclude:

COROLLARY 4.4.4.13. *Both $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ and $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ are complete and cocomplete.* □

4.4.5. Cocompletion with relations. Let U be an internal class, and let C be a $\mathcal{B}$-category. In § 3.4.1, we constructed the *free* U-*cocompletion* $\underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ of C, i.e. the universal U-cocomplete $\mathcal{B}$-category that is equipped with a functor $\mathsf{C} \to \underline{\mathsf{PSh}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$. The goal of this section is to generalise this result by imposing that a chosen collection of cocones in C (that are indexed by objects of U) become colimit cocones in the free U-cocompletion. Our proof of this result is a straightforward adaptation of the discussion in [57, § 5.3.6].

4.4.5.1. Let us fix a small collection $R = (\overline{d}_i \colon \mathsf{K}_i^{\triangleright} \to \pi_{A_i}^* \mathsf{C})_{i \in I}$ of cocones with $A_i \in \mathcal{B}$ and $\mathsf{K}_i \in \mathsf{U}(A_i)$ for all $i \in I$. Let $\mathsf{S}_R \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the (non-full) subcategory that is spanned by the canonical maps $(f_i \colon \operatorname{colim} h_{\mathsf{C}} d_i \to h_{\mathsf{C}} \overline{d}_i(\infty))_{i \in I}$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (with each f_i being in context A_i for $i \in I$), where d_i denotes the restriction of $\overline{d}_i$ along the inclusion $\mathsf{K}_i \hookrightarrow \mathsf{K}_i^{\triangleright}$ and where $\infty \colon A_i \to \mathsf{K}_i^{\triangleright}$ denotes the cone point.

Let us set $\mathsf{D} = \mathsf{Loc}_{\mathsf{S}_R}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. By Corollary 4.4.1.8, the inclusion $i \colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ admits a left adjoint $L \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$. In particular, D is cocomplete by Proposition 3.2.2.6. We define the (large) $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})^{(\mathsf{U},R)}(\mathsf{C})$ as the smallest full subcategory of D that contains the essential image of $Lh_{\mathsf{C}} \colon \mathsf{C} \to \mathsf{D}$ and that is closed under U-colimits in D, and we let $j_{\mathsf{C}} \colon \mathsf{C} \to \underline{\mathsf{PSh}}^{(\mathsf{U},R)}_{\mathcal{B}}(\mathsf{C})$ be the map that is obtained by composing Lh_{C} with the inclusion.

REMARK 4.4.5.2. Given any object $A \in \mathcal{B}$, we denote by $\pi_A^* R$ the set of cocones $(\pi_A^*(\overline{d}_i))_{i\in I}$. We then obtain an equivalence $\pi_A^* \mathsf{S}_R \simeq \mathsf{S}_{\pi_A^* R}$ of subcategories in $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{C})$. Hence Remark 4.4.1.2 and the same argument as in the proof of Proposition 3.4.1.11 shows that one obtains a canonical equivalence of large $\mathcal{B}_{/A}$-categories

$$\pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}^{(\pi_A^*\mathsf{U},\pi_A^* R)}(\pi_A^* \mathsf{C})$$

with respect to which $\pi_A^* j_{\mathsf{C}}$ corresponds to the map $j_{\pi_A^* \mathsf{C}}$.

For any U-cocomplete large $\mathcal{B}$-category E, we will denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})$ that arises as the pullback

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}) \arrow[d, "(h_{\mathsf{C}})_!"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E})_{\mathsf{S}_R} \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}).
\end{tikzcd}$$

We now obtain:

PROPOSITION 4.4.5.3. *For every $i \in I$ the cocone $(j_{\mathsf{C}})_*(\overline{d}_i)$ is a colimit cocone in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$, and for every U-cocomplete large $\mathcal{B}$-category E, precomposition with j_{C} induces an equivalence*

$$j_{\mathsf{C}}^* \colon \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}),\mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R.$$

PROOF. Note that by construction of D, the map j_{C} carries each of the cocones $\overline{d}_i$ to a colimit cocone in $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$, hence the first claim follows immediately. The proof of the second claim employs a similar strategy as in the proof of Theorem 3.4.1.13. First, if E is an arbitrary U-cocomplete $\mathcal{B}$-category, note that the Yoneda embedding induces a U-cocontinuous functor $\mathsf{E} \hookrightarrow \mathsf{E}' = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{E},\Omega_{\mathcal{B}})^{\mathrm{op}}$ into a cocomplete $\mathcal{B}$-category. By Corollary 4.4.3.2 and the universal property of presheaf $\mathcal{B}$-categories, we now obtain an equivalence

$$(Lh_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}')_R \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}')_{\mathsf{S}_R} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{cc}}(\mathsf{D},\mathsf{E}')$$

As the inclusion $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}) \hookrightarrow \mathsf{D}$ is by construction U-cocontinuous, we therefore obtain an induced inclusion

$$(j_{\mathsf{C}})_! \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E}')_R \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}),\mathsf{E}').$$

Now if $f \colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}) \to \mathsf{E}'$ is a U-cocontinuous functor, precisely the same argument as the one employed in the proof of Theorem 3.4.1.13 shows that the adjunction counit $\epsilon \colon (j_{\mathsf{C}})_! j_{\mathsf{C}}^* f \to f$ is an equivalence and that f is therefore contained in the essential image of $(j_{\mathsf{C}})_!$. Together with Remark 4.4.5.2, this shows that $(j_{\mathsf{C}})_!$ is an equivalence. Finally, the same argumentation as in the proof of Theorem 3.4.1.13 also shows that this equivalence restricts to the desired equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})_R \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C}),\mathsf{E})$. ☐

REMARK 4.4.5.4. In the situation of Proposition 4.4.5.3, if U is assumed to be small (i.e. a *doctrine* in the terminology of § 4.1.3) implies that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$ is small as well. In fact, as D is locally small, the essential image of $Lh_{\mathsf{C}} \colon \mathsf{C} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$ is small [62, Lemma 4.6.5]. Hence we can make use of the same argument as in Remark 3.4.1.7 to deduce that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R)}(\mathsf{C})$ must also be small.

CONSTRUCTION 4.4.5.5. For the remainder of this section, let us fix $\mathcal{B}$-categories $\mathsf{C}^1,\dots,\mathsf{C}^n$ and a doctrine U. Let $\mathcal{G} \subset \mathcal{B}$ be a small subcategory of generators, and let us set

$$R_k = \bigsqcup_{G\in\mathcal{G}} \left\{\overline{d} \colon \mathsf{K}^{\triangleright} \to \pi_G^* \mathsf{C}^k \mid \mathsf{K} \in \mathsf{U}(G)\right\}$$

as well as

$$S_k = \bigsqcup_{G\in\mathcal{G}} \left\{\overline{d} \colon \mathsf{K}^{\triangleright} \to \pi_G^* \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_k)}(\mathsf{C}^k) \mid \mathsf{K} \in \mathsf{U}(G),\ \overline{d} \text{ is a colimit cocone}\right\}.$$

Furthermore, let $\square_{k=1}^n R_k$ be the set of all diagrams of the form

$$(c_1,\dots,c_{l-1},\mathrm{id},c_{l+1},\dots,c_n)\overline{d} \colon \mathsf{K}^{\triangleright} \to \pi_G^* \mathsf{C}^l \to \prod_{k=1}^n \pi_G^* \mathsf{C}^k$$

where $\overline{d}$ is an element of R_l and $c_k\colon G \to \mathsf{C}^k$ is an arbitrary object for each $k \neq l$. Let $\square_{k=1}^n S_k$ be defined analogously. Then for any internal class V that contains U, the composition

$$\mathsf{C}^1 \times \cdots \times \mathsf{C}^n \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n))$$

carries each cocone in $\square_{k=1}^n R_k$ to a colimit cocone, hence Proposition 4.4.5.3 determines a functor

$$\varphi\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n R_k)}(\mathsf{C}^1 \times \cdots \times \mathsf{C}^n) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n)).$$

PROPOSITION 4.4.5.6. *The canonical map*

$$\varphi\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n R_k)}(\mathsf{C}^1 \times \cdots \times \mathsf{C}^n) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n))$$

is an equivalence.

PROOF. Note that in light of Remark 4.4.5.4, the map φ is a well-defined morphism in the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}$ of V-cocomplete $\mathcal{B}$-categories and V-cocontinuous functors. By combining Yoneda's lemma with Remark 4.4.5.2 and Remark 3.2.3.2, the result thus follows once we verify that for every V-cocomplete $\mathcal{B}$-category E the restriction functor

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n S_k)}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n)), \mathsf{E}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V},\square_{k=1}^n R_k)}(\mathsf{C}^1 \times \cdots \times \mathsf{C}^n), \mathsf{E})$$

that we denote $\varphi^*$is an equivalence. Using Proposition 4.4.5.3, this is in turn equivalent to the map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n), \mathsf{E})_{\square_{k=1}^n S_k} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^1 \times \cdots \times \mathsf{C}^n, \mathsf{E})_{\square_{k=1}^n R_k}$$

being an equivalence. We will use induction on n to show that this functor is an equivalence. If $n = 1$, this is precisely the content of Proposition 4.4.5.3. For $n > 1$, the construction of $\square_{k=1}^n R_k$ and $\square_{k=1}^n S_k$ together with Lemma 4.6.1.3 imply that the above map can be identified with the morphism

$$\begin{array}{c}\underline{\mathsf{Fun}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_1)}(\mathsf{C}^1) \times \cdots \times \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_{n-1})}(\mathsf{C}^{n-1}), \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n), \mathsf{E}))_{\square_{k=1}^{n-1} S_k} \\ \big\downarrow \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^1 \times \cdots \times \mathsf{C}^{n-1}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^n, \mathsf{E})_{R_n})_{\square_{k=1}^{n-1} R_k}.\end{array}$$

As Proposition 4.4.5.3 implies that the map $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^n, \mathsf{E})_{R_n} \to \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},R_n)}(\mathsf{C}^n), \mathsf{E})$ is an equivalence, the claim thus follows by the induction hypothesis. $\square$

4.4.6. U-sheaves. The main goal in this section is to derived yet another characterisation of presentable $\mathcal{B}$-categories: that of $\mathcal{B}$-categories of U-*sheaves* on an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category. These are defined as follows:

DEFINITION 4.4.6.1. Let U be an internal class and suppose that C is an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category. For any (not necessarily small) U-complete $\mathcal{B}$-category E, we denote by $\underline{\mathsf{Sh}}_{\mathsf{E}}^{\mathsf{U}}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$ that is spanned by those presheaves $F\colon A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$ (in arbitrary context $A \in \mathcal{B}$) that are $\pi_A^*\mathsf{U}$-continuous when viewed as functors $\pi_A^*\mathsf{C}^{\mathrm{op}} \to \pi_A^*\mathsf{E}$. We refer to such presheaves as U-*sheaves*. For the case where $\mathsf{U} = \mathsf{Cat}_{\mathcal{B}}$, we will simply call them *sheaves*, and we will write $\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{C}) = \underline{\mathsf{Sh}}_{\mathsf{E}}^{\mathsf{Cat}_{\mathcal{B}}}(\mathsf{C})$ for the associated $\mathcal{B}$-category

REMARK 4.4.6.2. By Remark 3.2.3.2, if $A \in \mathcal{B}$ is an arbitrary object, we obtain a canonical equivalence $\pi_A^*\underline{\mathsf{Sh}}_{\mathsf{E}}^{\mathsf{U}}(\mathsf{C}) \simeq \underline{\mathsf{Sh}}_{\pi_A^*\mathsf{E}}^{\pi_A^*\mathsf{U}}(\pi_A^*\mathsf{C})$.

We first focus on Ω-valued U-sheaves.

PROPOSITION 4.4.6.3. *Let* D *be a presentable* $\mathcal{B}$*-category and let* $F\colon \mathsf{D}^{\mathrm{op}} \to \Omega$ *be a presheaf on* D*. Then* F *is representable if and only if* F *is continuous. In particular, the Yoneda embedding induces an equivalence* $\mathsf{D} \simeq \underline{\mathsf{Sh}}_{\Omega}(\mathsf{D})$.

Proof. By Remark 4.4.6.2, the first claim implies the second, and by Proposition 3.2.2.9, every representable functor is continuous, so that it suffices to prove that every continuous presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is representable. Now F being continuous is equivalent to $F^{\mathrm{op}}\colon \mathsf{D} \to \Omega^{\mathrm{op}}$ being cocontinuous, which by Proposition 4.4.3.1 is in turn equivalent to it being a left adjoint. Hence F is continuous if and only if F is a right adjoint. Let $l\colon \Omega \to \mathsf{D}^{\mathrm{op}}$ be the left adjoint of F. Since the final $\mathcal{B}$-groupoid $1_\Omega\colon 1 \to \Omega$ corepresents the identity on Ω, we find equivalences

$$F \simeq \mathrm{map}_\Omega(1_\Omega, F(-)) \simeq \mathrm{map}_{\mathsf{D}^{\mathrm{op}}}(l(1_\Omega), -) \simeq \mathrm{map}_{\mathsf{D}}(-, l(1_\Omega)),$$

hence F is represented by $l(1_\Omega)$. □

Next, we use Proposition 4.4.6.3 to deduce that whenever U is a *doctrine*, the $\mathcal{B}$-category of Ω-valued U-sheaves on a small $\mathcal{B}$-category is presentable, and that it satisfies a universal property:

Proposition 4.4.6.4. *For any doctrine* U*, the large* $\mathcal{B}$*-category* $\underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})$ *is presentable. Moreover, for any complete large* $\mathcal{B}$*-category* E*, restriction along the Yoneda embedding* h_{C} *induces an equivalence*

$$h_{\mathsf{C}}^*\colon \underline{\mathsf{Sh}}_{\mathsf{E}}(\underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})) \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$$

of large $\mathcal{B}$*-categories.*

Proof. Fix a small full subcategory $\mathcal{G} \hookrightarrow \mathcal{B}$ of generators, and define the small set

$$R = \bigsqcup_{G\in\mathcal{G}} \{f\colon \operatorname{colim} h_{\mathsf{C}} d \to h_{\mathsf{C}} \operatorname{colim} d \mid d\colon \mathsf{K} \to \pi_G^*\mathsf{C},\ \mathsf{K}^{\mathrm{op}} \in \mathsf{U}(G)\}$$

(where each f is to be considered as a map in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ in context $G \in \mathcal{G}$). We let $\mathsf{S}_R \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ be the subcategory that is spanned by R. Note that since R is a small set, the subcategory S_R is small, so that $\mathsf{D} = \mathsf{Loc}_{\mathsf{S}_R}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ is a presentable $\mathcal{B}$-category. Moreover, if E is an arbitrary complete large $\mathcal{B}$-category, the construction of S_R (together with the fact that the preservation of limits can be checked locally, see Remark 3.1.1.8) makes it evident that a cocontinuous functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{E}^{\mathrm{op}}$ carries the maps in S_R to equivalences precisely if its restriction to C is $\mathrm{op}(\mathsf{U})$-cocontinuous. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, the same assertion holds for any object $A \to \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}^{\mathrm{op}})$. As a consequence, the universal property of presheaf $\mathcal{B}$-categories implies that restriction along the Yoneda embedding h_{C} determines an equivalence of large $\mathcal{B}$-categories $h_{\mathsf{C}}^*\colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{E}^{\mathrm{op}})_{\mathsf{S}_R} \simeq \underline{\mathsf{Fun}}^{\mathrm{op}(\mathsf{U})\text{-}\mathrm{cc}}_{\mathcal{B}}(\mathsf{C}, \mathsf{E}^{\mathrm{op}})$. Upon taking opposite $\mathcal{B}$-categories and using Corollary 4.4.3.2, one thus obtains an equivalence $(Lh_{\mathsf{C}})^*\colon \underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D}) \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$. By plugging in $\mathsf{E} = \Omega$ into this equivalence and using proposition 4.4.6.3, one ends up with an equivalence $\mathsf{D} \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})$ of full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, which completes the proof. □

Whenever U is a *sound* doctrine, we can identify the $\mathcal{B}$-category of U-sheaves on an $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category C with the free $\mathsf{Filt}_{\mathsf{U}}$-cocompletion of C:

Proposition 4.4.6.5. *Let* U *be a sound internal class and let* C *be an* $\mathrm{op}(\mathsf{U})$*-cocomplete* $\mathcal{B}$*-category. Then there is an equivalence* $\underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C}) \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *of full subcategories of* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*.*

Proof. On account of Proposition 4.3.4.6 as well as Remarks 4.4.6.2 and 4.3.4.2, it suffices to show that a presheaf $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is U-flat if and only if F is U-continuous. As the inclusion $h_{\mathsf{C}^{\mathrm{op}}}\colon \mathsf{C}^{\mathrm{op}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \Omega)$ commutes with all limits that exist in C, the presheaf F being U-flat immediately implies that F is U-continuous. Conversely, suppose that F is U-continuous. By Proposition 4.3.4.6, it suffices to show that $\mathsf{C}_{/F}$ is weakly U-filtered. By applying Lemma 4.1.4.5 to the pullback square

$$\begin{tikzcd}
\mathsf{C}_{/F} \arrow[r] \arrow[d] & \Omega^{\mathrm{op}}_{/1_\Omega} \arrow[d] \\
\mathsf{C} \arrow[r, "F^{\mathrm{op}}"] & \Omega^{\mathrm{op}},
\end{tikzcd}$$

(which satisfies the conditions of the lemma by Proposition 3.1.6.3), we conclude that $\mathsf{C}_{/F}$ is $\mathrm{op}(\mathsf{U})$-cocomplete, hence the claim follows from Example 4.1.2.4. □

Corollary 4.4.6.6. *Let* U *be a sound doctrine and let* C *be an* $\mathrm{op}(\mathsf{U})$*-cocomplete* $\mathcal{B}$*-category. Then* $\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C})$ *is presentable. Moreover, for any cocomplete large* $\mathcal{B}$*-category* E*, restriction along the Yoneda embedding* h_{C} *induces an equivalence*

$$h_{\mathsf{C}}^*\colon \underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{C}),\mathsf{E}) \simeq \underline{\mathsf{Fun}}^{\mathrm{op}(\mathsf{U})\text{-cc}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})$$

of large $\mathcal{B}$*-categories.* □

Corollary 4.4.6.7. *Let* D *be a large* $\mathcal{B}$*-category. Then the following are equivalent:*

(1) D *is presentable;*
(2) there is a sound doctrine U *such that* D *is* U*-accessible and* $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ *is* $\mathrm{op}(\mathsf{U})$*-cocomplete;*
(3) there is a doctrine U *and a small* $\mathrm{op}(\mathsf{U})$*-cocomplete* $\mathcal{B}$*-category* C *such that one has an equivalence* $\mathsf{D} \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})$*.*

Proof. By combining Theorem 4.4.2.4 with Proposition 4.1.5.4, it is clear that (1) implies (2). If (2) is satisfied, Proposition 4.3.3.2 implies that $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is small and that there is an equivalence $\mathsf{D} \simeq \underline{\mathsf{Ind}}^{\mathsf{U}}_{\mathcal{B}}(\mathsf{D}^{\mathsf{U}\text{-cpt}})$. In light of Proposition 4.4.6.5, this shows that (3) is satisfied. Finally, Proposition 4.4.6.4 shows that (3) implies (1). □

We complete this section by noting that as a consequence of the results that we have established so far, we may deduce that the $\mathcal{B}$-category of sheaves between presentable $\mathcal{B}$-categories is presentable as well:

Corollary 4.4.6.8. *For every two presentable* $\mathcal{B}$*-categories* D *and* E*, the* $\mathcal{B}$*-category* $\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D})$ *is presentable as well.*

Proof. By Corollary 4.4.6.7, we may find a doctrine U and a small $\mathrm{op}(\mathsf{U})$-cocomplete $\mathcal{B}$-category C such that $\mathsf{D} \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})$. Consequently, Proposition 4.4.6.4 gives rise to an equivalence $\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D}) \simeq \underline{\mathsf{Sh}}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C})$. Therefore, it suffices to show that the right-hand side is presentable. Choose a small $\mathcal{B}$-category C' such that $\mathsf{E} \simeq \mathsf{Loc}_{\mathsf{S}'}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}'))$ for some $\mathsf{S}' \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')$ with S' small. We obtain a commutative square

$$\begin{tikzcd}
\underline{\mathsf{Sh}}^{\mathsf{U}}_{\mathsf{E}}(\mathsf{C}) \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E}) \arrow[d, hook] \\
\underline{\mathsf{Sh}}^{\mathsf{U}}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')}(\mathsf{C}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')).
\end{tikzcd}$$

We first claim that this square is a pullback. To see this, note that by Remarks 4.4.6.2 and 4.4.1.2, it will be enough to verify that a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{E}$ is U-continuous if $\mathsf{C}^{\mathrm{op}} \to \mathsf{E} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}')$ is U-continuous. This is a straightforward consequence of the fact that fully faithful functors are conservative. To proceed, note that by Corollary 4.4.2.6, the vertical map on the right in the above diagram defines a map in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$. Using Proposition 4.4.4.11, the proof is thus complete once we verify that the lower horizontal map is a map in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ as well. To see this, observe that by Lemma 4.6.1.3 below, we may identify this map with the inclusion

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}'))$$

that is induced by postcomposition with the inclusion $\underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. As the latter is a map in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ by Proposition 4.4.6.4, the claim thus follows by again appealing to Corollary 4.4.2.6. □

4.5. Digression: Aspects of internal higher algebra

The goal of this section is to set up the basic framework of higher algebra (in the sense of [56]) in the context of internal higher category theory. As our main goal is to use this framework to define tensor products of $\mathcal{B}$-categories in § 4.6, we will restrict our attention to a few selected results instead of giving a comprehensive account of this machinery. We begin in § 4.5.1 by defining symmetric monoidal $\mathcal{B}$-categories (and more generally $\mathcal{B}$-operads), and in § 4.5.2 we study algebras and modules in symmetric monoidal $\mathcal{B}$-categories. As a first application of this framework, § 4.5.4 contains a characterisation of dualisable objects in the $\mathcal{B}$-category of modules over an $\mathbb{E}_\infty$-ring in $\mathcal{B}$. The proof of this characterisation requires a few results about the notion of *stability* in the world of $\mathcal{B}$-categories, which we briefly discuss in § 4.5.3.

4.5.1. $\mathcal{B}$-operads, symmetric monoidal $\mathcal{B}$-categories and commutative monoids. Recall from [56, § 2.3.2] the definition of the presentable ∞-category $\mathrm{Op}_\infty^{\mathrm{gen}}$ of generalised ∞-operads. By construction, this ∞-category is the (non-full) subcategory of $(\mathrm{Cat}_\infty)_{/\,\mathrm{Fin}_*}$ (where Fin_* denotes the 1-category of finite pointed sets) that is spanned by the generalised ∞-operads and the morphisms of generalised ∞-operads. The full subcategory of $\mathrm{Op}_\infty^{\mathrm{gen}}$ that is spanned by the ∞-operads is denoted by Op_∞. By [56, Corollary 2.3.2.6] the inclusion $\mathrm{Op}_\infty \hookrightarrow \mathrm{Op}_\infty^{\mathrm{gen}}$ admits a left adjoint, and by [56, § 2.1.4] the ∞-category Op_∞ is presentable as well. Finally, recall from [56, Variant 2.1.4.13] that the (presentable) ∞-category $\mathrm{Cat}_\infty^\otimes$ is defined to be the subcategory of Op_∞ that is spanned by the symmetric monoidal ∞-categories and the symmetric monoidal functors. By [56, Proposition 2.2.4.9] the inclusion $\mathrm{Cat}_\infty^\otimes \hookrightarrow \mathrm{Op}_\infty$ also admits a left adjoint. In light of Construction 2.3.1.1 we may now define:

DEFINITION 4.5.1.1. A *generalised $\mathcal{B}$-operad* is an $\mathrm{Op}_\infty^{\mathrm{gen}}$-valued sheaf on $\mathcal{B}$, and the $\mathcal{B}$-category of generalised $\mathcal{B}$-operads is defined as the large $\mathcal{B}$-category $\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}} = \mathrm{Op}_\infty^{\mathrm{gen}} \otimes \Omega$. A generalised $\mathcal{B}$-operad is said to be a *$\mathcal{B}$-operad* if it takes values in Op_∞, and the large $\mathcal{B}$-category $\mathsf{Op}_{\mathcal{B}}$ of $\mathcal{B}$-operads is defined as the full subcategory of $\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$ that is spanned by the $\mathcal{B}_{/A}$-operads for each $A \in \mathcal{B}$. Finally, a (generalised) $\mathcal{B}$-operad is said to be a *symmetric monoidal $\mathcal{B}$-category* if it takes values in $\mathrm{Cat}_\infty^\otimes$, and similarly a morphism of symmetric monoidal $\mathcal{B}$-categories is said to be a symmetric monoidal functor if it takes values in $\mathrm{Cat}_\infty^\otimes$. The large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^\otimes$ of symmetric monoidal $\mathcal{B}$-categories is defined as the subcategory of $\mathsf{Op}_{\mathcal{B}}$ that is spanned by the symmetric monoidal functors between symmetric monoidal $\mathcal{B}$-categories, and the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\otimes,\mathrm{lax}}$ is defined as the essential image of the inclusion $\mathsf{Cat}_{\mathcal{B}}^\otimes \hookrightarrow \mathsf{Op}_{\mathcal{B}}$. We refer to the maps in $\mathsf{Cat}_{\mathcal{B}}^{\otimes,\mathrm{lax}}$ as *lax symmetric monoidal functors*.

REMARK 4.5.1.2. By construction, we have canonical equivalences $\mathsf{Op}_{\mathcal{B}} \simeq \mathrm{Op}_\infty \otimes \Omega$ and $\mathsf{Cat}_{\mathcal{B}}^\otimes \simeq \mathrm{Cat}_\infty^\otimes \otimes \Omega$ with respect to which the inclusions $\mathsf{Op}_{\mathcal{B}} \hookrightarrow \mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$ and $\mathsf{Cat}_{\mathcal{B}}^\otimes \hookrightarrow \mathsf{Op}_{\mathcal{B}}$ correspond to the image of the inclusions $\mathrm{Op}_\infty \hookrightarrow \mathrm{Op}_\infty^{\mathrm{gen}}$ and $\mathrm{Cat}_\infty^\otimes \hookrightarrow \mathrm{Op}_\infty$ under the functor $- \otimes \Omega \colon \mathrm{Pr}_\infty^{\mathrm{R}} \to \mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ from Example 4.4.4.8. Consequently, the chain of inclusions

$$\mathsf{Cat}_{\mathcal{B}}^\otimes \hookrightarrow \mathsf{Op}_{\mathcal{B}} \hookrightarrow \mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$$

defines morphisms in $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{R}}$.

REMARK 4.5.1.3. Taking the fibre over $\langle 1 \rangle \in \mathrm{Fin}_*$ defines a forgetful functor $\mathrm{Op}_\infty \to \mathrm{Cat}_\infty$. By the discussion in [56, § 2.1.4], this functor defines a map in $\mathrm{Pr}_\infty^{\mathrm{R}}$. Consequently, applying $- \otimes \Omega$ to this map yields a well-defined morphism $\mathsf{Op}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ in $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{R}}$ (cf. 4.4.4.8). Given a $\mathcal{B}$-operad $\mathsf{O}^\otimes$, we denote its image under this functor by O.

REMARK 4.5.1.4. By making use of [56, Proposition B.2.9], the inclusion $\mathrm{Op}_\infty^{\mathrm{gen}} \hookrightarrow (\mathrm{Cat}_\infty)_{/\,\mathrm{Fin}_*}$ admits a left adjoint and thus defines a morphism in $\mathrm{Pr}_\infty^{\mathrm{R}}$. Upon applying the functor $- \otimes \Omega$, we thus obtain a monomorphism $\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}} \hookrightarrow (\mathrm{Cat}_\infty)_{/\,\mathrm{Fin}_*} \otimes \Omega \simeq (\mathsf{Cat}_{\mathcal{B}})_{/\,\mathrm{Fin}_*}$. Unwinding the definitions, one finds that a functor $p \colon \mathsf{O}^\otimes \to \mathrm{Fin}_*$ is contained in $\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$ precisely if

(1) for all $A \in \mathcal{B}$ the pullback $(\eta^A)^* \mathsf{O}^\otimes(A) \to \mathrm{Fin}_*$ of $p(A)$ along the adjunction unit $\eta^A \colon \mathrm{Fin}_* \to \Gamma_{\mathcal{B}_{/A}} \mathrm{Fin}_*$ defines a generalised ∞-operad, and
(2) for all maps $s \colon B \to A$ in $\mathcal{B}$ the induced map $(\eta^A)^* \mathsf{O}^\otimes(A) \to (\eta^B)^* \mathsf{O}^\otimes(B)$ defines a morphism of generalised ∞-operads.

Similarly, a morphism $f \colon \mathsf{O}^\otimes \to \mathsf{O}'^\otimes$ in $(\mathsf{Cat}_{\mathcal{B}})_{/\,\mathrm{Fin}_*}$ between generalised $\mathcal{B}$-operads is contained in $\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$ precisely if for all $A \in \mathcal{B}$ the pullback $(\eta^A)^* f(A)$ defines a morphism of generalised ∞-operads. One can make analogous observations for the subcategories $\mathsf{Op}_{\mathcal{B}}$, $\mathsf{Cat}_{\mathcal{B}}^{\otimes,\mathrm{lax}}$ and $\mathsf{Cat}_{\mathcal{B}}^\otimes$.

Remark 4.5.1.5. By combining Remarks 4.5.1.4 and 2.3.2.5, we may identify $\mathsf{Cat}_{\mathcal{B}}^{\otimes}$ with the full subcategory of $\mathsf{Cocart}_{\mathrm{Fin}_*}$ that is spanned by those cocartesian fibrations $p\colon \mathsf{C}^{\otimes} \to \mathrm{Fin}_*$ (in arbitrary context $A \in \mathcal{B}$) for which the induced cocartesian fibration $(\eta^B)^*\mathsf{C}^{\otimes}(B) \to \mathrm{Fin}_*$ defines a symmetric monoidal ∞-category for every $B \in \mathcal{B}_{/A}$.

For later use, recall from [56, Proposition 2.3.2.9] that the functor $\mathcal{C} \mapsto \mathrm{Fin}_* \times \mathcal{C}$ determines a fully faithful map $\mathrm{Cat}_\infty \hookrightarrow \mathrm{Op}_\infty^{\mathrm{gen}}$ in $\mathrm{Pr}_\infty^{\mathrm{R}}$. By applying the functor $- \otimes \Omega$, we thus obtain:

Proposition 4.5.1.6. *The map* $\mathsf{C} \mapsto \mathrm{Fin}_* \times \mathsf{C}$ *determines a fully faithful functor* $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$. ☐

Recall that under the straightening equivalence, symmetric monoidal ∞-categories can be identified with commutative monoids in Cat_∞. Our next goal is to derive an analogous result internally. To that end, we let $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ be the pointed map that carries i to 1 and every other element to 0. Given any functor $C_\bullet\colon \mathrm{Fin}_* \to \mathcal{C}$, $\langle n\rangle \mapsto C_n$ with values in an arbitrary ∞-category $\mathcal{C}$, we will denote by $p_i\colon C_n \to C_1$ the image of ρ_i under $C_\bullet$. We may now define:

Definition 4.5.1.7. A *commutative monoid* in a $\mathcal{B}$-category C is a functor $M\colon \mathrm{Fin}_* \to \mathsf{C}$ such that for all $n \geq 0$ the functors $(p_i\colon M_n \to M_1)_{1\leq i\leq n}$ exhibit M_n as the product $\prod_{i=1}^n M_1$ in C. We define the $\mathcal{B}$-category $\mathsf{CMon}(\mathsf{C})$ as the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{Fin}_*, \mathsf{C})$ that is spanned by those functors $\mathrm{Fin}_* \to \pi_A^*\mathsf{C}$ that define commutative monoids in $\pi_A^*\mathsf{C}$ for all $A \in \mathcal{B}$.

Remark 4.5.1.8. Since the condition of a morphism in a $\mathcal{B}$-category to be an equivalence is local in $\mathcal{B}$, an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathrm{Fin}_*, \mathsf{C})$ is contained in $\mathsf{CMon}(\mathsf{Cat}_{\mathcal{B}})$ if and only if it defines a commutative monoid in $\pi_A^*\mathsf{C}$. In particular, one obtains a canonical equivalence $\pi_A^* \mathsf{CMon}(\mathsf{C}) \simeq \mathsf{CMon}(\pi_A^*\mathsf{C})$ for every $A \in \mathcal{B}$.

Remark 4.5.1.9. Evaluation at $\langle 1\rangle\colon 1 \to \mathrm{Fin}_*$ defines a forgetful functor $\mathsf{CMon}(\mathsf{C}) \to \mathsf{C}$. We will usually abuse notation and identify a commutative monoid M with its underlying object in C. By evaluating such a monoid M at the unique active map $\langle 2\rangle \to \langle 1\rangle$ in Fin_*, one obtains a multiplication map $\mu\colon M \times M \to M$ that is associative and commutative up to infinite coherent homotopies. Moreover, evaluation at the unique pointed map $\langle 0\rangle \to \langle 1\rangle$ induces a unit $e\colon 1_{\mathsf{C}} \to M$ such that the induced maps $\mu(e, -)$ and $\mu(-, e)$ are equivalences.

By combining Remark 4.5.1.5 with the straightening equivalence for cocartesian fibrations Theorem 2.3.2.7, one now finds:

Proposition 4.5.1.10. *The straightening equivalence restricts to an equivalence of large $\mathcal{B}$-categories* $\mathsf{Cat}_{\mathcal{B}}^{\otimes} \simeq \mathsf{CMon}(\mathsf{Cat}_{\mathcal{B}})$. ☐

Remark 4.5.1.11. Upon composing the equivalence $\mathsf{Cat}_{\mathcal{B}}^{\otimes} \simeq \mathsf{CMon}(\mathsf{Cat}_{\mathcal{B}})$ from Proposition 4.5.1.10 with the forgetful functor $\mathsf{CMon}(\mathsf{Cat}_{\mathcal{B}}) \to \mathsf{Cat}_{\mathcal{B}}$, one recovers the functor $\mathsf{Cat}_{\mathcal{B}}^{\otimes} \to \mathsf{Cat}_{\mathcal{B}}$ from Remark 4.5.1.3. If $\mathsf{C}^{\otimes}$ is a symmetric monoidal $\mathcal{B}$-category, the multiplication map thus defines a bifunctor $- \otimes -\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, and the unit is an object $1_{\otimes}\colon 1 \to \mathsf{C}$.

A rich source for symmetric monoidal $\mathcal{B}$-categories are those $\mathcal{B}$-categories that admit finite products. We will denote by $\mathsf{Cat}_{\mathcal{B}}^{\Pi}$ the subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the $\mathsf{LConst}_{\mathrm{Fin}}$-continuous functors between $\mathsf{LConst}_{\mathrm{Fin}}$-complete $\mathcal{B}_{/A}$-categories for all $A \in \mathcal{B}$, where Fin is the 1-category of finite sets. By the dual of Corollary 3.2.4.6 and the fact that $\mathrm{Cat}_\infty^{\Pi}$ is presentable [56, Lemma 4.8.4.2], we may identify $\mathsf{Cat}_{\mathcal{B}}^{\Pi} \simeq \mathrm{Cat}_\infty^{\Pi} \otimes \Omega$. Now recall from [56, Corollary 2.4.1.9] that there is a fully faithful functor $(-)^{\times}\colon \mathrm{Cat}_\infty^{\Pi} \hookrightarrow \mathrm{Cat}_\infty^{\otimes}$ that assigns to an ∞-category $\mathcal{C}$ with finite products the associated *cartesian* monoidal ∞-category $\mathcal{C}^{\times}$ and that fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{\Pi} \arrow[rr, hook, "(-)^{\times}"] \arrow[dr] & & \mathrm{Cat}_\infty^{\otimes} \arrow[dl] \\
& \mathrm{Cat}_\infty &
\end{tikzcd}$$

in which the right diagonal map is the forgetful functor. Note that this diagram takes values in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$. Upon applying the functor $- \otimes \Omega$, we thus obtain:

PROPOSITION 4.5.1.12. *There is a fully faithful functor* $(-)^{\times}\colon \mathsf{Cat}_{\mathcal{B}}^{\Pi} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}^{\otimes}$ *that carries a* $\mathcal{B}$*-category* C *to the* cartesian *monoidal* $\mathcal{B}$*-category* $\mathsf{C}^{\times}$ *and that fits into a commutative diagram*

$$\begin{tikzcd} \mathsf{Cat}_{\mathcal{B}}^{\Pi} \arrow[rr, hook, "(-)^{\times}"] \arrow[dr] & & \mathsf{Cat}_{\mathcal{B}}^{\otimes} \arrow[dl] \\ & \mathsf{Cat}_{\mathcal{B}} & \end{tikzcd}$$

of large $\mathcal{B}$*-categories.*

REMARK 4.5.1.13. Suppose that $\mathsf{C}^{\otimes}$ is a symmetric monoidal $\mathcal{B}$-category and that D is a full subcategory of C such that the tensor functor $- \otimes -\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ restricts to a functor $\mathsf{D} \times \mathsf{D} \to \mathsf{D}$ and the unit object $1_{\otimes}\colon 1 \to \mathsf{C}$ is contained in D. Then we can canonically equip D with the structure of a symmetric monoidal $\mathcal{B}$-category: indeed, $\mathsf{C}^{\otimes}$ is determined by a functor $\mathsf{C}^{\otimes}(-,-)\colon \mathcal{B}^{\mathrm{op}} \times \mathrm{Fin}_* \to \mathrm{Cat}_{\infty}$ and we may consider the full subfunctor that consists for $A \in \mathcal{B}^{\mathrm{op}}$ and $\langle n \rangle \in \mathrm{Fin}_*$ of the full subcategory of $\mathsf{C}^{\otimes}(A, \langle n \rangle)$ that corresponds under the equivalence

$$\mathsf{C}^{\otimes}(A, \langle n \rangle) \simeq \prod_{i=1}^{n} \mathsf{C}^{\otimes}(A, \langle 1 \rangle) \simeq \prod_{i=1}^{n} \mathsf{C}(A)$$

to the subcategory $\prod_{i=1}^{n} \mathsf{D}(A) \subseteq \prod_{i=1}^{n} \mathsf{C}(A)$. By our assumption on $- \otimes -$ this yields a well-defined functor $\mathsf{D}^{\otimes}(-,-)\colon \mathcal{B}^{\mathrm{op}} \times \mathrm{Fin}_* \to \mathrm{Cat}_{\infty}$, and by [56, Remark 2.2.1.12] the functor $\mathsf{D}^{\otimes}(A,-)$ defines a symmetric monoidal ∞-category for every $A \in \mathcal{B}$. So we get a functor $\mathsf{D}^{\otimes}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\otimes}$ whose composition with the forgetful functor $\mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Cat}_{\infty}$ recovers D and which therefore defines the desired monoidal structure on D.

4.5.2. Algebras and modules in symmetric monoidal $\mathcal{B}$-categories. Recall from [56, § 2.1.3] that a *commutative algebra* in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a map of ∞-operads $\mathrm{Fin}_* \to \mathcal{C}^{\otimes}$, and an *associative algebra* in $\mathcal{C}^{\otimes}$ is a map of ∞-operads $\mathrm{Assoc} \to \mathcal{C}^{\otimes}$, where Assoc denotes the associative operad. One obtains functors $\mathrm{Alg}\colon \mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Cat}_{\infty}$ and $\mathrm{CAlg}\colon \mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Cat}_{\infty}$ that assign to a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ the ∞-category $\mathrm{Alg}(\mathcal{C})$ of associative algebras in $\mathcal{C}^{\otimes}$ and the ∞-category $\mathrm{CAlg}(\mathcal{C})$ of commutative algebras in $\mathcal{C}^{\otimes}$, respectively. Note that either of these functors defines a map in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$. In light of Example 4.4.4.8, we may thus define:

DEFINITION 4.5.2.1. We define the maps $\mathsf{Alg}\colon \mathsf{Cat}_{\mathcal{B}}^{\otimes} \to \mathsf{Cat}_{\mathcal{B}}$ and $\mathsf{CAlg}\colon \mathsf{Cat}_{\mathcal{B}}^{\otimes} \to \mathsf{Cat}_{\mathcal{B}}$ in $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ as the maps that arise from applying $- \otimes \Omega$ to the functors $\mathrm{Alg}\colon \mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Cat}_{\infty}$ and $\mathrm{CAlg}\colon \mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Cat}_{\infty}$, respectively. For a symmetric monoidal $\mathcal{B}$-category, we refer to the $\mathcal{B}$-category $\mathsf{Alg}(\mathsf{C})$ ($\mathsf{CAlg}(\mathsf{C})$) as the $\mathcal{B}$-category of *associative (commutative) algebras* in C.

By construction, an associative/commutative algebra in $\mathsf{C}^{\otimes}$ is simply an associative/commutative algebra in the symmetric monoidal ∞-category $\Gamma(\mathsf{C}^{\otimes})$.

Recall from [56, Proposition 2.4.2.5] that there is a commutative triangle

$$\begin{tikzcd} \mathrm{Cat}_{\infty}^{\Pi} \arrow[rr, hook, "(-)^{\times}"] \arrow[dr, "\mathrm{CMon}"'] & & \mathrm{Cat}_{\infty}^{\otimes} \arrow[dl, "\mathrm{CAlg}"] \\ & \mathrm{Cat}_{\infty} & \end{tikzcd}$$

in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$. By applying $- \otimes \Omega$ to this diagram, we therefore obtain:

PROPOSITION 4.5.2.2. *Let* C *be a* $\mathcal{B}$*-category with finite products. Then there is a canonical equivalence*

$$\mathsf{CMon}(\mathsf{C}) \simeq \mathsf{CAlg}(\mathsf{C}^{\times})$$

of $\mathcal{B}$*-categories that is natural in* C. ☐

Our next goal is to define $\mathcal{B}$-categories of *modules* in a symmetric monoidal $\mathcal{B}$-category. To that end, let us briefly recall the setup from [56, § 3.3.3]: we let $\mathcal{K}$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Fin}_*)$ that is spanned by the semi-inert maps, and we denote by $\mathcal{K}^0 \hookrightarrow \mathcal{K}$ the full subcategory that is spanned by the null maps. We say that a morphism f in $\mathcal{K}$ or $\mathcal{K}^0$ is inert if both $d_0(f)$ and $d_1(f)$ are inert. By making use of the projections $d_1 \colon \mathcal{K} \to \mathrm{Fin}_*$ and $d_1 \colon \mathcal{K}^0 \to \mathrm{Fin}_*$, we may regard both $\mathcal{K}$ and $\mathcal{K}^0$ as preoperads (with the marked edges given by the inert maps). Now the two spans $\mathrm{Fin}_* \xleftarrow{d_1} \mathcal{K} \xrightarrow{d_0} \mathrm{Fin}_*$ and $\mathrm{Fin}_* \xleftarrow{d_1} \mathcal{K}^0 \xrightarrow{d_0} \mathrm{Fin}_*$ (viewed as spans of marked simplicial sets) satisfy the conditions of [56, Theorem B.4.2] and therefore determine left Quillen endofunctors $- \times_{\mathrm{Fin}_*} \mathcal{K}$ and $- \times_{\mathrm{Fin}_*} \mathcal{K}^0$ on the 1-category POp_∞ of preoperads with respect to the model structure for generalised ∞-operads (see in particular [56, Proposition 3.3.3.18] for a proof of the first case, the second case follows by analogous arguments). Passing to right adjoints thus gives rise to right Quillen endofunctors $\overline{\mathrm{Mod}}(-)^\otimes$ and ${}^{\mathsf{P}}\mathrm{CAlg}(-)$ on POp_∞, and the inclusion $\mathcal{K}^0 \hookrightarrow \mathcal{K}$ determines a morphism $\overline{\mathrm{Mod}}(-)^\otimes \to {}^{\mathsf{P}}\mathrm{CAlg}(-)$. Similarly, the span $\mathrm{Fin}_* \leftarrow \mathrm{Fin}_* \times \mathrm{Fin}_* \to \mathrm{Fin}_*$ of marked simplicial sets satisfies the conditions of [56, Theorem B.4.2] as well and therefore gives rise to a left Quillen endofunctor whose right adjoint recovers the functor $\mathrm{Fin}_* \times \mathrm{CAlg}(-) \colon \mathrm{POp}_\infty \to \mathrm{POp}_\infty$. By [56, Remark 3.3.3.7] the map $\mathcal{K}^0 \to \mathrm{Fin}_* \times \mathrm{Fin}_*$ induces a categorical equivalence $\mathrm{Fin}_* \times \mathrm{CAlg}(-) \to {}^{\mathsf{P}}\mathrm{CAlg}(-)$ and therefore in particular a weak equivalence in the model structure for generalised ∞-operads.

In total, these observations imply that upon passing to the underlying ∞-categories, one ends up with a map $\mathrm{Mod}(-)^\otimes \colon \mathrm{Op}_\infty^{\mathrm{gen}} \to \mathrm{Op}_\infty^{\mathrm{gen}}$ in $\mathrm{Pr}^{\mathrm{R}}_\infty$ together with a morphism $\mathrm{Mod}(-)^\otimes \to \mathrm{Fin}_* \times \mathrm{CAlg}(-)$, or equivalently a map $\mathrm{Op}_\infty^{\mathrm{gen}} \to \mathrm{Fun}(\Delta^1, \mathrm{Op}_\infty^{\mathrm{gen}})$ in $\mathrm{Pr}^{\mathrm{R}}_\infty$. By making use of the evident equivalence of large $\mathcal{B}$-categories $\mathrm{Fun}(\Delta^1, \mathrm{Op}_\infty^{\mathrm{gen}}) \otimes \Omega \simeq (\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}})^{\Delta^1}$, applying $- \otimes \Omega$ yields a functor $\mathsf{Mod}(-)^\otimes \colon \mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}} \to \mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}$ in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ together with a morphism $p \colon \mathsf{Mod}(-)^\otimes \to \mathrm{Fin}_* \times \mathsf{CAlg}(-)$ of generalised $\mathcal{B}$-operads. Given a symmetric monoidal $\mathcal{B}$-category $\mathsf{C}^\otimes$ and a commutative algebra $R \colon 1 \to \mathsf{CAlg}(\mathsf{C})$, we will denote by $\mathsf{Mod}_R(\mathsf{C})^\otimes \to \mathrm{Fin}_*$ the pullback of p along the map $(\mathrm{id}, R) \colon \mathrm{Fin}_* \to \mathrm{Fin}_* \times \mathsf{CAlg}(\mathsf{C})$. By [56, Theorem 3.3.3.9] and Remark 4.5.1.4, the map $\mathsf{Mod}_R(\mathsf{C})^\otimes \to \mathrm{Fin}_*$ defines a $\mathcal{B}$-operad.

Definition 4.5.2.3. Let C be a symmetric monoidal $\mathcal{B}$-category and let $R \colon 1 \to \mathsf{CAlg}(\mathsf{C})$ be a commutative algebra in C. We define the $\mathcal{B}$-category $\mathsf{Mod}_R(\mathsf{C})$ of *modules* over R as the underlying $\mathcal{B}$-category of the $\mathcal{B}$-operad $\mathsf{Mod}_R(\mathsf{C})^\otimes$.

Our next goal is to investigate the functoriality of $\mathsf{Mod}_R(\mathsf{C})^\otimes$ in R. To that end, note that the diagonal embedding $\mathrm{Fin}_* \hookrightarrow \mathcal{K}$ induces a forgetful functor $\overline{\mathrm{Mod}}(-)^\otimes \to \mathrm{id}_{\mathrm{POp}_\infty}$ and therefore, by the same procedure as above, a morphism $\mathsf{Mod}(-)^\otimes \to \mathrm{id}_{\mathsf{Op}_{\mathcal{B}}^{\mathrm{gen}}}$. It thus follows from [56, Corollary 3.4.3.4]:

Proposition 4.5.2.4. *For any symmetric monoidal $\mathcal{B}$-category C, the projection $\mathsf{Mod}(\mathsf{C})^\otimes \to \mathsf{CAlg}(\mathsf{C})$ is a cartesian fibration, and a map in $\mathsf{Mod}(\mathsf{C})^\otimes$ is cartesian if and only if its image along the forgetful functor $\mathsf{Mod}(\mathsf{C})^\otimes \to \mathsf{C}^\otimes$ is an equivalence.* □

Corollary 4.5.2.5. *For any symmetric monoidal $\mathcal{B}$-category C, the projection $\mathsf{Mod}(\mathsf{C}) \to \mathsf{CAlg}(\mathsf{C})$ is a cartesian fibration.* □

Next we would like to establish functoriality in the opposite direction, i.e. to construct base change functors $\mathsf{Mod}_R(\mathsf{C})^\otimes \to \mathsf{Mod}_S(\mathsf{C})^\otimes$ along any algebra map $R \to S$. The existence of these functors requires the existence of geometric realisations. For the remainder of this section, we shall therefore fix an internal class U of $\mathcal{B}$-categories that contains Δ^{op} and assume that $\mathsf{C}^\otimes$ is a symmetric monoidal $\mathcal{B}$-category such that C is U-cocomplete and the tensor functor $- \otimes - \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ is U-bilinear in the sense of § 4.6.1 below. We now obtain:

Proposition 4.5.2.6. *The projection $p \colon \mathsf{Mod}(\mathsf{C})^\otimes \to \mathrm{Fin}_* \times \mathsf{CAlg}(\mathsf{C})$ is a cocartesian fibration.*

Proof. By using [61, Remark 3.2.7], it suffices to show that for any map $\alpha \colon \langle m \rangle \to \langle n \rangle$ of finite pointed sets and any map $f \colon \Delta^1 \otimes A \to \mathsf{CAlg}(\mathsf{C})$, the pair $(\pi_A^*(\alpha), f)$ admits a cocartesian lift. By [56, Theorem 4.5.3.1], the map $\mathrm{Mod}(\mathsf{C}(A))^{\otimes} \to \mathrm{Fin}_* \times \mathrm{CAlg}(\mathsf{C}(A))$ is a cocartesian fibration of ∞-categories, so we may choose a cocartesian lift h of (α, f) in $\mathrm{Mod}(\mathsf{C}(A))$. By construction, this map defines a lift of $(\pi_A^*\alpha, f)$ in $\mathsf{Mod}(\mathsf{C})^{\otimes}$ in context A. To show that h is cocartesian in the internal sense, it now suffices to show that for any map $s\colon B \to A$ in $\mathcal{B}$ the map $s^*(h)$ defines a cocartesian morphism with respect to $\mathrm{Mod}(\mathsf{C}(B))^{\otimes} \to \mathrm{Fin}_* \times \mathrm{CAlg}(\mathsf{C}(B))$. According to the proof of [56, Theorem 4.5.3.1], we may find a factorisation $h \simeq h''h'$, where h' defines a cocartesian morphism of the fibre $\mathrm{Mod}_R(\mathsf{C}(A)) \to \mathrm{Fin}_*$ over some $R \in \mathrm{CAlg}(\mathsf{C}(A))$ and where h'' is contained in the fibre $\mathrm{Mod}(\mathsf{C}(A))^{\otimes}|_{\langle n\rangle}$ and satisfies the following property: for any $1 \leq i \leq n$, there is a commutative diagram

$$\begin{tikzcd} y \arrow[r, "h''"] \arrow[d] & z \arrow[d] \\ y_i \arrow[r, "h_i''"] & z_i \end{tikzcd}$$

where the vertical maps are inert morphisms lying over $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ and h_i'' is a cocartesian morphism of $\mathrm{Mod}(\mathsf{C}(A)) \to \mathrm{CAlg}(\mathsf{C}(A))$. Since every map that admits such a factorisation must be cocartesian, it is enough to show that $s^*(h')$ is a cocartesian morphism of $\mathrm{Mod}_{s^*R}(\mathsf{C}(B))^{\otimes} \to \mathrm{Fin}_*$ and that $s^*(h_i'')$ is a cocartesian morphism of $\mathrm{Mod}(\mathsf{C}(B)) \to \mathrm{CAlg}(\mathsf{C}(B))$. To see the first case we equivalently have to see that the functor $\mathrm{Mod}_R(\mathsf{C}(A))^{\otimes} \to \mathrm{Mod}_{s*R}(\mathsf{C}(B))^{\otimes}$ induced by s^* is symmetric monoidal. This in turn follows from the description of the symmetric monoidal structure on $\mathrm{Mod}_R(\mathsf{C}(A))^{\otimes}$ via the bar-construction [56, Theorem 4.5.2.1, Propositions 4.4.3.12 and 4.4.2.8] because s^* commutes with Δ^{op}-indexed colimits. Similarly, the second case follows from the fact that the cocartesian maps in $\mathrm{Mod}(\mathsf{C}(A)) \to \mathrm{CAlg}(\mathsf{C}(A))$ are given by the relative tensor product (see e.g. [56, Lemma 4.5.3.5, Proposition 4.6.2.17 and Corollary 4.5.1.6]) and the latter is again described via the Bar-construction. $\square$

Corollary 4.5.2.7. *For every commutative algebra A in C, the $\mathcal{B}$-operad $\mathsf{Mod}_A(\mathsf{C})^{\otimes}$ is a symmetric monoidal $\mathcal{B}$-category, and the cocartesian fibration $\mathsf{Mod}(\mathsf{C})^{\otimes} \to \mathrm{Fin}_* \times \mathsf{CAlg}(\mathsf{C})$ straightens to a functor*

$$\mathsf{CAlg}(\mathsf{C}) \to \mathsf{CMon}(\mathsf{Cat}_{\mathcal{B}}) \simeq \mathsf{Cat}_{\mathcal{B}}^{\otimes}$$

that maps A to $\mathsf{Mod}_A(\mathsf{C})^{\otimes}$. $\square$

For later use we note the following observation:

Proposition 4.5.2.8. *Suppose that C is a complete and cocomplete symmetric monoidal $\mathcal{B}$-category and that the tensor product $-\otimes-\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C}$ is bilinear. Let $R\colon 1 \to \mathsf{CAlg}(\mathsf{C})$ be a commutative algebra. Then*

(1) The $\mathcal{B}$-category $\mathsf{Mod}_R(\mathsf{C})$ is complete and cocomplete.
(2) The canonical functor $F\colon \mathsf{Mod}_R(\mathsf{C}) \to \mathsf{C}$ is continuous and cocontinuous.
(3) The tensor product $-\otimes_R-\colon \mathsf{Mod}_R(\mathsf{C}) \times \mathsf{Mod}_R(\mathsf{C}) \to \mathsf{Mod}_R(\mathsf{C})$ is bilinear.

Furthermore if C is presentably symmetric monoidal (in the sense of Definition 4.6.2.10) then so is $\mathsf{Mod}_R(\mathsf{C})$.

Proof. Since by Corollary 4.5.2.5 and Proposition 4.5.2.6 the functor $\mathsf{Mod}(\mathsf{C}) \to \mathsf{CAlg}(\mathsf{C})$ is both cartesian and cocartesian, the essentially unique map $1_{\otimes} \to R$ of algebras induces an adjunction

$$-\otimes R\colon \mathsf{C} \simeq \mathsf{Mod}_{1_{\otimes}} \leftrightarrows \mathsf{Mod}_R(\mathsf{C})\colon F.$$

Therefore, the functor F preserves all limits that exist in $\mathsf{Mod}_R(\mathsf{C})$. For fixed $A \in \mathcal{B}$ the functor $F(A)$ is equivalent to the forgetful functor $\mathrm{Mod}_{\pi_A^*R}(\mathsf{C}(A)) \to \mathsf{C}(A)$, which creates colimits by [56, Corollary 3.4.4.6]. Furthermore $F(A)$ has a left adjoint given by $-\otimes \pi_A^* R$. Since the diagram

$$\begin{tikzcd} \mathrm{Mod}_{\pi_A^*R}(\mathsf{C}(A)) \arrow[r, "s^*"] \arrow[d, "F(A)"'] & \mathrm{Mod}_{\pi_B^*R}(\mathsf{C}(B)) \arrow[d, "F(B)"] \\ \mathsf{C}(A) \arrow[r, "s^*"'] & \mathsf{C}(B) \end{tikzcd}$$

commutes for any map $s\colon B \to A$ in $\mathcal{B}$, this shows that $\mathsf{Mod}_R(\mathsf{C})$ is LConst-cocomplete and F is LConst-cocontinuous. Next we show that $s^*\colon \mathrm{Mod}_{\pi_A^* R}(\mathsf{C}(A)) \to \mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B))$ admits a left adjoint $s_!$ that is compatible with F in the obvious way. By [56, Remark 4.7.3.15] we may find a functor $G\colon \mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B)) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C}(B))$ such that the composite

$$\mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B)) \xrightarrow{G} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C}(B)) \xrightarrow{-\otimes \pi_B^* R} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B))) \xrightarrow{\mathrm{colim}} \mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B))$$

is equivalent to the identity. Note that since C is cocomplete, the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ admits a left adjoint $s_!$. Therefore, the composition

$$\mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B)) \xrightarrow{G} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C}(B)) \xrightarrow{((-\otimes \pi_A^* R)\circ s_!)_*} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\pi_A^* R}(\mathsf{C}(A))) \xrightarrow{\mathrm{colim}} \mathrm{Mod}_{\pi_A^* R}(\mathsf{C}(A))$$

defines a left adjoint of the transition functor s^*. Next, we claim that the canonical natural transformation $\alpha\colon s_! F(B) \to F(A) s_!$ is an equivalence. Since all functors commute with colimits it suffices to check this for objects of the form $M \otimes \pi_B^* R$ for some $M \in \mathsf{C}(B)$. In this case α identifies with the projection formula transformation

$$s_!(M \otimes \pi_B^* R) \to s_! M \otimes \pi_A^* R$$

which is an equivalence since the tensor products commutes with Ω-indexed colimits in the first variable. It follows that for any pullback square

$$\begin{tikzcd} Q \arrow[r, "t"] \arrow[d, "q"'] & P \arrow[d, "p"] \\ B \arrow[r, "s"'] & A \end{tikzcd}$$

in $\mathcal{B}$ the canonical transformation $q_! t^* \to s^* p_!$ is an equivalence since we may check this after composing with the forgetful functor $F(B)$ and since C is cocomplete. Thus we have shown that $\mathsf{Mod}_R(\mathsf{C})$ is cocomplete and that the forgetful functor F is cocontinuous. Furthermore, we note that the functor $s^*\colon \mathrm{Mod}_{\pi_A^* R}(\mathsf{C}(A)) \to \mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B))$ admits a right adjoint given by the composite

$$\mathrm{Mod}_{\pi_B^* R}(\mathsf{C}(B)) \xrightarrow{s_*} \mathrm{Mod}_{s_* \pi_B^* R}(\mathsf{C}(A)) \to \mathrm{Mod}_{\pi_A^* R}(\mathsf{C}(A))$$

where the second functor is given restriction of scalars along $\pi_A^* R \to s_* s^* \pi_A^* R = s_* \pi_B^* R$. Since s_* is compatible with the forgetful functor F, the same argument as above shows that $\mathsf{Mod}_R(\mathsf{C})$ is Ω-complete and therefore complete. Thus we have shown (1) and (2).

We will now show (3). It follows from [56, Corollary 3.4.4.6] that for some fixed $M \in \mathsf{Mod}_R(\mathsf{C})$ in context A, the functor $-\otimes_R M$ is LConst-cocontinuous, so it suffices to show that it is also Ω-cocontinuous. For this we have to see that for any $s\colon C \to B$ in $\mathcal{B}_{/A}$ the canonical map

$$s_!(N) \otimes_{\pi_B^* R} \pi_B^* M \to s_!(N \otimes_{\pi_C^* R} \pi_C^* M)$$

is an equivalence, by Proposition 3.2.4.2. Again using that the relative tensor product is the colimit of the bar construction, it follows that we only have to see that the canonical map

$$s_!(N) \otimes (\pi_B^* R)^{\otimes n} \pi_B^* M \to s_!(N \otimes (\pi_C^* R)^{\otimes n} \pi_C^* M)$$

is an equivalence for all n This follows because π_B^* is symmetric monoidal and the projection formula holds for C .

Finally, if C is presentably symmetric monoidal then $\mathsf{Mod}_R(\mathsf{C})$ is section-wise presentable by [56, Theorem 3.4.4.2] and therefore presentably symmetric monoidal by what we have seen above. □

4.5.3. Stable $\mathcal{B}$-categories. In this section we will define and study a few basic properties of *stable* $\mathcal{B}$-categories. All definitions and results are straightforward adaptions of [56, §1.4.2].

DEFINITION 4.5.3.1. A $\mathcal{B}$-category C is called *pointed* if there is an object $0_{\mathsf{C}}\colon 1 \to \mathsf{C}$ which is both initial and final in C.

REMARK 4.5.3.2. It follows from Example 3.1.1.14 that C is pointed if and only if the associated functor $\mathsf{C}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through the subcategory $\mathrm{Cat}_{\infty,*}$ of pointed ∞-categories.

Definition 4.5.3.3. A pointed $\mathcal{B}$-category C is called *stable* if it is finitely complete and cocomplete and a square $\Delta^1 \times \Delta^1 \to \pi_A^* \mathsf{C}$ in any context $A \in \mathcal{B}$ is a pushout square if and only if it is a pullback square. A functor between stable $\mathcal{B}$-categories is said to be *exact* if it preserves finite colimits. We define the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}}$ of stable $\mathcal{B}$-categories as the subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the exact functors between stable $\mathcal{B}_{/A}$-categories, for every $A \in \mathcal{B}$.

Remark 4.5.3.4. Since existence and preservation of (co)limits are both local properties (see Remark 3.2.2.3) and since the collection of exact functors of stable $\mathcal{B}_{/A}$-categories is closed under composition and equivalences in the sense of Proposition 2.2.2.9, it follows as in Remark 3.2.3.2 that a functor $f \colon \mathsf{C} \to \mathsf{D}$ of $\mathcal{B}_{/A}$-categories is contained in $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}}(A)$ if and only if it is an exact functor between stable $\mathcal{B}_{/A}$-categories. In particular, one obtains a canonical equivalence $\pi_A^* \mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}} \simeq \mathsf{Cat}_{\mathcal{B}_{/A}}^{\mathrm{ex}}$ for every $A \in \mathcal{B}$.

Remark 4.5.3.5. It follows from Proposition 4.2.3.5 and Example 3.1.1.14 that a $\mathcal{B}$-category C is stable if and only of the associated sheaf $\mathsf{C} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through the subcategory $\mathrm{Cat}_\infty^{\mathrm{ex}} \hookrightarrow \mathrm{Cat}_\infty$ spanned by the stable ∞-categories and exact functors. Consequently, we obtain a canonical identification $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}} \simeq \mathrm{Cat}_\infty^{\mathrm{ex}} \otimes \Omega$.

Following the terminology of [56], we write $\mathcal{S}_*^{\mathrm{fin}}$ for the ∞-category of pointed finite ∞-groupoids.

Definition 4.5.3.6. Let C be a $\mathcal{B}$-category with finite limits. We let $\mathsf{Sp}(\mathsf{C})$ be the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{S}_*^{\mathrm{fin}}, \mathsf{C})$ spanned by those functors $\mathcal{S}_*^{\mathrm{fin}} \to \pi_A^* \mathsf{C}$ in context $A \in \mathcal{B}$ that preserve the final object and send pushout squares to pullbacks. We denote by $\Omega^\infty \colon \mathsf{Sp}(\mathsf{C}) \to \mathsf{C}$ the functor that is obtained by evaluation at $S^0 \in \mathcal{S}_*^{\mathrm{fin}}$.

Remark 4.5.3.7. Let $d \colon \Delta^1 \times \Delta^1 \to \pi_A^* \mathcal{S}_*^{\mathrm{fin}}$ be a square in the constant $\mathcal{B}$-category $\mathcal{S}_*^{\mathrm{fin}}$ in context $A \in \mathcal{B}$. Since $\Delta^1 \times \Delta^1$ is a finite $\mathcal{B}$-category we may find a cover $s \colon B \twoheadrightarrow A$ such that $s^* f$ is given by a square in the ∞-category $\mathcal{S}_*^{\mathrm{fin}}$, cf. Appendix 7.3.3. Since being a pullback square is a local condition, this implies that a functor $f \colon \mathcal{S}_*^{\mathrm{fin}} \to \mathsf{C}$ is contained in $\mathsf{Sp}(\mathsf{C})$ if and only if for every $A \in \mathcal{B}$, the functor $\mathcal{S}^{\mathrm{fin}} \to \mathsf{C}(A)$ that corresponds to $\pi_A^* f$ preserves finite limits.

Proposition 4.5.3.8. *The inclusion functor $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}} \to \mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}}$ from stable $\mathcal{B}$-categories to $\mathcal{B}$-categories with finite limits admits a right adjoint $\mathsf{Sp} \colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}} \to \mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}}$, that sends a $\mathcal{B}$-category C to $\mathsf{Sp}(\mathsf{C})$.*

Proof. By [56, Corollary 1.4.2.23] we have an adjunction

$$(i \dashv \mathrm{Sp}) \colon \mathrm{Cat}_\infty^{\mathrm{ex}} \leftrightarrows \mathrm{Cat}_\infty^{\mathrm{lex}},$$

hence Sp defines a morphism in $\mathrm{Pr}_\infty^{\mathrm{R}}$. By applying $- \otimes \Omega$, we thus obtain the desired adjunction of $\mathcal{B}$-categories $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{ex}} \leftrightarrows \mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}}$. Now Remark 4.5.3.7 implies that if C is a stable $\mathcal{B}$-category, the composition $\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathsf{C}} \mathrm{Cat}_\infty^{\mathrm{lex}} \xrightarrow{\mathrm{Sp}} \mathrm{Cat}_\infty^{\mathrm{ex}}$ is precisely $\mathsf{Sp}(\mathsf{C})$. $\square$

Definition 4.5.3.9. We call $\mathsf{Sp}(\Omega)$ the $\mathcal{B}$-category of $\mathcal{B}$-*spectra*.

Lemma 4.5.3.10. *Suppose that C is a presentable $\mathcal{B}$-category. Then $\mathsf{Sp}(\mathsf{C})$ is also presentable and the functor $\Omega^\infty \colon \mathsf{Sp}(\mathsf{C}) \to \mathsf{C}$ admits a left adjoint.*

Proof. Since $\mathsf{Sp}(\mathsf{C})$ is given by the composite $\mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}_\infty^{\mathrm{L}} \xrightarrow{\mathrm{Sp}} \mathrm{Pr}_\infty^{\mathrm{L}}$, the first claim is clear as the functor $\mathrm{Sp} = \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, -)$ is compatible with the mate construction. For the second claim we observe that for any $A \in \mathcal{B}$, the functor $\Omega^\infty(A)$ may be identified with $\Omega^\infty \colon \mathrm{Sp}(\mathsf{C}(A)) \to \mathsf{C}(A)$. Since the compatibility with étale base change is clear, it follows from Corollary 3.2.4.7 that Ω^∞ is continuous. Also $\Omega^\infty(A)$ is accessible since it admits a left adjoint [56, Proposition 1.4.4.4]. Thus the claim follows from Proposition 4.4.3.3. $\square$

Proposition 4.5.3.11. *The functor $\Omega^\infty \colon \mathrm{Sp}(\Omega) \to \Omega$ commutes with filtered colimits.*

Proof. By construction Ω^∞ is given as the composition

$$\mathsf{Sp}(\Omega) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{S}_*^{\mathrm{fin}}, \Omega) \xrightarrow{\mathrm{ev}_{S_0}} \Omega$$

and therefore it suffices to see that the inclusion $\mathsf{Sp}(\Omega) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{S}_*^{\mathrm{fin}}, \Omega)$ commutes with filtered colimits. By a similar argument as in the proof of Lemma 4.1.5.3, this follows from the fact that filtered colimits in Ω commute with finite limits. □

Remark 4.5.3.12. By construction, there is a canonical equivalence $\mathsf{Sp}(\Omega) \simeq \mathrm{Sp} \otimes \Omega$. Since the functor $- \otimes \Omega$ is symmetric monoidal (see the discussion before Proposition 4.6.3.7), it follows that $\mathsf{Sp}(\Omega)$ canonically admits the structure $\mathsf{Sp}(\Omega)^{\otimes}$ of a presentably symmetric monoidal $\mathcal{B}$-category.

Definition 4.5.3.13. A *commutative $\mathcal{B}$-ring spectrum* is a commutative algebra in $\mathsf{Sp}(\Omega)^{\otimes}$. We denote the unit object of $\mathsf{Sp}(\Omega)$ by S and call it the *$\mathcal{B}$-sphere spectrum.* If R is a commutative $\mathcal{B}$-ring spectrum we denote the $\mathcal{B}$-category of modules over R by $\mathsf{Mod}_R^{\mathcal{B}}$. We will write $\mathrm{Mod}_R^{\mathcal{B}}$ for the ∞-category of global sections of $\mathsf{Mod}_R^{\mathcal{B}}$.

Remark 4.5.3.14. Since $\mathsf{Sp}(\Omega)$ is a presentably symmetric monoidal $\mathcal{B}$-category, it receives a unique symmetric monoidal left adjoint functor $\Omega \to \mathsf{Sp}(\Omega)$, see Remark 4.6.2.6. By uniqueness, this functor automatically agrees with the functor that is given by applying $- \otimes \Omega$ to the symmetric monoidal functor $\Sigma^\infty_+ \colon \mathcal{S} \to \mathrm{Sp}$. It follows that the right adjoint of $\Sigma^\infty_+ \otimes \Omega$ is explicitly given by the map $\mathrm{Sh}_{\mathcal{B}_{/-}}(\mathrm{Sp}) \to \mathrm{Sh}_{\mathcal{B}_{/-}}(\mathcal{S})$ determined by precomposing with $(\Sigma^\infty_+)^{\mathrm{op}}$. Since we have a commutative square

$$\begin{tikzcd} \mathrm{Sh}_{\mathcal{B}_{/-}}(\mathrm{Sp}) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Sh}_{\mathcal{B}_{/-}}(\mathcal{S}) \arrow[d, "\simeq"] \\ \mathrm{Sp}(\mathcal{B}_{/-}) \arrow[r, "\Omega^\infty"'] & \mathcal{B}_{/-} \end{tikzcd}$$

it follows that $\Sigma^\infty_+ \otimes \Omega$ is a left adjoint of $\Omega^\infty \colon \mathsf{Sp}(\Omega) \to \Omega$. In particular Ω^∞ is the left adjoint of the unique colimit preserving functor $\Omega \to \mathsf{Sp}(\Omega)$ and thus there is a natural equivalence $\Omega^\infty \simeq \mathrm{map}_{\mathsf{Sp}(\Omega)}(S, -)$.

Remark 4.5.3.15. In [71], Denis Nardin develops a notion of stability in parametrised higher category theory, and he shows that if G is a finite group, the G-parametrised stabilisation of G-spaces recovers genuine G-spectra. However, Nardin's definition of stability is fundamentally different from ours: the key difference is that the class of parametrised ∞-category that he declares *finite* is much larger than ours. Consequently, the notion of finite limits and colimits are much more complicated than the finite limits and colimits we consider here, and as a result the associated notions of stability differ significantly. In particular, it is not possible to obtain genuine G-spectra by making use of our internal stabilisation procedure, which rather recovers *naive* G-spectra.

4.5.4. Application: dualisable objects in module $\mathcal{B}$-categories. If R is an $\mathbb{E}_\infty$-ring, let us denote by $\mathrm{Sh}(\mathcal{B}; R) = \mathcal{B} \otimes \mathrm{Mod}_R$ the ∞-category of sheaves of R-modules on $\mathcal{B}$. In many geometrically interesting situations, the ∞-category $\mathrm{Sh}(\mathcal{B}; R)$ is quite far from being compactly generated. For example, if M is a non-compact connected manifold of dimension at least 1, the only compact object in $\mathrm{Sh}(M; \mathbb{Z})$ is the zero object by [72, Theorem 0.1]. In this section we will see that one may fix this issue by working internally to the ∞-topos $\mathcal{B}$ itself: the $\mathcal{B}$-category $\mathsf{Mod}_R^{\mathcal{B}}$ (whose underlying ∞-category is $\mathrm{Sh}(\mathcal{B}; R)$) is always *internally* compactly generated (see Theorem 4.5.4.8). Furthermore, the internally compact objects can be completely characterised in terms of the symmetric monoidal structure on $\mathsf{Mod}_R^{\mathcal{B}}$, namely as the dualisable objects. In particular, this provides a classification result for the dualisable objects in $\mathrm{Sh}(\mathcal{B}; R)$ (see Corollary 4.5.4.12).

Definition 4.5.4.1. If C is a $\mathcal{B}$-category that admits filtered colimits (i.e. is $\mathsf{Filt}_{\mathsf{Fin}_{\mathcal{B}}}$-cocomplete), let us write $\mathsf{C}^{\mathrm{cpt}}$ for the full subcategory of $\mathsf{Fin}_{\mathcal{B}}$-compact objects. We will also simply refer to these as *internally compact* objects. We say that a cocomplete $\mathcal{B}$-category C is *compactly generated* if the canonical map $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{cpt}}) \to \mathsf{C}$ is an equivalence.

Remark 4.5.4.2. An internally compact object $d\colon 1 \to \mathsf{C}$ does not necessarily yield a compact object in the ∞-category $\mathsf{C}(1)$. For example, the final object $1_\Omega\colon 1 \to \Omega$ is always internally compact, but in general $1 \in \mathcal{B}$ is not compact. Since the mapping ∞-groupoid functor $\mathrm{map}_{\mathsf{C}(1)}(d, -)\colon \mathsf{C}(1) \to \mathcal{S}$ is given by composing $\Gamma(\mathrm{map}_{\mathsf{C}}(d, -))\colon \mathsf{C}(1) \to \mathcal{B}$ with the global sections functor $\Gamma \simeq \mathrm{map}_{\mathcal{B}}(1, -)$, we see that internal compactness implies compactness (for any $\mathcal{B}$-category C) if and only if $1 \in \mathcal{B}$ is compact.

Definition 4.5.4.3. An object c in context A of a symmetric monoidal $\mathcal{B}$-category C is called *dualisable* if it is dualisable in the symmetric monoidal ∞-category $\mathsf{C}(A)$. In other words, c is dualisable if there is an object $c^\vee\colon A \to \mathsf{C}$ together with maps $\eta\colon \pi_A^*(1_\otimes) \to c^\vee \otimes c$ and $\varepsilon\colon c^\vee \otimes c \to \pi_A^*(1_\otimes)$ such that $(c^\vee \otimes \epsilon) \circ (\eta \otimes c^\vee) \simeq \mathrm{id}$ and $(c \otimes \eta) \circ (\epsilon \otimes c) \simeq \mathrm{id}$. We denote the full subcategory of C that is spanned by the dualisable objects by $\mathsf{C}^{\mathrm{dual}}$.

Remark 4.5.4.4. It follows from [56, Proposition 4.6.1.11] that an object $c \in \mathsf{C}(A)$ is contained in $\mathsf{C}^{\mathrm{dual}}(A)$ if and only if it is dualisable.

A first easy consequence of the definitions is the following:

Lemma 4.5.4.5. *Let C be a symmetric monoidal $\mathcal{B}$-category and let $c\colon 1 \to \mathsf{C}$ be a dualisable object. Then the functor $- \otimes c^\vee\colon \mathsf{C} \to \mathsf{C}$ is a right adjoint of the functor $- \otimes c$.* □

Suppose now that R is an $\mathbb{E}_\infty$-ring object in $\mathcal{B}$. Since $\mathsf{Mod}_R^{\mathcal{B}}$ is a presentably symmetric monoidal $\mathcal{B}$-category by Proposition 4.5.2.8, for every object $x\colon 1 \to \mathsf{Mod}_R^{\mathcal{B}}$ the functor $- \otimes x$ admits a right adjoint

$$\underline{\mathrm{Hom}}_R(x, -)\colon \mathsf{Mod}_R^{\mathcal{B}} \to \mathsf{Mod}_R^{\mathcal{B}}.$$

Also, recall from Corollary 4.5.2.5 and Proposition 4.5.2.6 that the forgetful functor $F\colon \mathsf{Mod}_R^{\mathcal{B}} \to \mathsf{Mod}_S^{\mathcal{B}} \simeq \mathsf{Sp}(\Omega)$ has a symmetric monoidal left adjoint $- \otimes R\colon \mathsf{Sp}(\Omega) \to \mathsf{Mod}_R^{\mathcal{B}}$. We may therefore consider the composite

$$G\colon \mathsf{Mod}_R^{\mathcal{B}} \xrightarrow{\underline{\mathrm{Hom}}_R(x,-)} \mathsf{Mod}_R^{\mathcal{B}} \xrightarrow{F} \mathsf{Sp}(\Omega) \xrightarrow{\Omega^\infty} \Omega.$$

We claim that G is equivalent to the functor $\mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(x, -)$. Indeed we have a chain of natural equivalences

$$G \simeq \mathrm{map}_\Omega(1_\Omega, G(-)) \simeq \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(R, \underline{\mathrm{Hom}}_R(x, -)) \simeq \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(R \otimes x, -) \simeq \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(x, -).$$

Now if x is furthermore dualisable, the functor $\underline{\mathrm{Hom}}_R(x, -)$ is of the form $- \otimes x$ and therefore in particular cocontinuous. Since both F and Ω^∞ are $\mathsf{Filt}_{\mathsf{Fin}_{\mathcal{B}}}$-cocontinuous by Propositions 4.5.2.8 and 4.5.3.11, we have shown:

Lemma 4.5.4.6. *Let $x\colon 1 \to \mathsf{Mod}_R^{\mathcal{B}}$ be dualisable. Then x is internally compact.* □

Definition 4.5.4.7. We define the $\mathcal{B}$-category $\mathsf{Perf}_R^{\mathcal{B}}$ to be the smallest full subcategory of $\mathsf{Mod}_R^{\mathcal{B}}$ that is closed under finite colimits, retracts and contains the monoidal unit R. We call the objects in $\mathsf{Perf}_R^{\mathcal{B}}$ *perfect objects*.

We now obtain:

Theorem 4.5.4.8. *The $\mathcal{B}$-category $\mathsf{Mod}_R^{\mathcal{B}}$ is compactly generated, and the following full subcategories of $\mathsf{Mod}_R^{\mathcal{B}}$ are equivalent :*

(1) The full subcategory $\mathsf{Mod}_R^{\mathcal{B},\mathrm{dual}}$ spanned by the dualisable objects.
(2) The full subcategory $\mathsf{Mod}_R^{\mathcal{B},\mathrm{cpt}}$ spanned by the internally compact objects.
(3) The full subcategory $\mathsf{Perf}_R^{\mathcal{B}}$ spanned by the perfect objects.

Proof. In light of Remarks 4.1.5.2 and 4.5.4.4, Lemma 4.5.4.6 implies that we have an inclusion $\mathsf{Mod}_R^{\mathcal{B},\mathrm{dual}} \hookrightarrow \mathsf{Mod}_R^{\mathcal{B},\mathrm{cpt}}$. Furthermore, dualisable objects form a stable full subcategory that is closed under retracts, so that $\mathsf{Perf}_R^{\mathcal{B}} \hookrightarrow \mathrm{Mod}_R^{\mathcal{B},\mathrm{dual}}$ is clear. It remains to see that every compact object is perfect. On account of the inclusion $\mathsf{Perf}_R^{\mathcal{B}} \hookrightarrow \mathsf{Mod}_R^{\mathcal{B},\mathrm{cpt}}$, we deduce from Proposition 3.4.2.4 and Corollary 4.4.6.6 that the inclusion $\mathsf{Perf}_R^{\mathcal{B}} \hookrightarrow \mathsf{Mod}_R^{\mathcal{B}}$ extends to a fully faithful cocontinuous functor $i\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{Perf}_R^{\mathcal{B}}) \hookrightarrow \mathsf{Mod}_R^{\mathcal{B}}$.

If we can show that i is an equivalence we are done by Proposition 4.3.3.1. Since $\mathsf{Mod}_R^{\mathcal{B}}$ is cocomplete, we may now apply the adjoint functor theorem (Proposition 4.4.3.1) to deduce the existence of a right adjoint r of i. We complete the proof by showing that the counit $\varepsilon\colon ir \to \mathrm{id}$ is an equivalence. For this we pick an object $X \in \mathsf{Mod}_R^{\mathcal{B}}(A)$ and consider the fibre sequence

$$\mathrm{fib}(\varepsilon) \to irX \xrightarrow{\varepsilon} X$$

in $\mathsf{Mod}_R^{\mathcal{B}}(A)$. We need to show that $\mathrm{fib}(\varepsilon) \simeq 0$. For every n, we obtain a fibre sequence

$$\mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(\Sigma^n(\pi_A^* R), \mathrm{fib}(\varepsilon)) \to \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(\Sigma^n(\pi_A^* R), irX) \xrightarrow{\varepsilon_*} \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(\Sigma^n(\pi_A^* R), X)$$

in $\mathcal{B}_{/A}$. As $\Sigma^n \pi_A^* R$ is in the essential image of i, the map ε_* is an equivalence. Therefore, we conclude that we have equivalences

$$1 \simeq \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(\Sigma^n(\pi_A^* R), \mathrm{fib}(\varepsilon)) \simeq \mathrm{map}_{\mathsf{Mod}_R^{\mathcal{B}}}(\pi_A^* R, \Omega^n \mathrm{fib}(\varepsilon)) \simeq \Omega^\infty \Omega^n \mathrm{fib}(\varepsilon)$$

for any n. Thus $\mathrm{fib}(\varepsilon) \simeq 0$, which shows that ε is an equivalence, as desired. □

Remark 4.5.4.9. Let $\mathcal{B}_0$ be a 1-topos and R a ring object in $\mathcal{B}_0$. Let us denote the hypercomplete ∞-topos associated to $\mathcal{B}_0$ by $\mathcal{B}$. Then by [58, Theorem 2.1.2.2] there is an equivalence of ∞-categories

$$\mathbf{D}(\mathrm{Sh}(\mathcal{B}_0; R))) \simeq \mathrm{Mod}_R^{\mathcal{B}}$$

where we consider R as a discrete ring object in $\mathcal{B}$. Furthermore one can check that the induced symmetric monoidal structure on the homotopy category of $\mathbf{D}(\mathrm{Sh}(\mathcal{B}_0; R))$ is indeed the usual one. With these identifications in mind, the result that an object in $\mathrm{Mod}_R^{\mathcal{B}}$ is dualisable if and only if it is perfect appears as [Stacks, Tag 0FPV].

Example 4.5.4.10. Let X be a (spectral/derived) scheme and consider the Zariski-topos X_{Zar} of X. The structure sheaf $\mathcal{O}_X$ of X is a sheaf of ($\mathbb{E}_\infty$-)rings on X and thus gives rise to an object in $\mathrm{CAlg}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathrm{Sp}))$. Thus we may consider the X_{Zar}-category $\mathsf{Mod}_{\mathcal{O}_X}^{X_{\mathrm{Zar}}}$. By Theorem 4.5.4.8 an object $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}^{X_{\mathrm{Zar}}}(*) = \mathrm{Mod}(\mathcal{O}_X)$ is internally compact if and only if it is perfect, i.e. if and only if there is an open covering $X = \bigcup_i U_i$ by affines such that $\mathcal{F}_{|U_i}$ is contained in the smallest stable idempotent complete subcategory containing $(\mathcal{O}_X)_{|U_i} = \mathcal{O}_{U_i}$. Thus the full subcategory of $\mathrm{Mod}_{\mathcal{O}_X}$ spanned by the internally compact objects is equivalent to the usual category of perfect complexes on X.

Remark 4.5.4.11. Let R be an $\mathbb{E}_\infty$-ring. The stable constant sheaf functor $\mathrm{const}\colon \mathrm{Sp} \to \mathrm{Sh}_{\mathrm{Sp}}(\mathcal{B})$ is symmetric monoidal, therefore $\underline{R} \coloneqq \mathrm{const}\, R$ defines a commutative $\mathcal{B}$-ring spectrum. Let us denote the ∞-category of sheaves of R-modules by $\mathrm{Sh}(\mathcal{B}; R)$ Then by [56, Theorem 4.8.4.6] there is a canonical equivalence of symmetric monoidal ∞-categories $\mathrm{Sh}(\mathcal{B}; R) \simeq \mathrm{Mod}_{\underline{R}}(\mathrm{Sh}_{\mathrm{Sp}}(\mathcal{B})) = \mathrm{Mod}_{\underline{R}}^{\mathcal{B}}$.

Combining Theorem 4.5.4.8 with Corollary A.4 we arrive at the following classification result for dualisable objects:

Corollary 4.5.4.12. *Let R be an $\mathbb{E}_\infty$-ring. Then an object in* $\mathrm{Sh}(\mathcal{B}; R)$ *is dualisable if and only if it is locally constant with perfect values.* □

Remark 4.5.4.13. In the case of étale hypersheaves on a scheme X and where R is a discrete ring, the above corollary already appears in [18, Remark 6.3.27] and for pro-étale sheaves on X this is shown in [41, Corollary 3.4.3]. The proofs in both references rely on features of the specific geometric situation, more specifically on the existence of enough points and w-contractible objects, respectively. Having a sufficient amount of machinery from internal higher category theory at our disposal, we can recover these observations with a now completely formal proof.

4.6. The tensor product of presentable $\mathcal{B}$-categories

In [56], Lurie establishes a symmetric monoidal structure on the ∞-category of $\mathcal{K}$-cocomplete ∞-categories with $\mathcal{K}$-cocontinuous functors, for any class $\mathcal{K}$ of ∞-categories. In particular, his construction gives rise to a symmetric monoidal structure on the ∞-category $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ of presentable ∞-categories. In this section, our goal is to obtain an internal analogue of these results, i.e. to construct a symmetric monoidal structure on the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ of U-cocomplete $\mathcal{B}$-categories and U-cocontinuous functors, for any choice of internal class U. Our construction will be entirely analogous to the one in [56]: we will define the desired symmetric monoidal $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ as the subcategory of the cartesian monoidal $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\times} \to \mathrm{Fin}_*$ that is spanned by what we call U-*multilinear* functors. We define and study this concept in § 4.6.1, before we go on and discuss the symmetric monoidal structure on $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ in § 4.6.2. In particular, our construction will yield a symmetric monoidal structure on the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$. In § 4.6.3, we make use of this structure to identify $\mathcal{B}$-*modules* as a full subcategory of the ∞-category $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ of presentable $\mathcal{B}$-categories.

4.6.1. Bilinear functors. Recall that a bilinear functor of cocomplete ∞-categories $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a functor that preserves small colimits separately in each variable. We will now introduce this notion in the internal setting. It will be useful to consider functors that only preserve certain (internal) classes of colimits in each variable, so that we arrive at the following general definition:

Definition 4.6.1.1. Let U and V be two internal classes of $\mathcal{B}$-categories. Suppose that C, D and E are $\mathcal{B}$-categories such that C is U-cocomplete, D is V-cocomplete and E is $\mathsf{U}\cup\mathsf{V}$-cocomplete. We will say that a functor $f\colon \mathsf{C}\times\mathsf{D}\to\mathsf{E}$ is (U,V)-*bilinear* if for any $A\in\mathcal{B}$ and any two objects $c\colon A\to\mathsf{C}$ and $d\colon A\to\mathsf{D}$ the functor

$$f(c,-)\colon \pi_A^*\mathsf{D} \xrightarrow{c\times\mathrm{id}} \pi_A^*\mathsf{C}\times\pi_A^*\mathsf{D} \xrightarrow{\pi_A^* f} \pi_A^*\mathsf{E}$$

is $\pi_A^*\mathsf{V}$-cocontinuous and the functor

$$f(-,d)\colon \pi_A^*\mathsf{C} \xrightarrow{\mathrm{id}\times d} \pi_A^*\mathsf{C}\times\pi_A^*\mathsf{D} \xrightarrow{\pi_A^* f} \pi_A^*\mathsf{E}$$

is $\pi_A^*\mathsf{U}$-cocontinuous. We write $\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C}\times\mathsf{D},\mathsf{E})$ for the full subcategory spanned by the $(\pi_A^*\mathsf{U},\pi_A^*\mathsf{V})$-bilinear functors for every $A\in\mathcal{B}$. If $\mathsf{U}=\mathsf{V}=\mathsf{Cat}_{\mathcal{B}}$ (and C,D and E are large), we will simply say that f is *bilinear* and write $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{bil}}(\mathsf{C}\times\mathsf{D},\mathsf{E})$ for the associated $\mathcal{B}$-category of bilinear functors.

Remark 4.6.1.2. In the situation of Definition 4.6.1.1, the fact that U- and V-cocontinuity are local conditions by Remark 3.2.2.3, implies that for any cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$, a functor f is (U,V)-bilinear if and only if for each i the functor $\pi_{A_i}^* f$ is $(\pi_{A_i}^*\mathsf{U},\pi_{A_i}^*\mathsf{V})$-bilinear. In particular, an object $A\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})$ in context $A\in\mathcal{B}$ is contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})^{(\mathsf{U},\mathsf{V})}$ if and only if it defines a $(\pi_A^*\mathsf{U},\pi_A^*\mathsf{V})$-bilinear functor, and there consequently is a canonical equivalence

$$\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C}\times\mathsf{D},\mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{(\pi_A^*\mathsf{U},\pi_A^*\mathsf{V})}(\pi_A^*\mathsf{C}\times\pi_A^*\mathsf{D},\pi_A^*\mathsf{E})$$

of $\mathcal{B}_{/A}$-categories.

Lemma 4.6.1.3. *Let* U *and* V *be two internal classes and let* C,D *and* E *be* $\mathcal{B}$*-categories such that* C *is* U*-cocomplete,* D *is* V*-cocomplete and* E *is* $\mathsf{U}\cup\mathsf{V}$*-cocomplete. Then* $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})\subseteq\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ *is closed under* U*-colimits,* $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{E})$ *is closed under* V*-colimits, and there are natural equivalences*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C}\times\mathsf{D},\mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\mathsf{E})).$$

Proof. By symmetry, it is enough to show that $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})$ is closed under U-colimits and to construct the first of the two equivalences. To begin with, we claim that a functor $f\colon\mathsf{C}\times\mathsf{D}\to\mathsf{E}$ is (U,V)-bilinear if and only if its transpose $f'\colon\mathsf{C}\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ is U-cocontinuous and takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})$. To see this, note that for any $A\in\mathcal{B}$ and any object $c\colon A\to\mathsf{C}$, the functor $f'(c)\colon\pi_A^*\mathsf{D}\to\pi_A^*\mathsf{E}$ is by definition given by $f(c,-)$, which in turn implies that $f(c,-)$ is V-cocontinuous if and only if f' factors through the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{C},\mathsf{E})$. Moreover, given any object $d\colon A\to\mathsf{D}$ in context $A\in\mathcal{B}$, note that the functor $f(-,d)$ is given by the composite

$$\pi_A^*\mathsf{C}\xrightarrow{\pi_A^*f'}\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})\simeq\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{D},\pi_A^*\mathsf{E})\xrightarrow{d^*}\pi_A^*\mathsf{E}.$$

Therefore, Proposition 3.1.3.2 implies that f' is U-cocontinuous if and only if $f(-,d)$ is U-cocontinuous for all $d\colon A\to\mathsf{D}$ and all $A\in\mathcal{B}$. Hence the claim follows. In light of Remarks 4.6.1.2 and 3.2.3.2, this already implies that the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})\simeq\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}))$ induces a pullback square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}^{(\mathsf{U},\mathsf{V})}(\mathsf{C}\times\mathsf{D},\mathsf{E}) \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\mathsf{C},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})) \arrow[d, hook] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E}).
\end{tikzcd}$$

To complete the proof, it is now enough to show that $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})$ is closed under U-colimits. In light of Remark 3.2.3.2, this follows once we show that for any $\mathsf{I}\in\mathsf{U}(1)$ the colimit functor

$$\operatorname*{colim}_{\mathsf{I}}\colon\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}))\to\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$$

restricts to a functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E}))\to\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E})$. As $\operatorname{colim}_{\mathsf{I}}$ is cocontinuous, we get a commutative diagram

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{E})) \arrow[r, hook] \arrow[d, "(\operatorname{colim}_{\mathsf{I}})_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{E})) \arrow[d, "(\operatorname{colim}_{\mathsf{I}})_*"] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E}) \arrow[r, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}).
\end{tikzcd}$$

By what we have already shown above, the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}))\simeq\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{E}))$ restricts to an equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\mathsf{E}))\simeq\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{V}\text{-cc}}(\mathsf{D},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{E}))$. Hence the previous diagram shows that the colimit functor $\operatorname{colim}_{\mathsf{I}}$ restricts as desired. $\square$

We will now generalise the above situation to so-called *multilinear* functors. For the sake of simplicity, we will only do this in the case of one fixed internal class.

Definition 4.6.1.4. Let U be an internal class of $\mathcal{B}$-categories and suppose that $\mathsf{C}_1,\dots,\mathsf{C}_n,\mathsf{E}$ are U-cocomplete $\mathcal{B}$-categories. A functor $f\colon\mathsf{C}_1\times....\times\mathsf{C}_n\to\mathsf{E}$ is said to be U-*multilinear* if for every $i=1,\dots,n$ and all objects $c_j\colon A_j\to\mathsf{C}_j$ in context $A\in\mathcal{B}$ for $i\neq j$ the functor

$$\pi_A^*\mathsf{C}_i\xrightarrow{(c_1,\dots,\mathrm{id},\dots,c_n)}\prod_{k=1}^n\pi_A^*\mathsf{C}_k\xrightarrow{f}\pi_A^*\mathsf{D}$$

is $\pi_A^*\mathsf{U}$-cocontinuous. We will write $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^n\mathsf{C}_k,\mathsf{E})$ for the full subcategory spanned by the $\pi_A^*\mathsf{U}$-multilinear functors for all $A\in\mathcal{B}$.

Remark 4.6.1.5. By a similar argument as in Remark 4.6.1.2, the condition of a functor as in Definition 4.6.1.4 to be U-multilinear is local in $\mathcal{B}$, which implies that there is a canonical equivalence $\pi_A^*\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^n\mathsf{C}_k,\mathsf{E})\simeq\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{U}\text{-mult}}(\prod_{k=1}^n\pi_A^*\mathsf{C}_k,\pi_A^*\mathsf{E})$ for each $A\in\mathcal{B}$.

Remark 4.6.1.6. In the situation of Definition 4.6.1.4 we can construct the universal U-multilinear functor using Proposition 4.4.5.3. Namely, we may consider the collection of cocones $\square_{k=1}^n R_k$ from Construction 4.4.5.5 with respect to the internal class U. Then by construction a functor $\prod_{k=1}^n\mathsf{C}_k\to\mathsf{E}$ is U-multilinear if and only if it is contained in the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}(\prod_{k=1}^n\mathsf{C}_i,\mathsf{E})_{\square_{k=1}^n R_k}$. By Proposition 4.4.5.3, we thus have a canonical functor $j\colon\prod_{k=1}^n\mathsf{C}_i\to\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},\square_{k=1}^n R_k)}(\prod_{k=1}^n\mathsf{C}_i)$ that induces an equivalence

$$j^*\colon\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}(\underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{U},\square_{k=1}^n R_k)}(\prod_{i=1}^n\mathsf{C}_i),\mathsf{E})\xrightarrow{\simeq}\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-mult}}(\prod_{k=1}^n\mathsf{C}_k,\mathsf{E}).$$

4.6.2. The tensor product of U-cocomplete $\mathcal{B}$-categories. The goal of this section is to extend the results from [56, § 4.8.1] to the setting of $\mathcal{B}$-categories. Namely, we will construct a symmetric monoidal structure $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$ on the large $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$ of U-cocomplete $\mathcal{B}$-categories. For this we will roughly follow the arguments in [56].

Recall that the (large) $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}$ is complete. By Proposition 4.5.1.12, we therefore obtain a symmetric monoidal structure $p\colon \mathsf{Cat}_{\mathcal{B}}^{\times} \to \mathrm{Fin}_*$ on $\mathsf{Cat}_{\mathcal{B}}$. By construction, the pullback of $p(A)$ along the adjunction unit $\mathrm{Fin}_* \to \Gamma_{\mathcal{B}_{/A}} \mathrm{Fin}_*$ yields the cocartesian fibration classifying the symmetric monoidal ∞-category $\mathrm{Cat}(\mathcal{B}_{/A})^{\times}$.

Construction 4.6.2.1. We define a subcategory $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$ of $\mathsf{Cat}_{\mathcal{B}}^{\times}$ as follows: Let $f\colon x \to y$ be a morphism in $\mathsf{Cat}_{\mathcal{B}}^{\times}$ in context $A \in \mathcal{B}$, and assume that $p(f)$ is contained in the image of the functor $\mathrm{const}_{\mathcal{B}_{/A}}$ and thus given by a map $\alpha\colon \langle n\rangle \to \langle m\rangle$ in the 1-category Fin_*. We now obtain equivalences $x \simeq (\mathsf{C}_1, \dots, \mathsf{C}_n)$ and $y \simeq (\mathsf{D}_1, \dots, \mathsf{D}_m)$ where the C_i and D_j are $\mathcal{B}_{/A}$-categories, and the map f is determined by a collection of maps $f_j\colon \prod_{i \in \alpha^{-1}(j)} \mathsf{C}_i \to \mathsf{D}_j$ for $j = 1, \dots, m$. We shall say that f is U-*multilinear* if the C_i and D_j are $\pi_A^*\mathsf{U}$-cocomplete and the functors f_j are $\pi_A^*\mathsf{U}$-multilinear. Finally, we say that an arbitrary map $g\colon x \to y$ in $\mathsf{Cat}_{\mathcal{B}}^{\times}$ in context $A \in \mathcal{B}$ is *locally* U-multilinear if there is a cover $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ such that $s_i^*(g)$ is U-multilinear for each i. We let $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$ be the subcategory of $\mathsf{Cat}_{\mathcal{B}}^{\times}$ that is spanned by the locally U-multilinear maps.

Remark 4.6.2.2. Since every map in the constant $\mathcal{B}$-category Fin_* is *locally* of the form $\alpha\colon \langle n\rangle \to \langle m\rangle$ (i.e. is locally contained in the image of the constant sheaf functor Proposition A.2), for every map $g\colon x \to y$ in $\mathsf{Cat}_{\mathcal{B}}^{\times}$ in context $A \in \mathcal{B}$ there is a cover $(s_i)\colon \bigsqcup A_i \twoheadrightarrow A$ in $\mathcal{B}$ such that $p(s_i^* g)$ is given by a map $\alpha\colon \langle n\rangle \to \langle m\rangle$ of pointed finite sets. As moreover the condition that a functor is $\pi_A^*\mathsf{U}$-multilinear is local in $\mathcal{B}$ (see Remark 4.6.1.5), one finds that g is locally U-multilinear if and only if $s_i^*(g)$ is U-multilinear.

Lemma 4.6.2.3. *The inclusion* $(\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_1 \hookrightarrow (\mathsf{Cat}_{\mathcal{B}}^{\times})_1$ *identifies* $(\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_1$ *with the subobject of* $(\mathsf{Cat}_{\mathcal{B}}^{\times})_1$ *that is spanned by the locally* U*-multilinear maps.*

Proof. We first show that U-multilinear maps are closed under composition. To that end, suppose that $f\colon x \to y$ and $f'\colon y \to z$ are U-multilinear maps in context $A \in \mathcal{B}$, and consider the commutative diagram

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "h_y"'] & & y \arrow[rr, "f'"] \arrow[dr, "h_z"'] & & z \\ & x' \arrow[ur, "g_y"'] \arrow[dr, "h'_y"'] & & z' \arrow[ur, "g_z"'] & \\ & & y' \arrow[ur, "g'_y"'] & & \end{tikzcd}$$

in which h_y, h'_y and h_z are cocartesian and the maps g_y, g'_y and g_z are sent to identity maps in Fin_*. Then $f'f$ being U-multilinear precisely means that $g_z g'_y$ is determined by a tuple of $\pi_A^*\mathsf{U}$-multilinear functors between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}_{/A}$-categories. Unwinding the definitions, this follows immediately from the observation that $\pi_A^*\mathsf{U}$-multilinear functors compose in the expected way. Together with the fact that equivalences between $\pi_A^*\mathsf{U}$-cocomplete $\mathcal{B}$-categories are automatically $\pi_A^*\mathsf{U}$-cocontinuous, this already implies that the subobject of $(\mathsf{Cat}_{\mathcal{B}}^{\times})_1$ that is spanned by the locally U-multilinear maps is closed under composition and equivalences in the sense of Proposition 2.2.2.9, hence the very same proposition proves the claim. $\square$

To proceed, recall that the three maps id_0, $0 < 1$ and id_1 of the poset Δ^1 give rise to a decomposition $\Delta^1_1 \simeq 1 \sqcup 1 \sqcup 1$, both when viewing Δ^1 as an ∞-category and as a constant $\mathcal{B}$-category. Therefore, if C is an arbitrary $\mathcal{B}$-category, we obtain an induced decomposition $(\Delta^1 \otimes \mathsf{C})_1 \simeq \mathsf{C}_1 \sqcup \mathsf{C}_1 \sqcup \mathsf{C}_1$. By applying this observation to the case where $\mathsf{C} = \mathsf{Cat}_{\mathcal{B}}^{\times}$, we may thus define a subcategory $\mathsf{M}_{\mathsf{U}}^{\otimes} \hookrightarrow \Delta^1 \otimes \mathsf{Cat}_{\mathcal{B}}^{\times}$ via the subobject of morphisms

$$(\mathsf{Cat}_{\mathcal{B}}^{\times})_1 \sqcup (\mathsf{Cat}_{\mathcal{B}}^{\times})_1|_{(\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_0} \sqcup (\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_1 \hookrightarrow \Delta^1 \otimes (\mathsf{Cat}_{\mathcal{B}}^{\times})_1$$

where the middle summand denotes the pullback of $d_0\colon (\mathsf{Cat}_{\mathcal{B}}^{\times})_1 \to (\mathsf{Cat}_{\mathcal{B}}^{\times})_0$ along the inclusion of the subobject $(\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_0 \hookrightarrow (\mathsf{Cat}_{\mathcal{B}}^{\times})_0$. Evidently, this subobject is closed under composition and equivalences in the sense of Proposition 2.2.2.9, hence the inclusion $(\mathsf{M}_{\mathsf{U}}^{\otimes})_1 \hookrightarrow \Delta^1 \otimes (\mathsf{Cat}_{\mathcal{B}}^{\times})_1$ gives rise to an equivalence

$$(\mathsf{M}_{\mathsf{U}}^{\otimes})_1 \simeq (\mathsf{Cat}_{\mathcal{B}}^{\times})_1 \sqcup (\mathsf{Cat}_{\mathcal{B}}^{\times})_1|_{(\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_0} \sqcup (\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_1.$$

By construction, the pullback of the composition $q\colon \mathsf{M}_{\mathsf{U}}^{\otimes} \hookrightarrow \Delta^1 \otimes \mathsf{Cat}_{\mathcal{B}}^{\times} \to \Delta^1 \times \mathrm{Fin}_*$ along the inclusion $d^1\colon \mathrm{Fin}_* \hookrightarrow \Delta^1 \times \mathrm{Fin}_*$ recovers the cocartesian fibration $p\colon \mathsf{Cat}_{\mathcal{B}}^{\times} \to \mathrm{Fin}_*$, and the pullback of q along $d^0\colon \mathrm{Fin}_* \hookrightarrow \Delta^1 \times \mathrm{Fin}_*$ recovers the restriction of p to the subcategory $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$. We now obtain:

Proposition 4.6.2.4. *The composition $q\colon \mathsf{M}_{\mathsf{U}}^{\otimes} \hookrightarrow \Delta^1 \otimes \mathsf{Cat}_{\mathcal{B}}^{\times} \to \Delta^1 \times \mathrm{Fin}_*$ is a cocartesian fibration.*

Proof. Let us begin by fixing maps $\alpha\colon \langle n\rangle \to \langle m\rangle$ in the 1-category Fin_* and $\epsilon \leq \delta$ in the poset Δ^1, and let $x\colon A \to \mathsf{M}_{\mathsf{U}}^{\otimes}|_{(\epsilon,\langle n\rangle)}$ be an arbitrary object in context $A \in \mathcal{B}$. Let us write $\mathsf{V}_0 =$ and $\mathsf{V}_1 = \mathsf{U}$. By construction of $\mathsf{M}_{\mathsf{U}}^{\otimes}$, the object x corresponds to a tuple $(\pi_A^*\epsilon, \mathsf{C}_1, \dots, \mathsf{C}_n)$ where $\mathsf{C}_1, \dots, \mathsf{C}_n$ are $\pi_A^*\mathsf{V}_\epsilon$-cocomplete $\mathcal{B}_{/A}$-categories. Let $f\colon x \to y$ be a cocartesian lift of α in $\mathsf{Cat}_{\mathcal{B}}^{\times}$. For each $j = 1, \dots, m$, the construction in § 4.4.5 now allows us to define a map

$$g_j\colon \prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i \to \mathsf{D}_j = \underline{\mathsf{PSh}}_{\mathcal{B}}^{(\mathsf{V}_\delta,\square_i R_i)}(\prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i),$$

and by setting $z = (\delta, \mathsf{D}_1, \dots, \mathsf{D}_m)$, precomposing the tuple $g = (\epsilon \leq \delta, g_1, \dots, g_m)$ with $(\mathrm{id}_\epsilon, f)$ defines a lift of $(\epsilon \leq \delta, \alpha)$ in $\mathsf{M}_{\mathsf{U}}^{\otimes}$. By Remark 4.6.1.6, precomposition with each g_j induces an equivalence

$$g_j^*\colon \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{V}_\epsilon\text{-mult}}(\prod_i \mathsf{C}_i, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}^{\pi_A^*\mathsf{V}_\delta\text{-cc}}(\mathsf{D}_k, \mathsf{E})$$

for every $\pi_A^*\mathsf{V}_\delta$-cocomplete $\mathcal{B}_{/A}$-category E. By construction of $\mathsf{M}_{\mathsf{U}}^{\otimes}$ and Lemma 4.6.2.3, the underlying core $\mathcal{B}_{/A}$-groupoids of both domain and codomain of g_j^* recover the mapping $\mathcal{B}_{/A}$-groupoids in the pullback of q along the map $\mathrm{id} \times \langle 1\rangle\colon \Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$. As $A \in \mathcal{B}$ was chosen arbitrarily and in light of Remark 4.4.5.2, this shows that the functor $\mathsf{M}_{\mathsf{U}}^{\otimes} \times_{\Delta^1\times\mathrm{Fin}_*} \Delta^1 \to \Delta^1$ that is obtained as the pullback of q along $(\epsilon \leq \delta, \alpha)\colon \Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$ must be a cocartesian fibration (see [61, Lemma 6.5.2]). As every map in the constant $\mathcal{B}$-category $\Delta^1 \times \mathrm{Fin}_*$ is *locally* contained in the image of the constant sheaf functor (see Proposition A.2), the pullback of q along *any* map $\Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$ in $\mathrm{Cat}(\mathcal{B})$ is a cocartesian fibration after passing to a suitable cover $\bigsqcup_i A \twoheadrightarrow 1$ in $\mathcal{B}$ and must therefore be a cocartesian fibration itself, using that $\mathsf{Cocart}_{\Delta^1}$ is a sheaf by Theorem 2.3.2.7. In particular, $q(A)$ is a locally cocartesian fibration of ∞-categories for every $A \in \mathcal{B}$, and since it follows from Proposition 4.4.5.6 that the locally cocartesian maps are closed under composition, we conclude that $q(A)$ is a cocartesian fibration. Since cocompletions with relations are compatible with étale base change (Remark 4.4.5.2), the transition functors $s^*\colon \mathsf{M}_{\mathsf{U}}^{\otimes}(A) \to \mathsf{M}_{\mathsf{U}}^{\otimes}(B)$ preserve cocartesian morphisms, so that q is a cocartesian fibration, as claimed. □

Corollary 4.6.2.5. *The functor $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ is a cocartesian fibration that gives rise to a symmetric monoidal structure on the $\mathcal{B}$-category $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}}$.*

Proof. Since the map $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes} \to \mathrm{Fin}_*$ is a pullback of the functor q from Proposition 4.6.2.4, the same proposition immediately implies the first claim. Moreover, the straightforward observation that for every $n \geq 0$ the equivalence

$$(\mathsf{Cat}_{\mathcal{B}}^{\times})_n \simeq \prod_{i=1}^n (\mathsf{Cat}_{\mathcal{B}}^{\times})_1$$

restricts to an equivalence

$$(\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_n \simeq \prod_{i=1}^n (\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes})_1$$

shows the second claim. $\square$

REMARK 4.6.2.6. By unstraightening the cocartesian fibration q from Proposition 4.6.2.4 we get a functor $\Delta^1 \to \mathsf{CMon}(\mathsf{Cat}_{\mathcal{B}})$ and therefore a morphism of symmetric monoidal $\mathcal{B}$-categories $L\colon \mathsf{Cat}_{\mathcal{B}}^{\times} \to \mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$. Note that the pullback $\Delta^1 \times_{\Delta^1\times \mathrm{Fin}_*} \mathsf{M}_{\mathsf{U}}^{\otimes} \to \Delta^1$ of q along $\mathrm{id}\times\langle 1\rangle\colon \Delta^1 \to \Delta^1 \times \mathrm{Fin}_*$ is also a *cartesian* fibration: in fact, by making use of [61, Proposition 6.5.1] this follows from the straightforward observation that the adjunction $(d^1 \dashv s^0)\colon \Delta^1 \otimes \mathsf{Cat}_{\mathcal{B}} \leftrightarrows \mathsf{Cat}_{\mathcal{B}}$ restricts to an adjunction $\Delta^1 \times_{\Delta^1\times\mathrm{Fin}_*} \mathsf{M}_{\mathsf{U}}^{\otimes} \leftrightarrows \mathsf{Cat}_{\mathcal{B}}$. By [61, Corollary 6.5.5], this means that L is the left adjoint of the inclusion $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ as provided by Corollary 3.4.1.15. In particular we see that the $\mathcal{B}$-category underlying the tensor unit of $\mathsf{Cat}_{\mathcal{B}}^{\mathsf{U}\text{-cc},\otimes}$ is equivalent to the free U-cocompletion of the point $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$.

REMARK 4.6.2.7. By a similar argument as in Remark 4.6.2.6, the projection $\mathsf{M}_{\mathsf{U}}^{\otimes} \to \Delta^1$ is both cartesian and cocartesian. Therefore, one also obtains an adjunction $(L \dashv i)\colon \mathsf{Cat}_{\mathcal{B}}^{\mathsf{U},\otimes\text{-cc}} \leftrightarrows \mathsf{Cat}_{\mathcal{B}}^{\times}$ in which i is simply the inclusion. Since the projection $\mathsf{M}_{\mathsf{U}}^{\otimes} \to \mathrm{Fin}_*$ carries every map in $\mathsf{M}_{\mathsf{U}}^{\otimes}$ that is cartesian over Δ^1 to an equivalence, taking global sections and pulling back along the map $\mathrm{Fin}_* \to \Gamma\,\mathrm{Fin}_*$ yields a relative adjunction $\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc},\otimes} \leftrightarrows \mathrm{Cat}(\mathcal{B})^{\times}$ over Fin_*. As both maps are morphisms of ∞-operads, we obtain an induced adjunction

$$(L \dashv i)\colon \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}}) \leftrightarrows \mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})) \simeq \mathrm{Cat}(\mathcal{B})^{\otimes}$$

of ∞-categories. By unwinding the definitions, we see that a symmetric monoidal $\mathcal{B}$-category $\mathsf{C}^{\otimes}$ lies in $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\mathsf{U}\text{-cc}})$ if and only if its underlying $\mathcal{B}$-category is U-cocomplete and the functor $-\otimes-\colon \mathsf{C}\times\mathsf{C} \to \mathsf{C}$ is U-bilinear. In particular it follows from Remark 4.6.2.6 that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ can be canonically equipped with the structure of a symmetric monoidal $\mathcal{B}$-category $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)^{\otimes}$ such that $-\otimes-\colon \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ is U-bilinear and that the canonical functor $1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)^{\otimes}$ induced by the adjunction unit is symmetric monoidal. So in particular we have a commutative diagram

$$\begin{tikzcd}
1\times 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \arrow[r, "-\otimes-"] & \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1).
\end{tikzcd}$$

By the universal property of $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$ and Lemma 4.6.1.3 there is a unique such functor $-\otimes-$, which must therefore coincide with the product functor $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)\times\underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1) \to \underline{\mathsf{PSh}}_{\mathcal{B}}^{\mathsf{U}}(1)$, see Proposition 3.4.3.8.

EXAMPLE 4.6.2.8. Let $\mathcal{B} = \mathrm{PSh}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ and let $P \subseteq \mathcal{C}$ be a subcategory that is closed under pullbacks. Then P generates a local class W in $\mathrm{PSh}(\mathcal{C})$ and therefore a full subcategory $\Omega_W \hookrightarrow \Omega$. Then Propositions 3.2.4.2 and 3.2.4.5 imply together with Remark 4.6.2.7 that the ∞-category $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\Omega_W\text{-cc}})$ is equivalent to the ∞-category of *pullback formalisms* in the sense of [23, §2.2]. By Remarks 4.6.2.6 and 4.6.2.7 the initial object of $\mathrm{CAlg}(\mathrm{Cat}(\mathcal{B})^{\Omega_W\text{-cc}})$ is equivalent to the free Ω_W-cocompletion of the point $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_W}(1)$ equipped with the cartesian monoidal structure. Since Example 3.4.3.6 shows that $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_W}(1)$ agrees with the geometric pullback formalism constructed in [23, §4], this gives a new proof of [23, Theorem 3.25]. Furthermore our proof yields a slightly more general result because in loc. cit. the assumption that $\mathcal{C}$ is a 1-category was made.

We will now move one universe up and consider the case where $\mathsf{U} = \mathrm{Cat}_{\mathcal{B}}$ is the internal class of small $\mathcal{B}$-categories in $\mathrm{Cat}_{\widehat{\mathcal{B}}}$. By the above we obtain a symmetric monoidal structure $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc},\otimes}$ on the very large $\mathcal{B}$-category $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc}}$ of cocomplete large $\mathcal{B}$-categories and cocontinuous functors.

PROPOSITION 4.6.2.9. *The tensor product $-\otimes-\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{cc}}\times\mathsf{Cat}_{\mathcal{B}}^{\mathrm{cc}} \to \mathsf{Cat}_{\mathcal{B}}^{\mathrm{cc}}$ of cocomplete $\mathcal{B}$-categories restricts to a functor $-\otimes-\colon \mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}\times\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}} \to \mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$. Therefore $\mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ inherits the structure of a symmetric monoidal $\mathcal{B}$-category from $\mathsf{Cat}(\mathcal{B})^{\mathrm{cc},\otimes}$.*

PROOF. In light of the observation that the tensor unit in $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{cc}}$ is given by the presentable $\mathcal{B}$-category Ω by Remark 4.6.2.6, the second claim follows from Remark 4.5.1.13, so it suffices to show the first one. It will be enough to see that if D and E are presentable then so is their tensor product $\mathsf{D}\otimes\mathsf{E}$. By Corollary 4.4.6.7, we may find a sound doctrine U and U-cocomplete (small) $\mathcal{B}$-categories C and C' such that $\mathsf{D}\simeq\underline{\mathsf{Sh}}_{\Omega}^{\mathrm{op}(\mathsf{U})}(\mathsf{C})$ and $\mathsf{E}\simeq\underline{\mathsf{Sh}}_{\Omega}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}')$. If X is an arbitrary cocomplete large $\mathcal{B}$-category, we compute

$$\begin{aligned}\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D}\otimes\mathsf{E},\mathsf{X})&\simeq\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D},\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{E},\mathsf{X}))\\&\simeq\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C},\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C}',\mathsf{X}))\\&\simeq\underline{\mathsf{Fun}}^{\mathsf{U}\text{-mult}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{C}',\mathsf{X})\\&\simeq\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C}\otimes^{\mathsf{U}}\mathsf{C}',\mathsf{X}),\end{aligned}$$

where the first and third equivalence are consequences of Lemma 4.6.1.3, the second equivalence follows from Corollary 4.4.6.6 and where $-\otimes^{\mathsf{U}}-$ denotes the tensor product in $\mathsf{Cat}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}$. Now U being a doctrine implies that the tensor product $\mathsf{C}\otimes^{\mathsf{U}}\mathsf{C}'$ is small (see Remark 4.4.5.4), hence another application of Corollary 4.4.6.6 gives rise to an equivalence $\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cc}}_{\mathcal{B}}(\mathsf{C}\otimes^{\mathsf{U}}\mathsf{C}',\mathsf{X})\simeq\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\Omega}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}\otimes^{\mathsf{U}}\mathsf{C}'),\mathsf{X})$. As the same corollary shows that $\underline{\mathsf{Sh}}_{\Omega}^{\mathrm{op}(\mathsf{U})}(\mathsf{C}\otimes^{\mathsf{U}}\mathsf{C}')$ is presentable and as all of the above equivalences are natural in X, the result follows. ☐

Definition 4.6.2.10. A symmetric monoidal $\mathcal{B}$-category D is called *presentably* symmetric monoidal if D is contained in the image of the inclusion $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))\hookrightarrow\mathrm{CAlg}(\mathrm{Cat}(\widehat{\mathcal{B}})\simeq\mathrm{Cat}(\widehat{\mathcal{B}})^{\otimes}$. In other words, D is presentably symmetric monoidal if D is a presentable $\mathcal{B}$-category and the tensor functor $-\otimes-$ is bilinear.

Proposition 4.6.2.11. *Let D and E be presentable $\mathcal{B}$-categories. Then there is an equivalence of $\mathcal{B}$-categories*

$$\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D}),\mathsf{X})\simeq\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{E},\mathsf{X})$$

that is natural in $\mathsf{X}\in\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ and therefore in particular an equivalence $\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D})\simeq\mathsf{D}\otimes\mathsf{E}$.

Proof. Let us denote by $\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(-,-)\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ the full subcategory spanned by the continuous functors. We claim that we have a chain of equivalences

$$\begin{aligned}\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{D}\times\mathsf{E},\mathsf{X})&\simeq\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{D},\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{E},\mathsf{X}))\\&\simeq\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}},\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\mathsf{E},\mathsf{X})^{\mathrm{op}})^{\mathrm{op}}\\&\simeq\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}},\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X},\mathsf{E}))^{\mathrm{op}}\\&\simeq\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X},\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}},\mathsf{E}))^{\mathrm{op}}\\&\simeq\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}},\mathsf{E}),\mathsf{X})\\&\simeq\underline{\mathsf{Fun}}^{\mathrm{L}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D}),\mathsf{X})\end{aligned}$$

that are natural in E. The first equivalence follows from Lemma 4.6.1.3 and the last equivalence is obvious. The second equivalence follows from Proposition 4.4.3.1 and the third and fifth equivalences follow from Corollary 2.4.3.8 and Proposition 4.4.3.3. Therefore it remains to argue that the fourth equivalence holds.

We may choose a sound doctrine U such that $\mathsf{D}\simeq\underline{\mathsf{Sh}}^{\mathsf{U}}_{\Omega}(\mathsf{C})$ for some small U-cocomplete $\mathcal{B}$-category C (cf. Corollary 4.4.6.7). Using Corollary 4.4.6.6, we only need to see that the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X},\mathsf{E}))\simeq\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X},\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E}))$ restricts to an equivalence

$$\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X},\mathsf{E}))\simeq\underline{\mathsf{Fun}}^{\mathrm{R}}_{\mathcal{B}}(\mathsf{X},\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E}))$$

(where $\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(-,-)$ denotes the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ that is spanned by the $\pi_A^*\mathsf{U}$-continuous functors of $\mathcal{B}_{/A}$-categories, for all $A\in\mathcal{B}$). We already know from (the dual version of) Lemma 4.6.1.3 that we have an equivalence

$$\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{X},\mathsf{E}))\simeq\underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathcal{B}}(\mathsf{X},\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E})).$$

Furthermore note that because $\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E})\hookrightarrow\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E})$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous by Proposition 4.4.6.5 a functor $f\colon\mathsf{X}\to\underline{\mathsf{Fun}}^{\mathsf{U}\text{-cont}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}},\mathsf{E})$ is accessible if and only if it is so after composing with $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{U}\text{-cont}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$. Hence Proposition 4.4.3.3 together with Remark 3.2.3.2 and Remark 4.3.2.3 implies that the proof is finished once we verify that a functor $f\colon \mathsf{X} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{E})$ is accessible if only if its transpose $f'\colon \mathsf{C}^{\mathrm{op}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \mathsf{E})$ takes values in the full subcategory $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{X}, \mathsf{E})$ spanned by the accessible functors. If f is accessible there is some sound doctrine V such that f is $\mathsf{Filt}_{\mathsf{V}}$-cocontinuous. But then it follows from Lemma 4.6.1.3 that f' takes values in the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{V}}\text{-cc}}(\mathsf{X}, \mathsf{E}) \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{X}, \mathsf{E})$, as desired. For the converse, suppose that f' takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{acc}}(\mathsf{X}, \mathsf{E})$. Let $z\colon \mathsf{C}_0 \to \mathsf{C}$ be the tautological object. Then $f'(z)\colon \pi_{\mathsf{C}_0}^* \mathsf{X} \to \pi_{\mathsf{C}_0}^* \mathsf{E}$ is $\pi_{\mathsf{C}_0}^*\mathsf{V}$-accessible for some sound doctrine V. Since *every* object in C is a pullback of z, this already shows that f' takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathsf{Filt}_{\mathsf{V}}\text{-cc}}(\mathsf{X}, \mathsf{E})$, hence Lemma 4.6.1.3 shows that f is accessible. □

For later use we also record the following explicit description of the universal bilinear functor if one of the factors is presheaf category:

Lemma 4.6.2.12. *Let C be small $\mathcal{B}$-category and D a presentable $\mathcal{B}$-category. Then under the identification $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \otimes \mathsf{D} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{D})$ the universal bilinear functor $\tau\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \mathsf{D} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{D})$ is given by the transpose of the composite*

$$\tau'\colon \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \mathsf{D} \xrightarrow{\mathrm{ev}} \Omega_{\mathcal{B}} \times \mathsf{D} \xrightarrow{-\otimes-} \mathsf{D}.$$

Proof. We at first prove the claim when $\mathsf{D} = \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0)$ as well. In this case it is an easy consequence of Theorem 3.4.1.1 that the universal bilinear functor τ is the unique bilinear functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0)) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}_0)$ such that the composite

$$\mathsf{C}_0 \times \mathsf{D}_0 \xrightarrow{h_{\mathsf{C}} \times h_{\mathsf{D}_0}} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}_0)$$

is given by the Yoneda-embedding $h_{\mathsf{C}\times\mathsf{D}_0}$ (see also the proof of Proposition 4.6.2.9). Recall that the functor $-\otimes-$ is the unique bilinear functor that corresponds to the identity under the equivalence

$$\mathrm{Fun}_{\mathcal{B}}^{\mathrm{bil}}(\Omega_{\mathcal{B}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0), \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0)) \simeq \mathrm{Fun}_{\mathcal{B}}^{L}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0), \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0)).$$

From this it follows that the composite

$$\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \xrightarrow{\mathrm{ev}} \Omega_{\mathcal{B}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \xrightarrow{-\otimes-} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0)$$

transposes to the functor

$$\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \mathsf{D}_0^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \xrightarrow{\mathrm{ev}\times\mathrm{ev}} \Omega \times \Omega \xrightarrow{-\times-} \Omega_{\mathcal{B}}.$$

But after composing with $\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \times {\mathsf{D}_0}^{\mathrm{op}} \times \mathsf{D}_0 \xrightarrow{\mathrm{id}\times h_{\mathsf{C}} \times \mathrm{id} \times h_{\mathsf{D}}} \mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \mathsf{D}_0^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0)$ this functor yields $\mathrm{map}_{\mathsf{C}\times\mathsf{D}_0}(-,-)$ and thus transposes to the Yoneda-embedding $h_{\mathsf{C}\times\mathsf{D}_0}$, as desired. Now for the case of a general presentable $\mathcal{B}$-category D we pick a Bousfield localization $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \to \mathsf{D}$. Note that we have two commutative squares

$$\begin{tikzcd}
\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \arrow[r, shift left, "\tau"] \arrow[r, shift right, "\tau'"'] \arrow[d, "\mathrm{id}\times L"'] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}_0) \arrow[d, "L"'] \\
\mathsf{C}^{\mathrm{op}} \times \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \times \mathsf{D} \arrow[r, shift left, "\tau"] \arrow[r, shift right, "\tau'"'] & \mathsf{D}
\end{tikzcd}$$

with and without the prime. But by the first part of the proof the upper two functors agree and thus so do the lower two because $\mathrm{id}\times L$ has a section . □

4.6.3. $\mathcal{B}$-modules as presentable $\mathcal{B}$-categories. By the discussion in the previous section, there is a symmetric monoidal functor $L\colon \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\times} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc},\otimes}$ whose underlying functor of very large $\mathcal{B}$-categories is the left adjoint of the inclusion $\mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$. Upon taking global sections, we thus deduce from [56, Corollary 7.3.2.7] that the inclusion determines a lax symmetric monoidal functor $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc},\otimes} \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})^{\times}$ (of symmetric monoidal ∞-categories, i.e. a map of ∞-operads) and therefore a fortiori a lax symmetric monoidal functor $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^{\otimes} \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}})^{\times}$. Moreover, as the global sections functor Γ preserves limits, it defines a symmetric monoidal functor $\mathrm{Cat}(\widehat{\mathcal{B}})^{\times} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}$. Since a multilinear functor in $\mathrm{Cat}(\mathcal{B})$ induces a multilinear functor on the underlying ∞-categories of global sections, it is evident that the induced map $\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc},\otimes} \to \widehat{\mathrm{Cat}}_\infty^\times$ takes values in $\widehat{\mathrm{Cat}}_\infty^{\mathrm{cc},\otimes} \hookrightarrow \widehat{\mathrm{Cat}}_\infty^\times$ and therefore defines a lax symmetric monoidal functor $\Gamma^{\mathrm{cc},\otimes}\colon \mathrm{Cat}(\widehat{\mathcal{B}})^{\mathrm{cc},\otimes} \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{cc},\otimes}$. Upon restricting this functor to presentable $\mathcal{B}$-categories, we now end up with a lax symmetric monoidal functor $\Gamma^{\mathrm{cc},\otimes}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^\otimes \to (\mathrm{Pr}^{\mathrm{L}}_\infty)^\otimes$ that in turn induces a map

$$\Gamma^{\mathrm{lin}}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \simeq \mathrm{Mod}_\Omega(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$$

(where $\mathcal{B}$ is regarded as the algebra in $\mathrm{Pr}^{\mathrm{L}}_\infty$ that is given by image of the trivial algebra Ω in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ along $\Gamma^{\mathrm{cc},\otimes}$). Note that this is precisely the *cartesian* monoidal structure on $\mathcal{B}$ as the product bifunctor $\Omega \times \Omega \to \Omega$ is bilinear, cf. [61, Lemma 6.2.7].

The main goal of this section is to show that Γ^{lin} admits a fully faithful left adjoint that embeds $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$ into $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ and to give an explicit description of this embedding. As a preliminary step, we need to show that the global sections functor $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_\infty$ admits a left adjoint. To that end, recall from Example 4.4.4.8 that there is a functor $-\otimes\Omega\colon \mathrm{Pr}^{\mathrm{R}}_\infty \to \mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ that assigns to a presentable ∞-category $\mathcal{D}$ the presentable $\mathcal{B}$-category that is given by the sheaf $\mathcal{D}\otimes\mathcal{B}_{/-}$. Using Proposition 4.4.4.7, we may equivalently regard this map as a functor $\mathrm{Pr}^{\mathrm{L}}_\infty \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. We now obtain:

PROPOSITION 4.6.3.1. *The functor $-\otimes\Omega$ is left adjoint to the global sections functor $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_\infty$.*

PROOF. The composite $\Gamma\circ(-\otimes\Omega)$ is by definition given by the endofunctor $-\otimes\mathcal{B}\colon \mathrm{Pr}^{\mathrm{L}}_\infty \to \mathrm{Pr}^{\mathrm{L}}_\infty$ so that the functor $\Gamma_*\colon \mathrm{Sh}_{\mathcal{B}}(-) \to \mathrm{id}_{\mathrm{Pr}^{\mathrm{R}}_\infty}$ defines a natural transformation $\eta\colon \mathrm{id}_{\mathrm{Pr}^{\mathrm{L}}_\infty} \to -\otimes\mathcal{B}$ upon passing to opposite ∞-categories. We need to show that the composition

$$(*)\qquad \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})}(\mathcal{D}\otimes\Omega, \mathsf{E}) \to \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}_\infty}(\mathcal{D}\otimes\mathcal{B}, \Gamma\mathsf{E}) \xrightarrow{\eta_{\mathcal{D}}^*} \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}_\infty}(\mathcal{D}, \Gamma\mathsf{E})$$

is an equivalence. Choose a regular cardinal κ such that $\mathcal{D}\simeq\mathrm{Sh}^\kappa_{\mathcal{S}}(\mathcal{C})$ for some small ∞-category $\mathcal{C}$ that admits κ-small colimits. Using Proposition 4.4.6.4, we obtain an equivalence $\mathcal{D}\otimes\Omega \simeq \underline{\mathsf{Sh}}_\Omega^{\mathsf{LConst}_\kappa}(\mathcal{C})$ with respect to which the map $\eta_{\mathcal{D}}$ corresponds to the left adjoint of $\Gamma_*\colon \mathrm{Sh}^\kappa_{\mathcal{B}}(\mathcal{C}) \to \mathrm{Sh}^\kappa_{\mathcal{S}}(\mathcal{C})$. Again using Proposition 4.4.6.4, we have equivalences

$$\mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})}(\mathcal{D}\otimes\Omega, \mathsf{E}) \xrightarrow{(h^{\mathcal{B}}_{\mathcal{C}})^*} \mathrm{map}_{\mathrm{Cat}(\widehat{\mathcal{B}})^{\mathsf{LConst}_\kappa\text{-cc}}}(\mathcal{C}, \mathsf{E}) \simeq \mathrm{map}_{\widehat{\mathrm{Cat}}_\infty^{\kappa\text{-cc}}}(\mathcal{C}, \Gamma\mathsf{E}) \xrightarrow{(h^{\mathcal{S}}_{\mathcal{C}})_!} \mathrm{map}_{\mathrm{Pr}^{\mathrm{L}}_\infty}(\mathcal{D}, \Gamma\mathsf{E})$$

where $h^{\mathcal{B}}_{\mathcal{C}}$ is the Yoneda embedding in $\mathrm{Cat}(\widehat{\mathcal{B}})$ and $h^{\mathcal{S}}_{\mathcal{C}}$ is the Yoneda embedding in $\widehat{\mathrm{Cat}}_\infty$. On account of the commutative square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "h^{\mathcal{S}}_{\mathcal{C}}"] \arrow[d] & \mathrm{Sh}^\kappa_{\mathcal{S}}(\mathcal{C}) \arrow[d, "\eta_{\mathcal{D}}"] \\
\Gamma\mathcal{C} \arrow[r, hook, "\Gamma(h^{\mathcal{B}}_{\mathcal{C}})"] & \mathrm{Sh}^\kappa_{\mathcal{B}}(\mathcal{C})
\end{tikzcd}$$

in which the vertical map on the left is the unit of the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma$ (see [61, Lemma 6.4.5]), the composition of the above chain of equivalences recovers the map in $(*)$, hence the claim follows. $\square$

PROPOSITION 4.6.3.2. *The functor $\Gamma^{\mathrm{lin}}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$ admits a fully faithful left adjoint.*

PROOF. Note that since $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \simeq \mathrm{Pr}^{\mathrm{R}}(\mathcal{B})^{\mathrm{op}}$ it follows from Proposition 4.4.4.11 that the global sections functor $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_\infty$ preserves colimits. So in light of Proposition 4.6.3.1 we may apply [56, Corollary 4.7.3.16] to the commutative triangle

$$\begin{tikzcd}
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \arrow[rr, "\Gamma^{\mathrm{lin}}"] \arrow[dr, "\Gamma"'] & & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \arrow[dl, "U"] \\
& \mathrm{Pr}^{\mathrm{L}}_\infty &
\end{tikzcd}$$

(where U denotes the forgetful functor), which yields the claim. $\square$

We will now give a more explicit description of the left adjoint from Proposition 4.6.3.2. To that end, observe that the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,\Omega)\colon \Omega^{\mathrm{op}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ takes values in $\mathsf{Pr}^{\mathrm{R}}_{\mathcal{B}}$ and therefore determines a limit-preserving map $\mathcal{B}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}(\mathcal{B}) \simeq (\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))^{\mathrm{op}}$ which by postcomposition with Γ^{lin} results in a limit-preserving functor $\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}}$. We now get a map

$$\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \times \mathcal{B}^{\mathrm{op}} \xrightarrow{(-\otimes_{\mathcal{B}}\mathcal{B}_{/-})^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to (\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}_{\infty} \hookrightarrow \widehat{\mathrm{Cat}}_{\infty}$$

and hence by adjunction a functor $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to \mathrm{PSh}_{\widehat{\mathrm{Cat}}_{\infty}}(\mathcal{B})$.

Lemma 4.6.3.3. *The functor* $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to \mathrm{PSh}_{\widehat{\mathrm{Cat}}_{\infty}}(\mathcal{B})$ *factors through* $\mathrm{Pr}^{\mathrm{R}}(\mathcal{B})$ *and thus defines a functor*

$$-\otimes_{\mathcal{B}} \Omega\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}).$$

Proof. First we prove that the functor factors through $\mathrm{Cat}(\widehat{\mathcal{B}})$. This amounts to showing that the functor $\mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$ is continuous for every $\mathcal{D} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$. As the functor $\mathcal{B}_{/-}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}}$ preserves limits, this follows from the fact that $\mathcal{D} \otimes_{\mathcal{B}} -$, viewed as an endofunctor on $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}}$, preserve limits as well [56, Corollary 4.4.2.15]. Next, we show that the resulting $\mathcal{B}$-category $\mathcal{D} \otimes_{\mathcal{B}} \Omega$ is presentable. As it by construction takes values in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$, Theorem 4.4.2.4 implies that it suffices to show that $\mathcal{D} \otimes_{\mathcal{B}} \Omega$ is Ω-cocomplete and that the transition functors are cocontinuous. Both statements follow from the observation that the functor $\mathcal{D} \otimes_{\mathcal{B}} -\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ can be upgraded to an $(\infty,2)$-functor (see [45, §4.4] for details) and that for any $s\colon B \to A$ in $\mathcal{B}$ the adjunction $s_! \dashv s^*$ is $\mathcal{B}$-linear, see [56, Corollary 7.3.2.7]. To finish the proof, it remains to see that for any map of $\mathcal{B}$-modules $\mathcal{D} \to \mathcal{E}$ the induced map $\mathcal{E} \otimes_{\mathcal{B}} \Omega \to \mathcal{D} \otimes_{\mathcal{B}} \Omega$ admits a left adjoint. By construction, it has one section-wise, so it suffices to check that for any map $s\colon B \to A$ in $\mathcal{B}$ the vertical mate of

$$\begin{tikzcd}
\mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/B} \arrow[r] & \mathcal{E} \otimes_{\mathcal{B}} \mathcal{B}_{/B} \\
\mathcal{D} \otimes_{\mathcal{B}} \mathcal{B}_{/A} \arrow[r] \arrow[u] & \mathcal{E} \otimes_{\mathcal{B}} \mathcal{B}_{/A} \arrow[u]
\end{tikzcd}$$

commutes. Using again $(\infty,2)$-functoriality of the relative tensor product, this follows by essentially the same argument as in the proof of Lemma 3.1.2.10. ☐

Remark 4.6.3.4. It also seems natural to consider the functor

$$\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \times \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathrm{id}\times\mathcal{B}_{/-}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \times \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \xrightarrow{-\otimes_{\mathcal{B}}-} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Cat}_{\infty}$$

which by transposition also gives rise to a functor $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat}_{\infty})$. We expect that this functor takes values in $\mathrm{Cat}(\mathcal{B})$ and is equivalent to $-\otimes_{\mathcal{B}} \Omega$. It is easy to see that for fixed $\mathcal{C} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ the two resulting presheaves of categories on $\mathcal{B}$ have the same value on objects and morphisms. However, a proof that they agree as functors seems two require $(\infty,2)$-categorical techniques that are not quite available yet.

Lemma 4.6.3.5. *The functor* $-\otimes_{\mathcal{B}} \Omega\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *preserves colimits.*

Proof. As limits in $\mathrm{Cat}(\widehat{\mathcal{B}})$ are computed section-wise, it suffices to show that for every $A \in \mathcal{B}$ the functor

$$\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \xrightarrow{(-\otimes_{\mathcal{B}}\mathcal{B}_{/A})^{\mathrm{op}}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \to (\mathrm{Pr}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}_{\infty} \to \widehat{\mathrm{Cat}}_{\infty}$$

preserves limits, which is obvious. ☐

Proposition 4.6.3.6. *The functor* $-\otimes_{\mathcal{B}} \Omega$ *defines a left adjoint of* Γ^{lin}.

Proof. We show that $-\otimes_{\mathcal{B}} \Omega$ is equivalent to the left adjoint L of Γ^{lin} from Proposition 4.6.3.2. Let us denote by $-\otimes \mathcal{B}\colon \mathrm{Pr}^{\mathrm{L}}_{\infty} \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ the left adjoint to the forgetful functor. Then by the associativity of the relative tensor product ([56, Proposition 4.4.3.14] we have equivalences

$$(*) \qquad (-\otimes_{\mathcal{B}} \Omega) \circ (-\otimes \mathcal{B}) \simeq -\otimes \Omega \simeq L \circ (-\otimes \mathcal{B})$$

of functors from $\mathrm{Pr}^{\mathrm{L}}_\infty$ to $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. By [56, Remark 4.7.3.15] we may find a functor

$$F\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}_\infty)$$

such that the composite

$$\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \xrightarrow{F} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}_\infty) \xrightarrow{(-\otimes\mathcal{B})_*} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)) \xrightarrow{\mathrm{colim}_{\Delta^{\mathrm{op}}}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)$$

is equivalent to the identity. From $(*)$ and Lemma 4.6.3.5 it follows that the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}_\infty) \arrow[r, "(-\otimes\mathcal{B})_*"] \arrow[d, "(-\otimes\mathcal{B})_*"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)) \arrow[r, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] \arrow[d, "(-\otimes_{\mathcal{B}}\Omega)_*"] & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \arrow[d, "-\otimes_{\mathcal{B}}\Omega"] \\
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)) \arrow[r, "L_*"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})) \arrow[r, "\mathrm{colim}_{\Delta^{\mathrm{op}}}"] & \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})
\end{tikzcd}$$

commutes. Since L commutes with colimits as well, we get an equivalence $L \simeq (-\otimes_{\mathcal{B}} \Omega)$, as desired. □

The functor $-\otimes_{\mathcal{B}} \Omega$ can be naturally extended to a strong monoidal functor. To see this, observe that since the global sections functor $\Gamma\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}_\infty$ admits an extension to a lax monoidal functor $\Gamma^{\mathrm{cc},\otimes}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^\otimes \to (\mathrm{Pr}^{\mathrm{L}})^\otimes$, the commutative diagram

$$\begin{tikzcd}
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \arrow[rr, "\Gamma^{\mathrm{lin}}"] \arrow[dr, "\Gamma"'] & & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \arrow[dl, "U"] \\
& \mathrm{Pr}^{\mathrm{L}}_\infty &
\end{tikzcd}$$

can be naturally extended to a diagram of lax monoidal functors. By passing to left adjoints, we thus obtain a commutative triangle

$$\begin{tikzcd}
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^\otimes & & \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty)^\otimes \arrow[ll, "-\otimes_{\mathcal{B}}\Omega"'] \\
& (\mathrm{Pr}^{\mathrm{L}})^\otimes \arrow[ul, "-\otimes\Omega"] \arrow[ur, "-\otimes\mathcal{B}"'] &
\end{tikzcd}$$

of *oplax* monoidal functors, see [39]. In order to show that the functor $-\otimes_{\mathcal{B}} \Omega$ is strong monoidal, it thus suffices to show that the natural map

$$(-\otimes_{\mathcal{B}} -) \otimes_{\mathcal{B}} \Omega \to (-\otimes_{\mathcal{B}} \Omega) \otimes (-\otimes_{\mathcal{B}} \Omega)$$

is an equivalence. As both sides of this map preserve colimits in both variables and since every $\mathcal{B}$-module can be written as a colimit of objects that are contained in the image of $-\otimes \mathcal{B}$, it suffices to show that the natural map

$$(-\otimes -) \otimes \Omega \to (-\otimes \Omega) \otimes (-\otimes \Omega)$$

is an equivalence, i.e. that $-\otimes \Omega$ is strong monoidal. Recall (e.g. from Remark 4.4.4.12) that every presentable ∞-category can be obtained as a pushout (in $\mathrm{Pr}^{\mathrm{L}}_\infty$) of presheaf ∞-categories. The claim therefore follows from the observation that $-\otimes \Omega$ fits into a commutative square

$$\begin{tikzcd}
\mathrm{Cat}(\widehat{\mathcal{B}})^\times \arrow[d, "L"] & \widehat{\mathrm{Cat}}_\infty^\times \arrow[l, "\mathrm{const}_{\mathcal{B}}"'] \arrow[d, "L"] \\
\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})^\otimes & (\mathrm{Pr}^{\mathrm{L}})^\otimes \arrow[l, "-\otimes\Omega"']
\end{tikzcd}$$

of oplax monoidal functors (which is again constructed from the associated commutative square of lax monoidal functors by passing to left adjoints) in which both vertical maps as well as $\mathrm{const}_{\mathcal{B}}$ are strong monoidal. We conclude:

PROPOSITION 4.6.3.7. *The functor* $-\otimes_{\mathcal{B}} \Omega\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_\infty) \hookrightarrow \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *admits a natural enhancement to a strong monoidal functor.* □

The functor $- \otimes_{\mathcal{B}} \Omega$ being fully faithful raises the question what can be said about its essential image. First, we observe that there is an explicit criterion when a presentable $\mathcal{B}$-category arises from a $\mathcal{B}$-module:

REMARK 4.6.3.8. Let C be a presentable $\mathcal{B}$-category. Then the unit of the adjunction $- \otimes_{\mathcal{B}} \Omega \dashv \Gamma^{\mathrm{lin}}$ gives a canonical map $\Gamma^{\mathrm{lin}}(\mathsf{C}) \otimes_{\mathcal{B}} \Omega \to \mathsf{C}$. For $A \in \mathcal{B}$ the induced map $\varepsilon(A) \colon \mathcal{B}_{/A} \otimes_{\mathcal{B}} \Gamma^{\mathrm{lin}}(\mathsf{C}) \to \mathsf{C}(A)$ is the map underlying the essentially unique map of $\mathcal{B}_{/A}$-modules that makes the diagram

$$\begin{tikzcd} \mathcal{B}_{/A} \otimes_{\mathcal{B}} \Gamma^{\mathrm{lin}}(\mathsf{C}) \arrow[r, "\varepsilon(A)"] & \mathsf{C}(A) \\ \Gamma^{\mathrm{lin}}(\mathsf{C}) \arrow[r, "\mathrm{id}"] \arrow[u] & \Gamma^{\mathrm{lin}}(\mathsf{C}) \arrow[u, "\pi_A^*"'] \end{tikzcd}$$

of $\mathcal{B}$-modules commute. It follows that a presentable $\mathcal{B}$-category is in the essential image of $- \otimes_{\mathcal{B}} \Omega$ if and only if $\varepsilon(A)$ is an equivalence for all $A \in \mathcal{B}$.

Using the criterion from Remark 4.6.3.8, we are now able to write down an example of a presentable $\mathcal{B}$-category that is *not* in the essential image of $- \otimes_{\mathcal{B}} \Omega$. We learned about this example from David Gepner and Rune Haugseng.

EXAMPLE 4.6.3.9. Let us write Fin for the category of finite sets and let $\mathcal{B} = \mathrm{PSh}(\mathrm{Fin})$. Let X be a set with more than one element that we consider as an object in $\mathcal{B}$ via the Yoneda embedding. Then $\underline{\mathsf{Fun}}_{\mathcal{B}}(X, \Omega_{\mathcal{B}})$ is a presentable $\mathcal{B}$-category that is not in the essential image of $- \otimes_{\mathcal{B}} \Omega$. In fact, by Remark 4.6.3.8 this would imply that the canonical map $\varepsilon(X)$ being an equivalence. In our specific situation $\varepsilon(X)$ is the canonical left adjoint functor $\mathrm{PSh}(\mathrm{Fin}_{/X}) \otimes_{\mathrm{PSh}(\mathrm{Fin})} \mathrm{PSh}(\mathrm{Fin}_{/X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X \times X})$. Explicitly this functor is constructed by applying $\mathrm{PSh}(-)$ to the augmented cosimplicial diagram

$$\mathrm{Fin}_{/X \times X} \to \mathrm{Fin}_{/X} \times \mathrm{Fin}_{/X} \rightrightarrows \mathrm{Fin}_{/X} \times \mathrm{Fin} \times \mathrm{Fin}_{/X} \cdots$$

and then taking the induced map $\mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X \times X})$ in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. Thus, upon passing to right adjoints, we conclude that if the $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(X, \Omega_{\mathcal{B}})$ is contained in the essential image of $- \otimes_{\mathcal{B}} \Omega$, the cosimplicial diagram

$$\mathrm{PSh}(\mathrm{Fin}_{/X \times X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin}_{/X}) \rightrightarrows \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin} \times \mathrm{Fin}_{/X}) \cdots$$

in $\mathrm{Pr}^{\mathrm{R}}_{\infty}$ must be a limit diagram. We show that this cannot be true. Let us denote the map $\mathrm{Fin}_{/X \times X} \to \mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X}$ by f_n. It is given explicitly by the assignment

$$(A \to X \times X) \mapsto (A \to X \times X \xrightarrow{\mathrm{pr}_0} X, A, \dots, A, A \to X \times X \xrightarrow{\mathrm{pr}_1} X).$$

Now for any $n \geq 1$ the map $\mathrm{PSh}(\mathrm{Fin}_{/X \times X}) \to \mathrm{PSh}(\mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X})$ is the functor of right Kan extension $(f_n^{\mathrm{op}})_*$ along f_n^{op}. Hence, if the above cosimplicial diagram is a limit cone, the counit of the adjunctions $(f_n^{\mathrm{op}})^* \dashv (f_n^{\mathrm{op}})_*$ yields an equivalence $\mathrm{colim}_{n \in \Delta^{op}} (f_n^{\mathrm{op}})^* (f_n^{\mathrm{op}})_* F \to F$ for any $F \in \mathrm{PSh}(\mathrm{Fin}_{/X \times X})$. For any object $a = (A \to X, B_1, ..., B_n, C \to X) \in \mathrm{Fin}_{/X} \times \mathrm{Fin}^n \times \mathrm{Fin}_{/X}$ we can compute $(f_n^{\mathrm{op}})_* F(a)$ via the point-wise formula for right Kan extensions as a limit indexed by $(\mathrm{Fin}_{/X \times X})^{\mathrm{op}}_{a/}$. But $A \times B_1 \times ... \times B_n \times X \to X \times X$ defines an initial object of this category, hence we find

$$F(A \to X \times X) \simeq \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} F(A \times A^n \times A \to X \times X).$$

In particular, this shows that the map $F(A \times A \to X \times X) \to F(A \to X \times X)$ induced by $A \to A \times A$ is a cover in $\mathcal{S}$. By taking F to be the presheaf represented by the diagonal $X \to X \times X$, it in turn follows that the map

$$\mathrm{map}_{\mathrm{Fin}_{/X \times X}}(X \times X, X) \to \mathrm{map}_{\mathrm{Fin}_{/X \times X}}(X, X)$$

is surjective. In particular, there is a preimage of the identity $X \to X$. But since X has at least two elements there is no map α making the diagram

$$\begin{tikzcd} X \times X \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{id}"'] & & X \arrow[dl, "\Delta"] \\ & X \times X & \end{tikzcd}$$

commute, which yields the desired contradiction.

There is, however, a class of ∞-topoi $\mathcal{B}$ for which the functor $-\otimes_{\mathcal{B}}$ turns out to be essentially surjective, namely those that are generated by (-1)-truncated objects:

Proposition 4.6.3.10. *Assume that $\mathcal{B}$ is generated under colimits by (-1)-truncated objects. Then $-\otimes_{\mathcal{B}} \Omega$ is an equivalence.*

Proof. By Propositions 4.6.3.2 and 4.6.3.6 it remains to show essential surjectivity. Since $-\otimes_{\mathcal{B}} \Omega$ preserves colimits and every presentable $\mathcal{B}$-category is a pushout of presheaf $\mathcal{B}$-categories (see Remark 4.4.4.12) it suffices to see that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is in the essential image for any small $\mathcal{B}$-category C. Furthermore, we can write C as a colimit of $\mathcal{B}$-categories of the form $\Delta^n \otimes U$, where $U \in \mathcal{B}$ is (-1)-truncated. Since the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)\colon \mathrm{Cat}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ that is determined by the universal property of presheaf $\mathcal{B}$-categories is a (partial) left adjoint (see Corollary 3.4.1.15) and therefore preserves colimits, it suffices to see that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n \otimes U)$ is in the essential image. Since $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)$ is also symmetric monoidal by Remark 4.6.2.6, we have a canonical equivalence

$$\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n \otimes U) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n) \otimes \underline{\mathsf{PSh}}_{\mathcal{B}}(U).$$

Furthermore, we have $\underline{\mathsf{PSh}}_{\mathcal{B}}(\Delta^n) \simeq \mathrm{PSh}(\Delta^n) \otimes \Omega \simeq (\mathrm{PSh}(\Delta^n) \otimes \Omega) \otimes_{\mathcal{B}} \Omega$, and since $-\otimes_{\mathcal{B}} \Omega$ is symmetric monoidal by Proposition 4.6.3.7, it thus suffices to see that $\underline{\mathsf{PSh}}_{\mathcal{B}}(U)$ is in the essential image. By Remark 4.6.3.8, it follows that we need to check that for any $A \in \mathcal{B}$ the canonical map

$$\mathcal{B}_{/A} \otimes_{\mathcal{B}} \underline{\mathsf{PSh}}_{\mathcal{B}}(U)(1) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(U)(A)$$

of $\mathcal{B}_{/A}$-modules is an equivalence. Since $\mathcal{B}$ is generated under colimits by (-1)-truncated objects, we may assume that $A = V$ is also (-1)-truncated. Thus, we have to show that the canonical map

$$\mathcal{B}_{/V} \otimes_{\mathcal{B}} \mathcal{B}_{/U} \to \mathcal{B}_{/U \times V}$$

is an equivalence. For this, note that because U is (-1)-truncated, we have a canonical commutative square

$$\begin{tikzcd} \Delta^1 \arrow[r, "U \to 1"] \arrow[d, "\mathrm{id}_1"'] & \mathcal{B} \arrow[d, "- \times U"] \\ \mathcal{B} \arrow[r, "- \times U"'] & \mathcal{B}_{/U} \end{tikzcd}$$

By adjunction and the universal property of presheaf ∞-categories, this induces a commutative square

$$\begin{tikzcd} \mathrm{PSh}(\Delta^1) \otimes \mathcal{B} \arrow[r, "(U \to 1) \otimes \mathcal{B}"] \arrow[d, "(\mathrm{id}_1) \otimes \mathcal{B}"'] & \mathcal{B} \arrow[d, "- \times U"] \\ \mathcal{B} \arrow[r, "- \times U"'] & \mathcal{B}_{/U} \end{tikzcd}$$

in $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$. We claim that this square is a pushout. For this it suffices to see that the underlying square in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$ is a pushout, i.e. it is a pullback after passing to right adjoints. The right adjoint of $\mathrm{id}_1 \otimes \mathcal{B}$ is simply the diagonal map $\mathcal{B} \to \mathcal{B}^{\Delta^1}$, and the right adjoint of $(U \to 1) \otimes \mathcal{B}$ sends an object $A \in \mathcal{B}$ to the arrow

$$A \to \underline{\mathrm{Hom}}_{\mathcal{B}}(U, A).$$

Thus, we may identify the pullback, with the full subcategory of $\mathcal{B}$ spanned by those objects for which the canonical map $A \to \underline{\mathrm{Hom}}_{\mathcal{B}}(U, A)$ is an equivalence. But because U is (-1)-truncated, this subcategory is canonically equivalent to $\mathcal{B}_{/U}$, so that the above square is indeed a pushout. Repeating the same argument with $\mathcal{B}_{/V}$ in place of $\mathcal{B}$ and $U \times V$ in place of U, we get a similar pushout in $\mathrm{Mod}_{\mathcal{B}_{/U}}(\mathrm{Pr}^{\mathrm{L}}_{\infty})$ with $\mathcal{B}_{/U\times V}$ in the lower right corner. But applying $- \otimes_{\mathcal{B}} \mathcal{B}_{/V}$ to the above square, we also get a pushout

$$\begin{tikzcd}
\mathrm{PSh}(\Delta^1) \otimes \mathcal{B}_{/V} \arrow[r, "(U\times V \to V)\otimes \mathcal{B}_{/V}"] \arrow[d, "(\mathrm{id}_1)\otimes \mathcal{B}_{/V}"'] & \mathcal{B}_{/V} \arrow[d] \\
\mathcal{B}_{/V} \arrow[r] & \mathcal{B}_{/U} \otimes_{\mathcal{B}} \mathcal{B}_{/V}
\end{tikzcd}$$

and thus an equivalence of $\mathcal{B}_{/V}$-modules $\mathcal{B}_{/U\times V} \simeq \mathcal{B}_{/U} \otimes_{\mathcal{B}} \mathcal{B}_{/V}$. Furthermore this equivalence is by construction compatible with the canonical map from $\mathcal{B}$. Thus it is indeed the map of Remark 4.6.3.8, and the claim follows. $\square$

Our next goal will be to show that if R is a commutative $\mathcal{B}$-ring spectrum, the presentable $\mathcal{B}$-category $\mathsf{Mod}_R^{\mathcal{B}}$ constructed in § 4.5.2 is in the essential image of $- \otimes_{\mathcal{B}} \Omega$. For this we need the following observation. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and let $A \in \mathrm{CAlg}(\mathcal{C})$. The the commutative square

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathcal{C}) \arrow[r] & \mathrm{Mod}_{f(A)}(\mathcal{D}) \\
\mathcal{C} \arrow[u] \arrow[r] & \mathcal{D} \arrow[u]
\end{tikzcd}$$

shows that there is a unique map of $\mathcal{D}$-modules $\psi\colon \mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D} \to \mathrm{Mod}_{f(A)}(\mathcal{D})$ making the triangle of $\mathcal{C}$-module maps commute

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D} \arrow[rr] & & \mathrm{Mod}_{f(A)}(\mathcal{D}) \\
& \mathcal{D} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

LEMMA 4.6.3.11. *The map* $\psi\colon \mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D} \to \mathrm{Mod}_{f(A)}(\mathcal{D})$ *is an equivalence.*

PROOF. This can be extracted from the discussion in [56, §4.8.4 and 4.8.5]. More specifically, we observe that the $\mathcal{D}$-module $\mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D}$ together with the object $A \otimes 1_{\mathcal{D}}$ satisfies the conditions (1)-(6) of [56, p. 4.8.5.8]. Indeed, we have a $\mathcal{C}$-linear adjunction in Pr^{L}

$$- \otimes A\colon \mathcal{C} \leftrightarrows \mathrm{Mod}_A : G$$

so that after applying $- \otimes_{\mathcal{C}} \mathcal{D}$ we get a $\mathcal{D}$-linear adjunction and thus all conditions except (5) are obvious. To see that (5) also holds, one can argue as in the proof of [56, Theorem 4.8.4.6]. By [56, Proposition 4.8.5.8] there is thus an equivalence of $\mathcal{D}$-modules sending

$$\varphi\colon \mathrm{Mod}_{f(A)}(\mathcal{D}) \to \mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D}$$

$f(A)$ to the unit of $\mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D}$. Furthermore the construction of the equivalence in [56, Theorem 4.8.5.8] shows that φ makes the triangle

$$\begin{tikzcd}
\mathrm{Mod}_{f(A)}(\mathcal{D}) \arrow[rr, "\varphi"] & & \mathrm{Mod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{D} \\
& \mathcal{D} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

of $\mathcal{D}$-modules commute and thus the inverse of φ has to agree with our map ψ from above. $\square$

PROPOSITION 4.6.3.12. *Let R be a commutative $\mathcal{B}$-ring spectrum. Then* $\mathsf{Mod}_R^{\mathcal{B}}$ *is in the essential image of* $- \otimes_{\mathcal{B}} \Omega_{\mathcal{B}}$.

PROOF. Unwinding the definitions, we need to see that for any $A \in \mathcal{B}$ the canonical map of $\mathcal{B}_{/A}$-modules

$$\mathrm{Mod}_R(\mathrm{Sp}(\mathcal{B})) \otimes_{\mathcal{B}} \mathcal{B}_{/A} \to \mathrm{Mod}_{\pi_A^* R}(\mathrm{Sp}(\mathcal{B}_{/A}))$$

is an equivalence. Since Sp is an idempotent algebra in Pr^{L} we may identify the above map with the canonical map

$$\mathrm{Mod}_R(\mathrm{Sp}(\mathcal{B})) \otimes_{\mathrm{Sp}(\mathcal{B})} \mathrm{Sp}(\mathcal{B}_{/A}) \to \mathrm{Mod}_{\pi_A^* R}(\mathrm{Sp}(\mathcal{B}_{/A}))$$

which is an equivalence by Lemma 4.6.3.11. $\square$

PROPOSITION 4.6.3.13. *The functor* $\Gamma^{\mathrm{lin}}\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}})$ *is* $\mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}})$*-linear. In other words for* $\mathsf{C} \in \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ *and* $\mathcal{D} \in \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}})$*, the canonical map*

$$\gamma_{\mathcal{D},\mathsf{C}}\colon \mathcal{D} \otimes_{\mathcal{B}} \Gamma^{\mathrm{lin}}(\mathsf{C}) \to \Gamma^{\mathrm{lin}}((\mathcal{D} \otimes_{\mathcal{B}} \Omega) \otimes^{\mathcal{B}} \mathsf{C})$$

given by composing the map $\Gamma^{\mathrm{lin}}(-) \otimes \Gamma^{\mathrm{lin}}(-) \to \Gamma^{\mathrm{lin}}(- \otimes -)$ *with the unit* $\mathrm{id} \to \Gamma^{\mathrm{lin}}(- \otimes_{\mathcal{B}} \Omega)$*, is an equivalence*

PROOF. Note that by the proof of Proposition 4.6.3.2 the functor Γ^{lin} commutes with all colimits. Since $- \otimes_{\mathcal{B}} -$ and $- \otimes^{\mathcal{B}} -$ are both bilinear and because Γ is conservative, we may reduce to the case where $\mathcal{D}$ is a free $\mathcal{B}$-module on some $\mathcal{E} \in \mathrm{Pr}^{\mathrm{L}}$ and only have to show that the canonical map

$$\mathcal{E} \otimes \Gamma(\mathsf{C}) \to \Gamma(\mathcal{E} \otimes \Omega) \otimes \Gamma(\mathsf{C}) \to \Gamma((\mathcal{E} \otimes \Omega) \otimes^{\mathcal{B}} \mathsf{C})$$

is an equivalence. Since any presentable ∞-category is a pushout of presheaf categories, we may further reduce to the case where $\mathcal{E} = \mathrm{PSh}(\mathcal{C}_0)$. Recall from the proof of Proposition 4.6.3.1 that we have a canonical equivalence $\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{C}_0, \Omega)$. Therefore it follows that we have an equivalence $(\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \otimes^{\mathcal{B}} \mathsf{C} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{C})$ and we claim that the composite

$$\mathrm{PSh}(\mathcal{C}_0) \otimes \Gamma(\mathsf{C}) \xrightarrow{\gamma_{\mathrm{PSh}(\mathcal{C}_0),\mathsf{C}}} \Gamma((\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \otimes^{\mathcal{B}} \mathsf{C}) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{C})$$

is an equivalence. For this observe that by construction, the canonical map

$$\Gamma(\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \otimes \Gamma(\mathsf{C}) \to \Gamma((\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \otimes^{\mathcal{B}} \mathsf{C})$$

is the unique colimit preserving functor corresponding to the bilinear functor

$$\Gamma(\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \times \Gamma(\mathsf{C}) \xrightarrow{\simeq} \Gamma((\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \times \mathsf{C}) \xrightarrow{\Gamma(\tau)} \Gamma((\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \otimes^{\mathcal{B}} \mathsf{C}).$$

Here τ is the universal bilinear functor of $\mathcal{B}$-categories. We now consider the commutative square

$$\begin{tikzcd}
\mathcal{C}_0^{\mathrm{op}} \times \mathrm{PSh}(\mathcal{C}_0) \times \Gamma(\mathsf{D}) \arrow[r, "\mathrm{id} \times (\mathrm{const}_{\mathcal{B}})_* \times \mathrm{id}"] \arrow[d, "\mathrm{ev} \times \mathrm{id}"'] & \mathcal{C}_0^{\mathrm{op}} \times \mathrm{PSh}(\mathcal{C}_0) \times \Gamma(\mathsf{D}) \arrow[d, "\mathrm{ev} \times \mathrm{id}"] \\
\mathcal{S} \times \Gamma(\mathsf{D}) \arrow[r, "\mathrm{const}_{\mathcal{B}} \times \mathrm{id}"] \arrow[d, "- \otimes -"'] & \mathcal{B} \times \Gamma(\mathsf{D}) \arrow[d, "- \otimes -"] \\
\Gamma(\mathsf{D}) \arrow[r, "\mathrm{id}"'] & \Gamma(\mathsf{D})
\end{tikzcd}$$

By Lemma 4.6.2.12, the composite of the left vertical maps transposes to the universal bilinear functor $\mathrm{PSh}(\mathcal{C}_0) \times \Gamma(\mathsf{C}) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \Gamma(\mathsf{C}))$. But combining Lemma 4.6.2.12 with the equivalence $\mathrm{Fun}_{\mathcal{B}}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{D}) \simeq \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \Gamma(\mathsf{D}))$, the composite of the right vertical maps transposes to $\Gamma(\tau)$, where $\tau\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathcal{C}_0) \times \mathsf{D} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{D})$ is the universal bilinear functor of $\mathcal{B}$-categories. Since the unit map $\mathrm{PSh}(\mathcal{C}_0) \to \Gamma(\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega)$ can be identified with $(\mathrm{const}_{\mathcal{B}})_*\colon \mathrm{PSh}(\mathcal{C}_0) \to \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \mathcal{B})$ (see the proof of Proposition 4.6.3.1) it follows that the composite

$$\mathrm{PSh}(\mathcal{C}_0) \times \Gamma(\mathsf{C}) \to \mathrm{PSh}(\mathcal{C}_0) \otimes \Gamma(\mathsf{C}) \xrightarrow{\gamma_{\mathrm{PSh}(\mathcal{C}_0),\mathsf{C}}} \Gamma((\mathrm{PSh}(\mathcal{C}_0) \otimes \Omega) \otimes^{\mathcal{B}} \mathsf{C}) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{C}_0^{\mathrm{op}}, \mathsf{C}) \simeq \mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}, \Gamma(\mathsf{C}))$$

can be identified with the universal bilinear functor and therefore $\gamma_{\mathrm{PSh}(C_0),\mathsf{C}}$ is an equivalence, as desired. $\square$

CHAPTER 5

$\mathcal{B}$-topoi

Our definition of a $\mathcal{B}$-topos will be that of a presentable $\mathcal{B}$-category that satisfies the descent property. Therefore, we will begin by setting up the theory of descent for $\mathcal{B}$-categories in Section 5.1. We also relate the notion of descent to universality and disjointness of colimits.

In Section 5.2, we then proceed by developing the main concepts of $\mathcal{B}$-topos theory. Using the results from Section 5.1, we characterize $\mathcal{B}$-topoi in terms of an internal version of the Giraud axioms, see Theorem 5.2.1.5. We also prove the expected result that any $\mathcal{B}$-topos is a left exact accessible localisation of a presheaf $\mathcal{B}$-category in Theorem 5.2.3.1. As a consequence, we show another main result of this chapter, the equivalence of $\mathcal{B}$-topoi and relative topoi over $\mathcal{B}$, see Theorem 5.2.5.1. We then also give some applications of these results in ∞-topos theory, see for example Proposition 5.2.7.1.

Finally, Section 5.3 is dedicated to the discussion of localic $\mathcal{B}$-topoi. The main result of this section is that if $\mathcal{B}$ is itself a localic ∞-topos, then there is an equivalence between localic $\mathcal{B}$-topoi and locales over the locale of subobjects of the terminal object $\mathrm{Sub}_{\mathcal{B}}$. We also study a number compactness conditions for $\mathcal{B}$-locales, which we will be useful in the next chapter.

5.1. Descent

Recall that if $\mathcal{C}$ is an ∞-category with finite limits, the codomain fibration $d_0\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a cartesian fibration and therefore classified by a functor $\mathcal{C}_{/-}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$. If $\mathcal{C}$ furthermore has all colimits, one says that $\mathcal{C}$ satisfies *descent* if $\mathcal{C}_{/-}$ preserves limits [57, § 6.1.3]. The goal of this section is to discuss an analogues concept for $\mathcal{B}$-categories. We begin in § 5.1.1 with defining the descent property. In § 5.1.2, we bring this condition into a more explicit form using the notion of cartesian transformations. As we later want to compare descent with a $\mathcal{B}$-categorical version of the Giraud axioms, we use the remainder of this section to relate the descent property with the notions of universality (§ 5.1.3) and disjointness (§ 5.1.4) of colimits as well as effectivity of groupoid objects (§ 5.1.5).

5.1.1. The definition of descent. In order to define the descent property of a $\mathcal{B}$-category C, we first need to construct the functor $\mathsf{C}_{/-}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$. As we have the straightening equivalence for cartesian fibrations at our disposal [61, Theorem 6.3.1], we may proceed in the same fashion as in [57]. We begin with the following lemma:

LEMMA 5.1.1.1. *For any $\mathcal{B}$-category C, the codomain fibration $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ is a cocartesian fibration.*

PROOF. This is an immediate consequence of the analogous statement for ∞-categories and the explicit description of the cocartesian edges [57, Corollary 2.4.7.12]. □

By the $\mathcal{B}$-categorical straightening equivalence, the cocartesian fibration $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ gives rise to a functor $\mathsf{C}_{/-}\colon \mathsf{C} \to \mathsf{Cat}_{\mathcal{B}}$. Note that the map $(d_1, d_0)\colon \mathsf{C}^{\Delta^1} \to \mathsf{C} \times \mathsf{C}$ can be regarded as a morphism of cocartesian fibrations over C, where we regard the codomain as a cocartesian fibration over C by virtue of the projection onto the second factor. Therefore, one obtains an induced map $\mathsf{C}_{/-} \to \mathrm{diag}_{\mathsf{C}}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{Cat}_{\mathcal{B}})$, where $\mathrm{diag}_{\mathsf{C}}$ is the constant functor with value C. Alternatively, we may regard $\mathsf{C}_{/-}$ as a functor $\mathsf{C} \to (\mathsf{Cat}_{\mathcal{B}})_{/\mathsf{C}}$. By construction, if $c\colon A \to \mathsf{C}$ is an arbitrary object in context $A \in \mathcal{B}$, the induced map $\mathsf{C}_{/c} \to \pi_A^* \mathsf{C}$ is precisely given by the projection $(\pi_c)_!$ and therefore in particular a right fibration. Thus, the functor $\mathsf{C}_{/-}$ takes values in $\mathsf{RFib}_{\mathsf{C}}$. In particular, this implies that for any map $f\colon c \to d$ in C (in arbitrary context), the induced functor $\mathsf{C}_{/c} \to \mathsf{C}_{/d}$ is a right fibration. On account of the orthogonality between right fibrations and final functors, this map is uniquely determined by the image of the final object id_c. As it is moreover evident from the construction of $\mathsf{C}_{/-}$ that the image of id_c is given by f, we thus conclude that $\mathsf{C}_{/-}$ acts on maps by carrying f to the functor $f_!\colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ that is obtained as the image of f under the Yoneda embedding $\mathsf{C} \hookrightarrow \mathsf{RFib}_{\mathsf{C}}$.

PROPOSITION 5.1.1.2. *Let* C *be a* $\mathcal{B}$*-category. Then the following are equivalent:*

(1) The codomain fibration $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ *is a cartesian fibration;*
(2) for every map $f\colon c \to d$ *in* C*, the functor* $f_!\colon \mathsf{C}_{/c} \to \mathsf{C}_{/d}$ *admits a right adjoint* f^**;*
(3) C *admits pullbacks.*

PROOF. The fact that (1) and (3) are equivalent is an immediate consequence of the analogous statements and the explicit description of the cocartesian edges [57, Lemma 6.1.1.1]. That (2) and (3) are equivalent is the content of Corollary 3.1.7.5. □

REMARK 5.1.1.3. It follows from [57, Lemma 6.1.1.1], that a morphism $\Delta^1 \otimes A \to \mathsf{C}^{\Delta}$ is cocartesian with respect to d_0 if and only if it defines a pullback square in $\mathsf{C}(A)$.

REMARK 5.1.1.4. If C is a $\mathcal{B}$-category with pullbacks, straightening the cartesian fibration $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ yields a functor $\mathsf{C}_{/-}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$. By the discussion in [61, § 7.2], this functor is equivalently obtained by observing that the straightening of the *cocartesian* fibration $\mathsf{C}^{\Delta^1} \to \mathsf{C}$ takes values in $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}}$ and by applying the equivalence $\mathsf{Cat}^{\mathrm{L}}_{\mathcal{B}} \simeq (\mathsf{Cat}^{\mathrm{R}}_{\mathcal{B}})^{\mathrm{op}}$ from [61, Proposition 7.2.1].

We may now define:

DEFINITION 5.1.1.5. Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with finite limits. We say that C *satisfies* U*-descent* if the functor $\mathsf{C}_{/-}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is $\mathrm{op}(\mathsf{U})$-continuous. If X is a cocomplete large $\mathcal{B}$-category, we simply say that C satisfies descent if C satisfies $\mathsf{Cat}_{\mathcal{B}}$-descent.

REMARK 5.1.1.6. The property of a U-cocomplete $\mathcal{B}$-category C with pullbacks to satisfy U-descent is local in $\mathcal{B}$: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then C satisfies U-descent if and only if $\pi_{A_i}^*\mathsf{C}$ satisfies $\pi_{A_i}^*\mathsf{U}$-descent for all i. This follows immediately from the locality of U-continuity, see Remark 3.2.2.3, and from Remark 2.3.1.3.

EXAMPLE 5.1.1.7. Let $\mathcal{K}$ be a class of ∞-categories and let C be an $\mathsf{LConst}_{\mathcal{K}}$-cocomplete $\mathcal{B}$-category with pullbacks (where $\mathsf{LConst}_{\mathcal{K}}$ is the essential image of the functor $\mathrm{const}_{\mathcal{B}}\colon \mathcal{K} \to \mathsf{Cat}_{\mathcal{B}}$). Then C satisfies $\mathsf{LConst}_{\mathcal{K}}$-descent if and only if for all $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ satisfies $\mathcal{K}$-descent. In fact, by [61, Corollary 6.4.10] the composition $\Gamma_{\mathcal{B}_{/A}} \circ \mathsf{C}_{/-}(A)$ recovers the functor $\mathsf{C}(A)_{/-}\colon \mathsf{C}(A)^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$. Consequently, if $s\colon B \to A$ is an arbitrary map in $\mathcal{B}$, postcomposing $\mathsf{C}_{/-}(A)$ with the evaluation functor $\mathrm{ev}_B\colon \mathrm{Cat}(\mathcal{B}_{/A}) \to \widehat{\mathrm{Cat}}_{\infty}$ recovers the composition $\mathsf{C}_{/-}(B) \circ s^*$. Using that C is $\mathsf{LConst}_{\mathcal{K}}$-cocomplete, the functor $s^*\colon \mathsf{C}(A)^{\mathrm{op}} \to \mathsf{C}(B)^{\mathrm{op}}$ is $\mathrm{op}(\mathcal{K})$-continuous, and since limits in $\mathrm{Cat}(\mathcal{B}_{/A})$ are detected section-wise, the claim follows.

5.1.2. Cartesian transformations. The main goal of this section is to obtain a more explicit description of the descent property which will rely on the notion of *cartesian* morphisms of functors:

DEFINITION 5.1.2.1. Let I and C be $\mathcal{B}$-categories such that C admits pullbacks. We say that a map $\varphi\colon d \to d'$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ in context $1 \in \mathcal{B}$ is *cartesian* if for every map $i \to i'$ in I in context $A \in \mathcal{B}$ the induced commutative square

$$\begin{tikzcd} d(i) \arrow[r] \arrow[d] & d'(i) \arrow[d] \\ d(i') \arrow[r] & d'(i') \end{tikzcd}$$

is a pullback in $\mathsf{C}(A)$. A map $d \to d'$ in context $A \in \mathcal{B}$ is called cartesian if it is cartesian when viewed as a map in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{I}, \pi_A^*\mathsf{C})$ in context $1 \in \mathcal{B}_{/A}$. We denote by $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ that is spanned by the cartesian maps in arbitrary context $A \in \mathcal{B}$.

REMARK 5.1.2.2. In the situation of Definition 5.1.2.1, the property of a map $\varphi d \to d'$ in context $A \in \mathcal{B}$ being cartesian is local in $\mathcal{B}$: if $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ is a cover in $\mathcal{B}$, then φ is cartesian if and only if each $s_i^*(\varphi)$ is. In fact, by unwinding the definition, this follows from the fact that the property of a commutative square being a pullback is local in that sense. As a consequence, every object of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/d}^{\mathrm{cart}}$ in context A encodes a cartesian map $d \to d'$, and there is a canonical equivalence $\pi_A^* \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/d}^{\mathrm{cart}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{I}, \pi_A^* \mathsf{C})_{/\pi_A^* d}^{\mathrm{cart}}$ of $\mathcal{B}_{/A}$-categories.

LEMMA 5.1.2.3. *Let* I *and* C *be* $\mathcal{B}$*-categories, and suppose that* C *admits pullbacks. Then a map* $d \to d'$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ *(in arbitrary context) is cartesian if and only if the associated object in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}^{\Delta^1})$ *is contained in* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, (\mathsf{C}^{\Delta^1})_\sharp)$*, where* $(\mathsf{C}^{\Delta^1})_\sharp \hookrightarrow \mathsf{C}^{\Delta^1}$ *is the subcategory that is spanned by the cartesian morphisms over* $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$.

PROOF. This follows immediately from the description of the cartesian morphisms in C^{Δ^1} as pullback squares in C, see Remark 5.1.1.3. $\square$

The main goal of this section is to prove the following description of the descent property:

PROPOSITION 5.1.2.4. *Let* C *be a cocomplete* $\mathcal{B}$*-category with pullbacks and let* $d\colon \mathsf{I} \to \mathsf{C}$ *be a diagram that admits a colimit in* C. *Let* $\overline{d}\colon \mathsf{I}^{\triangleright} \to \mathsf{C}$ *be the corresponding colimit cocone. Then the functor* $\mathsf{C}_{/-}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$ *carries* $\overline{d}$ *to a limit cone in* $\mathsf{Cat}_{\mathcal{B}}$ *if and only if the restriction map* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/d}$ *restricts to an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}}^{\mathrm{cart}} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/d}^{\mathrm{cart}}$$

of $\mathcal{B}$*-categories.*

The main idea for the proof of Proposition 5.1.2.4 is to identify the left-hand side of the equivalence with $\mathsf{C}_{/\operatorname{colim} d}$ and the right-hand side with $\lim \mathsf{C}_{/d(-)}$. In order to do so, we will need the formula for limits in $\mathsf{Cat}_{\mathcal{B}}$ derived in [61, Proposition 7.1.2]. For the convenience of the reader, we will briefly recall the main setup from [61]. The ∞-topos of *marked simplicial objects* in $\mathcal{B}$ is defined as $\mathcal{B}_{\Delta}^{+} = \mathrm{Fun}(\Delta_{+}^{\mathrm{op}}, \mathcal{B})$, where Δ_+ denotes the marked simplex 1-category. Precomposition with the inclusion $\Delta \hookrightarrow \Delta_+$ induces a forgetful functor $(-)|_{\Delta}\colon \mathcal{B}_{\Delta}^{+} \to \mathcal{B}_{\Delta}$ which admits a left adjoint $(-)^{\flat}$ and a right adjoint $(-)^{\sharp}$. Every cartesian fibration $p\colon \mathsf{P} \to \mathsf{C}$ can be equivalently encoded by a *marked cartesian fibration* $p^{\natural}\colon \mathsf{P}^{\natural} \to \mathsf{C}^{\sharp}$, where $\mathsf{P}^{\natural}$ is the marked simplicial object that is obtained from P by marking the cartesian arrows and where a marked cartesian fibration is by definition a map that is internally right orthogonal to the collection of *marked right anodyne maps* (see [61, Definition 4.2.1]). Now if $d\colon \mathsf{I}^{\mathrm{op}} \to \mathsf{Cat}_{\mathcal{B}}$ is a functor and if $p\colon \mathsf{P} \to \mathsf{I}$ is the associated cartesian fibration, one obtains a canonical equivalence

$$\lim d \simeq (\underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(\mathsf{I}^{\sharp}, \mathsf{P}^{\natural})_{/\mathsf{I}^{\sharp}})|_{\Delta},$$

where the right-hand side is defined via the pullback diagram

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(\mathsf{I}^{\sharp}, \mathsf{P}^{\natural})_{/\mathsf{I}^{\sharp}})|_{\Delta} \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(\mathsf{I}^{\sharp}, \mathsf{P}^{\natural})|_{\Delta} \arrow[d] \\
1 \arrow[r, "\mathrm{id}_{\mathsf{I}^{\sharp}}"] & \underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(\mathsf{I}^{\sharp}, \mathsf{I}^{\sharp})|_{\Delta}
\end{tikzcd}$$

(in which $\underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(-,-)$ denotes the internal hom in $\mathcal{B}_{\Delta}^{+}$).

LEMMA 5.1.2.5. *Let* K *be a simplicial object in* $\mathcal{B}$ *and let* $p\colon \mathsf{P} \to \mathsf{C}$ *be a cartesian fibration. Then the canonical map* $K^{\flat} \to K^{\sharp}$ *of marked simplicial objects in* $\mathcal{B}$ *induces a fully faithful functor* $\underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(K^{\sharp}, \mathsf{P}^{\natural})|_{\Delta} \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{B}_{\Delta}^{+}}(K^{\flat}, \mathsf{P}^{\natural})|_{\Delta}$ *of* $\mathcal{B}$*-categories.*

PROOF. Let M be the marked simplicial object in $\mathcal{B}$ that fits into the pushout square

$$\begin{tikzcd}
(\Delta^0 \sqcup \Delta^0)^\flat \otimes K^\flat \arrow[r] \arrow[d] & (\Delta^0 \sqcup \Delta^0)^\flat \otimes K^\sharp \arrow[d] \\
(\Delta^1)^\flat \otimes K^\flat \arrow[r] & M.
\end{tikzcd}$$

Unwinding the definitions, we need to show that $\mathsf{P}^\natural$ is internally local with respect to the induced map $\varphi\colon M \to (\Delta^1)^\flat \otimes K^\sharp$. Since $\mathsf{C}^\sharp$ is easily seen to be internally local with respect to φ, this follows once we show that $p^\natural$ is internally right orthogonal to this map. We therefore need to verify that φ is marked right anodyne. Writing K as a colimit of objects of the form $\Delta^n \otimes A$, we may assume that $K = \Delta^n \otimes A$. Moreover, since marked right anodyne morphisms are closed under products, we can assume that $A \simeq 1$. Using that the two maps $(I^n)^\flat \hookrightarrow (\Delta^n)^\flat$ and $(I^n)^\sharp \hookrightarrow (\Delta^n)^\sharp$ that are induced by the spine inclusions are marked right anodyne, we may further reduce this to $K = \Delta^1$. In this case, one can apply [61, Lemma 4.2.3] to deduce that φ is an equivalence. Hence the claim follows. $\square$

LEMMA 5.1.2.6. *Let $s\colon B \to A$ be a map in $\mathcal{B}$, and let $P \to A$ be an arbitrary map. Let $\eta_s\colon \mathrm{id}_{\mathcal{B}_{/A}} \to s_*s^*$ be the adjunction unit. Then the value of the natural transformation $(\pi_A)_* \xrightarrow{(\pi_A)_*\eta_s} (\pi_A)_*s_*s^* \simeq (\pi_B)_*s^*$ at an object $p\colon P \to A$ in $\mathcal{B}_{/A}$ can be identified with the map*

$$\underline{\mathrm{Hom}}_{\mathcal{B}}(A, P)_{/A} \to \underline{\mathrm{Hom}}_{\mathcal{B}}(B, s^*P)_{/B}$$

that is induced by precomposition with s. Here $\underline{\mathrm{Hom}}_{\mathcal{B}}(-,-)$ denotes the internal hom in $\mathcal{B}$, $\underline{\mathrm{Hom}}_{\mathcal{B}}(A,P)_{/A}$ is the fibre of the map $\underline{\mathrm{Hom}}_{\mathcal{B}}(A,P) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(A,A)$ over id_A, and $\underline{\mathrm{Hom}}_{\mathcal{B}}(B, s^*P)_{/B}$ is the fibre of the map $\underline{\mathrm{Hom}}_{\mathcal{B}}(B, s^*P) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(B,B)$ over id_B.

PROOF. Since the morphism $\mathrm{id} \times s\colon - \times B \to - \times A$ can be identified with the composition

$$(\pi_B)_!\pi_B^* \xrightarrow{\simeq} (\pi_A)_!s_!s^*\pi_A^* \xrightarrow{(\pi_A)_!\epsilon_s\pi_A^*} (\pi_A)_!\pi_A^*$$

(in which ϵ_s is the counit of the adjunction $s_! \dashv s^*$), it follows by adjunction that the map $(\pi_A)_*\eta_s\pi_A^*$ can be identified with $s^*\colon \underline{\mathrm{Hom}}_{\mathcal{B}}(A,-) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(B,-)$. Now if $p\colon P \to A$ is any map, the unique morphism $p \to \mathrm{id}_{\mathcal{B}_{/A}}$ in $\mathcal{B}_{/A}$ is the pullback of $\pi_A^*(\pi_A)_!p \to \pi_A^*(\pi_A)_!1_{\mathcal{B}_{/A}}$ along the unit $1_{\mathcal{B}_{/A}} \to \pi_A^*(\pi_A)_!1_{\mathcal{B}_{/A}}$. Together with naturality of η_s, this implies that the map $(\pi_A)_*\eta_s(p)$ fits into a commutative diagram

$$\begin{tikzcd}
 & s_*s^*(\pi_A)_*(p) \arrow[rr] \arrow[dd] & & \underline{\mathrm{Hom}}_{\mathcal{B}}(B,P) \arrow[dd, "p_*"] \\
(\pi_A)_*(p) \arrow[ur, "(\pi_A)_*\eta_s(p)"] \arrow[rr] \arrow[dd] & & \underline{\mathrm{Hom}}_{\mathcal{B}}(A,P) \arrow[ur, "s^*"] \arrow[dd, "p_*", near start] & \\
 & 1 \arrow[rr, "s", near start] & & \underline{\mathrm{Hom}}_{\mathcal{B}}(B,A) \\
1 \arrow[ur] \arrow[rr, "\mathrm{id}_A"] & & \underline{\mathrm{Hom}}_{\mathcal{B}}(A,A) \arrow[ur, "s^*"] &
\end{tikzcd}$$

in which the front and the back square are pullbacks. As the fibre of

$$p_*\colon \underline{\mathrm{Hom}}_{\mathcal{B}}(B,P) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(B,A)$$

over s can be identified with $\underline{\mathrm{Hom}}_{\mathcal{B}}(B, s^*P)_{/B}$, the claim follows. $\square$

PROOF OF PROPOSITION 5.1.2.4. Let $\iota\colon \mathsf{I} \hookrightarrow \mathsf{I}^{\triangleleft}$ be the inclusion. Since $\mathsf{Cat}_{\mathcal{B}}$ is complete, the theory of Kan extensions gives rise to an adjunction

$$(\iota^* \dashv \iota_*)\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleleft}, \mathsf{Cat}_{\mathcal{B}}) \leftrightarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{Cat}_{\mathcal{B}}),$$

see § 3.3.3. Given any diagram $\overline{h}\colon \mathsf{I}^{\triangleleft} \to \mathsf{Cat}_{\mathcal{B}}$, we will let $h = \iota^*\overline{h}$, and we denote by $\eta_{\overline{h}}\colon \overline{h} \to \iota_*h$ the adjunction unit. Now let us set $\overline{h} = \mathsf{C}_{/\overline{d}(-)}$, so that we get $h = \mathsf{C}_{/d(-)}$. Furthermore, let $p\colon \mathsf{P} \to \mathsf{I}^{\triangleright}$ be the pullback of $d_0\colon \mathsf{C}^{\Delta^1} \to \mathsf{C}$ along $\overline{d}$, and let $q\colon \mathsf{Q} \to \mathsf{I}$ be the pullback of d_0 along d. According to [61, Proposition 7.1.2] and Lemma 5.1.2.6 (applied to the ∞-topos $\mathcal{B}_\Delta^+$ and the map $\iota^\sharp$), the canonical map $\lim \eta_{\overline{h}}\colon \lim \overline{h} \to \lim \iota_* h$ can be identified with the functor

$$(*)\qquad \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}((\mathsf{I}^\triangleright)^\sharp, \mathsf{P}^\natural)_{/(\mathsf{I}^\triangleright)^\sharp}|_\Delta \to \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{I}^\sharp, \mathsf{Q}^\natural)_{/\mathsf{I}^\sharp}|_\Delta$$

that is induced by precomposition with the inclusion $\iota\colon \mathsf{I} \hookrightarrow \mathsf{I}^\triangleright$. As $\mathsf{C}_{/-}$ preserving the limit of d is therefore equivalent to $(*)$ being an equivalence, we only need to identify this map with the functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$.

To see this, first note that there is an equivalence $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}((-)^\flat, -)|_\Delta \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(-, (-)|_\Delta)$. Therefore, we obtain a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}((\mathsf{I}^\triangleright)^\sharp, \mathsf{P}^\natural)_{/(\mathsf{I}^\triangleright)^\sharp}|_\Delta \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}((\mathsf{I}^\triangleright)^\sharp, \mathsf{P}^\natural)|_\Delta \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}((\mathsf{I}^\triangleright)^\sharp, (\mathsf{C}^{\Delta^1})^\natural)|_\Delta \arrow[d, hook] \\
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}} \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{P}) \arrow[r] \arrow[d] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C}^{\Delta^1}) \arrow[d] \\
1 \arrow[r, "\mathrm{id}_{\mathsf{I}^\triangleright}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{I}^\triangleright) \arrow[r, "\overline{d}_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})
\end{tikzcd}$$

in which the upper three vertical maps are induced by precomposition with the canonical map $(\mathsf{I}^\triangleright)^\flat \to (\mathsf{I}^\triangleright)^\sharp$. By Lemma 5.1.2.5, they are fully faithful. Furthermore, all but the upper right square are pullbacks. But since the map $(I^\triangleright)^\flat \to (I^\triangleright)^\sharp$ is internally right orthogonal to every map that is contained in the image of $(-)^\sharp\colon \mathcal{B}_\Delta \hookrightarrow \mathcal{B}_\Delta^+$, it must also be internally right orthogonal to $\mathsf{P}^\natural \to (\mathsf{C}^{\Delta^1})^\natural$ as the latter is the pullback of $\overline{d}^\sharp$. Therefore, the upper right square must also be a pullback. Note, furthermore, that since the map $(-)^\flat \to (-)^\sharp$ is an equivalence when restricted along the inclusion $\mathcal{B} \hookrightarrow \mathcal{B}_\Delta$, the upper right inclusion in the above diagram identifies the domain with the essential image of the map $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{I}^\triangleright, (\mathsf{C}^{\Delta^1})_\natural) \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{I}^\triangleright, \mathsf{C}^{\Delta^1})$. Therefore, Lemma 5.1.2.3 implies that there is an equivalence $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}((\mathsf{I}^\triangleright)^\sharp, \mathsf{P}^\natural)_{/(\mathsf{I}^\triangleright)^\sharp}|_\Delta \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$. By an analogous argument, one obtains an equivalence $\underline{\mathrm{Hom}}_{\mathcal{B}_\Delta^+}(\mathsf{I}^\sharp, \mathsf{Q}^\natural)_{/\mathsf{I}^\sharp}|_\Delta \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$, hence the claim follows. □

REMARK 5.1.2.7. In the situation of Proposition 5.1.2.4, let $\infty\colon 1 \to \mathsf{I}^\triangleleft$ be the cone point, and let

$$(\infty^* \dashv \infty_*)\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleleft, \mathsf{Cat}_{\mathcal{B}}) \leftrightarrows \mathsf{Cat}$$

be the induced adjunction. Let $\eta\colon \mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleleft, \mathsf{Cat}_{\mathcal{B}})} \to \infty_*\infty^*$ be the adjunction unit. By the same argument as in the proof of Proposition 5.1.2.4, evaluating $\lim_{\mathsf{I}^\triangleleft} \eta$ at the cone $\mathsf{C}_{/\overline{d}}$ recovers the restriction map

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \mathsf{C}_{/c}.$$

Since $\lim_{\mathsf{I}^\triangleleft}$ can be identified with ∞^* (owing to $\infty\colon 1 \to \mathsf{I}^\triangleleft$ being initial), the triangle identities imply that this map must be an equivalence. Furthermore, note that by Corollary 3.1.7.6 the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}} \to \mathsf{C}_{/c}$ admits a right adjoint that is is given by the composition

$$\mathsf{C}_{/c} \xrightarrow{\mathrm{diag}_{/c}} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\operatorname{diag}(c)} \xrightarrow{\eta^*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}}$$

(where $\eta\colon \overline{d} \to \operatorname{diag}(c)$ denotes the adjunction unit). Now if $c' \to c$ is a map in C, Corollary 3.1.7.5 implies that the counit $\eta_!\eta^* \operatorname{diag}(c') \to \operatorname{diag}(c')$ of the adjunction $\eta_! \dashv \eta^*$ is given by the pullback of η along $\operatorname{diag}(c') \to \operatorname{diag}(c)$. Since evaluation at ∞ preserves pullbacks and since $\mathrm{diag}_{/c}$ is fully faithful, we conclude that evaluating the counit of the adjunction $\mathsf{C}_{/c} \leftrightarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}}$ at $c' \to c$ must result in an equivalence. Upon replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ and repeating the same argument, we conclude that the entire counit must be an equivalence, so that the functor $\mathsf{C}_{/c} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}}$ is fully faithful. Now combining the evident observation that this map takes values in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$ with the fact that the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \mathsf{C}_{/c}$ is an equivalence, one concludes that the inclusion $\mathsf{C}_{/c} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}}$ identifies $\mathsf{C}_{/c}$ with $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$. In particular, the inclusion $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^\triangleright, \mathsf{C})_{/\overline{d}}$ admits a left adjoint.

REMARK 5.1.2.8. In the situation of Remark 5.1.2.7, let $\overline{d}' \to \overline{d}$ be a map in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})$, and let us set $c' = \infty^*(\overline{d}')$. Then the unit of the adjunction $\mathsf{C}_{/c} \leftrightarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}}$ evaluates at $\overline{d}'$ to the natural map $\overline{d}' \to \eta^* \operatorname{diag}(c')$. Therefore, the map $\overline{d}' \to \overline{d}$ is cartesian precisely if the square

$$\begin{tikzcd} \overline{d}' \arrow[r] \arrow[d] & \operatorname{diag}(c') \arrow[d] \\ \overline{d} \arrow[r] & \operatorname{diag}(c) \end{tikzcd}$$

is a pullback. As the functor $(\iota^*, \infty^*) \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \times \mathsf{C}$ is conservative (on account of the map $(\iota, \infty) \colon \mathsf{I} \sqcup 1 \twoheadrightarrow \mathsf{I}^{\triangleright}$ being essentially surjective) and as the image of the above square along ∞^* is always a pullback, the map $\overline{d}' \to \overline{d}$ is cartesian precisely if the square

$$\begin{tikzcd} d' \arrow[r] \arrow[d] & \operatorname{diag}(c') \arrow[d] \\ d \arrow[r] & \operatorname{diag}(c) \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})$ is a pullback.

Suppose that U is an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. Given any $\mathsf{I} \in \mathsf{U}(1)$ and any diagram $d \colon \mathsf{I} \to \mathsf{C}$ with colimit cocone $\overline{d} \colon \mathsf{I}^{\triangleright} \to \mathsf{C}$, Proposition 3.1.7.1 implies that the functor

$$\iota^*_{/\overline{d}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})_{/d}$$

has a left adjoint that is given by $(\iota_!)_{/d}$. Combining this observation with Remark 5.1.2.7, we thus end up with a left adjoint $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$ to the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ that we will refer to as the *glueing functor*. In light of Proposition 5.1.2.4, the functor $\mathsf{C}_{/-}$ preserves the limit of d precisely if both unit and counit of this adjunction are equivalences, i.e. if both the restriction functor and the glueing functor are fully faithful. We may therefore split up the notion of U-descent into two separate conditions:

DEFINITION 5.1.2.9. Let U be an internal class and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. We say that C has *faithful* U*-descent* if for every $A \in \mathcal{B}$, every $\mathsf{I} \in \mathsf{U}(A)$ and every diagram $d \colon \mathsf{I} \to \pi_A^* \mathsf{C}$ with colimit cocone $\overline{d} \colon \mathsf{I}^{\triangleright} \to \pi_A^* \mathsf{C}$, the restriction functor $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\triangleright}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/d}$ is fully faithful. We say that C has *effective* U*-descent* if for every $A \in \mathcal{B}$, every $\mathsf{I} \in \mathsf{U}(A)$ and every diagram $d \colon \mathsf{I} \to \pi_A^* \mathsf{C}$ with colimit cocone $\overline{d} \colon \mathsf{I}^{\triangleright} \to \pi_A^* \mathsf{C}$, the glueing functor $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/d} \to \underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\triangleright}, \pi_A^* \mathsf{C})^{\mathrm{cart}}_{/\overline{d}}$ is fully faithful. If C is a cocomplete large $\mathcal{B}$-category, we simply say that C has faithful/effective descent if it has faithful/effective $\mathsf{Cat}_{\mathcal{B}}$-descent.

REMARK 5.1.2.10. As a consequence of Remark 5.1.2.2, the property of C having faithful/effective U-descent is local in $\mathcal{B}$, in the sense that whenever $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, the $\mathcal{B}$-category C satisfies faithful/effective U-descent precisely if for each i the $\mathcal{B}_{/A_i}$-category $\pi_{A_i}^* \mathsf{C}$ has faithful/effective $\pi_{A_i}^* \mathsf{U}$-descent.

By unwinding how the unit and counit of the adjunction $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{C})^{\mathrm{cart}}_{/\overline{d}} \leftrightarrows \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C})^{\mathrm{cart}}_{/d}$ are computed (cf. Remark 3.1.7.4) and by using Remark 5.1.2.8, we may characterise the notion of faithful and effective U-descent as follows:

PROPOSITION 5.1.2.11. *Let U be an internal class and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. Then the following are equivalent:*

(1) C has faithful U-descent;

(2) for every $A \in \mathcal{B}$, every $\mathsf{I} \in \mathsf{U}(A)$ and every cartesian map $\overline{d}' \to \overline{d}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}^{\triangleright}, \pi_A^* \mathsf{C})$ in which $\overline{d}$ is a colimit cocone, $\overline{d}'$ is a colimit cocone as well;

(3) for every $A \in \mathcal{B}$, every $\mathsf{I} \in \mathsf{U}(A)$ and every pullback diagram

$$\begin{tikzcd} d' \arrow[r] \arrow[d] & \operatorname{diag}(c') \arrow[d, "\operatorname{diag}(g)"] \\ d \arrow[r, "\eta"] & \operatorname{diag}\operatorname{colim} d \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})$ *in which* η *is the unit of the adjunction* $\operatorname{colim} \dashv \operatorname{diag}$*, the transpose map* $\operatorname{colim} d' \to c'$ *is an equivalence.* □

Proposition 5.1.2.12. *Let* U *be an internal class and let* C *be a* U*-cocomplete* $\mathcal{B}$*-category with pullbacks. Then the following are equivalent:*

(1) C *has effective* U*-descent;*
(2) for every $A \in \mathcal{B}$*, every* $\mathsf{I} \in \mathsf{U}(A)$ *and every cartesian map* $d' \to d$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})$*, the induced map between colimit cocones* $\overline{d}' \to \overline{d}$ *is cartesian as well;*
(3) for every $A \in \mathcal{B}$*, every* $\mathsf{I} \in \mathsf{U}(A)$ *and every cartesian map* $d' \to d$ *in* $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \mathsf{C})$*, the naturality square*

$$\begin{tikzcd} d' \arrow[r, "\eta"] \arrow[d] & \operatorname{diag}(\operatorname{colim} d') \arrow[d] \\ d \arrow[r, "\eta"] & \operatorname{diag}\operatorname{colim} d \end{tikzcd}$$

is a pullback. □

Corollary 5.1.2.13. *Let* S *be a local class of maps in* $\mathcal{B}$ *and let* C *be an* Ω_S*-cocomplete* $\mathcal{B}$*-category with pullbacks. Then the following are equivalent:*

(1) C *has* Ω_S*-descent;*
(2) for every map $p\colon P \to A$ *in* S *the functor* $p_!\colon \mathsf{C}(P) \to \mathsf{C}(A)$ *is a right fibration;*
(3) for every map $p\colon P \to A$ *in* S *the functor* $(p_!)_{/1_{\mathsf{C}(P)}}\colon \mathsf{C}(P) \to \mathsf{C}(A)_{/p_!(1_{\mathsf{C}(P)})}$ *is an equivalence.*

Proof. Since $(p_!)_{/1_{\mathsf{C}(P)}}$ is always final, this functor is an equivalence if and only if it is a right fibration, which is in turn equivalent to $p_!$ being a right fibration. Hence (2) and (3) are equivalent conditions. Now in light of the adjunction $p_! \dashv p^*$, a map $f\colon c' \to c$ in $\mathsf{C}(P)$ is cartesian with respect to $p_!$ precisely if the naturality square

$$\begin{tikzcd} c' \arrow[r] \arrow[d, "f"] & p^* p_! c' \arrow[d, "p^* p_! f"] \\ c \arrow[r] & p^* p_! c \end{tikzcd}$$

is a pullback. Therefore, Proposition 5.1.2.12 and the fact that *every* map of diagrams indexed by a $\mathcal{B}$-groupoid is cartesian imply that C has effective Ω_S-descent if and only if for every map $p\colon P \to A$ in S, every morphism in $\mathsf{C}(P)$ is cartesian with respect to $p_!$. By the same observation, Proposition 5.1.2.11 shows that C has faithful Ω_S-descent if and only if for every map $p\colon P \to A$ in S, every object $c \in \mathsf{C}(P)$ and every morphism $g\colon c'' \to p_!(c)$ in $\mathsf{C}(A)$, the pullback of $p^*(g)$ along the adjunction unit $c \to p^* p_! c$ defines a cartesian lift of g. In other words, C has faithful Ω_S-descent if and only if $p_!$ is a cartesian fibration. Hence (1) and (2) are equivalent. □

For the next corollary, recall from § 4.2.1 that if $\mathcal{K}$ is a class of ∞-categories and S is a local class of morphisms in $\mathcal{B}$, we denote by $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$ the *left regularisation* of the internal class $\mathsf{LConst}_{\mathcal{K}} \cup \Omega_S$, i.e. the smallest internal class that contains $\Delta \cup \mathsf{LConst}_{\mathcal{K}} \cup \Omega_S$ and that is closed under $\mathsf{LConst}_{\mathrm{op}(\mathcal{K})} \cup \Omega_S$-colimits in $\mathsf{Cat}_{\mathcal{B}}$ (where $\mathrm{op}(\mathcal{K})$ is the image of $\mathcal{K}$ under the equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat}_\infty \simeq \mathrm{Cat}_\infty$). We now obtain:

Corollary 5.1.2.14. *Let* $\mathcal{K}$ *be a class of* ∞*-categories and let* S *be a local class of maps in* $\mathcal{B}$*. Let* C *be a* $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$*-cocomplete* $\mathcal{B}$*-category with pullbacks. Then* $\mathsf{Cat}_{\mathcal{B}}$ *satisfies* $\mathsf{Cat}_{\mathcal{B}}^{\langle \mathcal{K}, S\rangle}$*-descent if and only if*

(1) for all $A \in \mathcal{B}$ *the* ∞*-category* $\mathsf{C}(A)$ *satisfies* $\mathcal{K}$*-descent, and*
(2) for every map $p\colon P \to A$ *in* S *the functor* $p_!$ *is a right fibration.*

Proof. By Proposition 4.1.3.4, the $\mathcal{B}$-category C has $\mathsf{Cat}_{\mathcal{B}}^{\langle\mathcal{K},S\rangle}$-descent precisely if it satisfies both $\mathsf{LConst}_{\mathcal{K}}$- and Ω_S-descent. By Example 5.1.1.7 the first condition is equivalent to (1), and by Corollary 5.1.2.13 the second one is equivalent to (2). □

Example 5.1.2.15. Let S be a local class of morphisms in $\mathcal{B}$ that is closed under pullbacks in $\operatorname{Fun}(\Delta^1, \mathcal{B})$ and that is *left cancellable*, i.e. satisfies the condition that whenever there is a composable pair of morphisms f and g in $\mathcal{B}$ for which g is contained in S, then gf is contained in S if and only if f is. Then the associated subuniverse $\Omega_S \hookrightarrow \Omega$ is closed under pullbacks and under Ω_S-colimits, and for every map $s\colon B \to A$ in S the functor $s_!\colon \Omega_S(B) \to \Omega_S(A)$ is a right fibration. Hence Ω_S has Ω_S-descent. These conditions are for example satisfied if S is the right complement of a factorisation system.

5.1.3. Universality of colimits. The goal of this section is to establish that the notion of faithful U-descent is equivalent to *universality* of U-colimits:

Definition 5.1.3.1. Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. We say that U-colimits are *universal* in C if for every map $f\colon c \to d$ in C in context $A \in \mathcal{B}$ the functor $f^*\colon \mathsf{C}_{/d} \to \mathsf{C}_{/c}$ is $\pi_A^*\mathsf{U}$-cocontinuous. If C is a cocomplete large $\mathcal{B}$-category, we simply say that colimits are universal in C if $\mathsf{Cat}_{\mathcal{B}}$-colimits are universal in C.

Remark 5.1.3.2. In the situation of Definition 5.1.3.1, note that by Corollary 3.2.2.11 the $\mathcal{B}_{/A}$-category $\mathsf{C}_{/c} \simeq (\pi_A^*\mathsf{C})_{/\bar{c}}$ (where $\bar{c}\colon 1 \to \pi_A^*\mathsf{C}$ is the transpose of c) is $\pi_A^*\mathsf{U}$-cocomplete for every $c\colon A \to \mathsf{C}$. Therefore, asking for f^* to be $\pi_A^*\mathsf{U}$-cocontinuous makes sense.

Remark 5.1.3.3. The condition that U-colimits are universal in C is local in $\mathcal{B}$: if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then U-colimits are universal in C if and only if $\pi_{A_i}^*\mathsf{U}$-colimits are universal in $\pi_{A_i}^*\mathsf{C}$ for each i. This is easily seen using the fact that U-cocontinuity is a local condition by Remark 3.2.2.3.

Example 5.1.3.4. Let $\mathcal{K}$ be a class of ∞-categories and let C be an $\mathsf{LConst}_{\mathcal{K}}$-cocomplete $\mathcal{B}$-category with pullbacks. Then $\mathsf{LConst}_{\mathcal{K}}$-colimits are universal in C if and only if $\mathcal{K}$-colimits are universal in $\mathsf{C}(A)$ for all $A \in \mathcal{B}$. In fact, this follows immediately from the observation that for every map $f\colon c \to d$ in C in context $A \in \mathcal{B}$ and for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $f^*(B)$ can be identified with $(s^*f)^*\colon \mathsf{C}(B)_{/d} \to \mathsf{C}(B)_{/c}$.

Proposition 5.1.3.5. *Let U be an internal class of $\mathcal{B}$-categories and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks. Then U-colimits are universal in C if and only if C has faithful U-descent.*

Proof. Let I be an object in $\mathsf{U}(1)$, and let $f\colon c' \to c$ be an arbitrary map in C in context $1 \in \mathcal{B}$. Suppose that $d\colon \mathsf{I} \to \mathsf{C}_{/c}$ is a diagram with colimit cocone $\overline{d}\colon d \to \operatorname{diag}\operatorname{colim} d$, which we may equivalently regard as a diagram $\overline{d}\colon \mathsf{I} \to \mathsf{C}_{/\operatorname{colim} d}$. Let $g\colon \operatorname{colim} d \to c$ be the induced map. On account of the fact that the (vertical) mate of the commutative square

$$\begin{tikzcd}
\mathsf{C}_{/f^*(\operatorname{colim} d)} \arrow[r, "(f^*(g))_!"] \arrow[d, "(g^*f)_!"] & \mathsf{C}_{/c'} \arrow[d, "f_!"] \\
\mathsf{C}_{/\operatorname{colim} d} \arrow[r, "g_!"] & \mathsf{C}_{/c}
\end{tikzcd}$$

commutes and since the horizontal maps in this diagram are conservative and U-cocontinuous, the functor f^* preserves the colimit of d if and only if the functor $(g^*f)^*$ preserves the colimit of $\overline{d}\colon \mathsf{I} \to \mathsf{C}_{/\operatorname{colim} d}$. By (the proof of) Proposition 3.1.6.3, the colimit of the latter is the final object in $\mathsf{C}_{/\operatorname{colim} d}$. Therefore, in order to show that C has U-universal colimits, it suffices to consider those diagrams in $\mathsf{C}_{/c}$ whose colimit is the final object.

Now on account of the commutative square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})_{/\operatorname{diag}(c')} \arrow[r, "\simeq"] \arrow[d, "\operatorname{diag}(f)_!"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}_{/c'}) \arrow[d, "(f_!)_*"] \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C})_{/\operatorname{diag}(c)} \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I},\mathsf{C}_{/c}) \end{tikzcd}$$

we may identify $\operatorname{diag}(f)^*$ with $(f^*)_*$. Therefore, if $d \to \operatorname{diag}(c)$ is a colimit cocone, the upper horizontal equivalence in the above diagram identifies its pullback $d' \to \operatorname{diag}(c')$ along $\operatorname{diag}(f)$ with the composition $\mathsf{I} \xrightarrow{\overline{d}} \mathsf{C}_{/c} \xrightarrow{f^*} \mathsf{C}_{/c'}$. Thus, by again using Proposition 3.1.6.3, the map $d' \to \operatorname{diag}(c')$ is a colimit cocone if and only if the colimit of $f^*\overline{d}$ is the final object in $\mathsf{C}_{/c'}$, which is in turn equivalent to f^* preserving the colimit of $\overline{d}$. As replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$ allows us to arrive at the same conclusion for any $\mathsf{I} \in \mathsf{U}(A)$, Proposition 5.1.2.11 yields the claim. $\square$

Example 5.1.3.6. Let S be a local class of morphisms in $\mathcal{B}$ and let C be a Ω_S-cocomplete $\mathcal{B}$-category with pullbacks. Then Ω_S-colimits are universal in C if and only if for every map $p\colon P \to A$ in S the functor $p_!\colon \mathsf{C}(P) \to \mathsf{C}(A)$ is a cartesian fibration. In fact, in light of Proposition 5.1.3.5 this follows from the argument in the proof of Corollary 5.1.2.13.

We end this section by relating universality of colimits with the property of being *locally cartesian closed*:

Proposition 5.1.3.7. *Let X be a presentable $\mathcal{B}$-category. Then X has Ω-universal colimits if and only if for every object $x\colon A \to \mathsf{X}$, the $\mathcal{B}_{/A}$-category $(\pi_A^*\mathsf{X})_{/x}$ is cartesian closed, which is to say that there exists a bifunctor $\underline{\mathrm{Hom}}_{\pi_A^*\mathsf{X}}(-,-)\colon (\pi_A^*\mathsf{X})_{/x}^{\mathrm{op}} \times (\pi_A^*\mathsf{X})_{/x} \to (\pi_A^*\mathsf{X})_{/x}$ that fits into an equivalence*

$$\operatorname{map}_{(\pi_A^*\mathsf{X})_{/x}}(- \times -, -) \simeq \operatorname{map}_{(\pi_A^*\mathsf{X})_{/x}}(-, \underline{\mathrm{Hom}}_{\pi_A^*\mathsf{X}}(-,-)).$$

Proof. It will be enough to show that X is cartesian closed if and only if for every $y\colon A \to \mathsf{X}$ the functor $(\pi_y)^*\colon \pi_A^*\mathsf{X} \to (\pi_A^*\mathsf{X})_{/y}$ is $\Omega_{\mathcal{B}_{/A}}$-cocontinuous. Using Remark 5.1.3.3, we can assume that $A \simeq 1$. Recall that the forgetful functor $(\pi_y)_!\colon \mathsf{X}_{/y} \to \mathsf{X}$ is Ω-cocontinuous (Corollary 3.2.2.11). As this functor is moreover a right fibration and therefore in particular conservative, we find that π_y^* is Ω-cocontinuous if and only if the composition $(\pi_y)_!\pi_y^*$ is. Together with Proposition 3.1.7.3, this shows that π_y^* being Ω-cocontinuous is equivalent to $y \times -$ being Ω-cocontinuous. As X is presentable, this is in turn equivalent to $y \times -$ having a right adjoint $\underline{\mathrm{Hom}}_{\mathsf{X}}(y,-)$ (see Proposition 4.4.3.1). Clearly, this holds if X is cartesian closed. Conversely, if $y \times -$ admits a right adjoint for all $y\colon A \to \mathsf{X}$, then $\operatorname{map}_{\mathsf{X}}(- \times -, -)$, viewed as a functor $\mathsf{X}^{\mathrm{op}} \times \mathsf{X} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{X})$, takes values in $\mathsf{X} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{X})$ and therefore gives rise to the desired internal hom. $\square$

5.1.4. Disjoint colimits. If $\mathcal{C}$ is an ∞-category with pullbacks and finite coproducts, one says that a coproduct $c_0 \sqcup c_1$ in $\mathcal{C}$ is *disjoint* if the fibre product $c_i \times_{c_0 \sqcup c_1} c_j$ is equivalent to c_i if $i = j$ and the initial object otherwise. In this section, our goal is to study an internal analogue of this concept. In fact, we will define what it means for arbitrary $\mathcal{B}$-*groupoidal* colimits to be disjoint. To that end, recall that if S is an arbitrary local class of morphisms in $\mathcal{B}$, the associated subuniverse Ω_S is contained in the free Ω_S-cocompletion $\underline{\mathsf{PSh}}_{\mathcal{B}}^{\Omega_S}(1)$, see Example 3.4.3.5. Therefore, if C is an arbitrary Ω_S-cocomplete $\mathcal{B}$-category, the tensoring bifunctor

$$- \otimes -\colon \Omega_S \times \mathsf{C} \to \mathsf{C}$$

(see Proposition 3.4.3.8) is well-defined. Furthermore, note that S is closed under diagonals if and only if for every $\mathsf{G} \in \Omega_S(A)$ and every pair of objects $g, g'\colon A \rightrightarrows \mathsf{G}$ the mapping $\mathcal{B}_{/A}$-groupoid $\operatorname{map}_{\mathsf{G}}(g, g')$ is contained in $\Omega_S(A)$ as well. Therefore, we may define:

Definition 5.1.4.1. Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Ω_S-cocomplete $\mathcal{B}$-category with pullbacks. If $\mathsf{G} \in \Omega_S(1)$ is an arbitrary object, we say that

G-*indexed colimits are disjoint in* C if for all diagrams $d\colon \mathsf{G} \to \mathsf{C}$ and for every pair of objects g, g' in G in context $1 \in \mathcal{B}$ the diagram

$$\begin{tikzcd} \operatorname{map}_{\mathsf{G}}(g, g') \otimes d(g) \arrow[r] \arrow[d] & d(g') \arrow[d] \\ d(g) \arrow[r] & \operatorname{colim} d \end{tikzcd}$$

is a pullback. We say that Ω_S-*colimits are disjoint in* C if for all $A \in \mathcal{B}$ and all $\mathsf{G} \in \mathsf{U}(A)$ all G-indexed colimits are disjoint in $\pi_A^* \mathsf{C}$.

REMARK 5.1.4.2. In the situation of Definition 5.1.4.1, let $\overline{d}\colon \mathsf{G}^{\triangleright} \to \mathsf{C}$ be the colimit cocone associated with d. Then the commutative square in the definition is obtained by transposing the commutative diagram

$$\begin{tikzcd} \operatorname{map}_{\mathsf{G}^{\triangleright}}(g, g') \arrow[r, "\overline{d}"] \arrow[d] & \operatorname{map}_{\mathsf{C}}(d(g), d(g')) \arrow[d] \\ \operatorname{map}_{\mathsf{G}^{\triangleright}}(g, \infty) \arrow[r, "\overline{d}"] & \operatorname{map}_{\mathsf{C}}(d(g), \operatorname{colim} d) \end{tikzcd}$$

across the equivalence $\operatorname{map}_{\mathsf{C}}(- \otimes -, -) \simeq \operatorname{map}_{\Omega}(-, \operatorname{map}_{\mathsf{C}}(-, -))$, noting that since $\iota\colon \mathsf{G} \hookrightarrow \mathsf{G}^{\triangleright}$ is fully faithful the upper left corner can be identified with $\operatorname{map}_{\mathsf{G}}(g, g')$ and since $\infty\colon 1 \to \mathsf{G}^{\triangleright}$ is final the lower left corner is equivalent to the final object.

EXAMPLE 5.1.4.3. Let us unwind Definition 5.1.4.1 in the case where $\mathcal{B} = \mathcal{S}$ and where $\mathsf{G} = \{0, 1\}$. Then a diagram $d\colon \{0, 1\} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is simply given by a pair (c_0, c_1) of objects in $\mathcal{C}$, and its colimit is the coproduct $c_0 \sqcup c_1$. Furthermore, the square in Definition 5.1.4.1 is explicitly given by

$$\begin{tikzcd} \operatorname{map}_{\{0,1\}}(i, j) \times c_i \arrow[r] \arrow[d] & c_j \arrow[d] \\ c_i \arrow[r] & c_0 \sqcup c_1 \end{tikzcd}$$

(for $i, j \in \{0, 1\}$) and is therefore a pullback for all pairs (i, j) precisely if coproducts are disjoint in $\mathcal{C}$ in the usual sense.

REMARK 5.1.4.4. The property of Ω_S-colimits to be disjoint in C is a local condition. More precisely, if $\mathsf{G} \in \mathsf{U}(1)$ is an arbitrary object and if $\bigsqcup_i A_i \twoheadrightarrow 1$ is a cover in $\mathcal{B}$, then G-indexed colimits are disjoint in C if and only if $\pi_{A_i}^* \mathsf{G}$-indexed colimits are disjoint in $\pi_{A_i}^* \mathsf{C}$ for all i. This follows immediately from the fact that both limits and colimits are determined locally by Remark 3.1.1.8. As a consequence, if Ω_S is generated by a family of objects $(\mathsf{G}_i\colon A_i \to \Omega_S)_i$, then Ω_S-colimits are disjoint in C precisely if G_i-indexed colimits are disjoint in $\pi_{A_i}^* \mathsf{C}$ for all i.

EXAMPLE 5.1.4.5. Let S be the local class of morphisms in $\mathcal{B}$ that is generated by $\varnothing, 1$ and $2 = 1 \sqcup 1$. Then S is closed under diagonals. By using Remark 5.1.4.4 and Example 5.1.4.3, one finds that Ω_S-colimits are disjoint in C if and only if coproducts are disjoint in $\mathsf{C}(A)$ for all $A \in \mathcal{B}$.

EXAMPLE 5.1.4.6. Let G be a finite group. Then BG colimits are disjoint in some ∞-category $\mathcal{C}$ if for any object $X \in \operatorname{Fun}(BG, \mathcal{C})$ the canonical square

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{mult}"] & X \arrow[d] \\ X \arrow[r] & X_{hG} \end{tikzcd}$$

in $\mathcal{C}$ is a pullback.

The main goal of this section is to show:

Proposition 5.1.4.7. *Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Ω_S-cocomplete $\mathcal{B}$-category with pullbacks in which Ω_S-colimits are universal. Then C has effective Ω_S-descent if and only if Ω_S-colimits are disjoint.*

In order to prove Proposition 5.1.4.7, we will need a more explicit description of the notion of disjoint Ω_S-colimits. The key input is the following construction:

Construction 5.1.4.8. Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Ω_S-cocomplete $\mathcal{B}$-category with pullbacks. Suppose that $p\colon P \to A$ is a map in S, and let $c\colon P \to \mathsf{C}$ be an arbitrary object. Let $\eta\colon c \to p^* p_! c$ be the adjunction unit, and consider the pullback square

$$\begin{tikzcd} z \arrow[r] \arrow[d] & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\eta)"] \\ \mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^*(\eta)"] & \mathrm{pr}_0^* p^* p_!(c) \end{tikzcd}$$

in $\mathsf{C}(P \times_A P)$ (where we implicitly identify $\mathrm{pr}_0^* p^* \simeq \mathrm{pr}_1^* p^*$). Note that if $\Delta_p\colon P \to P \times_A P$ is the diagonal map, the pullback of the above square along Δ_p yields the pullback square

$$\begin{tikzcd} c \times_{p^* p_!(c)} c \arrow[r] \arrow[d] & c \arrow[d] \\ c \arrow[r] & p^* p_!(c) \end{tikzcd}$$

in $\mathsf{C}(P)$. Therefore, the diagonal map $c \to c \times_{p^* p_!(c)} c$ transposes to a map $\delta_p(c)\colon (\Delta_p)_!(c) \to z$.

Proposition 5.1.4.9. *Let S be a local class of morphisms in $\mathcal{B}$ that is closed under diagonals, and let C be an Ω_S-cocomplete $\mathcal{B}$-category with pullbacks. Then Ω_S-colimits are disjoint in C if and only if for all maps $p\colon P \to A$ and all objects $c\colon P \to \mathsf{C}$ the map $\delta_p(c)$ from Construction 5.1.4.8 is an equivalence.*

Proof. By identifying $p\colon P \to A$ with a $\mathcal{B}_{/A}$-groupoid G, the object $c\colon P \to \mathsf{C}$ corresponds to a diagram $d\colon \mathsf{G} \to \pi_A^* \mathsf{C}$. Also, the two maps $\mathrm{pr}_0, \mathrm{pr}_1\colon P \times_A P \rightrightarrows P$ correspond to objects g and g' in G in context $P \times_A P$. In light of these identifications, the two cospans

$$\begin{tikzcd} & \mathrm{pr}_1^*(c) \arrow[d, "\mathrm{pr}_1^*(\eta_P)"] \\ \mathrm{pr}_0^*(c) \arrow[r, "\mathrm{pr}_0^*(\eta_P)"] & \mathrm{pr}_0^* p^* p_!(c) \end{tikzcd} \qquad \begin{tikzcd} & d(g') \arrow[d] \\ d(g) \arrow[r] & \mathrm{pr}_0^* p^*(\operatorname{colim} d) \end{tikzcd}$$

(in context $P \times_A P$) are translated into each other. Next, we note that pulling back g and g' along the diagonal $\Delta\colon P \to P \times_A P$ recovers the tautological object τ of G (i.e. the one corresponding to id_P). As there is a section $\mathrm{id}_\tau\colon P \to \mathrm{map}_{\mathsf{G}}(\tau, \tau)$, we thus obtain a commutative diagram

$$\begin{tikzcd} d(\tau) \arrow[dr] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & \mathrm{map}_{\mathsf{G}}(\tau,\tau) \otimes d(\tau) \arrow[r] \arrow[d] & d(\tau) \arrow[d] \\ & d(\tau) \arrow[r] & p^*(\operatorname{colim} d) \end{tikzcd}$$

in context P. Observe that value of the unit of the adjunction $(\Delta_! \dashv \Delta^*)\colon \mathcal{B}_{/P} \leftrightarrows \mathcal{B}_{/P \times_A P}$ at the final object precisely recovers the map $\mathrm{id}_\tau\colon P \to \mathrm{map}_{\mathsf{G}}(\tau, \tau)$. As the functor

$$- \otimes d(\tau)\colon \pi_P^* \Omega \to \pi_P^* \mathsf{C}$$

is by construction $\pi_P^* \mathsf{U}$-cocontinuous, one thus finds that the map $d(\tau) \to \mathrm{map}_{\mathsf{G}}(\tau, \tau) \otimes d(\tau)$ can be identified with the unit of the adjunction $(\Delta_! \dashv \Delta^*)\colon \mathsf{C}(P) \leftrightarrows \mathsf{C}(P \times_A P)$. But this precisely means that the transpose map $\Delta_! d(\tau) \to \operatorname{map}_{\mathsf{G}}(g, g') \otimes d(g)$. must be an equivalence. As $d(\tau)$ is simply c, we therefore find that the two diagrams

$$\begin{tikzcd} \Delta_!(c) \arrow[r] \arrow[d] & \operatorname{pr}_1^*(c) \arrow[d, "\operatorname{pr}_1^*(\eta)"] \\ \operatorname{pr}_0^*(c) \arrow[r, "\operatorname{pr}_0^*(\eta)"] & \operatorname{pr}_0^* p^* p_!(c) \end{tikzcd} \qquad \begin{tikzcd} \operatorname{map}_{\mathsf{G}}(g, g') \otimes d(g) \arrow[r] \arrow[d] & d(g') \arrow[d] \\ d(g) \arrow[r] & \operatorname{pr}_0^* p^*(\operatorname{colim} d) \end{tikzcd}$$

are equivalent. To complete the proof, we still need to show that the right square being cartesian is equivalent to Ω_S-colimits being disjoint in C. Certainly, this is a necessary condition since this square is precisely of the form as in Definition 5.1.4.1 (after identifying $\operatorname{pr}_0^* p^*(\operatorname{colim} d)$ with $\operatorname{colim} \operatorname{pr}_0^* p^* d$ and regarding g and g' as objects of $\operatorname{pr}_0^* p^* \mathsf{G}$ in context $1 \in \mathcal{B}_{/P \times_A P}$). The converse follows from the observation that *every* pair of objects $h, h' \colon A \rightrightarrows \mathsf{G}$ must be a pullback of g and g', i.e. that the above diagram is the *universal* one. □

Proof of Proposition 5.1.4.7. We will freely make use of the setup from Proposition 5.1.4.9. Therefore, let us fix a map $p \colon P \to A$ in S, and let us denote the unit and counit of the associated adjunction $p_! \dashv p^*$ by η_p and ϵ_p, respectively. We first assume that C has effective Ω_S-descent. Choose an arbitrary object $c \colon P \to \mathsf{C}$ and consider the pullback

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d] & \operatorname{pr}_1^*(c) \arrow[d, "\operatorname{pr}_1^*(\eta_p)"] \\ \operatorname{pr}_0^*(c) \arrow[r, "\operatorname{pr}_0^*(\eta_p)"] & \operatorname{pr}_0^* p^* p_!(c) \end{tikzcd}$$

in $\mathsf{C}(P \times_A P)$. By making use of the commutative diagram

$$\begin{tikzcd} \operatorname{pr}_0^*(c) \arrow[r, "\operatorname{pr}_0^* \eta_p c"] \arrow[dr, "\eta_{\operatorname{pr}_1} \operatorname{pr}_0^*(c)"'] & \operatorname{pr}_1^* p^* p_!(c) \arrow[d, "\operatorname{pr}_1^*(\alpha)"] \\ & \operatorname{pr}_1^*(\operatorname{pr}_1)_! \operatorname{pr}_0^*(c), \end{tikzcd}$$

(where α is an equivalence owing to C having Ω_S-colimits), we may identify the above square with the pullback square

$$\begin{tikzcd} z \arrow[r, "g"] \arrow[d] & \operatorname{pr}_1^*(c) \arrow[d, "\operatorname{pr}_1^*(\alpha\eta_p)"] \\ \operatorname{pr}_0^*(c) \arrow[r, "\eta_{\operatorname{pr}_1} \operatorname{pr}_0^*(c)"] & \operatorname{pr}_1^*(\operatorname{pr}_1)_! \operatorname{pr}_0^*(c). \end{tikzcd}$$

Since by assumption Ω_S-colimits are universal in C, Proposition 5.1.3.5 implies that C has faithful Ω_S-descent. Hence Proposition 5.1.2.11 implies that the transpose $(\operatorname{pr}_1)_!(z) \to c$ of g must be an equivalence. Together with the commutative diagram

$$\begin{tikzcd} (\operatorname{pr}_1)_!\Delta_!(c) \arrow[r] \arrow[d, "\simeq"] \arrow[rr, bend left, "(\operatorname{pr}_1)_!(\delta_p(c))"] & (\operatorname{pr}_1)_!\Delta_!\Delta^*(z) \arrow[r, "(\operatorname{pr}_1)_!\epsilon_\Delta z"] \arrow[d, "\simeq"] & (\operatorname{pr}_1)_!(z) \arrow[d] \\ c \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \Delta^*(z) \arrow[r, "\Delta^*(g)"] & c, \end{tikzcd}$$

this observation implies that $(\operatorname{pr}_1)_!(\delta_p(c))$ is an equivalence. But since C has effective and faithful Ω_S-descent, Proposition 5.1.2.13 implies that $(\operatorname{pr}_1)_!$ is a right fibration and therefore in particular conservative. Hence $\delta_p(c)$ is already an equivalence.

Conversely, suppose that Ω_S-colimits in C are disjoint and let $f\colon c \to d$ be an arbitrary map in C in context P. Consider the diagram

$$\begin{tikzcd}
c \arrow[dr, "\varphi"] \arrow[drr, bend left, "\eta_P c"] \arrow[ddr, bend right, "f"'] & & \\
& e \arrow[r] \arrow[d] & p^*p_!(c) \arrow[d, "p^*p_!(f)"] \\
& d \arrow[r, "\eta d"] & p^*p_!(d)
\end{tikzcd}$$

in $\mathsf{C}(P)$ in which the square is a pullback. By Proposition 5.1.2.12, the result follows once we show that φ is an equivalence. We now obtain a pullback diagram

$$(*)\qquad\begin{tikzcd}
& x \arrow[rr] \arrow[dl] \arrow[dd] & & \mathrm{pr}_1^*(c) \arrow[dl] \arrow[dd] \\
\mathrm{pr}_0^*(e) \arrow[rr, crossing over] \arrow[dd] & & \mathrm{pr}_0^* p^*p_!(c) & \\
& y \arrow[rr] \arrow[dl] & & \mathrm{pr}_1^*(d) \arrow[dl] \\
\mathrm{pr}_0^*(d) \arrow[rr] & & \mathrm{pr}_0^* p^*p_!(d) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in $\mathrm{Fun}(\Delta^1, \mathsf{C}(P \times_A P))$ in which the front square is obtained by applying pr_0^* to the pullback square in the previous diagram and the right square is given by applying pr_1^* to the outer square in the previous diagram. Note that by applying the functor Δ^* to this cube, we obtain a commutative diagram

$$\begin{tikzcd}
& c \arrow[dr] \arrow[ddl, bend right, "\varphi"'] \arrow[ddd] \arrow[drr, bend left, "\mathrm{id}"] & & \\
& & \Delta^*(x) \arrow[r] \arrow[dll] \arrow[ddd] & c \arrow[dl] \arrow[ddd] \\
e \arrow[rr, crossing over] \arrow[ddd] & & p^*p_!(c) & \\
& d \arrow[dr] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddl, bend right, "\mathrm{id}"'] & & \\
& & \Delta^*(y) \arrow[r] \arrow[dll] & d \arrow[dl] \\
d \arrow[rr] & & p^*p_!(d) \arrow[from=uuu, crossing over] &
\end{tikzcd}$$

in which the cube defines a pullback in $\mathrm{Fun}(\Delta^1, \mathsf{C}(P))$. Now by disjointness of Ω_S-colimits, the map $\Delta_!(d) \to y$ must be an equivalence. Since this map fits into a commutative diagram

$$\begin{tikzcd}
d \arrow[r, "\eta' d"] \arrow[dr] & \Delta^*\Delta_!(c) \arrow[d, "\simeq"] \\
& \Delta^*(y)
\end{tikzcd}$$

(in which η' denotes the unit of the adjunction $\Delta_! \dashv \Delta^*$) and since we have a pullback square

$$\begin{tikzcd}
c \arrow[r] \arrow[d] & \Delta^*(x) \arrow[d] \\
d \arrow[r] & \Delta^*(y),
\end{tikzcd}$$

the assumption that Ω_S-colimits are universal in C and Proposition 5.1.2.11 imply that the transpose map $\Delta_!(c) \to x$ is an equivalence as well. By the argument in the beginning of the proof, applied to the top square in $(*)$, the commutative diagram

$$\begin{tikzcd} \mathrm{pr}_1^*(c) \arrow[r, "\mathrm{pr}_1^*\,\eta_p c"] \arrow[dr, "\eta_{\mathrm{pr}_0}\,\mathrm{pr}_1^*(c)"'] & \mathrm{pr}_0^*\,p^*p_!(c) \arrow[d, "\simeq"] \\ & \mathrm{pr}_0^*(\mathrm{pr}_0)_!\,\mathrm{pr}_1^*(c) \end{tikzcd}$$

implies that the map $(\mathrm{pr}_0)_!(x) \to e$ is an equivalence too. Taken together, we thus conclude that the composition

$$c \xrightarrow{\simeq} (\mathrm{pr}_0)_!\Delta_!(c) \to (\mathrm{pr}_0)_!(x) \to e$$

is an equivalence. By its very construction, this map can be identified with φ, hence the claim follows. $\square$

5.1.5. Effective groupoid objects. In this section we briefly review the notion of *groupoid objects* and their relation to descent (as discussed in [57, § 6.1]) in the context of $\mathcal{B}$-category theory.

DEFINITION 5.1.5.1. Let C be a $\mathcal{B}$-category with pullbacks. A *groupoid object* in C is a functor $G_\bullet\colon \Delta^{\mathrm{op}} \to \mathsf{C}$ such that for all $n \geq 0$ and every decomposition $\langle n\rangle \simeq \langle k\rangle \sqcup_{\langle 0\rangle} \langle l\rangle$ the map $G_n \to G_k \times_{G_0} G_l$ is an equivalence. We denote by $\mathsf{Seg}^{\simeq}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C})$ spanned by the groupoid objects in $\pi_A^*\mathsf{C}$ for every $A \in \mathcal{B}$.

DEFINITION 5.1.5.2. Let C be a $\mathcal{B}$-category that admits Δ^{op}-indexed colimits and pullbacks. We say that a groupoid object $G_\bullet$ in C is *effective* if the map $G_1 \to G_0 \times_{\operatorname{colim} G_\bullet} G_0$ is an equivalence in C. We denote by $\mathsf{Seg}^{\simeq}_{\mathrm{eff}}(\mathsf{C})$ the full subcategory of $\mathsf{Seg}^{\simeq}(\mathsf{C})$ that is spanned by the effective groupoid objects in $\pi_A^*\mathsf{C}$ for every $A \in \mathcal{B}$. We say that *groupoid objects are effective* in C if the inclusion $\mathsf{Seg}^{\simeq}_{\mathrm{eff}}(\mathsf{C}) \hookrightarrow \mathsf{Seg}^{\simeq}(\mathsf{C})$ is an equivalence.

REMARK 5.1.5.3. Since the property of a map being an equivalence is local in $\mathcal{B}$, it follows immediately from the definition that an object $A \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C})$ is contained in $\mathsf{Seg}^{\simeq}(\mathsf{C})$ if and only if it encodes a groupoid object in $\pi_A^*\mathsf{C}$, which is in turn equivalent to its transpose $\Delta^{\mathrm{op}} \to \mathsf{C}(A)$ being a groupoid object in the conventional sense. An analogous remark can be made for effective groupoid objects. In particular, groupoid objects are effective in C if and only if they are effective in $\mathsf{C}(A)$ for each $A \in \mathcal{B}$.

Let Pos be the 1-category of posets, which we always identify with 0-categories. Observe that the functor $(-)^{\triangleright}\colon$ Pos $\to$ Pos that freely adjoins a final object to a partially ordered set restricts to a functor $(-)^{\triangleright}\colon \Delta^{\triangleleft} \to \Delta$, and the map $\mathrm{id}_{\mathrm{Pos}} \hookrightarrow (-)^{\triangleright}$ restricts to a map $\mathrm{id}_\Delta \to (-)^{\triangleright}\iota$, where $\iota\colon \Delta \hookrightarrow \Delta^{\triangleleft}$ is the inclusion. By precomposition, we thus obtain a functor $(-)_{+1}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}((\Delta^{\mathrm{op}})^{\triangleright}, \mathsf{C})$ together with a morphism $\iota^*(-)_{+1} \to \mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^{\mathrm{op}}, \mathsf{C})}$. Now using Remark 5.1.5.3, one finds:

PROPOSITION 5.1.5.4 ([57, Lemma 6.1.3.7 and Remark 6.1.3.18]). *Let C be a $\mathcal{B}$-category that admits Δ^{op}-indexed colimits and pullbacks, and let $G_\bullet\colon \Delta^{\mathrm{op}} \to \mathsf{C}$ be a simplicial object. Then $G_{\bullet+1}$ is a colimit cocone, and $G_\bullet$ is a groupoid object if and only if the morphism of functors $\iota^* G_{\bullet+1} \to G_\bullet$ is cartesian.* $\square$

By combining Proposition 5.1.5.4 with Proposition 5.1.2.12, we conclude:

COROLLARY 5.1.5.5. *Let U be the internal class that is spanned by $\Delta^{\mathrm{op}}\colon 1 \to \mathsf{Cat}_{\mathcal{B}}$ and let C be a U-cocomplete $\mathcal{B}$-category with pullbacks that has effective U-descent. Then groupoid objects are effective in C.* $\square$

5.2. Foundations of $\mathcal{B}$-topos theory

In this section we develop the basic theory of $\mathcal{B}$-topoi. We begin in § 5.2.1 by giving an axiomatic definition of this concept using the notion of descent that has been established in the previous section. By unwinding the descent condition, we furthermore establish an explicit characterisation of $\mathcal{B}$-topoi in terms of the underlying $\widehat{\mathrm{Cat}}_\infty$-valued sheaves on $\mathcal{B}$. In § 5.2.2, we construct the *free* $\mathcal{B}$-topos on an arbitrary $\mathcal{B}$-category, which we use in § 5.2.3 to establish a characterisation of $\mathcal{B}$-topoi as left exact and accessible

Bousfield localisations of presheaf $\mathcal{B}$-categories. In § 5.2.4, we make use of this characterisation to show that the $\mathcal{B}$-category of $\mathcal{B}$-topoi is tensored and powered over $\mathsf{Cat}_{\mathcal{B}}$. In § 5.2.5, we prove that $\mathcal{B}$-topoi are entirely determined by their global sections, in the sense that the ∞-category of $\mathcal{B}$-topoi is equivalent to that of geometric morphisms of ∞-topoi with codomain $\mathcal{B}$. Having this simple description of $\mathcal{B}$-topoi at our disposal, it is straightforward to construct limits and colimits of $\mathcal{B}$-topoi, which is the topic of § 5.2.6. Also, we provide an explicit formula for the coproduct of $\mathcal{B}$-topoi in § 5.2.7, which in particular yields a formula for the pushout in $\mathrm{Top}^{\mathrm{L}}_{\infty}$. In § 5.2.8, we discuss a $\mathcal{B}$-categorical version of Diaconescu's theorem for $\mathcal{B}$-topoi, from which we deduce a universal property of étale $\mathcal{B}$-topoi in § 5.2.9. Lastly, we discuss *subterminal* $\mathcal{B}$-topoi in § 5.2.10, where we derive a general formula for left exact localisations in terms of internal colimits.

5.2.1. Definition and characterisation of $\mathcal{B}$-topoi. In this section we introduce the notion of a $\mathcal{B}$-*topos* and prove several equivalent characterisations of this concept.

Recall from Proposition 4.2.3.5 that a $\mathcal{B}$-category C admits finite limits if and only if for all $A \in \mathcal{B}$ the ∞-category $\mathsf{C}(A)$ admits finite limits and for each map $s\colon B \to A$ in $\mathcal{B}$ the functor $s^*\colon \mathsf{C}(A) \to \mathsf{C}(B)$ preserves finite limits. Similarly, a functor $f\colon \mathsf{C} \to \mathsf{D}$ between such $\mathcal{B}$-categories preserves finite limits precisely if it does so section-wise. We may now define:

Definition 5.2.1.1. A large $\mathcal{B}$-category X is a $\mathcal{B}$-*topos* if it is presentable and satisfies descent. A functor $f^*\colon \mathsf{X} \to \mathsf{Y}$ between $\mathcal{B}$-topoi is called an *algebraic morphism* if f is cocontinuous and preserves finite limits. A functor $f_*\colon \mathsf{Y} \to \mathsf{X}$ between $\mathcal{B}$-topoi is called a *geometric morphism* if f_* admits a left adjoint f^* that defines an algebraic morphism.

The large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ of $\mathcal{B}$-topoi is defined as the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is spanned by the algebraic morphisms between $\mathcal{B}_{/A}$-topoi, for all $A \in \mathcal{B}$. Dually, the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ of $\mathcal{B}$-topoi is defined as the subcategory of $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is spanned by the geometric morphisms between $\mathcal{B}_{/A}$-topoi, for all $A \in \mathcal{B}$. We denote by $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ and $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$, respectively, the underlying ∞-categories of global sections.

If X and Y are $\mathcal{B}$-topoi, we will denote by $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y})$ the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y})$ that is spanned by the algebraic morphisms $\pi_A^*\mathsf{X} \to \pi_A^*\mathsf{Y}$ for each $A \in \mathcal{B}$. We define the $\mathcal{B}$-category $\underline{\mathsf{Fun}}^{\mathrm{geom}}_{\mathcal{B}}(\mathsf{Y},\mathsf{X})$ of geometric morphisms in the evident dual way.

Remark 5.2.1.2. The fact that both $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ and $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ are large and not very large follows from Remark 4.4.4.3.

Remark 5.2.1.3. The subobject of $(\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ that is spanned by the algebraic morphisms between $\mathcal{B}_{/A}$-topoi (for each $A \in \mathcal{B}$) is stable under composition and equivalences in the sense of Proposition 2.2.2.9. Since moreover cocontinuity and the property that a functor preserves finite limits are local conditions by Remark 3.2.2.3 and on account of Remark 5.2.1.8 below, we conclude that a map $A \to (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ is contained in $(\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})_1$ if and only if it defines an algebraic morphism between $\mathcal{B}_{/A}$-topoi. In particular, if X and Y are $\mathcal{B}_{/A}$-topoi, the image of the monomorphism

$$(*) \qquad \operatorname{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{X},\mathsf{Y}) \hookrightarrow \operatorname{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{X},\mathsf{Y})$$

is spanned by the algebraic morphisms. Moreover, the sheaf associated to $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is given by sending $A \in \mathcal{B}$ to the subcategory $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \hookrightarrow \mathrm{Cat}(\widehat{\mathcal{B}}_{/A})$, and there is consequently a canonical equivalence $\pi_A^*\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}_{/A}}$. Analogous observations can be made for the $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$.

By the same argument, we have a canonical equivalence $\pi_A^*\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y}) \simeq \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{X},\pi_A^*\mathsf{Y})$ for all $\mathcal{B}$-topoi X and Y and all $A \in \mathcal{B}$. Furthermore, by using Corollary 3.1.5.4, we deduce that the inclusion in $(*)$ is obtained by applying the core $\mathcal{B}$-groupoid functor to the inclusion of $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y})$ into $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y})$. Again, analogous observations can be made for geometric morphisms.

By restricting the equivalence $\mathsf{Cat}^{\mathrm{R}}_{\widehat{\mathcal{B}}} \simeq (\mathsf{Cat}^{\mathrm{L}}_{\widehat{\mathcal{B}}})^{\mathrm{op}}$ from [61, Proposition 7.2.1], one finds:

PROPOSITION 5.2.1.4. *There is an equivalence* $(\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \simeq \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ *that acts as the identity on objects and that carries an algebraic morphism to its right adjoint.* □

Let us denote by $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty}$ the subcategory of $\mathrm{Top}^{\mathrm{L}}_{\infty}$ that is spanned by the étale algebraic morphisms (i.e. those that are of the form $\pi_U^* \colon \mathcal{X} \to \mathcal{X}_{/U}$ for some ∞-topos $\mathcal{X}$ and some $U \in \mathcal{X}$). By [57, Theorem 6.3.5.13], this ∞-category admits small limits, and the inclusion $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty} \hookrightarrow \mathrm{Top}^{\mathrm{L}}_{\infty}$ preserves small limits. The main goal of this section is to prove the following characterisation of $\mathcal{B}$-topoi:

THEOREM 5.2.1.5. *For a large $\mathcal{B}$-category X, the following are equivalent:*

(1) X *is a $\mathcal{B}$-topos;*
(2) X *satisfies the* internal Giraud axioms*:*
  (a) X *is presentable;*
  (b) X *has universal colimits;*
  (c) groupoid objects in X *are effective;*
  (d) Ω*-colimits in* X *are disjoint.*
(3) X *is Ω-cocomplete and takes values in* $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty}$*;*
(4) X *is a* $\mathrm{Top}^{\mathrm{L,\acute{e}t}}_{\infty}$*-valued sheaf that preserves pushouts.*

REMARK 5.2.1.6. It is crucial to include the condition that all Ω-groupoidal colimits are disjoint into the internal Giraud axioms, instead of just all coproducts. As a concrete example, let κ be an uncountable regular cardinal and let $\mathcal{C} \hookrightarrow \mathrm{Cat}_{\infty}$ be the subcategory spanned by the κ-small ∞-categories and *cocartesian* fibrations between them. Let us set $\mathcal{B} = \mathrm{PSh}(\mathcal{C})$ and let $\mathsf{X} \in \mathrm{Cat}(\widehat{\mathcal{B}})$ be the large $\mathcal{B}$-category that is determined by the presheaf $\mathrm{PSh}(-) \colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}_{\infty}$. Since X takes values in $\mathrm{Top}^{\mathrm{L}}_{\infty}$ and since cocartesian fibrations are *smooth* [57, Proposition 4.1.2.15], we deduce from Theorem 4.4.2.4, Remark 5.1.5.3 and Example 5.1.4.5 that X is presentable, has effective groupoid objects and that coproducts in X are disjoint. Moreover, again by using that cocartesian fibrations are smooth, one easily finds that X has universal colimits. Yet, the $\mathcal{B}$-category X cannot be a $\mathcal{B}$-topos since the transition functors are in general not étale.

Before we prove Theorem 5.2.1.5, let us us first record the following immediate consequence:

COROLLARY 5.2.1.7. *The universe Ω is a $\mathcal{B}$-topos.* □

REMARK 5.2.1.8. As another consequence of Theorem 5.2.1.5, a large $\mathcal{B}$-category X is a $\mathcal{B}$-topos if and only if there is a cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ such that for all i the large $\mathcal{B}_{/i}$-category $\pi_{A_i}^* \mathsf{X}$ is a $\mathcal{B}_{/A_i}$-topos. In fact, this most easily follows from part (3) of the theorem, together with the fact that Ω-cocompleteness can be checked locally by Remark 3.2.2.3.

The proof of Theorem 5.2.1.5 requires the following lemma:

LEMMA 5.2.1.9. *Let*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "h^*"] \arrow[d, "f^*"'] & \mathcal{Z} \arrow[d, "g^*"] \\ \mathcal{Y} \arrow[r, "k^*"] & \mathcal{W} \end{tikzcd}$$

be a commutative square in $\mathrm{Top}^{\mathrm{L}}_{\infty}$*, and suppose that h^* and k^* are étale. Then the square is a pushout in* $\mathrm{Top}^{\mathrm{L}}_{\infty}$ *if and only if the mate transformation $k_! g^* \to f^* h_!$ is an equivalence.*

PROOF. As h^* is étale, we may replace $\mathcal{Z}$ with $\mathcal{X}_{/U}$ and h^* with π_U^*, where $U = h_!(1_{\mathcal{Z}})$. By using [57, Remark 6.3.5.8], the pushout of π_U^* along f^* is given by the commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi_U^*"] \arrow[d, "f^*"'] & \mathcal{X}_{/U} \arrow[d, "f^*_{/U}"] \\ \mathcal{Y} \arrow[r, "\pi_{f^*(U)}^*"] & \mathcal{Y}_{/f^*(U)}. \end{tikzcd}$$

It is immediate that the mate of this square is an equivalence, so it suffices to prove the converse. Since k^* is étale, we may replace k^* with $\pi_V^*\colon \mathcal{Y} \to \mathcal{Y}_{/V}$, where $V = k_!(1_{\mathcal{W}})$. By [57, Remark 6.3.5.7], the induced map $\mathcal{Y}_{/f^*(U)} \to \mathcal{Y}_{/V}$ is uniquely determined by a morphism $V \to f^*(U)$ in $\mathcal{Y}$. Unwinding the definitions, this map is precisely the value of the mate transformation $(\pi_V)_! g^* \to f^*(\pi_U)_!$ at $1_{\mathcal{X}_{/U}}$ and therefore an equivalence. Hence the functor $\mathcal{Y}_{/f^*(U)} \to \mathcal{Y}_{/V}$ must be an equivalence as well, which finishes the proof. □

Proof of Theorem 5.2.1.5. Let X be a $\mathcal{B}$-topos. By combining Propositions 5.1.3.5 and 5.1.4.7 with Corollary 5.1.5.5, we find that X satisfies the internal Giraud axioms, so that (1) implies (2). If X satisfies the internal Giraud axioms, then X being presentable implies that it is Ω-cocomplete. Moreover, Examples 5.1.3.4 and 5.1.4.5 together with Remark 5.1.5.3 imply that $\mathsf{X}(A)$ satisfies the ∞-categorical Giraud axioms in the sense of [57] for all $A \in \mathcal{B}$, so that each $\mathsf{X}(A)$ is an ∞-topos. Now by Propositions 5.1.4.7 and 5.1.3.5, the $\mathcal{B}$-category X has Ω-descent, hence Corollary 5.1.2.13 implies that for every map $s\colon B \to A$ in $\mathcal{B}$ the functor $s_!$ is a right fibration. This implies that s^* is an étale geometric morphism, hence (3) holds. The equivalence between (3) and (4), on the other hand, is an immediate consequence of Lemma 5.2.1.9. Finally, if X satisfies condition (3), then X is presentable (see Theorem 4.4.2.4), hence the claim follows from Corollary 5.1.2.14. □

5.2.2. Free $\mathcal{B}$-topoi. The goal of this section is to construct a partial left adjoint to the forgetful functor $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ that is defined on the full subcategory $\mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ and that carries a $\mathcal{B}$-category C to the associated *free* $\mathcal{B}$-topos $\Omega_{\mathcal{B}}[\mathsf{C}]$. To that end, first note that if $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ denotes the subcategory spanned by the left exact (i.e. $\mathsf{Fin}_{\mathcal{B}_{/A}}$-continuous) functors between $\mathcal{B}_{/A}$-categories with finite limits for all $A \in \mathcal{B}$, then the dual of Corollary 3.4.1.15 implies that the inclusion admits a left adjoint $(-)^{\mathrm{lex}}\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}}$ that carries a $\mathcal{B}$-category C to its free $\mathsf{Fin}_{\mathcal{B}}$-completion $\mathsf{C}^{\mathrm{lex}}$. Moreover, the same result implies that we have a functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(-)\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$ that is obtained by restricting the free cocompletion functor $\mathsf{Cat}_{\widehat{\mathcal{B}}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{cc}}$ in the appropriate way. By combining these two constructions, we thus end up with a well-defined functor $\Omega_{\mathcal{B}}[-] = \underline{\mathsf{PSh}}_{\mathcal{B}}((-)^{\mathrm{lex}})\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Pr}_{\mathcal{B}}^{\mathrm{L}}$. Our goal is to show:

Proposition 5.2.2.1. *For any $\mathcal{B}$-category C, the large $\mathcal{B}$-category $\Omega_{\mathcal{B}}[\mathsf{C}]$ is a $\mathcal{B}$-topos. Moreover, if X is another $\mathcal{B}$-topos, precomposition with the canonical map $\mathsf{C} \to \Omega_{\mathcal{B}}[\mathsf{C}]$ induces an equivalence*

$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{alg}}(\Omega_{\mathcal{B}}[\mathsf{C}], \mathsf{X}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$$

of $\mathcal{B}$-categories.

The proof of Proposition 5.2.2.1 requires a few preparations and will be given at the end of this section. For now, let us record a few consequences of this result.

Corollary 5.2.2.2. *The functor $\Omega_{\mathcal{B}}[-]$ takes values in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$ and fits into an equivalence*

$$\mathrm{map}_{\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}}(\Omega_{\mathcal{B}}[-], -) \simeq \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-,-)$$

of bifunctors $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Top}_{\mathcal{B}}^{\mathrm{L}} \to \Omega_{\widehat{\mathcal{B}}}$.

Proof. Note that if $A \in \mathcal{B}$ is an arbitrary object, we deduce from Proposition 3.4.1.11 that the base change of the canonical map $\mathsf{C} \to \Omega_{\mathcal{B}}[\mathsf{C}]$ along π_A^* can be identified with the canonical map $\pi_A^*\mathsf{C} \to \Omega_{\mathcal{B}_{/A}}[\pi_A^*\mathsf{C}]$. Thus, in light of Remark 5.2.1.3, the result is an immediate consequence of Proposition 5.2.2.1. □

Corollary 5.2.2.2 already implies the existence of certain colimits in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$:

Corollary 5.2.2.3. *For any diagram $d\colon \mathsf{I} \to \mathsf{Cat}_{\mathcal{B}}$, the induced cocone $\Omega_{\mathcal{B}}[d(-)] \to \Omega_{\mathcal{B}}[\operatorname{colim} d]$ is a colimit cocone in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$.*

Proof. Combine Proposition 5.2.2.1 with Proposition 3.1.4.9. □

By combining Corollary 5.2.2.3 with the evident equivalence $1 \simeq \varnothing^{\mathrm{lex}}$, we in particular obtain:

COROLLARY 5.2.2.4. *The universe* Ω *defines an initial object in* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. □

In light of Corollary 5.2.2.4, we may now define:

DEFINITION 5.2.2.5. Let X be a $\mathcal{B}$-topos. The unique algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \Omega \to \mathsf{X}$ is referred to as the *constant sheaf functor*, and its right adjoint $\Gamma_{\mathsf{X}} \colon \mathsf{X} \to \Omega$ is called the *global sections functor*.

REMARK 5.2.2.6. If X is a $\mathcal{B}$-topos, then the global sections functor Γ_{X} is equivalent to $\mathrm{map}_{\mathsf{X}}(1_{\mathsf{X}}, -)$, where 1_{X} denotes the final object in X. In fact, since the unique algebraic morphism $\mathrm{const}_{\mathsf{X}} \colon \Omega_{\mathcal{B}} \to \mathsf{X}$ is left exact and since $\mathrm{map}_{\Omega_{\mathcal{B}}}(1_{\Omega}, -) \simeq \mathrm{id}_{\Omega}$ by [62, Proposition 4.6.3], this follows immediately from the adjunction $\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}$.

We now turn to the proof of Proposition 5.2.2.1. As a first step, we need to establish that presheaf $\mathcal{B}$-categories are $\mathcal{B}$-topoi:

PROPOSITION 5.2.2.7. *For every* $\mathcal{B}$*-category* C*, the large* $\mathcal{B}$*-category* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *is a* $\mathcal{B}$*-topos.*

PROOF. Since $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is presentable, we only need to show that it satisfies descent. Let us first show that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ has universal colimits. Let therefore $f \colon F \to G$ be an arbitrary map of presheaves on C in context $A \in \mathcal{B}$. By [62, Lemma 4.7.14], we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$, so that we can assume that $A \simeq 1$. By Lemma 3.3.1.5 there are equivalences $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F})$ and $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/G} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/G})$ with respect to which the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/G})$ that corresponds to $f_!$ carries the final presheaf on $\mathsf{C}_{/F}$ to the presheaf that classifies the right fibration $f_! \colon \mathsf{C}_{/F} \to \mathsf{C}_{/G}$. As the functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/F}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/G})$ is a morphism of right fibrations over $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, this map is uniquely specified by the image of the final object. We thus conclude that this functor must be equivalent to the functor of left Kan extension along $f_! \colon \mathsf{C}_{/F} \to \mathsf{C}_{/G}$. Its right adjoint is simply given by precomposition with $f_!$, which defines a cocontinuous functor. Hence $f^* \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/G} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$ must be cocontinuous as well, and we conclude that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ has universal colimits.

To conclude the proof, we need to show that $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ has effective descent. By Proposition 5.1.2.12, this is equivalent to the condition that for every $A \in \mathcal{B}$, every small $\mathcal{B}_{/A}$-category I and every cartesian map $d' \to d$ in $\underline{\mathsf{Fun}}_{\mathcal{B}_{/A}}(\mathsf{I}, \pi_A^* \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$, the naturality square

$$\begin{tikzcd} d' \arrow[r, "\eta"] \arrow[d] & \mathrm{diag}(\operatorname{colim} d') \arrow[d] \\ d \arrow[r, "\eta"] & \mathrm{diag}(\operatorname{colim} d) \end{tikzcd}$$

is a pullback. Upon replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$, we may assume without loss of generality $A \simeq 1$. Moreover, since limits and colimits in functor $\mathcal{B}$-categories are detected object-wise by Proposition 3.1.3.2, we can reduce to $\mathsf{C} \simeq 1$. In this case, the result follows from Corollary 5.2.1.7. □

Next, we need to establish an internal analogue of the well-known statement that left exact functors with values in an ∞-topos are equivalently *flat* functors [57, Proposition 6.1.5.2]. The key ingredient to this result is the following lemma:

LEMMA 5.2.2.8. *Let* C *be a* $\mathcal{B}$*-category, let* X *be a* $\mathcal{B}$*-topos and let* $f \colon \mathsf{C} \to \mathsf{X}$ *be a functor. Suppose that the Yoneda extension* $h_!(f) \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ *of this functor preserves the limit of every cospan in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *(in arbitrary context* $A \in \mathcal{B}$*) that is contained in the essential image of the Yoneda embedding* $h \colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$*. Then* $h_!(f)$ *preserves pullbacks.*

PROOF. Suppose that

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ G \arrow[r] & F \end{tikzcd}$$

is a pullback square in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. We need to show that the image of this square along $(h_{\mathsf{C}})_!(f)$ is a pullback in X. By combining [62, Lemma 4.7.14] with Remark 3.3.3.2, we may assume without loss of generality that the above square is in context $1 \in \mathcal{B}$.

Let us first show that the claim is true whenever F is representable by an object $c\colon 1 \to \mathsf{C}$. In this case, Lemma 3.3.1.5 implies that there is an equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(c)} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/c})$ with respect to which the composition $(h_! f)(\pi_{h(c)})_!$ can be identified with the left Kan extension of $f(\pi_c)_!$ along the Yoneda embedding $\mathsf{C}_{/c} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/c})$. Therefore, by replacing C with $\mathsf{C}_{/c}$, one can assume that $F \simeq 1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}$. Now the product functor $G \times -\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ being cocontinuous (by Proposition 5.2.2.7) implies that the canonical map $h_! f(G \times -) \to h_! f(G) \times h_! f(-)$ is a morphism between cocontinuous functors. On account of the universal property of presheaf $\mathcal{B}$-categories, this means that we may further reduce to the case where H is representable. By the same argument, the presheaf G can also be assumed to be representable. In this case, the claim follows from the assumption on $h_!(f)$.

We now turn to the general case. By Proposition 3.3.1.1, there is a diagram $d\colon \mathsf{I} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ such that $F \simeq \operatorname{colim} d$ and such that d takes values in $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Let us write $\overline{d}$ for the associated colimit cocone. In light of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}}$ from Remark 5.1.2.7 and by identifying the above pullback square with a diagram in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/F}$, we obtain a pullback diagram

$$\begin{tikzcd} \overline{q} \arrow[r] \arrow[d] & \overline{p} \arrow[d] \\ \overline{g} \arrow[r] & \overline{d} \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}}$. By the above and the fact that limits in functor $\mathcal{B}$-categories can be computed object-wise by Proposition 3.1.3.2, the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/d} \xrightarrow{(h_! f)_*} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{X})_{/(h_! f)_* d}$$

carries the above pullback diagram of cocones to a pullback and therefore in particular to a diagram in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{X})^{\mathrm{cart}}_{/(h_! f)_* d}$. By using descent in X and in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (cf. Proposition 5.2.2.7) together with the fact that $h_!(f)$ is cocontinuous, this implies that the functor $(h_! f)_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))^{\mathrm{cart}}_{/\overline{d}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}^{\triangleright}, \mathsf{X})_{/\overline{d}}$ preserves the above pullback. Upon evaluating the latter at the cone point $\infty\colon 1 \to \mathsf{I}^{\triangleright}$, we recover the image of the original pullback square along $h_!(f)$, hence the claim follows. □

Proposition 5.2.2.9. *Let* C *be a* $\mathcal{B}$*-category with finite limits, and let* X *be a* $\mathcal{B}$*-topos. Then a functor* $f\colon \mathsf{C} \to \mathsf{X}$ *preserves finite limits if and only if its left Kan extension* $h_!(f)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ *preserves finite limits.*

Proof. Since the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ preserves finite limits, it is clear that the condition is sufficient. Conversely, suppose that f preserves finite limits. Since the final object in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is contained in C, it is clear that $h_!(f)$ preserves final objects. We therefore only need to show that this functor also preserves pullbacks, which is an immediate consequence of Lemma 5.2.2.8. □

By combining Proposition 5.2.2.9 with the universal property of presheaf $\mathcal{B}$-categories, Remark 5.2.1.3 and Remark 3.2.3.2, we now conclude:

Corollary 5.2.2.10. *For any* $\mathcal{B}$*-category* C *with finite limits and any* $\mathcal{B}$*-topos* X*, the functor of left Kan extension along the Yoneda embedding* $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ *gives rise to an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}), \mathsf{X}),$$

where $\underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ *is the full subcategory of* $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ *that is spanned by the left exact functors in arbitrary context.* □

Proof of Proposition 5.2.2.1. Combine Proposition 5.2.2.7 with Corollary 5.2.2.10 and the universal property of free $\mathsf{Fin}_{\mathcal{B}}$-completion, see Theorem 3.4.1.13. □

5.2.3. Presentations of $\mathcal{B}$-topoi. Recall from Definition 4.3.3.4 that if C is a $\mathcal{B}$-category, a Bousfield localisation $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{D}$ is said to be *accessible* if the inclusion $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous for some choice of *sound doctrine* U (see Definitions 4.1.3.8 and 4.1.2.7). We will say that the localisation is *left exact* if L preserves finite limits. The main goal of this section is to prove the following characterisation of $\mathcal{B}$-topoi:

THEOREM 5.2.3.1. *A large $\mathcal{B}$-category X is a $\mathcal{B}$-topos if and only if there is a $\mathcal{B}$-category C such that X arises as a left exact and accessible localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$.*

The proof of Theorem 5.2.3.1 relies on the following two lemmas:

LEMMA 5.2.3.2. *Suppose that*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi_U^*"] \arrow[d, "L"] & \mathcal{X}_{/U} \arrow[d, "L'"] \\ \mathcal{Y} \arrow[r, "h^*"] & \mathcal{Z} \end{tikzcd}$$

is a commutative square in $\mathrm{Top}_\infty^{\mathrm{L}}$ in which L and L' are Bousfield localisations. Suppose furthermore that h^* admits a left adjoint $h_!$ and that the mate transformation $\varphi\colon h_!L' \to L(\pi_U)_!$ is an equivalence. Then h^* is étale.

PROOF. We would like to apply [57, Proposition 6.3.5.11], which says that the functor h^* is étale precisely if $h_!$ is conservative and if for every map $f\colon W \to V$ in $\mathcal{Y}$ and every object $P \in \mathcal{Z}_{/h^*(V)}$, the canonical map $\alpha\colon h_!(h^*(W) \times_{h^*(V)} P) \to W \times_V h_!(P)$ is an equivalence.

Let us begin by showing that $h_!$ is conservative. To that end, note that if $f\colon V \to W$ is a map in $\mathcal{X}_{/U}$ such that $L(\pi_U)_!(f)$ is an equivalence, then $L'(f)$ is an equivalence. In fact, since the adjunction unit of $(\pi_U)_! \dashv \pi_U^*$ exhibits f as a pullback of $\pi_U^*(\pi_U)_!(f)$, the localisation functor L' being left exact implies that $L'(f)$ is a pullback of $L'\pi_U^*(\pi_U)_!(f) \simeq h^*L(\pi_U)_!(f)$. Since the latter is an equivalence, the claim follows. Applying this observation to a map f that is contained in $\mathcal{Z}$ and using the assumption that the mate transformation $\varphi\colon h_!L' \to L(\pi_U)_!$ is an equivalence, we deduce that $h_!$ is indeed conservative.

To conclude the proof, we show that the map α is an equivalence. From the map $f\colon W \to V$ in $\mathcal{Y}$ we obtain a commutative diagram

$$\begin{tikzcd} & \mathcal{X}_{/V} \arrow[rr, "(\pi_U^*)_{/V}"] \arrow[dl, "L_{/V}"'] \arrow[dd, "f^*"] & & \mathcal{X}_{/U\times V} \arrow[dl, "L'_{/\pi_U^*(V)}"'] \arrow[dd, "(\pi_U^* f)^*"] \\ \mathcal{Y}_{/V} \arrow[rr, "h^*_{/V}"] \arrow[dd, "f^*"] & & \mathcal{Z}_{/h^*(V)} \arrow[dd, "(h^*f)^*"] & \\ & \mathcal{X}_{/W} \arrow[rr, "(\pi_U^*)_{/W}"] \arrow[dl, "L_{/W}"'] & & \mathcal{X}_{/U\times W} \arrow[dl, "L'_{/\pi_U^*(W)}"'] \\ \mathcal{Y}_{/W} \arrow[rr, "h^*_{/W}"] & & \mathcal{Z}_{/h^*(W)} & \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{L}}$ in which all of the four maps pointing to the right admit a left adjoint. Note that α being an equivalence for all $P \in \mathcal{Z}_{/h^*(V)}$ precisely means that the front square is left adjointable (i.e. has an invertible mate transformation). Now since the mate $\varphi\colon h_!L' \to L(\pi_U)_!$ is by assumption an equivalence, it follows that both the top and the bottom square in the above diagram are left adjointable. Since π_U^* is an étale algebraic morphism, the back square is left adjointable as well. Therefore, by combining the functoriality of the mate construction with the fact that the four maps in the above diagram pointing to the front are localisation functors and thus in particular essentially surjective, we conclude that the front square must be left adjointable as well, as desired. $\square$

LEMMA 5.2.3.3. *Let D be a presentable $\mathcal{B}$-category. Then there exists a sound doctrine U such that D is U-accessible and $\mathsf{D}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in D.*

PROOF. Since D is presentable, there exists a $\mathcal{B}$-category C and a sound doctrine U such that D arises as a U-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ (cf. Theorem 4.4.2.4). In particular, for every sound doctrine V that contains U, the $\mathcal{B}$-category D is V-accessible (Corollary 4.3.3.3). Therefore, for any cardinal κ we can always find a $\mathcal{B}$-regular cardinal $\tau \geq \kappa$ such that D is $\mathsf{Cat}_{\mathcal{B}}^{\tau}$-accessible. By Remark 4.2.2.5, we can always choose τ such that $\tau \gg \kappa$. Now D being presentable implies that D is section-wise accessible (Theorem 4.4.2.4). Therefore, if $\mathcal{G} \hookrightarrow \mathcal{B}$ is a small generating subcategory, we may find a regular cardinal κ such that $\mathsf{D}(G)$ is κ-accessible for all $G \in \mathcal{G}$. Let us choose a $\mathcal{B}$-regular cardinal $\tau \gg \kappa$ such that

(1) $\mathcal{G}$ is contained in $\mathcal{B}^{\tau\text{-cpt}}$;
(2) D is $\mathsf{Cat}_{\mathcal{B}}^{\tau}$-accessible;
(3) $\mathsf{D}(G)^{\kappa\text{-cpt}}$ is τ-small for all $G \in \mathcal{G}$.

Then [57, Proposition 5.4.7.4] implies that the inclusion $\mathsf{D}(G)^{\tau\text{-cpt}} \hookrightarrow \mathsf{D}(G)$ is closed under finite limits for all $G \in \mathcal{G}$. Recall from [62, Corollary 4.6.8] that for every object $d\colon A \to \mathsf{D}$ the mapping functor $\mathrm{map}_{\mathsf{D}(A)}(d, -)$ can be identified with the composition

$$\mathsf{D}(A) \xrightarrow{\mathrm{map}_{\mathsf{D}}(d,-)(A)} \mathcal{B}_{/A} \xrightarrow{\Gamma_{\mathcal{B}_{/A}}} \mathcal{S}.$$

By combining this observation with Proposition 4.3.2.4 and the fact that $\mathcal{B}$ is generated by $\mathcal{G}$, we find that for any $G \in \mathcal{G}$ an object $d\colon G \to \mathsf{D}$ is contained in $\mathsf{D}^{\mathsf{Cat}_{\mathcal{B}}^{\tau}\text{-cpt}}$ if and only if for every $H \in \mathcal{G}$ and every map $s\colon H \to G$ the object $s^*(d) \in \mathsf{D}(H)$ is contained in $\mathsf{D}(H)^{\tau\text{-cpt}}$. Since s^* commutes with limits, this implies that the inclusion $\mathsf{D}^{\mathsf{Cat}_{\mathcal{B}}^{\tau}\text{-cpt}} \hookrightarrow \mathsf{D}$ is closed under finite limits. □

PROOF OF THEOREM 5.2.3.1. Suppose that X is a left exact and U-accessible localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, and let us show that X is a $\mathcal{B}$-topos. We would like to apply Theorem 5.2.1.5. First, note that by choosing a $\mathcal{B}$-regular cardinal κ such that $\mathsf{U} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}^{\kappa}$, we may assume that X is a $\mathsf{Cat}_{\mathcal{B}}^{\kappa}$-accessible Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$. Therefore, for every $A \in \mathcal{B}$ the ∞-category $\mathsf{X}(A)$ is a κ-accessible and left exact Bousfield localisation of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})(A)$, and since the latter is an ∞-topos by Proposition 5.2.2.7, it follows that $\mathsf{X}(A)$ is an ∞-topos as well. Moreover, if $s\colon B \to A$ is a map in $\mathcal{B}$, the fact that X is a presentable $\mathcal{B}$-category (see Theorem 4.4.2.4) implies that $s^*\colon \mathsf{X}(A) \to \mathsf{X}(B)$ is continuous and cocontinuous and therefore in particular an algebraic morphism that admits a left adjoint $s_!\colon \mathsf{X}(B) \to \mathsf{X}(A)$. We are therefore in the situation of Lemma 5.2.3.2 and may thus conclude that s^* is an étale algebraic morphism. Theorem 5.2.1.5 thus implies that X is a $\mathcal{B}$-topos.

Conversely, suppose that X is a $\mathcal{B}$-topos. Then X is presentable, hence Lemma 5.2.3.3 implies that there exists a sound doctrine U such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. Then we may identify $\mathsf{X} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}^{\mathsf{U}}(\mathsf{X}^{\mathsf{U}\text{-cpt}})$, and since X is cocomplete the induced inclusion $\mathsf{X} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{X}^{\mathsf{U}\text{-cpt}})$ admits a left adjoint $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{X}^{\mathsf{U}\text{-cpt}}) \to \mathsf{X}$ which is obtained as the left Kan extension of the inclusion $\mathsf{X}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{X}$ (see Corollary 3.4.1.14). By Proposition 5.2.2.9, the functor L is left exact, hence the claim follows. □

COROLLARY 5.2.3.4. *For any $\mathcal{B}$-topos X and any $\mathcal{B}$-category D, the functor $\mathcal{B}$-category $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{X})$ is again a $\mathcal{B}$-topos.*

PROOF. Choose a left exact and accessible Bousfield localisation $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$. Then the postcomposition functor $L_*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C} \times \mathsf{D}^{\mathrm{op}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, \mathsf{X})$ is again an accessible and left exact Bousfield localisation, hence the claim follows. □

COROLLARY 5.2.3.5. *A large $\mathcal{B}$-category X is a $\mathcal{B}$-topos if and only if there is a $\mathcal{B}$-category C and a left exact and accessible Bousfield localisation $L\colon \Omega_{\mathcal{B}}[\mathsf{C}] \to \mathsf{X}$.*

PROOF. By Theorem 5.2.3.1, it suffices to show that every presheaf $\mathcal{B}$-topos arises as a left exact and accessible Bousfield localisation of a free $\mathcal{B}$-topos. But if C is a $\mathcal{B}$-category, the fact that $i\colon \mathsf{C} \hookrightarrow \mathsf{C}^{\mathrm{lex}}$ is fully faithful implies that $i_*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \hookrightarrow \Omega_{\mathcal{B}}[\mathsf{C}]$ is fully faithful as well (see the dual of Theorem 3.3.3.5), hence i^* is a left exact and accessible Bousfield localisation. □

COROLLARY 5.2.3.6. *Any $\mathcal{B}$-topos X is a pushout of free $\mathcal{B}$-topoi.*

PROOF. By Corollary 5.2.3.5, we may choose a small $\mathcal{B}$-category C and a left exact and accessible Bousfield localisation $L\colon \Omega_{\mathcal{B}}[\mathsf{C}] \to \mathsf{X}$. We can therefore find a small subcategory $\mathsf{W} \hookrightarrow \Omega_{\mathcal{B}}[\mathsf{C}]$ such that L induces an equivalence $\mathsf{Loc}_{\mathsf{W}}(\Omega_{\mathcal{B}}[\mathsf{C}]) \simeq \mathsf{X}$ (see Theorem 4.4.2.4). Note that a functor $\mathsf{X} \to \mathsf{Y}$ between $\mathcal{B}$-topoi is an algebraic morphism if and only if its precomposition with L is one (this is easily deduced from Remark 5.2.1.3 and the explicit computation of colimits in a Bousfield localisation, see Proposition 3.1.2.12). Therefore, by combining Corollary 4.4.3.2 with Proposition 5.2.2.1, we deduce that the induced square

$$\begin{tikzcd} \Omega_{\mathcal{B}}[\mathsf{W}] \arrow[r] \arrow[d] & \Omega_{\mathcal{B}}[\mathsf{C}] \arrow[d] \\ \Omega_{\mathcal{B}}\left[\mathsf{W}^{\mathrm{gpd}}\right] \arrow[r] & \mathsf{X} \end{tikzcd}$$

is a pushout in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. □

5.2.4. The $\mathsf{Cat}_{\widehat{\mathcal{B}}}$-enrichment of $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. Recall from Proposition 3.1.5.2 that $\mathsf{Cat}_{\widehat{\mathcal{B}}}$ is *cartesian closed*, i.e. that forming functor $\mathcal{B}$-categories defines a bifunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)\colon \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{op}} \times \mathsf{Cat}_{\widehat{\mathcal{B}}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ and therefore in particular a bifunctor $(\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$. Let $p\colon \mathsf{P} \to (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ be the unstraightening of the latter (in the sense of [61]). Explicitly, an object $A \to \mathsf{P}$ is given by a functor $\pi_A^*\mathsf{X} \to \pi_A^*\mathsf{Y}$ between $\mathcal{B}_{/A}$-topoi. Let $\mathsf{Q} \hookrightarrow \mathsf{P}$ be the full subcategory that is spanned by those objects that correspond to algebraic morphisms. By Lemma 5.2.4.1 below, the induced functor $q\colon \mathsf{Q} \to (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is a cocartesian fibration as well and therefore classified by a bifunctor $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(-,-)\colon (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$.

LEMMA 5.2.4.1. *Let $p\colon \mathsf{P} \to \mathsf{C}$ be a cocartesian fibration of $\mathcal{B}$-categories. Let $\mathsf{Q} \hookrightarrow \mathsf{P}$ be a full subcategory such that for each map $f\colon c \to d$ in C in context $A \in \mathcal{B}$ the induced functor $f_!\colon \mathsf{P}|_c \to \mathsf{P}|_d$ restricts to a functor $\mathsf{Q}|_c \to \mathsf{Q}|_d$. Then the induced functor $q\colon \mathsf{Q} \to \mathsf{C}$ is a cocartesian fibration as well, and the inclusion $\mathsf{Q} \hookrightarrow \mathsf{P}$ is a cocartesian functor.*

PROOF. It will be enough to show that for any cocartesian lift $\varphi\colon x \to y$ of f in P in which x is contained in $\mathsf{Q}|_c$, the object y is contained in $\mathsf{Q}|_d$. This immediately follows from the assumptions. □

DEFINITION 5.2.4.2. We define the *functor of $\mathcal{B}$-points* as the functor $\mathsf{Pt}_{\mathcal{B}} = \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(-,\Omega)\colon \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$.

Recall that by Corollary 5.2.3.4, if C is a $\mathcal{B}$-category and X is a $\mathcal{B}$-topos, then $\mathsf{X}^{\mathsf{C}} = \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C},\mathsf{X})$ is a $\mathcal{B}$-topos as well. Moreover, as precomposition and postcomposition preserves all limits and colimits, the bifunctor $\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)$ restricts to a bifunctor $(-)^{(-)}\colon \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ which we refer to as the *powering* of $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ over $\mathsf{Cat}_{\mathcal{B}}$. This terminology is justified by the following proposition:

PROPOSITION 5.2.4.3. *The powering bifunctor $(-)^{(-)}$ fits into an equivalence*

$$\mathrm{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(-,(-)^{(-)}) \simeq \mathrm{map}^{\mathrm{alg}}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-,\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)).$$

PROOF. If C is a $\mathcal{B}$-category and X and Y are $\mathcal{B}$-topoi, then Lemma 4.6.1.3 and its dual imply that a functor $\mathsf{X} \to \mathsf{Y}^{\mathsf{C}}$ defines an algebraic morphism if and only if the transpose functor $\mathsf{C} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y})$ takes values in $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X},\mathsf{Y})$. By replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$ (which is made possible by Remark 5.2.1.3 and [62, Lemma 4.2.3]) , one obtains that the same is true for any object $A \to \mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. Hence, the equivalence

$$\mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(i(-),\underline{\mathsf{Fun}}_{\mathcal{B}}(-,-)) \simeq \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-,\underline{\mathsf{Fun}}_{\mathcal{B}}(i(-),-))$$

of functors $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{op}} \times \mathsf{Cat}_{\widehat{\mathcal{B}}}^{\mathrm{op}} \times \mathsf{Cat}_{\widehat{\mathcal{B}}} \to \Omega_{\widehat{\mathcal{B}}}$ (where $i\colon \mathsf{Cat}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ is the inclusion) restricts in the desired way. □

COROLLARY 5.2.4.4. *The functor of $\mathcal{B}$-points* Pt *is a partial right adjoint of the functor*

$$\Omega^{(-)}\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}},$$

in the sense that there is an equivalence

$$\operatorname{map}_{\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}}(\Omega^{(-)}, -) \simeq \operatorname{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(-, \mathsf{Pt}(-))$$

of functors $\mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}} \to \Omega_{\widehat{\mathcal{B}}}$. ☐

In light of Corollary 5.2.4.4, it is reasonable to define:

DEFINITION 5.2.4.5. If C is a $\mathcal{B}$-category, we refer to the $\mathcal{B}$-topos $\mathsf{C}^{\mathrm{disc}} = \Omega^{\mathsf{C}}$ as the *discrete* $\mathcal{B}$-topos associated with C.

Lastly, we note that the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is also *tensored* over $\mathsf{Cat}_{\mathcal{B}}$:

PROPOSITION 5.2.4.6. *There is a bifunctor* $- \otimes -\colon \mathsf{Cat}_{\mathcal{B}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ *that fits into an equivalence*

$$\operatorname{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(- \otimes -, -) \simeq \operatorname{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(-, (-)^{(-)})$$

of functors $\mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times (\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})^{\mathrm{op}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \Omega_{\widehat{\mathcal{B}}}$.

PROOF. As an immediate consequence of the constructions, if C and D are $\mathcal{B}$-categories, we obtain a chain of equivalences

$$\operatorname{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(\Omega_{\mathcal{B}}[\mathsf{C}], (-)^{\mathsf{D}}) \simeq \operatorname{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{C}, \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}, -)) \simeq \operatorname{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{C} \times \mathsf{D}, -) \simeq \operatorname{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(\Omega_{\mathcal{B}}[\mathsf{C} \times \mathsf{D}], -),$$

which implies that the functor $\operatorname{map}_{\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{X}, (-)^{\mathsf{D}})$ is representable whenever X is in the image of $\Omega_{\mathcal{B}}[-]$. But since every $\mathcal{B}$-topos is a pushout of such $\mathcal{B}$-topoi (see Corollary 5.2.3.6), this functor must be representable for *any* $\mathcal{B}$-topos X. As by Remark 5.2.1.3 the same argument shows that this is the case for every object $\mathsf{X}\colon A \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ and every $\mathcal{B}_{/A}$-category D, the result follows. ☐

5.2.5. Relative ∞-topoi as $\mathcal{B}$-topoi. By Theorem 5.2.1.5 and the evident fact that an algebraic morphism between $\mathcal{B}$-topoi induces an algebraic morphism of ∞-topoi upon taking global sections, we obtain a functor $\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Top}^{\mathrm{L}}_{\infty}$. By making use of the fact that the universe Ω is an initial object in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ (Corollary 5.2.2.4), this functor factors through the projection $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/} \to \mathrm{Top}^{\mathrm{L}}$, so that we end up with a functor

$$\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}.$$

The main goal in this section is to prove:

THEOREM 5.2.5.1. *The global sections functor* $\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}$ *is an equivalence of* ∞-*categories.*

REMARK 5.2.5.2. Theorem 5.2.5.1 implies that the datum of a $\mathcal{B}$-topos X is equivalent to that of a geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$. We will refer to the latter as the geometric morphism that is *associated* with X.

REMARK 5.2.5.3. One can describe the inverse to the equivalence $\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \simeq (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/}$ from Theorem 5.2.5.1 explicitly as follows: Given any algebraic morphism $f^*\colon \mathcal{B} \to \mathcal{X}$, recall that we get an induced functor $f_*\colon \mathrm{Cat}(\widehat{\mathcal{X}}) \to \mathrm{Cat}(\widehat{\mathcal{B}})$. Then Theorem 5.2.1.5 easily implies that the large $\mathcal{B}$-category $\mathsf{X} = f_*\Omega_{\mathcal{X}}$ is a $\mathcal{B}$-topos (since the associated sheaf on $\mathcal{B}$ is simply given by $\mathcal{X}_{/f^*(-)}$). Moreover, the functor f^* induces a map $\mathcal{B}_{/-} \to \mathcal{X}_{/f^*(-)}$ of sheaves on $\mathcal{B}$ that recovers the unique algebraic morphism $\mathrm{const}_{\mathsf{X}}\colon \Omega_{\mathcal{B}} \to \mathsf{X}$. This implies that X is the image of $f^*\colon \mathcal{B} \to \mathcal{X}$ under the equivalence from Theorem 5.2.5.1.

The proof of Theorem 5.2.5.1 requires a few preparations. We begin with the following lemma:

LEMMA 5.2.5.4. *Let $\mathcal{C}$ be an ∞-category with an initial object $\varnothing_{\mathcal{C}}$, and let $\mathcal{D}$ be an ∞-category that admits pushouts. Then the evaluation functor $\mathrm{ev}_{\varnothing_{\mathcal{C}}} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ is a cocartesian fibration. Moreover, a morphism $\varphi \colon F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is cocartesian if and only if for every map $f \colon c \to c'$ in $\mathcal{C}$ the induced commutative square*

$$\begin{tikzcd} F(c) \arrow[r, "\alpha(c)"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha(c')"] & G(c') \end{tikzcd}$$

is a pushout in $\mathcal{D}$.

PROOF. Note that the diagonal functor $\mathrm{diag} \colon \mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ defines a left adjoint to $\mathrm{ev}_{\varnothing_{\mathcal{C}}}$. Therefore, we deduce from [38, Proposition 4.51] that a map $\alpha \colon F \to G$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is cocartesian if and only if for every $c \in \mathcal{C}$ the square

$$\begin{tikzcd} F(\varnothing_{\mathcal{C}}) \arrow[r, "\alpha(\varnothing_{\mathcal{C}})"] \arrow[d] & G(\varnothing_{\mathcal{C}}) \arrow[d] \\ F(c) \arrow[r, "\alpha(c)"] & G(c) \end{tikzcd}$$

is a pushout in $\mathcal{D}$. The assumption that $\mathcal{D}$ admits pushouts guarantees that there are enough such cocartesian maps, see [38, Corollary 4.52]. □

By Lemma 5.2.5.4, the global sections functor $\Gamma \colon \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B}) \to \mathrm{Top}^{\mathrm{L}}_{\infty}$ is a cocartesian fibration and therefore induces a left fibration $\Gamma \colon \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \to \mathrm{Top}^{\mathrm{L}}_{\infty}$, where $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ is the subcategory that is spanned by the cocartesian morphisms. Moreover, observe that by Theorem 5.2.1.5 we may regard the ∞-category $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ as a (non-full) subcategory of $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$. Now the key step towards the proof of Theorem 5.2.5.1 consists of the following proposition:

PROPOSITION 5.2.5.5. *The (non-full) inclusion $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ fits into a commutative diagram*

$$\begin{tikzcd} \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \arrow[r, hook] \arrow[dr, hook] & \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \arrow[d, hook] \\ & \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B}) \end{tikzcd}$$

in which the horizontal map is fully faithful. Moreover, if X is a $\mathcal{B}$-topos and if $\mathsf{X} \to F$ is a map in $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}}$, then F is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$.

PROOF. If $f^* \colon \mathsf{X} \to \mathsf{Y}$ is an algebraic morphism between $\mathcal{B}$-topoi, Lemma 5.2.1.9 and the fact that f^* is cocontinuous imply that for every map $s \colon B \to A$ in $\mathcal{B}$ the induced commutative square

$$\begin{tikzcd} \mathsf{X}(A) \arrow[r, "f^*(A)"] \arrow[d, "s^*"] & \mathsf{Y}(A) \arrow[d, "s^*"] \\ \mathsf{X}(B) \arrow[r, "f^*(B)"] & \mathsf{Y}(B) \end{tikzcd}$$

is a pushout in $\mathrm{Top}^{\mathrm{L}}_{\infty}$. By Lemma 5.2.5.4, this means that the underlying map of $\mathrm{Top}^{\mathrm{L}}_{\infty}$-valued presheaves on $\mathcal{B}$ defines a cocartesian morphism over $\mathrm{Top}^{\mathrm{L}}_{\infty}$. Hence the inclusion $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$ factors through the inclusion $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})^{\mathrm{cocart}} \hookrightarrow \mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_{\infty}}(\mathcal{B})$. To finish the proof, it now suffices to show that for any cocartesian morphism $f \colon \mathsf{X} \to F$ of $\mathrm{Top}^{\mathrm{L}}_{\infty}$-valued presheaves on $\mathcal{B}$, the presheaf F is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ and the map f defines an algebraic morphism of $\mathcal{B}$-topoi. Since f is a cocartesian morphism and since étale algebraic morphisms are closed under pushouts in $\mathrm{Top}^{\mathrm{L}}_{\infty}$, we find that for every $s \colon B \to A$ in $\mathcal{B}$ the induced functor $s^* \colon F(A) \to F(B)$ is an étale algebraic morphism of ∞-topoi. Moreover, the pasting lemma for pushouts (and the fact that X is a $\mathcal{B}$-topos) imply that $F \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L,ét}}_{\infty}$ preserves pushouts. Hence Theorem 5.2.1.5 implies that F must be contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ whenever F is a sheaf. But if

$d\colon I \to \mathcal{B}$ is an arbitrary diagram, then we deduce from [56, Corollary 4.7.4.18] that the commutative square

$$\begin{tikzcd} \mathsf{X}(\operatorname{colim} d) \arrow[r] \arrow[d] & F(\operatorname{colim} d) \arrow[d] \\ \lim \mathsf{X} \circ d \arrow[r] & \lim F \circ d \end{tikzcd}$$

is left adjointable and therefore a pushout in $\mathrm{Top}^{\mathrm{L}}_\infty$, using Lemma 5.2.1.9. Hence, since the left vertical map is an equivalence, so is the right one, which means that F is a sheaf. Finally, since f is already section-wise given by an algebraic morphism of ∞-topoi, the map defines an algebraic morphism of $\mathcal{B}$-topoi precisely if it is Ω-cocontinuous, which again follows from Lemma 5.2.1.9. □

Corollary 5.2.5.6. *The global sections functor* $\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Top}^{\mathrm{L}}_\infty$ *is a left fibration.*

Proof. By the first part of Proposition 5.2.5.5, every map in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ is cocartesian. By its second part, if X is a $\mathcal{B}$-topos and $f^*\colon \Gamma(\mathsf{X}) \to \mathcal{Z}$ is an arbitrary algebraic morphism, the codomain of the cocartesian lift $\mathsf{X} \to F$ of f^* in $\mathrm{PSh}_{\mathrm{Top}^{\mathrm{L}}_\infty}(\mathcal{B})$ is again a $\mathcal{B}$-topos. Hence the claim follows. □

Proof of Theorem 5.2.5.1. By Corollary 5.2.5.6, the global sections functor $\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Top}^{\mathrm{L}}_\infty$ is a left fibration, hence so is the functor $\Gamma\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \to (\mathrm{Top}^{\mathrm{L}}_\infty)_{\mathcal{B}/}$. Since this functor carries the initial object Ω to the initial object $\mathrm{id}_{\mathcal{B}}$, it must be an initial functor as well. Hence Γ is an equivalence. □

5.2.6. Limits and colimits of $\mathcal{B}$-topoi. In this section, we discuss how one can construct limits and colimits in the $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ of $\mathcal{B}$-topoi. The construction of *limits* in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is rather easy: they are simply computed in $\mathsf{Cat}_{\widehat{\mathcal{B}}}$. This is analogous to how limits are computed in the $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories, cf. Proposition 4.4.4.10. The proof of this statement follows along similar lines as well.

Proposition 5.2.6.1. *The large $\mathcal{B}$-category* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ *is complete, and the inclusion* $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ *is continuous.*

Proof. As in the proof of Proposition 4.4.4.10, it will be enough to show that whenever K is either given by the constant $\mathcal{B}$-category Λ^2_0 or by a $\mathcal{B}$-groupoid, the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ admits K-indexed limits and the inclusion $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\widehat{\mathcal{B}}}$ preserves K-indexed limits.

We begin with the case were K is a $\mathcal{B}$-groupoid. Let us set $A = \mathsf{K}_0$. Since $(\pi_A)_*\colon \mathrm{Cat}(\widehat{\mathcal{B}}_{/A}) \to \mathrm{Cat}(\widehat{\mathcal{B}})$ is given by precomposition with π_A^*, Theorem 5.2.1.5 implies that $(\pi_A)_*$ takes objects in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A})$ to objects in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. Furthermore it easily follows from Propositions 3.2.4.2 and 3.2.4.5 that $(\pi_A)_*$ therefore defines a functor $\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. Moreover, since the adjunction unit $\mathrm{id}_{\mathrm{Cat}(\widehat{\mathcal{B}})} \to (\pi_A)_*\pi_A^*$ is given by precomposition with the adjunction counit $(\pi_A)_!\pi_A^* \to \mathrm{id}_{\mathcal{B}}$ and vice versa for the adjunction counit, the same argument shows together with the fact that $\mathcal{B}$-topoi are Ω-cocomplete that these two maps must also restrict in the desired way. Hence $(\pi_A)_*\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ defines a right adjoint of π_A^*.

Now let us assume that $\mathsf{K} = \Lambda^2_0$, i.e. let

$$\begin{tikzcd} \mathsf{X} \times_{\mathsf{Z}} \mathsf{Y} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & \mathsf{Y} \arrow[d, "g"] \\ \mathsf{X} \arrow[r, "f"] & \mathsf{Z} \end{tikzcd}$$

be a pullback square in $\mathrm{Cat}(\widehat{\mathcal{B}})$ in which the cospan in the lower right corner is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. By Proposition 4.4.4.10 this square defines a pullback in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$, and [57, Proposition 6.3.2.3] implies that both pr_0 and pr_1 preserve finite limits. Hence the above pullback square is contained in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$ whenever $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ satisfies descent. But the codomain fibration $(\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y})^{\Delta^1} \to \mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ can be identified with the pullback of the cospan $\mathrm{pr}_0^*(\mathsf{X}^{\Delta^1}) \to \mathrm{pr}_0^* f^*(\mathsf{Z}^{\Delta^1}) \leftarrow \mathrm{pr}_1^*(\mathsf{Y}^{\Delta^1})$ of cartesian fibrations over $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$, which implies that we may identify $(\mathsf{X} \times_{\mathsf{Z}} \mathsf{y})_{/-}$ with the pullback $\mathsf{X}_{/\mathrm{pr}_0^*(-)} \times_{\mathsf{Z}_{/\mathrm{pr}_0^* f^*(-)}} \mathsf{Y}_{/\mathrm{pr}_1^*(-)}$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y})^{\mathrm{op}}, \mathsf{Cat}_{\widehat{\mathcal{B}}})$. Since all four functors in the initial pullback square are continuous, we conclude that $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y}$ satisfies descent provided that continuous functors are closed under pullbacks in $\underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y})^{\mathrm{op}}, \mathsf{Cat}_{\widehat{\mathcal{B}}})$, which follows immediately from the fact that limit functors are themselves continuous (see the proof of Lemma 4.1.5.3 for more details). We complete the proof by showing that if we are given another $\mathcal{B}$-topos E and algebraic morphisms $h \colon \mathsf{E} \to \mathsf{X}$ and $k \colon \mathsf{E} \to \mathsf{Z}$ together with an equivalence $f \circ h \simeq g \circ k$, the induced map $\mathsf{E} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{Z}$ is an algebraic morphism as well. That this map is cocontinuous follows from Proposition 4.4.4.10, and that it preserves finite limits is a consequence of the fact that this property can be checked section-wise. □

As a consequence of Proposition 5.2.6.1, we can now upgrade the equivalence from Theorem 5.2.5.1 to a *functorial* one:

COROLLARY 5.2.6.2. *Let $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{(\mathcal{B}_{/-})/}$ be the $\widehat{\mathrm{Cat}}_{\infty}$-valued presheaf on $\mathcal{B}$ whose associated cocartesian fibration on $\mathcal{B}^{\mathrm{op}}$ is given by the pullback of $d_1 \colon \mathrm{Fun}(\Delta^1, \mathrm{Top}^{\mathrm{L}}_{\infty}) \to \mathrm{Top}^{\mathrm{L}}_{\infty}$ along $\mathcal{B}_{/-} \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}_{\infty}$. Then this presheaf is a sheaf whose associated large $\mathcal{B}$-category is equivalent to $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$.*

PROOF. To begin with, note that the functor $\Omega_{/-} \colon \Omega^{\mathrm{op}} \to \mathsf{Cat}_{\widehat{\mathcal{B}}}$ takes values in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ (see the discussion before Definition 5.2.9.1 below). Thus, by combining descent with Proposition 5.2.6.1, we obtain an Ω-continuous functor $\Omega^{\mathrm{op}} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. Hence, the underlying map of $\widehat{\mathrm{Cat}}_{\infty}$-valued presheaves on $\mathcal{B}$ can be regarded as a morphism in $\mathrm{Fun}^{\mathrm{LAdj}}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty})$ (in the sense of [56, § 4.7.4]). On account of the equivalence $\mathrm{Fun}^{\mathrm{LAdj}}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty}) \simeq \mathrm{Fun}^{\mathrm{RAdj}}(\mathcal{B}, \widehat{\mathrm{Cat}}_{\infty})$ from [56, Corollary 4.7.4.18] that is furnished by passing to right adjoints, we thus obtain a morphism of functors $\varphi \colon (\mathcal{B}_{/-})^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/-})$ in which the functoriality on both sides is given by the right adjoints of the transition functors. Let $\eta \colon \varphi \to \mathrm{diag}_{\mathcal{B}}(\varphi(1))$ be the commutative square in $\mathrm{Fun}(\mathcal{B}, \widehat{\mathrm{Cat}}_{\infty})$ that is obtained from the unit of the adjunction $\mathrm{ev}_1 \dashv \mathrm{diag}_{\mathcal{B}} \colon \mathrm{Fun}(\mathcal{B}, \widehat{\mathrm{Cat}}_{\infty}) \leftrightarrows \widehat{\mathrm{Cat}}_{\infty}$. We may regard η as a morphism in $\mathrm{Fun}(\mathcal{B}, (\widehat{\mathrm{Cat}}_{\infty})^{\Delta^1})$. Note that for every map $s \colon B \to A$ in $\mathcal{B}$ one has a commutative triangle

$$\begin{tikzcd}
\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/B}) \arrow[rr, "s_*"] \arrow[dr, "\Gamma_{\mathcal{B}_{/B}}"'] & & \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \arrow[dl, "\Gamma_{\mathcal{B}_{/A}}"] \\
& \mathrm{Top}^{\mathrm{L}}_{\infty}, &
\end{tikzcd}$$

hence Corollary 5.2.5.6 implies that s_* is a left fibration. As the functor $s_!^{\mathrm{op}} \colon \mathcal{B}^{\mathrm{op}}_{/B} \to \mathcal{B}^{\mathrm{op}}_{/A}$ is a left fibration too, the map η thus defines a morphism in $\mathrm{Fun}(\mathcal{B}, \mathrm{LFib})$ (where LFib is the full subcategory of $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_{\infty})$ that is spanned by the left fibrations). Explicitly, this morphism carries $A \in \mathcal{B}$ to the commutative square $\eta(A) \colon \varphi(A) \to \varphi(1)$. Now observe that the domain of η is contained in the fibre $\mathrm{LFib}(\mathcal{B}^{\mathrm{op}}) \hookrightarrow \mathrm{LFib}$ and the codomain is contained in the fibre $\mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B})) \hookrightarrow \mathrm{LFib}$. Moreover, for each $A \in \mathcal{B}$ the functor $\varphi(A) \colon (\mathcal{B}_{/A})^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A})$ carries the final object in $\mathcal{B}_{/A}$ to the initial object $\mathcal{B}_{/A} \in \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A})$ (see Corollary 5.2.2.4), hence φ is section-wise initial. Altogether, these observations imply that

$$\mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/-}) \colon \mathcal{B} \to \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B}))$$

is equivalent to the composition of $(\mathcal{B}_{/-})^{\mathrm{op}} \colon \mathcal{B} \to \mathrm{LFib}(\mathcal{B}^{\mathrm{op}})$ (which is just the Yoneda embedding) with the functor of left Kan extension $\varphi(1)_! \colon \mathrm{LFib}(\mathcal{B}^{\mathrm{op}}) \to \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B}))$ along $\varphi(1) \colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B})$. But the latter composition is equivalent to the composition of $\varphi(1)^{\mathrm{op}} \colon \mathcal{B} \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B})^{\mathrm{op}}$ with the Yoneda embedding $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})^{\mathrm{op}} \hookrightarrow \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B}))$. By making use of the commutative diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\varphi(1)"] \arrow[rr, bend left, "\mathcal{B}_{/-}"] & \mathrm{Top}^{\mathrm{L}}(\mathcal{B})^{\mathrm{op}} \arrow[r, "\Gamma"] \arrow[d, hook] & (\mathrm{Top}^{\mathrm{L}}_{\infty})^{\mathrm{op}} \arrow[d, hook] \\
& \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}(\mathcal{B})) \arrow[r, "\Gamma_!"] & \mathrm{LFib}(\mathrm{Top}^{\mathrm{L}}_{\infty}),
\end{tikzcd}$$

the claim now follows. $\square$

REMARK 5.2.6.3. Corollary 5.2.6.2 in particular implies that for any map $s\colon B \to A$ in $\mathcal{B}$ the transition functor $s^*\colon \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/A}) \to \mathrm{Top}^{\mathrm{L}}(\mathcal{B}_{/B})$ can be identified with the pushout functor

$$- \sqcup_{\mathcal{B}_{/A}} \mathcal{B}_{/B}\colon (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/A}/} \to (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/B}/}.$$

As opposed to limits in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$, general colimits of $\mathcal{B}$-topoi can *not* be computed on the underlying $\mathcal{B}$-categories, not even after passing to the opposite $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$. The existence of constant colimits follows easily from Theorem 5.2.5.1:

LEMMA 5.2.6.4. *The large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is LConst-cocomplete.*

PROOF. In light of Remark 5.2.6.3, this follows from the fact that for any map $s\colon B \to A$ in $\mathcal{B}$ the ∞-categories $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/A}/}$ and $(\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}_{/B}/}$ have colimits by [57, Proposition 6.3.4.6] and $- \sqcup_{\mathcal{B}_{/A}} \mathcal{B}_{/B}$ preserves all colimits. $\square$

LEMMA 5.2.6.5. *The $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is Ω-cocomplete.*

PROOF. By Remark 5.2.1.3, it suffices to show that whenever G is a $\mathcal{B}$-groupoid and $d\colon \mathsf{G} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is a diagram, the functor $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})}(d, \mathrm{diag}(-))$ is corepresentable. Note that we have an equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, -) \simeq (\pi_{\mathsf{G}})_* \pi_{\mathsf{G}}^*$. Therefore, Corollary 5.2.3.6 implies that we can assume that d is in the image of $\Omega_{\mathcal{B}}[-]_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Cat}_{\widehat{\mathcal{B}}}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}})$. In this case, the claim follows from Corollary 5.2.2.3. $\square$

PROPOSITION 5.2.6.6. *The $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ is cocomplete.*

PROOF. By Proposition 3.2.4.1, this follows from, Lemmas 5.2.6.4 and 5.2.6.5. $\square$

5.2.7. A formula for the coproduct of $\mathcal{B}$-topoi. The goal of this section is to give an explicit description of the coproduct in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. To that end, recall that by the discussion in § 4.6.2 the large $\mathcal{B}$-category $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ of presentable $\mathcal{B}$-categories is symmetric monoidal. Explicitly, if D and E are presentable $\mathcal{B}$-categories, their tensor product $\mathsf{D} \otimes \mathsf{E}$ is equivalent to the $\mathcal{B}$-category $\underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D})$ of E-valued sheaves on D (i.e. the full subcategory of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{E})$ spanned by the continuous functors $\pi_A^* \mathsf{D}^{\mathrm{op}} \to \pi_A^* \mathsf{E}$ for each $A \in \mathcal{B}$). In light of this identification, the proof of Proposition 4.6.2.11 shows that if $f^*\colon \mathsf{D} \to \mathsf{D}'$ and $g^*\colon \mathsf{E} \to \mathsf{E}'$ are maps in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ with right adjoints f_* and g_*, then the functor $\mathrm{id} \otimes f^*\colon \mathsf{D} \otimes \mathsf{E} \to \mathsf{D} \otimes \mathsf{E}'$ can be identified with the left adjoint of $(f_*)_*\colon \underline{\mathsf{Sh}}_{\mathsf{E}'}(\mathsf{D}) \to \underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D})$, and the functor $g^* \otimes \mathrm{id}\colon \mathsf{D} \otimes \mathsf{E} \to \mathsf{D}' \otimes \mathsf{E}$ can be identified with the left adjoint of $(g^*)^*\colon \underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D}') \to \underline{\mathsf{Sh}}_{\mathsf{E}}(\mathsf{D})$.

PROPOSITION 5.2.7.1. *If X and Y are $\mathcal{B}$-topoi, then their tensor product $\mathsf{X} \otimes \mathsf{Y}$ is a $\mathcal{B}$-topos as well, and the functors $\mathrm{id} \otimes \mathrm{const}_{\mathsf{Y}}\colon \mathsf{X} \simeq \mathsf{X} \otimes \Omega \to \mathsf{X} \otimes \mathsf{Y}$ and $\mathrm{const}_{\mathsf{X}} \otimes \mathrm{id}\colon \mathsf{Y} \simeq \Omega \otimes \mathsf{Y} \to \mathsf{X} \otimes \mathsf{Y}$ exhibit $\mathsf{X} \otimes \mathsf{Y}$ as the coproduct of X and Y in $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$.*

Combining the above result with Proposition 4.6.3.10, we obtain the following generalisation of [7, Corollary 1.10]:

COROLLARY 5.2.7.2. *Assume that $\mathcal{B}$ is generated under colimits by (-1)-truncated objects. Then for $\mathcal{X}, \mathcal{Y} \in \mathrm{Top}^{\mathrm{L}}_{\mathcal{B}/}$ the canonical map*

$$\mathcal{X} \otimes_{\mathcal{B}} \mathcal{Y} \to \mathcal{X} \sqcup_{\mathcal{B}} \mathcal{Y}$$

is an equivalence.

The proof of Proposition 5.2.7.1 requires a few preparations and will be given at the end of this section. First, let us observe that this result provides an explicit formula for the pushout of ∞-topoi:

COROLLARY 5.2.7.3. *Given a cospan $\mathcal{X} \xleftarrow{f^*} \mathcal{Z} \xrightarrow{g^*} \mathcal{Y}$ in $\mathrm{Top}^{\mathrm{L}}_{\infty}$, there is a canonical equivalence*

$$\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y} \simeq \mathrm{Fun}^{\mathrm{cont}}_{\mathcal{Z}}(f_*(\Omega_{\mathcal{X}})^{\mathrm{op}}, g_* \Omega_{\mathcal{Y}})$$

in which the right-hand side denotes the full subcategory of $\mathrm{Fun}_{\mathcal{Z}}(f_*(\Omega_{\mathcal{X}})^{\mathrm{op}}, g_* \Omega_{\mathcal{Y}})$ that is spanned by the continuous functors. $\square$

REMARK 5.2.7.4. In light of Corollary 5.2.7.3, the ∞-topos $\mathcal{X} \sqcup_{\mathcal{Z}} \mathcal{Y}$ admits the following explicit description: It is the full subcategory of the ∞-category of natural transformations between the two $\widehat{\mathrm{Cat}}_\infty$-valued sheaves $\mathcal{X}_{/f^*(-)}$ and $\mathcal{Y}_{/g^*(-)}$ on $\mathcal{Z}$ that is spanned spanned by those maps $\varphi\colon (\mathcal{X}_{/f^*(-)})^{\mathrm{op}} \to \mathcal{Y}_{/g^*(-)}$ which satisfy that

(1) the functor $\varphi(A)$ preserves limits for all $A \in \mathcal{Z}$, and
(2) for any map $s\colon B \to A$ in $\mathcal{Z}$ the canonical lax square
$$\begin{tikzcd} (\mathcal{X}_{/f^*(B)})^{\mathrm{op}} \arrow[r, "\varphi(B)"] \arrow[d, "f^*(s)_!"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{Y}_{/g^*(B)} \arrow[d, "g^*(s)_*"] \\ (\mathcal{X}_{/f^*(A)})^{\mathrm{op}} \arrow[r, "\varphi(A)"'] & \mathcal{Y}_{/g^*(A)} \end{tikzcd}$$
commutes. $\square$

Admittedly, the description of the pushout of ∞-topoi in Remark 5.2.7.4 is rather unwieldy in general. However, we can paint a more concrete picture in the following case:

EXAMPLE 5.2.7.5. Let X be a $\mathcal{B}$-topos and let C be an arbitrary $\mathcal{B}$-category. Then Proposition 5.2.7.1 implies that the commutative square
$$\begin{tikzcd} \Omega \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{const}_{\mathsf{X}}"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "(\mathrm{const}_{\mathsf{X}})_*"] \\ \mathsf{X} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}) \end{tikzcd}$$
is a pushout in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{L}}$. Furthermore, if $f\colon \mathcal{X} \to \mathcal{B}$ is the geometric morphism associated to X, then the lower horizontal map can be identified with the image of $\mathrm{diag}\colon \Omega_{\mathcal{X}} \to \underline{\mathsf{PSh}}_{\mathcal{X}}(f^*\mathsf{C})$ along f_*.

We now turn to the proof of Proposition 5.2.7.1. It is a straightforward adaption of the proof presented in [3, §2.3] to the setting of $\mathcal{B}$-categories. We begin with the following lemma:

LEMMA 5.2.7.6. *Let C and D be $\mathcal{B}$-categories with finite limits. Then precomposition with the canonical maps $\mathrm{id}_{\mathsf{C}} \times 1_{\mathsf{D}}\colon \mathsf{C} \to \mathsf{C} \times \mathsf{D}$ and $1_{\mathsf{C}} \times \mathrm{id}_{\mathsf{D}}\colon \mathsf{D} \to \mathsf{C} \times \mathsf{D}$ induces an equivalence*
$$\underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C}, \mathsf{E}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{D}, \mathsf{E})$$
for any $\mathcal{B}$-category E with finite limits. In other words, these two maps exhibit $\mathsf{C} \times \mathsf{D}$ as the coproduct of C and D in $\mathsf{Cat}_{\mathcal{B}}^{\mathrm{lex}}$.

PROOF. The composition
$$\begin{aligned} \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C}, \mathsf{E}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{D}, \mathsf{E}) &\xrightarrow{\mathrm{pr}_0^* \times \mathrm{pr}_1^*} \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \\ &\xrightarrow{\simeq} \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C} \times \mathsf{D}, \mathsf{E} \times \mathsf{E}) \\ &\xrightarrow{(-\times-)_*} \underline{\mathsf{Fun}}_{\mathcal{B}}^{\mathrm{lex}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \end{aligned}$$
defines an inverse. $\square$

The rough strategy of the proof of Proposition 5.2.7.1 is to first prove the claim for free $\mathcal{B}$-topoi, which will follow from Lemma 5.2.7.6. In order to reduce the general case to this setting we need to understand the compatibility of tensor products with localisations:

LEMMA 5.2.7.7. *Suppose that C and D are presentable $\mathcal{B}$-categories and that $\mathsf{W} \hookrightarrow \mathsf{C}$ and $\mathsf{S} \hookrightarrow \mathsf{D}$ are small subcategories. Let $\mathsf{C}' \hookrightarrow \mathsf{C}$ be a small full subcategory that exhibits C as the free $\mathsf{Filt}_{\mathsf{U}}$-cocompletion of C' for some sound doctrine U. Let $\mathsf{D}' \hookrightarrow \mathsf{D}$ be chosen similarly. We write $\tau\colon \mathsf{C} \times \mathsf{D} \to \mathsf{C} \otimes \mathsf{D}$ for the universal bilinear functor. Let us set $\mathsf{W} \boxtimes \mathsf{S} = (\mathsf{W} \times (\mathsf{D}')^{\simeq}) \sqcup ((\mathsf{C}')^{\simeq} \times \mathsf{S})$. Then the canonical map $\mathsf{C} \otimes \mathsf{D} \to \mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \otimes \mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ induces an equivalence*
$$\mathsf{Loc}_{\mathsf{W} \boxtimes \mathsf{S}}(\mathsf{C} \otimes \mathsf{D}) \xrightarrow{\simeq} \mathsf{Loc}_{\mathsf{W}}(\mathsf{C}) \otimes \mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),$$

where the left-hand side is the $\mathcal{B}$-category of local objects with respect to $(\tau,\tau)\colon \mathsf{W}\boxtimes\mathsf{S}\to\mathsf{C}\otimes\mathsf{D}$.

PROOF. Let E be any other presentable $\mathcal{B}$-category and let us denote by $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})_{\mathsf{W}\boxtimes\mathsf{S}}$ the full subcategory of $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})$ that is spanned by those bilinear functors $\pi_A^*\mathsf{C}\times\pi_A^*\mathsf{D}\to\pi_A^*\mathsf{E}$ (in arbitrary context $A\in\mathcal{B}$) whose precomposition with $\pi_A^*(\mathsf{W}\boxtimes\mathsf{S})\to\pi_A^*\mathsf{C}\times\pi_A^*\mathsf{D}$ factors through $\pi_A^*\mathsf{E}^{\simeq}$. By combining the universal property of the tensor product with Corollary 4.4.3.2, we now obtain a chain of equivalences

$$\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}\boxtimes\mathsf{S}}(\mathsf{C}\otimes\mathsf{D}),\mathsf{E})\simeq\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C}\otimes\mathsf{D},\mathsf{E})_{\mathsf{W}\boxtimes\mathsf{S}}\simeq\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})_{\mathsf{W}\boxtimes\mathsf{S}}.$$

Note that a bilinear functor $f\colon\mathsf{C}\times\mathsf{D}\to\mathsf{E}$ is contained in $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})_{\mathsf{W}\boxtimes\mathsf{D}}$ if and only if

(1) for any $c\colon A\to\mathsf{C}'$ in context $A\in\mathcal{B}$ the functor $\pi_A^*\mathsf{S}\hookrightarrow\pi_A^*\mathsf{D}\xrightarrow{f(c,-)}\pi_A^*\mathsf{E}$ factors through $\pi_A^*\mathsf{E}^{\simeq}$, and
(2) for any $d\colon A\to\mathsf{D}'$ in context $A\in\mathcal{B}$ the functor $\pi_A^*\mathsf{W}\hookrightarrow\pi_A^*\mathsf{C}\xrightarrow{f(-,d)}\pi_A^*\mathsf{E}$ factors through $\pi_A^*\mathsf{E}^{\simeq}$.

Let $f'\colon\mathsf{C}\to\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ be the image of f under the equivalence $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})\simeq\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C},\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D},\mathsf{E}))$ from Lemma 4.6.1.3. Now the first condition is equivalent to the composition $\mathsf{C}'\hookrightarrow\mathsf{C}\xrightarrow{f'}\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ taking values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),\mathsf{E})$. Note that the inclusion $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),\mathsf{E})\hookrightarrow\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{D},\mathsf{E})$ is given by precomposition with $\mathsf{D}\to\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})$ and is therefore cocontinuous. Since C' generates C under $\mathsf{Filt}_{\mathsf{U}}$-colimits, it thus follows that (1) is equivalent to f' being contained in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C},\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),\mathsf{E}))$. Similarly, if $f''\colon\mathsf{D}\to\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C},\mathsf{E})$ is the other transpose of f, condition (2) is equivalent to f'' taking values in $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}}(\mathsf{C}),\mathsf{E})$. Thus the naturality of the equivalence in Lemma 4.6.1.3 implies that f satisfies (1) and (2) if and only if f is contained in $\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}}(\mathsf{C})\times\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),\mathsf{E})$. As the same argument can be carried out for bilinear functors in arbitrary context, this shows that the equivalence $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{C}\otimes\mathsf{D},\mathsf{E})\simeq\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{C}\times\mathsf{D},\mathsf{E})$ restricts to an equivalence $\underline{\mathsf{Fun}}^{\mathrm{cc}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}\boxtimes\mathsf{S}}(\mathsf{C}\otimes\mathsf{D}),\mathsf{E})\simeq\underline{\mathsf{Fun}}^{\mathrm{bil}}_{\mathcal{B}}(\mathsf{Loc}_{\mathsf{W}}(\mathsf{C})\times\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}),\mathsf{E})$, which proves the claim. ☐

A similar argument as above shows the following:

LEMMA 5.2.7.8. *Let C and D be presentable $\mathcal{B}$-categories and let $\mathsf{W}\hookrightarrow\mathsf{C}$ and $\mathsf{S}\hookrightarrow\mathsf{D}$ be small subcategories. Then the commutative square*

$$\begin{tikzcd}
\mathsf{C}\otimes\mathsf{D} \arrow[r] \arrow[d] & \mathsf{C}\otimes\mathsf{Loc}_{\mathsf{S}}(\mathsf{D}) \arrow[d] \\
\mathsf{Loc}_{\mathsf{W}}(\mathsf{C})\otimes\mathsf{D} \arrow[r] & \mathsf{Loc}_{\mathsf{W}}(\mathsf{C})\otimes\mathsf{Loc}_{\mathsf{S}}(\mathsf{D})
\end{tikzcd}$$

is a pushout in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$. ☐

PROOF OF PROPOSITION 5.2.7.1. To simplify notation, we shall write $i_0=\mathrm{id}\otimes\mathrm{const}_{\mathsf{Y}}$ as well as $i_1=\mathrm{const}_{\mathsf{X}}\otimes\mathrm{id}$. First, let us show the claim in the special case where $\mathsf{X}=\Omega_{\mathcal{B}}[\mathsf{C}]$ and $\mathsf{Y}=\Omega_{\mathcal{B}}[\mathsf{D}]$. In this situation, we have an equivalence $\mathsf{X}\otimes\mathsf{Y}\simeq\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{lex}}\times\mathsf{D}^{\mathrm{lex}})$ with respect to which the functors i_0 and i_1 are given by left Kan extension along $\mathrm{id}\times 1_{\mathsf{D}^{\mathrm{lex}}}\colon\mathsf{C}^{\mathrm{lex}}\to\mathsf{C}^{\mathrm{lex}}\times\mathsf{D}^{\mathrm{lex}}$ and $1_{\mathsf{C}^{\mathrm{lex}}}\times\mathrm{id}\colon\mathsf{D}^{\mathrm{lex}}\to\mathsf{C}^{\mathrm{lex}}\times\mathsf{D}^{\mathrm{lex}}$, respectively. By Lemma 5.2.7.6, the latter two functors exhibit $\mathsf{C}^{\mathrm{lex}}\times\mathsf{D}^{\mathrm{lex}}$ as the coproduct $\mathsf{C}^{\mathrm{lex}}\sqcup\mathsf{D}^{\mathrm{lex}}$ in $\mathsf{Cat}^{\mathrm{lex}}_{\mathcal{B}}$. As the functor $(-)^{\mathrm{lex}}$ is a left adjoint and thus preserves coproducts, we end up with an equivalence $\mathsf{X}\otimes\mathsf{Y}\simeq\Omega_{\mathcal{B}}[\mathsf{C}\sqcup\mathsf{D}]$ with respect to which i_0 and i_1 correspond to the image of the inclusions $\mathsf{C}\hookrightarrow\mathsf{C}\sqcup\mathsf{D}$ and $\mathsf{D}\hookrightarrow\mathsf{C}\sqcup\mathsf{D}$ along the functor $\Omega_{\mathcal{B}}[-]$. The claim thus follows from Corollary 5.2.2.3.

In the general case, we may choose left exact and accessible Bousfield localisations $L\colon\Omega_{\mathcal{B}}[\mathsf{C}]\to\mathsf{X}$ and $L'\colon\Omega_{\mathcal{B}}[\mathsf{D}]\to\mathsf{Y}$, cf. Corollary 5.2.3.5. By Lemma 5.2.7.8 we have a pushout square

$$\begin{tikzcd}
\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\Omega_{\mathcal{B}}[\mathsf{D}] \arrow[r] \arrow[d] & \mathsf{X}\otimes\Omega_{\mathcal{B}}[\mathsf{D}] \arrow[d] \\
\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\mathsf{Y} \arrow[r] & \mathsf{X}\otimes\mathsf{Y}
\end{tikzcd}$$

in $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$. The upper horizontal functor is equivalent to the functor

$$L_*\colon \underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{D}^{\mathrm{lex}})^{\mathrm{op}}, \Omega_{\mathcal{B}}[\mathsf{C}]) \to \underline{\mathsf{Fun}}_{\mathcal{B}}((\mathsf{D}^{\mathrm{lex}})^{\mathrm{op}}, \mathsf{X})$$

Thus $\mathsf{X}\otimes\mathsf{Y}$ is equivalent to the intersection of two accessible and left exact Bousfield localisations of $\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\Omega_{\mathcal{B}}[\mathsf{D}]$ and therefore by [57, Lemma 6.3.3.4] in particular a $\mathcal{B}$-topos. and therefore a left exact and accessible Bousfield localisation. By symmetry, the same holds for the left vertical functor. Since the square

$$\begin{tikzcd}
\Omega_{\mathcal{B}}[\mathsf{C}] \arrow[r] \arrow[d, "i_0"'] & \mathsf{X} \arrow[d, "i_0"'] \\
\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\Omega_{\mathcal{B}}[\mathsf{D}] \arrow[r] & \mathsf{X}\otimes\Omega_{\mathcal{B}}[\mathsf{D}]
\end{tikzcd}$$

commutes, it follows that $i_0\colon \mathsf{X}\to\mathsf{X}\otimes\Omega_{\mathcal{B}}[\mathsf{D}]$ is left exact. Since $i_0\colon\mathsf{X}\to\mathsf{X}\otimes\mathsf{Y}$ factors as the composite $\mathsf{X}\xrightarrow{i_0}\mathsf{X}\otimes\Omega_{\mathcal{B}}[\mathsf{D}]\to\mathsf{X}\otimes\mathsf{Y}$ it is therefore also left exact. The same argument shows that $i_1\colon\mathsf{Y}\to\mathsf{X}\otimes\mathsf{Y}$ is left exact. Finally, note that L and L' induce a commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X}\otimes\mathsf{Y},\mathsf{Z}) \arrow[r, "(i_0^*,i_1^*)"] \arrow[d] & \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{X},\mathsf{Z})\times\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{Y},\mathsf{Z}) \arrow[d] \\
\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\Omega_{\mathcal{B}}[\mathsf{D}],\mathsf{Z}) \arrow[r, "\simeq"] & \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{\mathcal{B}}[\mathsf{C}],\mathsf{Z})\times\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{\mathcal{B}}[\mathsf{D}],\mathsf{Z})
\end{tikzcd}$$

for any $\mathcal{B}$-topos Z. As the lower horizontal map being an equivalence implies that (i_0^*, i_1^*) is fully faithful, it thus suffices to see that this functor is also essentially surjective. Using Remark 5.2.1.3, it will be enough to show that for any two algebraic morphisms $f\colon\mathsf{X}\to\mathsf{Z}$ and $g\colon\mathsf{Y}\to\mathsf{Z}$ the induced map $\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\Omega_{\mathcal{B}}[\mathsf{D}]\to\mathsf{Z}$ factors through $\Omega_{\mathcal{B}}[\mathsf{C}]\otimes\Omega_{\mathcal{B}}[\mathsf{D}]\to\mathsf{X}\otimes\mathsf{Y}$. This is a direct consequence of Lemma 5.2.7.7. □

5.2.8. Diaconescu's theorem. In classical category theory, Diaconescu's theorem states that for any 1-category $\mathcal{C}$ and any 1-topos $\mathcal{X}$, a functor $f\colon\mathcal{C}\to\mathcal{X}$ is *internally flat* if and only if its left Kan extension $h_!f\colon\ \mathrm{PSh}_{\mathrm{Set}}(\mathcal{C})\to\mathcal{X}$ preserves finite limits, see for example [46, Theorem B.3.2.7]. Here f being internally flat precisely means that its internal unstraightening results in a *filtered* internal category in $\mathcal{X}$. For ∞-categories, a comparable result has been proved by Lurie [57, Proposition 6.1.5.2] in the special case where the ∞-category $\mathcal{C}$ already admits finite limits. In the general case, Raptis and Schäppi proved Diaconescu's theorem under the assumption that the codomain $\mathcal{X}$ is a hypercomplete ∞-topos [75].

The main goal of this section is to establish a general version of Diaconescu's theorem for $\mathcal{B}$-topoi and therefore also a general version of Diaconescu's theorem for ∞-topoi, without any hypercompleteness assumptions. To that end, let us say that a presheaf $F\colon\mathsf{C}^{\mathrm{op}}\to\Omega$ on an arbitrary $\mathcal{B}$-category C is *flat* if it is $\mathsf{Fin}_{\mathcal{B}}$-flat in the sense of Definition 4.3.4.1. We will denote by $\underline{\mathsf{Flat}}_{\mathcal{B}}(\mathsf{C})\subseteq\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ the associated $\mathcal{B}$-category of flat functors. Recall from Proposition 4.2.3.7 that the doctrine $\mathsf{Fin}_{\mathcal{B}}$ is sound. Therefore, Proposition 4.3.4.6 implies:

Proposition 5.2.8.1. *For any $\mathcal{B}$-category C, a functor $F\colon\mathsf{C}^{\mathrm{op}}\to\Omega$ is flat if and only if the $\mathcal{B}$-category $\mathsf{C}_{/F}$ is filtered.* □

Diaconescu's theorem for $\mathcal{B}$-topoi can now be stated as follows:

Theorem 5.2.8.2. *Let X be a $\mathcal{B}$-topos with associated geometric morphism $f_*\colon\mathcal{X}\to\mathcal{B}$ and let C be an arbitrary $\mathcal{B}$-category. The precomposition with the Yoneda embedding induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}),\mathsf{X})\simeq f_*\underline{\mathsf{Flat}}_{\mathcal{X}}(f^*\mathsf{C}^{\mathrm{op}}).$$

Specialising to the case where $\mathcal{B}\simeq\mathcal{S}$, Theorem 5.2.8.2 implies:

Corollary 5.2.8.3. *For any small ∞-category $\mathcal{C}$, a functor $f\colon\mathcal{C}\to\mathcal{B}$ is flat if and only if its Yoneda extension $h_!f\colon\ \mathrm{PSh}_{\mathcal{S}}(\mathcal{C})\to\mathcal{B}$ preserves finite limits. In particular, the functor of left Kan extension along $h_{\mathcal{C}}$ induces an equivalence*

$$h_!\colon\ \mathrm{Flat}_{\mathcal{B}}(\mathcal{C}^{\mathrm{op}})\simeq\mathrm{Fun}^{\mathrm{alg}}(\mathrm{PSh}_{\mathcal{S}}(\mathcal{C}),\mathcal{B})$$

of ∞-categories. $\square$

REMARK 5.2.8.4. Corollary 5.2.8.3 can be used to define morphisms of general ∞-sites: if $\mathcal{C}$ and $\mathcal{D}$ are ∞-sites, a functor $f\colon \mathcal{C} \to \mathcal{D}$ is a morphism of ∞-sites if the associated functor $f'\colon \mathcal{C} \to \mathrm{Sh}(\mathcal{D})$ (which is obtained by composing f with the sheafified Yoneda embedding $Lh\colon \mathcal{D} \to \mathrm{Sh}(\mathcal{D})$) is flat and if for every covering $(c_i \to c)_{i\in I}$ in $\mathcal{C}$ the induced functor $\bigsqcup_i f'(c_i) \to f'(c)$ is a cover in $\mathrm{Sh}(\mathcal{D})$. Using this definition, Corollary 5.2.8.3 and [57, Lemma 6.2.3.19] imply that every morphism of ∞-sites $f\colon \mathcal{C} \to \mathcal{D}$ induces an algebraic morphism $F\colon \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$.

The proof of Theorem 5.2.8.2 relies on the following two elementary lemmas:

LEMMA 5.2.8.5. *Let X be a $\mathcal{B}$-topos and let $f\colon \mathcal{X} \to \mathcal{B}$ be the corresponding geometric morphism. Suppose that $p\colon \mathsf{P} \to \mathsf{C}$ is a left fibration of $\mathcal{X}$-categories that is classified by a functor $g\colon \mathsf{C} \to \Omega_{\mathcal{X}}$. Then the left fibration $f_*(p)$ of $\mathcal{B}$-categories is classified by the composition $f_*\mathsf{C} \xrightarrow{f_*(g)} \mathsf{X} \xrightarrow{\Gamma_{\mathsf{X}}} \Omega_{\mathcal{B}}$.*

PROOF. Since the functor f_* commutes with pullbacks and with powering by ∞-categories, the image of the universal left fibration $(\Omega_{\mathcal{X}})_{1/} \to \Omega_{\mathcal{X}}$ along f_* can be identified with $(\pi_{1_{\mathsf{X}}})_!\colon \mathsf{X}_{1_{\mathsf{X}}/} \to \mathsf{X}$ and is therefore classified by $\mathrm{map}_{\mathsf{X}}(1_{\mathsf{X}}, -) \simeq \Gamma_{\mathsf{X}}$. Hence the claim follows. $\square$

LEMMA 5.2.8.6. *Let X be a $\mathcal{B}$-topos and let $f\colon \mathcal{X} \to \mathcal{B}$ be the corresponding geometric morphism. Then for any $\mathcal{B}$-category C, there is a commutative square*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "h_{\mathsf{C}}"] \arrow[d, "\eta"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "(\mathrm{const}_{\mathsf{X}})_*"] \\
f_*f^*\mathsf{C} \arrow[r, "f_*(h_{f^*\mathsf{C}})"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}).
\end{tikzcd}$$

PROOF. Transposing the Yoneda embedding $h_{f^*\mathsf{C}}\colon f^*\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{X}}(f^*\mathsf{C})$ across the adjunction $f^* \dashv f_*$ yields the composition

$$\mathsf{C} \xrightarrow{\eta} f^*f_*\mathsf{C} \xhookrightarrow{f_*(h_{f^*\mathsf{C}})} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X})$$

in which η is the adjunction unit. By transposing the above map across the adjunction $\mathsf{C}^{\mathrm{op}} \times - \dashv \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, -)$, one ends up with the functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \xrightarrow{\eta} f_*f^*(\mathsf{C}^{\mathrm{op}} \times \mathsf{C}) \xrightarrow{f_*(\mathrm{map}_{f^*\mathsf{C}})} \mathsf{X}.$$

On the other hand, the transpose of the composition $\mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X})$ yields

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \xrightarrow{\mathrm{map}_{\mathsf{C}}} \Omega_{\mathcal{B}} \xrightarrow{\mathrm{const}_{\mathsf{X}}} \mathsf{X},$$

so it suffices to show that these two functors are equivalent. By Lemma 5.2.8.5 the functor $\mathrm{map}_{f_*f^*\mathsf{C}}$ is equivalent to the composition $\Gamma_{\mathsf{X}} \circ f_*(\mathrm{map}_{f^*\mathsf{C}})\colon f_*\mathsf{C}^{\mathrm{op}} \times f_*\mathsf{C} \to \mathsf{X} \to \Omega_{\mathcal{B}}$. As a consequence, the morphism of functors $\mathrm{map}_{\mathsf{C}} \to \mathrm{map}_{f_*f^*\mathsf{C}} \circ \eta$ that is induced by the action of η on mapping $\mathcal{B}$-groupoids determines a morphism $\mathrm{map}_{\mathsf{C}} \to \Gamma_{\mathsf{X}} \circ f_*(\mathrm{map}_{f^*\mathsf{C}}) \circ \eta$ which in turn transposes to a map

$$\mathrm{const}_{\mathsf{X}} \circ \mathrm{map}_{\mathsf{C}} \to f_*(\mathrm{map}_{f^*\mathsf{C}}) \circ \eta.$$

To show that this is an equivalence, it will be enough to show that it induces an equivalence when evaluated at (τ, τ), where τ is the tautological object in C, i.e. the one given by the identity of C_0. But by construction, the resulting map is simply the transpose of $\eta\colon \mathsf{C}_1 \to f_*f^*\mathsf{C}_1$ across the adjunction $f^* \dashv f_*$ and therefore an equivalence, as desired. $\square$

PROOF OF THEOREM 5.2.8.2. To begin with, we note that the universal property of presheaf $\mathcal{B}$-categories together with Remarks 5.2.1.3 and 4.3.4.2 implies that it suffices to show that a functor $g\colon \mathsf{C} \to \mathsf{X}$ transposes to a flat functor $g'\colon f^*\mathsf{C} \to \Omega_{\mathcal{X}}$ if and only if its Yoneda extension $(h_{\mathsf{C}})_!(g)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$

preserves finite limits. Note that by Lemma 5.2.8.6 (and the fact that base change along f_* preserves cocontinuity), we have a commutative diagram

$$\begin{tikzcd} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "(\mathrm{const}_{\mathsf{X}})_*"] \arrow[dr, bend left, "(h_{\mathsf{C}})_!(g)"] & \\ \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}) \arrow[r, "f_*(h_{f^*\mathsf{C}})_!(g')"] & \mathsf{X}. \end{tikzcd}$$

Therefore, g' being flat immediately implies that $(h_{\mathsf{C}})_!(g)$ is an algebraic morphism, so it suffices to consider the converse implication. Suppose therefore that the left Kan extension $(h_{\mathsf{C}})_!(g)$ preserves finite limits. We wish to show that the functor $(h_{f^*\mathsf{C}})_!(g')$ preserves finite limits as well. In light of the previous commutative diagram and the fact that $(\mathrm{const}_{\mathsf{X}})_*$ preserves finite limits, it is clear that it preserves the final object, so we only need to consider the case of pullbacks. By Lemma 5.2.2.8, we may reduce to pullbacks of cospans in $\underline{\mathsf{PSh}}_{\mathcal{X}}(f^*\mathsf{C})$ (in arbitrary context $U \in \mathcal{X}$) which are contained in the essential image of the Yoneda embedding $h_{f^*\mathsf{C}}$. Since any such cospan is determined by a map $U \to (f^*\mathsf{C})^{\Lambda^2_0}$, it factors through the core inclusion $\tau_{f^*\mathsf{C}}\colon f^*(\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1) \to (f^*\mathsf{C})^{\Lambda^2_0}$, which we may regard as the *tautological* cospan. Therefore, it is enough to show that the pullback of $\tau_{f^*\mathsf{C}}$ is preserved by $(h_{f^*\mathsf{C}})_!(g')$. As this diagram is in context $f^*(\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1)$, we may make use of the adjunction $f^* \dashv f_*$ to regard $\tau_{f^*\mathsf{C}}$ as a cospan in $f_*f^*\mathsf{C}$ in context $\mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1$. As such, it is precisely the cospan that arises as the image of the tautological cospan τ_{C} in C (i.e. the one given by the core inclusion $\tau_{\mathsf{C}}\colon \mathsf{C}_1 \times_{\mathsf{C}_0} \mathsf{C}_1 \hookrightarrow \mathsf{C}^{\Lambda^2_0}$) along $\eta\colon \mathsf{C} \to f_*f^*\mathsf{C}$. By again making use of Lemma 5.2.8.6, we thus conclude that the image of $\tau_{f^*\mathsf{C}}$ along $f_*(h_{f^*\mathsf{C}})\colon f_*f^*\mathsf{C} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X})$ can be identified with the image of τ_{C} along the composition $(\mathrm{const}_{\mathsf{X}})_* \circ h_{\mathsf{C}}$. In particular, the cospan $f_*(h_{f^*\mathsf{C}})(\tau_{f^*\mathsf{C}})$ is contained in the image of $(\mathrm{const}_{\mathsf{X}})_*$, hence the above commutative diagram yields the claim. $\square$

In the remainder of this section we will explain how our version of Diaconescu's theorem for ∞-topoi (Corollary 5.2.8.3) relates to that of Raptis and Schäppi [75] when $\mathcal{B}$ is *hypercomplete*. More precisely, in [75, Theorem 1.1 (3)] Raptis and Schäppi give an explicit characterisation of flat functors $\mathcal{C} \to \mathcal{X}$ valued in a hypercomplete ∞-topos $\mathcal{X}$, and a priori it is not clear how to relate this description to our substantially less explicit characterisation of flat functors in terms of internal filteredness (Proposition 5.2.8.1). Therefore, our goal is to recover the description in [75, Theorem 1.1 (3)] from Proposition 5.2.8.1. To that end, suppose that there is a left exact and accessible localisation $L\colon \mathrm{PSh}(\mathcal{D}) \to \mathcal{B}$ for some small ∞-category $\mathcal{D}$, and let $i\colon \mathcal{B} \hookrightarrow \mathrm{PSh}(\mathcal{D})$ be its right adjoint. We denote by $\mathcal{C}_{f/} \to \mathcal{C}$ the left fibration (in $\mathrm{Cat}(\mathcal{B})$) that is classified by $f\colon \mathcal{C} \to \Omega$. By definition, it sits inside a pullback square

$$\begin{tikzcd} \mathcal{C}_{f/} \arrow[r] \arrow[d] & \Omega_{1/} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \Omega \end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$. If $\mathcal{B}$ is hypercomplete, we deduce from Propositions 5.2.8.1 and 4.2.3.14 that f being flat is equivalent to $(\mathcal{C}_{f/})^{\mathrm{op}}$ being quasi-filtered. In order to obtain a more explicit understanding of the latter condition, let us first consider the constant presheaf $\underline{\mathcal{C}}\colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ with value $\mathcal{C}$ and compute the pullback

$$\begin{tikzcd} \underline{\mathcal{C}}_{f'/} \arrow[r] \arrow[d] & i(\Omega_{1/}) \arrow[d] \\ \underline{\mathcal{C}} \arrow[r, "f'"] & i(\Omega) \end{tikzcd}$$

in $\mathrm{Cat}(\mathrm{PSh}(\mathcal{D})) \simeq \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. Here f' is the transpose of $f\colon \mathcal{C} \to \Omega$ across the adjunction $L \dashv i$. Note that $L(\underline{\mathcal{C}}_{f'/}) \simeq \mathcal{C}_{f/}$ since L is left exact. Upon evaluating the previous pullback square at any $d \in \mathcal{D}$, we obtain a commutative rectangle

$$\begin{tikzcd} \underline{\mathcal{C}}_{f'/}(d) \arrow[r] \arrow[d] & \mathcal{B}_{Ld/} \arrow[r] \arrow[d] & \mathcal{B}_{L(d)/\!\!/L(d)} \arrow[d] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{B} \arrow[r, "\pi^*_{L(d)}"] & \mathcal{B}_{/Ld} \end{tikzcd}$$

where the lower composite is is equivalent to $f'(d)$ and all squares are pullback squares. Here $\mathcal{B}_{L(d)/\!\!/L(d)}$ denotes the ∞-category of pointed objects in $\mathcal{B}_{/L(d)}$. It follows that we can explicitly describe $\underline{\mathcal{C}}_{f'/}(d)$ as the pullback in the left square and that for any map $s\colon d \to e$ in $\mathcal{D}$ the functor $s^*\colon \underline{\mathcal{C}}_{f'/}(e) \to \underline{\mathcal{C}}_{f'/}(d)$ is induced by pulling back the canonical functor $s^*\colon \mathcal{B}_{Le/} \to \mathcal{B}_{Ld/}$ along f. To proceed, we now need the following lemma that characterises those Cat_∞-valued presheaves on $\mathcal{D}$ which yield quasi-filtered $\mathcal{B}$-categories upon sheafification:

LEMMA 5.2.8.7. *Let $\mathsf{C} \in \mathrm{Cat}(\mathrm{PSh}(\mathcal{D}))$. Then $L\mathsf{C}$ is a quasi-filtered $\mathcal{B}$-category if and only if for any finite ∞-category $\mathcal{K}$, any $d \in \mathcal{D}$ and any map $\beta\colon \mathcal{K} \to \mathsf{C}(d)$ there exist morphisms $(s_i\colon d_i \to d)_i$ such that $(Ls_i)\colon \bigsqcup_i L(d_i) \twoheadrightarrow L(d)$ is a cover in $\mathcal{B}$, and there are maps $\alpha_i\colon \mathcal{K}^{\triangleright} \to \mathsf{C}(d_i)$ for every i that fit into commutative diagrams*

$$\begin{tikzcd} \mathcal{K}^{\triangleright} \arrow[r, "\alpha_i"] & \mathsf{C}(d_i) \\ \mathcal{K} \arrow[u] \arrow[r, "\beta"] & \mathsf{C}(d) \arrow[u, "s_i^*"'] \end{tikzcd}$$

of ∞-categories.

PROOF. The if part of the statement is a direct consequence of Proposition A.2. For the converse we note that for any finite ∞-category $\mathcal{K}$ the canonical map $L\underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}, \mathsf{C})^{\simeq} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, L\mathsf{C})^{\simeq}$ is an equivalence. Now for some $\beta\colon \mathcal{K} \to \mathsf{C}(d)$ corresponding via Yoneda's lemma to a morphism $d \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathcal{K}, \mathsf{C})^{\simeq}$ this shows that the projection map $\mathrm{pr}_1\colon d \times_{\underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K},\mathsf{C})^{\simeq}} \underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}^{\triangleright}, \mathsf{C})^{\simeq} \to d$ becomes a cover after applying L. We now pick a cover $(t_i)\colon \bigsqcup_i d_i \to d \times_{\underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K},\mathsf{C})^{\simeq}} \underline{\mathsf{Fun}}_{\mathrm{PSh}(\mathcal{D})}(\mathcal{K}^{\triangleright}, \mathsf{C})^{\simeq}$ in $\mathrm{PSh}(\mathcal{D})$ by representables. Then the $s_i = \mathrm{pr}_1 \circ t_i$ yield a cover after applying L, and by Yoneda's lemma every s_i gives a commutative square as in the claim. $\square$

By combining Lemma 5.2.8.7 with the discussion preceding it, we recover the following characterisation of flat functors in the hypercomplete case:

PROPOSITION 5.2.8.8 ([75, Definition 3.1 and Theorem 3.5]). *Suppose that $\mathcal{B}$ is hypercomplete, and let $f\colon \mathcal{C} \to \mathcal{B}$ be a functor. Then f is flat if and only if for every $d \in \mathcal{D}$, every functor $\alpha\colon \mathcal{K} \to \mathcal{C}$ (where $\mathcal{K}$ is a finite ∞-category) and every map $\overline{\beta}\colon \mathcal{K}^{\triangleleft} \to \mathcal{B}$ with cone point $L(d)$ such that $f\alpha \simeq \overline{\beta}|_{\mathcal{K}}$, there are maps $(s_i\colon d_i \to d)_i$ in $\mathcal{D}$ such that*

(1) $L(s_i)\colon \bigsqcup_i L(d_i) \twoheadrightarrow d$ is a cover in $\mathcal{B}$;
(2) for each i there is a cocone $\overline{\alpha}_i\colon \mathcal{K}^{\triangleleft} \to \mathcal{C}$ extending α, together with a morphism of cones $h\colon \Delta^1 \diamond \mathcal{K} \to \mathcal{B}$ from the cocone $\overline{\beta} \circ s_i$ (which is given by composing the cone point of β with s_i) to $f \circ \overline{\alpha}_i$. $\square$

5.2.9. Étale $\mathcal{B}$-topoi. By Theorem 5.2.5.1, geometric morphisms $f_*\colon \mathcal{X} \to \mathcal{B}$ are in correspondence with $\mathcal{B}$-topoi $f_*(\Omega_{\mathcal{X}})$. In this section, we study those $\mathcal{B}$-topoi that correspond to *étale* geometric morphisms. To prepare our discussion, note that Corollary 5.2.4.4 implies that the functor $(-)^{\mathrm{disc}} = \Omega^{(-)}\colon \Omega \hookrightarrow \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ from Definition 5.2.4.5 is cocontinuous. Moreover, as this functor carries the final object 1_Ω to Ω itself, the universal property of Ω implies that we have a functorial equivalence $(-)^{\mathrm{disc}} \simeq \Omega_{/-}$. In particular, the functor $\Omega_{/-}$ takes values in $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ too. We may therefore define:

DEFINITION 5.2.9.1. A $\mathcal{B}$-topos X is *étale* if there is an equivalence $\mathsf{X} \simeq \Omega_{/\mathsf{G}}$ for some $\mathcal{B}$-groupoid G.

In [57, Proposition 6.3.5.5], Lurie proved a universal property for étale geometric morphisms of ∞-topoi. In light of Theorem 5.2.5.1, such étale geometric morphisms precisely correspond to étale $\mathcal{B}$-topoi. The main goal of this section is to discuss how Lurie's result can also be deduced from Diaconescu's theorem. To that end, note that if X is a $\mathcal{B}$-topos with associated geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ and if G is a $\mathcal{B}$-groupoid, the fact that we may identify $\mathsf{X} \simeq f_*\Omega_{\mathcal{X}}$ implies that precomposition with the Yoneda embedding $h_{\mathsf{G}}\colon \mathsf{G} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \Omega) \simeq \Omega_{/\mathsf{G}}$ induces a map $\underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega_{/\mathsf{G}}, \mathsf{X}) \to f_*\underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{G}, \Omega_{\mathcal{X}}) \simeq \mathsf{X}_{/\operatorname{const}_{\mathsf{X}}\mathsf{G}}$. The universal property of étale $\mathcal{B}$-topoi can now be formulated as follows:

Proposition 5.2.9.2. *Let G be a $\mathcal{B}$-groupoid and let X be a $\mathcal{B}$-topos. Precomposition with the Yoneda embedding h_{G} induces a fully faithful functor $h_{\mathsf{G}}^*\colon \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{/\mathsf{G}}, \mathsf{X}) \hookrightarrow \mathsf{X}_{/\operatorname{const}_{\mathsf{X}}\mathsf{G}}$ that fits into a pullback square*

$$\begin{tikzcd} \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{/\mathsf{G}}, \mathsf{X}) \arrow[r, hook, "h_{\mathsf{G}}^*"] \arrow[d] & \mathsf{X}_{/\operatorname{const}_{\mathsf{X}}\mathsf{G}} \arrow[d, "(\pi_{\operatorname{const}_{\mathsf{X}}\mathsf{G}})_!"] \\ 1 \arrow[r, hook, "1_{\mathsf{X}}"] & \mathsf{X}. \end{tikzcd}$$

In particular, there is a canonical equivalence $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{/\mathsf{G}}, \mathsf{X}) \simeq \operatorname{map}_{\mathsf{X}}(1_{\mathsf{X}}, \operatorname{const}_{\mathsf{X}}\mathsf{G})$.

The proof of Proposition 5.2.9.2 requires the following lemma:

Lemma 5.2.9.3. *For any $\mathcal{B}$-groupoid G, the full embedding $\mathsf{G} \hookrightarrow \Omega_{/\mathsf{G}}$ that is obtained by combining the Yoneda embedding h_{G} the equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \Omega) \simeq \Omega_{/G}$ fits into a pullback square*

$$\begin{tikzcd} \mathsf{G} \arrow[r, hook] \arrow[d] & \Omega_{/\mathsf{G}} \arrow[d, "(\pi_{\mathsf{G}})_!"] \\ 1 \arrow[r, hook, "1_{\Omega}"] & \Omega. \end{tikzcd}$$

Proof. Since we have a commutative diagram

$$\begin{tikzcd} \Omega_{/\mathsf{G}} \arrow[r, "\simeq"] \arrow[dr, "(\pi_{\mathsf{G}})_!"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \Omega) \arrow[d, "\operatorname{colim}_{\mathsf{G}}"] \\ & \Omega, \end{tikzcd}$$

the claim follows once we show that a copresheaf $F\colon \mathsf{G} \to \Omega$ is representable if and only if $\operatorname{colim}_{\mathsf{G}} F \simeq 1_{\Omega}$. But F is representable if and only if $\mathsf{G}_{/F}$ admits an initial object, and since the latter is a $\mathcal{B}$-groupoid, this is in turn equivalent to $\mathsf{G}_{/F} \simeq 1$. Since by Proposition 3.1.4.1 we have $\mathsf{G}_{/F} \simeq \operatorname{colim}_{\mathsf{G}} F$, the claim follows. □

Proof of Proposition 5.2.9.2. Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be the geometric morphism that corresponds to the $\mathcal{B}$-topos $\mathcal{X}$. Since for every $U \in \mathcal{X}$ an $\mathcal{X}_{/U}$-groupoid is filtered if and only if it is final (see Remark 4.2.3.3), the Yoneda embedding $h_{f^*\mathsf{G}}\colon f^*\mathsf{G} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{G}, \Omega_{\mathcal{X}})$ induces an equivalence $f^*\mathsf{G} \simeq \underline{\mathsf{Flat}}_{\mathcal{X}}(f^*\mathsf{G})$. By combining this observation with Theorem 5.2.8.2, we thus find that precomposition with the Yoneda embedding $\mathsf{G} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \Omega)$ yields an equivalence

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega_{/\mathsf{G}}, \mathsf{X}) \simeq f_*(f^*\mathsf{G}).$$

Hence the claim follows from Lemma 5.2.9.3. □

Corollary 5.2.9.4. *The functor $\Omega_{/-}\colon \Omega \to \mathrm{Pr}^{\mathrm{R}}_{\mathcal{B}}$ factors through a cocontinuous and fully faithful embedding $\Omega_{/-}\colon \Omega \hookrightarrow \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ whose essential image is spanned by the étale $\mathcal{B}$-topoi.*

Proof. It is clear that this functor takes values in $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$, and by combining the descent property of Ω with Proposition 5.2.6.1, this functor must be cocontinuous. It therefore suffices to show that it is fully faithful. As we have seen above, we may identify $\Omega_{/-}$ with the restriction of the functor $(-)^{\mathrm{disc}}\colon \mathsf{Cat}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ from § 5.2.4 along the inclusion $\Omega \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$. Using Corollary 5.2.4.4, the claim thus follows once we show that for every $\mathcal{B}$-groupoid G the (partial) adjunction unit $\mathsf{G} \to \mathsf{Pt}_{\mathcal{B}}(\mathsf{G}^{\mathrm{disc}})$

is an equivalence. By construction, this map is obtained by the transpose of the evaluation map $\mathrm{ev}\colon \mathsf{G} \times \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \Omega) \to \Omega$, which by Yoneda's lemma is precisely the inverse of the equivalence from Proposition 5.2.9.2. This finishes the proof. $\square$

REMARK 5.2.9.5. The functor $\Omega_{/-}\colon \Omega \hookrightarrow \mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ also preserves finite limits. In fact, this is clear for the final object, and the case of binary products is an immediate consequence of the formula from Example 5.2.7.5 (together with the fact that the étale base change of this functor along π_A^* recovers the functor $(\Omega_{\mathcal{B}_{/A}})_{/-}$). This is already enough to deduce that $\Omega_{/-}$ preserves pullbacks: in fact, since Corollary 5.2.6.2 and Corollary 5.1.2.13 imply that $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ has Ω-descent, this follows from the argument in the second part of the proof of Lemma 5.2.2.8.

5.2.10. Subterminal $\mathcal{B}$-topoi. The goal of this section is to study *subterminal* $\mathcal{B}$-topoi. To begin with, observe that if $f_*\colon \mathcal{X} \to \mathcal{B}$ and $g_*\colon \mathcal{Y} \to \mathcal{B}$ are geometric morphism where f_* is fully faithful, then the formula that we derived in § 5.2.7 immediately implies that the geometric morphism $\mathcal{X} \times_{\mathcal{B}} \mathcal{Y} \to \mathcal{Y}$ (whose domain is the pullback in $\mathrm{Top}^{\mathrm{R}}_{\infty}$) is fully faithful as well. Thus, we may define:

DEFINITION 5.2.10.1. A $\mathcal{B}$-topos X is said to be *subterminal* if the global sections functor Γ_{X} is fully faithful, or equivalently if the associated geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ is fully faithful.

By Theorem 5.2.5.1, any subterminal $\mathcal{B}$-topos X determines is determined by a left exact and accessible Bousfield localisation of $\mathcal{B}$ and therefore in particular by a class of maps S in $\mathcal{B}$ for which $\Gamma(\mathsf{X}) \simeq \mathrm{Loc}_S(\mathcal{B})$. The main goal of this section is to characterise those collections of maps S that arise from and give rise to a subterminal $\mathcal{B}$-topos X in this way, and to describe the associated endofunctor

$$\Gamma_{\mathsf{X}}\,\mathrm{const}_{\mathsf{X}}\colon \Omega \to \Omega$$

by an explicit colimit formula in terms of S, akin to Lurie's sheafification formula from [57, § 6.2.2].

We begin with the following definition:

DEFINITION 5.2.10.2. Let $d\colon \mathsf{I} \to \Omega$ be a functor of $\mathcal{B}$-categories, where I is small. We define the *+-construction* $(-)^+_d\colon \Omega \to \Omega$ relative to d as the composition

$$\Omega \xhookrightarrow{h_\Omega} \underline{\mathsf{PSh}}_{\mathcal{B}}(\Omega) \xrightarrow{d^*} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I}) \xrightarrow{\mathrm{colim}_{\mathsf{I}^{\mathrm{op}}}} \Omega,$$

i.e. by the formula $(-)^+_d = \mathrm{colim}_{\mathsf{I}^{\mathrm{op}}}\,\mathrm{map}_\Omega(d(-), -)$.

REMARK 5.2.10.3. If I is *cofiltered*, i.e. if I^{op} is filtered, then the +-construction $(-)^+_d$ is left exact.

REMARK 5.2.10.4. If I is cofiltered, then the diagonal functor $\mathrm{diag}_{\mathsf{I}^{\mathrm{op}}}\colon \Omega \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I})$ is fully faithful (which follows from I being weakly contractible, see Remark 4.2.3.3, and from the explicit formula of the colimit in Ω from Proposition 3.1.4.1). Therefore, by applying the limit functor $\lim_{\mathsf{I}^{\mathrm{op}}}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I}) \to \Omega$ to the adjunction unit $\mathrm{id} \to \mathrm{diag}_{\mathsf{I}^{\mathrm{op}}}\,\mathrm{colim}_{\mathsf{I}^{\mathrm{op}}}$, we end up with a natural map $\lim_{\mathsf{I}^{\mathrm{op}}} \to \mathrm{colim}_{\mathsf{I}^{\mathrm{op}}}$. Now suppose furthermore that the colimit of $d\colon \mathsf{I} \to \Omega$ is the final object $1_\Omega\colon 1_{\mathcal{B}} \to \Omega$. Then the composition

$$\Omega \xhookrightarrow{h_\Omega} \underline{\mathsf{PSh}}_{\mathcal{B}}(\Omega) \xrightarrow{d^*} \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{I}) \xrightarrow{\lim_{\mathsf{I}^{\mathrm{op}}}} \Omega$$

is equivalent to the identity: in fact, this follows from the observation that its left adjoint is given by the composition of $\mathrm{diag}_{\mathsf{I}^{\mathrm{op}}}$ with the Yoneda extension of d (see Remark 3.4.1.4) and therefore preserves final objects. Thus, we obtain a natural map $\varphi\colon \mathrm{id}_\Omega \to (-)^+_d$.

To proceed, let us fix a (small) cofiltered $\mathcal{B}$-category I and a functor $d\colon \mathsf{I} \to \Omega$ whose colimit is the final object. Since I is small, there is a $\mathcal{B}$-regular cardinal κ such that the essential image of d is contained in the full subcategory $\Omega^\kappa_{\mathcal{B}} \hookrightarrow \Omega$ determined by the local class of relatively κ-compact objects in $\mathcal{B}$ (cf. Proposition 4.2.2.11). We will call such a $\mathcal{B}$-regular cardinal κ *adapted to d*. We will identify κ with the linearly ordered set of ordinals $< \kappa$. Using transfinite induction, we may now construct a diagram $T^d_\bullet\colon \kappa \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)$ by setting $T^d_0 = \mathrm{id}$, by defining the map $T_\tau \to T_{\tau+1}$ to be the morphism

$\varphi\colon T^d_\tau \to (T^d_\tau)^+_d$ from Remark 5.2.10.4 and finally by setting $T^d_\tau = \operatorname{colim}_{\tau'<\tau} T^d_{\tau'}$ whenever τ is a limit ordinal.

DEFINITION 5.2.10.5. Let $d\colon \mathsf{I} \to \Omega$ be a functor whose colimit is the final object and whose domain is a cofiltered small $\mathcal{B}$-category. We define the *sheafification functor* $(-)^{\mathrm{sh}}_d$ relative to the functor $d\colon \mathsf{I} \to \Omega$ as the colimit $(-)^{\mathrm{sh}}_d = \operatorname{colim}_{\tau<\kappa} T^d_\tau$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)$, where κ is an arbitrary $\mathcal{B}$-regular cardinal that is adapted to d.

REMARK 5.2.10.6. A priori, the sheafification functor $(-)^{\mathrm{sh}}_d$ depends on the choice of $\mathcal{B}$-regular cardinal κ. However, since d takes values in Ω^κ and therefore in κ-compact objects in Ω (see Corollary 4.2.2.23), and since κ (when viewed as a linearly ordered set) is κ-filtered, one can show that whenever $\tau \geq \kappa$ is another $\mathcal{B}$-regular cardinal, the sheafification functor that is constructed with respect to τ is equivalent to the one constructed with respect to κ.

REMARK 5.2.10.7. In the situation of Definition 5.2.10.5, the sheafification functor $(-)^{\mathrm{sh}}_d$ is left exact since by Remark 5.2.10.4 it is a filtered colimit of left exact functors (see the argument in the proof of Lemma 4.1.5.3).

EXAMPLE 5.2.10.8. Let S be a bounded local class of morphisms in $\mathcal{B}$ which is closed under finite limits in $\operatorname{Fun}(\Delta^1, \mathcal{B})$, and let $\iota\colon \Omega_S \hookrightarrow \Omega$ be the associated inclusion. Then Ω_S is small and closed under finite limits in Ω. In particular, Ω_S is cofiltered by Proposition 4.2.3.7 and contains the final object of Ω, so that the sheafification functor $(-)^{\mathrm{sh}}_\iota$ is well-defined.

EXAMPLE 5.2.10.9. Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism, and let S and ι be as in Example 5.2.10.8. Then the functor $\operatorname{const}_{f_*(\Omega_{\mathcal{X}})} \iota\colon \Omega_S \to f_*(\Omega_{\mathcal{X}})$ transposes to a map $\iota'\colon f^*(\Omega_S) \to \Omega_{\mathcal{X}}$ of $\mathcal{X}$-categories. As $\operatorname{const}_{f_*(\Omega_{\mathcal{X}})} \iota$ preserves the final object, its colimit is $1_{f_*(\Omega_{\mathcal{X}})}$, hence the colimit of ι' is the final object as well. Moreover, the fact that Ω_S is a cofiltered $\mathcal{B}$-category implies that $f^*\Omega_{\mathcal{X}}$ is a cofiltered $\mathcal{X}$-category: in fact, the colimit functor $\operatorname{colim}_{f^*(\Omega_S)^{\mathrm{op}}}\colon \underline{\mathsf{PSh}}_{\mathcal{X}}(f^*(\Omega_S)) \to \Omega_{\mathcal{X}}$ preserves finite limits if and only if the underlying functor of ∞-categories $\operatorname{PSh}_{\mathcal{X}}(f^*(\Omega_S)) \to \mathcal{X}$ preserves finite limits, and as the latter can be identified with the global sections of

$$\operatorname{colim}_{\Omega_S^{\mathrm{op}}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, f_*(\Omega_{\mathcal{X}})) \to f_*(\Omega_{\mathcal{X}}),$$

the claim follows from the fact that filtered colimits commute with finite limits in every $\mathcal{B}$-topos (which one can see by reducing to the case of a presheaf $\mathcal{B}$-topos where it readily follows from the definitions). Thus, we are in the situation of Definition 5.2.10.5, so that $(-)^{\mathrm{sh}}_{\iota'}$ is well-defined.

CONSTRUCTION 5.2.10.10. Suppose that S is a bounded local class of maps in $\mathcal{B}$. Since S is bounded, there is a $\mathcal{B}$-regular cardinal κ that is adapted to $\iota\colon \Omega_S \hookrightarrow \Omega$. Let $S^\kappa \subset S$ be the class of maps in S between κ-compact objects. We let $E \hookrightarrow \Omega_1$ be the subobject that is spanned by the maps $f\colon P \to Q$ in $\mathcal{B}_{/A}$ (for arbitrary $A \in \mathcal{B}$) for which $(\pi_A)_!(f)$ is contained in S and the two maps $P \to A$ and $Q \to A$ are relatively κ-compact. Since both S and the class of relatively κ-compact maps are local, a map f in Ω in context A is contained in E if and only if $(\pi_A)_!(f) \in S$ and both $P \to A$ and $Q \to A$ are relatively κ-compact. In particular, E is small. We define $\mathsf{W} \hookrightarrow \Omega$ as the subcategory that is generated by E in the sense of § 2.2.3. Note that as E is small, the subcategory W is small as well.

We can now state the first main result of this section:

PROPOSITION 5.2.10.11. *Let S be a bounded local class of morphisms in $\mathcal{B}$ such that S is closed under finite limits in* $\operatorname{Fun}(\Delta^1, \mathcal{B})$*. Let $\mathsf{W} \hookrightarrow \Omega$ be as in Construction 5.2.10.10. Then* $\mathsf{X} = \mathsf{Loc}_{\mathsf{W}}(\Omega)$ *is a subterminal $\mathcal{B}$-topos with the property that* $\Gamma(\mathsf{X}) \simeq \operatorname{Loc}_S(\mathcal{B})$*. Moreover, the adjunction unit $\eta\colon$* $\mathrm{id} \to \Gamma_{\mathsf{X}} \operatorname{const}_{\mathsf{X}}$ *can be identified with the map* $\mathrm{id} \to (-)^{\mathrm{sh}}_\iota$*, where $\iota\colon \Omega_S \hookrightarrow \Omega$ is the inclusion.*

LEMMA 5.2.10.12. *Let S be a bounded local class of morphisms in $\mathcal{B}$. Let $\mathsf{W} \hookrightarrow \Omega$ be as in Construction 5.2.10.10. Then there is an equivalence*

$$\Gamma(\mathsf{Loc}_{\mathsf{W}}(\Omega)) \simeq \mathrm{Loc}_S(\mathcal{B})$$

of full subcategories in $\mathcal{B}$.

PROOF. By Corollary 4.4.1.8, the inclusion $i\colon \mathsf{Loc}_{\mathsf{W}}(\Omega) \hookrightarrow \Omega$ admits a left adjoint L that exhibits $\mathsf{Loc}_{\mathsf{W}}(\Omega)$ as an accessible Bousfield localisation of Ω. Note that every map in S can be written as a colimit of maps in S^{κ}, which implies that every map in S is inverted by L. Consequently, we have an inclusion $\Gamma(\mathsf{Loc}_{\mathsf{W}}(\Omega)) \hookrightarrow \mathrm{Loc}_S(\mathcal{B})$, so that the claim follows once we verify that every S-local object $G \in \mathcal{B}$ is W-local. This amounts to showing that for every $A \in \mathcal{B}$ and and every map $s\colon P \to Q$ in $\mathcal{B}_{/A}$ for which $(\pi_A)_!(f) \in S$ and both $P \to A$ and $Q \to A$ are relatively κ-compact, the map

$$s^*\colon \operatorname{map}_{\Omega}(Q, \pi_A^* G) \to \operatorname{map}_{\Omega}(P, \pi_A^* G)$$

is an equivalence (cf. Remark 4.4.1.5). By Proposition 3.1.4.12, we may identify this morphism with the map

$$\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(Q, \pi_A^* G) \to \underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, \pi_A^* G).$$

By evaluating the latter at any object $B \to A$ in $\mathcal{B}_{/A}$, we recover the morphism

$$\operatorname{map}_{\mathcal{B}}(Q \times_A B, G) \to \operatorname{map}_{\mathcal{B}}(P \times_A B, G),$$

which is indeed an equivalence as the maps in S are closed under base change. Hence G is W-local, as claimed. □

Before we can prove Proposition 5.2.10.11, we first need to make a few remarks on the *internal hom* of a $\mathcal{B}$-topos X. Recall from Proposition 5.1.3.7 that colimits being universal in X precisely means that X is *cartesian closed.* We denote by

$$\underline{\mathrm{Hom}}_{\mathsf{X}}(-,-)\colon \mathsf{X}^{\mathrm{op}} \times \mathsf{X} \to \mathsf{X}$$

the internal hom of X that results from this observation. Note that if $f_*\colon \mathcal{X} \to \mathcal{B}$ is the geometric morphism associated with X, we deduce from combining Proposition 3.1.4.11 with Corollary 2.4.1.9 and [61, Lemma 5.2.1] that $\underline{\mathrm{Hom}}_{\mathsf{X}}(-,-)$ is explicitly given by the image of the bifunctor of $\mathcal{X}$-categories

$$\operatorname{map}_{\Omega_{\mathcal{X}}}(-,-)\colon \Omega_{\mathcal{X}}^{\mathrm{op}} \times \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}}$$

along f_*.

REMARK 5.2.10.13. If X is a $\mathcal{B}$-topos, then the composition $\Gamma_{\mathsf{X}} \circ \underline{\mathrm{Hom}}_{\mathsf{X}}(-,-)$ recovers the mapping bifunctor $\operatorname{map}_{\mathsf{X}}(-,-)$. In fact, as Remark 5.2.2.6 implies that Γ_{X} is corepresented by 1_{X}, we deduce that there is a pullback square

$$\begin{tikzcd}
\mathsf{X}_{1_{\mathsf{X}}/} \arrow[r] \arrow[d] & (\Omega_{\mathcal{B}})_{1_{\Omega_{\mathcal{B}}}/} \arrow[d] \\
\mathsf{X} \arrow[r, "\Gamma_{\mathsf{X}}"] & \Omega_{\mathcal{B}}.
\end{tikzcd}$$

On the other hand, [61, Lemma 5.2.1] implies that there also is a pullback square

$$\begin{tikzcd}
\mathsf{Tw}(\mathsf{X}) \arrow[r] \arrow[d, "p_{\mathsf{X}}"] & \mathsf{X}_{1_{\mathsf{X}}/} \arrow[d] \\
\mathsf{X}^{\mathrm{op}} \times \mathsf{X} \arrow[r, "\underline{\mathrm{Hom}}_{\mathsf{X}}^{\mathcal{B}}(-,-)"] & \mathsf{X}.
\end{tikzcd}$$

By pasting these two pullback squares together, the claim follows.

Proof of Proposition 5.2.10.11. Let κ and $\mathsf{W} \hookrightarrow \Omega$ be as in Construction 5.2.10.10, and let us denote by

$$(L \dashv i)\colon \Omega \leftrightarrows \mathsf{Loc}_{\mathsf{W}}(\Omega)$$

the associated Bousfield localisation provided by Corollary 4.4.1.8. In light of Lemma 5.2.10.12, we only need to show that $\mathsf{Loc}_{\mathsf{W}}(\Omega)$ is a subterminal $\mathcal{B}$-topos and to identify the adjunction unit $\mathrm{id} \to iL$ with the sheafification map $\mathrm{id} \to (-)^{\mathrm{sh}}_{\iota}$. In light of Theorem 5.2.3.1 and Example 5.2.10.8, the former claim is implied by the latter one, so that the proof is finished once we identify $\mathrm{id} \to iL$ with $\mathrm{id} \to (-)^{\mathrm{sh}}_{\iota}$.

We only need to show that for every object $G\colon A \to \Omega$ in arbitrary context $A \in \mathcal{B}$, the object G^{sh}_{ι} is W-local and the map $G \to G^{\mathrm{sh}}_{\iota}$ is inverted by the localisation functor L. Note that as $\mathcal{B}$ is generated by its κ-compact objects (as κ is assumed to be $\mathcal{B}$-regular), we may assume that A is κ-compact. In this case, note that κ is also $\mathcal{B}_{/A}$-regular and adapted to $\pi_A^*(\iota)$ (by Remark 4.2.2.6) and that we may identify the base change of $(-)^+_{\iota}$ along π_A^* with $(-)^+_{\pi_A^*(\iota)}\colon \Omega_{\mathcal{B}_{/A}} \to \Omega_{\mathcal{B}_{/A}}$. Therefore, we may also identify the base change of $(-)^{\mathrm{sh}}_{\iota}$ along π_A^* with $(-)^{\mathrm{sh}}_{\pi_A^*(\iota)}$. Together with Remark 4.4.1.2, this implies that we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and G with its transpose $\hat{G}\colon 1_{\mathcal{B}_{/A}} \to \Omega_{\mathcal{B}_{/A}}$, so that we may assume without loss of generality that $A \simeq 1_{\mathcal{B}}$.

We first show that the map $G \to G^{\mathrm{sh}}_{\iota}$ is inverted by L, for which it will be enough to show that the map $\varphi\colon G \to G^+_{\iota}$ is inverted by L, or equivalently that the map

$$\varphi^*\colon \operatorname{map}_{\Omega}(G^+_{\iota}, i(-)) \to \operatorname{map}_{\Omega}(G, i(-))$$

is an equivalence. Note that by the triangle identities, the map $\lim_{\Omega_S^{\mathrm{op}}} \to \operatorname{colim}_{(\Omega_S)^{\mathrm{op}}}$ can be identified with the composition

$$\lim_{\Omega_S^{\mathrm{op}}} \simeq \operatorname*{colim}_{\Omega_S^{\mathrm{op}}} \operatorname{diag}_{\Omega_S^{\mathrm{op}}} \lim_{\Omega_S^{\mathrm{op}}} \to \operatorname*{colim}_{\Omega_S^{\mathrm{op}}}$$

where the first map is induced by the inverse of counit of $\operatorname{colim}_{(\Omega_S)^{\mathrm{op}}} \dashv \operatorname{diag}_{\Omega_S^{\mathrm{op}}}$ and the second map is induced by the counit of $\operatorname{diag}_{\Omega_S^{\mathrm{op}}} \dashv \lim_{\Omega_S^{\mathrm{op}}}$. Moreover, observe that as $\lim_{\Omega_S^{\mathrm{op}}}$ is given by evaluation at the final object $1_{\Omega}\colon 1 \to \Omega_S$, this functor is cocontinuous and therefore given by the left Kan extension of its restriction along the Yoneda embedding $h_{\Omega_S}\colon \Omega_S \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\Omega_S)$. As the restriction of $\lim_{\Omega_S^{\mathrm{op}}}$ along h_{Ω_S} can be identified with $\operatorname{map}_{\Omega_S}(1_{\Omega}, -)$ and the latter is equivalent to the inclusion ι, it follows that the left adjoint of $\lim_{\Omega_S^{\mathrm{op}}}$ is given by $\iota^* h_{\Omega}$ (see Remark 3.4.1.4). Altogether, these observations imply that we may decompose φ^* into the chain of morphisms

$$\begin{aligned}
\operatorname{map}_{\Omega}(G^+_{\iota}, i(-)) &\simeq \operatorname{map}_{\Omega}(\operatorname*{colim}_{\Omega_S^{\mathrm{op}}} \iota^* h_{\Omega}(G), i(-)) \\
&\simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\Omega_S)}(\iota^* h_{\Omega}(G), \operatorname{diag}_{\Omega_S^{\mathrm{op}}} i(-)) \\
&\to \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\Omega_S)}(\iota^* h_{\Omega}(G), \iota^* h_{\Omega} \lim_{\Omega_S^{\mathrm{op}}} \operatorname{diag}_{\Omega_S^{\mathrm{op}}} i(-)) \\
&\simeq \operatorname{map}_{\Omega}(G, i(-))
\end{aligned}$$

in which the penultimate map is induced by the adjunction unit $\mathrm{id} \to \iota^* h_{\Omega} \lim_{\Omega_S^{\mathrm{op}}}$ and where the last equivalence follows from both $\operatorname{diag}_{\Omega_S^{\mathrm{op}}}$ and $\iota^* h_{\Omega}$ being fully faithful functors. Hence, it suffices to show that the map

$$\operatorname{diag}_{\Omega_S^{\mathrm{op}}} i \to \iota^* h_{\Omega} \lim_{\Omega_S^{\mathrm{op}}} \operatorname{diag}_{\Omega_S^{\mathrm{op}}} i$$

is an equivalence, i.e. that $\operatorname{diag}_{\Omega_S^{\mathrm{op}}} i$ takes value in the essential image of $\iota^* h_{\Omega}$. To see this, note that since the restriction of the localisation functor $L\colon \Omega \to \mathsf{Loc}_{\mathsf{W}}(\Omega)$ to Ω_S factors through the inclusion $1_{\Omega}\colon 1 \hookrightarrow \mathsf{Loc}_{\mathsf{W}}(\Omega)$, it follows that we may identify $\iota^* h_{\Omega} i \simeq \operatorname{map}_{\Omega}(\iota(-), i(-))$ with the transpose of the composition

$$\Omega_S^{\mathrm{op}} \times \mathsf{Loc}_{\mathsf{W}}(\Omega) \xrightarrow{\mathrm{pr}_1} \mathsf{Loc}_{\mathsf{W}}(\Omega) \xrightarrow{\operatorname{map}_{\mathsf{Loc}_{\mathsf{W}}}(1_{\Omega}, -)} \Omega,$$

which is precisely $\operatorname{diag}_{\Omega_S^{\mathrm{op}}} i$, as desired.

We finish the proof by showing that G_ι^{sh} is W-local. By the same reduction steps as above, it is enough to show that for every map $s\colon P \to Q$ in S between κ-compact objects, the map $s^*\colon \operatorname{map}_\Omega(Q, G_\iota^{\mathrm{sh}}) \to \operatorname{map}_\Omega(P, G_\iota^{\mathrm{sh}})$ is an equivalence. Note that in light of the adjunction $(\pi_Q)_! \dashv \pi_Q^*\colon \Omega_{/Q} \leftrightarrows \Omega$, the map s^* can be interpreted as the morphism $s^*\colon \operatorname{map}_{\Omega_{/Q}}(Q, \pi_Q^* G_\iota^{\mathrm{sh}}) \to \operatorname{map}_{\Omega_{/Q}}(P, \pi_Q^* G_\iota^{\mathrm{sh}})$. By Remark 5.2.10.13, we can identify $\operatorname{map}_{\Omega_{/Q}}(-,-)$ with the global sections of the internal hom $\underline{\operatorname{Hom}}_{\Omega_{/Q}}(-,-)$. Hence we may as well show that the map

$$s^*\colon \underline{\operatorname{Hom}}_{\Omega_{/Q}}(Q, \pi_Q^* G_\iota^{\mathrm{sh}}) \to \underline{\operatorname{Hom}}_{\Omega_{/Q}}(P, \pi_Q^* G_\iota^{\mathrm{sh}})$$

is an equivalence. In other words, by replacing $\mathcal{B}$ with $\mathcal{B}_{/Q}$, we can reduce to the case where $Q \simeq 1$. Thus, what is left to show is that $t^*\colon G_\iota^{\mathrm{sh}} \to \operatorname{map}_\Omega(P, G_\iota^{\mathrm{sh}})$ is an equivalence for every κ-compact $P\colon 1 \to \Omega_S$, where $t\colon G \to 1_\Omega$ is the terminal map. Note that by Corollary 4.2.2.23, the fact that P is κ-compact even implies that P is $\mathsf{Cat}_{\mathcal{B}}^\kappa$-compact when viewed as an object of Ω. Hence, the map t^* can be identified with the colimit

$$\operatorname*{colim}_{\tau<\kappa} t_\tau^*\colon \operatorname*{colim}_{\tau<\kappa} T_\tau^\iota G \to \operatorname*{colim}_{\tau<\kappa} \operatorname{map}_\Omega(P, T_\tau^\iota G).$$

To show that this map is an equivalence, it will be sufficient to prove that for every ordinal $\tau < \kappa$ the map $\operatorname{colim}_{n\in\mathbb{N}} t_{\tau+n}^*$ is one. To see the latter claim, observe that for every $H \in \mathcal{B}$ we have a commutative diagram

$$\begin{tikzcd}
H \arrow[rr, "t^*"] \arrow[d] & & \operatorname{map}_\Omega(P, H) \arrow[d, "\alpha"] \\
\operatorname{colim}_{(\Omega_S^{\mathrm{op}})_{P/}} \operatorname{map}_\Omega(\iota(\pi_P)_!(-), H) \arrow[rr, "\simeq"] \arrow[d, "\simeq"] & & \operatorname{colim}_{(\Omega_S)^{\mathrm{op}}} \operatorname{map}_\Omega(\iota(-), H) \arrow[d, "\beta"] \\
\operatorname{colim}_{(\Omega_S)^{\mathrm{op}}} \operatorname{map}_\Omega(\iota(-), H) \arrow[rr, "t^*"] & & \operatorname{map}_\Omega(P, \operatorname{colim}_{(\Omega_S)^{\mathrm{op}}} \operatorname{map}_\Omega(\iota(-), H)).
\end{tikzcd}$$

Here the composition of the two vertical maps on the left and right can be identified by φ and φ_*, respectively. Moreover, the equivalences in this diagram are induced by the initial map $(\pi_P)_!\colon (\Omega_S)_{/P} \to \Omega_S$, the map α is induced by $P\colon 1 \to \Omega_S$ and β is given by the composition

$$\begin{aligned}
\operatorname*{colim}_{(\Omega_S)^{\mathrm{op}}} \operatorname{map}_\Omega(\iota(-), H) &\to \operatorname*{colim}_{(\Omega_S)^{\mathrm{op}}} \operatorname{map}_\Omega(P, \operatorname{map}_\Omega(\iota(-), H)) \\
&\to \operatorname{map}_\Omega(P, \operatorname*{colim}_{(\Omega_S)^{\mathrm{op}}} \operatorname{map}_\Omega(\iota(-), H)).
\end{aligned}$$

By substituting $H = T_{\tau+n}^\iota G$ for any $n \in \mathbb{N}$, we deduce that the map $t_{\tau+n}^* \to t_{\tau+n+1}^*$ factors through an equivalence and thus $\operatorname{colim}_{n\in\mathbb{N}} t_{\tau+n}^*$ is an equivalence by an easy cofinality argument. ☐

We finish this section with the following converse of Proposition 5.2.10.11:

PROPOSITION 5.2.10.14. *Let* X *be a subterminal* $\mathcal{B}$*-topos. Then there is a bounded local class* S *that is closed under finite limits in* $\operatorname{Fun}(\Delta^1, \mathcal{B})$ *such that* $\Gamma(\mathsf{X}) \simeq \operatorname{Loc}_S(\mathcal{B})$*. Moreover, for* any *such local class* S*, we obtain an equivalence* $\mathsf{X} \simeq \mathsf{Loc}_{\mathsf{W}}(\Omega)$*, where* W *is as in Construction 5.2.10.10, and the adjunction unit* $\eta\colon \mathrm{id} \to \Gamma_{\mathsf{X}} \operatorname{const}_{\mathsf{X}}$ *can be identified with the map* $\mathrm{id} \to (-)_\iota^{\mathrm{sh}}$*, where* $\iota\colon \Omega_S \hookrightarrow \Omega$ *is the inclusion.*

PROOF. Let us denote by $j_*\colon \mathcal{X} \hookrightarrow \mathcal{B}$ the geometric morphism associated with X. We begin by proving the second statement, i.e. suppose that S is a bounded local class that is closed under finite limits in $\operatorname{Fun}(\Delta^1, \mathcal{B})$ such that we have $\mathcal{X} \simeq \operatorname{Loc}_S(\mathcal{B})$. Let $\mathsf{W} \hookrightarrow \Omega$ be as in Construction 5.2.10.10. By Lemma 5.2.10.12, we may identify $\Gamma(\mathsf{Loc}_{\mathsf{W}}(\Omega))$ with $\mathcal{X}$. As Proposition 5.2.10.11 moreover implies that $\mathsf{Loc}_{\mathsf{W}}(\Omega)$ is a subterminal $\mathcal{B}$-topos, Theorem 5.2.5.1 implies that we must necessarily have $\mathsf{Loc}_{\mathsf{W}}(\Omega) \simeq \mathsf{X}$. Hence the same proposition gives rise to the desired identification of the adjunction unit $\eta\colon \mathrm{id} \to \Gamma_{\mathsf{X}} \operatorname{const}_{\mathsf{X}}$.

To complete the proof, it is therefore enough to show that there always exists a bounded local class S that is closed under finite limits in $\operatorname{Fun}(\Delta^1, \mathcal{B})$ such that $\mathcal{X} \simeq \operatorname{Loc}_S(\mathcal{B})$. To that end, choose a $\mathcal{B}$-regular cardinal κ for which Γ_{X} is $\mathsf{Filt}_{\mathsf{Cat}_{\mathcal{B}}^\kappa}$-cocontinuous. We let S be the class of relatively κ-compact maps in $\mathcal{B}$ that are inverted by j^*. Since by Proposition 4.2.2.11 the class of relatively κ-compact maps in $\mathcal{B}$

is local, we find that S is local as well. Moreover, S is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$ as j^* is left exact and as κ-compact objects in $\mathcal{B}$ are closed under finite limits (by choice of κ). Since S is inverted by j, we already have an inclusion $\mathcal{X} \hookrightarrow \mathrm{Loc}_S(\mathcal{B})$, so that it suffices to prove that every S-local object in $\mathcal{B}$ is contained in $\mathcal{B}$. Since $\mathcal{X}$ is a κ-accessible localisation of $\mathcal{B}$ (using Proposition 4.3.2.4) and $\mathcal{B}$ itself is κ-accessible, we deduce from the proof of [57, Proposition 5.5.4.2] (or alternatively the proof of Proposition 4.4.1.6 applied in the case $\mathcal{B} = \mathcal{S}$) that $\mathcal{X}$ is the Bousfield localisation at the class S' of those maps in $\mathcal{B}$ between κ-compact objects which are inverted by j. Since every such map must be relatively κ-compact (using that κ-compact objects are closed under finite limits in $\mathcal{B}$), every such map is contained in S. Hence the claim follows. □

Remark 5.2.10.15. We can use our understanding of subterminal $\mathcal{B}$-topoi to obtain a quite explicit understanding of pushouts in $\mathrm{Top}_\infty^{\mathrm{L}}$ in which one of the two maps is a Bousfield localisation. In fact, suppose that $f_*\colon \mathcal{X} \to \mathcal{B}$ and $i_*\colon \mathcal{Z} \hookrightarrow \mathcal{B}$ be geometric morphisms, where i_* is fully faithful. Then $i_*(\Omega_{\mathcal{Z}})$ is a subterminal $\mathcal{B}$-topos, so that Proposition 5.2.10.14 implies that we can find a bounded local class S that is closed under compositions and finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{B})$ such that $\mathcal{Z} = \mathrm{Loc}_S(\mathcal{B})$. Let $\overline{f^*S}$ denote the smallest local class of maps in $\mathcal{X}$ that contains $f^*(S)$. Then we claim that the functor $j_*\colon \mathcal{Z} \times_{\mathcal{B}} \mathcal{X} \hookrightarrow \mathcal{X}$ exhibits $\mathcal{Z} \times_{\mathcal{B}} \mathcal{X}$ as the Bousfield localisation of $\mathcal{X}$ at $\overline{f^*S}$. To see this, note that Proposition 5.2.7.1 and Lemma 5.2.7.7 imply that the morphism in $\mathsf{Top}_{\mathcal{B}}^{\mathrm{R}}$ corresponding to j_* is given by $\mathsf{Loc}_{\mathsf{W} \boxtimes f_*(\Omega_{\mathcal{X}})^{\simeq}}(\Omega \otimes f_*(\Omega_{\mathcal{X}})) \hookrightarrow f_*(\Omega_{\mathcal{X}})$, where $\mathsf{W} \hookrightarrow \Omega$ is the subcategory from Construction 5.2.10.10. Since $\Omega \otimes f_*(\Omega_{\mathcal{X}}) \simeq f_*(\Omega_{\mathcal{X}})$, the left-hand side can be identified with the full subcategory of local objects with respect to

$$\mathsf{W} \times f_*(\Omega_{\mathcal{X}})^{\simeq} \xrightarrow{\mathrm{const}_{i_*\Omega_{\mathcal{X}}}(-)\times -} f_*(\Omega_{\mathcal{X}}).$$

By the same argument as in the proof of Lemma 5.2.10.12, this means that an object $U \in \mathcal{X}$ is contained in $\mathcal{Z} \times_{\mathcal{B}} \mathcal{X}$ if and only if it is local with respect to every map in $\mathcal{X}$ of the form

$$f^*(s) \times_{f^*A} X\colon f^*(P) \times_{f^*A} X \to f^*(Q) \times_{f^*A} X$$

where $s\colon P \to Q$ is a map in $\mathsf{W}(A)$ and X is an arbitrary object in $\mathcal{X}_{/f^*A}$. By construction of W, the map $(\pi_A)_!(f)$ is contained in S, which in turn implies that $f^*(s) \times_{f^*A} X$ is in $\overline{f^*S}$. Hence the claim follows.

5.3. Localic $\mathcal{B}$-topoi

In higher topos theory, the 1-category of *locales* (with left exact left adjoints as maps) arises as a coreflective subcategory of the ∞-category $\mathrm{Top}_\infty^{\mathrm{L}}$ of ∞-topoi. The inclusion is given by sending a locale L to the ∞-topos $\mathrm{Sh}(L)$ of sheaves on L, and the coreflection carries an ∞-topos $\mathcal{X}$ to the locale $\mathrm{Sub}(\mathcal{X})$ of subterminal (i.e. (-1)-truncated) objects in $\mathcal{X}$. An ∞-topos $\mathcal{X}$ is said to be localic if it is equivalent to $\mathrm{Sh}(\mathrm{Sub}(\mathcal{X}))$.

In this section, we give a brief exposition of the analogous story in the world of $\mathcal{B}$-topoi. We do not aim to provide a comprehensive study of localic $\mathcal{B}$-topoi, but rather restrict our attention to those aspects of the theory that allow us to *define* the notion of a localic $\mathcal{B}$-topos and to provide an external characterisation of this concept in the case where $\mathcal{B}$ is itself localic.

We begin in § 5.3.1 and 5.3.2 by providing the necessary background material on $\mathcal{B}$-posets. In § 5.3.3 we define and characterise $\mathcal{B}$-locales, and in § 5.3.4 we construct the $\mathcal{B}$-topos of sheaves on a $\mathcal{B}$-locale, which we use in § 5.3.5 to show that $\mathcal{B}$-locales are a coreflective localisation of $\mathcal{B}$-topoi. In § 5.3.6 we discuss how localic $\mathcal{B}$-topoi correspond to localic ∞-topoi over $\mathcal{B}$ in the case where $\mathcal{B}$ is itself localic. Lastly, § 5.3.7 discusses some compactness conditions of $\mathcal{B}$-locales and how they are inherited by the associated $\mathcal{B}$-topoi of sheaves. Finally in § 5.3.8, we study $\mathcal{B}$-locales arising from maps of topological spaces.

5.3.1. $\mathcal{B}$-posets. Recall that an ∞-category $\mathcal{C}$ is (equivalent to) a poset precisely if if for all objects c and d in $\mathcal{C}$ the mapping ∞-groupoid $\mathrm{map}_{\mathcal{C}}(c,d)$ is (-1)-truncated. In this section we discuss a generalisation of this concept to $\mathcal{B}$-categories.

Recall that the class of (-1)-truncated maps in $\mathcal{B}$ is precisely the collection of morphisms that are internally right orthogonal to the map $1 \sqcup 1 \to 1$ (in the sense of [62, § 2.5]). In particular, this class is local, so that we may define:

DEFINITION 5.3.1.1. The subuniverse $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \Omega_{\mathcal{B}}$ of *subterminal $\mathcal{B}$-groupoids* is the full subcategory of $\Omega_{\mathcal{B}}$ that is determined by the local class of (-1)-truncated morphisms in $\mathcal{B}$. A $\mathcal{B}_{/A}$-groupoid G is said to be a subterminal $\mathcal{B}_{/A}$-groupoid if it defines an object of $\mathsf{Sub}_{\mathcal{B}}$ (in context A).

REMARK 5.3.1.2. As the functor $(\pi_A)_!\colon \mathcal{B}_{/A} \to \mathcal{B}$ creates pullbacks, a map $P \to B$ in $\mathcal{B}_{/A}$ defines an object $B \to \mathsf{Sub}_{\mathcal{B}_{/A}}$ if and only if the underlying map in $\mathcal{B}$ defines an object $(\pi_A)_!(B) \to \mathsf{Sub}_{\mathcal{B}}$. Thus, the equivalence $\pi_A^*\Omega \simeq \Omega_{\mathcal{B}_{/A}}$ restricts to an equivalence $\pi_A^* \mathsf{Sub}_{\mathcal{B}} \simeq \mathsf{Sub}_{\mathcal{B}_{/A}}$ for every $A \in \mathcal{B}$.

By [6, Example 3.4.2] the internal saturation of the map $1 \sqcup 1 \to 1$ (i.e. the class of covers) in $\mathcal{B}$ is closed under base change, so that the factorisation system between these and (-1)-truncated maps even forms a *modality*. As a consequence, one finds:

PROPOSITION 5.3.1.3. *The $\mathcal{B}$-category $\mathsf{Sub}_{\mathcal{B}}$ is an accessible Bousfield localisation of $\Omega_{\mathcal{B}}$ (in the sense of Definition 4.3.3.4) and therefore in particular presentable.*

PROOF. By Example 5.1.2.15 the subcategory $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \Omega_{\mathcal{B}}$ is reflective. Since the inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \Omega_{\mathcal{B}}$ is section-wise accessible, Corollary 4.3.2.5 implies that the localisation must be accessible. $\square$

DEFINITION 5.3.1.4. A $\mathcal{B}$-category C is said to be a *$\mathcal{B}$-poset* if the mapping bifunctor $\mathrm{map}_{\mathsf{C}}$ takes values in $\mathsf{Sub}_{\mathcal{B}}$. The full subcategory of $\mathsf{Cat}_{\mathcal{B}}$ that is spanned by the $\mathcal{B}_{/A}$-posets for each $A \in \mathcal{B}$ is denoted by $\mathsf{Pos}_{\mathcal{B}}$ and its underlying ∞-category of global sections by $\mathrm{Pos}(\mathcal{B})$.

REMARK 5.3.1.5. A $\mathcal{B}$-category C is a $\mathcal{B}$-poset precisely if the map $\mathsf{C}_1 \to \mathsf{C}_0 \times \mathsf{C}_0$ is (-1)-truncated. In fact, since this map exhibits C_1 as the mapping $\mathcal{B}_{/\mathsf{C}_0 \times \mathsf{C}_0}$-groupoid between the two objects $\mathrm{pr}_0, \mathrm{pr}_1\colon \mathsf{C}_0 \times \mathsf{C}_0 \rightrightarrows \mathsf{C}$, this is clearly a necessary condition. The fact that it is also sufficient follows from the observation that every mapping $\mathcal{B}_{/A}$-groupoid of C is a pullback of this map.

REMARK 5.3.1.6. Since the class of (-1)-truncated maps in $\mathcal{B}$ is local, we deduce from Remark 5.3.1.5 that if $(s_i)\colon \bigsqcup_i A_i \twoheadrightarrow A$ is a cover in $\mathcal{B}$, a $\mathcal{B}_{/A}$-category $\mathsf{C}\colon A \to \mathsf{Cat}_{\mathcal{B}}$ defines a $\mathcal{B}_{/A}$-poset if and only if the $\mathcal{B}_{/A_i}$-category $s_i^*\mathsf{C}$ defines a $\mathcal{B}_{/A_i}$-poset for every i. In particular, *every* object of $\mathsf{Pos}_{\mathcal{B}}$ in context $A \in \mathcal{B}$ encodes a $\mathcal{B}_{/A}$-poset, and one has a canonical equivalence $\pi_A^* \mathsf{Pos}_{\mathcal{B}} \simeq \mathsf{Pos}_{\mathcal{B}_{/A}}$.

REMARK 5.3.1.7. Remark 5.3.1.5 and the fact that a map is (-1)-truncated precisely if it is so section-wise imply that a $\mathcal{B}$-category C is a $\mathcal{B}$-poset if and only if $\mathsf{C}(A)$ is an poset for every $A \in \mathcal{B}$. Together with Remark 5.3.1.6, this implies that we obtain an equivalence $\mathsf{Pos}_{\mathcal{B}} \simeq \mathrm{Pos} \otimes \Omega$ (where Pos is the 1-category of posets).

REMARK 5.3.1.8. Recall that if C is a $\mathcal{B}$-category, then the map $s_0\colon \mathsf{C}_0 \to \mathsf{C}_1$ is a monomorphism in $\mathcal{B}$. Together with Remark 5.3.1.5, this implies that if P is a $\mathcal{B}$-poset, then each P_0 is contained in the underlying 1-topos $\mathrm{Disc}(\mathcal{B}) \hookrightarrow \mathcal{B}$ of 0-truncated objects. Consequently, we may identify $\mathrm{Pos}(\mathcal{B})$ with the full subcategory of $\mathrm{Disc}(\mathcal{B})_\Delta$ that is spanned by the complete Segal objects P for which the map $\mathsf{P}_1 \to \mathsf{P}_0 \times \mathsf{P}_0$ is a monomorphism. Hence $\mathrm{Pos}(\mathcal{B})$ is equivalent to the 1-category of internal posets in the 1-topos $\mathrm{Disc}(\mathcal{B})$ in the sense of [46, § B2.3].

5.3.2. Presentable $\mathcal{B}$-posets. In this section we study *presentable* $\mathcal{B}$-posets. We begin with the following definition:

DEFINITION 5.3.2.1. If C is a $\mathcal{B}$-category, we define the full subcategory $\mathsf{Sub}(\mathsf{C}) \hookrightarrow \mathsf{C}$ of *subterminal objects* as the pullback $\mathsf{C} \times_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$.

Remark 5.3.2.2. If C is a $\mathcal{B}$-category and $A \in \mathcal{B}$ is an arbitrary object, then there is a canonical equivalence $\pi_A^* \mathsf{Sub}(\mathsf{C}) \simeq \mathsf{Sub}(\pi_A^*\mathsf{C})$ of full subcategories in $\pi_A^*\mathsf{C}$.

Proposition 5.3.2.3. *For every presentable $\mathcal{B}$-category D there is a canonical equivalence $\mathsf{Sub}(\mathsf{D}) \simeq \mathsf{D} \otimes \mathsf{Sub}_{\mathcal{B}}$ of full subcategories in D. In particular, $\mathsf{Sub}(\mathsf{D})$ is an accessible Bousfield localisation of D and therefore presentable as well.*

Proof. Recall from Proposition 4.4.6.3 that the Yoneda embedding $\mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ identifies D with the full subcategory $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{D}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ that is spanned by the continuous functors (in arbitrary context). Therefore, it will be enough to show that the square

$$\begin{tikzcd} \mathsf{D} \otimes \mathsf{Sub}_{\mathcal{B}} \arrow[r, hook] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \arrow[d, hook] \\ \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \end{tikzcd}$$

is a pullback. Together with the usual base change arguments, this means that we only need to check that a functor $F\colon \mathsf{D}^{\mathrm{op}} \to \mathsf{Sub}_{\mathcal{B}}$ is continuous if and only if its postcomposition with the inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \Omega$ is. This follows immediately from the fact that the inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \Omega$ has a left adjoint and is therefore continuous and conservative. ☐

For every presentable $\mathcal{B}$-category D, we will denote the left adjoint of the inclusion $\mathsf{Sub}(\mathsf{D}) \hookrightarrow \mathsf{D}$ by $(-)^{\mathrm{Sub}}$ and refer to it as the *subterminal truncation functor*.

Example 5.3.2.4. If C is an arbitrary $\mathcal{B}$-category, then Proposition 5.3.2.3 provides us with an equivalence of $\mathcal{B}$-categories $\mathsf{Sub}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})) \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$. Moreover, in light of the equivalence $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \simeq \mathsf{RFib}_{\mathsf{C}}$, we may identify $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ with the full subcategory of $\mathsf{RFib}_{\mathsf{C}}$ that is spanned by the right fibrations $p\colon \mathsf{P} \to \pi_A^*\mathsf{C}$ (in arbitrary context $A \in \mathcal{B}$) which are fully faithful, i.e. which are *sieves* in the $\mathcal{B}_{/A}$-poset $\pi_A^*\mathsf{C}$. To see the latter claim, first note that by Remark 5.3.2.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ so that we can assume that $A \simeq 1$. In this case, since $p_0\colon \mathsf{P}_0 \to \mathsf{C}_0$ can be identified with the image of the tautological object $\mathsf{C}_0 \to \mathsf{C}$ along the functor $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ that classifies p and since every object in C (in arbitrary context) arises as a pullback of the tautological object, F takes values in $\mathsf{Sub}_{\mathcal{B}}$ if and only if p_0 is a monomorphism. Therefore it suffices to show that p_0 is monic if and only if p is fully faithful. This follows from considering the commutative diagram

$$\begin{tikzcd} \mathsf{P}_1 \arrow[r, "\mathrm{id}"] \arrow[d, "(d_1, d_0)"] & \mathsf{P}_1 \arrow[r] \arrow[d] & \mathsf{C}_1 \arrow[d, "(d_1, d_0)"] \\ \mathsf{P}_0 \times \mathsf{P}_0 \arrow[r, "p_0 \times \mathrm{id}"] & \mathsf{C}_0 \times \mathsf{P}_0 \arrow[r, "\mathrm{id} \times p_0"] & \mathsf{C}_0 \times \mathsf{C}_0 \end{tikzcd}$$

in which the right square is a pullback as p is a right fibration and where the left square is a pullback since p_0 is a monomorphism.

Furthermore, the above observation implies that we may compute the subterminal truncation functor $(-)^{\mathrm{Sub}}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{Sub}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$ on the level of right fibrations by taking essential images: If $F\colon \mathsf{C}^{\mathrm{op}} \to \Omega$ is a presheaf, then F^{Sub} classifies the essential image of the right fibration $\mathsf{C}_{/F} \to \mathsf{C}$. In fact, this is a consequence of the straightforward observation that the essential image is still a right fibration.

By definition, if C is a $\mathcal{B}$-category, then $\mathsf{Sub}(\mathsf{C})$ is a $\mathcal{B}$-poset. Our next goal is to show that if C is presentable, then $\mathsf{Sub}(\mathsf{C})$ can be characterised as the *largest* accessible Bousfield localisation of C with that property.

Lemma 5.3.2.5. *Let $(l \dashv r)\colon \mathsf{D} \leftrightarrows \mathsf{C}$ be an adjunction of $\mathcal{B}$-categories. Then there is a commutative square*

$$\begin{tikzcd} \mathsf{Sub}(\mathsf{D}) \arrow[r, "r"] \arrow[d, hook] & \mathsf{Sub}(\mathsf{C}) \arrow[d, hook] \\ \mathsf{D} \arrow[r, "r"] & \mathsf{C} \end{tikzcd}$$

which is a pullback when r is fully faithful.

PROOF. Unwinding the definitions, it will be enough to show that we have a commutative square

$$\begin{tikzcd}
\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \arrow[r, hook, "r_!"] \arrow[d, hook] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \arrow[d, hook] \\
\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D}) \arrow[r, hook, "r_!"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})
\end{tikzcd}$$

that is a pullback when r is fully faithful. The existence of this square follows from observing that $r_!$ can be identified with l^*. If r is moreover fully faithful, then l is a localisation and therefore in particular essentially surjective. Thus, the second claim follows. $\square$

PROPOSITION 5.3.2.6. *Let* D *be a presentable* $\mathcal{B}$*-category and* P *be a presentable* $\mathcal{B}$*-poset. Then composition with the inclusion* $\mathsf{Sub}_{\mathcal{B}}(\mathsf{D}) \hookrightarrow \mathsf{D}$ *induces an equivalence*

$$\underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{P}, \mathsf{Sub}(\mathsf{D})) \hookrightarrow \underline{\mathsf{Fun}}^R_{\mathcal{B}}(\mathsf{P}, \mathsf{D}).$$

PROOF. By Remark 5.3.2.2, it will be enough to show that every right adjoint functor $\mathsf{P} \to \mathsf{D}$ factors through $\mathsf{Sub}(\mathsf{D})$. This follows immediately from Lemma 5.3.2.5. $\square$

In light of Remark 5.3.2.2, Proposition 5.3.2.6 implies:

COROLLARY 5.3.2.7. *The full subcategory of* $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *that is spanned by the presentable* $\mathcal{B}_{/A}$*-posets for all* $A \in \mathcal{B}$ *is reflective, with the left adjoint given by sending a presentable* $\mathcal{B}$*-category* D *to* $\mathsf{Sub}(\mathsf{D})$. $\square$

REMARK 5.3.2.8. Lemma 5.3.2.5 furthermore implies that if D is a presentable $\mathcal{B}$-category and $d\colon 1 \to \mathsf{D}$ is an arbitrary object, then d is subterminal if and only if the diagonal $d \to d \times d$ is an equivalence. In fact, by choosing a presentation of D as an accessible Bousfield localisation of a presheaf $\mathcal{B}$-category and making use of Lemma 5.3.2.5, we may assume that $\mathsf{D} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ and hence that d can be identified with a right fibration over C. Then the claim follows immediately from Example 5.3.2.4. In particular, this observation implies that $\mathsf{Sub}(\mathsf{D})$ can be identified with the sheaf $\mathrm{Sub}(\mathsf{D}(-))$ on $\mathcal{B}$.

Finally, we arrive at the following characterisation of presentable $\mathcal{B}$-posets:

PROPOSITION 5.3.2.9. *For an (a priori large)* $\mathcal{B}$*-category* D*, the following are equivalent:*

(1) D *is a presentable* $\mathcal{B}$*-poset;*
(2) $\mathsf{D} \simeq \mathsf{Sub}(\mathsf{E})$ *for some presentable* $\mathcal{B}$*-category* E*;*
(3) D *is small and cocomplete;*
(4) D *is small, and the Yoneda embedding* $h\colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ *admits a left adjoint;*
(5) D *is a small* $\mathcal{B}$*-poset, and the Yoneda embedding* $h\colon \mathsf{D} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ *admits a left adjoint.*

PROOF. (1) and (2) are equivalent by Proposition 5.3.2.3. If D is a presentable $\mathcal{B}$-poset, then Lemma 5.3.2.5 combined with Example 5.3.2.4 shows that there is a small $\mathcal{B}$-category C such that D arises as a Bousfield localisation of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ for some small $\mathcal{B}$-category C. Hence, as the latter is small, so is D. Therefore (1) implies (3). Conversely, every small and cocomplete $\mathcal{B}$-category is presentable (this follows from our characterisation of presentable $\mathcal{B}$-categories as the accessible and cocomplete ones, see Corollary 4.4.6.7), and since every small and cocomplete ∞-category is a poset, we conclude by employing Remark 5.3.1.7 that (3) implies (1). Furthermore, (3) and (4) are equivalent by the universal property of presheaf $\mathcal{B}$-categories (see Theorem 3.4.1.1). Finally, (4) implies (5) by Lemma 5.3.2.5 (and since we already know that (3) forces D to be a $\mathcal{B}$-poset), and (5) implies (3) since $\mathsf{Sub}_{\mathcal{B}}$ is cocomplete. $\square$

We end this section with the observation that all colimits in a presentable $\mathcal{B}$-poset are $\mathcal{B}$-groupoidal and can be computed by an explicit formula:

Proposition 5.3.2.10. *Let* D *be a presentable* $\mathcal{B}$*-poset and let* $d\colon \mathsf{I} \to \mathsf{D}$ *be a diagram. Then the inclusion* $\mathsf{I}^{\simeq} \to \mathsf{I}$ *induces an equivalence* $\operatorname{colim} d|_{\mathsf{I}^{\simeq}} \simeq \operatorname{colim} d$*. Moreover, this colimit can be explicitly computed as*

$$\operatorname{colim} d \simeq \bigvee_{\substack{i\colon G \to \mathsf{I} \\ G \in \mathcal{G}}} (\pi_G)_!(d(i)),$$

where $\mathcal{G} \hookrightarrow \mathcal{B}$ *is a small dense full subcategory.*

Proof. Consider the full subcategory $\mathcal{E} \subset \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ spanned by the diagrams $d\colon \mathsf{I} \to \mathsf{D}$ for which the inclusion $\mathsf{I}^{\simeq} \to \mathsf{I}$ induces an equivalence $\operatorname{colim} d|_{\mathsf{I}^{\simeq}} \simeq \operatorname{colim} d$. To prove the first statement, we need to show that $\mathcal{E} = \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$.

To that end, first observe that if $h_{\mathsf{D}}\colon \mathsf{D} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{D})$ denotes the Yoneda embedding, then $\operatorname{colim} h_{\mathsf{D}} d$ classifies the right fibration $p\colon \mathsf{P} \to \mathsf{D}$ that arises from factoring d into a final functor and a right fibration. In other words, p is the image of d under the localisation functor $L\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}} \to \mathrm{RFib}(\mathsf{D})$, and we may compute the colimit of d by applying the left adjoint $l\colon \mathsf{RFib}(\mathsf{D}) \to \mathsf{D}$ (which exists by Proposition 5.3.2.9) to p. Since both l and L are cocontinuous, it follows that for every ∞-category $\mathcal{K}$ and every diagram $\varphi\colon \mathcal{K} \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ with colimit $d\colon \mathsf{I} \to \mathsf{D}$, we have a canonical equivalence $\operatorname{colim} lL\varphi \simeq \operatorname{colim} d$.

Now let $\varphi^{\simeq}\colon \mathcal{K} \to \mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$ be the composition of φ with the core $\mathcal{B}$-groupoid functor $(-)^{\simeq}$. We then obtain a natural comparison map $\operatorname{colim} \varphi^{\simeq} \to \mathsf{I}^{\simeq}$ which has the property that the composition of this map with the inclusion $\mathsf{I}^{\simeq} \to \mathsf{I}$ can be identified with the colimit of the canonical morphism $\varphi^{\simeq} \to \varphi$. As a consequence, we obtain maps

$$\operatorname{colim} lL\varphi^{\simeq} \to \operatorname{colim} d|_{\mathsf{I}^{\simeq}} \to \operatorname{colim} d$$

in which the composition can be identified with $\operatorname{colim} lL\varphi^{\simeq} \to \operatorname{colim} lL\varphi$. Hence, if φ takes values in $\mathcal{E}$, the latter map is an equivalence, which implies that the map $\operatorname{colim} d|_{\mathsf{I}^{\simeq}} \to \operatorname{colim} d$ is one as well since D is a poset. Thus, we conclude that $\mathcal{E}$ is closed under colimits in $\mathrm{Cat}(\mathcal{B})_{/\mathsf{D}}$.

Consequently, since every $\mathcal{B}$-category can be written as a colimit of $\mathcal{B}$-categories of the form $G \otimes \Delta^n$ for $G \in \mathcal{B}$ and $n \in \mathbb{N}$ (cf. [62, Lemma 4.5.2]), it suffices to see that every diagram of the form $d\colon G \otimes \Delta^n \to \mathsf{D}$ is in $\mathcal{E}$. To that end, note that the colimit of a diagram $d\colon G \otimes \Delta^n \to \mathsf{D}$ is given by applying $(\pi_G)_!\colon \mathsf{D}(G) \to \mathsf{D}(1)$ to the colimit of the transposed map $d'\colon \Delta^n \to \mathsf{D}(G)$, which is simply $d'(n)$. Likewise, the colimit of the induced diagram $\bigsqcup_n G = (G \otimes \Delta^n)^{\simeq} \to \mathsf{D}$ is given by applying $(\pi_G)_!$ to the supremum of the objects $d'(i)$ for $i \in \Delta^n$. Since $d'(i) \leq d'(n)$ for all $i \in \Delta^n$, we deduce that $d \in \mathcal{E}$, as desired.

As for the second statement of the proposition, note that we have an equivalence

$$\mathsf{I}^{\simeq} \simeq \operatorname*{colim}_{\substack{i\colon G \to \mathsf{I} \\ G \in \mathcal{G}}} G$$

since $\mathcal{G}$ is dense in $\mathcal{B}$. Thus the description of $\operatorname{colim} d|_{\mathsf{I}^{\simeq}}$ follows from the observation at the beginning of the proof. □

5.3.3. $\mathcal{B}$-locales. In this section we define what it means for a $\mathcal{B}$-poset to be a $\mathcal{B}$-*locale* and provide a first characterisation of this notion.

Definition 5.3.3.1. A $\mathcal{B}$-category L is said to be a $\mathcal{B}$-*locale* if

(1) L is a $\mathcal{B}$-poset,
(2) L is presentable, and
(3) colimits are universal in L (in the sense of § 5.1.3).

A functor $f\colon K \to L$ between $\mathcal{B}$-locales is called an algebraic morphism of $\mathcal{B}$-locales if it is cocontinuous and preserves finite limits. We denote by $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Pos}_{\mathcal{B}}$ the subcategory that is spanned by the algebraic morphisms of $\mathcal{B}_{/A}$-locales for all $A \in \mathcal{B}$.

Remark 5.3.3.2. In the situation of Definition 5.3.3.1, note that colimits are universal in L if and only if for every $A \in \mathcal{B}$ and every $U\colon A \to \mathsf{L}$ the functor $U \times -\colon \pi_A^* \mathsf{L} \to \pi_A^* \mathsf{L}$ is cocontinuous. In fact, since for every $V\colon A \to \mathsf{L}$ with a map $U \to V$ the diagram

$$\begin{tikzcd} (\pi_A^*\mathsf{L})_{/V} \arrow[r, "U\times_V -"] \arrow[d, "(\pi_V)_!"] & (\pi_A^*\mathsf{L})_{/V} \arrow[d, "(\pi_V)_!"] \\ \pi_A^*\mathsf{L} \arrow[r, "U\times -"] & \pi_A^*\mathsf{L} \end{tikzcd}$$

commutes, the composition $(\pi_V)_!(U \times_V -)$ is cocontinuous. As $(\pi_V)_!$ is conservative, this implies that $U \times_V -$ is already cocontinuous.

Remark 5.3.3.3. Since the property of a $\mathcal{B}$-category L to be a presentable $\mathcal{B}$-poset is local in $\mathcal{B}$ (combine Remark 4.4.2.9 with Remark 5.3.1.6) and since the property of a functor to be cocontinuous is local in $\mathcal{B}$ as well Remark 3.2.2.3, one concludes that for every cover $\bigsqcup A_i \twoheadrightarrow 1$ in $\mathcal{B}$, the $\mathcal{B}$-category L is a $\mathcal{B}$-locale if and only if the $\mathcal{B}_{/A_i}$-category $\pi_{A_i}^* \mathsf{L}$ is a $\mathcal{B}_{/A_i}$-locale.

Remark 5.3.3.4. The subobject of $(\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ that is spanned by the algebraic morphisms between $\mathcal{B}_{/A}$-locales (for each $A \in \mathcal{B}$) is stable under composition and equivalences in the sense of Proposition 2.2.2.9. Since moreover cocontinuity and the property that a functor preserves finite limits are local conditions and on account of Remark 5.3.3.3, we conclude that a map $A \to (\mathsf{Cat}_{\widehat{\mathcal{B}}})_1$ is contained in $(\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}})_1$ if and only if it defines an algebraic morphism between $\mathcal{B}_{/A}$-locales. In particular, if L and M are $\mathcal{B}_{/A}$-locales, the image of the monomorphism

$$\mathrm{map}_{\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}}(\mathsf{L}, \mathsf{M}) \hookrightarrow \mathrm{map}_{\mathsf{Cat}_{\widehat{\mathcal{B}}}}(\mathsf{L}, \mathsf{M})$$

is spanned by the algebraic morphisms, and there is a canonical equivalence $\pi_A^* \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}} \simeq \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}_{/A}}$.

Remark 5.3.3.5. In light of Remark 5.3.1.8, it is easy to see that $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{Pos}(\mathcal{B})$ can be identified with the category of internal locales in $\mathrm{Disc}(\mathcal{B})$ in the sense of [46, § C1.6]. In other words, our notion of an internal locale coincides with the classical one.

Lemma 5.3.3.6. *Let D be a presentable $\mathcal{B}$-category with universal colimits, and let $l\colon \mathsf{D} \to \mathsf{L}$ be a Bousfield localisation that preserves binary products. Suppose furthermore that L is a $\mathcal{B}$-poset. L is a $\mathcal{B}$-locale.*

Proof. We need to show that colimits are universal in L, i.e. that for every $A \in \mathcal{B}$ and every object $U\colon A \to \mathsf{L}$ the functor $U \times -\colon \pi_A^* \mathsf{L} \to \pi_A^* \mathsf{L}$ is cocontinuous, or equivalently has a right adjoint. Now l preserving binary products implies that $U \times -$ carries every map in D (in arbitrary context) that is inverted by l to one that is inverted by l as well. Hence the functor $\underline{\mathrm{Hom}}_{\mathsf{D}}(i(U), i(-))$ (where $\underline{\mathrm{Hom}}_{\mathsf{D}}(-,-)$ is the internal hom in D) takes values in L, which yields the claim. $\square$

Proposition 5.3.3.7. *For a $\mathcal{B}$-category L, the following are equivalent:*

(1) L is a $\mathcal{B}$-locale.
(2) (a) L takes values in the 1-category $\mathrm{Loc}^{\mathrm{L}}$ of locales;
 (b) L is $\Omega_{\mathcal{B}}$-cocomplete;
 (c) for every map $s\colon B \to A$ in $\mathcal{B}$, the functor $s_!\colon \mathsf{L}(B) \to \mathsf{L}(A)$ is a cartesian fibration.
(3) L is small, and the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ admits a left adjoint which preserves finite products;
(4) L is a small $\mathcal{B}$-poset, and the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ admits a left exact left adjoint.

Proof. First, we show that (1) and (2) are equivalent. To that end, if L is a $\mathcal{B}$-locale, then for each $A \in \mathcal{B}$ the ∞-category $\mathsf{L}(A)$ is a presentable poset in which colimits are universal. Therefore, $\mathsf{L}(A)$ is a locale. Moreover, L being cocomplete implies that for every map $s\colon B \to A$ in $\mathcal{B}$ the transition functor

$s^*\colon \mathsf{L}(A) \to \mathsf{L}(B)$ is cocontinuous. Likewise, L having finite limits implies that s^* preserves finite limits. Therefore (2a) follows. Moreover, condition (2b) is part of the definition of a $\mathcal{B}$-locale, and condition (2c) is a reformulation of the condition that Ω-colimits are universal in L (see Example 5.1.3.6). Conversely, if the three conditions in (2) are satisfied, then L is both Ω- and LConst-cocomplete and section-wise presentable. Hence Theorem 4.4.2.4 implies that L is presentable. By Remark 5.3.1.7, the assumption that L is section-wise given by a poset implies that L is a $\mathcal{B}$-poset. Finally, the fact that L takes values in Loc implies that LConst-colimits are universal in L, so that it suffices to verify that Ω-colimits are universal in L as well. Again, this is a consequence of Example 5.1.3.6.

Next, if L is a $\mathcal{B}$-locale, then Proposition 5.3.2.9 implies that the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ has a left adjoint l. Moreover, as colimits are universal in L and as $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ is generated by L under colimits, the comparison map $l(- \times -) \to l(-) \times l(-)$ is an equivalence already when its restriction to L is one, which is trivially true. Hence (1) implies (3). If we assume (3), then Proposition 5.3.2.9 implies that L is a small $\mathcal{B}$-poset and that the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ has a left adjoint. Explicitly, this left adjoint arises as the restriction of the left adjoint $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \mathsf{L}$ and therefore preserves finite products. But since pullbacks in $\mathcal{B}$-posets coincide with binary products, this is already enough to conclude that this functor is left exact. Hence (4) follows. Finally, if (4) holds, then L is presentable by Proposition 5.3.2.9. Moreover, using Lemma 5.3.3.6 it will be enough to show that the subterminal truncation functor $(-)^{\mathrm{Sub}}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ preserves binary products, which is an immediate consequence of Example 5.3.2.4. □

Using Proposition 5.3.3.7, it is now easy to show that the $\mathcal{B}$-poset of subterminal objects in a $\mathcal{B}$-topos is a $\mathcal{B}$-locale. More precisely, one has:

Proposition 5.3.3.8. *The functor* $\mathsf{Sub}\colon \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$ *from Corollary 5.3.2.7 restricts to a functor* $\mathsf{Sub}\colon \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$.

Proof. By combining Remark 5.3.2.2 and Remark 5.3.3.4, it is enough to show that for every algebraic morphism $f^*\colon \mathsf{X} \to \mathsf{Y}$ of $\mathcal{B}$-topoi the induced map $\mathsf{Sub}(f^*)\colon \mathsf{Sub}(\mathsf{X}) \to \mathsf{Sub}(\mathsf{Y})$ is an algebraic morphism of $\mathcal{B}$-locales. First, let us show that $\mathsf{Sub}(\mathsf{X})$ (and therefore by symmetry also $\mathsf{Sub}(\mathsf{Y})$) is a $\mathcal{B}$-locale. To that end, choose a presentation of X as a left exact and accessible Bousfield localisation $L\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$. Then Remark 5.3.2.8 implies that L restricts to a left exact and accessible Bousfield localisation $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \to \mathsf{Sub}(\mathsf{X})$, hence the claim follows from Proposition 5.3.3.7. Second, since we already know that $\mathsf{Sub}(f^*)$ is cocontinuous, it is enough to show that it is left exact as well. But on account of Remark 5.3.2.8, this functor arises as the restriction of f^* to subterminal objects, which immediately yields the claim. □

5.3.4. Sheaves on a $\mathcal{B}$-locale. In this section we introduce and study the $\mathcal{B}$-category of *sheaves* on a $\mathcal{B}$-locale.

Definition 5.3.4.1. Let L be a $\mathcal{B}$-locale and let $U\colon A \to \mathsf{L}$ be an object. A *covering* of U is a diagram $d\colon \mathsf{G} \to \pi_A^*\mathsf{L}$ with colimit U, where G is a $\mathcal{B}_{/A}$-groupoid.

Remark 5.3.4.2. Explicitly, a covering of U is given by a map $s\colon B \to A$ in $\mathcal{B}$ together with an object $V\colon B \to \mathsf{L}$ such that $s_!(V) \simeq U$.

Example 5.3.4.3. Let L be a $\mathcal{B}$-locale and $U\colon A \to \mathsf{L}$ be an object. Then every covering $(j_i\colon U_i \to U)_{i\in I}$ in $\mathsf{L}(A)$ (in the conventional sense) can be regarded as a covering in the sense of Definition 5.3.4.1 by setting $\mathsf{G} = I$ and $d = (j_i)_{i\in I}$.

Recall from Proposition 5.3.3.7 that if L is a $\mathcal{B}$-locale, the Yoneda embedding $h_{\mathsf{L}}\colon \mathsf{L} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ admits a (left exact) left adjoint l. We denote by $\eta\colon \mathrm{id}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})} \to h_{\mathsf{L}} l$ the adjunction unit.

DEFINITION 5.3.4.4. Let L be a $\mathcal{B}$-locale and let $d\colon \mathsf{G} \to \pi_A^* \mathsf{L}$ be a covering of an object $U\colon A \to \mathsf{L}$. Then the induced map $\eta \operatorname{colim} h_{\mathsf{L}} d\colon S_d = \operatorname{colim} h_{\mathsf{L}} d \hookrightarrow h_{\mathsf{L}}(U)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ is referred to as the *covering sieve* associated with d.

REMARK 5.3.4.5. Let L be a $\mathcal{B}$-locale and $d\colon \mathsf{G} \to \pi_A^* \mathsf{L}$ be a covering of an object $U\colon A \to \mathsf{L}$. Then, for every map $s\colon B \to A$ in $\mathcal{B}$, we obtain an equivalence $s^* S_d \simeq S_{s^* d}$ that commutes with the canonical equivalence $s^* h_{\mathsf{L}}(U) \simeq h_{\mathsf{L}}(s^* U)$.

DEFINITION 5.3.4.6. If L is a $\mathcal{B}$-locale, we denote by $\mathsf{Cov} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ the subcategory that is spanned by the covering sieves in arbitrary context.

REMARK 5.3.4.7. For every $A \in \mathcal{B}$, one may identify $\pi_A^* \mathsf{Cov} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\pi_A^* \mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}_{/A}})$ with the subcategory of covering sieves in $\pi_A^* \mathsf{L}$.

REMARK 5.3.4.8. The $\mathcal{B}$-category Cov is small. In fact, first note that by Remark 5.3.4.5, the subcategory $\mathsf{Cov} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ is already spanned by all covering sieves of objects in context $G \in \mathcal{G}$, where $\mathcal{G}$ is a small full subcategory of $\mathcal{B}$ that generates $\mathcal{B}$ under colimits. Furthermore, since L is small, the collection of all coverings of objects in fixed context G is parametrised by a small set. Hence, the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(L)^{\Delta^1}$ that is spanned by the covering sieves must be small. In light of the very construction of a subcategory from a collection of morphisms (see § 2.2.2), the claim thus follows from the fact that the 1-image of a small $\mathcal{B}$-category in a locally small $\mathcal{B}$-category must also be small (see for example [62, Lemma 4.7.5]).

DEFINITION 5.3.4.9. Let L be a $\mathcal{B}$-locale. We define the $\mathcal{B}$-category $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ of *sheaves* on L to be the Bousfield localisation $\mathsf{Loc}_{\mathsf{Cov}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}))$. We will furthermore denote the underlying ∞-category of global sections of $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ by $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$.

REMARK 5.3.4.10. By Remark 5.3.4.7, for every $A \in \mathcal{B}$ there is a canonical equivalence $\pi_A^* \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{L})$ of full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{L})$.

REMARK 5.3.4.11. If L be a $\mathcal{B}$-locale, then Proposition 5.3.3.7 implies that the Yoneda embedding $\mathsf{L} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ admits a left adjoint $l\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \mathsf{L}$. By construction, this functor carries Cov into $\mathsf{L}^{\simeq}$. In other words, l factors through the sheafification functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$. By passing to right adjoints, this implies that the Yoneda embedding factors through the inclusion $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$, which means that every representable presheaf on L is already a sheaf.

The main goal of this section is to prove that $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is a $\mathcal{B}$-topos. More precisely, we will show:

PROPOSITION 5.3.4.12. *For any $\mathcal{B}$-locale L, the localisation functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ preserves finite limits. In particular, $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is a $\mathcal{B}$-topos.*

The proof of Proposition 5.3.4.12 is based on the following three lemmas:

LEMMA 5.3.4.13. *For every $\mathcal{B}$-locale L, the ∞-category $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ is the Bousfield localisation of $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$ at the set*

$$W = \{(\pi_A)_!(S) \hookrightarrow (\pi_A)_! h(U) \mid A \in \mathcal{B},\ U\colon A \to \mathsf{L},\ S \hookrightarrow h(U) \text{ covering sieve}\}$$

of morphisms in $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$.

PROOF. A presheaf $F\colon \mathsf{L}^{\mathrm{op}} \to \Omega$ is a sheaf if and only if for every $A \in \mathcal{B}$ and every covering sieve $S \hookrightarrow h(U)$ in context A the morphism $\varphi\colon \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(h(U), \pi_A^* F) \to \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(S, \pi_A^* F)$ is an equivalence in $\mathcal{B}_{/A}$. Recall from [62, Corollary 4.6.8] that if $s\colon B \to A$ is a map in $\mathcal{B}$, then on local sections over A the map φ recovers the morphism

$$\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(B)}(s^* h(U), \pi_B^* F) \to \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(B)}(s^* S, \pi_B^* F)$$

of mapping ∞-groupoids, which by adjunction can in turn be identified with the map

$$\mathrm{map}_{\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})}((\pi_B)_! s^* h(U), F) \to \mathrm{map}_{\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})}((\pi_B)_! s^* S, F).$$

Hence F is a sheaf precisely if the latter map is an equivalence for every covering sieve $S \hookrightarrow h(U)$ in context A and every map $s \colon B \to A$ in $\mathcal{B}$. Together with Remark 5.3.4.5, this yields the claim. ☐

LEMMA 5.3.4.14. *Let L be a locale and let $S \hookrightarrow h(U)$ be a covering sieve on an object $U \colon A \to \mathsf{L}$. Then for every map $V \to U$ in $\mathsf{L}(A)$ the map $h(V) \times_{h(U)} S \hookrightarrow h(V)$ is a covering sieve.*

PROOF. We may assume without loss of generality that $A \simeq 1$. Now if $d \colon \mathsf{G} \to \mathsf{L}$ is a covering of U giving rise to the covering sieve S, then universality of colimits in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ and the fact that h preserves limits implies that $h(V) \times_{h(U)} S$ is the colimit of the diagram

$$\mathsf{G} \xrightarrow{d} \mathsf{L} \xrightarrow{-\times V} \mathsf{L} \overset{h}{\hookrightarrow} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}).$$

Since universality of colimits in L implies that the diagram $d(-) \times V \colon \mathsf{G} \to \mathsf{L}$ is a covering of V, the claim follows. ☐

LEMMA 5.3.4.15. *Let L be a $\mathcal{B}$-locale and let $S_0 \hookrightarrow h(U)$ and $S_1 \hookrightarrow h(U)$ be covering sieves on an object $U \colon A \to \mathsf{L}$. Then $S_0 \times_{h(U)} S_1 \hookrightarrow h(U)$ is a covering sieve as well.*

PROOF. We may assume without loss of generality that $A \simeq 1$. Let $d_0 \colon \mathsf{G}_0 \to \mathsf{L}$ be a covering giving rise to the covering sieve S_0, and let $d_1 \colon \mathsf{G}_1 \to \mathsf{L}$ be a covering giving rise to S_1. Define $\mathsf{G} = \mathsf{G}_0 \times \mathsf{G}_1$ and let $d \colon \mathsf{G} \to \mathsf{L}$ be the diagram given by the composition

$$\mathsf{G}_0 \times \mathsf{G}_1 \xrightarrow{d_0 \times d_1} \mathsf{L} \times \mathsf{L} \xrightarrow{-\times-} \mathsf{L}.$$

Then we have $\operatorname{colim} d \simeq U$ since colimits are universal in L and since $U \times U \simeq U$ in L. Therefore, it is enough to show that the induced map $\operatorname{colim} h_{\mathsf{L}} d \hookrightarrow h(U)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ can be identified with $S_0 \times_{h(U)} S_1 \hookrightarrow h(U)$ This follows from the fact that $h_{\mathsf{L}} d$ is given by the composition

$$\mathsf{G}_0 \times \mathsf{G}_1 \xrightarrow{h_{\mathsf{L}} d_0 \times h_{\mathsf{L}} d_1} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \times \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \xrightarrow{-\times-} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$$

and the very same argument as above, using that colimits are universal in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ as well. ☐

PROOF OF PROPOSITION 5.3.4.12. Since the localisation is already accessible (being a Bousfield localisation at a small subcategory, see Theorem 4.4.2.4), the second claim follows from the first by Theorem 5.2.3.1. To prove the first, let $T'(A)$ be the class of monomorphisms $f \colon G \hookrightarrow H$ in the ∞-topos $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(A)$ (for arbitrary $A \in \mathcal{B}$) satisfying the condition that for every map $s \colon B \to A$ in $\mathcal{B}$, every $U \colon B \to \mathsf{L}$ and every map $h(U) \to s^* H$ the pullback $s^* G \times_{s^* H} h(U) \hookrightarrow h(U)$ is a covering sieve in context B. Then $T'(A)$ has the following properties:

(1) the maps in $T'(A)$ are closed under pullbacks in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(A)$;
(2) the maps in $T'(A)$ are closed under finite limits in $\mathrm{Fun}(\Delta^1, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(A))$;
(3) every map in $T'(A)$ is inverted by the localisation functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(A) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})(A)$;
(4) every covering sieve in context $A \in \mathcal{B}$ is contained in $T'(A)$.

In fact, the first property is evident, and the second property follows from combining the first one with Lemma 5.3.4.15. Property (3) follows from the observation that by descent in the $\mathcal{B}_{/A}$-topos $\underline{\mathsf{PSh}}_{\mathcal{B}_{/A}}(\pi_A^* \mathsf{L})$, every map in $T'(A)$ is a ($\mathcal{B}_{/A}$-internal) colimit of covering sieves, which implies (using Remark 5.3.4.10) that it is inverted by the localisation functor $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(A) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})(A)$. The last property is an immediate consequence of Lemma 5.3.4.14.

Let us set $T' = \bigcup_{A \in \mathcal{B}} (\pi_A)_! T'(A)$ and let T be the smallest local class of morphisms in $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$ that contains T'. Then T is bounded since it only contains monomorphisms. Moreover, T is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathrm{PSh}_{\mathcal{B}}(\mathsf{L}))$. To see this, the fact that every map in T is locally (in the ∞-topos $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$) contained in T' implies that it suffices to show that for every cospan $f_0 \to f \leftarrow f_1$ with f_0 and f_1 in T', their pullback is in T' as well. Note that if $s \colon B \to A$ is a map in $\mathcal{B}$ and if g is a map in

$\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})(B)$ such that $s_!(g) \in T'(A)$, we have $g \in T'(B)$. Therefore, we may assume that both f_0 and f_1 are in $T'(A)$ for some $A \in \mathcal{B}$. In this case, the claim immediately follows from properties (1) and (2) of $T'(A)$. The same argument moreover shows that every map in T is inverted by the localisation functor $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L}) \to \mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ as it can be written as a Δ^{op}-indexed colimits of maps in T'.

By employing Lemma 5.3.4.13 and property (4) above, we now conclude that there is an equivalence $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L}) \simeq \mathrm{Loc}_T(\mathrm{PSh}_{\mathcal{B}}(\mathsf{L}))$ of Bousfield localisations of $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L})$. Using Proposition 5.2.10.11, we thus obtain hat $\mathrm{PSh}_{\mathcal{B}}(\mathsf{L}) \to \mathrm{Sh}[\mathcal{B}](\mathsf{L})$ is left exact. In light of Remark 5.3.4.10, this is already sufficient to conclude that the entire functor of $\mathcal{B}$-categories $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ is left exact. □

5.3.5. The localic reflection of $\mathcal{B}$-topoi. In the previous section, we introduces the $\mathcal{B}$-topos of sheaves on a $\mathcal{B}$-locale. In this section, we show that this construction is the universal way to attach a $\mathcal{B}$-topos to a $\mathcal{B}$-locale. More precisely, we show:

PROPOSITION 5.3.5.1. *Let L be a $\mathcal{B}$-locale. Then the Yoneda embedding $h\colon \mathsf{L} \hookrightarrow \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ induces an equivalence $\mathsf{L} \simeq \mathsf{Sub}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}))$, and for every $\mathcal{B}$-topos X precomposition with h induces an equivalence*

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}), \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{L}, \mathsf{Sub}(\mathsf{X})),$$

where the left-hand side denotes the $\mathcal{B}$-category of algebraic morphisms between the $\mathcal{B}$-topoi $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ and X and the right-hand side denotes the $\mathcal{B}$-category of algebraic morphisms between the $\mathcal{B}$-locales L and $\mathsf{Sub}(\mathsf{X})$.

PROOF. We begin by showing the first claim. To that end, note that by Lemma 5.3.2.5 a sheaf $F\colon \mathsf{L}^{\mathrm{op}} \to \Omega$ is subterminal if and only if it takes values in $\mathsf{Sub}_{\mathcal{B}}$. Together with the usual base change arguments, this implies that the first claim follows once we verify that every such sheaf $F\colon \mathsf{L}^{\mathrm{op}} \to \mathsf{Sub}_{\mathcal{B}}$ is representable. Note that by Example 5.3.2.4, the associated right fibration $p\colon \mathsf{L}_{/F} \to \mathsf{L}$ is fully faithful. Let $U\colon 1 \to \mathsf{L}$ be the colimit of p. We then obtain a canonical map $F \to h(U)$ in $\mathsf{Sub}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}))$. To show the claim, it is therefore enough to produce a map in the opposite direction, which by Yoneda's lemma is equivalent to show that $F(U) \simeq 1_{\Omega}$. To see this, note that by Proposition 5.3.2.10 the object U is the colimit of the restriction of p to $\mathsf{G} = (\mathsf{L}_{/F})^{\simeq}$. In other words, we have a covering of U given by $p|_{\mathsf{G}}$. Let $S \hookrightarrow h(U)$ be the associated covering sieve. Then, since F is a sheaf, we obtain an equivalence $F(U) \simeq \mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(S, F)$. To complete the proof of the first claim, we therefore need to show that the right-hand side can be identified with 1_{Ω}. But as F is subterminal, we may in turn identify $\mathrm{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(S, F)$ with $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})}(S, F) \simeq \lim Fp|_{\mathsf{G}}$. Thus, the claim follows once we show that $Fp|_{\mathsf{G}}\colon \mathsf{G} \to \mathsf{Sub}_{\mathcal{B}}$ is final in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{G}, \mathsf{Sub}_{\mathcal{B}})$. Note that the associated object $P \hookrightarrow \mathsf{G}$ in $\mathrm{Sub}(\mathcal{B}_{/\mathsf{G}})$ is explicitly obtained as the fibre of $p\colon \mathsf{L}_{/F} \to \Omega$ over $p|_{\mathsf{G}}$. Thus, the inclusion $\mathsf{G} \hookrightarrow \mathsf{L}_{/F}$ induces a section $\mathsf{G} \to P$, which immediately yields the claim.

We now show the second claim. Let $l\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ be the localisation functor. We now have maps

$$\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}), \mathsf{X}) \xhookrightarrow{l^*} \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}), \mathsf{X}) \simeq \underline{\mathsf{Fun}}^{\mathrm{lex}}_{\mathcal{B}}(\mathsf{L}, \mathsf{Sub}(\mathsf{X})) \hookleftarrow \underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\mathsf{L}, \mathsf{Sub}(\mathsf{X}))$$

in which the fact that l^* is fully faithful follows from the universal property of localisations and where the equivalence in the middle follows from Diaconescu's theorem 5.2.2.10 and the straightforward observation that by Remark 5.3.2.8 every left exact functor $\pi_A^*\mathsf{L} \to \pi_A^*\mathsf{X}$ necessarily factors through $\pi_A^*\,\mathsf{Sub}(\mathsf{X})$. Thus, by using Remarks 5.3.4.10 and 5.3.2.2 together with Remark 3.2.2.3, the claim follows once we show that a left exact functor $f\colon \mathsf{L} \to \mathsf{Sub}(\mathsf{X})$ is cocontinuous if and only if the left Kan extension $h_!(if)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \mathsf{X}$ (where $i\colon \mathsf{Sub}(\mathsf{X}) \hookrightarrow \mathsf{X}$ is the inclusion) carries Cov into $\mathsf{X}^{\simeq}$. To see this, note that as $h_!(if)$ is an algebraic morphism, it restricts to a functor $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \to \mathsf{Sub}(\mathsf{X})$ which is cocontinuous as well. Consequently, for every object $U\colon A \to \mathsf{L}$ and every covering $d\colon \mathsf{G} \to \pi_A^*\mathsf{L}$ of U, the image of the associated covering sieve $S_d \hookrightarrow h(U)$ along $h_!(if)$ can be identified with the canonical morphism $\operatorname{colim} fd \to f(U)$. In other words, $h_!(if)$ carries Cov into $\mathsf{X}^{\simeq}$ precisely if f is Ω-cocontinuous. But by using Proposition 5.3.2.10, this already implies that f is cocontinuous. Hence the claim follows. □

COROLLARY 5.3.5.2. *The $\mathcal{B}$-category $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$ is a coreflective subcategory of $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$, where the inclusion is given by carrying a $\mathcal{B}$-locale to its associated sheaf $\mathcal{B}$-topos and the coreflection sends a $\mathcal{B}$-topos to its underlying $\mathcal{B}$-locale of subterminal objects.*

PROOF. Combine Proposition 5.3.3.8 with Proposition 5.3.5.1. $\square$

DEFINITION 5.3.5.3. A $\mathcal{B}$-topos X is *localic* if it is contained in the essential image of $\underline{\mathsf{Sh}}_{\mathcal{B}}(-)\colon \mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}} \hookrightarrow \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$.

5.3.6. Localic $\mathcal{B}$-topoi as relative locales. If $\mathcal{B}$ is a localic ∞-topos, then $\mathcal{B}$-locales precisely correspond to locales under $\mathrm{Sub}(\mathcal{B})$. More precisely, note that as a consequence of Corollary 5.3.5.2, the $\mathcal{B}$-locale $\mathsf{Sub}_{\mathcal{B}}$ is the *initial* $\mathcal{B}$-locale. Since moreover the global sections functor $\Gamma_{\mathcal{B}}$ restricts to a functor $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Loc}^{\mathrm{L}}$, we thus obtain an induced functor $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Loc}^{\mathrm{L}}_{\mathrm{Sub}(\mathcal{B})/}$.

PROPOSITION 5.3.6.1. *If $\mathcal{B}$ is a localic ∞-topos, then the functor $\Gamma\colon \mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Loc}^{\mathrm{L}}_{\mathrm{Sub}(\mathcal{B})/}$ is an equivalence of ∞-categories.*

PROOF. Since by Remark 5.3.3.5 the ∞-category $\mathrm{Loc}^{\mathrm{L}}(\mathcal{B})$ can be identified with the 1-category of internal locales in $\mathrm{Disc}(\mathcal{B})$, the statement reduces to the analogous result in 1-topos theory, see [46, Theorem C1.6.3]. $\square$

COROLLARY 5.3.6.2. *For every $\mathcal{B}$-locale L, the ∞-topos $\mathrm{Sh}_{\mathcal{B}}(\mathsf{L})$ can be canonically identified with $\mathrm{Sh}(\Gamma\mathsf{L})$.*

PROOF. As a result of Remark 5.3.2.8, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Top}^{\mathrm{L}}(\mathcal{B}) \arrow[r, "\Gamma"] \arrow[d, "\mathsf{Sub}"] & (\mathrm{Top}^{\mathrm{L}}_{\infty})_{\mathcal{B}/} \arrow[d, "\mathrm{Sub}"] \\
\mathrm{Loc}^{\mathrm{L}}(\mathcal{B}) \arrow[r, "\Gamma"] & \mathrm{Loc}^{\mathrm{L}}_{\mathrm{Sub}(\mathcal{B})/}
\end{tikzcd}$$

(where we note that as $\mathrm{Loc}^{\mathrm{L}}$ is a 1-category coherence issues do not arise). Therefore, the claim follows from Proposition 5.3.6.1 and the fact that by Theorem 5.2.5.1 the upper horizontal map is an equivalence as well. $\square$

REMARK 5.3.6.3. The inverse to the equivalence from Proposition 5.3.6.1 can be made explicit as follows: Given an algebraic morphism of locales $f^*\colon \mathrm{Sub}(\mathcal{B}) \to L$, let $f^*\colon \mathcal{B} \to \mathrm{Sh}(L)$ be the associated algebraic morphism of ∞-topoi. Then $f_* \mathsf{Sub}_{\mathrm{Sh}(L)}$ is a $\mathcal{B}$-locale (as can be easily verified using Proposition 5.3.3.7) whose underlying locale recovers L and is therefore the $\mathcal{B}$-locale associated to L. Explicitly, this $\mathcal{B}$-locale can be described as the sheaf $L_{/f^*(-)}$ on $\mathrm{Sub}(\mathcal{B})$, i.e. the $\widehat{\mathrm{Cat}}_{\infty}$-valued functor that is classified by the cartesian fibration $\mathrm{Sub}(\mathcal{B}) \times_L \mathrm{Fun}(\Delta^1, \mathsf{L}) \to \mathrm{Sub}(\mathcal{B})$.

5.3.7. Compactness conditions for $\mathcal{B}$-locales. In this section, we study how certain compactness properties of $\mathcal{B}$-locales are inherited by their associated localic $\mathcal{B}$-topoi. To that end, recall that if D is a presentable $\mathcal{B}$-category, we say that D is *compactly generated* if the inclusion $\mathsf{D}^{\mathrm{cpt}} \hookrightarrow \mathsf{D}$ of the full subcategory of compact objects induces via left Kan extension an equivalence $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{D}^{\mathrm{cpt}}) \simeq \mathsf{D}$. We will furthermore say that D is *compactly assembled* if D is a retract (in $\mathsf{Pr}^{\mathrm{L}}_{\mathcal{B}}$) of a compactly generated $\mathcal{B}$-category. We may now define:

DEFINITION 5.3.7.1. A $\mathcal{B}$-locale L is said to be *locally coherent* if it is compactly generated and if $\mathsf{L}^{\mathrm{cpt}}$ is closed under binary products in L. We say that L is *coherent* if it is locally coherent and 1_{L} is compact.

Furthermore, L is said to be *(locally) stably compact* if it is a retract in $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$ of a (locally) coherent $\mathcal{B}$-locale.

REMARK 5.3.7.2. Since the existence and preservation of limits is local in $\mathcal{B}$ by Remark 3.2.2.3 and one has $\pi_A^*\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}}) \simeq \underline{\mathsf{Ind}}_{\mathcal{B}_{/A}}(\pi_A^*\mathsf{L}^{\mathrm{cpt}})$ by Remark 4.1.5.2 and Remark 4.3.1.2 for every $A \in \mathcal{B}$, we deduce that for every cover $(\pi_{A_i})\colon : \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$, a $\mathcal{B}$-locale L is (locally) coherent if and only if $\pi_{A_i}^* \mathsf{L}$ is a (locally) coherent $\mathcal{B}_{/A_i}$-locale for every i.

Then main goal of this section is to relate sheaves on a locally coherent locale with *finitary sheaves* on its compact objects:

Definition 5.3.7.3. Let P be a $\mathcal{B}$-poset with finite colimits and binary products. A presheaf $F\colon \mathsf{P}^{\mathrm{op}} \to \Omega$ is said to be a *finitary sheaf* if

(1) $F(\varnothing_{\mathsf{P}}) \simeq 1_\Omega$;
(2) for every two objects $U, V\colon A \rightrightarrows \mathsf{P}$ in arbitrary context $A \in \mathcal{B}$, the commutative square

$$\begin{tikzcd} F(U \vee V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \wedge V) \end{tikzcd}$$

is a pullback.

We let $\underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{P})$ be the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ that is spanned by those presheaves $\pi_A^* \mathsf{P}^{\mathrm{op}} \to \Omega_{\mathcal{B}_{/A}}$ (in arbitrary context $A \in \mathcal{B}$) which are finitary sheaves on $\pi_A^* \mathsf{P}$.

Remark 5.3.7.4. As preservation of (co)limits is a local property Remark 3.1.1.8, we deduce that for every cover $\bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ a presheaf $F\colon \mathsf{P}^{\mathrm{op}} \to \Omega$ is a finitary sheaf if and only if the presheaf $\pi_{A_i}^*(F)$ is a finitary sheaf on $\pi_A^* \mathsf{P}$. In particular, an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ is contained in $\underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{P})$ if and only if it transposes to a finitary sheaf on $\pi_A^* \mathsf{P}$, and we obtain a canonical equivalence of $\mathcal{B}$-categories $\pi_A^* \underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{P}) \simeq \underline{\mathsf{Sh}}_{\mathcal{B}_{/A}}^{\mathrm{fin}}(\pi_A^* \mathsf{P})$ for every $A \in \mathcal{B}$.

Recall from § 4.4.6 that if C is a Filt-cocomplete $\mathcal{B}$-category, we denote by $\underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathsf{Filt}}(\mathsf{C})$ the full subcategory of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ that is spanned by the Filt-*sheaves*, i.e. by those functors $\pi_A^* \mathsf{C}^{\mathrm{op}} \to \Omega_{\mathcal{B}_{/A}}$ (in arbitrary context $A \in \mathcal{B}$) whose opposite is Filt-cocontinuous. We now obtain the following characterisation of sheaves on a $\mathcal{B}$-locale:

Proposition 5.3.7.5. *Let* L *be a* $\mathcal{B}$*-locale. Then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathrm{fin}}(\mathsf{L}) \cap \underline{\mathsf{Sh}}_{\mathcal{B}}^{\mathsf{Filt}}(\mathsf{L})$ *as full subcategories in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$.

Proof. By combining Remarks 5.3.7.4 and 5.3.4.10 with Remark 3.2.2.3, we need to show that for every $A \in \mathcal{B}$, a presheaf $F\colon \pi_A^* \mathsf{L}^{\mathrm{op}} \to \Omega_{\mathcal{B}_{/A}}$ is a sheaf if and only if

(1) $F^{\mathrm{op}}\colon \pi_A^* \mathsf{L} \to \Omega_{\mathcal{B}_{/A}}^{\mathrm{op}}$ is π_A^* Filt-cocontinuous;
(2) $F(\varnothing_{\pi_A^* \mathsf{L}}) \simeq 1_{\Omega_{\mathcal{B}_{/A}}}$;
(3) for every two objects $U, V\colon B \rightrightarrows \mathsf{L}$ in arbitrary context $B \in \mathcal{B}_{/A}$, the commutative square

$$\begin{tikzcd} F(U \vee V) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \wedge V) \end{tikzcd}$$

is a pullback.

By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may assume that $A \simeq 1$. Now suppose first that F is a sheaf. To show that (1) is satisfied, we need to verify that for every diagram $d\colon \mathsf{I} \to \pi_A^* \mathsf{L}$ where I is a filtered $\mathcal{B}_{/A}$-category and $A \in \mathcal{B}$ is arbitrarily chosen, the natural map $(\pi_A^* F)(\operatorname{colim} d) \to \lim(\pi_A^* F) d$ is an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may again assume without loss of generality that $A \simeq 1$. As I is filtered, the functor $\operatorname{colim}_{\mathsf{I}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})) \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ preserves finite limits and therefore (by Remark 5.3.2.8) subterminal objects. Therefore, we deduce that $\operatorname{colim} h_{\mathsf{L}} d$ is subterminal. Now by Proposition 5.3.2.10, we may replace I by $\mathsf{I}^{\simeq}$ and can thus assume that I is a $\mathcal{B}$-groupoid. Therefore, the sheaf condition implies that we obtain an equivalence

$$F(\operatorname{colim} d) \simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(\operatorname{colim} h_{\mathsf{L}} d, F) \simeq \lim F d.$$

This shows that F is a Filt-sheaf. Condition (2) follows from the observation that as $\varnothing_{\mathsf{L}}$ is the colimit of the unique diagram $\varnothing \to \mathsf{L}$, we obtain an equivalence $F(\varnothing_{\mathsf{L}}) \simeq \operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(\varnothing_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}, F) \simeq 1_{\Omega}$. Lastly, to show that condition (3) is met, we may again replace $\mathcal{B}$ with $\mathcal{B}_{/B}$ and can therefore assume that $B \simeq 1$. Now as $U \vee V$ is the coproduct of U and V in L, the claim follows from the fact that the pushout $h_{\mathsf{L}}(U) \sqcup_{h_{\mathsf{L}}(U \wedge V)} h_{\mathsf{L}}(V)$ in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$ computes the coproduct of $h_{\mathsf{L}}(U)$ and $h_{\mathsf{L}}(V)$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$.

Conversely, suppose that F satisfies the three conditions. To show that F is a sheaf, we need to verify that for every covering $d \colon \mathsf{G} \to \pi_A^* \mathsf{L}$ of an object $U \colon A \to \mathsf{L}$ the functor $\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(-, F)$ carries the induces covering sieve $S_d \hookrightarrow h(U)$ to an equivalence. By replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$, we may again assume that $A \simeq 1$. First, let us show the claim in the case where G is finite, i.e. a locally constant sheaf of finite ∞-groupoids. Upon passing to a suitable cover, we can even assume that G is (the constant $\mathcal{B}$-category associated with) a finite ∞-groupoid. Since L is a $\mathcal{B}$-poset, we can even assume that G is a finite set. By induction, it suffices to cover the cases $\mathsf{G} = \varnothing$ and $\mathsf{G} = 1 \sqcup 1$. By the above argumentation, these two cases follow immediately from conditions (2) and (3).

For the general case, let $\mathsf{Fin}_{\mathcal{B}}$ be the internal class of finite $\mathcal{B}$-categories. Since $\mathsf{Fin}_{\mathcal{B}}$ has the decomposition property (see § 4.1.4 and § 4.2.3), we may find a filtered $\mathcal{B}$-category I and a diagram $k \colon \mathsf{I} \to \mathsf{Fin}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ such that $\mathsf{G} \simeq \operatorname{colim} k$. Note that since G is a $\mathcal{B}$-groupoid and since the groupoidification of a finite $\mathcal{B}$-category is a finite $\mathcal{B}$-groupoid, postcomposing k with the groupoidification functor yields a diagram $k' \colon \mathsf{I} \to \Omega \cap \mathsf{Fin}_{\mathcal{B}} \hookrightarrow \mathsf{Cat}_{\mathcal{B}}$ that also has colimit G. Therefore, we deduce from Proposition 3.4.4.3 and by making use of the subterminal truncation functor $(-)^{\mathrm{Sub}} \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}) \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ that there is a diagram $d' \colon \mathsf{I} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ such that (a) we have $\operatorname{colim} d' \simeq \operatorname{colim} h_{\mathsf{L}} d$ and such that (b) for every object $i \colon A \to \mathsf{I}$ in arbitrary context $A \in \mathcal{B}$ there is a finite $\mathcal{B}_{/A}$-groupoid H_i together with a diagram $d_i \colon \mathsf{H}_i \to \pi_A^* \mathsf{L}$ such that $d'(i) \simeq \operatorname{colim} h_{\mathsf{L}} d_i$. From (a) we deduce that if $l \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \to \mathsf{L}$ is the left adjoint of the Yoneda embedding, the unit of the adjunction $l \dashv h_{\mathsf{L}}$ determines morphisms

$$\operatorname{colim} h_{\mathsf{L}} d \simeq \operatorname{colim} d' \xrightarrow{\alpha} \operatorname{colim} h_{\mathsf{L}} l d' \xrightarrow{\beta} h_{\mathsf{L}}(\operatorname{colim} l d') \simeq h_{\mathsf{L}}(\operatorname{colim} d)$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$. As I is filtered, the same argumentation as above implies that the colimit in the middle is already the colimit in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$. Thus condition (1) implies that $\operatorname{map}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})}(-, F)$ carries β to an equivalence. To finish the proof, it is therefore enough to show that this functor also sends α to an equivalence. For this, we only need to show that for every object $i \colon A \to \mathsf{I}$ in arbitrary context $A \in \mathcal{B}$ the map $d'(i) \to h_{\mathsf{L}} l d'(i)$ is sent to an equivalence. By (b), we find that $d'(i)$ is of the form $\operatorname{colim} h_{\mathsf{L}} d_i$ for some diagram $d_i \colon \mathsf{H}_i \to \pi_A^* \mathsf{L}$ where H_i is a finite $\mathcal{B}_{/A}$-groupoid. Since this case has already been shown above, the result follows. ☐

LEMMA 5.3.7.6. *Let L be a locally coherent $\mathcal{B}$-locale and let $F \colon \mathsf{L}^{\mathrm{op}} \to \Omega$ be a Filt-sheaf on L. Then F is a sheaf on L if and only if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ is a finitary sheaf on $\mathsf{L}^{\mathrm{cpt}}$.*

PROOF. By Proposition 5.3.7.5, we need to show that F is a finitary sheaf on L if and only if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ is a finitary sheaf on $\mathsf{L}^{\mathrm{cpt}}$. As L is locally coherent and therefore $\mathsf{L}^{\mathrm{cpt}}$ is closed under binary products in L, it is clear that the condition is necessary. Moreover, as $\mathsf{L}^{\mathrm{cpt}}$ contains the initial object, it is clear that F satisfies condition (1) of the definition of a finitary sheaf if and only if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ does. Therefore, we only need to show that if $F|_{\mathsf{L}^{\mathrm{cpt}}}$ is a finitary sheaf, then for every pair of objects $U, V \colon A \rightrightarrows \mathsf{L}$, the map $F(U \vee V) \to F(U) \times_{F(U \wedge V)} F(V)$ is an equivalence. Using Remark 5.3.7.4 and Remark 4.1.5.2, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. Note that it follows from the bifunctoriality of $- \wedge -$ that for a fixed U, both the map $U \wedge V \to V$ and the map $U \wedge V \to U$ are natural in V, i.e. define morphisms in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$. Therefore, we obtain a cospan $\operatorname{diag}(U) \leftarrow U \wedge - \to \mathrm{id}_{\mathsf{L}}$ in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$ (where $\operatorname{diag} \colon \mathsf{L} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}, \mathsf{L})$ is the diagonal map). By taking the colimit of this diagram, we end up with a commutative square

$$\begin{tikzcd} U \wedge - \arrow[r] \arrow[d] & \mathrm{id}_{\mathsf{L}} \arrow[d] \\ \operatorname{diag}(U) \arrow[r] & U \vee - \end{tikzcd}$$

in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L},\mathsf{L})$. Since colimits are universal in L, the functor $U \wedge -$ is cocontinuous. Furthermore, the functor $\mathrm{diag}(U)$ is Filt-cocontinuous: in fact, as it can be identified with $U \wedge \mathrm{diag}(1_{\mathsf{L}})(-)$, it suffices to see that $\mathrm{diag}(1_{\mathsf{L}}) \simeq 1_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L},\mathsf{L})}$ is Filt-cocontinuous. As in the proof of Lemma 4.1.5.3, this is a consequence of the fact that filtered colimits in L are left exact, which is easily shown using Lemma 5.3.8.7 below and the fact that L is a left exact localisation of $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$, see Proposition 5.3.3.7. Thus, as Filt-cocontinuous functors are clearly closed under pushouts in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{L},\mathsf{L})$, the above commutative diagram is a square of Filt-cocontinuous functors. By again using that filtered colimits in L are left exact, this observation now implies that by postcomposition with (the opposite of) F, we end up with a morphism $F(U \vee -) \to F(U) \times_{F(U\wedge -)} F(-)$ of Filt-cocontinuous functors $\mathsf{L} \to \Omega^{\mathrm{op}}$. Since $\mathsf{L} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$, the universal property of $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ thus implies that this morphism is an equivalence already when its restriction to $\mathsf{L}^{\mathrm{cpt}}$ is one. Together with our assumption on F, it follows that if U is compact, then the map $F(U \vee V) \to F(U) \times_{F(U\wedge V)} F(V)$ is an equivalence for *all* $V \colon 1 \to \mathsf{L}$. By symmetry and the fact that the context of U and V has been arbitrarily chosen, this now implies that the morphism $F(- \vee V) \to F(-) \times_{F(-\wedge V)} F(V)$ is an equivalence when restricted to $\mathsf{L}^{\mathrm{cpt}}$ and must therefore be an equivalence on all of L. Hence the claim follows. □

For later use we also record the following Lemma:

LEMMA 5.3.7.7. *Let* P *be a poset with finite colimits and binary products. Then* $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ *is closed under* Filt*-colimits in* $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$*.*

PROOF. We need to show that for every $A \in \mathcal{B}$ and every diagram $d \colon \mathsf{I} \to \pi_A^* \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ where I is a filtered $\mathcal{B}_{/A}$-category, the colimit of d is contained in $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$. Using Remark 5.3.7.4, we may replace $\mathcal{B}$ with $\mathcal{B}_{/A}$ and can therefore assume that $A \simeq 1$. We may compute the colimit of d as the composition

$$\mathsf{P}^{\mathrm{op}} \xrightarrow{d'} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \xrightarrow{\mathrm{colim}_{\mathsf{I}}} \Omega$$

where d' is the transpose of $d \colon \mathsf{I} \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$. As I is filtered, the functor on the right preserves finite limits. Moreover, the assumption that d takes values in $\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{P})$ and the fact that limits in functor $\mathcal{B}$-categories can be computed objectwise imply that d' is a $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega)$-valued finitary sheaf on P. Hence the claim follows. □

The main result of this section is the following description of sheaves on a locally coherent locale:

PROPOSITION 5.3.7.8. *Let* L *be a locally coherent locale. Then there is a canonical equivalence of* $\mathcal{B}$*-topoi*

$$\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}}).$$

PROOF. By Proposition 5.3.7.5, we have an identification $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}) \cap \underline{\mathsf{Sh}}^{\mathsf{Filt}}_{\mathcal{B}}(\mathsf{L})$ of full subcategories of $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L})$. In particular, we obtain an inclusion $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \hookrightarrow \underline{\mathsf{Sh}}^{\mathsf{Filt}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ (where we use that $\mathsf{L} \simeq \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ and the universal property of $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$). Using Remark 5.3.4.10 together with Lemma 5.3.7.6, we now find that an object $A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ is contained in $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ if and only if it transposes to a finitary sheaf on $\pi_A^* \mathsf{L}^{\mathrm{cpt}}$, proving the claim. □

COROLLARY 5.3.7.9. *Let* L *be a* $\mathcal{B}$*-locale.*

(1) If L *is locally coherent then* $\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L})$ *is compactly generated.*
(2) If L *is locally stably compact then* $\mathrm{Sh}_{\mathcal{B}}(L)$ *is compactly assembled.*

PROOF. Note that (2) is an immediate consequence of the definitions and (1). To see (1) note that we have an inclusion

$$\underline{\mathsf{Sh}}_{\mathcal{B}}(\mathsf{L}) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$$

which by Lemma 5.3.7.7 preserves filtered colimits. Thus the claim follows from Corollary 4.3.3.3. □

5.3.8. The internal locale of a locally proper and separated map. The goal of this section is to show that for a proper and separated map of topological spaces $f\colon Y \to X$, the $\mathrm{Sh}(X)$-locale given by $U \mapsto \mathcal{O}(f^{-1}(U))$ is a stably compact locale. This is a direct consequence of [47] but we decided to also provide a separate proof of Johnstone's result in the language of $\mathrm{Sh}(X)$-locales that we developed above. With future applications in mind, we will prove a slightly more general statement about *locally* proper and separated maps of topological spaces (which Johnstone also mentions in [46, § C4.1] but never explicitly spells out).

We begin by recalling the definition of a locally proper map from [81]:

Definition 5.3.8.1. A continuous map $f\colon Y \to X$ of topological spaces is said to be *locally proper* if for every $y \in Y$ and every open neighbourhood V of y there is a neighbourhood $K \subset V$ of y and an open neighbourhood U of $f(y)$ such that $f(K) \subset U$ and such that the induced map $K \to U$ is proper (i.e. universally closed).

Remark 5.3.8.2. The property of a map $f\colon Y \to X$ to be locally proper and separated is local in the target: if $X = \bigcup_i U_i$ is an open covering, then f is locally proper and separated if and only if each of the restrictions $f^{-1}(U_i) \to U_i$ has that property [81, Lemma 2.7].

Remark 5.3.8.3. Every proper and separated morphism is also locally proper [81, Proposition 2.12]. This is the relative version of the fact that compact Hausdorff spaces are locally compact as well.

Remark 5.3.8.4. In the situation of Definition 5.3.8.1, if f is separated and locally proper, then for every $y \in Y$ and every open neighbourhood V of y there is an open neighbourhood $V' \subset V$ and an open neighbourhood U of $f(y)$ such that $f(V') \subset U$ and such that the closure of V' in $f^{-1}(U)$ is proper over U. In fact, f being separated implies that its restriction $f^{-1}(U) \to U$ is separated as well. Therefore, [81, Lemma 9.12] implies that if $K \subset V$ is as in Definition 5.3.8.1, then K is closed in $f^{-1}(U)$. Hence the closure of the interior of K (again in $f^{-1}(U)$) is a closed subset of K and therefore also proper over U.

To proceed, recall that if $f\colon Y \to X$ is a map of topological spaces, we obtain an algebraic morphism of locales $f^*\colon \mathcal{O}(X) \to \mathcal{O}(Y)$, where $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ denote the locales of open subsets of X and Y, respectively. By Proposition 5.3.6.1, f^* gives rise to a $\mathrm{Sh}(X)$-locale $\mathsf{O}_X(Y)$ that is explicitly given by the sheaf on X that carries an open $U \in \mathcal{O}(X)$ to the locale $\mathcal{O}(f^{-1}(U))$ (see Remark 5.3.6.3). Recall, furthermore, that we refer to a $\mathcal{B}$-locale L as (locally) stably compact if it arises as a retract in $\mathsf{Loc}^{\mathrm{L}}_{\mathcal{B}}$ of a (locally) coherent $\mathcal{B}$-locale (see Definition 5.3.7.1). Now the main goal of this section is to show:

Proposition 5.3.8.5. *If $f\colon Y \to X$ is a locally proper and separated morphism of topological spaces, then $\mathsf{O}_X(Y)$ is a locally stably compact $\mathrm{Sh}(X)$-locale. If f is even proper, then $\mathsf{O}_X(Y)$ is stably compact.*

The proof of Proposition 5.3.8.5 requires a few preparations first. To begin with, we need to construct a candidate for a (locally) coherent $\mathrm{Sh}(X)$-locale of which $\mathsf{O}_X(Y)$ is a retract. We will use the following general observation:

Proposition 5.3.8.6. *Let L be a $\mathcal{B}$-locale and let $j\colon \mathsf{P} \hookrightarrow \mathsf{L}$ be a full subposet that is closed under binary products and finite colimits. Then*

(1) *the left Kan extension $h_!(j)\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \to \mathsf{L}$ is cocontinuous;*
(2) *$\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ is a locally coherent $\mathcal{B}$-locale which is coherent if P contains the final object of L;*
(3) *a right fibration over P (in arbitrary context $A \in \mathcal{B}$) is contained in the essential image of the inclusion $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \hookrightarrow \mathsf{RFib}_{\mathsf{P}}$ if and only if it the inclusion of a sieve in $\pi_A^*\mathsf{P}$ (i.e. a fully faithful right fibration) that is closed under finite colimits.*

The proof of Proposition 5.3.8.6 requires the following two lemmas:

Lemma 5.3.8.7. *The inclusion $\mathsf{Sub}_{\mathcal{B}} \hookrightarrow \Omega$ preserves filtered colimits*

Proof. Using Remark 5.3.2.2 and Example 5.3.2.4, it suffices to show that for every filtered $\mathcal{B}$-category I, the functor $\operatorname{colim}_{\mathsf{I}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega) \to \Omega$ restricts to subterminal objects. By Remark 5.3.2.8, this is an immediate consequence of $\operatorname{colim}_{\mathsf{I}}$ being left exact. $\square$

Lemma 5.3.8.8. *Let C be a $\mathcal{B}$-poset with finite colimits and let $p\colon \mathsf{P} \hookrightarrow \mathsf{C}$ be a sieve (i.e. a fully faithful right fibration). Then P is filtered if and only if it is closed under finite colimits in C.*

Proof. It will be sufficient to show that whenever $d\colon \mathsf{K} \to \mathsf{P}$ is a finite diagram, then $\mathsf{P}_{d/}$ admits an initial object which is carried to the initial object in $\mathsf{C}_{pd/}$ along the induced functor $p_*\colon \mathsf{P}_{d/} \to \mathsf{C}_{pd/}$. Note that $p\colon \mathsf{P} \to \mathsf{C}$ being a sieve implies that p_* is one as well. Now since C has finite colimits, $\mathsf{C}_{pd/}$ admits an initial object $\operatorname{colim}(pd)$. Since p_* is a right fibration, the inclusion $\mathsf{P}_{d/}|_{\operatorname{colim}(pd)} \hookrightarrow \mathsf{P}_{d/}$ of p_* over $\operatorname{colim}(pd)$ is initial [62, Proposition 4.4.7]. Since P is assumed to be filtered, we furthermore have $(\mathsf{P}_{d/})^{\mathrm{gpd}} \simeq 1$. Therefore, we must have $\mathsf{P}_{d/}|_{\operatorname{colim}(pd)} \simeq 1$ as this is already a subterminal $\mathcal{B}$-groupoid (since p is fully faithful, see Example 5.3.2.4). Hence $\mathsf{P}_{d/}$ admits an initial object which is preserved by p_*. $\square$

Proof of Proposition 5.3.8.6. The fact that P has finite colimits implies that the $\mathcal{B}$-category $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ is presentable and that the left Kan extension $h_!(j)\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \to \mathsf{L}$ is cocontinuous by Corollary 4.4.6.6, which shows (1).

To show (2), since $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ is by definition compactly generated and since we may always identify $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})^{\mathrm{cpt}} \simeq \mathsf{P}$ (as P is a $\mathcal{B}$-poset), we only need to verify that $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ is indeed a $\mathcal{B}$-locale. To that end, note that $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ being presentable implies that the inclusion $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ admits a left adjoint $l\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P}) \to \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ by Corollary 3.4.1.14. Moreover, Lemma 5.3.8.7 implies that the inclusion $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{P}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ preserves filtered colimits. Therefore, $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ must be contained in $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{P}^{\mathrm{op}}, \mathsf{Sub}_{\mathcal{B}})$ and is therefore in particular a $\mathcal{B}$-poset. Hence, we only need to check that l preserves binary products (see Lemma 5.3.3.6). This is equivalent to $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ being an exponential ideal in $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$, i.e. that for every object $F\colon A \to \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ and every object $G\colon A \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ (in arbitrary context $A \in \mathcal{B}$), the internal hom $\underline{\operatorname{Hom}}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}(G, F)$ is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$. By using Proposition 3.4.1.11, we can assume that $A \simeq 1$. Upon writing G as a colimit of representables and using that the inclusion $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})$ is continuous, we may assume without loss of generality that G is itself representable by an object $U\colon 1 \to \mathsf{P}$. Thus, Yoneda's lemma and the fact that the Yoneda embedding is continuous imply that $\underline{\operatorname{Hom}}_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{P})}(G, F)$ can be identified with the presheaf $F(U \times -)$. Note that by Proposition 4.4.6.5 a presheaf is contained in $\underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{P})$ if and only if it carries finite colimits in P to limits. Thus, as F by assumption has this property and since colimits are universal in L, the claim follows.

Lastly, in light of Example 5.3.2.4, statement (3) is an immediate consequence of Lemma 5.3.8.8. $\square$

In light of Proposition 5.3.8.6, our task is now to find a full subposet of $\mathsf{O}_X(Y)$ that is closed under binary products and finite colimits. To that end, note that the datum of an object in $\mathsf{O}_X(Y)$ in context $U \subset X$ is precisely given by an open subset $V \subset f^{-1}(U)$. With that in mind, we may now define:

Definition 5.3.8.9. Let $f\colon Y \to X$ be a locally proper and separated map of topological spaces. We say that an object $V \subset f^{-1}(U)$ has *proper closure* if its closure $\overline{V}$ in $f^{-1}(U)$ is proper over U. We define the subposet $\mathsf{O}_X^{\mathrm{pc}}(Y) \hookrightarrow \mathsf{O}_X(Y)$ as the full subposet of $\mathsf{O}_X(Y)$ that is spanned by these objects.

Remark 5.3.8.10. In the situation of Definition 5.3.8.9, note that f being separated implies that if $\overline{V}$ is proper over U, then $\overline{V}$ is also closed in Y (see [81, Lemma 9.12]). Therefore, $\overline{V}$ is also the closure of V in Y in this case.

A priori, the subposet $\mathsf{O}_X^{\mathrm{pc}}(Y)$ is only *spanned* by the objects with proper closure, so there could potentially be more objects. Our next result shows that this cannot happen:

Lemma 5.3.8.11. *An object $V \subset f^{-1}(U)$ in $\mathsf{O}_X(Y)$ is contained in $\mathsf{O}_X^{\mathrm{pc}}(Y)$ if and only if it has proper closure.*

Proof. By definition, the condition is sufficient, so it suffices to prove that it is also necessary. This amounts to showing that the property of having proper closure is *local* on the target: if $U = \bigcup_i U_i$ is a covering and if $\overline{V \cap f^{-1}(U_i)} \subset f^{-1}(U_i)$ is proper over U_i, then $\overline{V} \subset f^{-1}(U)$ is proper over U. Since properness is local on the target [81, § 9.5], this follows from the identity $\overline{V \cap f^{-1}(U_i)} = \overline{V} \cap f^{-1}(U_i)$. $\square$

Remark 5.3.8.12. Note that if $U \subset X$ is an arbitrary open subset, we may identify $\mathrm{Sh}(X)_{/U} \simeq \mathrm{Sh}(U)$. In light of this identification, the $\mathrm{Sh}(U)$-locale $\pi_U^* \mathsf{O}_X(Y)$ can be identified with $\mathsf{O}_U(f^{-1}(U))$. Moreover, Lemma 5.3.8.11 implies that we obtain a canonical equivalence $\pi_U^* \mathsf{O}_X^{\mathrm{pc}}(Y) \simeq \mathsf{O}_U^{\mathrm{pc}}(f^{-1}(U))$ of full subposets in $\mathsf{O}(f^{-1}(U)/U)$ (see also Remark 5.3.8.2).

Having an explicit description of the full subposet $\mathsf{O}_X^{\mathrm{pc}}(Y) \hookrightarrow \mathsf{O}_X(Y)$, we now proceed by showing that it satisfies the conditions of Proposition 5.3.8.6:

Lemma 5.3.8.13. *$\mathsf{O}_X^{\mathrm{pc}}(Y)$ is closed under binary products and finite colimits in $\mathsf{O}_X(Y)$.*

Proof. Since the map $\varnothing \to X$ is always proper, Lemma 5.3.8.11 implies that it is enough to show that for every two objects $V \subset f^{-1}(U)$ and $V' \subset f^{-1}(U)$ whose closure (in $f^{-1}(U)$) is proper over U, both $\overline{V \cup V'} \to U$ and $\overline{V \cap V'} \to U$ are proper. The first map is proper by [81, § 9.7] and the fact that union and closure commute. The second map is proper as it can be decomposed into the composition $\overline{V \cap V'} \hookrightarrow \overline{V} \to U$ where the first map is a closed embedding (hence proper) and the second map is proper by assumption. $\square$

Proposition 5.3.8.14. *The $\mathrm{Sh}(X)$-category $\underline{\mathsf{Ind}}_{\mathrm{Sh}(X)}(\mathsf{O}_X^{\mathrm{pc}}(Y))$ is a locally coherent $\mathrm{Sh}(X)$-locale, and the left Kan extension $\underline{\mathsf{Ind}}_{\mathrm{Sh}(X)}(\mathsf{O}_X^{\mathrm{pc}}(Y)) \to \mathsf{O}_X(Y)$ of the inclusion is a Bousfield localisation.*

Proof. In light of Lemma 5.3.8.13, the first claim follows from Proposition 5.3.8.6, so that it suffices to show the second one. We need to prove that the counit of the adjunction $\mathsf{O}_X(Y) \leftrightarrows \underline{\mathsf{Ind}}_{\mathrm{Sh}(X)}(\mathsf{O}_X^{\mathrm{pc}}(Y))$ is an equivalence. Using Remark 5.3.8.12, it will be enough to check this on a global object $V \subset Y$. By Remark 3.3.3.6, this amounts to showing that V is the colimit of the diagram $\mathsf{O}_X^{\mathrm{pc}}(Y)_{/V} \to \mathsf{O}_X(Y)$. Using Proposition 5.3.2.10, we only need to verify that $V \simeq \bigcup_{V' \subset f^{-1}(U) \cap V} V'$, where U runs though all open subsets of X and V' runs through all objects in $\mathsf{O}_X^{\mathrm{pc}}(Y)(U)$ which are contained in V. This is an immediate consequence of the fact that Y is locally proper and separated over X (see Remark 5.3.8.4). $\square$

The following Lemma is a suitable relative analogue of the fact that in a locally compact Hausdorff space, every open covering of a compact subset has a finite refinement:

Lemma 5.3.8.15. *Let $V \subset f^{-1}(U)$ be an object in $\mathsf{O}_X^{\mathrm{pc}}(Y)(U)$, let $(V'_j \subset f^{-1}(U_j))_{j \in J}$ be a family of objects in $\mathsf{O}_X(Y)$ and suppose that $\overline{V} \subset \bigcup_{j \in J} V'_j$. Then there is a covering $U = \bigcup_i U_i$ in X such that for each i there is a finite subset $J_i \subset J$ such that $U_i \subset U_j$ for all $j \in J_i$ and such that $\overline{V} \cap f^{-1}(U_i) \subset \bigcup_{j \in J_i} V'_j$.*

Proof. In light of Remark 5.3.8.12, we may replace Y/X by $f^{-1}(U)/U$ and each object $V'_j \subset f^{-1}(U_j)$ by its intersection $V'_j \cap f^{-1}(U) \subset f^{-1}(U_j \cap U)$ and can thus assume without loss of generality that $U = X$. Now since $\overline{V}$ is proper over X, its fibre $\overline{V}|_x$ over every $x \in X$ is compact (as being proper is stable under base change). Therefore, for each $x \in X$ we have a finite subset $J_x \subset J$ such that $\overline{V}|_x \subset \bigcup_{j \in J_x} V'_j$. We can assume that $x \in U_j$ for all $j \in J_x$, since otherwise $V'_j|_x$ would be empty. Now let Z be the complement of $\bigcup_{j \in J_x} V'_j$ in Y. Then $\overline{V} \cap Z$ is closed in $\overline{Y}$, hence $f(\overline{V} \cap Z)$ is closed in X (as proper maps are always closed). By construction, a point $x' \in X$ is contained in $f(\overline{V} \cap Z)$ precisely if $\overline{V}|_{x'}$ is not contained in $\bigcup_{j \in J_x} V'_j$. Therefore, if U is the complement of $f(\overline{V} \cap Z)$ in X, then U contains precisely those points $x' \in X$ for which $\overline{V}|_{x'} \subset \bigcup_{j \in J_x} V'_j$. In other words, we have $\overline{V} \cap f^{-1}(U) \subset \bigcup_{j \in J_x} V'_j$. Since $x \in U$, we may shrink U if necessary so that it is contained in $\bigcap_{j \in J_x} U_x$. Now the claim follows. $\square$

Proof of Proposition 5.3.8.5. By Proposition 5.3.8.14, the left Kan extension $l\colon \underline{\mathsf{Ind}}_{\mathcal{B}}(\mathsf{O}_X^{\mathrm{pc}}(Y)) \to \mathsf{O}_X(Y)$ is a Bousfield localisation. Therefore, we only need to show that l admits a left adjoint λ which preserves finite limits.

We begin by showing that $l(X)$ has a left adjoint λ_X. On account of Proposition 5.3.8.6, this amounts to showing that for every $V \subset Y$, there is sieve $\lambda_X(V)\colon \mathsf{P} \hookrightarrow \mathsf{O}_X^{\mathrm{pc}}(Y)$ which is closed under finite colimits such that for every other sieve $q\colon \mathsf{Q} \hookrightarrow \mathsf{O}_X^{\mathrm{pc}}(Y)$ with the same property and for which $V \subset \operatorname{colim} q$, we have $\mathsf{P} \hookrightarrow \mathsf{Q}$. We define P to be the full subposet of $\mathsf{O}_X^{\mathrm{pc}}(Y)$ which is spanned by those $V' \subset f^{-1}(U)$ whose closure is contained in V. This property is clearly local in X, so that every object of P in context $U \subset X$ will be of this form. Moreover, if $V'' \subset V'$ and V' is in $\mathsf{P}(U)$, so is V''. Therefore, $\mathsf{P} \hookrightarrow \mathsf{O}_X^{\mathrm{pc}}(Y)$ is a sieve. Furthermore, P is closed under finite colimits. Now let $V' \subset f^{-1}(U)$ be an arbitrary object in P in context $U \subset X$ and let $q\colon \mathsf{Q} \hookrightarrow \mathsf{O}_X^{\mathrm{pc}}(Y)$ be a sieve which is closed under finite colimits such that $V \subset \operatorname{colim} q$. We need to show that V' is contained in Q. By assumption, the closure $\overline{V'}$ is contained in $\operatorname{colim} q$. Using Proposition 5.3.2.10, we may identify

$$\operatorname{colim} q \simeq \bigcup_{\substack{V'' \in \mathsf{Q}(U)\\ U \subset X}} V''.$$

Therefore, Lemma 5.3.8.15 implies that there is a covering $U = \bigcup_{i \in I} U_i$ such that for each i there are finitely many $V''_{i_1}, \dots, V''_{i_n} \in \mathsf{Q}(U_i)$ with the property that $V' \cap f^{-1}(U_i) \subset \bigcup_{j=1}^n V''_{i_j}$. As Q is closed under finite colimits, the right-hand side is contained in $\mathsf{Q}(U_i)$. Consequently, V' is locally contained in Q and must therefore also be globally contained in Q.

Now by carrying out the above argument with $f|_{f^{-1}(U)}$ in place of f, Remark 5.3.8.12 implies that $l(U)$ admits a left adjoint λ_U for every $U \subset X$. Furthermore, for every pair of opens $U \subset U' \subset X$ and every $V' \subset f^{-1}(U')$, it follows readily from the constructions that the restriction of $\lambda_{U'}(V')$ to U can be identified with $\lambda_U(V' \cap f^{-1}(U))$. Therefore, we deduce from Corollary 2.4.2.11 that l admits a left adjoint, as desired. It is then clear from its explicit construction that this left adjoint preserves finite limits.

Lastly, if f is proper, then $\mathsf{O}_X^{\mathrm{pc}}(Y)$ contains the final object of $\mathsf{O}_X(Y)$, which immediately implies that $\mathsf{O}_X(Y)$ is stably compact. $\square$

CHAPTER 6

Application: Smooth and proper morphisms of ∞-topoi

In this chapter we will use the results of the previous chapters to study geometric properties of morphisms of ∞-topoi. We begin by studying *smooth* geometric morphism in § 6.1. Recall that a geometric morphism is called smooth if it satisfies smooth base change (see Definition 6.1.3.1 for a precise statement). The main goal of § 6.1.1 will be to show that a geometric morphism is smooth if and only if it is *locally contractible*. Here a geometric morphism $p_*\colon \mathcal{X} \to \mathcal{B}$ is called locally contractible if the associated constant sheaf functor $\mathrm{const}\colon \Omega_{\mathcal{B}} \to p_*\Omega_{\mathcal{X}}$ of $\mathcal{B}$-topoi admits a further left adjoint.

We continue by studying the notion of a *proper* geometric morphism, which is dual to the notion of smoothness, in § 6.2. Our main result will then be that a geometric morphism is proper if and only if it is *compact*. Here we call a geometric morphism $p_*\colon \mathcal{X} \to \mathcal{B}$ compact if the associated global section functor $p_*\Omega_{\mathcal{X}} \to \Omega_{\mathcal{B}}$ commutes with filtered colimits. Combining this criterion with the results of § 5.3 it will follow that a proper and separated morphism of arbitrary topological spaces induces a proper geometric morphism of ∞-topoi, generalizing [57, p. 7.3.1.16].

6.1. Smooth and locally contractible geometric morphisms

An ∞-topos $\mathcal{X}$ is said to be *locally contractible* if the constant sheaf functor $\mathrm{const}_{\mathcal{X}}\colon \mathcal{S} \to \mathcal{X}$ admits a left adjoint $\pi_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{S}$ which is to be thought of as the functor that carries an object $U \in \mathcal{X}$ to its *homotopy type* (or *shape*) $\pi_{\mathcal{X}}(U)$. In 1-topos theory, the corresponding notion is that of a *locally connected* 1-topos $\mathcal{E}$, in which the additional left adjoint carries an object $U \in \mathcal{E}$ to its set of connected components $\pi_0(E)$. In this section, we study the analogous concept for $\mathcal{B}$-topoi.

We begin in § 6.1.1 by recalling the notion of a locally contractible $\mathcal{B}$-topos and providing a few characterisations of this concept. In § 6.1.2, we show that every locally contractible $\mathcal{B}$-topos is generated by its *contractible objects* in a quite strong sense. Finally, in § 6.1.3 we provide a characterisation of locally contractible $\mathcal{B}$-topoi in terms of *smoothness* of the associated geometric morphisms.

6.1.1. Local contractibility. The goal of this section is to define the condition of a $\mathcal{B}$-topos to be locally contractible and to derive a few explicit characterisations of this concept. Locally contractible geometric morphisms have been introduced and studied, mostly in the context of sheaves of topological spaces, in [2, §3.2] and [86, §3.2]. We give the following equivalent definition, which is a straightforward generalisation of the notion of a locally contractible ∞-topos to the world of $\mathcal{B}$-topoi:

DEFINITION 6.1.1.1. A $\mathcal{B}$-topos X is *locally contractible* if the unique algebraic morphism $\mathrm{const}_{\mathsf{X}}\colon \Omega \to \mathsf{X}$ admits a left adjoint $\pi_{\mathsf{X}}\colon \mathsf{X} \to \Omega$. We call a geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ *locally contractible* if $f_*(\Omega_{\mathcal{X}})$ is a locally contractible $\mathcal{B}$-topos, in which case we denote by $f_!$ the additional left adjoint of f^* (i.e. the functor $\Gamma(\pi_{f_*(\Omega_{\mathcal{X}})})$).

REMARK 6.1.1.2. As the property of a functor being a right adjoint is local in $\mathcal{B}$ (see Remark 2.4.3.6) and by making use of Remark 5.2.1.3, we find that for any cover $\bigsqcup_i A_i \to 1$ in $\mathcal{B}$, the $\mathcal{B}$-topos X is locally contractible if and only if for every i the $\mathcal{B}_{/A_i}$-topos $\pi_{A_i}^*\mathsf{X}$ is locally contractible.

REMARK 6.1.1.3. Explicitly, a geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ is locally contractible precisely if $f^*\colon \mathcal{B} \to \mathcal{X}$ admits a left adjoint $f_!\colon \mathcal{X} \to \mathcal{B}$ such that for every map $s\colon B \to A$ in $\mathcal{B}$ the induced map

$$f_!(f^*B \times_{f^*A} -) \to B \times_A f_!(-)$$

is an equivalence. In fact, this follows from the section-wise characterisation of internal adjunctions (Proposition 2.4.2.9) together with the observation that if f^* admits a left adjoint $f_!$, one obtains an induced left adjoint $(f_!)_A$ of $f^*_{/A}\colon \mathcal{B}_{/A} \to \mathcal{X}_{/f^*A}$ for every $A \in \mathcal{B}$ which is simply given by the composition

$$\mathcal{X}_{/f^*A} \xrightarrow{(f_!)_{/A}} \mathcal{B}_{/f_!f^*A} \xrightarrow{\epsilon_!} \mathcal{B}_{/A}$$

(in which $\epsilon\colon f_!f^* \to \mathrm{id}_{\mathcal{B}}$ is the adjunction counit).

Example 6.1.1.4. Every étale $\mathcal{B}$-topos is locally contractible. More precisely, one can characterise the class of étale $\mathcal{B}$-topoi as those locally contractible $\mathcal{B}$-topoi X for which the additional left adjoint π_{X} is a conservative functor. This is an immediate consequence of [55, Proposition 6.3.5.11].

Recall from Theorem 5.2.5.1 and Remark 5.2.5.3 that every $\mathcal{B}$-topos X corresponds uniquely to a geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$ such that X can be recovered by $f_*\Omega_{\mathcal{X}}$. The goal of this section is to characterise the property that X is locally contractible in terms of the geometric morphism f_*. To that end, recall that a product-preserving functor $g\colon \mathcal{C} \to \mathcal{D}$ between cartesian closed ∞-categories is said to be cartesian closed if the natural map $g(\underline{\mathrm{Hom}}(-,-)) \to \underline{\mathrm{Hom}}(g(-), g(-))$ (in which $\underline{\mathrm{Hom}}(-,-)$ denotes the internal hom in $\mathcal{C}$ and $\mathcal{D}$, respectively) is an equivalence. If $\mathcal{C}$ and $\mathcal{D}$ are even *locally* cartesian closed and g preserves finite limits, one says that g is locally cartesian closed if the induced functor $g_{/c}\colon \mathcal{C}_{/c} \to \mathcal{D}_{/g(d)}$ is cartesian closed for every $c \in \mathcal{C}$. We now obtain:

Proposition 6.1.1.5. *Let X be a $\mathcal{B}$-topos and let $f_*\colon \mathcal{X} \to \mathcal{B}$ be the associated ∞-topos over $\mathcal{B}$. Then the following are equivalent:*

1. *X is locally contractible;*
2. *the unique algebraic morphism $\mathrm{const}_{\mathsf{X}}\colon \Omega_{\mathcal{B}} \to \mathsf{X}$ is Ω-continuous;*
3. *the functor $\mathrm{const}_{\mathsf{X}}\colon f^*(\Omega_{\mathcal{B}}) \to \Omega_{\mathcal{X}}$ (which is obtained by transposing the algebraic morphism $\mathrm{const}_{\mathsf{X}}\colon \Omega_{\mathcal{B}} \to \mathsf{X}$ across the adjunction $f^* \dashv f_*$) is fully faithful.*
4. *the functor $f^*\colon \mathcal{B} \to \mathcal{X}$ is locally cartesian closed.*

Before we can prove Proposition 6.1.1.5, we need the following lemma:

Lemma 6.1.1.6. *Let X be a $\mathcal{B}$-topos and let $f_*\colon \mathcal{X} \to \mathcal{B}$ be the associated ∞-topos over $\mathcal{B}$. Let $A \in \mathcal{B}$ be an arbitrary object and let $P \to A$ and $Q \to A$ be two $\mathcal{B}_{/A}$-groupoids. Then the morphism of $\mathcal{X}_{/f^*A}$-groupoids $\mathrm{map}_{f^*(\Omega_{\mathcal{B}})}(P, Q) \to \mathrm{map}_{\Omega_{\mathcal{X}}}(\mathrm{const}_{\mathsf{X}}(P), \mathrm{const}_{\mathsf{X}}(Q))$ that is induced by the action of $\mathrm{const}_{\mathsf{X}}$ recovers the canonical map $f^*(\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P, Q)) \to \underline{\mathrm{Hom}}_{\mathcal{X}_{/f^*A}}(f^*P, f^*Q)$.*

Proof. Using Remark 5.2.1.3, we may assume without loss of generality that $A \simeq 1$. Furthermore, by transposing across the adjunction $f^* \dashv f_*$, it suffices to show that the map $\underline{\mathrm{Hom}}_{\mathcal{B}}(P, Q) \to f_*\underline{\mathrm{Hom}}_{\mathcal{X}}(f^*P, f^*Q)$ can be identified with the morphism of $\mathcal{B}$-groupoids

$$\mathrm{map}_{\Omega_{\mathcal{B}}}(\mathsf{G}, \mathsf{H}) \to \mathrm{map}_{\mathsf{X}}(\mathrm{const}_{\mathsf{X}}(\mathsf{G}), \mathrm{const}_{\mathsf{X}}(\mathsf{H})).$$

Now the latter can be identified with $\eta_*\colon \mathrm{map}_{\Omega_{\mathcal{B}}}(\mathsf{G}, \mathsf{H}) \to \mathrm{map}_{\Omega_{\mathcal{B}}}(\mathsf{G}, \Gamma_{\mathsf{X}} \mathrm{const}_{\mathsf{X}} \mathsf{H})$ (where $\Gamma_{\mathsf{X}}\colon \mathsf{X} \to \Omega$ denotes the unique geometric morphism and where η is the adjunction unit). In light of the equivalence $f_*\underline{\mathrm{Hom}}(f^*P, f^*Q) \simeq \underline{\mathrm{Hom}}(P, f_*f^*P)$, we can also identify the former map with $\eta_*\colon \underline{\mathrm{Hom}}_{\mathcal{B}}(P, Q) \to \underline{\mathrm{Hom}}_{\mathcal{X}}(P, f_*f^*Q)$. Therefore, the claim follows from Proposition 2.1.10.3. $\square$

Proof of Proposition 6.1.1.5. Since $\mathrm{const}_{\mathsf{X}}$ is cocontinuous and preserves finite limits, one deduces from Proposition 3.4.4.1 and the adjoint functor theorem (Proposition 4.4.3.1) that (1) and (2) are equivalent. In light of Lemma 6.1.1.6, it is moreover clear that (3) and (4) are equivalent. To complete the proof, we will show that (2) and (4) are equivalent. To that end, given any map $p\colon P \to A$ in $\mathcal{B}$, consider the commutative diagram

$$\begin{tikzcd} \mathcal{X}_{/f^*A} \arrow[r, "p^*"] & \mathcal{X}_{/f^*P} \arrow[r, "p_!"] & \mathcal{X}_{/f^*A} \\ \mathcal{B}_{/A} \arrow[u, "f^*_{/A}"] \arrow[r, "p^*"] & \mathcal{B}_{/P} \arrow[u, "f^*_{/P}"] \arrow[r, "p_!"] & \mathcal{B}_{/A}. \arrow[u, "f^*_{/A}"] \end{tikzcd}$$

Given $q\colon Q \to A$, the natural map $f^*\underline{\mathrm{Hom}}_{\mathcal{B}_{/A}}(P,Q) \to \underline{\mathrm{Hom}}_{\mathcal{X}_{/f^*A}}(f^*P, f^*Q)$ is precisely obtained by evaluating the (horizontal) mate of the composite square at q. Since the horizontal mate of the left square being an equivalence (for every such map p) precisely means that $\mathrm{const}_{\mathsf{X}}$ is Ω-continuous, we only need to show that the mate of the left square is an equivalence if and only if the mate of the composite square is one. One direction is trivial. As for the other direction, if we know that the map $\varphi\colon f^*_{/A}p_* \to p_*f^*_{/P}$ is an equivalence for every object in the image of $p^*\colon \mathcal{B}_{/A} \to \mathcal{B}_{/P}$, then the entire map has to be an equivalence since every object in $\mathcal{B}_{/P}$ can be written as a pullback of such objects and since both domain and codomain of φ preserves finite limits. $\square$

6.1.2. Contractible objects. A topological space X is by definition locally contractible if it admits a basis of contractible open subsets. A priori, the definition of a locally contractible $\mathcal{B}$-topos does not appear to be related to this condition at all. In this section, we reconcile the two notions by showing that local contractibility of a $\mathcal{B}$-topos can be characterised by the property of it being generated under colimits by its *contractible objects*. We begin with the following definition:

DEFINITION 6.1.2.1. Let X be a $\mathcal{B}$-topos. An object $U\colon A \to \mathsf{X}$ is said to be *contractible* if the functor $\mathrm{map}_{\mathsf{X}}(U, \mathrm{const}_{\mathsf{X}}(-))\colon \Omega_{\mathcal{B}_{/A}} \to \Omega_{\mathcal{B}_{/A}}$ is an equivalence. We define the full subcategory $\mathsf{Contr}(\mathsf{X}) \hookrightarrow \mathsf{X}$ as the fibre of the functor

$$\mathrm{const}_{\mathsf{X}}^* \, h^{\mathrm{op}}_{\mathsf{X}^{\mathrm{op}}}\colon \mathsf{X} \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{X}, \Omega)^{\mathrm{op}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)^{\mathrm{op}}$$

over the identity $\mathrm{id}\colon \Omega \to \Omega$.

REMARK 6.1.2.2. Note that as Ω is the initial $\mathcal{B}$-topos, we find that the inclusion of the identity $\mathrm{id}_\Omega\colon 1 \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)$ determines a fully faithful functor that identifies the domain with the full subcategory $\underline{\mathsf{Fun}}^{\mathrm{alg}}_{\mathcal{B}}(\Omega, \Omega)$. Therefore, the functor $\mathsf{Contr}(\mathsf{X}) \hookrightarrow \mathsf{X}$ is indeed fully faithful. Moreover, as the universal property of $\Omega_{\mathcal{B}_{/A}}$ implies that a functor $\Omega_{\mathcal{B}_{/A}} \to \Omega_{\mathcal{B}_{/A}}$ is an equivalence if and only if it is equivalent to the identity, we find that an object $U\colon A \to \mathsf{X}$ is contained in $\mathsf{Contr}(\mathsf{X})$ if and only if it is contractible.

REMARK 6.1.2.3. If $A \in \mathcal{B}$ is an arbitrary object, we may combine Remark 5.2.1.3 with [62, Lemma 4.7.14] and [62, Lemma 4.2.3] to deduce that there is a canonical equivalence $\pi_A^* \mathsf{Contr}(\mathsf{X}) \simeq \mathsf{Contr}(\pi_A^*\mathsf{X})$ of full subcategories in $\pi_A^*\mathsf{X}$.

REMARK 6.1.2.4. In the situation of Definition 6.1.2.1, suppose that X is locally contractible. Then we obtain an equivalence $\mathrm{const}_{\mathsf{X}}^* \, h^{\mathrm{op}}_{\mathsf{X}^{\mathrm{op}}} \simeq h^{\mathrm{op}}_{\Omega^{\mathrm{op}}}\pi_{\mathsf{X}}$. Since $h^{\mathrm{op}}_{\Omega^{\mathrm{op}}}$ is fully faithful and since the identity on Ω is corepresented by 1_Ω (see [62, Proposition 4.6.3]), we thus find that $\mathsf{Contr}(\mathsf{X})$ arises as the fibre of $\pi_{\mathsf{X}}\colon \mathsf{X} \to \Omega$ over $1_\Omega\colon 1 \hookrightarrow \Omega$. In particular, this means that an object $U\colon A \to \mathsf{X}$ is contractible precisely if $\pi_{\mathsf{X}}(U)\colon A \to \Omega$ transposes to the final object in $\Omega_{\mathcal{B}_{/A}}$.

For the remainder of this section, let us fix a $\mathcal{B}$-topos X. Recall from Lemma 5.2.3.3 that we may always find a sound doctrine U such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. We will denote by $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ the intersection of $\mathsf{Contr}(\mathsf{X})$ with $\mathsf{X}^{\mathsf{U}\text{-cpt}}$. The main goal of this section is to prove the following proposition:

PROPOSITION 6.1.2.5. *Let X be a $\mathcal{B}$-topos and let U be a sound doctrine such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. Then the following are equivalent:*

(1) X is locally contractible;

(2) the left Kan extension $h_!(j)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})) \to \mathsf{X}$ of $j\colon \mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ along the Yoneda embedding defines a left exact and accessible Bousfield localisation;

(3) X *is generated by* $\mathsf{Contr}(\mathsf{X})$ *under colimits.*

The proof of Proposition 6.1.2.5 is based on the following two lemmas:

LEMMA 6.1.2.6. *Let* $j\colon \mathsf{C} \hookrightarrow \mathsf{X}$ *be a (small) full subcategory such that the identity on* X *is the left Kan extension of* j *along itself. Then* j *is flat.*

PROOF. We need to show that $h_!(j)\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \to \mathsf{X}$ preserves finite limits. By virtue of Proposition 3.3.1.1, the final object $1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})}\colon 1 \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$ is given by the colimit of the Yoneda embedding $h\colon \mathsf{C} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})$, hence $h_!(j)(1_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})})$ is the colimit of j. But as the left Kan extension of j along itself is by assumption the identity on X, the formula from Remark 3.3.3.6 implies that every object $U\colon 1 \to \mathsf{X}$ is the colimit of the composition $\mathsf{C}_{/U} \hookrightarrow \mathsf{X}_{/U} \to \mathsf{X}$. In particular, the final object in X must be the colimit of j itself. Hence $h_!(j)$ preserves the final object. To complete the proof, it therefore suffices to show that $h_!(j)$ also preserves pullbacks. By Lemma 5.2.2.8 and in light of Remark 3.3.3.2 and [62, Lemma 4.7.14], it will be enough to show that if σ is an arbitrary cospan in C in context $1 \in \mathcal{B}$, the functor $h_!(j)$ preserves the pullback P of $h(\sigma)$. In other words, we need to prove that the induced functor $h_!(j)_*\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(\sigma)} \to \mathsf{X}_{/j(\sigma)}$ preserves the final object. Let $Q\colon 1 \to \mathsf{X}$ be the pullback of $j(\sigma)$. We then have a commutative diagram

$$\begin{tikzcd}
\mathsf{C}_{/P} \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/P} \arrow[r] \arrow[d, "\simeq"] & \mathsf{X}_{/Q} \arrow[d, "\simeq"] \\
\mathsf{C}_{/\sigma} \arrow[r, hook] & \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/h(\sigma)} \arrow[r, "h_!(j)_*"] & \mathsf{X}_{/j(\sigma)}
\end{tikzcd}$$

in which the upper right horizontal functor can be identified with the composition of $h_!(j)_{/P}\colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/P} \to \mathsf{X}_{/h_!(j)(P)}$ with the forgetful functor $\mathsf{X}_{/h_!(j)(P)} \to \mathsf{X}_{/Q}$ along the natural map $h_!(j)(P) \to Q$. As both of these functors are cocontinuous (see Corollary 3.1.7.6 and Proposition 3.1.7.3), the upper right horizontal functor must be cocontinuous as well. By combining this observation with the identification $\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C})_{/P} \simeq \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}_{/P})$ from Lemma 3.3.1.5 and the universal property of presheaf $\mathcal{B}$-categories, we conclude that this functor is equivalent to the Yoneda extension of the inclusion $\mathsf{C}_{/P} \hookrightarrow \mathsf{X}_{/Q}$. Therefore, by using the first part of the proof, it will be enough to show that the identity on $\mathsf{X}_{/j(\sigma)}$ is the left Kan extension of the inclusion $\mathsf{C}_{/\sigma} \hookrightarrow \mathsf{X}_{/j(\sigma)}$ along itself. By using the criterion from Remark 3.3.3.6 together with the fact that any slice $\mathcal{B}$-category over $\mathsf{X}_{/j(\sigma)}$ can be identified with a slice $\mathcal{B}$-category over X, this in turn follows from the assumption that the identity on X is the left Kan extension of j along itself. □

LEMMA 6.1.2.7. *If* X *is locally contractible, the identity on* X *is the left Kan extension of the inclusion* $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ *along itself.*

PROOF. Note that we have inclusions $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{X}$ in which the left Kan extension of the second inclusion along itself is the identity on X. Consequently, it will be enough to show that the left Kan extension of the inclusion $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}$ along the inclusion $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}) \hookrightarrow \mathsf{X}^{\mathsf{U}\text{-cpt}}$ recovers the inclusion $\mathsf{X}^{\mathsf{U}\text{-cpt}} \hookrightarrow \mathsf{X}$. By using Remark 3.3.3.6 together with Remark 6.1.2.3, this follows once we verify that for any U-compact object $U\colon 1 \to \mathsf{X}$, the colimit of the induced inclusion $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U} \hookrightarrow \mathsf{X}_{/U}$ is the final object. Now observe that the functor $(\pi_{\mathsf{X}})_{/U}\colon \mathsf{X}_{/U} \to \Omega_{/\pi_{\mathsf{X}}(U)}$ restricts to a functor $(\pi_{\mathsf{X}})_{/U}\colon \mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U} \to \pi_{\mathsf{X}}(U)$. We claim that the right adjoint $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}\colon \Omega_{/\pi_{\mathsf{X}}(U)} \to \mathsf{X}_{/U}$ (which is constructed by composing the functor $(\mathrm{const}_{\mathsf{X}})_{/\pi_{\mathsf{X}}(U)}\colon \Omega_{/\pi_{\mathsf{X}}(U)} \to \mathsf{X}_{/\mathrm{const}_{\mathsf{X}}\pi_{\mathsf{X}}(U)}$ with the pullback morphism $\eta^*\colon \mathsf{X}_{/\mathrm{const}_{\mathsf{X}}\pi_{\mathsf{X}}(U)} \to \mathsf{X}_{/U}$ along the adjunction unit $\eta\colon U \to \mathrm{const}_{\mathsf{X}}\pi_{\mathsf{X}}(U)$, see § 3.1.7) restricts to a map $\pi_{\mathsf{X}}(U) \to \mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U}$. In fact, by making use of Remark 6.1.2.3, it will be enough to verify that if $x\colon 1 \to \pi_{\mathsf{X}}(U)$ is an arbitrary object in context $1 \in \mathcal{B}$, its image along $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}$ is U-compact and contractible. By construction, this object fits into a pullback square

$$\begin{tikzcd}
(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x) \arrow[r] \arrow[d] & 1_{\mathsf{X}} \arrow[d, "\mathrm{const}_{\mathsf{X}}(x)"] \\
U \arrow[r, "\eta"] & \mathrm{const}_{\mathsf{X}}\pi_{\mathsf{X}}(U).
\end{tikzcd}$$

Note that both π_{X} and $\mathrm{const}_{\mathsf{X}}$ are left adjoint to $\mathsf{Filt}_{\mathsf{U}}$-cocontinuous functors and therefore preserve U-compact objects. In combination with our assumption that the full subcategory of U-compact objects in X is closed under finite limits, we thus find that $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x)$ is U-compact too. Furthermore, note that we may regard η as an object in $\mathcal{X}_{/f^*(\pi_{\mathsf{X}}(U))} = \mathsf{X}(\pi_{\mathsf{X}}(U))$, i.e. as an *object* in X in context $\pi_{\mathsf{X}}(U)$. As such, η is contractible: in fact, by Remark 6.1.1.3 the object $\pi_{\mathsf{X}}(\eta) \in \Omega_{\mathcal{B}}(\pi_{\mathsf{X}}(U))$ is explicitly computed as the composition

$$\pi_{\mathsf{X}}(U) \xrightarrow{\pi_{\mathsf{X}}(\eta)} \pi_{\mathsf{X}} \, \mathrm{const}_{\mathsf{X}} \, \pi_{\mathsf{X}}(U) \xrightarrow{\epsilon} \pi_{\mathsf{X}}(U)$$

(where in the first map η is regarded as a *morphism* in X in context $1 \in \mathcal{B}$ and where ϵ is the counit of the adjunction $\pi_{\mathsf{X}} \dashv \mathrm{const}_{\mathsf{X}}$), hence the claim follows from the triangle identities. Now viewing η as an object in X in context $\pi_{\mathsf{X}}(U)$, the above pullback square exhibits the global object $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x) \in \mathsf{X}(1) = \mathcal{X}$ as the image of $\eta \in \mathsf{X}(\pi_{\mathsf{X}}(U))$ along the transition map $x^* \colon \mathsf{X}(\pi_{\mathsf{X}}(U)) \to \mathsf{X}(1)$. Therefore, η being a contractible object implies that $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x)$ must be contractible as well. Thus we conclude that $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}(x)$ is contained in $\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U}$, as claimed.

So far, our arguments have shown that we have a commutative square

$$\begin{tikzcd}
\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U} \arrow[r, hook] & \mathsf{X}_{/U} \\
\pi_{\mathsf{X}}(U) \arrow[r, hook] \arrow[u] & \Omega_{/\pi_{\mathsf{X}}(U)}. \arrow[u, "(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}"']
\end{tikzcd}$$

Since the vertical maps in this diagram are right adjoints, they are in particular final. Since furthermore $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}$ is cocontinuous, the colimit of the upper horizontal map is the image of the colimit of the lower horizontal map along $(\mathrm{const}_{\mathsf{X}})_{\pi_{\mathsf{X}}(U)}$. To complete the proof, it is therefore enough to prove that the colimit of the lower horizontal map is the final object in $\Omega_{/\pi_{\mathsf{X}}(U)}$. But this is simply the statement that $\pi_{\mathsf{X}}(U)$ is the colimit of the constant diagram $\pi_{\mathsf{X}}(U) \to 1 \hookrightarrow \Omega$ with value 1_{Ω}, which is clear. ☐

Proof of Proposition 6.1.2.5. Suppose first that X is locally contractible. By combining Lemmas 6.1.2.7 and 6.1.2.6, the map $h_!(j) \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})) \to \mathsf{X}$ is left exact, so it suffices to show that this functor is a Bousfield localisation. Since it is cocontinuous, it has a right adjoint r (which is automatically accessible). The counit of this adjunction carries an object $U \colon A \to \mathsf{X}$ to the canonical map from the colimit of $\pi_A^* \mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})_{/U} \to \pi_A^* \mathsf{X}$ to U. By again using Lemma 6.1.2.7, this map is an equivalence, hence the claim follows. If (2) is satisfied, the map $h_!(j) \colon \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})) \to \mathsf{X}$ being a (left exact and accessible) Bousfield localisation in particular implies that X is generated under colimits by $\mathsf{Contr}^{(\text{-cpt}\,\mathsf{U})}(\mathsf{X})$, in the sense that the smallest full subcategory of X that contains $\mathsf{Contr}^{(\text{-cpt}\,\mathsf{U})}(\mathsf{X})$ and that is closed under $\mathsf{Cat}_{\mathcal{B}}$-colimits in X must already be X itself. In particular, (3) follows. Lastly, if (3) is satisfied, consider the commutative diagram

$$\begin{tikzcd}
\mathsf{Contr}(\mathsf{X}) \arrow[r, hook] \arrow[d] & \mathsf{P} \arrow[r, hook] \arrow[d] & \mathsf{X} \arrow[d, "\mathrm{const}_{\mathsf{X}}^* h_{\mathsf{X}^{\mathrm{op}}}^{\mathrm{op}}"] \\
1 \arrow[r, hook, "1_{\Omega}"] & \Omega \arrow[r, hook, "h_{\Omega^{\mathrm{op}}}^{\mathrm{op}}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)^{\mathrm{op}}
\end{tikzcd}$$

in which both squares are pullbacks. Since $h_{\Omega^{\mathrm{op}}}^{\mathrm{op}}$ and $h_{\mathsf{X}^{\mathrm{op}}}^{\mathrm{op}}$ are cocontinuous functors by Proposition 3.2.2.9, the inclusion $\mathsf{P} \hookrightarrow \mathsf{X}$ is cocontinuous (see Lemma 4.1.4.5) and must therefore be an equivalence. Hence $\mathrm{const}_{\mathsf{X}}^* h_{\mathsf{X}^{\mathrm{op}}}^{\mathrm{op}}$ takes values in $\Omega \hookrightarrow \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega, \Omega)^{\mathrm{op}}$, which precisely means that $\mathrm{const}_{\mathsf{X}}$ has a left adjoint π_{X}. Hence X is locally contractible. ☐

6.1.3. Classification of smooth geometric morphisms. In 1-topos theory, it is well-known (see [46, Corollary C.3.3.16]) that locally connected geometric morphisms are precisely the *smooth* maps, i.e. those that satisfy *smooth base change*. Our main goal in this section is to prove the ∞-toposic analogue of this result. We begin with the following definition:

Definition 6.1.3.1. Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. We say that f_* is *smooth* if for every diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "k'_*"] \arrow[d, "g'_*"] & \mathcal{Y} \arrow[r, "k_*"] \arrow[d, "g_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{A}' \arrow[r, "h'_*"] & \mathcal{A} \arrow[r, "h_*"] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ in which both squares are pullbacks, the mate transformation $g^* h'_* \to k'_*(g')^*$ is an equivalence.

We may now formulate the main result of this section as follows:

Theorem 6.1.3.2. *Let X be a $\mathcal{B}$-topos and let $f_*\colon \mathcal{X} \to \mathcal{B}$ be the associated geometric morphism. Then X is locally contractible if and only if f_* is smooth.*

The proof of Theorem 6.1.3.2 relies on a few reduction steps. Our first goal is to establish that the property of a geometric morphism to be locally contractible is stable under taking *powers* by $\mathcal{B}$-categories: Recall from Proposition 5.2.4.3 that the large $\mathcal{B}$-category $\mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$ admits a powering bifunctor $(-)^{(-)}\colon \mathsf{Cat}^{\mathrm{op}}_{\mathcal{B}} \times \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}} \to \mathsf{Top}^{\mathrm{L}}_{\mathcal{B}}$. We now find:

Lemma 6.1.3.3. *Let C be a $\mathcal{B}$-category and let X be a locally contractible $\mathcal{B}$-topos. Then the geometric morphism $(\Gamma_{\mathsf{X}})_*\colon \mathsf{X}^{\mathsf{C}} \to \Omega^{\mathsf{C}}$ exhibits $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$ as a locally contractible $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega)$-topos.*

Proof. Since the algebraic morphism associated with $(\Gamma_{\mathsf{X}})_*$ is given by $(\mathrm{const}_{\mathsf{X}})_*$, the functor $(\pi_{\mathsf{X}})_*$ defines a further left adjoint of $(\mathrm{const}_{\mathsf{X}})_*$. Therefore, Remark 6.1.1.3 implies that we only need to show that for every map $F \to G$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega)$ and every map $H \to (\mathrm{const}_{\mathsf{X}})_*(G)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \mathsf{X})$, the canonical morphism

$$(\pi_{\mathsf{X}})_*((\mathrm{const}_{\mathsf{X}})_* F \times_{(\mathrm{const}_{\mathsf{X}})_* G} H) \to F \times_H (\pi_{\mathsf{X}})_* H$$

is an equivalence. It will be enough to show that this map becomes an equivalence after being evaluated at an arbitrary object $c\colon A \to \mathsf{C}$ in context $A \in \mathcal{B}$. In light of Remark 6.1.1.2 and [62, Lemma 4.2.3], we can replace $\mathcal{B}$ with $\mathcal{B}_{/A}$, so that we can reduce to the case $A \simeq 1$. But as pullbacks in functor $\mathcal{B}$-categories are computed object-wise by Proposition 3.1.3.2 and as evaluating the unit and counit of the adjunction $(\pi_{\mathsf{X}})_* \dashv (\mathrm{const}_{\mathsf{X}})_*$ at c recovers the unit and counit of the adjunction $\pi_{\mathsf{X}} \dashv \mathrm{const}_{\mathsf{X}}$, the claim follows from the assumption that X is locally contractible and Remark 6.1.1.3. ☐

Before we can prove Theorem 6.1.3.2, we also need the following result:

Lemma 6.1.3.4. *Let X be a locally contractible $\mathcal{B}$-topos and let U be a sound doctrine such that X is U-accessible and $\mathsf{X}^{\mathsf{U}\text{-cpt}}$ is closed under finite limits in X. Then the diagonal map $\mathrm{diag}\colon \Omega \to \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}))$ takes values in $\mathsf{X} \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X}))$.*

Proof. We need to show that if $\eta\colon \mathrm{id} \to iL$ is the unit of the adjunction

$$L \dashv i\colon \mathsf{X} \leftrightarrows \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})),$$

then the induced morphism $\eta\,\mathrm{diag}\colon \mathrm{diag} \to iL\,\mathrm{diag}$ is an equivalence. As we may check this object-wise and by using Remarks 6.1.1.2, 4.1.5.2, 4.3.1.2 and 6.1.2.3 together with [62, Lemma 4.2.3], we only need to show that for any object $U\colon 1 \to \mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})$ the map $U^*\eta\,\mathrm{diag}\colon U^*\,\mathrm{diag} \to U^* iL\,\mathrm{diag}$ is an equivalence in Ω. Note that we have a chain of equivalences

$$U^* iL\,\mathrm{diag} \simeq \mathrm{map}_{\mathsf{X}}(U, \mathrm{const}_{\mathsf{X}}) \simeq \mathrm{map}_{\Omega}(\pi_{\mathsf{X}}(U), -).$$

As $\pi_{\mathsf{X}}(U) \simeq 1_\Omega$, we thus find that $U^* iL\,\mathrm{diag} \simeq \mathrm{id}$. Since also $U^*\,\mathrm{diag}$ is equivalent to the identity and since the universal property of Ω implies that $\mathrm{map}_{\underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega,\Omega)}(\mathrm{id}, \mathrm{id}) \simeq 1_\Omega$, the claim follows. ☐

Proof of Theorem 6.1.3.2. Suppose first that f_* is smooth. Then f_* in particular satisfies condition (2) of Proposition 6.1.1.5 and is therefore locally contractible. To prove the converse direction, suppose that we have two pullback squares

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "k'_*"] \arrow[d, "g'_*"] & \mathcal{Y} \arrow[r, "k_*"] \arrow[d, "g_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{A}' \arrow[r, "h'_*"] & \mathcal{A} \arrow[r, "h_*"] & \mathcal{B} \end{tikzcd}$$

of ∞-topoi in which f_* is locally contractible. By viewing $\mathcal{A}$ as a $\mathcal{B}$-topos and using Theorem 5.2.3.1, we may factor h_* into a composition $\mathcal{A} \hookrightarrow \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \Omega_{\mathcal{B}}) \to \mathcal{B}$. Since the commutative square

$$\begin{tikzcd} \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \mathsf{X}) \arrow[r, "\lim"] \arrow[d, "(\Gamma_{\mathsf{X}})_*"] & \mathsf{X} \arrow[d, "\Gamma_{\mathsf{X}}"] \\ \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{C}) \arrow[r, "\lim"] & \Omega \end{tikzcd}$$

is a pullback in $\mathsf{Top}^{\mathrm{R}}_{\mathcal{B}}$ (see Example 5.2.7.5) and on account of Lemma 6.1.3.3, this allows us to reduce to the case where h_* is already an embedding. But then k_* must be an embedding as well, so that the mate of the left square is an equivalence if and only if the mates of the right one and the composite one are equivalences. Hence, to complete the proof, it will be enough to show that if we are given any pullback square

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "k_*"] \arrow[d, "g_*"] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{A} \arrow[r, "h_*"] & \mathcal{B} \end{tikzcd}$$

in which f_* is locally contractible, the mate transformation $f^* h_* \to k_* g^*$ is an equivalence. By the same argument as above (and the fact that $\mathrm{const}_{\mathsf{X}}$ is continuous), we can moreover still assume that h_* and k_* are embeddings. To proceed, we make use of Proposition 6.1.2.5 to obtain a commutative diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "k_*"] \arrow[d, hook] & \mathcal{X} \arrow[d, hook] \\ \mathrm{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})^{\mathrm{op}}, h_*(\Omega_{\mathcal{A}})) \arrow[r, hook] \arrow[d, "\lim"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})^{\mathrm{op}}, \Omega_{\mathcal{B}}) \arrow[d, "\lim"] \\ \mathcal{A} \arrow[r, hook, "h_*"] & \mathcal{B} \end{tikzcd}$$

in which both squares are pullbacks. Since the mate of the lower square is evidently an equivalence and since Lemma 6.1.3.4 implies that the diagonal map $\mathrm{diag}\colon \mathcal{B} \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})^{\mathrm{op}}, \Omega_{\mathcal{B}})$ factors through $\mathcal{X}$, it will be enough to show that the diagonal map $\mathrm{diag}\colon \mathcal{A} \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})^{\mathrm{op}}, h_*\Omega_{\mathcal{A}})$ factors through $\mathcal{Y}$ as well. Let us therefore pick an arbitrary object $A \in \mathcal{A}$. By [57, Lemmas 6.3.3.4], the upper square in the above diagram is a pullback square of ∞-categories, hence it suffices to show that the image of $\mathrm{diag}(A)$ in $\mathrm{Fun}_{\mathcal{B}}(\mathsf{Contr}^{\mathsf{U}\text{-cpt}}(\mathsf{X})^{\mathrm{op}}, \Omega_{\mathcal{B}})$ is contained in $\mathcal{X}$. But as the mate of the lower square is an equivalence, this latter object is equivalent to $\mathrm{diag}\, h_*(A)$, hence another application of Lemma 6.1.3.4 yields the claim. □

6.2. Proper geometric morphisms

In this section, we study the relationship between *proper* and *compact* geometric morphisms of ∞-topoi. Most of the chapter is devoted to the proof of Theorem 6.2.1.12, which states that a geometric morphism $\mathcal{X} \to \mathcal{B}$ is proper precisely if it is compact. We begin by defining these two notions in § 6.2.1. In § 6.2.2 and § 6.2.3, we discuss two auxiliary steps that are required for the proof of Theorem 6.2.1.12: the ∞-toposic cone construction and the compatibility of pullbacks with localisations of ∞-topoi. Lastly, we put everything together in § 6.2.4 to finish the proof.

In § 6.2.5 we discuss how we can apply Theorem 6.2.1.12 to show that the geometric morphism associated with a proper and separated map of topological spaces is proper. Finally, in § 6.2.6 we discuss a variant of Theorem 6.2.1.12 in which we allow coefficients in an arbitrary compactly generated ∞-category.

6.2.1. Compactness and properness. In this section we will introduce the two main properties, properness and compactness, of a geometric morphism that we want to study in this section. Let us begin with the notion of properness, which is due to Lurie:

Definition 6.2.1.1 ([57, Definition 7.3.1.4]). Let $p_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. We say that f_* is *proper* if for every commutative diagram

$$\begin{tikzcd} \mathcal{Y}' \arrow[r, "g'_*"] \arrow[d, "q'_*"'] & \mathcal{Y} \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{A}' \arrow[r, "f'_*"'] & \mathcal{A} \arrow[r, "f_*"'] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}^{\mathrm{R}}_\infty$ in which both squares are cartesian, the left square is left adjointable, in the sense that the mate transformation $(f')^* q_* \to q'_*(g')^*$ is an equivalence.

Example 6.2.1.2. It follows from [57, Proposition 7.3.12 and Corollary 7.3.2.13] that any *closed immersion* of ∞-topoi, in the sense of [57, Definition 7.3.2.6], is proper.

Example 6.2.1.3. Let $p\colon Y \to X$ be a proper and separated morphism of topological spaces. In § 6.2.5 we will prove that the geometric morphism $p_*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is proper, generalising a result of Lurie [57, Theorem 7.3.1.16].

Recollection 6.2.1.4. Recall from Definition 4.2.3.1 that a $\mathcal{B}$-category I is called *filtered* if and only if the functor $\mathrm{colim}_{\mathsf{I}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) \to \Omega_{\mathcal{B}}$ is left exact. Let us denote by Filt the internal class of filtered $\mathcal{B}$-categories. We will say that a functor $f\colon \mathsf{C} \to \mathsf{D}$ of $\mathcal{B}$-categories preserves filtered colimits, if it is Filt-cocontinuous.

Recall that one may call an ∞-topos compact if the global sections functor preserves filtered colimits. Relativizing this condition we obtain the following definition

Definition 6.2.1.5. A $\mathcal{B}$-topos X is said to be *compact* if the global sections functor $\Gamma_{\mathsf{X}}\colon \mathsf{X} \to \Omega_{\mathcal{B}}$ preserves filtered colimits. We say that a geometric morphism $p_*\colon \mathcal{X} \to \mathcal{B}$ is compact if the associated $\mathcal{B}$-topos $p_*\Omega_{\mathcal{X}}$ is compact.

Example 6.2.1.6. If $A \in \mathcal{B}$ is an arbitrary object, the étale geometric morphism $(\pi_A)_*\colon \mathcal{B}_{/A} \to \mathcal{B}$ is compact if and only if A is *internally compact* in $\mathcal{B}$, i.e. if the functor $\mathrm{map}_\Omega(A, -)\colon \Omega \to \Omega$ preserves filtered colimits. To see this, first note that $(\pi_A)_*$ corresponds to the étale $\mathcal{B}$-topos $\underline{\mathsf{Fun}}_{\mathcal{B}}(A, \Omega_{\mathcal{B}})$ (see § 5.2.9) and the unique geometric morphism into $\Omega_{\mathcal{B}}$ is given by the limit functor $\lim_A\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(A, \Omega_{\mathcal{B}}) \to \Omega_{\mathcal{B}}$ (as its left adjoint diag_A is again a right adjoint and therefore preserves all limits Proposition 3.1.2.11). Moreover, since $A \simeq \mathrm{colim}_A \mathrm{diag}_A 1_\Omega$, where $1_\Omega\colon 1_{\mathcal{B}} \to \Omega$ is the section encoding the final object $1_{\mathcal{B}} \in \mathcal{B}$ (see Proposition 3.1.4.1), the adjunctions $\mathrm{colim}_A \dashv \mathrm{diag}_A \dashv \lim_A$ imply that we obtain an identification $\mathrm{map}_\Omega(A, -) \simeq \mathrm{map}_\Omega(1_\Omega, \lim_A \mathrm{diag}_A) \simeq \lim_A \mathrm{diag}_A$ (since $\mathrm{map}_\Omega(1_\Omega, -)$ is equivalent to the identity, see [62, Proposition 4.6.3]). Hence $\lim_A$ preserving filtered colimits implies that A is internally compact. To see the converse, note that by Corollary 4.2.3.8 A being internally compact is equivalent to A being locally constant with compact values. If this is the case, then the fact that Filt-cocontinuity can be checked locally in $\mathcal{B}$ (see Remark 3.2.2.3) allows us to reduce to the case where A is constant with compact value. In other words, A is a retract of a finite $\mathcal{B}$-groupoid, so that we may further reduce to the case where A is already finite. In this case, $\lim_A$ preserves filtered colimits by the very definition of filteredness.

Warning 6.2.1.7. In the context of Definition 6.2.1.5, it is essential that we require $\Gamma_{\mathsf{X}}\colon \mathsf{X} \to \Omega_{\mathcal{B}}$ to be Filt-cocontinuous instead of just asking for the underlying functor p_* to preserve ordinary filtered colimits. In fact, if $A \in \mathcal{B}$ is an arbitrary object, we saw in Example 6.2.1.6 that $(\pi_A)_* \colon \mathcal{B}_{/A} \to \mathcal{B}$ is compact if and only if A is internally compact. On the other hand, $(\pi_A)_*$ preserves ordinary filtered colimits if and only if the functor $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, -) \colon \mathcal{B} \to \mathcal{B}$ does (where $\underline{\mathrm{Hom}}_{\mathcal{B}}$ denotes the internal hom of $\mathcal{B}$). By Proposition 3.1.4.11, A being internally compact implies that $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, -)$ preserves ordinary filtered colimits, but the converse is not true in general. For example, if X is a coherent topological space, then any quasi-compact open $U \subset X$ defines an object in the ∞-topos $\mathrm{Sh}(X)$ satisfying the latter condition (since quasi-compact opens in X define compact objects in $\mathrm{Sh}(X)$ and generate this ∞-topos under colimits). On the other hand, U is in general quite far from being locally constant and can therefore not always be internally compact.

Remark 6.2.1.8. Let $\mathcal{X}$ be a 1-localic ∞-topos. If $\mathcal{X}$ is compact, the associated 1-topos $\mathrm{Disc}(\mathcal{X})$ of 0-truncated objects in $\mathcal{X}$ is *tidy* in the sense of [69]. However, the converse it not true in general. For example, any coherent 1-topos is tidy, but 1-localic coherent ∞-topoi are not compact in general. An explicit counterexample is $\mathrm{Spec}(\mathbb{R})_{\text{ét}} \simeq \mathrm{Fun}(B(\mathbb{Z}/2\mathbb{Z}), \mathcal{S})$, which cannot be tidy since $B(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{RP}^{\infty}$ is not compact in $\mathcal{S}$.

Example 6.2.1.9. Any ∞-topos of the form $\mathrm{Sh}^{\tau}(\mathcal{C})$ where $\mathcal{C}$ is an ∞-category with an initial and a terminal object and τ a topology generated by a cd-structure is compact. This follows since under these assumption $\mathrm{Sh}^{\tau}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ commutes with filtered colimits and $\mathrm{PSh}(\mathcal{C})$ is always compact if $\mathcal{C}$ has a terminal object. Example of such topologies from algebraic geometry include the Zariski-, Nisnevich- and cdh-topology.

Lemma 6.2.1.10. *Compact $\mathcal{B}$-topoi are stable under retracts in* $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$.

Proof. Let X be a compact $\mathcal{B}$-topos and

$$\mathsf{Y} \xrightarrow{s_*} \mathsf{X} \xrightarrow{r_*} \mathsf{Y}$$

a retract diagram of $\mathcal{B}$-topoi. We thus get a retract diagram of functors

$$\Gamma_{\mathsf{Y}} \simeq \mathrm{map}_{\mathsf{Y}}(1, -) \xrightarrow{r^*} \mathrm{map}_{\mathsf{X}}(r^*1, r^*(-)) \xrightarrow{s^*} \mathrm{map}_{\mathsf{Y}}(s^*r^*1, s^*r^*(-)) \simeq \Gamma_{\mathsf{Y}}.$$

Furthermore $\mathrm{map}_{\mathsf{X}}(r^*1, r^*(-)) \simeq \Gamma_{\mathsf{X}} \circ r^*$ and thus Γ_{Y} preserves filtered colimits as a retract of a filtered colimit preserving functor. □

Example 6.2.1.11. Let L be a stably compact $\mathcal{B}$-locale. Then $\underline{\mathsf{Sh}}_{\mathcal{B}}(L)$ is a compact $\mathcal{B}$-topos. Indeed, by it follows that $\underline{\mathsf{Sh}}_{\mathcal{B}}(L)$ is a retract of $\underline{\mathsf{Sh}}_{\mathcal{B}}(L')$ for some coherent locale L'. Using Lemma 6.2.1.10, it follows that we may assume that L is coherent. In this case recall from Proposition 5.3.7.8 that there is an equivalence $\underline{\mathsf{Sh}}_{\mathcal{B}}(L) \simeq \underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$ and by Lemma 5.3.7.7 the embedding

$$\underline{\mathsf{Sh}}^{\mathrm{fin}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}}) \hookrightarrow \underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})$$

preserves colimits. So it suffices to see that $\Gamma_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})}$ preserves filtered colimits. Now note that because $\mathsf{L}^{\mathrm{cpt}}$ has a final object 1, it follows that $\Gamma_{\underline{\mathsf{PSh}}_{\mathcal{B}}(\mathsf{L}^{\mathrm{cpt}})} \simeq \mathrm{ev}_1$ by Proposition 3.1.8.1. In particular it preserves filtered colimits.

We are now ready to state the main result of this paper:

Theorem 6.2.1.12. *A geometric morphism $p_* \colon \mathcal{X} \to \mathcal{B}$ is proper if and only if it is compact.*

The full proof of Theorem 6.2.1.12 is rather involved and will be given in § 6.2.4. One implication, however, is straightforward:

Lemma 6.2.1.13. *Let $p_* \colon \mathcal{X} \to \mathcal{B}$ be a proper geometric morphism. Then p_* is compact.*

Proof. Let us denote by $\mathsf{X} = p_*\Omega_{\mathcal{X}}$ the $\mathcal{B}$-topos that corresponds to p_*. Note that if $A \in \mathcal{B}$ is an arbitrary object, the induced morphism $\mathcal{X}_{/p^*A} \to \mathcal{B}_{/A}$ is proper as well. As this is the geometric morphism which corresponds to the $\mathcal{B}_{/A}$-topos $\pi_A^*\mathsf{X}$, we may (after replacing $\mathcal{B}$ with $\mathcal{B}_{/A}$) reduce to the case where we have to show that if I is a filtered $\mathcal{B}$-category, then Γ_{X} preserves I-filtered colimits. By definition of filteredness, the colimit functor $\operatorname{colim}_{\mathsf{I}}\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) \to \Omega_{\mathcal{B}}$ is left exact, which implies that $\operatorname{diag}\colon \Omega_{\mathcal{B}} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}})$ defines a geometric morphism of $\mathcal{B}$-topoi. We may therefore consider the two pullback squares

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\operatorname{diag}"] \arrow[d, "\Gamma_{\mathsf{X}}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{X}) \arrow[r, "\lim"] \arrow[d, "(\Gamma_{\mathsf{X}})_*"'] & \mathsf{X} \arrow[d, "\Gamma_{\mathsf{X}}"] \\
\Omega_{\mathcal{B}} \arrow[r, "\operatorname{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) \arrow[r, "\lim"] & \Omega_{\mathcal{B}}
\end{tikzcd}$$

in $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$ (see Example 5.2.7.5). As for every $A \in \mathcal{B}$ the geometric morphism $\Gamma_{\mathsf{X}}(A)\colon \mathcal{X}_{/p^*A} \to \mathcal{B}_{/A}$ is proper, it follows that the mate of the left square is an equivalence. This precisely means that Γ_{X} commutes with I-indexed colimits, as desired. $\square$

6.2.2. The toposic cone. Every topological space X admits a closed immersion into a contractible and locally contractible space Y, for example by setting $Y = C(X)$, where $C(X)$ is the *cone* of X. In this section, we discuss a $\mathcal{B}$-toposic analogue of this observation. To that end, if X is a $\mathcal{B}$-topos, recall that the *comma $\mathcal{B}$-category* $\mathsf{X} \downarrow_{\mathsf{X}} \Omega$ is defined via the pullback square

$$\begin{tikzcd}
\mathsf{X} \downarrow_{\mathsf{X}} \Omega \arrow[r, "e^*"] \arrow[d, "j^*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X}) \arrow[d, "d_0"] \\
\Omega \arrow[r, "\operatorname{const}_{\mathsf{X}}"'] & \mathsf{X}
\end{tikzcd}$$

in $\mathrm{Cat}(\mathcal{B})$, where $\operatorname{const}_{\mathsf{X}}$ denotes the unique algebraic morphism $\Omega_{\mathcal{B}} \to \mathsf{X}$, i.e. the left adjoint of Γ_{X}. By Proposition 5.2.6.1, this is a pullback diagram in $\mathrm{Top}^{\mathrm{L}}(\mathcal{B})$, so that $\mathsf{X} \downarrow_{\mathsf{X}} \Omega$ is a $\mathcal{B}$-topos and j^* and e^* are algebraic morphisms.

Definition 6.2.2.1. For any $\mathcal{B}$-topos X, we refer to the $\mathcal{B}$-topos $\mathsf{X} \downarrow_{\mathsf{X}} \Omega$ as its *$\mathcal{B}$-toposic right cone* and denote it by $\mathsf{X}^{\triangleright}$.

If X is a $\mathcal{B}$-topos, let $i^*\colon \mathsf{X}^{\triangleright} \to \mathsf{X}$ be the algebraic morphism that is obtained by composing the functor $d_1\colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X}) \to \mathsf{X}$ with the upper horizontal map in the defining pullback square of $\mathsf{X}^{\triangleright}$.

Remark 6.2.2.2. Suppose that X is a $\mathcal{B}$-topos and let $f_*\colon \mathcal{X} \to \mathcal{B}$ be the associated geometric morphism of ∞-topoi. Then the ∞-topos $\mathrm{Cone}(f) = \Gamma_{\mathcal{B}}(\mathsf{X}^{\triangleright})$ recovers the comma ∞-category $\mathcal{X} \downarrow_{\mathcal{X}} \mathcal{B}$ and is therefore the *recollement* of $\mathcal{B}$ and $\mathcal{X}$ along f^* in the sense of [56, § A.8]. In particular, $j_*\colon \mathcal{B} \to \mathrm{Cone}(f)$ is an open and $i_*\colon \mathcal{X} \to \mathrm{Cone}(f)$ a closed immersion of ∞-topoi. Note that this in particular implies that $i_*\colon \mathsf{X} \to \mathsf{X}^{\triangleright}$ and $j_*\colon \Omega \to \mathsf{X}^{\triangleright}$ are (section-wise) fully faithful.

Remark 6.2.2.3. In the situation of Remark 6.2.2.2, the ∞-topos $\mathrm{Cone}(f)$ sits inside a pushout square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f_*"] \arrow[d, "\mathrm{id} \otimes s^0"'] & \mathcal{B} \arrow[d] \\
\mathcal{X} \otimes \Delta^1 \arrow[r] & \mathrm{Cone}(f)
\end{tikzcd}$$

in $\mathrm{Top}^{\mathrm{R}}_{\infty}$, where $\mathcal{X} \otimes \Delta^1$ denotes the tensoring in $\mathrm{Top}^{\mathrm{R}}_{\infty}$ over Cat_{∞}. Therefore $\mathrm{Cone}(f)$ is to be thought of as the mapping cone of f_*.

Recall from § 6.1.1 that a $\mathcal{B}$-topos X is said to be *locally contractible* if $\operatorname{const}_{\mathsf{X}}\colon \Omega_{\mathcal{B}} \to \mathsf{X}$ has a left adjoint π_{X}. The following proposition expresses the fact that the $\mathcal{B}$-toposic right cone $\mathsf{X}^{\triangleright}$ is contractible and locally contractible (in the $\mathcal{B}$-toposic sense):

Proposition 6.2.2.4. *For every $\mathcal{B}$-topos X, the $\mathcal{B}$-topos $\mathsf{X}^{\triangleright}$ is locally contractible, and the additional left adjoint π_{X} of $\operatorname{const}_{\mathsf{X}^{\triangleright}}$ is equivalent to j^*. In particular, π_{X} preserves finite limits.*

PROOF. Since $s_0 \colon \mathsf{X} \to \underline{\mathsf{Fun}}_{\mathcal{B}}(\Delta^1, \mathsf{X})$ is right adjoint to d_0 (for example by using Proposition 2.4.1.15) and by the dual of Lemma 3.3.3.9, the functor j_* is the pullback of s_0 along ϵ^*. Since s_0 is cocontinuous and as $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \hookrightarrow \mathrm{Cat}(\mathcal{B})$ preserves limits, this implies that j_* must be cocontinuous as well and therefore equivalent to $\mathrm{const}_{\mathsf{X}^{\triangleright}}$ (by the universal property of Ω). As this shows that j^* is left adjoint to $\mathrm{const}_{\mathsf{X}^{\triangleright}}$, the claim follows. □

REMARK 6.2.2.5. It follows from Proposition 6.2.2.4 that if $f_* \colon \mathcal{X} \to \mathcal{B}$ is any geometric morphism, we may factor it as $h_* \circ i_*$ where i_* is a closed immersion and h_* is locally contractible and the additional left adjoint $h_\sharp$ is left exact. For 1-topoi, the factorization constructed above appears in the proof of [46, Theorem C.3.3.14].

6.2.3. Compatibility of pullbacks with localisations. The goal of this section is to establish the main technical step towards the proof of Theorem 6.2.1.12, the fact that compact geometric morphisms commute with localisations of subtopoi:

PROPOSITION 6.2.3.1. *Consider a pullback square in* $\mathrm{Top}^{\mathrm{R}}_{\infty}$

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "p'_*"] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{Z} \arrow[r, "j_*"'] & \mathcal{B} \end{tikzcd}$$

where j_* is fully faithful and p_* is compact. Then the mate natural transformation $p'_* j'^* \to j^* p_*$ is an equivalence.

Intuitively, Proposition 6.2.3.1 should hold because the localisation functor $(j')^* \colon \mathcal{X} \to \mathcal{Z}'$ is given by an (internally) filtered colimit. Indeed, by Proposition 5.2.10.14 we may pick a bounded local class S' of morphism in $\mathcal{X}$ (in the sense of [62, Definition 3.9.10]) that is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{X})$ such that there is a natural equivalence

$$j'_*(j')^*(X) \simeq X^{\mathrm{sh}}_{\iota'} = \operatorname*{colim}_{\tau<\kappa} T^{\iota'}_{\tau} X$$

where we denote by $\iota' \colon \Omega_{S'} \hookrightarrow \Omega_{\mathcal{X}}$ the associated full subcategory. Here the functors $(-)^{\mathrm{sh}}_{\iota'}$ and $T^{\iota'}_{\tau}$ are the functors constructed in § 5.2.10 and κ is a suitably large cardinal. Furthermore recall that $T^{\iota'}_{\tau} X$ is defined recursively by the condition that we have $T^{\iota'}_{\tau+1} X = \mathrm{colim}_{\Omega^{\mathrm{op}}_{S'}} \mathrm{map}_{\Omega_{\mathcal{X}}}(\iota'(-), T^{\iota'}_{\tau} X)$ and that $T^{\iota'}_{\tau} X = \mathrm{colim}_{\tau'<\tau} T^{\iota'}_{\tau'} X$ when τ is a limit ordinal. Since $\Omega_{S'}$ is cofiltered because S is closed under finite limits, it follows that the endofunctor $j'_*(j')^*$ is given by an (iterated) *filtered* colimit. So intuitively, p_* being compact should imply that this functor carries $j'_*(j')^*$ to $j_* j^*$, which precisely means that the mate transformation $p'_*(j')^* \to j^* p_*$ is an equivalence. However, we have to be a bit careful at this point: the above formula for $j'_*(j')^*$ exhibits $j'_*(j')^*(X)$ as a filtered colimit *internal to* $\mathcal{X}$, whereas p_* being compact only implies that this functor commutes with filtered colimits *internal to* $\mathcal{B}$. Hence, the main challenge is to rewrite the above formula in terms of a filtered colimit internal to $\mathcal{B}$.

OBSERVATION 6.2.3.2. Let $f_* \colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism, I a $\mathcal{B}$-category and C an $\mathcal{X}$-category. Since the adjunction $f^* \dashv f_* \colon \mathrm{Cat}(\mathcal{B}) \leftrightarrows \mathrm{Cat}(\mathcal{X})$ yields an equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(-, f_*(-)) \simeq f_* \underline{\mathsf{Fun}}_{\mathcal{X}}(f^*(-), -)$, we obtain a commutative diagram

$$\begin{tikzcd} f_*\mathsf{C} \arrow[r, "\mathrm{diag}_{\mathsf{I}}"] \arrow[dr, "f_*(\mathrm{diag}_{f^*\mathsf{I}})"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\mathsf{C}) \arrow[d, "\simeq"] \\ & f_*\underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{I}, \mathsf{C}) \end{tikzcd}$$

of $\mathcal{B}$-categories. Hence the $\mathcal{B}$-category $f_*\mathsf{C}$ admits I-indexed colimits if and only if the $\mathcal{X}$-category C admits $f^*\mathsf{I}$-indexed colimits, and we may identify the colimit functor $\mathrm{colim}_{\mathsf{I}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\mathsf{C}) \to f_*\mathsf{C}$ with the composition

$$\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\mathsf{C}) \simeq f_*\underline{\mathsf{Fun}}_{\mathcal{X}}(f^*\mathsf{I}, \mathsf{C}) \xrightarrow{f_*(\mathrm{colim}_{f^*\mathsf{I}})} f_*\mathsf{C}.$$

By passing to global sections, this implies that for every diagram $d\colon \mathsf{I} \to f_*\mathsf{C}$ with transpose $\bar{d}\colon f^*\mathsf{I} \to \mathsf{C}$, we have a canonical equivalence $\operatorname{colim}^{\mathcal{B}}_{\mathsf{I}} d \xrightarrow{\simeq} \operatorname{colim}^{\mathcal{X}}_{f^*\mathsf{I}} \bar{d}$ in the ∞-category $\Gamma_{\mathcal{X}}(\mathsf{C}) = \Gamma_{\mathcal{B}}(f_*\mathsf{C})$ (where the superscripts emphasise internal to which ∞-topos the colimits are taken). We will repeatedly use this observation throughout this chapter.

Suppose that S is a bounded local class of morphisms in $\mathcal{B}$ that is closed under finite limits in $\operatorname{Fun}(\Delta^1, \mathcal{B})$ and let $\iota\colon \Omega_S \hookrightarrow \Omega$ be the associated full subcategory. The assumption on S makes sure that ι is closed under finite limits, so that in particular Ω_S is cofiltered. Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism and let us denote by $\iota'\colon f^*\Omega_S \to \Omega_{\mathcal{X}}$ the functor of $\mathcal{X}$-categories that arises from transposing $\operatorname{const}_{f_*\Omega_{\mathcal{X}}} \iota\colon \Omega_S \to f_*\Omega_{\mathcal{X}}$ across the adjunction $f^* \dashv f_*$. By Example 5.2.10.9, $f^*\Omega_S$ is a cofiltered $\mathcal{X}$-category and the colimit of ι' is the final object. Therefore, we are in the situation of Definition 5.2.10.5 and thus obtain an endofunctor $(-)^{\mathrm{sh}}_{\iota'}\colon \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}}$ via $(-)^{\mathrm{sh}}_{\iota'} = \operatorname{colim}_{\tau<\kappa} T^{\iota'}_{\tau}$, where κ is a suitable regular cardinal and where $T^{\iota'}_{\bullet}\colon \kappa \to \operatorname{Fun}_{\mathcal{X}}(\Omega_{\mathcal{X}}, \Omega_{\mathcal{X}})$ is defined via transfinite induction by setting $T^{\iota'}_0 = \mathrm{id}$, by defining the map $T^{\iota'}_{\tau} \to T^{\iota'}_{\tau+1}$ to be the morphism $\varphi\colon T^{\iota'}_{\tau} \to (T_{\tau})^{+}_{\iota'} = \operatorname{colim}_{f^*\Omega_S^{\mathrm{op}}} \operatorname{map}_{\Omega_{\mathcal{X}}}(\iota'(-), -)$ from Remark 5.2.10.4 and finally by setting $T^{\iota'}_{\tau} = \operatorname{colim}_{\tau'<\tau} T^{\iota'}_{\tau'}$ whenever τ is a limit ordinal. We will slightly abuse notation and also denote by $(-)^{\mathrm{sh}}_{\iota'}$ the underlying endofunctor on $\mathcal{X}$ that is obtained from $(-)^{\mathrm{sh}}_{\iota}\colon \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}}$ upon passing to global sections. It will always be clear from the context which variant we refer to.

Proposition 6.2.3.3. *Consider a pullback square $\mathcal{Q}$ in* $\mathrm{Top}^{\mathrm{R}}_{\infty}$

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "g_*"'] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z} \arrow[r, "j_*"'] & \mathcal{B} \end{tikzcd}$$

where j_* (and therefore also j'_*) is fully faithful. Let S be a bounded local class in $\mathcal{B}$ that is closed under finite limits in $\operatorname{Fun}(\Delta^1, \mathcal{B})$ *such that j^* is Bousfield localization at S (such a class always exists by Proposition 5.2.10.14). Let $\iota\colon \Omega_S \hookrightarrow \Omega_{\mathcal{B}}$ be the associated full subcategory. Then we obtain an equivalence $j'_*(j')^* \simeq (-)^{\mathrm{sh}}_{\iota'}$, where $\iota'\colon f^*\Omega_S \to \Omega_{\mathcal{X}}$ is the transpose of* $\operatorname{const}_{f_*\Omega_{\mathcal{X}}} \iota$.

Remark 6.2.3.4. The above proposition can be thought of as an ∞-toposic version of [46, Theorem C.3.3.14].

We first prove this proposition in a special case:

Lemma 6.2.3.5. *Consider a pullback square $\mathcal{Q}$ in* $\mathrm{Top}^{\mathrm{R}}_{\infty}$

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "h'_*"'] & \mathcal{X} \arrow[d, "h_*"] \\ \mathcal{Z} \arrow[r, "j_*"'] & \mathcal{B} \end{tikzcd}$$

where j_* is fully faithful and h_* is locally contractible such that the additional left adjoint $h_!$ of h^* preserves finite limits. Let S and ι be as in Proposition 6.2.3.3. Then there is an equivalence $j'_*(j')^* \simeq (-)^{\mathrm{sh}}_{\iota'}$, where $\iota'\colon h^*\Omega_S \to \Omega_{\mathcal{X}}$ is the transpose of $\operatorname{const}_{h_*\Omega_{\mathcal{X}}} \iota$.

Proof. By Proposition 6.1.1.5, the functor ι' is fully faithful, and since h_* is locally contractible the $\mathcal{X}$-category $h^*\Omega_S$ is given by the sheaf $\Omega_S(h_!(-))$. It follows that a map $s\colon X \to Y$ in $\mathcal{X}$ defines an object of $h^*\Omega_S(Y)$ if and only if $h_!(s) \in S$ and the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & h^*h_!X \arrow[d] \\ Y \arrow[r] & h^*h_!Y \end{tikzcd}$$

is a pullback. Let W be the class of maps in $\mathcal{X}$ that satisfies these two conditions. Then, since $h_!$ is cocontinuous and preserves finite limits, it easily follows that W is local. Hence we find $h^*\Omega_S = \Omega_W$ as full subcategories of $\Omega_{\mathcal{X}}$. Moreover, by the explicit description of W, it is clear that W is closed under finite limits in $\mathrm{Fun}(\Delta^1, \mathcal{X})$. Thus, by appealing to Proposition 5.2.10.14, we only need to verify that $\mathcal{Z}'$ is the Bousfield localisation of $\mathcal{X}$ at W. We know from Remark 5.2.10.15 that $\mathcal{Z}' \hookrightarrow \mathcal{X}$ is obtained as the Bousfield localisation of $\mathcal{X}$ at the smallest local class $\overline{h^*S}$ that contains the image h^*S of S along h^*. Since we clearly have $h^*S \subset W$, this immediately implies $W = \overline{h^*S}$, hence the claim follows. □

Lemma 6.2.3.6. *Let $p_* \colon \mathcal{X} \to \mathcal{B}$ be a compact geometric morphism and let $\iota \colon \mathsf{I} \to \Omega_{\mathcal{B}}$ be a functor where I is cofiltered and where $\operatorname{colim} \iota$ is the final object. Let $\iota' \colon p^*\mathsf{I} \to \Omega_{\mathcal{X}}$ be the transpose of $\mathrm{const}_{p_*\Omega_{\mathcal{X}}} \iota \colon \mathsf{I} \to p_*\Omega_{\mathcal{X}}$. Then there is an equivalence $p_*(-)^{\mathrm{sh}}_{\iota'} \simeq (-)^{\mathrm{sh}}_{\iota} p_*$.*

Proof. Since $(-)^{\mathrm{sh}}_{\iota'}$ and $(-)^{\mathrm{sh}}_{\iota}$ are obtained as filtered colimits of iterations of $(-)^+_{\iota'}$ and $(-)^+_{\iota}$, respectively, and as p_* commutes with filtered colimits, it suffices to produce an equivalence $p_*(-)^+_{\iota'} \simeq (-)^+_{\iota} p_*$. Now for every $X \in \mathcal{X}$, we have a natural chain of equivalences

$$
\begin{aligned}
(p_*X)^+_{\iota} &= \operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{\Omega_{\mathcal{B}}}(\iota(-), \Gamma_{p_*\Omega_{\mathcal{X}}} X) \\
&\simeq \operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{p_*\Omega_{\mathcal{X}}}(\mathrm{const}_{p_*\Omega_{\mathcal{X}}} \iota(-), X) \\
&\simeq \operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \Gamma_{p_*\Omega_{\mathcal{X}}}(\underline{\mathrm{Hom}}^{\mathcal{B}}_{p_*\Omega_{\mathcal{X}}}(\mathrm{const}_{p_*\Omega_{\mathcal{X}}} \iota(-), X)) \\
&\simeq \Gamma_{p_*\Omega_{\mathcal{X}}}(\operatorname{colim}^{\mathcal{B}}_{\mathsf{I}^{\mathrm{op}}} \underline{\mathrm{Hom}}^{\mathcal{B}}_{p_*\Omega_{\mathcal{B}}}(\mathrm{const}_{p_*\Omega_{\mathcal{X}}} \iota(-), X)) \\
&\simeq \Gamma_{p_*\Omega_{\mathcal{X}}}(\operatorname{colim}^{\mathcal{X}}_{p^*\mathsf{I}^{\mathrm{op}}} \operatorname{map}_{p_*\Omega_{\mathcal{X}}}(\iota'(-), X)) \\
&\simeq p_* X^+_{\iota'}
\end{aligned}
$$

where the third step follows from Remark 5.2.10.13, the fourth step is a consequence of the fact that $\Gamma_{p_*\Omega_{\mathcal{X}}}$ preserves filtered colimits and the fifth step follows from Observation 6.2.3.2. Hence the result follows. □

Proof of Proposition 6.2.3.3. Using Proposition 6.2.2.4, we may factor the pullback square $\mathcal{Q}$ into two squares

$$
\begin{tikzcd}
\mathcal{Z}' \arrow[r, "j'_*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d, "i_*"] \\
\mathcal{Z}'' \arrow[r, "j''_*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d, "h_*"] \\
\mathcal{Z} \arrow[r, "j_*"] & \mathcal{B}
\end{tikzcd}
$$

where h_* is as in Lemma 6.2.3.5 and i_* is a closed immersion (and therefore proper, see Example 6.2.1.2). By Lemma 6.2.3.5, we have an equivalence $j''_*(j'')^* \simeq (-)^{\mathrm{sh}}_{\iota''}$, where $\iota'' \colon h^*\Omega_S \to \Omega_{\mathcal{Y}}$ is the transpose of $\mathrm{const}_{h_*\Omega_{\mathcal{Y}}} \iota$. Furthermore, since i_* is a closed immersion and therefore proper, the upper square is horizontally left adjointable. Therefore, we have an equivalence $j'_*j'^* \simeq i^*j''_*j''^*i_*$ and therefore $j'_*j'^* \simeq i^*(-)^{\mathrm{sh}}_{\iota''}i_*$. Now as i_* being proper implies that i_* is compact (by Lemma 6.2.1.13), we may apply Lemma 6.2.3.6 to deduce $(-)^{\mathrm{sh}}_{\iota''}i_* \simeq i_*(-)^{\mathrm{sh}}_{\iota'}$, which yields the claim. □

We are finally ready to prove Proposition 6.2.3.1:

Proof of Proposition 6.2.3.1. It suffices to show that there is a natural equivalence $p_*j'_*j'^* \simeq j_*j^*p_*$. We pick a local class S in $\mathcal{B}$, as in Proposition 6.2.3.3, and we let $\iota \colon \Omega_S \hookrightarrow \Omega_{\mathcal{B}}$ be the associated full subcategory. Furthermore, we let $\iota' \colon p^*\Omega_S \to \Omega_{\mathcal{X}}$ be the transpose of $\mathrm{const}_{p_*\Omega_{\mathcal{X}}} \iota \colon \Omega_S \to p_*\Omega_{\mathcal{X}}$. We then have equivalences $j_*j^* \simeq (-)^{\mathrm{sh}}_{\iota}$ (by Proposition 5.2.10.11) and $j'_*j'^* \simeq (-)^{\mathrm{sh}}_{\iota'}$ (by Proposition 6.2.3.3). Hence the claim follows from Lemma 6.2.3.6. □

6.2.4. The proof of Theorem 6.2.1.12. We now turn to the proof of the main theorem. We begin with the following small but useful observation:

LEMMA 6.2.4.1. *Let*

$$\begin{tikzcd} \mathsf{Q} \arrow[r, "g_*"] \arrow[d, "q_*"] & \mathsf{P} \arrow[d, "p_*"] \\ \mathsf{Y} \arrow[r, "f_*"] & \mathsf{X} \end{tikzcd}$$

be a commutative square in $\mathrm{Top}^{\mathrm{R}}(\mathcal{B})$. *Then the mate transformation* $\varphi\colon f^*p_* \to g_*q^*$ *is an equivalence if and only if it induces an equivalence on global sections.*

PROOF. Since the condition is clearly necessary, it suffices to show that it is sufficient too. To that end, we need to show that for any object $A \in \mathcal{B}$, the horizontal mate $\varphi(A)$ of the back square in the commutative diagram

$$\begin{tikzcd} & \mathsf{Q}(A) \arrow[rr, "g_*(A)"] \arrow[dl, "(\pi_A)_*"'] \arrow[dd, "q_*(A)" near end] & & \mathsf{P}(A) \arrow[dl, "(\pi_A)_*"] \arrow[dd, "p_*(A)"] \\ \mathsf{Q}(1) \arrow[rr, "g_*(1)" near start, crossing over] \arrow[dd, "q_*(1)"'] & & \mathsf{P}(1) & \\ & \mathsf{Y}(A) \arrow[rr, "f_*(A)"' near start] \arrow[dl, "(\pi_A)_*"] & & \mathsf{X}(A) \arrow[dl, "(\pi_A)_*"] \\ \mathsf{Y}(1) \arrow[rr, "f_*(1)"'] & & \mathsf{X}(1) \arrow[from=uu, "p_*(1)" near start, crossing over] & \end{tikzcd}$$

is an equivalence, given that the mate $\varphi(1)$ of the front square is one. But since the horizontal mate of both the left and the right square is an equivalence, it follows that $\varphi(A)$ is an equivalence when evaluated at any object in the image of π_A^*. Since $\mathsf{P}(A)$ is étale over $\mathsf{P}(1)$, every object in $\mathsf{P}(A)$ is a pullback of objects that are contained in the image of π_A^*. Therefore, the claim follows from the fact that $\varphi(A)$ is a morphism of left exact functors. □

In order to prove Theorem 6.2.1.12, we in particular need to show that compact morphisms are stable under pullback. In fact it will suffice to prove this in a special case (see Corollary 6.2.4.4), which we will turn to now.

LEMMA 6.2.4.2. *Let* $f_*\colon \mathcal{X} \to \mathcal{B}$ *be a geometric morphism of* ∞*-topoi. Suppose we are given a commutative square*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "q_*"] \arrow[d, "g_*"'] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z} \arrow[r, "p_*"'] & \mathcal{B} \end{tikzcd}$$

whose horizontal mate is an equivalence and such that g_* *is compact. Then, for every filtered* $\mathcal{B}$*-category* I*, the functor* $p^*\colon \mathcal{B} \to \mathcal{Z}$ *carries the horizontal mate of the commutative square*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathrm{diag}"] \arrow[d, "f_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, f_*\Omega_{\mathcal{X}}) \arrow[d, "(\Gamma_{f_*\Omega_{\mathcal{X}}})_*"] \\ \mathcal{B} \arrow[r, "\mathrm{diag}"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) \end{tikzcd}$$

to an equivalence.

Proof. Note that we have a commutative diagram of ∞-topoi

$$\begin{tikzcd}
& \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, f_*\Omega_{\mathcal{X}}) \arrow[dd, "(f_*)_*"'] & & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, p_*g_*\Omega_{\mathcal{W}}) \arrow[ll, "(q_*)_*"'] \arrow[dd, "(g_*)_*"] \\
\mathcal{X} \arrow[ur, "\mathrm{diag}"] \arrow[dd, "f_*"'] & & \mathcal{W} \arrow[ll, "q_*"', near start] \arrow[ur, "\mathrm{diag}"] \arrow[dd, "g_*", near start] & \\
& \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}}) & & \mathrm{Fun}_{\mathcal{B}}(\mathsf{I}, p_*\Omega_{\mathcal{Z}}) \arrow[ll, "(p_*)_*", near start] \\
\mathcal{B} \arrow[ur, "\mathrm{diag}"] & & \mathcal{Z} \arrow[ll, "p_*"] \arrow[ur, "\mathrm{diag}"] &
\end{tikzcd}$$

where the horizontal mates of the front and the back square are invertible (the latter using Lemma 6.2.4.1). Furthermore, the adjunction $p^* \dashv p_*$ allows us to identify the right square with

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\mathrm{diag}"] \arrow[d, "g_*"] & \mathrm{Fun}_{\mathcal{Z}}(p^*\mathsf{I}, g_*\Omega_{\mathcal{W}}) \arrow[d, "(g_*)_*"] \\
\mathcal{Z} \arrow[r, "\mathrm{diag}"] & \mathrm{Fun}_{\mathcal{Z}}(p^*\mathsf{I}, \Omega_{\mathcal{Z}})
\end{tikzcd}$$

whose horizontal mate is invertible since g_* was assumed to be compact and $p^*\mathsf{I}$ is filtered. Therefore, the functoriality of mates implies that the functor $p^* \colon \mathcal{B} \to \mathcal{Z}$ carries the horizontal mate of the left square to the mate of the right square, so an equivalence. □

As a consequence of Lemma 6.2.4.2, we obtain that compactness can be checked locally on the base in the following strong sense:

Proposition 6.2.4.3. *Let $f_* \colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. Assume that there exists a family of commutative squares*

$$\begin{tikzcd}
\mathcal{W}_i \arrow[r] \arrow[d, "g^i_*"] & \mathcal{X} \arrow[d, "f_*"] \\
\mathcal{Z}_i \arrow[r, "p^i_*"] & \mathcal{B}
\end{tikzcd}$$

whose mate is an equivalence such that the $(p^i)^*$ are jointly conservative and each g^i_* is compact. Then f_* is compact.

Proof. First, let us verify that for any filtered $\mathcal{B}$-category I the mate of the commutative square

$$\begin{tikzcd}
f_*\Omega_{\mathcal{X}} \arrow[r, "\mathrm{diag}"] \arrow[d, "f_*"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\Omega_{\mathcal{X}}) \arrow[d, "(\Gamma_{f_*\Omega_{\mathcal{X}}})_*"] \\
\Omega_{\mathcal{B}} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}})
\end{tikzcd}$$

is an equivalence. By Lemma 6.2.4.1 it suffices to see this on global sections. Our assumptions guarantee that we can check that the mate is an equivalence after applying $(p^i)^* \colon \mathcal{B} \to \mathcal{Z}_i$ for every i. But then the claim follows from Lemma 6.2.4.2. Now if $A \in \mathcal{B}$ and I is a filtered $\mathcal{B}_{/A}$-category, we observe that the family of squares obtained by pulling back along $(\pi_A)_* \colon \mathcal{B}_{/A} \to \mathcal{B}$ again satisfy the assumptions of the proposition. Thus we can replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ in the first part of the proof and the result follows. □

Corollary 6.2.4.4. *Let $p_* \colon \mathcal{X} \to \mathcal{B}$ be a compact geometric morphism and let C be a $\mathcal{B}$-category. Then the geometric morphism $(\Gamma_{p_*\Omega_{\mathcal{X}}})_* \colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, p_*\Omega_{\mathcal{X}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})$ is again compact.*

Proof. The core inclusion $\iota \colon \mathsf{C}^{\simeq} \hookrightarrow \mathsf{C}$ gives rise to a geometric morphism

$$\iota_* \colon \mathcal{B}_{/\mathsf{C}^{\simeq}} \simeq \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\simeq}, \Omega_{\mathcal{B}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})$$

whose left adjoint is given by restriction along ι and is therefore conservative (which is easily seen using Theorem 2.1.11.5 together with [62, Proposition 4.1.18]). Since in the commutative diagram

$$\begin{tikzcd} \mathcal{X}_{/p^*(\mathsf{C}^\simeq)} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, p_*\Omega_{\mathcal{X}}) \arrow[r, "\lim"] \arrow[d] & \mathcal{X} \arrow[d, "p_*"'] \\ \mathcal{B}_{/\mathsf{C}^\simeq} \arrow[r] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}}) \arrow[r, "\lim"] & \mathcal{B} \end{tikzcd}$$

both squares are pullbacks (the one on the right by Example 5.2.7.5), it follows that the left vertical morphism is compact as an étale base change of a compact morphism. As a consequence, the left square satisfies the assumptions of Proposition 6.2.4.3, which immediately yields the claim. □

Proof of Theorem 6.2.1.12. Suppose that $p_*\colon \mathcal{X} \to \mathcal{B}$ is a compact geometric morphism. First, we show that for any pullback square

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{Z} \arrow[r, "f_*"] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ the mate natural transformation $q_*g^* \to f^*p_*$ is invertible. To see this, we factor the above square as

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "q_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, p_*\Omega_{\mathcal{X}}) \arrow[r, "\lim_{\mathsf{C}^{\mathrm{op}}}"] \arrow[d, "(p_*)_*"] & \mathcal{X} \arrow[d, "p_*"'] \\ \mathcal{Z} \arrow[r, "j_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \Omega_{\mathcal{B}}) \arrow[r, "\lim_{\mathsf{C}^{\mathrm{op}}}"] & \mathcal{B}. \end{tikzcd}$$

(using again Example 5.2.7.5). It is clear that the mate of the right square is an equivalence, hence it suffices to show the claim for the left square. In other words, by Corollary 6.2.4.4 we may reduce to the case where f_* is already fully faithful, which follows from Proposition 6.2.3.1.

To complete the proof, we now have to show that given a second pullback

$$\begin{tikzcd} \mathcal{W}' \arrow[r, "s_*"] \arrow[d, "\bar{q}_*"] & \mathcal{Z}' \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{W} \arrow[r, "r_*"] & \mathcal{Z} \arrow[r, "f_*"] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ the mate of the left square is an equivalence. For this we again use the factorisation from above and consider the diagram

$$\begin{tikzcd} \mathcal{W}' \arrow[r, "s_*"] \arrow[d, "q'_*"] & \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d, "q_*"'] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, p_*\Omega_{\mathcal{X}}) \arrow[d, "(p_*)_*"] \\ \mathcal{W} \arrow[r, "r_*"] & \mathcal{Z} \arrow[r, "j_*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\mathrm{op}}, \Omega_{\mathcal{B}}) \end{tikzcd}$$

By Corollary 6.2.4.4 the geometric morphism $(p_*)_*$ is compact. Together with what we have already shown so far, this implies that both the outer square and the right square is left adjointable. As j'_* is fully faithful it now immediately follows that the left square is also left adjointable, as desired. □

6.2.5. Proper maps in topology. Recall that a map $p\colon Y \to X$ of topological spaces is called *proper* if it is universally closed, and p is called *separated* if the diagonal $Y \to Y \times_X Y$ is a closed embedding. These are the relative versions of compactness and of being Hausdorff, respectively. Our main goal in this section is to prove the following result about proper separated maps:

Theorem 6.2.5.1. *Let $p\colon Y \to X$ be a proper and separated map of topological spaces. Then the induced geometric morphism $p_*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is proper.*

Remark 6.2.5.2. A continuous map $p\colon Y \to X$ is separated as soon as Y is Hausdorff. Since any completely regular topological space is Hausdorff, it follows that Theorem 6.2.5.1 includes [57, Theorem 7.3.1.6].

EXAMPLE 6.2.5.3. It follows from [46, Example C.3.4.1] that the separatedness assumption in Theorem 6.2.5.1 cannot be dropped. We briefly recall the example for the convenience of the reader. Consider the topological space Y that is given by taking two copies of the interval $[0,1]$ and identifying both copies of x for $0 < x < 1$. Then Y is compact, but $\mathrm{Sh}(Y)$ is not. Indeed, consider the sequence that takes $n \in \mathbb{N}$ to the sheaf represented by the map $Y_n \to Y$ in which Y_n is given by two copies of $[0,1]$ where we identify both copies of x for $2^{-n} < x < 1 - 2^{-n}$. We note that all the maps $Y_n \to Y_{n+1}$ and $Y_n \to Y$ are local homeomorphisms, which implies that the colimit of the sheaves represented by $(Y_n \to Y)_{n\in\mathbb{N}}$ is the sheaf represented by $\operatorname{colim}_n Y_n = Y$. In particular, we have $\Gamma_Y(\operatorname{colim}_n Y_n) = 1$, but since $\operatorname{colim}_n \Gamma_Y(Y_n) = \varnothing$, the global sections functor Γ_Y does not commute with filtered colimits.

Before we prove Theorem 6.2.5.1, let us record that it implies the proper base change theorem in topology, at least for sober spaces:

COROLLARY 6.2.5.4. *For every pullback square*

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "q"] & P \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

of sober topological spaces in which p is proper and separated, the induced commutative square

$$\begin{tikzcd} \mathrm{Sh}(Q) \arrow[r, "g_*"] \arrow[d, "q_*"] & \mathrm{Sh}(P) \arrow[d, "p_*"] \\ \mathrm{Sh}(Y) \arrow[r, "f_*"] & \mathrm{Sh}(X) \end{tikzcd}$$

is horizontally left adjointable.

PROOF. Using Theorem 6.2.5.1, it suffices to show that the second square is a pullback in $\mathrm{Top}_\infty^{\mathrm{R}}$, or equivalently that the underlying square of locales is a pullback. The latter fact follows from combining [47, Corollary 3.6] with [47, Lemmas 2.1]. □

Using the results from § 5.3.8 on localic $\mathcal{B}$-topoi arising from maps of topological spaces, the proof of Theorem 6.2.5.1 is now remarkably short:

PROOF OF THEOREM 6.2.5.1. The map p being proper and separated implies that it is also *locally proper* in the sense of Definition 5.3.8.1 (see Remark 5.3.8.3). Therefore, Proposition 5.3.8.5 shows that $\mathsf{O}_X(Y)$ is a *stably compact* $\mathrm{Sh}(X)$*-locale* in the sense of Definition 5.3.7.1. Thus the claim follows from Example 6.2.1.11 and Theorem 6.2.1.12. □

REMARK 6.2.5.5. If one assumes that the topological space Y is *completely regular* (see [57, Definition 7.3.1.12]), one can alternatively apply a number of geometric reduction steps, as in the proof of [57, Theorem 7.3.16], to reduce to the case where $X = *$ and then use that any compact Hausdorff space is a retract of a coherent topological space. This proof strategy for Theorem 6.2.5.1 was explained to us by Ko Aoki. In comparison, Lurie shows that $\mathrm{Sh}(Y)$ is compact in [57, Corollary 7.3.4.12] by using the theory of $\mathcal{K}$-sheaves.

REMARK 6.2.5.6. If $p\colon Y \to X$ is only assumed to be *locally* proper (see Definition 5.3.8.1 for a precise definition), the same argumentation as in the proof of Theorem 6.2.5.1 shows that the $\mathrm{Sh}(X)$-topos $\underline{\mathsf{Sh}}_{\mathrm{Sh}(X)}(\mathsf{O}_X(Y))$ is *compactly assembled.* Therefore, by suitably internalising the arguments in [58, § 21.1.6] (or alternatively those in [3]), one can deduce that p_* is *exponentiable* (i.e. that $- \times_{\mathrm{Sh}(X)} \mathrm{Sh}(Y)\colon \mathrm{Top}_\infty^{\mathrm{R}} \to \mathrm{Top}_\infty^{\mathrm{R}}$ has a right adjoint) and that, as a consequence, the stable ∞-category $\mathrm{Sh}_{\mathrm{Sp}}(Y)$ of sheaves of spectra on Y is a dualisable $\mathrm{Sh}_{\mathrm{Sp}}(X)$-module.

6.2.6. Proper base change with coefficients. The goal of this section is to discuss a generalisation of Theorem 6.2.1.12 where we allow coefficients in an arbitrary compactly generated ∞-category $\mathcal{E}$. The proof is is essentially the same as the one of Theorem 6.2.1.12, however this level of generality allows us to apply the result to a wider range of examples.

DEFINITION 6.2.6.1. Let $\mathcal{E}$ be a presentable ∞-category. Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi. We say that f_* is *$\mathcal{E}$-proper* if for every diagram

$$\begin{tikzcd} \mathcal{W}' \arrow[r, "t'_*"] \arrow[d, "h_*"'] & \mathcal{W} \arrow[r, "t_*"] \arrow[d, "g_*"'] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z}' \arrow[r, "s'_*"'] & \mathcal{Z} \arrow[r, "s_*"'] & \mathcal{B} \end{tikzcd}$$

in $\mathrm{Top}_\infty^{\mathrm{R}}$ in which both squares are pullbacks, the square

$$\begin{tikzcd} \mathcal{W}' \otimes \mathcal{E} \arrow[r, "g_* \otimes \mathcal{E}"] \arrow[d, "p'_* \otimes \mathcal{E}"] & \mathcal{W} \otimes \mathcal{E} \arrow[d, "f_* \otimes \mathcal{E}"] \\ \mathcal{Z}' \otimes \mathcal{E} \arrow[r, "q_* \otimes \mathcal{E}"] & \mathcal{Z} \otimes \mathcal{E} \end{tikzcd}$$

is horizontally left adjointable. Here $- \otimes -\colon \mathrm{Pr}_\infty^{\mathrm{R}} \times \mathrm{Pr}_\infty^{\mathrm{R}} \to \mathrm{Pr}_\infty^{\mathrm{R}}$ denotes Lurie's tensor product of presentable ∞-categories.

There is an natural way to enhance Lurie's tensor products to $\mathcal{B}$-categories, that we will need to formulate our version of compactness with coefficients:

CONSTRUCTION 6.2.6.2. In Construction 2.3.1.1 we constructed a functor $- \otimes \Omega_{\mathcal{B}}\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ that sends a presentable ∞-category $\mathcal{E}$ to the $\mathcal{B}$-category

$$\mathcal{E} \otimes \Omega_{\mathcal{B}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty; \quad A \mapsto \mathcal{E} \otimes \mathcal{B}_{/A}.$$

For a presentable $\mathcal{B}$-category C, we can therefore consider the $\mathcal{B}$-category $\mathsf{C} \otimes \mathcal{E} \coloneqq \mathsf{C} \otimes^{\mathcal{B}} (\mathcal{E} \otimes \Omega_{\mathcal{B}})$. Here $- \otimes^{\mathcal{B}} -$ denotes the tensor product of presentable $\mathcal{B}$-categories introduced in § 4.6.2. In particular $- \otimes \mathcal{E}$ defines a functor $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \to \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$.

REMARK 6.2.6.3. If I is a $\mathcal{B}$-category, C a presentable $\mathcal{B}$-category and $\mathcal{E}$ is a presentable ∞-category, it follows from the explicit description of the tensor product of presentable $\mathcal{B}$-categories Proposition 4.6.2.11 that we have a canonical equivalence $\underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C}) \otimes \mathcal{E} \simeq \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \mathsf{C} \otimes \mathcal{E})$. In combination with Proposition 4.6.3.13, we in particular get an equivalence $\mathsf{C}(A) \otimes \mathcal{E} \simeq (\mathsf{C} \otimes \mathcal{E})(A)$ for every $A \in \mathcal{B}$.

DEFINITION 6.2.6.4. Let $p_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism and $\mathcal{E}$ a presentable ∞-category. Then p_* is called $\mathcal{E}$-proper if $\Gamma_{p_*\Omega_{\mathcal{X}}} \otimes \mathcal{E}\colon p_*\Omega_{\mathcal{X}} \otimes \mathcal{E} \to \Omega_{\mathcal{B}} \otimes \mathcal{E}$ commutes with filtered colimits.

We now come to the main result of this section, the $\mathcal{E}$-linear version of Theorem 6.2.1.12:

THEOREM 6.2.6.5. *Let $p_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism and $\mathcal{E}$ a compactly generated ∞-category. Then p_* is $\mathcal{E}$-proper if and only if it is $\mathcal{E}$-compact.*

REMARK 6.2.6.6. More generally one could define that $p_*\colon \mathcal{X} \to \mathcal{B}$ is E-compact for a presentable $\mathcal{B}$-category E, whenever $p_*\Omega_{\mathcal{X}} \otimes^{\mathcal{B}} \mathsf{E} \to \mathsf{E}$ commutes with filtered colimits. Similarly one can also define a notion of E-properness. Then the analogue of Theorem 6.2.6.5 still holds whenever E is compactly generated (in a suitable $\mathcal{B}$-categorical sense). We decided to only prove the result in the case where $\mathsf{E} = \Omega_{\mathcal{B}} \otimes \mathcal{E}$, since the proof is slightly less technical and since this case already contains most examples of interest.

REMARK 6.2.6.7. Let $\mathcal{E}$ be a compactly generated ∞-category and X a $\mathcal{B}$-topos. Then for any $A \in \mathcal{B}$ we may identify the tensor product $\mathsf{X}(A) \otimes \mathcal{E}$ with the ∞-category $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathsf{X}(A))$ (where $\mathcal{E}^{\mathrm{cpt}} \hookrightarrow \mathcal{E}$ is the full subcategory of compact objects). Furthermore, since for any map $s\colon B \to A$ the transition functors $s^*\colon \mathsf{X}(A) \to \mathsf{X}(B)$ is a left exact left adjoint, it follows that we may identify the transition map $s^* \otimes \mathcal{E}\colon (\mathsf{X} \otimes \mathcal{E})(A) \to (\mathsf{X} \otimes \mathcal{E})(B)$ with the functor

$$\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathsf{X}(A)) \to \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathsf{X}(B))$$

given by postcomposition with s^*. Now let $f_*\colon \mathsf{X} \to \mathsf{Y}$ be a geometric morphism of $\mathcal{B}$-topoi. Since f_* and f^* are both left exact if follows as in [32, Observation 2.9] that the induced morphism $f_* \otimes \mathcal{E}$ is given by pointwise postcomposition with f_* and its left adjoint is given by postcomposition with f^*.

We begin by establish the $\mathcal{E}$-linear analogue of Corollary 6.2.4.4. This requires a few preparations:

Proposition 6.2.6.8. *Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be a geometric morphism of ∞-topoi and let $\mathcal{E}$ be a compactly generated ∞-category. Assume that there exists a family of commutative squares*

$$\begin{tikzcd} \mathcal{W}_i \arrow[r] \arrow[d, "(g^i)_*"'] & \mathcal{X} \arrow[d, "f_*"] \\ \mathcal{Z}_i \arrow[r, "p^i_*"] & \mathcal{B} \end{tikzcd}$$

such that for every $A \in \mathcal{B}$ the functor $(- \times_{\mathcal{B}} \mathcal{B}_{/A}) \otimes \mathcal{E}$ carries these squares to left adjointable squares, the $p_i^* \otimes \mathcal{E}$ are jointly conservative and each (g^i_*) is $\mathcal{E}$-compact. Then f_* is $\mathcal{E}$-compact.

Proof. The proof is essentially the same as the one of Proposition 6.2.4.3. We first check that for every filtered $\mathcal{B}$-category I the mate of the commutative square

$$\begin{tikzcd} f_*\Omega_{\mathcal{X}} \otimes \mathcal{E} \arrow[r, "\mathrm{diag}"] \arrow[d, "f_* \otimes \mathcal{E}"'] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, f_*\Omega_{\mathcal{X}} \otimes \mathcal{E}) \arrow[d, "(\Gamma_{f_*\Omega_{\mathcal{X}}} \otimes \mathcal{E})_*"] \\ \Omega_{\mathcal{B}} \otimes \mathcal{E} \arrow[r, "\mathrm{diag}"] & \underline{\mathsf{Fun}}_{\mathcal{B}}(\mathsf{I}, \Omega_{\mathcal{B}} \otimes \mathcal{E}) \end{tikzcd}$$

is an equivalence. Let us first show that the mate $\mathrm{colim}_{\mathsf{I}}(\Gamma_{f_*\Omega_{\mathcal{X}}} \otimes \mathcal{E})_* \to (f_* \otimes \mathcal{E})\,\mathrm{colim}_{\mathsf{I}}$ is an equivalence after passing to global sections. For this, it suffices to see that the mate is an equivalence after composing with the maps $p_i^* \otimes \mathcal{E}\colon \mathcal{B} \otimes \mathcal{E} \to \mathcal{Z}_i \otimes \mathcal{E}$ for all i. But now the claims follows from an $\mathcal{E}$-linear version of Lemma 6.2.4.2, which is proved in exactly the same way. To see that the mate is an equivalence after evaluating at $A \in \mathcal{B}$ we may replace $\mathcal{B}$ by $\mathcal{B}_{/A}$ and the above square by its base change along π_A^* to reduce to the case treated above. Finally, we have to see that for every $A \in \mathcal{B}$ the functor of $\mathcal{B}_{/A}$-categories $\pi_A^*(f_*\Omega_{\mathcal{X}} \otimes \mathcal{E})\colon \pi_A^*(f_*\Omega_{\mathcal{X}} \otimes \mathcal{E}) \to \pi_A^*(\Omega_{\mathcal{B}} \otimes \mathcal{E})$ commutes with colimits indexed by filtered $\mathcal{B}_{/A}$-categories. But this follows again from the above after replacing $\mathcal{B}$ by $\mathcal{B}_{/A}$. $\square$

Remark 6.2.6.9. Note that in Proposition 6.2.6.8, we require that the assumptions also hold locally on $\mathcal{B}$, while for the version without coefficients (Proposition 6.2.4.3) this was automatic due to Lemma 6.2.4.1. To illustrate why Lemma 6.2.4.1 may fail when using coefficients, consider the example where $\mathcal{E} = \mathrm{Sub}(\mathcal{S}) \simeq \Delta^1$ is the ∞-category of (-1)-truncated spaces. Then, a square

$$\begin{tikzcd} \mathcal{W}_i \arrow[r] \arrow[d, "(g^i)_*"'] & \mathcal{Y} \arrow[d, "f_*"] \\ \mathcal{Z}_i \arrow[r, "p^i_*"] & \mathcal{B} \end{tikzcd}$$

being horizontally left adjointable after tensoring with $\mathrm{Sub}(\mathcal{S})$ simply means that the mate transformation is an equivalence on (-1)-truncated objects in $\mathcal{Y}$. However, after passing to a slice $\mathcal{X}_{/X}$, the mate transformations now involves (-1)-truncated objects in $\mathcal{X}_{/X}$, i.e. subobjects of X. These need not be (-1)-truncated in general, therefore there is no reason for the mate transformation to be an equivalence.

Remark 6.2.6.10. The proof of Proposition 6.2.6.8 shows that more generally we do not need the existence of such squares for every $A \in \mathcal{B}$, but it suffices to find these for a set of objects $A_i \in \mathcal{B}$ that generates $\mathcal{B}$ under colimits.

Lemma 6.2.6.11. *For every $\mathcal{B}$-category C and every geometric morphism $f_*\colon \mathcal{X} \to \mathcal{B}$, the straightening equivalences $\mathrm{Fun}(\mathsf{C}, \Omega_{\mathcal{B}}) \simeq \mathrm{LFib}_{\mathcal{B}}(\mathsf{C})$ and $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*\mathsf{C}) \simeq \mathrm{Fun}_{\mathcal{X}}(f^*\mathsf{C}, \Omega_{\mathcal{X}}) \simeq \mathrm{LFib}_{\mathcal{X}}(f^*\mathsf{C})$ fit into a commutative square*

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{const}_{f_*\Omega_{\mathcal{X}}}"] & \mathrm{LFib}_{\mathcal{B}}(\mathsf{C}) \arrow[d, "f^*"] \\ \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*\Omega_{\mathcal{X}}) \arrow[r, "\simeq"] & \mathrm{LFib}_{\mathcal{X}}(f^*\mathsf{C}) \end{tikzcd}$$

where f^* is the restriction of $f^*\colon \mathrm{Cat}(\mathcal{B})_{/\mathsf{C}} \to \mathrm{Cat}(\mathcal{X})_{/f^*\mathsf{C}}$ to left fibrations. Moreover, this commutative square is natural in C.

Proof. This is shown in exactly the same fashion as [61, Lemma 4.6.4]. $\square$

Corollary 6.2.6.12. *Let $f_*\colon \mathcal{X} \to \mathcal{B}$ be an $\mathcal{E}$-compact geometric morphism and C a $\mathcal{B}$-category. Then the geometric morphism $(\Gamma_{f_*\Omega_{\mathcal{X}}})_*\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*\Omega_{\mathcal{X}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})$ is $\mathcal{E}$-compact.*

Proof. Let $F \in \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})$ be an arbitrary functor and let us set $G = (\mathrm{const}_{f_*\Omega_{\mathcal{X}}})_*(F)$. Furthermore, let $\mathsf{C}_{F/} \to \mathsf{C}$ be the left fibration associated to F via the straightening equivalence. We then deduce from Lemma 6.2.6.11 and the fact that left fibrations satisfy the left cancellation property (being determined by a factorisation system) that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})_{/F} \arrow[r, "\simeq"] \arrow[d, "(\mathrm{const}_{f_*\Omega_{\mathcal{X}}})_*"] & \mathrm{LFib}_{\mathcal{B}}(\mathsf{C})_{/(\mathsf{C}_{F/})} \arrow[r, "\simeq"] \arrow[d, "f^*"] & \mathrm{LFib}_{\mathcal{B}}(\mathsf{C}_{F/}) \arrow[r, "\simeq"] \arrow[d, "f^*"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, \Omega_{\mathcal{B}}) \arrow[d, "(\mathrm{const}_{f_*\Omega_{\mathcal{X}}})_*"] \\ \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*\Omega_{\mathcal{X}})_{/G} \arrow[r, "\simeq"] & \mathrm{LFib}_{\mathcal{X}}(f^*\mathsf{C})_{/f^*(\mathsf{C}_{F/})} \arrow[r, "\simeq"] & \mathrm{LFib}_{\mathcal{X}}(f^*(\mathsf{C}_{F/})) \arrow[r, "\simeq"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, f_*\Omega_{\mathcal{X}}) \end{tikzcd}$$

which is natural in C. Thus, by passing to right adjoints, the base change of $(\Gamma_{f_*\Omega_{\mathcal{X}}})_*$ along the geometric morphism $(\pi_F)_*\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})_{/F} \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}})$ can be identified with with the geometric morphism $(\Gamma_{f_*\Omega_{\mathcal{X}}})\colon \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, f_*\Omega_{\mathcal{X}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, \Omega_{\mathcal{B}})$. Also, the base change of the right adjoint of the restriction functor $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}}) \to \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}^{\simeq}, \Omega_{\mathcal{B}}) \simeq \mathcal{B}_{/\mathsf{C}^{\simeq}}$ along $(\pi_A)_*$ can be identified with the right adjoint of the restriction $\mathrm{Fun}_{\mathcal{B}}(\mathsf{C}_{F/}, \Omega_{\mathcal{B}}) \to \mathcal{B}_{/(\mathsf{C}_{F/})^{\simeq}}$ (using that the pullback of $\mathsf{C}_{F/} \to \mathsf{C}$ along $\mathsf{C}^{\simeq} \to \mathsf{C}$ is $(\mathsf{C}_{F/})^{\simeq}$, see [62, Corollary 4.1.16]). Consequently, we conclude that the pullback square

$$\begin{tikzcd} \mathcal{X}_{/f^*\mathsf{C}^{\simeq}} \arrow[r] \arrow[d] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, f_*\Omega_{\mathcal{X}}) \arrow[d] \\ \mathcal{B}_{/\mathsf{C}^{\simeq}} \arrow[r] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{C}, \Omega_{\mathcal{B}}) \end{tikzcd}$$

satisfies the assumptions of Proposition 6.2.6.8. Thus the claim follows. $\square$

Proof of Theorem 6.2.6.5: By Remark 6.2.6.3, the same proof as in Lemma 6.2.1.13 shows that an $\mathcal{E}$-proper morphism is $\mathcal{E}$-compact. Hence it remains to prove the converse. By Corollary 6.2.6.12, the same reduction steps as in the proof of Theorem 6.2.1.12 imply that it suffices to see that for every pullback square of $\mathcal{B}$-topoi

$$\begin{tikzcd} \mathcal{Z}' \arrow[r, "j'_*"] \arrow[d] & \mathcal{X} \arrow[d, "p_*"] \\ \mathcal{Z} \arrow[r, "j_*"] & \mathcal{B} \end{tikzcd}$$

in which j_* is fully faithful, the square is left adjointable after tensoring with $\mathcal{E}$. By Remark 6.2.6.7, it suffices to see that the square

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \arrow[r, "(j'_*j'^*)_*"] \arrow[d, "(p_*)_*"'] & \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \arrow[d, "(p_*)_*"] \\ \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \arrow[r, "(j_*j^*)_*"'] & \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \end{tikzcd}$$

commutes. We pick a local class S of maps in $\mathcal{B}$ as in Proposition 6.2.3.3, so that we obtain equivalences $j'_*j'^* \simeq (-)^{\mathrm{sh}}_{\iota'}$ and $j_*j^* \simeq (-)^{\mathrm{sh}}_{\iota}$. Now since the inclusions $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \hookrightarrow \mathrm{Fun}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X})$ and $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \hookrightarrow \mathrm{Fun}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})$ preserve filtered colimits and since colimits in functor ∞-categories are computed object-wise, it follows that $(j'_*j'^*)_*$ and $(j_*j^*)_*$ are given by the κ-fold iteration of postcomposition with the functors $(-)^+_{\iota'}$ and $(-)^+_{\iota}$, respectively. Therefore, it suffices to provide an equivalence $(p_*(-)^+_{\iota'})_* \simeq ((-)^+_{\iota}p_*)_*$.

To obtain such an equivalence, note that Remark 6.2.6.3 implies that we may identify the map $\mathrm{colim}_{\Omega_S^{\mathrm{op}}} \otimes \mathcal{E}$ with $\mathrm{colim}_{\Omega_S^{\mathrm{op}}} \colon \underline{\mathsf{Fun}}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, \Omega \otimes \mathcal{E}) \to \Omega \otimes \mathcal{E}$. Therefore, we deduce that postcomposition with $(-)^+_{\iota}$ is equivalently given by the composition

$$\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \to \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{PSh}(\Omega_S)) \simeq \mathrm{Fun}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, \Omega \otimes \mathcal{E}) \xrightarrow{\mathrm{colim}} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})$$

in which the first functor is given by postcomposition with (the global sections of) $\mathrm{map}_{\Omega}(\iota(-), -)$. Similarly, postcomposition with $(-)^+_{\iota'}$ can be identified with the composition

$$\begin{aligned}\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X}) \to \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{Fun}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, p_*\Omega_{\mathcal{X}})) &\simeq \mathrm{Fun}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, p_*\Omega_{\mathcal{X}} \otimes \mathcal{E}) \\ &\xrightarrow{\mathrm{colim}} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{X})\end{aligned}$$

where the first functor is given by postcomposing with (the global sections of) $\underline{\mathrm{Hom}}_{p_*\Omega_{\mathcal{X}}}(\mathrm{const}_{p_*\Omega_{\mathcal{X}}}\,\iota(-), -)$ (since this is precisely the map we obtain when composing $\Gamma_{\mathcal{X}}(\mathrm{map}_{\Omega_{\mathcal{X}}}(\iota'(-), -)) \colon \mathcal{X} \to \mathrm{PSh}_{\mathcal{X}}(p^*\Omega_S)$ with the equivalence $\mathrm{PSh}_{\mathcal{X}}(p^*\Omega_S) \simeq \mathrm{Fun}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, p_*\Omega_{\mathcal{X}})$). Thus, since $\Gamma_{p_*\Omega_{\mathcal{X}}} \otimes \mathcal{E}$ commutes with $\mathrm{colim}_{\Omega_S^{\mathrm{op}}}$, it is enough to provide a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\underline{\mathrm{Hom}}_{p_*\Omega_{\mathcal{X}}}(\mathrm{const}_{p_*\Omega_{\mathcal{X}}}\,\iota(-),-)"] \arrow[d, "p_*"] & \mathrm{Fun}_{\mathcal{B}}(\Omega_S^{\mathrm{op}}, p_*\Omega_{\mathcal{X}}) \arrow[d, "(\Gamma_{p_*\Omega_{\mathcal{X}}})_*"] \\ \mathcal{B} \arrow[r, "\mathrm{map}_{\Omega_{\mathcal{X}}}(\iota(-),-)"] & \mathrm{PSh}_{\mathcal{B}}(\Omega_S), \end{tikzcd}$$

which is evident from Remark 5.2.10.13. □

Example 6.2.6.13. For a scheme X let us denote by $X^{\mathrm{hyp}}_{\text{ét}}$ the ∞-topos of étale hypersheaves of spaces on X. If $f \colon X \to S$ is a proper morphism of schemes, then the geometric morphism $f_* \colon X^{\mathrm{hyp}}_{\text{ét}} \to S^{\mathrm{hyp}}_{\text{ét}}$ is $\mathbf{D}(R)$-proper for any torsion ring R. In fact, since $X^{\mathrm{hyp}}_{\text{ét}}$ has enough points by [58, Theorem A.4.0.5], the family of all points $\bar{s}_* \colon \mathcal{S} \to X^{\mathrm{hyp}}_{\text{ét}}$ yields a family of jointly conservative functors $\bar{s}^* \otimes \mathbf{D}(R)$. Furthermore, proper base change for unbounded derived categories of étale sheaves (see [18, Theorem 1.2.1]) implies that the squares

$$\begin{tikzcd} X^{\mathrm{hyp}}_{\bar{s},\text{ét}} \arrow[r] \arrow[d] & X^{\mathrm{hyp}}_{\text{ét}} \arrow[d] \\ \mathcal{S} \arrow[r] & S^{\mathrm{hyp}}_{\text{ét}} \end{tikzcd}$$

are left adjointable after applying $- \otimes \mathbf{D}(R)$. Finally, [18, Corollary 1.1.15] implies that $X^{\mathrm{hyp}}_{\bar{s},\text{ét}}$ is $\mathbf{D}(R)$-compact so that we may apply Proposition 6.2.6.8 and Theorem 6.2.6.5 to conclude that f_* is $\mathbf{D}(R)$-proper.

Definition 6.2.6.14. We call a geometric morphism $f_* \colon \mathcal{Y} \to \mathcal{X}$ n-proper if it is $\mathcal{S}_{\leq n}$-proper, where $\mathcal{S}_{\leq n}$ denotes the ∞-category of n-truncated spaces. We call f_* almost proper if it is n-proper for all n.

Example 6.2.6.15. Recall that by [56, Example 4.8.1.22] one may identify $\mathcal{X} \otimes \mathcal{S}_{\leq n} \simeq \mathcal{X}_{\leq n}$. Thus it follows from [58, Proposition A.2.3.1] and Theorem 6.2.6.5 that for an n-coherent ∞-topos $\mathcal{X}$ the geometric morphism $\Gamma_* \colon \mathcal{X} \to \mathcal{S}$ is n-proper. In particular it is almost proper if $\mathcal{X}$ is coherent. However it is not proper in general (see Remark 6.2.1.8).

Example 6.2.6.16. A geometric morphism $f_* \colon \mathcal{X} \to \mathcal{B}$ is Set-proper if and only if the underlying morphism of 1-topoi is tidy in the sense of [69, §3].

CHAPTER 7

Applications in étale homotopy theory

The goal of this chapter is to present some applications of our results in algebraic geometry and more specifically in *étale homotopy theory.* For this we will mostly work internally to the ∞-topos of *condensed* or *pyknotic* spaces. In § 7.1, we begin this chapter by recalling the definition of pyknotic spaces and pyknotic objects from [20],[10]. We also review a number of useful results about how to embed pro-finite spaces or categories into pyknotic spaces/categories from [9]. Furthermore we give a brief recollection on spectral ∞-topoi, that were also developed in [9].

In § 7.2 we prove that the ∞-topos of pro-étale hypersheaves on a qcqs scheme X, as introduced by Bhatt-Scholze in [14], is equivalent to a presheaf topos when considered as $\mathrm{Pyk}(\mathcal{S})$-category. The main ingredients for our proof are the *Exodromy equivalence* for constructible étale sheaves from [9] and some of the machinery developed in the previous chapters of this thesis.

Finally in § 7.3 we use the results of § 7.2 to introduce the pro-étale homotopy type of scheme, which is a pyknotic refinement of the usual étale homotopy type. We prove an internal version of Quillen's Theorem B which we then employ to understand fibres of maps between pro-étale homotopy types of schemes.

7.1. Preliminaries on pyknotic objects and spectral ∞-topoi

7.1.1. Background on pyknotic objects. In this section we recall a few basics and notations about *condensed* or *pyknotic* mathematics, introduced by Clausen-Scholze in [20] and Barwick-Haine in [10]. The only real difference between the two approaches is the way in which set-theoretic issues are dealt with. Since we will directly build on the results in [9] which are all formulated in the framework of [10], we will do the same in this chapter.

Notation 7.1.1.1. We fix a *tiny* and a *small* universe, respectively determined by two strongly inaccessible cardinals

$$\delta_0 < \delta_1.$$

For a strongly inaccessible cardinal δ, we write $\mathcal{S}_\delta$) for the ∞-category of δ-small spaces. By default all higher categorical notions are taken to be with respect to the cardinal δ_1. For example a δ-∞-topos is a left exact accessible localization of $\mathrm{Fun}(\mathcal{C}, \mathcal{S}_\delta)$ for a δ-small ∞-category $\mathcal{C}$. We will simply say ∞-topos instead of δ_1-∞-topos.

Definition 7.1.1.2. Let $\mathrm{Pro}^{\delta_0}(\mathrm{Set}^{\mathrm{fin}})$ denote the category of δ_0-small pro-finite sets. We equip this category with the Grothendieck-topology eff generated by finite families of jointly surjective maps. We define the ∞-topos of *pyknotic spaces* to be the ∞-category $\mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}^{\delta_0}(\mathrm{Set}^{\mathrm{fin}}), \mathcal{S}_{\delta_1})$ of hypersheaves on $\mathrm{Pro}^{\delta_0}(\mathrm{Set}^{\mathrm{fin}})$.

Observation 7.1.1.3 ([10, § 2.2]). Let us denote by Proj the full subcategory of those (δ_0-small) pro-finite sets K that have the property that any surjection $T \to K$ from another pro-finite set has a section. We call the pro-finite sets in Proj *projective.* We equip the category Proj with the topology τ, generated by coverings of the form $\{U \hookrightarrow U \amalg V, V \hookrightarrow U \amalg V\}$. Since the Stone-Čech compactification of a discrete set is projective, it follows that any pro-finite set admits a surjection from an extremally disconnected (δ_0-small) pro-finite set. Thus the ∞-categorical comparison lemma for hypersheaves (see e.g. [73, Proposition 2.22]) implies that we have an equivalence

$$\mathrm{Pyk}(\mathcal{S}) \simeq \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathrm{Proj}, \mathcal{S}) \simeq \mathrm{Fun}^\times(\mathrm{Proj}^{\mathrm{op}}, \mathcal{S}).$$

Here $\mathrm{Fun}^\times(-,-)$ denotes the ∞-category of product preserving functors. In particular $\mathrm{Pyk}(\mathcal{S})$ is projectively generated.

Definition 7.1.1.4. Let $\mathcal{C}$ be any ∞-category with finite products. Then we define the category of *pyknotic objects in* $\mathcal{C}$

$$\mathrm{Pyk}(\mathcal{C}) = \mathrm{Fun}^\times(\mathrm{Proj}^{\mathrm{op}}, \mathcal{C}).$$

Remark 7.1.1.5. Note that we evidently have a canonical equivalence $\mathrm{Pyk}(\mathrm{Cat}_\infty) \simeq \mathrm{Cat}(\mathrm{Pyk}(\mathcal{S}))$ between pyknotic ∞-categories and $\mathrm{Pyk}(\mathcal{S})$-categories.

Notation 7.1.1.6. As for any ∞-topos, we get a canonical adjunction $\mathcal{S} \rightleftarrows \mathrm{Pyk}(\mathcal{S})$. We denote the left adjoint by $(-)^{\mathrm{disc}}$ and the right adjoint by Un. More generally for any presentable ∞-category $\mathcal{C}$, applying $- \otimes \mathcal{C}$ yields an adjunction

$$(-)^{\mathrm{disc}} \colon \mathcal{C} \underset{\perp}{\rightleftarrows} \mathrm{Pyk}(\mathcal{C}) : \mathrm{Un}\,.$$

Definition 7.1.1.7. The left exact functor $(-)^{\mathrm{disc}} \colon \mathcal{S} \to \mathrm{Pyk}(\mathcal{S})$ extends to a right adjoint functor

$$\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pyk}(\mathcal{S})$$

whose left adjoint we denote by H. For $X \in \mathrm{Pyk}(\mathcal{S})$, we call $H(X) \in \mathrm{Pro}(\mathcal{S})$ the *homotopy type* of X.

Notation 7.1.1.8. We write $\mathcal{S}_{<\infty} \subset \mathcal{S}$ for the full subcategory spanned by those spaces which are n-truncated for some n. We write $\mathcal{S}_\pi$ for the full subcategory of $\mathcal{S}_{<\infty}$ spanned by those spaces whose homotopy sets are all finite. If Σ is a set of primes, we write $\mathcal{S}_\Sigma$ for the full subcategory of $\mathcal{S}_\pi$ spanned by the spaces all of whose homotopy groups are finite Σ-groups (i.e. whose order is a product of powers of primes in Σ).

7.1.1.9. The inclusion $\iota \colon \mathcal{S}_{<\infty} \to \mathcal{S}$ induces an adjunction

$$\tau_{<\infty} : \mathrm{Pro}(\mathcal{S}) \rightleftarrows \mathrm{Pro}(\mathcal{S}_{<\infty}) : \mathrm{Pro}(\iota).$$

We call the right adjoint $\tau_{<\infty}$ the *pro-truncation functor*. Similarly for a set of primes Σ, the inclusion $\mathcal{S}_\Sigma \to \mathcal{S}$ yields an adjunction

$$(-)^\wedge_\Sigma \colon \mathrm{Pro}(\mathcal{S}) \rightleftarrows \mathrm{Pro}(\mathcal{S}_\Sigma).$$

We call $(-)^\wedge_\Sigma$ the *pro-Σ-completion* functor. In the case where Σ is the set of all primes, we write $\mathrm{Pro}(\mathcal{S}_\pi) = \mathcal{S}_\pi^\wedge$ and $(-)^\wedge_\Sigma = (-)^\wedge_\pi$ and say *pro-finite completion functor*.

Definition 7.1.1.10. Composing the adjunction in Definition 7.1.1.7, with the above adjunctions we obtain adjunctions

$$H_{<\infty} \colon \mathrm{Pyk}(\mathcal{S}) \rightleftarrows \mathrm{Pro}(\mathcal{S}_{<\infty}) \quad \text{and} \quad H^\wedge_\Sigma \colon \mathrm{Pyk}(\mathcal{S}) \rightleftarrows \mathrm{Pro}(\mathcal{S}_\Sigma).$$

For $X \in \mathrm{Pyk}(\mathcal{S})$ we call $H_{<\infty}(X)$ the *pro-truncated homotopy of X* and $H^\wedge_\Sigma(X)$ the *pro-Σ-completed homotopy type of X*. If Σ is the set of all primes we write $H^\wedge_\Sigma = H^\wedge_\pi$ and call $H^\wedge_\pi(X)$ the *pro-finite homotopy type of X*.

We will also the use the following result later:

Theorem 7.1.1.11 ([9, §13.4]). *Extending the functor $(-)^{\mathrm{disc}} \colon \mathcal{S}_\pi \to \mathrm{Pyk}(\mathcal{S})$ to pro-objects gives a fully faithful functor*

$$\mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pyk}(\mathcal{S}).$$

Recollection 7.1.1.12. Recall from [9, Definition 2.3.7] that an ∞-category $\mathcal{C}$ is called *layered*, if every endomorphism in $\mathcal{C}$ is an equivalence. An ∞-category $\mathcal{C}$ is called *π-finite* if it has finitely many objects up to equivalence and all its mapping spaces are π-finite. We write Lay_π for the full subcategory spanned by the π-finite and layered ∞-categories.

Also recall that a π-finite *stratified space* is a π-finite ∞-category Π together with a conservative functor $\Pi \to P$ to a finite poset P. In particular Π is layered. We write Str_π for the full subcategory of $\mathrm{Cat}_\infty^{\Delta^1}$ spanned by the π-finite stratified spaces. Note that evaluating at 0 induces an evident functor $\mathrm{Str}_\pi \to \mathrm{Lay}_\pi$.

If $\mathcal{C}$ is a π-finite layered ∞-category, it follows that $\mathcal{C}^{\simeq} \in \mathcal{S}_\pi$. Since $\mathcal{C}^{\Delta^1}$ is again π-finite and layered, it follows that the nerve functor $N\colon \mathrm{Cat}_\infty \to \mathrm{CSS}(\mathcal{S})$ restricts to a fully faithful functor

$$N\colon \mathrm{Lay}_\pi \to \mathrm{CSS}(\mathcal{S}_\pi).$$

We thus have the following Corollary of Theorem 7.1.1.11:

Corollary 7.1.1.13 ([9, §13.5]). *Extending the functor* $(-)^{\mathrm{disc}}\colon \mathrm{Lay}_\pi \to \mathrm{Pyk}(\mathrm{Cat})$ *to pro-objects, yields a fully faithful functor*

$$\mathrm{Pro}(\mathrm{Lay}_\pi) \to \mathrm{Pyk}(\mathrm{Cat}_\infty).$$

Proof. The functor factors as

$$\mathrm{Pro}(\mathrm{Lay}_\pi) \xrightarrow{\mathrm{Pro}(N(-))} \mathrm{Pro}(\mathrm{CSS}(\mathcal{S}_\pi)) \to \mathrm{CSS}(\mathcal{S}_\pi^\wedge) \to \mathrm{CSS}(\mathrm{Pyk}(\mathcal{S})) = \mathrm{Pyk}(\mathrm{Cat}_\infty),$$

where the second functor is the extension of the inclusion $\mathrm{CSS}(\mathcal{S}_\pi) \hookrightarrow \mathrm{CSS}(\mathcal{S}_\pi^\wedge)$ to pro-objects. By Theorem 7.1.1.11, the third functor is fully faithful and thus the claim follows because the second functor also is by [9, p. 13.1.3]. $\square$

We also recall the following useful observation for later:

Lemma 7.1.1.14. *The functor* $\lim\colon \mathrm{Pro}(\mathrm{Lay}_\pi) \to \mathrm{Cat}_\infty$ *is conservative.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Pro}(\mathrm{Lay}_\pi) \arrow[r, "\lim"] \arrow[d] & \mathrm{Cat}_\infty \arrow[d, "\simeq"] \\
\mathrm{CSS}(\mathcal{S}_\pi^\wedge) \arrow[r, "\lim_*"'] & \mathrm{CSS}(\mathcal{S})
\end{tikzcd}$$

and thus the claim follows from the fact that $\lim\colon \mathcal{S}_\pi^\wedge \to \mathcal{S}$ is conservative by [58, Theorem E.3.1.6] $\square$

7.1.2. Background on spectral ∞-topoi. In this section we briefly recall the notion of a *spectral* ∞-topos from [9].

Recollection 7.1.2.1. Recall from [58, Definition A.6.1.1] that an ∞-category $\mathcal{X}_0$ is called an *∞-pretopos* if

(1) The ∞-category $\mathcal{X}_0$ has finite limits.
(2) Finite coproducts exist in $\mathcal{X}_0$ and are universal and disjoint.
(3) Groupoid objects in $\mathcal{X}_0$ are effective and their geometric realizations are universal.

An ∞-pretopos $\mathcal{X}_0$ is called *bounded* if $\mathcal{X}_0$ is small and every object in $\mathcal{X}_0$ is n-truncated for some integer n. If $\mathcal{X}$ is a *coherent* ∞-topos in the sense of [58, Definition A.2.0.12], the full subcategory of $\mathcal{X}_{<\infty}^{\mathrm{coh}}$ spanned by the truncated and coherent objects is a bounded ∞-pretopos [58, Example 7.4.4]. In fact any bounded ∞-pretopos arises this way by [58, Theorem 7.5.3]. Conversely, any bounded coherent ∞-topos $\mathcal{X}$ is of the form $\mathrm{Sh}_{\mathrm{eff}}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is some bounded ∞-pretopos and eff is the topology generated by finite families $U_i \to U$ such that $\amalg_i U_i \to U$ is an effective epimorphism. Furthermore in this case we have

$$\mathrm{Sh}_{\mathrm{eff}}(\mathcal{X}_0)_{<\infty}^{\mathrm{coh}} \simeq \mathcal{X}_0.$$

Also recall that a geometric morphism $f_*\colon \mathcal{X} \to \mathcal{Y}$ of coherent ∞-topoi is called *coherent* if f^* sends Y^{coh} into $\mathcal{X}^{\mathrm{coh}}$. We denote the ∞-category of tiny bounded coherent ∞-topoi and coherent geometric morphisms between them by $\mathrm{Top}_\infty^{\mathrm{R,coh}}$. For more background on ∞-pretopoi and coherent ∞-topoi, the reader may consult [58, Appendix A] or [9, §3].

EXAMPLE 7.1.2.2. If X is a qcqs scheme, the ∞-topos $X_{\text{ét}}$ of étale sheaves on X is bounded coherent.

EXAMPLE 7.1.2.3. If $\mathcal{X}$ is a coherent ∞-topos, the terminal geometric morphism $\Gamma\colon \mathcal{X} \to \mathcal{S}$ is coherent. To see this recall from [58, Example A.2.17] that an object $A \in \mathcal{S}$ is coherent if and only if all its homotopy sets are finite. By [58, Lemma E.1.6.5] it follows that A is equivalent to the geometric realization of a Kan-complex $A_\bullet$ where all A_n are finite sets. If follows that $\operatorname{const} A \simeq |\operatorname{const} A_\bullet|$. Since the full subcategory $\mathcal{X}^{\mathrm{coh}} \subseteq \mathcal{X}$ is closed under coproducts and geometric realizations of groupoid objects (see the proof of [56, Proposition A.6.1.6]) the claim follows.

DEFINITION 7.1.2.4. ([9, Definition 9.2.1]) Let S be a spectral topological space. An *S-stratified ∞-topos* is a bounded coherent ∞-topos $\mathcal{X}$, together with coherent geometric morphism $\mathcal{X} \to \mathrm{Sh}(S)$. We will say that a S-stratified ∞-topos $\mathcal{X}$ is *spectral*, if the induced functor $\mathrm{Pt}(\mathcal{X}) \to \mathrm{Pt}(\mathrm{Sh}(S))$ on ∞-categories of points is conservative. We define the ∞-category of spectral ∞-topoi $\operatorname{Spec}\mathrm{Top}_\infty^{\mathrm{R}}$ to be the full subcategory of $(\mathrm{Top}_\infty^{\mathrm{R,coh}})^{\Delta^1}$ spanned by the spectral ∞-topoi.

EXAMPLE 7.1.2.5. If X is a qcqs scheme, the ∞-topos $X_{\text{ét}}$ together with it's canonical geometric morphism $X_{\text{ét}} \to \mathrm{Sh}(X_{\mathrm{Zar}})$ is spectral [9, Example 9.2.4]. In this case the ∞-category $X^{\mathrm{coh}}_{\text{ét},<\infty}$ of truncated coherent objects is equivalent to the ∞-category $X^{\mathrm{constr}}_{\text{ét}}$ of constructible étale sheaves on X, which is therefore an ∞-pretopos. In fact we more generally have an equivalence

$$\mathcal{X}^{\mathrm{coh}}_{<\infty} \simeq \mathcal{X}^{S\text{-constr}}$$

for any spectral S-stratified ∞-topos $\mathcal{X} \to \mathrm{Sh}(S)$ [9, Corollary 9.5.5].

Sending a π-finite stratified space $\Pi \to P$ to the spectral ∞-topos $\mathrm{Fun}(\Pi, \mathcal{S}) \to \mathrm{Fun}(P, \mathcal{S})$ defines a functor

$$\lambda\colon\ \mathrm{Str}_\pi \to (\mathrm{Top}_\infty^{\mathrm{R,coh}})^{\Delta^1}.$$

Extending this functor to pro-objects, we obtain a functor $\lambda^\wedge\colon\ \mathrm{Pro}(\mathrm{Str}_\pi) \to (\mathrm{Top}_\infty^{\mathrm{R,coh}})^{\Delta^1}$.

THEOREM 7.1.2.6 ([9, Theorem 9.3.1]). *The functor $\lambda^\wedge\colon\ \mathrm{Pro}(\mathrm{Str}_\pi) \to (\mathrm{Top}_\infty^{\mathrm{R,coh}})^{\Delta^1}$ is fully faithful and the essential image is given by the full subcategory $\operatorname{Spec}\mathrm{Top}_\infty^{\mathrm{R}}$ of the spectral ∞-topoi.*

DEFINITION 7.1.2.7 ([9, § 10.1]). Let $\mathcal{X}$ be a spectral S-stratified ∞-topos. We denote the inverse under the equivalence of Theorem 7.1.2.6 by $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}) = \lambda^{-1}(\mathcal{X})$. We call $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ the *pro-finite stratified shape* of $\mathcal{X}$. If X is a qcqs scheme, we simply write $\mathrm{Gal}(X) = \hat{\Pi}^{X_{\mathrm{Zar}}}_{(\infty,1)}(X_{\text{ét}})$ and call $\mathrm{Gal}(X)$ the *Galois-category* of X.

Via the composite $\mathrm{Pro}(\mathrm{Str}_\pi) \to \mathrm{Pro}(\mathrm{Lay}_\pi) \hookrightarrow \mathrm{Pyk}(\mathrm{Cat}_\infty)$, we may consider $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ as a condensed $\mathrm{Pyk}(\mathcal{S})$-category. Theorem 7.1.2.6 allows us to give an explicit description of this $\mathrm{Pyk}(\mathcal{S})$-category.

PROPOSITION 7.1.2.8. *The $\mathrm{Pyk}(\mathcal{S})$-category underlying $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ is given by the sheaf*

$$\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})\colon\ \mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}}) \to \mathrm{Cat}_\infty, \quad K \mapsto \mathrm{Fun}^{\mathrm{coh}}_*(\mathrm{Sh}(K), \mathcal{X}).$$

Here $\mathrm{Fun}^{\mathrm{coh}}_*(\mathrm{Sh}(K), \mathcal{X})$ denotes the full subcategory of $\mathrm{Fun}_*(\mathrm{Sh}(K), \mathcal{X})$ spanned by the coherent *geometric morphisms, i.e. those geometric morphism $f_*\colon\ \mathrm{Sh}(K) \to \mathcal{X}$ whose left adjoint f^* preserves coherent objects*

PROOF. For a pro-finite set K, we may regard $\mathrm{Sh}(K)$ as a K-stratified spectral ∞-topos via the identity. By Theorem 7.1.2.6, we obtain a chain of natural equivalences

$$\begin{aligned}\mathrm{map}_{\mathrm{Pro}(\mathrm{Cat}_\infty)}(K, \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})) \simeq \mathrm{map}_{\mathrm{Str}_\pi}(K, \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})) &\simeq \mathrm{map}_{\mathrm{Spec\,Top}^{\mathrm{R}}_\infty}(\mathrm{Sh}(K), \mathcal{X}) \\ &\simeq \mathrm{map}_{\mathrm{Top}^{\mathrm{R,coh}}_\infty}(\mathrm{Sh}(K), \mathcal{X}).\end{aligned}$$

Furthermore replacing K with $K \times \Delta^n$ and using the naturality of the above equivalence in the first variable we obtain a natural equivalence

$$\mathrm{Fun}_{\mathrm{Pro}(\mathrm{Cat}_\infty)}(K, \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})) \simeq \mathrm{Fun}^{\mathrm{coh}}_*(\mathrm{Sh}(K), \mathcal{X})$$

of ∞-categories. Since the composite $\mathrm{Pro}(\mathrm{Str}_\pi) \to \mathrm{Pro}(\mathrm{Lay}_\pi) \hookrightarrow \mathrm{Pyk}(\mathrm{Cat}_\infty)$ by construction sends $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ to the sheaf

$$\mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}}) \to \mathrm{Cat}_\infty, K \mapsto \mathrm{Fun}_{\mathrm{Pro}(\mathrm{Cat}_\infty)}(K, \hat{\Pi}^S_{(\infty,1)}(\mathcal{X}))$$

the claim follows. □

REMARK 7.1.2.9. As a consequence of [9, Proposition 9.5.7] it follows that for a spectral ∞-topos $\mathcal{X}$, any point $x_*\colon S \to \mathcal{X}$ is automatically coherent. In particular the underlying ∞-category of $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ is simply the category $\mathrm{Pt}(\mathcal{X})$ of points of $\mathcal{X}$ by Proposition 7.1.2.8. If X is a qcqs scheme the category of points $\mathrm{Pt}(X_{\text{ét}})$ admits the following explicit description [15, Exposé VIII, Théorème 7.9]. The objects are given by geometric points of X, so morphisms of schemes $\bar{x} \to X$, where $\bar{x} = \mathrm{Spec}(\bar{k})$ for some separably closed field $\bar{x}$. A morphism between two geometric points $\bar{x}$ and $\bar{y}$ is given by an étale specialization $\bar{y} \to \bar{x}$, so by a morphism of X-schemes

$$X_{(\bar{y})} \to X_{(\bar{x})}.$$

Here we denote by $X_{(\bar{x})} = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{x}})$ the strict henselisation of Y at $\bar{y}$ (see [Stacks, Tag 0BSK]).

7.2. The pro-étale topos as pyknotic presheaves

The main goal of this section is to prove the following theorem:

THEOREM 7.2.0.1. *Let X be a qcqs scheme. Then the exodromy equivalence induces an equivalence of ∞-topoi*

$$X^{\mathrm{hyp}}_{\text{proét}} \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S})).$$

In fact we will prove a more general result for the *pyknotification* $\mathcal{X}^{\mathrm{pyk}} = \mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty}))$ of a spectral ∞-topos $\mathcal{X}$. We introduce the pyknotification of a coherent ∞-topos in § 7.2.1. The two crucial examples to keep in mind are that $\mathcal{S}^{\mathrm{pyk}} = \mathrm{Pyk}(\mathcal{S})$ and $X^{\mathrm{pyk}}_{\text{ét}} = X^{\mathrm{hyp}}_{\text{proét}}$, see Examples 7.2.1.8 and 7.2.1.9.

Let us now briefly describe our strategy to prove Theorem 7.2.0.1. By the Exodromy Theorem [9, Theorem 13.2.10] we have that $(X_{\text{ét}})^{\mathrm{coh}}_{<\infty} \simeq \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathcal{S}^{\mathrm{disc}}_\pi)$. And via the fully faithful embedding $\mathcal{S}_\pi \to \mathrm{Pyk}(\mathcal{S})$ we thus obtain an embedding

$$(X_{\text{ét}})^{\mathrm{coh}}_{<\infty} \to \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S})).$$

As a next step we identify the essential image of this functor in terms of the corresponding left fibrations of $\mathrm{Pyk}(\mathcal{S})$-categories and conclude that the above functor remains fully faithful after extending it to pro-objects. This is the content of § 7.2.2.

By a general ∞-toposic principle it suffices to show that any object in $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S}))$ admits an effective epimorphism form a coproduct of objects in $\mathrm{Pro}((X_{\text{ét}})^{\mathrm{coh}}_{<\infty})$. Since the ∞-category $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S}))$ is a presheaf category and $\mathrm{Pyk}(\mathcal{S})$ is generated under colimits by pro-finite sets it follows that any object admits admits an effective epimorphism from a coproduct of objects that are colimits of diagrams of the form $K \to \mathrm{Gal}(X)$, where K is some pro-finite set. Thus it remains to see that these objects are contained in $\mathrm{Pro}((X_{\text{ét}})^{\mathrm{coh}}_{<\infty})$, which we prove in § 7.2.3.

The contents of this section originally appeared in [91]. However, using results from internal higher category theory some parts of the proof are now simplified. Furthermore we use the tensor product of presentable $\mathrm{Pyk}(\mathcal{S})$-categories, developed in § 4.6 to extend the equivalence of Theorem 7.2.0.1 to more general coefficients.

7.2.1. Pro-objects and Pyknotification. If $\mathcal{X}_0$ is an ∞-pretopos, the ∞-category $\mathrm{Pro}(\mathcal{X}_0)$ is in general not an ∞-pretopos. The goal of this section is to show that even though this is case, the notion of effective epimorphism still yields a reasonable Grothendieck-topology on $\mathrm{Pro}(\mathcal{X}_0)$. The majority of the material presented here has been worked out in the 1-categorical case by Lurie in [59, §6.1] and most of the arguments in this section are just very straight-forward adaptions of the ones presented there.

LEMMA 7.2.1.1. *Let $\mathcal{X}_0$ be a tiny bounded ∞-pretopos. Let $f_\bullet\colon I \to \mathrm{Fun}(\Delta^1, \mathrm{Pro}(\mathcal{X}_0))$ be a tiny cofiltered diagram of effective epimorphisms. Then $\lim_i f_i$, considered as a morphism in $\mathrm{Pro}(\mathcal{X}_0)$, is an effective epimorphism.*

PROOF. This is a straight-forward adaption of the argument given in [58, Prop. E.5.5.3]: Let us denote the source and target of $f_\bullet$ by $X_\bullet$ and $Y_\bullet$, respectively. Let us write $U_\bullet$ for the Čech-nerve of $f = \lim_i f_i$. We would like to show that, for every $C \in \mathrm{Pro}(\mathcal{X}_0)$, the induced morphism

$$\mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(\lim_i X_i, C) \longrightarrow \lim_{n\in\mathbf{\Delta}} \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_n, C)$$

is an equivalence. We observe that we may immediately assume that $C \in \mathcal{X}_0 \subseteq \mathrm{Pro}(\mathcal{X}_0)$. We now write $U_{\bullet,i}$ for the Čech-nerve of f_i. Since we assumed that C is cocompact, the above map may be identified with the composite

$$\begin{aligned}\operatorname*{colim}_i \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(X_i, C) &\longrightarrow \operatorname*{colim}_i \lim_{n\in\mathbf{\Delta}} \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_{i,n}, C)\\ &\xrightarrow{\alpha} \lim_{n\in\mathbf{\Delta}} \operatorname*{colim}_i \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_{i,n}, C).\end{aligned}$$

Since every f_i was assumed to be an effective epimorphism, the first map is an equivalence. Thus it suffices to see that α is an equivalence. Now, since $\mathcal{X}_0$ is bounded, there is an $n \in \mathbb{N}$ such that C is n-truncated. We may thus replace the $U_{i,n}$ with their n-truncations $\tau_{\leq n}(U_{i,n})$. It follows from [42, Proposition A.1] that in the commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_i \lim_{n\in\mathbf{\Delta}} \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_{i,n}, C) \arrow[r, "\alpha"] \arrow[d] & \lim_{n\in\mathbf{\Delta}} \operatorname{colim}_i \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_{i,n}, C) \arrow[d] \\
\operatorname{colim}_i \lim_{n\in\mathbf{\Delta}^{\leq n}} \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_{i,n}, C) \arrow[r] & \lim_{n\in\mathbf{\Delta}^{\leq n}} \operatorname{colim}_i \mathrm{map}_{\mathrm{Pro}(\mathcal{X}_0)}(U_{i,n}, C)
\end{tikzcd}$$

the horizontal arrows are equivalences and the bottom vertical arrow is an equivalence as well since taking limits over $\Delta^{\leq n}$ commutes with filtered colimits. This completes the proof. $\square$

Let us quickly recall the following from [58, Proposition A.6.2.1]:

PROPOSITION 7.2.1.2. *Let $\mathcal{X}_0$ be an ∞-pretopos. Then $\mathcal{X}_0$ admits a factorization system (S_L, S_R) (in the sense of [57, §5.2.8]), where S_L is the collection of effective epimorphisms and S_R the collection of (-1)-truncated morphisms.*

PROPOSITION 7.2.1.3. *Let $\mathcal{X}_0$ be a tiny bounded ∞-pretopos. Then the following hold:*

(1) The collection of effective epimorphisms and (-1)-truncated morphisms form a factorization system on $\mathrm{Pro}(\mathcal{X}_0)$.

(2) A morphism in $\mathrm{Pro}(\mathcal{X}_0)$ is an effective epimorphism if and only if it can be written as a tiny inverse limit of effective epimorphisms in $\mathcal{X}_0$.

(3) Effective epimorphisms are stable under pullback in $\mathrm{Pro}(\mathcal{X}_0)$.

PROOF. It is clear that effective epimorphisms and (-1)-truncated morphisms are stable under retracts. Furthermore we observe that effective epimorphisms are left orthogonal to (-1)-truncated morphisms in any ∞-category with finite limits and geometric realizations. So let $f\colon X \to Z$ be a morphism in $\mathrm{Pro}(\mathcal{X}_0)$. By (the dual of) [58, Proposition E.4.2.2] we can write $f \simeq \lim_i h_i \circ \lim_j g_j$, where

h_i is a (-1)-truncated morphism in $\mathcal{X}_0$ and g_j is an effective epimorphism in $\mathcal{X}_0$. By Lemma 7.2.1.1 it follows that $\lim_i g_i$ is an effective epimorphism. Furthermore the diagonal $\lim_i Y_i \to \lim_i Y_i \times_{\lim_i Z_i} \lim_i Y_i$ may be identified with the limit of the diagonals $Y_i \to Y_i \times_{Z_i} Y_i$ and is thus an equivalence. This proves i).

One direction of ii) is simply Lemma 7.2.1.1 combined with the observation that the inclusion $\mathcal{X}_0 \hookrightarrow \mathrm{Pro}(\mathcal{X}_0)$ preserves effective epimorphisms. For the other direction assume that $f\colon X \to Z$ is an effective epimorphism. We now pick a factorization $f \simeq (\lim_i h_i) \circ (\lim_i g_i)$ as above. By i) it follows from [57, Proposition 5.2.8.6] that $\lim_i h_i$ is both an effective epimorphism and (-1)-truncated. Thus it is left orthogonal to itself, so it is an equivalence, which completes the proof of ii).

For iii) we may use [57, Proposition 5.3.5.15] again to assume that we are given a cofiltered diagram $I \to \mathrm{Fun}(\Lambda^2_2, \mathcal{X}_0)$ depicted as

$$\begin{tikzcd} & X_\bullet \arrow[d, "f_\bullet"] \\ T_\bullet \arrow[r, "\gamma_\bullet"] & Z_\bullet \end{tikzcd}$$

such that the induced map $\lim_i X_i \to \lim_i Z_i$ is an effective epimorphism. We have to show that the induced map

$$\lim_i T_i \times_{\lim_i Z_i} \lim_i X_i \longrightarrow \lim_i T_i$$

is an effective epimorphism. Again we get a functorial factorization

$$X_i \xrightarrow{g_i} Y_i \xrightarrow{h_i} Z_i$$

where g_i is an effective epimorphism and h_i is (-1)-truncated for all i. We now consider the diagram

$$\begin{tikzcd} \lim_i(T_i \times_{Z_i} X_i) \arrow[r] \arrow[d, "g'"] & \lim_i X_i \arrow[d, "g=\lim_i g_i"] \\ \lim_i(T_i \times_{Z_i} Y_i) \arrow[r] \arrow[d, "h'"] & \lim_i Y_i \arrow[d, "h=\lim_i h_i"] \\ \lim_i T_i \arrow[r] & \lim_i Z_i\,. \end{tikzcd}$$

Since $f = h \circ g$ is an effective epimorphism, it follows like in the proof of ii) that h is an equivalence. Thus h' is an equivalence. Since $\mathcal{X}_0$ is an ∞-pretopos, it follows that g' is an inverse limit of effective epimorphisms and so the claim follows from Lemma 7.2.1.1. $\square$

Lemma 7.2.1.4. *Let $\mathcal{X}_0$ be a tiny ∞-pretopos. Then finite coproducts are universal in* $\mathrm{Pro}(\mathcal{X}_0)$.

Proof. Again we may use [57, Proposition 5.3.5.15] to reduce to the case where we are given a cofiltered family of diagrams

$$\begin{tikzcd} & X_i \amalg X'_i \arrow[d, "f_i\ f'_i"] \\ Y_i \arrow[r, "\gamma_i"] & Z_i \end{tikzcd}$$

and have to show that the induced map

$$\Big(\lim_i Y_i \times_{\lim_i Z_i} \lim_i X_i\Big) \amalg \Big(\lim_i Y_i \times_{\lim_i Z_i} \lim_i X'_i\Big) \longrightarrow \lim_i Y_i \times_{\lim_i Z_i} \lim_i \Big(X_i \coprod X'_i\Big)$$

is an equivalence. But this map can be identified with the limit of the induced morphisms

$$(Y_i \times_{Z_i} X_i) \amalg (Y_i \times_{Z_i} X'_i) \longrightarrow Y_i \times_{Z_i} (X_i \amalg X'_i)$$

which are equivalences, as $\mathcal{X}_0$ is an ∞-pretopos. $\square$

Finally we observe that, as a consequence of [56, Proposition 5.2.8.6] and Proposition 7.2.1.3, the collection of effective epimorphisms in $\mathrm{Pro}(\mathcal{X}_0)$ is closed under composition and it is clearly closed under finite coproducts. We may thus apply [58, Proposition A.3.2.1] to get the following:

COROLLARY 7.2.1.5. *Let $\mathcal{X}_0$ be a bounded ∞-pretopos. Define a collection of morphisms $\{C_i \to D\}_{i\in I}$ in* $\mathrm{Pro}(\mathcal{X}_0)$ *to be covering if and only if there is a finite subset $J \subseteq I$ such that the induced map*

$$\coprod_{j\in J} C_j \longrightarrow D$$

is an effective epimorphism in $\mathrm{Pro}(\mathcal{X}_0)$. *This defines a topology on* $\mathrm{Pro}(\mathcal{X}_0)$.

DEFINITION 7.2.1.6. Let $\mathcal{X}_0$ be a tiny bounded ∞-pretopos. We call the topology on $\mathrm{Pro}(\mathcal{X}_0)$ from Corollary 7.2.1.5 the *effective epimorphism topology.* For a bounded coherent δ_0-∞-topos $\mathcal{X}$, we define the *pyknotification* of $\mathcal{X}$ to be the ∞-topos

$$\mathcal{X}^{\mathrm{pyk}} = \mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}\left(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})\right).$$

REMARK 7.2.1.7. The pyknotification of a bounded coherent ∞-topos appeared first in [10, Construction 3.3.2] under the name *solidification.* Since the word solidification is also used in [20] in an unrelated way, a different name is used here.

EXAMPLE 7.2.1.8. In the case where $\mathcal{X} = \mathcal{S}$, the ∞-category of bounded coherent objects in $\mathcal{S}$ is the ∞-category of π-finite spaces $\mathcal{S}_\pi$. It is shown in [9, Proposition 13.4.9] that any profinite space admits an effective epimorphism from a profinite set. In other words the full subcategory $\mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}}) \subseteq \mathcal{S}_\pi^\wedge$ is a basis for the effective epimorphism topology. It follows that we have an equivalence of ∞-topoi

$$\mathcal{S}^{\mathrm{pyk}} \simeq \mathrm{Pyk}(\mathcal{S}).$$

EXAMPLE 7.2.1.9. More generally for a qcqs scheme X, any object in $\mathrm{Pro}(X^{\mathrm{constr}}_{\acute{e}t})$ admits an effective epimorphism from an object in $\mathrm{Pro}(X^{\mathrm{constr}}_{\leq 0})$ by [10, Proposition 3.3.8]. It follows that $X^{\mathrm{pyk}}_{\acute{e}t}$ is the hypercomplete ∞-topos associated to the 1-topos $\mathrm{Sh}_{\mathrm{eff}}(\mathrm{Pro}(X^{\mathrm{constr}}_{\leq 0}), \mathrm{Set})$ introduced in [59, §7.1]. Thus [59, Example 7.1.7] shows that $X^{\mathrm{pyk}}_{\acute{e}t}$ is equivalent to the hypercomplete ∞-topos $X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}}$ of pro-étale sheaves on X defined by Bhatt-Scholze in [14].

REMARK 7.2.1.10. In principle we can also consider a version of the pyknotification where we consider the ∞-topos $\mathrm{Sh}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}_0))$ of all sheaves with respect to the effective epimorphism topology instead of just hypersheaves. However there are reasons to prefer the hypercomplete version in Definition 7.2.1.6. For example in many cases of interest the ∞-topos $\mathcal{X}^{\mathrm{pyk}}$ will be postnikov complete and even have a set of compact projective generators (see Corollary 7.2.3.13), which makes it convenient to work with. This will not hold for the non-hypercomplete version in general [10, Warning 2.2.2].

7.2.1.11. The construction of $(-)^{\mathrm{pyk}}$ defines a functor

$$\mathrm{Top}^{\mathrm{R,coh}}_\infty \to \mathrm{Top}^{\mathrm{R}}_\infty\,.$$

In particular it follows from Example 7.1.2.3 that the terminal geometric morphism $\Gamma_*\colon \mathcal{X} \to \mathcal{S}$ induces adjunction

$$\Gamma^*_{\mathrm{pyk}}\colon\ \mathrm{Pyk}(\mathcal{S}) \to \mathcal{X}^{\mathrm{pyk}}\colon \Gamma^{\mathrm{pyk}}_*$$

where Γ^*_{pyk} is left exact. Via the equivalence of Theorem 5.2.5.1 we therefore get an induced $\mathrm{Pyk}(\mathcal{S})$-topos that we denote by $\underline{\mathcal{X}}^{\mathrm{pyk}}$.

7.2.2. Embedding pro-constructible sheaves. The goal of this section is to show that for a spectral ∞-topos $\mathcal{X}$, we have a fully faithful embedding

$$\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty}) \hookrightarrow \mathrm{Fun}_{\,\mathrm{Pyk}(\mathcal{S})}((\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})).$$

We will begin by showing that there is a fully faithful embedding if one removes the $\mathrm{Pro}(-)$ above.

LEMMA 7.2.2.1. *Let $q\colon \mathcal{F} \to \Pi$ be a left fibration with π-finite fibres and let $\Pi \to P$ be a π-finite stratified space. Then $\mathcal{F} \to P$ is a π-finite stratified space.*

Proof. The only thing that is not obvious is that the mapping space $\mathrm{map}_{\mathcal{F}}(x,y)$ is π-finite for all $x,y\in\mathcal{F}$. Since q is a left fibration every morphism in $\mathcal{F}$ is q-cocartesian. Thus for every $f\colon x\to y$ in $\mathrm{map}_{\mathcal{F}}(x,y)$, the square

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{F}_{p(y)}}(y,y) \arrow[r, "f^*"] \arrow[d] & \mathrm{map}_{\mathcal{F}}(x,y) \arrow[d] \\
* \arrow[r, "q(f)"] & \mathrm{map}_{\Pi}\big(f(x),f(y)\big)
\end{tikzcd}$$

is a pullback in $\mathcal{S}$ by [57, Proposition 2.4.4.2]. By assumption, the bottom right and the top left corner are π-finite and thus the claim follows. $\square$

Lemma 7.2.2.2. *Consider the continuous functor $j\colon \mathcal{S}_\pi^{\mathrm{disc}}\to\mathbf{Pyk}(\mathcal{S})$ corresponding under adjunction to the embedding $(-)^{\mathrm{disc}}\colon\mathcal{S}_\pi\to\mathrm{Pyk}(\mathcal{S})$. Then j is a fully faithful functor of $\mathrm{Pyk}(\mathcal{S})$-categories.*

Proof. We need to check that $j(K)\colon \mathcal{S}_\pi^{\mathrm{disc}}(K)\to\mathrm{Pyk}(\mathcal{S})_{/K}$ is fully faithful for every $K\in\mathrm{Proj}$. Write $K=\lim_{i\in I}K_i$ as an inverse limit of finite sets. Recall from [9, Construction 13.3.10] that

$$\mathcal{S}_\pi^{\mathrm{disc}}(K)\simeq\operatorname*{colim}_i\mathcal{S}_\pi^{K_i}.$$

For $i\in I$, the functor $j(K)(x)$ sends a π-finite space x over K_i to the pyknotic space over K given by the pullback

$$\begin{tikzcd}
j(K)(x) \arrow[r] \arrow[d] & x \arrow[d] \\
K \arrow[r] & K_i\,.
\end{tikzcd}$$

In particular $j(K)$ factors through the full subcategory $\mathcal{S}^\wedge_{\pi/K}\subseteq\mathrm{Pyk}(\mathcal{S})_{/K}$ spanned by the profinite spaces over K. Now let $x,y\in\mathcal{S}_\pi^{\mathrm{disc}}(K)$. Since I is filtered we may assume that there is some $i\in I$ such that $x,y\in\mathcal{S}_\pi^{K_i}$. Replacing I by $I_{/i}$, we may furthermore assume that i is the final object. For a map $j\to i$ in I, let us denote the pullback $x\times_{K_i}K_j$ by x_j and analogously for y. We have to see that the map

$$\operatorname*{colim}_j\mathrm{map}_{\mathcal{S}_\pi^{K_j}}(x_j,y_j)\to\mathrm{map}_{\mathcal{S}^\wedge_{\pi/K}}\big(j(K)(x),j(K)(y)\big)$$

induced by j_K is an equivalence. Composing with the projection $j(K)(y)\to y$ induces an equivalence

$$\mathrm{map}_{\mathcal{S}^\wedge_{\pi/K}}\big(j(K)(x),j(K)(y)\big)\xrightarrow{\simeq}\mathrm{map}_{\mathcal{S}^\wedge_{\pi/K_i}}\big(j(K)(x),y\big).$$

Analogously composing with the projections $y_j\to y$ induces an equivalence

$$\operatorname*{colim}_j\mathrm{map}_{\mathcal{S}_\pi^{K_j}}(x_j,y_j)\xrightarrow{\simeq}\operatorname*{colim}_j\mathrm{map}_{\mathcal{S}_\pi^{K_i}}(x_j,y)$$

We obtain a commutative square

$$\begin{tikzcd}
\mathrm{colim}_j\,\mathrm{map}_{\mathcal{S}_\pi^{K_j}}(x_j,y_j) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{map}_{\mathcal{S}^\wedge_{\pi/K}}\big(j(K)(x),j(K)(y)\big) \arrow[d, "\simeq"] \\
\mathrm{colim}_j\,\mathrm{map}_{\mathcal{S}_\pi^{K_i}}(x_j,y) \arrow[r] & \mathrm{map}_{\mathcal{S}^\wedge_{\pi/K_i}}\big(j(K)(x),y\big)\,.
\end{tikzcd}$$

But it is clear that the lower horizontal map is an equivalence, since $j(K)(x)\simeq\lim_j x_j$ in $\mathcal{S}^\wedge_{\pi/K_i}$ and y is cocompact in $\mathcal{S}^\wedge_{\pi/K_i}$. $\square$

Remark 7.2.2.3. One can give an alternative prove of Lemma 7.2.2.2 as follows. Let K be an extremally disconnected set. By [9, Proposition 4.4.18] the ∞-category $\mathcal{S}_\pi^{\mathrm{disc}}(K)$ is equivalent to the full subcategory $\mathrm{Lcc}(K)$ of the ∞-topos $\mathrm{Sh}(K)$ of sheaves on K spanned by the locally constant constructible sheaves in the sense of [58, Definition 2.5.1]. Then [33, Corollary 4.4] provides a fully faithful embedding

$$\mathrm{Sh}(K)\hookrightarrow\mathrm{Pyk}(\mathcal{S})_{/K}.$$

Therefore we also have a fully faithful embedding $\mathcal{S}_\pi^{\mathrm{disc}}(K)\hookrightarrow\mathrm{Pyk}(\mathcal{S})_{/K}$ and one can check that this embedding agrees with the functor $j(K)$.

Corollary 7.2.2.4. *Let $\mathcal{C}$ be a pyknotic ∞-category. Then composition with $j\colon \mathcal{S}_\pi^{\mathrm{disc}} \to \mathbf{Pyk}(\mathcal{S})$ induces a fully faithful functor*

$$j_*\colon \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathcal{C}, \mathcal{S}_\pi^{\mathrm{disc}}) \to \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathcal{C}, \mathbf{Pyk}(\mathcal{S})).$$

Remark 7.2.2.5. Let $\mathcal{C}$ be an ordinary ∞-category and $p\colon P \to \mathcal{C}$ a left fibration classifying a functor $f\colon \mathcal{C} \to \mathcal{S}$. Thus we also get an induced left fibration $p^{\mathrm{disc}}\colon P^{\mathrm{disc}} \to \mathcal{C}^{\mathrm{disc}}$ of $\mathrm{Pyk}(\mathcal{S})$-categories. Recall that the functoriality of the Grothendieck-construction (Lemma 5.2.8.5) shows that the diagram

$$\begin{tikzcd}
\mathrm{LFib}(\mathcal{C}) \arrow[r, "\simeq"] & \operatorname{Fun}(\mathcal{C}, \mathcal{S}) \\
\mathrm{LFib}(\mathcal{C}^{\mathrm{disc}}) \arrow[u, "\mathrm{Un}"] \arrow[r, "\simeq"] & \operatorname{Fun}(\mathcal{C}, \mathbf{Pyk}(\mathcal{S})) \arrow[u, "\mathrm{Un}"']
\end{tikzcd}$$

commutes. After passing to left adjoints vertically, it follows that the left fibration p^{disc} classifies the composite

$$\mathcal{C} \xrightarrow{f} \mathcal{S} \xrightarrow{(-)^{\mathrm{disc}}} \mathrm{Pyk}(\mathcal{S}).$$

Notation 7.2.2.6. Let S be a spectral topological space and $\mathcal{X}$ a spectral S-stratified ∞-topos [9, Definition 9.2.1]. Consider the profinite stratified shape $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}) \in \mathrm{Pro}(\mathrm{Str}_\pi)$ of $\mathcal{X}$ [9, Definition 10.1.4]. Say that $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}) \simeq \lim_i(\mathcal{C}_i \to S_i)$ where $\mathcal{C}_i \to S_i$ is a π-finite stratified space.

Recall that, by [9, Lemma 13.6.1], the canonical functor

$$\operatorname*{colim}_{i\in I} \operatorname{Fun}\left(\mathcal{C}_i, \mathcal{S}_\pi\right) \longrightarrow \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}\right)$$

is an equivalence. Thus, using Remark 7.2.2.5 we get the following explicit description of the essential image of the functor j_*:

Lemma 7.2.2.7. *With notations as in 7.2.2.6, consider the fully faithful functor*

$$j_*\colon \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}) \to \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})).$$

Then a continuous functor $F\colon \hat{\Pi}^S_{(\infty,1)}(\mathcal{X}) \to \mathbf{Pyk}(\mathcal{S})$ is in the essential image of j_* if and only if there is a left fibration of usual ∞-categories $\mathcal{F}_i \longrightarrow \mathcal{C}_i$ with π-finite fibers such that the left fibration of $\mathrm{Pyk}(\mathcal{S})$*-categories*

$$\mathcal{F} = \lim_{j\in I_{/i}} \mathcal{F}_i^{\mathrm{disc}} \times_{\mathcal{C}_i^{\mathrm{disc}}} \mathcal{C}_j^{\mathrm{disc}} \simeq \mathcal{F}_i^{\mathrm{disc}} \times_{\mathcal{C}_i^{\mathrm{disc}}} \hat{\Pi}^S_{(\infty,1)}(\mathcal{X}) \to \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$$

classifies F. In particular $\mathcal{F} \to S$ is a profinite stratified space by Lemma 7.2.2.1.

We can now prove the desired fully faithfulness:

Proposition 7.2.2.8. *The embedding* $\operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}) \to \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S}))$ *extends to a fully faithful tiny limit preserving embedding*

$$\iota\colon \mathrm{Pro}\Big(\operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}\right)\Big) \hookrightarrow \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right).$$

Proof. We have to see that, for a cofiltered diagram $Y_\bullet\colon J \to \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}})$ and and object $X \in \operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}})$, the canonical map

$$\operatorname*{colim}_j \operatorname{map}_{\operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}\right)}(Y_j, X) \longrightarrow \operatorname{map}_{\operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(S)\right)}\left(\lim_j Y_j, X\right) \tag{7.2.2.9}$$

is an equivalence. Using Lemmas 7.2.2.1 and 7.2.2.7, we see that the embedding

$$\operatorname{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}\right) \hookrightarrow \mathrm{Pyk}(\mathrm{Cat}_\infty)_{/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})}$$

factors through the full subcategory $\mathrm{Pro}(\mathrm{Lay}_\pi)_{/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})} \subseteq \mathrm{Pyk}(\mathrm{Cat}_\infty)_{/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})}$ (see [9, Proposition 13.5.2]). Again by Lemma 7.2.2.7, there is a map of stratified spaces

$$\begin{tikzcd} \hat{\Pi}^S_{(\infty,1)}(\mathcal{X}) \arrow[r] \arrow[d] & \mathcal{C}_i \arrow[d] \\ S \arrow[r] & P_i \end{tikzcd}$$

and a discrete left fibration $X_i \to \mathcal{C}_i$ with π-finite fibres such that the map (7.2.2.9) is given by the canonical map

$$\begin{aligned} &\operatorname{colim}_j \mathrm{map}_{\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}),\mathcal{S}^{\mathrm{disc}}_\pi)}(Y_j, X) \simeq \\ &\qquad \operatorname{colim}_j \mathrm{map}_{\mathrm{Pro}(\mathrm{Lay}_\pi)_{/\mathcal{C}_i}}(Y_j, X_i) \longrightarrow \mathrm{map}_{\mathrm{Pro}(\mathrm{Lay}_\pi)_{/\mathcal{C}_i}}(\lim_j Y_j, X_i) \\ &\qquad\qquad \simeq \mathrm{map}_{\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}),\mathbf{Pyk}(S))}(\lim_j Y_j, X), \end{aligned}$$

which is an equivalence as X_i and $\mathcal{C}_i$ are in $\mathrm{Lay}_\pi \subseteq \mathrm{Pro}(\mathrm{Lay}_\pi)$. □

7.2.3. The Proof of Theorem 7.2.3.1. We will now start with the proof the main result of this section.

THEOREM 7.2.3.1. *Let S be a spectral topological space and $\mathcal{X}$ an S-stratified spectral ∞-topos. Then the exodromy equivalence induces an equivalence of* $\mathrm{Pyk}(\mathcal{S})$*-topoi*

$$\underline{\mathcal{X}}^{\mathrm{pyk}} \xrightarrow{\simeq} \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}\big(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\big).$$

In order to prove Theorem 7.2.3.1 we will apply the following general topos-theoretic result [58, Proposition A.3.4.2]:

THEOREM 7.2.3.2. *Let $\mathcal{T}$ be a hypercomplete ∞-topos. Let $\mathcal{C} \subseteq \mathcal{T}$ be a full subcategory satisfying the following:*

(1) The ∞-category $\mathcal{C}$ is essentially small.
(2) All objects in $\mathcal{C}$ are coherent.
(3) The subcategory $\mathcal{C} \subseteq \mathcal{T}$ is closed under finite coproducts and under fibre products.
(4) Every object in $\mathcal{T}$ admits a cover by objects in $\mathcal{C}$.

Then the composite

$$\mathcal{T} \xrightarrow{h} \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\text{restrict}} \mathrm{PSh}(\mathcal{C})$$

induces an equivalence of ∞-topoi

$$\mathcal{T} \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{can}}(\mathcal{C}).$$

Here can *denotes the topology given by declaring a family $\{U_i \to X\}_i$ in $\mathcal{C}$ to be covering if there is a finite subset $J \subseteq I$ such that the induced morphism*

$$\coprod_{j \in J} U_j \longrightarrow X$$

is an effective epimorphism in $\mathcal{T}$.

In order to apply Theorem 7.2.3.2 we need the following:

LEMMA 7.2.3.3. *The ∞-topos* $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\big(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\big)$ *is postnikov-complete, so in particular hypercomplete.*

PROOF. The canonical functor $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq} \to \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ induces an algebraic morphism of $\mathrm{Pyk}(\mathcal{S})$-topoi

$$\underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}\big(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\big) \to \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}\big(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}, \mathbf{Pyk}(\mathcal{S})\big)$$

that is continuous by Proposition 3.1.3.1 and conservative (this is e.g. an easy consequence of Theorem 2.1.11.5). It follows that the induced morphism of underlying ∞-topoi

$$\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right) \to \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X})^{\simeq}, \mathbf{Pyk}(\mathcal{S})\right) \simeq \mathrm{Pyk}(\mathcal{S})_{/\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X})^{\simeq}}$$

is conservative and in particular commutes with arbitrary small limits and colimits. Thus it also commutes with the truncation functors $\tau_{\leq n}$ by [57, Proposition 5.5.6.28]. Since $\mathrm{Pyk}(\mathcal{S})_{/\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X})^{\simeq}}$ is postnikov-complete (see [10, Lemma 2.4.10]), this implies that also $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right)$ is postnikov-complete. $\square$

7.2.3.4. We will now show that the full subcategory

$$\iota\,\mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathcal{S}^{\mathrm{disc}}_{\pi}\right)\right) \subseteq \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right)$$

satisfies the conditions above. Note that i) and iii) are obvious.

We will now show that iv) of Theorem 7.2.3.2 is satisfied.

LEMMA 7.2.3.5. *Let* C *be any* $\mathrm{Pyk}(\mathcal{S})$*-category. Then any object in* $F \in \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathbf{Pyk}(\mathcal{S}))$ *admits an effective epimorphism from a small coproducts of objects that arise as colimit of diagrams*

$$d\colon K \to \mathsf{C}^{\mathrm{op}} \xrightarrow{h} \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathbf{Pyk}(\mathcal{S}))$$

where K *is a pro-finite set.*

PROOF. By the co-Yoneda Lemma (Proposition 3.3.1.1), F is the colimit of some diagram $d'\colon \mathsf{I} \to \mathsf{C}^{\mathrm{op}} \xrightarrow{h} \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathbf{Pyk}(\mathcal{S}))$. Since $\mathrm{Pyk}(\mathcal{S})$ is generated under colimits by pro-finite sets, we can find some effective epimorphism

$$\alpha\colon\ \amalg_i K_i \to F$$

where every K_i is a profinite set. Therefore by Corollary 3.1.4.4 we get an effective epimorphism

$$\operatorname{colim} d' \circ \alpha \to F.$$

Finally, (the proof of) Proposition 3.1.9.3 shows that $\operatorname{colim} d' \circ \alpha$ is equivalent to $\amalg_i \operatorname{colim}_i(d' \circ \alpha_{|K_i})$ and we are done. $\square$

To simplify notation we from now on write $\mathsf{C} = \hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}) \in \mathrm{Cat}(\mathrm{Pyk}(\mathcal{S}))$ and furthermore we fix a cofiltered diagram of π-finite stratified spaces $\mathcal{C}_i$ such that $\mathsf{C} \simeq \lim_i \mathcal{C}_i^{\mathrm{disc}}$.

PROPOSITION 7.2.3.6. *Let* K *be a pro-finite set. Then for any diagram of the form*

$$d\colon K \xrightarrow{k} \mathsf{C}^{\mathrm{op}} \xrightarrow{h} \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathbf{Pyk}(\mathcal{S}))$$

the colimit $\operatorname{colim} d$ *is contained in* $\iota\,\mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\mathsf{C}, \mathcal{S}^{\mathrm{disc}}_{\pi}\right)\right)$.

PROOF. Consider the pullback functor

$$k^*\colon \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathbf{Pyk}(\mathcal{S}))) \to \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(K, \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathbf{Pyk}(\mathcal{S})))$$

and its left adjoint $k_!$. Then $\operatorname{colim} d$ is given by applying the $\operatorname{colim}_{\mathsf{C}}$ to the functor $k_! k^* h$. Via the straightening equivalence Theorem 2.1.11.5, we may thus describe $\operatorname{colim} d$ explicitly by forming the pullback

$$\begin{tikzcd} \mathsf{P} \arrow[r] \arrow[d] & \mathsf{Tw}(\mathsf{C}) \arrow[d] \\ K \times \mathsf{C} \arrow[r, "d \times \mathrm{id}"'] & \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \end{tikzcd}$$

and then composing with the projection $K \times \mathsf{C} \to \mathsf{C}$. First we observe that, since the twisted arrow construction is compatible with limits, it follows that the canonical map

$$\mathsf{Tw}(\mathsf{C}) \longrightarrow \lim_i \mathsf{Tw}(\mathcal{C}_i^{\mathrm{disc}}) \times_{(\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}}} \mathsf{C}^{\mathrm{op}} \times \mathsf{C}$$

is an equivalence. Since $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathcal{C}, \mathcal{S}_\pi^{\mathrm{disc}}))$ is closed under cofiltered limits, it suffices to see that P_i, given by the pullback square

$$\begin{tikzcd} \mathsf{P}_i \arrow[r] \arrow[d] & \mathsf{Tw}(\mathcal{C}_i^{\mathrm{disc}}) \arrow[d] \\ K \times \mathsf{C} \arrow[r] & (\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}} \end{tikzcd}$$

lies in $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathcal{S}_\pi^{\mathrm{disc}}))$ when composed with the projection $K \times \mathsf{C} \to \mathsf{C}$. Furthermore note that because $(-)^{\mathrm{disc}}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}(\mathrm{Pyk}(\mathcal{S}))$ is compatible with $(-)^{\mathrm{op}}$ and $\mathrm{Tw}(-)$ it follows that the right vertical map is $(-)^{\mathrm{disc}}$ of an ordinary left fibration. Let us say that $K = \{K_j\}_{j \in J}$ as a profinite set. Then the map $K \to \mathsf{C}^{\mathrm{op}} \to \mathcal{C}_i^{\mathrm{disc}}$ factors through some $K \to K_j$. In particular we have a canonical equivalence

$$P_i \longrightarrow \lim_{j \in J_{/j_0}} \left(K_j \times \mathcal{C} \times_{(\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}}} \mathrm{Tw}(\mathcal{C}_i^{\mathrm{disc}})\right)$$

is an equivalence. Thus it suffices to see that the composite

$$K_j \times \mathsf{C} \times_{(\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}}} \mathrm{Tw}(\mathcal{C}_i^{\mathrm{disc}}) \longrightarrow K_j \times \mathsf{C} \longrightarrow \mathsf{C}$$

is contained $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathsf{C}, \mathcal{S}_\pi^{\mathrm{disc}}))$ for all j. But by construction all squares in the diagram

$$\begin{tikzcd} K_j \times \mathsf{C} \times_{(\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}}} \mathrm{Tw}(\mathcal{C}_i^{\mathrm{disc}}) \arrow[r] \arrow[d] & K_j \times \mathcal{C}_i^{\mathrm{disc}} \times_{(\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}}} \mathrm{Tw}(\mathcal{C}_i^{\mathrm{disc}}) \arrow[d] \\ K_j \times \mathsf{C} \arrow[r] \arrow[d] & K_j \times \mathcal{C}_i^{\mathrm{disc}} \arrow[d] \\ \mathsf{C} \arrow[r] & \mathcal{C}_i^{\mathrm{disc}} \end{tikzcd}$$

are pullback squares. Thus the claim follows from Lemma 7.2.2.7, as the map

$$K_j \times \mathcal{C}_i^{\mathrm{disc}} \times_{(\mathcal{C}_i^{\mathrm{disc}})^{\mathrm{op}} \times \mathcal{C}_i^{\mathrm{disc}}} \mathrm{Tw}(\mathcal{C}_i)^{\mathrm{disc}} \longrightarrow K_j \times \mathcal{C}_i^{\mathrm{disc}} \longrightarrow \mathcal{C}_i^{\mathrm{disc}}$$

is given by applying $(-)^{\mathrm{disc}}$ to a discrete left fibration with π-finite fibres. $\square$

We will now show that ii) of Theorem 7.2.3.2 is satisfied:

PROPOSITION 7.2.3.7. *The fully faithful embedding*

$$\iota\colon \mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}\right)\right) \hookrightarrow \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right)$$

factors through the full subcategory spanned by the coherent objects.

PROOF. We will show that all objects in $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}))$ are n-coherent using induction on n. Let us start with $n = 0$. Recall that the functor

$$U\colon \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right) \longrightarrow \mathrm{Pyk}(\mathcal{S})_{/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}}$$

is conservative and preserves all limits and colimits see the proof of Lemma 7.2.3.3. Thus it suffices to see that, for an object $\mathcal{F} \in \mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}))$, the pyknotic space $U(\mathcal{F})$ over $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}$ is quasi-compact. We now observe that the functor U takes objects in $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}})$ to objects in $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}, \mathcal{S}_\pi^{\mathrm{disc}}) \subseteq \mathcal{S}^\wedge_{\pi/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}}$. Since the inclusion

$$\mathcal{S}^\wedge_{\pi/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}} \longrightarrow \mathrm{Pyk}(\mathcal{S})_{/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}}$$

preserves limits, it follows that U takes objects in $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}))$ to objects in the full subcategory $\mathcal{S}^\wedge_{\pi/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}}$. Now [9, Corollary 13.4.10] and [58, Remark 2.0.5, Proposition 2.2.2 and Proposition 3.1.3] imply that all object in $\mathcal{S}^\wedge_{\pi/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}}$ are coherent and thus in particular quasi-compact. This completes the case $n = 0$. The induction step is now clear from [58, Corollary A.2.1.4]. $\square$

To complete the proof of Theorem 7.2.3.1, we will need a few more technical details:

LEMMA 7.2.3.8. *Let* $X_\bullet\colon I \to \mathbf{bPretop}_\infty^{\delta_0}$ *be a tiny filtered diagram in the* ∞*-category of tiny bounded* ∞*-pretopoi. Let* $X = \operatorname{colim}_i X_i$ *denote the colimit in* $\mathbf{bPretop}_\infty^{\delta_0}$ *and let* $f\colon C \to D$ *be an effective epimorphism in* X*. Then there is an* $i_0 \in I$ *and an effective epimorphism* $f_{i_0}\colon C_{i_0} \to D_{i_0}$ *mapping to* f *under the canonical functor*

$$k_{i_0}\colon X_{i_0} \longrightarrow X.$$

PROOF. By [58, Proposition A.8.3.1], the inclusion $\mathbf{bPretop}_\infty^{\delta_0} \to \mathrm{Cat}_\infty^{\delta_0}$ preserves filtered colimits. Thus we may find a morphism $f_{j_0}\colon C_{j_0} \to D_{j_0}$ such that $k_{j_0}(f_{j_0})$ is equivalent to f. Since k_{j_0} is a morphism of ∞-pretopoi, we get an equivalence

$$k_{j_0}(\check{C}(f_{j_0})_\bullet) \simeq \check{C}(f)_\bullet$$

of simplicial objects in X. Furthermore, since k_{j_0} is a morphism of ∞-pretopoi, it preserves finite limits and effective epimorphisms and thus geometric realizations of groupoid objects. It follows that the canonical map

$$c\colon |\check{C}(f_{j_0})_\bullet| \longrightarrow D_{j_0}$$

becomes an equivalence after applying k_{j_0}. Thus there is a map $\gamma\colon j_0 \to i_0$ such that $X_\gamma(c)$ is an equivalence and since X_γ is a morphism of ∞-pretopoi, it follows that $X_\gamma(f_{j_0})$ is an effective epimorphism, as desired. □

COROLLARY 7.2.3.9. *Let* $K = \{K_i\}_i$ *be a profinite space. Then the fully faithful functor*

$$\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(K, \mathcal{S}_\pi^{\mathrm{disc}}) \longrightarrow \mathrm{Pyk}(\mathcal{S})_{/K}$$

preserves effective epimorphisms.

PROOF. By [9, Lemma 13.6.1] and Lemma 7.2.3.8, it suffices to see that, for all i, every effective epimorphism f in $\mathrm{Fun}(K_i, \mathcal{S}_\pi)$ maps to an effective epimorphism in $\mathrm{Pyk}(\mathcal{S})_{/K}$. Denote by $p_i\colon K \to K_i$ the projection. Since we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(K_i, \mathcal{S}_\pi^{\mathrm{disc}}) \arrow[r, "\varphi"] \arrow[d, "p_i^*"] & \mathrm{Pyk}(\mathcal{S})_{/K_i} \arrow[d, "p_i^*"] \\
\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(K, \mathcal{S}_\pi^{\mathrm{disc}}) \arrow[r] & \mathrm{Pyk}(\mathcal{S})_{/K}
\end{tikzcd}$$

it suffices to see that the top horizontal functor φ preserves effective epimorphism. Picking a section of the canonical morphism $K_i \to \pi_0(K_i)$ and precomposing with it, we may assume that K_i is a finite set. In this case φ is identified with the product of finitely many copies of the inclusion

$$\mathcal{S}_\pi \longrightarrow \mathrm{Pyk}(\mathcal{S}),$$

which clearly preserves effective epimorphisms. This completes the proof. □

We finally arrive at the following:

PROPOSITION 7.2.3.10. *A morphism* $f\colon X \to Y$ *in* $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi)^{\mathrm{disc}})$ *is an effective epimorphism if and only if* $\iota(f)$ *is an effective epimorphism in* $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\big(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\big)$.

PROOF. Let us first assume that $\iota(f)$ is an effective epimorphism. We then may factor $f \simeq g \circ h$, where h is an effective epimorphism and g is (-1)-truncated. Since the inclusion ι preserves finite limits, it follows that $\iota(g)$ is (-1)-truncated as well. But by [57, Corollary 6.2.3.12], the map $\iota(g)$ is an effective epimorphism because $\iota(f)$ is. This implies that $\iota(g)$, and thus g, is an equivalence, as desired.

Now we show that ι preserves effective epimorphisms. Again we consider the inclusion $K = \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq} \hookrightarrow \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$. Pre-composing with this inclusion induces a morphism of ∞-pretopoi

$$\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\big(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathcal{S}_\pi^{\mathrm{disc}}\big) \longrightarrow \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(K, \mathcal{S}_\pi^{\mathrm{disc}}).$$

So it follows from Proposition 7.2.1.3 that the induced functor

$$\mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathcal{S}^{\mathrm{disc}}_{\pi}\right)\right) \longrightarrow \mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(K, \mathcal{S}^{\mathrm{disc}}_{\pi})\right)$$

preserves effective epimorphisms. We may thus reduce to showing that the induced functor

$$j\colon \mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(K, \mathcal{S}^{\mathrm{disc}}_{\pi})\right) \longrightarrow \mathrm{Pyk}(\mathcal{S})_{/K}$$

preserves effective epimorphisms. By Corollary 7.2.3.9, the inclusion

$$k\colon \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(K, \mathcal{S}^{\mathrm{disc}}_{\pi}) \hookrightarrow \mathrm{Pyk}(\mathcal{S})_{/K}.$$

preserves effective epimorphisms. Furthermore k factors through the full subcategory $\mathcal{S}^{\wedge}_{\pi}{}_{/K}$ and hence so does j. By Lemma 7.2.1.1 and since effective epimorphisms in slice categories are detected by the projections, it suffices to see that the inclusion $\mathcal{S}^{\wedge}_{\pi} \to \mathrm{Pyk}(\mathcal{S})$ preserves effective epimorphisms, which is clear by [9, Corollary 13.4.10]. □

We have finally collected all the necessary ingredients that are needed to prove our main theorem:

Proof of 7.2.3.1: Let us begin by showing that there is an equivalence of underlying ∞-topoi

$$\mathcal{X}^{\mathrm{pyk}} \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right).$$

that is natural in the spectral ∞-tops $\mathcal{X}$. The Exodromy Theorem [9, Theorem 13.2.11] provides a natural equivalence of tiny ∞-pretopoi

$$\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathcal{S}^{\mathrm{disc}}_{\pi}\right) \simeq \mathcal{X}^{\mathrm{coh}}_{<\infty}.$$

We have seen above that the full subcategory

$$\mathrm{Pro}\left(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathcal{S}^{\mathrm{disc}}_{\pi}\right)\right) \hookrightarrow \mathrm{Fun}^{\mathrm{cts}}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right)$$

satisfies the assumptions of Theorem 7.2.3.2 and thus we get a natural equivalence

$$\mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{can}}\left(\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathcal{S}^{\mathrm{disc}}_{\pi})\right)\right) \simeq \mathrm{Fun}^{\mathrm{cts}}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right).$$

Thus it remains to see that the topologies can and eff on $\mathrm{Pro}(\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathcal{S}_{\pi}))$ agree, but this is just a reformulation of Proposition 7.2.3.10. To prove the general claim about $\mathrm{Pyk}(\mathcal{S})$-topoi note that by naturality of the above equivalence the triangle

$$\begin{tikzcd}
\mathcal{X}^{\mathrm{pyk}} \arrow[rr, "\simeq"] & & \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\left(\hat{\Pi}^{S}_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})\right) \\
& \mathrm{Pyk}(\mathcal{S}) \arrow[ul, "\Gamma^{*}_{\mathrm{pyk}}"] \arrow[ur, "\mathrm{diag}"'] &
\end{tikzcd}$$

commtes. This proves the claim by Theorem 5.2.5.1. □

In the case where $\mathcal{X} = X_{\text{ét}}$ for some qcqs scheme X, we obtain using Example 7.2.1.9:

Corollary 7.2.3.11. *Let X be a qcqs scheme. Then the exodromy equivalence induces an equivalence of ∞-topoi*

$$X^{\mathrm{hyp}}_{\text{proét}} \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S})).$$

Let us obtain some easy consequence:

Corollary 7.2.3.12. *Let $f : X \to Y$ be any morphism of schemes. Then the induced pull-back functor*

$$f^{*} : Y^{\mathrm{hyp}}_{\text{proét}} \longrightarrow X^{\mathrm{hyp}}_{\text{proét}}$$

has both a left and a right adjoint.

PROOF. By the adjoint functor theorem [57, Corollary 5.5.2.9] it suffices to see that f^* preserves all limits and colimits. We may cover X by affine opens $j_i\colon U_i \to X$ such that for every i we have an affine open $t_i\colon V_i \to Y$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U_i \arrow[u, hook, "j_i"] \arrow[r, "f_i"'] & V_i \arrow[u, hook, "t_i"'] . \end{tikzcd}$$

Since the j_i^* are jointly conservative and restrictions to open subschemes commute with all limits and colimits, it suffices to see that each $j_i^* \circ f^*$ preserves limits. Because $j_i^* \circ f^* \simeq f_i^* \circ t_i^*$, it suffices to see that f_i^* preserves all limits and colimits and we may therefore assume that X and Y are affine, so in particular quasi-compact. In this case f^* corresponds to the functor

$$\mathrm{Gal}(f)^*\colon \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\big(\mathrm{Gal}(Y), \mathbf{Pyk}(\mathcal{S})\big) \longrightarrow \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}\big(\mathrm{Gal}(X), \mathbf{Pyk}(\mathcal{S})\big)$$

given by precomposing with $\mathrm{Gal}(f)\colon \mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ via Corollary 7.2.3.11 and the claim follows. ☐

COROLLARY 7.2.3.13. *Let $\mathcal{X}$ be a spectral ∞-topos. Then $\mathcal{X}^{\mathrm{pyk}}$ is projectively generated (in the sense of [57, Definition 5.5.8.23]).*

PROOF. By Theorem 7.2.3.1, we have to see that $\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S}))$ is projectively generated. As in the proof of Lemma 7.2.3.3 the inclusion $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq} \to \hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ induces a conservative limit and colimit preserving functor

$$\mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})) \to \mathrm{Pyk}(\mathcal{S})_{/\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\simeq}}.$$

Since the domain is projectively generated, because $\mathrm{Pyk}(\mathcal{S})$ is (see Observation 7.1.1.3), the claim follows from [56, Corollary 4.7.3.18]. ☐

Using the methods developed in Section 4.6 we can now easily extend Theorem 7.2.3.1 to arbitrary coefficients.

COROLLARY 7.2.3.14. *Let C be any presentable $\mathrm{Pyk}(\mathcal{S})$-category and $\mathcal{X}$ a spectral ∞-topos. Then there is a natural equivalence*

$$\underline{\mathcal{X}}^{\mathrm{pyk}} \otimes^{\mathrm{Pyk}(\mathcal{S})} \mathsf{C} \simeq \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathsf{C}).$$

PROOF. By Theorem 7.2.3.1 and Proposition 4.6.2.11

$$\underline{\mathcal{X}}^{\mathrm{pyk}} \otimes^{\mathrm{Pyk}(\mathcal{S})} \mathsf{C} \simeq \underline{\mathsf{Fun}}_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})) \otimes \mathsf{C} \simeq \underline{\mathsf{Fun}}^{\mathrm{cont}}_{\mathrm{Pyk}(\mathcal{S})}(\underline{\mathsf{PSh}}_{\mathrm{Pyk}(S)}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Pyk}(\mathcal{S})).$$

Thus the claim follows from the universal property of $\underline{\mathsf{PSh}}_{\mathrm{Pyk}(\mathcal{S})}(-)$, Theorem 3.4.1.1. ☐

Let us also record the following more explicit variant. For this recall that if R is a condensed ring, we constructed in Section 4.5.2 a presentable $\mathrm{Pyk}(\mathcal{S})$-category $\mathsf{Mod}^R_{\mathrm{Pyk}(\mathcal{S})}$, that can be explicitly described as the sheaf

$$K \mapsto \mathrm{Mod}_{\pi_K^* R}(\mathrm{Pyk}(\mathcal{S})_{/K} \otimes \mathrm{Sp}).$$

COROLLARY 7.2.3.15. *Let R be a pyknotic ring and X a qcqs scheme. We denote by $\mathbf{D}_{\text{proét}}(X, R)$ the derived category of the abelian category of $\Gamma^*_{\mathrm{pyk}}(R)$-modules in pro-étale sheaves on X. Then there is a natural equivalence*

$$\mathbf{D}_{\text{proét}}(X, R) \simeq \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\mathrm{Gal}(X), \mathsf{Mod}^R_{\mathrm{Pyk}(\mathcal{S})}).$$

PROOF. This is an immediate consequence of the previous Corollary if we show that

$$\mathbf{D}_{\text{proét}}(X, R) \simeq \Gamma(X^{\mathrm{hyp}}_{\text{proét}} \otimes^{\mathrm{Pyk}(\mathcal{S})} \mathsf{Mod}^R_{\mathrm{Pyk}(\mathcal{S})}).$$

But by Propositions 4.6.3.13 and 4.6.3.12 we find that

$$\Gamma(X^{\mathrm{hyp}}_{\text{proét}} \otimes^{\mathrm{Pyk}(\mathcal{S})} \mathsf{Mod}^R_{\mathrm{Pyk}(\mathcal{S})}) \simeq X^{\mathrm{hyp}}_{\text{proét}} \otimes_{\mathrm{Pyk}(\mathcal{S})} \mathrm{Mod}_R(\mathrm{Pyk}(\mathrm{Sp}))$$

and furthermore Lemma 4.6.3.11 implies that

$$X^{\mathrm{hyp}}_{\mathrm{pro\acute{e}t}} \otimes_{\mathrm{Pyk}(\mathcal{S})} \mathrm{Mod}_R(\mathrm{Pyk}(\mathrm{Sp})) \simeq \mathrm{Mod}_{\Gamma^*_{\mathrm{pyk}}(R)}(\mathrm{Sh}^{\mathrm{hyp}}(X^{\mathrm{pro\acute{e}t}}, \mathrm{Sp})).$$

Thus the claim follows from [58, Theorem 2.1.2.2]. □

REMARK 7.2.3.16. We will now roughly sketch how to circumvent the enlargement of universes which appears in our results, following [10, §1.4]. Let $\mathcal{X}$ be a spectral ∞-topos and let β be an uncountable regular cardinal such that $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ is a β-small inverse limit of π-finite layered ∞-categories. Let $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})_\beta$ denote the small subcategory spanned by the β-cocompact objects. We define

$$\mathcal{X}^{\mathrm{pyk},\beta} = \mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})_\beta)$$

and observe that $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ naturally defines a sheaf of ∞-categories on $\mathrm{Pro}(\mathcal{S}_\pi)_\beta$, i.e. an ∞-category object in $\mathcal{S}^{\mathrm{pyk},\beta}$. Furthermore let us write $\mathrm{Pyk}(\mathcal{S})^\beta = \mathcal{S}^{\mathrm{pyk},\beta}$ We can then reproduce the results of §3 and §4 in this framework to obtain an equivalence

$$\mathcal{X}^{\mathrm{pyk},\beta} \simeq \mathrm{Fun}_{\mathrm{Pyk}(S)^\beta}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})^\beta)$$

of ∞-topoi. Considering the left Kan-extensions along $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})_{\lambda_0} \hookrightarrow \mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})_{\lambda_1}$ for any $\beta < \lambda_0 < \lambda_1$, we obtain an equivalence

$$\mathcal{X}^{\mathrm{pyk,acc}} := \operatorname*{colim}_{\lambda>\beta} \mathcal{X}^{\mathrm{pyk},\lambda} = \mathrm{Sh}^{\mathrm{hyp,acc}}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})) \xrightarrow{\simeq} \operatorname*{colim}_{\lambda>\beta} \mathrm{Fun}^\lambda_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})^\lambda).$$

Furthermore the filtered colimit

$$\operatorname*{colim}_{\lambda>\beta} \mathrm{Pyk}(\mathcal{S})^\lambda = \mathrm{Sh}^{\mathrm{hyp,acc}}_{\mathrm{eff}}(\mathcal{S}^\wedge_\pi)$$

is given by the ∞-category of accessible sheaves on $\mathcal{S}^\wedge_\pi$, which we can further identify with the ∞-category of *condensed spaces* $\mathrm{Cond}(\mathcal{S})$ of Clausen and Scholze. We may consider $\mathrm{Cond}(\mathcal{S})$ as a hypersheaf with respect to the effective epimorphism topology on $\mathcal{S}^\wedge_\pi$ as follows. Denote by $\mathbf{Cond}(\mathcal{S})^\lambda$ the sheaf given by left Kan-extension of

$$\mathbf{Pyk}(\mathcal{S})^\lambda \colon \mathrm{Pro}(\mathcal{S}_\pi)_\lambda \to \mathrm{Cat}_\infty; \quad K \mapsto \mathrm{Pyk}(\mathcal{S})^\lambda_{/K}$$

along $(\mathcal{S}^\wedge_\pi)_\lambda \to \mathcal{S}^\wedge_\pi$ and define

$$\mathbf{Cond}(\mathcal{S}) \simeq \operatorname*{colim}_{\lambda} \mathbf{Cond}(\mathcal{S})^\lambda.$$

We may also consider $\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})$ as a sheaf of ∞-categories on $\mathcal{S}^\wedge_\pi$ via left Kan-extension. The resulting sheaf is therefore an accessible sheaf and thus κ-compact for some regular cardinal κ. It follows that the ∞-category of natural transformations

$$\mathrm{Fun}^{\mathrm{Cond}}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Cond}(\mathcal{S})) = \int_{K\in\mathcal{S}^\wedge_\pi} \mathrm{Fun}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X})(K), \mathbf{Cond}(\mathcal{S})(K))$$

is equivalent to the filtered colimit

$$\operatorname*{colim}_{\lambda>\kappa} \mathrm{Fun}^\lambda_{\mathrm{Pyk}(\mathcal{S})}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Pyk}(\mathcal{S})^\lambda).$$

Thus the above equivalence shows that we have an equivalence

$$\mathcal{X}^{\mathrm{pyk,acc}} \simeq \mathrm{Fun}^{\mathrm{Cond}}(\hat{\Pi}^S_{(\infty,1)}(\mathcal{X}), \mathbf{Cond}(\mathcal{S})).$$

7.3. The pro-étale homotopy type of a scheme

We now apply the results of the last section to introduce and study the pro-étale homotopy type $\Pi^{\mathrm{pro\acute{e}t}}_\infty(X)$ of a scheme X. We give the main definition and prove some first elementary results in § 7.3.1. Also, we show that $\Pi^{\mathrm{pro\acute{e}t}}_\infty(X)$ agrees with the *condensed shape* of X, introduced in [41, Appendix A].

In § 7.3.2 we prove an internal version of Quillen's Theorem B (see Theorem 7.3.2.11) up to localization at a class of morphisms. While the result may be of independent interest, we mainly want to apply it in the case where we work internally to $\mathrm{Pyk}(\mathcal{S})$ and localize at the class of morphisms that get inverted by pro-finite completion.

Finally we apply Theorem 7.3.2.11 to the functor of $\mathrm{Pyk}(\mathcal{S})$-categories $\mathrm{Gal}(f)\colon \mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ induced by a smooth and proper morphism of schemes $f\colon X \to Y$. Using the invariance under specialization for étale homotopy types [35, Proposition 2.49], we deduce that the fibre of the map $\Pi_\infty^{\mathrm{proét}}(f)\colon \Pi_\infty^{\mathrm{proét}}(X) \to \Pi_\infty^{\mathrm{proét}}(Y)$ agrees with the pro-étale homotopy type of the geometric fiber, up to pro-finite completion.

7.3.1. Fundamentals of the pro-étale homotopy type. Let X be a qcqs scheme. It follows from Theorem 7.2.0.1 that the $\mathrm{Pyk}(\mathcal{S})$-topos $X_{\mathrm{proét}}^{\mathrm{hyp}}$ is locally contractible. In particular the canonical morphism $\Gamma_{\mathrm{pyk}}^*\colon \mathrm{Pyk}(\mathcal{S}) \to X_{\mathrm{proét}}^{\mathrm{hyp}}$ admits a left adjoint that we will denote by $\pi_\sharp$.

Definition 7.3.1.1. Let X be a qcqs scheme. We define the *pro-étale homotopy type of* X to be $\Pi_\infty^{\mathrm{proét}}(X) \coloneqq \pi_\sharp(1)$.

Proposition 7.3.1.2. *There is canonical equivalence*

$$\Pi_\infty^{\mathrm{proét}}(X) \simeq \mathrm{Gal}(X)^{\mathrm{gpd}} \in \mathrm{Pyk}(\mathcal{S}).$$

Proof. This an immediate consequence of Theorem 7.2.0.1 and Proposition 3.1.4.1. □

Recollection 7.3.1.3. Let X be a scheme. We define the étale homotopy type $\Pi_\infty^{\mathrm{ét}}(X) \in \mathrm{Pro}(\mathcal{S})$ of X to be the shape (in the sense of [57, Definition 7.1.6.1]) of the étale ∞-topos $X_{\mathrm{ét}}$. We call $\Pi_{<\infty}^{\mathrm{ét}}(X) = \tau_{<\infty}\Pi_\infty^{\mathrm{ét}}(X)$ the *pro-truncated étale homotopy type of* X and similarly we call $\widehat{\Pi}_\infty^{\mathrm{ét}}(X) = \Pi_\infty^{\mathrm{ét}}(X)_\pi^\wedge$ the *pro-finite étale homotopy type of* X.

Remark 7.3.1.4. Since the shape of $X_{\mathrm{ét}}$ and $X_{\mathrm{ét}}^{\mathrm{hyp}}$ agree up to pro-truncation, it follows from [44, Corollary 5.6] that $\Pi_\infty^{\mathrm{ét}}(X)$ and Friedlander's étale topological type [24] agree up to pro-truncation.

Our next goal will be the promised result that the pro-étale homotopy type is a refinement of the usual pro-truncated étale homotopy type.

7.3.1.5. Let X be a qcqs scheme. Then the inclusion $j\colon \mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}} \to \mathrm{Pro}(\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}})$ is a morphism of sites and therefore induces an algebraic morphism of ∞-topoi

$$\nu^*\colon \mathrm{Sh}_{\mathrm{eff}}^{\mathrm{hyp}}(\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}}) \simeq X_{\mathrm{ét}}^{\mathrm{hyp}} \to X_{\mathrm{proét}}^{\mathrm{hyp}} \simeq \mathrm{Sh}_{\mathrm{eff}}^{\mathrm{hyp}}(\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}}).$$

Proposition 7.3.1.6. *Let $n \in \mathbb{N}$. The functor ν^* restricts to a fully faithful functor $X_{\mathrm{ét},\leq n} \to X_{\mathrm{proét},\leq n}$ on n-truncated objects.*

Proof. The functor ν^* is given by left Kan-extension $j_!$ along the inclusion $\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}} \to \mathrm{Pro}(\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}})$ and then sheafifying the resulting presheaf. Since left Kan-extension along a fully faithful functor is fully faithful, it suffices to show that $j_!(\mathcal{F})$ is already a sheaf whenever $\mathcal{F}$ is an n-truncated sheaf. So let $f\colon \lim_i U_i \to \lim_i V_i$ be an effective epimorphism in $\mathrm{Pro}(\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}})$ and using Proposition 7.2.1.3, we may assume that $f = \lim_i f_i$ where f_i is an effective epimorphism in $\mathcal{X}_{\mathrm{ét}}^{\mathrm{constr}}$. Let us write $U_\bullet$ for the Čech-nerve of f and $U_{i,\bullet}$ for the Čech-nerve of f_i. We want to show that the canonical map

$$j_!(\mathcal{F})(\lim_i V_i) \to \lim_{n\in\Delta} j_!(\mathcal{F})(U_n)$$

is an equivalence. But because $j_!(\mathcal{F})$ is by definition left Kan-extended, this map may be identified with the composite

$$\operatorname*{colim}_i \mathcal{F}(V_i) \to \operatorname*{colim}_i \lim_{n\in\Delta} \mathcal{F}(U_{i,n}) \to \lim_{n\in\Delta} \operatorname*{colim}_i \mathcal{F}(U_{i,n}).$$

Now the first map is an equivalence because $\mathcal{F}$ is a sheaf on $X_{\mathrm{ét}}^{\mathrm{constr}}$ by assumption and the second map is an equivalence by [42, Proposition A.1], since all spaces involved are n-truncated. □

Proposition 7.3.1.7. *Let X be a qcqs scheme. Then there is a canonical equivalence*

$$H_{<\infty}(\Pi_\infty^{\mathrm{proét}}(X)) \simeq \Pi_{<\infty}^{\mathrm{ét}}(X).$$

In particular we also have an equivalence $H_\Sigma^\wedge(\Pi_\infty^{\mathrm{proét}}(X)) \simeq \Pi_\infty^{\mathrm{ét}}(X)_\Sigma^\wedge$ for any set of primes Σ.

PROOF. Recall that under the identification $\mathrm{Pro}(\mathcal{S}_{<\infty}) \simeq \mathrm{Fun}^{\mathrm{lex,acc}}(\mathcal{S}_{<\infty}, \mathcal{S})^{\mathrm{op}}$, the pro-truncated shape $\Pi^{\text{ét}}_{<\infty}(X)$ is given by the composite

$$\mathcal{S}_{<\infty} \xrightarrow{\mathrm{const}} X^{\mathrm{hyp}}_{\text{ét}} \xrightarrow{\Gamma_*} \mathcal{S}.$$

Note that the functor ν^* evidently fits into a commutative square

$$\begin{tikzcd} X^{\mathrm{hyp}}_{\text{ét}} \arrow[r, "\nu^*"] & X^{\mathrm{hyp}}_{\text{proét}} \\ \mathcal{S} \arrow[u, "\mathrm{const}"] \arrow[r, "(-)^{\mathrm{disc}}"'] & \mathrm{Pyk}(\mathcal{S}). \arrow[u, "\Gamma^*_{\mathrm{pyk}}"'] \end{tikzcd}$$

Since ν^* is fully faithful on truncated objects we have equivalences

$$\Gamma_* \circ \mathrm{const} \simeq \Gamma_* \circ \nu_* \circ \nu^* \circ \mathrm{const} \simeq \mathrm{Un} \circ \Gamma^{\mathrm{pyk}}_* \circ \Gamma^*_{\mathrm{pyk}} \circ (-)^{\mathrm{disc}}$$

in $\mathrm{Pro}(\mathcal{S}_{<\infty})$. Since the functor $\mathrm{Un} \circ \Gamma^{\mathrm{pyk}}_* \circ \Gamma^*_{\mathrm{pyk}}$ is representable by $\Pi^{\text{proét}}_\infty(X)$, the claim follows. □

RECOLLECTION 7.3.1.8. Recall that an affine scheme W is w-contractible if every weakly étale cover (in the sense of [14, Definition 1.2]) $Y \to W$ has a section.

In general the pro-étale homotopy type can be very hard (in fact nearly impossible) to compute. However we have the following useful observation:

LEMMA 7.3.1.9. *Let W be a w-contractible affine scheme. Then there is a canonical equivalence $\Pi^{\text{proét}}_\infty(W) \to \pi_0 W \in \mathrm{Pyk}(\mathcal{S})$. Here $\pi_0 W$ is the set of connected components of W, equipped with the topology induced by W.*

PROOF. Let us write $W^{\text{proét}}$ for the full subcategory of $\mathrm{Sch}_{/W}$ spanned by pro-étale affine W-schemes. Note that because W is affine, $\pi_0 W$ is pro-finite. Recall from [14, Lemma 2.2.8] that there is an adjunction

$$\pi_0 \colon W^{\text{proét}} \rightleftarrows \mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}})_{/\pi_0(W)} \colon W \times_{\pi_0(W)} -$$

where the right adjoint is fully faithful. In fact $W \times_{\pi_0(W)} -$ is a morphism of sites and therefore induces a geometric morphism

$$\psi_* \colon W^{\mathrm{hyp}}_{\text{proét}} \to \mathrm{Pyk}(\mathcal{S})_{/K}.$$

Also $W \times_{\pi_0(W)} -$ commutes with the obvious functors from $\mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}})$ and therefore ψ_* refines to a geometric morphism over $\mathrm{Pyk}(\mathcal{S})$. Note that the Lemma now immediately follows if the left adjoint ψ^* is fully faithful. For this we observe that by [14, Example 2.2.2 and Lemma 2.4.8], the above adjunction restricts to the full subcategories $W^{\mathrm{proj}} \subseteq W^{\text{proét}}$ of w-contractible pro-étale X-schemes and $\mathrm{Proj}_{/\pi_0(W)} \subseteq \mathrm{Pro}(\mathrm{Set}^{\mathrm{fin}})_{/\pi_0(W)}$. Since both subcategories are bases for the respective topologies it follows that we may identify ψ^* with the functor

$$\mathrm{Pyk}(\mathcal{S})_{/\pi_0(W)} \simeq \mathrm{Fun}^{\times}(\mathrm{Proj}^{\mathrm{op}}_{/\pi_0(W)}, \mathcal{S}) \to \mathrm{Fun}^{\times}((W^{\mathrm{proj}})^{\mathrm{op}}, \mathcal{S}) \simeq W^{\mathrm{hyp}}_{\text{proét}}$$

given by left Kan-extension along $\mathrm{Proj}_{/\pi_0(W)} \to W^{\mathrm{proj}}$. Thus ψ^* is fully faithful as left Kan-extension along a fully faithful functor. □

REMARK 7.3.1.10. Note that if $f \colon V \to W$ is a map of w-contractible schemes we get an induced commutative square

$$\begin{tikzcd} V^{\mathrm{hyp}}_{\text{proét}} \arrow[r, "\psi_*"] \arrow[d, "f_*"] & \mathrm{Pyk}(\mathcal{S})_{/\pi_0(V)} \arrow[d, "\pi_0(f)_*"] \\ W^{\mathrm{hyp}}_{\text{proét}} \arrow[r, "\psi_*"] & \mathrm{Pyk}(\mathcal{S})_{/\pi_0(W)} \end{tikzcd}$$

and thus the isomorphism in Lemma 7.3.1.9 is natural in W.

Recall that for a scheme X, Hemo-Richarz-Scholbach defined the *condensed shape* of X in [41]. It may be computed by taking a hypercover by w-contractibles $W_\bullet \to X$ and then taking the geometric realization $\Pi^{\mathrm{cond}}(X) = \mathrm{colim}_{\Delta^{\mathrm{op}}} \pi_0(W_\bullet)$ in $\mathrm{Pyk}(\mathcal{S})$. As a consequence of Lemma 7.3.1.9 we therefore obtain:

PROPOSITION 7.3.1.11. *Let X be a qcqs scheme. Then there is an equivalence* $\Pi^{\mathrm{cond}}(X) \simeq \Pi_\infty^{\mathrm{proét}}(X)$.

PROOF. Let $W_\bullet \to X$ be a hypercover by w-contractible objects. Considering these schemes as objects in $X^{\mathrm{hyp}}_{\mathrm{proét}}$ via the Yoneda-embedding we get that $\mathrm{colim}_{\Delta^{\mathrm{op}}} W_\bullet \simeq X$ by the definition of hyperdescent. Since $\pi_\sharp \colon X^{\mathrm{hyp}}_{\mathrm{proét}} \to \mathrm{Pyk}(\mathcal{S})$ preserves colimits, we get that

$$\Pi_\infty^{\mathrm{proét}}(X) \simeq \pi_\sharp(\operatorname*{colim}_{\Delta^{\mathrm{op}}} W_\bullet) \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} \pi_\sharp(W_\bullet).$$

Therefore the claim follows from Lemma 7.3.1.9 and the obvious equivalence $\pi_\sharp(W_\bullet) \simeq \Pi_\infty^{\mathrm{proét}}(W_\bullet)$. $\square$

Thus [41, Proposition A.1] also shows:

PROPOSITION 7.3.1.12. *Let R be condensed ring. Then there is an equivalence of ∞-categories*

$$\mathbf{D}_{\mathrm{proét}}(X,R)^{\mathrm{dual}} \simeq \mathrm{Fun}_{\mathrm{Pyk}(\mathcal{S})}(\Pi_\infty^{\mathrm{proét}}(X), \mathsf{Perf}_R^{\mathrm{Pyk}(\mathcal{S})}).$$

REMARK 7.3.1.13. The above proposition gives a reason to prefer the pro-étale homotopy type $\Pi_\infty^{\mathrm{proét}}(X)$ over the classical étale homotopy type. Even for $R = \mathbb{Q}_l$ it is in general not true that arbitrary dualisable $\mathbb{Q}_l$-sheaves on a scheme X can be recovered as representations of the classical étale homotopy type. See [14, Example 7.4.9] for a concrete example. The pro-étale homotopy type fixes this issue, at the cost of being significantly harder to compute.

7.3.2. Internal Theorem B up to completions. The goal of this section is to prove a general version of Quillen's Theorem B internal to any ∞-topos $\mathcal{B}$. The main application that we have in mind is in the case $\mathcal{B} = \mathrm{Pyk}(\mathcal{S})$, however the general statement may also be of independent interest.

DEFINITION 7.3.2.1. A map $p\colon P \to K$ in $\mathcal{B}_\Delta$ is a *Kan-fibration* if it is both a left and a right fibration.

LEMMA 7.3.2.2. *Let $f\colon \mathsf{P} \to \mathsf{C}$ be a left fibration of $\mathcal{B}$-categories and $\tilde{f}\colon \mathsf{C} \to \Omega_{\mathcal{B}}$ the associated functor. Then for any morphism α in $\mathsf{C}(A)$, given by $\alpha\colon \Delta^1 \times A \to \mathsf{C}$, the map $\tilde{f}(\alpha)$ in $\mathcal{B}_{/A}$ is given by composing*

$$(\{0\} \times A) \times_{\mathsf{C}} \mathsf{P} \to ((\Delta^1 \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}}$$

with the inverse of the equivalence $(\{1\} \times A) \times_{\mathsf{C}} \mathsf{P} \to ((\Delta^1 \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}}$.

PROOF. By pulling back along α we may assume that α is the identity. Also we have an equivalence

$$\mathrm{LFib}_{\mathcal{B}}(\Delta^1 \times A) \simeq \mathrm{Fun}_{\mathcal{B}}(\Delta^1 \times A, \mathcal{B}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{B}_{/A})$$

Now observe that $\tilde{f}(\alpha)$ can be computed as

$$\mathrm{ev}_1(\mathrm{diag}\,\mathrm{ev}_0\,\tilde{f} \to \tilde{f})$$

Here η denotes the unit of the adjunction $\mathrm{ev}_0 \dashv \mathrm{id}\,(-)$. Translating to the fibrational perspective via Theorem 2.1.11.5 we obtain a rectangle

$$\begin{tikzcd}
\{1\} \times \mathsf{P}_{\{0\}} \arrow[r] \arrow[d, "\tilde{f}(\alpha)"] \arrow[dr, phantom, "\lrcorner"] & \mathsf{P}_{\{0\}} \times_{\{0\}\times A} (\Delta^1 \times A) \simeq \Delta^1 \times \mathsf{P}_{\{0\}} \arrow[d, "\eta"] \\
\mathsf{P}_{\{1\}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{P} \arrow[d] \\
\{1\} \times A \arrow[r] & \Delta^1 \times A
\end{tikzcd}$$

and we are done once we see that the composite $\mathsf{P}_{\{0\}} \to \mathsf{P}_{\{0\}} \times_{\{0\}\times A} (\Delta^1 \times A) \to \mathsf{P}$ is equivalent to the inclusion $\mathsf{P}_{\{0\}} \to \mathsf{P}$ after applying $(-)^{\mathrm{gpd}}$. But this is clear, since the two inclusions $\{\varepsilon\} \times \mathsf{P}_{\{0\}} \hookrightarrow \Delta^1 \times \mathsf{P}_{\{0\}}$ become equivalent after applying $(-)^{\mathrm{gpd}}$ and the composite

$$\{0\} \times \mathsf{P}_{\{0\}} \hookrightarrow \Delta^1 \times \mathsf{P}_{\{0\}} \to \mathsf{P}$$

yields the inclusion $\mathsf{P}_{\{0\}} \to \mathsf{P}$ by construction. □

REMARK 7.3.2.3. In the situation of Lemma 7.3.2.2, we may more generally consider a map $\alpha\colon \Delta^n \times A \to \mathsf{C}$ corresponding to a composable sequence of n arrows in $\mathsf{C}(A)$. Let us denote by $j\colon \Delta^1 \to \Delta^n$ the map that sends 0 to 0 and 1 to n. We then get a commutative square

$$\begin{tikzcd}
(\{0\} \times A) \times_{\mathsf{C}} \mathsf{P} \arrow[r] \arrow[d, "\mathrm{id}"'] & ((\Delta^1 \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}} \arrow[d] & (\{1\} \times A) \times_{\mathsf{C}} \mathsf{P} \arrow[l, "\simeq"'] \arrow[d, "\mathrm{id}"] \\
(\{0\} \times A) \times_{\mathsf{C}} \mathsf{P} \arrow[r] & ((\Delta^n \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}} & (\{n\} \times A) \times_{\mathsf{C}} \mathsf{P} \arrow[l, "\simeq"']
\end{tikzcd}$$

where the map in the middle is induced by j. Since left fibrations are *smooth* [62, Proposition 4.4.7], the right horizontal maps are equivalences and thus also the vertical map in the middle is an equivalence. It follows that the composite of the lower left map with the inverse of the lower right map is equivalent to $\tilde{f}$ applied to the composite of the n arrows determined by α.

LEMMA 7.3.2.4. *Let $p\colon \mathsf{P} \to \mathsf{C}$ be a Kan-fibration. Then the classified functor $\tilde{p}\colon \mathsf{C} \to \Omega_{\mathcal{B}}$ factors through the inclusion of the maximal subgroupoid $\Omega_{\mathcal{B}}^{\simeq} \to \Omega_{\mathcal{B}}$.*

PROOF. By Lemma 7.3.2.2, it follows that we only need to see that the map

$$(\{0\} \times A) \times_{\mathsf{C}} \mathsf{P} \to ((\Delta^1 \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}}$$

is an equivalence. But this follows because $(\{0\} \times A) \to (\Delta^1 \times A)$ is initial and a pullback of a initial map along a right fibration is initial by [62, Proposition 4.4.7]. □

LEMMA 7.3.2.5. *Consider a cartesian square of $\mathcal{B}$-categories*

$$\begin{tikzcd}
\mathsf{Q} \arrow[r] \arrow[d] & \mathsf{P} \arrow[d, "p"] \\
\mathsf{D} \arrow[r] & \mathsf{C}
\end{tikzcd}$$

and assume that p is a Kan-fibration. Then the induced square

$$\begin{tikzcd}
\mathsf{Q}^{\mathrm{gpd}} \arrow[r] \arrow[d] & \mathsf{P}^{\mathrm{gpd}} \arrow[d, "p^{\mathrm{gpd}}"] \\
\mathsf{D}^{\mathrm{gpd}} \arrow[r] & \mathsf{C}^{\mathrm{gpd}}
\end{tikzcd}$$

in $\mathcal{B}$ is also cartesian.

PROOF. The Kan-fibration p classifies a functor $\tilde{p}\colon \mathsf{C} \to \Omega_{\mathcal{B}}$ that by Lemma 7.3.2.4 factors through the canonical map $\mathsf{C} \to \mathsf{C}^{\mathrm{gpd}}$. It follows that there is some map of $\mathcal{B}$-groupoids $\mathsf{T} \to \mathsf{C}^{\mathrm{gpd}}$ and a rectangle

$$\begin{tikzcd}
\mathsf{Q} \arrow[r] \arrow[d] & \mathsf{P} \arrow[r] \arrow[d, "p"] & \mathsf{T} \arrow[d] \\
\mathsf{D} \arrow[r] & \mathsf{C} \arrow[r] & \mathsf{C}^{\mathrm{gpd}}
\end{tikzcd}$$

in which all squares are cartesian. Since geometric realizations are universal in $\mathcal{B}$, the localization functor $(-)^{\mathrm{gpd}}\colon \mathrm{Cat}(\mathcal{B}) \to \mathcal{B}$ is locally cartesian (in the sense of [28, § 1.2]) and the claim follows. □

The main technical input for our version of Theorem B is the following Proposition, which may be seen as a variation of [68, Theorem 5.1]

Proposition 7.3.2.6. *Let C be a $\mathcal{B}$-category and $L\colon \mathcal{B} \to \mathcal{C}$ be a colimit preserving functor to some ∞-category $\mathcal{C}$. Furthermore fix a collection of objects $\mathcal{G}$ in $\mathcal{B}$ that generate $\mathcal{B}$ under colimits. Let $f\colon \mathsf{P} \to \mathsf{C}$ be a left fibration and $\tilde{f}\colon \mathsf{C} \to \Omega_{\mathcal{B}}$ be the corresponding functor. Suppose that for any $A \in \mathcal{G}$, the composite*

$$\mathsf{C}(A) \xrightarrow{\tilde{f}(A)} \mathcal{B}_{/A} \to \mathcal{B} \xrightarrow{L} \mathcal{C}$$

sends all morphisms to equivalences. Then for any $K \in \mathcal{B}$ and any map $d\colon K \to \mathsf{C}$ the induced map

$$\tilde{f}(d) = K \times_{\mathsf{C}} \mathsf{P} \to K \times_{\mathsf{C}^{\mathrm{gpd}}} \mathsf{P}^{\mathrm{gpd}}$$

is an equivalence after applying L.

Proof. Since colimits in $\mathcal{B}_\Delta$ are universal, we may assume that $K \in \mathcal{G}$. We factor $K \to \mathsf{C}$ as $K \xrightarrow{i} T \xrightarrow{p} \mathsf{C}$ where i is contained in the smallest saturated class in $(\mathcal{B}_\Delta)_{/\mathsf{C}}$ containing all maps of the form

$$\begin{tikzcd} \{\varepsilon\} \times A \arrow[rr] \arrow[dr] & & \Delta^n \times A \arrow[dl] \\ & \mathsf{C} & \end{tikzcd}$$

where $\varepsilon \in \{0, n\}$, $A \in \mathcal{G}$ and p is right orthogonal to these maps. It follows from [62, Lemma 4.1.2] that p is a Kan-fibration. Therefore we have a canonical equivalence

$$K \times_{\mathsf{C}^{\mathrm{gpd}}} \mathsf{P}^{\mathrm{gpd}} \simeq T^{\mathrm{gpd}} \times_{\mathsf{C}^{\mathrm{gpd}}} \mathsf{P}^{\mathrm{gpd}} \simeq (T \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}}$$

by Lemma 7.3.2.5. Thus it suffices to see that the induced map $K \times_{\mathsf{C}} \mathsf{P} \to T \times_{\mathsf{C}} \mathsf{P}$ induces an equivalence after applying $L \circ (-)^{\mathrm{gpd}}$. We note that by universality of colimits in $\mathcal{B}_\Delta$, the class S of all maps $s\colon A \to B$ in $(\mathcal{B}_\Delta)_{/\mathsf{C}}$ that have the property that

$$L \operatorname*{colim}_{\Delta^{\mathrm{op}}}(A \times_{\mathsf{C}} \mathsf{P}) \to L \operatorname*{colim}_{\Delta^{\mathrm{op}}}(B \times_{\mathsf{C}} \mathsf{P})$$

is an equivalence is a saturated class. To see that i is contained in S it therefore suffices to check this for the maps $\{\varepsilon\} \times A \to \Delta^n \times A$, where $A \in \mathcal{G}$ and $\varepsilon \in \{0, n\}$. Note that since the pulled back functor $(\Delta^n \times A) \times_{\mathsf{C}} \mathsf{P} \to \Delta^n \times A$ is again a left fibration the induced map $(\{n\} \times A) \times_{\mathsf{C}} \mathsf{P} \to (\Delta^n \times A) \times_{\mathsf{C}} \mathsf{P}$ is final as a pullback of a final map along a left fibration by [62, Proposition 4.4.7]. In particular

$$((\{n\} \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}} \to ((\Delta^n \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}}$$

is an equivalence, so $\{n\} \times A \to \Delta^n \times A$ is in S. Furthermore under this equivalence the induced map

$$(\{0\} \times A) \times_{\mathsf{C}} \mathsf{P} \to ((\Delta^n \times A) \times_{\mathsf{C}} \mathsf{P})^{\mathrm{gpd}}$$

is equivalent to the map $(\{0\} \times A) \times_{\mathsf{C}} \mathsf{P} \to (\{n\} \times A) \times_{\mathsf{C}} \mathsf{P}$ induced by $0 \to n$ (see Lemma 7.3.2.2 and Remark 7.3.2.3). But this map is an L-equivalence by assumption. Therefore i is contained in S and the proof is complete. $\square$

Next we need to produce a suitable left fibration, to which we can apply the above proposition.

7.3.2.7. Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories. We consider the comma category $\mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D}$ defined via the pullback

$$\begin{tikzcd} \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}^{\Delta^1} \arrow[d] \\ \mathsf{C} \times \mathsf{D} \arrow[r, "f \times \mathrm{id}"'] & \mathsf{D} \times \mathsf{D} \end{tikzcd}$$

Furthermore the commutative diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathrm{diag} \circ f"] \arrow[d, "(\mathrm{id}, f)"] & \mathsf{D}^{\Delta^1} \arrow[d] \\ \mathsf{C} \times \mathsf{D} \arrow[r, "f \times \mathrm{id}"'] & \mathsf{D} \times \mathsf{D} \end{tikzcd}$$

induces a functor $j\colon \mathsf{C} \to \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D}$ that composed with $\mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \to \mathsf{C} \times \mathsf{D} \xrightarrow{\mathrm{pr}_2} \mathsf{D}$ recovers f.

PROPOSITION 7.3.2.8. *The composite* $\mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \to \mathsf{C} \times \mathsf{D} \xrightarrow{\mathrm{pr}_2} \mathsf{D}$ *is a cocartesian fibration. Furthermore for any morphism* $s\colon d \to d' \in \mathsf{D}(A)$ *in context* A *the induced functor on fibres is the canonical functor*

$$\mathsf{C}_{/d} = \mathsf{C} \times_{\mathsf{D}} \mathsf{D}_{/x} \to \mathsf{C} \times_{\mathsf{D}} \mathsf{D}_{/y} = \mathsf{C}_{/d'}$$

in $\mathrm{Cat}(\mathcal{B}_{/A})$ *induced by* $s_!\colon \mathsf{D}_{/d} \to \mathsf{D}_{/d'}$.

PROOF. The fact that $\mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \to \mathsf{C} \times \mathsf{D} \to \mathsf{D}$ is a cocartesian fibration follows immediately from [57, Corollary 2.4.7.12]. The second claim follows because the pullback square

$$\begin{tikzcd} \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}^{\Delta^1} \arrow[d] \\ \mathsf{C} \times \mathsf{D} \arrow[r, "f \times \mathrm{id}"'] & \mathsf{D} \times \mathsf{D} \end{tikzcd}$$

is in fact a pullback in $\mathsf{Cocart}_{\mathsf{D}}$. Under the equivalence of Theorem 2.3.2.7, it therefore corresponds to a cartesian square of functors $\mathsf{D} \to \mathsf{Cat}_{\mathcal{B}}$

$$\begin{tikzcd} \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}_{/-} \arrow[d] \\ \operatorname{diag} \mathsf{C} \arrow[r, "f"'] & \operatorname{diag} \mathsf{D} \end{tikzcd}$$

which proves the claim. $\square$

LEMMA 7.3.2.9. *The functor* $j\colon \mathsf{C} \to \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D}$ *is a left adjoint, so in particular initial.*

PROOF. The functor j sits inside the commutative diagram

$$\begin{tikzcd} \mathsf{C} \arrow[r, "f"] \arrow[d, "j"'] & \mathsf{D} \arrow[d, "\operatorname{diag}"] \\ \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \arrow[r] \arrow[d] & \mathsf{D}^{\Delta^1} \arrow[d, "\mathrm{ev}_0"] \\ \mathsf{C} \arrow[r, "f"] & \mathsf{D} \end{tikzcd}$$

in which all squares are cartesian. Since diag is the fully faithful left adjoint of ev_0, the proof of Lemma 3.3.3.9 shows that j is also a fully faithful left adjoint. $\square$

7.3.2.10. Let C be a $\mathcal{B}$-category. Composing with the inclusion $\Omega_{\mathcal{B}} \to \mathsf{Cat}_{\mathcal{B}}$ fits into a commutative square

$$\begin{tikzcd} \mathrm{Fun}_{\mathcal{B}}(\mathsf{D}, \Omega_{\mathcal{B}}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{Fun}_{\mathcal{B}}(\mathsf{D}, \mathsf{Cat}_{\mathcal{B}}) \arrow[d, "\simeq"] \\ \mathrm{LFib}(\mathsf{D}) \arrow[r, hook] & \mathrm{Cocart}(\mathsf{D}) \end{tikzcd}$$

Let us denote the left adjoint of the lower horizontal functor by F. Recall that by general facts about factorization systems, F is given by factoring a cocartesian fibration into an initial functor followed by a left fibration. If follows by passing to left adjoints, that for a cocartesian fibration $p\colon \mathsf{P} \to \mathsf{D}$ classifying a functor $\tilde{p}\colon \mathsf{D} \to \mathsf{Cat}_{\mathcal{B}}$, the left fibration $F(p)$ classifies the composite

$$\mathsf{D} \to \mathsf{Cat}_{\mathcal{B}} \xrightarrow{(-)^{\mathrm{gpd}}} \Omega_{\mathcal{B}}.$$

We can now finally formulate the following version of Quillen's Theorem B:

THEOREM 7.3.2.11. *Let $f\colon \mathsf{C} \to \mathsf{D}$ be a functor of $\mathcal{B}$-categories and let $L\colon \mathcal{B} \to \mathcal{C}$ be a colimit preserving functor to some ∞-category $\mathcal{C}$. Furthermore fix a collection of objects $\mathcal{G}$ in $\mathcal{B}$ that generate $\mathcal{B}$ under colimits. Suppose that for any $A \in \mathcal{G}$, and any map $s\colon d \to d'$ in $\mathsf{D}(A)$ the induced functor*

$$(\mathsf{C}_{/d})^{\mathrm{gpd}} \to (\mathsf{C}_{/d'})^{\mathrm{gpd}}$$

is an equivalence after applying L. Then for any $K \in \mathcal{B}$ and any object $d\colon K \to \mathsf{D}$ in context K, the canonical map

$$(\mathsf{C}_{/d})^{\mathrm{gpd}} \to K \times_{\mathsf{D}^{\mathrm{gpd}}} \mathsf{C}^{\mathrm{gpd}}$$

induced by the cartesian square

$$\begin{tikzcd} \mathsf{C}/d \arrow[r] \arrow[d] & \mathsf{C} \arrow[d, "f"] \\ \mathsf{D}/d \arrow[r, "(\pi_d)_!"] & \mathsf{D} \end{tikzcd}$$

and the equivalence $K \simeq (\mathsf{D}_{/d})^{\mathrm{gpd}}$, is an equivalence after inverting L.

PROOF. By Proposition 7.3.2.8 and Lemma 7.3.2.9 we have a factorization

$$\mathsf{C} \xrightarrow{j} \mathsf{C} \downarrow_{\mathsf{D}} \mathsf{D} \xrightarrow{p} \mathsf{D}$$

into an initial functor followed by a cocartesian fibration. Therefore also the induced left fibration $F(p)\colon \mathsf{P} \to \mathsf{D}$ fits into a factorization

$$\mathsf{C} \xrightarrow{j'} \mathsf{P} \xrightarrow{F(p)} \mathsf{D}$$

where j' is initial. Now our assumptions and Proposition 7.3.2.8 guarantee that we may apply Proposition 7.3.2.6 to the left fibration $F(p)$. Thus the canonical map

$$(\mathsf{C}_{/d})^{\mathrm{gpd}} \to K \times_{\mathsf{D}^{\mathrm{gpd}}} \mathsf{P}^{\mathrm{gpd}}$$

is an equivalence after applying L. Finally, we consider the commutative diagram

$$\begin{tikzcd} & \mathsf{C}_{/d} \arrow[r] \arrow[d, "i"'] & \mathsf{C} \arrow[d, "j'"] \arrow[dd, bend left, "f"] \\ \mathsf{C}_{/d} \arrow[r, "i'"] \arrow[d] & \mathsf{D}_{/d} \times_{\mathsf{D}} \mathsf{P} \arrow[r] \arrow[d] & \mathsf{P} \arrow[d] \\ K \arrow[r, "\mathrm{id}_d"'] & \mathsf{D}_{/d} \arrow[r] & \mathsf{D} \end{tikzcd}$$

Since j' is initial and $\mathsf{D}_{/d} \times_{\mathsf{D}} \mathsf{P} \to \mathsf{P}$ is a right fibration, i is initial as well. Also, since $\mathsf{D}_{/d} \times_{\mathsf{D}} \mathsf{P} \to \mathsf{D}_{/d}$ is a left fibration, i' is final as the pullback of id_d. It follows that id_d, i', i and j' all induce equivalences after applying $(-)^{\mathrm{gpd}}$. Thus we may identify the map $(\mathsf{C}_{/d})^{\mathrm{gpd}} \to K \times_{\mathsf{D}^{\mathrm{gpd}}} \mathsf{P}^{\mathrm{gpd}}$ from above with the canonical map

$$(\mathsf{C}_{/d})^{\mathrm{gpd}} \to K \times_{\mathsf{D}^{\mathrm{gpd}}} \mathsf{C}^{\mathrm{gpd}},$$

which completes the proof. □

7.3.3. Comparing geometric and homotopy theoretic fibres. One of the most fundamental, but also quite difficult questions in étale homotopy theory is concerned with computing fibres of a map of étale homotopy types $\Pi_\infty^{\text{ét}}(X) \to \Pi_\infty^{\text{ét}}(Y)$ induced by a map of schemes $f\colon X \to Y$. The goal of this section is to compare the homotopy theoretic fibre of the induced map $\Pi_\infty^{\text{proét}}(f)\colon \Pi_\infty^{\text{proét}}(X) \to \Pi_\infty^{\text{proét}}(Y)$ with the scheme theoretic fibre if f is smooth an proper.

Whenever $\mathcal{E} \in \mathrm{Pro}(\mathrm{Lay}_\pi)$ we will simply write $\mathcal{E}^{\mathrm{gpd}} \in \mathrm{Pyk}(\mathcal{S})$ for the groupoidification of the $\mathrm{Pyk}(\mathcal{S})$-category, associated with $\mathcal{E}$ via the embedding $\mathrm{Pro}(\mathrm{Lay}_\pi) \to \mathrm{Cat}(\mathrm{Pyk}(\mathcal{S}))$. We then have the following special case of our Theorem B:

THEOREM 7.3.3.1. *Let Σ be a set of prime numbers Suppose that $f\colon \mathcal{C} \to \mathcal{D}$ is a functor in* $\mathrm{Pro}(\mathrm{Lay}_\pi)$ *such that for any map $d \to d'$ in $\mathcal{D}$ the induced morphism*

$$H_\Sigma^\wedge((\mathcal{C}_{/d})^{\mathrm{gpd}}) \to H_\Sigma^\wedge((\mathcal{C}_{/d'})^{\mathrm{gpd}})$$

in $\mathrm{Pro}(\mathcal{S}_\Sigma)$ *is an equivalence. Then for any $d \in \mathcal{D}$ the induced map $H_\Sigma^\wedge((\mathcal{C}_{/d})^{\mathrm{gpd}}) \to H_\Sigma^\wedge(\mathrm{fib}_d(f^{\mathrm{gpd}}))$ is an equivalence.*

For its proof we need the following small additional input:

LEMMA 7.3.3.2. *Consider a cartesian square*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ L \arrow[r] & K \end{tikzcd}$$

in $\mathrm{Pyk}(\mathcal{S})$ *where $K, L \in$* $\mathrm{Pro}(\mathcal{S}_\Sigma)$ *and X is the colimit of a simplicial diagram $\Delta^{\mathrm{op}} \to \mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pyk}(\mathcal{S})$. Then $H_\Sigma^\wedge$ preserves this pullback square.*

PROOF. Since geometric realizations are universal in both $\mathrm{Pyk}(\mathcal{S})$ and $\mathrm{Pro}(\mathcal{S}_\Sigma)$ (see [34, Corollary 1.13 & Example 1.9]), we can assume that X is in the image of $\mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pyk}(\mathcal{S})$. Now observe that because $\mathrm{Pro}(\mathcal{S}_\pi) \hookrightarrow \mathrm{Pyk}(\mathcal{S})$ is fully faithful, the composite

$$\mathrm{Pro}(\mathcal{S}_\pi) \hookrightarrow \mathrm{Pyk}(\mathcal{S}) \xrightarrow{H_\Sigma^\wedge} \mathrm{Pro}(\mathcal{S}_\Sigma)$$

is equivalent to the left adjoint of the inclusion $\mathrm{Pro}(\mathcal{S}_\Sigma) \to \mathrm{Pro}(\mathcal{S}_\pi)$. Since this left adjoint is locally cartesian closed, as a consequence of [36, Proposition 3.18], the claim follows. $\square$

EXAMPLE 7.3.3.3. Note that since the inclusion $\mathrm{Pro}(\mathrm{Lay}_\pi) \to \mathrm{Cat}_{\mathrm{Pyk}(\mathcal{S})}$ factors through the full subcategory $\mathrm{CSS}(\mathrm{Pro}(\mathcal{S}_\pi)) \subseteq \mathrm{Cat}_{\mathrm{Pyk}(\mathcal{S})}$ (see the proof of Corollary 7.1.1.13) it follows that for $\mathcal{C} \in \mathrm{Pro}(\mathrm{Lay}_\pi)$ its groupoidification $\mathcal{C}^{\mathrm{gpd}} \in \mathrm{Pyk}(\mathcal{S})$ satisfies the assumption of Lemma 7.3.3.2.

PROOF OF THEOREM 7.3.3.1. We consider $\mathcal{C} \to \mathcal{D}$ as a functor of $\mathrm{Pyk}(\mathcal{S})$-categories. We want to apply Theorem 7.3.2.6 in the case where $\mathcal{B} = \mathrm{Pyk}(\mathcal{S})$, L is the functor $H_{\hat{\Sigma}}$

$$(-)_\Sigma^\wedge\colon \mathrm{Pyk}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\Sigma),$$

and $\mathcal{G} = \mathrm{Proj}$ is the collection of projective pro-finite spaces. Thus we need to see that for any $K \in \mathrm{Proj}$ and map $d \to d' \in \mathcal{D}(K)$ the induced map

$$(\mathcal{C}_{/d})^{\mathrm{gpd}} \to (\mathcal{C}_{/d'})^{\mathrm{gpd}}$$

in $\mathrm{Pyk}(\mathcal{S})_{/K}$ induces an equivalence after applying $H_\Sigma^\wedge$. Since equivalences of pro-finite spaces can be checked fibrewise, it follows from Lemma 7.3.3.2 (which we may apply thanks to Example 7.3.3.3) that it suffices to see that for any point $x\colon * \to K$ the map

$$(\mathcal{C}_{/d})_x^{\mathrm{gpd}} \to (\mathcal{C}_{/d'})_x^{\mathrm{gpd}}$$

on fibres is an equivalence after applying $H_\Sigma^\wedge$. But this map is by construction the map

$$(\mathcal{C}_{/d \circ x})^{\mathrm{gpd}} \to (\mathcal{C}_{/d' \circ x})^{\mathrm{gpd}}$$

induced by the morphism $d \circ x \to d' \circ x$ in $\mathcal{D}$. Thus it is an equivalence after applying $H_\Sigma^\wedge$ by assumption. $\square$

LEMMA 7.3.3.4. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor in* $\mathrm{Pro}(\mathrm{Lay}_\pi)$ *considered as a functor of* $\mathrm{Pyk}(\mathcal{S})$*-categories. If the underlying functor of ∞-categories is a left fibration, then f is a left fibration of* $\mathrm{Pyk}(\mathcal{S})$*-categories.*

PROOF. The functor f is a left fibration if and only if for every profinite set K and every map $x\colon K \to \mathcal{C}$ the induced functor $f_{x/}\colon \mathcal{C}_{x/} \to \mathcal{D}_{f(x)/}$ is an equivalence of $\mathrm{Pyk}(\mathcal{S})$-categories.

Recall that $\mathcal{C}_{x/}$ is defined via the pullback

$$\begin{tikzcd} \mathcal{C}_{x/} \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d, "\mathrm{ev}_0"] \\ K \arrow[r, "x"'] & \mathcal{C}, \end{tikzcd}$$

and $\mathcal{D}_{f(x)/}$ is defined likewise. Observe that the functor $f_{x/}$ is induced by a map in $\mathrm{Pro}(\mathrm{Lay}_\pi)$ and thus it suffices to see that the functor on underlying ∞-categories is an equivalence by Lemma 7.1.1.14. This map in turn can be identified with the functor induced by $f_{x/}$ upon passing to global sections. Hence, since the functor $f_{x/}$ is a map of $\mathrm{Pyk}(\mathcal{S})$-categories over K, it suffices to show that for every global section $k\colon * \to K$ the map of fibres $(\mathcal{C}_{x/})|_k \to (\mathcal{D}_{f(x)/})|_k$ induces an equivalence on global sections. The latter functor can in turn be identified with the map $\mathcal{C}_{x\circ k/} \to \mathcal{D}_{f(x\circ k)/}$ and is therefore an equivalence by assumption. $\square$

LEMMA 7.3.3.5. *Let $\mathcal{C} \in \mathrm{Pro}(\mathrm{Lay}_\pi)$, considered as a $\mathrm{Pyk}(\mathcal{S})$-category. Then an object $x\colon 1 \to \mathcal{C}$ is initial if and only if it is an initial object of the underlying ∞-category of $\mathcal{C}$.*

PROOF. Recall that x is initial if and only if the map of $\mathrm{Pyk}(\mathcal{S})$-categories $\mathcal{C}_{x/} \to \mathcal{C}$ is an equivalence. Thus the claim follows from Proposition 7.1.1.14. $\square$

NOTATION 7.3.3.6. If Y is a scheme and $\bar{y} \to Y$ a geometric point, we denote by $Y_{(\bar{y})} = \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{Y,\bar{y}})$ the strict henselisation of Y at $\bar{y}$ (see [Stacks, Tag 0BSK]). If $f\colon X \to Y$ is a morphism of schemes, we write $X_{(\bar{y})} = X \times_Y Y_{(\bar{y})}$ for the *Milnor fibre* of f at $\bar{y}$.

PROPOSITION 7.3.3.7. *Let X be a qcqs scheme and let $\bar{x} \to X$ be a geometric point of X. Then the functor $\mathrm{Gal}(X_{(\bar{x})}) \to \mathrm{Gal}(X)$, induced by the map of schemes $X_{(x)} \to X$, induces an equivalence of $\mathrm{Pyk}(\mathcal{S})$-categories*

$$\mathrm{Gal}(X_{(\bar{x})}) \to \mathrm{Gal}(X)_{\bar{x}/}$$

over $\mathrm{Gal}(X)$.

PROOF. The claim follows immediately if we can show that $\mathrm{Gal}(X_{(x)}) \to \mathrm{Gal}(X)$ is a left fibration and that the closed point $x\colon \bar{x} \to X_{(x)}$ in an initial object. Thus the claim follows from the corresponding statements about the underlying categories by the last two lemmas. These are proven in [15, Expsoé VIII, Corollary 7.6]. $\square$

7.3.3.8. Let $\bar{x}, \bar{y}$ be geometric points of X and $\eta\colon \bar{x} \to \bar{y}$ an étale specialization, so a map of X-schemes $\eta\colon X_{(\bar{x})} \to X_{(\bar{y})}$. Note that in this case $X_{(\bar{x})}$ is also the strict henselisation of $X_{(\bar{y})}$ at the image of $\bar{x}$ along η. Applying Proposition 7.3.3.7 twice, we get a commutative diagram

$$\begin{tikzcd} \mathrm{Gal}(X_{(\bar{x})}) \arrow[r, "\eta"] \arrow[d, "\simeq"'] & \mathrm{Gal}(X_{(\bar{y})}) \arrow[d, "\simeq"'] \\ \mathrm{Gal}(X)_{\bar{x}/} \arrow[r] & \mathrm{Gal}(X)_{\bar{y}/} \end{tikzcd}$$

of $\mathrm{Pyk}(\mathcal{S})$-categories where the lower horizontal map is given by composition with the map $\bar{y} \to \bar{x} \in \mathrm{Gal}(X)$ given by η (see also Remark 7.1.2.9).

In order to apply our abstract Theorem B, we need the following additional input:

Proposition 7.3.3.9 ([36, Corollary 2.4]). *Let $f\colon X \to Y$ be a morphism of qcqs schemes and $\bar{y} \to Y$ a geometric point. Then the canonical commutative square*

$$\begin{tikzcd} \mathrm{Gal}(X_{(\bar{y})}) \arrow[r] \arrow[d] & \mathrm{Gal}(X) \arrow[d] \\ \mathrm{Gal}(Y_{(\bar{y})}) \arrow[r] & \mathrm{Gal}(Y) \end{tikzcd}$$

of $\mathrm{Pyk}(\mathcal{S})$*-categories is cartesian.*

Theorem 7.3.3.10. *Let $f\colon X \to Y$ be a smooth proper morphism of qcqs schemes and $\bar{y} \to Y$ a geometric point of Y. Let Σ be a set of primes invertible on Y. Then the induced morphism*

$$\Pi_\infty^{\text{proét}}(X_{\bar{y}}) \to \mathrm{fib}_{\bar{y}}(\Pi_\infty^{\text{proét}}(f))$$

is an equivalence after applying $H_\Sigma^\wedge$.

Proof. We want apply Theorem 7.3.3.1 to the functor of pro-finite categories

$$\mathrm{Gal}(f)^{\mathrm{op}}\colon \mathrm{Gal}(X)^{\mathrm{op}} \to \mathrm{Gal}(Y)^{\mathrm{op}}.$$

Let $s\colon \bar{y} \to \bar{y}'$ be a map in $\mathrm{Gal}(Y)$ corresponding to a specialization $\eta\colon Y_{(\bar{y}')} \to Y_{(\bar{y})}$. By combining 7.3.3.8 and Proposition 7.3.3.9, it follows that the induced functor

$$\mathrm{Gal}(X)_{\bar{y}'/} \to \mathrm{Gal}(X)_{\bar{y}/}$$

is identified with the functor

$$\mathrm{Gal}(X_{(\bar{y}')}) \to \mathrm{Gal}(X_{(\bar{y})})$$

induced by the morphism of Milnor fibres $X_{(\bar{y}')} \to X_{(\bar{y})}$. Hence Proposition 7.3.1.7 shows that after applying $H_\Sigma^\wedge$, the morphism $\mathrm{Gal}(X_{(x)})^{\mathrm{gpd}} \to \mathrm{Gal}(X_{(y)})^{\mathrm{gpd}}$ is identified with the map on Σ-complete étale homotopy types

$$\Pi_\infty^{\text{ét}}(X_{(\bar{x})})_\Sigma^\wedge \to \Pi_\infty^{\text{ét}}(X_{(\bar{y})})_\Sigma^\wedge.$$

Therefore it is an equivalence by the invariance of the étale homotopy type under specialization [35, Proposition 2.49]. Thus Theorem 7.3.3.1 implies that the canonical map

$$\Pi_\infty^{\text{proét}}(X_{(\bar{y})}) \to \mathrm{fib}_{\bar{y}}(\Pi_\infty^{\text{proét}}(f))$$

is an equivalence after applying $H_\Sigma^\wedge$. Since the canonical map $\Pi_\infty^{\text{proét}}(X_{\bar{y}}) \to \Pi_\infty^{\text{proét}}(X_{(\bar{y})})$ induced by the inclusion of the fibre $X_{\bar{y}} \to X_{(\bar{y})}$ is an equivalence after applying $H_\Sigma^\wedge$ by Proposition 7.3.1.7 and [35, Corollary 2.39], the theorem follows. ☐

Remark 7.3.3.11. Theorem 7.3.3.10 is a variant of Friedlander's fibre sequence [25, Theorem 3.7] for the pro-finite étale homotopy type. However, note that Friedlander in particular assumes that Y is normal and that all geometric fibres of f are connected. These assumptions are not necessary for our result.

Also observe that Theorem 7.3.3.10 does not directly imply Friedlander's result, since $H_\Sigma^\wedge$ typically does not preserve fibre sequences.

Appendix: Locally constant sheaves

This appendix is devoted to the study of locally constant sheaves in ∞-topoi. For the entire section, let us fix a compactly generated ∞-category $\mathcal{E}$. Recall that we write $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B}) = \mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \mathcal{E}) = \mathcal{B} \otimes \mathcal{E}$, where $- \otimes -$ denotes the tensor product in $\mathrm{Pr}^{\mathrm{L}}_{\infty}$. By applying $- \otimes \mathcal{E}$ to the constant sheaf functor $\mathrm{const}_{\mathcal{B}} : \mathcal{S} \to \mathcal{B}$ we obtain an adjunction

$$(\mathrm{const}_{\mathcal{B}} \dashv \Gamma_{\mathcal{B}}) \colon \mathcal{E} \leftrightarrows \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}).$$

Similarly, by applying $- \otimes \mathcal{E}$ to the adjunction $(\pi_A^* \dashv (\pi_A)_*) \colon \mathcal{B} \leftrightarrows \mathcal{B}_{/A}$ for some $A \in \mathcal{B}$, we obtain an induced adjunction

$$(\pi_A^* \dashv (\pi_A)_*) \colon \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}) \leftrightarrows \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}_{/A}).$$

Furthermore, if there is an accessible left exact localisation $L \dashv i \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ we get an induced localisation $L_{\mathcal{E}} \dashv i_{\mathcal{E}} \colon \mathrm{PSh}_{\mathcal{E}}(\mathcal{C}) \to \mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$.

Definition A.1. Let us fix the following terminology:

(1) We call $\mathrm{const}_{\mathcal{B}}(K)$ the *constant sheaf associated to* $K \in \mathcal{E}$. The objects in the essential image of $\mathrm{const}_{\mathcal{B}}$ are called *constant $\mathcal{E}$-valued sheaves*.
(2) We call an $\mathcal{E}$-valued sheaf F *constant with compact values if* it is of the form $\mathrm{const}_{\mathcal{B}}(K)$ for some compact object $K \in \mathcal{E}$.
(3) An $\mathcal{E}$-valued sheaf F is called *locally constant* if there is a cover $(\pi_{A_i}) \colon \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ such that for every i the $\mathcal{E}$-valued sheaf $\pi_{A_i}^* F \in \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}_{/A_i})$ is constant.
(4) We call an $\mathcal{E}$-valued sheaf F *locally constant with compact values* if we can find a cover $(s_i)_i \colon \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ such that $s_i^* F$ is constant with compact values.
(5) We denote by $\mathrm{LConst}^{\mathcal{E}}(\mathcal{B})$ the full subcategory of $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ spanned by the locally constant sheaves, and by $\mathrm{LConst}^{\mathcal{E}}_{\mathrm{cpt}}(\mathcal{B})$ the full subcategory spanned by the locally constant sheaves with compact values.

The key result that we will show in this section is the following:

Proposition A.2. *Suppose that $L \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ is a left exact and accessible localisation, and let F be an $\mathcal{E}$-valued presheaf on $\mathcal{C}$. Then for any $c \in \mathcal{C}$, any $K \in \mathcal{E}^{\omega}$ and any map $K \to L_{\mathcal{E}} F(c)$ there is a collection of morphisms $(s_i \colon c_i \to c)_{i \in I}$ in $\mathcal{C}$ such that $(L s_i) \colon \bigsqcup_i L(c_i) \twoheadrightarrow L(c)$ is a cover in $\mathcal{B}$ and for any $i \in I$ the composite $K \to L_{\mathcal{E}} F(c) \xrightarrow{s_i^*} L_{\mathcal{E}} F(c_i)$ factors as a composite $K \xrightarrow{m_i} F(c_i) \xrightarrow{\varepsilon_F(c_i)} L_{\mathcal{E}} F(c_i)$ for some $m_i \colon K \to F(c_i)$.*

As an immediate consequence we obtain the following:

Corollary A.3. *Let $f \colon \mathrm{const}_{\mathcal{B}}(K) \to \mathrm{const}_{\mathcal{B}}(M)$ be a morphism in $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ where K is compact. Then there is a cover $(\pi_{A_i}) \colon \bigsqcup_i A_i \twoheadrightarrow 1$ in $\mathcal{B}$ and maps $f_i \colon K \to M$ in $\mathcal{E}$ for each i such that $\pi_{A_i}^* f$ is equivalent to $\mathrm{const}_{\mathcal{B}_{/A_i}}(f_i)$.*

Proof. We may pick a left exact accessible localisation $L \colon \mathrm{PSh}(\mathcal{C}) \to \mathcal{B}$ where $\mathcal{C}$ has a final object 1. The morphism f corresponds to a map $\tilde{f} \colon K \to \Gamma\, \mathrm{const}_{\mathcal{B}}(M) = \mathrm{const}_{\mathcal{B}}(M)(1)$. By Proposition A.2

we may now find a covering $(\pi_{Lc_i})\colon \bigsqcup_i L(c_i) \twoheadrightarrow 1$ and commutative squares

$$\begin{tikzcd} M = \underline{M}(c_i) \arrow[r] & \mathrm{const}_{\mathcal{B}}(M)(c_i) \\ K \arrow[u, "m_i"] \arrow[r, "\tilde{f}"] & \mathrm{const}_{\mathcal{B}}(M)(1) \arrow[u, "\pi^*_{Lc_i}"'] \end{tikzcd}$$

where $\underline{M}$ denotes the constant M-valued presheaf. Let $f_i = \mathrm{const}_{\mathcal{B}}(m_i)$. Then the above square translates to the statement that $\pi^*_{Lc_i}(f_i)$ is equivalent to $\mathrm{const}_{\mathcal{B}_{/Lc_i}}(f_i)$, and the claim follows. □

Corollary A.4. *The full subcategory* $\mathrm{LConst}^{\mathcal{E}}_{\mathrm{cpt}}(\mathcal{B}) \hookrightarrow \mathcal{E} \otimes \mathcal{B}$ *is closed under finite colimits and retracts.*

Proof. We start by showing the claim about finite colimits. Since $\mathrm{LConst}^{\mathcal{E}}_{\mathrm{cpt}}(\mathcal{B})$ contains the initial object it suffices to see that it is closed under pushouts. So let us consider a span $F \leftarrow G \rightarrow H$ in $\mathrm{LConst}^{\mathcal{E}}_{\mathrm{cpt}}(\mathcal{B})$. We may pass to a cover in $\mathcal{B}$ to assume that F, G and H are constant. Thus by proposition A.3 we may further pass to a cover so that we can assume that the span above is given by applying $\mathrm{const}_{\mathcal{B}}$ to a span in $\mathcal{E}^{\mathrm{cpt}}$. So the claim follows since $\mathcal{E}^{\mathrm{cpt}}$ is closed under finite colimits and $\mathrm{const}_{\mathcal{B}}$ preserves finite colimits. The proof that $\mathrm{LConst}^{\mathcal{E}}_{\mathrm{cpt}}(\mathcal{B})$ is closed under retracts proceeds in the same way. □

In order to prove Proposition A.2, we first need to treat the special case where $\mathcal{E} = \mathcal{S}$ and $K = 1$:

Lemma A.5. *Let* $F \in \mathrm{PSh}(\mathcal{C})$ *and let* $f\colon 1 \to LF(c)$ *be a map for some* $c \in \mathcal{C}$. *Then there is a collection of morphisms* $(s_i\colon c_i \to c)_{i \in I}$ *in* $\mathcal{C}$ *such that* $(Ls_i)\colon \bigsqcup_i L(c_i) \twoheadrightarrow L(c)$ *is a cover in* $\mathcal{B}$ *and maps* $m_i\colon 1 \to F(c_i)$ *for each* i *such that* $s_i^* f$ *is equivalent to the composite* $1 \xrightarrow{m_i} F(c_i) \xrightarrow{\varepsilon_F(c_i)} LF(c_i)$.

Proof. We pick a cover $(t_j)\colon \bigsqcup_j d_j \twoheadrightarrow F$ in $\mathrm{PSh}(\mathcal{C})$. Consider the pullback square

$$\begin{tikzcd} \bigsqcup_j A_j \arrow[r] \arrow[d] & \bigsqcup_i d_j \arrow[d] \\ c \arrow[r, "f"'] & LF \end{tikzcd}$$

in $\mathrm{PSh}(\mathcal{C})$. Covering each $A_j \in \mathrm{PSh}(\mathcal{C})$ with representables c^j_k then yields the desired collection of maps $(s^j_k)\colon \bigsqcup_{j,k} c^j_k \twoheadrightarrow c$. □

To reduce the general case to the above lemma we use the ideas of [32, §2]. Indeed, the fact that $\mathcal{E}$ is by assumption compactly generated means that we may identify $\mathcal{E} \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{S})$. Consequently, we obtain an equivalence $\mathrm{Sh}_{\mathcal{E}}(\mathcal{B}) \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})$. In light of these identifications, the adjunction $L_{\mathcal{E}} \dashv i_{\mathcal{E}}\colon \mathrm{PSh}_{\mathcal{E}}(\mathcal{C}) \to \mathrm{Sh}_{\mathcal{E}}(\mathcal{B})$ translates into the adjunction $\mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathrm{PSh}(\mathcal{C})) \leftrightarrows \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B})$ that is obtained by postcomposition with $L \dashv i$. An analogous observation shows that for $c \in \mathcal{C}$ the evaluation functor $\mathrm{ev}^{\mathcal{E}}_c\colon \mathrm{Sh}_{\mathcal{E}}(\mathcal{B}) \to \mathcal{E}$ is equivalent to the functor $\mathrm{ev}_{c,*}\colon \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{B}) \to \mathrm{Fun}^{\mathrm{lex}}((\mathcal{E}^{\mathrm{cpt}})^{\mathrm{op}}, \mathcal{S})$ given by composing with $\mathrm{ev}_c\colon \mathcal{B} \to \mathcal{S}$.

Proof of Proposition A.2. Since K is compact, the above discussion and Yoneda's lemma allow us to identify $K \to LF(c)$ with a map $f\colon 1 \to LF(c)(K) \simeq L(F(K))(c)$. Therefore we are in the situation of Lemma A.5 and get a collection of morphisms $(s_i\colon c_i \to c)_{i \in I}$ in $\mathcal{C}$ such that $(Ls_i)\colon \bigsqcup_i L(c_i) \twoheadrightarrow L(c)$ is a cover in $\mathcal{B}$ and maps $n_i\colon 1 \to F(K)(c)$ such that for each i we have a commutative square

$$\begin{tikzcd} F(K)(c_i) \arrow[r] & L(F(K))(c_i) \\ 1 \arrow[u, "n_i"'] \arrow[r, "f"] & L(F(K))(c). \arrow[u, "s_i^*"] \end{tikzcd}$$

Via Yoneda's lemma the maps n_i now yield the desired maps $m_i\colon K \to F(c_i)$. □

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