Convexification of differentiable functions with applications in relaxation models in soil mechanics

DISSERTATION

ZUR ERLANGUNG DES DOKTORGRADES DER NATURWISSENSCHAFTEN (DR. RER. NAT.) DER FAKULTÄT FÜR MATHEMATIK DER UNIVERSITÄT REGENSBURG

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im Jahr 2024

Promotionsgesuch eingereicht am: 10.07.2024

Die Arbeit wurde angeleitet von: Prof. Dr. Georg Dolzmann

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1 Introduction

Calculating the convex envelope of multivariate functions plays an important role in optimization and hence in many fields of applied mathematics, physics and mechanics. It is well known that the convex envelope of some lower semi-continuous, superlinear growing function $f : \mathbb{R}^d \to \mathbb{R}$ in some point $x \in \mathbb{R}$ can be obtained by affine interpolation of function values f in not necessarily unique points $x^{(1)}, ..., x^{(q)} \in \mathbb{R}^d$ with $q \leq d+1$. If these points are unique and affine independent, the simplex $\text{conv}\{x^{(1)}, ..., x^{(q)}\}$ is called the phase simplex of x. Seeking to calculate the convex envelope of f in a neighbourhood of x, a natural approach is to vary the vertices of the phase simplex $\text{conv}\{x^{(1)}, ..., x^{(q)}\}$ of x within respective neighbourhoods $U^{(1)}, ..., U^{(q)}$ of $x^{(1)}, ..., x^{(q)}$, looking for further phase simplices. We show that this procedure succeeds whenever the phase simplex of x is maximal in the sense, that it is not the face of a larger phase simplex of some other point x', and f is differentiable and strictly convex in each neighbourhood $U^{(1)}, ..., U^{(q)}$. We derive a continuous parametrization of a neighbourhood of the simplex $\text{conv}\{x^{(1)}, ..., x^{(q)}\}$, such that we can give an expression of the convex envelope in this neighbourhood in terms of the parametrization. This parametrization especially characterizes all involved phase simplices of any dimension. Additionally, we show that the regularity of this parametrization improves to Lipschitz-continuity, if the restriction of f to each neighbourhood $U^{(1)}, ..., U^{(q)}$ has Lipschitz-continuous gradient and is strongly convex.

This work is organized as follows: In Chapter 2, some frequently used facts about Lipschitz continuity and generalized derivatives in the sense of Clarke are collected. Chapter 3 concerns convexity and the connection of the subdifferential and generalized derivatives, especially in view of characterizing strong convexity. We prove a duality result for the Legendre transform of strongly convex functions with Lipschitz-continuous gradient similar to the Fenchel-duality for the convex conjugate. The last part of the chapter presents some useful properties and tools for the calculation of convex envelopes, embedded in the framework of Griewank and Rabier [18]. Since their work already contains a similar approach of characterizing phase simplices near a known maximal phase simplex, we dedicate the first part of Chapter 4 to motivate this work by pointing out the differences to the assumptions of [18]. Then we first consider a specialized setting in which one of the points $x^{(1)}, ..., x^{(q)}$ is the origin and the other points are the first $q-1$ unit vectors, simplifying the algebraic calculations. We characterize all phase simplices containing a vertex near the origin and give a suitable parametrization of a set, in which we can give an expression for the convex envelope in terms of the parametrization. The general case is covered by applying for

$i \in \{1, ..., q\}$ an affine transformation, which maps $x^{(i)}$ to the origin and $x^{(1)}, ..., x^{(i-1)}, x^{(i+1)}, ..., x^{(q)}$ to the first $q-1$ unit vectors. Applying the results from the specialized setting to the transformed function and reverting the affine transformation afterwards, we are able to obtain a characterization of all phase simplices containing a vertex near $x^{(i)}$. To make the parametrizations compatible with each other, a recursive re-parametrization is required in order to combine them to one paramterization of a whole neighbourhood of $\mathrm{conv}\{x^{(1)}, ..., x^{(q)}\}$ capturing all phase simplices of any dimension with vertices near $x^{(1)}, ..., x^{(q)}$ (or a subset of these points). Finally, the convex envelope of f can be given in the parametrized set in terms of the parametrization and is proven to admit the expected regularity. In the last chapter, three examples of functions are presented, for which the convex envelope can be calculated explicitly. The first two examples are designed to illustrate the regularity of the derived parametrization. The last example arose from our joint work [2] concerning relaxation models in soil mechanics. Although the example does not completely match the assumptions of the main theorem, it illustrates the interaction between one- and two-dimensional phase simplices and gave the inspiration for the investigation of convexification of functions with locally Lipschitz continuous derivative.

Acknowledgements I would like to thank my advisor Prof. Dr. Georg Dolzmann for his constant support as well as Prof. Dr. Klaus Hackl and Ghina Jezdan from Ruhr University Bochum for the productive cooperation within our project. I also gratefully acknowledge the support of the Priority Program 2256 „Variational Methods for Predicting Complex Phenomena in Engineering Structures and Materials“ of the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation).

1.1 Notation

For any $n \in \mathbb{N}$, the Euclidean vector space $\mathbb{R}^n$ will be always equipped with the Euclidean norm $||\cdot||_2$ and for any $m \in \mathbb{N}$, the vector space of $m \times n$-matrices $\mathbb{R}^{m\times n}$ will be always equipped with the induced matrix norm $||\cdot||_2$. In both cases, we simply write $||\cdot||$ instead of $||\cdot||_2$. Furthermore the following conventions are used:

$$\begin{aligned}
\mathbb{R}_+ &:= \{x \in \mathbb{R} \mid x > 0\} \\
\mathbb{R}_{\geq t} &:= \{x \in \mathbb{R} \mid x \geq t\},\ (t \in \mathbb{R}) \\
\mathbb{R}^{d\times d}_{sym} &:= \{A \in \mathbb{R}^{d\times d} \mid A^T = A\}, \\
\mathrm{GL}(d) &:= \{A \in \mathbb{R}^{d\times d} \mid \det(A) \neq 0\}, \\
\mathrm{PD}(d) &:= \{A \in \mathbb{R}^{d\times d}_{sym} \mid \forall h \in \mathbb{R}^d \setminus \{0\} : h^T A h > 0\}.
\end{aligned}$$

The set of affine functions and affine transformations is denoted as follows:

$$\operatorname{Aff}(m,n) := \{h : \mathbb{R}^m \to \mathbb{R}^n \mid \underset{A\in\mathbb{R}^{n\times m}}{\exists} \ \underset{b\in\mathbb{R}^n}{\exists} \ \underset{x\in\mathbb{R}^m}{\forall} \ h(x) = b + Ax\}\,,$$
$$\operatorname{AffT}(d) := \{h : \mathbb{R}^d \to \mathbb{R}^d \mid \underset{A\in\operatorname{GL}(d)}{\exists} \ \underset{b\in\mathbb{R}^d}{\exists} \ \underset{x\in\mathbb{R}^d}{\forall} \ h(x) = b + Ax\}\,.$$

For any $A \in \operatorname{GL}(d)$ we write $A^{-T} := (A^T)^{-1} = (A^{-1})^T$. The d-dimensional identity matrix is written as Id_d and for $x \in \mathbb{R}^d$, we define $\operatorname{diag}(x) \in \mathbb{R}^{d\times d}$ as the unique diagonal matrix satisfying for any $i, j \in \{1, ..., d\}$, $\operatorname{diag}(x)_{i,j} = x_i \cdot \chi_{\{j\}}(i)$.
For any subset $M \subset \mathbb{R}^d$, the affine hull and the convex hull are defined as

$$\operatorname{aff}(M) := \{\sum_{i=1}^{m} t_i x_i \mid m \in \mathbb{N}, \underset{i\in\{1,\dots,m\}}{\forall} (t_i, x_i) \in \mathbb{R} \times M, \sum_{i=1}^{m} t_i = 1\} \subset \mathbb{R}^d\,,$$
$$\operatorname{conv}(M) := \{\sum_{i=1}^{m} t_i x_i \mid m \in \mathbb{N}, \underset{i\in\{1,\dots,m\}}{\forall} (t_i, x_i) \in [0,1] \times M, \sum_{i=1}^{m} t_i = 1\} \subset \mathbb{R}^d\,.$$

A set $C \subset \mathbb{R}^n$ is called affine if and only if $C = \operatorname{aff}(C)$ and C is called convex if and only if $C = \operatorname{conv}(C)$.
For some set $M \subset \mathbb{R}^d$, $\operatorname{relint}(M)$ denotes the interior of M with respect to the subspace topology of $\operatorname{aff}(M)$.
We denote with $0_d \in \mathbb{R}^d$ and $0_{m\times n} \in \mathbb{R}^{m\times n}$ the respective neutral elements of addition and omit the index if the dimension is clear from the context.
For $d \in \mathbb{N}$ and $x_0 \in \mathbb{R}^d$ we denote with

$$B_r(x_0) := \{x \in \mathbb{R}^d \mid ||x - x_0||_2 < r\}$$

the open ball with radius r centred at x_0 and with

$$C_r(x_0) := \{x \in \mathbb{R}^d \mid ||x - x_0||_\infty \leq r\}$$

the closed hypercube with edge length $2r$ centred at x_0. If $x_0 = 0_d$ then we abbreviate $B_r^d := B_r(0_d)$ and $C_r^d := C_r(0_d)$.
For a set X and a subset $A \subset X$, we denote with $\operatorname{id}_X : X \to X,\ x \mapsto x$ the identity and with

$$\chi_A : X \to \mathbb{R},\ \chi_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \in X \setminus A \end{cases}$$

the characteristic function of A.
For two sets X, Y and functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$, we write $g \leq f$

if for any $z \in X \cap Y$ the inequality $g(z) \leq f(z)$ is satisfied. If $X \subset Y$ and $g \leq f$, we say that g minorizes f on X.
For a finite set X, $|X| \in \mathbb{N}_0$ denotes the number of elements in X.
Any vector $x \in \mathbb{R}^d$ can be interpreted as an element of $\mathbb{R}^{d\times 1}$ which can be transposed to $x^T \in \mathbb{R}^{1\times d}$.
For any $i \in \{1, ..., d\}$, we denote with $\mathrm{pr}_i : \mathbb{R}^d \to \mathbb{R}$ the projection onto the i-th component and with $\mathrm{pr}_i^\perp : \mathbb{R}^d \to \mathbb{R}^{d-1}$ the mapping, which deletes the i-th component.
For any $I \subset \{1, ..., d\}$ we define

$$\begin{aligned}\mathrm{pr}_I :&\mathbb{R}^d \to \mathbb{R}^d,\ \mathrm{pr}_I(x) = (x_i \cdot \chi_I(i))_{i\in\{1,...,d\}}\,,\\ \mathrm{pr}_I^\perp :&\mathbb{R}^d \to \mathbb{R}^d,\ \mathrm{pr}_I(x) = (x_i \cdot \chi_{\{1,...,d\}\setminus I}(i))_{i\in\{1,...,d\}}\,,\end{aligned}$$

where pr_I replaces all entries of $x \in \mathbb{R}^d$ by zero except those indexed with an element of I, while $\mathrm{pr}_I^\perp$ produces zero exactly at the entries indexed with an element of I, i.e. $x = \mathrm{pr}_I(x) + \mathrm{pr}_I^\perp(x)$ for any $I \subset \{1, ..., d\}$. Notice for $i \in \{1, ..., d\}$ the difference between $\mathrm{pr}_i^\perp$ and $\mathrm{pr}_{\{i\}}^\perp$, since the former reduces the dimension by deleting the i-th component while the latter retains the dimension by setting the i-th component to zero.
For any $k \in \mathbb{N}$ and $\delta \geq 0$, we define

$$\begin{aligned}\Delta_k^\delta &= \{t \in [-\delta, \infty)^k \mid \sum_{i=1}^k t_i = 1\}\,,\\ \tilde{\Delta}_{k-1}^\delta &= \{t \in [-\delta, \infty)^{k-1} \mid \sum_{i=1}^{k-1} t_i \leq 1\}\,.\end{aligned}$$

and abbreviate $\Delta_k := \Delta_k^0$ and $\tilde{\Delta}_{k-1} := \tilde{\Delta}_{k-1}^0$. Notice that for any $i \in \{1, ..., k\}$ the projection $\mathrm{pr}_i^\perp$ maps Δ_k^0 bijective and bi-Lipschitz onto $\tilde{\Delta}_{k-1}^0$.

For any $\alpha \in \mathbb{R} \cup \{\infty\}$ we set $\alpha + \infty := \infty$ and for any $\beta \in \mathbb{R}_+$ we set $\beta \cdot \infty := \infty$. Other arithmetic calculations involving ∞ will not appear.
If $U \subset \mathbb{R}^d$ and $f : U \to \mathbb{R}^n$ is differentiable at $x \in \mathrm{int}(U)$, then $Df(x) \in \mathbb{R}^{n\times d}$ denotes the Jacobian matrix of f in x. If $U \subset \mathbb{R}^d$ is open and $f : U \to \mathbb{R}^n$ is a differentiable function (i.e. differentiable at every $x \in U$), then we set $Df : U \to \mathbb{R}^{n\times d},\ x \mapsto Df(x)$.
If $n = 1$ and f is differentiable at $x \in \mathrm{int}(U)$, then the gradient of f at x is defined as $\nabla f(x) := Df(x)^T \in \mathbb{R}^d$ identifying $\mathbb{R}^{d\times 1} \cong \mathbb{R}^d$ and the first order Taylor polynomial is denoted by $T_x f : \mathbb{R}^d \to \mathbb{R},\ x' \mapsto f(x) + \langle \nabla f(x), x' - x\rangle$ describing the tangent plane of f at x. If $U \subset \mathbb{R}^d$ is open and $f : U \to \mathbb{R}$ is a differentiable function, then we set $\nabla f : U \to \mathbb{R}^d,\ x \mapsto \nabla f(x)$.

2 Lipschitz Calculus

2.1 Basics

2.1 Definition *Fix two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : X \to Y$.
Then f *is called Lipschitz, if there exists some* $L > 0$*, such that for any* $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq L \cdot d_X(x, x') \,. \tag{2.1}$$

Any such $L > 0$ *will be called a Lipschitz constant of* f.
Furthermore, f *is called bi-Lipschitz, if there exists some* $M > 0$*, such that for any* $x, x' \in X$,

$$\frac{1}{M} \cdot d_X(x, x') \leq d_Y(f(x), f(x')) \leq M \cdot d_X(x, x') \,. \tag{2.2}$$

For $x_0 \in X$, f *is called locally Lipschitz (locally bi-Lipschitz) in* x_0*, if there exists some open neighbourhood* $U \subset X$ *of* x_0*, such that the restriction of* f *to* U *is Lipschitz (bi-Lipschitz). We call* f *locally Lipschitz (locally bi-Lipschitz), if* f *is locally Lipschitz (locally bi-Lipschitz) in any point.*

An immediate consequence of the definition is, that for metric spaces (X, d_X) and (Y, d_Y) a function $f : X \to Y$ is bi-Lipschitz, if and only if f is Lipschitz, injective and $f^{-1} : f(X) \to X$ is Lipschitz. Since any Lipschitz function is especially continuous, any bi-Lipschitz function is an embedding (see Definition 6.1).

2.2 Proposition *Assume* $(X, d_X), (Y, d_Y), (Z, d_Z)$ *are metric spaces and the functions* $f, f_1, f_2 : X \to Y$ *and* $g : Y \to Z$ *are Lipschitz.*
Then $g \circ f$ *is Lipschitz and if* g *and* f *are bi-Lipschitz, then* $g \circ f$ *is bi-Lipschitz.*
If Y *is a normed space, for any* $\alpha \in \mathbb{R}$ *the functions* $f_1 + f_2$ *and* $\alpha \cdot f$ *are Lipschitz. If* $Y = \mathbb{R}$ *and* f_1, f_2 *are bounded, then* $f_1 \cdot f_2$ *is Lipschitz. If* $Y = \mathbb{R}$ *and there exists some* $m > 0$*, such that for any* $x \in X$ *we have* $|f(x)| \geq m$*, then* $1/f$ *is Lipschitz.*

Proof. [5, Propositions 2.3.1, 2.3.3, 2.3.4, 2.3.7] □

It is straightforward, that the previous Proposition remains valid, if we replace everywhere „Lipschitz“ by „locally Lipschitz“ and „bi-Lipschitz“ by „locally bi-Lipschitz“.

2.3 Proposition *Assume $(X, d_X), (Y, d_Y)$ are metric spaces. If (X, d_X) is compact, then each locally Lipschitz function $f : X \to Y$ is Lipschitz.*

Proof. [5, Theorem 2.1.6] $\square$

2.4 Corollary *Assume $(X, d_X), (Y, d_Y)$ are metric spaces. If (X, d_X) is compact, then each locally bi-Lipschitz injection $f : X \to Y$ is bi-Lipschitz.*

Proof. By Proposition 6.2, f is an embedding, i.e. $f^{-1} : f(X) \to X$ is continuous. According to Proposition 2.3 the function f is Lipschitz and for any $y = f(x) \in f(X)$, there is a neighbourhood $U_x \subset X$ of x, such that the restriction of f to U_x is bi-Lipschitz. Now $V_y := f(U_x) = (f^{-1})^{-1}(U_x)$ is by continuity of f^{-1} a neighbourhood of y, such that the restriction of f^{-1} to V_y is Lipschitz. So f^{-1} is locally Lipschitz and by compactness of $f(X)$, Proposition 2.3 implies that f^{-1} is Lipschitz. Altogether, f is bi-Lipschitz. $\square$

2.5 Lemma *If $C \subset \mathbb{R}^m$ is convex, $f \in C \to \mathbb{R}^n$ is continuous and $\{C_i \mid i \in I\}$ is a finite cover of C, i.e. $\bigcup\limits_{i \in I} C_i = C$, such that for any $i \in I$ the restriction $f|_{C_i}$ is Lipschitz, then f is Lipschitz.*

Proof. Denote for any $i \in I$ the Lipschitz constant of $f|_{C_i}$ with $L_i > 0$ and set $L := \max\{L_i \mid i \in I\}$. Fix $x, y \in C$. There is some $i_1 \in I$ with $x_0 := x \in C_{i_1}$. Set $t_0 := 0$ and $t_1 := \sup\{t \in [0, 1] \mid (1 - t)x + ty \in C_{i_1}\} \geq t_0$ and $x_1 := (1 - t_1)x + t_1 y$. If $n \in \{1, ..., |I| - 1\}$ and for any $m \in \{1, ..., n\}$ the elements $i_m \in I$, $t_m \in [0, 1]$ and $x_m \in C$ are constructed and $t_n < 1$, then there is some $i_{n+1} \in I \setminus \{i_1, ..., i_n\}$, such that $t_n = \inf\{t \in [t_n, 1] \mid (1 - t)x + ty \in C_{i_{n+1}}\}$. Set

$$t_{n+1} := \sup\{t \in [t_n, 1] \mid (1 - t)x + ty \in C_{i_{n+1}}\} \geq t_n \,,$$
$$x_{n+1} := (1 - t_{n+1})x + t_{n+1} y \,.$$

Since I is finite and $i_{n+1} \in I \setminus \{i_1, ..., i_n\}$, this process terminates at some $i_N \in \{1, ..., |I|\}$ with $t_N = 1$ and $x_N = y$, since y lies in some of the sets $C_i, i \in I$ (not necessarily in C_{i_N}). For any $n \in \{1, ..., N\}$ there exist sequences $(t_l^-)_{l \in \mathbb{N}}, (t_l^+)_{l \in \mathbb{N}} \subset [t_{n-1}, t_n]$ with $x_l^\pm := (1 - t_l^\pm)x + t_l^\pm y \in C_{i_n}$ and $t_l^- \xrightarrow{l \to \infty} t_{n-1}$ and $t_l^+ \xrightarrow{l \to \infty} t_n$. Then $||f(x_l^+) - f(x_l^-)|| \leq L_{i_n} ||x_l^+ - x_l^-||$ and taking the limit $l \to \infty$ on both sides gives us

$$||f(x_n) - f(x_{n-1})|| \leq L_{i_n} ||x_n - x_{n-1}||$$

by continuity of f. Altogether we get

$$\begin{aligned} ||f(y)-f(x)|| &\leq \sum_{n=1}^{N} ||f(x_n)-f(x_{n-1})|| \leq \sum_{n=1}^{N} L_{i_n} ||x_n - x_{n-1}|| \\ &\leq L \sum_{n=1}^{N} ||(t_n - t_{n-1})(y-x)|| = L \sum_{n=1}^{N} (t_n - t_{n-1}) ||y-x|| \\ &= L(t_N - t_0)||y-x|| = L \cdot ||y-x|| \end{aligned}$$

and hence the Lipschitz continuity of f with constant L. □

2.6 Corollary *If $C \subset \mathbb{R}^m$ is convex, $f : C \to \mathbb{R}^n$ is an embedding and $\{C_i \mid i \in I\}$ is a finite partition of C, such that for any $i \in I$ the restriction $f|_{C_i}$ is bi-Lipschitz, then for any compact set $K \subset C$ with $f(K) \subset \mathrm{int}(f(C))$ the restriction f_K is bi-Lipschitz.*

Proof. The function f is Lipschitz by Lemma 2.5 and $f(K)$ is compact, hence according to Proposition 2.3 it suffices to show that the restriction of f^{-1} to $f(K)$ is locally Lipschitz. Fix $y = f(x) \in f(K) \subset \mathrm{int}(f(C))$ and some $r > 0$ with $B_r(y) \subset f(C)$. Since $f^{-1}|_{B_r(y)}$ is continuous and $\{f(C_i) \cap B_r(y) \mid i \in I\}$ is a partition of the convex set $B_r(y)$, such that for any $i \in I$ the restriction of f^{-1} to $f(C_i) \cap B_r(y)$ is Lipschitz, by Lemma 2.5 the restriction of f^{-1} to $B_r(y)$ is Lipschitz. By the fact that $y \in f(K)$ was arbitrary, the restriction of f^{-1} to $f(K)$ is locally Lipschitz. □

2.2 Generalized derivatives

2.7 Definition *For $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ Lipschitz denote with $N_f \subset \Omega$ the set of points, in which f is not differentiable.*

For $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ locally Lipschitz, $N_f \subset \Omega$ has measure zero by Rademacher's theorem. For any sequence $(x_l)_{l\in\mathbb{N}} \subset \Omega \backslash N_f$ converging to some $x_0 \in \Omega$, the sequence $(Df(x_l))_{l\in\mathbb{N}} \subset \mathbb{R}^{m\times n}$ is for sufficiently large l bounded by the local Lipschitz constant of f in a neighbourhood of x_0. Therefore we can extract a subsequence, which converges to some $M \in \mathbb{R}^{m\times n}$. This observation allows the definition of the generalized derivative in the sense of Clarke (denoted by ∂_c to distinguish it from the subdifferential), see for example [3, Chapter 2].

2.8 Definition *Assume* $\Omega \subset \mathbb{R}^n$ *is open and* $f : \Omega \to \mathbb{R}^m$ *is Lipschitz near* $x_0 \in \Omega$. *We define*

$$\mathcal{J}_f(x_0) := \{\lim_{l\to\infty} Df(x_l) \in \mathbb{R}^{m\times n} \mid (x_l)_{l\in\mathbb{N}} \subset \Omega \setminus N_f,\quad x_l \to x_0, \lim_{l\to\infty} Df(x_l) \text{ exists}\}$$

and denote the generalized derivative of f *at* x_0 *by* $\partial_c f(x_0) := \operatorname{conv}(\mathcal{J}_f(x_0))$.

The generalized derivative is indeed a generalization of the concept of continuously differentiable functions. If $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is continuously differentiable, then $N_f = \emptyset$ and for any sequence $(x_l)_{l\in\mathbb{N}} \subset \Omega$ converging to some $x_0 \in \Omega$ the sequence $Df(x_l)$ converges to $Df(x_0)$ by continuity of the derivative. The generalized derivative in any point $x_0 \in \Omega$ therefore contains only one element, namely the classical derivative of f in x_0, i.e. $\mathcal{J}_f(x_0) = \{Df(x_0)\} = \operatorname{conv}(\{Df(x_0)\}) = \operatorname{conv}(\mathcal{J}_f(x_0)) = \partial_c f(x_0)$.

In [3], Clarke first defines for a (possibly infinite dimensional) Banach space X and a function $f : X \to \mathbb{R}$, which is Lipschitz near some point $x_0 \in X$, the generalized gradient as

$$\partial_c f(x_0) := \{v \in X^* \mid \underset{h\in X}{\forall} \limsup_{x\to x_0,\ t\searrow 0} \frac{f(x+th) - f(x)}{t} \geq v(h)\},$$

where X^* denotes the topological dual of X.
He showed in [3, Theorem 2.5.1], that for $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ Lipschitz near $x_0 \in \Omega$, for any set $S \subset \Omega$ with measure zero the gerenalized gradient can be characterized by

$$\partial_c f(x_0) = \operatorname{conv}(\{\lim_{l\to\infty} \nabla f(x_l)^T \in \mathbb{R}^{1\times n} \mid (x_l)_{l\in\mathbb{N}} \subset \Omega \setminus (N_f \cup S),\quad x_l \to x_0, \lim_{l\to\infty} \nabla f(x_l) \text{ exists}\}),$$

where the transposition of the gradient (although not appearing in [3]) provides consistency with Definition 2.8. This raises the question, whether also for a locally Lipschitz function $f : \Omega \to \mathbb{R}^m$ with $m > 1$ the generalized derivative in some point $x_0 \in \Omega$ remains unchanged, if we avoid with the approximating sequences $(x_l)_{l\in\mathbb{N}} \subset \Omega \setminus N_f$ an additional set $S \subset \Omega$ of measure zero. Clarke already proved in [3, Proposition 2.4.6], that for any $h \in \mathbb{R}^n$ the image set $\partial_c f(x_0)h \subset \mathbb{R}^m$ remains unchanged, later on Fabián showed in [7] that even the generalized derivative $\partial_c f(x_0)$ itself is not altered by this modification. This especially implies, that the generalized derivative introduced by Pourciau in [16] coincides with Clarke's generalized derivative, as Fabián pointed out in [7, Remark 1].

We quote some basic properties of the generalized derivative, which are stated in [3, Proposition 2.6.2]:

2.9 Proposition *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}^m$ be Lipschitz near $x_0 \in \Omega$. Then the generalized derivative has the following properties:*

(i) $\partial_c f(x_0) \subset \mathbb{R}^{m\times n}$ *is non-empty, convex and compact,*

(ii) if $\alpha \in \mathbb{R}$, *then* $\partial_c(\alpha \cdot f)(x_0) = \alpha \cdot \partial_c f(x_0)$,

(iii) if $g : \Omega \to \mathbb{R}^m$ *is Lipschitz near* x_0, *then* $\partial_c(f+g)(x_0) \subset \partial_c f(x_0) + \partial_c g(x_0)$.

The following Proposition was already mentioned without a proof in [17, Eq. (4.20)], the special case $m_1 = \ldots = m_k = 1$ can be found in [3, Proposition 2.6.2 (e)].

2.10 Proposition *Fix $k \in \mathbb{N}$, $m_1, \ldots, m_k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open and $x_0 \in \Omega$. If for any $i \in \{1, \ldots, k\}$ the function $f_i : \Omega \to \mathbb{R}^{m_i}$ is Lipschitz near x_0, then*

$$f : \Omega \to \mathbb{R}^{m_1+\ldots+m_k},\ f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix}$$

is Lipschitz near x_0 with

$$\partial_c f(x_0) \subset \{ \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix} \mid M_i \in \partial_c f_i(x_0)\} .$$

Proof. For any $i \in \{1, \ldots, k\}$ let $U_i \subset \Omega$ be an open neighbourhood of x_0, such that f_i is Lipschitz on U_i with some constant $L_i > 0$. Then for any $x, y \in \bigcap_{i=1}^{k} U_k$ and $L := (L_1, ..., L_k)^T \in \mathbb{R}^k$ we can estimate

$$||f(x) - f(y)||_2^2 = \sum_{i=1}^{k} ||f_i(x) - f_i(y)||_2^2 \leq \sum_{i=1}^{k} L_i^2 ||x-y||_2^2 \leq ||L||_\infty^2 ||x-y||_2^2 .$$

f is differentiable in some point $x \in \Omega$ if and only if any f_i is differentiable in x, i.e. $N_f = \bigcup_{i=1}^{k} N_{f_i}$. For any sequence $(x_l)_{l\in\mathbb{N}} \subset \Omega \setminus N_f$, the sequence $(Df(x_l))_{l\in\mathbb{N}}$ converges if and only if for any $i \in \{1, ..., k\}$ the sequence $(Df_i(x_l))_{l\in\mathbb{N}}$ converges. This implies for any $x_0 \in \Omega$ the inclusion

$$\mathrm{conv}(\mathcal{J}_f(x_0)) \subset \mathrm{conv}\{ \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix} \mid M_i \in \mathcal{J}_{f_i}(x_0)\} \subset \{ \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix} \mid M_i \in \partial_c f_i(x_0)\} .$$

as asserted. □

Before we formulate Clarke's inverse function theorem, we first state the following result concerning the generalized Jacobian of Lipschitz inverse functions, proven by the author in [1, Theorem 2.1].

2.11 Lemma *Assume that $U, V \subset \mathbb{R}^d$ are open and that $f : U \to \mathbb{R}^d$ and $g : V \to \mathbb{R}^d$ are Lipschitz continuous with $g \circ f = \mathrm{Id}_U$ and $f \circ g = \mathrm{Id}_V$. Then $f(N_f) = N_g$, $g(N_g) = N_f$ and for any $x_0 \in U$ the set $\mathcal{J}_f(x_0)$ is invertible with*

$$\mathcal{J}_f(x_0)^{-1} = \mathcal{J}_g(f(x_0)) \,.$$

In particular

$$\partial_c g(f(x_0)) = \mathrm{conv}(\mathcal{J}_g(f(x_0))) = \mathrm{conv}(\mathcal{J}_f(x_0)^{-1}) \,.$$

Now we can state the inverse function theorem proven by Clarke in [4] and supplement the conclusion by a statement about the generalized derivative of the inverse function.

2.12 Theorem *If $\Omega \subset \mathbb{R}^d$ is open, $f : \Omega \to \mathbb{R}^d$ Lipschitz and for $x_0 \in \Omega$ the generalized Jacobian $\partial_c f(x_0)$ is of maximal rank, i.e. $\partial_c f(x_0) \subset \mathrm{GL}(d)$, then there exist neighbourhoods U and V of x_0 and $f(x_0)$ respectively, and a Lipschitz function $g : V \to \mathbb{R}^d$, such that*

(a) $\underset{u \in U}{\forall}\, g(f(u)) = u$,

(b) $\underset{v \in V}{\forall}\, f(g(v)) = v$.

Furthermore, we have $\partial_c g(f(x_0)) = \mathrm{conv}((\mathcal{J}_f(x_0))^{-1}) \subset \mathrm{conv}((\partial_c f(x_0)^{-1})$.

Proof. See [3, Theorem 7.1.1] for the existence of the inverse function g. Then

$$\partial_c g(f(x_0)) = \mathrm{conv}(\mathcal{J}_g(f(x_0))) = \mathrm{conv}((\mathcal{J}_f(x_0))^{-1}) \subset \mathrm{conv}((\partial_c f(x_0)^{-1})$$

follows by Lemma 2.11 and the fact, that $\partial_c f(x_0)$ has maximal rank. □

In [8, Theorem 2], under the assumptions of Clarke's inverse function theorem $\partial_c f(x_0) \subset \mathrm{GL}(d)$, the inclusion $\mathcal{J}_f(x_0)^{-1} \subset \mathcal{J}_g(f(x_0))$ (eq. (10)) and the formula $\partial_c g(f(x_0)) = \mathrm{conv}(\mathcal{J}_f(x_0)^{-1})$ (eq. (11)) was proven. Notice that the authors of [8] denote the generalized derivative $\partial_c f(x_0)$ with $\mathcal{J} f(x_0)$ and the set $\mathcal{J}_f(x_0)$ with $\underrightarrow{\mathcal{J} f}(x_0)$. Lemma 2.11 shows the stronger assertion $\mathcal{J}_f(x_0)^{-1} = \mathcal{J}_g(f(x_0))$, which gives $\partial_c g(f(x_0)) = \mathrm{conv}(\mathcal{J}_f(x_0)^{-1})$ immediately by taking the convex hull, even under the weaker assumption that

g is a Lipschitz continuous inverse of f in respective open neighbourhoods $U \subset \mathbb{R}^d$ of x_0 and $V \subset \mathbb{R}^d$ of $f(x_0)$. [1, Example 2.1] provides an example of a Lipschitz continuous inverse function where Lemma 2.11 allows us to calculate the generalized derivative of the inverse function, while [8, Theorem 2] is not applicable due to singular matrices in the generalized derivative. This example is a special case of the one constructed in [15, Example 2.2] and [9, Example 3.9] and provides a piecewise linear bi-Lipschitz function with the generalized derivative containing the zero matrix.

Given open sets $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^m$ and Lipschitz continuous functions $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^n$ with $f(U) \subset V$, then we seek for a chain rule at least giving us for $x \in U$ an upper estimate for $\partial_c(g \circ f)(x)$ in terms of $\partial_c f(x)$ and $\partial_c g(f(x))$. A natural starting point characterizing the elements of $\mathcal{J}_{g\circ f}(x)$ is applying the classical chain rule at every point $x \in U$, where f is differentiable in x and g is differentiable in $f(x)$. Unfortunately, it is possible that no such point exists, as a simple example like $f : \mathbb{R} \to \mathbb{R},\ f(x) = 0$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = |x|$ shows. Then $f \circ g = 0$ is differentiable everywhere, but by $f^{-1}(N_g) = \mathbb{R}$ we cannot apply the classical chain at any single point. Nevertheless the following chain rule is available for the composition of two Lipschitz functions:

2.13 Proposition ([14, Theorem 4]) *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz near $x \in \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ is Lipschitz near $f(x)$, then*

$$\partial_c(g \circ f)(x) \subset \operatorname{conv}(\partial_c g(f(x))\partial_c f(x))\,.$$

*If g is continuously differentiable near $f(x)$, then equality holds (see [17, Theorem 4.3]) and if f or g is continuously differentiable, the respective generalized derivative reduces to a singleton and „*conv*“ can be omitted.*

A weaker version of this chain rule was already proven in [3, Corollary to Proposition 2.6.5], with both sides of the inclusion applied to some arbitrary but fixed vector $h \in \mathbb{R}^n$.

3 Convexity

3.1 Convex functions

We start with the different notions of convex functions.

3.1 Definition *Fix a convex set $\Omega \subset \mathbb{R}^d$ and a function $f : \Omega \to \mathbb{R}$. Then f is called convex with modulus $\mu \geq 0$, if for all $x, y \in \Omega$ and $t \in (0, 1)$:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\mu}{2}t(1-t)||x-y||^2\,.$$

f is called strictly convex if the above inequality is strict for $\mu = 0$ and $x \neq y$. For some point $x_0 \in \operatorname{int}(\Omega)$, f is called locally convex with modulus $\mu \geq 0$ in x_0, if there is a convex neighbourhood $U \subset \Omega$ of x_0, such that $f|_U$ is convex with modulus μ. We call f locally convex with modulus $\mu \geq 0$, if Ω is open and f is locally convex with modulus $\mu \geq 0$ in any $x_0 \in \Omega$.
If f is (locally) convex with modulus 0, then f is called (locally) convex, and if f is (locally) convex with some modulus $\mu > 0$, then f is called (locally) strongly convex.

It follows immediately from the definition, that any (locally) strongly convex function is (locally) strictly convex and that any (locally) strictly convex function is (locally) convex.

Convex functions have some special regularity properties.

3.2 Proposition *If $\Omega \subset \mathbb{R}^d$ is open, convex and $f : \Omega \to \mathbb{R}$ is convex, then f is locally Lipschitz and if additionally f is differentiable at any point $x \in \Omega$, then f is continuously differentiable.*

Proof. The first part is [19, Corollary 10.4] and the second part is [19, Corollary 25.5.1]. □

3.3 Definition *For a set $\Omega \subset \mathbb{R}^d$ and a function $f : \Omega \to \mathbb{R}$, the subdifferential of f in some point $x_0 \in \Omega$ is defined as*

$$\partial f(x_0) := \{v \in \mathbb{R}^d \mid \underset{x \in \Omega}{\forall} f(x) \geq f(x_0) + \langle v, x - x_0 \rangle\} .$$

It is a straightforward consequence of the definition, that the subdifferential is a convex set. There is a close relationship between the generalized derivative and the subdifferential, for convex functions (which are necessarily locally Lipschitz) the generalized derivative coincides with the subdifferential [3, Proposition 2.2.7]:

3.4 Proposition *If $U \subset \mathbb{R}^d$ is open, convex and $f : U \to \mathbb{R}$ is convex, then for any $x \in U$, $\partial_c f(x) = \partial f(x)$.*

The subdifferential of a convex function can be empty in some point if the function becomes arbitrarily steep, as the example $f : [0, \infty) \to \mathbb{R},\ f(x) = -\sqrt{x}$ with $\partial f(0) = \emptyset$ illustrates. However, this is not possible for points lying in the relative interior of Ω.

3.5 Proposition *If $\Omega \subset \mathbb{R}^d$ is convex and $f : \Omega \to \mathbb{R}$ is convex, then for any $x \in \operatorname{relint}(\Omega)$ we have $\partial f(x) \neq \emptyset$.*

Proof. [11, IV, Proposition 2.5.1] □

If the subdifferential of a convex function reduces to a singleton at some point of the interior of the domain, the function is differentiable at this point.

3.6 Proposition *If $\Omega \subset \mathbb{R}^d$ is convex and $f : \Omega \to \mathbb{R}$ is convex, then for any $x \in \mathrm{int}(\Omega)$ the subdifferential $\partial f(x)$ is a singleton if and only if f is differentiable in x.*

Proof. [19, Theorem 25.1] □

The next Proposition gives useful equivalent characterizations of convexity with modulus $\mu \geq 0$ or strict convexity in terms of the subdifferential [11, VI, Theorem 6.1.2 & Proposition 6.1.3].

3.7 Proposition *Let $\Omega \subset \mathbb{R}^d$ be convex and $f : \Omega \to \mathbb{R}$. Then the following statements are equivalent*

(i) f *is convex with modulus* $\mu \geq 0$,

(ii) $\underset{x,y\in\Omega}{\forall} \ \underset{v\in\partial f(x)}{\forall} \ f(y) \geq f(x) + \langle v, y-x\rangle + \frac{\mu}{2}||x-y||^2$,

(iii) $\underset{x,y\in\Omega}{\forall} \ \underset{v_x\in\partial f(x), v_y\in\partial f(y)}{\forall} \ \langle v_y - v_x, y-x\rangle \geq \mu||y-x||^2$.

Furthermore, strict convexity of f is equivalent to (ii) and to (iii) with the respective inequalities assumed to be strict for $\mu = 0$ and $x \neq y$.

In view of Proposition 3.4 it is not surprising that for locally Lipschitz functions the subdifferential can be replaced by the generalized derivative.

3.8 Corollary *Let $\Omega \subset \mathbb{R}^d$ be open, convex and $f : \Omega \to \mathbb{R}$ be locally Lipschitz. Then, all equivalences of Proposition 3.7 remain true with the subdifferential ∂ replaced by the generalized derivative ∂_c.*

Proof. It suffices to show that each condition (i), (ii) and (iii) implies the convexity of f, since then by Proposition 3.4 the generalized derivative and the subdifferential coincide and the equivalences follow by Proposition 3.7.
If f is convex with modulus $\mu \geq 0$ or strictly convex, then f is especially convex.
(ii) implies (iii), since for $x, y \in \Omega$ and $v_x \in \partial_c f(x)$, $v_y \in \partial_c f(y)$, (ii) gives us the two inequalities

$$f(y) \geq f(x) + \langle v_x, y-x\rangle + \frac{\mu}{2}||x-y||^2\,,$$
$$f(x) \geq f(y) + \langle v_y, x-y\rangle + \frac{\mu}{2}||x-y||^2$$

and (iii) follows by addition of both inequalities. If $\mu = 0$ and the inequality in (ii) is strict whenever $x \neq y$, then the inequality in (ii) with $\mu = 0$ is also strict whenever $x \neq y$.
If (iii) is satisfied with $\mu = 0$, then f is convex by [3, Proposition 2.2.9]. This is especially the case if (iii) is satisfied with $\mu > 0$ or with $\mu = 0$ and the inequality being strict whenever $x \neq y$. $\square$

For continuously differentiable functions, the generalized derivatives in Corollary 3.8 reduce to singletons containing only the respective gradient and we obtain as a special case the equivalences for convexity with modulus $\mu \geq 0$ shown in [11, IV, Theorem 4.1.1, Theorem 4.1.4].

For $\Omega \subset \mathbb{R}^d$ open, convex and $f \in C^2(\Omega)$, it is well known that f is convex if and only if for any $x \in \Omega$ the Hessian $\nabla^2 f(x) \in \mathbb{R}^{d\times d}$ is positively semi-definite. An analogue statement for $f \in C^{1,1}_{loc}(\Omega)$ is mentioned without proof in [13, Example 2.2], here we give a proof for an equivalent characterization of convexity with modulus $\mu \geq 0$ in terms of generalized second derivatives.

3.9 Definition *For* $0 \leq \mu, L \leq \infty$ *define*

$$\mathrm{PD}^L_\mu(d) := \{A \in \mathbb{R}^{d\times d}_{sym} \mid \underset{h\in\mathbb{R}^d}{\forall} \mu||h||^2 \leq h^T A h \leq L||h||^2\}\,.$$

Notice that for $L < \mu$ the set $\mathrm{PD}^L_\mu(d)$ is empty, and that for $\mu = 0$ and $L = \infty$ we obtain the set of symmetric positively semi-definite matrices.

3.10 Proposition *Let* $\Omega \subset \mathbb{R}^d$ *be open, convex and* $f \in C^{1,1}_{loc}(\Omega)$.
Then f *is convex with modulus* $\mu \geq 0$*, if and only if*

$$\underset{x_0\in\Omega}{\forall} \partial_c \nabla f(x_0) \subset \mathrm{PD}^\infty_\mu(d)\,.$$

Proof. Assume f is convex with modulus $\mu \geq 0$. Fix $x_0 \in \Omega$, an open neighbourhood $U \subset \Omega$ of x_0 on which f is Lipschitz with constant $L > 0$ and $x \in U \setminus N_{\nabla f}$. Then f is twice differentiable in x with $\nabla^2 f(x) \in \mathbb{R}^{d\times d}_{sym}$. For any $h \in \mathbb{R}^d$ and sufficiently small $t > 0$ we have $x + th \in U$ and

$$h^T \nabla^2 f(x) h = \langle \nabla^2 f(x)h, h\rangle = \lim_{t\to 0} \frac{1}{t^2}(\langle \nabla f(x+th) - \nabla f(x), th\rangle)\,.$$

By Proposition 3.7 (iii), the Cauchy-Schwarz inequality and the Lipschitz continuity of ∇f we get

$$\begin{aligned}\mu||h||^2 &= \frac{1}{t^2}(\mu||th||^2) \leq \frac{1}{t^2}(\langle \nabla f(x+th) - \nabla f(x), th\rangle) \\ &\leq \frac{1}{t^2}||\nabla f(x+th) - \nabla f(x)|| \cdot ||th|| \leq \frac{1}{t^2}(L||th||^2) = L||h||^2\,.\end{aligned}$$

This implies $\nabla^2 f(x) \in \mathrm{PD}_\mu^L(d)$ and

$$\begin{aligned}\partial_c \nabla f(x_0) &= \mathrm{conv}(\mathcal{J}_{\nabla f}(x_0)) \subset \mathrm{conv}(\mathrm{clos}(\{\nabla^2 f(x) \mid x \in U \setminus N_{\nabla f}\})) \\ &\subset \mathrm{conv}(\mathrm{clos}(\mathrm{PD}_\mu^L(d))) = \mathrm{PD}_\mu^L(d)\end{aligned}$$

where the first inclusion follows by definition of $\mathcal{J}_{\nabla f}(x_0)$, the second inclusion by monotonicity of closure and convex hull and the last equality by Proposition 6.5.
Now assume that for all $x_0 \in \Omega$ the inclusion $\partial_c \nabla f(x_0) \subset \mathrm{PD}_\mu^\infty(d)$ is satisfied. Fix $x, y \in \Omega$ and consider $\gamma : [0,1] \to \Omega,\ \gamma(t) = (1-t)\cdot x + t \cdot y$ and $g : [0,1] \to \mathbb{R},\ g(t) = f(\gamma(t))$. The function g is continuously differentiable with

$$g' : [0,1] \to \mathbb{R},\ g'(t) = \langle \nabla f(\gamma(t)), \gamma'(t)\rangle = \langle \nabla f(\gamma(t)), y - x\rangle$$

and since g' is locally Lipschitz as the composition of Lipschitz functions and $[0,1]$ is compact, g' is Lipschitz with some parameter $L > 0$. For any $t \in (0,1)$,

$$\begin{aligned}\partial_c g'(t) &\subset \mathrm{conv}((y-x)^T \cdot \partial_c(\nabla f)(\gamma(t)) \cdot \gamma'(t)) \subset (y-x)^T \cdot \mathrm{PD}_\mu^L(d) \cdot (y-x) \\ &\subset \{z \in \mathbb{R} \mid z \geq \mu ||y-x||^2\}\end{aligned}$$

by Proposition 2.13, especially $g''(t) \geq \mu||y-x||^2$ whenever $t \in (0,1) \setminus N_{g'}$. Since g' is especially absolutely continuous and differentiable almost everywhere, by the fundamental theorem of Lebesgue integral calculus

$$\begin{aligned}f(y) = g(1) &= g(0) + \int_0^1 g'(t)\ dt = g(0) + \int_0^1 \Big(g'(0) + \int_0^t g''(s)\ ds\Big)\ dt \\ &\geq f(\gamma(0)) + \langle \nabla f(\gamma(0)), \gamma'(0)\rangle + \int_0^1 \int_0^t \mu ||y-x||^2\ ds\ dt \\ &= f(x) + \langle \nabla f(x), y - x\rangle + \frac{\mu}{2}||y-x||^2\end{aligned}$$

and the argument is completed by Proposition 3.7 (ii). □

A main feature of a strictly convex function $f \in C^1(U)$ defined on an open convex set $U \subset \mathbb{R}^d$ is, that the gradient mapping is an embedding.

3.11 Lemma *Assume $U \subset \mathbb{R}^d$ is an open, convex set and $f : U \to \mathbb{R}$ is differentiable and strictly convex.*
Then $V := \nabla f(U)$ is open, and $\nabla f : U \to \mathbb{R}^d$ is an embedding.

Proof. By Proposition 3.2, f is continuously differentiable. For $x, y \in U$ with $x \neq y$, we have $\partial_c f(x) = \{\nabla f(x)\}$ and $\partial_c f(y) = \{\nabla f(y)\}$, therefore by strict convexity of f, Corollary 3.8 implies $\langle \nabla f(y) - \nabla f(x), y - x\rangle > 0$ and especially $\nabla f(x) \neq \nabla f(y)$. Now the assertion follows by injectivity and continuity of ∇f with Proposition 6.3. □

If $f \in C^{1,1}(U)$ and f is convex with modulus $\mu > 0$, then ∇f is even bi-Lipschitz. Lemma 2.11 allows us furthermore to estimate the generalized derivative of the inverse of the gradient mapping.

3.12 Lemma *Assume $U \subset \mathbb{R}^d$ is an open, convex set and $f : U \to \mathbb{R}$ is convex with parameter $\mu > 0$ and differentiable with Lipschitz continuous gradient with constant $L > 0$.*
Then $V := \nabla f(U)$ is open, $\nabla f : U \to \mathbb{R}^d$ is injective and the inverse function $(\nabla f)^{-1} : V \to \mathbb{R}^d$ is Lipschitz with constant μ^{-1}. Especially, ∇f and $(\nabla f)^{-1}$ are bi-Lipschitz and the generalized derivative of $(\nabla f)^{-1}$ in any point $v \in V$ can be estimated by

$$\partial_c(\nabla f)^{-1}(v) \subset \mathrm{PD}_{L^{-1}}^{\mu^{-1}}(d)\,.$$

Proof. For any $u \in U$ the generalized derivative of f in u can be estimated with Proposition 3.10 by $\partial(\nabla f)(u) \subset \mathrm{PD}_{\mu}^{L}(d) \subset \mathrm{GL}(d)$. Theorem 2.12 gives us the local invertibility of ∇f in any $u \in U$, especially $V = \nabla f(U)$ is open. By Proposition 3.7 (iii) and the Cauchy-Schwarz inequality we get

$$||\nabla f(y) - \nabla f(x)|| \cdot ||y - x|| \geq \langle \nabla f(y) - \nabla f(x), y - x\rangle \geq \mu ||y - x||^2 \quad (3.1)$$

to see that ∇f is one-to-one. The inverse function $(\nabla f)^{-1} : V \to \mathbb{R}^d$ is Lipschitz with constant μ^{-1} by (3.1). The inclusion of the generalized derivative of the $(\nabla f)^{-1}$ in some $v \in V$ is due to Lemma 2.11 and Proposition 6.5 with

$$\begin{aligned}\partial_c(\nabla f)^{-1}(v) &\subset \mathrm{conv}\big(\big(\partial(\nabla f)((\nabla f)^{-1}(v))\big)^{-1}\big) \subset \mathrm{conv}((\mathrm{PD}_{\mu}^{L}(d))^{-1}) \\ &= \mathrm{conv}(\mathrm{PD}_{L^{-1}}^{\mu^{-1}}(d)) = \mathrm{PD}_{L^{-1}}^{\mu^{-1}}(d)\,,\end{aligned}$$

which completes the proof. □

3.2 The Legendre transform

A useful concept in convex analysis is the Legendre transform, see for example [19, Section 26].

3.13 Definition *Let $U \subset \mathbb{R}^d$ be open and $f : U \to \mathbb{R}$ be differentiable with*

$$\underset{u,u' \in U}{\forall} \big(\nabla f(u) = \nabla f(u') \Rightarrow \langle u, \nabla f(u)\rangle - f(u) = \langle u', \nabla f(u')\rangle - f(u')\big)\,. \tag{3.2}$$

Then, for $V := \nabla f(U)$ the function

$$f^* : V \to \mathbb{R},\ f^*(v) = \langle u, v\rangle - f(u), \textit{if } u \in (\nabla f)^{-1}(\{v\}) \tag{3.3}$$

is well-defined and f^* is called the Legendre transform of f.

Any differentiable and convex function defined on a convex set satisfies (3.2), as well as any differentiable function whose gradient mapping is injective. If the gradient mapping of $f : U \to \mathbb{R}$ is injective, (3.3) simplifies to

$$f^*(v) = \langle (\nabla f)^{-1}(v), v\rangle - f((\nabla f)^{-1}(v))\,.$$

For convex functions, the Legendre transform is closely related to the convex conjugate, see for example [20, Theorem 26.4]. The following Proposition is the counterpart of Fenchel's inequality in terms of the Legendre transform.

3.14 Proposition *If $U \subset \mathbb{R}^d$ is open, convex and $f \in C^1(U)$ is convex, then for any $v_0 \in V := \nabla f(U)$ the affine function*

$$h : \mathbb{R}^d \to \mathbb{R},\ h(u) = \langle v_0, u\rangle - f^*(v_0)$$

minorizes f and for any $u_0 \in (\nabla f)^{-1}(\{v_0\})$ we have $f(u_0) = h(u_0)$.

Proof. For some arbitrary $u_0 \in (\nabla f)^{-1}(\{v_0\})$, by Proposition 3.7 (ii), we have for any $u \in U$,

$$f(u) \geq f(u_0) + \langle \nabla f(u_0), u - u_0\rangle = \langle v_0, u\rangle - f^*(v_0) = h(u)\,,$$

with equality for $u = u_0$. □

The following Proposition describes, how the Legendre transform behaves under an affine transformation of the argument. In [19, Theorem 12.3] it was formulated for the convex conjugate, the analogue statement for the Legendre transform is an immediate consequence of the relation [20, Theorem 26.4] between the Legendre transform and the convex conjugate.

3.15 Proposition *Let $\tilde{U} \subset \mathbb{R}^d$ be open and convex, $\tilde{f} : \tilde{U} \to \mathbb{R}$ be a strictly convex function, $A \in \mathrm{GL}(d)$, $a, a^* \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$. Then with $\alpha^* := -\alpha - \langle a, a^* \rangle$, $\tilde{V} := \nabla f(\tilde{U})$, $U := a + A^{-1} \cdot \tilde{U}$ and $V := a^* + A^T \cdot \tilde{V}$ the Legendre transform of the function*

$$f : U \to \mathbb{R},\ f(x) = \tilde{f}(A(x-a)) + \langle x, a^* \rangle + \alpha$$

is given by

$$f^* : V \to \mathbb{R},\ f^*(x^*) = \tilde{f}^*((A^T)^{-1}(x^* - a^*)) + \langle x^*, a \rangle + \alpha^* .$$

3.16 Proposition *Assume $U \subset \mathbb{R}^d$ is an open, convex set and $f : U \to \mathbb{R}$ is continuously differentiable and strictly convex.*
Then the Legendre transform f^* is continuously differentiable with $\nabla f^* = (\nabla f)^{-1}$ and strictly convex on each convex subset C of $\nabla f(U)$.

Proof. Fix $x^* \in \nabla f(U) =: V$ and $x := (\nabla f)^{-1}(x^*)$. Then $x \in \partial f^*(x^*)$, since for any $y^* \in V$ with $y := (\nabla f)^{-1}(y^*)$ we can estimate

$$\begin{aligned} f^*(y^*) - f^*(x^*) - \langle x, y^* - x^* \rangle &= \langle y, y^* \rangle - f(y) - \langle x, x^* \rangle + f(x) - \langle x, y^* - x^* \rangle \\ &= f(x) - f(y) - \langle \nabla f(y), x - y \rangle \geq 0 , \end{aligned}$$

with the inequality being strict for $x^* \neq y^*$ (since then $x \neq y$).
Now fix $v \in \partial f^*(x^*)$. Since U is open, we can choose $t > 0$ small enough, such that $z := (1-t)x + tv \in U$ and by convexity of $\partial f^*(x^*)$, $z \in \partial f^*(x^*)$. Defining $z^* := \nabla f(z)$, calculate for any $w \in U$

$$\begin{aligned} f(w) \geq f(x) + \langle x^*, w - x \rangle &= f(x) - \langle x^*, x \rangle + \langle x^*, z \rangle + \langle x^*, w - z \rangle \\ &= -f^*(x^*) - \langle z, z^* - x^* \rangle + \langle z^*, z \rangle + \langle x^*, w - z \rangle \\ &\geq -f^*(z^*) + \langle z^*, z \rangle + \langle x^*, w - z \rangle \\ &= f(z) + \langle x^*, w - z \rangle , \end{aligned}$$

giving $x^* \in \partial f(z)$. By differentiability of f, $\partial f(z) = \{\nabla f(z)\}$ and strict convexity implies $z = x$, especially $v = x$ and since $v \in \partial f^*(x^*)$ was arbitrary, $\partial f^*(x^*) = \{x\}$. For any convex subset $C \subset V$, Proposition 3.7 (ii) implies the strict convexity of f^* on C and choosing C as a convex neighbourhood of x^*, Proposition 3.6 gives us the differentiability of f^* in x^*. According to Proposition 3.2, ∇f^* is continuous in any convex open subset of the open set V, therefore ∇f^* is continuous. □

3.17 Theorem *Assume $U \subset \mathbb{R}^d$ is an open, convex set and $f : U \to \mathbb{R}$ is convex with modulus $\mu > 0$ and differentiable with Lipschitz continuous*

gradient with constant $L > 0$.
Then the Legendre transform f^* is strongly convex with modulus L^{-1} on each convex subset C of V and has Lipschitz continuous gradient $(\nabla f)^{-1}$ with constant μ^{-1}.

Proof. According to Proposition 3.16, the Legendre transform f^* is continuously differentiable with $\nabla f^* = (\nabla f)^{-1}$. By Lemma 3.12, $(\nabla f)^{-1}$ is Lipschitz with constant μ^{-1} and for any $v \in V$ the generalized derivative can be estimated by $\partial(\nabla f)^{-1}(v) \subset \mathrm{PD}^{\mu^{-1}}_{L^{-1}}(d)$. The assertion follows by-Proposition 3.10 applied to $f^*|_C$. □

One may wonder, whether there is a subclass of convex functions, on which the Legendre transform acts as an involution. Unfortunately the set of gradients, which is the domain of the Legrendre transform is not necessarily convex, as Rockafellar pointed out in his counterexample [20, Section 4]:

$$f_{ce} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}, \; f_{ce}(x_1, x_2) = \frac{1}{4}\left(\frac{x_1^2}{x_2} + x_1^2 + x_2^2\right).$$

This problem can be solved, assuming that $U \subset \mathbb{R}^d$ is open and $f : U \to \mathbb{R}$ is strictly convex and differentiable with $\lim_{n\to\infty} ||\nabla f(x_n)|| = \infty$ whenever $x_n \xrightarrow{n\to\infty} x \in \partial U$. Such a function is called a function of Legendre type and the Legendre transform of such a function is again a function of Legendre type, whose Legendre transform is (U, f) [19, Theorem 26.5].
Notice that for $U \neq \mathbb{R}^d$, no $f \in C^{1,1}(U)$ is of Legendre type, since any such function is required to be arbitrarily steep near the boundary of U.

3.3 Convex envelopes

In this section, $\Omega \subset \mathbb{R}^d$ is always a non-empty set and $f : \Omega \to \mathbb{R}$ is an arbitrary function.

3.18 Proposition *If f is minorized by some affine function, the function*

$$\begin{aligned} &\mathrm{conv}(f) : \mathrm{conv}(\Omega) \to \mathbb{R}, \\ &\mathrm{conv}(f)(x) = \sup\{h(x) \mid h : \mathrm{conv}(\Omega) \to \mathbb{R} \cup \{\infty\} \text{ convex}, \; h \le f\} \end{aligned} \tag{3.4}$$

is well-defined and convex.

Proof. The fact, that f is minorized by some affine function $h \in \mathrm{Aff}(d, 1)$ ensures that the set in the supremum is non-empty. If $x \in \mathrm{conv}(\Omega)$ and

$x = \sum_{i=1}^{q} \lambda_i x^{(i)}$ is a convex combination with $x^{(1)}, ..., x^{(q)} \in \Omega$, then for any convex function $h : \mathrm{conv}(\Omega) \to \mathbb{R}$ minorizing f we have

$$h(x) \leq \sum_{i=1}^{q} \lambda_i h(x^{(i)}) \leq \sum_{i=1}^{q} \lambda_i f(x^{(i)})$$

and hence $\mathrm{conv}(f)$ is well-defined, since the supremum is bounded from above. For any $x, y \in \Omega$ and $\lambda \in [0,1]$ there is a sequence $(h_n)_{n\in\mathbb{N}}$ of convex functions minorizing f with

$$\begin{aligned}\mathrm{conv}(f)((1-\lambda)x + \lambda y) &= \limsup_{n\to\infty} \big(h_n((1-\lambda)x + \lambda y)\big) \\ &\leq \limsup_{n\to\infty} \big((1-\lambda)h_n(x) + \lambda h_n(y)\big) \\ &\leq (1-\lambda)\,\mathrm{conv}(f)(x) + \lambda\,\mathrm{conv}(f)(y)\,,\end{aligned}$$

which implies the convexity of $\mathrm{conv}(f)$. □

3.19 Definition *If f is minorized by some affine function,* $\mathrm{conv}(f)$ *from Proposition 3.18 is called the convex envelope of f.*

In [11, IV, Proposition 2.5.1], for any $x \in \mathrm{conv}(\Omega)$ the following expression for the convex envelope was given:

$$\begin{aligned}\mathrm{conv}(f)(x) = \inf\{\sum_{i=1}^{q} \lambda_i f(x^{(i)}) \mid q \in \mathbb{N},\ \lambda \in \Delta_q,\ x^{(1)}, ..., x^{(q)} \in \Omega, \\ \sum_{i=1}^{q} \lambda_i x^{(i)} = x\}\,.\end{aligned} \tag{3.5}$$

In (3.4), the convex envelope is obtained by approximation from below with minorizing convex functions and in (3.5), the convex envelope is obtained by approximation from above by convex combinations. If $x \in \Omega$ with $\partial f(x) \neq \emptyset$, then for $v \in \partial f(x)$ the affine function $h : \mathbb{R}^d \to \mathbb{R},\ h(x') = f(x') + \langle v, x' - x\rangle$ minorizes f with $f(x) = h(x) \leq (\mathrm{conv}(f))(x)$ by (3.4) and $(\mathrm{conv}(f))(x) \leq f(x)$ by (3.5) for $q = 1$. Therefore, $(\mathrm{conv}(f))(x) = f(x)$.

3.20 Definition *If f is minorized by some affine function and $x \in \mathrm{conv}(\Omega)$, a convex combination*

$$x = \sum_{i=1}^{q} \lambda_i x^{(i)}$$

with $\lambda \in \operatorname{relint}(\Delta_q)$ *and pairwise distinct points* $x^{(1)}, ..., x^{(q)} \in \Omega$ *is called a stable phase splitting of* x, *if*

$$(\operatorname{conv}(f))(x) = \sum_{i=1}^{q} \lambda_i f(x^{(i)}) \,,$$

i.e. the infimum in (3.5) *is a minimum achieved at* $(\lambda_1, x^{(1)}), ..., (\lambda_q, x^{(q)})$. *Furthermore, the points* $x^{(1)}, ..., x^{(q)}$ *are called phases of the stable phase splitting and the stable phase splitting is called a unique stable phase splitting of* x, *if any other stable phase splitting* $x = \sum_{i=1}^{q'} \lambda'_i x'^{(i)}$ *with* $x'^{(1)}, ..., x'^{(q')} \in \Omega$ *and* $\lambda' \in \operatorname{relint}(\Delta_{q'})$ *satisfies* $q' = q$, $x'^{(i)} = x^{(i)}$ *and* $\lambda'_i = \lambda_i$ *for any* $i \in \{1, ..., q\}$.

Given a stable phase splitting of $x \in \operatorname{conv}(\Omega)$, the subdifferential of the convex envelope in x can be computed as the intersection of the subdifferentials of the interpolating points [12, Theorem 1.5.6].

3.21 Theorem *If* f *is minorized by some affine function and* $x = \sum_{i=1}^{q} \lambda_i \cdot x^{(i)}$ *is a stable phase splitting, i.e.* $\lambda \in \operatorname{relint}(\Delta_q)$ *and* $x^{(1)}, ..., x^{(q)} \in \Omega$ *pairwise distinct with*

$$(\operatorname{conv}(f))(x) = \sum_{i=1}^{q} \lambda_i f(x^{(i)}) \,,$$

then

$$\partial(\operatorname{conv}(f))(x) = \bigcap_{i=1}^{q} \partial f(x^{(i)}) \,.$$

Theorem 3.21 has a natural conversion. Given some points $x^{(1)}, ..., x^{(q)} \in \Omega$ with a common subgradient, we can give on $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ an expression for the convex envelope of f by convex interpolation of the values of f in $x^{(1)}, ..., x^{(q)}$.

3.22 Proposition *Assume* $x^{(1)}, ..., x^{(q)} \in \Omega$ *with* $\bigcap_{i=1}^{q} \partial f(x^{(i)}) \neq \emptyset$. *Then* f *is minorized by some affine function and for each* $\lambda \in \Delta_q$, *the convex envelope of* f *in* $x := \sum_{i=1}^{q} \lambda_i x^{(i)} \in \operatorname{conv}(\{x^{(1)}, ..., x^{(q)}\})$ *is given by*

$$(\operatorname{conv}(f))(x) = \sum_{i=1}^{q} \lambda_i f(x^{(i)}) \,.$$

Proof. For $v \in \bigcap_{i=1}^{q} \partial f(x^{(i)})$, the function

$$h : \mathbb{R}^d \to \mathbb{R},\ h(x) = f(x^{(1)}) + \langle v, x - x^{(1)} \rangle$$

is by $v \in \partial f(x^{(1)})$ an affine function minorizing f. For any $i \in \{1, ..., q\}$, by $v \in \partial f(x^{(i)})$ we have

$$\begin{aligned} h(x^{(i)}) &= f(x^{(1)}) + \langle v, x^{(i)} - x^{(1)} \rangle \geq f(x^{(i)}) + \langle v, x^{(1)} - x^{(i)} \rangle + \langle v, x^{(i)} - x^{(1)} \rangle \\ &= f(x^{(i)}) \geq h(x^{(i)}) \,. \end{aligned}$$

Now Proposition 6.8, equations (3.4) and (3.5) imply

$$\begin{aligned} \sum_{i=1}^{q} \lambda_i f(x^{(i)}) &= \sum_{i=1}^{q} \lambda_i h(x^{(i)}) = h\Big(\sum_{i=1}^{q} \lambda_i x^{(i)}\Big) = h(x) \leq (\mathrm{conv}(f))(x) \\ &= (\mathrm{conv}(f))(\sum_{i=1}^{q} \lambda_i x^{(i)}) \leq \sum_{i=1}^{q} \lambda_i f(x^{(i)}) \end{aligned}$$

and hence the claimed equality. □

We can derive a corollary, which relates the representations (3.4) and (3.5) with each other:

3.23 Corollary *If $f : \Omega \to \mathbb{R}$ is a function minorized by $h \in \mathrm{Aff}(d, 1)$, then for any $x^{(1)}, ..., x^{(q)} \in \{x \in \Omega \mid f(x) = h(x)\}$ and $\lambda_1, ..., \lambda_q \in [0, 1]$ with $\sum_{i=1}^{q} \lambda_i = 1$, the convex envelope of f in $x := \sum_{i=1}^{q} \lambda_i x^{(i)}$ is given by*

$$\mathrm{conv}(f)(x) = \sum_{i=1}^{q} \lambda_i f(x^{(i)}) = h(x) \,.$$

3.24 Proposition *If $x^{(1)}, ..., x^{(q)} \in \Omega$ are pairwise distinct and $\lambda \in \mathrm{relint}(\Delta_q)$, such that $x = \sum_{i=1}^{q} \lambda_i x^{(i)} \in \mathrm{conv}(\Omega)$ is a unique stable phase splitting, then the points $x^{(1)}, ..., x^{(q)}$ are affinely independent and for any $v \in \bigcap_{i=1}^{q} \partial f(x^{(i)})$ and $x' \in \mathrm{aff}\{x^{(1)}, ..., x^{(q)}\} \setminus \{x^{(1)}, ..., x^{(q)}\}$ we have*

$$f(x') > (\mathrm{conv}(f))(x) + \langle v, x' - x \rangle \,.$$

Proof. Assume $x = \sum_{i=1}^{q} \lambda_i x^{(i)}$ is a unique stable phase splitting, especially

$$(\text{conv}(f))(x) = \sum_{i=1}^{q} \lambda_i f(x^{(i)}) .$$

Fix $\lambda'_1, ..., \lambda'_q \in \mathbb{R}$ with $\sum_{i=1}^{q} \lambda'_i = 0$ and $\sum_{i=1}^{q} \lambda'_i x^{(i)} = 0$. Fix $t' > 0$ small enough, such that for any $i \in \{1, ..., q\}$ we have $t' \cdot |\lambda'_i| < \lambda_i$. Then

$$x = \sum_{i=1}^{q} \lambda_i x^{(i)} = \sum_{i=1}^{q} (\lambda_i + t' \cdot \lambda'_i) x^{(i)}$$

is by $\sum_{i=1}^{q} (\lambda_i + t' \cdot \lambda'_i) = 1$ and Proposition 3.22 a stable phase splitting and uniqueness implies $\lambda'_1 = ... = \lambda'_q = 0$, i.e. $x^{(1)}, ..., x^{(q)}$ are affine independent according to Proposition 6.7.
By Theorem 3.21 and convexity of $\text{conv}(f)$,

$$\emptyset \neq \partial(\text{conv}(f))(x) = \bigcap_{i=1}^{q} \partial f(x^{(i)}) .$$

Assume $v \in \bigcap_{i=1}^{q} \partial f(x^{(i)})$ and $x' \in \text{aff}\{x^{(1)}, ..., x^{(q)}\}$ with

$$f(x') \leq (\text{conv}(f))(x) + \langle v, x' - x \rangle .$$

Since $v \in \partial(\text{conv}(f))(x)$, for any $x'' \in \Omega$ we have

$$f(x'') \geq (\text{conv}(f))(x) + \langle v, x'' - x \rangle \geq f(x') + \langle v, x'' - x' \rangle$$

and hence $v \in \partial f(x')$ and $v \in \partial f(x') \cap \Big(\bigcap_{i=1}^{q} \partial f(x^{(i)}) \Big)$.
Now fix $\lambda'_1, ..., \lambda'_q \in \mathbb{R}$ with $\sum_{i=1}^{q} \lambda'_i = 1$ and $\sum_{i=1}^{q} \lambda'_i x^{(i)} = x'$. Choose $t' > 0$ small enough, such that for any $i \in \{1, ..., q\}$ we have $t' \cdot |\lambda'_i| < \lambda_i$. Then

$$x = \sum_{i=1}^{q} \lambda_i x^{(i)} = \sum_{i=1}^{q} (\lambda_i - t' \cdot \lambda'_i) x^{(i)} + t' \cdot x'$$

is by $\sum_{i=1}^{q} (\lambda_i - t' \cdot \lambda'_i) + t' = 1$ and Proposition 3.22 a stable phase splitting, which implies by uniqueness $x' \in \{x^{(1)}, ..., x^{(q)}\}$. □

If a stable phase splitting $x = \sum_{i=1}^{q} \lambda_i x^{(i)} \in \operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is unique, $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is a $(q-1)$-dimensional simplex, which is called the phase simplex of x without ambiguity. Any face of this phase simplex is the $(q-2)$-dimensional phase simplex of each point in its relative interior. Conversely, it is also possible that $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is the face of a larger, q-dimensional phase simplex. If there is some $x^{(q+1)} \in \Omega \setminus \operatorname{aff}\{x^{(1)}, ..., x^{(q)}\}$ with $\bigcap_{i=1}^{q+1} \partial f(x^{(i)}) \neq \emptyset$, then for any $\lambda'_1, ..., \lambda'_{q+1} > 0$, $x' := \sum_{i=1}^{q+1} \lambda'_i x^{(i)}$ is a stable phase splitting of x' according to Proposition 3.22. If this stable phase splitting is unique, then the $(q-1)$-dimensional phase simplex $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is a face of the q-dimensional phase simplex $\operatorname{conv}\{x^{(1)}, ..., x^{(q+1)}\}$ of x'. This leads us to the notion of a maximal phase simplex as one, for which no such $x^{(q+1)}$ exists. By Theorem 3.21, the affinely independent vertices $x^{(1)}, ..., x^{(q)}$ share at least one common subgradient and each common subgradient uniquely defines a hyperplane minorizing f and touching the graph of f at $x^{(1)}, ..., x^{(q)}$. Consequently, if any such hyperplane strictly minorizes f except at $x^{(1)}, ..., x^{(q)}$, no $x^{(q+1)} \in \Omega \setminus \{x^{(1)}, ..., x^{(q)}\}$ with $\bigcap_{i=1}^{q+1} \partial f(x^{(i)}) \neq \emptyset$ exists.

3.25 Definition *For $x^{(1)}, ..., x^{(q)} \in \Omega$, the simplex $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is called a maximal phase simplex of f, if the points $x^{(1)}, ..., x^{(q)}$ are affinely independent, $\bigcap_{i=1}^{q} \partial f(x^{(i)}) \neq \emptyset$ and for any $v \in \bigcap_{i=1}^{q} \partial f(x^{(i)})$, $i \in \{1, ..., q\}$ and $x' \in \Omega \setminus \{x^{(1)}, ..., x^{(q)}\}$ we have*

$$f(x') > f(x^{(i)}) + \langle v, x' - x^{(i)} \rangle .$$

3.26 Corollary *For affinely independent points $x^{(1)}, ..., x^{(q)} \in \Omega$ satisfying $\bigcap_{i=1}^{q} \partial f(x^{(i)}) \neq \emptyset$, $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is a maximal phase simplex of f if and only if for any $x' \in \Omega \setminus \{x^{(1)}, ..., x^{(q)}\}$ we have $\partial f(x') \cap \left(\bigcap_{i=1}^{q} \partial f(x^{(i)}) \right) = \emptyset$.*

Proof. We show both directions of the equivalence by contraposition.
If the simplex $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$ is not maximal, then there is $v \in \bigcap_{i=1}^{q} \partial f(x^{(i)})$, $i \in \{1, ..., q\}$ and $x' \in \Omega \setminus \{x^{(1)}, ..., x^{(q)}\}$ with

$$f(x') \leq f(x^{(i)}) + \langle v, x' - x^{(i)} \rangle .$$

By $v \in \partial f(x^{(i)})$ we have $f(x') = f(x^{(i)}) + \langle v, x' - x^{(i)} \rangle$ and hence for any $x \in \Omega$,

$$f(x) \geq f(x^{(i)}) + \langle v, x - x^{(i)} \rangle = f(x') + \langle v, x - x' \rangle ,$$

therefore $v \in \partial f(x')$.
Conversely, if there is some $x' \in \Omega \setminus \{x^{(1)}, ..., x^{(q)}\}$ such that there exists $v \in \partial f(x') \cap \Big(\bigcap_{i=1}^{q} \partial f(x^{(i)}) \Big)$, then $\mathrm{conv}\{x^{(1)}, ..., x^{(q)}\}$ is not maximal, since for some arbitrary $i \in \{1, ..., q\}$ we have

$$f(x^{(i)}) \geq f(x') + \langle v, x^{(i)} - x' \rangle \,.$$

Therefore the equivalence is shown. □

At this point, it is worth mentioning that our notion of a maximal phase simplex differs from the one given by Griewank and Rabier in the introduction (or again before Theorem 5.2) of [18], as one which is not the face of a larger phase simplex. The reason is, that Griewank and Rabier only consider functions, for which any stable phase splitting is unique. A maximal phase simplex (according to our definition) cannot be the face of a larger phase simplex, since our definition ensures that there is no $x^{(q+1)} \in \Omega$ satisfying $\bigcap_{i=1}^{q+1} \partial f(x^{(i)}) \neq \emptyset$. Conversely, given a phase simplex which is not the face of a larger phase simplex, without uniqueness of stable phase splittings we cannot conclude that this phase simplex is maximal (according to our definition). Consider for example an arbitrary function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(1,1) = f(1,-1) = f(-1,1) = f(-1,-1) = 0$ and $f > 0$ otherwise. Then $\mathrm{conv}\{(1,1),(1,-1)\}$ is the phase simplex of $(1,0)$, which is not the face of a larger phase simplex, since the potential larger phase simplices $\mathrm{conv}\{(1,1),(1,-1),(-1,1)\}$ and $\mathrm{conv}\{(1,1),(1,-1),(-1,-1)\}$ disqualify due to non-uniqueness of the corresponding stable phase splittings. Therefore we stick with our notion of maximal phase simplices, emphasizing that our definition is equivalent to the one of Griewank and Rabier for functions only admitting unique stable phase splittings.

3.27 Proposition *Fix $x^{(1)}, ..., x^{(q)} \in \Omega$, such that $\mathrm{conv}\{x^{(1)}, ..., x^{(q)}\}$ is a maximal phase simplex of f. Then, for any $\lambda \in \Delta_q$ and $x := \sum_{i=1}^{q} \lambda_i x^{(i)}$ we have*

$$(\mathrm{conv}(f))(x) = \sum_{i=1}^{q} \lambda_i f(x^{(i)})$$

and for $I := \{i \in \{1, ..., q\} \mid \lambda_i > 0\}$, $x = \sum_{i \in I} \lambda_i x^{(i)}$ is a unique stable phase splitting.

Proof. Since $\bigcap_{i=1}^{q} \partial f(x^{(i)}) \neq \emptyset$ by definition of maximality of the phase simplex $\operatorname{conv}\{x^{(1)}, ..., x^{(q)}\}$, Proposition 3.22 implies the expression for the convex envelope and especially, that $x = \sum_{i \in I} \lambda_i x^{(i)}$ is a stable phase splitting. Assume J is a finite set and $x = \sum_{j \in J} \lambda'_j x'^{(j)}$ is another stable phase splitting. Fix $v \in \bigcap_{i=1}^{q} \partial f(x^{(i)}) \subset \bigcap_{i \in I} \partial f(x^{(i)}) = \partial f(x)$ and some $i \in I$. If there was some $j \in J$ with $x'^{(j)} \notin \{x^{(i)} \mid i \in I\}$, then

$$\begin{aligned}\sum_{j \in J} \lambda'_j f(x'^{(j)}) > \sum_{j \in J} \lambda'_j \big(f(x^{(i)}) + \langle v, x'^{(j)} - x^{(i)} \rangle\big) \\ = f(x^{(i)}) + \langle v, x - x^{(i)} \rangle \geq (\operatorname{conv}(f))(x^{(i)}) + \langle v, x - x^{(i)} \rangle \\ \geq (\operatorname{conv}(f))(x) + \langle v, x^{(i)} - x \rangle + \langle v, x - x^{(i)} \rangle = (\operatorname{conv}(f))(x)\,,\end{aligned}$$

a contradiction. Therefore, we can assume without loss of generality $J \subset I$ and for any $j \in J$, $x'^{(j)} = x^{(j)}$. This implies

$$0 = \sum_{i \in I} \lambda_i x^{(i)} - \sum_{j \in J} \lambda'_j x^{(j)} = \sum_{i \in I \setminus J} \lambda_i x^{(i)} + \sum_{j \in J} (\lambda_i - \lambda'_j) x^{(j)}$$

and with $\sum_{i \in I \setminus J} \lambda_i + \sum_{j \in J} (\lambda_i - \lambda'_j) = 0$ and Proposition 6.7, we obtain $\lambda'_j = \lambda_i$ whenever $j \in J$ and $\lambda_i = 0$ whenever $i \in I \setminus J$, which is by the assumption on I only possible if $J = I$. □

3.28 Corollary *Under the assumptions and hypotheses of Theorem 3.21, for any $v \in \partial(\operatorname{conv}(f))(x)$ and $j \in \{1, ..., q\}$ with $\lambda_j > 0$ we have*

$$(\operatorname{conv}(f))(x) - \langle v, x \rangle = f(x^{(j)}) - \langle v, x^{(j)} \rangle\,.$$

If additionally $x^{(j)} \in \operatorname{int}(\Omega)$ and f is differentiable at $x^{(j)}$, then $x \in \operatorname{int}(\Omega)$ and $\operatorname{conv}(f)$ is differentiable in x with $v = \nabla(\operatorname{conv}(f))(x) = \nabla f(x^{(j)})$ and $T_x(\operatorname{conv}(f)) = T_{x^{(j)}} f$.

Proof. Fix $v \in \partial(\operatorname{conv}(f))(x)$, which is non-empty by convexity of $\operatorname{conv}(f)$, and fix $j \in \{1, ..., q\}$ with $\lambda_j > 0$. By $v \in \partial(\operatorname{conv}(f))(x) \subset \partial f(x^{(j)})$, we have

$$f(x^{(j)}) \geq (\operatorname{conv}(f))(x^{(j)}) \geq (\operatorname{conv}(f))(x) + \langle v, x^{(j)} - x \rangle\,.$$

Since $v \in \partial(\operatorname{conv}(f))(x) \subset \partial f(x^{(j)})$, the affine (especially convex) function $h : \mathbb{R}^d \to \mathbb{R},\ h(x') = f(x^{(j)}) + \langle v, x' - x^{(j)} \rangle$ minorizes f and the definition of the convex envelope implies

$$(\operatorname{conv}(f))(x) \geq h(x) = f(x^{(j)}) + \langle v, x - x^{(j)} \rangle\,.$$

The claimed equality follows by

$$\begin{aligned}(\operatorname{conv}(f))(x) - \langle v, x\rangle &\geq f(x^{(j)}) + \langle v, x - x^{(j)}\rangle - \langle v, x\rangle = f(x^{(j)}) - \langle v, x^{(j)}\rangle \\ &\geq (\operatorname{conv}(f))(x) + \langle v, x^{(j)} - x\rangle - \langle v, x^{(j)}\rangle \\ &= (\operatorname{conv}(f))(x) - \langle v, x\rangle\,.\end{aligned}$$

Now assume that $x^{(j)} \in \operatorname{int}(\Omega)$ and f is differentiable at $x^{(j)}$. The mapping $T : \mathbb{R}^d \to \mathbb{R}^d$, $T(x) = \sum\limits_{i\in J\setminus\{j\}} \lambda_i x^{(i)} + \lambda_j x$ is by $\lambda_j > 0$ an affine transformation and for $r > 0$ small enough, such that $B_r(x^{(j)}) \subset \Omega$, the image of $B_r(x^{(j)})$ is an open subset of $\operatorname{conv}(\Omega)$. Since $\operatorname{conv}(f)$ is convex, by Proposition 3.5 and Theorem 3.21 we have $\emptyset \neq \partial(\operatorname{conv}(f))(x) \subset \partial f(x^{(j)})$ and Proposition 3.6 implies the differentiability of $\operatorname{conv}(f)$ in x with $\nabla(\operatorname{conv}(f))(x) = \nabla f(x^{(j)})$. Finally, for any $x' \in \mathbb{R}^d$ we obtain

$$\begin{aligned}&(\operatorname{conv}(f))(x) + \langle\nabla(\operatorname{conv}(f))(x), x' - x\rangle = (\operatorname{conv}(f))(x) + \langle v, x' - x\rangle \\ &= f(x^{(j)}) + \langle v, x' - x^{(j)}\rangle = f(x^{(j)}) + \langle\nabla f(x^{(j)}), x' - x\rangle\end{aligned}$$

and therefore $T_x(\operatorname{conv}(f)) = T_{x^{(j)}} f$. □

3.29 Definition *The function f is said to have a common tangent plane in a set $\{x^{(1)}, ..., x^{(q)}\} \subset \operatorname{int}(\Omega)$, if for any $i \in \{1, ..., q\}$ the function f is differentiable at $x^{(i)}$ and*

$$\begin{gathered}\nabla f(x^{(1)}) = ... = \nabla f(x^{(q)})\,, \\ f(x^{(1)}) - \langle\nabla f(x^{(1)}), x^{(1)}\rangle = ... = f(x^{(q)}) - \langle\nabla f(x^{(q)}), x^{(q)}\rangle\,.\end{gathered} \tag{3.6}$$

Corollary 3.28 especially implies, that for $x^{(1)}, ..., x^{(q)} \in \operatorname{int}(\Omega)$ with f differentiable at each point $x^{(1)}, ..., x^{(q)}$ and $\lambda_1, ..., \lambda_q > 0$ with $x = \sum\limits_{i=1}^{q} \lambda_i x^{(i)}$ and $(\operatorname{conv}(f))(x) = \sum\limits_{i=1}^{q} \lambda_i f(x^{(i)})$, necessarily f has a common tangent plane in the set $\{x^{(1)}, ..., x^{(q)}\}$, see [18, Remark 2.1].

The condition, that f has a common tangent plane in $\{x^{(1)}, ..., x^{(q)}\} \subset \operatorname{int}(\Omega)$ is not sufficient for $\bigcap\limits_{i=1}^{q} \partial f(x^{(i)}) \neq \emptyset$, since a gradient is in general not a subgradient. However, since gradients are local objects while the calculation of subgradients involve the evaluation of the function on the whole domain, for a differentiable function it can be more convenient to solve (3.6) first and check afterwards, whether the solutions also satisfy $\bigcap\limits_{i=1}^{q} \partial f(x^{(i)}) \neq \emptyset$ or not.

In general, the infimum in (3.5) is not necessarily a minimum, as the example $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + \chi_{\{0\}}(x)$ shows, where $(\mathrm{conv}(f))(0) = 0$ cannot be represented as the convex combination of values of f due to the lack of lower semicontinuity of f.

3.30 Definition *A function $f : \Omega \to \mathbb{R}$ is called closed, if for any $r \in \mathbb{R}$ the sublevel-set $f^{-1}((-\infty, r])$ is closed in $\mathbb{R}^d$.*

By [11, IV, Proposition 1.2.2] a function $f : \Omega \to \mathbb{R}$ is closed if and only if the extended function $f_{ext} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ coinciding with f on Ω and taking the value ∞ outside Ω (which has the same sublevel-sets) is lower semi-continuous, i.e. if for any $x \in \mathbb{R}^d$ we have $\liminf_{y \to x} f_{ext}(y) \geq f_{ext}(x)$. It is not enough for f to be lower semi-continuous in order to be closed, since for $\Omega \subset \mathbb{R}^d$ open with $\Omega \neq \mathbb{R}^d$ and f lower semi-continuous and bounded, for sufficiently large r the sublevel set $f^{-1}((-\infty, r])$ equals Ω, which is not closed.

Since $\mathrm{conv}(f)$ is closed, according to [12, IV, Proposition 1.2.8] it is sufficient to take in (3.4) the supremum over all affine functions $h : \mathbb{R}^d \to \mathbb{R}$ with $h \leq f$. Nevertheless, closedness alone is not enough to ensure that the infimum in (3.5) is a minimum, for example consider $g : \mathbb{R} \to \mathbb{R},\ g(x) = e^{-x^2}$ with $\mathrm{conv}(g) \equiv 0$. In this case it is the absence of superlinear growth preventing the minimum to exist.

3.31 Definition *We say that a function $f : \Omega \to \mathbb{R}$ satisfies the superlinear growth condition, if for any sequence $(x_n)_{n \in \mathbb{N}} \subset \Omega$ with $||x_n|| \xrightarrow{n \to \infty} \infty$ we have*

$$\lim_{n \to \infty} \frac{f(x_n)}{||x_n||} = \infty \,. \tag{3.7}$$

For an extended-valued function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ we write $\mathrm{dom}(f) := \{x \in \mathbb{R}^d \mid f(x) < \infty\}$ and note, that if f is minorized by an affine function, the convex envelope of f can be defined by the same expression as in (3.4). The convex envelope of such a function is exactly the extension of $\mathrm{conv}(f|_{\mathrm{dom}(f)})$ with the value ∞ outside of $\mathrm{dom}(\mathrm{conv}(f)) = \mathrm{conv}(\mathrm{dom}(f))$.

Griewank and Rabier showed in [18, Theorem 2.1, Theorem 2.3], that the convex envelope of a proper ($\mathrm{dom}(f) \neq \emptyset$), lower semi-continuous function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ satisfying the superlinear growth condition is again a proper, lower semi-continuous and convex function with $\mathrm{dom}(\mathrm{conv}(f)) = \mathrm{conv}(\mathrm{dom}(f))$. Moreover, any $x \in \mathrm{dom}(\mathrm{conv}(f))$ admits a stable phase splitting $\sum_{i=1}^{q} \lambda_i x^{(i)} = x$ with $q \in \{1, ..., d+1\}$, $\lambda_1, ..., \lambda_q > 0$ and pairwise distinct

$x^{(1)}, ..., x^{(q)} \in \mathrm{dom}(f)$. We can assign to each $x \in \mathrm{conv}(\mathrm{dom}(f))$ the minimal number $q(x) \in \{1, ..., d+1\}$, for which a stable phase splitting of x with $q(x)$ phases exists, and call this function the phase-number function of f. This phase-number function is according to [18, Theorem 2.2] lower semi-continuous on the relative interior of $\mathrm{conv}(\Omega)$ and a point in which the phase-number function is not locally constant is called a point of phase bifurcation. If some point of a phase simplex is a point of phase bifurcation, then by [18, Theorem 5.2] every point of the phase simplex is a point of phase bifurcation and we will call the simplex a phase simplex of phase bifurcation.

Our construction of the convex envelope of a function $f : \Omega \to \mathbb{R}$ in the subsequent section relies crucially on finding points in which the tangent plane of f lies below the graph of f. If the tangent plane of f in some point lies strictly below the graph of f outside of a neighbourhood of this point, closedness and superlinear growth ensure that this property is stable under small variations of the point.

3.32 Proposition *Assume $f : \Omega \to \mathbb{R}$ is a closed function, which satisfies the superlinear growth condition* (3.7) *and denote for any $y \in \mathrm{int}(\Omega)$, at which f is differentiable, with $T_y f : \mathbb{R}^d \to \mathbb{R}$, $T_y f(y') = f(y) + \langle \nabla f(y'), y' - y \rangle$ the first-order Taylor polynomial. If f is continuously differentiable near some $\bar{y} \in \mathrm{int}(\Omega)$ and $U \subset \Omega$ is an open neighbourhood of $\bar{y}$ with $T_{\bar{y}} f < f|_{\Omega \setminus U}$, then there exists a neighbourhood $\widehat{U} \subset U$ of $\bar{y}$, such that for any $\hat{y} \in \widehat{U}$ we have $T_{\hat{y}} f < f|_{\Omega \setminus U}$ on $\Omega \setminus U$.*

Proof. In this proof, it is more convenient to work with lower semi-continuity instead of closedness, so denote with $f_{ext} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ the extension of f coinciding with f on Ω and taking the value ∞ outside Ω, which is lower semi-continuous.
Fix a sequence $(U_n)_{n\in\mathbb{N}}$ of open neighbourhoods of $\bar{y}$ with $U_1 \subset U$, $U_{n+1} \subset U_n$ for any $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} U_n = \{\bar{y}\}$. Assume for any $n \in \mathbb{N}$ there exists an $\hat{y}_n \in U_n$ and some $y_n \in \mathbb{R}^d \backslash U$ such that $f_{ext}(y_n) \leq T_{\hat{y}_n} f_{ext}(y_n)$. Then $\hat{y}_n \xrightarrow{n\to\infty} \bar{y}$ and the sequence $(y_n)_{n\in\mathbb{N}}$ is either bounded or unbounded. If $(y_n)_{n\in\mathbb{N}}$ was unbounded, then we could extract a subsequence (without relabelling) satisfying $||y_n|| \xrightarrow{n\to\infty} \infty$. But this would imply

$$\begin{aligned} f_{ext}(y_n) \leq T_{\hat{y}_n} f_{ext}(y_n) = T_{\hat{y}_n} f(y_n) &= f(\hat{y}_n) + \langle \nabla f(\hat{y}_n), y_n - \hat{y}_n \rangle \\ &\leq |f(\hat{y}_n) - \langle \nabla f(\hat{y}_n), \hat{y}_n \rangle| + ||\nabla f(\hat{y}_n)|| \cdot ||y_n|| , \end{aligned}$$

contradicting the superlinear growth condition of f by $\hat{y}_n \to \bar{y}$, $f(\hat{y}_n) \to f(\bar{y})$ and $\nabla f(\hat{y}_n) \to \nabla f(\bar{y})$ for $n \to \infty$ together with the fact that $f_{ext}(y_n) = f(y_n)$ by finiteness of the right hand side. So assume $(y_n)_{n\in\mathbb{N}}$ is bounded. Then there exists a subsequence, which converges to some $y^* \in \mathbb{R}^d \setminus U$. By lower semi-continuity of f and $\nabla f_{ext}(\hat{y}_n) \to \nabla f_{ext}(\bar{y})$ for $n \to \infty$, this would imply

$$f_{ext}(y^*) \leq \liminf_{n\to\infty} f_{ext}(y_n) \leq \liminf_{n\to\infty} T_{\hat{y}_n} f_{ext}(y_n) = T_{\bar{y}}(y^*)\,,$$

a contradiction. Consequently there exists some $n_0 \in \mathbb{N}$, such that for any $\hat{y} \in U_{n_0} =: \widehat{U}$ and any $y \in \Omega \setminus U$ we have

$$f(y) = f_{ext}(y) > T_{\hat{y}} f_{ext}(y) = T_{\hat{y}} f(y)\,.$$

4 Construction of the convex envelope

4.1 Motivation

The purpose of this subsection is to illustrate the ideas of our construction of the convex envelope near a known maximal phase simplex and to explain, how those ideas are related to the work of Griewank and Rabier in [18].

The approach of Griewank and Rabier

Griewank and Rabier investigated in [18] the convexification of smooth functions and classified the points at which phase bifurcation occurs.

They restricted their analysis of convex envelopes to the case, where $\Omega \subset \mathbb{R}^d$ is non-empty and open, $f \in C^\infty(\Omega)$ is closed, satisfies the superlinear growth condition (3.7) and additionally f is generic, in the sense that any stable phase splitting is unique. For a detailed characterization of those generic functions, see [18, Section 4]. Assume $\bar{y} \in \operatorname{conv}(\Omega)$ and $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)} \in \Omega$, such that $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$ is the phase simplex of $\bar{y}$. Necessarily, the set $\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ satisfies (3.6) and equivalently, for $p : \Omega \to \mathbb{R}$, $p(y) = f(y) - \langle \nabla f(y), y \rangle$ the $(k+1)$-tuple $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$ is a zero of

$$
\begin{aligned}
&F : \Omega^{k+1} \to (\mathbb{R}^{d+1})^k, \\
&F(y^{(1)}, ..., y^{(k+1)}) = (\nabla f(y^{(i)}) - \nabla f(y^{(1)}), p(y^{(i)}) - p(y^{(1)}))_{i=2,...,k+1} \, .
\end{aligned}
$$

If $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$ is the face of a larger phase simplex then it is necessarily a phase simplex of phase bifurcation, since any neighbourhood of some point in $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$ contains points of the relative interior of the larger phase simplex having a greater phase number.

If $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$ is a maximal phase simplex of phase bifurcation, then for arbitrarily small neighbourhoods $U^{(1)}, ..., U^{(k+1)} \subset \Omega$ of $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}$ there are $(\tilde{y}^{(1)}, ..., \tilde{y}^{(k+1)}) \in \prod_{i=1}^{k+1} U^{(i)}$, $j \in \{1, ..., k+1\}$ and $\tilde{z}^{(j)} \in U^{(j)} \setminus \{\tilde{y}^{(j)}\}$ with $F(\tilde{y}^{(1)}, ..., \tilde{y}^{(k+1)}) = 0$, $\nabla f(\tilde{z}^{(j)}) = \nabla f(\tilde{y}^{(j)})$ and $p(\tilde{z}^{(j)}) = p(\tilde{y}^{(j)})$ (see [18, page 374]). Roughly speaking, there is a sequence of higher dimensional phase simplices, whose vertices accumulate at the vertices of $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$ having at least one extra vertex in one of the neighbourhoods $U^{(i)}$. This situation can be prevented assuming that f is strictly locally convex near each $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}$, since by injectivity of any $\nabla f|_{U^{(i)}}$, $i = 1, ..., k+1$, no $\tilde{z}^{(j)}$ arbitrarily close to $\tilde{y}^{(j)}$ with $\nabla f(\tilde{z}^{(j)}) = \nabla f(\tilde{y}^{(j)})$ exists.

In [18, Definition 5.1], they call the given phase simplex $\operatorname{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ non-degenerate, if either $k = 0$ or the Jacobian of F at $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$ has full rank of $(d+1)\cdot k$, which guarantees the solvability of $F(y^{(1)}, ..., y^{(k+1)}) = 0$ in a neighbourhood of $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$ by the implicit function theorem. Those solutions occupy a manifold V of dimension $(d-k)$ near $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}) \in \Omega^{k+1}$, see the discussion before [18, Theorem 5.3]. If $\mathrm{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$ is a non-degenerate maximal phase simplex, which is not a phase simplex of phase bifurcation, then according to [18, Theorem 5.3] any zero of F close enough to $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$ defines the vertices of a phase simplex, which is not a simplex of phase bifurcation. Furthermore, in the proof of [18, Theorem 5.3] they constructed for a small neighbourhood $N \subset \Omega$ of $\bar{y}$ a parametrization of the phase simplices of the elements of N. More precisely, they constructed for any $i \in \{1, ..., k+1\}$ neighbourhoods $N^{(i)} \subset \Omega$ of $\bar{y}^{(i)}$ and continuous functions $v^{(i)} : N \to N^{(i)}$ and $l_i : N \to \mathbb{R}$, such that for any $y \in N$,

$$(v^{(1)}(y), ..., v^{(k+1)}(y), l_1(y), ..., l_{k+1}(y)) \in V \times \mathrm{relint}(\Delta_{k+1})$$

and $\mathrm{conv}\{v^{(1)}(y), ..., v^{(k+1)}(y)\}$ is the phase simplex of y, i.e. $y = \sum\limits_{i=1}^{k+1} l_i(y) \cdot v^{(i)}(y)$ and

$$(\mathrm{conv}(f))(y) = \sum_{i=1}^{k+1} l_i(y) \cdot f(v^{(i)}(y)) \,.$$

The „vertex functions“ $v^{(1)}, ..., v^{(k+1)}$ determine the phase simplex, while the coordinate functions $l_1, ..., l_{k+1}$ determine the barycentric coordinates of the corresponding stable phase splitting. Altogether, the mapping

$$N \ni y \mapsto (v^{(1)}(y), ..., v^{(k+1)}(y), l_1(y), ..., l_{k+1}(y)) \in V \times \mathrm{relint}(\Delta_{k+1})$$

is continuous, injective and open and the inverse mapping can be viewed as a parametrization of the neighbourhood N, for which an expression of the convex envelope in terms of the parametrization is available.

Our approach

We also consider some non-empty set $\Omega \subset \mathbb{R}^d$ (not necessarily open) and some closed function $f : \Omega \to \mathbb{R}$, which satisfies the superlinear growth condition (3.7). Fix $\bar{y} \in \mathrm{conv}(\Omega)$ and a unique stable phase splitting of $\bar{y}$, i.e. $k \in \{0, ..., d\}$, pairwise distinct points $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)} \in \Omega$ and $t \in \mathrm{relint}(\Delta_{k+1})$ with $\sum\limits_{i=1}^{k+1} t_i \bar{y}^{(i)} = \bar{y}$ and

$$(\mathrm{conv}(f))(\bar{y}) = \sum_{i=1}^{k+1} t_i f(\bar{y}^{(i)}) \,.$$

Furthermore, assume that $\operatorname{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ is a maximal phase simplex of $\bar{y}$. Taking $k+1$ instead of k as the number of phases simplifies the indexing later on.

Just as the construction of Griewank and Rabier, our construction also crucially relies on finding further solutions of (3.6) near our known solution $\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$. The definition of non-degeneracy of a phase simplex requires the existence of second derivatives, which we do not want to assume. Instead, we assume that $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)} \in \operatorname{int}(\Omega)$ and that f is (once) differentiable and locally strictly convex near each point $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}$, i.e. for any $i \in \{1, ..., k+1\}$ there exists an open convex neighbourhood $U^{(i)} \subset \Omega$ of $\bar{y}^{(i)}$, such that the restriction $f|_{U^{(i)}}$ is strictly convex and differentiable. Notice that differentiability and strict convexity of the respective restrictions already imply $f|_{U^{(i)}} \in C^1(U^{(i)})$ by Proposition 3.2. Then we are able to derive for some sufficiently small $\delta > 0$ a continuous parametrization $\Phi : \Delta_{k+1}^{\delta} \times C_{\delta}^{d-k} \to \mathbb{R}^d$ of a whole neighbourhood of $\operatorname{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$, on which we can give an expression for the convex envelope via a stable phase splitting in terms of the parametrization. The idea is, that the d-dimensional set $\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}$ can be partitioned into sets $\{S_I \mid \emptyset \neq I \subset \{1, ..., k+1\}\}$, such that for each $\emptyset \neq I \subset \{1, ..., k+1\}$ the restriction of Φ to S_I provides a parametrization of exactly those phase simplices, whose corners lie in the neighbourhoods $U^{(i)}, i \in I$.

In the first step, we focus on solving (3.6) and parametrizing the solutions. This will be done in a specialized setting, in which a known solution of (3.6) is given by the origin 0_d and the first k unit vectors $e^{(1)}, ..., e^{(k)}$. This simplifies the calculations while solving (3.6) and deriving a parametrization of those phase simplices including a vertex near the origin. Afterwards, we are going back to the general setting described above and use affine transformation in the argument of f to create the situation of the specialized setting. Making the above mentioned parametrization Φ continuous at the interfaces of the partitioning sets S_I requires several re-parametrizations.

Finally, we not only obtain a parametrization of all phase simplices in a neighbourhood of $\operatorname{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$, we additionally observe that the parametrization Φ is bi-Lipschitz, if any restriction $f|_{U^{(i)}}$, $i \in \{1, ..., k+1\}$ is strongly convex and has Lipschitz continuous gradient.

4.2 Finding points with a common tangent plane

This section is dedicated to solve the system (3.6) in a neighbourhood of a known solution and to parametrize the set consisting of the simplices of those solutions. More precisely, assume throughout this section that $\widetilde{\Omega} \subset \mathbb{R}^d$ is a non-empty set and $\tilde{f} : \widetilde{\Omega} \to \mathbb{R}$. For the sake of notational convenience we write $e^{(0)} := 0_d$ and assume that $k \in \{1, ..., d\}$, $e^{(0)}, ..., e^{(k)} \in \text{int}(\widetilde{\Omega})$ and for any $i \in \{0, ..., k\}$, $\widetilde{U}_i \subset \widetilde{\Omega}$ is an open convex neighbourhood of $e^{(i)}$, such that the restriction $\tilde{f}|_{\widetilde{U}_i}$ is differentiable and strictly convex. Finally, assume that $\tilde{f}(e^{(0)}) = ... = \tilde{f}(e^{(k)})$ and $\nabla \tilde{f}(e^{(0)}) = ... = \nabla \tilde{f}(e^{(k)}) = 0$, especially, $\{e^{(0)}, ..., e^{(k)}\}$ is a solution of (3.6). Strict convexity of the restrictions allows us to reformulate the problem of solving (3.6) in terms of the gradients, since for any $i \in \{0, ..., k\}$ by Lemma 3.11 the gradient mapping $\nabla \tilde{f}|_{\widetilde{U}^{(i)}}$ is an embedding.

4.1 Proposition *For any* $I \subset \{1, ..., k\}$ *and* $(y^{(i)})_{i \in I \cup \{0\}} \in \prod_{i \in I \cup \{0\}} \widetilde{U}_i$ *the set* $\{y^{(i)} \mid i \in I \cup \{0\}\}$ *satisfies* (3.6)*, if and only if there is some* $v \in \bigcap_{i \in I \cup \{0\}} \nabla \tilde{f}(\widetilde{U}_i)$*, such that for any* $i \in I \cup \{0\}$ *we have* $y^{(i)} = (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(v)$ *and*

$$\underset{i \in I}{\forall} (\tilde{f}|_{\widetilde{U}_i})^*(v) = (\tilde{f}|_{\widetilde{U}_0})^*(v)\,, \tag{4.1}$$

where for any $i \in I \cup \{0\}$ *the function* $(\tilde{f}|_{\widetilde{U}_i})^*$ *denotes the Legendre transform of* $\tilde{f}|_{\widetilde{U}_i}$ *as defined in Definition 3.13.*

Proof. The first equation of (3.6) implies, that for any $i \in I$, $v := \nabla \tilde{f}(y^{(0)}) = \nabla \tilde{f}(y^{(i)})$, especially $v \in \bigcap_{i \in I \cup \{0\}} \nabla \tilde{f}(\widetilde{U}_i)$ and for any $i \in I \cup \{0\}$, $y^{(i)} = (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(v)$. Therefore replacing any $y^{(i)}$ by $(\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(v)$ in the second equality of (3.6) gives the equality of the (negative) Legrendre transforms and therefore implies (4.1). Conversely, if $v \in \bigcap_{i \in I \cup \{0\}} \nabla \tilde{f}(\widetilde{U}_i)$ satisfies (4.1), then by definition of the Legendre transform the points $\{(\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(v) \mid i \in I \cup \{0\}\}$ satisfy both equalities of (3.6). □

4.2 Remark In the previous Proposition, for $I = \emptyset$ any $y^{(0)} \in \tilde{U}_0$ satisfies trivially (4.1) with $v := \nabla f(y^{(0)})$, as well as $y^{(0)}$ satisfies trivially (3.6) with $q = 1$ and $x^{(1)} = y^{(0)}$. *

Define $\widetilde{V} := \bigcap_{i=0}^{k} \nabla \tilde{f}(\widetilde{U}_i)$ and

$$\widetilde{H} : \widetilde{V} \to \mathbb{R}^k,\ \widetilde{H}(v) = \begin{pmatrix} (\tilde{f}|_{\widetilde{U}_1})^*(v) - (\tilde{f}|_{\widetilde{U}_0})^*(v) \\ \vdots \\ (\tilde{f}|_{\widetilde{U}_k})^*(v) - (\tilde{f}|_{\widetilde{U}_0})^*(v) \end{pmatrix}.$$

For fixed $I \subset \{1, ..., k\}$ and $v \in \widetilde{V}$, (4.1) is satisfied if and only if for any $i \in I$ we have $\widetilde{H}_i(v) = 0$. Especially $\widetilde{H}(0) = 0$, since $\{e^{(0)}, ..., e^{(k)}\}$ satisfies (3.6) with $\nabla \tilde{f}(e^{(0)}) = ... = \nabla \tilde{f}(e^{(k)}) = 0$. The next Theorem allows us, to encode all solutions of (4.1) near the origin for arbitrary $I \subset \{1, ..., k\}$ in one diffeomorphism, if the restrictions $\tilde{f}|_{\widetilde{U}_i}$, $i \in \{0, ..., k\}$, are strictly convex.

4.3 Theorem *Assume that for any $i \in \{0, ..., k\}$ the restriction $\tilde{f}|_{\widetilde{U}_i}$ is strictly convex.*
Then $\widetilde{V}$ is an open neighbourhood of the origin and there exists some open neighbourhood $\widetilde{W} \subset \mathbb{R}^d$ of the origin and a diffeomorphism $\tilde{\xi} : \widetilde{W} \to \widetilde{V}$ onto its image with $\tilde{\xi}(0) = 0$ and $D\tilde{\xi}(0) = \mathrm{Id}_d$, such that

$$\underset{w \in \widetilde{W}}{\forall} \ \underset{i \in \{1,...,k\}}{\forall} : \widetilde{H}(\tilde{\xi}(w))_i = 0 \Leftrightarrow w_i = 0 \, .$$

For any $i \in \{0, ..., k\}$ the mapping $\tilde{g}^{(i)} := (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1} \circ \tilde{\xi} : \widetilde{W} \to \widetilde{U}_i$ is an embedding and for any $I \subset \{1, ..., k\}$ and $(y^{(i)})_{i \in I \cup \{0\}} \in \prod_{i \in I \cup \{0\}} \tilde{g}^{(i)}(\widetilde{W})$ the set $\{y^{(i)} \mid i \in I \cup \{0\}\}$ satisfies (3.6), if and only if

$$(y^{(i)})_{i \in I \cup \{0\}} \in \{(\tilde{g}^{(i)}(w))_{i \in I \cup \{0\}} \mid w \in \widetilde{W}, \underset{i \in I}{\forall} w_i = 0\} \, .$$

Proof. For any $i \in \{1, ..., k\}$, by Proposition 3.16 the i-th component of $\widetilde{H}$ is continuously differentiable at the origin with the derivative given by

$$\begin{aligned} D(\mathrm{pr}_i \circ \widetilde{H})(0) &= (\nabla (\tilde{f}|_{\widetilde{U}_i})^*(0) - \nabla (\tilde{f}|_{\widetilde{U}_0})^*(0))^T \\ &= ((\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(0) - (\nabla \tilde{f}|_{\widetilde{U}_0})^{-1}(0))^T = (e^{(i)})^T \, . \end{aligned}$$

If $d = 1$, set $\tilde{h}_1 : \{0\} \to \mathbb{R}, \ 0 \mapsto 0$.
If $d > 1$, then by the implicit function theorem there exist open neighbourhoods $K_i \subset \mathbb{R}^{d-1}$ of the origin and $J_i \subset \mathbb{R}$ of the origin with $\widetilde{V}_i' := \{v \in \mathbb{R}^d \mid \mathrm{pr}_i^{\perp}(v) \in K_i, v_i \in J_i\} \subset \widetilde{V}$ and a continuously differentiable function $\tilde{h}_i : K_i \to J_i$, such that for any $v \in \widetilde{V}_i'$ we have $\widetilde{H}(v)_i = 0$ if and only if $v_i = \tilde{h}_i(\mathrm{pr}_i^{\perp}(v))$. Furthermore, by the formula for the derivative of implicit functions, $D\tilde{h}_i(0) = 1^{-1} \cdot 0_{d-1}^T = 0_{d-1}^T$.

For the open neighbourhood $\widetilde{V}' := \bigcap_{i=1}^{k} \widetilde{V}'_i$ of the origin, the function

$$\tilde{\eta} : \widetilde{V}' \to \mathbb{R}^d,\ \tilde{\eta}(v) = \begin{pmatrix} v_1 - \tilde{h}_1(\mathrm{pr}_1^\perp(v)) \\ \vdots \\ v_k - \tilde{h}_k(\mathrm{pr}_k^\perp(v)) \\ v_{k+1} \\ \vdots \\ v_d \end{pmatrix}$$

is continuously differentiable with $\tilde{\eta}(0) = 0$ by $\tilde{h}_i(0) = 0$ for any $i \in \{1, ..., k\}$ and $D\tilde{\eta}(0) = \mathrm{Id}_d$. Hence there exist by the inverse function theorem an open neighbourhood $\widetilde{W} \subset \mathbb{R}^d$ of the origin and a continuously differentiable function $\tilde{\xi} : \widetilde{W} \to \mathbb{R}^d$ with $\tilde{\xi}(\widetilde{W}) \subset \widetilde{V}'$, $\tilde{\xi}(0) = 0$ and $D\tilde{\xi}(0) = (D\tilde{\eta}(0))^{-1} = \mathrm{Id}_d$, such that for any $w \in \widetilde{W}$ we have $\tilde{\eta}(\tilde{\xi}(w)) = w$ and for any $v \in \tilde{\xi}(\widetilde{W})$ we have $\tilde{\xi}(\tilde{\eta}(v)) = v$. Especially $\tilde{\xi}$ is a diffeomorphism onto its image.
For any $w \in \widetilde{W}$ and $i \in \{1, ..., k\}$ we have $\widetilde{H}(\tilde{\xi}(w))_i = 0$ if and only if $\tilde{\xi}(w)_i = \tilde{h}_i(\mathrm{pr}_i^\perp(\tilde{\xi}(w)))$ which is equivalent to

$$w_i = \tilde{\eta}(\tilde{\xi}(w))_i = \tilde{\xi}(w)_i - \tilde{h}_i(\mathrm{pr}_i^\perp(\tilde{\xi}(w))) = 0\,.$$

For any $i \in \{0, ..., k\}$ the mapping $\tilde{g}^{(i)} = (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1} \circ \tilde{\xi}$ is well-defined by $\tilde{\xi}(\widetilde{W}) \subset \widetilde{V} \subset \nabla \tilde{f}(\widetilde{U}_i)$ and the composition of the diffeomorphism $\tilde{\xi}$ and the embedding $(\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}$ according to Lemma 3.11, hence $\tilde{g}^{(i)}$ is an embedding.
Now fix $I \subset \{1, ..., k\}$ and $(y^{(i)})_{i \in I \cup \{0\}} \in \prod_{i \in I \cup \{0\}} \tilde{g}^{(i)}(\widetilde{W})$. Then

$$v := \nabla \tilde{f}(y^{(0)}) \in \nabla \tilde{f}(\tilde{g}^{(0)}(\widetilde{W})) = \nabla \tilde{f}\big((\nabla \tilde{f}|_{\widetilde{U}_0})^{-1}(\tilde{\xi}(\widetilde{W}))\big) = \tilde{\xi}(\widetilde{W})\,,$$

and there exists some $w \in \widetilde{W}$ with $v = \tilde{\xi}(w)$. By Proposition 4.1, the set $\{y^{(i)} \mid i \in I \cup \{0\}\}$ satisfies (3.6), if and only if v satisfies (4.1) and for any $i \in I \cup \{0\}$ we have $y^{(i)} \in (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(\{v\})$, which is for $\tilde{f}|_{\widetilde{U}_i}$ strictly convex equivalent to $y^{(i)} = (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(v)$. Now v satisfies (4.1) and only if for any $i \in I$ we have $\widetilde{H}(\tilde{\xi}(w))_i = \widetilde{H}(v)_i = 0$, which is by the first part of this Theorem the case if and only if $w_i = 0$. Therefore the set $\{y^{(i)} \mid i \in I \cup \{0\}\}$ satisfies (3.6), if and only if for any $i \in I \cup \{0\}$ we have

$$y^{(i)} = (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(v) = (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(\tilde{\xi}(w)) = \tilde{g}^{(i)}(w)$$

and $\forall_{i \in I}\, w_i = 0$. □

4.4 Corollary *If $\delta > 0$ is small enough, such that $C_\delta^d \subset \widetilde{W}$, then for any $w \in C_\delta^d$ and $i \in \{1, ..., k\}$ we have*

$$\operatorname{sign}(\widetilde{H}(\tilde{\xi}(w)))_i = \operatorname{sign}(w_i)\,.$$

Proof. For any $i \in \{1, ..., k\}$ the sets $\{w \in C_\delta^d \mid w_i < 0\}$ and $\{w \in C_\delta^d \mid w_i > 0\}$ are connected as well as the images $(\operatorname{pr}_i \circ \widetilde{H} \circ \tilde{\xi})(\{w \in C_\delta^d \mid w_i < 0\})$ and $(\operatorname{pr}_i \circ \widetilde{H} \circ \tilde{\xi})(\{w \in C_\delta^d \mid w_i > 0\})$ by continuity of $\operatorname{pr}_i \circ \widetilde{H} \circ \tilde{\xi}$. Since for any $w \in C_\delta^d$ we have $(\widetilde{H} \circ \tilde{\xi})(w)_i = 0$ if and only if $w_i = 0$, both images are connected subsets of $\mathbb{R} \setminus \{0\}$. With $(\widetilde{H} \circ \tilde{\xi})_i(0) = 0$ and calculating

$$D_i(\operatorname{pr}_i \circ \widetilde{H} \circ \tilde{\xi})(0) = D(\operatorname{pr}_i \circ \widetilde{H})(\tilde{\xi}(0)) \cdot D\tilde{\xi}(0)e^{(i)} = (e^{(i)})^T \operatorname{Id}_d e^{(i)} = 1\,,$$

we can conclude $(\operatorname{pr}_i \circ \widetilde{H} \circ \tilde{\xi})(\{w \in C_\delta^d \mid w_i < 0\}) \subset \mathbb{R}_-$ as well as $(\operatorname{pr}_i \circ \widetilde{H} \circ \tilde{\xi})(\{w \in C_\delta^d \mid w_i > 0\}) \subset \mathbb{R}_+$. □

4.5 Corollary *If in the situation of Theorem 4.3 for $i \in \{0, ..., k\}$ the restriction $\tilde{f}|_{\widetilde{U}_i}$ is strongly convex and $\nabla \tilde{f}|_{\widetilde{U}_i}$ is Lipschitz, then $\tilde{g}^{(i)}$ is locally bi-Lipschitz satisfying*

$$\partial_c \tilde{g}^{(i)}(0) \subset \operatorname{PD}(d)\,.$$

Proof. Since $\tilde{\xi}$ is a diffeomorphism, $\tilde{\xi}$ is locally bi-Lipschitz. If $\mu > 0$ is the modulus of convexity of $\tilde{f}|_{\widetilde{U}_i}$ and $L > 0$ is a Lipschitz constant of $\nabla \tilde{f}|_{\widetilde{U}_i}$, then by Lemma 3.12 the function $(\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}$ is bi-Lipschitz, $\tilde{g}^{(i)}$ is locally bi-Lipschitz as the composition of locally bi-Lipschitz functions and by Proposition 2.13,

$$\partial_c \tilde{g}^{(i)}(0) \subset \partial_c (\nabla \tilde{f}|_{\widetilde{U}_i})^{-1}(\tilde{\xi}(0)) \cdot D\tilde{\xi}(0) \subset \operatorname{PD}_{L^{-1}}^{\mu^{-1}}(d) \cdot \operatorname{Id}_d \subset \operatorname{PD}(d)\,,$$

where the last inclusion is due to Proposition 6.5. □

4.6 Definition *For any $I \subset \{1, ..., k\}$ and $\delta > 0$ define the set*

$$\tilde{S}_I^\delta := \{w \in \mathbb{R}^d \mid \operatorname{pr}_I(w) \in \tilde{\Delta}_d,\ \operatorname{pr}_I^\perp(w) \in C_\delta^d\}\,.$$

4.7 Theorem *For any $\delta > 0$ with $C_\delta^d \subset \widetilde{W}$ and any $I \subset \{1, ..., k\}$ the restriction of the function*

$$\begin{aligned} &\tilde{\phi}_I : (\operatorname{pr}_I^\perp)^{-1}(\widetilde{W}) \to \mathbb{R}^d, \\ &\tilde{\phi}_I(w) = \sum_{i \in I} w_i \cdot \tilde{g}^{(i)}(\operatorname{pr}_I^\perp(w)) + (1 - \sum_{i \in I} w_i) \cdot \tilde{g}^{(0)}(\operatorname{pr}_I^\perp(w)) \end{aligned}$$

to the set $\tilde{S}_I^\delta$ is an embedding.

Proof. Fix $\delta > 0$ with $C^d_\delta \subset \widetilde{W}$, $I \subset \{1, ..., k\}$. Since $\tilde{S}^\delta_I$ is compact and $\tilde{\phi}_I$ is continuous as the composition and sum of continuous functions, by Proposition 6.2, it suffices to show that $\tilde{\phi}_I$ is injective.
For any $w \in \tilde{S}^\delta_I$ the set $\{\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w)) \mid i \in I \cup \{0\}\}$ satisfies (3.6), i.e. for all $i \in I \cup \{0\}$ we have $v := \nabla \tilde{f}(\tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))) = \nabla \tilde{f}(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w)))$ and

$$\tilde{f}(\tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))) - \langle v, \tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))\rangle = \tilde{f}(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))) - \langle v, \tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))\rangle\,.$$

Therefore, the affine function

$$h : \mathbb{R}^d \to \mathbb{R},\ h(y) = \tilde{f}(\tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))) + \langle v, y - \tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))\rangle$$

satisfies for any $i \in I \cup \{0\}$ the equality $h(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))) = \tilde{f}(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w)))$ and for all $y \in \widetilde{U}_i$ by convexity of $\tilde{f}|_{\widetilde{U}_i}$ the inequality

$$\begin{aligned}\tilde{f}(y) &\geq \tilde{f}(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))) + \langle \nabla \tilde{f}(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))), y - \tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))\rangle \\ &= \tilde{f}(\tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))) + \langle \nabla \tilde{f}(\tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))), y - \tilde{g}^{(0)}(\mathrm{pr}^\perp_I(w))\rangle = h(y)\,.\end{aligned}$$

Defining $\widetilde{U} := \bigcup\limits_{i \in I \cup \{0\}} \widetilde{U}_i$ and $w_0 := 1 - \sum\limits_{i \in I} w_i$, the affine function h minorizes $\tilde{f}|_{\widetilde{U}}$ and Proposition 3.23 implies

$$\mathrm{conv}(\tilde{f}|_{\widetilde{U}})(\tilde{\phi}_I(w)) = \sum_{i \in I \cup \{0\}} w_i \cdot \tilde{f}\big(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))\big)\,. \tag{4.2}$$

Since there is at least one $i \in I \cup \{0\}$ with $w_i > 0$, by Corollary 3.28 $\mathrm{conv}(\tilde{f}|_{\widetilde{U}})$ is differentiable in $\tilde{\phi}_I(w)$ with

$$\nabla\, \mathrm{conv}(\tilde{f}|_{\widetilde{U}})(\tilde{\phi}_I(w)) = \nabla \tilde{f}(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w))) = \tilde{\xi}(\mathrm{pr}^\perp_I(w))\,.$$

Now assume $w' \in \tilde{S}^\delta_I$ with $\tilde{\phi}_I(w) = \tilde{\phi}_I(w')$. With $w'_0 := 1 - \sum\limits_{i \in I} w'_i$, by the same argument as above we obtain

$$\mathrm{conv}(\tilde{f}|_{\widetilde{U}})(\tilde{\phi}_I(w')) = \sum_{i \in I \cup \{0\}} w'_i \cdot \tilde{f}\big(\tilde{g}^{(i)}(\mathrm{pr}^\perp_I(w'))\big)$$

and

$$\nabla\, \mathrm{conv}(\tilde{f}|_{\widetilde{U}})(\tilde{\phi}_I(w')) = \tilde{\xi}(\mathrm{pr}^\perp_I(w'))\,.$$

Since $\tilde{\xi}$ is injective it follows $\mathrm{pr}^\perp_I(w) = \mathrm{pr}^\perp_I(w')$, i.e. for all $i \in \{1, ..., d\} \setminus I$, $w_i = w'_i$.

Since the points $(\tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w)))_{i\in I}$ are affinely independent by assumption on the sets $\widetilde{U}_i$, $i \in I \cup \{0\}$,

$$\begin{aligned}
0 &= \tilde{\phi}_I(w) - \tilde{\phi}_I(w') \\
&= \sum_{i\in I} w_i \cdot \tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w)) + (1 - \sum_{i\in I} w_i) \cdot \tilde{g}^{(0)}(\mathrm{pr}_I^\perp(w)) \\
&\quad - \sum_{i\in I} w_i' \cdot \tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w')) - (1 - \sum_{i\in I} w_i') \cdot \tilde{g}^{(0)}(\mathrm{pr}_I^\perp(w')) \\
&= \sum_{i\in I} (w_i - w_i') \cdot \tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w)) - \sum_{i\in I} (w_i - w_i') \cdot \tilde{g}^{(0)}(\mathrm{pr}_I^\perp(w)) \\
&= \sum_{i\in I} (w_i - w_i') \cdot (\tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w)) - \tilde{g}^{(0)}(\mathrm{pr}_I^\perp(w)))
\end{aligned}$$

implies $w_i = w_i'$ for all $i \in I$. Altogether, $w = w'$ and since w and w' were arbitrary elements of $\tilde{S}_I^\delta$, the restriction of $\tilde{\phi}_I$ to $\tilde{S}_I^\delta$ is injective. □

4.8 Theorem *If for all $i \in \{0, ..., k\}$ the restriction $\tilde{f}|_{\widetilde{U}_i}$ is convex with parameter $\mu > 0$ and $\nabla \tilde{f}|_{\widetilde{U}_i}$ is Lipschitz with parameter $L > 0$, then there is some $\delta > 0$, such that $C_\delta^d \subset \widetilde{W}$ and for any $I \subset \{1, ..., k\}$ the restriction of $\tilde{\phi}_I$ from the previous theorem to the set $\tilde{S}_I^\delta$ is bi-Lipschitz.*

Proof. In this proof, denote for any subset $J \subset \{1, ..., d\}$ the complement of J in $\{1, ..., d\}$ with $J^c := \{1, ..., d\} \setminus J$ and set $\mathrm{Id}_J := \mathrm{diag}(\sum_{j\in J} e_j) \in \mathbb{R}^{d\times d}$, which is the diagonal matrix associated to the projection pr_J. Especially for any $J' \subset \{1, ..., d\}$ we have $\mathrm{Id}_J \cdot \mathrm{Id}_{J'} = \mathrm{Id}_{J\cap J'}$.
Fix $I \subset \{1, ..., k\}$ and $w_0 \in \{w \in \mathbb{R}^d \mid \mathrm{pr}_I(w) \in \tilde{\Delta}_d,\ \mathrm{pr}_I^\perp(w) = 0\} \subset (\mathrm{pr}_I^\perp)^{-1}(\widetilde{W})$. We want to use Theorem 2.12 to show that $\tilde{\phi}_I$ is locally bi-Lipschitz in w_0. In order to estimate $\partial_c \tilde{\phi}_I(w_0) \subset \mathrm{GL}(d)$, we write $\tilde{\phi}_I$ as the composition of two other functions and use Proposition 2.13.
Define

$$\tilde{G}_I : (\mathrm{pr}_I^\perp)^{-1}(\widetilde{W}) \to (\mathbb{R}^d)^{k+2},\ \tilde{G}_I(w) = \begin{pmatrix} \mathrm{pr}_I(w) \\ \tilde{g}^{(0)}(\mathrm{pr}_I^\perp(w)) \\ \vdots \\ \tilde{g}^{(k)}(\mathrm{pr}_I^\perp(w)) \end{pmatrix}$$

and

$$\tilde{\psi} : (\mathbb{R}^d)^{k+2} \to \mathbb{R}^d,\ \tilde{\psi}\begin{pmatrix} t \\ y^{(0)} \\ \vdots \\ y^{(k)} \end{pmatrix} = \sum_{i=1}^k t_i \cdot y^{(i)} + (1 - \sum_{i=1}^k t_i) \cdot y^{(0)}\,.$$

Now $\tilde{\phi}_I = \tilde{\psi} \circ \tilde{G}_I$, since for any $w \in (\mathrm{pr}_I^\perp)^{-1}(\widetilde{W})$ and any $i \in \{1, ..., k\} \setminus I$ we get $(\tilde{G}_I(w))_i = \mathrm{pr}_I(w)_i = 0$. The projections pr_I and $\mathrm{pr}_I^\perp$ are continuously differentiable with $D\,\mathrm{pr}_I \equiv \mathrm{Id}_I$ and $D\,\mathrm{pr}_I^\perp \equiv \mathrm{Id}_{I^c}$. For any $i \in \{0, ..., k\}$ the function $\tilde{g}^{(i)}$ is by Corollary 4.5 locally (bi-)Lipschitz with $\partial_c \tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w_0)) = \partial_c \tilde{g}^{(i)}(0) \subset \mathrm{PD}(d)$, hence the composition $\tilde{g}^{(i)} \circ \mathrm{pr}_I^\perp$ is locally Lipschitz. According to Proposition 2.10, $\tilde{G}_I$ is locally Lipschitz with

$$\begin{aligned}
\partial_c \tilde{G}_I(w_0) &\subset \left\{ \begin{pmatrix} D\,\mathrm{pr}_I(w_0) \\ M_0 \\ \vdots \\ M_k \end{pmatrix} \;\middle|\; \underset{i\in\{0,\dots,k\}}{\forall} M_i \in \partial_c(\tilde{g}^{(i)} \circ \mathrm{pr}_I^\perp)(w_0) \right\} \\
&\subset \left\{ \begin{pmatrix} D\,\mathrm{pr}_I(w_0) \\ M_0 \\ \vdots \\ M_k \end{pmatrix} \;\middle|\; \underset{i\in\{0,\dots,k\}}{\forall} M_i \in \partial_c \tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w_0)) \cdot D\,\mathrm{pr}_I^\perp(w_0) \right\} \\
&\subset \left\{ \begin{pmatrix} \mathrm{Id}_I \\ M_0 \cdot \mathrm{Id}_{I^c} \\ \vdots \\ M_k \cdot \mathrm{Id}_{I^c} \end{pmatrix} \;\middle|\; \underset{i\in\{0,\dots,k\}}{\forall} M_i \in \mathrm{PD}(d) \right\}.
\end{aligned}$$

$\tilde{\psi}$ is continuously differentiable and with $t_0 := 1 - \sum_{i=1}^{k} t_i$ we obtain for any $t, y^{(0)}, ..., y^{(k)} \in \mathbb{R}^d$:

$$D\tilde{\psi} \begin{pmatrix} t \\ y^{(0)} \\ \vdots \\ y^{(k)} \end{pmatrix} = \left(y^{(1)} - y^{(0)} \mid \cdots \mid y^{(k)} - y^{(0)} \mid 0_{d\times(d-k)} \mid t_0 \cdot \mathrm{Id}_d \mid \cdots \mid t_k \cdot \mathrm{Id}_d \right)$$

and if $y^{(i)} = e^{(i)} = \tilde{g}^{(i)}(0) = \tilde{g}^{(i)}(\mathrm{pr}_I^\perp(w_0))$ for all $i \in \{0, ..., k\}$, this expression simplifies to

$$D\tilde{\psi}(\tilde{G}_I(w_0)) = \left(\mathrm{Id}_{\{1,\dots,k\}} \mid t_0 \cdot \mathrm{Id}_d \mid \cdots \mid t_k \cdot \mathrm{Id}_d \right).$$

The function $\tilde{\phi}_I$ is locally Lipschitz as composition of locally Lipschitz functions and defining $t := \mathrm{pr}_I(w_0) \in \tilde{\Delta}_d$, we can use again Proposition 2.13 to calculate the generalized derivative of $\tilde{\phi}_I$ in w_0 by

$$
\begin{aligned}
\partial_c\tilde{\phi}_I(w_0) &= D\tilde{\psi}(\tilde{G}_I(w_0)) \cdot \partial_c\tilde{G}_I(w) \\
&\subset \{\mathrm{Id}_{\{1,\dots,k\}} \cdot \mathrm{Id}_I + \big(\sum_{i=0}^{k} t_i \cdot M_i\big) \cdot \mathrm{Id}_{I^c} \mid \forall_{i\in\{0,\dots,k\}} M_i \in \mathrm{PD}(d)\} \\
&\subset \{\mathrm{Id}_I + M \cdot \mathrm{Id}_{I^c} \mid M \in \mathrm{PD}(d)\}\,.
\end{aligned}
$$

Now we show, that for any $M \in \mathrm{PD}(d)$ the matrix $\mathrm{Id}_I + M \cdot \mathrm{Id}_{I^c}$ is invertible, which especially implies $\partial_c\tilde{\phi}_I(w) \subset \mathrm{GL}(d)$.
Fix $x \in \mathbb{R}^d$ with $0 = (\mathrm{Id}_I + M \cdot \mathrm{Id}_{I^c})x = \mathrm{pr}_I(x) + M \cdot \mathrm{pr}_{I^c}(x)$. Multiplying with $(\mathrm{pr}_{I^c}(x))^T$ from the left, leads to $0 = (\mathrm{pr}_{I^c}(x))^T \cdot M \cdot \mathrm{pr}_{I^c}(x)$, which is by positive definiteness of M only possible if $\mathrm{pr}_{I^c}(x) = 0$. Then $0 = \mathrm{pr}_I(x) + M \cdot \mathrm{pr}_{I^c}(x) = \mathrm{pr}_I(x)$ implies $x = 0$ and injectivity as well as invertibility of $\mathrm{Id}_I + M \cdot \mathrm{Id}_{I^c}$. By Theorem 2.12, $\tilde{\phi}_I$ is locally Lipschitz in w_0 and since $w_0 \in \{w \in \mathbb{R}^d \mid \mathrm{pr}_I(w) \in \tilde{\Delta}_d,\ \mathrm{pr}_I^{\perp}(w) = 0\}$ was arbitrary, there exists some open neighbourhood $O_I \subset (\mathrm{pr}_I^{\perp})^{-1}(\widetilde{W})$ of $\{w \in \mathbb{R}^d \mid \mathrm{pr}_I(w) \in \tilde{\Delta}_d,\ \mathrm{pr}_I^{\perp}(w) = 0\}$, such that $\tilde{\phi}_I|_{O_I}$ is locally bi-Lipschitz. Now choose for any $I \subset \{1,...,k\}$ some $\delta_I > 0$ small enough, such that $C^d_{\delta_I} \subset \widetilde{W}$ and $\tilde{S}_I^{\delta_I} \subset O_I$. If no such δ_I exists, then for any $n \in \mathbb{N}$ there would be some

$$x^{(n)} \in \{w \in \mathbb{R}^d \mid \mathrm{pr}_I(w) \in \tilde{\Delta}_d,\ \mathrm{pr}_I^{\perp}(w) \in C^d_{1/n}\} =: K_n$$

with $x^{(n)} \notin O_I$ for all $m \in \mathbb{N}$. The sets K_n are compact with $K_{n+1} \subset K_n$. By compactness of K_1 there exists some subsequence $(x^{(n_m)})_{m\in\mathbb{N}}$ converging to some $x^* \in K_1$ and by closedness of the sets K_n, necessarily

$$x^* \in \bigcap_{n\in\mathbb{N}} K_n = \{w \in \mathbb{R}^d \mid \mathrm{pr}_I(w) \in \tilde{\Delta}_d,\ \mathrm{pr}_I^{\perp}(w) = 0\} \subset O_I\,,$$

a contradiction.
Set $\delta := \min\{\delta_I \mid I \subset \{1,...,k\}\}$. Then, for any $I \subset \{1,...,k\}$ the restriction of $\tilde{\phi}_I$ to the compact set $\tilde{S}_I^{\delta} \subset O_I \subset (\mathrm{pr}_I^{\perp})^{-1}(\widetilde{W})$ is locally bi-Lipschitz by $\tilde{S}_I^{\delta} \subset O_I$ and injective by Theorem 4.7, hence bi-Lipschitz by Corollary 2.4.□

4.3 General case

Now we recapitulate the assumptions described in the motivation of this chapter, which from now on will be kept throughout the rest of this chapter.

Assumptions

(i) $\Omega \subset \mathbb{R}^d$ is a non-empty set and $f : \Omega \to \mathbb{R}$ is closed and satisfies the superlinear growth condition (3.7),

(ii) $\bar{y} \in \operatorname{conv}(\Omega)$ is some point with a unique stable phase splitting, given by $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)} \in \operatorname{int}(\Omega)$ and $t \in \operatorname{relint}(\Delta_{k+1})$ with $\bar{y} = \sum_{i=1}^{k+1} t_i \cdot \bar{y}^{(i)}$ and

$$(\operatorname{conv}(f))(\bar{y}) = \sum_{i=1}^{k+1} t_i \cdot f(\bar{y}^{(i)}) \,, \tag{4.3}$$

(iii) the phase simplex $\operatorname{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ of $\bar{y}$ is maximal,

(iv) for any $i \in \{1, ..., k+1\}$ there exists an open neighbourhood $U^{(i)} \subset \Omega$ of $\bar{y}^{(i)}$, such that the restriction $f|_{U^{(i)}}$ is strictly convex and differentiable.

Corollary 3.28 implies (3.6) with $(x^{(1)}, ..., x^{(q)}) = (\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$, that $\bar{y} \in \operatorname{int}(\operatorname{conv}(\Omega))$, the differentiability of $\operatorname{conv}(f)$ at $\bar{y}$ and that for any $i \in \{1, ..., k+1\}$ we have $T_{\bar{y}}(\operatorname{conv}(f)) = T_{\bar{y}^{(i)}} f$. The points $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}$ are affinely independent according to Proposition 3.24 and by Proposition 3.27, for any $y' \in \Omega \backslash \{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ we have $f(y') > T_{\bar{y}}(\operatorname{conv}(f))(y')$. The geometrical meaning is, that the common tangent plane of the points $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}$ lies strictly below the graph of f except at $\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}$, since otherwise the stable phase splitting would not be unique or the phase simplex would not be maximal.

4.9 Proposition *There exists for any $i \in \{1, ..., k+1\}$ a neighbourhood $\widehat{U}^{(i)} \subset U^{(i)}$ of $\bar{y}^{(i)}$, such that for any $\hat{y}^{(i)} \in \widehat{U}^{(i)}$ and $y \in \Omega \setminus \bigcup_{i=1}^{k+1} U^{(i)}$ we have*

$$f(y) > f(\hat{y}^{(i)}) + \langle \nabla f(\hat{y}^{(i)}), y - \hat{y}^{(i)} \rangle$$

and for any $(y^{(1)}, ..., y^{(k+1)}) \in \prod_{i=1}^{k+1} \widehat{U}^{(i)}$ the points $y^{(1)}, ..., y^{(k+1)}$ are affinely independent.

Proof. The mapping

$$r : (\mathbb{R}^d)^{k+1} \to \{1, ..., k+1\}, \; (y^{(1)}, ..., y^{(k+1)}) \mapsto \operatorname{rank}(y^{(1)}, ..., y^{(k+1)})$$

is lower semi-continuous, hence the sublevel-set $r^{-1}((-\infty, k])$ is closed and $r^{-1}(\{k+1\}) = r^{-1}((k, \infty))$ is an open neighbourhood of $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$. Defining $U := \bigcup_{i=1}^{k+1} U^{(i)}$, by Proposition 3.32 there exists for each $i \in \{1, ..., k+1\}$ some neighbourhood $\widehat{U}^{(i)} \subset U^{(i)}$ of $\bar{y}^{(i)}$, such that for any $\hat{y}^{(i)} \in \widehat{U}^{(i)}$ and $y \in \Omega \setminus U$ we have

$$f(y) > f(\hat{y}^{(i)}) + \langle \nabla f(\hat{y}^{(i)}), y - \hat{y}^{(i)} \rangle \,.$$

Since $r^{-1}(\{k+1\})$ is an open neighbourhood of $(\bar{y}^{(1)}, ..., \bar{y}^{(k+1)})$, the neighbourhoods $\widehat{U}^{(i)}$ can be chosen small enough, such that $\prod_{i=1}^{k+1} \widehat{U}^{(i)} \subset r^{-1}(\{k+1\})$, which implies for $I \subset \{1, ..., k+1\}$ the affine independence of any points $y^{(1)}, ..., y^{(k+1)}$ with $(y^{(1)}, ..., y^{(k+1)}) \in \prod_{i=1}^{k+1} \widehat{U}^{(i)}$. $\square$

Defining $V := \bigcap_{i=1}^{k+1} \nabla f(U^{(i)})$ any information, which is needed for the construction of the convex envelope in a neighbourhood of $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$, is encoded in the function

$$
\begin{aligned}
&H : V \to \mathbb{R}^{(k+1)\times(k+1)},\\
&H(v)_{i,j} = (f|_{U^{(i)}})^*(v) - (f|_{U^{(j)}})^*(v),\ i,j = 1, ..., k+1\,.
\end{aligned}
$$

For each $v \in V$, $H(v)$ is a skew-symmetric matrix and the same calculation as in Proposition 4.1 shows that $\bar{v} := \nabla f(\bar{y}^{(1)}) = ... = \nabla f(\bar{y}^{(k+1)})$ is a zero of H, i.e. $H(\bar{v}) = 0_{(k+1)\times(k+1)}$.

The challenge of constructing the desired continuous (or bi-Lipschitz) parametrization of a neighbourhood of $\operatorname{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ is twofold. First, the construction in the previous section assumes that the corners of the given phase simplex coincide with the origin and the first k unit vectors. For any $l \in \{1, ..., k+1\}$ we can create this situation by applying an affine transformation in the argument of f, which maps $e^{(0)}$ to $\bar{y}^{(l)}$ and $\{e^{(i)} \mid i \in \{1, ..., k\}\}$ to $\{\bar{y}^{(i)} \mid i \in \{1, ..., k+1\} \setminus \{l\}\}$. After applying Theorem 4.7 (or Theorem 4.8) for the transformed function, we can transform back and get for any $I \subset \{1, ..., k+1\}$ containing l a parametrization of those potential phase simplices, whose corners lie in the neighbourhoods $U^{(i)}$, $i \in I$. The second step is, to make those parametrizations compatible with each other, in order to „glue“ them together to one continuous (or bi-Lipschitz) parametrization of a neighbourhood of $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$. The result is a parametrization of a whole neighbourhood of $\operatorname{conv}(\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\})$, in which we are able to give an expression of the convex envelope of f in terms of the parametrization.

Fix $l \in \{1, ..., k+1\}$. Then there is some (not necessarily unique) invertible matrix $A_l \in \mathrm{GL}(d)$ with

$$
A_l \cdot e^{(i)} = \begin{cases} \bar{y}^{(i)} - \bar{y}^{(l)} & \text{, if } i \in \{1, ..., l-1\} \\ \bar{y}^{(i+1)} - \bar{y}^{(l)} & \text{, if } i \in \{l, ..., k\} \end{cases} .
$$

The function

$$
h_l : \mathbb{R}^d \to \mathbb{R}^d,\ h_l(y) = \bar{y}^{(l)} + A_l \cdot y
$$

is an affine transformation satisfying for any $i \in \{0, ..., k\}$

$$h_l(e^{(i)}) = \begin{cases} \bar{y}^{(l)} & \text{, if } i = 0 \\ \bar{y}^{(i)} & \text{, if } i \in \{1, ..., l-1\} \\ \bar{y}^{(i+1)} & \text{, if } i \in \{l, ..., k\} \end{cases} .$$

We can define $\Omega_l := (h_l)^{-1}(\Omega)$ and with $\bar{v} = \nabla f(\bar{y}^{(1)}) = ... = \nabla f(\bar{y}^{(k+1)})$ as above,

$$f_l : \Omega_l \to \mathbb{R}, \; f_l(y) = f(h_l(y)) - \langle h_l(y), \bar{v} \rangle .$$

The function f_l satisfies for any $i \in \{1, ..., k\}$,

$$\begin{aligned} f_l(e^{(i)}) &= f(h_l(e^{(i)})) - \langle h_l(e^{(i)}), \bar{v} \rangle \\ &= \begin{cases} f(\bar{y}^{(i)}) - \langle \bar{y}^{(i)}, \bar{v} \rangle & \text{, if } i \in \{1, ..., l-1\} \\ f(\bar{y}^{(i+1)}) - \langle \bar{y}^{(i+1)}, \bar{v} \rangle & \text{, if } i \in \{l, ..., k\} \end{cases} \\ &= f(\bar{y}^{(l)}) - \langle \bar{y}^{(l)}, \bar{v} \rangle = f(h_l(e^{(0)})) - \langle h_l(e^{(0)}), \bar{v} \rangle = f_l(e^{(0)}) , \end{aligned}$$

since $\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$ satisfies (3.6) for f.
By Proposition 6.8, for any $i \in \{0, ..., k\}$ the set

$$U_l^{(i)} := \begin{cases} h_l^{-1}(U^{(l)}) & \text{, if } i = 0 \\ h_l^{-1}(U^{(i)}) & \text{, if } i \in \{1, ..., l-1\} \\ h_l^{-1}(U^{(i+1)}) & \text{, if } i \in \{l, ..., k\} \end{cases}$$

is a convex neighbourhood of $e^{(i)}$ and $f_l|_{U_l^{(i)}}$ is strictly convex, which implies by Proposition 3.11 that $\nabla f_l|_{U_l^{(i)}}$ is invertible. If $f|_{U^{(i)}}$ is strongly convex, then $f_l|_{U_l^{(i)}}$ is by Proposition 6.8 also strongly convex.
For any $y \in \Omega_l$ we can calculate

$$\nabla f_l(y) = Df_l(y)^T = \big((Df(h_l(y)) - \bar{v}^T) Dh_l(y) \big)^T = (A_l)^T (\nabla f(h_l(y)) - \bar{v}) , \tag{4.4}$$

especially for any $i \in \{0, ..., k\}$, since $\nabla f(\bar{y}^{(1)}) = ... = \nabla f(\bar{y}^{(k+1)}) = \bar{v}$ and $h_l(e^{(i)}) \in \{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$,

$$\nabla f_l(e^{(i)}) = (A_l)^T \cdot (\nabla f(h_l(e^{(i)})) - \bar{v}) = 0 .$$

With the affine transformation

$$T_l : \mathbb{R}^d \to \mathbb{R}^d, \; T_l(v) = \bar{v} + (A_l)^{-T} v$$

we have for any $y \in \bigcup_{i=1}^{k+1} U_l^{(i)}$, $(T_l \circ \nabla f_l)(y) = (\nabla f \circ h_l)(y)$.
Defining

$$V_l := \bigcap_{i=1}^{k+1} \nabla f_l(U_l^{(i)}) = \bigcap_{i=1}^{k+1} (T_l)^{-1}(\nabla f(h_l(U_l^{(i)}))) = (T_l)^{-1}\big(\bigcap_{i=1}^{k+1} (\nabla f(U^{(i)}))\big)$$
$$= (T_l)^{-1}(V)$$

and since for any $i \in \{0, ..., k\}$ the restrictions $\nabla f|_{U^{(i)}}$ and $\nabla f_l|_{h_l^{-1}(U^{(i)})}$ are invertible, for any $v \in V_l$:

$$(\nabla f|_{U^{(i)}})^{-1}(T_l(v)) = h_l((\nabla f_l|_{h_l^{-1}(U^{(i)})})^{-1}(v)) \,. \tag{4.5}$$

The next proposition describes, how the function

$$H_l : V_l \to \mathbb{R}^k,\ H_l(v) = \begin{pmatrix} (f_l|_{U_l^{(1)}})^*(v) - (f_l|_{U_l^{(0)}})^*(v) \\ \vdots \\ (f_l|_{U_l^{(k)}})^*(v) - (f_l|_{U_l^{(0)}})^*(v) \end{pmatrix}$$

is related to the l-th column of H.

4.10 Proposition *For any $v \in V_l$ we have $H_l(v) = \mathrm{pr}_l^{\perp}(H(T_l(v)) \cdot e^{(l)})$.*

Proof. For any $y \in \Omega_l$ the function f_l can be written as

$$f_l(y) = f(h_l(y)) - \langle h_l(y), \bar{v}\rangle = f(A_l y + \bar{y}^{(l)}) - \langle A_l y + \bar{y}^{(l)}, \bar{v}\rangle$$
$$= f\big(A_l(y + (A_l)^{-1}\bar{y}^{(l)})\big) + \langle y, -A_l^T \bar{v}\rangle - \langle \bar{y}^{(l)}, \bar{v}\rangle$$

and applying Proposition 3.15 with $A = A_l$, $a = -(A_l)^{-1}\bar{y}^{(l)}$, $a^* = -A_l^T\bar{v}$, $\alpha = -\langle \bar{y}^{(l)}, \bar{v}\rangle$ and $\alpha^* = -\alpha - \langle a, a^*\rangle = 0$ leads for any $i \in \{0, ..., k\}$ and any $v \in V_l$ to

$$(f_l|_{U_l^{(i)}})^*(v) = (f|_{h_l(U_l^{(i)})})^*((A_l)^{-T}\big(v - (-A_l^T\bar{v})\big)) + \langle v, -(A_l)^{-1}\bar{y}^{(l)}\rangle$$
$$= (f|_{h_l(U_l^{(i)})})^*(T_l(v)) - \langle v, (A_l)^{-1}\bar{y}^{(l)}\rangle \,.$$

This implies for any $i \in \{1, ..., k\}$ the equation

$$\begin{aligned}(H_l(v))_i &= (f_l|_{U_l^{(i)}})^*(v) - (f_l|_{U_l^{(0)}})^*(v) \\ &= (f|_{h_l(U_l^{(i)})})^*(T_l(v)) - (f|_{h_l(U_l^{(0)})})^*(T_l(v)) \\ &= \begin{cases} (f|_{U^{(i)}})^*(T_l(v)) - (f|_{U^{(l)}})^*(T_l(v)) & \text{, if } i \in \{1, ..., l-1\} \\ (f|_{U^{(i+1)}})^*(T_l(v)) - (f|_{U^{(l)}})^*(T_l(v)) & \text{, if } i \in \{l, ..., k\} \end{cases} \\ &= \begin{cases} H(T_l(v))_{i,l} & \text{, if } i \in \{1, ..., l-1\} \\ H(T_l(v))_{i+1,l} & \text{, if } i \in \{l, ..., k\} \end{cases} \\ &= \mathrm{pr}_l^{\perp}(H(T_l(v)) \cdot e^{(l)}) \,,\end{aligned}$$

which completes the proof. $\square$

Since f_l satisfies the assumptions of the previous section, we can apply Theorem 4.3 and Theorem 4.7 or Theorem 4.8 to f_l and then use the affine transformation T_l to obtain analogue results for f.

4.11 Theorem *There exists $\delta^{(0)} > 0$ and for any $l \in \{1, ..., k+1\}$ a bi-Lipschitz function $\xi_l^{(0)} : \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1}) \to V$ with $\xi_l^{(0)}(0) = \bar{v}$, such that for any $w \in \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1})$ we have*

$$\underset{j\in\{1,...,k+1\}}{\forall} \mathrm{sign}(H(\xi_l^{(0)}(w))_{j,l}) = \mathrm{sign}(w_j) \tag{4.6}$$

and for any $I \subset \{1, ..., k+1\}$ containing l the mapping

$$\begin{aligned}&\phi_{l,I}^{(0)} : \{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(0)}}^{d+1},\ \mathrm{pr}_I(w) \in \Delta_{d+1}\} \to \mathbb{R}^d,\\ &\phi_{l,I}^{(0)}(w) = \sum_{i\in I} w_i \cdot (\nabla f|_{U^{(i)}})^{-1}(\xi_l^{(0)}(\mathrm{pr}_I^{\perp}(w)))\end{aligned}$$

is an embedding. If for any $i \in \{1, ..., k+1\}$ the restriction $f|_{U^{(i)}}$ is strongly convex and $\nabla f|_{U^{(i)}}$ is Lipschitz, then $\delta^{(0)}$ can be chosen small enough such that for any $I \subset \{1, ..., k+1\}$ containing l the mapping $\phi_{l,I}^{(0)}$ is bi-Lipschitz.

Proof. Fix $l \in \{1, ..., k+1\}$. Then $f_l \in C^1(\Omega_l)$ satisfies $f_l(e^{(0)}) = ... = f_l(e^{(k)})$, $0 = \nabla f_l(e^{(0)}) = ... = \nabla f_l(e^{(k)})$ and for any $i \in \{0, ..., k\}$ the restriction $f_l|_{U_l^{(i)}}$ is strictly convex. By Theorem 4.3, there exists some open neighbourhood $W_l \subset \mathbb{R}^d$ of the origin and a diffeomorphism $\xi_l : W_l \to V_l$ onto its image with $\xi_l(0) = 0$ and $D\xi_l(0) = \mathrm{Id}_d$, such that

$$\underset{w\in W_l}{\forall}\ \underset{j\in\{1,...,k\}}{\forall} H_l(\xi_l(w))_j = 0 \Leftrightarrow w_j = 0\,.$$

For $\delta_l > 0$ small enough, such that $C_{\delta_l}^d \subset W_l$, we have by Corollary 4.4

$$\underset{w\in C_{\delta_l}^d}{\forall}\ \underset{j\in\{1,...,k\}}{\forall} \mathrm{sign}(H_l(\xi_l(w))_j) = \mathrm{sign}(w_j) \tag{4.7}$$

and for any $I_l \subset \{1, ..., k\}$ with $\tilde{S}_{I_l}^{\delta_l}$ from Definition 4.6 the function

$$\begin{aligned}&\tilde{\phi}_{l,I_l} : \tilde{S}_{I_l}^{\delta_l} \to \mathbb{R}^d,\\ &\tilde{\phi}_{l,I_l}(w) = \sum_{i\in I_l} w_i \cdot (\nabla f_l|_{U_l^{(i)}})^{-1}(\xi_l(\mathrm{pr}_{I_l}^{\perp}(w)))\\ &\qquad + (1 - \sum_{i\in I_l} w_i) \cdot (\nabla f_l|_{U_l^{(0)}})^{-1}(\xi_l(\mathrm{pr}_{I_l}^{\perp}(w)))\end{aligned}$$

is according to Theorem 4.7 an embedding. If for any $i \in \{1, ..., k+1\}$ the restriction $f|_{U^{(i)}}$ is strongly convex and $\nabla f|_{U^{(i)}}$ Lipschitz, then for any $i \in \{0, ..., k\}$ the restriction $f_l|_{U_l^{(i)}}$ is strongly convex and $\nabla f_l|_{U_l^{(i)}}$ is Lipschitz and by Theorem 4.8 δ_l can be chosen small enough such that $\tilde{\phi}_{l,I_l}$ is bi-Lipschitz.
With δ_l and ξ_l being constructed for any $l \in \{1, ..., k+1\}$ we can define $\delta^{(0)} := \min\{\delta_l \mid l \in \{1, ..., k+1\}\} > 0$ and for any $l \in \{1, ..., k+1\}$ the function

$$\xi_l^{(0)} : \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1}) \to V,\ \xi_l^{(0)}(w) = T_l(\xi_l(\mathrm{pr}_l^{\perp}(w))),$$

which is well-defined by

$$T_l(\xi_l(\mathrm{pr}_l^{\perp}(\mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1})))) = T_l(\xi_l(C_{\delta^{(0)}}^{d})) \subset T_l(\xi_l(W_l)) \subset T_l(V_l) = V.$$

The function $\xi_l^{(0)}$ is bi-Lipschitz, since the restriction of $\mathrm{pr}_l^{\perp}$ to $\mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1})$ is bi-Lipschitz (it eliminates the l-th component, which is zero), the restriction of the diffeomorphism ξ_l to the compact set $\mathrm{pr}_l^{\perp}(\mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1})) = C_{\delta^{(0)}}^{d}$ is bi-Lipschitz and the affine transformation T_l is bi-Lipschitz whenever $I_l \subset \{1, ..., k\}$.
Now fix $j \in \{1, ..., k+1\}$ and $w \in \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1})$. If $j = l$, then

$$\mathrm{sign}(H(\xi_l^{(0)}(w))_{j,l}) = \mathrm{sign}(H(\xi_l^{(0)}(w))_{l,l}) = 0 = \mathrm{sign}(w_l) = \mathrm{sign}(w_j),$$

if $j < l$, then with Proposition 4.10 and (4.7),

$$\begin{aligned}\mathrm{sign}(H(\xi_l^{(0)}(w))_{j,l}) &= \mathrm{sign}((H(\xi_l^{(0)}(w)) \cdot e^{(l)})_j) \\ &= \mathrm{sign}(\mathrm{pr}_l^{\perp}(H(T_l(\xi_l(\mathrm{pr}_l^{\perp}(w)))) \cdot e^{(l)})_j) \\ &= \mathrm{sign}(H_l(\xi_l(\mathrm{pr}_l^{\perp}(w)))_j) = \mathrm{sign}(\mathrm{pr}_l^{\perp}(w)_j) = \mathrm{sign}(w_j)\end{aligned}$$

and if $j > l$, then with Proposition 4.10 and (4.7),

$$\begin{aligned}\mathrm{sign}(H(\xi_l^{(0)}(w))_{j,l}) &= \mathrm{sign}((H(\xi_l^{(0)}(w)) \cdot e^{(l)})_j) \\ &= \mathrm{sign}(\mathrm{pr}_l^{\perp}(H(T_l(\xi_l(\mathrm{pr}_l^{\perp}(w)))) \cdot e^{(l)})_{j-1}) \\ &= \mathrm{sign}(H_l(\xi_l(\mathrm{pr}_l^{\perp}(w)))_{j-1}) = \mathrm{sign}(\mathrm{pr}_l^{\perp}(w)_{j-1}) = \mathrm{sign}(w_j).\end{aligned}$$

Fix $I \subset \{1, ..., k+1\}$ containing l and $w \in \mathbb{R}^{d+1}$ with $\mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(0)}}^{d+1}$ and $\mathrm{pr}_I(w) \in \Delta_d$. Setting

$$I_l := \{i - \chi_{\{l+1,...,k+1\}}(i) \mid i \in I \setminus \{l\}\} \subset \{1, ..., k\}$$

we want to show

$$\phi^{(0)}_{l,I}(w) = (h_l \circ \tilde{\phi}_{l,I_l} \circ \mathrm{pr}_l^\perp)(w)\,.$$

Now $\mathrm{pr}_l^\perp \circ \mathrm{pr}_I^\perp = \mathrm{pr}_{I_l}^\perp \circ \mathrm{pr}_l^\perp$ as mappings from $\mathbb{R}^{d+1}$ to $\mathbb{R}^d$ and for any $i \in \{1, ..., k+1\}$ we can use (4.5) to infer

$$\begin{aligned}(\nabla f|_{U^{(i)}})^{-1}(\xi_l^{(0)}(\mathrm{pr}_I^\perp(w))) &= (\nabla f|_{U^{(i)}})^{-1}\big(T_l(\xi_l(\mathrm{pr}_l^\perp(\mathrm{pr}_I^\perp(w))))\big) \\ &= h_l\Big((\nabla f_l|_{h_l^{-1}(U^{(i)})})^{-1}\big(\xi_l(\mathrm{pr}_{I_l}^\perp(\mathrm{pr}_l^\perp(w)))\big)\Big)\,.\end{aligned}$$

With Proposition 6.8 we have

$$\begin{aligned}\phi^{(0)}_{l,I}(w) &= \sum_{i\in I} w_i \cdot (\nabla f|_{U^{(i)}})^{-1}(\xi_l^{(0)}(\mathrm{pr}_I^\perp(w))) \\ &= \sum_{i\in I} w_i \cdot h_l\Big((\nabla f_l|_{h_l^{-1}(U^{(i)})})^{-1}\big(\xi_l(\mathrm{pr}_{I_l}^\perp(\mathrm{pr}_l^\perp(w)))\big)\Big) \\ &= h_l\Big(\sum_{i\in I} w_i \cdot (\nabla f_l|_{h_l^{-1}(U^{(i)})})^{-1}\big(\xi_l(\mathrm{pr}_{I_l}^\perp(\mathrm{pr}_l^\perp(w)))\big)\Big)\,.\end{aligned}$$

Writing $* := \xi_l(\mathrm{pr}_{I_l}^\perp(\mathrm{pr}_l^\perp(w)))$ to shorten the notation, we obtain

$$\begin{aligned}&\sum_{i\in I} w_i \cdot (\nabla f_l|_{h_l^{-1}(U^{(i)})})^{-1}\big(\xi_l(\mathrm{pr}_{I_l}^\perp(\mathrm{pr}_l^\perp(w)))\big) \\ =& \sum_{i\in I, i<l} w_i \cdot (\nabla f_l|_{U_l^{(i)}})^{-1}(*) + w_l \cdot (\nabla f_l|_{U_l^{(0)}})^{-1}(*) + \sum_{i\in I, i>l} w_i \cdot (\nabla f_l|_{U_l^{(i-1)}})^{-1}(*) \\ =& \sum_{i\in I, i<l} (\mathrm{pr}_l^\perp(w))_i \cdot (\nabla f_l|_{U_l^{(i)}})^{-1}(*) + \sum_{i\in I, i>l} (\mathrm{pr}_l^\perp(w))_{i-1} \cdot (\nabla f_l|_{U_l^{(i-1)}})^{-1}(*) \\ &+ \big(1 - \sum_{i\in I\setminus\{l\}} w_i\big) \cdot (\nabla f_l|_{U_l^{(0)}})^{-1}(*) \\ =& \sum_{i\in I_l} (\mathrm{pr}_l^\perp(w))_i \cdot (\nabla f_l|_{U_l^{(i)}})^{-1}(*) + \big(1 - \sum_{i\in I_l} (\mathrm{pr}_l^\perp(w))_i\big) \cdot (\nabla f_l|_{U_l^{(0)}})^{-1}(*) \\ =& \tilde{\phi}_{l,I_l}(\mathrm{pr}_l^\perp(w))\end{aligned}$$

giving

$$\phi^{(0)}_{l,I}(w) = (h_l \circ \tilde{\phi}_{l,I_l} \circ \mathrm{pr}_l^\perp)(w)\,.$$

The function $\mathrm{pr}_l^\perp$ maps the set $\{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I^\perp(w) \in C^d_{\delta^{(0)}}, \mathrm{pr}_I(w) \in \Delta_d\}$ bi-Lipschitz onto $\tilde{S}^{\delta^{(0)}}_{I_l}$, $\tilde{\phi}_{l,I_l}$ is an embedding and h_l is an affine transformation, hence also bi-Lipschitz. Consequently $\phi_{l,I}^{(0)}$ is an embedding and if for any $i \in \{1,...,k+1\}$ the restriction $f|_{U^{(i)}}$ is strongly convex and $\nabla f|_{U^{(i)}}$ is Lipschitz, then $\delta^{(0)}$ can be chosen small enough such that any $\tilde{\phi}_{l,I_l}$ is bi-Lipschitz and then $\phi_{l,I}^{(0)}$ is bi-Lipschitz as a composition of bi-Lipschitz mappings. □

The mappings $\xi_l^{(0)}$ are not compatible with each other, in the sense that for $I \subset \{1,...,k+1\}$ and $j,l \in I$ we cannot conclude that for any $w \in C_{\delta^{(0)}}^{d+1}$ we have $\xi_j(\mathrm{pr}_I^\perp(w)) = \xi_l(\mathrm{pr}_I^\perp(w))$. In order to achieve this property, we have to modify the functions ξ_l without violating (4.6).

4.12 Theorem *There exists $\delta^* > 0$ and for any $l \in \{1,...,k+1\}$ a bi-Lipschitz mapping $\xi_l^* : \mathrm{pr}_{\{l\}}^\perp(C_{\delta^*}^{d+1}) \to V$ with $\xi_l^*(0) = \bar{v}$, such that the following properties are satisfied:*

(i) $\underset{l,j\in\{1,...,k+1\}}{\forall}\ \underset{w\in\mathrm{pr}_{\{l\}}^\perp(C_{\delta^*}^{d+1})}{\forall} : \mathrm{sign}(H(\xi_l^*(w))_{j,l}) = \mathrm{sign}(w_j),$

(ii) $\underset{l,j\in\{1,...,k+1\}}{\forall}\ \underset{w\in\mathrm{pr}_{\{l,j\}}^\perp(C_{\delta^*}^{d+1})}{\forall} : \xi_l^*(w) = \xi_j^*(w),$

(iii) for any $l \in \{1,...,k+1\}$ and $I \subset \{1,...,k+1\}$ containing l the mapping

$$\phi_{l,I}^* : \{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I^\perp(w) \in C_{\delta^*}^{d+1},\ \mathrm{pr}_I(w) \in \Delta_{d+1}\} \to \mathbb{R}^d,$$
$$\phi_{l,I}^*(w) = \sum_{i\in I} w_i \cdot (\nabla f|_{U^{(i)}})^{-1}(\xi_l^*(\mathrm{pr}_I^\perp(w)))$$

is an embedding/bi-Lipschitz.

The first condition says, that the sign of the j-th component of w determines, whether the tangent plane of f in $(\nabla f|_{U_j})^{-1}(\xi_l^*(w))$ lies below (positive sign), lies above (negative sign) or coincides (sign equals zero) with the tangent plane of f in $(\nabla f|_{U_l})^{-1}(\xi_l^*(w))$. The fact that ξ_l^* is defined on $\mathrm{pr}_{\{l\}}^\perp(C_{\delta^*}^{d+1})$ ensures compatibility with the case $j = l$.
The second condition ensures, that the value of the function $\phi_{l,I}^*$ from (iii) does not depend on the choice of $l \in I$.
The last condition provides a bi-Lipschitz parametrization of a neighbourhood of relint(conv$\{\bar{y}^{(i)} \mid i \in I\}$), which will be used later on to construct the convex envelope in this neighbourhood.

The construction of δ^* and the functions ξ_l^* will be done successively, constructing for any $i \in \{0,...,k+1\}$ some $\delta^{(i)} > 0$ and bi-Lipschitz mappings $\xi_l^{(i)} : \mathrm{pr}_{\{l\}}^\perp(C_{\delta^{(i)}}^{d+1}) \to \mathbb{R}^d$, such that conditions (i)-(iii) are satisfied (with δ^* and ξ_l^* replaced by $\delta^{(i)}$ and $\xi_l^{(i)}$) with the restriction, that (ii) is only required to be satisfied whenever $l,j \leq i$. The functions $\xi_l^{(i+1)}$ are then constructed from $\xi_l^{(i)}$ by a bi-Lipschitz transformation of the argument, which ensures the validity of (ii) whenever $j \leq i+1$ without violating (i) and (iii). Finally, $\delta^* := \delta^{(k+1)}$ and $\xi_l^* := \xi_l^{(k+1)}$ for any $l \in \{1,...,k+1\}$ will satisfy (i)-(iii).

Proof. We show that for any $i \in \{0,...,k+1\}$ there exists $\delta^{(i)} > 0$ and for any $l \in \{1,...,k+1\}$ some bi-Lipschitz function $\xi_l^{(i)} : \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i)}}^{d+1}) \to V$ with $\xi_l^{(i)}(0) = \bar{v}$, such that the following conditions are satisfied:

C1(i): $\underset{l,j\in\{1,...,k+1\}}{\forall} \ \underset{w\in\mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i)}}^{d+1})}{\forall} : \mathrm{sign}(H(\xi_l^{(i)}(w))_{j,l}) = \mathrm{sign}(w_j)$,

C2(i): $\underset{l,j\in\{1,...,k+1\}}{\forall} \ \underset{w\in\mathrm{pr}_{\{l,j\}}^{\perp}(C_{\delta^{(i)}}^{d+1})}{\forall} : (l,j \leq i \Rightarrow \xi_l^{(i)}(w) = \xi_j^{(i)}(w))$,

C3(i): for any $l \in \{1,...,k+1\}$ and $I \subset \{1,...,k+1\}$ containing l the mapping

$$\phi_{l,I}^{(i)} : \{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(i)}}^{d+1},\ \mathrm{pr}_I(w) \in \Delta_d\} \to \mathbb{R}^d,$$
$$\phi_{l,I}^{(i)}(w) = \sum_{j\in I} w_j \cdot (\nabla f|_{U^{(j)}})^{-1}(\xi_l^{(i)}(\mathrm{pr}_I^{\perp}(w)))$$

is an embedding/bi-Lipschitz.

For $i = 0$, by Theorem 4.11 there exists $\delta^{(0)}$ and for any $l \in \{1,...,k+1\}$ a bi-Lipschitz function $\xi_l^{(0)} : \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(0)}}^{d+1}) \to V$ with $\xi_l^{(0)}(0) = \bar{v}$, such that C1(0) and C3(0) are satisfied, C2(0) is also trivially satisfied, since no $l,j \in \{1,...,k+1\}$ with $l,j \leq i = 0$ exist.
Now assume that for some $i \in \{0,...,k\}$, $\delta^{(i)} > 0$ and $\xi_l^{(i)} : \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i)}}^{d+1}) \to V$, $l \in \{1,...,k+1\}$, satisfying C1(i)-C3(i) are already constructed.
Since for any $l \in \{1,...,k+1\}$ the set $\mathrm{pr}_{\{l\}}^{\perp}(\mathrm{int}(C_{\delta^{(i)}}^{d+1}))$ is a relatively open neighbourhood of 0, by Corollary 6.4 the set $\xi_l^{(i)}(\mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i)}}^{d+1}))$ is a neighbourhood of $\xi_l^{(i)}(0) = \bar{v}$. Hence by continuity of $\xi_{i+1}^{(i)}$, there exists $0 < \delta^{(i+1)} \leq \delta^{(i)}$ small enough, such that for any $l \in \{1,...,k+1\}$ we have $\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(C_{\delta^{(i+1)}}^{d+1})) \subset \xi_l^{(i)}(\mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i)}}^{d+1}))$. This ensures, that for any $l \in \{1,...,k+1\}$ the mapping

$$\theta_l^{(i)} : \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i+1)}}^{d+1}) \to \mathbb{R}^{d+1},\ \theta_l^{(i)}(w) = (\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w))) + \mathrm{pr}_{\{i+1\}}(w)$$

is well-defined.
By C1(i), we have for any $l,j \in \{1,...,k+1\}$ and $w \in \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i+1)}}^{d+1})$ using

$H(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))_{l,i+1}) = \mathrm{sign}((\mathrm{pr}^{\perp}_{\{i+1\}}(w))_l) = 0$:

$$\begin{aligned}
\mathrm{sign}\left((\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))_j\right) &= \mathrm{sign}\left(H(\xi^{(i)}_l((\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))))_{j,l}\right) \\
&= \mathrm{sign}\left(H(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))_{j,l})\right) \\
&= \mathrm{sign}\left(H(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))_{j,l})\right. \\
&\quad \left. + H(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))_{l,i+1})\right) \\
&= \mathrm{sign}\left(H(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))_{j,i+1})\right) \\
&= \mathrm{sign}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)_j) \\
&= \begin{cases} \mathrm{sign}(w_j) & , \text{ if } j \neq i+1 \\ 0 & , \text{ if } j = i+1 \end{cases} .
\end{aligned}$$

Especially, $(\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))_{i+1} = 0$ and therefore

$$\theta^{(i)}_l(w)_j = \begin{cases} (\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))_j & , \text{ if } j \neq i+1 \\ w_{i+1} & , \text{ if } j = i+1 \end{cases}$$

as well as

$$\begin{aligned}
\mathrm{sign}(\theta^{(i)}_l(w)_j) &= \begin{cases} \mathrm{sign}\left((\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))_j\right) & , \text{ if } j \neq i+1 \\ \mathrm{sign}(w_{i+1}) & , \text{ if } j = i+1 \end{cases} \\
&= \begin{cases} \mathrm{sign}(w_j) & , \text{ if } j \neq i+1 \\ \mathrm{sign}(w_{i+1}) & , \text{ if } j = i+1 \end{cases} \\
&= \mathrm{sign}(w_j) \qquad (4.8)
\end{aligned}$$

and in particular $\theta^{(i)}_l(0) = 0$. The restriction of $(\xi^{(i)}_l)^{-1} \circ \xi^{(i)}_{i+1}$ to $\mathrm{pr}^{\perp}_{\{l,i+1\}}(C^{d+1}_{\delta^{(i+1)}})$ is bi-Lipschitz as the composition of bi-Lipschitz functions, hence there exist $0 < c \leq C$, such that for all $w, w' \in \mathrm{pr}^{\perp}_{\{l,i+1\}}(C^{d+1}_{\delta^{(i+1)}})$ we have:

$$c \cdot ||w - w'||_1 \leq ||(\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(w)) - (\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(w'))||_1 \leq C \cdot ||w - w'||_1 .$$

This implies for all $w, w' \in \mathrm{pr}^{\perp}_{\{l\}}(C^{d+1}_{\delta^{(i+1)}})$ the estimate

$$
\begin{aligned}
&\min\{c,1\} \cdot ||w - w'||_1 \\
&\leq c \cdot ||\,\mathrm{pr}^{\perp}_{\{i+1\}}(w) - \mathrm{pr}^{\perp}_{\{i+1\}}(w')||_1 + |w_{i+1} - w'_{i+1}| \\
&\leq ||(\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))) - (\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w')))||_1 + |w_{i+1} - w'_{i+1}| \\
&= ||\theta^{(i)}_l(w) - \theta^{(i)}_l(w')||_1 \\
&= ||(\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))) - (\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w')))||_1 + |w_{i+1} - w'_{i+1}| \\
&\leq C \cdot ||\,\mathrm{pr}^{\perp}_{\{i+1\}}(w) - \mathrm{pr}^{\perp}_{\{i+1\}}(w')||_1 + |w_{i+1} - w'_{i+1}| \\
&\leq \max\{C,1\} \cdot ||w - w'||_1 \,,
\end{aligned} \tag{4.9}
$$

hence $\theta^{(i)}_l$ is bi-Lipschitz. By $(\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))) \in \mathrm{pr}^{\perp}_{\{l\}}(C^{d+1}_{\delta^{(i)}})$ with $(\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))_{i+1} = 0$ and $|w_{i+1}| \leq \delta^{(i+1)} \leq \delta^{(i)}$ we can conclude $\theta^{(i)}_l(w) \in \mathrm{pr}^{\perp}_{\{l\}}(C^{d+1}_{\delta^{(i)}})$. The mapping

$$\xi^{(i+1)}_l : \mathrm{pr}^{\perp}_{\{l\}}(C^{d+1}_{\delta^{(i+1)}}) \to V,\ \xi^{(i+1)}_l(w) = \xi^{(i)}_l(\theta^{(i)}_l(w))$$

is well-defined and bi-Lipschitz as composition of bi-Lipschitz mappings with $\xi^{(i+1)}_l(0) = \xi^{(i)}_l(\theta^{(i)}_l(0)) = \xi^{(i)}_l(0) = \bar{v}$.
Now we show that the mappings $\xi^{(i+1)}_l$, $l \in \{1, ..., k+1\}$, satisfy C1(i+1)-C3(i+1).
C1(i+1):
For any $l, j \in \{1, ..., k+1\}$ and $w \in \mathrm{pr}^{\perp}_{\{l\}}(C^{d+1}_{\delta^{(i+1)}})$ we have

$$\mathrm{sign}(H(\xi^{(i+1)}_l(w))_{j,l}) = \mathrm{sign}(H(\xi^{(i)}_l(\theta^{(i)}_l(w))_{j,l}) = \mathrm{sign}(\theta^{(i)}_l(w)_j) = \mathrm{sign}(w_j)\,.$$

C2(i+1):
Assume $l, j \in \{1, ..., i+1\}$ and $w \in \mathrm{pr}^{\perp}_{\{l,j\}}(C^{d+1}_{\delta^{(i+1)}})$.
If $j = i+1$, then $\mathrm{pr}_{\{i+1\}}(w) = \mathrm{pr}_{\{j\}}(w) = 0$, $\mathrm{pr}^{\perp}_{\{i+1\}}(w) = \mathrm{pr}^{\perp}_{\{j\}}(w) = w$ and

$$
\begin{aligned}
\xi^{(i+1)}_j(w) &= \xi^{(i+1)}_{i+1}(w) = \xi^{(i+1)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)) = \xi^{(i)}_l\big((\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w)))\big) \\
&= \xi^{(i)}_l\big((\xi^{(i)}_l)^{-1}(\xi^{(i)}_{i+1}(\mathrm{pr}^{\perp}_{\{i+1\}}(w))) + \mathrm{pr}_{\{i+1\}}(w)\big) = \xi^{(i)}_l(\theta^{(i)}_l(w)) \\
&= \xi^{(i+1)}_l(w)
\end{aligned}
$$

and if $l = i+1$, analogously $\xi^{(i+1)}_l(w) = \xi^{(i+1)}_j(w)$.
If $j, l \leq i$, then by $\mathrm{sign}(\theta^{(i)}_l(w)_j) = w_j = 0 = w_l = \mathrm{sign}(\theta^{(i)}_l(w)_l)$ we have $\theta^{(i)}_l(w) \in \mathrm{pr}^{\perp}_{\{l,j\}}(C^{d+1}_{\delta^{(i)}})$ and according to C2(i):

$$\xi^{(i+1)}_l(w) = \xi^{(i)}_l(\theta^{(i)}_l(w)) = \xi^{(i)}_j(\theta^{(i)}_l(w))\,.$$

Furthermore, by

$$\begin{aligned}
&\operatorname{sign}\big((\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w)))_l\big) = \operatorname{sign}(w_l) = 0\,,\\
&\operatorname{sign}\big((\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w)))_j\big) = \operatorname{sign}(w_j) = 0\,,
\end{aligned}$$

we can conclude $(\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w))) \in \mathrm{pr}_{\{l,j\}}^{\perp}(C_{\delta^{(i)}}^{d+1})$ and again by C2(i),

$$\begin{aligned}
\theta_l^{(i)}(w) &= (\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w))) + \mathrm{pr}_{\{i+1\}}(w)\\
&= (\xi_j^{(i)})^{-1}\big(\xi_j^{(i)}((\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w))))\big) + \mathrm{pr}_{\{i+1\}}(w)\\
&= (\xi_j^{(i)})^{-1}\big(\xi_l^{(i)}((\xi_l^{(i)})^{-1}(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w))))\big) + \mathrm{pr}_{\{i+1\}}(w)\\
&= (\xi_j^{(i)})^{-1}\big(\xi_{i+1}^{(i)}(\mathrm{pr}_{\{i+1\}}^{\perp}(w))\big) + \mathrm{pr}_{\{i+1\}}(w) = \theta_j^{(i)}(w),
\end{aligned}$$

which gives us $\xi_l^{(i+1)}(w) = \xi_j^{(i)}(\theta_l^{(i)}(w)) = \xi_j^{(i)}(\theta_j^{(i)}(w)) = \xi_j^{(i+1)}(w)$ as asserted.
C3(i+1):
Fix $l \in \{1, ..., k+1\}$ and $I \subset \{1, ..., k+1\}$ containing l. Then the mapping

$$\begin{aligned}
&\Theta_{l,I}^{(i)} : \{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(i+1)}}^{d+1},\ \mathrm{pr}_I(w) \in \Delta_{d+1}\} \to \mathbb{R}^{d+1},\\
&\Theta_{l,I}^{(i)}(w) = \mathrm{pr}_I(w) + \theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w))
\end{aligned}$$

is well-defined, since for any $w \in \mathbb{R}^{d+1}$, $\mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(i+1)}}^{d+1}$ and $l \in I$ imply $\mathrm{pr}_I^{\perp}(w) \in \mathrm{pr}_I^{\perp}(C_{\delta^{(i+1)}}^{d+1}) \subset \mathrm{pr}_{\{l\}}^{\perp}(C_{\delta^{(i+1)}}^{d+1})$.
For any $w \in \mathbb{R}^{d+1}$ with $\mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(i+1)}}^{d+1}$ and $\mathrm{pr}_I(w) \in \Delta_{d+1}$, equation (4.8) implies $\mathrm{pr}_I^{\perp}\big(\theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w))\big) = \theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w))$ and with

$$\begin{aligned}
\mathrm{pr}_I^{\perp}(\Theta_{l,I}^{(i)}(w)) &= \mathrm{pr}_I^{\perp}\big(\,\mathrm{pr}_I(w) + \theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w))\big) = \theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w)) \in C_{\delta^{(i)}}^{d+1}\\
\mathrm{pr}_I(\Theta_{l,I}^{(i)}(w)) &= \mathrm{pr}_I\big(\,\mathrm{pr}_I(w) + \theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w))\big) = \mathrm{pr}_I(w) \in \Delta_{d+1}
\end{aligned}$$

the image of $\Theta_{l,I}^{(i)}$ is a subset of $\{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I^{\perp}(w) \in C_{\delta^{(i)}}^{d+1},\ \mathrm{pr}_I(w) \in \Delta_{d+1}\}$.
Therefore we obtain $\phi_{l,I}^{(i+1)} = \phi_{l,I}^{(i)} \circ \Theta_{l,I}^{(i)}$, since

$$\xi_l^{(i)}(\mathrm{pr}_I^{\perp}(\Theta_{l,I}^{(i)}(w))) = \xi_l^{(i)}(\theta_l^{(i)}(\mathrm{pr}_I^{\perp}(w))) = \xi_l^{(i+1)}(\mathrm{pr}_I^{\perp}(w))$$

and for any $j \in I$ we have $\Theta_{l,I}^{(i)}(w)_j = \mathrm{pr}_I(w)_j = w_j$.
The function $\Theta_{l,I}^{(i)}$ is bi-Lipschitz, since for any w, w' in the domain of $\Theta_{l,I}$ we

can use (4.9) to calculate:

$$\begin{aligned}
&\min\{c,1\}\cdot||w-w'||_1\\
\le\ & ||\operatorname{pr}_I(w)-\operatorname{pr}_I(w')||_1+\min\{c,1\}\cdot||\operatorname{pr}_I^\perp(w)-\operatorname{pr}_I^\perp(w')||_1\\
\le\ & ||\operatorname{pr}_I(w)-\operatorname{pr}_I(w')||_1+||\theta_l^{(i)}(\operatorname{pr}_I^\perp(w))-\theta_l^{(i)}(\operatorname{pr}_I^\perp(w'))||_1\\
=\ & ||\operatorname{pr}_I(\Theta_{l,I}^{(i)}(w)-\Theta_{l,I}^{(i)}(w'))||_1+||\operatorname{pr}_I^\perp(\Theta_{l,I}^{(i)}(w)-\Theta_{l,I}^{(i)}(w'))||_1\\
=\ & ||\Theta_{l,I}^{(i)}(w)-\Theta_{l,I}^{(i)}(w')||_1\\
=\ & ||\operatorname{pr}_I(\Theta_{l,I}^{(i)}(w)-\Theta_{l,I}^{(i)}(w'))||_1+||\operatorname{pr}_I^\perp(\Theta_{l,I}^{(i)}(w)-\Theta_{l,I}^{(i)}(w'))||_1\\
=\ & ||\operatorname{pr}_I(w)-\operatorname{pr}_I(w')||_1+||\theta_l^{(i)}(\operatorname{pr}_I^\perp(w))-\theta_l^{(i)}(\operatorname{pr}_I^\perp(w'))||_1\\
\le\ & ||\operatorname{pr}_I(w)-\operatorname{pr}_I(w')||_1+\max\{C,1\}\cdot||\operatorname{pr}_I^\perp(w)-\operatorname{pr}_I^\perp(w')||_1\\
\le\ & \max\{C,1\}\cdot||w-w'||_1
\end{aligned}$$

Since $\Theta_{l,I}^{(i)}$ is bi-Lipschitz and $\phi_{l,I}^{(i)}$ is an embedding/bi-Lipschitz, $\phi_{l,I}^{(i+1)}$ is an embedding/bi-Lipschitz.
Finally, for $\delta^* := \delta^{(k+1)}$ and $\xi_l^* := \xi_l^{(k+1)}$ for any $l \in \{1,...,k+1\}$ the conditions C1(k+1)- C3(k+1) are eqivalent to (i)-(iii). □

Recall for any $i \in \{1,...,k+1\}$ the set $\widehat{U}^{(i)}$ from Proposition 4.9 with the property, that for any $\hat{y}^{(i)} \in \widehat{U}^{(i)}$ and $y \in \Omega \setminus \bigcup_{i=1}^{k+1} U^{(i)}$ we have

$$f(y) > f(\hat{y}^{(i)}) + \langle \nabla f(\hat{y}^{(i)}), y - \hat{y}^{(i)} \rangle$$

and for any $(y^{(i)})_{i\in I} \in \prod_{i\in I} \widehat{U}^{(i)}$ the points $y^{(1)}, ..., y^{(k+1)}$ are affinely independent. For any $i \in \{1,...,k+1\}$ the mapping $(\nabla f|_{U^{(i)}})^{-1} \circ \xi_i^*$ is continuous.

4.13 Definition *Fix $0 < \delta \le \delta^*$ small enough, such that for any $i \in \{1,...,k+1\}$ we have $(\nabla f|_{U^{(i)}})^{-1}(\xi_i^*(\operatorname{pr}_{\{i\}}^\perp(C_{\delta^*}^{d+1}))) \subset \widehat{U}^{(i)}$ and define the function*

$$g^{(i)} : \operatorname{pr}_{\{i\}}^\perp(C_{\delta^*}^{d+1}) \to \widehat{U}^{(i)},\ g^{(i)}(s) := (\nabla f|_{U^{(i)}})^{-1}(\xi_i^*(s)).$$

Furthermore, for any non-empty $I \subset \{1,...,k+1\}$ define the set

$$S_I^\delta := \{w \in \Delta_{k+1}^\delta \times C_\delta^{d-k} \mid \underset{i\in\{1,...,k+1\}}{\forall} w_i \ge 0 \Leftrightarrow i \in I\} \subset \mathbb{R}^{d+1}.$$

Due to the ⇔-relation in the definition of S_I^δ, the sets S_I^δ form a partition of $\Delta_{k+1}^\delta \times C_\delta^{d-k} \subset \mathbb{R}^{d+1}$.

4.14 Proposition *For any* $w \in S_I^\delta$*, the simplex* $\operatorname{conv}\{g^{(i)}(\operatorname{pr}_I^\perp(w)) \mid i \in I\}$ *is a maximal phase simplex of* f*.*

Proof. For any $i \in I$, $\nabla f(g^{(i)}(\operatorname{pr}_I^\perp(w))) = \xi_i^*(\operatorname{pr}_I^\perp(w)) \in V$ takes by Theorem 4.12 (ii) the same value, denoted by v. For any $i \in I$ and $j \in \{1, ..., k+1\}$, we have by definition of S_I^δ and Theorem 4.12 (i),

$$\operatorname{sign}(H(v)_{j,i}) = \operatorname{sign}(H(\xi_i^*(\operatorname{pr}_I^\perp(w)))_{j,i}) = \operatorname{sign}(\operatorname{pr}_I^\perp(w)_j) = \begin{cases} 0 & \text{, if } j \in I \\ -1 & \text{, if } j \notin I \end{cases} .$$

According to Proposition 3.7, for any $y^{(j)} \in U^{(j)}$ we can estimate using formula (3.3)

$$\begin{aligned} f(y^{(j)}) &\geq f(g^{(j)}(\operatorname{pr}_I^\perp(w))) + \langle \nabla f(g^{(j)}(\operatorname{pr}_I^\perp(w))), y^{(j)} - g^{(j)}(\operatorname{pr}_I^\perp(w))\rangle \\ &= f((\nabla f|_{U^{(j)}})^{-1}(v)) - \langle v, (\nabla f|_{U^{(j)}})^{-1}(v)\rangle + \langle v, y^{(j)}\rangle \\ &= -(f|_{U^{(j)}})^*(v) + \langle v, y^{(j)}\rangle = -(f|_{U^{(i)}})^*(v) - H(v)_{j,i} + \langle v, y^{(j)}\rangle \\ &= f((\nabla f|_{U^{(i)}})^{-1}(v)) - \langle v, (\nabla f|_{U^{(i)}})^{-1}(v)\rangle + \langle v, y^{(j)}\rangle - H(v)_{j,i} \\ &= f(g^{(i)}(\operatorname{pr}_I^\perp(w))) + \langle v, y^{(j)} - g^{(i)}(\operatorname{pr}_I^\perp(w))\rangle - H(v)_{j,i} \\ &\geq f(g^{(i)}(\operatorname{pr}_I^\perp(w))) + \langle v, y^{(j)} - g^{(i)}(\operatorname{pr}_I^\perp(w))\rangle , \end{aligned}$$

with the first inequality being strict whenever $y^{(j)} \neq g^{(j)}(\operatorname{pr}_I^\perp(w))$ and the second inequality being strict whenever $j \notin I$.
By Proposition 4.9 and $g^{(i)}(\operatorname{pr}_I^\perp(w)) \in \widehat{U}^{(i)}$, for any $y \in \mathbb{R}^d \setminus \bigcup_{i=1}^{k+1} U^{(i)}$ we have

$$\begin{aligned} f(y) &> f(g^{(i)}(\operatorname{pr}_I^\perp(w))) + \langle \nabla f(g^{(i)}(\operatorname{pr}_I^\perp(w))), y - g^{(i)}(\operatorname{pr}_I^\perp(w))\rangle \\ &= f(g^{(i)}(\operatorname{pr}_I^\perp(w))) + \langle v, y - g^{(i)}(\operatorname{pr}_I^\perp(w))\rangle . \end{aligned}$$

Altogether, we conclude for any $y' \in \Omega$,

$$f(y') \geq f(g^{(i)}(\operatorname{pr}_I^\perp(w))) + \langle v, y - g^{(i)}(\operatorname{pr}_I^\perp(w))\rangle ,$$

with equality if and only if $y' \in \{g^{(i)}(\operatorname{pr}_I^\perp(w)) \mid i \in I\}$. Especially, $v \in \bigcap_{i \in I} \partial f(g^{(i)}(\operatorname{pr}_I^\perp(w)))$ and by differentiability of f in any $g^{(i)}(\operatorname{pr}_I^\perp(w))$, $i \in I$, we have $\{v\} = \bigcap_{i \in I} \partial f(g^{(i)}(\operatorname{pr}_I^\perp(w)))$ and $\operatorname{conv}\{g^{(i)}(\operatorname{pr}_I^\perp(w)) \mid i \in I\}$ is a maximal phase simplex. □

In S_I^δ the previous Proposition provides a parametrization of maximal phase simplices whose vertices lie in $U^{(i)}$, $i \in I$, using only the $d + 1 -$

$|I|$ components whose index is not an element of I. The components with index in I (which are non-negative) are now supposed to parametrize the position within the respective maximal phase simplex in terms of barycentric coordinates, but first we have to do a normalization since in general they do not sum up to 1.

4.15 Lemma *For any $\delta > 0$ and $\emptyset \neq I \subset \{1, ..., k+1\}$ the mapping*

$$\kappa_I : S_I^\delta \to \{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I(w) \in \Delta_{d+1},\ \mathrm{pr}_I^\perp(w) \in C_\delta^{d+1}\},$$
$$(\kappa_I(w))_i = \begin{cases} \Big(\sum_{j\in I} w_j\Big)^{-1} \cdot w_i & , i \in I \\ w_i & , i \in \{1, ..., d+1\} \setminus I \end{cases}$$

is well-defined and a bi-Lipschitz bijection.

Proof. First recognise that by definition of S_I^δ we have $\sum_{i=1}^{k+1} w_i = 1$ with $w_i < 0$ whenever $i \in \{1, ..., k+1\} \setminus I$, which implies $\sum_{j\in I} w_j \geq 1$ and prevents a division by zero. Furthermore, κ_I is well defined, since for any $i \in \{1, ..., k+1\} \setminus I$, $\kappa_I(w)_i = w_i \in [-\delta, 0)$ and

$$\sum_{i=1}^{d+1} \mathrm{pr}_I(\kappa_I(w))_i = \sum_{i\in I} \kappa_I(w)_i = \sum_{i\in I} \Big(\sum_{j\in I} w_j\Big)^{-1} \cdot w_i = 1\,.$$

The Lipschitz continuity follows by

$$\begin{aligned} 1 = \sum_{j=1}^{k+1} w_j \leq \sum_{j\in I} w_j = 1 - \sum_{j\in\{1,...,k+1\}\setminus I} w_j \leq 1 + \sum_{j\in\{1,...,k+1\}\setminus I} |w_j| \\ \leq 1 + (k+1)\cdot\delta \end{aligned}$$

and Proposition 2.2.
We prove that κ_I is bijective by giving the inverse function

$$\kappa_I^{-1} : \{w \in \mathbb{R}^{d+1} \mid \mathrm{pr}_I(w) \in \Delta_{d+1},\ \mathrm{pr}_I^\perp(w) \in C_\delta^{d}\} \to S_I^\delta,$$
$$(\kappa_I^{-1}(w))_i = \begin{cases} \Big(1 - \sum_{j\in\{1,...,k+1\}\setminus I} w_j\Big) \cdot w_i & , i \in I \\ w_i & , i \in \{1, ..., d+1\} \setminus I \end{cases},$$

which is well-defined, since for any $i \in \{1, ..., k+1\} \setminus I$, $\kappa_I(w)_i = w_i \in [-\delta, 0)$ and

$$\sum_{i=1}^{d+1} \kappa_I^{-1}(w)_i = \sum_{i\in I} \Big(1 - \sum_{j\in\{1,...,k+1\}\setminus I} w_j\Big) \cdot w_i + \sum_{i\in\{1,...,k+1\}\setminus I} w_i$$
$$= \sum_{i=1}^{k+1} \mathrm{pr}_I(w)_i \cdot \Big(1 - \sum_{j\in\{1,...,k+1\}\setminus I} w_j\Big) + \sum_{i\in\{1,...,k+1\}\setminus I} w_i = 1\,.$$

By $\sum_{j\in I} w_j = 1 - \sum_{j\in\{1,...,k+1\}\setminus I} w_j$, the function κ_I^{-1} is indeed inverse to κ_I and Lipschitz, since each component is Lipschitz as the product of two bounded Lipschitz functions according to Proposition 2.2. □

We are now able to prove the main Theorem, a parametrization of a neighbourhood of the maximal phase simplex $\mathrm{conv}\{\bar{y}^{(1)}, ..., \bar{y}^{(k+1)}\}$, such that an expression for the convex envelope of f can be given in this neighbourhood in terms of the parametrization. The proof follows similar lines as the one of Theorem 4.7.

4.16 Theorem *Recall Definition 4.13 and consider the mapping $\Phi : \Delta_{k+1}^{\delta} \times C_{\delta}^{d-k} \to \mathbb{R}^d$, which is given for $\emptyset \neq I \subset \{1, ..., k+1\}$ on S_I^{δ} by*

$$\Phi(w) = \sum_{i\in I} \Big(\sum_{j\in I} w_j\Big)^{-1} w_i \cdot g^{(i)}(\mathrm{pr}_I^{\perp}(w))\,.$$

The restriction of Φ to the set $\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}$ is an embedding and for any $\emptyset \neq I \subset \{1, ..., k+1\}$ and $w \in S_I^{\delta}$ the convex envelope of f in $\Phi(w)$ is given by the convex combination

$$\mathrm{conv}(f)(\Phi(w)) = \sum_{i\in I} \Big(\sum_{j\in I} w_j\Big)^{-1} w_i \cdot f\big(g^{(i)}(\mathrm{pr}_I^{\perp}(w))\big) \qquad (4.10)$$

of function values of f.

Proof. For any non-empty $I \subset \{1, ..., k+1\}$ and $w \in S_I^{\delta}$, (4.10) follows immediately by Proposition 4.14 and Proposition 3.27, since for any $i \in I$ we have $w_i \geq 0$ and $\sum_{i\in I} \Big(\sum_{j\in I} w_j\Big)^{-1} w_i = 1$.

By Proposition 6.2 and compactness of $\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}$, it suffices to show that Φ is a continuous injection.

We first prove the injectivity of Φ. Fix non-empty sets $I, I' \subset \{1, ..., k+$

$1\}$ and $w \in S_I$ and $w' \in S_{I'}$ with $\Phi(w) = \Phi(w') := y$. Define $J := \{j \in \{1, ..., k+1\} \mid w_j > 0\} \subset I$ and $J' := \{j \in \{1, ..., k+1\} \mid w'_j > 0\} \subset I'$. Then $\mathrm{pr}_I^\perp(w) = \mathrm{pr}_J^\perp(w)$ and $\mathrm{pr}_{I'}^\perp(w) = \mathrm{pr}_{J'}^\perp(w')$. By Proposition 4.14, $\mathrm{conv}\{g^{(i)}(\mathrm{pr}_I^\perp(w)) \mid i \in I\}$ and $\mathrm{conv}\{g^{(i)}(\mathrm{pr}_{I'}^\perp(w')) \mid i \in I'\}$ are maximal phase simplices of f and Proposition 3.27 implies, that both $y = \sum_{i\in J}\Big(\sum_{j\in J} w_j\Big)^{-1} w_i \cdot g^{(i)}(\mathrm{pr}_I^\perp(w))$ and $y = \sum_{i\in J'}\Big(\sum_{j\in J'} w_j\Big)^{-1} w'_i \cdot g^{(i)}(\mathrm{pr}_{I'}^\perp(w'))$ are unique stable phase splittings. Since the sets $U^{(i)}$ are pairwise disjoint, uniqueness implies $J = J'$ and for any $i \in J$, $g^{(i)}(\mathrm{pr}_I^\perp(w)) = g^{(i)}(\mathrm{pr}_{I'}^\perp(w'))$ and $\Big(\sum_{j\in J} w_j\Big)^{-1} w_i = \Big(\sum_{j\in J} w'_j\Big)^{-1} w'_i$. For any $i \in J$, $g^{(i)}$ is bi-Lipschitz as the composition of bi-Lipschitz functions and hence $\mathrm{pr}_J^\perp(w) = \mathrm{pr}_I^\perp(w) = \mathrm{pr}_{I'}^\perp(w') = \mathrm{pr}_{J'}^\perp(w') = \mathrm{pr}_J^\perp(w')$. With the bi-Lipschitz bijection κ_J from Lemma 4.15, we have $\kappa_J(w) = \kappa_J(w')$ and hence $w = w'$.

Next we show that Φ is continuous. Fix $I, J \subset \{1, ..., k\}$, $w \in S_I^\delta$ and a sequence $(w^{(n)})_{n\in\mathbb{N}} \subset S_J^\delta$ converging to w. For any $i \in J$, the component w_i is non-negative as the limit of the sequence $(w^{(n)})_{n\in\mathbb{N}}$ of non-negative numbers, hence $J \subset I$. Furthermore, for any $i \in I \setminus J$ the component w_i is zero, since it is non-negative and the limit of the sequence $(w_i^{(n)})_{n\in\mathbb{N}}$ of negative numbers. Therefore $\mathrm{pr}_I^\perp(w) = \mathrm{pr}_J^\perp(w)$ and

$$
\begin{aligned}
\Phi(w) &= \sum_{i\in I}\Big(\sum_{j\in I} w_j\Big)^{-1} w_i \cdot g^{(i)}(\mathrm{pr}_I^\perp(w)) = \sum_{i\in J}\Big(\sum_{j\in J} w_j\Big)^{-1} w_i \cdot g^{(i)}(\mathrm{pr}_J^\perp(w)) \\
&= \sum_{i\in J} \kappa_J(w)_i \cdot g^{(i)}(\mathrm{pr}_J^\perp(\lim_{n\to\infty} w^{(n)})) = \lim_{n\to\infty} \sum_{i\in J} \kappa_J(w^{(n)})_i \cdot g^{(i)}(\mathrm{pr}_J^\perp(w^{(n)})) \\
&= \lim_{n\to\infty}\Big(\sum_{i\in J}\Big(\sum_{j\in J} w_j^{(n)}\Big)^{-1} w_i^{(n)} \cdot g^{(i)}(\mathrm{pr}_J^\perp(w^{(n)}))\Big) = \lim_{n\to\infty} \Phi(w^{(n)}),
\end{aligned}
$$

where the continuity of the functions κ_J and $g^{(i)}$ for $i \in J$ was used. Notice that here was no loss in generality assuming that the whole sequence $(w^{(n)})_{n\in\mathbb{N}}$ lies in the same set S_J^δ of the finite partition of $\Delta_{k+1}^\delta \times C_\delta^{d-k}$. If not, the sequence can be partitioned into subsequences with each subsequence lying entirely in one of the sets S_J^δ. Since the values of Φ in any such subsequence with infinitely many elements converge to $\Phi(w)$, the whole sequence $\Phi(w^{(n)})$ converges to $\Phi(w)$. □

4.17 Corollary *If for any $i \in \{1, ..., k+1\}$ the restriction $f|_{U^{(i)}}$ is strongly convex and has Lipschitz continuous gradient, then for any $0 < \delta' < \delta$ the restriction of Φ from Theorem 4.16 to the set $\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}$ is bi-Lipschitz.*

Proof. Fix $0 < \delta' < \delta$. The restriction of Φ to the convex set $\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}$ is an embedding and $\{S_I^{\delta} \mid \emptyset \neq I \subset \{1, ..., k+1\}\}$ is a finite partition of $\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}$, such that the restriction of Φ to S_I^{δ} is given by the bi-Lipschitz function $\phi_I^* \circ \kappa_I$. Since $\Phi(\mathrm{relint}(\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k})) \subset \mathbb{R}^d$ is open by Corollary 6.4 and $\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}$ is compact with $\Phi(\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}) \subset \Phi(\mathrm{relint}(\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}))$, the restriction of Φ to $\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}$ is bi-Lipschitz by Corollary 2.6. □

The convex envelope of a closed and continuously differentiable function satisfying the superlinear growth condition (3.7) is known to be continuously differentiable [10, Theorem 3.2]. Local Lipschitz-continuity of the gradient (even local Hölder-continuity with $0 < \alpha \leq 1$) is also inherited by the convex envelope [10, Theorem 4.2]. We close this section by showing, that the convex envelope $\mathrm{conv}(f)$ constrcuted in Theorem 4.16 admits on $\mathrm{int}(\Phi(\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}))$ the expected regularity. It is worth noting, that global regularity of f is not required.

4.18 Corollary *Under the hypothesis of Theorem 4.16, the function* $\mathrm{conv}(f)$ *is continuously differentiable on* $\mathrm{int}(\Phi(\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}))$*. If additionally for any* $i \in \{1, ..., k+1\}$ *the restriction* $f|_{U^{(i)}}$ *is strongly convex and has Lipschitz continuous gradient, then for any* $0 < \delta' < \delta$ *the gradient of* $\mathrm{conv}(f)$ *is locally Lipschitz on* $\mathrm{int}(\Phi(\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}))$.

Proof. For the sake of convenience, we set throughout this proof $f_c := \mathrm{conv}(f)$.
For the first part, fix $y_0 \in \mathrm{int}(\Phi(\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}))$. By $\Phi^{-1}(y_0) \in \Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}$, there is some $i_0 \in \{1, ..., k+1\}$ with $\mathrm{pr}_{i_0}(\Phi^{-1}(y_0)) > 0$. Since pr_{i_0} and Φ^{-1} are continuous mappings, we can choose $r > 0$, such that $B_r(y_0) \subset \mathrm{int}(\Phi(\Delta_{k+1}^{\delta} \times C_{\delta}^{d-k}))$ and $\mathrm{pr}_{i_0}(\Phi^{-1}(B_r(y_0))) \subset \mathbb{R}_+$. For any $y \in B_r(y_0)$ let $I \subset \{1, ..., k+1\}$ be the unique non-empty subset with $w := \Phi^{-1}(y) \in S_I^{\delta}$. Theorem 4.7 gives

$$f_c(y) = \sum_{i \in I} \Big(\sum_{j \in I} w_j \Big)^{-1} w_i \cdot f\big(g^{(i)}(\mathrm{pr}_I^{\perp}(w))\big)$$

and defining $I' := \{i \in I \mid w_i > 0\}$ Theorem 3.21 implies with $w_{i_0} > 0$,

$$\partial f_c(y) = \bigcap_{i \in I'} \partial f\big(g^{(i)}(\mathrm{pr}_I^{\perp}(w))\big) \subset \partial f\big(g^{(i_0)}(\mathrm{pr}_I^{\perp}(w))\big) = \{\nabla f\big(g^{(i_0)}(\mathrm{pr}_I^{\perp}(w))\big)\}\,.$$

Proposition 3.5 and convexity of f_c imply $\partial f_c(y) \neq \emptyset$, hence $\partial f_c(y) = \{\nabla f\big(g^{(i_0)}(\mathrm{pr}_I^{\perp}(w))\big)\}$ and by Proposition 3.6 f_c is differentiable at y. Since $y \in B_r(y_0)$ was arbitrary, $f_c|_{B_r(y_0)}$ is convex and differentiable, therefore continuously differentiable according to Proposition 3.2.

Concerning the second part, assume $y_0 \in \text{int}(\Phi(\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}))$ and r is small enough, such that $B_r(y_0) \subset \text{int}(\Phi(\Delta_{k+1}^{\delta'} \times C_{\delta'}^{d-k}))$. For any $\emptyset \neq I \subset \{1, ..., k+1\}$, on $B_r(y_0) \cap \Phi(S_I^\delta)$ the gradient mapping of f_c is given by $\nabla f_c = \nabla f \circ g^{(i_0)} \circ \text{pr}_I^\perp \circ \Phi^{-1}$, which is Lipschitz as the composition of Lipschitz functions. Since $\{B_r(y_0) \cap \Phi(S_I^\delta) \mid \emptyset \neq I \subset \{1, ..., k+1\}\}$ is a partition of $B_r(y_0)$, continuity of ∇f_c on $B_r(y)$ implies that ∇f_c is Lipschitz on $B_r(y)$ by Lemma 2.5. $\square$

5 Examples

5.1 Example *Consider the functions* $f_0 : \mathbb{R}^2 \to \mathbb{R},\ f_0(x) = \frac{1}{2} \cdot (x_1^2 + x_2^2)$ *and*

$$f_1 : \mathbb{R}^2 \to \mathbb{R},$$
$$f_1(x) = \begin{cases} \frac{1}{2} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} & , \text{if } x_2 \leq x_1 - 1 \\ \frac{1}{2} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} & , \text{if } x_2 > x_1 - 1 \end{cases}.$$

The function f_0 is smooth with $\nabla f_0 = \text{id}_\mathbb{R}$ and $H_{f_0} \equiv \text{Id}_2$, especially has Lipschitz continuous gradient and is convex with modulus 1 according to Proposition 3.10. By $(\nabla f_0)^{-1} = \text{id}_{\mathbb{R}^2}$, the Legendre transform of f_0 is given by $f_0^* = f_0$.
The function f_1 is continuously differentiable with

$$\nabla f_1 : \mathbb{R}^2 \to \mathbb{R},\ \nabla f_1(x) = \begin{cases} \begin{pmatrix} x_1 - 1 \\ x_2 \end{pmatrix} & , \text{if } x_2 \leq x_1 - 1 \\ \begin{pmatrix} 2(x_1 - 1) - x_2 \\ -(x_1 - 1) + 2x_2 \end{pmatrix} & , \text{if } x_2 > x_1 - 1 \end{cases}$$

and has Lipschitz continuous gradient by Lemma 2.5. Especially ∇f_1 is locally Lipschitz and we can calculate the generalized Hessian by

$$\partial_c \nabla f_1(x) = \begin{cases} \{\text{Id}_2\} & , \text{if } x_2 < x_1 - 1 \\ \{\begin{pmatrix} 1+t & -t \\ -t & 1+t \end{pmatrix} \mid t \in [0,1]\} & , \text{if } x_2 = x_1 - 1 \\ \{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\} & , \text{if } x_2 > x_1 - 1 \end{cases}.$$

For any $x \in \mathbb{R}^2$ *we have* $\partial_c \nabla f_1(x) \subset \mathrm{PD}_1^3(2)$ *(see Definition 3.9) and by Proposition 3.10,* f_1 *is convex with modulus* 1*. By*

$$(\nabla f_1)^{-1} : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$(\nabla f_1)^{-1}(v) = \begin{cases} \begin{pmatrix} v_1 + 1 \\ v_2 \end{pmatrix} & , \text{ if } v_2 \le v_1 \\ \frac{1}{3} \cdot \begin{pmatrix} 3 + 2v_1 + v_2 \\ v_1 + 2v_2 \end{pmatrix} & , \text{ if } v_2 > v_1 \end{cases},$$

a straightforward calculation shows that the Legendre transform of f_1 *is given by*

$$f_1^* : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$f_1^*(v) = \begin{cases} \frac{1}{2}v_1^2 + v_1 + \frac{1}{2}v_2^2 & , \text{ if } v_2 \le v_1 \\ \frac{1}{3} \cdot (v_1^2 + 3v_1 + v_1 v_2 + v_2^2) & , \text{ if } v_2 > v_1 \end{cases}.$$

Now define $g : \mathbb{R} \to \mathbb{R},\ t \mapsto 1/2 + \chi_{\mathbb{R}_+}(t) \cdot (t - \sqrt{t})$ *and the sets* $\Omega_0 := \{x \in \mathbb{R}^2 \mid x_1 < g(x_2)\}$*,* $\Omega_1 := \{x \in \mathbb{R}^2 \mid x_1 > g(x_2)\}$ *and* $\Omega := \Omega_0 \cup \Omega_1$*. See Figure 1 (a) for a visualization of the shape of* Ω_0 *and* Ω_1*. The function*

$$f : \Omega \to \mathbb{R},\ f(x) = \begin{cases} f_0(x) & , \text{ if } x \in \Omega_0 \\ f_1(x) & , \text{ if } x \in \Omega_1 \end{cases}$$

is continuously differentiable with locally Lipschitz continuous gradient and is locally convex with modulus 1*. Furthermore,* $(f|_{\Omega_0})^* = (f_0^*)|_{\Omega_0}$ *and* $(f|_{\Omega_1})^* = (f_1^*)|_{\Omega_1}$*. Solving for* $v \in \mathbb{R}$ *the equation* $f_0^*(v) = f_1^*(v)$ *leads for* $v_2 \le v_1$ *to* $v_1 = 0$ *and for* $v_2 > v_1$ *to* $v_1 = 3 + v_2 - \sqrt{9 + 6v_2}$*, hence with*

$$h : \mathbb{R} \to \mathbb{R},\ \chi_{\mathbb{R}_+}(s) \cdot (3 + s - \sqrt{9 + 6s})$$

we have

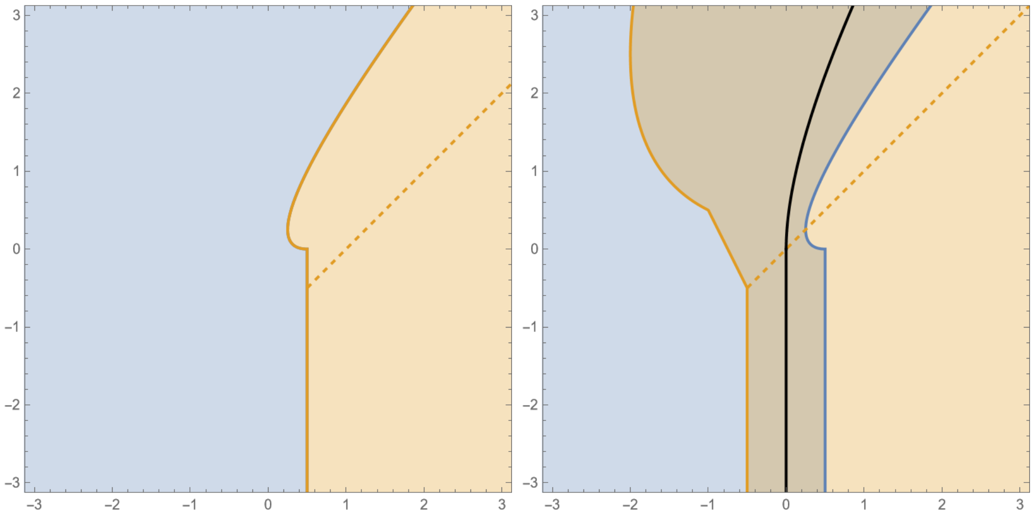
$$f_0^*(v) = f_1^*(v) \Leftrightarrow v_1 = h(v_2)\,.$$

With Proposition 3.16 we can calculate

$$\langle \nabla f_1^*(0) - \nabla f_0^*(0), e_1 \rangle = \langle (\nabla f_1)^{-1}(0) - (\nabla f_0)^{-1}(0), e_1 \rangle = \langle e_1, e_1 \rangle = 1$$

and conclude

$$f_1^*(v) - f_0^*(v) \begin{cases} > 0 & , \text{ if } v_1 > h(v_2) \\ = 0 & , \text{ if } v_1 = h(v_2) \\ < 0 & , \text{ if } v_1 < h(v_2) \end{cases}. \tag{5.1}$$



(a) blue: Ω_0, orange: Ω_1

(b) blue: $\nabla f(\Omega_0)$, orange: $\nabla f(\Omega_1)$, black line: $\{v \in \mathbb{R}^2 \mid (f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)\}$

Figure 1: Visualization of the domain of f (left) and the gradient set of $f|_{\Omega_0}$ and $f|_{\Omega_1}$ (right) with the solution curve of $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$. The dashed orange line indicates the discontinuous second derivative.

In order to show $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v) \Leftrightarrow v_1 = h(v_2)$, we need to prove for any $s \in \mathbb{R}$, $(h(s), s) \in \nabla f(\Omega_0) \cap \nabla f(\Omega_1)$ (see Figure 1 (b)).
For $s \leq 0$ we have $h(s) = 0 < 1/2 = g(s)$, hence $(h(s), s) \in \Omega_0 = \nabla f(\Omega_0)$, as well as $(h(s), s) = \nabla f(1, s)$ and hence $(h(s), s) \in \nabla f(\Omega_1)$. Now we assume $s > 0$ and infer $(h(s), s) \in \nabla f(\Omega_0)$ by

$$\begin{aligned} & h(s) < g(s) \\ \Leftrightarrow\ & 3 + s - \sqrt{9+6s} < \frac{1}{2} + s - \sqrt{s} \\ \Leftrightarrow\ & \frac{5}{2} + \sqrt{s} < \sqrt{9+6s} \\ \Leftrightarrow\ & \frac{25}{4} + 5\sqrt{s} + s < 9 + 6s \\ \Leftrightarrow\ & 0 < \frac{11}{4} - 5\sqrt{s} + 5s \\ \Leftrightarrow\ & 0 < \frac{3}{2} + 5 \cdot \left(\frac{1}{2} - \sqrt{s}\right)^2 . \end{aligned}$$

In order to show $(h(s), s) \in \nabla f(\Omega_1)$, we define the affine transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(v) = \frac{1}{3} \cdot \begin{pmatrix} 3 + 2v_1 + v_2 \\ v_1 + 2v_2 \end{pmatrix}$$

and show $T(h(s),s) \in \Omega_1$ *with* $T(h(s),s)_2 > T(h(s),s)_1 - 1$*, then the assertion follows by* $\nabla f(T(h(s),s)) = (h(s),s)$*. Now calculate*

$$T(h(s),s) = \frac{1}{3} \cdot \begin{pmatrix} 3 + 2h(s) + s \\ h(s) + 2s \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 9 + 3s - 2\sqrt{9+6s} \\ 3 + 3s - \sqrt{9+6s} \end{pmatrix}$$

and by $s > 0$ *and* $h(s) = 3 + s - \sqrt{9+6s} < s$ *we have*

$$T(h(s),s)_1 - 1 = \frac{1}{3}(2h(s) + s) < \frac{1}{3}(h(s) + 2s) = T(h(s),s)_2 \,.$$

Substituting $s' := \frac{1}{3} \cdot (3 + 3s - \sqrt{9+6s}) > \frac{1}{3} \cdot (3 + 3s - \sqrt{(3+s)^2}) = \frac{2}{3}s > 0$ *leads to* $s = \frac{1}{3} \cdot (-2 + 3s' + \sqrt{4+6s'})$ *and therefore to*

$$\begin{aligned} \sqrt{9+6s} &= \sqrt{9 - 4 + 6s' + 2\sqrt{4+6s'}} = \sqrt{1 + 2\sqrt{4+6s'} + 4 + 6s'} \\ &= 1 + \sqrt{4+6s'} \end{aligned}$$

and

$$T(h(s),s) = \begin{pmatrix} 2 + s' - \frac{1}{3}\sqrt{9+6s} \\ s' \end{pmatrix} = \begin{pmatrix} \frac{5}{3} + s' - \frac{1}{3}\sqrt{4+6s'} \\ s' \end{pmatrix} .$$

We can check $T(h(s),s) \in \Omega_1$ *by*

$$\begin{aligned} & T(h(s),s)_1 > g(T(h(s),s)_2) \\ \Leftrightarrow\ & \frac{5}{3} + s' - \frac{1}{3}\sqrt{4+6s'} > \frac{1}{2} + s' - \sqrt{s'} \\ \Leftrightarrow\ & \frac{7}{2} + 3\sqrt{s'} > \sqrt{4+6s'} \\ \Leftrightarrow\ & \frac{49}{4} + 21\sqrt{s'} + 9s' > 4 + 6s' \\ \Leftrightarrow\ & \frac{33}{4} + 21\sqrt{s'} + 3s' > 0 \,. \end{aligned}$$

The curves

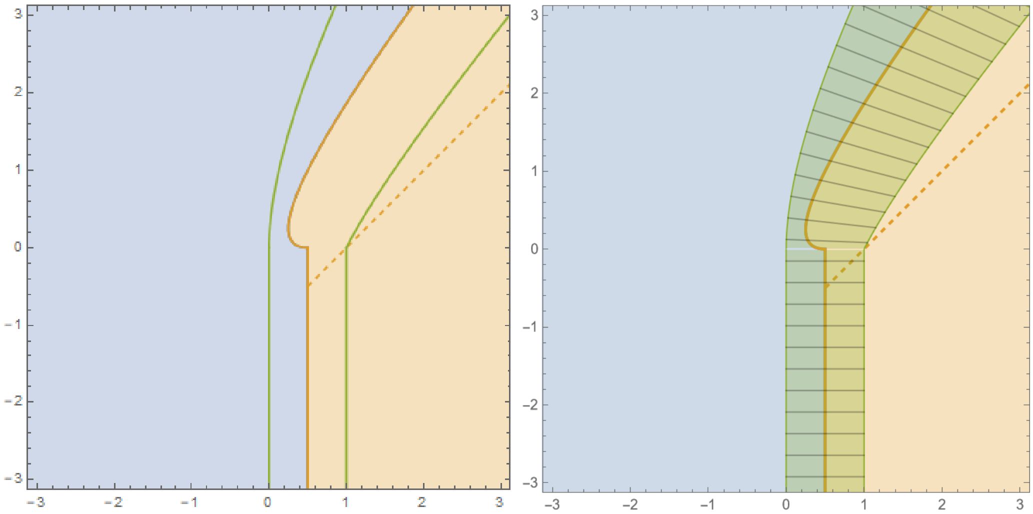
$$\alpha : \mathbb{R} \to \mathbb{R}^2,\ \alpha(s) = (\nabla f|_{\Omega_0}^{-1})(h(s),s)$$

and

$$\beta : \mathbb{R} \to \mathbb{R}^2,\ \beta(s) = (\nabla f|_{\Omega_1}^{-1})(h(s),s)$$

are well-defined, as well as the function

$$\begin{aligned} & \phi : [0,1] \times \mathbb{R} \to \mathbb{R}^2, \\ & \phi(t,s) = (1-t) \cdot \alpha(s) + t \cdot \beta(s) \,, \end{aligned}$$



(a) blue: Ω_0, orange: Ω_1, green: α (left) and β (right)

(b) blue: Ω_0, orange: Ω_1, green: $\phi([0,1] \times \mathbb{R})$

Figure 2: The boundary of the green area is given by the curves α and β and the connecting straight lines represent the sets $\{\phi(t,s) \mid t \in [0,1]\}$ for several fixed values of s.

visualized in Figure 2. Since $f|_{\Omega_0}$ and $f|_{\Omega_1}$ are restrictions of strictly convex and continuously differentiable functions respectively, for any $s \in \mathbb{R}$ the equality $(f|_{\Omega_0})^*(h(s),s) = (f|_{\Omega_1})^*(h(s),s)$ implies $\{(h(s),s)\} = \partial f(\alpha(s)) \cap \partial f(\beta(s))$. For any $(t,s) \in [0,1] \times \mathbb{R}$ Proposition 3.22 gives

$$(\mathrm{conv}(f))(\phi(t,s)) = (1-t) \cdot f(\alpha(s)) + t \cdot f(\beta(s))$$

and hence $\partial(\mathrm{conv}(f))(\phi(t,s)) = \{(h(s),s)\}$.
If $(t,s),(t',s') \in [0,1] \times \mathbb{R}$ with $\phi(t,s) = \phi(t',s')$, then

$$\{(h(s),s)\} = \partial(\mathrm{conv}(f))(\phi(t,s)) = \partial(\mathrm{conv}(f))(\phi(t',s')) = \{(h(s'),s')\}$$

and therefore $s = s'$ as well as $t = t'$ by $\alpha(s) \neq \beta(s)$ and ϕ is injective.
Assume $x \in \mathbb{R}^2 \setminus \phi([0,1] \times \mathbb{R}) \subset \Omega_0 \cup \Omega_1$. By Proposition 3.14 the function $h_0 : \mathbb{R}^2 \to \mathbb{R}$, $h_0(u) = \langle \nabla f(x), u \rangle - f_0^*(\nabla f(x))$ minorizes f_0 with $h_0(x) = f_0(x)$ and the function $h_1 : \mathbb{R}^2 \to \mathbb{R}$, $h_1(u) = \langle \nabla f(x), u \rangle - f_1^*(\nabla f(x))$ minorizes f_1 with $h_1(x) = f_1(x)$. If $x \in \Omega_0$, then $\nabla f(x) \in \{v \in \mathbb{R}^2 \mid v_1 < h(v_2)\}$ and by (5.1) we have $f_0^*(x) > f_1^*(x)$. Now $f(x) = f_0(x)$ and $h_0 < h_1$ implies $\nabla f(x) \in \partial f(x) \neq \emptyset$ and $(\mathrm{conv}(f))(x) = f(x)$. Analogously, if $x \in \Omega_0$ we conclude $\nabla f(x) \in \{v \in \mathbb{R}^2 \mid v_1 > h(v_2)\}$, $f_0^*(x) < f_1^*(x)$, $h_0 > h_1$ and

$(\mathrm{conv}(f))(x) = f(x)$ *by* $\nabla f(x) \in \partial f(x) \neq \emptyset$. *This finally proves*

$$\mathrm{conv}(f)(x) = \begin{cases} (1-t)\cdot\alpha(s) + t\cdot\beta(s) & , \text{ if } x = \phi(t,s) \in \phi([0,1]\times\mathbb{R}) \\ f(x) & , \text{ if } x \notin \phi([0,1]\times\mathbb{R}) \end{cases} .$$

5.2 Remark The function f in the previous example is not closed, but for $\varepsilon > 0$ the restriction of f to the set $\{x \in \mathbb{R}^2 \mid |x_1 - g(x_2)| \geq \varepsilon\}$ is closed and with ε sufficiently small the construction of the convex envelope remains unchanged. $*$

5.3 Example *Consider the sets* $\Omega_0 = \mathbb{R}^2 \setminus B_{1.8}(2,0)$, $\Omega_1 = \overline{B_{1.5}(2,0)}$ *and define for* $\Omega := \Omega_0 \cup \Omega_1$ *the function*

$$f : \Omega \to \mathbb{R},$$

$$f(x_1,x_2) = \begin{cases} \frac{1}{2}(x_1^2 + x_2^2) & , \text{ if } x \in \Omega_0 \\ \frac{1}{4}(x_1 - 1)^4 + x_2^2 & , \text{ if } x \in \Omega_1 \end{cases} ,$$

which is closed, smooth on $\mathrm{int}(\Omega)$ *and strictly convex on each convex subset of its domain. Calculate*

$$\nabla f(x) = \begin{cases} (x_1,x_2)^T & , \text{ if } x \in \mathrm{int}(\Omega_0) \\ ((x_1-1)^3, 2x_2)^T & , \text{ if } x \in \mathrm{int}(\Omega_1) \end{cases} ,$$

$$H_f(x) = \begin{cases} \mathrm{Id}_2 & , \text{ if } x \in \mathrm{int}(\Omega_0) \\ \begin{pmatrix} 3(x_1-1)^2 & 0 \\ 0 & 1 \end{pmatrix} & , \text{ if } x \in \mathrm{int}(\Omega_1) \end{cases}$$

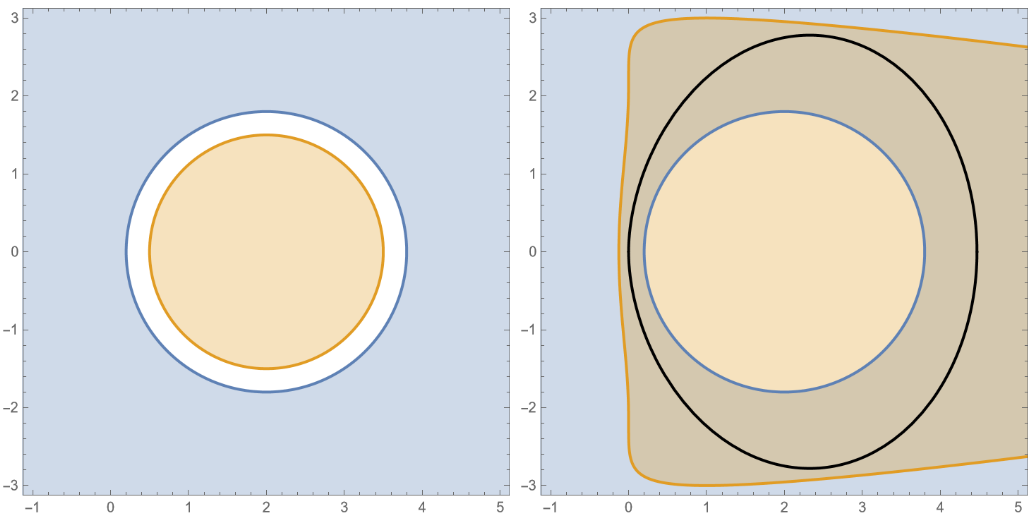
in order to see, that f *is not locally strongly convex in* $\mathrm{e}^{(1)} \in \Omega_1$, *since* $H_f(\mathrm{e}^{(1)})$ *is singular. Since* $f|_{\Omega_0}$ *and* $f|_{\Omega_1}$ *are restrictions of strictly convex functions respectively, the gradient mappings* $\nabla f|_{\mathrm{int}(\Omega_0)}$ *and* $\nabla f|_{\mathrm{int}(\Omega_1)}$ *are injective, the Legendre transforms are well-defined and given by*

$$(f|_{\mathrm{int}(\Omega_0)})^* : \nabla f(\mathrm{int}(\Omega_0)) \to \mathbb{R},\ (f|_{\mathrm{int}(\Omega_0)})^*(v) = \frac{1}{2}(v_1^2 + v_2^2)\,,$$

$$(f|_{\mathrm{int}(\Omega_1)})^* : \nabla f(\mathrm{int}(\Omega_1)) \times \mathbb{R},\ (f|_{\mathrm{int}(\Omega_1)})^*(v) = v_1 + \frac{3}{4}v_1^{4/3} + \frac{1}{4}v_2^2\,.$$

The equation $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$ *can be solved explicitly in this situation and the solutions are described by*

$$v_2^2 = v_1 \cdot (-2v_1 + 3v_1^{1/3} + 4) \tag{5.2}$$



(a) blue: Ω_0, orange: Ω_1

(b) blue: $\nabla f(\Omega_0)$, orange: $\nabla f(\Omega_1)$, black line: $\{v \in \mathbb{R}^2 \mid (f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)\}$

Figure 3: Visualization of the domain of f (left) and the gradient set of $f|_{\Omega_0}$ and $f|_{\Omega_1}$ (right) with the solution curve of $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$.

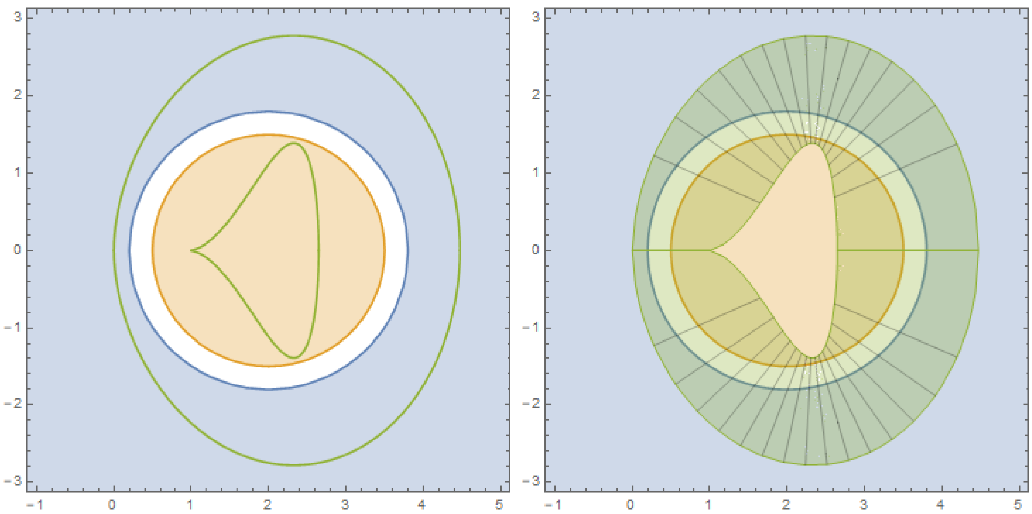
with $v_1 \cdot (-2v_1 + 3v_1^{1/3} + 4) \geq 0$. *Solving a cubic equation, one sees that*

$$-2v_1 + 3v_1^{1/3} + 4 \geq 0 \Leftrightarrow v_1 \leq 2 + \frac{3}{4}\left(8 - 2\sqrt{14}\right)^{1/3} + \frac{3}{4}\left(2(4+\sqrt{14})\right)^{1/3} =: c$$

with $c \approx 4.47 > 0$ *and therefore necessarily* $0 \leq v_1 \leq c$. *The analytical proof, that any* (v_1, v_2) *satisfying* $v_1 \in [0, c]$ *and* (5.2) *lies in the intersection* $\nabla f(\Omega_0) \cap \nabla f(\Omega_1)$ *(and is therefore a solution of* $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$*) is not very enlightening and therefore omitted here, but it should be emphasized that* Ω_0 *and* Ω_1 *are chosen particularly to ensure this, see Figure 3 (b). We now obtain a continuous parametrization of the solutions of* $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$ *via*

$$v : [-c, c] \to \mathbb{R}^2,\ v(s) = \left(|s|, \operatorname{sign}(s) \cdot \sqrt{|s| \cdot (-2|s| + 3|s|^{1/3} + 4)}\right),$$

which is in fact a loop since $v(c) = (c, 0) = v(-c)$ *(Figure 3 (b)). The image* $v([-c, c])$ *is a differentiable manifold, since it can be represented locally as the graph of a differentiable function. This is a consequence of Theorem 4.7 and can be seen in this case, by the fact that* $v|_{(0,c)}$ *and* $v|_{(-c,0)}$ *are graphs of differentiable functions and* $\lim_{s \nearrow 0} v_2'(s) = \infty = \lim_{s \searrow 0} v_2'(s)$ *as well as* $\lim_{s \nearrow c} v_2'(s) = -\infty = \lim_{s \searrow c} v_2'(s)$. *If we had solved* $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$ *for* v_1 *instead for* v_2 *(which is much harder), then we would have obtained another parametrization*



(a) blue: Ω_0, orange: Ω_1,
green: $f|_{\mathrm{int}(\Omega_0)})^{-1} \circ v$ (outer loop)
and $f|_{\mathrm{int}(\Omega_1)})^{-1} \circ v$ (inner loop)

(b) blue: Ω_0, orange: Ω_1,
green: $\phi([0,1] \times [-c,c]/\{c \sim -c\})$

Figure 4: The boundary of the green area in the right picture is given by the loops $(\nabla f|_{\mathrm{int}(\Omega_0)})^{-1} \circ v$ and $(\nabla f|_{\mathrm{int}(\Omega_1)})^{-1} \circ v$ and the connecting straight lines represent the sets $\{\phi(t,s) \mid t \in [0,1]\}$ for several fixed values of s.

of the same loop. The compositions $(\nabla f|_{\mathrm{int}(\Omega_0)})^{-1} \circ v$ and $(\nabla f|_{\mathrm{int}(\Omega_1)})^{-1} \circ v$ are continuous loops, whose images are manifolds, but since f is not strongly convex in $e^{(1)}$, the Lipschitz-continuity is lost as one can see in Figure 4. Similar arguments like in Theorem 4.16 show, that the mapping

$$\begin{aligned}&\phi : [0,1] \times [-c,c]/\{c \sim -c\} \to \mathbb{R}^2,\\ &\phi(t,s) = (1-t) \cdot (\nabla f|_{\mathrm{int}(\Omega_0)})^{-1}(v(s)) + t \cdot (\nabla f|_{\mathrm{int}(\Omega_1)})^{-1}(v(s))\end{aligned}$$

is injective and hence an embedding. For any $x = \phi(t,s) \in \phi([0,1] \times [-c,c]/\{c \sim -c\})$, the convex envelope of f is given by

$$(\mathrm{conv}(f))(x) = (1-t) \cdot f((\nabla f|_{\mathrm{int}(\Omega_0)})^{-1}(v(s))) + t \cdot f((\nabla f|_{\mathrm{int}(\Omega_1)})^{-1}(v(s)))\,,$$

since for any $s \in [-c,c]/\{c \sim -c\}$ by strict convexity of the functions $\frac{1}{2}(x_1^2 + x_2^2)$ and $\frac{1}{4}(x_1-1)^4 + x_2^2$ we have $v(s) \in \partial f|_{\Omega_0}\big((\nabla f|_{\mathrm{int}(\Omega_0)})^{-1}(v(s))\big)$ and $v(s) \in \partial f|_{\Omega_1}\big((\nabla f|_{\mathrm{int}(\Omega_1)})^{-1}(v(s))\big)$, hence by $(f|_{\Omega_0})^*(v) = (f|_{\Omega_1})^*(v)$ also $v(s) \in \partial f\big((\nabla f|_{\mathrm{int}(\Omega_0)})^{-1}(v(s))\big) \cap \partial f\big((\nabla f|_{\mathrm{int}(\Omega_0)})^{-1}(v(s))\big)$.

5.4 Remark The reasons for the lack of smoothness of the curves in the previous two examples are different. In Example 5.1, the discontinuity of the second derivative causes a jump in one of the eigenvalues of the Hessian although the modulus of convexity, which is determined by the smallest eigenvalue, remains constant on Ω_1. Since f is locally convex with modulus 1, Corollary 4.17 guarantees Lipschitz continuity of the parametrization ϕ and the example shows, that we cannot expect more regularity. In Example 5.3, f is strictly but not strongly convex near $e^{(1)}$, since one of the eigenvalues of the Hessian is zero. The parametrization ϕ is continuous according to Theorem 4.16, but not Lipschitz. $*$

In [2] we investigated relaxation models in soil mechanics, which involved the convexification of a condensed energy of the form

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(y) = \frac{1}{2}||y||_2^2 - \frac{(y_2 - r(y_1))_+^2}{2(b+1)} \tag{5.3}$$

with $b > 0$, $r : \mathbb{R} \to \mathbb{R}$ compactly supported and concave on its support and $(\cdot)_+$ denoting the positive part.
In the special case $\sqrt{b} \cdot (y_{max} - y_{min})/2 \leq r(y_{min} + y_{max})/2$ we were able to give an explicit expression for the convex envelope [2, Theorem 2]. Since the detailed calculations can be found in this paper, we just give a short sketch of the construction omitting extensive calculations or proofs.

5.5 Example *Fix $y_{min}, y_{max} \in \mathbb{R}$, with $y_{min} < y_{max}$,*

$$\mathcal{R} := \{r \in C(\mathbb{R}) \mid \operatorname{supp}(r) = [y_{min}, y_{max}], r|_{[y_{min}],y_{max}]} \ \textit{concave}\}$$

and consider for $b > 0$ and $r \in \mathcal{R}$ the function f as in (5.3).
The function f is continuous, partially differentiable with respect to the second argument and continuously differentiable in $(\mathbb{R} \setminus \{y_{min}, y_{max}\}) \times \mathbb{R}$ with locally Lipschitz continuous gradient. In [2, Theorem 2], the convex envelope of f is constructed assuming that

$$\sqrt{b} \cdot \frac{y_{max} - y_{min}}{2} \leq r\Big(\frac{y_{min} + y_{max}}{2}\Big). \tag{5.4}$$

The strategy is, to find pairs of points $y, \tilde{y} \in \mathbb{R}^2$ with $\partial f(y) \cap \partial f(\tilde{y}) \neq \emptyset$, then Proposition 3.22 gives the convex envelope of f on $\operatorname{conv}\{y, \tilde{y}\}$ by affine interpolation of $f(y)$ and $f(\tilde{y})$. Since $g : \mathbb{R}^2 \to \mathbb{R}$, $g(y) = \frac{1}{2}y_1^2 + \frac{b}{2(b+1)}y_2^2$ is a strictly convex function minorizing f and coinciding on $\mathbb{R}^2 \setminus (y_{min}, y_{max})$ with f, it is sufficient to consider $y, \tilde{y} \in [y_{min}, y_{max}] \times \mathbb{R}$. Assuming $y_1 \in (y_{min}, y_{max})$ and $\tilde{y}_1 \in \{y_{min}, y_{max}\}$, we recognize that $D_2 f(y) = D_2 f(\tilde{y})$ and $f(\tilde{y}) = T_y f(\tilde{y})$ are necessary conditions for $\partial f(y) \cap \partial f(\tilde{y}) \neq \emptyset$. If $y_2 \leq$

$r(y_1)$, the second term in the definition of $f(y)$ vanishes and these necessary conditions simplify to

$$\frac{b}{b+1}\tilde{y}_2 = y_2$$
$$(\tilde{y}_1 - y_1)^2 + (\tilde{y}_2 - y_2)^2 = \frac{1}{b+1}\tilde{y}_2\,.$$

The solution pairs can be parametrized by the following four pairs of curves

$$\alpha^{\pm}_{min}(s) = \begin{pmatrix} y_{min} + s \\ \pm\sqrt{b}\cdot s \end{pmatrix},\ \beta^{\pm}_{min}(s) = \begin{pmatrix} y_{min} \\ \pm\frac{b+1}{\sqrt{b}}\cdot s \end{pmatrix},$$
$$\alpha^{\pm}_{max}(s) = \begin{pmatrix} y_{max} - s \\ \pm\sqrt{b}\cdot s \end{pmatrix},\ \beta^{\pm}_{max}(s) = \begin{pmatrix} y_{max} \\ \pm\frac{b+1}{\sqrt{b}}\cdot s \end{pmatrix},$$

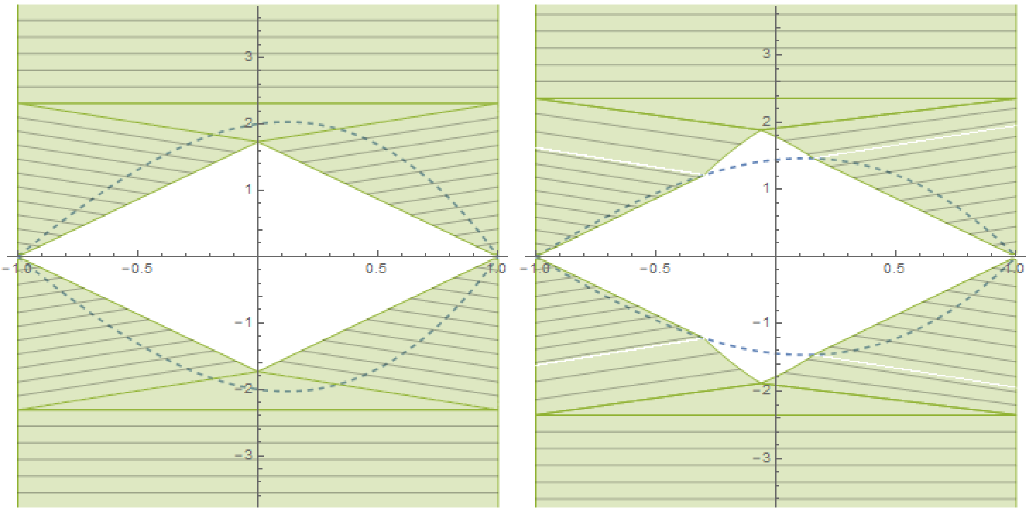
with $s \in (y_{min}, y_{max})$ taking any value, such that $\mathrm{pr}_2(\alpha^{\pm}_{min}(s)) \leq r(\mathrm{pr}_1(\alpha^{\pm}_{min}(s))$ (or $\mathrm{pr}_2(\alpha^{\pm}_{max}(s)) \leq r(\mathrm{pr}_1(\alpha^{\pm}_{max}(s)))$. The curves $\alpha^{\pm}_{min}$ and $\alpha^{\pm}_{max}$ intersect for $s = (y_{min} + y_{max})/2$, therefore only $0 < s < (y_{min} + y_{max})/2$ are considered as possible vertices of one-dimensional phase simplices and the triples

$$\begin{pmatrix} \frac{y_{min}+y_{max}}{2} \\ \pm\sqrt{b}\cdot\frac{y_{max}-y_{min}}{2} \end{pmatrix},\ \begin{pmatrix} y_{min} \\ \pm\frac{b+1}{\sqrt{b}}\cdot\frac{y_{max}-y_{min}}{2} \end{pmatrix},\ \begin{pmatrix} y_{max} \\ \pm\frac{b+1}{\sqrt{b}}\cdot\frac{y_{max}-y_{min}}{2} \end{pmatrix}$$

are reasonable candidates for the vertices of two-dimensional phase simplices. Finally, we want to find $y^{(1)}, y^{(2)} \in \mathbb{R}^2$ with $y^{(1)}_1, y^{(2)}_1 \in \{y_{min}, y_{max}\}$ satisfying $\partial f(y^{(1)}) \cap \partial f(y^{(2)}) \neq \emptyset$. The necessary condition $D_2 f(y^{(1)}) = D_2 f(y^{(2)})$ enforces $y^{(1)}_2 = y^{(2)}_2$ and assuming $y^{(1)} \neq y^{(2)}$ gives another two pairs of curves

$$\alpha^{\pm}_{\infty}(s) = \begin{pmatrix} y_{min} \\ \pm s \end{pmatrix},\ \beta^{\pm}_{\infty}(s) = \begin{pmatrix} y_{max} \\ \pm s \end{pmatrix},$$

where only $s > \frac{b+1}{\sqrt{b}}\frac{y_{max}-y_{min}}{2}$ will lead to further one-dimensional phase simplices.
By affine interpolation along each potential phase simplex constructed above, the resulting function can be proven to be convex and minorizing f, hence coinciding with the convex envelope of f [2, Theorem 2]. The construction scheme is visualized in Figure 5 (a). The construction crucially relies on the fact, that the curves $\alpha^{\pm}_{min}$ and $\alpha^{\pm}_{max}$ do not leave the part of the domain of f (until they intersect), in which the second term of f vanishes. This is ensured by (5.4) and concavity of $r|_{[y_{min},y_{max}]}$. If we drop this assumption, the curves can cross the line $y_2 = r(y_1)$, at which the second derivative



(a) blue dashed line: Graph of r and $-r$, green: Area on which the convex envelope of f differs from f

(b) blue dashed line: Graph of r and $-r$, green: Area on which the convex envelope of f differs from f

Figure 5: Construction schemes for $r(y_1) = c \cdot (1 - y_1^2) \cdot (y_1 + 4)$ with $c = 0.36$ (left) and $c = 0.5$ (right). The straight lines and triangles illustrate the one- and two-dimensional phase simplices.

of f is discontinuous and the local modulus of convexity suddenly changes. For $y \in (y_{min}, y_{max}) \times \mathbb{R}$ and $\tilde{y} \in \{y_{min}, y_{max}\} \times \mathbb{R}$ the necessary conditions $D_2 f(y) = D_2 f(\tilde{y})$ and $f(\tilde{y}) = T_y f(\tilde{y})$ still can be solved and the corresponding curves $\alpha^{\pm}_{min}$ and $\alpha^{\pm}_{max}$ show a similar non-smooth behaviour as in Example 5.1. For illustration purposes, we give without a proof in Figure 5 (b) the construction scheme for a function $r \in \mathcal{R}$ not satisfying (5.1).

6 Appendix

6.1 Topology

6.1 Definition *For topological spaces* (X, τ_X) *and* (Y, τ_Y)*, a mapping* $f : X \to Y$ *is called an embedding, if it is a homeomorphism onto its image, i.e. if* $f : X \to f(X),\ x \mapsto f(x)$ *is a homeomorphism with* $f(X)$ *carrying the subspace topology* $\tau_{f(X)} := \{O \cap f(X) \mid O \in \tau_Y\}$*. Equivalently,* f *is continuous, injective and the inverse mapping* $f^{-1} : f(X) \to X$ *is also continuous.*

A continuous injection is not necessarily an embedding, since the inverse function in general does not need to be continuous. Nevertheless, any continuous injection from a compact space into a Hausdorff-space is an embedding [6, Proposition 1.4.3].

6.2 Proposition *Assume* (X, τ_X) *is a compact topological space,* (Y, τ_Y) *is a Hausdorff-space and* $f : X \to Y$ *is a continuous injections, then* f *is an embedding.*

Another situation, in which a continuous injection is automatically an embedding is the Theorem of invariance of domain. It states that a continuous injection of an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ is an embedding, which is even an open mapping (maps open sets to open sets) [6, Theorem 10.3.7].

6.3 Proposition *If* $U \subset \mathbb{R}^n$ *is an open subset and* $f : U \to \mathbb{R}^n$ *is an injective continuous map, then* $V := f(U)$ *is open and* f *is a homeomorphism between* U *and* V*.*

6.4 Corollary *If* $X \subset \mathbb{R}^d$ *is an affine subspace of* $\mathbb{R}^d$ *with* $\dim(X) = n$*,* $U \subset X$ *is relatively open and* $f : U \to \mathbb{R}^n$ *is an injective continuous map, then* $V := f(U)$ *is open and* f *is a homeomorphism between* U *and* V*.*

6.2 Linear Algebra

Recall the notation of Definition 3.9.

6.5 Proposition *For* $0 \leq \mu \leq L \leq \infty$*, the set* $\mathrm{PD}_\mu^L(d)$ *is convex, if* $L < \infty$ *it is compact and if additionally* $\mu > 0$ *it is a subset of* $\mathrm{PD}(d)$ *with* $(\mathrm{PD}_\mu^L(d))^{-1} = \mathrm{PD}_{L^{-1}}^{\mu^{-1}}(d)$*.*

Proof. The set $\mathrm{PD}_\mu^L(d)$ is convex, since for any $A, B \in \mathrm{PD}_\mu^L(d)$ and $t \in [0,1]$:

$$\begin{aligned}\mu||h||^2 &= t\mu||h||^2 + (1-t)\mu||h||^2 \le t \cdot h^T A h + (1-t) \cdot h^T B h \\ &= h^T(t \cdot A + (1-t) \cdot B)h = t \cdot h^T A h + (1-t) \cdot h^T B h \\ &\le tL||h||^2 + (1-t)L||h||^2 = L||h||^2 \,.\end{aligned}$$

Any matrix $A \in \mathbb{R}^{d\times d}_{sym}$ has d real eigenvalues $\lambda_1, ..., \lambda_d \in \mathbb{R}$ and we can denote with $\lambda_{min}(A), \lambda_{max}(A) \in \mathbb{R}$ the smallest and largest eigenvalue of A. We now prove

$$\mathrm{PD}_\mu^L(d) = \{A \in \mathbb{R}^{d\times d}_{sym} \mid \mu \le \lambda_{min}(A) \le \lambda_{max}(A) \le L\} \,.$$

For $A \in \mathbb{R}^{d\times d}_{sym}$ the spectral theorem states that there exists an orthogonal matrix $Q \in \mathbb{R}^{d\times d}$ and a diagonal matrix whose diagonal entries are $\lambda_1, ..., \lambda_d$ with $A = Q^T D Q$. The equality

$$\begin{aligned}\{h^T A h \mid ||h|| = 1\} &= \{h^T Q^T D Q h \mid ||h|| = 1\} = \{h^T D h \mid ||h|| = 1\} \\ &= \{\sum_{i=1}^d \lambda_i h_i^2 \mid \sum_{i=1}^d h_i^2 = 1\} = [\lambda_{min}(A), \lambda_{max}(A)] \,.\end{aligned}$$

shows, that $A \in \mathrm{PD}_\mu^L(d)$ if and only if $\mu \le \lambda_{min}(A) \le \lambda_{max}(A) \le L$. Now assume $L < \infty$. By $A^T = A$ the spectral norm of A is given by $||A|| = |\lambda_{max}(A)|$, hence $\mathrm{PD}_\mu^L(d)$ is bounded. Since $\mathbb{R}^{d\times d}_{sym}$ is a closed linear subspace of $\mathbb{R}^{d\times d}$ and for any $h \in \mathbb{R}^d$ the mapping $\varphi_h : \mathbb{R}^{d\times d} \to \mathbb{R},\ A \mapsto h^T A h$ is continuous,

$$\mathrm{PD}_\mu^L(d) = \mathbb{R}^{d\times d}_{sym} \cap \Big(\bigcap_{h\in\mathbb{R}^d} \varphi_h^{-1}\big([\mu||h||^2, L||h||^2]\big)\Big)$$

is closed as intersection of closed sets and consequently $\mathrm{PD}_\mu^L(d)$ is compact. If additionally $\mu > 0$, then all eigenvalues of any element in $\mathrm{PD}_\mu^L(d)$ are positive, hence $\mathrm{PD}_\mu^L(d) \subset \mathrm{PD}(d) \subset \mathrm{GL}(d)$. This leads to

$$\begin{aligned}(\mathrm{PD}_\mu^L(d))^{-1} &= \{A \in \mathbb{R}^{d\times d}_{sym} \mid \mu \le \lambda_{min}(A^{-1}) \le \lambda_{max}(A^{-1}) \le L\} \\ &= \{A \in \mathbb{R}^{d\times d}_{sym} \mid \mu \le \lambda_{max}(A)^{-1} \le \lambda_{min}(A)^{-1} \le L\} \\ &= \{A \in \mathbb{R}^{d\times d}_{sym} \mid L^{-1} \le \lambda_{max}(A) \le \lambda_{min}(A) \le \mu^{-1}\} = \mathrm{PD}_{L^{-1}}^{\mu^{-1}}(d)\end{aligned}$$

as asserted. □

6.6 Definition *For $q \in \{1, ..., d+1\}$, q points $x^{(1)}, ..., x^{(q)} \in \mathbb{R}^d$ are called affinely independent, if the points $x^{(2)} - x^{(1)}, ..., x^{(q)} - x^{(1)}$ are linearly independent.*

6.7 Proposition *For* $q \in \{1, ..., d+1\}$ *and* $x^{(1)}, ..., x^{(q)} \in \mathbb{R}^d$*, the following statements are equivalent:*

(i) $x^{(1)}, ..., x^{(q)}$ *are affinely independent,*

(ii) $\dim(\text{span}(\{x^{(1)}, ..., x^{(q)}\})) = q - 1$*,*

(iii) if $\lambda_1, ..., \lambda_q \in \mathbb{R}$ *with* $\sum_{i=1}^{q} \lambda_i = 0$ *and* $\sum_{i=1}^{q} \lambda_i x^{(i)} = 0$*, then* $\lambda_1 = ... = \lambda_q = 0$*.*

6.8 Proposition *If* $h \in \text{Aff}(m, n)$*, then for any* $q \in \mathbb{N}$*,* $x^{(1)}, ..., x^{(q)} \in \mathbb{R}^m$ *and* $\lambda_1, ..., \lambda_q$ *with* $\sum_{i=1}^{q} \lambda_i = 1$ *we have*

$$h(\sum_{i=1}^{q} \lambda_i \cdot x^{(i)}) = \sum_{i=1}^{q} \lambda_i \cdot h(x^{(i)})\,.$$

Especially, if $U \subset \mathbb{R}^m$ *is convex,* $f : U \to \mathbb{R}$ *is (strictly/strongly) convex and* $h \in \text{AffT}(d)$*, then* $h(U)$ *is convex and* $f \circ h^{-1} : h(U) \to \mathbb{R}$ *is convex.*

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