Masterarbeit Mathematik

Manifolds with Special Holonomy

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Zusammenfassung

Der Begriff der Holonomie ist von großer Bedeutung in der Differentialgeometrie. Betrachtet man ein Hauptfaserbündel mit Zusammenhang oder ein Vektorbündel mit Zusammenhang, so formen die Endomorphismen der Fasern, welche durch den Paralleltransport entlang geschlossener Schleifen gegeben sind, eine Gruppe. Diese wird die Holonomiegruppe des Zusammenhangs genannt und steht in enger Verbindung zu dessen Krümmung. In dieser Arbeit werden diese Gruppen betrachtet, insbesondere für den Spezialfall des Levi-Civita-Zusammenhangs auf einer Riemannschen Mannigfaltigkeit.

Im ersten Abschnitt werden grundlegende Eigenschaften der Holonomiegruppen, reduzible Riemannsche Holonomien und lokale und infinitesimale Holonomiegruppen studiert. Im darauffolgenden Teil werden invariante Zusammenhänge auf Riemannschen homogenen und symmetrischen Räumen betrachtet und es wird bewiesen, dass die Holonomiegruppe letzterer (unter geeigneten Annahmen) durch die lineare Isotropiegruppe gegeben ist. Daraufhin wird das Reduktions-Theorem sowie das Ambrose-Singer-Theorem bewiesen, welche für allgemeine Zusammenhänge auf Hauptfaserbündeln gelten. Im vierten Abschnitt werden als Beispiel für Riemannsche Mannigfaltigkeiten mit spezieller Holonomie solche mit Holonomie enthalten in $\mathrm{Sp}(m)\mathrm{Sp}(1)$ behandelt, die Quaternion-Kähler-Mannigfaltigkeiten. Abschließend werden die Holonomiegruppen metrischer Zusammenhänge auf Vektorbündeln betrachtet und die Änderung der Holonomiegruppe bei einer Änderung des Zusammenhangs studiert. Insbesondere wird ein Monotonie-Resultat für die Holonomiegruppe bei Betrachtung von Limites bewiesen.

Summary

The notion of holonomy is an important topic in differential geometry. Given a principal bundle with connection or a vector bundle with connection, the endomorphisms of the fibres induced by parallel transport along closed curves form a group, the holonomy group, which is closely related to the curvature of the connection. In this work, these groups are studied in more detail, in particular with respect to the Levi-Civita connection on a Riemannian manifold.

In the first section, we study basic properties of holonomy groups, reducible Riemannian holonomy groups and local and infinitesimal holonomy groups. In the second part, we examine more closely the properties of invariant connections on Riemannian homogeneous spaces and symmetric spaces and prove that the holonomy group of the latter coincides (under suitable assumptions) with the linear isotropy group. Afterwards, we prove the well-known reduction theorem and the Ambrose-Singer theorem which hold true in general for connections on principal bundles. In the fourth section, we study Riemannian manifolds with holonomy contained in $\mathrm{Sp}(m)\mathrm{Sp}(1)$, the quaternion-Kähler manifolds, as an example of manifolds with special holonomy. Finally, we consider the holonomy groups of metric connections on vector bundles and consider the changes of the holonomy group under a variation of connection. In particular, we prove a monotony result for the holonomy group in limits.

1. Introduction to holonomy groups

In this document, we assume for simplicity that all manifolds are orientable and connected.

We start out with the definition of the holonomy group. More general statements about differential geometry and the theory of connections which are occasionally used in this document can be found in Appendix A.

Definition 1.1 ([16], Section 2.3, Section 2.4, [18], Chapter II, Section 4)**.** Let M be a manifold, let G be a Lie group and let $x \in M$. Let P be a principal G-bundle over M with connection Γ and let E be a vector bundle over M with standard fibre $\mathbb{R}^k$, $k \in \mathbb{N}$, and connection ∇.

1. We denote by $\Omega_{\mathrm{p}-C^1}(x)$ the set of all piecewise-differentiable loops $c : [0,1] \to M$ based at x.
2. Let $c \in \Omega_{\mathrm{p}-C^1}(x)$. We define the *parallel transport* P_c^Γ along c (with respect to the connection Γ on P) as the following endomorphism of P_x: Given any $p \in P_x$, let $c_p : [0,1] \to P$ be the horizontal lift of c starting at p. Then, we set $P_c^\Gamma(p) := c_p(1)$. Analogously we define $P_c^\Gamma : P_x \to P_y$ for a piecewise-differentiable curve in M from x to y.

   Given $c, d \in \Omega_{\mathrm{p}-C^1}(x)$, their concatenation $c * d$ is again an element of $\Omega_{\mathrm{p}-C^1}(x)$, and by construction we have $P_c^\Gamma \circ P_d^\Gamma = P_{c*d}^\Gamma$. Moreover, if we denote the loop c traversed in the opposite direction by $\overline{c}$, then $(P_c^\Gamma)^{-1} = P_{\overline{c}}^\Gamma$. Therefore, the set

   $$\mathrm{Hol}_x(\Gamma) := \left\{ P_c^\Gamma \,|\, c \in \Omega_{\mathrm{p}-C^1}(x) \right\}$$

   forms a group, the *holonomy group* of Γ (at x).
3. The *restricted holonomy group* of Γ at x, which is a subgroup of $\mathrm{Hol}_x(\Gamma)$, is defined by

   $$\mathrm{Hol}_x^0(\Gamma) := \left\{ P_c^\Gamma \,\middle|\, c \in \Omega_{\mathrm{p}-C^1}(x) \text{ such that } c \text{ is contractible} \right\}.$$

4. Let $p \in P_x$. Given $c \in \Omega_{\mathrm{p}-C^1}(x)$, we have $P_c^\Gamma(p) \in P_x$ by construction, hence $P_c^\Gamma(p) = pg$ for some $g \in G$. We set

   $$\mathrm{Hol}_p(\Gamma) := \left\{ g \in G \,\middle|\, \exists c \in \Omega_{\mathrm{p}-C^1}(x) \text{ such that } P_c^\Gamma(p) = pg \right\},$$

   the *holonomy group of* Γ *with reference point* $p \in P$. (This is indeed a group.) Put differently, we may also regard $\mathrm{Hol}_p(\Gamma)$ as the group of all elements $g \in G$ such that there exists a horizontal path in P from p to pg.

   There exists a natural isomorphism $\mathrm{Hol}_x(\Gamma) \to \mathrm{Hol}_p(\Gamma)$. We will use this isomorphism to regard $\mathrm{Hol}_x(\Gamma)$ as a Lie subgroup of G (in general not an embedded one) if it is convenient. However, this embedding $\mathrm{Hol}_x(\Gamma) \hookrightarrow G$ depends on the choice of p. In fact, if $q = pg$ for $g \in G$, then $\mathrm{Hol}_q(\Gamma) = \mathrm{ad}(g^{-1})(\mathrm{Hol}_p(\Gamma))$.[2] In this sense, $\mathrm{Hol}_x(\Gamma)$ is a subgroup of G defined up to conjugation. Moreover, one may prove that

[2] Indeed: Assume that $h \in \mathrm{Hol}_p(g)$, i. e. p and ph can be joined by a horizontal curve. Then, (since the right-translation of a horizontal curve is again a horizontal curve), $q = pg$ can be joined with phg by a horizontal curve. Since $phg = qg^{-1}hg$, we obtain $g^{-1}hg = \mathrm{ad}(g^{-1})(h) \in \mathrm{Hol}_q(\Gamma)$. Thus, $\mathrm{ad}(g^{-1})(\mathrm{Hol}_p(\Gamma)) \subseteq \mathrm{Hol}_q(\Gamma)$, and since for the same reason $\mathrm{ad}(g)(\mathrm{Hol}_q(\Gamma)) \subseteq \mathrm{Hol}_p(\Gamma)$ and $\mathrm{ad}(g), \mathrm{ad}(g^{-1})$ are inverses of each other, in fact in both cases equality must hold true.

up to conjugation it is also independent of the basepoint $x \in M$.[3] For this reason, we will in some cases omit the reference to x and simply write $\mathrm{Hol}(\Gamma)$.

Analogously we define the restricted holonomy group of Γ with reference point $p \in P$; the same remarks made above hold also true for this group and we can define $\mathrm{Hol}^0(\Gamma)$ in the obvious way as a subgroup of G defined up to conjugation.

5. The holonomy group $\mathrm{Hol}_p(\Gamma)$, $p \in P_x$, is a Lie subgroup of G ([18], Chapter II, Theorem 4.2; the proof uses that by the theorem [32] of Yamabe, an arcwise-connected subgroup of a Lie group is a Lie group). We define $\mathfrak{hol}_p(\Gamma)$ to be its Lie algebra. Carrying over the Lie group structure of $\mathrm{Hol}_p(\Gamma)$ to $\mathrm{Hol}_x(\Gamma)$, also $\mathrm{Hol}_x(\Gamma)$ becomes a Lie group; we let $\mathfrak{hol}_x(\Gamma)$ be its Lie algebra, the *holonomy algebra* of Γ (at x). Analogous results hold for the restricted holonomy groups.[4]
6. Let $c \in \Omega_{\mathrm{p}-C^1}(x)$. We define the *parallel transport* P_c^∇ along c (with respect to the connection ∇ on E) as follows: Given any $e \in E_x$, there exists a unique section s of $E|_{c([0,1])}$ with $\nabla_{\dot{c}(t)} s(t) = 0$ for all $t \in [0,1]$, and we let $P_c^\nabla(e) := s(1)$.[5] Then, P_c^∇ is a linear automorphism of E_x. Analogously we define $P_c^\nabla : E_x \to E_y$ for a piecewise-differentiable curve in M from x to y.

   As in the case of a principal bundle, we note that
   $$\mathrm{Hol}_x(\nabla) := \left\{ P_c^\nabla \,\middle|\, c \in \Omega_{\mathrm{p}-C^1}(x) \right\}$$
   forms a subgroup of $\mathrm{GL}(E_x)$, the holonomy group of ∇ (at x). We define also the restricted holonomy group of ∇ at x,
   $$\mathrm{Hol}_x^0(\nabla) := \left\{ P_c^\nabla \,\middle|\, c \in \Omega_{\mathrm{p}-C^1}(x) \text{ such that } c \text{ is contractible} \right\} \subseteq \mathrm{Hol}_x(\nabla).$$
   The *holonomy representation* of $\mathrm{Hol}_x(\nabla)$ is the inclusion $\mathrm{Hol}_x(\nabla) \hookrightarrow \mathrm{GL}(E_x)$ defining the holonomy group.
7. Choosing any basis of E_x, we may identify E_x with $\mathbb{R}^k$ and consequently view $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_x^0(\nabla)$ as subgroups of $\mathrm{GL}(k)$. However, analogously to the case of a principal bundle, this is only up to conjugation; the choice of different bases yields subgroups conjugate in $\mathrm{GL}(k)$.
8. As for the case of a principal bundle, one can define a Lie group structure on $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_x^0(\nabla)$ (i. e. one proves that they are Lie subgroups of $\mathrm{GL}(k)$ after choosing a basis of E_x). We define $\mathfrak{hol}_x(\nabla)$ to be the Lie algebra of $\mathrm{Hol}_x(\nabla)$.

Remark 1.2. One may equivalently define the holonomy group of a connection on a principal or vector bundle by only considering parallel transport along piecewise-smooth loops. This yields the same group, see e. g. [18], Chapter II, Theorem 7.2.

The probably best-known connection on a manifold is the Levi-Civita connection on the tangent bundle of an Riemannian manifold, and in fact a large part of this work is dedicated to study the holonomy group of this special case.

[3] To this end, together with an application of Footnote 2 it suffices to prove that if p, p' can be connected in P by a horizontal curve, then $\mathrm{Hol}_p(\Gamma) = \mathrm{Hol}_{p'}(\Gamma)$. This follows directly from the definitions.

[4] In fact, the holonomy algebra is the Lie algebra of the restricted holonomy group as well, since we will see in Lemma 1.4 below that $\mathrm{Hol}_x^0(\Gamma)$ is the identity component of $\mathrm{Hol}_x(\Gamma)$.

[5] Using the relation between connections on principal bundles and vector bundles, Remark A.6, the two notions of parallel transport can also be related to each other.

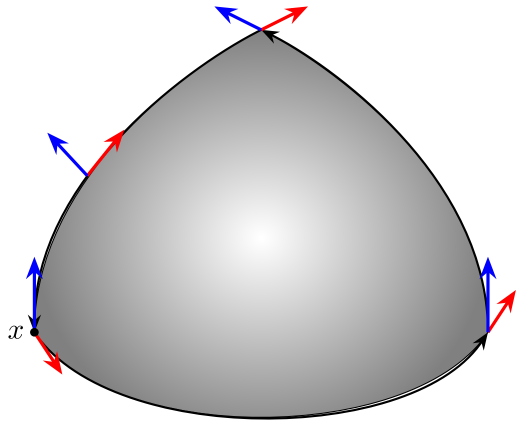

Figure 1: Illustration of the holonomy on the tangent bundle

Definition 1.3. Let (M, g) be a Riemannian manifold of dimension n with Levi-Civita connection ∇ and $x \in M$. For $c \in \Omega_{\text{p}-C^1}(x)$, we will also write $P_c^g := P_c^\nabla$. Moreover, we define $\text{Hol}_x(g) := \text{Hol}_x(\nabla)$, the *holonomy group of* g (at x). Analogously we set $\text{Hol}_x^0(g) := \text{Hol}_x^0(\nabla)$, the restricted holonomy group of g and $\mathfrak{hol}_x(g) := \mathfrak{hol}_x(\nabla)$, the holonomy algebra of g (at x).

For the Levi-Civita connection, parallel transport is an isometry. Hence, if we choose an *orthonormal* basis of T_xM to identify it with $\mathbb{R}^n$, then with respect to this identification we have $\text{Hol}_x(g) \subseteq \text{SO}(n)$.[6] In this sense, we may regard $\text{Hol}_x(g)$ as a subgroup of $\text{SO}(n)$ defined up to conjugation within $\text{SO}(n)$.

The effect of the holonomy on the tangent bundle is attempted to be illustrated in Figure 1.

As a first statement concerning holonomy groups, we note the following lemma which finds frequent (often implicit) application.

Lemma 1.4 ([1], Section 2, Lemma 4)**.** *Let M be a manifold, G be a Lie group and $P \to M$ be a principal G-bundle equipped with a connection Γ. Let $x \in M$. Then, the restricted holonomy group $\text{Hol}_x^0(\Gamma)$, consisting of parallel transport along piecewise smooth contractible loops in M, is the connected component of the identity $1 \in \text{Hol}_x(\Gamma)$.*

Proof. Let us temporarily denote the connected component of the identity $1 \in \text{Hol}_x(\Gamma)$ by $\text{Hol}_x^1(\Gamma)$. If $c \in \Omega_{\text{p}-C^1}(x)$ is contractible and $P_c^\Gamma \in \text{Hol}_x^0(\Gamma)$ denotes the parallel transport along c, then any nullhomotopy $H : [0,1] \times [0,1] \to M$ (which we may choose such that $H(\cdot, s)$ is piecewise differentiable for all $s \in [0,1]$, see the construction in [18], Chapter II, proof of Theorem 4.2) provides a path $s \mapsto P_{H(\cdot,s)}^\Gamma$ in $\text{Hol}_x^0(\Gamma)$ from P_c^Γ to the identity. Therefore, we see that $\text{Hol}_x^0(\Gamma) \subseteq \text{Hol}_x^1(\Gamma)$. Moreover, both groups are path-connected subgroups of a Lie group and hence by the theorem [32] of Yamabe Lie groups themselves. Thus, $\text{Hol}_x^0(\Gamma)$ is a connected Lie subgroup of $\text{Hol}_x^1(\Gamma)$.

Since $\text{Hol}_x^1(\Gamma)$ is connected, it now suffices to prove that $\text{Hol}_x^0(\Gamma) \subset \text{Hol}_x^1(\Gamma)$ is both open and closed. Moreover, since $\text{Hol}_x^1(\Gamma)$ is locally compact and every open subgroup of a locally compact topological group is closed as well (see [5], Lemma 1.1.7), it suffices to prove

[6] Recall that for simplicity, we assume M to be orientable.

that $\mathrm{Hol}^0_x(\Gamma)$ is open in $\mathrm{Hol}^1_x(\Gamma)$. To do so, we first show:

Claim 1. If a subgroup $H \subseteq \mathrm{Hol}^1_x(\Gamma)$ is not open, then it has zero Haar measure.

Proof of claim 1. Note that the Haar measure on a Lie group can equivalently be described by use of the volume form of a left-invariant Riemannian metric. In particular, its expression in charts is given by a (position-dependent) multiple of the Lebesgue measure. From the properties of the latter (and the fact that every manifold being second countable, it can be covered by a countable set of chart domains), it follows that any submanifold of $\mathrm{Hol}^1_x(\Gamma)$ which has nonzero codimension is of zero Haar measure.

In fact, this even holds true for the Lie subgroup $H \subseteq \mathrm{Hol}^1_x(\Gamma)$ (which, as is well known, need not be an embedded submanifold in the usual sense of the word). For if the same argument shows that a suitable H-neighbourhood of the identity[7] has zero Haar measure in $\mathrm{Hol}^1_x(\Gamma)$, and then H is (again, by second countability) a countable union of left translates of this unit neighbourhood.

Now, if $H \subseteq \mathrm{Hol}^1_x(\Gamma)$ is not open, by considering charts again, one sees that it must have nonzero codimension. Thus, by the above, it has zero Haar measure.

By claim 1, it is sufficient to show that $\mathrm{Hol}^0_x(\Gamma) \subseteq \mathrm{Hol}^1_x(\Gamma)$ has nonzero Haar measure. Now, we note the following: If $\mathrm{Hol}^0_x(\Gamma)$ has at most countably many cosets in $\mathrm{Hol}^1_x(\Gamma)$, then it must have nonzero Haar measure. Indeed, if there existed a subgroup $H \subseteq \mathrm{Hol}^1_x(\Gamma)$ with zero Haar measure and countably many cosets in $\mathrm{Hol}^1_x(\Gamma)$, one could write $\mathrm{Hol}^1_x(\Gamma)$ as a countable union of sets with zero measure. Hence, it would have zero measure itself, a contradiction. Therefore, at this point it suffices to prove:

Claim 2. $\mathrm{Hol}^0_x(\Gamma)$ has at most countably many cosets in $\mathrm{Hol}^1_x(\Gamma)$.

Proof of claim 2. By a theorem of Borsuk and Hanner ([10], Theorem 6.1), there exists a countable CW-complex C and two maps $\phi : M \to C, \psi : C \to M$ such that $\psi \circ \phi$ is homotopic to the identity of M.[8] Since the fundamental group of a countable CW-complex is countable (as can be seen by considering its description in terms of the 1- and 2-cells), $\pi_1(C)$ is countable. From functoriality of the homotopy groups and $\psi \circ \phi \simeq \mathrm{Id}_M$, one sees that $\pi_1(M)$ must be countable as well.

Using this fact, at this point it suffices to show that there exists a surjection from $\pi_1(M)$ onto the set of cosets of $\mathrm{Hol}^0_x(\Gamma) \subseteq \mathrm{Hol}^1_x(\Gamma)$. This surjection can be defined as follows: For every $\alpha \in \pi_1(M)$, let $E(\alpha) := \{P^\Gamma_c \in \mathrm{Hol}^1_x(\Gamma) \,|\, [c] = \alpha \in \pi_1(M)\}$. Then, this map descends to a map from $\pi_1(M)$ to the left cosets of $\mathrm{Hol}^0_x(\Gamma) \subseteq \mathrm{Hol}^1_x(\Gamma)$: To show this, it suffices to show that for each $\alpha \in \pi_1(M)$, $E(\alpha)$ is contained in a left coset of $\mathrm{Hol}^0_x(\Gamma)$ respectively for each $P^\Gamma_c, P^\Gamma_d \in E(\alpha)$ we have $(P^\Gamma_c)^{-1} P^\Gamma_d \in \mathrm{Hol}^0_x(\Gamma)$. But this is follows since $[c] = [d] = \alpha$ and thus $(P^\Gamma_c)^{-1} P^\Gamma_d = P^\Gamma_{\bar{c} * d}$ is given by parallel transport along a contractible loop based at x.

That the assignment $\pi_1(M) \to \mathrm{Hol}^1_x(\Gamma)/\mathrm{Hol}^0_x(\Gamma), \alpha \mapsto [E(\alpha)]$, is surjective, is realized by

[7] With "suitable", it is meant that it is so small that it is an embedded submanifold of $\mathrm{Hol}^1_x(\Gamma)$. Such a neighbourhood can be shown to exist using the properties of the exponential map.

[8] Since M is a manifold, one can show that it is an absolute neighbourhood retract as considered in the quoted reference. Note that in the theorem no statement concerning the countability is made, but it follows directly from its proof that the constructed CW-complex is in fact countable.

The theorem was originally proved by Borsuk in the compact case and subsequently generalized by Hanner to the noncompact case.

taking for any left coset of $\mathrm{Hol}_x^0(\Gamma)$ some loop c whose parallel transport lies in this coset; then the left coset is the image of $[c]$. $\square$

The holonomy group of a connection is intimately related to the existence of a reduction of the structure group and the existence of a principal subbundle, as is the statement of the following reduction theorem.

Theorem 1.5 ([16], Theorem 2.3.6, [18], Chapter II, Theorem 7.1)**.** *Let M be a manifold, G be a Lie group and $P \to M$ be a principal G-bundle. Let Γ be a connection in P. For $p \in P$, let $H := \mathrm{Hol}_p(\Gamma)$. We define an equivalence relation $\sim$ on P by setting $q \sim r$ if and only if q and r can be connected by a piecewise-differentiable horizontal curve in P, and we set $P(p) := \{q \in P \,|\, q \sim p\}$. Then, $P(p)$ is an injectively immersed principal subbundle of P with fibre H (i. e. a reduction of P), and the connection Γ on P restricts to a connection Γ' on $P(p)$.*

Definition 1.6. In the situation of Theorem 1.5, we call $P(p)$ the *holonomy bundle* of Γ through p.

We will prove Theorem 1.5 in Section 3. Intuitively, it states that the holonomy group is the smallest subgroup H of the structure group of a principal bundle P such that P may be reduced to a principal H-bundle compatible with the connection.

If the connection under consideration is of class C^1, one may consider its curvature. It is intuitive to assume that holonomy is related to curvature, and indeed the restricted holonomy group of the connection may be determined from the curvature form. This is the content of the following well-known theorem, originally due to Ambrose and Singer.

Theorem 1.7 ([1], Theorem 2)**.** *Let M be a smooth manifold, let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi : P \to M$ be a principal G-bundle. Let $x \in M$ and $p \in P_x$. Let ω be the connection 1-form of a C^1-connection Γ on E; let Ω denote the corresponding curvature form. Let $P(p)$ be the holonomy bundle of this connection through p. Let $L(p)$ be the subalgebra of $\mathfrak{g}$ generated by all $\Omega(X, Y)$, where X and Y run through all pairs of tangent vectors to P at all points of $P(p)$. Then, $L(p) = \mathfrak{hol}_p(\Gamma)$, that is, $\mathrm{Hol}_p^0(\Gamma)$ is generated by $L(p)$.*

Also Theorem 1.7 shall be proven in Section 3. We remark that using the relationship between connections on principal bundles and vector bundles described in Remark A.6, the following can be deduced from it for vector bundles: If E is a vector bundle over M with connection ∇ and $x \in M$, then $\mathfrak{hol}_x(\nabla)$ equals the span of all elements of the form $P_c^\nabla R_{c(0)}(X, Y)(P_c^\nabla)^{-1}$ for all piecewise-differentiable curves $c : [0, 1] \to M$ with $c(1) = x$.

The holonomy group of a Riemannian manifold (M, g) can be related to the existence of constant tensors, as is indicated by the following theorem. It may be regarded as a generalization of the fact that it follows from $\nabla g = 0$ that $\mathrm{Hol}(g) \subseteq \mathrm{SO}(\dim(M))$ and serves as an important tool for the actual computation of Riemannian holonomy groups.

Theorem 1.8 ([3], Fundamental Principle 10.19)**.** *Let (M, g) be a connected Riemannian manifold and $(r, s) \in \mathbb{N} \times \mathbb{N}$. Then, the following three properties are equivalent:*

1. *There exists on (M, g) a tensor field α of type (r, s) which is invariant by parallel transport.*

2. *There exists on* (M,g) *a tensor field* α *of type* (r,s) *which has a vanishing covariant derivative,* $\nabla\alpha = 0$.
3. *There exists* $p \in M$ *and* $\alpha_0 \in \mathrm{T}^{r,s}_p(M)$ *which is invariant by the tensorial extension (of type* (r,s)*) of the holonomy representation* $\mathrm{Hol}_p(g)$.

Proof. The implication "1. ⇒ 3." follows directly from the definition of the holonomy group. To prove "3. ⇒ 1.", we assume that we are given $p \in M, \alpha_0 \in \mathrm{T}^{r,s}(M)$ as in 3 and construct explicitly a tensor field α which is invariant by parallel transport.

Let $q \in M$ and $c : [0,1] \to M$ be a smooth path from p to q. We define $\alpha(q) := P^g_c(\alpha_0)$. Since α_0 is invariant under the tensorial extension of $\mathrm{Hol}_p(g)$, $\alpha(q)$ is invariant under the tensorial extension of $\mathrm{Hol}_p(q)$. Using this, one sees that $\alpha(q)$ does also not depend on the particular choice of c. Finally, differentiability of α follows from the smooth dependency of solutions of ordinary differential equations on initial data (see [22], Theorem D.6, applied to the curve $(c(t), \alpha(c(t))) \subset \mathrm{T}^{r,s}(M)$, respectively its expression in suitable coordinates).

To show "2. ⇒ 1.", it suffices to note that a tensor field α is parallel along some differentiable curve $c : [0,1] \to M$ if and only if $\nabla_{\dot{c}(t)}\alpha(c(t)) = 0$ for all $t \in [0,1]$. The implication "1. ⇒ 2." follows from the same observation, and the fact that for any $X \in TM$ there exists a curve $c : [0,1] \to M$ with $\dot{c}(0) = X$. □

Now, the classification of holonomy groups shall be examined. Preliminarily, we note the following two propositions.

Proposition 1.9 ([16], Proposition 2.4.2)**.** *Let* M *be a manifold,* $\pi : P \to M$ *a principal bundle with fibre* G*, and* Γ *a connection on* P*. Then, for each* $p \in P$ *the curvature form* $\Omega_p(\Gamma)$ *of* Γ *at* p *lies in* $\Lambda^2 T^*_p P \otimes \mathfrak{hol}_p(\Gamma)$.

Proof. This is directly implied by the Ambrose-Singer theorem. □

Proposition 1.10 ([16], Theorem 3.1.7)**.** *Let* (M,g) *be a Riemannian manifold. At each* $x \in M$*, the curvature* $(4,0)$*-tensor* R_x *of* g *at* x *lies in the subspace* $\mathrm{S}^2(\mathfrak{hol}_x(g))$ *of* $\Lambda^2 T^*_x M \otimes \Lambda^2 T^*_x M$*, where we use the musical isomorphism induced by the metric* g *to identify* $\mathfrak{hol}_x(g)$ *with a subspace of* $\Lambda^2 T^*_x M$.

Proof. As a corollary of Proposition 1.9, $R_x \in \Lambda^2 T^*_x M \otimes \mathfrak{hol}_x(g)$. However, by a well-known symmetry of the curvature tensor also $R_x \in \mathrm{S}^2(\Lambda^2 T^*_x M)$ and therefore $R_x \in \mathrm{S}^2(\mathfrak{hol}_x(g))$. □

Together with the Bianchi identities, this gives restrictions on the curvature tensor of a metric with a prescribed holonomy group. This already hints at the fact that not all subgroups of $\mathrm{SO}(n)$ can occur as holonomy groups of Riemannian metrics and is useful for proving the classification of holonomy groups.

The main classification result for Riemannian holonomy groups is due to Berger. Here, however, the slightly refined "standard textbook"-version from [16], Theorem 3.4.1 is quoted. In order to state it, we need first to introduce two easy definitions.

Definition 1.11. Let (M,g) be a Riemannian manifold.

1. We call M or g *reducible* if (M, g) is a *Riemannian product*, i. e. there exist Riemannian manifolds $(M_1, g_1), (M_2, g_2)$ of nonzero dimension such that (M, g) is isometric to $(M_1 \times M_2, g_1 \oplus g_2)$. We call it *locally reducible* if every point in M has a reducible open neighbourhood. If M is not reducible, we call it *irreducible*, analogously with local reducibility.
2. We call (M, g) a *(Riemannian) symmetric space* and g *symmetric* if for every $p \in M$ there exists a *symmetry at* p, i. e. an isometry $s_p : M \to M$ that is an involution (that is, $s_p^2 = \mathrm{Id}_M$), such that p is an isolated fixed point of s_p.

Theorem 1.12 ([16], Theorem 3.4.1)**.** *Let (M, g) be a simply-connected Riemannian manifold of dimension n and assume that g is irreducible and nonsymmetric. Then, exactly one of the following seven cases holds true:*

1. $\mathrm{Hol}(g) = \mathrm{SO}(n)$.
2. *$n = 2m$ with $m \geq 2$, and* $\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(2m)$. *In this case, we call (M, g) a* Kähler manifold.
3. *$n = 2m$ with $m \geq 2$, and* $\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(2m)$. *In this case, we call (M, g) a* Calabi-Yau manifold.
4. *$n = 4m$ with $m \geq 2$, and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\mathrm{Sp}(1) \subset \mathrm{SO}(4m)$. *In this case, we call (M, g) a* quaternion-Kähler manifold.
5. *$n = 4m$ with $m \geq 2$, and* $\mathrm{Hol}(g) = \mathrm{Sp}(m) \subset \mathrm{SO}(4m)$. *In this case, we call (M, g) a* hyperkähler manifold.
6. *$n = 7$ and* $\mathrm{Hol}(g) = G_2 \subset \mathrm{SO}(7)$.
7. *$n = 8$ and* $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$.

In proposition 1.13 below, we will show that for a reducible Riemannian metric, the holonomy group is a product of the holonomy groups of its factors. Moreover, the holonomy groups of (simply-connected) symmetric spaces are understood, see Theorem 2.22. Finally, the restricted holonomy group of a space which is not simply connected coincides with the holonomy group of its universal covering. Thus, Berger's classification is in fact a classification of all "interesting" restricted Riemannian holonomy groups. It is one of the greatest achievements in the theory of holonomy groups; unfortunately, its proof is too complicated to be shown here.

For the construction of manifolds with special holonomy, in particular the two "exotic" cases $\mathrm{Hol}(g) = G_2$ and $\mathrm{Hol}(g) = \mathrm{Spin}(7)$, see e. g. the textbook by Joyce ([16]).

1.1. Reducible Holonomies

In this subsection we treat Riemannian holonomy groups, and in particular the case that the holonomy representation is reducible. We prove that the manifold in this case is locally a Riemannian product. In fact, if the manifold in question is simply connected, it even is globally a product (by the de Rham decomposition theorem, see below).

First, we note the easier converse of this theorem:

Proposition 1.13 ([16], Proposition 3.2.1)**.** *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Then, the product metric $g_1 \times g_2$ on $M_1 \times M_2$ has the holonomy group* $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.

Proof. Let $p : M_1 \times M_2 \to M_1$, $q : M_1 \times M_2 \to M_2$ denote the canonical projections. The proposition follows directly from the definition of the holonomy group and the observation that a section X of $T(M_1 \times M_2)$ along a curve c is parallel if and only if $dp \circ X$ is parallel along $p \circ c$ and $dq \circ X$ is parallel along $q \circ c$. □

We turn to the proof of the (local) decomposition result. First we note:

Proposition 1.14 ([16], Proposition 3.2.2). *Let (M, g) be a Riemannian manifold and let $x \in M$, so that $\mathrm{Hol}_x(g)$ acts on T_xM. Suppose that $T_xM = V_x \oplus W_x$, where V_x, W_x are proper vector subspaces of T_xM preserved by $\mathrm{Hol}_x(g)$, and orthogonal with respect to g. Then, there are natural vector subbundles V, W of TM with fibres V_x, W_x at x. These subbundles V, W are orthogonal with respect to g and closed under parallel translation. Moreover, they satisfy $TM = V \oplus W$ and $T^*M = V^* \oplus W^*$.*

Proof. We apply the fundamental principle 1.8 to the orthogonal projections $P_{V_x} : T_xM \to V_x$ and $P_{W_x} : T_xM \to W_x$. This yields two parallel tensor fields P_V, P_W on M. Using the explicit construction of these parallel tensor fields in Theorem 1.8 via parallel transport, it is recognized that they satisfy $P_V + P_W = \mathrm{Id}_{TM}$, $P_V \circ P_W = P_W \circ P_V = 0$ and $P_V \circ P_V = P_V, P_W \circ P_W = P_W$.[9] Summarizing, since also the rank of P_V, P_W must be the same at each point, we see that P_V, P_W are projections onto two orthogonal subbundles V, W of TM with $TM = V \oplus W$. The decomposition $T^*M = V^* \oplus W^*$ follows from dualizing this orthogonal decomposition. □

Proposition 1.15 ([16], Proposition 3.2.3). *In the situation of Proposition 1.14, let R denote the Riemann curvature tensor of g. Then, R is a section of the subbundle $\mathrm{S}^2(\Lambda^2 V^*) \oplus \mathrm{S}^2(\Lambda^2 W^*)$ of $\mathrm{S}^2(\Lambda^2 T^*M)$. Also, the restricted holonomy group $\mathrm{Hol}^0_x(g)$ is a product group $H_V \times H_W$, where H_V is a subgroup of $\mathrm{SO}(V_x)$ and acts trivially on W_x, and H_W is a subgroup of $\mathrm{SO}(W_x)$ and acts trivially on V_x.*

Proof. Since $T^*M \cong V^* \oplus W^*$, we have a decomposition $\Lambda^2 T^*M \cong \Lambda^2 V^* \oplus \Lambda^2 W^* \oplus V^* \otimes W^*$.[10] By Proposition 1.10, we know that $R \in \mathrm{S}^2(\mathfrak{hol}_x(g))$, where we regard $\mathfrak{hol}_x(g)$ as a subspace of $\Lambda^2 T^*M$ by using the musical isomorphism induced by the metric. Now, since the action of $\mathrm{Hol}_x(g)$ preserves V_x and W_x, so does the action of $\mathfrak{hol}_x(g)$, and using this one sees that $\mathfrak{hol}_x(g) \subseteq \Lambda^2 V_x^* \oplus \Lambda^2 W_x^*$ under the musical isomorphism. Therefore (and applying the same reasoning to any other point in M), the Riemannian curvature $(4, 0)$-tensor is a section of

$$\Lambda^2 V^* \otimes \Lambda^2 V^* \oplus \Lambda^2 W^* \otimes \Lambda^2 W^* \oplus \Lambda^2 V^* \otimes \Lambda^2 W^* \oplus \Lambda^2 W^* \otimes \Lambda^2 V^*.$$

Now, by using the first Bianchi identity, we see that the components of R in the last two summands above must in fact vanish.[11] Since R is symmetric in its two $\Lambda^2 T^*M$-factors, we see that, in fact, it must be a section of $\mathrm{S}^2(\Lambda^2 V^*) \oplus \mathrm{S}^2(\Lambda^2 W^*)$, as claimed.

[9] To be precise, here we use that Id_{TM} and 0 are trivially parallel, and the composition of two tensors can be written as a tensor contraction and hence is also preserved by parallel translation.

[10] The isomorphism can best be described in terms of its inverse, which is given by $(X \wedge X', Y \wedge Y', X'' \otimes Y'') \mapsto X \wedge X' + Y \wedge Y' + X'' \wedge Y''$.

[11] This is easiest to see in index notation: Consider a component R_{abcd} of the curvature tensor such that, for example, the indices a, b belong to the V^*-component of T^*M and the indices c, d belong to the W^*-component. Now, $R_{abcd} = -R_{cabd} - R_{bcad}$. In both terms on the right-hand side, one of the first two indices belong to the V^*-component of T^*M and the other one belongs to the W^*-component, hence both must vanish. Thus, also $R_{abcd} = 0$, as we wanted to show.

From this, we furthermore deduce that the Riemannian curvature $(3,1)$-tensor is a section of

$$V \otimes V^* \otimes \Lambda^2 V^* \oplus W \otimes W^* \otimes \Lambda^2 W^*.$$

In other words, for all $y \in M$, $X, Y \in T_yM$, we have $R(X,Y) \in A_y \oplus B_y$, where A_y is a subspace of $V_y \otimes V_y^*$ and B_y is a subspace of $W_y \otimes W_y^*$. But by the Ambrose-Singer theorem, $\mathfrak{hol}_x(g)$ is spanned by the elements of T_xM of the form $(P_c^g)^{-1} R(X,Y) P_c^g$, for all $y \in M$, all piecewise smooth paths c from x to y and all $X, Y \in T_yM$. Since V, W are preserved by parallel transport, we see that $(P_c^g)^{-1} A_y P_c^g \subseteq V_x \otimes V_x^*$ and $(P_c^g)^{-1} B_y P_c^g \subseteq W_x \otimes W_x^*$. In other words, we have shown that for a fixed piecewise smooth path $c : [0,1] \to M$, the subspace spanned by elements of the form $(P_c^g)^{-1} R(X,Y) P_c^g$ (for all $X, Y \in T_{c(1)}M$) splits as direct sum of a piece in $V_x \otimes V_x^*$ and a piece in $W_x \otimes W_x^*$. Therefore, the span of these elements for all piecewise smooth $c : [0,1] \to$M splits in the same way, and the Ambrose-Singer theorem yields that $\mathfrak{hol}_x(g) = \mathfrak{h}_V \oplus \mathfrak{h}_W$, where $\mathfrak{h}_V$ is a subspace of $V_x \otimes V_x^*$ and $\mathfrak{h}_W$ is a subspace of $W_x \otimes W_x^*$.

In fact, since $V_x \otimes V_x^*$ and $W_x \otimes W_x^*$ are closed under the Lie bracket, we see that $\mathfrak{h}_V, \mathfrak{h}_W$ are Lie subalgebras of $\mathfrak{hol}_x(g)$. There exist unique connected Lie subgroups H_V, H_W of $\mathrm{Hol}_x^0(g)$ such that $\mathfrak{h}_V$ is the Lie algebra of H_V and $\mathfrak{h}_W$ is the Lie algebra of H_W (by the well-known subgroups-subalgebras theorem from Lie group theory, see [11], Chapter II, Theorem 2.1). Moreover, then $H_V \subseteq \mathrm{SO}(V_x)$ and $H_W \subseteq \mathrm{SO}(W_x)$ (since the corresponding Lie algebra inclusions also hold true). Since (again by the subgroups-subalgebras theorem) there exists a unique connected Lie subgroup of $\mathrm{Hol}_x^0(g)$ with Lie algebra $\mathfrak{hol}_x(g) = \mathfrak{h}_V \oplus \mathfrak{h}_W$, but on the other hand such one is given by $H_V \times H_W$, we conclude that $\mathrm{Hol}_x^0(g) = H_V \times H_W$ and the proof is achieved. $\square$

Now, we are able to prove the local decomposition.

Proposition 1.16 ([16], Proposition 3.2.4, [18], Chapter IV, Proposition 5.2)**.** *In the situation of Proposition 1.15, there is a connected open neighbourhood N of x in M and a diffeomorphism $N \cong X \times Y$ for manifolds X, Y, such that under the isomorphism $T(X \times Y) \cong TX \oplus TY$, we have $V|_N = TX$ and $W|_N = TY$. There are Riemannian metrics g_X on X and g_Y on Y such that $g|_N$ is isometric to the product metric $g_X \times g_Y$. Thus, g is locally reducible, and moreover $\mathrm{Hol}_p^0(g_X) \subseteq H_V$ and $\mathrm{Hol}_q^0(g_Y) \subseteq H_W$ where $x \in N$ is identified with $(p,q) \in X \times Y$.*

Proof. First, we show that V and W, viewed as distributions, are involutive. Indeed: It suffices to prove the statement for V. So let $A, B \in V$. Since V is closed under parallel translation, we have $\nabla_A B \in V$ and $\nabla_B A \in V$. But since ∇ is torsion-free, we deduce that then $[A,B] = \nabla_A B - \nabla_B A \in V$, proving that V is involutive.

Let r, s denote the fibre dimensions of V respectively W, accordingly $r+s = \dim(M)$. Since V is involutive, by the Frobenius theorem (see e. g. [22], Theorem 19.21 for a textbook account) there exists a local coordinate system $\phi = (y^1, \ldots, y^r, x^{r+1}, \ldots, x^{r+s}) : U \to U' \subseteq \mathbb{R}^{\dim(M)}$ with $\phi(x) = 0$ such that $\left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^r}\right)$ form at every $y \in U$ a basis of $V|_y$. That is, the equations $x^j = c^j$, $r+1 \le j \le r+s$, define (locally, for suitable c^j) an integral manifold of V. Similarly, there exists a local coordinate system $\psi = (x^1, \ldots, x^r, y^{r+1}, \ldots, y^{r+s})$ with $\psi(x) = 0$, such that the equations $x^i = c^i$, $1 \le i \le r$, define (locally, for suitable c^i) an integral manifold of W. Then, the coordinate system $\mathbf{x} := (x^1, \ldots, x^r, x^{r+1}, \ldots, x^{r+s})$ satisfies that for any set of constants $(c^1, \ldots, c^r, c^{r+1}, \ldots, c^{r+s})$ the equations $x^i = c^i$,

$1 \leq i \leq r$, respectively $x^j = c^j$, $r+1 \leq i \leq r+s$, define an integral manifold of W respectively V.

Using the coordinate system $\mathbf{x}$, we can now prove the local product structure of M. Let $N := \mathbf{x}^{-1}((-c,c)^{\dim(M)})$, where $c > 0$ is so small that $(-c,c)^{\dim(M)} \subseteq U'$. Let $X := \{y \in N \,|\, \forall r+1 \leq j \leq r+s : x^j(y) = 0\}$ and $Y := \{y \in N \,|\, \forall 1 \leq i \leq r : x^i(y) = 0\}$. Then, X respectively Y is an integral submanifold of V respectively W through x, and by construction the coordinates $\mathbf{x}$ (together with the projections onto the first r respectively the last s coordinates) yield a diffeomorphism $N \cong X \times Y$. We shall prove that this diffeomorphism is an isometry, that is, $g|_N = g_X \times g_Y$. To this end, it suffices to prove the following claim:

Claim.

1. The function $g_{ij}(\mathbf{x}(y)) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})(y)$ is independent of $x^{r+1}(y), \ldots, x^{r+s}(y)$ if $1 \leq i,j \leq r$.
2. The function $g_{ij}(\mathbf{x}(y))$ is independent of $x^1(y), \ldots, x^r(y)$ if $r+1 \leq i,j \leq r+s$.
3. We have $g_{ij}(\mathbf{x}(y)) = 0$ if $1 \leq i \leq r, r+1 \leq j \leq r+s$ or $r+1 \leq i \leq r+s, 1 \leq j \leq r$.

Proof of the claim. Assertion 3 follows directly from the fact that V and W are orthogonal at each point of M, since $\frac{\partial}{\partial x^i}$ lies in V if $1 \leq i \leq r$ and in W if $r+1 \leq i \leq r+s$.

Assertion 1 can be proven by showing that $\frac{\partial}{\partial x^k} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = 0$ (as functions on N) if $1 \leq i,j \leq r, r+1 \leq k \leq s$. Now, note first that we have $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}$ (since all coordinate vector fields commute, and ∇ is torsion-free). Furthermore, since W is closed under parallel translations, we have $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \in W$. The analogous observation holds true with i replaced by j. Thus, we can compute, using that ∇ is a metric connection and that V, W are orthogonal,

$$\begin{aligned} \frac{\partial}{\partial x^k} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= g\left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right) \\ &= g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\right) = 0. \end{aligned}$$

This proves assertion 1, assertion 2 follows analogously and the claim is proven.

Thus, g is locally reducible. The statement about the holonomy groups follows from the observation that the product decomposition of $\mathrm{Hol}_x(g|_N)$ from Proposition 1.13 and the product structure $N \cong X \times Y$ is compatible with the inclusion $\mathrm{Hol}_x(g|_N) \subseteq \mathrm{Hol}_x(g)$ and the product decomposition of $\mathrm{Hol}_x(g)$ from Proposition 1.15. □

This proposition can be generalized as follows.

Proposition 1.17 ([16], Theorem 3.2.6, [18], Chapter IV, Theorem 5.4)**.** *Let (M,g) be a Riemannian manifold of dimension n, let $x \in M$. Then, there is a splitting $T_xM \cong T_x^{(0)} \oplus \cdots \oplus T_x^{(k)}$, where $\dim(T_x^{(0)}) \geq 0$, $\dim(T_x^{(j)}) > 0$ for $1 \leq j \leq k$, and a corresponding isomorphism $\mathrm{Hol}_x^0(g) = H_0 \times \cdots \times H_k$, where $H_0 = \{1\}$ and H_j is a connected Lie subgroup of $\mathrm{SO}(T_x^{(j)})$ acting irreducibly on $T_x^{(j)} \cong \mathbb{R}^{\dim(T_x^{(j)})}$ for $1 \leq j \leq k$.*

Proof. Let $T^{(0)}$ denote the involutive distribution on M consisting of all parallel vector fields. We may inductively apply Proposition 1.15 and Proposition 1.16 (starting with the integral manifold of the orthogonal complement of $T^{(0)}$, which is closed under parallel transport and hence involutive as well) to find that there exist involutive distributions $T^{(1)}, \dots, T^{(k)}$ on M, an open neighbourhood U of x and an isometry $U \cong U_0 \times \dots \times U_k$, where U_0 is an integral manifold of $T^{(j)}$ through x. Moreover, there exists a decomposition $\mathrm{Hol}_x(g) \cong H_0 \times \dots \times H_k$ with the desired properties. □

Remark 1.18. Proposition 1.17 shows that if the holonomy representation of $\mathrm{Hol}^0_x(g)$ is reducible, then $\mathrm{Hol}^0_x(g)$ is a product group. However, it does not follow that the individual factors are themselves holonomy groups (e. g. of the factors U_j in the proof of Proposition 1.17). The reason for this is that in Proposition 1.16, we have merely inclusions $\mathrm{Hol}^0_p(g|_X) \subseteq H_X$ and $\mathrm{Hol}^0_q(g|_Y) \subseteq H_Y$, not equality. This is clear from the local nature of the product decomposition.

In the simply-connected and complete case, de Rham proved a much stronger statement, the *de Rham decomposition theorem*. Here, the proof is omitted due to its complexity.

Theorem 1.19 ([16], Theorem 3.2.7, [29], Théorème II)**.** *Let (M, g) be a complete, simply-connected Riemannian manifold and let $x \in M$. Then, there exist complete, simply-connected Riemannian manifolds $(M_0, g_0), \dots, (M_k, g_k)$, such that $\mathrm{Hol}_{\mathrm{pr}_0(x)}(g_0) = \{1\}$ and the holonomy representation of $\mathrm{Hol}_{\mathrm{pr}_j(x)}(g_j)$ is irreducible for $1 \leq j \leq k$, and the manifold M is isometric to the Riemannian product $(M_0 \times \dots \times M_k, g_1 \times \dots \times g_k)$. Moreover, $\mathrm{Hol}_x(g) \cong \mathrm{Hol}_{\mathrm{pr}_0(x)}(g_0) \times \dots \times \mathrm{Hol}_{\mathrm{pr}_k(x)}(g_k)$.*

1.2. Local and infinitesimal holonomy groups

In this subsection, we return to the general study of holonomy groups on principal bundles and introduce local and infinitesimal holonomy groups. They are subgroups of the restricted holonomy groups, which capture in a precise way the holonomy in the neighbourhood of a specific point. It can be shown that under assumptions on the dimension of these groups depending on the basepoint, these subgroups in fact coincide with the full holonomy group and hence encode already all information about the holonomy of the given manifold. This will be useful for the study of symmetric spaces.

Definition 1.20 ([18], Chapter II, Section 10)**.** Let M be a connected manifold, G be a Lie group and $\pi : P \to M$ be a principal G-bundle. Let Γ be a connection on P. For every connected open subset U of M, let $\Gamma_U := \Gamma|_U$ be the restriction of Γ to $P|_U = \pi^{-1}(U)$. For each $u \in P|_U$, we denote by $\mathrm{Hol}^0_u(\Gamma_U)$ and $P(u, U)$ the restricted holonomy group with reference point u and the holonomy bundle through u of the connection Γ_U, respectively.

We define the *local holonomy group* $\mathrm{Hol}^*_u(\Gamma)$ with reference point u of Γ by

$$\mathrm{Hol}^*_u(\Gamma) := \bigcap_{\substack{U \text{ connected open neigh-} \\ \text{bourhood of } \pi(u)}} \mathrm{Hol}^0_u(\Gamma_U).$$

If $(U_k)_{k\in\mathbb{N}}$ is a sequence of connected open neighbourhoods of $\pi(u)$ with $\overline{U_{k+1}} \subseteq U_k$ for $k \geq 1$ and $\bigcap_{k=0}^{\infty} U_k = \{\pi(u)\}$, then we have $\mathrm{Hol}^0_u(\Gamma_{U_0}) \supseteq \mathrm{Hol}^0_u(\Gamma_{U_1}) \supseteq \dots$. Since for every connected open neighbourhood U of $\pi(u)$ there exists $K \in \mathbb{N}$ with $U_k \subseteq U$ for $k \geq K$, it can be shown that $\mathrm{Hol}^*_u(\Gamma) = \bigcap_{k=0}^{\infty} \mathrm{Hol}^0_u(\Gamma_{U_k})$. Since each group $\mathrm{Hol}^0_u(\Gamma_{U_k})$ is a connected Lie subgroup of G, it follows that $\dim(\mathrm{Hol}^0_u(\Gamma_{U_k}))$ is eventually constant, hence, since they are connected, by the subgroups-subalgebras theorem the holonomy groups $\mathrm{Hol}^0_u(\Gamma_{U_k})$ are all the same for $k \geq K$ if K is sufficiently large. We conclude that $\mathrm{Hol}^*_u(\Gamma) = \mathrm{Hol}^0_u(\Gamma_{U_k})$ for $k \geq K$.

Proposition 1.21 ([18], Chapter II, Proposition 10.1)**.** *In the situation of Definition 1.20, let $\mathfrak{hol}_u(\Gamma)$ and $\mathfrak{hol}^*_u(\Gamma)$ be the Lie algebras of $\mathrm{Hol}^0_u(\Gamma)$ and $\mathrm{Hol}^*_u(\Gamma)$, respectively. Then, $\mathfrak{hol}_u(\Gamma)$ is spanned by all $\mathfrak{hol}^*_v(\Gamma)$, $v \in P(u)$, and $\mathrm{Hol}^0_u(\Gamma)$ is generated by all $\mathrm{Hol}^*_v(\Gamma)$, $v \in P(u)$.*

Proof. By the Ambrose-Singer theorem, we have

$$\mathfrak{hol}_u(\Gamma) = \{\Omega_v(X,Y) \,|\, v \in P(u), X, Y \text{ horizontal vectors at } v\}$$

for Ω the curvature form of Γ. For $v \in P(u)$, X, Y horizontal vectors at v we observe that $\Omega_v(X,Y)$ is contained in the Lie algebra of $\mathrm{Hol}^0_v(\Gamma_V)$ for every connected open neighbourhood V of $\pi(v)$, hence it is contained in $\mathfrak{hol}^*_v(\Gamma)$ and consequently $\mathfrak{hol}_u(\Gamma)$ is spanned by all $\mathfrak{hol}^*_v(\Gamma)$, $v \in P(u)$. This shows the first part of the proposition.

That then $\mathrm{Hol}^0_u(\Gamma)$ is generated by all $\mathrm{Hol}^*_v(\Gamma)$, $v \in P(u)$, is implied by the following claim.

Claim. If the Lie algebra $\mathfrak{g}$ of a connected Lie group G is generated by a family of subspaces $\{\mathfrak{m}_\lambda\}_{\lambda \in L}$, then every element of G can be written as a product $\exp(X_1) \ldots \exp(X_k)$ for $k \in \mathbb{N}$, where each X_i is contained in some $\mathfrak{m}_\lambda$.

Proof of the claim. Let

$$H := \{g \in G \,|\, g = \exp(X_1) \ldots \exp(X_k), k \in \mathbb{N}, \forall_{i \leq k} \exists_{\lambda \in I} : X_i \in \mathfrak{m}_\lambda\}.$$

Then, H is obviously a path-connected subgroup of G. By the theorem [32] of Yamabe, H is a connected Lie subgroup of G. Its Lie algebra contains all $\mathfrak{m}_\lambda$ and therefore is equal to $\mathfrak{g}$. Hence, by the subgroups-subalgebras theorem from Lie group theory, $H = G$, implying the claim. By the above observations, this concludes the proof of the proposition. □

Next we introduce infinitesimal holonomy groups, which we do by first constructing their Lie algebra.

Definition 1.22 ([18], Chapter II, Section 10)**.** Let

$$\mathfrak{m}_0(u) := \{\Omega_u(X,Y) \,|\, X, Y \text{ horizontal vectors at } u\}.$$

We define a *function of the form* (I_k) to be a $\mathfrak{g}$-valued function f of the form

$$f = V_k(\ldots(V_1(\Omega(X,Y)))), \tag{1.1}$$

where $X, Y, V_1, \ldots, V_k$ are arbitrary horizontal vector fields on P. Let $\mathfrak{m}_k(u)$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions of type (I_k). Let $\mathfrak{hol}'_u(\Gamma) := \bigcup_{k \in \mathbb{N}} \mathfrak{m}_k(u)$.

By the following lemma 1.23, $\mathfrak{hol}'_u(\Gamma)$ is a subalgebra of $\mathfrak{hol}^*_u(\Gamma)$. We define the *infinitesimal holonomy group at u*, $\mathrm{Hol}'_u(\Gamma)$, to be the connected Lie subgroup of G generated by $\mathfrak{hol}'_u(\Gamma)$.[12]

[12] This subgroup exists by the subgroups-subalgebras theorem.

Lemma 1.23. *Let* $\mathfrak{hol}'_u(\Gamma) \subseteq \mathfrak{g}$ *be defined as above. Then,* $\mathfrak{hol}'_u(\Gamma)$ *is a subalgebra of* $\mathfrak{hol}^*_u(\Gamma)$*.*

Proof. First we show that $\mathfrak{hol}'_u(\Gamma) \subseteq \mathfrak{hol}^*_u(\Gamma)$, which we do by showing that $\mathfrak{m}_k(u) \subseteq \mathfrak{hol}^*_u(\Gamma)$ for all $k \in \mathbb{N}$. This is done by induction on k: The case $k = 0$ is clear by Theorem 1.7. For the induction step, assume that $\mathfrak{m}_{k-1}(u) \subseteq \mathfrak{hol}^*_u(\Gamma)$ for every $u \in P$. By definition of $\mathfrak{m}_{k+1}(u)$, it is sufficient to show that for every horizontal vector field X and every function f of the form (I_k), we have $X_u(f) \in \mathfrak{hol}^*_u(\Gamma)$. Let $u_t, |t| < \epsilon, \epsilon > 0$ be a horizontal curve through u with $\dot{u}_0 = X$. As observed in Definition 1.20, we have $\mathrm{Hol}^*_u(\Gamma) = \mathrm{Hol}^0(\Gamma_U)$ for some sufficiently small neighbourhood U of u. Since the curve $t \mapsto u_t$ lies in U for $|t|$ small, it follows directly from the definitions that therefore we have $\mathfrak{hol}^*_{u_t}(\Gamma) \subseteq \mathfrak{hol}^*_u(\Gamma)$ for sufficiently small $|t|$. Now, $X_u(f) = \lim_{t\to 0} \frac{1}{t}(f(u_t) - f(u))$, thus $X_u(f) \in \mathfrak{hol}^*_u(\Gamma)$ since $\mathfrak{hol}^*_u(\Gamma)$ is a subvectorspace of $\mathfrak{g}$ and hence closed. Taking the union over all $\mathfrak{m}_k$, we obtain $\mathfrak{hol}'_u(\Gamma) \subseteq \mathfrak{hol}^*_u(\Gamma)$.

In order to prove that $\mathfrak{hol}'_u(\Gamma)$ is a subalgebra of $\mathfrak{hol}^*_u(\Gamma)$, we need the following two claims.

Claim 1. Let f be a $\mathfrak{g}$-valued function of type $\mathrm{ad}\, G$ on P. Then

1. For any vector field X on P, we have $VX_u(f) = -[\omega_u(X), f(u)]$.
2. For any horizontal vector fields X and Y on P, we have
$$V([X,Y]_u)(f) = [\Omega_u(X,Y), f(u)].$$
3. If X and Y are vector fields on P which are invariant by all R_a, $a \in G$, then $\Omega(X,Y)$ and $X(f)$ are functions of type $\mathrm{ad}\, G$.

Proof of claim 1. Ad 1. This follows from a direct computation: Let $A := \omega_u(X) \in \mathfrak{g}$ and let $a_t := \exp(tA)$ be the integral curve of A. Then, by definition of ω and since f is of type $\mathrm{ad}\, G$, we have

$$\begin{aligned}
VX_u(f) = A^*_u(f) &= \lim_{t\to 0} \frac{1}{t}(f(ua_t) - f(u)) \\
&= \lim_{t\to 0} \frac{1}{t}(\mathrm{ad}(a_t^{-1})(f(u)) - f(u)) \\
&= - \lim_{t\to 0} \frac{1}{t}(\mathrm{ad}(a_t)(f(u)) - f(u)) \\
&= - \frac{d}{dt}|_{t=0} \mathrm{ad}(a_t)(f(u)) \\
&= - \mathrm{ad}(A)(f(u)) = -[A, f(u)] = -[\omega_u(X), f(u)].
\end{aligned}$$

Ad 2. This follows from Lemma A.9 and inserting $[X,Y]$ instead of X into part 1.

Ad 3. Since ω is of type $\mathrm{ad}\, G$, also Ω is of type $\mathrm{ad}\, G$. Therefore,

$$\Omega_{ua}(dR_a(X), dR_a(Y)) = \mathrm{ad}(a^{-1})(\Omega_u(X,Y)),$$

which means that $\Omega(X,Y)$ is of type $\mathrm{ad}\, G$, if $X = dR_a(X)$ and $Y = dR_a(Y)$. Furthermore,

$$(X(f))_{ua} = X_{ua}(f) = (dR_a(X_u))(f) = X_u(f \circ R_a) = \mathrm{ad}(a^{-1})(X_u(f)) = \mathrm{ad}(a^{-1})(X(f))_u$$

if f is of type $\mathrm{ad}\, G$ and X is invariant by R_a. This completes the proof of part 3 of claim 1 and hence of the claim.

Let $X_i := \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$, where $(x^1, \ldots, x^n)$ is a chart of M in a neighbourhood U of $x := \pi(u)$. Let $\overline{X}_i$, $i = 1, \ldots, n$, be the horizontal lift of X_i. We define a *function of the form* (II_k) on $\pi^{-1}(U)$ to be a $\mathfrak{g}$-valued function f of the form

$$f = \overline{X}_{j_k}(\ldots(\overline{X}_{j_1}(\Omega(\overline{X}_i, \overline{X}_l)))), \tag{1.2}$$

where $i, l, j_1, \ldots, j_k \in \{1, \ldots, n\}$. Note that it follows in particular from claim 1, part 3 (applied iteratively), that every function of the form (II_k) is of type $\operatorname{ad} G$. We claim:

Claim 2. For each k, $\mathfrak{m}_k(u)$ is spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions f of the form (II_k).

Proof of claim 2. The proof is done by induction on k. The base case $k = 0$ follows directly from the definition. Now, assume that for $k \in \mathbb{N}_{>1}$, the claim holds true for all $s \leq k - 1$. If X is any horizontal vector field in $\pi^{-1}(U)$, we can write it as a linear combination of $\overline{X}_1, \ldots, \overline{X}_n$ with real-valued functions as coefficients. By the properties of the derivative, this means that the restriction of every function f of the form (I_k) to U is a linear combination of functions of the form (II_s) with $s \leq k$, with real valued functions as coefficients. Claim 2 now follows from the induction assumption and the definition of $\mathfrak{m}_k(u)$.

Now, it will be implied by the following claim that $\mathfrak{hol}'_u(\Gamma) \subseteq \mathfrak{hol}^*_u(\Gamma)$ is a Lie subalgebra; hence the proof of this claim concludes the proof of the lemma.

Claim 3. For $k, s \in \mathbb{N}$ we have $[\mathfrak{m}_k(u), \mathfrak{m}_s(u)] \subseteq \mathfrak{m}_{k+s+2}(u)$.

Proof of claim 3. In view of claim 2, it suffices to prove that for every function f of the form (I_s) and every function g of the form (II_k), the function $[f, g](u) = [f(u), g(u)]$ is a linear combination of functions of the form (I_r), $r \leq k + s + 2$, with real valued functions as coefficients. This can be shown by induction on s.

For the base case $s = 0$, we have $f(u) = \Omega_u(X, Y)$, where X, Y are horizontal vector fields on P. Since g is of type $\operatorname{ad} G$ (as it is of the form (II_k)), we have by part 2 of claim 1

$$[\Omega_u(X, Y), g(u)] = V([X, Y])_u(g).$$

On the other hand

$$V([X, Y])_u(g) = [X, Y]_u(g) - H([X, Y])_u(g) = X_u(Y(g)) - Y_u(X(g)) - H([X, Y])_u(g),$$

and as X, Y are horizontal, the first two terms here of the form (I_{k+2}), the last one of the form (I_{k+1}). This proves claim 3 for $s = 0$, k arbitrary.

To show the induction step, let $s \geq 1$ and assume that claim 3 holds true for $s - 1$ and every k. Let f be any function of the form (I_s). Then, we can write $f = X(h)$, where h is a function of the form (I_{s-1}) and X is a horizontal vector field. Let g be an arbitrary function of the form (II_k). Then,

$$[f(u), g(u)] = [X_u(h), g(u)] = X_u([f, g]) - [f(u), X_u(g)]$$

(by the product rule for the Lie bracket). The function $[f, X(g)]$ is by induction assumption a linear combination of functions of the form (I_r), $r \leq k + s + 2$. The function $X([f, g])$ is by induction assumption a linear combination of functions of the form (I_r), $r \leq k + s + 2$. Thus, $[X(f), g]$ is a linear combination of functions of the form (I_r), $r \leq k + s + 2$. This finishes the proof of claim 3 and hence the proof of the lemma. $\square$

The Lie algebra $\mathfrak{hol}'_u(\Gamma)$ appears in some respect "smaller" than the full holonomy algebra $\mathfrak{hol}_u(\Gamma)$, and consequently in general $\mathrm{Hol}'_u(\Gamma)$ is strictly contained in $\mathrm{Hol}^0_u(\Gamma)$. In the following proposition, we will however show that in the case that the dimension of the infinitesimal holonomy group and of the local holonomy group are constant, then all three groups coincide. This will be applied to the case of symmetric spaces, where these dimensions are locally constant by homogeneity.

Proposition 1.24.

1. *If* $u \mapsto \dim \mathrm{Hol}^*(u)$ *is constant on* P*, then* $\mathrm{Hol}^0_u(\Gamma) = \mathrm{Hol}^*_u(\Gamma)$ *for all* $u \in P$.
2. *If* $u \mapsto \dim \mathrm{Hol}'(u)$ *is constant on* P*, then* $\mathrm{Hol}^*_u(\Gamma) = \mathrm{Hol}'_u(\Gamma)$ *for all* $u \in P$.

Proof. Ad 1. Let $u \in P$. As in Definition 1.20 we observe that $\mathrm{Hol}^*_u(\Gamma) = \mathrm{Hol}_u(\Gamma_U)$ for some suitable connected open neighbourhood U of $\pi(u)$. For $v \in U$, it follows then from the definition of $\mathrm{Hol}^*_v(\Gamma)$ that $\mathrm{Hol}^*_v(\Gamma) \subseteq \mathrm{Hol}^*_u(\Gamma)$. Furthermore, both are connected Lie groups and since by assumption their dimensions are the same, we in fact have $\mathrm{Hol}^*_v(\Gamma) = \mathrm{Hol}^*_u(\Gamma)$. Now, let $u \in P$ and $v \in P(u)$; this means in particular that u, v can be joined by a curve. By compactness of the unit interval and the above observation applied a finite number of times, it follows that $\mathrm{Hol}^*_v(\Gamma) = \mathrm{Hol}^*_u(\Gamma)$. Then, $\mathrm{Hol}^0_u(\Gamma) = \mathrm{Hol}^*_u(\Gamma)$ follows from this fact applied for every $v \in P(u)$ and Proposition 1.21.

Ad 2. We start out with a preliminary claim.

Claim 1. Let $v \in P$, $a \in G$. Then, $\mathrm{Hol}'_{va}(\Gamma) = \mathrm{ad}(a^{-1})(\mathrm{Hol}_v(\Gamma))$.

Proof of claim 1. It suffices to show that we have the equality $\mathfrak{m}_k(va) = \mathrm{ad}(a^{-1})(\mathfrak{m}_k(v))$ for all $k \in \mathbb{N}$, where $\mathfrak{m}_k(\cdot)$ are as in the definition of the infinitesimal holonomy group. This follows by induction on k: The base case $k = 0$ follows from the fact that Ω is of type $\mathrm{ad}\, G$. To show the induction step, let $k \in \mathbb{N}_{\geq 1}$ and assume that the assertion holds true for $k - 1$. By part 3 of claim 1 of the proof of Lemma 1.23, every function of the form (II_k) (defined in eq. (1.2)) is of the form $\mathrm{ad}\, G$. Claim 1 now follows from the induction assumption and claim 2 of the proof of Lemma 1.23.

Now, let $u \in P$. We claim:

Claim 2. There exists an open neighbourhood U of $x = \pi(u)$ such that $\mathfrak{hol}'_u(\Gamma) = \mathfrak{hol}'_v(\Gamma)$ for all $v \in P(u, U)$.

Proof of the claim. We may assume that the domain under consideration can be covered by a single chart. We define functions of the form (II_k) as in eq. (1.2) (using this chart). Let $f_1, \ldots, f_s$ be a finite number of functions of the form (II_k), such that $f_1(u), \ldots, f_s(u)$ form a basis of $\mathfrak{hol}'_u(\Gamma)$. Let V be a small neighbourhood of u such that for all $v \in V$, $f_1(v), \ldots, f_s(v)$ are linearly independent. By assumption they then, form a basis of $\mathfrak{hol}'_v(\Gamma)$. Since $f_1, \ldots, f_s$ are of type $\mathrm{ad}\, G$ by claim 1 from the proof of Lemma 1.23, $f_1(va), \ldots, f_s(va)$ form a basis of $\mathfrak{hol}'_{va}(\Gamma) = \mathrm{ad}(a^{-1})(\mathfrak{hol}'_v(\Gamma))$ (the equality follows by claim 1). This means that there exists an open neighbourhood U of x such that $f_1(v), \ldots, f_s(v)$ form a basis for $\mathfrak{hol}'_v(\Gamma)$ at every $v \in \pi^{-1}(U)$.

Now, let $v \in P(u, U)$ and u_t, $0 \leq t \leq 1$, be a horizontal curve in $\pi^{-1}(U)$ such that $u_0 = u, u_1 = v$. We need to show that $\mathfrak{hol}'_u(\Gamma) = \mathfrak{hol}'_v(\Gamma)$; to this end, we may assume that

u_t is differentiable (the piecewise-differentiable case follows from applying the differentiable case a finite number of times). Let $g_i(t) := f_i(u_t)$, $i = 1, \ldots, s$, $t \in [0, 1]$, and $X = \dot{u}_t$. Then, X is horizontal and consequently

$$\frac{dg_i}{dt} = (X(f_i))(u_t) \in \mathfrak{hol}'_{u_t}(\Gamma), \qquad i = 1, \ldots, s.$$

Since $g_1(t), \ldots, g_s(t)$ form a basis of $\mathfrak{hol}'_{u_t}(\Gamma)$, we may write

$$\frac{dg_i}{dt} = \sum_{j=1}^{s} A_{ij}(t) g_j(t),$$

where $A_{ij} : [0, 1] \to \mathbb{R}$, $i, j = 1, \ldots, s$ are continuous functions of t. It follows from the theory of linear ordinary differential equations (see e. g. [28], Korollar 2.5.2, applied to expressions in a suitable chart) that there exists a unique curve $(a_{ij}(t))_{i,j=1,\ldots,s}$ in $\mathrm{GL}(s)$ such that

$$\frac{da_{ij}}{dt} = \sum_{k=1}^{s} A_{ik} a_{kj} \qquad \text{and} \qquad a_{ij}(0) = \delta_{ij}.$$

Let $(b_{ij}(t))$, $t \in [0, 1]$, be the inverse matrix of $(a_{ij}(t))$, such that $\frac{db_{ij}}{dt} = -\sum_{k=1}^{s} b_{ik} A_{kj}$ by a well-known formula for the differential of a matrix inverse. Then,

$$\frac{d}{dt}\left(\sum_{j=1}^{s} b_{ij} g_j\right) = \sum_{j=1}^{s} \frac{db_{ij}}{dt} g_j + \sum_{k=1}^{s} b_{ik} \frac{dg_k}{dt} = \sum_{j=1}^{s} \left(\frac{db_{ij}}{dt} + \sum_{k=1}^{s} b_{ik} A_{kj}\right) g_j = 0.$$

Since $b_{ij}(0) = \delta_{ij}$, this implies that $\sum_{j=1}^{s} b_{ij}(t) g_j(t) = g_i(0)$. In particular, the basis $(g_j(1))_{j=1,\ldots,s}$ of $\mathfrak{hol}'_v(\Gamma) = \mathfrak{hol}'_{u_1}(\Gamma)$ is related via the invertible matrix $b_{ij}(1)$ to the basis $(g_j(0))_{j=1,\ldots,s}$ of $\mathfrak{hol}'_u(\Gamma)$. This means that $\mathfrak{hol}'_u(\Gamma) = \mathfrak{hol}'_v(\Gamma)$, proving claim 2.

In the situation of claim 2, taking U sufficiently small, we may assume that

$$\mathfrak{hol}_u(\Gamma) \supseteq \mathfrak{hol}^*_v(\Gamma) \supseteq \mathfrak{hol}'_v(\Gamma) \supseteq \mathfrak{m}_0(v) \qquad \text{for all } v \in P(u, U).$$

By the Ambrose-Singer theorem, $\mathfrak{hol}^0_u(\Gamma_U)$ is spanned by all $\mathfrak{m}_0(v)$, $v \in P(u, U)$ and a fortiori $\mathfrak{hol}^*_u(\Gamma)$ is spanned by all $\mathfrak{hol}'_v(\Gamma)$, $v \in P(u, U)$. Since by claim 2 we have $\mathfrak{hol}'_v(\Gamma) = \mathfrak{hol}'_u(\Gamma)$ for all $v \in P(u, U)$, one concludes $\mathfrak{hol}^*_u(\Gamma) = \mathfrak{hol}'_u(\Gamma)$ and $\mathrm{Hol}^*_u(\Gamma) = \mathrm{Hol}'_u(\Gamma)$ as both are connected Lie subgroups of G. $\square$

2. Homogeneous and symmetric spaces

Symmetric spaces have been defined in Definition 1.11. We will show in Proposition 2.3 that every symmetric space is in fact a special case of a homogeneous space, defined below. In this section, we will study the holonomy groups of these important classes of Riemannian manifolds.

Definition 2.1. Let (M, g) be a Riemannian manifold.

1. We call (M, g) a *(Riemannian) homogeneous space* if there exists a Lie group G and a transitive smooth isometric G-action on M. We use the convention that G acts from the left. Given any $p \in M$, the stabilizer H_p of p is a closed subgroup of G and M is canonically diffeomorphic to the coset space G/H_p by the map $gH_p \to gp$ for $g \in G$ ([11], Chapter II, Theorem 3.2, Theorem 4.2, Proposition 4.3). (In fact, by the closed-subgroup theorem from Lie group theory, see [11], Chapter II, Theorem 2.3, H_p is actually even a closed Lie subgroup of G.)
2. Let G be a Lie group acting on M. We call the action *effective* if the set of all elements of G which act trivially on M consists of $1 \in G$ only. We call it *almost effective* if the set of all elements of G which act trivially on M is discrete.

Lemma 2.2 ([16], Lemma 3.3.2)**.** *Let (M, g) be a complete Riemannian manifold, let $p \in M$ and suppose that $s_p : M \to M$ is an involutive isometry with isolated fixed point p. Then, $s_p(\exp_p(v)) = \exp_p(-v)$ for all $v \in T_pM$. In particular, restricted to a normal coordinate neighbourhood of p, s_p is nothing else but the geodesic reflection at p.*

Proof. Since $s_p : M \to M$ is an involution, $ds_p : T_pM \to T_pM$ is an involution. Moreover, $\exp_p \circ ds_p = s_p \circ \exp_p$, since s_p is an isometry which maps a geodesic with tangent vector v at p onto a geodesic with tangent vector $ds_p(v)$ at p. Since $\exp_p$ is injective on a neighbourhood of 0, we see from this that ds_p has no fixed point in a neighbourhood of 0 except for 0 itself. As ds_p is a linear involution, the only possibility that remains is $ds_p = -\mathrm{Id}_{T_pM}$. Now, the claimed formula follows directly from $\exp_p \circ ds_p = s_p \circ \exp_p$. □

The geometry of symmetric spaces turns out to be largely determined by the group generated by the symmetries. We have:

Proposition 2.3 ([16], Proposition 3.3.3)**.** *Let (M, g) be a connected, simply-connected Riemannian symmetric space. Then, M is complete. Let G be the group of isometries of (M, g) generated by elements of the form $s_q \circ s_r$ for $q, r \in M$. Then, G is a connected Lie group acting transitively on M. Let $p \in M$ and denote its stabilizer subgroup by $H_p \subseteq G$. Then, H_p is a closed, connected Lie subgroup of G, and M is the homogeneous space G/H_p.*

Proof. By the Hopf-Rinow theorem, M is complete if we can show that all geodesics can be defined on the whole of $\mathbb{R}$. So let $\gamma : (-\epsilon, \epsilon) \to M$ be a geodesic segment in M; it suffices to show that γ can be extended to any larger interval. We may assume that γ is parametrized by arc-length. Then, $(s_{\gamma(0)} \circ \gamma)(t) = \gamma(-t)$ for $t \in (-\epsilon, \epsilon)$ and $(s_{\gamma(\epsilon/4)} \circ \gamma)(t) = \gamma(\epsilon/2 - t)$ for $t \in (-\epsilon/2, \epsilon)$ by Lemma 2.2 and so $(s_{\gamma(\epsilon/4)} \circ s_{\gamma(0)} \circ \gamma)(t) = \gamma(\epsilon/2 + t)$ for $t \in (-\epsilon, \epsilon/2)$. On the other hand, clearly the map $(s_{\gamma(\epsilon/4)} \circ s_{\gamma(0)} \circ \gamma)$ is defined on the interval $(-\epsilon, \epsilon)$ and is a geodesic because $s_{\gamma(0)}$ and $s_{\gamma(\epsilon/4)}$ are isometries. Thus, upon setting $\gamma(t) := (s_{\gamma(\epsilon/4)} \circ s_{\gamma(0)} \circ \gamma)(t - \epsilon/2)$ for $t \in (-\epsilon/2, 3\epsilon/2)$, we can extend the geodesic γ to the interval

$(-\epsilon, 3\epsilon/2)$. Iterating this procedure with different reflection points, it is seen that γ can in fact be extended to the whole of $\mathbb{R}$, hence M is complete.

Now, given any geodesic γ and $t \in \mathbb{R}$, $u := \gamma(0), v := \gamma(t) \in M$, we see that $v = (s_{\gamma(t/2)} \circ s_{\gamma(0)})(u)$. In other words, there exists $g \in G$ with $gu = v$ in the case that u and v can be connected by a geodesic. But by the Hopf-Rinow theorem again, this holds true for any $u, v \in M$ and thus G acts transitively on M.

To show that G is connected, it suffices to show that all the generating elements $s_q \circ s_r$ can be connected to the identity by a continuous path. This is indeed the case, for if γ is a geodesic and $t \in \mathbb{R}$ (w.l.o.g. greater than zero) with $\gamma(0) = q, \gamma(t) = r$, the isometry $s_q \circ s_r$ can be connected via the path given by $s_q \circ s_{\gamma(s)}$, $s \in [0, t]$, with the identity. Hence, G is an arcwise-connected subgroup of $\mathrm{Isom}(M)$, which is a Lie group that acts smoothly on M (by the Myers-Steenrod theorem). So by the theorem [32] of Yamabe, G is a Lie group itself.

Finally, since the action of G on M is smooth, the stabilizer subgroup H_p is a closed Lie subgroup (as observed in Definition 2.1), and since G acts transitively, we have $M \cong G/H_p$. Since M is simply connected, G/H_p is simply-connected; moreover we recall that G is connected. Therefore, the long exact sequence of homotopy groups of a Serre fibration (see [6], Theorem (6.3.2)) implies that H_p is connected. This concludes the proof of the proposition. □

Proposition 2.4 ([16], Proposition 3.3.4, [19], Chapter XI, Proposition 2.2)**.** *Let M, g, G, p, H_p be as in Proposition 2.3, and let $\mathfrak{g}$ be the Lie algebra of G. Then, there is an involutive Lie group isomorphism $\sigma : G \to G$ which descends to a well-defined diffeomorphism of G/H_p. This diffeomorphism coincides with s_p under the identification $G/H_p \cong M$. Moreover, there is a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H_p and $\mathfrak{h}, \mathfrak{m}$ are the eigenspaces of the involution $d\sigma : \mathfrak{g} \to \mathfrak{g}$ with eigenvalues $+1, -1$, respectively. These subspaces satisfy*

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \qquad \text{and} \qquad [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}. \tag{2.1}$$

There is a natural isomorphism $\mathfrak{m} \cong T_pM$.

The adjoint action of H_p on $\mathfrak{g}$ induces a representation of H_p on $\mathfrak{m}$, or equivalently T_pM, and this representation is faithful. We call it the linear isotropy representation $\lambda : H_p \to \mathrm{GL}(\mathfrak{m})$*. Finally, H_p is the identity component of the fixed point set of σ.*

Proof. We define $\sigma : G \to G$ by $\sigma(\alpha) := s_p \circ \alpha \circ s_p$. This is a well-defined automorphism of G; moreover, since $s_p^2 = \mathrm{Id}_M$ we have $\sigma^2 = \mathrm{Id}_G$. It follows directly from the definitions that σ preserves H_p and thus descends to a diffeomorphism $\overline{\sigma} : G/H_p \to G/H_p$. From a direct computation it follows now that $\overline{\sigma} = s_p$ under the identification $G/H_p = M$: Given $\alpha \in G$ we have

$$\overline{\sigma}([\alpha]) = [\sigma(\alpha)] = [s_p \circ \alpha \circ s_p] = (s_p \circ \alpha \circ s_p)H_p = (s_p \circ \alpha)H_p = s_p([\alpha]),$$

since $s_pH_p = H_p$.

Since $\sigma^2 = \mathrm{Id}_G$, we have $(d\sigma)^2 = \mathrm{Id}_{\mathfrak{g}}$, i. e. $d\sigma$ is a linear involution. Since every such one is known to be diagonalizable with eigenvalues ± 1, we obtain a decomposition $\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{m}$ into the $+1$-eigenspace $\mathfrak{h}'$ and the -1-eigenspace $\mathfrak{m}$ of σ.[13]

[13] The statement about diagonalizability follows from linear algebra: Since we have $(d\sigma)^2 - \mathrm{Id} = 0$, the

Now, we show that in fact $\mathfrak{h}' = \mathfrak{h}$. To show the inclusion "$\supseteq$", it is sufficient to show that H_p is preserved by σ (since upon differentiation we then obtain that $d\sigma$ acts as the identity on $\mathfrak{h}$). To show this, at first we note that for $\alpha \in G$ and $q \in M$ we have $\alpha \circ s_q \circ \alpha^{-1} = s_{\alpha(q)}$: Indeed, we observe that both the left-hand side and the right-hand side are involutive isometries which have q as an isolated fixed point. By Lemma 2.2, the left and the right-hand side must coincide on the set of points in M which can be connected to q by a geodesic, which (due to the Hopf-Rinow theorem) is all of M. Now, let $h \in H_p$. Then, by the above observation we have $h \circ s_p \circ h^{-1} = s_p$ and consequently $\sigma(h) = s_p \circ h \circ s_p = h \circ s_p \circ h^{-1} \circ h \circ s_p = h$. Thus, h is preserved by σ and therefore $\mathfrak{h}' \supseteq \mathfrak{h}$.

Now, the identification $M = G/H_p$ gives the claimed isomorphism $T_pM \cong \mathfrak{g}/\mathfrak{h}$. Above, we showed that σ descends to the action s_p on M. Thus, $d\sigma$ descends to an action on $T_pM \cong \mathfrak{g}/\mathfrak{h}$, which, by Lemma 2.2, is given by minus the identity. But then, it is clear that $d\sigma$ cannot act as the identity on any subspace of $\mathfrak{g}$ which is not contained in $\mathfrak{h}$. Therefore, $\mathfrak{h}' = \mathfrak{h}$ and we have obtained the claimed decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ into the eigenspaces of $d\sigma$. The bracket relations eq. (2.1) now follow from the fact that $d\sigma$, being the differential of a smooth map, is a Lie algebra isomorphism.[14] The natural isomorphism $\mathfrak{m} \cong T_pM$ is given as the composition of the isomorphism $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$ (which exists since $\mathfrak{m}$ is a direct complement of $\mathfrak{h}$) and the isomorphism $\mathfrak{g}/\mathfrak{h} \cong T_pM$ from above.

To show that the adjoint action of H_p on $\mathfrak{g}$ induces a representation on $\mathfrak{m}$, it suffices to show that $\mathfrak{m}$ is preserved by the adjoint action of H_p on $\mathfrak{g}$. Indeed: It suffices to prove that for $X \in \mathfrak{m}$ and $h \in H_p$, we have $d\sigma(\mathrm{ad}(h)(X)) = -\mathrm{ad}(H)(X)$. But this follows since σ is a Lie group homomorphism and H_p is preserved by σ: Therefore, $d\sigma(\mathrm{ad}(h)(X)) = \sigma(\mathrm{ad}(h))(d\sigma(X)) = \sigma(\mathrm{ad}(h))(-X) = -\mathrm{ad}(h)(X)$.

This representation of H_p on T_pM can also be described by $h \mapsto (d_ph : T_pM \to T_pM)$ (which is well-defined because every $h \in H_p$ fixes p).[15] As $\exp_p(d_ph(A)) = h(\exp_p(A))$ for all $h \in H_p$ and $A \in T_pM$ (since h acts isometrically), this shows that the representation of H_p on the range of $\exp_p$ is determined by its representation on T_pM. However, since M is connected, this range is the whole of M. In particular, if $d_ph = \mathrm{Id}_{T_pM}$, then $h = \mathrm{Id}_M$ and therefore, the representation of H_p on T_pM respectively on $\mathfrak{m}$ is faithful.

Finally, we already know that H_p is part of the fixed point of σ, that H_p is connected, and that the subspace of $\mathfrak{g}$ fixed by $d\sigma$ is $\mathfrak{h}$. Together these yield that H_p is the identity component of the fixed point set of σ.´ □

The next result proves that in many cases, it suffices to consider irreducible symmetric spaces.

Proposition 2.5 ([19], Chapter X, Theorem 6.6). *Let (M, g) be a simply connected symmetric space. Let $M = M_0 \times \cdots \times M_k$ be its de Rham decomposition as in Theorem 1.19. Then, each factor M_i, $i = 0, \ldots, k$, is a Riemannian symmetric space.*

Proof. Note first that M is complete by Proposition 2.3, hence the application of Theorem

minimal polynomial of $d\sigma$ (in $\mathbb{R}[x]$) divides the polynomial $x^2 - 1 = (x+1)(x-1)$. It can be shown (using induction on the degree of the minimal polynomial) that if a matrix has a minimal polynomial which splits into distinct linear factors, it is diagonalizable. Thus, $d\sigma$ is diagonalizable with eigenvalues ± 1.

[14]For example, for $X, Y \in \mathfrak{h}$, i. e. X, Y are eigenvectors of $d\sigma$ with eigenvalue 1, we have $d\sigma([X, Y]) = [d\sigma(X), d\sigma(Y)] = [X, Y]$, hence also $[X, Y]$ is an eigenvector of $d\sigma$ with eigenvalue 1.

[15]This follows from Lemma A.3 applied to the diffeomorphism $\phi = h : G/H_p \to G/H_p$ and any vector field $X \in \mathfrak{m}$.

1.19 is valid. An argument by induction shows that it suffices to prove the following claim.

Claim. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. If their Riemannian direct product $(M_1 \times M_2, g_1 \oplus g_2)$ is Riemannian symmetric, then both M_1 and M_2 are Riemannian symmetric.

Proof of the claim. Let $q \in M_1, r \in M_2, p := (q, r)$. Let s_p the symmetry of $M_1 \times M_2$ at p. Let $X_1 \in T_q(M_1)$ and $X := (X_1, 0) \in T_p(M_1 \times M_2)$. Then, s_p maps the geodesic $\exp(tX) = (\exp(tX_1), r)$ onto the geodesic $\exp(-tX) = (\exp(-tX_1), r)$. Therefore, s_p sends $M_1 \times \{r\}$ onto itself, inducing a symmetry of M_1 at q. Thus, M_1 is Riemannian symmetric. Analogously, we can prove that M_2 is Riemannian symmetric. □

In the following, the holonomy group of a connection on a Riemannian symmetric space $M \cong G/H_p$ will be shown to equal the Lie group H_p. In order to do so, we will first prove some results on invariant connections on homogeneous spaces which hold true in greater generality. To this end, we need to introduce a definition.

Definition 2.6 ([18], Chapter VI, Proposition 2.1, [19], Chapter X, Section 1)**.** Let M be a manifold.

1. Let X be a vector field on M. We define the *natural lift* $\tilde{X}$ to $L(M)$ of X as follows: Given $x \in M$, let ϕ_t be a local 1-parameter group of local transformations generated by X in a neighbourhood U of x. For each t, ϕ_t induces a transformation $\tilde{\phi}_t$ of $\pi^{-1}(U)$ onto $\pi^{-1}(\phi_t(U))$ by locally mapping $(x, E) \mapsto (\phi_t(x), d\phi_t \circ E)$ (and $\pi : L(M) \to M$ is the projection). Therefore, we obtain a 1-parameter group of local transformations $\tilde{\phi}_t : \pi^{-1}(U) \to L(M)$; we define $\tilde{X}|_{\pi^{-1}(U)}$ to be the corresponding vector field (i. e. $\tilde{\phi}_t$ is the flow of $\tilde{X}|_{\pi^{-1}(U)}$). These locally defined vector fields glue together to a global one on $L(M)$.
2. Let J be a Lie group. Consider a J-structure P on M and assume that ϕ is transformation of M such that the induced transformation of $L(M)$ maps P onto itself. In this case we call ϕ an *automorphism of the J-structure* P. In particular, assume that a Lie group G acts on M. In the case that each such transformation is an automorphism of the J-structure P, we call P a *G-invariant J-structure.*
3. Now, let G be a Lie group with Lie algebra $\mathfrak{g}$ and assume that $M = G/H_p$ is a homogeneous space. Then, given any $X \in \mathfrak{g}$, we obtain a vector field X^+ on M as follows: The one-parameter subgroup $\exp(tX)$ of G is by definition a one-parameter group of transformations of M. This one-parameter group of transformations generates in the usual way a vector field on M, and we set X^+ to be this vector field.

   Under the identification $\mathfrak{m} \cong T_pM$, the assignment $X \mapsto X^+|_p$ is nothing but the projection from $\mathfrak{g}$ onto $\mathfrak{m}$ in the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In particular, it is surjective.

   In the following, we will abbreviate $\tilde{X} := \widetilde{X^+}$ for a vector field $X \in \mathfrak{g}$ and refer to $\tilde{X}$ also as the natural lift of X.

Remark 2.7. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $M = T/H_p$ be a homogeneous space and $X, Y \in \mathfrak{g}$. Since the action of G on M is from the *left*, we have

$$[X^+, Y^+] = -[X, Y]^+$$

(note the sign), see [11], Theorem 3.4. Similarly we obtain an equation for the bracket of the natural lifts of X, Y,

$$[\tilde{X}, \tilde{Y}] = -\widetilde{[X, Y]}.$$

Definition 2.8 ([19], Chapter X, Section 1)**.** Given a homogeneous space $M = G/H_p$, we may define the linear isotropy representation $\lambda : H_p \to \mathrm{GL}(T_pM)$ by the assignment $h \mapsto d_ph$ as observed in the proof of Proposition 2.4. We call the image $\lambda(H_p) \subseteq \mathrm{GL}(T_pM)$ the *linear isotropy group* at p.

In the following, it shall be shown that invariant connections can be described entirely in terms of a certain function Λ, defined below. As a first step, we prove the following proposition. For the proof, we first note an alternative characterization of the linear isotropy representation.

Remark 2.9. Let $M = G/H_p$ be a homogeneous space of dimension n, where $H_p \subseteq G$ is the stabilizer of some $p \in M$. We choose a frame $u_0 \in L(M)$ in the fibre over p as a basepoint and use it to identify T_pM with $\mathbb{R}^n$. Then, we may describe the linear isotropy representation $\lambda : H_p \to \mathrm{GL}(T_pM)$ alternatively as the unique representation such that for all $h \in H_p$, we have $h(u_0) = u_0\lambda(h)$. Indeed: It follows immediately from the definitions that the isomorphism $u_0 : \mathbb{R}^n \to T_pM$ turns the representation $\lambda : H_p \to \mathrm{GL}(T_pM)$ into the representation $\lambda : H_p \to J \subseteq \mathrm{GL}(\mathbb{R}^n), h \mapsto u_0^{-1} \circ d_ph \circ u_0$ (see also the proof of Proposition 2.4).[16] On the other hand, $h(u_0) = d_ph \circ u_0$ for $h \in H_p$ by definition of the action of H_p on $L(M)$.

Furthermore, if we let h_t be a one-parameter subgroup of H_p with $\dot{h}_t|_{t=0} =: X$, from differentiating the equation $h_t(u_0) = u_0\lambda(h_t)$ and the definition of the natural lift $\tilde{X}$ of X we obtain the infinitesimal version of this statement, namely $\tilde{X}_{u_0} = u_0\lambda(X)$.

Proposition 2.10 ([18], Chapter II, Proposition 11.3)**.** *Let $M = G/H_p$ be a homogeneous space, where $H_p \subseteq G$ is the stabilizer of some $p \in M$. Let $\mathfrak{g}, \mathfrak{h}$ denote the Lie algebras of G, H_p, respectively. Let J be a Lie subgroup of $\mathrm{GL}(\dim(M))$ with Lie algebra $\mathfrak{j}$.[17] Let $\pi : P \to M$ be a G-invariant J-structure on M. We choose a frame $u_0 \in P \subseteq L(M)$ with $\pi(u_0) = p$ as a basepoint. Let ω be the connection form of a G-invariant connection on P. We define a map $\Lambda : \mathfrak{g} \to \mathfrak{j}$ by*

$$\Lambda(X) = \omega_{u_0}(\tilde{X}_{u_0}),$$

where $\tilde{X}$ denotes the natural lift to $L(M)$ of a vector field $X \in \mathfrak{g}$.[18] Then, Λ has the following two properties:

1. *$\Lambda(X) = \lambda(X)$ for $X \in \mathfrak{h}$,*
2. *$\Lambda(\mathrm{ad}(h)(X)) = \mathrm{ad}(\lambda(h))(\Lambda(X))$ for $h \in H_p$ and $X \in \mathfrak{g}$.*

Here, λ denotes the linear isotropy representation $H_p \to J$, where we use the frame u_0 to identify J with a subgroup of $\mathrm{GL}(T_pM)$, and also the induced Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{j}$,[19] The assignment $h \mapsto \mathrm{ad}(h)$ denotes the adjoint representation of H_p in $\mathfrak{g}$ and $\lambda(h) \mapsto \mathrm{ad}(\lambda(h))$ denotes the adjoint representation of J in $\mathfrak{j}$.

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[16]For simplicity, we use the same symbol for this representation of H_p on $\mathbb{R}^n$ and the representation on T_pM.

[17]The proposition will be applied to $J = \mathrm{SO}(\dim(M))$.

[18]Note that $\tilde{X}$ is tangent to P (since its flow maps P into itself).

[19]So if the notation were consistent, if the former is denoted λ, the latter should be denoted $d_1\lambda$. This is not the case, in order to avoid cluttered notation.

Proof. Ad 1. Let $X \in \mathfrak{h}$ and let ϕ_t be the one-parameter subgroup of G generated by X. Since $\phi_t \subseteq H_p$, we have $\pi(\phi_t(u_0)) = p$ for all t. Hence, $\phi_t(u_0) = u_0\lambda(\phi_t)$, by definition of the linear isotropy representation $\lambda : H_p \to J$ (see Remark 2.9). In particular, $\frac{d}{dt}\phi_t(u_0)|_{t=0} = \frac{d}{dt}u_0\lambda(\phi_t)|_{t=0} = u_0\lambda(X)$. Then, property 1 follows from the following claim:

Claim. If we write $\phi_t(u_0) = u_0 a_t$ with $a_t \in J$ for all t, then a_t is the 1-parameter subgroup of J generated by $A = \omega_{u_0}(\tilde{X}_{u_0})(=: \Lambda(X))$.

Proof of the claim. We have $\frac{d}{dt}\phi_t(u_0) = u_0\dot{a}_t$ and thus $\omega(\frac{d}{dt}\phi_t(u_0)) = dL_{a_t^{-1}}(\dot{a}_t)$ by definition of ω.[20] On the other hand, $\frac{d}{dt}\phi_t(u_0) = \tilde{X}_{\phi_t(u_0)} = dL_{\phi_t}(\tilde{X}_{u_0})$ and hence $\omega(\frac{d}{dt}\phi_t(u_0)) = \omega_{u_0}(\tilde{X}_{u_0}) = A$, since the connection form ω is G-invariant. Thus, $dL_{a_t^{-1}}(\dot{a}_t) = A$ respectively $\dot{a}_t = dL_{a_t}(A)$, which by definition of the generated 1-parameter subgroup means that a_t is generated by A.

Ad 2. Let $X \in \mathfrak{g}$ and $h \in H_p$. We set $Y := \mathrm{ad}(h)(X)$. Then, by Lemma A.3, Y generates the one-parameter subgroup $h\phi_t h^{-1}$, where ϕ_t denotes again the one-parameter subgroup generated by X. This maps u_0 into $h\phi_t h^{-1}(u_0) = h\phi_t(u_0\lambda(h^{-1})) = h(R_{\lambda(h^{-1})}\phi_t(u_0))$.[21] Differentiating, we obtain $\tilde{Y}_{u_0} = dh(dR_{\lambda(h^{-1})}(\tilde{X}_{u_0}))$. Since the connection form ω is invariant by h and is of type $\mathrm{ad}\, J$, we get

$$\begin{aligned}\Lambda(\mathrm{ad}(h)(X)) =&\omega_{u_0}(\tilde{Y}_{u_0}) = \omega_{u_0}(dh(dR_{\lambda(h^{-1})}(\tilde{X}_{u_0}))) = \omega_{h^{-1}u_0}(dR_{\lambda(h^{-1})}(\tilde{X}_{u_0}))\\ =&\mathrm{ad}(\lambda(h))(\omega_{u_0}(\tilde{X}_{u_0})) = \mathrm{ad}(\lambda(h))(\Lambda(X)).\end{aligned}$$

Thus, Λ satisfies property 2. □

The following result can be seen as a converse to this. The proof given here was taken from [18]; originally the theorem is due to Wang. In the proof, we will need the following observation: By definition, for a homogeneous space $M = G/H_p$ the Lie group G acts transitively on M and consequently G acts fibre-transitively on P, i. e. for all $x, y \in M$ there exists a transformation $g \in G$ of M such that the induced transformation of P maps P_x onto P_y.

Theorem 2.11 ([18], Chapter II, Theorem 11.5). *Let $M, G, p, H_p, J, P, u_0, \mathfrak{g}, \mathfrak{j}$ be as in Proposition 2.10. Then, there is a one-to-one correspondence between the set of G-invariant connections on J and the set of linear maps $\Lambda : \mathfrak{g} \to \mathfrak{j}$ which satisfy the properties 1 and 2 from Proposition 2.10. The correspondence is given by*

$$\Lambda(X) = \omega_{u_0}(\tilde{X}),$$

where $\tilde{X}$ denotes the natural lift of X to P.

Proof. In view of Proposition 2.10, it suffices to show that if $\Lambda : \mathfrak{g} \to \mathfrak{j}$ satisfies the properties 1 and 2 given there, then there exists a unique G-invariant connection form ω on P such that $\Lambda(X) = \omega_{u_0}(\tilde{X})$ for all $X \in \mathfrak{g}$. By fibre transitivity of the G-action and ω being G-invariant and of type $\mathrm{ad}\, J$, to show uniqueness of ω it suffices to prove uniqueness

[20] Indeed: By definition, $\omega_{u_0 a_t}(u_0\dot{a}_t)$ is the value at 1 of the unique left invariant vector field on J such that the induced fundamental vector field on P has value $u_0\dot{a}_t$ at $u_0 a_t$. Such one is given by $dL_{a_t^{-1}}(\dot{a}_t)$.

[21] Note that ϕ_t is a bundle automorphism, in particular J-equivariant, and therefore it commutes with right translation.

of ω_{u_0}. Now, uniqueness of ω_{u_0} follows from the fact that due to fibre-transitivity of the G-action the natural lifts of elements of $\mathfrak{g}$ evaluated at u_0 together with the vertical subspace at u_0 suffice to span the whole tangent space $T_{u_0}P$ (in fact, this holds true for every point of P).[22] On the former, ω_{u_0} is defined uniquely by the claimed correspondence, on the latter it is defined uniquely because it is a connection form. Thus, ω_{u_0} is unique.

We turn to the proof of existence. Let $u \in P$ and $\overline{X} \in T_uP$; we will construct $\omega_u(\overline{X})$ explicitly.

Since G acts fibre-transitively, we can write $u_0 = gua = g \circ R_a(u)$ for some $g \in G$, $a \in J$. We decompose

$$dg \circ dR_a(\overline{X}) = \tilde{X}_{u_0} + A^*_{u_0},$$

where $\tilde{X}$ is the natural lift corresponding to some $X \in \mathfrak{g}$ and A^* is the fundamental vertical vector field induced by some $A \in \mathfrak{j}$ (as observed above, this is always possible). Now, we define

$$\omega(\overline{X}) := \mathrm{ad}(a)(\Lambda(X) + A). \tag{2.2}$$

Claim 1. The one-form ω in eq. (2.2) is well-defined, i. e. $\omega(\overline{X})$ does not depend on the choices of g, a, X, A.

Proof of claim 1. At first, it shall be shown that it does not depend on the choices of X and A. Assume that $\tilde{X}_{u_0} + A^*_{u_0} = \tilde{Y}_{u_0} + B^*_{u_0}$ for some $Y \in \mathfrak{g}$ ($\tilde{Y}$ again denoting its natural lift) and $B \in \mathfrak{j}$. Then, $\tilde{X}_{u_0} - \tilde{Y}_{u_0} = B^*_{u_0} - A^*_{u_0}$. From the definition of the linear isotropy representation $\lambda : \mathfrak{h} \to \mathfrak{j}$, we get $\lambda(X - Y) = B - A$.[23] By condition 1 from Proposition 2.10, we obtain $\lambda(X - Y) = \Lambda(X - Y) = \Lambda(X) - \Lambda(Y)$, thus $\Lambda(X) + A = \Lambda(Y) + B$ and $\omega(\overline{X})$ is independent of the choice of X and A.

Furthermore, $\omega(\overline{X})$ does not depend on the choice of g and a: Assume that $u_0 = gua = g_1ua_1$ for some $g_1 \in G, a_1 \in J$. Then, $g_1g^{-1}u_0 = g_1ua = u_0a_1^{-1}a$. Note that $h := g_1g^{-1} \in H_p$ with $\lambda(h) = a_1^{-1}a$ by Remark 2.9. Thus

$$\begin{aligned} dg_1 \circ dR_{a_1}(\overline{X}) =& dh \circ dg \circ dR_{a\lambda(h^{-1})}(\overline{X}) = dh \circ dR_{\lambda(h^{-1})}(dg \circ dR_a(\overline{X})) \\ =& dh \circ dR_{\lambda(h^{-1})}(\tilde{X}_{u_0}) + dh \circ dR_{\lambda(h^{-1})}(A^*_{u_0}). \end{aligned}$$

We now compute the two terms on the right-hand side. We have $dh \circ dR_{\lambda(h^{-1})}(\tilde{X}_{u_0}) = dh(\tilde{X}_{u_0\lambda(h^{-1})}) = \tilde{Z}_{u_0}$, where $Z := \mathrm{ad}(h)(X)$ (this follows using Lemma A.3 as in the proof of Proposition 2.10, part 2).

Concerning the second term, we calculate

$$dh \circ dR_{\lambda(h^{-1})}(A^*_{u_0}) = dR_{\lambda(h^{-1})}(dh(A^*_{u_0})) = dR_{\lambda(h^{-1})}A^*_{hu_0} = dR_{\lambda(h^{-1})}A^*_{u_0\lambda(h)} = C^*_{u_0},$$

where $C := \mathrm{ad}(\lambda(h))(A)$, by Lemma A.4.[24] Therefore, $dg_1 \circ dR_{a_1}(\overline{X}) = \tilde{Z}_{u_0} + C^*_{u_0}$ and then

$$\begin{aligned} \mathrm{ad}(a_1)(\Lambda(Z) + C) =& \mathrm{ad}(a_1)(\Lambda(\mathrm{ad}(h)(X)) + \mathrm{ad}(\lambda(h))A) \\ =& \mathrm{ad}(a_1)(\mathrm{ad}(\lambda(h))(\lambda(X) + A)) \\ =& \mathrm{ad}(a)(\Lambda(X) + A) \end{aligned}$$

[22] To be precise, we have $T_{u_0}P \cong VT_{u_0}P \oplus HT_{u_0}P$, and similarly to the observation made in Definition 2.6, part 3, we see that the map $X \mapsto \tilde{X}_{u_0}$ has the second summand contained in its image.

[23] Indeed, by definition of the fundamental vector field it suffices to show that $u_0\lambda(X - Y) = B^*_{u_0} - A^*_{u_0}$. But by Remark 2.9 we have $u_0\lambda(X - Y) = \tilde{X}_{u_0} - \tilde{Y}_{u_0} = B^*_{u_0} - A^*_{u_0}$.

[24] Note furthermore that the right translation of vector fields commutes with the action of dh and dh maps fundamental vector fields onto themselves, since, as already used before, the action of h on P is a bundle automorphism. Intuitively, it follows since dh acts from the left.

by property 2 of Λ and since ad is a group action. Therefore, $\omega(\overline{X})$ does also not depend on the choices of g and a, but only on $\overline{X}$.

At this point, it suffices to prove:

Claim 2. The one-form ω defined in eq. (2.2) is a differentiable G-invariant connection form on P.

Proof of claim 2. First we prove that ω, defined as above, is indeed a connection form. Let $\overline{X} \in T_uP$ and $u_0 = gua$ as above. Let $b \in J$. We set

$$\overline{Y} = dR_b(\overline{X}) \in T_vP,$$

where $v = ub$, so that $u_0 = kub(b^{-1}a) = kv(b^{-1}a)$. Then,

$$dg \circ dR_{b^{-1}a}(\overline{Y}) = dg \circ dR_{b^{-1}a} \circ dR_b(\overline{X}) = dg \circ dR_a(\overline{X}) = \tilde{X}_{u_0} + A^*_{u_0}$$

with X, $\tilde{X}$, A^* as before, and hence

$$\begin{aligned}\omega(dR_b(\overline{X})) =& \omega(\overline{Y}) = \mathrm{ad}(b^{-1}a)(\Lambda(X) + A)\\ =& \mathrm{ad}(b^{-1})(\mathrm{ad}(a)(\Lambda(X) + A)) = \mathrm{ad}(b^{-1})(\omega(\overline{X})),\end{aligned}$$

and thus ω is of type ad J. We verify that ω yields the correct result on fundamental vector fields: Let $A \in \mathfrak{g}$, $u \in P$ and $g \in G$, $a \in J$ such that and $u_0 = gua$. Then,

$$dg \circ dR_a(A^*_u) = dR_a \circ dg(A^*_u) = dR_a(A^*_{gu}) = B^*_{u_0},$$

where $B = \mathrm{ad}(a^{-1})(A)$ (by Lemma A.4). Therefore, $\omega(A^*_u) = \mathrm{ad}(a)(B) = A$, as desired. Thus, ω is indeed a connection form.

To show that ω is differentiable, let $u_1 \in P$ and $g_1 \in G, a_1 \in J$ with $u_0 = g_1u_1a_1$. We have a fibre bundle $G \to G/H_p = M$. Let $\sigma : U \to G$ be a local cross section of this bundle defined in a neighbourhood U of $\pi(u_1)$, such that $\sigma(\pi(u_1)) = g_1$. For each $u \in \pi^{-1}(U)$, we can choose $g(u) \in G$ and $a(u) \in J$ such that

$$g(u) = \sigma(\pi(u)) \qquad \text{and} \qquad u_0 = g(u)ua(u).$$

Then, both $u \mapsto g(u)$ and $u \mapsto a(u)$ are differentiable functions. We choose a direct complement $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. For an arbitrary $\overline{X} \in T_u(P)$, we choose X, A such that

$$dg(u) \circ dR_{a(u)}(\overline{X}) = \tilde{X}_{u_0} + A^*_{u_0}$$

and $X \in \mathfrak{m}$. Then, both X and A are uniquely determined and depend differentiably on $\overline{X}$. Thus, $\omega(\overline{X}) = \mathrm{ad}(a)(\Lambda(X) + A)$ depends differentiably on $\overline{X}$ and hence ω is differentiable.

Finally, we show that ω is G-invariant: Let $u \in P$, $\overline{X} \in T_u(P)$ and $g \in G$, $a \in J$ such that $u_0 = gua$. Let $g_1 \in G$. Then, $dg_1(\overline{X}) \in T_{g_1u}(P)$ and $u_0 = gg_1^{-1}(g_1u)a$. Hence,

$$d(gg_1^{-1}) \circ dR_a(dg_1(\overline{X})) = dg \circ dR_a(\overline{X}).$$

From the definition of ω in eq. (2.2) it follows that $\omega(dg_1(\overline{X})) = \omega(\overline{X})$. □

The theorem describes all G-invariant connections on a given G-invariant J-structure on a homogeneous space. Often it is simpler to consider the following special case of homogeneous spaces.

Definition 2.12 ([19], Chapter X, Section 2.1)**.** Let $M = G/H_p$ be a homogeneous space. We say that it is *reductive* if the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H_p and $\mathrm{ad}(H_p)\mathfrak{m} \subseteq \mathfrak{m}$. Given any $X \in \mathfrak{g}$, we denote by $X_\mathfrak{h}$ and $X_\mathfrak{m}$ the components of X corresponding to this splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Corollary 2.13 ([18], Chapter II, Theorem 11.7, [19], Chapter X, Proposition 2.3)**.** *In the situation of Theorem 2.11, assume moreover that G/H_p is reductive, such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m}$ is $\mathrm{ad}(H_p)$-invariant. Then, we have:*

1. *There is a one-to-one correspondence between the set of G-invariant connections in P and the set of linear maps $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{j}$ such that*

$$\Lambda_\mathfrak{m}(\mathrm{ad}(h)(X)) = \mathrm{ad}(\lambda(h))(\Lambda_\mathfrak{m}(X))$$

for $X \in \mathfrak{m}$ and $h \in H_p$. The correspondence is given by setting

$$\Lambda(X) = \begin{cases} \lambda(X), & \text{if } X \in \mathfrak{h}, \\ \Lambda_\mathfrak{m}(X), & \text{if } X \in \mathfrak{m} \end{cases}$$

in Theorem 2.11 and extending this linearly to a map $\mathfrak{g} \to \mathfrak{j}$.

2. *The torsion tensor T of the invariant connection corresponding to $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{j}$ as in the preceding point is given at p by*

$$T(X, Y)_p = \Lambda_\mathfrak{m}(X)(Y) - \Lambda_\mathfrak{m}(Y)(X) - [X, Y]_\mathfrak{m} \tag{2.3}$$

for $X, Y \in \mathfrak{m}$.

3. *The curvature tensor R of the invariant connection corresponding to $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{j}$ as in point 1 is given at p by*

$$R(X, Y)_p = [\Lambda_\mathfrak{m}(X), \Lambda_\mathfrak{m}(Y)] - \Lambda_\mathfrak{m}([X, Y]_\mathfrak{m}) - \lambda([X, Y]_\mathfrak{h}) \tag{2.4}$$

for $X, Y \in \mathfrak{m}$.

Proof. Ad 1. By the preceding theorem, it suffices to show that the assignment $\Lambda_\mathfrak{m} \mapsto \Lambda$ gives a well-defined bijection of maps $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{j}$ satisfying the condition stated and maps $\Lambda : \mathfrak{g} \to \mathfrak{j}$ satisfying properties 1 and 2 from Proposition 2.10. Well-definedness follows directly from the definition. An inverse assignment is given by $\Lambda \mapsto \Lambda_\mathfrak{m} := \Lambda|_\mathfrak{m}$, where $\Lambda|_\mathfrak{m}$ denotes the restriction of Λ to $\mathfrak{m}$. Hence, the assignment is bijective.

Ad 2. By definition of T, we have

$$T(X, Y)_p = u_0(\Theta(\tilde{X}, \tilde{Y}))$$

for $X, Y \in \mathfrak{m}$, where Θ is the torsion form of the given connection and we use the frame u_0 as an isomorphism $\mathbb{R}^n \to T_pM$ as usual. Now, by the first structure equation (eq. (A.7)),

$$\Theta(\tilde{X}, \tilde{Y}) = d\theta(\tilde{X}, \tilde{Y}) - (\omega(\tilde{X})\theta(\tilde{Y}) - \omega(\tilde{Y})\theta(\tilde{X})),$$

where θ is the canonical form on $P \subseteq L(M)$. Since $\Lambda_\mathfrak{m}(X) = \Lambda(X) = \omega_{u_0}(\tilde{X})$ by point 1 and the definition of Λ, and $\theta(\tilde{Y}) = u_0^{-1}(Y)$ (and analogous statements with the roles of X

and Y interchanged), it now suffices to show that $d\theta(\tilde{X},\tilde{Y}) = -\theta(\widetilde{[X,Y]}_{\mathfrak{m}})$. This follows from the fact that by G-invariance of θ we have

$$\begin{aligned} d\theta(\tilde{X},\tilde{Y}) &= \underbrace{\tilde{X}(\theta(\tilde{Y})) - \theta([\tilde{X},\tilde{Y}])}_{=(\mathcal{L}_{\tilde{X}}\theta)(\tilde{Y})=0} - \tilde{Y}(\theta(\tilde{X})) \\ &= \underbrace{-\tilde{Y}(\theta(\tilde{X})) + \theta([\tilde{Y},\tilde{X}])}_{=-(\mathcal{L}_{\tilde{Y}}\theta)(\tilde{X})=0} - \theta([\tilde{Y},\tilde{X}]) \\ &= \theta([\tilde{X},\tilde{Y}]), \end{aligned}$$

and this equals $-\theta(\widetilde{[X,Y]}_{\mathfrak{m}})$ by Remark 2.7 and since $\widetilde{[X,Y]}_{\mathfrak{h}} \in \mathfrak{h} = \ker(d\pi)$.

Ad 3. This follows similarly to part 2, using the second structure equation and G-invariance of ω instead. To be precise: We have

$$R(X,Y)Z = u_0(\Omega(\tilde{X},\tilde{Y})(u_0^{-1}Z))$$

for $X,Y,Z \in \mathfrak{m}$, thus it suffices to prove that $\Omega(\tilde{X}_{u_0},\tilde{Y}_{u_0}) = [\Lambda_{\mathfrak{m}}(X),\Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \lambda([X,Y]_{\mathfrak{h}})$. Now, by eq. (A.6) and G-invariance of ω, we have

$$\begin{aligned} \Omega(\tilde{X},\tilde{Y}) &= d\omega(\tilde{X},\tilde{Y}) + [\omega(\tilde{X}),\omega(\tilde{Y})] \\ &= \underbrace{\tilde{X}(\omega(\tilde{Y})) - \omega([\tilde{X},\tilde{Y}])}_{=(\mathcal{L}_{\tilde{X}}\omega)(\tilde{Y})=0} - \tilde{Y}(\omega(\tilde{X})) + [\omega(\tilde{X}),\omega(\tilde{Y})] \\ &= \underbrace{-\tilde{Y}(\omega(\tilde{X})) + \omega([\tilde{Y},\tilde{X}])}_{=-(\mathcal{L}_{\tilde{Y}}\omega)(\tilde{X})=0} - \omega([\tilde{Y},\tilde{X}]) + [\omega(\tilde{X}),\omega(\tilde{Y})]. \end{aligned}$$

Part 3 now follows analogously to part 2 from the definition of $\Lambda_{\mathfrak{m}}$ and the fact that $[\tilde{Y},\tilde{X}] = -\widetilde{[Y,X]} = \widetilde{[X,Y]}$. □

Remark 2.14. It follows immediately from Proposition 2.4 that every symmetric space is a reductive homogeneous space. Moreover, we then have $[X,Y]_{\mathfrak{h}} = [X,Y]$, $[X,Y]_{\mathfrak{m}} = 0$ for all $X,Y \in \mathfrak{m}$. Therefore, eqs. (2.3) and (2.4) in Corollary 2.13 simplify to

$$T(X,Y)_p = \Lambda_{\mathfrak{m}}(X)(Y) - \Lambda_{\mathfrak{m}}(Y)(X), \tag{2.5}$$

$$R(X,Y)_p = [\Lambda_{\mathfrak{m}}(X),\Lambda_{\mathfrak{m}}(Y)] - \lambda([X,Y]) \tag{2.6}$$

for $X,Y \in \mathfrak{m}$.

Now, we put to use the results obtained previously in the subsection on local and infinitesimal holonomy groups: Proposition 1.24 implies in particular that if $u \mapsto \dim \mathrm{Hol}'(u)$ is constant on P, then $\mathrm{Hol}^0(u) = \mathrm{Hol}^*(u) = \mathrm{Hol}'(u)$. This will be used in the following to prove the announced result that the holonomy group of the simply connected symmetric space $M \cong G/H_p$ coincides with the isotropy subgroup H_p.

Theorem 2.15 ([18], Chapter II, Theorem 11.8, [19], Chapter X, Corollary 4.2)**.**

1. *In the situation of Theorem 2.11, the holonomy algebra $\mathfrak{hol}_{u_0}(\Gamma)$ of the G-invariant connection Γ defined by $\Lambda : \mathfrak{g} \to \mathfrak{j}$ respectively $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{j}$ is given by*

$$\mathfrak{m}_0 + [\Lambda(\mathfrak{g}),\mathfrak{m}_0] + [\Lambda(\mathfrak{g}),[\Lambda(\mathfrak{g}),\mathfrak{m}_0]] + \ldots,$$

where $\mathfrak{m}_0$ *is the subspace of* $\mathfrak{j}$ *spanned by*

$$\{[\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]) \mid X, Y \in \mathfrak{g}\}.$$

2. *Assume moreover the situation of Corollary 2.13, in particular that* G/H_p *is reductive. Then,* $\mathfrak{hol}_{u_0}(\Gamma)$ *is equivalently given by*

$$\mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] + \dots,$$

where $\mathfrak{m}_0$ *is the subspace of* $\mathfrak{j}$ *spanned by*

$$\{[\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \lambda([X,Y]_{\mathfrak{h}}) \mid X, Y \in \mathfrak{m}\}.$$

Proof. Ad 1. Since G acts fibre-transitively on P and maps functions of the form (I_k) (defined as in eq. (1.1)) onto functions of the form (I_k), the function $u \mapsto \dim \mathrm{Hol}'_u(\Gamma)$ is constant on P. Therefore, by Proposition 1.24 (and since M is simply-connected), we get $\mathrm{Hol}_{u_0}(\Gamma) = \mathrm{Hol}^0_{u_0}(\Gamma) = \mathrm{Hol}'_{u_0}(\Gamma)$. Define a series of subspaces $\mathfrak{n}_k$, $k \in \mathbb{N}$, of $\mathfrak{g}$ as follows:

$$\mathfrak{n}_0 = \mathfrak{m}_0, \qquad \mathfrak{n}_1 = \mathfrak{m}_0 + [\Lambda(\mathfrak{g}), \mathfrak{m}_0],$$
$$\mathfrak{n}_2 = \mathfrak{m}_0 + [\Lambda(\mathfrak{g}), \mathfrak{m}_0] + [\Lambda(\mathfrak{g}), [\Lambda(\mathfrak{g}), \mathfrak{m}_0]], \qquad \dots .$$

Since $\mathfrak{hol}_{u_0}(\Gamma) = \mathfrak{hol}'_{u_0}(\Gamma)$ by the above observation, it now is, by definition of $\mathfrak{hol}'_{u_0}(\Gamma)$, sufficient to show that with the notation from Definition 1.22, we have $\mathfrak{m}_k(u_0) = \mathfrak{n}_k$ for all $k \in \mathbb{N}$.

By the proof of Corollary 2.13, $\mathfrak{n}_0$ is spanned by all elements of the form $\Omega_{u_0}(\tilde{X}, \tilde{Y})$, where $X, Y \in \mathfrak{g}$. Since $\Omega_{u_0}(\tilde{X}, \tilde{Y}) = \Omega_{u_0}(H\tilde{X}, H\tilde{Y})$, we get $\mathfrak{m}_0(u_0) = \mathfrak{n}_0$. In order to prove $\mathfrak{n}_k = \mathfrak{m}_k(u_0)$ for $k \geq 1$, we need the following two claims.

Claim 1. If Y is a horizontal vector field on P and $\tilde{X}$ is the natural lift to P of an element $X \in \mathfrak{g}$, then $[\tilde{X}, Y]$ is horizontal.

Proof of claim 1. We have

$$\tilde{X}(\omega(Y)) = (\mathcal{L}_{\tilde{X}}\omega)(Y) + \omega([\tilde{X}, Y]).$$

Now, since Y is horizontal, the left-hand side vanishes, and the first term on the right-hand side also vanishes as ω is G-invariant. Thus, $\omega([\tilde{X}, Y]) = 0$, which proves the claim.

Claim 2. Let $V, W, Y_1, \dots, Y_r$ be arbitrary horizontal vector fields on P and let $\tilde{X}$ be the vector field on P induced by an element $X \in \mathfrak{g}$. Then,

$$\tilde{X}_{u_0}(Y_r(\dots(Y_1(\Omega(V,W))))) \in \mathfrak{m}_r(u_0).$$

Proof of claim 2. We have

$$\begin{aligned}\tilde{X}_{u_0}(Y_r(\dots(Y_1(\Omega(V,W))))) =& [\tilde{X}, Y_r]_{u_0}(Y_{r-1}(\dots(Y_1(\Omega(V,W))))) \\ &+ (Y_r)_{u_0}(\tilde{X}(Y_{r-1}(\dots(Y_1(\Omega(V,W)))))).\end{aligned}$$

Since $[\tilde{X}, Y_r]_{u_0}$ is horizontal by claim 1, the first term here lies in $\mathfrak{m}_r(u_0)$ and it only remains to show that the second one lies in $\mathfrak{m}_r(u_0)$. Iterating this argument and the procedure of commuting $\tilde{X}$ with the Y_i, $i = r-1, \dots, 1$, one sees that, in fact, it only needs to be shown that $(Y_r)_{u_0}(Y_{r-1}(\dots(Y_1(\tilde{X}(\Omega(V,W)))))) \in \mathfrak{m}_r(u_0)$. Now, by the Leibniz rule for the Lie derivative on tensors (noting that insertion of a vector into a tensor can be written as a tensor contraction), we have

$$\tilde{X}(\Omega(V,W)) = (\mathcal{L}_{\tilde{X}}\Omega)(V,W) + \Omega([\tilde{X},V],W) + \Omega(V,[\tilde{X},W]).$$

Since Ω is G-invariant, $\mathcal{L}_{\tilde{X}}\Omega = 0$ and we get

$$\begin{aligned}(Y_r)_{u_0}(Y_{r-1}(\dots(Y_1(\tilde{X}(\Omega(V,W)))))) =&(Y_r)_{u_0}(Y_{r-1}(\dots(Y_1(\Omega([\tilde{X},V],W)))))\\ &+(Y_r)_{u_0}(Y_{r-1}(\dots(Y_1(\Omega(V,[\tilde{X},W]))))).\end{aligned}$$

Since $[\tilde{X},V]$ and $[\tilde{X},W]$ are horizontal by claim 1, the two terms on the right-hand side here belong to $\mathfrak{m}_r(u_0)$. This was what remained to show claim 2.

Let $(x^1,\dots,x^n)$ be a chart in the neighbourhood of p. Let $X_i := \frac{\partial}{\partial x^i}$ and X_i^* be the horizontal lift of X_i, $i=1,\dots,n$. Let $f = \overline{X}_{j_r}(\dots(\overline{X}_{j_1}(\Omega(\overline{X}_i,\overline{X}_l))))$ be a function of the form (II_r) as defined in eq. (1.2). As seen in the proof of Lemma 1.23, Claim 1, we have $-(V\tilde{X})_{u_0}(f) = [\omega_{u_0}(\tilde{X}), f(u_0)]$ and thus

$$(H\tilde{X})_{u_0}(f) = -(V\tilde{X})_{u_0}(f) + \tilde{X}_{u_0}(f) = [\omega_{u_0}(\tilde{X}), f(u_0)] + \tilde{X}_{u_0}(f).$$

Now, $\tilde{X}_{u_0}(f) \in \mathfrak{m}_r(u_0)$ by claim 2 and $\omega_{u_0}(\tilde{X}) = \Lambda(X)$, thus

$$(H\tilde{X})_{u_0}(f) \equiv [\Lambda(X), f(u_0)] \mod \mathfrak{m}_r(u_0).$$

Now, we can prove inductively $\mathfrak{n}_s = \mathfrak{m}_s(u_0)$: The base case $s=0$ has been dealt with above. Assume for the induction step that $\mathfrak{n}_r = \mathfrak{m}_r(u_0)$ for all $r<s$. Since G is fibre-transitive on P, every horizontal vector at u_0 is of the form $(H\tilde{X})_{u_0}$ for some $X \in \mathfrak{g}$. Hence, $\mathfrak{m}_s(u_0)$ is spanned by $\mathfrak{m}_{s-1}(u_0)$ and the set of all $(H\tilde{X})_{u_0}(f)$, where $X \in \mathfrak{g}$ and f is a form of type (II_{s-1}). On the other hand, $\mathfrak{n}_s$ is spanned by $\mathfrak{n}_{s-1} = \mathfrak{m}_{s-1}(u_0)$ and by $[\Lambda(\mathfrak{g}), \mathfrak{n}_{s-1}] = [\Lambda(\mathfrak{g}), \mathfrak{m}_{s-1}(u_0)]$. In other words, $\mathfrak{n}_s$ is spanned by $\mathfrak{m}_{s-1}(u_0)$ and the set of all $[\Lambda(X), f(u_0)]$, where $X \in \mathfrak{g}$ and f is a function of the form (II_{s-1}). Then, $\mathfrak{m}_{s-1}(u_0) = \mathfrak{n}_s(u_0)$ follows from the fact that $(H\tilde{X})_{u_0}(f) \equiv [\Lambda(X), f(u_0)] \mod \mathfrak{m}_{s-1}(u_0)$.

Ad 2. In order to make the statement not a notational catastrophe, let us start out the proof by showing that the subspace named $\mathfrak{m}_0$ in the first part of the theorem coincides with the one named $\mathfrak{m}_0$ in the second part. To this end it suffices to show that for $X, Y \in \mathfrak{g}$ we have the equality $[\Lambda(X),\Lambda(Y)] - \Lambda([X,Y]) = [\Lambda_{\mathfrak{m}}(X_{\mathfrak{m}}), \Lambda_{\mathfrak{m}}(Y_{\mathfrak{m}})] - \Lambda_{\mathfrak{m}}([X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}}) - \lambda([X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{h}})$. This follows from the construction of Λ from $\Lambda_{\mathfrak{m}}$ described in Corollary 2.13 and the computation

$$\begin{aligned}[\Lambda(X),\Lambda(Y)] - \Lambda([X,Y]) =&[\lambda(X_{\mathfrak{h}}) + \Lambda_{\mathfrak{m}}(X_{\mathfrak{m}}), \lambda(Y_{\mathfrak{h}}) + \Lambda_{\mathfrak{m}}(Y_{\mathfrak{m}})]\\ &-\lambda([X_{\mathfrak{h}}, Y_{\mathfrak{h}}] + [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{h}})\\ &-\Lambda_{\mathfrak{m}}([X_{\mathfrak{h}}, Y_{\mathfrak{m}}] + [X_{\mathfrak{m}}, Y_{\mathfrak{h}}] + [X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}})\\ =&[\Lambda_{\mathfrak{m}}(X_{\mathfrak{m}}), \Lambda_{\mathfrak{m}}(Y_{\mathfrak{m}})] - \Lambda_{\mathfrak{m}}([X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}}) - \lambda([X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{h}}).\end{aligned}$$

Here the second equality follows from the fact that λ is a Lie algebra homomorphism and the equality

$$[\lambda(X_{\mathfrak{h}}), \Lambda_{\mathfrak{m}}(Y_{\mathfrak{m}})] = \mathrm{ad}(\lambda(X_{\mathfrak{h}}))(\Lambda_{\mathfrak{m}}(Y_{\mathfrak{m}})) = \Lambda_{\mathfrak{m}}(\mathrm{ad}(X_{\mathfrak{h}})(Y_{\mathfrak{m}})) = \Lambda_{\mathfrak{m}}([X_{\mathfrak{h}}, Y_{\mathfrak{m}}])$$

(and a similar one with the roles of X and Y interchanged).

Now, it shall be shown that

$$\mathfrak{m}_0 + [\Lambda(\mathfrak{g}), \mathfrak{m}_0] + [\Lambda(\mathfrak{g}), [\Lambda(\mathfrak{g}), \mathfrak{m}_0] + \cdots = \mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + \ldots.$$

This equality, together with the first part of the theorem, will show that the second part holds true and thus conclude the proof.

Note first that the inclusion "$\supseteq$" holds true trivially, hence it suffices to show the reverse inclusion. To this end, let $X, Y \in \mathfrak{m}$. We abbreviate $A(X,Y) := [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \lambda([X,Y]_{\mathfrak{h}}) (= [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda([X,Y]))$. It suffices to prove the following claim:

Claim 3. For all $r \geq 1$, $Z_1, \ldots, Z_r \in \mathfrak{g}$ we have $[\Lambda(Z_r), [\ldots[\Lambda(Z_1), A(X,Y)]]] \in \mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + \cdots + [\underbrace{\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\ldots, [\Lambda_{\mathfrak{m}}(\mathfrak{m})}_{r \text{ times}}, \mathfrak{m}_0]]]$.

Proof of claim 3. Consider first the case $r = 1$; let $Z := Z_1$. Then, using the properties of Λ and $\Lambda_{\mathfrak{m}}$ and the Jacobi identity, we obtain

$$\begin{aligned}
[\Lambda(Z), A(X,Y)] =& [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), A(X,Y)] + [\lambda(Z_{\mathfrak{h}}), A(X,Y)] \\
=& [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), A(X,Y)] + [\lambda(Z_{\mathfrak{h}}), [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)]] - [\lambda(Z_{\mathfrak{h}}), \Lambda([X,Y])] \\
=& [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), A(X,Y)] - [\Lambda_{\mathfrak{m}}(Y), [\lambda(Z_{\mathfrak{h}}), \Lambda_{\mathfrak{m}}(X)]] - [\Lambda_{\mathfrak{m}}(X), [\Lambda_{\mathfrak{m}}(Y), \lambda(Z_{\mathfrak{h}})]] \\
&- \Lambda([Z_{\mathfrak{h}}, [X,Y]]) \\
=& [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), A(X,Y)] - [\Lambda_{\mathfrak{m}}(Y), \Lambda_{\mathfrak{m}}([Z_{\mathfrak{h}}, X])] - [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}([Y, Z_{\mathfrak{h}}])] \\
&+ \Lambda([Y, [Z_{\mathfrak{h}}, X]]) + \Lambda([X, [Y, Z_{\mathfrak{h}}]]) \\
=& [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), A(X,Y)] - A(Y, [Z_{\mathfrak{h}}, X]) - A(X, [Y, Z_{\mathfrak{h}}])
\end{aligned}$$

This lies in $\mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]$, proving the claimed inclusion for $r = 1$.

The case $r \geq 2$ can be proven using the case $r = 1$ and an iterative argument, which shall be illustrated here for $r = 2$: For $Z \in \mathfrak{g}$ as before we obtain by the Jacobi identity

$$\begin{aligned}
[\Lambda(Z), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \subseteq& [\lambda(Z_{\mathfrak{h}}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] + [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \\
\subseteq& [[\lambda(Z_{\mathfrak{h}}), \Lambda_{\mathfrak{m}}(\mathfrak{m})], \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\lambda(Z_{\mathfrak{h}}), \mathfrak{m}_0]] + [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \\
\subseteq& [\Lambda_{\mathfrak{m}}([Z_{\mathfrak{h}}, \mathfrak{m}]), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \\
&+ [\Lambda_{\mathfrak{m}}(Z_{\mathfrak{m}}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \\
\subseteq& [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]]
\end{aligned}$$

and thus

$$[\Lambda(\mathfrak{g}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]] \subseteq [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]].$$

By the inclusion already proven for $r = 1$, this shows that

$$[\Lambda(\mathfrak{g}), [\Lambda(\mathfrak{g}), \mathfrak{m}_0]] \subseteq \mathfrak{m}_0 + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0] + [\Lambda_{\mathfrak{m}}(\mathfrak{m}), [\Lambda_{\mathfrak{m}}(\mathfrak{m}), \mathfrak{m}_0]],$$

as desired. Iterating this argument we obtain claim 3, finishing the proof. □

In the following lemma and proposition, we study a special connection obtained from this corollary.

Lemma 2.16 ([19], Chapter X, Proposition 2.4)**.** *In the situation of Corollary 2.13, we define the* canonical connection *on M by letting the function $\Lambda_{\mathfrak{m}}$ identically vanish. Then, we may alternatively characterize it as follows: The canonical connection on P is the unique G-invariant connection in P possessing the following property: If $f_t = \exp(tX)$ is the one-parameter subgroup of G generated by an arbitrary element $X \in \mathfrak{m}$ and $\tilde{f}_t$ is the one-parameter group of transformations of P induced by f_t, then the orbit $\tilde{f}_t(u_0)$ is horizontal.*

Proof. For $X \in \mathfrak{m}$, let $\tilde{X}$ be the vector field corresponding to the induced transformation $\tilde{f}_t$ on P (i. e. the natural lift of X). As seen in Corollary 2.13, any G-invariant connection in P with connection form ω can be described by the function $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{j}$ with $\Lambda_{\mathfrak{m}}(X) = \omega_{u_0}(\tilde{X})$. It follows that $\Lambda_{\mathfrak{m}}(X) = 0$ for the connection with connection form ω if and only if $\tilde{X}$ is horizontal at u_0. Since $\tilde{f}_t$ leaves by definition $\tilde{X}$ invariant and we are considering G-invariant connections, $\tilde{X}$ is horizontal at each $\tilde{f}_t(u_0)$ if it is horizontal at u_0. Hence, $\Lambda_{\mathfrak{m}}(X) = 0$ if and only if $\tilde{X}$ is horizontal at each $\tilde{f}_t(u_0)$, in other words if and only if the integral curve $\tilde{f}_t(u_0)$ of $\tilde{X}$ through u_0 is horizontal. This implies the lemma. □

In view of Theorem 2.15, the holonomy algebra of the canonical connection on a reductive homogeneous space assumes a particularly simple form: $\mathfrak{hol}_{u_0}(\Gamma) = \{-\lambda([X,Y]_{\mathfrak{h}}) \,|\, X, Y \in \mathfrak{m}\}$.

In the following proposition, it shall be shown that for a Riemannian homogeneous space the canonical connection in fact coincides with the Levi-Civita connection on M. Note that by Corollary 2.13, part 2 (cf. also Remark 2.14), the canonical connection is torsion-free, hence by the uniqueness properties of the Levi-Civita connection it suffices to prove that g is parallel with respect to the canonical connection.

Proposition 2.17 ([19], Proposition 2.4, Proposition 2.7)**.** *In the situation of Corollary 2.13, consider the canonical connection corresponding to $\Lambda_{\mathfrak{m}} \equiv 0$. Then, every G-invariant tensor on M is parallel with respect to this connection; in particular, the Riemannian metric g is parallel. Since, as observed before, the canonical connection is moreover torsion-free, the canonical connection must equal the Levi-Civita connection of g.*

Proof. Let T be a G-invariant tensor of rank (r,s) on M, i. e. a G-invariant section of the tensor bundle $\mathrm{T}^{r,s}M \to M$. Let $\pi : P \to M$ denote the projection. By definition of the covariant derivative on an associated vector bundle, it suffices to show that the $\pi^*(T^*M) \otimes \pi^*(\mathrm{T}^{r,s}M)$-component of $d(\pi^*T) \in T^*P \otimes \pi^*(\mathrm{T}^{r,s}) \cong \mathfrak{j}^* \otimes \pi^*(T^*M) \oplus \pi^*(T^*M) \otimes \pi^*(\mathrm{T}^{r,s}M)$ vanishes.[25] In other words, it suffices to show that the derivative of π^*T into the direction of any horizontal vector on P vanishes.

Now, by Lemma 2.16, every horizontal vector at u_0 can be represented by a curve of the form $\tilde{f}_t(u_0)$, where $\tilde{f}_t$ is an automorphism of P leaving π^*T invariant.[26] Thus, the derivative of π^*T in the direction of this vector is zero at u_0, implying $(\nabla T)_p = 0$ as observed before (here ∇ is the connection on $\mathrm{T}^{r,s}M$ induced by the canonical connection on P). Since T and ∇ are G-invariant, this yields $\nabla T \equiv 0$ on M and the proposition is proven. □

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[25]It may be recalled that the isomorphism here is induced by the isomorphism $T_pP = VT_pP \oplus HT_pP \cong \mathfrak{g} \oplus \pi^*(T_{\pi(p)}M)$ coming from the connection on P.

[26]Note that π^*T is G-invariant if T is G-invariant, by definition of the action of G on P.

We now specialize again from the case of a homogeneous space to the case of a symmetric space (recall that we proved Theorem 2.15 mostly in order to study the holonomy groups of symmetric spaces, although it holds true in greater generality). We can assemble the information obtained so far in this section together in the following statements.

Proposition 2.18 ([16], Proposition 3.3.5)**.** *In the situation of Proposition 2.4, the Levi-Civita connection of g coincides with the canonical connection on the reductive homogeneous space $M \cong G/H_p$. The Riemann curvature tensor R_p of g at p is given by*

$$R_p(Y,Z)X = -\mathrm{ad}(Y,Z)(X) = [X,[Y,Z]] \tag{2.7}$$

for all $X, Y, Z \in \mathfrak{m}$ (identifying $\mathfrak{m}$ and T_pM in the natural way). The holonomy algebra $\mathfrak{hol}_p(\Gamma)$ equals

$$\mathfrak{hol}_p(\Gamma) = \{-\lambda([X,Y]) \mid X, Y \in \mathfrak{m}\}. \tag{2.8}$$

Moreover, the Riemann curvature tensor is parallel, $\nabla R \equiv 0$.

Proof. In Proposition 2.17 we have proven that the Levi-Civita connection of g coincides with the canonical connection on $M \cong G/H_p$. The formula for R_p follows then directly from inserting $\Lambda_{\mathfrak{m}} \equiv 0$ into Corollary 2.13, part 3.

Moreover, by Theorem 2.15, part 2 (again inserting $\Lambda_{\mathfrak{m}} \equiv 0$), we have

$$\mathfrak{hol}_p(\Gamma) = \mathfrak{hol}_{u_0}(\Gamma) = \{-\lambda([X,Y]) \mid X, Y \in \mathfrak{m}\}$$

(here u_0 is again a frame on T_pM).

At this point, the only thing which remains to prove is $\nabla R = 0$. This can be seen most easily for the Riemannian curvature $(4,0)$-tensor, whose covariant derivative ∇R is a tensor of rank $(5,0)$. Then, since the derivative of the symmetry s_q, $q \in M$, is an isometry on T_qM, but acts as -1 there, we obtain

$$\begin{aligned}(\nabla R)_q(V,W,X,Y,Z) =& (\nabla R)_q(d_q s_q(V), d_q s_q(W), d_q s_q(X), d_q s_q(Y), d_q s_q(Z)) \\ =& -(\nabla R)_q(V,W,X,Y,Z).\end{aligned}$$

for all $V, W, X, Y, Z \in T_pM$. Thus, $\nabla R \equiv 0$. □

Corollary 2.19 ([7], Theorem 6)**.** *Assume the situation of Proposition 2.18 and that M is* isotropy irreducible*, that is, the linear isotropy representation of H_p is irreducible. Then, there exists a constant $\alpha \in \mathbb{R}$, such that for all orthonormal $X, Y \in \mathfrak{m}$ the sectional curvature of the plane spanned by X and Y is given by*

$$g(R(X,Y)Y,X) = \alpha B([X,Y],[X,Y]),$$

where B is the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{m}$.

Proof. First we need the following claim.

Claim. The Killing form B restricted to $\mathfrak{h} \subseteq \mathfrak{g}$ is negative definite.

Proof of the claim. Note first that H_p is compact, since it is a closed subgroup of the group of all isometries of M which keep p fixed and the latter one is compact (see [11], Chapter IV, Theorem 2.5). Therefore, one may define an $\mathrm{ad}(H_p)$-invariant scalar product on G by taking first any scalar product and then averaging over its H_p-orbits. For any $X \in \mathfrak{h}$, the endomorphism $\mathrm{ad}(X) = \frac{d}{dt}|_{t=0}\mathrm{ad}(\exp(tX))$ is skew-symmetric with respect to this scalar product. Therefore, its square $\mathrm{ad}(X)^2$ is symmetric with nonpositive eigenvalues and thus $B(X,X) = \mathrm{tr}(\mathrm{ad}(X)^2) < 0$ unless $\mathrm{ad}(X) = 0$. However, if $\mathrm{ad}(X) = 0$, then $X \in \mathfrak{z}$, the center of $\mathfrak{g}$. But this case is only possible for $X = 0 \in \mathfrak{h}$, since G acts (by definition as a subgroup of $\mathrm{Isom}(M)$) effectively on M and any $h \in H_p \cap Z(G)$ ($Z(G)$ denoting the center of G) acts trivially on G/H_p.[27] Therefore, $B|_{\mathfrak{h}}$ is negative definite.

By the isomorphism $\mathfrak{m} \cong T_pM$, the Riemannian metric on M induces a scalar product on $\mathfrak{m}$ which is invariant by the adjoint action of H_p on $\mathfrak{m}$ by construction. On the other hand, the restriction of B to $\mathfrak{m}$ is another symmetric bilinear form on $\mathfrak{m}$ which is invariant by the adjoint action of H_p. Since H_p acts irreducibly on $\mathfrak{m}$, by Schur's lemma ([5], Lemma 6.1.7), B must be a multiple of g.[28] That is, $\lambda g(A,B) = B(A,B)$ for all $A, B \in \mathfrak{m}$. Now

$$\lambda g(R(X,Y)Y,X) = B([Y,[X,Y]],X) = -B([X,Y],[Y,X]) = B([X,Y],[X,Y])$$

for all $X, Y \in \mathfrak{m}$, proving the claimed equation if $\lambda \neq 0$ if we set $\alpha := \lambda^{-1}$. If on the other hand $\lambda = 0$, then the above equation reads simply $B([X,Y],[X,Y]) = 0$. But $[X,Y] \in \mathfrak{h}$ by eq. (2.1), and on $\mathfrak{h}$, the Killing form is negative definite by the claim. Hence, in this case $[X,Y] = 0$ for all $X, Y \in \mathfrak{m}$, hence by eq. (2.7), the full curvature tensor vanishes and we obtain the claimed equation if we set $\alpha := 0$. □

Remark 2.20 ([7], Section 6)**.** As observed in the proof of Corollary 2.19, in the situation of this corollary the restriction of the Killing form B to $\mathfrak{h}$ is negative definite and its restriction to $\mathfrak{m}$ is a scalar multiple of the Riemannian metric g on $\mathfrak{m} \cong T_pM$, $B(X,Y) = \lambda g(X,Y)$ for all $X, Y \in \mathfrak{m}$. If $\lambda < 0$, we call M a symmetric space *of compact type*, if $\lambda > 0$ *of noncompact type*, if $\lambda = 0$ *of Euclidean type*. It follows from Corollary 2.19, the third inclusion from eq. (2.1) and the fact that $B|_{\mathfrak{h}}$ is negative definite that a symmetric space of compact type has nonnegative sectional curvature, one of noncompact type has nonpositive scalar curvature and one of Euclidean type is flat. In fact, for isotropy irreducible symmetric spaces it can be shown that if they are of compact or noncompact type, then they have strictly positive or negative sectional curvature ([7], Section 6).

Eq. (2.8) almost shows the desired equality $\mathrm{Hol}_p(g) = \lambda(H_p)$ for a simply-connected symmetric space $M = G/H_p$ as before (without flat factor in its de Rham-decomposition). Indeed, since M is simply connected we have $\mathrm{Hol}_p(g) = \mathrm{Hol}_p^0(g)$, and since H_p is connected (cf. Proposition 2.4) and $\mathrm{Hol}_p^0(g)$ is connected, by the subgroups-subalgebras theorem the only thing which remains to prove is $\mathfrak{hol}_p(\Gamma) = \lambda(\mathfrak{h})$ (where $\mathfrak{h}$ is the Lie algebra of H_p as before). But this will follow from eq. (2.8), provided that we can show that $[\mathfrak{m},\mathfrak{m}] = \mathfrak{h}$, that is, in the third inclusion in eq. (2.1), equality holds true.

[27] The last statement follows since $h(gH_p) = g(hH_p) = gH_p$ for $g \in G$.

[28] To be precise: As in [4], Lemma 7.5.5, we realize that by the Riesz representation theorem there exists a bounded linear operator $T : \mathfrak{m} \to \mathfrak{m}$ such that $g(TX,Y) = B(X,Y)$ for all $X, Y \in \mathfrak{m}$. It suffices to show that T is a multiple of the identity. Hence, by Schur's lemma, it suffices to show that T commutes with every $\mathrm{ad}(h)$, $h \in H_p$. This follows since

$$g(\mathrm{ad}(h)TX,Y) = g(TX,\mathrm{ad}(h)^{-1}Y) = B(X,\mathrm{ad}(h)^{-1}Y) = B(\mathrm{ad}(h)X,Y) = g(T\mathrm{ad}(h)X,Y)$$

for all $X, Y \in \mathfrak{m}$.

Lemma 2.21 (cf. [7], Theorem 7.1, Theorem 7.2)**.** *Assume the situation of Proposition 2.4 and that the de Rham-decomposition of M does not contain a flat factor. Then, we have*

$$[\mathfrak{m},\mathfrak{m}] = \mathfrak{h}.$$

Proof. The inclusion "$\subseteq$" was proven in Proposition 2.4. To show the inclusion "$\supseteq$", we need the following claim.

Claim. The Killing form B on $\mathfrak{g}$ is nondegenerate (that is, G is a *semi-simple* Lie group).

Proof of the claim. By Proposition 2.5 and the fact that the direct sum of nondegenerate bilinear forms is nondegenerate, it suffices to consider the irreducible factors in the de Rham-decomposition of M. So in the following, we may assume that M is irreducible and, by assumption, nonflat. In particular, the holonomy representation on T_pM is irreducible. By Proposition 2.18, the holonomy representation coincides with the isotropy representation (restricted to a subgroup of H_p which we are yet to show that it equals H_p). In particular, the isotropy representation must therefore also be irreducible.

Now, it follows from Remark 2.20 and the fact that M is nonflat that B is definite on the two B-orthogonal subspaces $\mathfrak{h}$ and $\mathfrak{m}$ spanning $\mathfrak{g}$, hence it is nondegenerate.[29]

At this point ,since B is nondegenerate, it suffices to prove that the orthogonal complement of $[\mathfrak{m},\mathfrak{m}]$ in $\mathfrak{h}$ is zero. So assume that $A \in \mathfrak{h}$ is B-perpendicular to $[\mathfrak{m},\mathfrak{m}] = \{[X,Y] \,|\, X,Y \in \mathfrak{m}\}$. Then, $0 = B(A,[X,Y]) = B([A,X],Y)$ for all $X,Y \in \mathfrak{m}$ by basic properties of the Killing form and thus (using the inclusions (2.1), and nondegeneracy) $0 = [A,X] = \mathrm{ad}(A)(X)$ for all $X \in \mathfrak{m}$. Thus, $\exp(tA)$ acts trivially on $\mathfrak{m}$ for all t, but since the adjoint representation of H_p on $\mathfrak{m}$ is faithful, $\exp(tA) = 1$ and $A = 0$. This, by the above observations, proves the lemma. $\square$

As observed before Lemma 2.21, together with Proposition 2.18 this finally proves the following theorem.

Theorem 2.22 ([16], Proposition 3.3.5)**.** *In the situation of Proposition 2.4, assume moreover that the de Rham-decomposition of M does not contain a flat factor. Then, the holonomy group $\mathrm{Hol}_p(g)$ coincides with the isotropy group H_p, with representation the adjoint representation on $\mathfrak{m} \cong T_pM$.*

By Theorem 2.22, the holonomy group of (simply-connected) symmetric spaces are completely understood, yielding many examples of manifolds with special holonomy. (In the case that the de Rham-decomposition of a simply-connected symmetric space does contain a flat factor, the holonomy group may still be computed by applying the de Rham decomposition theorem and noting that the holonomy group of the flat factor is trivial by the Ambrose-Singer theorem.) This great achievement is due to Cartan. On the other hand, for this reason today's research on holonomy groups is mostly concerned with nonsymmetric manifolds, for which different techniques needed and need to be invented.

Remark 2.23. There are many different ways to arrive at the conclusions of Theorem 2.22, and the one chosen here is admittedly not the simplest one. For example, as in [26] Lemma

[29] $\mathfrak{h}$ and $\mathfrak{m}$ are B-orthogonal since for all $X \in \mathfrak{h}$, $Y \in \mathfrak{m}$ we have $B(X,Y) = B(d\sigma(X), d\sigma(Y)) = B(X,-Y) = -B(X,Y)$. Note for the first equality that any automorphism of G leaves B invariant.

32, one may prove the formula (2.7) directly. Then, one may use the parallelism of the Riemannian curvature tensor and the Ambrose-Singer theorem to arrive at the conclusion $\mathrm{Hol}_p(g) = H_p$ more straightforwardly, for this approach see [16], Proposition 3.3.5. The way followed here has been chosen since the results on invariant connections give interesting information also on connections on homogeneous spaces for which ∇R does not necessarily vanish (although the further applications of these results have not been given in detail here).

Remark 2.24. Additionally to the groups G and H_p from Proposition 2.3, there often exist different Lie groups G' and subgroups H' such that $M \cong G'/H'$. Occasionally, therefore a symmetric space is defined as an explicit triple (G', H', σ), where G' is a Lie group, H' is a closed subgroup of G' and σ is an involutive automorphism of G' such that σ acts trivially on H' and H' contains the identity component of the subgroup of G' on which σ acts trivially. Since many theorems of this sections were proven for homogeneous spaces, they are directly applicable also for this case.

This is apparently a contradiction to Theorem 2.22, which computes the holonomy group in terms of H_p (since the holonomy group of a manifold is evidently unique). It is instructive to see what goes wrong for the example of $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$ as a homogeneous space, equipped with the standard metric g (which has Riemannian holonomy $\mathrm{SO}(2n-1)$).

In this case, the group $\mathrm{SU}(n)$ does not contain all isometries of the form $s_q \circ s_r$, $q, r \in S^{2n-1}$ (since these generate the whole group $\mathrm{SO}(2n)$). Consequently, by [19], Chapter XI, Section 4, Lemma 1, there exists no involution $\sigma : \mathrm{SU}(n) \to \mathrm{SU}(n)$ such that $(\mathrm{SU}(n), \mathrm{SU}(n-1), \sigma)$ is a symmetric space in the sense defined above. For this reason, also for any complement $\mathfrak{m}$ to the Lie algebra $\mathfrak{su}(n-1)$ in $\mathfrak{su}(n)$, the third inclusion $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{su}(n-1)$ need not hold true.[30] Consequently, by Corollary 2.13, the canonical connection on $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$ with respect to any $\mathrm{ad}(\mathrm{SU}(n-1))$-invariant decomposition $\mathfrak{su}(n) = \mathfrak{su}(n-1) \oplus \mathfrak{m}$ need not be torsion-free, and hence it need not equal the Levi-Civita connection of g.[31] Thus, Theorem 2.22 is not applicable.

[30] In fact, for the "obvious" direct complement (if one views $\mathfrak{su}(n)$ and $\mathfrak{su}(n-1)$ as explicit skew-Hermitian $n \times n$- and $(n-1) \times (n-1)$-matrices, embedding $\mathbb{C}^{(n-1)\times(n-1)}$ into $\mathbb{C}^{n\times n}$), one may directly verify (e. g. by computing the commutator of two particular generators of $\mathfrak{su}(2)$) that this inclusion does not hold true.

[31] This actually gives an indirect proof of the fact that there can not exist any $\mathrm{ad}(\mathrm{SU}(n-1))$-invariant complement $\mathfrak{m}$ to $\mathfrak{su}(n-1)$ in $\mathfrak{su}(n)$ with $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{su}(n-1)$, precisely because otherwise the canonical connection would equal the Levi-Civita connection of g and Proposition 2.18 would yield that $\mathrm{Hol}(g) \subseteq \mathrm{SU}(n-1)$ up to conjugation.

3. The holonomy bundle and the Ambrose-Singer theorem

In this section, we are going to prove the reduction theorem 1.5 and the Ambrose-Singer theorem 1.7 as announced in Section 1. We start out with the proof of Theorem 1.5, whose statement we recall for convenience:

Theorem 1.5 ([16], Theorem 2.3.6, [18], Chapter II, Theorem 7.1)**.** *Let M be a manifold, G be a Lie group and $P \to M$ be a principal G-bundle. Let Γ be a connection in P. For $p \in P$, let $H := \mathrm{Hol}_p(\Gamma)$. We define an equivalence relation $\sim$ on P by setting $q \sim r$ if and only if q and r can be connected by a piecewise-differentiable horizontal curve in P, and we set $P(p) := \{q \in P \,|\, q \sim p\}$. Then, $P(p)$ is an injectively immersed principal subbundle of P with fibre H (i. e. a reduction of P), and the connection Γ on P restricts to a connection Γ' on $P(p)$.*

For the proof, we will need the following two lemmas.

Lemma 3.1 ([1], Section 2, Lemma 1)**.** *In the situation of Theorem 1.5, let $(x^1, \dots, x^n) = \mathbf{x} : U \to V$, $n := \dim(M)$, be coordinates on a chart domain $U \subset M$ which trivializes P. Let $x_0 \in U$. Assume that V is an open ball in $\mathbb{R}^n$ and $\mathbf{x}(x_0) = 0$. Let $\phi : U \times G \to p^{-1}(U)$ be the inverse of a trivializing map for P. We define a map $\alpha : U \to G$ (depending on $\mathbf{x}$ and ϕ) as follows: For every $x \in U$, let $\rho_x : [0,1] \to M$, $t \mapsto \mathbf{x}^{-1}(t\mathbf{x}(x))$. Let $\sigma_x : [0,1] \to P$ be the horizontal lift of ρ_x going through $p_0 := \phi(x_0, 1)$ and let $\tau_x : [0,1] \to P$, $t \mapsto \phi(\rho_x(t), 1)$. There exists a unique $g \in G$ such that $\sigma_x(1) = \tau_x(1)g$; let $\alpha(x) := g$.*

Then, the map α is differentiable in a neighbourhood $\tilde{U}$ of x_0.

Proof. Let $\mathbf{v} = (v^1, \dots, v^k) : U' \to V'$, $k := \dim(G)$, be coordinates on a chart domain $U' \subset G$ which contains the identity. By definition of a differentiable map between manifolds, it suffices to prove that the maps $v^1 \circ \alpha, \dots v^k \circ \alpha : \tilde{U} \to V'$ are differentiable for some suitable neighbourhood $\tilde{U} \subset U$ of x_0. Note that $(x^1, \dots, x^n, v^1, \dots, v^k) \circ \phi^{-1} : \phi(U \times U') \to V \times V'$ define coordinates in a neighbourhood of p_0. We abbreviate these by $\mathbf{y} = (y^1, \dots, y^{n+k})$.

Let $x \in U$. By G-equivariance of ϕ we have

$$\phi(x, \alpha(x)) = \phi(x, 1)\alpha(x) = \tau_x(1)\alpha(x) = \sigma_x(1).$$

Since $\sigma_x(t) = \sigma_{\rho_x(t)}(1)$ for $t \in [0,1]$ (by definition of ρ_x and σ_x), we conclude that $\phi(\rho_x(t), \alpha(\rho_x(t))) = \sigma_x(t)$. Since $\sigma_x : [0,1] \to P$ is a differentiable curve and $\sigma_x(0) = 1$, we have $t_x := \sup\{t \in [0,1] \,|\, \sigma_x([0,t)) \subset \phi(U \times U')\} > 0$. Then, $v^j(\alpha(\rho_x(t))) = y^{j+n}(\phi(\rho_x(t), \alpha(\rho_x(t)))) = y^{j+n}(\sigma_x(t))$ for $t < t_x$ and $j = 1, \dots, k$.

By definition of the horizontal lift, σ_x is an integral curve of the vector field $x \mapsto H(d\mathbf{y}^{-1}((\mathbf{x}(x), 0)))$ (here we regard $(\mathbf{x}(x), 0)$ as a vector in $T_{\mathbf{y}(y)}\mathbb{R}^{n+k}$ for $y \in \phi(U \times U')$). Thus, by the chain rule

$$\frac{d}{dt}(y^{j+n} \circ \sigma_x) = d_{\sigma_x(t)} y^{j+n}(H(d_{\mathbf{y}(\sigma_x(t))}\mathbf{y}^{-1}((\mathbf{x}(x), 0))))$$

for $j = 1, \dots, k$. Now, the vector field $x \mapsto H(d_{\mathbf{y}(\sigma_x(t))}\mathbf{y}^{-1}((\mathbf{x}(x), 0)))$ is differentiable for all t by definition of a connection; thus the functions $t \mapsto (y^{j+n} \circ \sigma_x)(t)$ solve a system of ordinary differential equations whose initial conditions depend smoothly on x. Hence, the functions $(t,x) \mapsto (y^{j+n} \circ \sigma_x)(t)$ are defined in a neighbourhood of $(0, x_0)$ and differentiable themselves.[32]

Since $\sigma_x(t) = \sigma_{\mathbf{x}^{-1}(t\mathbf{x}(x))}(1)$, it follows that $\sigma_x(1) \in \phi(U \times U')$ in some neighbourhood $\tilde{U}$ of x_0. Then, $x \mapsto (v^j \circ \alpha)(x) = (y^{j+n} \circ \sigma_x)(1)$ is differentiable on $\tilde{U}$, proving the lemma. □

Lemma 3.2 ([1], Section 2, Lemma 2)**.** *In the situation of Theorem 1.5, let $x_0 \in M$, $p_0 \in P$ be any point of P lying over x_0 and $\mathbf{x} = (x^1, \ldots, x^n) : U \to V$ be a chart on M whose domain contains x_0 and with $\mathbf{x}(x_0) = 0$. Then, there exists a trivializing map $\tilde{\psi} : \tilde{\psi}^{-1}(\tilde{U} \times G) \to \tilde{U} \times G$ for P, where $\tilde{U}$ is an open ball with respect to the coordinates $\mathbf{x}$, with inverse $\psi := \tilde{\psi}^{-1}$, such that: We have $\psi(x_0, 1) = p_0$ and for any ray (with respect to the coordinates $\mathbf{x}$) $\rho_x : [0,1] \to \tilde{U}$, $t \mapsto \mathbf{x}^{-1}(t\mathbf{x}(x))$, the curve $t \mapsto \psi(\rho_x(t), 1) \subset P$ is horizontal. Such a trivializing map shall be called* canonical *with respect to the coordinates $\mathbf{x}$.*

Proof. Let $\tilde{\phi} : \tilde{\phi}^{-1}(O \times G) \to O \times G$ be any trivializing map for P with $\tilde{\phi}^{-1}(x_0, 1) = p_0$, where $O \subset U$ is open. Let $\phi := \tilde{\phi}^{-1}$. As in the previous lemma we define $\alpha : \tilde{U} \to G$ (depending on $\mathbf{x}$ and ϕ), where $\tilde{U}$ shall be chosen such that α is differentiable on $\tilde{U}$ and $\tilde{U}$ is a ball with respect to the coordinates $\mathbf{x}$. Let $\psi : \tilde{U} \times G \to P$; $(p, g) \mapsto \phi(p, \alpha(p)g)$, and $\tilde{\psi} := \psi^{-1}$.

Then, ψ is differentiable because ϕ and α are differentiable. Furthermore, for any other trivializing map $\tilde{\chi}$ defined on a subset of $\psi(\tilde{U} \times G)$, we have $(\tilde{\chi} \circ \psi)(x, g) = (\tilde{\chi} \circ \phi)(x, \alpha(x)g)$ for any point $\phi(x, g)$ on the common domain of $\tilde{\psi}$ and $\tilde{\chi}$. Since the map $\tilde{\chi} \circ \phi$ is the composition of the inverse of a trivializing map and another trivializing map, it is G-equivariant and thus $(\tilde{\chi} \circ \phi)(x, g) = (p, L_{h(x)} g)$ for a differentiable function $x \mapsto h(x)$. Thus, $(\tilde{\chi} \circ \psi)(x, g) = (x, L_{h(x)} \alpha(x) g)$. This shows that $\tilde{\chi} \circ \phi$ is G-equivariant, and furthermore $(x, g) \mapsto (\tilde{\psi} \circ \tilde{\chi}^{-1})(x, g) = (x, \alpha(x)^{-1} L_{h(x)^{-1}} g)$ is a smooth function. By definition of the smooth structure on P via trivializing maps, this shows that $\tilde{\psi}$ is also smooth, hence a diffeomorphism, and the G-equivariance of the transition map $\tilde{\chi} \circ \psi$ shows that $\tilde{\psi}$ is indeed itself a trivializing map.

Finally, for every ray $\rho_x : [0,1] \to \tilde{U}$; $t \mapsto \mathbf{x}^{-1}(t\mathbf{x}(x))$, the curve $t \mapsto \psi(\rho_x(t), 1) = \phi(\rho_x(t), \alpha(\rho_x(t))) = R_{\alpha(\rho_x(t))} \phi(\rho_x(t), 1)$ is horizontal by definition of α. □

Now, we can prove Theorem 1.5.

Proof of theorem 1.5. The projection $\pi : P \to M$ restricts to a projection $P(p) \to M$ and for $x \in M$ one sees that the fibre $P(p)_x$ consists of the orbit of any point $q \in P(p)_x$ under the action of H.[33] We recall that H is a Lie subgroup of G, in particular an injectively immersed submanifold.

We define a differentiable structure on $P(p)$ such that $P(p)$ is a submanifold of P and $P(p)$ is a principal fibre bundle over M with fibre H, as follows: For $c_0 \in P(p)$, we set $x_0 := \pi(c_0)$ and choose a trivializing chart neighbourhood $\tilde{U}$ of x_0 and a canonical trivializing map $\tilde{\psi} : \tilde{\psi}^{-1}(\tilde{U} \times G) \to \tilde{U} \times G$ as defined in Lemma 3.2.[34] Then, one can show

[32] Differentiability with respect to t follows directly from the definition, for the x-direction it follows from the smooth dependency of solutions of ordinary differential equations on the right-hand side. See e. g. [22], Theorem D.6.

[33] Clearly H acts fibre-preserving, thus the orbit of q is contained in $P(p)_x$. Conversely, any $q' \in P(p)_x$ can be connected to q by a horizontal path and thus by definition lies in the H-orbit of q.

[34] By restricting U in Lemma 3.2, we may assume that $U = \tilde{U}$, simplifying the notation.

that $\tilde{\psi}^{-1}$ maps $\tilde{U} \times H$ bijectively onto $\tilde{\psi}^{-1}(\tilde{U} \times G) \cap P(p)$ and we define a differentiable structure on $P(p)$ by requiring that these maps for all $c_0, \tilde{\psi}$ are diffeomorphisms.[35] Then, by construction $P(p)$ becomes an injectively immersed principal subbundle of P with fibre H.

It remains to prove the statement concerning the connections. By definition, for any $q \in P(p)$ the horizontal subspace HT_qM (with respect to Γ) at q is contained in $T_qP(p)$. Now, if V_q denotes the subspace of T_qP tangent to the fibre of P and V'_q denotes the subspace of $T_qP(p)$ tangent to the fibre of $P(p)$, one sees that from $T_qP = V_q \oplus HT_qM$ (which holds true by definition of a connection) it follows that $T_qP(p) = V'_q \oplus HT_qM$. Therefore, the restriction of Γ to $P(p)$ is a connection on $P(p)$ and the theorem is proven. $\square$

We continue with the proof of Theorem 1.7. First we recall the precise statement.

Theorem 1.7 ([1], Theorem 2)**.** *Let M be a smooth manifold, let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi : P \to M$ be a principal G-bundle. Let $x \in M$ and $p \in P_x$. Let ω be the connection 1-form of a C^1-connection Γ on E; let Ω denote the corresponding curvature form. Let $P(p)$ be the holonomy bundle of this connection through p. Let $L(p)$ be the subalgebra of $\mathfrak{g}$ generated by all $\Omega(X, Y)$, where X and Y run through all pairs of tangent vectors to P at all points of $P(p)$. Then, $L(p) = \mathfrak{hol}_p(\Gamma)$, that is, $\mathrm{Hol}^0_p(\Gamma)$ is generated by $L(p)$.*

The next lemma is of computational nature; it will be needed in the proof of the Ambrose-Singer theorem to show that a certain set of vector fields is an involutive distribution on P.

Lemma 3.3 ([1], Section 2, Lemma 3)**.** *In the situation of Theorem 1.7, let $Q \in \mathfrak{g}$, Q^* be the fundamental vector field corresponding to Q, $c, d \in P(p)$ and assume that $Q^*_c \in T_cP(p)$. Then, $Q^*_d \in T_dP(p)$.*

Proof. At first, the lemma shall be proven in two special cases. Assume for the first one that c and d lie in the same fibre of P, such that $d = cg$ for some $g \in G$. Since $c, d \in P(p)$, in fact $g \in \mathrm{Hol}_c(\omega)$. Now, let $\phi : \phi^{-1}(U \times G) \to U \times G$ be any trivializing map of P with $\phi(c) = (\pi(c), 1)$ and consider the map $\psi := \phi^{-1} \circ L_g \circ \phi : \phi^{-1}(U \times G) \to \phi^{-1}(U \times G)$ (note that deliberately here a *left* translation has been chosen). Then, $\psi(c) = d$ and, by definition of a fundamental vector field, $d_c\psi(Q_c) = Q_d$. Moreover, ψ maps $\phi^{-1}(U \times G) \cap P(p)$ diffeomorphically onto itself (since $P(p)$ is a principal subbundle of P with fiber $\mathrm{Hol}_{\pi(c)}(\Gamma)$). Thus, $Q^*_q \in T_dP(p)$.

For the second special case, assume that there exists a coordinate system $\mathbf{x} = (x^1, \dots, x^n)$ of M with center at $x := \pi(c)$ and a canonical trivializing map as in Lemma 3.2 (for this chart $\mathbf{x}$) $\tilde{\psi} : \tilde{\psi}^{-1}(P \times G) \to P \times G$, such that $d = \tilde{\psi}^{-1}(x', 1)$ for some $x' \in P$. Let $\mathbf{v} = (v^1, \dots, v^k)$ be coordinates of G adapted to $\mathrm{Hol}^0_c(\Gamma)$, that is, for given (suitable) constants $c^{l+1}, \dots, c^k$, $l := \dim(\mathrm{Hol}_c(\Gamma))$, the slice $\{h \in G \,|\, v^{l+1}(h) = c^{l+1}, \dots, v^k = c^k\}$ is contained in a left coset of $\mathrm{Hol}^0_c(\Gamma)$, and the slice $\{h \in G \,|\, v^{l+1}(h) = 0, \dots, v^k = 0\}$ is contained in $\mathrm{Hol}^0_c(\Gamma)$ itself. Together with the chart $\mathbf{x}$ this yields coordinates $\mathbf{y} = (y^1, \dots, y^{n+l}) := (x^1, \dots, x^n, v^1, \dots, v^l)$ on $P(p)$ whose domain is a $P(p)$-neighbourhood of c. Since Q^* is a fundamental vector field and $Q^*_c \in T_cP(p)$, we have $Q^*_c = \sum_{i=n+1}^{n+l} a_i \frac{\partial}{\partial y^i}|_c$ for some $a_i \in \mathbb{R}$, and then, $Q^*_d = \sum_{i=n+1}^{n+l} a_i \frac{\partial}{\partial y^i}|_d$ which lies in $T_dP(p)$ by its very definition.

[35] For the proof that this indeed turns $P(p)$ into a differentiable manifold, see the discussion in [18], Chapter II, Section 7, Lemma 1.

The case of arbitrary $c, d \in P(p)$ follows by applying the result for the two special cases subsequently a finite number of times. □

Now, we are able to prove the Ambrose-Singer theorem.

Proof of Theorem 1.7. In order to prove Theorem 1.7, we will describe the holonomy bundle $P(p)$ as the integral manifold of a certain distribution, i. e. we will describe the tangent spaces of $P(p)$ at each point, in terms of the curvature form on P. Afterwards, noting that the vertical part of the tangent spaces of $P(p)$ can be related to the Lie algebra of $\mathrm{Hol}^0_p(\Gamma)$, we can conclude the proof.

This idea shall now be carried out in more detail. Let $n := \dim(M)$, $k := \dim(G)$. Recall that $L(p)$ denotes the Lie subalgebra of $\mathfrak{g}$ generated by all $\Omega(X, Y)$, where X and Y run through the tangent vectors to P at all points of $P(p)$. Let q be the homomorphism from the Lie algebra $\mathfrak{g}$ into the Lie algebra of vertical vector fields on P given by assigning to an $A \in \mathfrak{g}$ the corresponding fundamental vector field A^*. Now, we describe a Lie algebra $\widetilde{L}(p) \subseteq \mathrm{im}(q)$ which will later be shown to be equal to $q(L(p))$.

Let K, K' be any horizontal vector fields defined on an open subset $U \subseteq P$ and given any $d \in U \cap P(p)$ consider the vector field $Q^* \in \mathrm{im}(q)$ generated by $V([K, K']|_d)$, where $V(\cdot)$ denotes the vertical part of a vector.[36] Then, Q^* is a vector field depending on K, K' and d. Let $\widetilde{R}$ denote the subset of $\mathrm{im}(q)$ given by all vector fields which can be described in this way for certain K, K' and d. Let $\widetilde{L}(p)$ denote the Lie algebra generated by $\widetilde{R}$.

Now, for every $c \in P(p)$, we define a vector subspace $\Delta_c \subset T_cP$ as follows: Let HT_cP denote the horizontal vectors at c. Let Δ_c be the subspace of T_cP containing HT_cP and the set of all vectors Q^*_c, where Q^* is a vector field belonging to $\widetilde{L}(p)$. We claim:

Claim 1. Δ is an involutive differentiable distribution and $\Delta_c = T_cP(p)$ for all $c \in P(p)$, so $P(p)$ is an integral manifold of Δ.

Proof of claim 1. At first, we show the inclusion "$\Delta_c \subseteq T_cP(p)$". Note that trivially $HT_cP \subseteq T_cP(p)$, thus it only remains to show that $T_cP(p)$ contains the set of all vectors Q^*_c with $Q^* \in \widetilde{L}(p)$. To this end, let $d \in P(p)$ and let K_1, K_2 be horizontal vector fields defined on a neighbourhood U of d in P. Let K'_1, K'_2 denote their restrictions to $P(p)$. Note that $K'_1|_{U\cap P(p)} = K_1|_{U\cap P(p)}$ is in fact a vector field in $TP(p)$, because K_1 is a horizontal vector field and $TP(p)$ contains all horizontal vectors at all points of $P(p)$; similarly for K'_2. Then, $[K_1, K_2]|_d = [K'_1, K'_2]|_d \in T_dP(p)$ as well (by definition $TP(p)$ is closed under Lie brackets). Since $T_dP(p)$ contains also $H([K_1, K_2]|_d) \in HT_cP$ and $[K_1, K_2]|_d = H([K_1, K_2]|_d) + V([K_1, K_2]|_d)$, $T_dP(p)$ must also contain $V([K_1, K_2]|_d)$ because it is a vector subspace of T_dP. Therefore, and by Lemma 3.3, $T_cP(p)$ contains also Q^*_c, if $Q^* \in \mathrm{im}(q)$ is such that there exists $d \in P(p)$ and horizontal K_1, K_2 defined in a neighbourhood of d with $V([K_1, K_2]|_d) = Q^*_d$.

In other words, all vectors of the form Q^*_c, where $Q^* \in \widetilde{R}$, belong to $T_cP(p)$. To prove that it even contains all vectors of the form Q^*_c, where $Q^* \in \widetilde{L}(p)$, by definition of the generated Lie algebra it suffices to show that if $Q^*_1, Q^*_2 \in \mathrm{im}(q)$ with $Q^*_1|_c, Q^*_2|_c \in T_cP(p)$, then also $[Q^*_1, Q^*_2]|_c \in T_cP(p)$. This holds true since if Q'_1, Q'_2 are the restrictions of Q^*_1, Q^*_2 to $P(p)$, then by Lemma 3.3 we know that $Q'_1|_d, Q'_2|_d \in T_dP(p)$ for all $d \in P(p)$, hence $[Q^*_1, Q^*_2]|_c = [Q'_1, Q'_2]|_c \in T_cP(p)$ (again, because $TP(p)$ is closed under Lie brackets). Together with $HT_cP \subseteq T_cP(p)$ this proves $\Delta_c \subseteq T_cP(p)$ for all $c \in P(p)$.

[36] In other words, $Q^* = \omega([K, K']|_d)^*$.

Now, we prove that Δ is differentiable (hence a distribution) and involutive (hence integrable by the Frobenius theorem). To this end, let $(Q_i^*)_{1\leq i\leq r}$ be a basis of $\widetilde{L}(p)$, let $c \in P(p)$ and $(K_i)_{1\leq i\leq n}$ be any differentiable basis for HTP in a neighbourhood of c. Let Q_i' and K_i' denote the restrictions of the Q_i^* and K_i to $P(p)$. Then, $(Q_1', \ldots, Q_r', K_1', \ldots, K_n')$ is a differentiable basis for Δ in a neighbourhood of c, hence Δ is differentiable. Furthermore, to prove that Δ is involutive it suffices to show that the Lie bracket of any pair of elements from such a basis belong to Δ (see e. g. [22], Lemma 19.4). This is proven by noting that $[Q_i', Q_j']$ $(1 \leq i,j \leq r)$ belongs to Δ because $\widetilde{L}(p)$ is closed under Lie brackets, $[K_i', K_j']$ $(1 \leq i,j \leq n)$ belongs to Δ by definition of Δ_d for $d \in P(p)$ and $[Q_i', K_j']$ $(1 \leq i \leq r, 1 \leq j \leq n)$ belongs to Δ because by Lemma A.10 it is horizontal.

Now, it shall be shown that for all $d \in P(p)$ we have $\Delta_d \supseteq T_dP(p)$, hence $\Delta_d = T_dP(p)$. To this end, let D be the integral manifold of Δ through p (which exists by the Frobenius theorem). We prove $\Delta_d \supseteq T_dP(p)$ by showing that $D \supseteq P(p)$, that is, for all $c \in P(p)$ we have $c \in D$. To this end, let ρ be a horizontal curve in $P(p)$ from p to c (which exists by definition of $P(p)$); assume at first that ρ is differentiable. We extend the tangent vectors to ρ in a neighbourhood of ρ to a horizontal vector field K. Note that $K|_D$ is a section of TD, since $\Delta_d \supseteq HT_dP$ for all $d \in D$. Thus, $K|_D$ must have an integral curve through p which, by the uniqueness of integral curves, must coincide with ρ. Therefore, $\rho \subseteq D$ and consequently $c \in D$. If ρ is merely a piecewise differentiable horizontal curve in $P(p)$ from p to c, we can still conclude $c \in D$ by applying the same reasoning a finite number of times. This concludes the proof of claim 1.

Now, we attempt to describe the elements of Δ in terms of the curvature of P. We claim:

Claim 2. $q(L(p)) = \widetilde{L}(p)$.

Proof of claim 2. By Lemma A.9 (and noting that $\omega(V[K_1,K_2]|_c) = \omega([K_1,K_2]|_c)$ for all $c \in P$ and vector fields K_1, K_2 by definition), we have

$$\begin{aligned} R :=& \{\Omega_c(X,Y) \,|\, c \in P(p), X, Y \in T_cP(p)\} \\ =& \left\{ \omega(V[K_1,K_2]|_c) \,\middle|\, c \in P(p), \begin{array}{c} K_1, K_2 \text{ horizontal vector fields} \\ \text{defined on a } P\text{-neighbourhood of } c \end{array} \right\}. \end{aligned}$$

To show that $q(L(p)) = \widetilde{L}(p)$, it suffices to show that $q(R) = \widetilde{R}$ (since q is a Lie algebra homomorphism). This is indeed the case: Assume first that $X \in R$. Accordingly, there exist $c \in P(p)$, K_1, K_2 horizontal vector fields defined on a P-neighbourhood of c such that $X = \omega(V[K_1,K_2]|_c)$. Thus, $q(X) = X^* = \omega(V[K_1,K_2]|_c)^* \in \widetilde{R}$. On the other hand, given any $X^* \in \widetilde{R}$, by definition of $\widetilde{R}$ and since $\omega(\overline{Y}_c)_c = \overline{Y}_c$ for all vertical vector fields $\overline{Y}$ on P, there exist $c \in P(p)$ and K_1, K_2 horizontal vector fields defined on a P-neighbourhood of c such that $X_c^* = V[K_1,K_2]|_c$. Thus, $Y := \omega(V[K_1,K_2]|_c) \in R$ satisfies $q(Y)_c = Y_c^* = X_c^*$, and since both X and Y are fundamental vector fields which must already be the same if they coincide at one point, we obtain $Y^* = X^*$. Therefore, $X^* \in q(R)$.

Now, by definition of the differentiable structure on $P(p)$ (see also the construction of coordinate systems on $P(p)$ in Lemma 3.3), given any inverse of a trivializing map $\phi : U \times G \to P$ with $\phi(x,1) = p$ (recall that $x = \pi(p)$), we know that $d_1\phi(x,\cdot)$ takes $S := \{A_1 \,|\, A \in \mathfrak{hol}_p(\Gamma)\}$ onto the vertical part of $T_pP(p)$. Hence, by claim 1, it takes S onto the vertical part of Δ_p, which by definition is the set of all Q_p where $Q \in \widetilde{L}(p)$. Since

$\widetilde{L}(p) = q(L(p))$ by claim 2 and q is an isomorphism, this shows that $L(p) = \mathfrak{hol}_p(\Gamma)$.[37] This concludes the proof of the theorem. $\square$

[37] Here we use again that two fundamental vector fields are the same if they coincide at one point, and that $d_1\phi(x, \cdot)(A) = A_p^*$ for $A \in \mathfrak{g}$.

4. Quaternion-Kähler manifolds

As an example of manifolds with special holonomy, in this section we study quaternion-Kähler manifolds in more detail. These are Riemannian manifolds with holonomy contained in $\mathrm{Sp}(m)\mathrm{Sp}(1)$. They are automatically Einstein manifolds, and it is conjectured that they need be symmetric in the case that they have positive scalar curvature. The archetypical example is $\mathbb{HP}^m$. Such manifolds have associated *twistor spaces*, which fibre over the manifold under consideration with fibres $\mathbb{CP}^1$. The twistor spaces can be shown to be themselves complex manifolds, which allows for the study of quaternion-Kähler manifolds through the means of complex algebraic geometry.

We recall some definitions from Theorem 1.12 and introduce a few new ones.

Definition 4.1 ([3], Chapter 2, Chapter 14)**.**

1. An *almost complex structure* on a smooth manifold is a smooth field of automorphisms J of the tangent bundle TM satisfying $J_x^2 = -\mathrm{Id}_{T_xM}$ for all $x \in M$. We call it a *complex structure* if its *Nijenhuis tensor*
$$N_J(X,Y) = \frac{1}{4}([X,Y] + J[JX,Y] + J[X,JY] - [JX,JY])$$
(for all $x \in M, X, Y \in T_xM$, extended to local vector fields) vanishes.

   The Newlander-Nirenberg theorem ([25], Theorem 1.1) states (in a weak formulation) that if a smooth manifold M admits a smooth complex structure, then it is in fact a complex manifold, that is, there exists a covering of M by open subsets which are homeomorphic to open subsets of $\mathbb{C}^{\dim(M)/2}$, such that for any two such homeomorphisms $\phi : U \to V$, $\tilde{\phi} : \tilde{U} \to \tilde{V}$, the map $\phi \circ \tilde{\phi}^{-1}$ is holomorphic on $\tilde{\phi}(U \cap \tilde{U})$.

2. A Riemannian metric g on a manifold with almost complex structure J is said to be *Hermitian* with respect to J if $g(JX, JY) = g(X,Y)$ for all $X, Y \in T_xM$, $x \in M$.

3. A *Kähler manifold* is a manifold M with a complex structure J and a Riemannian metric g such that g is Hermitian with respect to J and J is parallel with respect to the Levi-Civita connection of g. Equivalently, it satisfies $\mathrm{Hol}_x(M) \subseteq \mathrm{U}(\dim(M)/2)$ for all $x \in M$ (up to conjugation).

4. We denote by $\mathbb{H}$ the *quaternions*, i. e. the four-dimensional $\mathbb{R}$-algebra spanned by the elements $1, i, j, k$ satisfying the relations $i^2 = j^2 = k^2 = ijk = -1$. $\mathbb{H}$ is a skew-field. In the following, we will regard $\mathbb{H}^m$, $m \in \mathbb{N}$, as a *right* module over $\mathbb{H}$; then linear maps $\mathbb{H}^m \to \mathbb{H}^n$ can be described by *left* multiplication with matrices from $A \in \mathbb{H}^{n \times m}$.

5. Given $m \in \mathbb{N}$, $\mathrm{Sp}(m) := \{A \in \mathrm{GL}(m, \mathbb{H}) \,|\, \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{H}^m\} \subset \mathrm{GL}(m, \mathbb{H})$ shall denote the *symplectic group*, where $\langle x, y \rangle := \sum_{i=1}^m \overline{x}_i y_i$ is the standard Hermitian form on $\mathbb{H}^n$. Identifying $\mathbb{H}^m$ with $\mathbb{R}^{4m}$ (as will be done throughout this section), we may regard $\mathrm{Sp}(m)$ as a subgroup of $\mathrm{SO}(4m)$.

   We denote by $\mathrm{Sp}(m)\mathrm{Sp}(1)$ the group of $\mathbb{R}$-linear automorphisms of $\mathbb{R}^{4m}$ that is generated by the automorphisms given by left multiplication with matrices fom $\mathrm{Sp}(m)$ and right (i. e. scalar) multiplication with elements from $\mathrm{Sp}(1) \subset \mathbb{H}$. In other words, an element from $\mathrm{Sp}(m)\mathrm{Sp}(1)$ corresponds to an $\mathbb{R}$-linear map $\mathbb{R}^{4m} \to \mathbb{R}^{4m}$, $v \mapsto Avq^*$, where $A \in \mathrm{Sp}(m)$ and $q \in \mathrm{Sp}(1)$ ([30]).[38] Alternatively, we may describe $\mathrm{Sp}(m)\mathrm{Sp}(1)$

[38] It is not $v \mapsto Avq$ since this would not give a well-defined (left) action on $\mathbb{R}^{4m}$.

(up to isomorphism) as the quotient of $\mathrm{Sp}(m) \times \mathrm{Sp}(1)$ by its center $\{\pm 1\}$.

Since $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$ (which may not really be regarded as a "special" holonomy group), we will assume throughout the section that the (real) dimension of the vector spaces and manifolds under consideration is at least 8.

6. An *almost quaternionic manifold* is a $4m$-dimensional manifold which admits a $\mathrm{GL}(m,\mathbb{H})\mathrm{Sp}(1)$-structure (where we define $\mathrm{GL}(m,\mathbb{H})\mathrm{Sp}(1)$ analogously to $\mathrm{Sp}(m)\mathrm{Sp}(1)$ before). A *quaternionic manifold* is an almost quaternionic manifold, such that the corresponding $\mathrm{GL}(m,\mathbb{H})\mathrm{Sp}(1)$-bundle admits a torsion-free connection. A *quaternion-Kähler manifold* is a Riemannian manifold (M, g) of real dimension $4m$, such that $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)\mathrm{Sp}(1)$ (up to conjugation). Every quaternion-Kähler manifold is quaternionic ([3], Proposition 14.63).
7. An *Einstein manifold* is a Riemannian manifold (M, g) such that its Ricci tensor is proportional to the metric tensor,

$$\mathrm{ric} = \lambda g$$

for some $\lambda \in \mathbb{R}$.[39]

As a preliminary cautionary remark, it shall be noted that quaternion-Kähler manifolds are not Kähler, since $\mathrm{Sp}(m)\mathrm{Sp}(1)$ is in general not a subgroup of $\mathrm{U}(2m)$.

Proposition 4.2 ([3], Proposition 14.36). *A Riemannian manifold (M, g), $\dim(M) = 4m$, is quaternion-Kähler if and only if there exists a covering of M by open sets $(U_i)_{i\in S}$ and for each $i \in S$ two almost complex structures I_i and J_i such that for all $i \in S$, we have:*

1. *g is Hermitian for I_i and J_i on U_i.*
2. *$I_iJ_i = -J_iI_i$.*
3. *The Levi-Civita derivatives of I_i and J_i are linear combinations of I_i, J_i and $K_i := I_iJ_i$.*
4. *For any $j \in S$ and $x \in U_i \cap U_j$, the vector space of endomorphisms of T_xM generated by I_i, J_i and K_i is the same as the vector space generated by I_j, J_j and K_j.*

Proof. In order to prove the proposition, we will characterize $\mathrm{Sp}(m)\mathrm{Sp}(1)$ using the following claim.

Claim. Let $R \subset \mathrm{End}(\mathbb{R}^{4m})$ be the three-dimensional subvector space generated by the $\mathbb{R}$-linear maps that are given by right multiplication with i, j and k. As subgroups of $\mathrm{SO}(4m)$, we have

$$\mathrm{Sp}(m)\mathrm{Sp}(1) = \{A \in \mathrm{SO}(4m) \mid \forall B \in R : ABA^{-1} \in R\}.$$

In other words, $\mathrm{Sp}(m)\mathrm{Sp}(1)$ is the subgroup of $\mathrm{SO}(4m)$ such that its natural representation on $\mathrm{End}(\mathbb{R}^{4m})$ maps R onto itself.

Proof of the claim. To show the inclusion "$\subseteq$", note that right multiplication with any quaternion commutes with left multiplication by a matrix from $\mathrm{Sp}(m)$; since furthermore for all purely imaginary $q, r \in \mathbb{H}$ also $q^{-1}rq$ is purely imaginary, we obtain that also the $\mathrm{Sp}(1)$-action on $\mathbb{R}^{4m}$ leaves R invariant. Hence, $\mathrm{Sp}(m)\mathrm{Sp}(1)$ leaves R invariant.

[39] Einstein manifolds are very important in differential geometry and physics, and for the lack of space it shall not be dwelled upon this subject further. See e. g. [3] for a thorough treatment.

That the inclusion cannot be strict follows by noting that by a result of Gray ([8], Proposition 1), $\mathrm{Sp}(m)\mathrm{Sp}(1)$ is a maximal Lie subgroup of $\mathrm{SO}(4m)$ (for $m \geq 2$). Hence, equality must hold true.

Using the claim, now the proposition can be proven: Note that properties 1-4 may be reformulated as follows: There exists a subbundle E of $\mathrm{End}(TM)$ such that:

1. For all $S \in E$ with $g(S,S) = 4m$, g is Hermitian for S.
2. For all $x \in M$ and $A, B \in E_x$, we have $A \circ B + B \circ A = -\frac{2}{4m} g(A,B)\mathrm{Id}_{T_xM}$.
3. E is preserved by parallel transport.

The "if"-direction now follows from applying the holonomy principle, Theorem 1.8, to any $x \in M$ and the projection $\pi_x : \mathrm{End}(T_xM) \to E_x$. To this end, we choose an orthonormal frame on M to identify T_xM with $\mathbb{R}^{4m}$, such that under the induced isomorphism $\mathrm{End}(T_xM) \cong \mathrm{End}(\mathbb{R}^{4m})$, E_x gets mapped onto R.[40]

The "only if"-direction follows again from applying the holonomy principle to some $x \in M$ and the projection $\pi_x : \mathrm{End}(\mathbb{R}^{4m}) \to R$. Upon identifying T_xM with $\mathbb{R}^{4m}$ by an orthonormal frame as before, this yields the existence of a three-dimensional subbundle E of $\mathrm{End}(TM)$ which is preserved by parallel transport (i. e. having property 3 stated above). Moreover, since parallel transport is an isometry, one sees that for all $S \in E$ with $g(S,S) = 4m$, g is Hermitian for S, i. e. E satisfies property 1 stated above. Finally, since the tensor $A \circ B + B \circ A + \frac{2}{4m} g(A,B)\mathrm{Id}_{T_xM}$ is parallel for any parallel $A, B \in E$, we can deduce property 2 of E (by parallel-transporting the equation that this tensor vanishes). $\square$

Remark 4.3.

1. As in the Kähler case, one observes that the three local almost complex structures from Proposition 4.2 are automatically skew-symmetric with respect to g.
2. If one can find on the Riemannian manifold (M,g) even two globally defined complex structures I, J such that g is Hermitian for all of them and they are anticommuting and *parallel*, then (M, g, I, J, K) with $K := IJ$ is called a *hyperkähler* manifold and one even has $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ (up to conjugation).

Next it shall be shown that quaternion-Kähler manifolds need necessarily be Einstein manifolds. Thus, the knowledge of the holonomy group tells us in this case about restrictions on the curvature of the metric. By the Ambrose-Singer theorem, such restrictions are expected, but the knowledge that a manifold is Einstein gives more information on the metric.

Proposition 4.4 ([3], Theorem 14.39)**.** *Let (M,g) be a quaternion-Kähler manifold of dimension $4m \geq 8$. Then, M is Einstein,* $\mathrm{ric} = \lambda g$ *for some* $\lambda \in \mathbb{R}$.

Proof. As the Einstein condition may be verified locally, we may choose an open set $U \subseteq M$ where local almost complex structures I, J and K with the properties 1-4 from proposition 4.2 exist. Let X, Y, Z be vector fields on U and let $p \in U$. First we claim:

[40] This is not the case for a general orthonormal frame. But since the holonomy group is defined only up to conjugation and hence only lies up to conjugation in $\mathrm{Sp}(m)\mathrm{Sp}(1)$, we may choose a suitable one.

Claim 1. There exist local 1-forms p, q, r on U such that

$$\begin{aligned}\nabla_X I =& r(X)J - q(X)K,\\ \nabla_X J =& - r(X)I + p(X)K,\\ \nabla_X K =& q(X)I - p(X)J\end{aligned}$$

for all $X \in T_xM$, $x \in U$.

Proof of claim 1. Since the bundle spanned by I, J, K is preserved by covariant differentiation, there exist local 1-forms a_1, r_1, q_1 on U such that $\nabla_X I = a_1(X)I + r_1(X)J - q_1(X)K$ for all $X \in T_xM$, $x \in U$. However, a_1 must vanish, since

$$0 = \frac{1}{2}X(g(I,I)) = g(\nabla_X I, I) = a_1(X)g(I,I) = 4ma_1(X)$$

(again, for all X). Similarly we see that there exist local 1-forms r_2, p_2, q_3, p_3 such that for all $X \in T_xM$, $x \in U$ we have $\nabla_X J = -r_2(X)I + p_2(X)K$, $\nabla_X K = q_3(X)I - p_3(X)J$. However, these forms are not independent, since e. g.

$$\begin{aligned}0 =& X(g(I,J)) = g(\nabla_X I, J) + g(I, \nabla_X J)\\ =& r_1(X)g(J,J) - r_2(X)g(I,I) = 4m(r_1(X) - r_2(X))\end{aligned}$$

for all $X \in T_xM$, $x \in U$. Thus, $r_1 = r_2$. Similarly we see that $p_2 = p_3$ and $q_1 = q_3$. Setting $p := p_1 = p_2$, $q := q_1 = q_2$, $r := r_1 = r_2$, we obtain the claim.

Now, we note that

$$\begin{aligned}[R(X,Y), I]Z =& \nabla_X(\nabla_Y(IZ)) - \nabla_Y(\nabla_X(IZ)) - I\nabla_X(\nabla_Y Z)\\ &+ I\nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}(IZ) + I\nabla_{[X,Y]}Z\\ =& (\nabla_X I)\nabla_Y Z + \nabla_X((\nabla_Y I)Z) - (\nabla_Y I)\nabla_X Z\\ &- \nabla_Y((\nabla_X I)Z) - (\nabla_{[X,Y]}I)Z\\ =& (\nabla_X(\nabla_Y I))Z - (\nabla_Y(\nabla_X I))Z - (\nabla_{[X,Y]}I)Z,\end{aligned}$$

hence $[R(X,Y), I] = \nabla_X(\nabla_Y I) - \nabla_Y(\nabla_X I) - \nabla_{[X,Y]}I$.[41] Analogous formulas hold true for J and K. Using this we may prove that

$$\begin{aligned}[R(X,Y), I] =& \nabla_X(r(Y)J - q(Y)K) - \nabla_Y(r(X)J - q(X)K)\\ &- r([X,Y])J + q([X,Y])K\\ =& (X(r(Y)) - Y(r(X)) - r([X,Y]))J\\ &+ (-X(q(Y)) + Y(q(X)) + q([X,Y]))K\\ &+ (p(X)q(Y) - q(X)p(Y))J + (p(X)r(Y) - r(X)p(Y))K\\ =& \gamma(X,Y)J - \beta(X,Y)K,\end{aligned} \tag{4.1}$$

where $\beta := dq + r \wedge p$, $\gamma := dr + p \wedge q$. Similarly we see that

$$[R(X,Y), J] = - \gamma(X,Y)I + \alpha(X,Y)K, \tag{4.2}$$

$$[R(X,Y), K] = \beta(X,Y)I - \alpha(X,Y)J, \tag{4.3}$$

[41] This is nothing else but $\nabla^2_{X,Y}I - \nabla^2_{Y,X}I = R^{\mathrm{End}(TM)}(X,Y)I$.

where β, γ are as before and $\alpha := dp + q \wedge r$.

These forms shall now be related to the Ricci curvature.

Claim 2. If $m \geq 2$, then for the forms α, β, γ as above, we have

$$\alpha(X,Y) = -\frac{1}{m+2}\mathrm{ric}(IX,Y), \tag{4.4}$$

$$\beta(X,Y) = -\frac{1}{m+2}\mathrm{ric}(JX,Y), \tag{4.5}$$

$$\gamma(X,Y) = -\frac{1}{m+2}\mathrm{ric}(KX,Y). \tag{4.6}$$

Proof of claim 2. Applying the formula (4.3) to a vector field Z and applying $g(\cdot, JZ)$ to it, we obtain (using that g is Hermitian for J and K, $g(JA, JB) = g(KA, KB) = g(A, B)$)

$$g(R(X,Y)KZ, JZ) - g(KR(X,Y)Z, JZ) = \beta(X,Y)g(IZ, JZ) - \alpha(X,Y)g(JZ, JZ)$$

and thus

$$\alpha(X,Y)|Z|^2 = g(R(X,Y)JZ, KZ) + g(R(X,Y)Z, IZ). \tag{4.7}$$

Now, we choose a local orthonormal frame $(e_i)_{1 \leq i \leq 4m}$ such that for all e_i, up to a sign also Ie_i, Je_i and Ke_i are contained in the basis. Inserting these e_i into the above equation and summing i from 1 to $4m$, we see that the two terms on the right-hand side contribute equally to the sum. We obtain

$$4m\alpha(X,Y) = 2\sum_{i=1}^{4m} g(R(X,Y)e_i, Ie_i). \tag{4.8}$$

Dividing by two and applying the first Bianchi identity yields together with the symmetries of the curvature tensor

$$\begin{aligned} 2m\alpha(X,Y) &= \sum_{i=1}^{4m}(-g(R(e_i,X)Y, Ie_i) - g(R(Y,e_i)X, Ie_i)) \\ &= \sum_{i=1}^{4m}(g(R(X,e_i)Y, Ie_i) - g(R(X,Ie_i)Y, e_i). \end{aligned}$$

Again, since i is summed from 1 to $4m$ it can be shown that the two terms contribute actually the same to the sum. Thus

$$m\alpha(X,Y) = \sum_{i=1}^{4m} g(R(X,e_i)Y, Ie_i) = -\sum_{i=1}^{4m} g(IR(X,e_i)Y, e_i).$$

Using eq. (4.1), we obtain

$$m\alpha(X,Y) = -\sum_{i=1}^{4m}(g(R(X,e_i)IY, e_i) - \gamma(X,e_i)g(JY,e_i) + \beta(X,e_i)g(KY,e_i)),$$

thus $m\alpha(X,Y) = \mathrm{ric}(X,IY) + \gamma(X,JY) - \beta(X,KY)$ since the e_i form an orthonormal frame. Applying this formula to IY instead of Y gives

$$m\alpha(X,IY) = -\mathrm{ric}(X,Y) - \gamma(X,KY) - \beta(X,JY)$$

respectively

$$m\alpha(X, IY) + \beta(X, JY) + \gamma(X, KY) = -\mathrm{ric}(X, Y).$$

Repeating this computation starting instead with eq. (4.1) and KZ and with eq. (4.2) and IZ yields $\alpha(X, IY) + m\beta(X, JY) + \gamma(X, KY) = -\mathrm{ric}(X, Y)$, $\alpha(X, IY) + \beta(X, JY) + m\gamma(X, KY) = -\mathrm{ric}(X, Y)$. Solving this system of linear equations for $\alpha(X, IY)$, $\beta(X, JY)$, $\gamma(X, KY)$ we obtain

$$\alpha(X, IY) = \beta(X, JY) = \gamma(X, KY) = -\frac{1}{m+2}\mathrm{ric}(X, Y). \tag{4.9}$$

Now, since α is alternating and ric is symmetric, this implies $\alpha(X, Y) = -\alpha(Y, X) = \alpha(Y, I(IX)) = -\frac{1}{m+2}\mathrm{ric}(Y, IX) = -\frac{1}{m+2}\mathrm{ric}(IX, Y)$. Similarly one obtains the other two claimed formulas.

Now, using eqs. (4.7) (twice) and (4.9) (four times) and $J^2 = -1$ we compute for arbitrary X and Z:

$$\begin{aligned}
&\underbrace{g(R(X, IX)JZ, KZ) + g(R(X, IX)Z, IZ)}_{=\alpha(X,IX)|Z|^2} \\
&+ \underbrace{g(R(JX, KX)JZ, KZ) + g(R(JX, KX)Z, IZ)}_{=\alpha(JX,I(JX))|Z|^2} \\
=& -\frac{1}{m+2}\mathrm{ric}(X, X)|Z|^2 - \frac{1}{m+2}\mathrm{ric}(JX, JX)|Z|^2 \\
=& -\frac{1}{m+2}\mathrm{ric}(X, X)|Z|^2 - \beta(JX, X)|Z|^2 \\
=& -\frac{2}{m+2}\mathrm{ric}(X, X)|Z|^2.
\end{aligned}$$

We observe, using the symmetries of the curvature tensor, that the second and the third term on the left-hand side of the above equation are symmetric under the exchange of X and Z and the second and fourth are swapped under this exchange. Hence, the right-hand side must be symmetric under the exchange of X and Z and therefore $\mathrm{ric}(X, X)|Z|^2 = \mathrm{ric}(Z, Z)|X|^2$. In other words, $\frac{\mathrm{ric}(X,X)}{|X|^2}$ (for $X \neq 0$) does not depend on X, i. e. $\mathrm{ric}(X, X) = \lambda|X|^2$ for some $\lambda \in \mathbb{R}$. By polarization, we obtain $\mathrm{ric} = \lambda g$, thus (M, g) is an Einstein manifold. $\square$

The next proposition describes in more detail the geometry of quaternion-Kähler manifolds depending on their scalar curvature. The proof here originates from [3]; originally it is due to Berger.

Proposition 4.5 ([3], Theorem 14.45, cf. also [14], Lemma 2.1)**.**

1. *Let $m \geq 2$ and (M, g) be a $4m$-dimensional quaternion-Kähler manifold. Then, the scalar curvature of M is zero if and only if M is locally hyperkähler.*
2. *Let $m \geq 2$ and (M, g) be a $4m$-dimensional quaternion-Kähler manifold with nonvanishing scalar curvature. Then, M is not de Rham-reducible, even locally.*

Proof. Ad 1. By Proposition 4.4, M is an Einstein manifold, thus the scalar curvature of M is zero if and only if M is Ricci-flat. The “if”-direction follows from the following observation: If M is locally hyperkähler, then, the locally defined almost complex structures I, J, K satisfying the quaternionic relations as in Proposition 4.2 are in fact parallel (almost by definition, see Remark 4.3) In particular, they commute with the Riemann tensor, hence the 2-forms α, β, γ in the proof of Proposition 4.4 vanish. Since $\mathrm{ric}(X, Y) = (m+2)\alpha(IX, Y)$ for $X, Y \in TM$ by the claim from the proof of Proposition 4.4, M is Ricci-flat.

To show the "only if"-direction, we shall prove that if M is Ricci-flat, then one can choose locally three almost complex structures I, J, K which satisfy the quaternionic relations and which are parallel. To this end, it suffices to note that if $\mathrm{ric} \equiv 0$, then, due to eqs. (4.1)-(4.6), the curvature tensor R^E of the bundle E from the proof of Proposition 4.2 vanishes.[42] Now, using this fact, by applying successive parallel transports along coordinate lines of some chart one can construct three local sections I, J, K of E which have norm $\sqrt{4m}$ and are parallel. (The proof is analogous to the one that a Riemannian manifold with vanishing curvature tensor is locally isometric to Euclidean space, see [21], Theorem 7.10.) Moreover, we may choose them to satisfy the quaternionic relations at one point and such that g is Hermitian with respect to them at one point. Since these properties are preserved by parallel transport, as in the proof of Proposition 4.2 one recognizes that I, J, K satisfy then the quaternionic relations and g is Hermitian for them on their whole domain of definition. This proves the "only if"-direction of part 1.

Ad 2. We start out with a preliminary claim.

Claim. Let (M, g) be a $4m$-dimensional quaternion-Kähler manifold with nonvanishing scalar curvature and $m \geq 2$. Then, $\mathrm{Sp}(1) \subseteq \mathrm{Hol}(g)$ if we regard $\mathrm{Hol}(g)$ and $\mathrm{Sp}(1)$ as subgroups of $\mathrm{SO}(4m)$ by using an orthonormal frame at some basepoint, where $\mathrm{Sp}(1)$ is here the subgroup of unit imaginary quaternions acting on $\mathbb{R}^{4m}$ by right quaternion multiplication as before.

Proof of the claim. By eqs. (4.8) and (4.4) and the symmetries of the curvature tensor, we have with the notation of Proposition 4.4

$$\sum_{i=1}^{4m} g(R(e_i, Ie_i)X, Y) = 2m\alpha(X, Y) = -\frac{2m}{m+2}\mathrm{ric}(IX, Y) = -\frac{2m\lambda}{m+2} g(IX, Y)$$

for $X, Y \in TM$, where λ is the Einstein constant of M. Since $\lambda \neq 0$ by assumption, by nondegeneracy of g we obtain $I = \frac{m+2}{2m\lambda} \sum_{i=1}^{4m} R(e_i, Ie_i)$. Now, by Proposition 1.10 we have $R(e_i, Ie_i) \in \mathfrak{hol}(g)$ for all i, hence $I \in \mathfrak{hol}(g)$. Similarly one obtains that $J, K \in \mathfrak{hol}(g)$ and therefore the $\mathfrak{sp}(1)$-factor of the Lie algebra $\mathfrak{sp}(m) \oplus \mathfrak{sp}(1)$ of $\mathrm{Sp}(m)\mathrm{Sp}(1)$ is contained in $\mathfrak{hol}(g)$. Therefore, $\mathrm{Hol}(g)$ contains the connected subgroup of $\mathrm{SO}(4m)$ whose Lie algebra is this $\mathfrak{sp}(1)$, which is precisely $\mathrm{Sp}(1)$.

Now, to prove part 2, assume for a contradiction that there existed a quaternion-Kähler manifold (M, g) with nonvanishing scalar curvature and dimension $4m \geq 8$ that were locally de Rham-reducible. That is, locally $TM = E_1 \oplus E_2$ with two orthogonal subbundles E_1 and E_2 invariant by the holonomy group. Let X and Y be nonzero vectors of E_1 and E_2, respectively. Since $I, J, K \in \mathfrak{hol}(g)$ by the claim, we have $IX, JX, KX \in E_1$, $IY, JY, KY \in E_2$. In particular, $g(R(X, IX)Y, IY) = -g(R(Y, X)IX, IY) - g(R(IX, Y)X, IY) = 0$ and $g(R(X, IX)JY, KY) = -g(R(JY, X)IX, KY) - g(R(IX, JY)X, KY) = 0$ (by the

[42] See footnote 41.

Bianchi identity and since $R(Y,X), R(IX,Y), R(JY,X), R(IX,JY) \in \mathfrak{hol}(g)$ preserve E_1). However, by eqs. (4.4) and (4.7) we have

$$\begin{aligned} g(R(X,IX)Y,IY) + g(R(X,IX)JY,KY = &\alpha(X,IX)|Z|^2 = -\frac{1}{m+2}\mathrm{ric}(IX,IX)|Z|^2 \\ = &-\frac{\lambda}{m+2}g(IX,IX)|Z|^2 = -\frac{\lambda}{m+2}|X|^2|Z|^2, \end{aligned}$$

where λ denotes again the Einstein constant of M. Since the right-hand side is evidently nonzero, we have obtained a contradiction and therefore (M,g) must be de Rham-irreducible (even locally). $\square$

Since Einstein manifolds which are not Ricci-flat are often symmetric spaces, it has been conjectured by LeBrun and Salamon that complete quaternion-Kähler manifolds with positive scalar curvature necessarily need to be symmetric. In dimension 8, this has been proven by Poon and Salamon using the twistor space theory which will be described below (see [27]). At first, however, the known symmetric examples shall be given, the Wolf spaces.

Example 4.6 ([3], Theorem 14.51, Table 14.52)**.** We remark first that Theorem 2.22 also holds true if a symmetric space $(M,g) = (G/H_p, g)$ is written as a homogeneous space where G acts almost effectively (with a similar proof).

Now, let $(M,g) = (G/H_p, g)$ be a Riemannian symmetric space with dimension $4m \geq 8$, where G acts almost effectively on M and $H_p \subseteq G$ is the stabilizer of some $p \in M$. We fix an orthonormal frame to identify T_pM with $\mathbb{R}^{4m}$ and $\mathrm{Hol}_p(g)$ and H_p with subgroups of $\mathrm{SO}(4m)$ via the holonomy representation and the isotropy representation. Assume that M is quaternion-Kähler but not locally hyperkähler. By Theorem 2.22, we know that there is a natural isomorphism $\mathrm{Hol}_p(g) \cong H_p$. By the claim in the proof of part 2 of Proposition 4.5, we know that $\mathrm{Sp}(1) \subseteq H_p$. Since $H_p \subseteq \mathrm{Sp}(m)\mathrm{Sp}(1)$ by assumption and $\mathrm{Sp}(1)$ is normal in $\mathrm{Sp}(m)\mathrm{Sp}(1)$, $\mathrm{Sp}(1)$ is also normal in H_p.

Conversely, if the isotropy subgroup H_p of a Riemannian symmetric space of dimension $4m$ contains a normal subgroup $\mathrm{Sp}(1)$, then one may write $H_p = K\mathrm{Sp}(1)$, where $K \subseteq H_p$ is the centralizer of $\mathrm{Sp}(1)$, i. e. the normal subgroup of H_p consisting of elements commuting with all elements of $\mathrm{Sp}(1)$.[43]

In particular, $K \cap \mathrm{Sp}(1) = Z(\mathrm{Sp}(1)) = \{\pm 1\}$, the center of $\mathrm{Sp}(1)$. Furthermore, as the elements of K commute with the elements of $\mathrm{Sp}(1)$, they are in fact quaternion-right linear maps, i. e. $K \subseteq \mathrm{Sp}(m)$. Thus, $\mathrm{Hol}_p(g) \subseteq \mathrm{Sp}(m)\mathrm{Sp}(1)$ and M is quaternion-Kähler.

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[43] Clearly this is a subgroup. Furthermore, it is normal because it is the kernel of the homomorphism $H_p = N_{H_p}(\mathrm{Sp}(1)) \to \mathrm{Aut}(\mathrm{Sp}(1))$, $h \mapsto (s \mapsto hsh^{-1})$.

Surjectivity of the inclusion $K\mathrm{Sp}(1) \hookrightarrow H_p$ can be seen as follows: By the N/C-theorem ([13], Problem 3.9), H_p/K is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{Sp}(1))$. Now, $\mathrm{Aut}(\mathrm{Sp}(1)) = \mathrm{Inn}(\mathrm{Sp}(1))$, because the automorphism group $\mathrm{Aut}(\mathrm{Sp}(1))$ of the simply connected group $\mathrm{Sp}(1)$ is isomorphic to the automorphism group $\mathrm{Aut}(\mathfrak{sp}(1))$ of its Lie algebra, and under this isomorphism the inner automorphisms of $\mathrm{Sp}(1)$ are mapped to inner automorphisms of $\mathfrak{sp}(1)$. Now, $\mathfrak{sp}(1)$ has only inner automorphisms because these correspond to automorphisms of its Dynkin diagram, but that one is trivial. Hence, $\mathrm{Sp}(1)$ has only inner automorphisms as well. This means that for any $k \in K, h \in H_p$ there is a $s \in \mathrm{Sp}(1)$ such that the automorphism $t \mapsto (hk)t(hk)^{-1} (= hth^{-1})$ is the same as the automorphism $t \mapsto sts^{-1}$. The equivalence class of s in $\mathrm{Sp}(1)/Z(\mathrm{Sp}(1)) = \mathrm{Sp}(1)/(\mathrm{Sp}(1) \cap K)$ is unique. This yields a map $H_p/K \to \mathrm{Sp}(1)/(\mathrm{Sp}(1) \cap K)$ which by construction is an injection. Using this injection, one can choose, given $h \in H_p$, a $s \in \mathrm{Sp}(1)$ such that the equivalence classes $[h]$ and $[s]$ in H_p/K coincide. This means $h = sk$ for $k \in K$ and therefore the map $K\mathrm{Sp}(1) \hookrightarrow H_p$ is surjective, as h was arbitrary.

The idea of this argumentation was taken from the answer by Lawton in [20].

Now, the classification of symmetric quaternion-Kähler manifolds with nonvanishing scalar curvature follows from the above characterization of such ones (respectively the characterization of H_p) and the general classification of irreducible symmetric spaces (constructed by Cartan). One obtains a list of (irreducible) quaternion-Kähler symmetric spaces (with nonvanishing scalar curvature), the *Wolf spaces*, given below. The Wolf spaces can be joined together in two groups: The compact ones have positive Ricci curvature and can be written as G/K, where G, K are given in the table below. The noncompact ones have negative Ricci curvatures and can be written as G^*/K with G^*, K in the table below (they are the so-called dual symmetric spaces of the G/K).

G	G^*	K	$\dim(M)$
$\mathrm{Sp}(m+1)$	$\mathrm{Sp}(m,1)$	$\mathrm{Sp}(m)\mathrm{Sp}(1)$	$4m\ (m \geq 1)$
$\mathrm{SU}(m+2)$	$\mathrm{SU}(m,2)$	$\mathrm{S}(\mathrm{U}(m)\mathrm{U}(2))$	$4m\ (m \geq 1)$
$\mathrm{SO}(m+4)$	$\mathrm{SO}(m,4)$	$\mathrm{SO}(m)\mathrm{SO}(4)$	$4m\ (m \geq 3)$
G_2	G_2^2	$\mathrm{SO}(4)$	8
F_4	F_4^{-20}	$\mathrm{Sp}(3)\mathrm{Sp}(1)$	28
E_6	E_6^2	$\mathrm{SU}(6)\mathrm{Sp}(1)$	40
E_7	E_7^{-5}	$\mathrm{Spin}(12)\mathrm{Sp}(1)$	64
E_8	E_8^{-24}	$E_7\mathrm{Sp}(1)$	112

For example, we have $\mathrm{Sp}(m+1)/(\mathrm{Sp}(m)\mathrm{Sp}(1)) = \mathbb{H}\mathrm{P}^m$.[44]

Next we give a glimpse of the twistor space theory and its applications to quaternion-Kähler manifolds. We start out with the definition of the twistor space.

Definition 4.7 ([3], Definition 14.67)**.** Let (M, g) be a quaternion-Kähler manifold of dimension $4m$, $m \geq 2$. Recall that by Proposition 4.2, we can choose a family of domains $(U_i)_{i\in I}$ covering M and on each U_i three complex structures I_i, J_i and K_i that satisfy the quaternionic relations, such that the vector space spanned by $(I_i)_x$, $(J_i)_x$ and $(K_i)_x$ for some $x \in U_i \subseteq M$ does not depend on the choice of U_i (with $x \in U_i$). This gives rise to a globally defined bundle E over M whose fibre over $x \in M$ is spanned by $(I_i)_x, (J_i)_x$ and $(K_i)_x$, for i arbitrary (with $x \in U_i$). We denote by

$$Z := \{A \in E \,|\, g(A, A) = 4m\},$$

the sphere bundle of E (with radius $\sqrt{4m}$), the *twistor space* of M.[45]

The twistor space turns out to be a useful tool for studying quaternion-Kähler manifolds. It allows to translate problems on quaternion-Kähler manifolds into problems on complex manifolds, which can be studied using complex geometry. This is due to the following theorem.

Theorem 4.8 ([3], Theorem 14.68, [31], Theorem 4.1, [9], Theorem 4.12)**.** *Let (M, g) be a $4m$-dimensional quaternion-Kähler manifold, $m \geq 2$. Then, its twistor space Z admits*

[44] The notation might be slightly misleading, since $\mathrm{Sp}(m)\mathrm{Sp}(1) = (\mathrm{Sp}(m)\times\mathrm{Sp}(1))/\{\pm 1\}$ is not in a natural way a subgroup of $\mathrm{Sp}(m+1)$. However, the actual isometry group of $\mathbb{H}\mathrm{P}^m$ is $\mathrm{Sp}(m+1)/\{\pm 1\}$, since $\mathrm{Sp}(m+1)$ does act only almost effectively on $\mathbb{H}\mathrm{P}^m$. Of the latter group, $\mathrm{Sp}(m)\mathrm{Sp}(1)$ is a subgroup in the obvious way and can be divided out. Similar remarks hold true for the other groups on the list.

[45] The construction would also work similarly if M were merely a quaternionic manifold. In order to define the "sphere-subbundle" in this case, it suffices to note that for $x \in M$ the space E_x admits a natural metric where precisely the almost complex structures on T_xM have norm $\sqrt{4m}$.

a natural complex structure such that the fibres of the projection $\pi : Z \to M$ are compact complex curves of genus 0 *(i. e. Riemann spheres).*[46]

Proof. The Levi-Civita connection on M induces a torsion-free metric connection on $E \subset \mathrm{End}(TM)$, which thus in particular preserves Z. This yields a splitting of the tangent bundle $TZ = V \oplus H$, where V is the vertical distribution (tangent to the fibres of π) and H is a supplementary horizontal distribution, defined to consist of vectors which are tangent to parallel sections of Z.

Now, the parallel transport along integral curves of H preserves, by definition, the metric on the fibres of Z. Moreover, it preserves the orientation. Since each fibre of Z is a 2-sphere and the latter admits a canonical complex structure defined in terms of its metric and its orientation only,[47] we obtain for each $z \in Z$ an endomorphism $\widehat{J}$ of V_z, the subspace of T_zZ tangent to the fibre of z, with $\widehat{J}^2 = -\mathrm{Id}$: We choose a fibre and a complex structure on it; then we parallel-transport this along integral curves of H to all other fibres. By the above observation, the resulting endomorphism field $\widehat{J}$ does not depend on the choice of the fibre we started out with. Moreover, due to the definition via parallel transport one can see that $\widehat{J}$ is smooth.

On the other hand, by definition every $z \in Z$ is a complex structure on $T_{\pi(z)}M$. Now, π induces an isomorphism of H_z onto $T_{\pi(z)}M$ and using this (and z) we obtain an endomorphism $\overline{J}$ of H_z with $\overline{J}^2 = -\mathrm{Id}$. Thus, we are able to define an almost complex structure J on Z by setting

$$J(X) = \widehat{J}(VX) + \overline{J}(HX)$$

In fact, J does not depend on the Levi-Civita connection; it follows from [15], Corollary 3.3, that any other torsion-free $\mathrm{GL}(m,\mathbb{H})\mathrm{Sp}(1)$-connection would do as well.

At this point, to prove the integrability of J, by the Newlander-Nirenberg theorem it suffices to show that the Nijenhuis tensor of J vanishes. To do so, we consider the components of it in the splitting $TZ = V \oplus H$ into its vertical and horizontal subbundle.

Claim.

1. If U, V are vertical vector fields, then $N_J(U, V) = 0$.
2. If X is a basic vector field (that is, $X \in H$ and there exists a vector field $\overline{X}$ on M such that $d\pi(X) = \overline{X}$, [3], 9.22) and U is a vertical vector field, then $N_J(X, U) = 0$.
3. If X, Y are basic vector fields, then $N_J(X, Y) = 0$.

Summarizing, N_J vanishes (since it is tensorial and thus it suffices to plug in all possible combinations of basic and vertical vector fields, and it is antisymmetric).

Proof of the claim. Ad 1. Since U, V are vertical vector fields, so are JU, JV and all brackets of them. Then, we obtain $N_J(U, V) = 0$ from the fact that $\widehat{J}$ is a complex structure on each fibre.

Ad 2. Observe that since the horizontal transport along H preserves $\widehat{J}$, we have $[X, JU] =$

[46] The last part merely means that the complex structure restricts to one on the fibres, such that as complex manifolds they become isomorphic to the Riemann sphere $\mathbb{C}\mathrm{P}^1$.

[47] This is obtained by the observation $\mathrm{Hol}(S^2) \subseteq \mathrm{SO}(2) \cong \mathrm{U}(1)$ which turns $S^2 \cong \mathbb{C}\mathrm{P}^1$ into a Kähler manifold.

$J[X,U]$ (since $[X,U]$ is vertical[48]). Therefore

$$N_J(X,U) = \frac{1}{4}([X,U] + J[JX,U] + J[X,JU] - [JX,JU]) = \frac{1}{4}(J[JX,U] - [JX,JU]).$$

Now, JX is in general not a basic vector field. However, it is horizontal and for horizontal vectors, it follows from the product rule for the Lie bracket that the expression $VJ[JX,U] - V[JX,JU]$ is tensorial in JX. Thus, to prove the vanishing of $VN_J([X,U])$ at some point $z \in Z$, we may replace JX with a basic vector field $\overline{JX}$ coinciding with JX at z. Then, $VN_J(X,U)|_z = \frac{1}{4}(J[\overline{JX},U]|_z - [\overline{JX},JU]|_z) = 0$ by the same reasoning as above, replacing X with $\overline{JX}$. Since z was arbitrary, $VN_J(X,U) = 0$.

It remains to show that $HN_J(X,U) = 0$. Now, $d\pi$, restricted to the horizontal subspace at some point, is an isomorphism and hence it suffices to prove that $d\pi(HN_J(X,U)) = 0$. We compute

$$\begin{aligned} d_z\pi(HN_J(X,U)) =& \frac{1}{4}(d_z\pi(HJ[JX,U] - H[JX,JU])) \\ =& \frac{1}{4}(\phi(z)d_z\pi(H[JX,U]) - d_z\pi(H[JX,JU])) \end{aligned} \tag{4.10}$$

for $z \in Z$ by definition of J, where $\phi : Z_{\pi(z)} \to \mathrm{End}(T_{\pi(z)}M), z \mapsto z$ is the tautological map associating to a point in Z the complex structure on some tangent space of M to which it corresponds.

Now, we note that if we denote the local flow of U by ψ_t, we have at $z \in Z$

$$\begin{aligned} d_z\pi(H[JX,U]) =& d_z\pi([JX,U]) = -d_z\pi(\mathcal{L}_U(JX)) \\ =& -\lim_{t\to 0}\frac{1}{t}d_z\pi(d_{\psi_t(z)}\psi_{-t}(JX|_{\psi_t(z)}) - d_z\pi(JX_z)) \\ =& -\lim_{t\to 0}\frac{1}{t}d_{\psi_t(z)}\pi(JX|_{\psi_t(z)}) - d_z\pi(JX_z) \\ =& -\lim_{t\to 0}\frac{1}{t}\phi(\psi_t(z))d_{\psi_t(z)}\pi(X|_{\psi_t(z)}) - \phi(z)d_z\pi(X_z) \\ =& -\lim_{t\to 0}\frac{1}{t}\phi(\psi_t(z))d_z\pi(X_z) - \phi(z)d_z\pi(X_z) \\ =& -d_z\phi(U)(d_z\pi(X_z)) \end{aligned}$$

since U is vertical, X is basic and by definition of J on horizontal vectors.[49] Inserting this into eq. (4.10) and the same computation with U replaced by $JU = \widehat{J}U$ instead, we obtain

$$d_z\pi(HN_J(X,U)) = -\frac{1}{4}(\phi(z)d_z\phi(U)(d_z\pi(X)) - d_z\phi(\widehat{J}U)(d_z\pi(X))).$$

Therefore, $d_z\pi(HN_J(X,U))$ vanishes due to the fact that $\phi(z)d_z\phi(U) = d_z\phi(\widehat{J}U)$.[50]

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[48] We have $d\pi([X,U]) = [d\pi(X), d\pi(U)] = [d\pi(X), 0] = 0$. This shows that $[X,U]$ is vertical, as it is in the kernel of $d\pi$.

[49] Here we identify $T_z\mathrm{End}(T_{\pi(z)}M)$ with $\mathrm{End}(T_{\pi(z)}M)$.

[50] This in turn can be seen by an explicit computation: We choose three complex structures $I_{\pi(z)}, J_{\pi(z)}, K_{\pi(z)}$ on $T_{\pi(z)}M$ as in Proposition 4.2, such that $(I_{\pi(z)}, J_{\pi(z)}, K_{\pi(z)})$ is positively oriented. Without loss of generality we may assume that $z = I_{\pi(z)}$ and U is a linear combination of $J_{\pi(z)}$ and $K_{\pi(z)}$. Now, by definition of the complex structure on $T_{I_{\pi(z)}}S^2$, it maps $J_{\pi(z)}$ onto $K_{\pi(z)}$ and $K_{\pi(z)}$ onto $-J_{\pi(z)}$. This is the same as composition with $I_{\pi(z)}$ does. The claimed formula follows then directly from inserting this observation into the definition of J on vertical vectors.

Ad 3. First we show $HN_J(X,Y)=0$. Let $z \in Z$. By definition X,Y are elements of the horizontal distribution; since $\pi|_{H_z}$ is an isomorphism, as before we see that it suffices to prove $d_z\pi(N_J(X,Y))=0$. Now, since Z is a fibre bundle, we may extend z to a local section S over a neighbourhood of $\pi(z)$. In the usual way (by parallel transport along radial geodesics in a normal coordinate neighbourhood) we see that we may construct S such that $\nabla S|_{\pi(z)}=0$. Now, since X,Y are projectable, by definition of the Nijenhuis tensor in terms of Lie brackets and by definition of J on H, we see that

$$d_z\pi(N_J(X,Y)) = N_S(d_z\pi(X), d_z\pi(Y)).$$

Since ∇ is torsion-free, we may express all Lie brackets in the definition of the Nijenhuis tensor in terms of covariant derivatives and compute

$$\begin{aligned} N_S(d_z\pi(X), d_z\pi(Y))|_{\pi(z)} =& \frac{1}{4}(\nabla_{d\pi(X)}d\pi(Y) - \nabla_{d\pi(Y)}d\pi(X) + S(\nabla_{Sd\pi(X)}d\pi(Y) \\ &- \nabla_{d\pi(Y)}(Sd\pi(X))) + S(\nabla_{d\pi(X)}(Sd\pi(Y)) - \nabla_{Sd\pi(Y)}d\pi(X)) \\ &- \nabla_{Sd\pi(X)}(Sd\pi(Y)) + \nabla_{Sd\pi(Y)}Sd\pi(X))|_z = 0, \end{aligned}$$

since $\nabla S|_{\pi(z)}=0$.

It remains to show that $VN_J(X,Y)=0$. To do so, we first show a more generally applicable formula: If for $e \in E$ we identify VT_eE in the canonical way with $E_{\pi(E)}$ (also denoting the projection $E \to M$ by π), we have an equality of maps

$$R^E(d\pi(X), d\pi(Y)) = -V[X,Y] : E \to E. \tag{4.11}$$

That is, for all $e \in E$ we have $R^E(d\pi(X), d\pi(Y))e = V[X,Y]|_e$. Here R^E denotes the curvature tensor of the bundle $E \to M$, which is related to the Riemannian curvature tensor R of M by the formula $R^E(d\pi(X), d\pi(Y))e = [R(d\pi(X), d\pi(Y)), e]$ for all $X, Y \in TE$, $e \in E$. The claimed equality follows as a corollary of Lemma A.9.[51]

By eq. (4.11), the definition of N_J and the definition of J on horizontal vectors, it at this point suffices to show the following equation: For all $S = aI_x + bJ_x + cK_x$, where I_x, J_x, K_x are three complex structures on the tangent space of some $x \in M$ as in Proposition 4.2, and $a^2+b^2+c^2=1$, we have

$$[R(SX,SY),S] - S[R(SX,Y),S] - S[R(X,SY),S] - [R(X,Y),S] = 0. \tag{4.12}$$

By applying a suitable rotation of the sphere of complex structures, we see that it suffices to prove eq. (4.12) for $S = I_x$. Now, using eqs. (4.1), (4.5) and (4.6) and the fact that M

[51]Indeed: Let P be the frame bundle of E and $V := \mathbb{R}^3$, such that E is the associated bundle $\rho(P)$ to the standard representation ρ of $\mathrm{GL}(3,\mathbb{R})$ on $\mathbb{R}^3$. Let X', Y' denote the horizontal lifts to P of the vector fields $d\pi(X), d\pi(Y)$ and extend them trivially to $P \times V$. Then, $d\Pi(X') = X$, $d\Pi(Y') = Y$ if we denote the map from $P \times V$ to E by Π. Moreover, then $d\Pi([X',Y']) = [d\Pi(X), d\Pi(Y)] = [X,Y]$, because $d\Pi$, as the differential of a smooth map, maps brackets onto brackets. Now, by Lemma A.9, if u is any element of $P_{\pi(e)}$,

$$R^E(X,Y)e = u\Omega(X',Y')u^{-1}e = -u\omega([X',Y'])u^{-1}e,$$

and this equals $-V(d\Pi([X',Y']))|_e = -V[X,Y]|_e$ by definition of the action of the element $u\omega([X',Y'])u^{-1} \in \mathrm{End}(E_{\pi(e)})$ on $E_{\pi(e)}$ and the construction of Π. (Indeed $uAu^{-1}e = d\Pi(A^*)|_e$ for any $A \in \mathfrak{gl}(3,\mathbb{R})$ and A^* the corresponding fundamental vector field on P, as follows from the fact that for any curve $g_t : [0,1] \to \mathrm{SO}(3)$ and $W \in \mathbb{R}^3$ the curves $\Pi((ug_t, W^T))$ and $\Pi((u, W^T g_t^{-1})) = \Pi((u, (g_tW)^T))$ coincide.)

is an Einstein manifold (with Einstein constant λ, say), we may compute

$$\begin{aligned}
&[R(I_xX, I_xY), I_x] - I_x[R(I_xX, Y), I_x] - I_x[R(X, I_xY), I_x] - [R(X, Y), I_x] \\
&\qquad = -\frac{1}{m+2}(\mathrm{ric}(K_xI_xX, I_xY)J_x - I_x\mathrm{ric}(K_xI_xX, Y)J_x - I_x\mathrm{ric}(K_xX, I_xY)J_x \\
&\qquad - \mathrm{ric}(K_xX, Y)J_x - \mathrm{ric}(J_xI_xX, I_xY)K_x + I_x\mathrm{ric}(J_xI_xX, Y)K_x \\
&\qquad +I_x\mathrm{ric}(J_xX, I_xY)K_x + \mathrm{ric}(J_xX, Y)K_x) \\
&\qquad = -\frac{\lambda}{m+2}(g(J_xX, I_xY)J_x - g(J_xX, Y)K_x - g(K_xX, I_xY)K_x \\
&\qquad - g(K_xX, Y)J_x + g(K_xX, I_xY)K_x + g(K_xX, Y)J_x \\
&\qquad -g(J_xX, I_xY)J_x + g(J_xX, Y)K_x) = 0.
\end{aligned}$$

The proof of eq. (4.12) finishes the proof of part 3 of the claim above. Summarizing again, we have shown that N_J vanishes. The Newlander-Nirenberg theorem now shows that J is in fact a complex structure on Z and the theorem is proven. □

Example 4.9 ([3], Examples 14.75)**.** For quaternionic projective space $\mathbb{H}\mathrm{P}^m$, the associated twistor space is the complex projective space $\mathbb{C}\mathrm{P}^{2m+1}$, and the fibration $\mathbb{C}\mathrm{P}^{2m+1} \to \mathbb{H}\mathrm{P}^m$ may be described as a Hopf fibration. For the proof, see Besse ([3], Examples 14.75).

Remark 4.10.

1. In the case that a quaternion-Kähler manifold has positive scalar curvature, the twistor space can even be equipped with a Riemannian metric such that with its canonical complex structure it becomes a Kähler manifold ([31], Theorem 6.1).
2. The twistor space theory allows to translate questions on quaternion-Kähler manifolds into questions on complex manifolds. Without proof, we state here some results whose proof involves the usage of complex geometry on the twistor space (see [3], Section 14.H).
   - Any quaternion-Kähler manifold is analytic.
   - A compact quaternion-Kähler manifold with positive Ricci curvature is simply connected and its odd-dimensional Betti numbers vanish.
   - If (M, g) is a compact quaternion-Kähler manifold with positive Ricci curvature and the second Stiefel-Whitney class of the $\mathfrak{sp}(1)$-bundle E vanishes, then (M, g) is isometric to $\mathbb{H}\mathrm{P}^n$ (up to a constant factor).
   - As stated before, any complete quaternion-Kähler manifold with positive scalar curvature and dimension 8 is a Wolf space (the LeBrun-Salamon conjecture in dimension 8, proven in [27]).

5. Monotony of holonomy groups in limits

In this final section, we study the effect on holonomy groups under a change of the connection under consideration. With an eye towards the application in Riemannian geometry, we consider vector bundles equipped with bundle metrics and compatible connections. We prove: Given a closed subgroup H of $\mathrm{SO}(l)$, if we consider a sequence of metrics converging at one point and a sequence of compatible connections, such that all their holonomies are contained in H, converging in C^0, then the limit connection also has holonomy contained in H. In particular, this finds an application for Riemannian holonomy groups, yielding that the map $g \mapsto \mathrm{Hol}(g)$ from the space of all metrics on some manifold M to the space of conjugacy classes of subgroups of $\mathrm{SO}(\dim(M))$ is monotonous in limits.

These questions have also been studied by Müller ([23]), using a somewhat different attempt. Müller also phrases the stated monotonicity in limits in terms of lower semicontinuity of a certain map assigning to a metric the conjugacy class of its holonomy group.

Situation 5.1.

– Let M be a manifold and $E \to M$ be an orientable vector bundle with standard fibre $\mathbb{R}^l$. Let $x \in M$. We denote by E^* the dual bundle of E and by S^2E^* the symmetrized tensor product of the dual bundle.

– Let $g \in C^1(M, \mathrm{S}^2E^*)$ be a bundle metric on E (that remains fixed throughout this section unless otherwise stated). Let ∇^g be a connection on E whose coefficients are continuous and that is compatible with g, i. e. $X(g(v,w)) = g(\nabla^g_X v, w) + g(v, \nabla^g_X w)$ for all $X \in TM$, $v, w \in C^1(M, E)$.

– Given any metric h on E, let

$$\mathrm{SO}_l(E_x)_h := \{A \in \mathrm{End}(E_x) \mid \det(A) = 1, \forall v, w \in E_x : h|_x(Av, Aw) = h|_x(v,w)\}$$

be the special orthogonal group of E_x with respect to the metric $h|_x$. Then, since ∇^g is metric, we have $P_c^{\nabla^g} \in \mathrm{SO}_l(E_x)_g$ for all $c \in \Omega_{\mathrm{p}-C^1}(x)$.[52]

– An orthonormal basis of E_x with respect to some metric h are vectors $e_1, \dots, e_l$ with

$$\forall i, j = 1, \dots, n : h(e_i, e_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. We choose a positively oriented orthonormal basis $(b_1, \dots, b_l)$ of E_x with respect to the metric g and use it to identify E_x with $\mathbb{R}^l$. This then identifies $\mathrm{SO}_l(E_x)_g$ with $\mathrm{SO}(l) \subset \mathrm{GL}(l)$ and consequently we may view $\mathrm{Hol}_x(\nabla^g)$ as an explicit subgroup of $\mathrm{SO}(l)$.

Now, given a (possibly different) metric $h \in C^1(M, \mathrm{S}^2E^*)$, we may still view $\mathrm{SO}_l(E_x)_h$ as a distinguished subgroup of $\mathrm{GL}(l)$, as follows:

Definition 5.2.

1. Using the basis $(b_1, \dots, b_l)$ of E_x, we may view all bilinear forms $Q \in \mathrm{S}^2E_x^*$ and all endomorphisms $Z \in \mathrm{End}(E_x)$ as $l \times l$-matrices over $\mathbb{R}$. In particular, we may associate to a bilinear form Q its representing matrix M_Q and via this an endomorphism Z_Q of $E_x \cong \mathbb{R}^l$.[53] This defines an embedding $\mathrm{S}^2E_x^* \to \mathrm{End}(E_x)$, $Q \mapsto Z_Q$. Now $\mathrm{End}(E_x)$

[52] We recall that $P_c^{\nabla^g}$ denotes the parallel transport along the curve c with respect to the connection ∇^g.

[53] To be very explicit, therefore $Q(X,Y) = g(X, Z_Q Y)$ for all $X, Y \in E_x$, and if we view X, Y also as their respective coordinate vectors with respect to the basis $(b_1, \dots, b_l)$, then this is nothing but $X^T M_Q Y$.

can be turned into a normed vector space by using the operator norm induced by the metric g. By pullback along the embedding described above, also $\mathrm{S}^2 E_x^*$ becomes a normed vector space. Unless otherwise stated, in the following these norms will be used. (As all vector spaces involved are finite-dimensional, the resulting topologies do not depend on g or the basis.)

2. Let $h \in C^1(M, \mathrm{S}^2 E^*)$ with $h|_x \in B_1(g|_x)$. Let M_h be the representing matrix of $h|_x$ and Z_h the corresponding endomorphism as in the previous point. We define

$$\mathrm{SO}(l)_h := \{A \in \mathrm{GL}(l) \,|\, A^T M_h A - M_h = 0\}$$

(the x-dependency is understood). Let W_h be a symmetric positive definite square root of Z_h as defined by the power series of $\sqrt{1+c}$ in $B_1(0)$. Then, the map $\mathrm{SO}(l) \to \mathrm{SO}(l)_h, A \mapsto W_h^{-1} A W_h$ is an isomorphism. (Here we again use the basis $(b_1, \ldots, b_l)$ of E_x to identify E_x with $\mathbb{R}^l$, $\mathrm{SO}(l)$ with a matrix group, Z_h with M_h and W_h also with an explicit $l \times l$-matrix over $\mathbb{R}$.) Indeed this follows from the following matrix computation: If $A \in \mathrm{SO}(l)$, then

$$\begin{aligned}(W_h^{-1} A W_h)^T h|_x (W_h^{-1} A W_h) - M_h &= W_h^T A^T W_h^{-T} W_h^T W_h W_h^{-1} A W_h - M_h \\ &= W_h^T A^T A W_h - M_h \\ &= W_h^T W_h - M_h = M_h - M_h = 0,\end{aligned}$$

thus $W_h^{-1} A W_h \in \mathrm{SO}(l)_h$; the inverse isomorphism is simply given by $A \mapsto W_h A W_h^{-1}$. We call this isomorphism the *nonstandard embedding* $\phi_h : \mathrm{SO}(l) \to \mathrm{SO}(l)_h \subset \mathrm{GL}(l)$. By the description of W_h in terms of a power series in h, we obtain that if we have a sequence $(h_k)_{k\in\mathbb{N}} \subset C^1(M, \mathrm{S}^2 E^*)$ with $h_k|_x \in B_1(g|_x) \cap \mathrm{S}^2 E_x^*$ for all k and $h_k|_x \to g|_x$ in $\mathrm{S}^2 E_x^*$, then $W_{h_k} \to I_l$ (the $l \times l$-identity matrix).

3. Given any two connections $\nabla, \tilde{\nabla}$ on E with continuous coefficients, its difference $\nabla - \tilde{\nabla}$ is a continuous section of $T^*M \otimes E \otimes E^*$. We equip the space $\Gamma(T^*M \otimes E \otimes E^*)$ of such sections with the *compact-open C^r-topology*, $r = 0$, which is defined as follows ([12], Chapter 2.1): Given any two charts $\mathbf{x} : U \to V, \mathbf{v} : U' \to V'$ of M respectively $T^*M \otimes E \otimes E^*$, a compact $K \subseteq U$, $\epsilon > 0$ and $f \in \Gamma(T^*M \otimes E \otimes E^*)$, we let

$$\begin{aligned}B(f, \mathbf{x}, \mathbf{v}, K, \epsilon) :=& \Big\{ g \in \Gamma(T^*M \otimes E \otimes E^*) \,|\, g(K) \subseteq V, \forall 0 \leq k \leq r : \\ & \Big\| D^k(\mathbf{v} \circ f \circ \mathbf{x}^{-1}) - D^k(\mathbf{v} \circ g \circ \mathbf{x}^{-1}) \Big\|_{C^0} < \epsilon \Big\}.\end{aligned}$$

[54] (Of course, the above is only relevant for $r = 0$.) The set of all such $B(f, \mathbf{x}, \mathbf{v}, K, \epsilon)$ has the basis property and thus defines a topology on $\Gamma(T^*M \otimes E \otimes E^*)$. Analogously, we define compact-open C^r-topologies on the spaces of sufficiently often differentiable sections of other bundles (also for other r).

Using this, we can speak of C^0-convergence in $\Gamma(T^*M \otimes E \otimes E^*)$, and we say that a sequence of connections ∇^k converges in C^0 to a connection ∇ on E if their difference converges to zero.[55] If this is the case, then the restrictions of $\nabla^k - \nabla$ to any compact $K \subseteq M$ converge to zero uniformly on K (with respect to any set of charts covering K).

Now, we can state the announced result.

[54] To be precise, the C^0-norm indicated here is the standard norm in the space $C^0(\mathbb{R}^n, \mathbb{R}^{(k+1)n+2l})$.

[55] The case $r = 1$ will be needed for the convergence of metrics in the subsequent corollary.

Theorem 5.3. *Assume Situation 5.1 and let $H \subseteq \mathrm{SO}(l)$ be closed. Moreover, for $k \in \mathbb{N}$, let $g_k \in C^1(M, \mathrm{S}^2 E^*)$ be a bundle metric on E and ∇^k be a connection with continuous coefficients compatible with g_k, such that $g_k|_x \to g|_x$ in $\mathrm{S}^2 E_x^*$ and $\nabla^k - \nabla^g \to 0$ in the compact-open C^0-topology on $\Gamma(M, T^*M \otimes E^* \otimes E)$.*

Assume that for all $k \in \mathbb{N}$, there exists a $g_k|_x$-orthonormal basis of E_x that is positively oriented and such that if we use this basis to identify E_x with $\mathbb{R}^l$, we have $\mathrm{Hol}_x(\nabla^k) \subseteq H$. Then, there exists a positively oriented g-orthonormal basis $(c_1, \dots, c_l)$ of E_x such that if we use this basis to identify E_x with $\mathbb{R}^l$, we have $\mathrm{Hol}_x(\nabla^g) \subseteq H$.

A word of warning: After the thesis has been submitted, it was recognized that its statement lacks the assumption that all connections involved need to be Lipschitz continuous. With this assumption, the proof works as before but the statement becomes slightly weaker. Similar remarks then also hold for the corollaries.

Remark 5.4. An examination of the proof shows that, if one works instead of in $\mathrm{SO}(l)$ in the coset space $\mathrm{GL}(l)/N_{\mathrm{GL}(l)}(H)$ to find the conjugating element V ($N_{\mathrm{GL}(l)}(H)$ denoting the normalizer of H in $\mathrm{GL}(l)$), one may relax the condition of metricity of the connections to the condition that $\mathrm{GL}(l)/N_{\mathrm{GL}(l)}(H)$ is compact.[56] However, due to the lack of examples for this case, the theorem is formulated in this weaker but more cleanly readable way.

Proof of Theorem 5.3. We note first that the conclusion of the theorem may be reformulated as follows: If we use our pre-chosen basis $(b_1, \dots, b_l)$ to identify E_x with $\mathbb{R}^l$, then there exists $V \in \mathrm{SO}(l)$ such that $V\mathrm{Hol}_x(\nabla^g)V^{-1} \subseteq H$.[57] Therefore, we fix once and for all the basis $(b_1, \dots, b_l)$ and use it to identify E_x with $\mathbb{R}^l$.

We may assume that $g_k|_x \in B_1(g|_x) \cap \mathrm{S}^2 E_x^*$ for all $k \in \mathbb{N}$ by dropping a finite number of elements from the sequence, hence we may define the matrices W_{g_k}, the subgroups $\mathrm{SO}(l)_{g_k}$ and the nonstandard embedding ϕ_{g_k} for $k \in \mathbb{N}$ as in Definition 5.2. Furthermore, we denote by H_{g_k} the image of the composition of the inclusion $H \hookrightarrow \mathrm{SO}(l)$ and the nonstandard embedding $\mathrm{SO}(l) \to \mathrm{SO}(l)_{g_k}$.

The assumption then means that for all $k \in \mathbb{N}$ there exists an $U_k \in \mathrm{SO}(l)_{g_k}$ such that $U_k \mathrm{Hol}_x(\nabla^k) U_k^{-1} \subseteq H_{g_k}$.[58] Now, the sequence $\phi_{g_k}^{-1}(U_k)$ is contained in the compact group $\mathrm{SO}(l)$ and hence possesses a convergent subsequence. Moreover, using the explicit description of ϕ_{g_k} in terms of conjugation by matrices converging to the identity matrix, we see that if $(\phi_{g_{k_m}}^{-1}(U_{k_m}))_{m \in \mathbb{N}}$ converges to some $V \in \mathrm{SO}(l)$, then also $(U_{k_m})_{m \in \mathbb{N}}$ converges to V.

Therefore, passing to the subsequence without changing the notation for brevity, we may assume that $(U_k)_{k \in \mathbb{N}}$ converges to some $V \in \mathrm{SO}(l)$. At this point, it suffices to prove the following claim:

Claim. We have $V\mathrm{Hol}_x(\nabla^g)V^{-1} \subseteq H$.

Proof of the claim. Let $c \in \Omega_{\mathrm{p}-C^1}(x)$. Note that by definition of the compact-open topology, the restrictions of the connections to the (compact) image of c converge uniformly in $\Gamma((T^*M \otimes E^* \otimes E)|_{c([0,1])})$. Therefore, the coefficients of the parallel transport equation defining $P_c^{\nabla^k}$ converge uniformly to the coefficients of the parallel transport equation

[56] One then needs to slightly adapt the proof to find an actual convergent sequence of conjugating elements instead of merely convergent equivalence classes, but this can be done without much difficulty.

[57] Indeed: It suffices to take V as the matrix such that $\sum_{k=1}^{l} V_{ki} c_k = b_i$ for $i = 1, \dots, l$.

[58] Note that since the connection ∇^k is metric with respect to g_k, we have $\mathrm{Hol}_x(\nabla^k) \subseteq \mathrm{SO}(l)_{g_k}$ for all $k \in \mathbb{N}$.

defining $P_c^{\nabla^g}$. As the solutions to ordinary differential equations depend continuously on their coefficients, we have $\lim\limits_{k\to\infty} P_c^{\nabla^k} = P_c^{\nabla^g}$.[59] Thus, also

$$VP_c^{\nabla^g}V^{-1} = \lim_{k\to\infty} U_k P_c^{\nabla^k} U_k^{-1} = \lim_{k\to\infty} W_{g_k} U_k P_c^{\nabla^k} U_k^{-1} W_{g_k}^{-1}.$$

Since $U_k P_c^{\nabla^k} U_k^{-1} \in H_{g_k}$ respectively $W_{g_k} U_k P_c^{\nabla^k} U_k^{-1} W_{g_k}^{-1} \in H$ for all $k \in \mathbb{N}$, we obtain $VP_c^{\nabla^g}V^{-1} \in H$ by continuity of matrix multiplication. This proves the claim and hence the theorem. □

The most useful use case of Theorem 5.3 is probably the case of oriented Riemannian manifolds and their Levi-Civita connection. In this case, the condition of convergence $\nabla^k \to \nabla^g$ is automatically satisfied if the metrics converge in C^1, and we get the following corollary.

Corollary 5.5. *Let (M, g) be an oriented n-dimensional Riemannian manifold with $g \in \Gamma(M, \mathrm{S}^2T^*M)$ of class C^1 and let ∇ be its Levi-Civita connection. Let $H \subseteq \mathrm{SO}(n)$ be a closed subgroup and let $x \in M$. For $k \in \mathbb{N}$, let g_k be a Riemannian metric of class C^1 on M and ∇^k its Levi-Civita connection. Assume that $g_k \to g$ in $\Gamma(M, \mathrm{S}^2T^*M)$ in the compact-open C^1-topology and for all $k \in \mathbb{N}$, we have $\mathrm{Hol}_x(\nabla^k) \subseteq H$ up to conjugation in $\mathrm{SO}(n)$. Then, $\mathrm{Hol}_x(\nabla) \subseteq H$ up to conjugation in $\mathrm{SO}(n)$.*

As a further corollary, we get:

Corollary 5.6. *Let M be an oriented n-dimensional Riemannian manifold, let $H \subseteq \mathrm{SO}(n)$ be a closed subgroup and let $x \in M$. Then, the set*

$$\{h \in \Gamma(\mathrm{S}^2T^*M) \,|\, h \text{ Riemannian metric with Levi-Civita connection } \nabla \\ \text{such that } \mathrm{Hol}_x(\nabla) \subseteq H \text{ up to conjugation}\}$$

is closed in the set of all C^1-Riemannian metrics on M, if this set is equipped with the compact-open C^1-topology.

Conversely, the conjugacy classes of holonomy groups are upper semicontinuous only in special cases. For example, for Ricci-flat deformations of a Ricci-flat metric the conjugacy class of the holonomy group is known to be constant if the universal covering of M admits a parallel spinor (see [2]). In general, full continuity is not true, as the following two examples show.

Example 5.7. Consider (subsets of) scaled Poincaré disk models, $M := B_{1/2}(0) \subset \mathbb{R}^2$ equipped with the family of metrics $g_{ab}^{(k)} := \frac{4}{(1-\|\mathbf{x}\|^2/k^2)^2}\delta_{ab}$ in cartesian coordinates (suppressing the coordinate dependence from the notation). These converge in $\Gamma(M, \mathrm{S}^2T^*M)$ in the compact-open C^1-topology to the limit metric $g_{ab} := 4\delta_{ab}$. The metrics g_k have sectional curvature $-1/k^2$, while g has sectional curvature zero. Thus, the Riemann tensors are $R_{abcd}^{(k)} = -\frac{1}{k^2}(g_{ac}^{(k)} g_{bd}^{(k)} - g_{ab}^{(k)} g_{cd}^{(k)})$ and $R_{abcd} = 0$, respectively.

[59] See e. g. [28] Satz 4.1.2; to be precise, the "continuous dependence on the coefficients" is the statement that if the coefficients converge in C^0 (and are all locally Lipschitz continuous), then the solutions converge in C^0. Note that we can assume that the coefficients of the parallel transport equations along c are continuous since we can parametrize c such that it has vanishing speed at the points where its parametrization by arc-length would not be differentiable.

As M is simply connected, for all $\mathbf{x} \in M$ the holonomy groups $\mathrm{Hol}_{\mathbf{x}}(\nabla^k)$ and $\mathrm{Hol}_{\mathbf{x}}(\nabla)$, where ∇^k and ∇ are the Levi-Civita connections corresponding to $g^{(k)}$ and g, respectively, coincide with the corresponding restricted holonomy groups $\mathrm{Hol}^0_{\mathbf{x}}(\nabla^k)$ and $\mathrm{Hol}^0_{\mathbf{x}}(\nabla)$. By the Ambrose-Singer theorem, the Lie algebras of the former are generated by the endomorphisms

$$(R^{(k)})_{a11}{}^d = 0, \qquad (R^{(k)})_{a12}{}^d = -\frac{4k^2}{(k^2-\|\mathbf{x}\|^2)^2}\epsilon_a{}^d,$$
$$(R^{(k)})_{a21}{}^d = \frac{4k^2}{(k^2-\|\mathbf{x}\|^2)^2}\epsilon_a{}^d, \qquad (R^{(k)})_{a22}{}^d = 0,$$

where $\epsilon_a{}^d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in matrix notation.[60] These generate the Lie algebra of $\mathrm{SO}(2)$, hence $\mathrm{Hol}_{\mathbf{x}}(\nabla^k) = \mathrm{SO}(2)$ for all $k \in \mathbb{N}$. However, $\mathrm{Hol}_{\mathbf{x}}(\nabla) = \{I_2\} \subset \mathrm{SO}(2)$ because g is flat.

Example 5.8. Let M be the Fermat quartic, which is a K3 surface in $\mathbb{C}\mathrm{P}^3$. Let ω be the Kähler form of some Kähler metric on M. Let $\alpha \in \ker(\partial)\setminus\ker(\overline{\partial}) \subset C^\infty(M, \Lambda^{1,0}T^*M)$.[61] Let $\rho := \overline{\partial}\alpha = d\alpha \in C^\infty(M, \Lambda^{1,1}T^*M)$; by construction ρ represents zero in cohomology.

By the Calabi conjecture (solved by Yau in [33], [34], for a textbook account see [16], chapter 5), there exists a unique Kähler metric g_1 on M with Ricci form ρ and such that its Kähler form is contained in the cohomology class of ω. Moreover, there exists a unique Ricci-flat Kähler metric g_∞ on M with Kähler form contained in the cohomology class of ω. Let $g_k := \frac{1}{k}g_1 + \frac{k-1}{k}g_\infty$ for $k \in \mathbb{N}_{\geq 2}$. Then, the metrics g_k are Kähler since convex combinations of Kähler metrics are again Kähler, and the sequence $(g_k)_{k\in\mathbb{N}}$ converges in $\Gamma(M, \mathrm{S}^2T^*M)$ in the compact-open C^1-topology to g_∞. Moreover, for all $k \in \mathbb{N}$ the metric g_k is not Ricci-flat.[62]

Thus, the metrics g_k all have holonomy group contained in $\mathrm{U}(2)$ (by e. g. [16], proposition 4.4.2), but not in $\mathrm{SU}(2)$ (see e. g. [16], proposition 6.1.1). However, g_∞ is Ricci-flat and hence has its holonomy group contained in $\mathrm{SU}(2) \subset \mathrm{U}(2)$.

[60] For every k, M is isometric to an open subset of a symmetric space, namely the Poincaré disk model scaled by a factor of k. In particular, this shows that the curvature endomorphisms are parallel (see Proposition 2.18) and hence in fact the holonomy algebra is at each point of M already generated by the curvature tensor at this point, without the necessity for considering parallel-transported curvature tensors at other points.

[61] For example, let $h \in C^\infty(M, \mathbb{R})$ be such that $\overline{\partial}\partial h$ is not identically zero; such a function can be defined in a chart. Then, define $\alpha := \partial h$.

[62] If it were, it would contradict the uniqueness statement from the Calabi conjecture: The metrics g_k also all have Kähler forms contained in the cohomology class of ω. However, there exists only a unique such one which is Ricci-flat, namely g_∞.

Conclusion

In this document, we have studied the holonomy groups of connections on principal bundles and vector bundles. At first, we examined many well-known facts from their theory, in particular concerning the Riemannian case. We considered reducible Riemannian holonomy groups and the general concept of local and infinitesimal holonomy groups. We studied invariant connections on Riemannian homogeneous and symmetric spaces and showed that their holonomy algebra is equal to the algebra of the linear isotropy group. Afterwards we proved the well-known reduction theorem and the holonomy theorem by Ambrose and Singer. In the following section, we specialized again to the case of Riemannian holonomy groups and considered in more detail the geometry of quaternion-Kähler manifolds, which we proved to be Einstein manifolds and to admit an associated complex manifold fibring over them with fibres $\mathbb{C}\mathrm{P}^1$, the twistor space.

In the last section, we considered metric connections on vector bundles and examined the changes in their holonomy group under changes of the connection. In particular, we proved that the assignment of the conjugacy class of the holonomy group to some Riemannian metric on a prescribed manifold is monotonous in limits. To the best of the knowledge of the author, this result is not contained in preceding published literature. Only in the preprint [23] by Müller, a similar but weaker theorem has been proven by different means.

There is much more which could be said about holonomy groups, and this work can at best give a glimpse of some of the best-known results. The study of holonomy has led to many fruitful developments in differential geometry and even in physics, and the author expects this to continue in the future. It is hoped that this document could give an introduction to the topic, maybe answer one or another question concerning holonomy groups and inspire the reader to further study and perhaps research.

A. Some definitions and lemmas from differential geometry

In this appendix, we list for the sake of completeness some general lemmas on differential geometry and connections which find application in the main body of the text. Moreover, we give precise definitions of the objects used which could be regarded as not interesting enough for the main part.

A.1. General differential geometry

Definition A.1 ([16], Section 2.1.1, [18], Chapter I, Section 1, Section 5, Chapter II, Section 1, Chapter III, Section 2, [11], Chapter II, Section 6, [17], Section 3.1)**.** Let M be a smooth manifold of dimension n and let G be a Lie group.

1. We denote the left translation map on G by L_g and the right translation map by R_g.

   The *Lie algebra* $\mathfrak{g}$ of G is the set of left-invariant vector fields on G, that is,

   $$\mathfrak{g} := \{X \in \Gamma(TG) \,|\, \forall g \in G : dL_g(X) = X\}.$$

   This is a Lie subalgebra of the Lie algebra of all vector fields on G with the usual bracket operation. Evaluation at the identity $1 \in G$ yields an isomorphism $\mathfrak{g} \cong T_1G$. We will use this isomorphism as an identification if it is convenient.

   The *adjoint representation* of G on $\mathfrak{g}$ is the map $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ satisfying $\mathrm{ad}(g)(X) = \frac{d}{dt}|_{t=0} g \exp(tX) g^{-1}$ for $g \in G$, $X \in T_1G \cong \mathfrak{g}$. Its differential, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (where $\mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of $\mathrm{GL}(\mathfrak{g})$) is denoted with the same symbol and is a Lie algebra representation. It can be shown that $\mathrm{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

   The *Killing form* on the Lie algebra $\mathfrak{g}$ is the symmetric bilinear form

   $$B : (X, Y) \mapsto \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

   By construction, for all $g \in G$ and $X, Y \in \mathfrak{g}$ we have $B(\mathrm{ad}(g)X, \mathrm{ad}(g)Y) = B(X, Y)$ (so $\mathrm{ad}(g)$ is an "isometry" with respect to B). Differentiating this equation, we obtain that for all $Z \in \mathfrak{g}$ we have $B(\mathrm{ad}(Z)X, Y) = -B(X, \mathrm{ad}(Z)Y)$.

   The *exponential map* is the map $\exp : \mathfrak{g} \to G$, $X \mapsto \gamma_X(1)$, where $\gamma_X : \mathbb{R} \to G$ is the *one-parameter subgroup* of G with derivative X at 0, i. e. the morphism of Lie groups $\gamma_X : \mathbb{R} \to G$ with $\dot{\gamma}_X(0) = X$.

2. A *principal G-bundle* over M is a smooth fiber bundle $\pi : P \to M$ such that there exists a smooth free G-action on P and the fibers are precisely the orbits of the G-action. Thus, for all $x \in M$, the fibre $P_x := \pi^{-1}(x)$ is diffeomorphic to G, and $M = P/G$ is the quotient of P by this G-action. We call G the *structure group* of P. We use the convention that G acts from the right and write the action $p \mapsto pg$ (for $g \in G$). Alternatively, we write R_g also for this "right translation" action. We denote by VT_pP or V_p, $p \in P$, the *vertical tangent space* of P at p, that is, the part tangent to the fibre of p, and by VTP the bundle consisting of the vertical tangent spaces at each point of P.

   Given $X \in \mathfrak{g}$, we denote by X^* the corresponding *fundamental vector field* on P induced by the one-parameter group of diffeomorphisms of P given by $t \mapsto R_{\exp(At)}$. The assignment $A \mapsto A^*_p$ yields for every $p \in P$ a canonical isomorphism from $\mathfrak{g}$ onto VT_pP.

A *pseudotensorial k-form of type* (ρ, V) on P, where V is a vector space and $\rho : G \to V$ is a representation on V, is a differential form $\phi : \Lambda^k T^*P \to V$ on P with values in V such that $R_g^*\phi = \rho(g^{-1})\phi$ for $g \in G$. If it is clear from the context, we omit the reference to V. In particular, a *differential form of type* $\mathrm{ad}\, G$ on P is a Lie algebra-valued differential form $\phi : \Lambda^k T^*P \to \mathfrak{g}$ (for $k \in \mathbb{N}$) such that for all $g \in G$ we have $R_g^*\phi = \mathrm{ad}(g^{-1})\phi$.

Given a submanifold $W \subseteq X$, we denote by $P|_W := \pi^{-1}(W)$ the restriction of P to W; the map $\pi : P|_W \to W$ is a principal G-bundle over W.

A *reduction* of P to a principal H-bundle, where $H \subseteq G$ is a Lie subgroup, is a subspace $Q \subseteq P$, together with the restriction of the projection $\pi|_Q : Q \to M$ such that Q is preserved by the action of H on P (obtained by the inclusion $H \subseteq G$ and the action of G on P) and H acts freely on Q such that precisely the fibers of Q over points in M are the orbits of the H-action.

3. We denote by $\pi : L(M) \to M$ the *bundle of linear frames* over M, i. e. the principal $\mathrm{GL}(n)$-bundle whose fibre over each $x \in M$ consists of all linear isomorphisms $\mathbb{R}^n \to T_xM$ (the action of $\mathrm{GL}(n)$ on it is given by composition from the right). An element $u \in L(M)$ is called a *frame* on M (or on $T_{\pi(u)}M$).

   The *canonical form* on $L(M)$ is the $\mathbb{R}^n$-valued 1-form on P defined by $\theta(X) = u^{-1}(\pi(X))$ for $X \in T_uP$, $u \in P$. This is a pseudotensorial form of type $(\rho, \mathbb{R}^n)$, where ρ is the standard representation of $\mathrm{GL}(n)$ on $\mathbb{R}^n$.

   A reduction of $L(M)$ to a principal H-bundle for $H \subseteq \mathrm{GL}(n)$ is called a *H-structure* on M.

4. A *vector bundle* on M is a smooth fibre bundle $\pi : E \to M$ such that for each $x \in M$, the fibre $E_x := \pi^{-1}(x)$ has the structure of a vector space and their is an open neighbourhood U_x of x such that $\pi^{-1}(U_x) \cong U_x \times V$, where V is the (standard) fibre of E.

   We denote by $\Gamma(E)$ the space of all sections of E, sufficiently often differentiable if necessary.

   Suppose P is a principal G-bundle over M with fibre G and that $\rho : G \to \mathrm{GL}(V)$ for some vector space V defines a group action from the right.[63] Then, G acts on $P \times V$ by the principal bundle action on P and ρ on V. Let $\rho(P) := (P \times V)/G$ be the quotient of $P \times V$ by this G-action. The projection $\rho(P) = (P \times V)/G \to P/G = M$ induced by the projection $P \times V \to P$ gives a projection from $\rho(P)$ onto M. Since G acts freely on P, this projection has fibre V, so $\rho(P)$ is a vector bundle over M with fibre V. The bundle $\rho(P)$ is called the *associated bundle* (to ρ and P).

   In fact, every vector bundle E may be described as the associated bundle to its *frame bundle*
   $$F^E := \{(x, \phi) \,|\, x \in M, \phi : \mathbb{R}^k \to E_x \text{ linear isomorphism}\},$$
   where k is the fibre dimension of E. This is a principal $\mathrm{GL}(k)$-bundle by letting $\mathrm{GL}(k)$ act from the right by composition. The bundle $L(M)$ described above is nothing else but the frame bundle corresponding to the tangent bundle.

   In particular, we consider the tensor bundles $\mathrm{T}^{r,s}(M)$ of r-times covariant and s-times contravariant tensors, the bundle S^2T^*M of symmetric $(2,0)$-tensors over M

[63] Often one is instead given a group action ρ' from the left, then one obtains a right action by setting $v\rho(g) := \rho'(g^{-1})v$ for $v \in V$, $g \in G$.

and the bundles $\Lambda^k T^*M$ of alternating forms (as well as similar bundles derived from other bundles over M than the tangent and cotangent bundle).

5. A *transformation* of M is a diffeomorphism from M onto itself. A 1*-parameter group of transformations* of M is a map $\mathbb{R} \times M \to M$, $\mathbb{R} \times M \ni (t,x) \mapsto \phi_t(x) \in M$ such that:

   – For each $t \in \mathbb{R}$, the map $\phi_t : M \to M$, $x \mapsto \phi_t(x)$ is a transformation of M.

   – For all $t, s \in \mathbb{R}$ and $x \in M$, $\phi_{t+s}(x) = \phi_t(\phi_s(x))$.

   A *local* 1*-parameter group of local transformations* is defined in the same way, except that $\phi_t(x)$ is defined only for t in a neighbourhood of 0 and x in an open set of M. Any local 1-parameter group of local transformations of M induces a vector field X by assigning to $x \in M$ the vector tangent to the curve $t \mapsto \phi_t(x)$ at 0. Conversely, for every vector field X on M and every $x \in M$, there exists an open neighbourhood U of x and a local 1-parameter group ϕ_t of local transformations of M defined on U such that X is the vector field induced by ϕ_t ([18], Chapter I, Proposition 1.5).

   The *isometry group* $\mathrm{Isom}(M)$ of M is the group of all transformations of M which are isometries, i. e. for $\phi \in \mathrm{Isom}(M)$, $X, Y \in T_xM$, $x \in M$ we have $g(d\phi(X), d\phi(Y)) = g(X, Y)$. The Myers-Steenrod theorem ([24]) states that this is a Lie group.

Now, we note some general facts from differential geometry.

Remark A.2. We recall the following two well-known formulas. Given a scalar or vector space-valued 1-form α on a manifold M and vector fields X, Y defined on (a subset of) M, then ([18], Chapter I, Proposition 3.11)[64]

$$(d\omega)(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \tag{A.1}$$

(Here and throughout the text, we use the standard notation $X(f)$ for the partial derivative of a function f in the direction of a vector X.)

For the Lie derivative, we have the following important formula (see [18], Chapter I, Proposition 1.9; in some cases this formula is used as a definition): Given two vector fields X, Y on a manifold M such that X generates a local 1-parameter group of transformations ϕ_t of M, we have

$$[X,Y]_x = \lim_{t\to 0} \frac{1}{t}(Y_x - (d\phi_t(Y))_x) \tag{A.2}$$

for all $x \in M$.

The next two lemmas deal with 1-parameter groups of diffeomorphisms, first in the general case, then concerning the special case of right translations on a principal bundle. In particular, the latter one might give a concise motivation for the importance of the adjoint representation of a Lie group.

Lemma A.3 ([18], Chapter I, Proposition 1.7)**.** *Let M, M' be manifolds and $\phi : M \to M'$ be a diffeomorphism. If a vector field X generates a local* 1*-parameter group of local transformations ϕ_t of M, then the vector field $d\phi(X)$ generates the local* 1*-parameter group of local diffeomorphisms $\phi \circ \phi_t \circ \phi^{-1}$ of M'.*

[64] Note however that ref. [18] uses a different convention for the exterior derivative (and for wedge products) than this document. Moreover, they give the proof only in the scalar case, but the vector space-valued case can be proven in the same manner (since for vector-space valued forms the exterior differential simply acts componentwise).

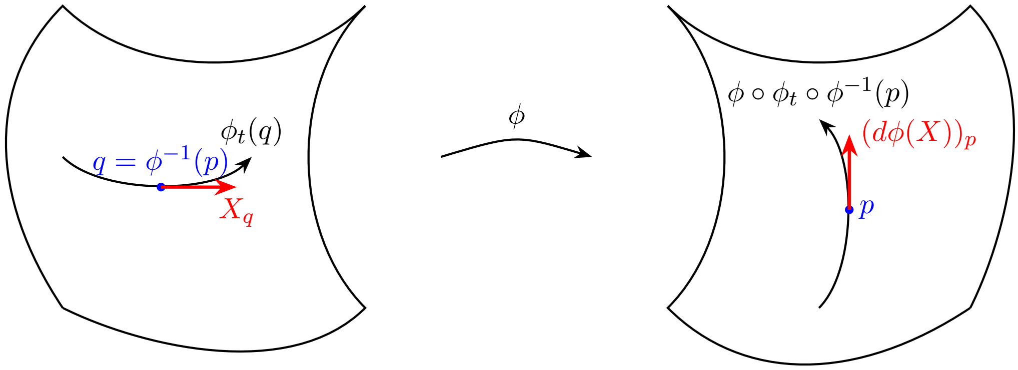

Figure 2: Illustration of the proof of Lemma A.3

Proof. The proof follows almost immediately from the definitions and may be illustrated by Figure 2.

To be precise: It follows from the definitions that $\phi \circ \phi_t \circ \phi^{-1}$ is a local 1-parameter group of local transformations, thus it suffices to show that it induces the vector field $d\phi(X)$. So let $p \in M'$; we denote $q := \phi^{-1}(p)$. Since ϕ_t induces the vector field X, the vector $X_q \in T_qM$ is tangent to the curve $x(t) = \phi_t(q)$ at $q = x(0)$. Therefore, the vector

$$(d\phi(X))_p = d\phi(X_q) \in T_pM'$$

is tangent to the curve $y(t) = \phi \circ \phi_t(q) = \phi \circ \phi_t \circ \phi^{-1}(p)$. $\square$

Lemma A.4 ([18], Chapter I, Proposition 5.1)**.** *Let M be a manifold, G be a Lie group and $P \to M$ be a principal G-bundle. Let A^* be the fundamental vector field on P corresponding to some $A \in \mathfrak{g}$. Then, for all $g \in G$, the vector field $dR_g(A^*)$ is the fundamental vector field corresponding to* $(\mathrm{ad}(g^{-1}))(A) \in \mathfrak{g}$.

Proof. By definition of the fundamental vector field, A^* is induced by the one-parameter group of transformations $t \mapsto R_{a_t}$, where $a_t := \exp(tA)$ (for all t). By Lemma A.3 (with $M = M' = P$), $dR_g(A^*)$ is induced by the one-parameter group of diffeomorphisms given by $t \mapsto R_g R_{a_t} R_{g^{-1}} = R_{g^{-1}a_t g}$. At this point, the lemma follows from the fact that $t \mapsto g^{-1}a_t g$ is precisely the one-parameter group of diffeomorphisms of G generated by $(\mathrm{ad}(g^{-1}))(A) \in \mathfrak{g}$. $\square$

A.2. Connections

In the following, we will state some general definitions and results from the theory of connections on principal and vector bundles.

Definition A.5 ([16], Section 2.1.2, Section 2.1.3, [18], Chapter II, Section 1, Chapter III, Section 2)**.** Let M be a manifold and let G be a Lie group with Lie algebra $\mathfrak{g}$.

1. A *connection* on a principal G-bundle $\pi : P \to M$ is a so-called *horizontal* vector subbundle HTP (with fibres HT_pP, $p \in P$, the *horizontal tangent space* of P at p) of TP that is invariant under the G-action on P, smooth and such that TP is the

direct sum of VTP and HTP, that is, at each $p \in M$, $VT_pP \oplus HT_pP = T_pP$. We will use the letter Γ to denote such a connection.

We call a vector vertical if it lies in the vertical tangent space and horizontal if it lies in the horizontal tangent space. Given any $X \in T_pP$, $p \in P$ (or a vector field X on P), we denote by VX its vertical component, i. e. the component corresponding to VT_pP (or VTP) in this splitting, and by HX its horizontal component, i. e. the component corresponding to HT_pP in this splitting. We call a piecewise-differentiable curve $c : [0,1] \to P$ horizontal if its tangent vectors lie in HTP for all $t \in [0,1]$. Note that given any smooth vector field X on P, since HTP is a smooth vector subbundle, the vector field HX is again smooth. Moreover, then also $VX = X - HX$ is smooth.

Given any $x \in M$, $p \in P_x$, the map $d_p\pi|_{HT_pP} : HT_pP \to T_xM$ is an isomorphism. Given any $X \in T_xM$, we call the inverse image of X under this isomorphism the *horizontal lift* of X to P; analogously we do this for vector fields. Given any piecewise-differentiable curve $c : [0,1] \to M$, there exists a unique horizontal lift of the curve c, i. e. a piecewise-differentiable curve $c_1 : [0,1] \to M$, such that for all $t \in [0,1]$, we have $\pi(c_1(t)) = c(t)$ and the vector $\dot{c}_1(t) \in T_{c_1(t)}P$ is the horizontal lift of $\dot{c}(t) \in T_{c(t)}M$.

If $H \subseteq G$ is a Lie subgroup and $Q \subseteq P$ a reduction of P to a principal H-bundle, and if moreover $HT_qP \subseteq T_qQ$ for all $q \in Q$, then the restriction of Γ to Q is a connection on Q. We call this restriction also a *reduction* of the connection Γ to Q. If $P = L(M)$, then Q is a H-structure on M and we call the connection Γ (respectively its restriction) then also a *H-connection*.

The *connection 1-form* of the connection Γ is the $\mathfrak{g}$-valued 1-form ω on P that is defined as follows: Given any $X \in T_pP$, $p \in P$, the vector VX is vertical. Now, $VT_pP \cong \mathfrak{g}$ by the assignment $Y \mapsto Y_p^*$, where Y^* is the fundamental vector field corresponding to $Y \in \mathfrak{g}$. Hence, $VX = Y*_p$ for some unique $Y \in \mathfrak{g}$ and we set $\omega(X) := Y$.

If moreover $P = L(M)$, for any $\xi \in \mathbb{R}^n$ the connection Γ defines a notion of a *standard horizontal vector field* $B(\xi)$ on P: Given any $u \in P$, we set $(B(\xi))_u$ to be the unique horizontal vector at u such that $\pi((B(\xi))_u) = u(\xi)$. Then, $\theta(B(\xi)) = \xi$ if θ is the canonical form on $L(M)$.

2. A *connection* on a vector bundle $\pi : E \to M$ is a linear map $\nabla^E : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the condition

$$\nabla^E(\alpha e) = \alpha\nabla^E e + d\alpha \otimes e,$$

whenever $e \in \Gamma(E)$ is a smooth section of E and α is a smooth function on E. If E is understood, we drop it from the notation. If ∇^E is a connection on E, $e \in \Gamma(E)$ and $x \in M$, $X \in T_xM$ is a vector or $X \in \Gamma(TM)$ is a vector field, then we also write $\nabla^E_X e = X \cdot \nabla^E e$, where the dot means tensor contraction of the TM-factor and the T^*M-factor.

We call section $c_1 : [0,1] \to E|_{c([0,1])}$ over a piecewise-differentiable curve $c : [0,1] \to M$ *horizontal* or *parallel* if $\nabla^E_{\dot{c}(t)} c_1(t) = 0$ for all t. We call a vector in TE horizontal if it can be written as the tangent vector to a horizontal curve. The connection ∇^E induces a splitting $TE = VTE \oplus HTE$, where VT_eE consists at each point $e \in E$ of the vectors tangent to the fiber of e (the vertical subbundle) and HT_eE consists of the horizontal vectors at e.

Remark A.6. Given a connection Γ on a principal G-bundle $\pi : P \to M$ and $\rho : G \to \mathrm{GL}(V)$ is a representation of G on a vector space V, we can define a connection ∇^E on the vector bundle $E := \rho(P)$ as follows: The map $\pi : P \to M$ extends by construction to a bundle map from $P \times V$ onto E. Since every bundle map between vector bundles is a pullback map ([6], Chapter 14.2), we may pull back any section $e \in \Gamma(E)$ along π to a section $\pi^*(e) \in \Gamma(P \times V)$ (over P). Then, the exterior derivative $d\pi^*(e)$ is a section of $T^*P \otimes (P \times V)$.[65]

Now, the connection Γ on P induces a splitting $TP = VTP \oplus HTP$ and by dualizing also a splitting $T^*P = VT^*P \oplus HT^*P$. Tensoring with $P \times V$, this yields a splitting $T^*P \otimes (P \times V) = V(T^*P) \otimes (P \times V) \oplus H(T^*P) \otimes (P \times V)$. Now, since $HTP \cong \pi^*(TM)$,[66] we also have $H(T^*P) \cong \pi^*(T^*M)$. Hence, $H(T^*P) \otimes (P \times V)$ is canonically isomorphic to $\pi^*(T^*M \otimes E)$.

Let us denote the G-invariant sections of a bundle F over P by $\Gamma_{G-\mathrm{inv}}(F)$. By definition of E, we have $\pi^*(f) \in \Gamma_{G-\mathrm{inv}}(P \times V)$ for all $f \in \Gamma(E)$ (that is, $\pi^*(f)|_{pg} = R_g^*(\pi^*(f)|_p)$ for all $g \in G$, $p \in P$). In fact, $\pi^* : \Gamma(E) \to \Gamma_{G-\mathrm{inv}}(P \times V)$ is an isomorphism, since an inverse is given by assigning to a G-invariant section f of $P \times V$ the map $x \mapsto \Pi(f(p))$, where Π is the bundle map from $P \times V$ to E and for any $x \in M$ one chooses an arbitrary p in the fibre of P over x. Analogous remarks hold true for $H(T^*P) \otimes (P \times V)$ and $T^*M \otimes E$.

Moreover, if we denote the projection from $T^*P \otimes (P \times V)$ to $H(T^*P) \otimes (P \times V)$ by pr, then it follows from the definition of the connection on the dual bundle that then $(\mathrm{pr} \circ d)(\pi^*(f)) \in \Gamma_{G-\mathrm{inv}}(H(T^*P) \otimes V)$, since for $p \in P$ and $g \in G$ we have

$$\mathrm{pr}(d(\pi^*(f))|_{pg}) = \mathrm{pr}(d(R_g^*(\pi^*(f)))|_p) = \mathrm{pr}(R_g^*(d\pi^*(f)|_p)) = R_g^*(\mathrm{pr}(d\pi^*(f)|_p)).$$

By the above, it makes sense to consider the following diagram:

$$\begin{tikzcd}
\Gamma_{G-\mathrm{inv}}(P \times V) \arrow[d, "\mathrm{pr}\circ d"] & \Gamma(E) \arrow[l, "\pi^*"'] \arrow[d, dashed, "\nabla^E"] \\
\Gamma_{G-\mathrm{inv}}(H(T^*P) \otimes (P \times V)) & \Gamma(T^*M \otimes E) \arrow[l, "\pi^*"]
\end{tikzcd}$$

Since the lower map is an isomorphism, we can now define ∇^E, as indicated, to be the unique map such that the diagram commutes.

By construction, if we define the horizontal subbundles HTP of P (w.r.t. Γ) and HTE of E (w.r.t. ∇^E) as above and consider $HT_pP \times V$ as a subspace of $T_{(p,v)}(P \times V)$ for $p \in P$, $v \in V$, we have $d\Pi(HTP \times V) = HTE$.

If $G = \mathrm{GL}(\mathbb{R}, k)$, $k \in \mathbb{N}$, and the representation ρ defining E is the standard representation of G on $\mathbb{R}^k$, then P is the frame bundle of E. In this case, the above assignment $\Gamma \mapsto \nabla^E$ of a connection on E to a connection on P is bijective. In this sense, a connection on a vector bundle is a special case of a connection on a suitably chosen principal bundle, and for this reason it often suffices to prove statements in the principal bundle case. For example, the Ambrose-Singer theorem 1.7 does for this reason analogously hold true for vector bundles.

[65] Here we act the exterior derivative simply act componentwise on the trivial bundle $P \times V$. This coincides with the notion of exterior derivative we would have obtained if we regarded $\pi^*(e)$ as a vector space-valued function $P \to V$.

[66] Here we use again that every bundle map is a pullback (note that in the terminiology of ref. [6], a bundle map is a fiberwise isomorphism).

We will often use Remark A.6, implicitly or explicitly.

Between connections on principal bundles and connection 1-forms, we have the following easy relation.

Lemma A.7 ([18], Chapter II, Proposition 1.1)**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and P be a principal G-bundle.*

1. *Let Γ be a connection on P. Then, the connection form ω corresponding to Γ has the following properties:*
    a) *$\omega(A^*) = A$ for every $A \in \mathfrak{g}$, where A^* denotes the fundamental vector field generated by A.*
    b) *$(R_a)^*\omega = \mathrm{ad}(a^{-1})\omega$, that is, $\omega(dR_a(X)) = \mathrm{ad}(a^{-1})(\omega(X))$ for every $a \in G$ and every vector field X on P, where* ad *denotes the adjoint representation of G in $\mathfrak{g}$.*[67]
2. *Conversely, given a $\mathfrak{g}$-valued 1-form ω on P satisfying conditions a) and b) above, there is a unique connection Γ in P whose connection form is ω.*

Proof. Ad 1. Property a) follows directly from the definition of the connection 1-form. Property b) can be deduced as follows: Since $TP = VTP \oplus HTP$, it suffices to consider the cases that X is a vertical horizontal vector field on P. If X is horizontal, then $dR_a(X)$ is also horizontal for every $a \in G$ since HTP is invariant under the G-action on P. Hence, $\omega(dR_a(X)) = 0 = \mathrm{ad}(a^{-1})(\omega(X))$. If X on the other hand is vertical, then we may further assume that X is a fundamental vector field A^* corresponding to some $A \in \mathfrak{g}$. Then, $dR_a(X)$ is the fundamental vector field corresponding to $\mathrm{ad}(a^{-1})(A)$ by Lemma A.4. Thus

$$(R_a^*\omega)_p(X) = \omega_{pa}(dR_a(X)) = \mathrm{ad}(a^{-1})A = \mathrm{ad}(a^{-1})(\omega_u(X))$$

for every $p \in P$, proving b).

Ad 2. Given the 1-form ω satisfying conditions a) and b) above, we set

$$HT_pP := \{X \in T_pP \,|\, \omega(X) = 0\}$$

for all $p \in P$. The assignment $p \mapsto HT_pP$ defines a connection Γ on P with connection form ω. Moreover, Γ with this property is unique. □

We continue by introducing the curvature form of a connection, as well as the torsion form of a connection on $L(M)$. Afterwards, we define curvature and torsion tensors as it is convenient for this document.

Definition A.8 ([18], Chapter II, Section 5, Chapter III, Section 2, Section 5)**.** Let M be a manifold of dimension n, let G be a Lie group and let $\pi : P \to M$ be a principal G-bundle with connection Γ.

1. Let ρ be a representation of G on a vector space V and ϕ be a pseudotensorial form on P of type (ρ, V). We define the *exterior covariant derivative* $D\phi$ of ϕ by the formula
$$D\phi(X_1, \dots, X_r) := (d\phi)(HX_1, \dots, HX_r)$$

[67] In other words, ω is a differential form of type $\mathrm{ad}\, G$.

for $r-1$ the degree of ϕ and $X_1, \dots, X_r$ vectors on P. This is a pseudotensorial form on P of type (ρ, V) which vanishes on vertical vectors by construction. We also call such forms *tensorial.*

2. Let ω be the connection form on P. We define the *curvature form* of Γ by

$$\Omega := D\omega.$$

This is a tensorial 2-form of type $\operatorname{ad} G$.

3. Assume moreover that $G = \mathrm{GL}(n)$, $P = L(M)$ and let θ be the canonical form on $L(M)$. We then define the *torsion form* of Γ by

$$\Theta := D\theta.$$

This is a tensorial 2-form of type $(\rho, \mathbb{R}^n)$, where ρ is the standard representation of $\mathrm{GL}(n)$ on $\mathbb{R}^n$.

4. If $G = \mathrm{GL}(k)$ for some $k \in \mathbb{N}$ and ρ is the identity representation, then the elements of P may be regarded as the frames on a vector bundle E on M (i. e. isomorphisms $\mathbb{R}^k \to E_x$, $x \in M$). We define then the *curvature tensor* R^E on M as follows: Given $x \in M$, $X, Y \in T_xM$, $Z \in E_x$, we choose $u \in P_x$, $X', Y' \in T_uP$ with $d\pi(X') = X$, $d\pi(Y') = Y$ and set

$$R^E(X,Y)Z := u((\Omega(X',Y'))(u^{-1}z)).$$

This is a section of $\Lambda^2 T^*M \otimes \mathrm{End}(E)$. If it is clear from the context, we drop the superscript.

If moreover $k = n$, $P = L(M)$, we define the *torsion tensor* T on M by setting

$$T(X,Y) := u(\Theta(X',Y')).$$

This is a section of $\Lambda^2 T^*M \otimes TM$.

By [18], Chapter III, Theorem 5.1 we obtain the following, perhaps better known formulas for the curvature and torsion tensors: For all vector fields X, Y, Z on M and a connection on $L(M)$ with covariant derivative ∇ on TM,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{A.3}$$

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z. \tag{A.4}$$

The following two lemmas consist of observations which are easy to prove, but find valuable application e. g. in the proof of the Ambrose-Singer theorem.

Lemma A.9 (cf. [1], proof of Theorem 2)**.** *Let Γ be a connection on a principal bundle P. Let ω be the corresponding connection 1-form and let Ω be its curvature form. Let X, Y be horizontal vector fields on P and $p \in P$. Then,*

$$\Omega(X_p, Y_p) = -\omega([X,Y]|_p). \tag{A.5}$$

Proof. By definition and eq. (A.1) we have

$$\begin{aligned} \Omega(X_p, Y_p) =& (D\omega)(X_p, Y_p) = (d\omega)(HX_p, HY_p) \\ =& HX_p(\omega(HY)) - HY_p(\omega(HX)) - \omega([HX, HY]|_p). \end{aligned}$$

Since ω vanishes by definition on all horizontal vectors and $HX = X$, $HY = Y$, this reduces to the claimed formula. $\square$

Lemma A.10 ([18], Chapter II, Section 5, Lemma)**.** *Let M be a manifold, G be a Lie group with Lie algebra $\mathfrak{g}$ and $P \to M$ be a principal G-bundle with connection. Let K be a horizontal vector field on P, $Q \in \mathfrak{g}$ and Q^* denote the fundamental vertical vector field on P generated by Q. Then, $[K, Q^*]$ is horizontal.*

Proof. Let $c \in P$. The flow of Q^* is given by $t \mapsto R_{\exp(tQ)}$ and hence maps the fiber of c onto itself. Now, it is well known (see e. g. [18], Chapter I, Proposition 1.9) that then

$$[K, Q^*] = \lim_{t\to 0} \frac{1}{t}(dR_{\exp(Qt)}(K) - K).$$

If K is horizontal, so is $dR_{\exp(Qt)}(K)$, hence $[K, Q^*]$ is horizontal. □

Finally, we prove the well-known structure equations for connections on a principal bundle.

Theorem A.11 ([18], Chapter II, Theorem 5.2, Chapter III, Theorem 2.4)**.** [68]*Let $\pi : P \to M$ be a principal bundle over a manifold M. Let ω be the connection form of a connection on P and let Ω be the corresponding curvature form. Then, the following* structure equation *holds true: For all $X, Y \in T_pP$, $p \in P$,*

$$d\omega(X, Y) = -[\omega(X), \omega(Y)] + \Omega(X, Y). \tag{A.6}$$

Moreover, if $P = L(M)$, θ denotes the canonical form on $L(M)$ and Θ the torsion form, then the following structure equation holds true:[69]

$$d\theta(X, Y) = -\omega(X)\theta(Y) + \omega(Y)\theta(X) + \Theta(X, Y). \tag{A.7}$$

Proof. We begin by proving eq. (A.6). Let $p \in P$ and $X, Y \in T_pP$. Every vector in TP is the sum of a vertical vector and a horizontal vector. Since both sides of eq. (A.6) are bilinear and skew-symmetric in X and Y, it suffices to prove the equality in the following three special cases.

(1) X and Y are horizontal. In this case $\omega(X) = \omega(Y) = 0$ and $d\omega(X, Y) = D\omega(X, Y) = \Omega(X, Y)$.

(2) X and Y are vertical. Let $A, B \in \mathfrak{g}$ with $A^*_p = X$ and $B^*_p = Y$. Then, by eq. (A.1)

$$\begin{aligned} d\omega(X, Y) =& d\omega(A^*_p, B^*_p) = A^*(\omega(B)) - B^*(\omega(A)) - \omega([A^*, B^*]) \\ =& -\omega([A, B]^*) - [A, B] = -[\omega(A^*_p), \omega(B^*_p)] = -[\omega(X), \omega(Y)] \end{aligned}$$

and $\Omega(X, Y) = 0$, thus eq. (A.6) holds true.

(3) X is horizontal and Y is vertical. We extend X to a horizontal vector field on P, still called X. Moreover, let $A \in \mathfrak{g}$ with $A^*_p = Y$. Note that the right-hand side of eq. (A.6) vanishes, thus it suffices to show that $d\omega(X, A^*) = 0$. Now, again by eq. (A.1)

$$d\omega(X, A^*) = X(\omega(A^*)) - A^*(\omega(X)) - \omega([X, A^*]) = -\omega([X, A^*]).$$

Eq. (A.6) follows now from Lemma A.10.

[68] Recall however again that ref. [18] uses different conventions for prefactors in wedge products and exterior derivatives.

[69] Here $\omega(X)$ respectively $\omega(Y)$ are used as endomorphisms of $\mathbb{R}^{\dim(M)}$.

In order to prove eq. (A.7), we will need the following claim.

Claim. If A^* is the fundamental vector field on P corresponding to some $A \in \mathfrak{g}$ and $B(\xi)$ is the standard horizontal vector field corresponding to $\xi \in \mathbb{R}^{\dim(M)}$, then

$$[A^*, B(\xi)] = B(A\xi). \tag{A.8}$$

Proof of the claim. Note first that for $g \in \mathrm{GL}(\dim(M))$, we have $dR_g(B(\xi)) = B(g^{-1}\xi)$, since if $p \in P$ and $X_p \in T_pM$ is horizontal, then $dR_g(X_p)$ is a horizontal vector at pg having the same projection to M as X_p. Now, let $a_t := \exp(tA)$ be the 1-parameter subgroup of $\mathrm{GL}(\dim(M))$ generated by A. Using eq (A.2), we compute

$$[A^*, B(\xi)] = \lim_{t\to 0} \frac{1}{t}(B(\xi) - dR_{a_t}(B(\xi))) = \lim_{t\to 0} \frac{1}{t}(B(\xi) - B(a_t^{-1}\xi)).$$

Now, for any $p \in P$, the map $\xi \mapsto (B(\xi))_p$ is a linear isomorphism from $\mathbb{R}^n$ onto HT_pP and therefore

$$\lim_{t\to 0}(B(\xi) - B(a_t^{-1}\xi)) = B\left(\lim_{t\to 0} \frac{1}{t}(\xi - a_t^{-1}\xi)\right) = B\left(\lim_{t\to 0} a_t\xi - \xi\right) = B(A\xi),$$

and we obtain eq. (A.8).

Now, we can prove eq. (A.7). Again, both sides are bilinear and skew-symmetric in X and Y, hence it suffices to consider the same three cases as in the proof of eq. (A.6).

(1) X and Y are horizontal. The terms containing ω on the right-hand side vanish and the equation becomes just the definition of Θ.

(2) X and Y are vertical. In this case, extending X and Y to fundamental vertical vector fields on P, the equation reduces to $0 = 0$.

(3) X is horizontal and Y is vertical. Let $\xi \in \mathbb{R}^{\dim(M)}$ and $A \in \mathfrak{g}$ such that $X = B(\xi)_p$ (i. e. X is the value at p of the standard horizontal vector field corresponding to ξ) and $Y = A_p^*$. Then, $\Theta(X,Y) = 0$, $\omega(X)\theta(Y) = 0$ and $\omega(Y)\theta(X) = \omega(A^*)\theta(B(\xi)) = A\xi$ since $\omega(A^*) = A$ and $\theta(B(\xi)) = \xi$. On the other hand,

$$\begin{aligned} d\theta(X,Y) =& B(\xi)(\theta(A^*)) - A^*(\theta(B(\xi))) - \theta([B(\xi), A^*]) \\ =& -\theta([B(\xi), A^*]) = \theta(B(A\xi)) = A\xi \end{aligned}$$

by the claim (and since $\theta(A^*)$ and $A^*(\theta(B(\xi)))$ are zero). This proves eq. (A.7) also in this case. □

It shall be remarked that if $P = L(M)$, then eq. (A.7) is occasionally called the *first* structure equation, while eq. (A.6) is called the *second* structure equation. This might be regarded as slightly misleading, since in the case of a general principal bundle with connection only the second exists, but not the first.

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Erklärung

Dies ist eine redigierte Version der Arbeit, in der Tippfehler und fachliche Fehler bereinigt wurden; somit ist sie nicht mehr identisch zu der ursprünglich eingereichten. Dennoch habe ich die Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt, sowie die Rückmeldung von Prof. Dr. Ammann.

Regensburg, den 13.5.2025

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