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An integral representation for the Dirac propagator in the Reissner–Nordström geometry in Eddington–Finkelstein coordinates

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Abstract

The Cauchy problem for the massive Dirac equation is studied in the Reissner–Nordström geometry in horizon-penetrating Eddington–Finkelstein-type coordinates. We derive an integral representation for the Dirac propagator involving the solutions of the ordinary differential equations which arise in the separation of variables. Our integral representation describes the dynamics of Dirac particles outside and across the event horizon, up to the Cauchy horizon.

Keyword Black holes · Dirac equation · Reissner-Nordström metric · horizon-penetrating coordinates

Mathematics Subject Classification 35Q75 · 83C60 · 81T20

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1 Introduction

The Dirac equation in curved spacetime describes the dynamics of quantum mechanical waves in the presence of classical gravitational fields. The propagation of Dirac waves in black hole geometries is of particular interest with respect to Hawking radiation, the stability of black holes and the fermionic signature operator. So far, this problem has been studied mainly in the exterior region of the black hole [6, 7]. Here we turn attention to the behavior of Dirac waves across and inside the event horizon. We consider the Reissner–Nordström geometry, which describes a spherical symmetric and charged black hole. Our starting point is the Dirac equation in Eddington–Finkelstein coordinates as derived in [16]. Our main result is to derive a corresponding integral representation of the propagator

$$\psi(\tau) = e^{-i\tau H}\psi_0 = \int_{\sigma(H)} e^{-i\omega\tau}\, \mathrm{d}E_\omega \psi_0 \,,$$

where $\mathrm{d}E_\omega$ is the spectral measure of the Dirac Hamiltonian H, and ψ_0 is smooth initial data with compact support. We express the spectral measure explicitly in terms of the fundamental solutions of the radial ordinary differential equation (ODE) arising in Chandrasekhar's separation of variables (see Theorem 5.5). We remark that similar results have been derived previously in the Kerr geometry [12]. The novel feature of the present paper is the charge of the black hole. Moreover, we work out the integral representation in more detail and simplify the formulas considerably. Our integral representation will be used in a follow-up paper to compute the spectrum of the fermionic signature operator [9].

The paper is organized as follows. In Section 2 we give the necessary preliminaries on the Dirac equation in globally hyperbolic spacetimes and on its separation in the Reissner–Nordström geometry in Eddington–Finkelstein-type coordinates. In Section 3 we introduce Dirichlet-type boundary conditions inside the Cauchy horizon and show that the resulting Hamiltonian is essentially self-adjoint. Moreover, we express the spectral measure via Stone's formula in terms of the resolvent. In Section 4 the resolvent is computed in terms of the fundamental solutions. To this end, we construct Jost solutions and use the conservation law for the radial flux in order to compute the Green's matrix. In Section 5 we use the obtained formulas for the resolvent in order to express the spectral measure explicitly in terms of the fundamental solutions. This gives the simple and useful analytic expression for the Dirac propagator as stated in Theorem 5.5.

2 Preliminaries

2.1 The Dirac equation in a globally hyperbolic spacetime

We begin with preliminaries on the Dirac equation in globally hyperbolic spacetimes, following the presentation in [10]. Thus, let $(\mathscr{M}, g)$ be a four-dimensional, smooth, globally hyperbolic Lorentzian spin manifold. For the signature of the metric we use the convention $(+, -, -, -)$. As proven in [1], $\mathscr{M}$ admits a smooth foliation $(\mathscr{N}_\tau)_{\tau\in\mathbb{R}}$ by Cauchy hypersurfaces. We denote the corresponding spinor bundle by $S\mathscr{M}$. Its fibers $S_x\mathscr{M}$ are endowed with an inner product $\prec.|.\succ_x$ of signature $(2, 2)$. The smooth sections of the spinor bundle are denoted by $C^\infty(\mathscr{M}, S\mathscr{M})$. Likewise, $C_0^\infty(\mathscr{M}, S\mathscr{M})$ are the smooth sections with compact support. We also refer to these sections as wave functions and usually denote them by ψ or ϕ. On the wave functions, we introduce the Lorentz invariant inner product

$$\begin{aligned} \langle \cdot | \cdot \rangle \ &: C^\infty(\mathscr{M}, S\mathscr{M}) \times C_0^\infty(\mathscr{M}, S\mathscr{M}) \longrightarrow \mathbb{C}, \\ \langle \psi | \phi \rangle &:= \int_{\mathscr{M}} \prec \psi | \phi \succ_x \, d\mu_{\mathscr{M}} . \end{aligned}$$

We consider the Dirac equation for a given mass parameter $m \geq 0$. We write the Dirac equation as

$$(\mathcal{D} - m)\psi = 0 , \tag{2.1}$$

where the Dirac operator takes the form

$$\mathcal{D} = iG^k \partial_k + \mathcal{B} \,:\, C^\infty(\mathscr{M}, S\mathscr{M}) \longrightarrow C^\infty(\mathscr{M}, S\mathscr{M}) ,$$

and $G^k : T_x\mathscr{M} \longrightarrow L(S_x\mathscr{M})$ are the Dirac matrices. They fulfill the anti-commutation relations

$$\{G^j, G^k\} = 2\, g^{jk}\, \mathbb{1}_{S_x\mathscr{M}} .$$

One can understand this map as a representation of the Clifford multiplication in components of the general Dirac matrices. We will use the Feynman dagger notation reading $\not{v} = G^j v_j$. The connection part of the covariant derivative is summarized in the term $\mathcal{B}$. We remark that the Dirac equation can be written alternatively as $\mathcal{D} = iG^j \nabla_j$, where ∇ is the Levi-Civita spin connection on $S\mathscr{M}$. For more details on the Dirac equation in curved spacetimes, we refer to [10] or [17].

Given initial data on a Cauchy surface, the Dirac equation admits unique global solutions. Choosing compactly supported initial data, due to finite propagation speed, the resulting solution also has compact support on any other Cauchy surface. Such solutions are referred to as being *spatially compact*. The smooth, spatially compact solutions are denoted by $C_{sc}^\infty(\mathscr{M}, S\mathscr{M})$. On such solutions, one has the scalar product

$$(\psi | \phi) = \int_{\mathscr{N}} \prec \psi \,|\, \nu^j G_j\, \phi \succ_x \, d\mu_{\mathscr{N}}(x) , \tag{2.2}$$

where ν is the future directed-normal on $\mathscr{N}$. (Due to current conservation, the scalar product is in fact independent of the choice of $\mathscr{N}$; for details see [10, Section 2].) Forming the completion gives the Hilbert space $(\mathcal{H}, (\cdot|\cdot))$.

In this paper, we restrict attention to *stationary* spacetimes, meaning that there is a Killing field K which is asymptotically timelike (for the general definition see [18]). We always choose the foliation $(\mathscr{N}_\tau)_{\tau\in\mathbb{R}}$ such that the Killing field is given by $K = \partial_\tau$. In this case, it is useful to write spacetime as a product $\mathscr{M} = \mathbb{R}\times\mathscr{N}$. Moreover, we can write the Dirac equation in the Hamiltonian form

$$i\partial_\tau\psi = H\psi \qquad \text{with} \qquad H := -(G^\tau)^{-1}\Big(i\sum_{\alpha=1}^{3} G^\alpha\partial_\alpha + \mathcal{B} - m\Big)\,, \tag{2.3}$$

where the Hamiltonian H is an operator acting on $\mathcal{H}$ with dense domain

$$\mathcal{D}(H) = C_0^\infty(\mathscr{N}, S\mathscr{M})\,.$$

Lemma 2.1 *In a stationary spacetime, the Hamiltonian H with domain $\mathcal{D}(H)$ is a symmetric operator on $\mathcal{H}$.*

Proof Since the scalar product in (2.2) is independent of the choice of the Cauchy surface, we know that for all $\psi, \phi \in \mathcal{D}(H)$,

$$0 = \partial_\tau(\psi|\phi)\,.$$

Since the Dirac matrices G^k, as well as the normal vector field ν and the volume form, do not depend on τ, we only need to differentiate the wave functions. We thus obtain

$$0 = (\partial_\tau\psi|\phi) + (\psi|\partial_\tau\phi) = -i\Big((H\psi|\phi) - (\psi|H\phi)\Big)\,.$$

This concludes the proof. □

2.2 The Dirac equation in the Reissner–Nordström geometry

We work in Eddington–Finkelstein coordinates $(\tau, r, \vartheta, \varphi)$ in the range $\mathbb{R}\times(0,\infty)\times(0,\pi)\times[0,2\pi)$ as defined in [2, 16]. In these coordinates, the metric takes the form

$$\begin{aligned} g = \frac{\Delta}{r^2}\,\mathrm{d}\tau\otimes\mathrm{d}\tau - \left[2-\frac{\Delta}{r^2}\right]\mathrm{d}r\otimes\mathrm{d}r - \left[1-\frac{\Delta}{r^2}\right]\Big(\mathrm{d}\tau\otimes\mathrm{d}r + \mathrm{d}r\otimes\mathrm{d}\tau\Big) \\ - r^2\,\mathrm{d}\vartheta\otimes\mathrm{d}\vartheta - r^2\,\sin(\vartheta)^2\,\mathrm{d}\varphi\otimes\mathrm{d}\varphi \end{aligned} \tag{2.4}$$

with $\Delta \equiv \Delta(r) = r^2 - 2Mr + Q^2$ and $\tau = t + u - r$. Here u is the Regge–Wheeler coordinate ("tortoise coordinate") which is defined in terms of r by

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{r^2}{\Delta}\,. \tag{2.5}$$

The zeros of the function Δ denoted by

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

describe the event horizon and Cauchy horizon, respectively. The region $\{r > r_-\}$ outside the Cauchy horizon is a globally hyperbolic spacetime. The interior of the Cauchy horizon $\{r < r_-\}$, however, is not globally hyperbolic because of the spacetime singularity at $r = 0$. Our spacetime has the topology $\mathscr{M} \cong \mathbb{R}^2 \times S^2$.

In [16] the Dirac equation was computed and separated in a specific gauge in which the Dirac matrices are in the Weyl representation. Starting from this representation, it is most convenient to transform the Dirac wave function and the Dirac operator as

$$\Psi = D\,\psi \tag{2.6}$$

$$\Gamma_{\text{trafo}}\, D\,(\mathcal{D} - m)\, D^{-1}\,\Psi = (\mathcal{R} + \mathcal{A})\,\Psi = 0 \tag{2.7}$$

with the transformation matrices

$$D := \frac{\sqrt{r}}{r_+}\begin{bmatrix} |\Delta|^{1/2} & 0 & 0 & 0 \\ 0 & r_+ & 0 & 0 \\ 0 & 0 & r_+ & 0 \\ 0 & 0 & 0 & |\Delta|^{1/2} \end{bmatrix}, \qquad \Gamma_{\text{trafo}} := r\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here $\mathcal{R}$ and $\mathcal{A}$ are the radial and angular operators given by

$$\mathcal{R} := \begin{bmatrix} irm & 0 & |\Delta|^{1/2}\mathfrak{D}_0 & 0 \\ 0 & -irm & 0 & |\Delta|^{-1/2}\mathfrak{D}_1 \\ |\Delta|^{-1/2}\mathfrak{D}_1 & 0 & -irm & 0 \\ 0 & |\Delta|^{1/2}\mathfrak{D}_0 & 0 & irm \end{bmatrix} \tag{2.8}$$

$$\mathcal{A} := \begin{bmatrix} 0 & 0 & 0 & \mathcal{L}_+ \\ 0 & 0 & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & 0 & 0 \\ -\mathcal{L}_- & 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

where the linear operators $\mathcal{D}_{0/1}$, $\mathcal{L}_\pm : C^\infty(\mathscr{M}, S\mathscr{M}) \longrightarrow C^\infty(\mathscr{M}, S\mathscr{M})$ have the from

$$\mathfrak{D}_0 := -\big(\partial_\tau - \partial_r\big) \tag{2.10}$$

$$\mathfrak{D}_1 := \big(2r^2 - \Delta\big)\partial_\tau + \Delta\,\partial_r \tag{2.11}$$

$$\mathcal{L}_\pm := \partial_\vartheta + \frac{\cot(\vartheta)}{2} \mp i\csc(\vartheta)\partial_\varphi\,. \tag{2.12}$$

In this formulation, the spin inner product takes the form

$$\prec\psi|\phi\succ_x = -\langle\Psi, \begin{pmatrix} 0 & \mathbb{1}_{\mathbb{C}^2} \\ \mathbb{1}_{\mathbb{C}^2} & 0 \end{pmatrix}\Phi\rangle_{\mathbb{C}^4}\,. \tag{2.13}$$

Employing the separation ansatz

$$\Psi = e^{-i(k+\frac{1}{2})\varphi} \frac{1}{r_+} \begin{bmatrix} X_+(\tau,r)\, Y_l(\vartheta)_+ \\ r_+ X_-(\tau,r)\, Y_l(\vartheta)_- \\ r_+ X_-(\tau,r)\, Y_l(\vartheta)_+ \\ X_+(\tau,r)\, Y_l(\vartheta)_- \end{bmatrix} \quad \text{with} \quad \omega \in \mathbb{R} \text{ and } k,\, l \in \mathbb{Z}\,, \tag{2.14}$$

we obtain the eigenvalue problems

$$\mathcal{R}\Psi = \xi\Psi \quad \text{and} \quad \mathcal{A}\Psi = -\xi\Psi$$

with a separation constant ξ. In this way, the Dirac equation decouples into a radial and angular ODE of the form

$$\begin{bmatrix} (2r^2 - \Delta)\,\partial_\tau + \Delta\partial_r & |\Delta|^{1/2}(imr - \xi) \\ -\epsilon(\Delta)|\Delta|^{1/2}(imr + \xi) & -\Delta(\partial_\tau - \partial_r) \end{bmatrix} \begin{pmatrix} X_+(\tau,r) \\ r_+ X_-(\tau,r) \end{pmatrix} = 0 \tag{2.15}$$

$$\left(\begin{bmatrix} 0 & \mathcal{L}_- \\ -\mathcal{L}_+ & 0 \end{bmatrix} - \xi\, \mathbb{1}_{\mathbb{C}^4} \right) \begin{pmatrix} Y_l(\vartheta)_+ \\ Y_l(\vartheta)_- \end{pmatrix} = 0 \tag{2.16}$$

where

$$\epsilon(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases} \tag{2.17}$$

is the sign function. In view of (2.12), the angular operator in (2.16) does not involve τ-derivatives. (Note that this is no longer the case in the Kerr geometry.) This angular operator is an essentially self-adjoint operator on $L^2(S^2, \mathbb{C}^2)$ with dense domain $C^\infty(S^2, \mathbb{C}^2)$, having a purely discrete spectrum (for details see [7, Section 3 and Appendix A]). More specifically, the angular operator is the spin-weighted spherical operator for $s = \frac{1}{2}$ as analyzed in [15]. We denote the corresponding orthonormal eigenvector basis by ${}_{\frac{1}{2}}Y_{kl} = e^{-i(k+\frac{1}{2})\varphi}\, Y_{kl}(\vartheta)$ with $l, k \in \mathbb{Z}$, i.e.,

$$\left\langle e^{-i(k+\frac{1}{2})\varphi}\, Y_{kl}(\vartheta),\, e^{-i(k'+\frac{1}{2})\varphi}\, Y_{k'l'}(\vartheta) \right\rangle_{L^2(S^2)} = \delta_{k,k'}\, \delta_{l,l'}\,. \tag{2.18}$$

Restricting attention to one angular momentum mode, it suffices to solve the PDE (2.15) in τ and r for $\xi = \lambda_{kl}$, where λ_{kl} denotes the corresponding angular eigenvalue. Since the angular eigenfunctions are orthonormal, the general Cauchy problem can be solved by decomposing the initial data into angular momentum modes, solving the PDE (2.15) for each mode and taking the superposition.

Similar to (2.3), the PDE (2.15) can again be written in the Hamiltonian form:

Lemma 2.2 *The radial equation* (2.15) *can be written in the Schrödinger-type form*

$$i\partial_\tau X(\tau, r) = H_\xi\, X(\tau, r) \tag{2.19}$$

with the Hamiltonian H_ξ given by

$$H_\xi := \underbrace{\begin{bmatrix} -\frac{i}{(2r^2-\Delta)} & 0 \\ 0 & \frac{i}{\Delta} \end{bmatrix}}_{=: C^{-1}(r) \in M(2,\mathbb{C})} \begin{bmatrix} \Delta\partial_r & |\Delta|^{1/2}(imr-\xi) \\ -\epsilon(\Delta)|\Delta|^{1/2}(imr+\xi) & \Delta\partial_r \end{bmatrix},$$

where $X(\tau, r)$ is the radial part of the wave function from (2.14).

Proof Follows by direct computation. □

The domain of the radial Hamiltonian in (2.19) will be discussed after Lemma 3.3 below.

Since ∂_τ is a Killing field, we can write $X(\tau, r) = e^{-i\omega\tau} X^{(\omega)}(r)$ and get a first-order radial ODE given by

$$\begin{bmatrix} -(2r^2-\Delta)i\omega + \Delta\partial_r & |\Delta|^{1/2}(imr-\xi) \\ -\epsilon(\Delta)\,|\Delta|^{1/2}(imr+\xi) & \Delta(i\omega+\partial_r) \end{bmatrix} \begin{pmatrix} X_+^{(\omega)}(r) \\ r_+ X_-^{(\omega)}(r) \end{pmatrix} = 0\,. \tag{2.20}$$

More details on the previous steps and the asymptotics of the solutions to the radial ODE (2.20) are worked out in [16]. We here state the main results, which can be obtained from [16, Theorem 1.1] by setting the angular momentum a equal to zero.

Lemma 2.3 *In the case $|\omega| < m$, one solution of the radial ODE (2.20) has exponential decay and the other one exponential growth for $u \to \infty$.*

In the case $|\omega| > m$, on the other hand, the solutions have following asymptotics:

(i) **(Asymptotics at infinity)** *Let $w_1 \in \mathbb{C}$ be the root of $\omega^2 - m^2$ contained in the convex hull of $\mathbb{R}_+$ and $\mathbb{R}_+ \cdot i$ and $w_2 = -w_1$ the other root, and let*

$$\Theta := \frac{1}{4}\ln\left(\frac{\omega-m}{\omega+m}\right),$$

then there is $f_\infty := (f_\infty^{(1)}, f_\infty^{(2)})^T \in \mathbb{R}^2\backslash\{0\}$ with

$$X^{(\omega)}(u) = \begin{bmatrix} \cosh(\Theta) & \sinh(\Theta) \\ \sinh(\Theta) & \cosh(\Theta) \end{bmatrix} \begin{pmatrix} f_\infty^{(1)} e^{i\Phi_+(u)} \\ f_\infty^{(2)} e^{-i\Phi_-(u)} \end{pmatrix} + E_\infty(u)$$

for the asymptotic phases

$$\Phi_\pm(u) := w_1\, u + M\left(\pm 2\omega + \frac{m^2}{w_1}\right)\ln(u) \tag{2.21}$$

and for an error function $E_\infty(u)$ with polynomial decay. More precisely, there is $c \in \mathbb{R}_+$ with

$$||E_\infty|| \leqslant \frac{c}{u}.$$

(ii) **(Asymptotics at the Cauchy horizon)** *For every non-trivial solution X,*

$$X^{(\omega)}(u) = \begin{pmatrix} h^{(1)}_{r_-} e^{2i\omega u} \\ h^{(2)}_{r_-} \end{pmatrix} + E_{r_-}(u)$$

with $h_{r_-} := (h^{(1)}_{r_-}, h^{(2)}_{r_-})^T \in \mathbb{R}^2 \backslash \{0\}$, *with* E_{r_-} *such that for r sufficiently close to* r_- *and suitable constants* $a, b \in \mathbb{R}_+$,

$$||E_-(u)|| \leq a e^{-bu} .$$

3 Functional analytic preparations

In this section we bring the scalar product (2.2) into a more explicit form. Moreover, we set up the Cauchy problem in a way where spectral methods in Hilbert space become applicable. In order to obtain a unitary time evolution, we must consider a spacetime region which includes the Cauchy horizon, and we must introduce reflecting boundary conditions on the timelike surface $r = r_0 < r_-$. (Our solution of the Cauchy problem outside the Cauchy horizon will not depend on the choice of r_0, as will be explained after Lemma 3.4.) Moreover, we must make sure that the surfaces $\{\tau = \text{const}\}$ are space-like. Noting that the metric coefficient g_{rr} in (2.4) has a zero at

$$r_{\min} = \sqrt{M^2 + Q^2} - M < r_- ,$$

we are led to choosing r_0 in the interval

$$r_{\min} < r_0 < r_- . \tag{3.1}$$

We thus consider the spacetime region $M \subset \mathscr{M}$ defined by

$$M := \{\tau, r > r_0, \vartheta, \varphi\} \quad \text{with timelike boundary} \quad \partial M := \{\tau, r = r_0, \vartheta, \varphi\} .$$

This spacetime is foliated by the space-like hypersurfaces $(N_\tau)_{\tau \in \mathbb{R}}$ given by

$$N_\tau := \{\tau = \text{const.}, r \geq r_0, \vartheta, \varphi\} .$$

Each hypersurface has the boundary $\partial N_\tau = \partial M \cap N_\tau \simeq S^2$. These $N_\tau \subset \mathscr{N}_\tau$ give rise to a space-like foliation of M. The Killing field $K = \partial_\tau$ is tangential to ∂M and is timelike on this hypersurface (because $g(K, K) = g_{00} = \Delta(r_0)/r^2 > 0$). We denote the corresponding spinor bundle by SM.

We first compute the scalar product in this spacetime region.

Lemma 3.1 *The scalar product* (2.2) *can be written as*

$$(\psi \,|\, \phi) = \int_{r_0}^{\infty} dr \int_{-\pi/2}^{\pi/2} d\vartheta \int_0^{2\pi} d\varphi \; \Psi^\dagger \, \Gamma \, \Phi \sin(\vartheta) \tag{3.2}$$

(where the capital letters always denote the transformed Dirac wave function (2.6)*) with*

$$\Gamma = \frac{r_+}{|\Delta|} \begin{bmatrix} 2r^2 - \Delta & 0 & 0 & 0 \\ 0 & |\Delta| & 0 & 0 \\ 0 & 0 & |\Delta| & 0 \\ 0 & 0 & 0 & 2r^2 - \Delta \end{bmatrix} . \tag{3.3}$$

Note that, in view of the lower bound in (3.1), the matrix Γ is strictly positive, showing that (3.2) is indeed a scalar product.

Proof By direct computation, we see that the volume form is

$$\sqrt{|\det g_{\mathcal{N}_\tau}|} = \sqrt{(2r^2 - \Delta)}\, r\, \sin\vartheta \; .$$

It remains to compute the combination $G^j \nu_j$. The normal ν is determined by the four equations

$$g(\nu, \nu) = 1, \quad g(\nu, \partial_r) = 0, \quad g(\nu, \partial_\varphi) = 0 \quad \text{and} \quad g(\nu, \partial_\vartheta) = 0 \; .$$

By direct computation, we find

$$\nu = -i\,\frac{\sqrt{\Delta - 2r^2}}{r}\,\partial_\tau - i\,\frac{\Delta + r^2}{r\sqrt{\Delta - 2r^2}}\,\partial_r \; .$$

This corresponds to the co-vector

$$\nu = \frac{r}{\sqrt{2r^2 - \Delta}}\, d\tau \; .$$

Reading off the transformed Gamma-matrix $\overline{G^\tau}$ from (2.8), we need to transform it back with the relation

$$G^\tau = D^{-1} \Gamma_{\text{trafo}}^{-1} \overline{G^\tau} D \, .$$

Using the form of the spin inner product (2.13), we obtain

$$\prec\psi | \nu\!\!\!/ \, \phi\succ \sqrt{|\det g_{\mathcal{N}_\tau}|} = \Psi^\dagger \underbrace{D^{-1} \gamma^0\, G^\tau D^{-1} r^2}_{=:\Gamma} \Phi \sin(\vartheta) = \Psi^\dagger \Gamma \Phi \sin(\vartheta) \; ,$$

concluding the proof. ◻

In order to obtain a Cauchy problem with a well-defined, unitary time evolution, we need to introduce suitable boundary conditions at $r = r_0$. Following the procedure in [11], we introduce the reflecting boundary conditions

$$(\nu\!\!\!/ - i)\, \psi\big|_{\partial M} = 0 \, ,$$

where $\not{\nu}$ is the inner normal on ∂M. For the Cauchy problem, we set $N = N_\tau|_{\tau=0}$. We choose initial data in the class

$$C^\infty_{\text{init}}(N) := \left\{ \psi_0 \in C^\infty_0(N\,, SM) \text{ with } (\not{\nu} - i)\,(H^p\,\psi_0)\big|_{\partial N} = 0 \text{ for all } p \in \mathbb{N} \right\}. \tag{3.4}$$

We denote the Hilbert space generated by these functions (with the scalar product computed in Lemma 3.1) by $\mathcal{H}_N$. The following lemma was proved in [11].

Lemma 3.2 *For initial data ψ_0 in the class (3.4), the Cauchy problem with boundary conditions*

$$i\partial_\tau \psi = H\,\psi\,, \qquad \psi|_N = \psi_0\,, \qquad (\not{\nu} - i)\,\psi\big|_{\partial M} = 0$$

has a unique, global solution $\psi \in C^\infty_{sc}(M, SM)$. Evaluating this solution at subsequent times τ and τ' gives rise to a unique unitary time evolution operator leaving the domain $C^\infty_{init}(N)$ invariant, i.e.,

$$U^{\tau';\tau} : C^\infty_{init}(N) \subset \mathcal{H}_N \longrightarrow C^\infty_{init}(N) \subset \mathcal{H}_N\,.$$

Having a dense domain which is invariant under the time evolution makes it possible to apply Chernoff's method [3] to obtain the following result. More details can be found in [11].

Lemma 3.3 *The Dirac Hamiltonian H in the Reissner–Nordström geometry in Eddington–Finkelstein coordinates with domain of definition*

$$\mathcal{D}(H) = C^\infty_{init}(N)$$

is essentially self-adjoint on the Hilbert space $\mathcal{H}_N$.

Having specified the domain of the Hamiltonian, one could go through the transformations in Section 2.2 to work out the corresponding domain of the radial Hamiltonian in (2.19). Since the details will not be needed for our results, we omit these computations.

For ease in notation, we denote the self-adjoint extension of the Hamiltonian again by H. By the spectral theorem for self-adjoint operators, we can express the solution of the Cauchy problem for any $\psi_0 \in \mathcal{H}_N$ as

$$\psi(\tau) = e^{-i\tau H}\psi_0 = \int_{\sigma(H)} e^{-i\omega\tau}\,\mathrm{d}E_\omega\psi_0\,,$$

where $\omega \in \sigma(H)$ are the spectral values, and $\mathrm{d}E_\omega$ is the corresponding spectral measure of H. As explained after (2.18), from now on we restrict attention to one angular momentum mode and consider the corresponding two-component Dirac equation (2.19). For the later explicit analysis, it is helpful to rewrite the spectral measure with the help of Stone's formula.

Lemma 3.4 *The solution of the Cauchy problem can be written as*

$$X(\tau, r) = \frac{1}{2\pi i} \lim_{a\to\infty} \lim_{\varepsilon\searrow 0} \int_{-a}^{a} e^{-i\omega\tau} \Big[(H_\xi - \omega - i\varepsilon)^{-1} - (H_\xi - \omega + i\varepsilon)^{-1} \Big] X_0(r)\, d\omega\,, \tag{3.5}$$

where $(H_\xi - \omega \mp i\varepsilon)^{-1}$ are the resolvents of the Dirac Hamiltonian H_ξ in the upper and lower half-planes and $X_0(r) \in C^\infty_{init}(N)$ is the initial data for a fixed angular mode k, l.

Proof Using the properties of the spectral measure, we obtain

$$\begin{aligned} X(\tau, r) &= e^{-iH_\xi\tau} \lim_{a\to\infty} E_{(-a,a)} X_0(r) \\ &= \frac{1}{2} e^{-iH_\xi\tau} \lim_{a\to\infty} \Big[E_{(-a,a)} + E_{[-a,a]} \Big] X_0(r) \end{aligned}$$

Applying Stone's formula (see [19, Theorem VII.13]) gives the result. □

We point out that in this lemma we solved the Cauchy problem in the region $M = \mathbb{R}^+ \times (r_0, \infty) \times S^2$ which extends behind the Cauchy horizon at $r = r_-$. By restricting this solution to the region $\mathbb{R}^+ \times (r_-, \infty) \times S^2$, we obtain the solution of the Cauchy problem outside the Cauchy horizon, for initial data on the hypersurface $(r_-, \infty) \times S^2$. Here we make use of the fact that, due to causality, a Dirac wave cannot cross the Cauchy horizon from inside to outside.

4 Computation of the resolvent

4.1 The Green's matrix

Our next task is to calculate the resolvent $(H_\xi - \omega \mp i\varepsilon)^{-1}$ appearing in Lemma 3.4 in the upper and lower half-planes. It is most convenient to work with the Hamiltonian for the radial equation (2.15) after separation of variables, denoted by H_ξ.

We introduce the abbreviation $\omega \mp i\varepsilon =: \omega_\varepsilon$. Moreover, it is useful for the computations to write the operator $H_\xi - \omega_\varepsilon \mathbb{1}_{\mathbb{C}^2}$ as

$$H_\xi - \omega_\varepsilon \mathbb{1}_{\mathbb{C}^2} = C(r)^{-1} \mathcal{R}(\partial_r;\, r) \tag{4.1}$$

with the matrix

$$C(r)^{-1} = \begin{bmatrix} -\frac{i}{(2r^2-\Delta)} & 0 \\ 0 & \frac{i}{\Delta} \end{bmatrix}$$

and the radial differential operator

$$\mathcal{R}(\partial_r;\, r) := \begin{bmatrix} \Delta\partial_r - i\omega_\varepsilon(2r^2 - \Delta) & |\Delta|^{1/2}(imr - \xi) \\ -\epsilon(\Delta)|\Delta|^{1/2}(imr + \xi) & \Delta(i\omega_\varepsilon + \partial_r) \end{bmatrix}.$$

This makes it possible to write the radial equation as

$$\Delta(r)\,\partial_r X(r) = \begin{bmatrix} i\omega_\varepsilon(2r^2-\Delta) & -|\Delta|^{1/2}(imr-\xi) \\ \epsilon(\Delta)|\Delta|^{1/2}(imr+\xi) & -i\Delta\omega_\varepsilon \end{bmatrix} X(r) \tag{4.2}$$

Our strategy is to invert this differential operator with the help of the so-called Green's matrix $G(r;r')_{\omega_\varepsilon}$, being a distributional solution to the equation

$$\mathcal{R}(\partial_r;r)\,G(r;r')_{\omega_\varepsilon} = \delta(r-r')\mathbb{1}_{\mathbb{C}^2}\,. \tag{4.3}$$

The Green's matrix can be expressed in terms of the fundamental solutions of the radial equations. We postpone the detailed computations to the next section (Section 4.2). Here we explain how the resolvent can be computed from the Green's matrix.

Lemma 4.1 *Let $X_0 \in C^\infty_{init}(N)$ be initial data for a fixed angular mode k, l. Then the resolvent acting on X_0 can be expressed in terms of the Green's matrix $G(r;r')_{\omega_\varepsilon}$ in* (4.3) *as*

$$R_{\omega_\varepsilon}\,X_0(r) = (H_\xi - \omega_\varepsilon\mathbb{1}_{\mathbb{C}^2})^{-1}X_0(r) = \int_{r_0}^\infty G(r;r')_{\omega_\varepsilon}\,C(r')\,X_0(r')\,dr'\,,$$

where $C(r)$ is the matrix

$$C(r) = \begin{bmatrix} i(2r^2-\Delta) & 0 \\ 0 & -i\Delta \end{bmatrix}.$$

Proof By direct computation using (4.3), one verifies that the operator $H_\xi - \omega_\varepsilon\mathbb{1}_{\mathbb{C}^2}$ in (4.1) has the inverse

$$\Big(\big(H_\xi - \omega_\varepsilon\mathbb{1}_{\mathbb{C}^2}\big)^{-1}X\Big)(r) = \int_{r_0}^\infty G(r;r')_{\omega_\varepsilon}\,C(r')\,X(r')\,\mathrm{d}r'\,.$$

This gives the result. □

4.2 The radial Jost solutions

We now define the fundamental solutions which we will later use to compute the Green's matrix. Our method is to construct Jost solutions $\mathcal{J}_\pm$. In preparation, we rewrite the first-order radial system of ODEs stemming from the matrix Dirac equation in (2.20) into two second-order scalar equations, sometimes referred to as the Jost equation. We consider the three regions

$$(r_0, r_-)\,,\qquad (r_-, r_+)\qquad \text{and}\qquad (r_+,\infty) \tag{4.4}$$

separately. Choosing the Regge–Wheeler coordinate $u \in \mathbb{R}$ in one of these regions, the radial solutions $X^{(\omega)}(r(u))$ satisfy the equations

$$\left[\partial_u^2 + E_{\omega,l}(u)\partial_u + K^+_{\omega,l}(u)\right] X_+^{(\omega)}(u) = 0 \tag{4.5}$$

$$\left[\partial_u^2 + E_{\omega,l}(u)\partial_u + K^-_{\omega,l}(u)\right] X_-^{(\omega)}(u) = 0 \tag{4.6}$$

where $E_{\omega,l}(u)$ and $K^\pm_{\omega,l}(u)$ are smooth functions. The explicit form of these functions is rather involved and will not be given here. However, for the construction of the Jost solutions, it suffices to analyze the asymptotic form of these functions as $u \to \pm\infty$. We recall the basic idea for $X_+^{(\omega)}$ and the asymptotics $u \to +\infty$. We rewrite (4.5) in the form

$$\left[-\partial_u^2 - \Omega_+^2\right]\mathcal{J}_+ = -W^+(u)\mathcal{J}_+$$

with a complex constant Ω_+ and a potential W^+. Having the complex coefficients, the differential operator on the left side can be inverted with the help of an explicitly given Green's kernel $S(u, v)$, i.e.,

$$\left[\partial_v^2 - \Omega_+^2\right]S(u, v) = \delta(u - v)$$

This makes it possible to formulate a Lippmann–Schwinger equation of the form

$$\mathcal{J}_+ = e^{i\Omega_+ u} + \int_u^\infty S(u, v)\ W^+(v)\ \mathcal{J}_+(v)\ \mathrm{d}v \tag{4.7}$$

Using that W decays as $u \to \infty$, one can perform an expansion in powers of W,

$$\mathcal{J}_+ = \sum_n^\infty \mathcal{J}_+^{(n)}\ . \tag{4.8}$$

Similar to the usual Picard–Lindelöf iteration (defined on a bounded interval), one can show that this series converges uniformly, giving rise to a unique solution with prescribed asymptotics as $u \to \infty$. Substituting the resulting solution $X_+^{(\omega)}$ into the first-order system (4.2), one can solve for $X_-^{(\omega)}$. We thus obtain a solution of the radial equation with prescribed asymptotics.

This method has been worked out for the Dirac and wave equations in the Kerr geometry in Boyer–Lindquist coordinates in [8, 13] as well in Eddington–Finkelstein coordinates in [12] (for basics and more details see also [4]). For simplicity, we state the result only in the region (r_+, ∞) outside the event horizon and note that the regions (r_0, r_-) and (r_-, r_+) are treated similarly.

Lemma 4.2 *For every angular momentum mode k and l and $\omega_\varepsilon \in \{\omega_\varepsilon \neq 0 \,|\, \omega_\varepsilon \in \mathbb{C}\}$, there are unique radial Jost solution to the complexified Schrödinger-type equations (4.5) in the region (r_+, ∞) with the asymptotic boundary conditions*

$$\lim_{u\to+\infty} e^{-i\Omega_+(\omega_\varepsilon)\,u - ic(\omega_\varepsilon)\ln u}\, \mathcal{J}_+(u) = 1\,, \qquad \lim_{u\to-\infty} e^{-i\Omega_-(\omega_\varepsilon)\,u}\, \mathcal{J}_-(u) = 1 \tag{4.9}$$

$$\lim_{u\to+\infty} \partial_u \Big(e^{-i\Omega_+(\omega_\varepsilon)\,u - ic(\omega_\varepsilon)\ln u}\, \mathcal{J}_+(u) \Big) = 0\,, \quad \lim_{u\to-\infty} \partial_u \Big(e^{-i\Omega_-(\omega_\varepsilon)\,u}\, \mathcal{J}_-(u) \Big) = 0 \tag{4.10}$$

for suitable complex numbers c and $\Omega_\pm$. Furthermore, these solutions are analytic and smooth in u and ω_ε.

Proof In the case $\operatorname{Im}\omega_\varepsilon > 0$, the Jost solutions have been constructed in [8, 13]. By complex conjugation of the Schrödinger-type equation, we obtain corresponding solutions for $\operatorname{Im}\omega_\varepsilon < 0$. □

We note that the asymptotics (4.9) and (4.10) for $\mathcal{J}_\pm$ can also be written as

$$\mathcal{J}_+(u) \simeq e^{i\Omega_+(\omega_\varepsilon)u + ic(\omega_\varepsilon)\ln u} + E_+(u) \quad \text{and} \quad \mathcal{J}_-(u) \simeq e^{i\Omega_-(\omega_\varepsilon)u} + E_-(u)$$

with a decaying error term $E_\pm(u)$. This shows that $\Omega(\omega_\varepsilon)_\pm$ and $c(\omega_\varepsilon)$ encode the asymptotic phase and amplitude of the wave.

We now collect the Jost solutions in the respective regions (4.4) and introduce a convenient notation. We can use the same notation for the Jost solutions constructed near the event horizon from inside and outside (and similarly near the Cauchy horizon), because these solutions have the same asymptotics in the Regge–Wheeler coordinates of the respective spacetime region. Note that the Regge–Wheeler coordinate u tends to $-\infty$ at the event horizon and to $+\infty$ at the Cauchy horizon.

In order to find Jost solutions, we must ensure that the integral in (4.7) and the series (4.8) converge. Depending on the signs of $\operatorname{Im}(\omega_\varepsilon)$ and $\operatorname{Re}(\omega_\varepsilon)$, we thus obtain different solutions as compiled in the next lemma.

Lemma 4.3 *We introduce Jost solutions with the following asymptotics:*

(i) Near spatial infinity in the case $|\omega| > m$ and sufficiently small $\varepsilon > 0$:

$$\acute{\mathcal{J}}_\infty(u) = f_{\infty,1}\, U_{\omega_\varepsilon} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\phi_+(u)} \left[1 + \mathcal{O}\Big(\frac{1}{u}\Big) \right] \tag{4.11}$$

$$\text{if } \operatorname{Im}(\omega_\varepsilon) < 0 \text{ with } \operatorname{Re}(\omega_\varepsilon) < 0 \text{ or } \operatorname{Im}(\omega_\varepsilon) > 0 \text{ with } \operatorname{Re}(\omega_\varepsilon) > 0$$

$$\grave{\mathcal{J}}_\infty(u) = f_{\infty,2}\, U_{\omega_\varepsilon} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\phi_-(u)} \left[1 + \mathcal{O}\Big(\frac{1}{u}\Big) \right] \tag{4.12}$$

$$\text{if } \operatorname{Im}(\omega_\varepsilon) > 0 \text{ with } \operatorname{Re}(\omega_\varepsilon) < 0 \text{ or } \operatorname{Im}(\omega_\varepsilon) < 0 \text{ with } \operatorname{Re}(\omega_\varepsilon) > 0$$

with

$$\phi_{\pm}(\mu) = \sqrt{\omega_\varepsilon^2 - m^2}\, u + M\left(\pm 2\omega_\varepsilon + \frac{m^2}{\sqrt{\omega_\varepsilon^2 - m^2}}\right) \ln(u)$$

and

$$U_{\omega_\varepsilon} = \begin{bmatrix} \cosh(\Theta) & \sinh(\Theta) \\ \sinh(\Theta) & \cosh(\Theta) \end{bmatrix} \quad \textit{with} \quad \Theta := \frac{1}{4} \ln\left(\frac{\omega_\varepsilon - m}{\omega_\varepsilon + m}\right).$$

(ii) Near spatial infinity in the case $|\omega| < m$ *and for sufficiently small* $\varepsilon > 0$*:*

$$\widehat{\mathcal{J}}_\infty(u) = V_{\omega_\varepsilon} \begin{pmatrix} f_\infty \\ 0 \end{pmatrix} e^{-\sqrt{m^2 - \omega_\varepsilon^2}\, u} \left[1 + \mathcal{O}\left(\frac{1}{u}\right)\right]$$

with

$$V_{\omega_\varepsilon} = \begin{bmatrix} \dfrac{im}{2\sqrt{m^2-\omega^2}} & \dfrac{1}{2}\left(1 + \dfrac{i\omega}{\sqrt{m^2-\omega^2}}\right) \\ -\dfrac{im}{2\sqrt{m^2-\omega^2}} & \dfrac{1}{2}\left(1 - \dfrac{i\omega}{\sqrt{m^2-\omega^2}}\right) \end{bmatrix}.$$

(iii) Near the event horizon r_+*:*

$$\check{\mathcal{J}}_+(u) = \begin{pmatrix} 0 \\ h_{+,1} \end{pmatrix} \left[1 + \mathcal{O}\left(e^{bu}\right)\right] \quad \textit{and} \quad \widehat{\mathcal{J}}_+(u) = \begin{pmatrix} h_{+,2} \\ 0 \end{pmatrix} e^{2i\omega_\varepsilon u}\left[1 + \mathcal{O}\left(e^{bu}\right)\right]$$

where, depending on $\mathrm{Im}(\omega)$*, the above functions are well-defined and in* L^2_{loc} *near the event horizon:*

$$\begin{aligned} \check{\mathcal{J}}_+(u),\quad \widehat{\mathcal{J}}_+(u) \quad &\textit{for} \quad \mathrm{Im}(\omega_\varepsilon) < 0 \\ \check{\mathcal{J}}_+(u) \quad &\textit{for} \quad \mathrm{Im}(\omega_\varepsilon) > 0 \end{aligned}$$

(iv) Near the Cauchy horizon r_-*:*

$$\check{\mathcal{J}}_-(u) = \begin{pmatrix} 0 \\ h_{-,1} \end{pmatrix} \left[1 + \mathcal{O}\left(e^{-bu}\right)\right] \textit{and}\ \widehat{\mathcal{J}}_-(u) = \begin{pmatrix} h_{-,2} \\ 0 \end{pmatrix} e^{2i\omega_\varepsilon u}\left[1 + \mathcal{O}\left(e^{-bu}\right)\right]$$

where, depending on $\mathrm{Im}(\omega)$*, the above functions are well-defined and in* L^2_{loc} *near the Cauchy horizon:*

$$\begin{aligned} \check{\mathcal{J}}_-(u),\quad \widehat{\mathcal{J}}_-(u) \quad &\textit{for} \quad \mathrm{Im}(\omega_\varepsilon) > 0 \\ \check{\mathcal{J}}_-(u) \quad &\textit{for} \quad \mathrm{Im}(\omega_\varepsilon) < 0 \end{aligned}$$

Here $f_{\infty,1/2} \neq 0$ *and* $h_{\pm,1/2} \neq 0$ *are constants and* $b \in \mathbb{R}_+$.

All these Jost solutions converge locally uniformly as $\varepsilon \searrow 0$, *respectively* $\varepsilon \nearrow 0$, *to smooth solutions of the ODE with* $\varepsilon = 0$.

We remark that, in the limit $\varepsilon \searrow 0$, the asymptotics of the solutions near infinity and near the horizons was worked out in [16, Theorem 1.1]. We also point out that, in the case $|\omega_\varepsilon| < m$, there is only one Jost solution, denoted by $\widehat{\mathcal{J}}_\infty$, which decays exponentially at infinity. In this case, we choose an arbitrary fundamental solution which is linearly independent of $\widehat{\mathcal{J}}_\infty$ and denote it by $\check{\mathcal{J}}_\infty$. This fundamental solution increases exponentially near infinity. It is not canonical. But our main theorem will not depend on this choice.

Proof of Lemma 4.3 We begin with the Jost solutions near spatial infinity. According to Lemma 2.3, the solutions have the plane wave asymptotics with asymptotic phases given by (2.21). It suffices to consider the linear term in (2.21), because for large u it dominates the logarithm. In the case $|\omega_\varepsilon| > |m|$, by a Taylor expansion of the square root we obtain

$$\sqrt{\omega_\varepsilon^2 - m^2} = \sqrt{\omega^2 - m^2} + i \,\mathrm{Im}\,(\omega_\varepsilon)\, \mathrm{Re}\,(\omega_\varepsilon) \frac{1}{\sqrt{\omega^2 - m^2}} + \mathcal{O}\big(\varepsilon^2\big) .$$

Since the imaginary term determines the decay behavior, it suffices to consider the second term. This shows that the combinations of real and imaginary parts in (4.11) and (4.12) ensure that the exponentials $e^{\pm i\phi_\pm(u)}$ decay at infinity. This property is precisely what is needed in order for the Jost solutions to be well-defined. In the remaining case $|\omega_\varepsilon| \leq |m|$, using that $\sqrt{\omega_\varepsilon^2 - m^2} = i\sqrt{m^2 - \omega_\varepsilon^2}$ and doing the same expansion in ε, we see the first term in u dominates the convergent behavior. It is also independent of $\mathrm{Im}(\omega_\varepsilon)$. Therefore, we only have one exponential decaying solution for this case at infinity.

Now, we look at the Jost functions near the event and Cauchy horizons. We only consider the Cauchy horizon, because the event horizon can be treated similarly.

Near the Cauchy horizon, the fundamental solutions have the plane wave asymptotics as worked out in Theorem 2.3. Using that the potential W in the Lippmann–Schwinger equation (4.7) decays exponentially, it turns out that there are two convergent Jost solutions $\widehat{\mathcal{J}}_-$ and $\check{\mathcal{J}}_-(u)$ defined for ω_ε close to the real axis, which form a fundamental system (for details see [8, Section 3]). □

Finally, for the boundary condition at $r = r_0$ as initial data for the radial ODE, we obtain a solution $\mathcal{J}_{\partial M}$ with the following boundary values.

(iv) At the boundary ∂M:

$$\mathcal{J}_{\partial M}(u) = \mathcal{J}^{(1)}_{\partial M}(u) \begin{pmatrix} 1 \\ \frac{\sqrt{|\Delta|}}{r_+} \end{pmatrix}$$

4.3 Construction of the Green's matrix

Our next goal is to express the Green's matrix (4.3) in terms of Jost solutions. To this end, we make the general "gluing ansätze" depending on r' and $\text{Im}(\omega_\varepsilon)$

$$\begin{aligned} G(r,r') &= \Theta(r-r') \times \big[\Phi_1(r) \otimes P_1(r') + \Phi_2(r) \otimes P_2(r')\big] \\ &\qquad \text{for } r' \in (r_-, r_+) \text{ and } \text{Im}(\omega_\varepsilon) < 0 \\ G(r,r') &= \Theta(r'-r) \times \big[\Phi_1(r) \otimes P_1(r') + \Phi_2(r) \otimes P_2(r')\big] \\ &\qquad \text{for } r' \in (r_-, r_+) \text{ and } \text{Im}(\omega_\varepsilon) > 0 \\ G(r,r') &= \Theta(r-r')\Phi_1(r) \otimes P_1(r') + \Theta(r'-r)\Phi_2(r) \otimes P_2(r') \\ &\qquad \text{for } r' \notin (r_-, r_+)\,, \end{aligned}$$

where Φ_1 and Φ_2 are the Jost solutions defined in the respective regions

$$\begin{cases} \Phi_1(r) \text{ for } r_0 \le r \le r' \\ \Phi_2(r) \text{ for } r' \le r < \infty\,. \end{cases}$$

Here and in what follows, it is more convenient to work again with the radial variable $r \in [r_0, \infty)$. The corresponding Regge–Wheeler coordinate is obtained in the respective regions (i.e., inside the Cauchy horizon, between the horizons, and in the asymptotic end) by (2.5). We point out that when extending the solutions Φ_1 or Φ_2 across the event or Cauchy horizons, we need to make sure that their L^2-norms are finite, i.e.

$$\Phi_1 \in L^2((r_0, r'), \mathbb{C}^2) \qquad \text{and} \qquad \Phi_2 \in L^2((r', \infty), \mathbb{C}^2)\,,$$

where we work with the L^2-scalar product in (3.2).

We now explain in detail how the functions Φ_1 and Φ_2 can be chosen. We consider the cases separately when ω_ε is in the upper and lower half plane.

(i) $\text{Im}(\omega_\varepsilon) < 0$ and $\text{Re}(\omega_\varepsilon) < 0$ for $|\omega_\varepsilon| > m$:
We begin with the case that r' is outside the event horizon. Due to the choice of ω_ε, $\widehat{\mathcal{J}_\infty}(r)$ decays as $r \to \infty$, and we can set

$$\Phi_1^\infty(r) = a\, \widehat{\mathcal{J}_\infty}(r)\,.$$

For $r < r'$ we have two possibilities. One is to take the solution $\widehat{\mathcal{J}_+}$. This solution decays exponentially at the event horizon. Extending it by zero up to $r = r_0$ gives a solution in L^2. This leads us to the ansatz

$$\Phi_2^\infty(r) = b\, \Theta(r'-r)\, \Theta(r-r_+)\, \widehat{\mathcal{J}_+}(r)\,. \tag{4.13}$$

The other possibility is to take another fundamental solution and to extend it across the event and Cauchy horizons up to the boundary at $r = r_0$. Because of the negative imaginary part of ω_ε, only the constant function $\check{\mathcal{J}}_-(r)$ is square integrable at the Cauchy horizon. However, for all values of ω, this solution does

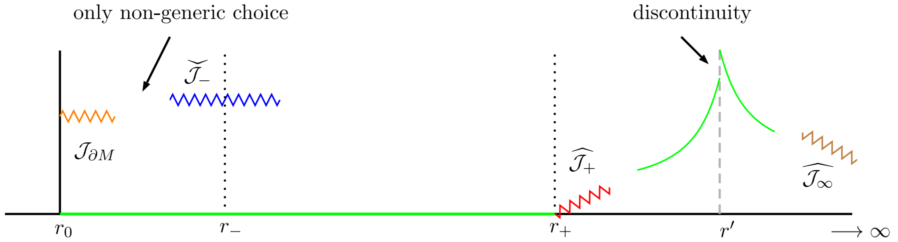

Fig. 1 Extending Jost functions across the horizons for $r_+ < r' < \infty$. At r_- no generic linear combinations are possible. The green lines indicate the extensions left and right of r'. Note that due to the symmetric behavior of the Regge–Wheeler coordinate we use the same notation for the constructed Jost solutions from inside and outside the horizons (see Lemma 4.2 and 4.3)

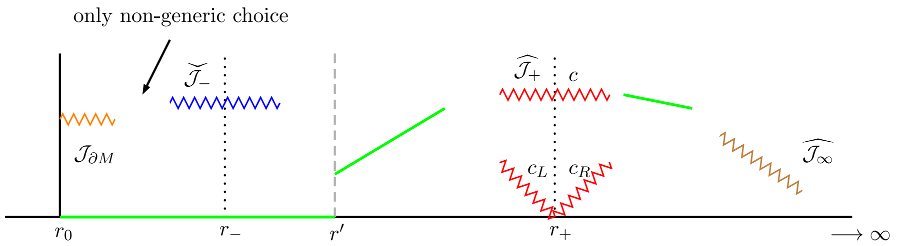

Fig. 2 Extending Jost functions across the horizons for $r_- < r' \leq r_+$. At r_+ generic linear combinations are possible which is not the case at r_- with only one Jost function. The green lines indicate the extensions left and right of r'. The zigzag lines imply possible Jost functions. Note that $\widehat{\mathcal{J}}_+$ stands for two possible Jost solutions for $\text{Im}(\omega_\varepsilon) < 0$ (see Lemma 4.3)

not satisfy the boundary conditions at r_0. For this reason, we are forced to choose the fundamental solution (4.13), leaving us with two free parameters $a, b \in \mathbb{C}$. (Here the superscript ∞ clarifies that r' lies in the asymptotic end.) These solutions are shown in Figure 1.

We next consider the case $r_- < r' \leq r_+$. By the same argumentation as above, there are no non-trivial solutions crossing the Cauchy horizon. Therefore, the wave needs to vanish at r' and is continuously extended by zero to r_0. This leads us to the above ansatz where both fundamental solutions are considered only in the region $r > r'$.

For $r > r_+$, on the other hand, we have two possible functions. Since $\check{\mathcal{J}}_+$ is a constant function it needs to be the same inside and outside the event horizon restricting $c = c'$, but the decaying solution can have two different prefactors in the linear combination for waves. This gives us enough freedom to match the solutions. The different fundamental solutions are illustrated in Figure 2. We begin with the most general linear combination and simplify it afterward. It has the form

$$\begin{aligned}\Phi^{r_+}(r) = \Theta(r - r')\Big[& c_L\, \Theta(r_+ - r)\widehat{\mathcal{J}}_+(r) + c\, \Theta(r_+ - r)\check{\mathcal{J}}_+(r) \\ & + c'\, \Theta(r - r_+)\, \check{\mathcal{J}}_+(r) + c_R\, \Theta(r - r_+)\widehat{\mathcal{J}}_+(r) \\ & + a\, \Theta(r - r_+)\, \widehat{\mathcal{J}}_\infty(r)\Big]. \end{aligned} \tag{4.14}$$

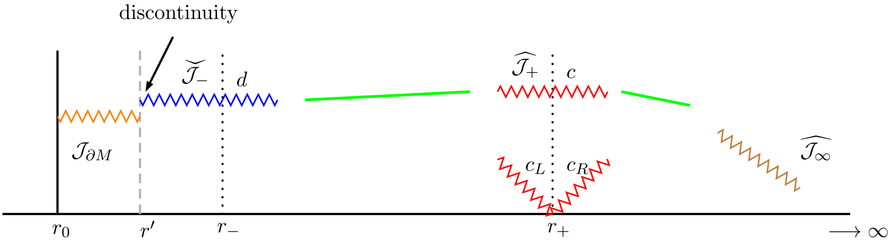

Fig. 3 Extending Jost functions across the horizons for $r_0 \leq r' \leq r_-$. Again, t r_+ generic linear combinations are possible. This is not the case at r_- with only one Jost function. The green lines indicate the extensions left and right of r'. This time, due to the discontinuity at $r = r'$ non-generic linear combinations are possible

Now we can express $\widehat{\mathcal{J}_\infty}(r)$ as a linear combination of $\widecheck{\mathcal{J}_+}(r)$ and $\widehat{\mathcal{J}_+}(r)$. This results in

$$\Phi^{r_+}(r) = \Theta(r - r')\Big[c\,\Theta(r_+ - r)\widecheck{\mathcal{J}_+}(r) + a\,\Theta(r - r_+)\widehat{\mathcal{J}_\infty}(r) + c_L\,\Theta(r_+ - r)\widehat{\mathcal{J}_+}(r)\Big]$$

with two free parameters $a, c_L \in \mathbb{C}$. (The parameter c, on the other hand, is determined by the matching conditions on the event horizon.) Thus, we have the solutions

$$\Phi_1^{r_+}(r) = c_L\,\Theta(r_+ - r)\widehat{\mathcal{J}_+}(r)$$
$$\Phi_2^{r_+}(r) = c\,\Theta(r_+ - r)\widecheck{\mathcal{J}_+}(r) + a\,\Theta(r - r_+)\widehat{\mathcal{J}_\infty}(r)$$

It remains to consider the case $r_0 \leq r' \leq r_-$. On the left side of r' the boundary conditions at r_0 admit, up to a multiple, a unique solution, i.e.

$$\Phi_1^{r_-}(r) = e\,\mathcal{J}_{\partial M}(r)\,.$$

For $r > r_-$, only the solution $\widecheck{\mathcal{J}_-}(r)$ can be extended in L^2 across the Cauchy horizon. This solution must be extended also across the event horizon and must be matched to the decaying solution (see Figure 3 for a detailed sketch). Now the matching across the Cauchy horizon determines the constants c_L and c, whereas the matching across the event horizon determines a and c_R. We conclude that

$$\Phi_1^{r_-}(r) = e\,\mathcal{J}_{\partial M}(r)$$
$$\begin{aligned}\Phi_2^{r_-}(r) = {} & d\,\Theta(r - r_-)\,\widehat{\mathcal{J}_-}(r) + c_L\,\Theta(r - r_-)\,\Theta(r_+ - r)\widehat{\mathcal{J}_+}(r) \\ & + c_R\,\Theta(r - r_+)\widehat{\mathcal{J}_+}(r) + c\,\Theta(r - r_-)\,\Theta(r_+ - r)\,\widecheck{\mathcal{J}_+}(r) \\ & + a\,\Theta(r - r_+)\widehat{\mathcal{J}_\infty}(r)\end{aligned}$$

with two free parameter $d, e \in \mathbb{C}$ (and c_L, c_R, c and a fixed by the matching conditions).

(ii) $\text{Im}(\omega_\varepsilon) > 0$ and $\text{Re}(\omega_\varepsilon) < 0$ for $|\omega_\varepsilon| > m$:
The procedure to find the fundamental solutions in this case is similar as in the first case. One only needs to keep in mind that the roles of the event and Cauchy horizons are interchanged, in the sense that at the Cauchy horizon, both fundamental solutions are in L^2, whereas at the event horizon, one fundamental solution is singular.

The remaining cases are similar. For $|\omega_\varepsilon| < m$ we need to remember that we only have the decaying solution $\mathcal{J}_\infty$ at infinity. We summarize the results in following lemma.

Lemma 4.4 *The extended Jost solutions Φ_1 and Φ_2 can be expressed as*

(i) $\text{Im}(\omega_\varepsilon) < 0$ *and* $\text{Re}(\omega_\varepsilon) < 0$ *and* $|\omega_\varepsilon| > m$*:*

- $\Phi_1^\infty(r) = c_1\, \Theta(r - r_+)\widehat{\mathcal{J}_+}(r)$
  $\Phi_2^\infty(r) = c_2\, \widehat{\mathcal{J}_\infty}(r)$

- $\Phi_1^{r_+}(r) = c_1\, \Theta(r_+ - r)\widehat{\mathcal{J}_+}(r)$
  $\Phi_2^{r_+}(r) = c_2\, \Theta(r - r_+)\widehat{\mathcal{J}_\infty}(r) + a_1\, \Theta(r_+ - r)\widecheck{\mathcal{J}_+}(r)$

- $\Phi_1^{r_-}(r) = c_1\, \mathcal{J}_{\partial M}(r)$
  $$\begin{aligned}\Phi_2^{r_-}(r) = {} & c_2\, \Theta(r - r_-)\widecheck{\mathcal{J}_-}(r) + a_1\, \Theta(r - r_-)\Theta(r_+ - r)\widehat{\mathcal{J}_+} \\ & + a_2\, \Theta(r - r_+)\widehat{\mathcal{J}_+}(r) + a_3\, \Theta(r - r_-)\Theta(r_+ - r)\widecheck{\mathcal{J}_+}(r) \\ & + a_4\, \Theta(r - r_+)\widehat{\mathcal{J}_\infty}(r)\end{aligned}$$

Here c_1 and c_2 are free parameters, and $a_1, \ldots, a_4$ are constants which depend on c_1, c_2 and ω_ε.

(ii) $\text{Im}(\omega_\varepsilon) > 0$ *and* $\text{Re}(\omega_\varepsilon) < 0$ *and* $|\omega_\varepsilon| > m$*:*

- $$\begin{aligned}\Phi_1^\infty(r) = {} & c_1\, \Theta(r - r_+)\widecheck{\mathcal{J}_+}(r) + a_1\, \Theta(r - r_-)\Theta(r_+ - r)\widecheck{\mathcal{J}_-}(r) \\ & + a_2\, \Theta(r - r_-)\Theta(r_+ - r)\widehat{\mathcal{J}_-}(r) + a_3\, \Theta(r_- - r)\widehat{\mathcal{J}_-}(r) \\ & + a_4\, \Theta(r_- - r)\mathcal{J}_{\partial M}(r)\end{aligned}$$
  $\Phi_2^\infty(r) = c_2\, \widecheck{\mathcal{J}_\infty}(r)$

- $\Phi_1^{r_+}(r) = c_1\, \Theta(r_- - r)\mathcal{J}_{\partial M} + a_1\, \Theta(r - r_-)\widecheck{\mathcal{J}_-}(r)$
  $\Phi_2^{r_+}(r) = c_2\, \Theta(r - r_-)\widehat{\mathcal{J}_-}(r)$

- $\Phi_1^{r_-}(r) = c_1\, \Theta(r_- - r)\mathcal{J}_{\partial M}(r)$
  $\Phi_2^{r_-}(r) = c_2\, \Theta(r_- - r)\widehat{\mathcal{J}_-}(r)$

Here c_1 and c_2 are again free parameters, and $a_1, \ldots, a_4$ are constants which depend on c_1, c_2 and ω_ε.

(iii) $\mathrm{Im}(\omega_\varepsilon) < 0$ *and* $\mathrm{Re}(\omega_\varepsilon) > 0$ *and* $|\omega_\varepsilon| > m$*:*

- *Similar to the first case, but with* $\widehat{\mathcal{J}_\infty}$ *and* $\widecheck{\mathcal{J}_\infty}$ *interchanged when* $|\omega_\varepsilon| > m$.

(iv) $\mathrm{Im}(\omega_\varepsilon) > 0$ *and* $\mathrm{Re}(\omega_\varepsilon) > 0$ *and* $|\omega_\varepsilon| > m$*:*

- *Similar to the second case, but with* $\widehat{\mathcal{J}_\infty}$ *and* $\widecheck{\mathcal{J}_\infty}$ *interchanged when* $|\omega_\varepsilon| > m$.

(v) $|\omega_\varepsilon| < m$ *with all previous combinations of* $\mathrm{Re}(\omega_\varepsilon)$ *and* $\mathrm{Im}(\omega_\varepsilon)$*:*

- *All four cases (i), (ii), (iii) and (iv) are repeated, but with* $\widehat{\mathcal{J}_\infty}$ *and* $\widecheck{\mathcal{J}_\infty}$ *replaced by* $\mathcal{J}_\infty$.

Remark 4.5 *(matching conditions and weak solutions)* We now explain how the matching conditions derived above can be understood from the perspective of weak solutions of the Dirac equation. We only consider the event horizon, noting that the Cauchy horizon can be treated similarly. In (4.14) we began with a general ansatz for the solution. Our matching conditions stated that this solution must be continuous across the horizon, meaning that

$$c = c' , \tag{4.15}$$

whereas the prefactors c_L and c_R of the solution $\widehat{\mathcal{J}_+}$ can be chosen arbitrarily and independently inside and outside the event horizon. This is illustrated in Figure 2. An alternative method for deriving these matching conditions is to work out the corresponding Dirac solution ψ in (2.1) by inserting the fundamental solutions into the separation ansatz 2.14 and (2.6). Evaluating the Dirac equation weakly across the event horizon (similar as is done in Schwarzschild and Boyer–Lindquist coordinates in [5, 14]), one would again get (4.15). For brevity we omit the details of this computation.

These matching conditions correspond to the following physical picture. The fundamental solution $\widecheck{\mathcal{J}_+}$ describes a wave which crosses the event horizon. Therefore, current conservation gives rise to a matching condition for this solution. The fundamental solution $\widehat{\mathcal{J}_+}$, however, describes a wave which propagates along the event horizon, but does not cross it. Therefore, we do not get a matching condition. ♢

4.4 Computation of the Green's matrix

In this subsection, we will use the global Jost solutions constructed above to compute the Green's matrix. For scalar Jost solutions, this construction is well-known and uses the conservation of the Wronskian. In our setting of two-component solutions, we make use instead of the conservation of the Dirac current $\prec \Phi, \Gamma^\mu \Phi \succ$ in radial directions within the given region (r_-, r_+) and (r_+, ∞). More precisely, we define a Gram matrix, referred to as the *radial flux matrix*. We want to highlight that the corresponding local conservation law holds only after taking the limit $\varepsilon \searrow 0$. (Otherwise, the currents on different time slices do not cancel each other.)

Lemma 4.6 *In the limit $\varepsilon \searrow 0$, the radial flux matrix $h_{ij} = h_{ij}(c_1, c_2, \omega_\varepsilon)$ defined by*

$$<\Phi_i(r), A\,\Phi_j(r)>_{\mathbb{C}^2} =: h_{ij} \quad \text{with} \quad i, j \in \{1, 2\} \tag{4.16}$$

and the matrix A given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\epsilon(\Delta) \end{bmatrix}$$

is independent of r in the respective regions (r_0, r_-), (r_-, r_+) and (r_+, ∞).

Proof We begin with the case $r \in (r_+, \infty)$. By acting with the partial derivative of r, we end up with

$$\Delta(r)\,\partial_r <\Phi(r), A\,\Phi(r)>_{\mathbb{C}^2} = \Delta(r)<\Phi(r), \underbrace{(V^\dagger A + AV)}_{=0}\,\Phi(r)>_{\mathbb{C}^2} = 0 \quad \forall r\,,$$

where V is the matrix on the right side of (4.2). A straightforward calculation gives the result. The steps in the other regions $r \in (r_0, r_-)$ and $r \in (r_-, r_+)$ are identical. □

We remark that the conservation of the radial flux (4.16) was first observed in [14] and used in order to rule out non-trivial time-periodic solutions of the Dirac equation in the exterior Reissner–Nordström geometry. Here we can use this conservation law in order to show that the extended Jost solutions Φ_1 and Φ_2 constructed in Section 4.2 are linearly independent:

Lemma 4.7 *In the limit $\varepsilon \searrow 0$, the two solutions Φ_1 and Φ_2 in Lemma 4.4 are linearly independent.*

Proof The case $|\omega| < m$ is trivial, because Φ_1 is exponentially decreasing at infinity, whereas Φ_2 exponentially increasing. As the remaining cases can be treated similarly, we only consider case (i) in Lemma 4.4. Then the radial flux of Φ_2 can be computed asymptotically as $r \to \infty$ using (4.11). The radial flux of Φ_1, on the other hand, can be computed at the event horizon using the asymptotics in Lemma 4.3 (iii). If Φ_1 and Φ_2 were linearly dependent, these radial fluxes would have opposite signs, a contradiction. □

Following up, we make the ansatz for $G(r; r')$ when $r' \in (r_+, \infty)$.

$$\begin{aligned} G(r, r') := \;& \frac{\Theta(r - r')}{\Delta(r')} \sum_{j=1}^{2} c_{1j}\; \Phi_1(r) \otimes \big(A(r')\Phi_j(r')\big)^\dagger \\ &+ \frac{\Theta(r' - r)}{\Delta(r')} \sum_{j=1}^{2} c_{2j}\; \Phi_2(r) \otimes \big(A(r')\Phi_j(r')\big)^\dagger \end{aligned} \tag{4.17}$$

and for $r' \in (r_-, r_+)$

$$G(r,r') := \frac{1}{\Delta(r')} \begin{cases} \Theta(r-r') \sum_{i,j=1}^{2} c_{ij} \Phi_i(r) \otimes \big(A(r')\Phi_j(r')\big)^\dagger & \text{for} \quad \operatorname{Im}\omega_\varepsilon < 0 \\ \Theta(r'-r) \sum_{i,j=1}^{2} c_{ij} \Phi_i(r) \otimes \big(A(r')\Phi_j(r')\big)^\dagger & \text{for} \quad \operatorname{Im}\omega_\varepsilon > 0 \end{cases} \tag{4.18}$$

with $c_{ij} \in \mathbb{C}$. Note that the matrix $A(r')$ is constant within the respective regions. Thus, in all follow-up computations we drop the r' dependency. Additionally, we want to highlight that the above tensor notation corresponds to the bra-ket notation

$$X \otimes Y^\dagger = |X\rangle\langle Y|.$$

Lemma 4.8 *The Green's matrix is well-defined and bounded in the three regions. Additionally, the coefficients c_{ij} for $r' \notin (r_-, r_+)$ are given by*

$$c_{ij} = \begin{bmatrix} h^{11} & h^{12} \\ -h^{21} & -h^{22} \end{bmatrix},$$

where h^{ij} is the inverse matrix of h_{ij} from Lemma 4.6. In the case $r' \in (r_-, r_+)$, one ends up with

$$c_{ij} = \begin{bmatrix} h^{11} & h^{12} \\ h^{21} & h^{22} \end{bmatrix} \textit{ for } \operatorname{Im}\omega_\varepsilon < 0 \quad \textit{and} \quad c_{ij} = \begin{bmatrix} -h^{11} & -h^{12} \\ -h^{21} & -h^{22} \end{bmatrix} \textit{ for } \operatorname{Im}\omega_\varepsilon > 0\,.$$

Proof We point out that the Gram matrix h_{ij} is invertible because the Φ_1 and Φ_2 in (4.16) are two fundamental solutions and A is a regular matrix. The proof is a straightforward computation. We want to solve the distributional equation (4.3)

$$\mathcal{R}(\partial_r; r)\, G(r; r')_{\omega_\varepsilon} = \delta(r-r')\mathbb{1}_{\mathbb{C}^2}$$

Inserting the ansatz for $r' \notin (r_-, r_+)$, we end up with

$$\begin{aligned} \mathcal{R}(\partial_r; r)\, G(r; r')_{\omega_\varepsilon} = \delta(r-r')\Big[& c_{11}\Phi(r)_1 \otimes \big(A\Phi(r)_1\big)^\dagger + c_{12}\Phi(r)_1 \otimes \big(A\Phi(r)_2\big)^\dagger \\ & - c_{21}\Phi(r)_2 \otimes \big(A\Phi(r)_1\big)^\dagger - c_{22}\Phi(r)_2 \otimes \big(A\Phi(r)_2\big)^\dagger \Big] \end{aligned}$$

Additionally, we have following completeness relation on our Hilbert space

$$\sum_{i,j} h^{ij}\ \Phi(r)_i \otimes \big(A\Phi(r)_j\big)^\dagger = \mathbb{1}_{\mathbb{C}^2}$$

with

$$h^{ij} = \frac{1}{\det h} \begin{bmatrix} h_{22} & -h_{12} \\ -h_{21} & h_{11} \end{bmatrix}.$$

Combining both parts gives the result. In the case $r' \in (r_-, r_+)$, the steps are identical.

By Lemma 4.4 we have globally well-defined Jost solutions which are all bounded. Together with the computed coefficients, one sees that all Green's matrices are bounded in the corresponding interval for r'. □

5 Integral representation of the Dirac propagator

5.1 Abstract representation

We can now state a first step toward our main result.

Proposition 5.1 *Let H be the Hamiltonian of the Dirac equation in the Reissner–Nordström geometry in Eddington–Finkelstein coordinates. Then the corresponding Dirac propagator has the integral representation*

$$X(\tau, r) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega\tau} \lim_{\varepsilon \searrow 0} \Big[\big(H_\xi - \mathbb{1}_{\mathbb{C}^2}(\omega - i\varepsilon)\big)^{-1} - \big(H_\xi + \mathbb{1}_{\mathbb{C}^2}(\omega - i\varepsilon)\big)^{-1} \Big] X_0(r)\, d\omega.$$

Here the resolvent can be expressed in forms of the Green's matrices $G(r; r')_{\omega_\varepsilon}$ from the ansatz (4.17) and (4.18) plus the coefficients from Lemma 4.8. It has the expression

$$(H_k - \omega \pm i\varepsilon)^{-1} X_0(r) = \int_{r_0}^{\infty} G(r; r')_{\omega_\varepsilon}\, C(r')\, X_0(r')\, dr'$$

with

$$C(r') = \begin{bmatrix} 2r'^2 - \Delta(r') & 0 \\ 0 & -\Delta(r') \end{bmatrix},$$

where $X_0 \in C^\infty_{init}(N)$ is the initial data for a fixed angular momentum mode k, l.

Proof Combining the results from Propositions 3.4, 4.1 and 4.8, the remaining task is to interchange the limit $\varepsilon \searrow 0$ with the integral and then to take the limit $a \to \infty$ in (3.5). Since all extended Jost functions are bounded (for details see Subsection 4.3), we can apply Lebesgue's dominated convergence theorem to take the limit $\varepsilon \searrow 0$ inside the integral. The limit $a \to \infty$ exists by Stone's theorem. □

5.2 Main theorem

By further calculations it is possible to bring the result from Proposition 5.1 in a much more handy form. We begin by computing the Green's matrices in the upper and lower complex plane separately and taking the limit $\varepsilon \searrow 0$ for $r' \in (r_+, \infty)$. Afterward, we will take the difference and find a more compact expression for the integral representation. In the end we can extend this to $r' \in (r_-, \infty)$.

Lemma 5.2 *For $r' \in (r_+, \infty)$ we can express the differences of the Green's function as*

$$\lim_{\varepsilon \searrow 0} G(r;r')_{>0} - \lim_{\varepsilon \nearrow 0} G(r;r')_{<0} = \sum_{i,j=1}^{2} \frac{g_{ij}}{\Delta(r')}\, \chi_i(r) \otimes \chi_j(r')^\dagger A$$

with coefficients g_{ij} of the form

$$g_{11} = g_{22} = 1,\ \ g_{12} = \frac{a}{b},\ \ g_{21} = \frac{d}{c} \qquad \text{if } |\omega| > m \tag{5.1}$$

$$g_{ij} = f\, \delta_{i,1}\delta_{j,1} \qquad \text{if } |\omega| < m\ , \tag{5.2}$$

where $a, b, c, d, f \in \mathbb{C}$ and $b, c \neq 0$. Moreover, A is again the matrix from Lemma 4.6. Additionally, $\chi(r) = (\chi_1(r), \chi_2(r))^T$ are the limits of the Jost solutions from Lemma 4.3 defined on the real axis and $G(r;r')_{>0}$ describes the Green's matrix on the upper, as well as $G(r;r')_{<0}$ on the lower complex half plane.

Proof We begin with the case $|\omega_\varepsilon| > m$. By looking at Lemma 4.4 with $r' \in (r_+, \infty)$ for $\text{Im}(\omega_\varepsilon) > 0$, we get two extended Jost solutions $\Phi_1(r)$ and $\Phi_2(r)$. Since one can express any Jost solutions as a linear combination of two others, we choose for $\Phi_2(r)$ a different ansatz to simplify the calculations

$$\Phi_1(r) = \widehat{\mathcal{J}_\infty}(r) \quad \text{and} \quad \Phi_2(r) = a\, \widehat{\mathcal{J}_\infty}(r) + b\, \widecheck{\mathcal{J}_\infty}(r)\ , \tag{5.3}$$

where the coefficients of the linear combinations are denoted by $a,\ b \in \mathbb{C}$. It follows from Lemma 4.7 that b is nonzero. First, we will calculate the coefficients c_{ij} with those functions. After that we substitute everything into the ansatz (4.17) and compute the result. We want to highlight that the matrix from lemma 4.3 is pseudo-orthonormal to our Wronskian product. Because we are in the complex planes, we need to treat the product $\langle U_{\omega_\varepsilon} | A\, U_{\omega_\varepsilon}\rangle$ more carefully. We can rewrite the hyperbolic functions in exponential functions and expand them in a power series

$$U_{\omega_\epsilon} \approx U_{|\omega_\varepsilon|} + i\varphi(\varepsilon) \begin{pmatrix} \sinh(|\omega_\varepsilon|) & \cosh(|\omega_\varepsilon|) \\ \cosh(|\omega_\varepsilon|) & \sinh(|\omega_\varepsilon|) \end{pmatrix}.$$

Thus, we end up with

$$(U_{\omega_\varepsilon})^\dagger\, A\, U_{\omega_\varepsilon} \approx \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2\, i\, \varphi(\varepsilon) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \varphi(\varepsilon)^2 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We are only interested in the term of zeroth order in ε because we will perform the limit $\varepsilon \to 0$ in the end. Having that in mind, we can compute the coefficients for the Green's function with our ansatz 5.3. (Note that in the considered region $r' \in (r_+, \infty)$, the function $\Delta(r')$ is positive.)

$$c_{ij} = -\frac{1}{|b|^2} \begin{bmatrix} |a|^2 - |b|^2 & -a \\ a^* & -1 \end{bmatrix}. \tag{5.4}$$

Now we begin by computing the first term from (4.17) (denoted by the superscript (1)),

$$G(r,r')^{(1)}_{>0} = \frac{\Theta(r-r')}{\Delta(r')} \sum_{j=1}^{2} c_{1j}\, \Phi_1(r) \otimes \big(A\Phi_j(r')\big)^\dagger$$
$$= \frac{\Theta(r-r')}{\Delta(r')} \left[\frac{a}{b}\, \widehat{\mathcal{J}_\infty}(r) \otimes \big(A\, \widecheck{\mathcal{J}_\infty}(r')\big)^\dagger + \widehat{\mathcal{J}_\infty}(r) \otimes \big(A\, \widehat{\mathcal{J}_\infty}(r')\big)^\dagger \right].$$

Repeating the steps for the second term gives

$$G(r,r')^{(2)}_{>0} = \frac{\Theta(r'-r)}{\Delta(r')} \sum_{j=1}^{2} c_{2j}\, \Phi_2(r) \otimes \big(A\Phi_j(r')\big)^\dagger$$
$$= \frac{\Theta(r'-r)}{\Delta(r')} \left[\widecheck{\mathcal{J}_\infty}(r) \otimes \big(A\, \widecheck{\mathcal{J}_\infty}(r')\big)^\dagger + \frac{a}{b}\, \widehat{\mathcal{J}_\infty}(r) \otimes \big(A\, \widecheck{\mathcal{J}_\infty}(r')\big)^\dagger \right].$$

Taking the sum of both terms leads to

$$G(r,r')_{>0} = \frac{1}{\Delta(r')}\, \frac{a}{b}\, \widehat{\mathcal{J}_\infty}(r) \otimes \big(A\, \widecheck{\mathcal{J}_\infty}(r')\big)^\dagger + \frac{\Theta(r-r')}{\Delta(r')}\, \widehat{\mathcal{J}_\infty}(r) \otimes \big(A\, \widehat{\mathcal{J}_\infty}(r')\big)^\dagger + \frac{\Theta(r'-r)}{\Delta(r')}\, \widecheck{\mathcal{J}_\infty}(r) \otimes \big(A\, \widecheck{\mathcal{J}_\infty}(r')\big)^\dagger\,.$$

Since the resolvent is bounded, we can take the ϵ-limit inside the integral from Proposition 5.1 and apply the limit on the Jost solutions from the upper and lower complex plane. More importantly, the Jost solutions from the lower and upper plane coincide on the real axis. We define following limits

$$\lim_{\varepsilon\searrow 0} \widehat{\mathcal{J}_\infty}(r) = \lim_{\varepsilon\nearrow 0} \widehat{\mathcal{J}_\infty}(r) =: \chi_1(r) \quad \text{and} \quad \lim_{\varepsilon\searrow 0} \widecheck{\mathcal{J}_\infty}(r) = \lim_{\varepsilon\nearrow 0} \widecheck{\mathcal{J}_\infty}(r) =: \chi_2(r),$$

Therefore, we end up with the result for the upper half plane

$$\lim_{\varepsilon\searrow 0} G(r,r')_{>0} = \frac{1}{\Delta(r')}\, \frac{a}{b}\, \chi_1(r) \otimes \chi_2(r')^\dagger A + \frac{\Theta(r-r')}{\Delta(r')}\, \chi_1(r) \otimes \chi_1(r')^\dagger A + \frac{\Theta(r'-r)}{\Delta(r')}\, \chi_2(r) \otimes \chi_2(r')^\dagger A\,. \tag{5.5}$$

In the lower half plane, we take the Jost solutions

$$\Phi_1(r) = \widecheck{\mathcal{J}_\infty}(r) \quad \text{and} \quad \Phi_2(r) = c\, \widehat{\mathcal{J}_\infty}(r) + d\, \widecheck{\mathcal{J}_\infty}(r) \qquad \text{if } |\omega| > m\,. \tag{5.6}$$

with $c,\ d\ \in \mathbb{C}$. It follows from Lemma 4.7 that c is nonzero. Repeating similar steps for the lower half plane, we end up with

$$\begin{aligned}\lim_{\varepsilon \nearrow 0} G(r,r')_{<0} = &-\frac{1}{\Delta(r')}\frac{d}{c}\,\chi_2(r)\otimes\chi_1(r')^\dagger A\\ &-\frac{\Theta(r'-r)}{\Delta(r')}\,\chi_1(r)\otimes\chi_1(r')^\dagger A\\ &-\frac{\Theta(r-r')}{\Delta(r')}\,\chi_2(r)\otimes\chi_2(r')^\dagger A\,.\end{aligned}$$

Taking the difference and setting the coefficients as the matrix entries gives the result (5.1).

In the case $|\omega_\varepsilon| < m$, however, we choose the same ansatz 5.3 for the positive, but take a different one for the negative complex plane, i.e.

$$\Phi_1(r) = \widehat{\mathcal{J}_\infty}(r) \quad \text{and} \quad \Phi_2(r) = a\,\widehat{\mathcal{J}_\infty}(r) + b\,\widetilde{\mathcal{J}_\infty}(r) \qquad \text{if } |\omega| < m\,.$$

Furthermore, we have the matrix V_{ω_ε} in front of our fundamental solution $\widehat{\mathcal{J}_\infty}(r)$. This matrix behaves differently in the Wronskian and results in a mixing of the components of our fundamental solutions

$$(V_{\omega_\varepsilon})^\dagger A\, V_{\omega_\varepsilon} = \underbrace{\frac{im}{2\sqrt{m^2-\omega^2}}}_{=:g(\omega,m)}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} + \mathcal{O}(\varepsilon).$$

Again, we are only interested in the term with zeroth order in ε. This time, we end up with a coefficients matrix of the form

$$c_{ij} = \frac{g}{|b|^2}\begin{bmatrix}a^*b - ab^* & -b\\ -b* & 0\end{bmatrix},$$

for the upper complex plane. Continuing with similar computations, the Green's function has the form

$$\begin{aligned}G(r,r')_{>0} = &-\frac{g}{\Delta(r')}\frac{a}{b}\,\widehat{\mathcal{J}_\infty}(r)\otimes\widehat{\mathcal{J}_\infty}(r')^\dagger A\\ &-\frac{g\,\Theta(r-r')}{\Delta(r')}\,\widehat{\mathcal{J}_\infty}(r)\otimes\widetilde{\mathcal{J}_\infty}(r')^\dagger A\\ &-\frac{g\,\Theta(r'-r)}{\Delta(r')}\,\widetilde{\mathcal{J}_\infty}(r)\otimes\widehat{\mathcal{J}_\infty}(r')^\dagger A\,.\end{aligned}$$

A similar computation gives the result for the lower complex plane. Taking the difference and the limit $\varepsilon \to 0$ gives the second result (5.2),

$$\lim_{\varepsilon \searrow 0} G(r, r')_{>0} - \lim_{\varepsilon \nearrow 0} G(r, r')_{<0} = \frac{1}{\Delta(r')} \underbrace{g\left(\frac{a}{b} - \frac{c}{d}\right)}_{=:f} \widehat{\mathcal{J}_\infty}(r) \otimes \widehat{\mathcal{J}_\infty}(r')^\dagger A \,.$$

This concludes the proof. □

In a next step we want to compute the spectral measure from the difference of the Green's matrices only outside the black hole. This is possible because the Green's matrix is evaluated in the resolvent only pointwise, making it possible to split the integral due to linearity.

Lemma 5.3 *The spectral measure of the Dirac Hamiltonian in Eddington–Finkelstein coordinates for $r' \in (r_+, \infty)$ on initial data $X_0 \in C^\infty_{init}(N)$ has the form*

$$dE_\omega(X_0)(r) = \sum_{i,j} t_{ij}\, \chi_i(r) \otimes \int_{r_+}^\infty \chi_j(r')^\dagger\, \Gamma(r')\, X_0(r')\, dr'\, d\omega \,,$$

where t_{ij} are the components of the matrix T given by

$$T = \begin{cases} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} & \text{if } |\omega| < m \\ \begin{pmatrix} 1 & \frac{a}{b} \\ \frac{d}{c} & 1 \end{pmatrix} & \text{if } |\omega| > m \,. \end{cases} \tag{5.7}$$

Moreover, $\Gamma := \Gamma\big|_{2\times 2}$ is the upper left 2×2-block of the matrix in (3.3).

Proof By linearity we can pull the difference of the resolvents and by dominated convergence the ε-limit from Proposition 5.1 into the integral over r'. Then we use the result from Lemma 5.2 and obtain the result by direct computation. □

We next extend the spectral projection to the region $r' \in (r_-, \infty)$.

Lemma 5.4 *The spectral measure of the Dirac Hamiltonian in Eddington–Finkelstein coordinates on initial data $X_0 \in C^\infty_{init}(N)$ from Lemma 5.3 extends to $r' \in (r_-, \infty)$, i.e.*

$$dE_\omega(X_0)(r) = \sum_{i,j} t_{ij}\, \chi_i(r) \otimes \int_{r_-}^\infty \chi_j(r')^\dagger\, \Gamma(r')\, X_0(r')\, dr'\, d\omega \,. \tag{5.8}$$

Moreover, the matrix T in (5.7) can be simplified to

$$T = \begin{cases} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \text{ with } f \in \mathbb{R} \text{ if } |\omega| < m \\ \begin{pmatrix} 1 & \frac{a}{b} \\ \frac{a^*}{b^*} & 1 \end{pmatrix} \qquad \text{if } |\omega| > m \,. \end{cases} \tag{5.9}$$

Proof Since the spectral projector is a symmetric operator, we can interchange the variables $r \leftrightarrow r'$. Therefore, the relation in Lemma 5.3 also holds if $r \in (r_+, \infty)$ and $r' \in (r_-, \infty)$. With this in mind, it remains to consider the case $r, r' \in (r_-, r_+)$. To this end, one repeats the computational steps in Lemma 5.2 for the ansatz (4.18), again with the same fundamental solutions Φ_1 and Φ_2 in (5.3) and (5.6). A straightforward computation shows that the spectral projector is again of the form as in (5.8).

It remains to show that $d/c = a^*/b^*$. To this end, we use that the Hamiltonian is symmetric with respect to the conserved scalar product from Lemma 3.1. By direct computation, one sees that the resolvent needs to be symmetric with respect to the adjoint operator of the underlying Hilbert space. Thus, the equation

$$\left(\Gamma R_{\omega_\varepsilon}\right)^\dagger = \Gamma R_{\overline{\omega_\varepsilon}}$$

needs to be satisfied, which determines the quotient $d/c = a^*/b^*$. (Here the dagger denotes the adjoint with respect to the scalar product $L^2(dr)$.) In the case $|\omega_\varepsilon| < m$, we can use the same argumentation from above. This time, we only have an entry in the diagonal of the matrix T. Therefore, f needs to be a real number. This gives the result. □

We are now in the position to state the main theorem of this paper.

Theorem 5.5 *The Dirac propagator in Proposition 5.1 can be expressed in terms of globally defined fundamental solutions $\chi_i(r, \omega)$ for $i \in \{1, 2\}$ and $r \in (r_-, \infty)$ as*

$$X(\tau, r) = \frac{1}{2} \int_{\mathbb{R}\setminus\{\pm m\}} e^{-i\omega\tau} \sum_{i=1}^{2} \widehat{X}_i(\omega)\, \chi_i(r, \omega)\, d\omega$$

with $\widehat{X}_i(\omega) : \mathbb{C} \to \mathbb{C}$ smooth functions defined by

$$\widehat{X}_i(\omega) = \frac{1}{(2\pi)^2} \sum_{j=1}^{2} t_{ij}\, (\chi_j(\omega) \,|\, X_0) \,.$$

Here $(\cdot|\cdot)$ denotes the conserved scalar product on the hypersurfaces defined in (2.2) and given more explicitly in (3.2), restricted to the upper left 2×2 block. Moreover, t_{ij} are again the entries of the matrix (5.9) and $X_0(r) \in C^\infty_{init}(N)$. Here $a, b \in \mathbb{C}$ are the transmission coefficients of the radial ODE defined in (5.3).

Proof We combine the results from Proposition 5.1, Lemma 5.3 and Lemma 5.4 and evaluate the integral over r' only in the range (r_-, ∞). This is possible because of the boundary conditions the wave is reflected on ∂M and never comes back through the Cauchy horizon. Thus, all interactions behind the Cauchy horizon do not contribute. Using the expression of the scalar product on the Cauchy hypersurfaces from Lemma 3.1 gives the final expression. □

We finally remark that this integral representation is independent of the choice of the prefactors of the fundamental solutions. Indeed, as demonstrated in the proof of Lemma 5.2, the difference of the Green's matrices depends solely on the quotient of the coefficients of the fundamental solutions. In this way, the prefactors are dropped out.

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