Received: 28 February 2024 | Revised: 17 February 2025 | Accepted: 28 April 2025

DOI: 10.1002/gamm.70003

ORIGINAL ARTICLE

GAMM Mitteilungen

Nonlocal-to-local convergence for a Cahn–Hilliard tumor growth model

Christoph Hurm[1] | **Maximilian Moser**[2]

[1]Fakultät für Mathematik, Universität Regensburg, Regensburg, Germany

[2]Institute of Science and Technology Austria, Klosterneuburg, Austria

Correspondence
Christoph Hurm, Fakultät für Mathematik, Universität Regensburg, Universitätsstraße 31, Regensburg D-93053, Germany.
Email: christoph.hurm@mathematik.uni-regensburg.de

Funding information
European Research Council, Grant/Award Number: 948819; Deutsche Forschungsgemeinschaft, Grant/Award Number: 321821685

Abstract

We consider a local Cahn–Hilliard-type model for tumor growth as well as a nonlocal model where, compared to the local system, the Laplacian in the equation for the chemical potential is replaced by a nonlocal operator. The latter is defined as a convolution integral with suitable kernels parametrized by a small parameter. For sufficiently smooth bounded domains in three dimensions, we prove convergence of weak solutions of the nonlocal model toward strong solutions of the local model together with convergence rates with respect to the small parameter. The proof is done via a Gronwall-type argument and a convergence result with rates for the nonlocal integral operator toward the Laplacian due to Abels and Hurm.

KEYWORDS

non-local and local Cahn–Hilliard equation, nonlocal to local convergence, tumor growth

MOS SUBJECT CLASSIFICATION

Primary 35K57; Secondary 35B40; 35K61; 35Q92

1 | INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded domain with C^3-boundary, $T > 0$ be fixed and $\Omega_T := \Omega \times (0, T)$ as well as $\partial\Omega_T := \partial\Omega \times (0, T)$. We consider the following local Cahn–Hilliard model for tumor growth,

$$\partial_t \varphi = \Delta\mu + (\mathcal{P}\sigma - \mathcal{A})h(\varphi) \qquad \text{in } \Omega_T, \tag{1}$$

$$\mu = -\Delta\varphi + \Psi'(\varphi) \qquad \text{in } \Omega_T, \tag{2}$$

$$\partial_t \sigma = \Delta\sigma + \mathcal{B}(\sigma_S - \sigma) - \mathcal{C}\sigma h(\varphi) \qquad \text{in } \Omega_T, \tag{3}$$

$$\partial_{\mathbf{n}}\varphi = \partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\sigma = 0 \qquad \text{on } \partial\Omega_T, \tag{4}$$

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \qquad \text{in } \Omega. \tag{5}$$

Here, $\varphi : \Omega_T \to \mathbb{R}$ is an order parameter distinguishing the healthy and tumor tissue, $\mu : \Omega_T \to \mathbb{R}$ is the chemical potential and $\sigma : \Omega_T \to \mathbb{R}$ the nutrient concentration. Moreover, $\Psi : \mathbb{R} \to \mathbb{R}$ is a double well potential with wells of equal depth and minima at ± 1 and $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C} \geq 0$ are constants representing tumor proliferation rate, tumor apoptosis rate, nutrient supply rate and nutrient consumption rate, respectively. Additionally, $h : \mathbb{R} \to [0,1]$ is an interpolation function only present in the tumor phase and the function $\sigma_S : \Omega_T \to \mathbb{R}$ is a preexisting nutrient concentration. Finally, $\partial_{\mathbf{n}}$ is the derivative in normal direction with respect to $\partial\Omega$.

The system (1) to (5) is a special case of the models derived in Garcke, Lam, Sitka, Styles,[1] in particular neglecting chemotaxis and active transport. Moreover, the system in Garcke, Lam, Rocca[2] reduces to our model when neglecting the control term. Indeed, this yields uniqueness and existence of strong solutions to our model, cf. Theorem 2.2 below. The interested reader may also consider References 1 and 2 for other Cahn–Hilliard-type models for tumor growth, for example the Cahn–Hilliard-Darcy variant and optimal control problems. Let us just mention the results in References 3–8 for similar systems as (1) to (5).

Next, for $\varepsilon > 0$ small we consider the nonlocal Cahn–Hilliard model for tumor growth,

$$\partial_t \varphi_\varepsilon = \Delta\mu_\varepsilon + (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon) \qquad \text{in } \Omega_T, \tag{6}$$

$$\mu_\varepsilon = \mathcal{L}_\varepsilon \varphi_\varepsilon + \Psi'(\varphi_\varepsilon) \qquad \text{in } \Omega_T, \tag{7}$$

$$\partial_t \sigma_\varepsilon = \Delta\sigma_\varepsilon + \mathcal{B}(\sigma_S - \sigma_\varepsilon) - \mathcal{C}\sigma_\varepsilon h(\varphi_\varepsilon) \qquad \text{in } \Omega_T, \tag{8}$$

$$\partial_{\mathbf{n}}\mu_\varepsilon = \partial_{\mathbf{n}}\sigma_\varepsilon = 0 \qquad \text{on } \partial\Omega_T, \tag{9}$$

$$\varphi_\varepsilon(0) = \varphi_{0,\varepsilon}, \quad \sigma_\varepsilon(0) = \sigma_{0,\varepsilon} \qquad \text{in } \Omega. \tag{10}$$

Here, the interpretation of the functions $\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon$ and Ψ, h, σ_S as well as the constants $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ is analogous to the local model (1) to (5) above. Moreover, for $\varepsilon > 0$ the nonlocal operator $\mathcal{L}_\varepsilon$ is defined by the following convolution integral,

$$\mathcal{L}_\varepsilon \psi(x) := \int_\Omega J_\varepsilon(x-y)(\psi(x) - \psi(y))\, dy \quad \text{for all } x \in \Omega, \tag{11}$$

for integrable $\psi : \Omega \to \mathbb{R}$ and suitable convolution kernels $J_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}_\varepsilon$ approximates $-\Delta$ as $\varepsilon \to 0$, cf. Theorem 2.1 below. In this setting, also the variational convergence

$$\mathcal{E}_\varepsilon(\psi) := \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\psi(x) - \psi(y)|^2\, dy\, dx \to \frac{1}{2}\int_\Omega |\nabla\psi|^2\, dx$$

as $\varepsilon \to 0$ for all $\psi \in H^1(\Omega)$ is well-known, cf. the results by Ponce.[9, 10]

The nonlocal system (6) to (10) was introduced in Scarpa, Signori,[11] where the authors considered a more general model with chemotaxis and active transport as well as relaxation parameters. Their paper yields an existence and uniqueness result for weak solutions of (6) to (10), cf. Theorem 2.3 below. The motivation to replace $-\Delta$ in (2) by $\mathcal{L}_\varepsilon$ in order to obtain (7) is to take into account long-range interactions, cf. also.[11] Here, note that the nonlocal system (6) to (10) is of second order compared to the fourth order system (1) to (5), hence there is no boundary condition for φ_ε in (9). For references in the direction of nonlocal Cahn–Hilliard-type models we refer to References 11 and 12 and Davoli et al.[13] Moreover, let us also note the results in Reference 14, where the author studied the optimal control problem for a viscous non-local tumor growth model.

In this contribution, we apply the results in Abels, Hurm,[15] in order to prove convergence of the weak solution of (6) to (10) to the strong solution of (1) to (5) for $\varepsilon \to 0$ together with rates of convergence with respect to ε. In the literature there are several results for nonlocal-to-local convergence. For example, the convergence (without rates) of weak solutions of the nonlocal Cahn–Hilliard equation, that is, (6) to (10) with $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C} = 0$, to the solution of the local Cahn–Hilliard equation, that is, (1) to (5) with $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C} = 0$, has already been shown in various settings such as periodic boundary conditions,[16, 17] Neumann boundary conditions[18, 19] and degenerate mobility.[20] Recently, Davoli et al. [13] proved the nonlocal-to-local limit for a viscous Cahn–Hilliard model for tumor growth. The authors in Reference 15 also derived precise rates of convergence.

The structure of this work is as follows. In Section 2, we recall some preliminaries and Section 3 contains the main result.

2 | PRELIMINARIES

2.1 | Notation

Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be open, $1 \le p \le \infty$ and $k \in \mathbb{N}$. We use the notation $L^p(\Omega)$ and $W^{k,p}(\Omega)$ for the standard Lebesgue and Sobolev spaces on Ω. The corresponding norms are denoted by $\|.\|_{L^p(\Omega)}$ and $\|.\|_{W^{k,p}(\Omega)}$, respectively. We also write $H^k(\Omega) := W^{k,2}(\Omega)$.

For any Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we use the notation X' for its dual space. The corresponding dual pairing is denoted by $\langle ., . \rangle_X : X' \times X \to \mathbb{K}$.

Moreover, we recall that the inverse of the negative Laplacian $-\Delta_N$ with Neumann boundary condition is a well-defined isomorphism

$$(-\Delta_N)^{-1} : \{c \in H^1(\Omega)' : c_\Omega = 0\} \to \{c \in H^1(\Omega) : c_\Omega = 0\}, \tag{12}$$

where $c_\Omega := \frac{1}{|\Omega|}\langle c, 1\rangle_{H^1(\Omega)}$ with $|\Omega|$ denoting the n-dimensional Lebesgue measure of Ω.

2.2 | Assumptions

We make the following general assumptions.

(A.1) Let $\Omega \subseteq \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded domain with C^3-boundary. Moreover, let $T > 0$ be fixed and $\Omega_T := \Omega \times (0, T)$ as well as $\partial\Omega_T := \partial\Omega \times (0, T)$.

(A.2) Let $J_\varepsilon : \mathbb{R}^n \to [0, \infty)$ be a non-negative function given by $J_\varepsilon(x) = \frac{\rho_\varepsilon(|x|)}{|x|^2}$ for all $x \in \mathbb{R}^n$ and $J_\varepsilon \in W^{1,1}(\mathbb{R}^n)$, where $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers satisfying

$$\rho : \mathbb{R} \to [0, \infty), \quad \rho \in L^1(\mathbb{R}), \quad \rho(r) = \rho(-r) \quad \text{for all } r \in \mathbb{R},$$

$$\rho_\varepsilon(r) = \varepsilon^{-n}\rho\left(\frac{r}{\varepsilon}\right) \quad \text{for all } r \in \mathbb{R}, \text{ for all } \varepsilon > 0,$$

$$\int_0^\infty \rho_\varepsilon(r) r^{n-1}\, dr = \frac{2}{C_n} \quad \text{for all } \varepsilon > 0,$$

$$\lim_{\varepsilon \searrow 0} \int_\delta^\infty \rho_\varepsilon(r) r^{n-1}\, dr = 0 \quad \text{for all } \delta > 0,$$

where $C_n := \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 \, d\mathcal{H}^{n-1}(\sigma)$.

(A.3) $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ are non-negative constants, $\sigma_S \in L^\infty(\Omega_T)$ and $0 \le \sigma_S \le 1$ a.e. in Ω_T.

(A.4) The function $h : \mathbb{R} \to [0, 1]$ is of class C^2, bounded and Lipschitz continuous.

(A.5) The potential $\Psi : \mathbb{R} \to [0, \infty)$ is of class C^3 and satisfies

$$|\Psi'(s)| \le k_0\Psi(s) + k_1, \tag{13}$$

$$\Psi(s) \ge k_2|s| - k_3, \tag{14}$$

$$-k_4 \le \Psi''(s) \le k_4(1 + |s|^2), \tag{15}$$

for all $s, t \in \mathbb{R}$ and some positive constants k_i, $i = 0, \dots, 4$.

(A.6) There are constants $C_1, C_2 > 0$ such that for all $s \in \mathbb{R}$

$$\Psi(s) \ge C_1|s|^4 - C_2$$

Remark 2.1. A typical example for a potential satisfying the Assumptions 2.2 and 2.2 is the classical double-well potential $\Psi(s) := \frac{1}{4}(1 - s^2)^2$ for all $s \in \mathbb{R}$.

Remark 2.2. Observe that functions $\Psi : \mathbb{R} \to [0, \infty)$ as in 2.2 also satisfy the inequality

$$|\Psi'(s) - \Psi'(t)| \leq k_5(1 + |s|^2 + |t|^2)|s - t| \tag{16}$$

for all $s, t \in \mathbb{R}$. This follows from the mean value theorem and (15).

2.3 | Inequalities

Lemma 2.1. *For every $\delta > 0$, there exist constants $C_\delta > 0$ and $\varepsilon_\delta > 0$ such that for every sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$ there holds*

$$\|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{L^2(\Omega)} \leq \delta \mathcal{E}_{\varepsilon_1}(f_{\varepsilon_1}) + \delta \mathcal{E}_{\varepsilon_2}(f_{\varepsilon_2}) + C_\delta \|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{H^1(\Omega)'} \tag{17}$$

for all $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_\delta)$.

Proof. For a proof, we refer to Reference 5 (Lemma 4(2)). ■

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, be a bounded domain with C^3-boundary. Moreover, let $\mathcal{L}_\varepsilon$ be defined as in (11) and J_ε satisfy 2.2 from Section 2.2. Then for all $c \in H^3(\Omega)$ with $\partial_{\mathbf{n}} c = 0$ on $\partial\Omega$ it holds for a constant $K > 0$ independent of ε*

$$\|\mathcal{L}_\varepsilon c + \Delta c\|_{L^2(\Omega)} \leq K\sqrt{\varepsilon}\|c\|_{H^3(\Omega)}. \tag{18}$$

Proof. A proof can be found in Reference 1 (Corollary 4.2). ■

2.4 | Existence and uniqueness results

Well-posedness of the local Cahn–Hilliard model (1) to (5) is available in a slightly more general setting due to,[2] where the authors considered the problem with an additional control function. In particular, we have the following well-posedness result for the system (1) to (5).

Theorem 2.2 (Well-posedness of the local model). *Let the Assumptions 2.2, 2.2–2.2 hold and let $n = 3$. Let the initial data (φ_0, σ_0) satisfy $\varphi_0 \in H^3(\Omega)$ with $\partial_{\mathbf{n}}\varphi_0 = 0$ on $\partial\Omega$ and $\sigma_0 \in H^1(\Omega)$ with $0 \leq \sigma_0 \leq 1$ a.e. in Ω.*

Then there is a unique solution (φ, μ, σ) of (1)–(5) with the regularity

$$\begin{aligned}
&\varphi \in L^\infty(0, T, H^2(\Omega)) \cap L^2(0, T, H^3(\Omega)) \cap H^1(0, T, L^2(\Omega)) \cap C^0(\overline{\Omega_T}),\\
&\mu \in L^2(0, T, H^2(\Omega)) \cap L^\infty(0, T, L^2(\Omega)),\\
&\sigma \in L^\infty(0, T, H^1(\Omega)) \cap L^2(0, T, H^2(\Omega)) \cap H^1(0, T, L^2(\Omega)), \quad 0 \leq \sigma \leq 1 \text{ a.e. in } \Omega_T,
\end{aligned}$$

such that $\varphi(0) = \varphi_0$, $\sigma(0) = \sigma_0$ and for a.e. $t \in (0, T)$ and all $\xi \in H^1(\Omega)$ it holds

$$0 = \int_\Omega \partial_t\varphi\xi + \nabla\mu \cdot \nabla\xi - (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\xi \, dx, \tag{19}$$

$$0 = \int_\Omega \mu\xi - \Psi'(\varphi)\xi - \nabla\varphi \cdot \nabla\xi \, dx, \tag{20}$$

$$0 = \int_\Omega \partial_t\sigma\xi + \nabla\sigma \cdot \nabla\xi + \mathcal{C}h(\varphi)\xi + \mathcal{B}(\sigma - \sigma_S)\xi \, dx. \tag{21}$$

Proof. We refer to Reference 15 (Theorem 2.1). ■

Existence of weak solutions of the nonlocal system (6) to (10) has already been shown in Reference 11, where the authors considered a more general Cahn–Hilliard system including chemotaxis and active transport, as well as possible relaxation terms for the Cahn–Hilliard equation. The following result is obtained:

Theorem 2.3 (Well-posedness of the non-local model). *Let the Assumptions 2.2–2.2 hold and let* $n = 3$. *Moreover, we assume that* $\varphi_{0,\varepsilon}, \sigma_{0,\varepsilon} \in L^2(\Omega)$. *Then for* $\varepsilon_0 > 0$ *small and all* $\varepsilon \in (0, \varepsilon_0]$ *there exists a unique weak solution* $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ *of (6)–(10) with the regularity*

$$\begin{aligned}
&\varphi_\varepsilon \in H^1(0,T,H^1(\Omega)') \cap L^2(0,T,H^1(\Omega)),\\
&\mu_\varepsilon \in L^2(0,T,H^1(\Omega)),\\
&\sigma_\varepsilon \in H^1(0,T,H^1(\Omega)') \cap L^2(0,T,H^1(\Omega)), \quad 0 \le \sigma_\varepsilon(t) \le 1 \text{ a.e. in } \Omega, \text{ for all } t \in [0,T],
\end{aligned}$$

such that $\varphi_\varepsilon(0) = \varphi_{0,\varepsilon}, \sigma_\varepsilon(0) = \sigma_{0,\varepsilon}$ *and for all* $\xi \in H^1(\Omega)$ *and a.e.* $t \in (0,T)$ *it holds*

$$0 = \langle \partial_t \varphi_\varepsilon, \xi \rangle_{H^1(\Omega)} + \int_\Omega \nabla \mu_\varepsilon \cdot \nabla \xi \, dx - \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A}) h(\varphi_\varepsilon) \xi \, dx, \tag{22}$$

$$\mu_\varepsilon = \mathcal{L}_\varepsilon \varphi_\varepsilon + \Psi'(\varphi_\varepsilon), \tag{23}$$

$$0 = \langle \partial_t \sigma_\varepsilon, \xi \rangle_{H^1(\Omega)} + \int_\Omega \nabla \sigma_\varepsilon \cdot \nabla \xi + \mathcal{C} h(\varphi_\varepsilon)\xi + \mathcal{B}(\sigma_\varepsilon - \sigma_S)\xi \, dx. \tag{24}$$

Proof. This follows from Reference 11, Theorem 2.14 and Theorem 2.15. Here, $\varepsilon_0 > 0$ is such that for some $c_0 > 0$ it holds

$$\Psi''(s) + \inf_{x\in\Omega} \int_\Omega J_\varepsilon(x-y)\, dy \ge c_0 \quad \text{for all } s \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0].$$

The existence of such an $\varepsilon_0 > 0$ follows from (15) and

$$\inf_{x\in\Omega} \int_\Omega J_\varepsilon(x-y) dy \ge \frac{c}{\varepsilon^2} \quad \text{for all } \varepsilon \in (0,1]$$

for some $c > 0$. Let us briefly remark the ideas for the latter estimate here. First, note that by applying the transformation rule and a rescaling argument together with the properties of J_ε from 2.2, we obtain

$$\int_{B_\delta(x)} J_\varepsilon(x-y) dy \ge \frac{c}{\varepsilon^2} \quad \text{for all } \varepsilon \in (0,1],$$

where $B_\delta(x) \subseteq \mathbb{R}^n$ is the ball with radius $\delta > 0$ around x. Here $c > 0$ is a positive constant that can be chosen uniformly for all $x \in \mathbb{R}^n$ and $\delta \ge \delta_0 > 0$, where δ_0 is any fixed positive constant. This holds analogously if the balls are replaced by sectors of such balls based on angles uniformly bounded away from zero. Hence the task reduces to assign to each point $x \in \Omega$ such an object contained in Ω. For points $x \in \Omega$ outside a tubular neighborhood of $\partial\Omega$ this is directly clear. For points close to the boundary one can employ the uniform interior ball condition for $\partial\Omega$. ∎

3 | CONVERGENCE RESULT

Theorem 3.1. *Let the Assumptions 2.2–2.2 hold, let* $n = 3$ *and* $\varepsilon_0 > 0$ *be as in Theorem 2.3. Moreover, for the initial data* (φ_0, σ_0) *to the local system (1)–(5) we assume* $\varphi_0 \in H^3(\Omega)$ *with* $\partial_{\mathbf{n}}\varphi_0 = 0$ *on* $\partial\Omega$ *and* $\sigma_0 \in H^1(\Omega)$ *with* $0 \le \sigma_0 \le 1$ *a.e. in* Ω. *Additionally, for* $\varepsilon \in (0, \varepsilon_0]$ *let the initial data for the nonlocal system (6)–(10) satisfy* $\varphi_{0,\varepsilon}, \sigma_{0,\varepsilon} \in L^2(\Omega)$, $\mathscr{E}_\varepsilon(\varphi_{0,\varepsilon}) \le C$, $\int_\Omega \Psi(\varphi_{0,\varepsilon})\, dx \le C$ *and*

$$\|\varphi_{0,\varepsilon} - \varphi_0\|_{H^1(\Omega)'} + \|\sigma_{0,\varepsilon} - \sigma_0\|_{L^2(\Omega)} + |(\varphi_{0,\varepsilon})_\Omega - (\varphi_0)_\Omega| \le C\sqrt{\varepsilon} \tag{25}$$

for some constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0]$.

Then there exists a constant $K > 0$ such that the weak solution $(\varphi_\varepsilon, \sigma_\varepsilon, \mu_\varepsilon)$ of the nonlocal model (6)–(10) for $\varepsilon \in (0, \varepsilon_0]$ from Theorem 2.3 and the strong solution (φ, σ, μ) of the local model (1)–(5) from Theorem 2.2 satisfy for some constant $K > 0$ independent of ε,

$$\sup_{t\in[0,T]} \|\varphi_\varepsilon(t) - \varphi(t)\|_{H^1(\Omega)'} + \|\varphi_\varepsilon - \varphi\|_{L^2(0,T;L^2(\Omega))} \le K\sqrt{\varepsilon}, \tag{26}$$

$$\sup_{t\in[0,T]} \|\sigma_\varepsilon(t) - \sigma(t)\|_{L^2(\Omega)} + \|\nabla\sigma_\varepsilon - \nabla\sigma\|_{L^2(0,T;L^2(\Omega))} \le K\sqrt{\varepsilon}. \tag{27}$$

Remark 3.1. We assumed $n = 3$ in the theorem because this is also the case in References 2 and 11. However, note that Theorem 2.2, Theorem 2.3 and Theorem 3.1 should also work in the case $n = 2$ because the required embeddings are improved.

Proof. We denote by $(\varphi_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ the weak solution to the nonlocal model (6) to (10) given by Theorem 2.3 and with (φ, μ, σ) the unique solution to the local model (1) to (5) provided by Theorem 2.2. Then, the functions $\tilde{\varphi} := \varphi_\varepsilon - \varphi, \tilde{\mu} := \mu_\varepsilon - \mu, \tilde{\sigma} := \sigma_\varepsilon - \sigma$ solve the system

$$\partial_t\tilde{\varphi} = \Delta\tilde{\mu} + (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(\varphi) \qquad \text{in } \Omega_T, \tag{28}$$

$$\tilde{\mu} = \mathcal{L}_\varepsilon\varphi_\varepsilon + \Delta\varphi + \Psi'(\varphi_\varepsilon) - \Psi'(\varphi) \qquad \text{in } \Omega_T, \tag{29}$$

$$\partial_t\tilde{\sigma} = \Delta\tilde{\sigma} - \mathcal{B}\tilde{\sigma} - \mathcal{C}\sigma_\varepsilon h(\varphi_\varepsilon) + \mathcal{C}\sigma h(\varphi) \qquad \text{in } \Omega_T. \tag{30}$$

in a weak sense. More precisely, the weak formulation is obtained by testing with functions in $H^1(\Omega)$, cf. the weak formulations (19) to (21) and (22) to (24).

Testing (28) with $(-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega) \in H^1(\Omega)$, cf. the property (12) for the inverse Neumann Laplacian above, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'} &= -\int_\Omega \tilde{\mu}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx \\ &\quad + \int_\Omega \big[(\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\big](-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx.\end{aligned} \tag{31}$$

For the second term on the right-hand side of (31), we have

$$\begin{aligned}&\int_\Omega \big[(\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\big](-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx \\ &= \int_\Omega \mathcal{P}\sigma_\varepsilon(h(\varphi_\varepsilon) - h(\varphi))(-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx \\ &\quad - \int_\Omega \mathcal{A}(h(\varphi_\varepsilon) - h(\varphi))(-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx \\ &\quad + \int_\Omega h(\varphi)\mathcal{P}\tilde{\sigma}(-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx =: I_1 + I_2 + I_3.\end{aligned}$$

Using the Lipschitz continuity and boundedness of h, cf. assumption 2.2, and $|\sigma_\varepsilon| \le 1$ a.e. in Ω_T due to Theorem 2.3, we obtain the following estimates, where L_h is the Lipschitz constant of h:

$$|I_1| \le \mathcal{P}L_h\|\tilde{\varphi}\|_{L^2(\Omega)}\|(-\Delta_N)^{-1}(\tilde{\varphi} - \tilde{\varphi}_\Omega))\|_{L^2(\Omega)} \tag{32}$$

$$\le \frac{1}{36}\|\tilde{\varphi}\|^2_{L^2(\Omega)} + \mathcal{P}K\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'}, \tag{33}$$

$$|I_2| \le \frac{1}{36}\|\tilde{\varphi}\|^2_{L^2(\Omega)} + \mathcal{A}K\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'}, \tag{34}$$

$$|I_3| \leq \frac{1}{2}\|\tilde{\sigma}\|^2_{L^2(\Omega)} + \mathcal{P}K\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'}. \tag{35}$$

In the next step, we test (29) with $\tilde{\varphi} - \tilde{\varphi}_\Omega$. This yields

$$\int_\Omega \tilde{\mu}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx = \int_\Omega (\mathcal{L}_\varepsilon\varphi_\varepsilon + \Delta\varphi)(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx + \int_\Omega \big(\Psi'(\varphi_\varepsilon) - \Psi'(\varphi)\big)(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx. \tag{36}$$

For the first term on the right-hand side of (36), it holds

$$\int_\Omega (\mathcal{L}_\varepsilon\varphi_\varepsilon + \Delta\varphi)(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx = \int_\Omega (\mathcal{L}_\varepsilon\varphi + \Delta\varphi)(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx + \int_\Omega \mathcal{L}_\varepsilon\tilde{\varphi}(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx. \tag{37}$$

Recalling that $\mathcal{L}_\varepsilon\tilde{\varphi}_\Omega = 0$, we can add the term $-\int_\Omega \mathcal{L}_\varepsilon\tilde{\varphi}_\Omega(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx$ to the right-hand side in (37). Then, by the symmetry of the interaction kernel J_ε, we observe that

$$\int_\Omega \mathcal{L}_\varepsilon(\tilde{\varphi} - \tilde{\varphi}_\Omega)(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx = 2\mathcal{E}_\varepsilon(\tilde{\varphi} - \tilde{\varphi}_\Omega).$$

For the second term in (36), we first observe that

$$\int_\Omega \big(\Psi'(\varphi_\varepsilon) - \Psi'(\varphi)\big)(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,dx = \int_\Omega \big(\Psi'(\varphi_\varepsilon) - \Psi'(\varphi)\big)\tilde{\varphi}\,dx - \int_\Omega \big(\Psi'(\varphi_\varepsilon) - \Psi'(\varphi)\big)\tilde{\varphi}_\Omega\,dx.$$

For the first term on the right-hand side, we use the Fundamental Theorem of Calculus. Then, the assumption $\Psi'' \geq -k_4$ yields

$$\int_\Omega \big(\Psi'(\varphi_\varepsilon) - \Psi'(\varphi)\big)\tilde{\varphi}\,dx \geq -k_4\|\tilde{\varphi}\|^2_{L^2(\Omega)}.$$

For the second term, we use assumption (16) and get

$$\int_\Omega \big(\Psi'(\varphi_\varepsilon) - \Psi'(\varphi)\big)\tilde{\varphi}_\Omega\,dx \leq k_5\int_\Omega (1 + |\varphi_\varepsilon|^2 + |\varphi|^2)|\tilde{\varphi}||\tilde{\varphi}_\Omega|\,dx.$$

Invoking the inequalities of Hölder and Young, we infer

$$\begin{aligned}
k_5\int_\Omega (1 + |\varphi_\varepsilon|^2 + |\varphi|^2)|\tilde{\varphi}||\tilde{\varphi}_\Omega|\,dx &\leq k_5\|\tilde{\varphi}\|_{L^2(\Omega)}\Big\|(1 + |\varphi_\varepsilon|^2 + |\varphi|^2)|\tilde{\varphi}_\Omega|\Big\|_{L^2(\Omega)} \\
&\leq k_5\|\tilde{\varphi}\|^2_{L^2(\Omega)} + K|\tilde{\varphi}_\Omega|^2\|(1 + |\varphi_\varepsilon|^2 + |\varphi|^2)\|^2_{L^2(\Omega)} \\
&\leq k_5\|\tilde{\varphi}\|^2_{L^2(\Omega)} + K|\tilde{\varphi}_\Omega|^2\Big(1 + \|\varphi_\varepsilon\|^4_{L^4(\Omega)} + \|\varphi\|^4_{L^4(\Omega)}\Big).
\end{aligned}$$

In the next step, we need to control the term $\|\varphi_\varepsilon\|_{L^4(\Omega)}$ uniformly in ε. To this end, we test equation (22) by μ_ε, (23) by $-\partial_t\varphi_\varepsilon$ and (24) by σ_ε and sum the resulting equations. This yields

$$\begin{aligned}
&\frac{d}{dt}\left(\mathcal{E}_\varepsilon(\varphi_\varepsilon) + \int_\Omega \Psi(\varphi_\varepsilon)\,\mathrm{d}x + \|\sigma_\varepsilon\|^2_{L^2(\Omega)}\right) + \|\nabla\mu_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\sigma_\varepsilon\|^2_{L^2(\Omega)} \\
&\quad = \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon)\mu_\varepsilon\,\mathrm{d}x - \int_\Omega \mathcal{C}h(\varphi_\varepsilon)\sigma_\varepsilon\,\mathrm{d}x - \int_\Omega \mathcal{B}(\sigma_\varepsilon - \sigma_S)\sigma_\varepsilon\,\mathrm{d}x.
\end{aligned}$$

Since the function h is bounded, we can control the terms on the right-hand side using the assumptions 2.2 and Young's inequality. In particular, we obtain

$$\frac{d}{dt}\left(\mathcal{E}_\varepsilon(\varphi_\varepsilon) + \int_\Omega \Psi(\varphi_\varepsilon)\,\mathrm{d}x + \|\sigma_\varepsilon\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|\nabla\mu_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\sigma_\varepsilon\|^2_{L^2(\Omega)} \leq C + C\|\sigma_\varepsilon\|^2_{L^2(\Omega)}.$$

Note that we also used the Poincaré-Wirtinger inequality, in order to absorb $\|\nabla\mu_\varepsilon\|_{L^2(\Omega)}$ into the left-hand side. In fact, testing (23) by 1 and using the properties of Ψ, one can show that μ_ε has bounded mean value. Finally, Gronwalls Lemma and our assumptions on the initial data give uniform estimates independent of ε. In particular, we have that

$$\int_\Omega \Psi(\varphi_\varepsilon)\,\mathrm{d}x \leq C.$$

Then, assumption 2.2 yields that the sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subset L^\infty(0,T;L^4(\Omega))$ is bounded.

Next, we test equation (30) with $\tilde\sigma$. Then, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde\sigma\|^2_{L^2(\Omega)} + \|\nabla\tilde\sigma\|^2_{L^2(\Omega)} &= \int_\Omega (\mathcal{B}(\sigma_S-\sigma_\varepsilon) - \mathcal{B}(\sigma_S-\sigma))\tilde\sigma\,dx \\ &\quad - \int_\Omega (\mathcal{C}\sigma_\varepsilon h(\varphi_\varepsilon) - \mathcal{C}\sigma h(\varphi))\tilde\sigma\,dx \\ &= -\mathcal{B}\|\tilde\sigma\|^2_{L^2(\Omega)} - \int_\Omega (\mathcal{C}\sigma_\varepsilon h(\varphi_\varepsilon) - \mathcal{C}\sigma h(\varphi))\tilde\sigma\,dx. \end{aligned} \tag{38}$$

For the second term in (38), we observe

$$\begin{aligned}\int_\Omega (\mathcal{C}\sigma_\varepsilon h(\varphi_\varepsilon) - \mathcal{C}\sigma h(\varphi))\tilde\sigma\,dx &= \int_\Omega \mathcal{C}\sigma_\varepsilon(h(\varphi_\varepsilon) - h(\varphi))\tilde\sigma\,dx + \int_\Omega \mathcal{C}h(\varphi)|\tilde\sigma|^2\,dx \\ &\leq K\mathcal{C}\|\tilde\sigma\|^2_{L^2(\Omega)} + \frac{1}{36}\|\tilde\varphi\|^2_{L^2(\Omega)}, \end{aligned} \tag{39}$$

where we again used $|\sigma_\varepsilon| \leq 1$ a.e. in Ω_T and the Lipschitz continuity and boundedness of h. Combining the previous estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{H^1(\Omega)'} + \|\tilde\sigma\|^2_{L^2(\Omega)}\right) + 2\mathcal{E}_\varepsilon(\tilde\varphi - \tilde\varphi_\Omega) + \|\nabla\tilde\sigma\|^2_{L^2(\Omega)} + \frac{1}{2}\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{12}\|\tilde\varphi\|^2_{L^2(\Omega)} + K\|\tilde\sigma\|^2_{L^2(\Omega)} - \int_\Omega (\mathcal{L}_\varepsilon\varphi + \Delta\varphi)(\tilde\varphi - \tilde\varphi_\Omega)\,dx + (k_4+k_5)\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{L^2(\Omega)} \\ &\quad + K\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{H^1(\Omega)'} + \frac{1}{2}\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{L^2(\Omega)} + K|\tilde\varphi_\Omega|^2. \end{aligned} \tag{40}$$

Finally, we test (28) with the mean value $\tilde\varphi_\Omega$ and obtain

$$\frac{|\Omega|}{2}\partial_t|\tilde\varphi_\Omega|^2 = \int_\Omega \partial_t\tilde\varphi\tilde\varphi_\Omega dx = \int_\Omega \left[(\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\right]\tilde\varphi_\Omega dx.$$

With analogous estimates as before, it follows that

$$\partial_t|\tilde\varphi_\Omega|^2 \leq \frac{1}{12}\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{L^2(\Omega)} + C(\|\tilde\sigma\|^2_{L^2(\Omega)} + |\tilde\varphi_\Omega|^2).$$

Moreover, Hölder's and Young's inequalities imply

$$\int_\Omega (\mathcal{L}_\varepsilon\varphi + \Delta\varphi)(\tilde\varphi - \tilde\varphi_\Omega)\,dx \leq \frac{1}{6}\|\tilde\varphi - \tilde\varphi_\Omega\|^2_{L^2(\Omega)} + K\|\mathcal{L}_\varepsilon\varphi + \Delta\varphi\|^2_{L^2(\Omega)}.$$

Owing to the inequality,

$$\|\tilde\varphi\|^2_{L^2(\Omega)} \leq \|\tilde\varphi - \tilde\varphi_\Omega\|^2_{L^2(\Omega)} + C|\tilde\varphi_\Omega|^2,$$

we can now collect the terms on the right-hand side in (40) and use inequality (17) with $\delta = \frac{6}{6(k_4+k_5)+5}$ to obtain

$$(k_4 + k_5 + \frac{5}{6})\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{L^2(\Omega)} \leq \mathcal{E}_\varepsilon(\tilde{\varphi} - \tilde{\varphi}_\Omega) + C\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'}.$$

Eventually, we then arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'} + \|\tilde{\sigma}\|^2_{L^2(\Omega)} + |\tilde{\varphi}_\Omega|^2\Big) + \mathcal{E}_\varepsilon(\tilde{\varphi} - \tilde{\varphi}_\Omega) + \|\nabla\tilde{\sigma}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{L^2(\Omega)} \\ \leq K\Big(\|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{H^1(\Omega)'} + \|\tilde{\sigma}\|^2_{L^2(\Omega)} + |\tilde{\varphi}_\Omega|^2\Big) + K\|\mathcal{L}_\varepsilon\varphi + \Delta\varphi\|^2_{L^2(\Omega)}.\end{aligned} \tag{41}$$

Thus, Gronwall's lemma implies

$$\begin{aligned}&\frac{1}{2}\sup_{t\in[0,T]}\|\tilde{\varphi}(t) - \tilde{\varphi}_\Omega(t)\|^2_{H^1(\Omega)'} + \frac{1}{2}\sup_{t\in[0,T]}\|\tilde{\sigma}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\sup_{t\in[0,T]}|\tilde{\varphi}_\Omega(t)|^2 \\ &\quad + \int_0^T \mathcal{E}_\varepsilon(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,\mathrm{d}t + \int_0^T \|\nabla\tilde{\sigma}\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\leq \left(\frac{1}{2}\|\tilde{\varphi}_0 - \tilde{\varphi}_{\Omega,0}\|^2_{H^1(\Omega)'} + \frac{1}{2}\|\tilde{\sigma}_0\|^2_{L^2(\Omega)} + \frac{1}{2}|\tilde{\varphi}_{\Omega,0}|^2 + \int_0^T \|\mathcal{L}_\varepsilon\varphi + \Delta\varphi\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)e^{CT}.\end{aligned}$$

Hence, recalling the assumptions on the initial data, cf. (25), and Theorem 2.1, we obtain

$$\begin{aligned}&\frac{1}{2}\sup_{t\in[0,T]}\|\tilde{\varphi}(t) - \tilde{\varphi}_\Omega(t)\|^2_{H^1(\Omega)'} + \frac{1}{2}\sup_{t\in[0,T]}\|\tilde{\sigma}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\sup_{t\in[0,T]}|\tilde{\varphi}_\Omega(t)|^2 \\ &\quad + \int_0^T \mathcal{E}_\varepsilon(\tilde{\varphi} - \tilde{\varphi}_\Omega)\,\mathrm{d}t + \int_0^T \|\nabla\tilde{\sigma}\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\tilde{\varphi} - \tilde{\varphi}_\Omega\|^2_{L^2(\Omega)}\,\mathrm{d}t \leq C\varepsilon.\end{aligned}$$

Finally, observe that it holds

$$\begin{aligned}\|\tilde{\varphi}\|_{L^\infty(0,T;H^1(\Omega)')} &\leq \|\tilde{\varphi} - \tilde{\varphi}_\Omega\|_{L^\infty(0,T;H^1(\Omega)')} + \|\tilde{\varphi}_\Omega\|_{L^\infty(0,T)} \\ &\leq \sup_{t\in[0,T]}\|\tilde{\varphi}(t) - \tilde{\varphi}_\Omega(t)\|^2_{H^1(\Omega)'} + |\tilde{\varphi}_{\Omega,0}| \leq C\sqrt{\varepsilon},\end{aligned}$$

which shows the convergence in (26). Hence, the proof is complete. ∎

ACKNOWLEDGEMENTS

C. Hurm was partially supported by the Graduiertenkolleg 2339 *IntComSin* of the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)–Project-ID 321821685. M. Moser has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 948819). The support is gratefully acknowledged. Finally, we thank Daniel Böhme and Jonas Stange for careful proofreading. Open Access funding enabled and organized by Projekt DEAL.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

Christoph Hurm https://orcid.org/0009-0009-4023-178X

REFERENCES

[1] H. Garcke, K. F. Lam, E. Sitka, and V. Styles, A Cahn–Hilliard–Darcy model for tumour growth with chemotaxis and active transport, Math. Models Methods Appl. Sci. **26** (2016), no. 6, 1095–1148.

[2] H. Garcke, K. F. Lam, and E. Rocca, Optimal control of treatment time in a diffuse interface model of tumor growth, Appl. Math. Optim. **78** (2018), 495–544.

[3] P. Colli, G. Gilardi, and D. Hilhorst, On a Cahn–Hilliard type phase field model related to tumor growth, Discrete Contin. Dyn. Syst. **35** (2015), no. 6, 2423–2442.

[4] P. Colli, G. Gilardi, E. Rocca, and J. Sprekels, Vanishing viscosities and error estimate for a Cahn–Hilliard type phase field system related to tumor growth, Nonlinear Anal. Real World Appl. **26** (2015), 93–108.

[5] P. Colli, G. Gilardi, E. Rocca, and J. Sprekels, Asymptotic analyses and error estimates for a Cahn–Hilliard type phase field system modelling tumor growth, Discrete Contin. Dyn. Syst. Ser. S **10** (2017), 37–54.

[6] S. Frigeri, M. Grasselli, and E. Rocca, On a diffuse interface model of tumor growth, Eur. J. Appl. Math. **26** (2015), 215–243.

[7] H. Garcke and K. F. Lam, Well-posedness of a Cahn–Hilliard system modelling tumour growth with chemotaxis and active transport, Eur. J. Appl. Math. **28** (2017), no. 2, 284–316.

[8] H. Garcke and K. F. Lam, Analysis of a Cahn–Hilliard system with non zero Dirichlet conditions modelling tumour growth with chemotaxis, Discrete Contin. Dyn. Syst. **37** (2017), no. 8, 4277–4308.

[9] A. C. Ponce, An estimate in the spirit of Poincaré's inequality, J. Eur. Math. Soc. **6** (2004), no. 1, 1–15.

[10] A. C. Ponce, A new approach to Sobolev spaces and connections to Γ-convergence, Calc. Var. **19** (2004), no. 3, 229–255.

[11] L. Scarpa and A. Signori, On a class of non-local phase-field models for tumor growth with possibly singular potentials, chemotaxis, and active transport, Nonlinearity **34** (2021), 3199–3250.

[12] S. Frigeri, K. F. Lam, and E. Rocca, "*On a diffuse interface model for tumour growth with non-local interactions and degenerate mobilities*," *Solvability, regularity, and optimal control of boundary value problems*, Vol **22**, Springer INdAM Series, Cham, 2017, pp. 217–254.

[13] E. Davoli, E. Rocca, L. Scarpa, and L. Trussardi. Local asymptotics and optimal control for a viscous Cahn-Hilliard-Reaction-Diffusion model for tumor growth. *Preprint* arXiv:2311.10457 2023.

[14] M. Fornoni, Optimal distributed control for a viscous non-local tumour growth model, Appl. Math. Optim. **89** (2024), 8.

[15] H. Abels and C. Hurm, Strong nonlocal-to-local convergence of the Cahn-Hilliard equation and its operator, J. Differ. Equ. **402** (2024), 593–624.

[16] E. Davoli, H. Ranetbauer, L. Scarpa, and L. Trussardi, Degenerate nonlocal Cahn-Hilliard equations: Well-posedness, regularity and local asymptotics, Ann. Inst. H. Poincaré Anal. Non Linéaire **37** (2020), no. 3, 627–651.

[17] S. Melchionna, H. Ranetbauer, L. Scarpa, and L. Trussardi, From nonlocal to local Cahn-Hilliard equation, Adv. Math. Sci. Appl. **28** (2019), 197–211.

[18] E. Davoli, L. Scarpa, and L. Trussardi, Local asymptotics for nonlocal convective Cahn-Hilliard equations with $W^{1,1}$-kernel and singular potential, J. Differ. Equ. **289** (2021), 35–58.

[19] E. Davoli, L. Scarpa, and L. Trussardi, Nonlocal-to-local convergence of Cahn–Hilliard Equations: neumann boundary conditions and viscosity terms, Arch. Ration. Mech. Anal. **239** (2021), 117–149.

[20] C. Elbar and J. Skrzeczkowski, Degenerate Cahn-Hilliard equation: From nonlocal to local, J. Differ. Equ. **364** (2023), 576–611.

How to cite this article: C. Hurm and M. Moser, *Nonlocal-to-local convergence for a Cahn–Hilliard tumor growth model*, GAMM-Mitteilungen. **48** (2025), e70003. https://doi.org/10.1002/gamm.70003