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Original Paper

Holographic Mixing and Fock Space Dynamics of Causal Fermion Systems

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Abstract. A limiting case is considered in which the causal action principle for causal fermion systems describing Minkowski space gives rise to the linear Fock space dynamics of quantum electrodynamics. The quantum nature of the bosonic field is a consequence of the stochastic description of a multitude of fluctuating fields coupled to non-commuting operators, taking into account dephasing effects. The scaling of all error terms is specified. Our analysis leads to the concept of holographic mixing, which is introduced and explained in detail.

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1. Introduction

This paper is part of the research program aimed at getting a precise connection between causal fermion systems and quantum field theory (QFT). This research program was initiated in [20,21] by the abstract construction of a quantum state of a causal fermion system for any fixed time. In [23], this construction was extended, allowing for the description of general entangled states, but the question remained of how to derive dynamical equations for the time evolution of this quantum state. In the present paper, we resolve this question by deriving these dynamical equations and formulating them in terms of a unitary time evolution on bosonic and fermionic Fock spaces. More precisely, we obtain standard quantum electrodynamics (QED) in Minkowski space with an ultraviolet cutoff in the limiting case where different types of error terms can be neglected. We specify the scaling behavior of all these error terms. This analysis opens the door for going beyond standard QED by working out correction terms (as will be discussed in the outlook in Sect. 8). We also plan to write a detailed survey article on the connection between causal fermion systems and QFT [4].

Compared to the previous papers cited above, where the goal was to relate the mathematical structures of a causal fermion system to corresponding objects of QFT, we now need to delve deeper into the dynamical equations of causal fermion systems. More precisely, the analytic backbone of the theory of causal fermion systems is the *causal action principle*, a variational principle for the measure describing the causal fermion system. The dynamics of the causal fermion system are then governed by the corresponding *Euler–Lagrange* (EL) *equations*, being nonlinear equations formulated in spacetime. By linearizing these equations, one obtains the so-called *linearized field equations*, which are the starting point for the detailed analysis of the causal action principle, including the resulting nonlinear dynamics. The present work is based on and makes essential use of results in [16], where the linearized field equations were analyzed in detail for causal fermion systems describing Minkowski space. Our main objective is to show that the dynamics of these linearized fields together with their coupling to matter can be described in a well-defined limiting case as a unitary time evolution of a quantum state on fermionic and bosonic Fock spaces.

We now give a brief outline of our methods and results. We take the following results from [16] as the basic input:

(i) The dynamics of the wave functions in a causal fermion system describing Minkowski space can be described by a Dirac equation of the form

$$\big(i\partial\!\!\!/ + \mathcal{B} - m\big)\psi = 0\,, \tag{1.1}$$

where $\mathcal{B}$ is a *nonlocal* potential, i.e., an integral operator of the form

$$\big(\mathcal{B}\psi\big)(x) = \int_M \mathcal{B}(x,y)\,\psi(y)\,d^4y\,. \tag{1.2}$$

The integral kernel $\mathcal{B}(x,y)$ is nonlocal on a scale $\ell_{\min}$ lying between the length scale ε of the ultraviolet regularization (which can be thought of as the Planck scale) and the length scale ℓ_{macro} of macroscopic physics (which can be thought of as the Compton scale),

$$\varepsilon \ll \ell_{\min} \ll \ell_{\text{macro}}\,. \tag{1.3}$$

In simple terms, $\ell_{\min}$ can be regarded as the minimal length scale on which the analysis on the light cone and the resulting formalism of the continuum limit as developed in [7,11] apply.

(ii) The nonlocal potential $\mathcal{B}$ is composed of a collection of vector potentials A_a with $a \in \{1, \ldots N\}$. More precisely,

$$\mathcal{B}(x,y) = \sum_{a=1}^N A\!\!\!/_a\Big(\frac{x+y}{2}\Big)\,L_a(y-x)\,, \tag{1.4}$$

where the L_a are fixed complex-valued kernels. The number N of these fields is very large and scales like

$$N \simeq \frac{\ell_{\min}}{\varepsilon}\,. \tag{1.5}$$

We note that, in the special case $L_a(y-x) = \delta^4(y-x)$, the potential A_a can be regarded as a classical electromagnetic potential. If L_a is a nonlocal kernel, the potential A_a can still be regarded as being classical, but its coupling to the wave functions is modified by L_a. We also remark that the ansatz (1.4) poses no restrictions on the form of the nonlocal potential (for details, see the approximation argument in Lemma 3.4), but the scaling of the number N of summands (1.5) clearly gives constraints for the form of the potential.

To summarize, the Dirac equation (1.1) describes the propagation of Dirac waves in the presence of a multitude of classical background fields, which are nonlocal on a small scale $\ell_{\min}$. Apart from these results from [16], our analysis is based on two additional assumptions. First, in order for the potentials A_a to have a notable effect, they must clearly be nonzero. It seems easiest and physically sensible to describe them stochastically by Gaussian fields:

(iii) The potentials A_a are non-vanishing and can be described stochastically by Gaussian fields.

As we shall see in detail later in this paper, the above assumptions do not yet give rise to bosonic quantum fields. One important ingredient which is still missing is related to the local gauge freedom of electrodynamics. Namely, a classical electromagnetic potential A_a leads to local phase transformations of the Dirac wave functions. (For basics, see the beginning of Sect. 4.2.) Likewise, the potentials in (1.4) also lead to local phase transformations, but only of those wave functions to which the corresponding potential couples. This gives rise to a decomposition of the wave functions into many components, each experiencing different relative phase transformations. We refer to these components as *holographic components* and the resulting mechanism as *holographic mixing*. It is one of the main tasks of the present paper to model holographic mixing mathematically and to work out the resulting effects. Here, we do not yet enter the details but merely state in general our second additional assumption:

(iv) The potentials A_a lead to dephasing effects.

Here, by *dephasing* we mean the common effect in quantum theory that a superposition of wave functions becomes small if the summands are "not in phase" (for example, by involving independently chosen or random phase factors). This effect is often described by the closely related notions of *destructive interference* or *decoherence*. We here prefer the more specific notion of dephasing.

The main lesson from the present paper is that implementing the above assumptions (i)–(iv) in a coherent mathematical setting gives rise to bosonic quantum fields. Moreover, in a specific limiting case, the coupling of the bosonic fields to the Dirac wave functions gives rise to QED. This limiting case involves a specific choice of the covariances of the Gaussian stochastic fields, in agreement with Lorentz invariance on macroscopic length scales. In this way, we obtain a derivation of QFT in the time-dependent setting with Lorentzian signature from the fundamental and broader causal action principle. Even more generally and independent of causal fermion systems, our methods and results apply to any physical theory satisfying the above assumptions (i)–(iv), thereby giving a connection to quantum field theory.

We proceed by explaining in words how the above assumptions are connected to the appearance of bosonic quantum fields. Let us begin by noting that studying similarities and connections between random processes and QFT has a long history. The striking similarities between the Wiener process and the Euclidean path integral (see, for example, [36, 42]) inspired attempts to describe the "quantization" of a classical system using stochastic notions. The most prominent approach in this direction is Nelson's mechanics [44]. From the more mathematical point of view, there is a close and well-established connection between a suitable class of random distributions and the n-point functions of Euclidean QFTs, as highlighted by the stochastic quantization program [43, 45]. On the other hand, the dynamics of a random distribution is often encoded by nonlinear stochastic partial differential equations with an additive or a multiplicative white noise. When the linear part of these equations is ruled either by a parabolic or by an elliptic partial differential operator, the solution theory is well-understood provided that the nonlinearity is sufficiently tame. Both para-controlled calculus [38] and regularity structures [40] are efficient frameworks to prove existence and uniqueness of solutions. For our purposes, it is not necessary to enter the details of these formulations, but it suffices to highlight that they rely on an algorithmic construction of the solution, based on a graph expansion. This is necessary in order to cope with ill-defined products of distributions which are a consequence of the singular structure of the white noise. We point out that this graph expansion is reminiscent and has close similarities to the expansion into Feynman diagrams, being at the heart of the textbook approach to perturbative QFT. These similarities and the presence of the same structural hurdles as in QFT (like, for example, renormalization) are not a mere accident, as has been studied more in depth in [2]. Herein, an algorithm was devised to construct the solution and the correlation functions of a stochastic, nonlinear partial differential equation adapting the language and the tools of the algebraic approach to QFT, in particular the Epstein–Glaser renormalization scheme.

In view of these connections and similarities, the reader may wonder what the difference between a quantum field and a classical stochastic field actually is. So, what is it that makes a field "quantum"? This questions can be answered in various ways. On the level of Feynman diagrams, one difference is that, for a classical stochastic field, the bosonic lines in the loops are described by fundamental solutions (meaning by solutions of the homogeneous equations; for details see again [34]), whereas quantum fields give rise to propagators (i.e., Green's operators like the Feynman propagator, being inhomogeneous solutions). Another important difference is that, in statistical physics, one takes the statistical average by integrating over the stochastic fields at the very end when computing the statistical mean. In quantum physics, however, the path integral, being an integral over field configurations, describes the time evolution of the state $|\Psi>$, and probabilities are obtained by first computing the path integral and then taking the expectation value of observables. Thus, denoting the path integral or the probability integral by $\mathcal{D}\phi$, the difference can be summarized symbolically as follows,

$$\text{Statistical physics:} \qquad \int \mathcal{D}\phi \, \langle \Psi | A | \Psi \rangle$$

$$\text{Quantum physics:} \qquad \left\langle \Big(\int \mathcal{D}\phi \; \Psi \Big) \,\Big|\, A \,\Big|\, \Big(\int \mathcal{D}\phi \; \Psi \Big) \right\rangle ,$$

where A denotes the observable. This means in particular that, in quantum physics, we have separate Feynman diagrams for bra and ket, but there are no bosonic lines connecting bra and ket.

One way of understanding how this basic difference between quantum fields and classical stochastic fields comes about is that in QFT one has *non-commuting operators*. For example, the Feynman propagator arises from a time ordering of the bosonic field operators which satisfy the canonical commutation relations (CCR). It is important to observe that with the above assumption (i) non-commutativity comes into play. Namely, regarding the nonlocal potential as a convolution operator (1.2), the convolution operators corresponding to different potentials A_a do in general not commute with each other. However, our analysis will reveal that the assumption (i) together with the stochasticity assumption (iii) alone is not sufficient for getting QFT. Instead, the fact that we have many stochastic fields (ii) which give rise to dephasing effects (iv) will be essential for obtaining QED.

We next explain in some more detail how bosonic quantum fields arise in our setting. This connection is easier to make in momentum space. Taking the Fourier transform of the kernel of the nonlocal potential (1.2) by setting

$$\hat{\mathcal{B}}(p,k) := \int_M d^4x \int_M d^4y \; \mathcal{B}(x,y)\, e^{ipx-iky} \,, \tag{1.6}$$

in operator products the adjacent momentum variables always coincide (like, for example, in products involving Green's operators (3.3)). Therefore, the variable $p-k$ tells us about the momentum change (or momentum transfer) of the operator $\mathcal{B}$. The field operator $\hat{\mathcal{B}}_q$ of momentum q is obtained by fixing this momentum change to q, i.e.,

$$\hat{\mathcal{B}}_q(p,k) := (2\pi)^4\, \delta^4(p-k-q)\, \hat{\mathcal{B}}(p,k).$$

(For simplicity of presentation, we here disregard the spinorial degrees of freedom; for more precise formulas for scalar and vector fields, see (3.8), (3.35) and (4.20).) By defining the field operators in this way, we have arranged that q gives the desired momentum transfer inside the Feynman diagrams. Note that the field operator is again a nonlocal operator. Moreover, the field operators for different momenta will in general not commute with each other. In this way, non-commutativity arises, being a basic feature of field operators. The question whether this non-commutativity indeed gives rise to the CCR is more subtle. As already mentioned, it can be answered affirmatively only at the end of our constructions using the assumptions (iii) and (iv) as additional ingredients (see Theorem 4.1). In this setting, the CCR even arise naturally (see Remark 4.2).

We next specify the structure of the error terms. The causal fermion systems being considered involve several length scales:

ε	regularization length (length scale of ultraviolet cutoff)

$\ell_{\min}$	length scale of nonlocality of potential $\mathcal{B}$
ℓ_Λ	length scale of holographic dephasing
ℓ_{macro}	length scale of macroscopic physics .

The regularization length ε can be thought of as a length scale as small as the Planck length. By the length scale of macroscopic physics, we mean the smallest length scale accessible to experiments. The length scales $\ell_{\min}$ and ℓ_Λ are largely unknown, except that they should lie between the two other length scales, i.e.,

$$\varepsilon \ll \ell_{\min}, \ell_\Lambda, \ll \ell_{\text{macro}} \,.$$

Our results apply to the two limiting cases $\ell_\Lambda \ll \ell_{\min}$ and $\ell_{\min} \ll \ell_\Lambda$. The corresponding relative error terms are

$$\times \Big(1 + \mathcal{O}\Big(\frac{\ell_\Lambda}{\ell_{\min}} \Big) \Big), \tag{1.7}$$

respectively,

$$\times \Big(1 + \mathcal{O}\Big(\frac{\ell_\Lambda}{\ell_{\text{macro}}} \Big) + \mathcal{O}\Big(\frac{\ell_{\min}}{\ell_\Lambda} \Big) \Big) \,. \tag{1.8}$$

The error terms (1.7) are easier to obtain. They will be derived in Sect. 4.5 (see Theorem 4.3). The error terms (1.8) are physically more convincing because they allow the dephasing effects to happen on larger length scales (i.e., lower frequencies and smaller momenta). They will be derived with a different method in Sect. 5.4 (see Theorem 5.1). We note that the present methods do not cover the case that ℓ_Λ and $\ell_{\min}$ are of the same order of magnitude. This mainly technical issue will be excluded from our analysis.

As a final remark, we note that dephasing effects were first considered in [23], and they were shown to be essential for the description of entangled quantum states. However, from the mathematical point of view, the treatment of dephasing in [23] is quite different from the methods in the present paper. More precisely, in [23] unitary group integrals were considered, and the different "dephased components" were recovered as saddle points of these group integrals. In the present paper, however, the dephasing is analyzed using stationary phase methods in position space. These methods seem so complementary that it is not yet clear if and how these techniques are related to each other (see Sect. 7 for a discussion of this point).

The paper is organized as follows. Sect. 2 provides the necessary background on causal fermion systems and the causal action principle. It also reviews the relevant results on the linearized field equations obtained in [16]. In Sect. 3, we outline our strategy for obtaining bosonic field operators, which should satisfy the CCR (Sect. 3.1). The general idea is that the nonlocality of the bosonic potentials also introduces a non-commutativity, giving rise to non-trivial commutation relations. The basic question is whether it is possible to arrange the CCR. This question is first analyzed for a single scalar stochastic field (Sect. 3.2). This stochastic field can be regarded as a classical bosonic background field. However, it is nonlocal on the scale $\ell_{\min}$. We show that this setting allows to arrange that the CCR are satisfied in the stochastic average, but not as operator equations. We proceed by analyzing if the situation improves if, instead of a single stochastic field, one considers a large number N of fields (thus modeling the findings of [16] outlined above). This gives additional freedom and flexibility, but not quite to the extent that the CCR can be satisfied as operator equations. In Sect. 4, the holographic phases are introduced. It is shown that, making use of resulting dephasing effects, it does become possible to realize the CCR. The errors of the dephasing effects are specified with the help of a stationary phase analysis. In Sect. 5, the Dirac dynamics in the presence of holographic phases is developed. It is a subtle question how to incorporate the phases into the Green's operators. We show that, by doing this properly, one can improve the scaling of the error terms in the stationary phase analysis. In Sect. 6, the dynamics is formulated in the language of Fock spaces. In Sect. 7, we explain how our constructions relate to the quantum state as constructed in [21]. Section 8 concludes the paper with a brief discussion of our results and of open problems. The appendices provide mathematical tools for a more computational analysis of Dirac waves in the presence of nonlocal potentials and holographic phases.

2. Preliminaries

This section provides the necessary background on causal fermion systems and the linearized field equations.

2.1. Causal Fermion Systems and the Causal Action Principle

We begin with the abstract setting.

Definition 2.1. (causal fermion systems) Given a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle .|. \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the *"spin dimension"*), we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all symmetric operators on $\mathcal{H}$ of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues. On $\mathcal{F}$, we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$). We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system*.

A causal fermion system describes a spacetime together with all structures and objects therein. In order to single out the physically admissible causal fermion systems, one must formulate physical equations. To this end, we impose that the measure ρ should be a minimizer of the causal action principle. which we now introduce. For brevity of the presentation, we only consider the *reduced causal action principle* where the so-called boundedness constraint has been built incorporated by a Lagrange multiplier term. This simplification is no loss of generality, because the resulting EL equations are the same as for the non-reduced action principle as introduced, for example, in [11, Section §1.1.1].

For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. However, in general it is no longer a symmetric operator because $(xy)^* = yx$, and this is different from xy unless x and y commute. As a consequence, the eigenvalues of the operator xy are in general complex. We denote the rank of xy

by $k \leq 2n$. Counting algebraic multiplicities, we choose $\lambda_1^{xy}, \ldots, \lambda_k^{xy} \in \mathbb{C}$ as all the nonzero eigenvalues and set $\lambda_{k+1}^{xy}, \ldots, \lambda_{2n}^{xy} = 0$. We refer to the resulting collection of complex numbers $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ as the *non-trivial eigenvalues* of xy. Given a parameter $\kappa > 0$ (which will be kept fixed throughout this paper), we introduce the κ-Lagrangian and the causal action by

$$\kappa - \text{Lagrangian:} \quad \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(\big|\lambda_i^{xy}\big| - \big|\lambda_j^{xy}\big| \Big)^2 + \kappa \,\bigg(\sum_{j=1}^{2n} \big|\lambda_j^{xy}\big| \bigg)^2 \tag{2.1}$$

$$\text{causal action:} \qquad \mathcal{S}(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y)\;. \tag{2.2}$$

The *reduced causal action principle* is to minimize $\mathcal{S}$ by varying the measure ρ under the following constraints,

$$\text{volume constraint:} \qquad \rho(\mathcal{F}) = 1 \tag{2.3}$$

$$\text{trace constraint:} \qquad \int_{\mathcal{F}} \operatorname{tr}(x)\, d\rho(x) = 1\;. \tag{2.4}$$

This variational principle is mathematically well-posed if $\mathcal{H}$ is finite-dimensional. For the existence theory and the analysis of general properties of minimizing measures, we refer to [1,8] or [25, Chapter 12]. In the existence theory, one varies in the class of regular Borel measures (with respect to the topology on $\mathrm{L}(\mathcal{H})$ induced by the operator norm), and the minimizing measure is again in this class. With this in mind, we always assume that ρ is a *regular Borel measure*.

We finally point out that the causal action principle is invariant under unitary transformations, i.e., to joint transformations $x \mapsto \mathcal{U} x \mathcal{U}^{-1}$ of all spacetime point operators with $\mathcal{U}$ a unitary linear operator on $\mathcal{H}$. The conservation laws corresponding to this symmetry will be considered in Sect. 2.4.

2.2. The Physical Wave Functions and the Wave Evaluation Operator

Let ρ be a *minimizing* measure. Defining *spacetime* M as the support of this measure,

$$M := \operatorname{supp} \rho \subset \mathcal{F}\;.$$

the spacetimes points are symmetric linear operators on $\mathcal{H}$. These operators contain a lot of information which, if interpreted correctly, gives rise to spacetime structures like causal and metric structures, spinors and interacting fields. (For details, see [11, Chapter 1].) Here, we restrict attention to those structures needed in what follows. We begin with a basic notion of causality.

Definition 2.2. (causal structure) The points x and y are called *spacelike* separated if all the non-trivial eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ have the same absolute value. They are said to be *timelike* separated if the non-trivial eigenvalues are all real and do not all have the same absolute value. In all other cases (i.e., if the λ_j^{xy} are not all real and do not all have the same absolute value), the points x and y are said to be *lightlike* separated.

Restricting the causal structure of $\mathcal{F}$ to M, we get corresponding causal relations in spacetime. Before going on, we point out that this "spectral definition" of causal structures is closely related but not equivalent to the standard notions of causality in classical spacetimes. The correspondence is obtained by considering so-called regularized Dirac sea configurations in Minkowski space, in which case the above notions agree with the corresponding causal notions in Minkowski space in the limiting case when the regularization length ε tends to zero. (This is worked out in detail in [11, §1.2.5].) This analysis applies similarly in Lorentzian spacetimes (see [12, 18]). On the fundamental level, the above definition can be supplemented by a functional distinguishing a time direction. (For details, see [11, eq. (1.1.11) and §1.2.5].) Other constructions of causal cone structures were given in [3, Section 4.1] and [30, Section 5]. In simple terms, this analysis shows that, on large scales, a causal fermion system has transitive causal order relations similar to a causal set. (For the general context, see the review [47].) On small scales and on the fundamental level, one still has causal relations, but they are not necessarily transitive.

Next, for every $x \in \mathcal{F}$ we define the *spin space* S_xM by $S_xM = x(\mathcal{H})$; it is a subspace of $\mathcal{H}$ of dimension at most $2n$. It is endowed with the *spin inner product* $\prec .|.\succ_x$ defined by

$$\prec u|v\succ_x = -\langle u|xv\rangle_{\mathcal{H}} \qquad \text{(for all } u, v \in S_xM) . \tag{2.5}$$

A *wave function* ψ is defined as a function which to every $x \in M$ associates a vector of the corresponding spin space,

$$\psi \,:\, M \to \mathcal{H} \qquad \text{with} \qquad \psi(x) \in S_xM \quad \text{for all } x \in M . \tag{2.6}$$

We remark that a wave function ψ is said to be *continuous* if for every $x \in M$ and $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big\| \sqrt{|y|}\, \psi(y) - \sqrt{|x|}\, \psi(x) \big\|_{\mathcal{H}} < \varepsilon \qquad \text{for all } y \in M \text{ with } \|y - x\| \leq \delta \tag{2.7}$$

(where $|x|$ is the absolute value of the symmetric operator x on $\mathcal{H}$, and $\sqrt{|x|}$ is the square root thereof). We denote the set of continuous wave functions by $C^0(M, SM)$.

It is an important observation that every vector $u \in \mathcal{H}$ of the Hilbert space gives rise to a distinguished wave function. In order to obtain this wave function, denoted by ψ^u, we simply project the vector u to the corresponding spin spaces,

$$\psi^u \,:\, M \to \mathcal{H} , \qquad \psi^u(x) = \pi_x u \in S_xM , \tag{2.8}$$

where $\pi_x : \mathcal{H} \to S_xM$ denotes the orthogonal projection operator to $S_xM \subset \mathcal{H}$. We refer to ψ^u as the *physical wave function* of $u \in \mathcal{H}$. A direct computation shows that the physical wave functions are continuous (in the sense (2.7)). Associating with every vector $u \in \mathcal{H}$, the corresponding physical wave function gives rise to the *wave evaluation operator*

$$\Psi \,:\, \mathcal{H} \to C^0(M, SM) , \qquad u \mapsto \psi^u . \tag{2.9}$$

The wave evaluation operator describes the whole family of physical wave functions. All the structures in spacetime are encoded in Ψ and therefore in the family of physical wave functions. This can be see, for example, by expressing the spacetime point operators $x \in M$ as (for the derivation, see [11, Lemma 1.1.3])

$$x = -\Psi(x)^* \, \Psi(x) \,. \tag{2.10}$$

Here, $\Psi(x) : \mathcal{H} \to S_x$, and its adjoint $\Psi(x)^* : S_x \to \mathcal{H}$ is taken with respect to the corresponding inner products, meaning that the relation

$$\prec\!\Psi(x)\, u \,|\, \phi\!\succ_x = \langle u \,|\, \Psi(x)^* \, \phi \rangle_{\mathcal{H}} \qquad \text{holds for all } u \in \mathcal{H} \text{ and } \phi \in S_x \,.$$

2.3. The Restricted Euler–Lagrange Equations

We now state the Euler–Lagrange equations.

Proposition 2.3. *Let ρ be a minimizer of the reduced causal action principle. Then, the local trace is constant in spacetime, meaning that*

$$\mathrm{tr}(x) = 1 \qquad \text{for all } x \in M \,.$$

Moreover, there are parameters $\mathfrak{r}, \mathfrak{s} > 0$ such that the function ℓ defined by

$$\ell \,:\, \mathcal{F} \to \mathbb{R} \,, \qquad \ell(x) := \int_M \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{r}\, \big(\mathrm{tr}(x) - 1 \big) - \mathfrak{s} \tag{2.11}$$

is minimal and vanishes in spacetime, i.e.,

$$\ell|_M \equiv \inf_{\mathcal{F}} \ell = 0 \,. \tag{2.12}$$

For the proof of the EL equations and more details, we refer, for example, to [16]. The parameter $\mathfrak{r}$ can be viewed as the Lagrange parameter corresponding to the trace constraint. Likewise, $\mathfrak{s}$ is the Lagrange parameter of the volume constraint.

We now work out what the EL equations mean for first variations of the spacetime points. The starting point of our consideration is the formula (2.10), which expresses the spacetime point operator in terms of the wave evaluation operator. Using this formula, first variations of the wave evaluation operator $\Psi(x)$ (see (2.9)) at a given spacetime point $x \in M$ give rise to corresponding variations of the spacetime point operator, i.e.,

$$\mathbf{u} := \delta x = -\delta\Psi(x)^* \, \Psi(x) - \Psi(x)^* \, \delta\Psi(x) \,. \tag{2.13}$$

The minimality of ℓ on M as expressed by (2.12) implies that the derivative of ℓ in the direction of $\mathbf{u}$ vanishes, i.e.,

$$D_{\mathbf{u}} \ell(x) = 0 \tag{2.14}$$

for all variations of the form (2.13) for which the directional derivative in (2.14) exists. Here, the derivative $D_{\mathbf{u}}$ can be understood geometrically as follows. The linear operators in $\mathcal{F}$ of maximal rank (the so-called *regular points*) form a manifold. (For details, see [24,31] or [25, Section 3.1].) The vector $\mathbf{u}$ in (2.13) is a tangent vector of this manifold at x, and the left side in (2.14) is the derivative of the function ℓ in the direction of this tangent vector.

For the computations, it is more convenient to reformulate the restricted EL equations in terms of variations of the kernel of the fermionic projector, as we now explain. In preparation, we use (2.11) in order to write (2.14) as

$$\int_M D_{1,\mathbf{u}}\mathcal{L}(x,y)\, d\rho(y) = \mathfrak{r}\, D_{\mathbf{u}}\, \mathrm{tr}(x)\;, \tag{2.15}$$

where the meaning of the index is that the directional derivative acts on the first argument of the Lagrangian. For the computation of the first variation of the Lagrangian, one can make use of the fact that for any $p \times q$-matrix A and any $q \times p$-matrix B, the matrix products AB and BA have the same nonzero eigenvalues, with the same algebraic multiplicities (as is explained in detail in [11, §1.1.2 after eq. (1.1.8)]). As a consequence, applying again (2.10),

$$xy = \Psi(x)^* \left(\Psi(x)\, \Psi(y)^* \Psi(y)\right) \simeq \left(\Psi(x)\, \Psi(y)^* \Psi(y)\right) \Psi(x)^*\;, \tag{2.16}$$

where $\simeq$ means that the operators are isospectral (in the sense that they have the same non-trivial eigenvalues with the same algebraic multiplicities). Thus, introducing the *kernel of the fermionic projector* $P(x,y)$ by

$$P(x,y) := -\Psi(x)\, \Psi(y)^* \;:\; S_yM \rightarrow S_xM\;, \tag{2.17}$$

we can write (2.16) as

$$xy \simeq P(x,y)\, P(y,x) \;:\; S_xM \rightarrow S_xM\;.$$

In this way, the eigenvalues of the operator product xy as needed for the computation of the Lagrangian (2.1) are recovered as the eigenvalues of a $2n \times 2n$-matrix. Since $P(y,x) = P(x,y)^*$, the Lagrangian $\mathcal{L}(x,y)$ in (2.1) can be expressed in terms of the kernel $P(x,y)$. Consequently, the first variation of the Lagrangian can be expressed in terms of the first variation of this kernel. Being real-valued and real-linear in $\delta P(x,y)$, it can be written as

$$\delta\mathcal{L}(x,y) = 2\, \mathrm{Re}\, \mathrm{Tr}_{S_xM}\big(Q(x,y)\, \delta P(x,y)^*\big)\;, \tag{2.18}$$

where $Q(x,y)$ is a kernel which is again symmetric (with respect to the spin inner product), i.e.,

$$Q(x,y) \;:\; S_yM \rightarrow S_xM \qquad \text{and} \qquad Q(x,y)^* = Q(y,x)\;. \tag{2.19}$$

Here, Tr_{S_xM} denotes the trace on the spin space S_xM, and $\delta P(x,y)$ is the first variation of $P(x,y)$ as a linear operator from S_yM to S_xM. More details on this method and many computations can be found in [11, Sections 1.4 and 2.6 as well as Chapters 3-5].

Expressing the variation of $P(x,y)$ in terms of $\delta\Psi$, the first variations of the Lagrangian can be written as

$$\begin{aligned} D_{1,\mathbf{u}}\mathcal{L}(x,y) &= -2\; \mathrm{Re}\, \mathrm{tr}\left(\delta\Psi(x)^*\, Q(x,y)\, \Psi(y)\right) \\ D_{2,\mathbf{u}}\mathcal{L}(x,y) &= -2\; \mathrm{Re}\, \mathrm{tr}\left(\Psi(x)^*\, Q(x,y)\, \delta\Psi(y)\right) \end{aligned}$$

(where tr denotes the trace of a finite-rank operator on $\mathcal{H}$). Using these formulas, the restricted EL equation (2.15) becomes

$$\mathrm{Re} \int_M \mathrm{tr}\left(\delta\Psi(x)^*\, Q(x,y)\, \Psi(y)\right) d\rho(y) = \mathfrak{r}\; \mathrm{Re}\, \mathrm{tr}\left(\delta\Psi(x)^*\, \Psi(x)\right)\;.$$

Using that the variation can be arbitrary at every spacetime point, we obtain

$$\int_M Q(x,y)\, \Psi(y)\, d\rho(y) = \mathfrak{r}\, \Psi(x) \qquad \text{for all } x \in M\,,$$

where $\mathfrak{r} \in \mathbb{R}$ is the Lagrange parameter of the trace constraint.

2.4. The Conserved Commutator Inner Product

The connection between symmetries and conservation laws made by Noether's theorem extends to causal fermion systems [26]. However, the conserved quantities of a causal fermion system have a rather different structure, being formulated in terms of so-called surface layer integrals. A *surface layer integral* is a double integral of the form

$$\int_\Omega \bigg(\int_{M \setminus \Omega} (\cdots)\, \mathcal{L}(x,y)\, d\rho(y) \bigg)\, d\rho(x)$$

where the two variables x and y are integrated over Ω and its complement, and $(\cdots)$ stands for variational derivatives acting on the Lagrangian. Since in typical applications, the Lagrangian is small if x and y are far apart, the main contribution to the surface layer integral is obtained when both x and y are near the boundary $\partial\Omega$. With this in mind, a surface layer integral can be thought of as a "thickened" surface integral, where we integrate over a spacetime strip of a certain width. For systems in Minkowski space as considered here, the length scale of this strip is the Compton scale m^{-1}. For more details on the concept of a surface layer integral, we refer to [25, Section 9.1].

There are various *Noether-like theorems* for causal fermion systems, which relate symmetries to conservation laws. (For an overview, see [28] or [25, Chapter 9].) The conserved quantity of relevance here is the commutator inner product (for more details, see [25, Section 9.4] or [26, Section 5] and [22, Section 3]): The causal action principle is invariant under unitary transformations of the measure ρ, i.e., under transformations

$$\rho \to \mathcal{U}\rho \qquad \text{with} \qquad (\mathcal{U}\rho)(\Omega) := \rho\big(\mathcal{U}^{-1}\, \Omega\, \mathcal{U}\big)\,,$$

where $\mathcal{U}$ is a unitary operator on $\mathcal{H}$ and $\Omega \subset \mathcal{F}$ is any measurable subset. The conserved quantity corresponding to this symmetry is the so-called *commutator inner product*

$$\begin{aligned} \langle \psi | \phi \rangle^\Omega := -2i \bigg(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x) \int_\Omega d\rho(y) \bigg) \\ \prec\!\psi(x)\,|\, Q(x,y)\, \phi(y)\!\succ_x\,, \end{aligned} \tag{2.20}$$

where ψ, ϕ are wave functions (2.6) and $\Omega \subset M$ describes a spacetime region ($Q(x,y)$ is again the kernel in (2.19)). Here, *conservation* means that the commutator inner product of any two physical wave functions ψ and ϕ (as defined by (2.8)) vanishes for any compact $\Omega \subset M$. If Ω is chosen to be non-compact, the commutator inner product is in general nonzero. But, taking exhaustions, the conservation law can be stated that the commutator inner product (2.20) does not depend on the choice of Ω within a certain class of sets. This "class of sets" can be specified systematically by working with equivalence classes (for details see [22, Section 3.2]). For our purposes, it suffices to restrict attention to *Minkowski-type spacetimes* as introduced in [21, Section 2.8]. In this case, there is a global time function $T : M \to \mathbb{R}$, and the commutator inner product is well defined if Ω is chosen as the past of any time t,

$$\Omega = \Omega_t := T^{-1}\big((-\infty, t]\big) .$$

The set Ω_t can be thought of as the past of the Cauchy surface at time t, so that the surface layer integral describes a "thickened" integral over the Cauchy surface. The conservation law states that the commutator product $\langle \psi | \phi \rangle^{\Omega_t}$ does not depend on $t \in \mathbb{R}$.

We finally remark that the commutator inner product is also independent of the choice of the time function, within a large class of time functions, making it possible to describe more general Cauchy surfaces. As this generalization will not be needed here, we refer for the details to [22].

2.5. The Linearized Field Equations in Minkowski Space

The linearized field equations describe variations of the measure ρ which preserve the EL equations. The linearized field equations play a central role in the analysis of causal fermion systems, both conceptually and computationally. From the conceptual point of view, the analysis of the linearized field equations reveals the causal nature of the dynamics and thereby clarifies the causal structure of spacetime itself. From the computational point of view, being a linear equation, it becomes possible to analyze the equations explicitly using methods of functional analysis and Fourier analysis. Moreover, the linearized field equations are an important first step toward the analysis of the nonlinear dynamics as described by the EL equations (for example, perturbatively using the methods developed in [15]). The linearized field equations were derived and formulated in [27]. They were first analyzed in [3] using energy methods. In [16], the EL equations and linearizations thereof (the so-called linearized field equations) were studied in detail for causal fermion systems describing Minkowski space. In this setting, the abstract structures of a causal fermion system become more concrete, opening the door for a detailed analysis.

We now explain what these findings mean for the structure of the dynamics in the presence of linearized fields for causal fermion systems describing Minkowski space. In this setting, the physical wave functions (2.8) can be represented by usual spinorial wave functions in Minkowski space. Moreover, the spin inner product (2.5) goes over to the usual pointwise inner product on Dirac spinors, i.e.,

$$\prec\!\psi|\phi\!\succ\!(x) = \overline{\psi(x)}\phi(x) = \psi(x)^\dagger \gamma^0 \phi(x) , \tag{2.21}$$

where $\overline{\psi(x)}$ is sometimes referred to as the adjoint spinor. (For more details on the correspondence of the abstract objects with objects in Minkowski space, see [11, Section 1.2].) Moreover, in the Minkowski vacuum, the linearized field equations can be described by a Dirac equation in Minkowski space

$$\big(i\partial\!\!\!/ - m\big)\psi(x) = 0 ,$$

where, for ease of notation, the superscript u of the wave function was omitted. In the interacting situation, one must insert potentials into the Dirac equation. It turns out that the linearized field equations do not allow only for homogeneous classical fields (like plane electromagnetic waves), but instead for a plethora of fields coupling to different wave packets propagating in different directions. The reason for this surprisingly large space of linearized solutions can be understood in non-technical terms from the fact that the causal Lagrangian (2.1) is invariant under phase transformations of the kernel of the fermionic projector (2.17) of the form

$$P(x,y) \rightarrow e^{i\lambda(x,y)}\, P(x,y) \tag{2.22}$$

with a real-valued function $\lambda(x,y)$ which is anti-symmetric (i.e., $\lambda(x,y) = -\lambda(y,x)$ for all $x,y \in M$). This invariance generalizes the local gauge invariance of electrodynamics, which is recovered by choosing $\lambda(x,y) = \Lambda(x) - \Lambda(y)$ with Λ a real-valued function. (For basics, see again the beginning of Section 4.2.) This more general invariance, which can be understood as a direction-dependent local gauge freedom, is the underlying reason for the appearance of many additional linearized fields. More details on this point and its mathematical underpinning can be found in [16].

We here proceed by stating how the dynamics in the presence of this multitude of linearized fields is described mathematically. As already mentioned in the introduction, the resulting Dirac equation involves a *nonlocal potential* $\mathscr{B}$ with integral kernel $\mathscr{B}(x,y)$ (see (1.1) and (1.2)). This integral kernel is of the form

$$\mathscr{B}(x,y) = \sum_{a=1}^{N} B_a\Big(\frac{x+y}{2}\Big)\, L_a(y-x)\,, \tag{2.23}$$

where $B_a(x)$ are multiplication operators acting on the spinors and the L_a are smooth complex-valued functions. The factors in the nonlocal potential are symmetric in the sense that

$$B_a(x)^* = B_a(x) \qquad \text{and} \qquad \overline{L_a(\xi)} = L_a(-\xi) \tag{2.24}$$

(where the star is the adjoint with respect to the spin inner product). The number N of these potentials is very large and scales like (1.5) with $\ell_{\min}$ in the range (1.3). The scale $\ell_{\min}$ also determines the scale of the nonlocality of the potential. It is an important consequence of (2.24) that the nonlocal potential is *symmetric*, meaning that

$$\mathscr{B}(x,y)^* = \mathscr{B}(x,y)\,. \tag{2.25}$$

The conserved commutator inner product (2.20) takes the following form

$$\begin{aligned} \langle \psi | \phi \rangle_t &:= \int \prec\!\psi \,|\, \gamma^0\, \phi\!\succ_{(t,\vec{x})}\, d^3x && (2.26)\\ &\quad - i \int_{x^0<t} d^4x \int_{y^0>t} d^4y\, \prec\!\psi(x) \,|\, \mathscr{B}(x,y)\, \phi(y)\!\succ_x && (2.27)\\ &\quad + i \int_{x^0>t} d^4x \int_{y^0<t} d^4y\, \prec\!\psi(x) \,|\, \mathscr{B}(x,y)\, \phi(y)\!\succ_x\,. && (2.28) \end{aligned}$$

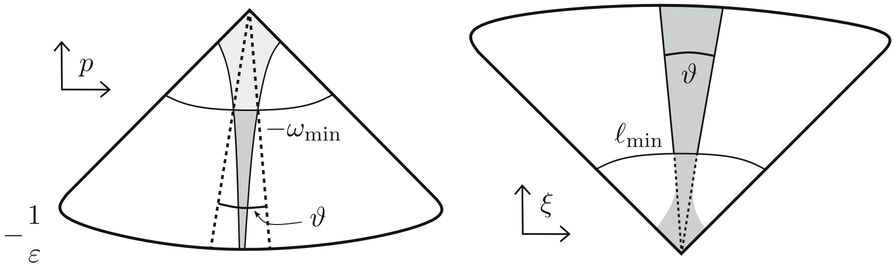

Figure 1. A homogeneous solution in momentum and position space

Note that (2.26) is the usual scalar product on Dirac wave functions. The additional summands (2.27) and (2.28) can be understood as correction terms which take into account the nonlocality of the potential $\mathcal{B}$ in (2.21). The mathematical structure of these additional terms is again that of a surface layer integral. Making use of the symmetry of the nonlocal potential (2.25), a straightforward computation shows that *current conservation* holds in the sense that the quantity is independent of the time t. (For the derivation, see [19, Proposition B.1].) We remark that this conservation law holds more generally for arbitrary Cauchy surfaces. (For more details, see [19, Appendix B].)

In [16], the structure of the potentials in (2.23) is specified in some more detail, as we now outline. Considering the kernel $L_a(y-x)$ as a convolution operator, its Fourier transform $\hat{L}_a$ defined by

$$\hat{L}_a(k) := \int_M L_a(\xi)\, e^{-i\xi k}\, d^4\xi \tag{2.29}$$

is a multiplication operator in momentum space. The symmetry property in (2.24) means that the function $\hat{L}_a$ is real-valued. In Fig. 1, the functions L_a and $\hat{L}_a$ are depicted in a typical example.

On the left side, the support of $\hat{L}_a$ is shown on the lower mass shell. Given that a parameter $\omega_{\min}$ is in the range

$$\frac{1}{\ell_{\min}} \lesssim \omega_{\min} \leq \frac{1}{\varepsilon}\,, \tag{2.30}$$

for frequencies smaller than $-\omega_{\min}$, the function $\hat{L}_a$ is supported inside a cone of opening angle

$$\vartheta = \frac{1}{\sqrt{\ell_{\min}\,\omega_{\min}}}\,.$$

This function is smooth in the sense that its derivatives have the scaling behavior

$$\big|D^s \hat{L}(p)\big| \lesssim \frac{1}{\big(\vartheta\,(\omega_{\min}+|\omega|)\big)^s} \tag{2.31}$$

(for any $s \in \mathbb{N}$). For frequencies larger than $-\omega_{\min}$, the support of the function $\hat{L}_a$ is a bit more spread out, as indicated by the light gray region on the left of Figure 1. Taking the Fourier transform, the corresponding function L_a can be thought of as a wave packet localized in space on the scale $\ell_{\min}$, as shown on the right of Figure 1.

Next, the potential B_a in (2.23) is vectorial. Thus, it can be written as

$$B_a = (A_a)_j \, \gamma^j \tag{2.32}$$

with potentials A_a which can be thought of as nonlocal generalizations of an electromagnetic potential. In a suitable gauge, they satisfy the homogeneous wave equation with an error term,

$$\Box A_a = \mathcal{O}\big(\ell_{\min} \, D^3 A_a\big) \,, \tag{2.33}$$

where $D^3 A_a$ denotes the third derivatives of A_a. Thus, the error term is of the multiplicative order $\ell_{\min}/\ell_{\text{macro}}$, where ℓ_{macro} is the length scale on which the potential A_a varies. These error terms and their scaling behavior are worked out in detail in [16]. In this paper, it is also shown that the functions L_a and A_a can be computed iteratively in an expansion in powers of $\ell_{\min}/\ell_{\text{macro}}$. Here, we do not need the details, but it suffices to work with the error term (2.33).

The symmetry of the nonlocal potential (2.25) can be expressed more conveniently by introducing the indefinite inner product

$$\begin{aligned} &<.|.> \;:\; C^\infty(M, SM) \times C^\infty_0(M, SM) \rightarrow \mathbb{C} \,, \\ &<\psi|\phi> = \int_M \prec \psi(x) \,|\, \phi(x) \succ d^4x \end{aligned} \tag{2.34}$$

(where $C^\infty(M, SM)$ denotes the smooth wave functions in Minkowski space, and the subscript zero indicates compact support). This indefinite inner product endows the wave functions with a Krein structure; we denote the corresponding Krein space by $(\mathcal{K}, <.|.>)$. The symmetry of the nonlocal potential implies that the Dirac operator in (1.1) is symmetric with respect to the Krein inner product. Taking the Fourier transform and using Plancherel's theorem, the Krein inner product can also be expressed in momentum space. In what follows, it will sometimes be convenient to consider wave functions in spacetime as vectors in the Krein space $(\mathcal{K}, <.|.>)$.

3. Dynamics in the Presence of Stochastic Nonlocal Potentials

3.1. Strategy for Obtaining Bosonic Field Operators

We consider the Dirac equation (1.1) in the presence of a nonlocal potential of the form (1.2) and (2.23). We again assume that the potential is symmetric (see (2.25) and (2.24)). Solutions of the Dirac equation can be constructed with the help of the retarded perturbation series

$$\tilde{\psi} = \sum_{n=0}^\infty \big(- s^\wedge_m \, \mathcal{B} \big)^n \, \psi \,, \tag{3.1}$$

where $s^\wedge_m$ is the retarded Dirac Green's operator defined as follows. As is common in QFT, we usually prefer to work in momentum space, denoting the four-momenta by p, k and q. Then, the retarded Dirac Green's operator is the operator of multiplication by the distribution

$$s_m^\wedge(p) = \lim_{\varepsilon \searrow 0} \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon p^0} . \tag{3.2}$$

The Fourier transform of $\mathcal{B}$ given by (1.6) has the corresponding symmetry property

$$\hat{\mathcal{B}}(p,k)^* = \hat{\mathcal{B}}(k,p) .$$

Now, the operator products in the perturbation series (3.1) can be built up of integral expressions of the form

$$\int \frac{d^4 p_2}{(2\pi)^4} \cdots s_m^\wedge(p_1)\, \hat{\mathcal{B}}(p_1,p_2)\, s_m^\wedge(p_2)\, \hat{\mathcal{B}}(p_2,p_3) \cdots . \tag{3.3}$$

We remark that this perturbation expansion is well defined to every order, provided that the dynamical potentials contained in $\mathcal{B}$ are smooth and fall off sufficiently fast at infinity. (This can be proved similar to [11, Lemma 2.1.2], noting that the nonlocal kernels give rise to additional convolutions.) Following the usual procedure in perturbative QFT, we shall not enter the question of convergence of this perturbation expansion. Instead, we consider the perturbation series as an asymptotic series. In simple situations (for example, if $\mathcal{B}$ is a smooth multiplication operator), the left side of (3.1) can be defined even non-perturbatively (for example, using the theory of linear hyperbolic systems; see, for example, [25, Chapter 13]). But, even in this case, it is not obvious whether the right side of (3.1) converges and coincides with the left side. In [32], it was shown that the series converges in suitable function spaces, provided that the nonlocal potential $\mathcal{B}$ is sufficiently small. Even if this result applies, the subtle issue remains that it is not clear whether the stochastic averages may be taken order by order in perturbation theory. As already mentioned, here we shall disregard these issues, taking the naive point of view that the perturbative description should make sense provided that the dynamical potentials are sufficiently small.

In order to get a connection to bosonic *quantum* fields, the factors $\hat{\mathcal{B}}$ in the above perturbation expansion would have to be replaced by field operators acting on the bosonic Fock space. Our general strategy is to show that, making use of the nonlocality of the potential $\hat{B}$, these factors can indeed be interpreted as such field operators, without the need to "quantize" them. The crucial step for making this strategy work is to show that these operators satisfy the CCR, as we now make precise. Decomposing $\mathcal{B}$ into the Dirac covariants,

$$\mathcal{B}(x,y) = \mathcal{B}_j(x,y)\, \gamma^j + \Phi(x,y)\, \mathbb{1} , \tag{3.4}$$

we can speak of the *vector* and *scalar components* of these fields. (The pseudoscalar, axial and bilinear contributions will not be considered here.)

Having a Maxwell field in mind, we will be mainly concerned with the vector component. Nevertheless, for ease of presentation, we begin with the scalar component. We are aiming at getting a connection to a *massless real* *scalar field*. The corresponding field operators, denoted by $\phi(x)$, satisfy the CCR

$$\big[\phi(x), \phi(y)\big] = K(x,y)\,, \tag{3.5}$$

where $K(x,y)$ is the *causal fundamental solution*

$$K(x,y) := \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2)\, \epsilon(k^0)\, e^{-ik(x-y)} = \frac{i}{4\pi^2}\, \delta\big(\xi^2\big)\, \epsilon(\xi^0) \tag{3.6}$$

(where we set $\xi := y - x$, and $\epsilon(t)$ is the usual sign function). Taking the Fourier transform,

$$\phi(x) = \int \frac{d^4q}{(2\pi)^4}\, \hat{\phi}_q\, e^{-iqx} \qquad \text{with} \qquad \hat{\phi}_q^* = \hat{\phi}_{-q}\,,$$

the CCR take the form

$$\big[\hat{\phi}_q, \hat{\phi}_{q'}\big] = (2\pi)^4\, \delta^4\big(q+q'\big)\, \delta(q^2)\, \epsilon(q^0)\,, \tag{3.7}$$

because then

$$\begin{aligned}\big[\phi(x), \phi(y)\big] &= \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4q'}{(2\pi)^4}\, \big[\hat{\phi}_q, \hat{\phi}_{q'}\big]\, e^{-iqx-iq'y} \\ &= \int \frac{d^4q}{(2\pi)^4}\, \delta(q^2)\, \epsilon(q^0)\, e^{-iqx+iqy} = K(x,y)\,,\end{aligned}$$

as desired. The coupling of the scalar quantum field to the Dirac field is described again by (3.1), but with $\mathcal{B}$ replaced by factors ϕ. Likewise, in momentum space, the operator products in (3.3) become

$$s_m^\wedge(p_1)\, \hat{\phi}_{p_1-p_2}\, s_m^\wedge(p_2)\, \hat{\phi}_{p_2-p_3}\, \cdots\,.$$

This motivates us to *define* the field operators by

$$\hat{\phi}_q(k_L, k_R) := (2\pi)^4\, \delta^4(k_L - k_R - q)\, \hat{\Phi}(k_L, k_R)\,. \tag{3.8}$$

Our goal is to show that, under suitable assumptions on $\mathcal{B}$, the so-defined field operators really satisfy the desired commutation relations (3.7). To this end, we proceed in several steps. We first try to arrange the commutation relations by choosing Φ as a nonlocal Gaussian field (Sect. 3.2). Our analysis will show that (3.7) can be arranged in the statistical mean, but not as an operator equation. Next, following (2.23), we consider instead of one scalar field a multitude of stochastic vector potentials (Sect. 3.3). This gives us more freedom, but, as we shall see, it will not be sufficient for satisfying the CCR as operator equations. Nevertheless, this analysis will be a preparation for the constructions in Sect. 4, where it will be shown that the CCR hold naturally once holographic mixing is taken into account.

3.2. Example of a Nonlocal Stochastic Scalar Field

We consider a nonlocal scalar potential. Thus, in (3.4) we set $\mathcal{B}_j$ to zero and choose Φ as

$$\Phi(x,y) = W\Big(\frac{x+y}{2}\Big)\, L(y-x)\,, \tag{3.9}$$

where the kernel L is complex-valued and symmetric in the sense that

$$\overline{L(\xi)} = L(-\xi)\,.$$

Moreover, we choose W as a real-valued Gaussian stochastic field with mean zero,

$$\ll W(x) \gg = 0\,, \qquad \ll W(x)\, W(y) \gg = h(y-x)\,, \tag{3.10}$$

with a covariance $h(y-x)$ which is real and symmetric, i.e.,

$$\overline{h(\xi)} = h(\xi) = h(-\xi) \qquad \text{for all } \xi \in M\,. \tag{3.11}$$

Transforming to momentum space, we obtain

$$\ll \hat{W}(q) \gg = 0\,, \qquad \ll \hat{W}(q)\, \hat{W}(q') \gg = (2\pi)^4\, \delta^4(q+q')\, \hat{h}(q)\,, \tag{3.12}$$

and (3.11) translates into

$$\overline{\hat{h}(q)} = \hat{h}(-q) = \hat{h}(q) \qquad \text{for all } q \in \hat{M}\,. \tag{3.13}$$

Moreover, being defined as the covariance of a real-valued field, the function h must have the positivity property that for any test function $f \in C^\infty_0(M,\mathbb{R})$,

$$0 \leq \ll \Big(\int_M f(x)\, W(x)\, d^4x \Big)^2 \gg = \int_M d^4x \int_M d^4y\; h(y-x)\, f(x)\, f(y)\,. \tag{3.14}$$

Using that convolution in position space corresponds to multiplication in momentum space, this inequality is equivalent to $\hat{h}$ being positive,

$$\hat{h}(q) \geq 0 \quad \text{for all } q \in \hat{M}\,. \tag{3.15}$$

We next transform the nonlocal potential to momentum space by taking the inverse transformation to (1.6),

$$\begin{aligned} \Phi(x,y) &= \int \frac{d^4k_L}{(2\pi)^4} \int \frac{d^4k_R}{(2\pi)^4}\, \hat{\Phi}\big(k_L, k_r\big)\, e^{-ik_L x + ik_R y} \\ &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \hat{\Phi}\Big(p+\frac{q}{2}, p-\frac{q}{2}\Big)\, e^{ip\xi + iq\zeta}\,, \end{aligned} \tag{3.16}$$

where we set

$$\xi := y-x\,, \qquad \zeta := \frac{y+x}{2}\,.$$

Using (3.9), we obtain

$$\hat{\Phi}\Big(p+\frac{q}{2}, p-\frac{q}{2}\Big) = \hat{W}(q)\, \hat{L}(p)\,. \tag{3.17}$$

3.2.1. The Statistical Mean of the Commutator. Taking (3.8) as the definition of the field operator $\hat{\phi}_q$, we would like to satisfy the CCR (3.7). The question is whether these commutation relations can be obtained by a suitable choice of the covariance h. As an intermediate step toward answering this question, we now compute the statistical mean of the commutator. First, using (3.8),

$$\begin{aligned}(\hat{\phi}_{q'}\,\hat{\phi}_q\,\hat{\psi})(k+q+q') &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4p'}{(2\pi)^4}\, \hat{\phi}_{q'}(k+q'+q,p)\,\hat{\phi}_q(p,p')\,\hat{\psi}(p') \\ &= \hat{\Phi}(k+q+q',k+q)\,\hat{\Phi}(k+q,k)\,\hat{\psi}(k)\,.\end{aligned}$$

For the operator product on the left side, we also use the short notation

$$\hat{\phi}_{q'}\,\hat{\phi}_q\big|_k\,\hat{\psi}(k)\,. \tag{3.18}$$

We thus obtain

$$\begin{aligned}&\big[\hat{\phi}_{q'},\hat{\phi}_q\big]\big|_k \\ &= \hat{\Phi}(k+q+q',k+q)\,\hat{\Phi}(k+q,k) - \hat{\Phi}(k+q+q',k+q')\,\hat{\Phi}(k+q',k) \\ &\overset{(3.17)}{=} \hat{W}(q)\,\hat{W}(q')\,\Big(\hat{L}\Big(k+q+\frac{q'}{2}\Big)\,\hat{L}\Big(k+\frac{q}{2}\Big) - \hat{L}\Big(k+q'+\frac{q}{2}\Big)\,\hat{L}\Big(k+\frac{q'}{2}\Big)\Big)\,.\end{aligned} \tag{3.19}$$

Taking the statistical mean with the help of (3.12) gives

$$\begin{aligned}&\ll\!\big[\hat{\phi}_{q'},\hat{\phi}_q\big]\big|_k\!\gg \\ &= (2\pi)^4\,\delta^4(q+q')\,\hat{h}(q)\,\Big(\hat{L}\Big(k+\frac{q}{2}\Big)\,\hat{L}\Big(k+\frac{q}{2}\Big) - \hat{L}\Big(k-\frac{q}{2}\Big)\,\hat{L}\Big(k-\frac{q}{2}\Big)\Big)\,.\end{aligned}$$

In order to get agreement with (3.7), we need to arrange that

$$\hat{h}(q)\,\Big(\hat{L}\Big(k+\frac{q}{2}\Big)^2 - \hat{L}\Big(k-\frac{q}{2}\Big)^2\Big) = \delta(q^2)\,\epsilon(q^0)\,. \tag{3.20}$$

These equations can be solved explicitly, as illustrated by the following simple example.

Proposition 3.1. *Choosing*

$$\hat{L}(p) = \Big(C+\frac{p^0}{C}\Big) \qquad \textit{and} \qquad \hat{h}(q) = \frac{1}{2\,|q^0|}\,\delta(q^2)\,, \tag{3.21}$$

the field operators $\hat{\phi}_q$ defined by (3.8) and (3.17) for W the Gaussian field with covariance (3.12) satisfy the CCR asymptotically for large C; more precisely,

$$\ll\!\big[\hat{\phi}_q,\hat{\phi}_{q'}\big]\!\gg\, = (2\pi)^4\,\delta^4\big(q+q'\big)\,\delta(q^2)\,\epsilon(q^0)\, + \mathcal{O}\Big(\frac{1}{C^2}\Big)\,. \tag{3.22}$$

Proof. By direct computation,

$$\begin{aligned}\hat{L}(p)^2 &= \Big(C+\frac{p^0}{C}\Big)^2 = C^2 + 2p^0 + \Big(\frac{p^0}{C}\Big)^2 \\ \hat{L}\Big(k+\frac{q}{2}\Big)^2 - \hat{L}\Big(k-\frac{q}{2}\Big)^2 &= 2q^0 + \frac{2kq}{C^2}\,.\end{aligned}$$

Using this result together with the formula for $\hat{h}(q)$ in (3.21) in (3.20) gives the result. $\square$

We note that the function $\hat{h}$ has the desired symmetry properties (3.13) and is positive (3.15). We also remark that the error term in (3.22) could be avoided by choosing $\hat{L}(p)$ as a function involving a square root. This has the disadvantage, however, that the computation of the Fourier transform becomes more difficult. This is why we here prefer to work with (3.21).

3.2.2. The Commutator in Position Space. In order to complete the picture, we proceed by rewriting the above findings to position space. We first compute the covariance in position space.

Lemma 3.2. *The Fourier transform of the distribution $\hat{h}(q)$ in (3.21) is given by*

$$h(x) = \frac{1}{16\pi^2} \frac{1}{|\vec{x}|} \Theta\big(|\vec{x}| - |x^0|\big) .$$

Proof. The following method was already used in [14, Section 5]. We first note that, for any $\varepsilon > 0$,

$$\begin{aligned} &\int_{-\infty}^{\infty} \epsilon(\tau)\, e^{-\varepsilon\,|\tau|}\, e^{-i\omega(t-\tau)} d\tau \\ &= \int_{-\infty}^{\infty} \epsilon(\tau)\, \frac{1}{i\omega - \varepsilon\,\epsilon(\tau)}\, \frac{d}{d\tau} e^{-\varepsilon\,|\tau|}\, e^{-i\omega(t-\tau)} d\tau = -\Big(\frac{1}{i\omega - \varepsilon} + \frac{1}{i\omega + \varepsilon} \Big)\, e^{-i\omega t} \end{aligned}$$

and thus

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \epsilon(\tau)\, e^{-\varepsilon\,|\tau|}\, e^{-i\omega(t-\tau)} d\tau = 2i\, \frac{\mathrm{PP}}{\omega}\, e^{-i\omega t}$$

(where "PP" denotes the principal part). Hence,

$$\frac{1}{2q^0}\, \delta(q^2)\, \epsilon(q^0)\, e^{-iqx} = -\frac{i}{4} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \epsilon(\tau)\, e^{-\varepsilon\,|\tau|}\, \delta(q^2)\, \epsilon(q^0)\, e^{-iqx + iq^0\tau}\, d\tau .$$

Integrating over q gives

$$h(x) = \frac{i}{4} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \epsilon(\tau)\, e^{-\varepsilon\,|\tau|}\, K_0(x^0 - \tau, \vec{x}) ,$$

where K_0 is again the causal fundamental solution (3.6), because

$$\begin{aligned} K_0(x) &= -\int \frac{d^4 q}{(2\pi)^4}\, \delta(q^2)\, \epsilon(q^0)\, e^{-iqx} \\ &= \int \frac{d^4 q}{(2\pi)^4}\, \delta(q^2)\, \epsilon(q^0)\, e^{iqx} = K(x) = \frac{i}{4\pi^2}\, \delta\big(x^2\big)\, \epsilon(x^0) . \end{aligned}$$

Thus,

$$h(x) = -\frac{1}{16\pi^2} \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \epsilon(\tau)\, e^{-\varepsilon\,|\tau|}\, \delta\Big((x^0 - \tau)^2 - |\vec{x}|^2 \Big)\, \epsilon(x^0 - \tau) ,$$

and carrying out the τ-integral gives

$$h(x) = -\frac{1}{16\pi^2}\;\frac{1}{2\,|\vec{x}|}\,(-2)\,\Theta\big(|\vec{x}| - |x^0|\big)\,.$$

This concludes the proof. □

Note that the function h has the desired symmetry properties (3.11).

Finally, it is instructive to see how the CCR arise in position space. To this end, we introduce the field operator $\phi(x)$ as the Fourier transform of $\hat{\phi}_q$ in the variable q,

$$\phi_x(k_L, k_R) = \int \frac{d^4q}{(2\pi)^4}\;\hat{\phi}_q(k_L, k_R)\; e^{-iqx}\,.$$

Using (3.8), (3.17) and (3.21), we obtain

$$\begin{aligned}
\phi_x(k_L, k_R) &= \hat{\Phi}(k_L, k_R)\, e^{-i(k_L-k_R)x} = \hat{W}(k_L-k_R)\;\hat{L}\Big(\frac{k_L+k_R}{2}\Big)\, e^{-i(k_L-k_R)x} \\
&= \hat{W}(k_L-k_R)\,\Big(C + \frac{k_L^0+k_R^0}{2C}\Big)\, e^{-i(k_L-k_R)x}\,.
\end{aligned}$$

Next, we also transform the variables k_L and k_R to position space by setting

$$\begin{aligned}
\phi_x(y_L, y_R) &= \int \frac{d^4k_L}{(2\pi)^4}\int \frac{d^4k_R}{(2\pi)^4}\;\phi_x(k_L, k_R)\; e^{-ik_L y_L + ik_R y_R} \\
&= \int \frac{d^4k_L}{(2\pi)^4}\int \frac{d^4k_R}{(2\pi)^4}\;\hat{W}(k_L-k_R)\,\Big(C + \frac{k_L^0+k_R^0}{2C}\Big)\, e^{-ik_L y_L + ik_R y_R}\, e^{-i(k_L-k_R)x}\,.
\end{aligned}$$

The term linear in C depends only on the difference of momenta $k_L - k_R$ and thus depends to the operator of multiplication by $W(x)$. Using that linear factors in momentum space correspond to partial derivatives in position space, we can carry out the Fourier integrals to obtain the simple formula

$$\phi(x) \equiv \phi_x = C\,W(x) + \frac{1}{2C}\,\big\{iD_t, W(x)\big\}\,, \tag{3.23}$$

where $D_t = \partial_t$ is a differential operator. Now, the commutation relations can be verified by direct computation.

Proposition 3.3. *The field operators $\phi(x)$ in (3.23) satisfy the canonical commutation relations in the statistical mean up to errors of the order $\mathcal{O}(C^{-2})$, i.e.,*

$$\ll\!\big[\phi(x), \phi(y)\big]\!\gg\; = K(x,y) + \mathcal{O}\Big(\frac{1}{C^2}\Big)\,.$$

Proof. Using (3.23), we obtain

$$\begin{aligned}
\big[\phi(x), \phi(y)\big] &= C^2\,\big[W(x), W(y)\big] + \frac{1}{2}\,\big[W(x), \{iD_t, W(y)\}\big] \\
&\quad + \frac{1}{2}\,\big[\{iD_t, W(x)\}, W(y)\big] \\
&\quad + \frac{1}{4C^2}\,\big[\{iD_t, W(x)\}, \{iD_t, W(y)\}\big] + \mathcal{O}\Big(\frac{1}{C^2}\Big) \\
&= i\,\big(W(x)\,\dot{W}(y) - W(y)\,\dot{W}(x)\big) + \mathcal{O}\Big(\frac{1}{C^2}\Big)\,.
\end{aligned} \tag{3.24}$$

Taking the statistical mean gives

$$\begin{aligned}
&\ll\big[\phi(x), \phi(y)\big]\gg = iD_{y^0} h(y-x) - i\, D_{x^0} h(y-x) + \mathcal{O}\Big(\frac{1}{C^2}\Big) \\
&= 2i\, \frac{\partial}{\partial \xi^0} h(\xi) + \mathcal{O}\Big(\frac{1}{C^2}\Big) = 2i\, \frac{1}{16\pi^2}\, \frac{1}{|\vec{\xi}|}\, \frac{\partial}{\partial \xi^0} \Theta\big(|\vec{\xi}| - \xi^0\big) + \mathcal{O}\Big(\frac{1}{C^2}\Big) \\
&= \frac{i}{8\pi^2}\, \frac{1}{|\vec{\xi}|}\, \frac{\partial}{\partial \xi^0} \Theta\big(|\vec{\xi}| - |\xi^0|\big) + \mathcal{O}\Big(\frac{1}{C^2}\Big) = \frac{i}{8\pi^2}\, \frac{1}{|\vec{\xi}|}\, \delta\big(|\vec{\xi}| - \xi^0\big)\big)\, \epsilon(\xi^0) + \mathcal{O}\Big(\frac{1}{C^2}\Big) \\
&= \frac{i}{4\pi^2}\, \delta\big(\xi^2\big)\, \epsilon(\xi^0) + \mathcal{O}\Big(\frac{1}{C^2}\Big) \overset{(3.6)}{=} K(x,y) + \mathcal{O}\Big(\frac{1}{C^2}\Big)\,,
\end{aligned}$$

as desired. □

3.2.3. Going Beyond the Statistical Mean. With the constructions so far, we have arranged that the CCR are satisfied for the *statistical mean* of the commutator (3.22). However, in order to get a connection to QFT, we need to make sure that the CCR hold without taking the statistical mean (because the statistical mean is taken only when computing the expectation value of an observable in a quantum measurement). To state it differently, we need to make sure that the CCR also hold when taking the statistical mean of composite expressions, i.e.,

$$\begin{aligned}
&\ll\phi(x_1) \cdots \phi(x_p)\, \big[\phi(x), \phi(y)\big]\, \phi(x_{p+1}) \cdots \phi(x_q)\gg \\
&\quad = K(x,y) \ll\phi(x_1) \cdots \phi(x_p)\, \phi(x_{p+1}) \cdots \phi(x_q)\gg\,,
\end{aligned} \tag{3.25}$$

where the dots stand for any combination of field operators. Let us verify whether this relation is satisfied. Applying the Wick rules gives pairings of the field operators. If the two field operators inside the commutator are paired with each other, we get the statistical mean as computed in Proposition 3.1. But we also need to take into account the contributions when the operators inside the commutator are paired with operators outside. In order to analyze these contributions in a clear setting, it is convenient to consider the statistical mean of the combination

$$\big[\phi(x), \phi(y)\big] \otimes \phi(x_1) \otimes \phi(x_2)\,,$$

where the tensor product means that we do not specify what these operators act on or are multiplied by. In this formulation, our task is to show that

$$\ll\big[\phi(x), \phi(y)\big] \otimes \phi(x_1) \otimes \phi(x_2)\gg = K(x,y)\, \mathbb{1} \otimes \ll\phi(x_1) \otimes \phi(x_2)\gg \tag{3.26}$$

(possibly up to certain error terms). This amounts to showing that the contributions by pairings of operators inside the commutator with operators outside are negligible. We refer to relations of the form (3.26) that we the CCR are satisfied *in the operator sense*.

Let us evaluate the condition (3.26) in the concrete example of Proposition 3.1. Using (3.23) and (3.24),

$$\begin{aligned}
&\ll\big[\phi(x), \phi(y)\big] \otimes \phi(x_1) \otimes \phi(x_2)\gg \\
&= -i\, C^2 \ll\big(W(x)\, \dot{W}(y) - W(y)\, \dot{W}(x)\big)\, W(x_1)\, W(x_2)\, \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}\gg + \mathcal{O}(C)\,,
\end{aligned}$$

and applying the Wick rules with covariance (3.10) gives

$$\ll\big[\phi(x),\phi(y)\big]\otimes\phi(x_1)\otimes\phi(x_2)\gg\, = C^2\,\Big(h\big(x_1-x_2\big)\,K(x,y)\Big) \tag{3.27}$$

$$-\,i\,C^2\,\Big(-h\big(x_1-x\big)\,\dot h\big(x_2-y\big)-h\big(x_2-x\big)\,\dot h\big(x_1-y\big) \tag{3.28}$$

$$+\,\dot h\big(x_1-x\big)\,h\big(x_2-y\big)+\dot h\big(x_2-x\big)\,h\big(x_1-y\big)\Big)+\mathcal{O}(C)\,. \tag{3.29}$$

The terms in the last two lines violate the CCR.

3.3. A Multitude of Nonlocal Vector Potentials

In the previous section, we saw that with a classical nonlocal potential with Gaussian distribution we can realize the CCR hold in the statistical mean (3.1). But it was impossible to arrange the CCR as operator equations (see the consideration after (3.25)). In order to improve the situation, we proceed in several steps. In this section, instead of a single potential we consider a multitude of potentials labeled by $a\in\{1,\ldots,N\}$. Moreover, we shall work with vector potentials denoted by A^j_a (with j a tensor index). This ansatz reflects the structure of the solutions of the linearized fields in Minkowski space as discovered in [16]. In Sect. 4, we will proceed by building in holographic phases.

Our starting point is again the nonlocal Dirac equation introduced in Section 2.5. We thus consider the Dirac equation (1.1) with a nonlocal potential $\mathcal{B}$ of the form (2.23), where the kernels L_a are complex-valued and symmetric (2.24). Before moving on, we point out that this ansatz by itself is no loss of generality, but that all the constraints on the form of the potential are imposed merely by the the scalings (1.5) and (2.31).

Lemma 3.4. *Every nonlocal potential $\mathcal{B}(x,y)$ can be approximated by the ansatz (1.4), with an error going to zero if $N\to\infty$.*

Proof. We represent the nonlocal potential similar to (3.16) as

$$\mathcal{B}(x,y)=\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\;\hat{\mathcal{B}}\Big(p+\frac{q}{2},p-\frac{q}{2}\Big)\;e^{ip\xi+iq\zeta}$$

(with $\hat{\mathcal{B}}$ as in (1.6)). Now, we approximate the integrand by a sum

$$\hat{\mathcal{B}}\Big(p+\frac{q}{2},p-\frac{q}{2}\Big)=\sum_{a=1}^N\hat W_a(q)\;\hat L_a(p)\,.$$

Similar to a Riemann sum, the error of this approximation can be made arbitrarily small by choosing the functions $\hat L_a$ to be supported in small cubes and letting the length of the sides of the cubes tend to zero and N to infinity. Transforming back to position space gives the ansatz (1.4). □

Next, we specialize the setting by assuming that the potentials B_a are vectorial (2.32), and that the corresponding potentials A^j_a are real-valued. This ansatz ensures in particular that the nonlocal potential is symmetric (2.25). This guarantees that a nonlocal version of current conservation holds. (For details, see [19, Proposition B.1].) We remark that this setup is similar to that in [29], except that here we restrict attention to the linear dynamics.

In analogy to (3.10), we treat the potentials A_a as Gaussian fields, i.e.,

$$\ll A^j_a(x) \gg = 0\,, \qquad \ll A^j_a(x)\, A^k_b(y) \gg = h^{jk}_{a,b}(y-x)$$

with covariance matrices $h^{jk}_{a,b}(y-x)$ which are real and symmetric, i.e.,

$$\overline{h^{jk}_{a,b}(\xi)} = h^{jk}_{a,b}(\xi) = h^{kj}_{b,a}(-\xi) \quad \text{for all } \xi \in M\,. \tag{3.30}$$

These covariances will be specified below. Transforming to momentum space, similar to (3.12) we obtain

$$\ll \hat{A}^j_a(q) \gg = 0\,, \qquad \ll \hat{A}^j_a(q)\, \hat{A}^k_b(q') \gg = (2\pi)^4\, \delta^4(q+q')\, \hat{h}^{jk}_{a,b}(q)\,, \tag{3.31}$$

and (3.30) translates into

$$\overline{\hat{h}^{jk}_{a,b}(q)} = \hat{h}^{jk}_{a,b}(-q) = \hat{h}^{kj}_{b,a}(q) \quad \text{for all } q \in \hat{M}\,. \tag{3.32}$$

Moreover, extending the positivity argument (3.14) to the vector-valued case, we find similarly to (3.15) that the matrices $\hat{h}^{jk}_{a,b}$ must be positive semi-definite, i.e.,

$$\sum_{j,k,a,b} \hat{h}^{jk}_{a,b}(q)\, \overline{u^a_j}\, u^b_k \geq 0 \qquad \text{for all } q \in \hat{M} \text{and } u \in M \times \mathbb{C}^N \simeq \mathbb{C}^{4N}\,. \tag{3.33}$$

Using (2.23) and (2.32), similar to (3.17), in momentum space the potential takes the form

$$\hat{\mathcal{B}}\Big(p+\frac{q}{2}, p-\frac{q}{2}\Big) = \sum_{a=1}^N \not{\hat{A}}_a(q)\, \hat{L}_a(p)\,. \tag{3.34}$$

The symmetry of the potential means that

$$\hat{\mathcal{B}}(k_L, k_R)^* = \hat{\mathcal{B}}(k_R, k_L)\,,$$

and therefore

$$\hat{A}(q)^* = \hat{A}(-q) \qquad \text{and} \qquad \hat{L}_a(p)^* = \hat{L}_a(p)\,.$$

Keeping in mind that we now have a vector potential, we define the corresponding field operators $\hat{\mathcal{B}}^j_q$ in analogy to (3.8) by

$$\hat{\mathcal{B}}^j_q(k_L, k_R) := (2\pi)^4\, \delta^4(k_L - k_R - q)\, \hat{\mathcal{B}}^j(k_L, k_R)\,, \tag{3.35}$$

where $\hat{\mathcal{B}}^j$ are the vector components of (3.34), i.e.,

$$\hat{\mathcal{B}}^j\Big(p+\frac{q}{2}, p-\frac{q}{2}\Big) := \sum_{a=1}^N \hat{A}^j_a(q)\, \hat{L}_a(p)\,.$$

Using (3.34), we obtain

$$\hat{\mathcal{B}}^j_q(k_L, k_R) = (2\pi)^4\, \delta^4(k_L - k_R - q) \sum_{a=1}^N \hat{A}^j_a(q)\, \hat{L}_a\Big(\frac{k_L+k_R}{2}\Big)\,. \tag{3.36}$$

3.3.1. Computation of the Commutator in Momentum Space. Similar to (3.26), our goal is to compute the statistical mean of the following tensor product,

$$\ll \big[\hat{\mathcal{B}}^i_q, \hat{\mathcal{B}}^j_{q'}\big]\big|_p \otimes \hat{\mathcal{B}}^k_r\big|_k \otimes \hat{\mathcal{B}}^l_{r'}\big|_{k'} \gg , \tag{3.37}$$

where we again used the notation (3.18). We begin by computing the commutator in momentum space similar to (3.19),

$$\begin{aligned}
\big[\hat{\mathcal{B}}^k_{q'}, \hat{\mathcal{B}}^j_q\big]\big|_k &= \hat{\mathcal{B}}^k(k+q+q', k+q)\, \hat{\mathcal{B}}^j(k+q, k) \\
&\quad - \hat{\mathcal{B}}^j(k+q+q', k+q')\, \hat{\mathcal{B}}^k(k+q', k) \\
&= \sum_{a,b=1}^N \hat{A}^j_a(q)\, \hat{A}^k_b(q') \\
&\quad \times \bigg(\hat{L}_b\Big(k+q+\frac{q'}{2}\Big)\, \hat{L}_a\Big(k+\frac{q}{2}\Big) - \hat{L}_a\Big(k+q'+\frac{q}{2}\Big)\, \hat{L}_b\Big(k+\frac{q'}{2}\Big)\bigg) .
\end{aligned}$$

We proceed with a Taylor expansion of the functions $\hat{L}_a$ and $\hat{L}_b$ in q and q' (this expansion is admissible in view of the smoothness assumption (2.31)),

$$\begin{aligned}
&\big[\hat{\mathcal{B}}^k_{q'}, \hat{\mathcal{B}}^j_q\big]\big|_k + \mathcal{O}\big(q^j q^k\big) + \mathcal{O}\big(q'^j q^k\big) + \mathcal{O}\big(q'^j q'^k\big) \\
&= \sum_{a,b=1}^N \hat{A}^j_a(q)\, \hat{A}^k_b(q')\, q^l \bigg((\partial_l \hat{L}_b)\, \hat{L}_a + \frac{1}{2}\, \hat{L}_b\, (\partial_l \hat{L}_a) - \frac{1}{2}\, (\partial_l \hat{L}_a)\, \hat{L}_b \bigg)\bigg|_k \\
&\quad + \sum_{a,b=1}^N \hat{A}^j_a(q)\, \hat{A}^k_b(q')\, q'^l \bigg(\frac{1}{2}\, (\partial_l \hat{L}_b)\, \hat{L}_a - (\partial_l \hat{L}_a)\, \hat{L}_b - \frac{1}{2}\, \hat{L}_a\, (\partial_l \hat{L}_b) \bigg)\bigg|_k \\
&= \sum_{a,b=1}^N \hat{A}^j_a(q)\, \hat{A}^k_b(q') \Big(\big(q^l\, \partial_l \hat{L}_b\big)\, \hat{L}_a - \big(q'^l\, \partial_l \hat{L}_a\big)\, \hat{L}_b \Big) \\
&= \sum_{a,b=1}^N \Big(q^l \hat{A}^j_a(q)\, \hat{A}^k_b(q') - \hat{A}^j_b(q)\, q'^l \hat{A}^k_a(q') \Big)\, \big(\partial_l \hat{L}_b\big)\, \hat{L}_a \,.
\end{aligned}$$

Noting that each partial derivative of $\hat{L}_b$ yields in position space a scaling actor $(y-x)_l \sim \ell_{\min}$, we can write the error terms in the shorter form

$$\begin{aligned}
\big[\hat{\mathcal{B}}^k_{q'}, \hat{\mathcal{B}}^j_q\big]\big|_k &= \sum_{a,b=1}^N \Big(q^l \hat{A}^j_a(q)\, \hat{A}^k_b(q') - \hat{A}^j_b(q)\, q'^l \hat{A}^k_a(q') \Big)\, \big(\partial_l \hat{L}_b\big)\, \hat{L}_a \\
&\qquad \times \Big(1 + \mathcal{O}\big(q\ell_{\min}\big) + \mathcal{O}\big(q'\ell_{\min}\big) \Big) \,.
\end{aligned} \tag{3.38}$$

When substituting this formula into (3.37) and writing the two last factors in this equation with the help of (3.36), we can take the statistical mean by forming Gaussian pairings using (3.31). When doing so, we get two different types of contributions: Those where the two factors $\hat{A}_a$ and $\hat{A}_b$ in (3.38) are paired with each other (so-called *inner pairings*), and those where each of these factors is paired with one of the additional factors in the tensor product (3.37) (*outer pairings*). For the contributions involving the inner pairings, the statistical mean factorizes as

$$\ll\big[\hat{\mathcal{B}}^i_q, \hat{\mathcal{B}}^j_{q'}\big]\big|_p \otimes \hat{\mathcal{B}}^l_r\big|_k \otimes \hat{\mathcal{B}}^{l'}_{r'}\big|_{k'}\gg \;=\; \ll\big[\hat{\mathcal{B}}^i_q, \hat{\mathcal{B}}^j_{q'}\big]\big|_p\gg \otimes \ll\hat{\mathcal{B}}^l_r\big|_k \otimes \hat{\mathcal{B}}^{l'}_{r'}\big|_{k'}\gg .$$

Therefore, it suffices to arrange that the statistical mean of the commutator is of the desired form (similar to what was accomplished for the scalar field in Proposition 3.1). Similar to the contribution (3.27) for the scalar field, the contributions involving inner pairings are the good terms which realize the CCR. For the contributions involving outer pairings, however, we are not allowed to take the statistical mean of the commutator. Similar to the contributions (3.28) and (3.29) for the scalar field, these contributions are *not* of the desired form.

Having a multitude of fields makes it possible to distinguish the inner and outer pairings also in a different way. For simplicity, we assume that the covariance $\hat{h}^{j,k}_{ab}(q)$ in (3.31) has is diagonal in the indices ab, i.e.,

$$\hat{h}^{j,k}_{ab}(q) = \delta_{ab}\, \hat{h}^{j,k}_a(q) \tag{3.39}$$

with new functions $\hat{h}^{j,k}_a(q)$. (This can indeed be arranged by diagonalizing the covariance, as will be explained in detail in the proof of Theorem 4.1].) Then, in view of the factor δ_{ab} in (3.39), for the inner pairings the two indices a and b in (3.38) coincide, whereas for outer parings these two indices will in general be different. In particular, the contributions with inner pairings become large compared to the contributions with outer pairings if

$$\big(\partial_l \hat{L}_b\big)\, \hat{L}_a \ll \hat{L}_a \hat{L}_a \qquad \text{for all } a \text{ and } b \neq a\,. \tag{3.40}$$

Here, it is a subtle point to give the symbol $\ll$ a precise meaning. Before explaining this point, let us clarify the general mechanism by specifying the scalings if for simplicity we assume that the left side of (3.40) is zero for all $a \neq b$. In this case, the contributions with inner pairings scale like

$$\sum_{a,b} \big(\cdots\big)_a \otimes (\cdots)_b \otimes (\cdots)_b \sim N^2\,,$$

whereas the contributions with outer pairings scale like

$$\sum_{a,b} \big(\delta_{ab} \cdots\big) \otimes (\cdots)_a \otimes (\cdots)_b \sim N\,.$$

Therefore, the contributions with outer pairings are smaller by a scaling factor $1/N$.

In order to specify what we mean by (3.40), we first have a closer look at the form the nonlocal potentials as shown in Fig. 1. As explained in Sect. 2.5, the function $\hat{L}_a$ is supported in the gray region on the left of Fig.e 1. Likewise, the product $\hat{L}_a \hat{L}_b$ is supported in the intersection of the corresponding gray regions, as shown in Fig. 2.

The derivative in (3.40) gives a scaling factor of one over the size of this intersection region, i.e.,

$$\big(\partial_l \hat{L}_b\big)\, \hat{L}_a \simeq \frac{1}{\omega_{\min}}\, \hat{L}_b \hat{L}_a \qquad \text{if } b \neq a\,. \tag{3.41}$$

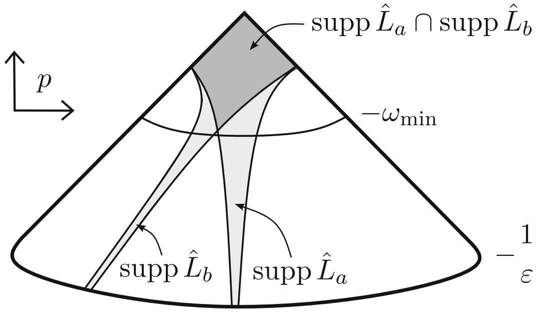

FIGURE 2. Estimate of $(\partial_l \hat{L}_b)\,\hat{L}_a$

In the case $a = b$, however, the function $\hat{L}_a^2$ is supported in the whole shaded region in Fig. 1. Since this shaded region has a spike-like form for large $|\omega|$, the derivative only gives a scaling factor $\ell_{\min}$,

$$(\partial_l \hat{L}_a)\,\hat{L}_a \simeq \ell_{\min}\,\hat{L}_a \hat{L}_a\,. \tag{3.42}$$

According to (2.30), the parameter $\omega_{\min}$ is typically much larger than $1/\ell_{\min}$, in which case (3.41) really is much smaller than (3.42), as desired.

This consideration points to a mechanism to the effect that the contributions with outer pairings become small. Note that this mechanism is based on two facts: That we have a multitude N of fields, and that these fields couple to the sea states as shown in Fig. 1. Although it might look promising at first sight, this mechanism turns out *not* to be sufficient for satisfying the CCR as operator equations. The reason for this shortcoming can be understood in words as follows. In order to satisfy the CCR as operator equations, we must make sure that the equation (3.40) holds in the operator sense. Having multiplication operators in momentum space, this means that (3.40) should hold for any $p \in \hat{M}$. However, the scaling behavior in (3.41) and (3.42) does not hold uniformly for all $p \in \hat{M}$. In fact, in (3.41) we considered p with $|p^0| \lesssim \omega_{\min}$ (otherwise the operator product vanishes), whereas in (3.42) we considered the high-frequency region $|p^0| \gg \omega_{\min}$ where the shaded region in Fig. 1 has a spike-like form. With this in mind, the scalings in (3.41) and (3.42) do *not* hold as operator equations.

This problem will be resolved in Sect. 4 by taking into account the gauge phases of the potentials A_a. Before explaining how this works, we conclude this section with a few clarifying considerations. In the next section, we make the just-described no-go result precise. In Sect. 3.3.3 we complement the picture by computations in position space.

3.3.2. Shortcoming of the Ansatz. We now explain in more detail why the above ansatz (2.23) with vectorial stochastic potentials (2.32), (3.31) is not quite sufficient for implementing the CCR. Our argument is more general than the consideration in the previous section, because we shall not make use of the specific form of the homogeneous solutions as depicted in Fig. 1. Beginning from (3.36), we compute the following statistical mean with the help of (3.31),

$$\ll \sum_{a,b=1}^{N} \hat{A}_a^j(q)\, \hat{L}_a(p_L)\, \hat{A}_b^k(q')\, \hat{L}_b(p_R) \gg = (2\pi)^4\, \delta^4(q+q')\, K^{jk}(q, p_L, p_R)$$

with

$$K^{jk}(q, p_L, p_R) := \sum_{a,b=1}^{N} \hat{h}_{a,b}^{jk}(q)\, \hat{L}_a(p_L)\, \hat{L}_b(p_R)\,. \tag{3.43}$$

Using the notation (3.18), we thus obtain

$$\ll \hat{\mathcal{B}}_q^j|_p\, \hat{\mathcal{B}}_{q'}^k|_{p'} \gg = (2\pi)^4\, \delta^4(q+q')\, K^{jk}\Big(q, p+\frac{q}{2}, p'-\frac{q}{2}\Big)\,.$$

Next, similar to (3.26) we consider the product

$$\begin{aligned} &\big[\hat{\mathcal{B}}_q^i, \hat{\mathcal{B}}_{q'}^j\big]\big|_p \otimes \hat{\mathcal{B}}_r^k\big|_k \otimes \hat{\mathcal{B}}_{r'}^l\big|_{k'} \\ &= \hat{\mathcal{B}}_q^i\big|_{p+q'}\, \hat{\mathcal{B}}_{q'}^j\big|_p\, \hat{\mathcal{B}}_r^k\big|_k\, \hat{\mathcal{B}}_{r'}^l\big|_{k'} - \hat{\mathcal{B}}_{q'}^j\big|_{p+q}\, \hat{\mathcal{B}}_q^i\big|_p\, \hat{\mathcal{B}}_r^k\big|_k\, \hat{\mathcal{B}}_{r'}^l\big|_{k'}\,. \end{aligned}$$

Its statistical mean can be computed with the Wick rules to

$$\frac{1}{(2\pi)^8} \ll \big[\hat{\mathcal{B}}_q^i, \hat{\mathcal{B}}_{q'}^j\big]\big|_p \otimes \hat{\mathcal{B}}_r^k\big|_k \otimes \hat{\mathcal{B}}_{r'}^l\big|_{k'} \gg$$

$$= \delta^4(q+q')\, \delta^4(r+r') \left(K^{ij}\Big(q, p-\frac{q}{2}, p-\frac{q}{2}\Big) \right.$$

$$\left. - K^{ij}\Big(q, p+\frac{q}{2}, p+\frac{q}{2}\Big)\right) K^{kl}\Big(r, k+\frac{r}{2}, k'-\frac{r}{2}\Big) \tag{3.44}$$

$$+ \delta^4(q+r)\, \delta^4(q'+r') \left(K^{ik}\Big(q, p+q'+\frac{q}{2}, k-\frac{q}{2}\Big)\, K^{jl}\Big(q', p+\frac{q'}{2}, k'-\frac{q'}{2}\Big) \right.$$

$$\left. - K^{ik}\Big(q, p+\frac{q}{2}, k-\frac{q}{2}\Big)\, K^{jl}\Big(q', p+q+\frac{q'}{2}, k'-\frac{q'}{2}\Big)\right) \tag{3.45}$$

$$+ \delta^4(q+r')\, \delta^4(q'+r) \left(K^{ik}\Big(q, p+q'+\frac{q}{2}, k'-\frac{q'}{2}\Big)\, K^{jl}\Big(q', p+\frac{q'}{2}, k-\frac{q}{2}\Big) \right.$$

$$\left. - K^{ik}\Big(q, p+\frac{q}{2}, k'-\frac{q}{2}\Big)\, K^{jl}\Big(q', p+q+\frac{q'}{2}, k-\frac{q'}{2}\Big)\right). \tag{3.46}$$

In (3.44), the two operators in the commutator are paired. This pairing gives the desired commutation relations. In (3.45) and (3.46), however, the operators of the commutator are paired with operators outside. These contributions should vanish. Since q, q' and k, k' can be chosen arbitrarily, it follows that (3.45) and (3.46) must vanish separately. Moreover, using that q and q' can be chosen independently, we obtain the separate conditions

$$\begin{aligned} K^{ik}\Big(q, p+q'+\frac{q}{2}, k-\frac{q}{2}\Big) &= K^{ik}\Big(q, p+\frac{q}{2}, k-\frac{q}{2}\Big) \\ K^{jl}\Big(q', p+\frac{q'}{2}, k'-\frac{q'}{2}\Big) &= K^{jl}\Big(q', p+q+\frac{q'}{2}, k'-\frac{q'}{2}\Big)\,, \end{aligned}$$

to be satisfied for all q, q' and k, k'. The first equation means that K^{ik} does not depend on the second variable. Since the function K^{jk} is symmetric in its second and third variables (as is obvious from (3.43)), we conclude that it is also independent of the third variable. Hence, the covariance depends only on the first variable. But then also (3.44) vanishes, so that the CCR are violated.

3.3.3. The Commutator in Position Space. Similar as in Sect. 3.2.2, we now compute the commutator in position space. We set

$$\begin{aligned}\mathcal{B}^j_x(k_L, k_R) &= \int \frac{d^4q}{(2\pi)^4}\, \hat{\mathcal{B}}^j_q(k_L, k_R)\, e^{-iqx} \\ &= \sum_{a=1}^N \hat{A}^j_a(k_L - k_R)\, \hat{L}_a\Big(\frac{k_L + k_R}{2}\Big)\, e^{-i(k_L - k_R)x}\,. \end{aligned} \tag{3.47}$$

Transforming to position space, we obtain (again using the notation (3.2.2))

$$\begin{aligned}\mathcal{B}^j_x(y_L, y_R) &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \mathcal{B}^j_x\Big(p + \frac{q}{2}, p - \frac{q}{2}\Big)\, e^{ip\xi + iq\zeta} \\ &= \sum_{a=1}^N \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \hat{A}^j_a(q)\, \hat{L}_a(p)\, e^{ip\xi+iq\zeta}\, e^{-iqx} \\ &= \sum_{a=1}^N A^j_a\Big(\frac{y_L + y_R}{2} + x\Big)\, L_a(y_R - y_L)\,.\end{aligned}$$

This formula shows that the argument x simply describes a translation of the nonlocal potential in position space.

Next, it is useful to introduce the notation

$$\big(\mathfrak{D}^j A\big)(x, y) = (y - x)^j\, A(x, y)\,.$$

This notation is motivated by the fact that the factor $(y - x)^j$ corresponds to a partial derivative in momentum space (as one verifies in detail by differentiating (2.29)). This also explains the product rule

$$\mathfrak{D}_j(AB) = (\mathfrak{D}_j A)\, B + A\, (\mathfrak{D}_j B)$$

(where AB is the product of two convolution operators).

Lemma 3.5. *For any $x, y \in M$, the commutator of the field operators defined by (3.47) is given in position space by*

$$\begin{aligned}\big[\mathcal{B}^j_x, \mathcal{B}^k_y\big](y_L, y_R) = \frac{1}{2} \sum_{a,b=1}^N \Big(&\partial_l A^j_a(\zeta + x)\, A^k_b(\zeta + y) - \partial_l A^k_a(\zeta + y)\, A^j_b(\zeta + x)\Big) \\ &\times \big(L_a\, (\mathfrak{D}^l L_b) + (\mathfrak{D}^l L_b)\, L_a\big)(y_R - y_L) + \mathcal{O}\big(\ell^2\big)\end{aligned}$$

with

$$\zeta = \frac{y_L + y_R}{2}\,.$$

Proof. First,

$$
\begin{aligned}
&\big[\mathcal{B}^j_x, \mathcal{B}^k_y\big](y_L, y_R) \\
&\quad = \sum_{a,b=1}^N \int d^4z \left\{ A^j_a\Big(\frac{y_L+z}{2} + x\Big)\, L_a(z-y_L)\, A^k_b\Big(\frac{z+y_R}{2} + y\Big)\, L_b(y_R - z) \right\} \\
&\qquad - \sum_{a,b=1}^N \int d^4z \left\{ A^k_b\Big(\frac{y_L+z}{2} + y\Big)\, L_b(z-y_L)\, A^j_a\Big(\frac{z+y_R}{2} + x\Big)\, L_a(y_R - z) \right\} .
\end{aligned}
$$

In the arguments of the potentials A_a and A_b, we use the transformations

$$
\frac{y_L+z}{2} = \frac{y_L+y_R}{2} + \frac{z-y_R}{2} \qquad \text{and} \qquad \frac{z+y_R}{2} = \frac{y_L+y_R}{2} + \frac{z-y_L}{2}
$$

and expand in powers of the last summands. We thus obtain

$$
\begin{aligned}
&\big[\mathcal{B}^j_x, \mathcal{B}^k_y\big](y_L, y_R) = \sum_{a,b=1}^N A^j_a(\zeta + x)\, A^k_b(\zeta + y) \\
&\quad \times \int d^4z \left\{ L_a(z-y_L)\, L_b(y_R-z) - L_b(z-y_L)\, L_a(y_R-z) \right\} \\
&\quad + \frac{1}{2} \sum_{a,b=1}^N \partial_l A^j_a(\zeta+x)\, A^k_b(\zeta+y) \\
&\quad \times \int d^4z \left\{ (z-y_R)^l\, L_a(z-y_L)\, L_b(y_R-z) - L_b(z-y_L)\, (z-y_L)^l\, L_a(y_R-z) \right\} \\
&\quad + \frac{1}{2} \sum_{a,b=1}^N A^j_a(\zeta+x)\, \partial_l A^k_b(\zeta+y) \\
&\quad \times \int d^4z \left\{ L_a(z-y_L)\, (z-y_L)^l\, L_b(y_R-z) - (z-y_R)^l\, L_b(z-y_L)\, L_a(y_R-z) \right\} \\
&\quad + \mathcal{O}\big(\ell^2\big)
\end{aligned}
$$

Being complex convolution operators, the operators L_a and L_b clearly commute. Therefore, the zero-order terms drop out. The remaining first-order terms can be written as

$$
\begin{aligned}
\big[\mathcal{B}^j_x, \mathcal{B}^k_y\big](y_L, y_R) = \;& \frac{1}{2} \sum_{a,b=1}^N \partial_l A^j_a(\zeta+x)\, A^k_b(\zeta+y) \\
& \times \Big(L_a\, (\mathfrak{D}^l L_b) + (\mathfrak{D}^l L_b)\, L_a \Big)(y_R - y_L) \\
& + \frac{1}{2} \sum_{a,b=1}^N A^j_a(\zeta+x)\, \partial_l A^k_b(\zeta+y) \\
& \times \Big(-(\mathfrak{D}^l L_a)\, L_b - L_b\, (\mathfrak{D}^l L_a) \Big)(y_R - y_L) + \mathcal{O}\big(\ell^2\big) ,
\end{aligned}
$$

giving the result. □

As is verified by a straightforward computation, the formula of this lemma is indeed the Fourier transform of (3.38). Also, the error terms agree, if one keeps in mind that in the last proof, every factor ℓ comes with a derivative acting on the potential, which in momentum space gives rise to a factor q or q'.

We finally remark that the above argument (3.41) and (3.42) in which we made use of smoothness properties of $\hat{L}_a$ and $\hat{L}_b$ can be recast in position space by saying that the operator product $L_a L_b$ is typically of much shorter range than the operator L_a^2.

4. Holographic Mixing

The analysis of the previous section showed that a nonlocal potential of the form (2.23) is not sufficient for realizing the CCR as operator equations. The general idea in order to overcome this problem is to take into account that, similar to gauge phases of classical electrodynamics, the potentials A_a in (1.4) give rise to phase factors. Having a multitude of N potentials yields N different phase factors. Consequently, relative phase factors should lead to dephasing effects. In this section, we will work out this mechanism and its consequences in detail. The mechanism is referred to as *holographic mixing*. Holographic mixing is a specific feature of causal fermion systems. For this reason, we introduce it in detail step by step. After a few general considerations (Sect. 4.1), we begin under simplifying assumptions with an analysis of momentum-dependent gauge transformations (Sect. 4.2). We proceed by specifying the form of the dynamical gauge potentials in the presence of holographic phases (Sect. 4.3). Then, we will show that these potentials naturally satisfy the CCR (Sects. 4.4 and 4.5). In Sect. 5, the dynamical equations including holographic phases will be derived.

4.1. Basic Concepts Behind Holographic Mixing

The concept of holographic mixing evolved in various steps over several years. Before entering the detailed constructions, we now give a brief outline of how holographic mixing developed. It is one of the basic features of causal fermion systems that one can go beyond classical spacetimes and allow for the description of spacetimes which have a non-smooth, possibly discrete, microstructure or which on small scales resemble a "quantum spacetime" with "quantum geometric" structures (as is made precise in [18]). Since in the setting of causal fermion systems, the geometric structures of spacetime are encoded in the family of all physical wave functions, such novel or generalized spacetime structures are implemented by working with physical wave functions having a non-trivial behavior on small scales. This concept was first introduced in [9, Section 4] under the name *microscopic mixing*. Refinements of this concept were used in [10, Section 3] for getting a first connection between causal fermion systems and QFT. Subsequently, in [11, §1.5.3] microscopic mixing came up from a somewhat different perspective by studying a decomposition of the measure ρ of the causal fermion system into many components, with the purpose of decreasing the causal action in the vacuum. This point of view was followed up in [13, Section 5], where also the term *fragmentation* was introduced. A more systematic study of fragmentation and the resulting dynamics of the causal fermion system was carried out in [15, Section 5].

A simple way of understanding fragmentation is the observation that the measure ρ of the causal fermion system does not necessarily need to be the push-forward of the volume measure of a classical spacetime. One method for constructing more general measures is to take convex combinations of measures, leading to the ansatz

$$\rho = \sum_{a=1}^N c_a\, \rho_a \qquad \text{with } c_a > 0 \text{ and } \quad \sum_{a=1}^N c_a = 1\,. \tag{4.1}$$

If each measure is constructed from a classical spacetime, then $M_a := \operatorname{supp} \rho_a$ is diffeomorphic to Minkowski space (or to a curved spacetime). In this case, the spacetime $M := \operatorname{supp} \rho$ of the causal fermion system is the union of all the sub-spacetimes M_a,

$$M = \bigcup_{a=1}^N M_a\,.$$

The resulting physical picture is that spacetime consists of N copies of classical spacetimes, each with its own physical wave functions and corresponding spacetime structures. This has similarities with taking a "superposition" of classical spacetimes, except that the coefficients c_a in the linear combination (4.1) must all be positive (in order to ensure that ρ is a positive measure). The decomposition of spacetime into sub-spacetimes becomes a bit clearer by writing the corresponding wave evaluation operator $\Psi : \mathcal{H} \to C^0(M, SM)$ (see (2.9)) as

$$\Psi(x) = \sum_{a=1}^N \chi_{M_a}(x)\, \Psi_a(x)\,, \tag{4.2}$$

where $\Psi_a : \mathcal{H} \to C^0(M_a, SM_a)$ are the wave evaluation operators of the sub-spacetimes.

Clearly, the fact that taking convex combinations is mathematically allowed does not necessarily mean that measures of the form (4.1) should occur in physics. One argument in favor of (4.1) is that this ansatz makes it possible to decrease the causal action. Indeed, rewriting the causal action as

$$\begin{aligned} \mathcal{S}(\rho) &= \sum_{a,b=1}^N c_a\, c_b \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho_a(x)\, d\rho_b(y) \\ &= \sum_{a=1}^N c_a^2\, \mathcal{S}(\rho_a) + \sum_{a\neq b} c_a\, c_b \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho_a(x)\, d\rho_b(y)\,, \end{aligned}$$

the first sum can be arranged to give a scaling factor $1/N$ (for example, by choosing $\mathcal{S}(\rho_a) = \mathcal{S}(\rho_b)$ and setting $c_a = 1/N$ for all $a, b \in \{1, \ldots, N\}$). The summands for $a \neq b$, on the other hand, can be made small using dephasing effects. This becomes clear by transforming the wave evaluation operators according to

$$\Psi_a \longrightarrow \Psi_a \,\mathcal{U}_a \;,$$

where $\mathcal{U}_a$ are unitary operators on $\mathcal{H}$. In this case, the kernel of the fermionic projector transforms according to

$$\begin{aligned}\chi_{M_a}(x)\, P(x,y)\, \chi_{M_b}(y) &= -\chi_{M_a}(x)\, \Psi_a(x)\, \Psi_b(y)^*\, \chi_{M_b}(y) \\ &\longrightarrow -\chi_{M_a}(x)\, \Psi_a(x)\, \mathcal{U}_a\, \mathcal{U}_b^*\, \Psi_b(y)^*\, \chi_{M_b}(y)\;.\end{aligned}$$

In the case $a = b$, the unitary operators drop out, showing that $\mathcal{S}(\rho_a)$ remains unchanged. In the case $a \neq b$, however, the unitary operator $\mathcal{U}_a\, \mathcal{U}_b^*$ can be arranged to be small due to dephasing. (For more details, see [11, §1.5.3].)

Considerations along this line explain why fragmentation occurs naturally when minimizing the causal action principle. This also means that dephasing effects between the physical wave functions in spacetime should be of physical relevance. Nevertheless, fragmentation is not a fully convincing concept for the following reasons. One difficulty is to describe fragmentation dynamically. How do the sub-spacetimes interact with each other? What is the resulting "effective" dynamics of the whole system? Another, more conceptual problem is that, thinking of a fully interacting situation, it does not seem to be a sensible concept to speak of individual sub-spacetimes. Should one not consider M instead as a whole, without decomposing it into sub-spacetimes? Thinking about these questions led to *holographic mixing* as a preferable method for describing the above dephasing effects. To this end, we generalize (4.2) by an ansatz for the wave evaluation operator (2.9) as a linear combination

$$\Psi(x) = \sum_{a=1}^{N} e^{i \Lambda_a(x)}\, \Psi_a(x) \qquad \text{with} \qquad \Psi_a : \mathcal{H} \to C^0(M, SM)\;, \tag{4.3}$$

and Λ_a are real-valued functions. Note that, in contrast to (4.2), we no longer have sub-spacetimes. Instead, all the mappings $\Psi_a(x)$ are defined in the whole spacetime $x \in M$. The dephasing effects are described by the phase factors $e^{i\Lambda_a}$ which may oscillate on small length scales, thereby implementing the original concept of a non-trivial microstructure of spacetime. The name "holographic mixing" is inspired by the similarity to a hologram in which several pictures are stored, each of which becomes visible only when looking at the hologram in the corresponding coherent light.

One advantage of holographic mixing (4.3) is that it fits better to the structure of the linearized field equations in Minkowski space as unveiled in [16], because the summands Ψ_a can be associated with the physical wave functions lying in the support of the functions $\hat{L}_a$ (i.e., the shaded regions in Fig. 1). These summands are no longer localized in separate sub-spacetimes (as in (4.2)), but rather correspond to "subsystems" in Minkowski space formed of wave functions propagating into a common null direction. Another advantage of this description is that the phase functions $e^{i\Lambda_a}$ can be associated with local phases corresponding to the potentials A_a. (This is also why we use the same notation with a subscript a.) This also gives an explanation for why these phase functions come about. Moreover, it will become possible to derive dynamical equations for the system including holographic phases. Finally, the

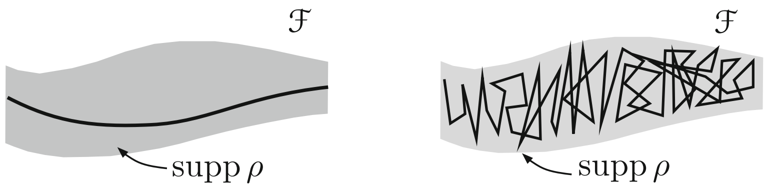

FIGURE 3. A measure obtained by fragmentation (left) and by holographic mixing with fluctuations (right)

concept of holographic mixing ties in nicely with the constructions in [23], where dephasing effects also play a crucial role.

The critical reader may wonder how fragmentation (4.1) and holographic mixing (4.3) are related to each other, and how it comes about that both methods can be used to describe interacting causal fermion systems. In non-technical terms, this can be understood as follows. (A more detailed explanation of this point from a slightly different perspective can be found in [25, Chapter 22].) In the limit when N gets large, the fragmented measure $\tilde{\rho}$ goes over to a measure with enlarged support. (See the gray region on the left of Fig. 3.) Integrating with respect to this measure does not only involve an integral over the four-dimensional spacetime (indicated by the black line on the left of Fig. 3), but it also involves an integration over the "internal degrees of freedom" corresponding to the directions which are transverse to the four-dimensional spacetime. This situation bears similarity to the path integral formulation of QFT if one identifies the above "internal degrees of freedom" with field configurations. As explained above, in the method of holographic mixing, spacetime $M := \operatorname{supp} \rho$ remains four-dimensional. But, keeping in mind that we allow for high-frequency fluctuations, the measure could still approximate the measure described by fragmentation (as shown by the black line on the right of Fig. 3, noting that high frequencies correspond to low regularity in position space, as drawn for simplicity by a non-smooth curve). With this intuitive picture, it becomes clear why, from the mathematical point of view, fragmentation and holographic mixing can be used equivalently. However, it seems that the method of holographic mixing is more suitable for the analysis of the causal action principle.

We finally remark that the concept of holographic mixing is also related to Everett's multiverse interpretation of quantum theory and the collapse of the wave function. We refer the interested reader to [29,33].

4.2. Momentum-Dependent Phase Transformations

In order to develop our methods, we begin with the behavior of Dirac wave functions under local gauge transformations. The local gauge transformations of electrodynamics are described by the joint transformation

$$\psi(x) \to e^{i\Lambda(x)}\, \psi(x)$$

of all wave functions, with Λ a real-valued function. The Dirac operator of the vacuum transforms according to

$$i\partial\!\!\!/ \to e^{i\Lambda} \big(i\partial\!\!\!/\big) e^{-i\Lambda} = i\partial\!\!\!/ + (\partial\!\!\!/\Lambda) \,.$$

The additional term $(\partial\!\!\!/\Lambda)$ is the standard coupling term to the gauge potentials $A_j = \partial_j \Lambda$. Before going on, we point out that, in order to capture dephasing effects, we need to cover the case that the phase functions like $e^{i\Lambda}$ have rapid oscillations on small length scales. Such oscillatory phase functions cannot be treated with a simple perturbation expansion. Therefore, it will not suffice for our purposes to take the Dirac operator $i\partial\!\!\!/ + (\partial\!\!\!/\Lambda)$ as the starting point. Instead, it is preferable to work with the phase factors $e^{\pm i\Lambda}$ where the oscillations are manifest. This observation will serve as a guiding principle for the following constructions.

As summarized in Sect. 2.5, it is the main conclusion of the analysis in [16] that the causal action principle admits a larger class of homogeneous fields. Rather than just the electromagnetic field, there is a plethora of fields described by nonlocal potentials (see (1.1), (1.2) and (2.23)). These fields are related to the corresponding local gauge invariance (2.22) of the causal Lagrangian. This suggests that nonlocal potentials should give rise to corresponding phase factors, also leading to dephasing effects. In order to analyze how such effects can be modeled mathematically, we proceed step by step and begin with the situation that the different phase factors act on orthogonal subspaces of the wave functions in spacetime. To this end, we consider a family $(L_a)_{1=1,\ldots,N}$ of homogeneous scalar operators, i.e.,

$$\big(L_a \psi\big)(x) = \int L_a(y-x)\, \psi(y)\, d^4y \qquad \text{and} \qquad L_a(y-x) \in \mathbb{C}\,, \tag{4.4}$$

which for simplicity we again assume to be smooth. Moreover, we assume that the L_a form a complete set of projection operators, i.e.,

$$L_a L_b = \delta_{ab}\, L_a \qquad \text{and} \qquad \sum_{a=1}^N L_a = \mathbb{1}\,. \tag{4.5}$$

Now, for each $a \in \{1, \ldots, N\}$ we introduce a real-valued phase function Λ_a and introduce the operator U by

$$(U\psi)(x) := \sum_{a=1}^N e^{i\Lambda_a(x)} \, \big(L_a \psi\big)(x)\,. \tag{4.6}$$

In order to simplify the situation further, we also assume that the images of the summands are orthogonal and again complete, i.e.,

$$L_a \, e^{-i\Lambda_a}\, e^{i\Lambda_b}\, L_b = \delta_{ab}\, L_a \qquad \text{and} \qquad \sum_{a=1}^N e^{i\Lambda_a}\, L_a\, e^{-i\Lambda_a} = \mathbb{1}\,. \tag{4.7}$$

Then, the above operator U is unitary on $L^2(M, \mathbb{C})$, because

$$UU^* = \sum_{a,b=1}^N e^{i\Lambda_a}\, L_a\, L_b\, e^{-i\Lambda_b} = \sum_{a=1}^N e^{i\Lambda_a}\, L_a\, e^{-i\Lambda_a} = \mathbb{1} \tag{4.8}$$

$$U^*U = \sum_{a,b=1}^{N} L_a \, e^{-i\Lambda_a} \, e^{i\Lambda_b} \, L_b = \sum_{a=1}^{N} L_a = \mathbb{1} \,. \tag{4.9}$$

Now, the vacuum Dirac operator transforms to

$$i\partial\!\!\!/ \to U(i\partial\!\!\!/)U^* = \sum_{a,b=1}^{N} e^{i\Lambda_a} \, L_a \, (i\partial\!\!\!/) \, L_b \, e^{-i\Lambda_b} = \sum_{a=1}^{N} e^{i\Lambda_a} \, L_a \, (i\partial\!\!\!/) \, e^{-i\Lambda_a} \tag{4.10}$$

$$\begin{aligned}
&= \frac{1}{2} \, i\partial\!\!\!/ \left(\sum_{a=1}^{N} e^{i\Lambda_a} \, L_a \, e^{-i\Lambda_a} \right) - \frac{1}{2} \sum_{a=1}^{N} \left[i\partial\!\!\!/, e^{i\Lambda_a} \right] L_a \, e^{-i\Lambda_a} \\
&\quad + \frac{1}{2} \left(\sum_{a=1}^{N} e^{i\Lambda_a} \, L_a \, e^{-i\Lambda_a} \right) i\partial\!\!\!/ + \frac{1}{2} \sum_{a=1}^{N} e^{i\Lambda_a} \, L_a \left[i\partial\!\!\!/, e^{-i\Lambda_a} \right] \\
&= i\partial\!\!\!/ + \frac{1}{2} \sum_{a=1}^{N} e^{i\Lambda_a} \Big((\partial\!\!\!/ \Lambda_a) \, L_a + L_a \, (\partial\!\!\!/ \Lambda_a) \Big) \, e^{-i\Lambda_a}.
\end{aligned} \tag{4.11}$$

(In the last step, we again used the completeness relation in (4.7).) With this transformation, we introduced an interaction operator into the Dirac operator. It can be written as in (1.1) with a nonlocal potential $\mathcal{B} = \mathcal{B}_\Lambda$ given by

$$\mathcal{B}_\Lambda(x,y) = \frac{1}{2} \sum_{a=1}^{N} e^{i\Lambda_a(x)} \Big(\big(\partial\!\!\!/ \Lambda_a(x)\big) \, L_a(y-x) + L_a(y-x) \, \big(\partial\!\!\!/ \Lambda_a\big)(y) \Big) \, e^{-i\Lambda_a(y)} \,. \tag{4.12}$$

Before going on, we explain the connection to the potential in (2.23). We first point out that the form of the potential (2.23) was derived in [16] for *linearized* fields in the Minkowski vacuum. Therefore, in order to make the connection, we also need to linearize (4.12) to obtain

$$\mathcal{B}_\Lambda(x,y) = \frac{1}{2} \sum_{a=1}^{N} \Big(\big(\partial\!\!\!/ \Lambda_a(x)\big) \, L_a(y-x) + L_a(y-x) \, \big(\partial\!\!\!/ \Lambda_a\big)(y) \Big) + \mathcal{O}\big(\Lambda_a^2\big) \,.$$

Next, we need to assume that the phase functions are smooth and do not oscillate on the length scale $\ell_{\min}$ of the nonlocality, i.e., if

$$\big| D^p \Lambda_a \big| \lesssim \frac{1}{\ell_\Lambda^p} \, |\Lambda_a| \qquad \text{with} \qquad \ell_\Lambda \gg \ell_{\min} \,. \tag{4.13}$$

Under this condition, we may expand the phase functions in a Taylor series. Indeed, writing

$$\Lambda_a(x) = \Lambda_a\Big(\frac{x+y}{2} - \frac{\xi}{2} \Big) \,, \qquad \Lambda_a(y) = \Lambda_a\Big(\frac{x+y}{2} + \frac{\xi}{2} \Big)$$

with $\xi := y - x$ and expanding in powers of ξ, the potential (4.12) can be written as

$$\mathcal{B}_\Lambda(x,y) = \sum_{a=1}^{N} \sum_{\kappa} (\partial\!\!\!/ \Lambda_{a,\kappa}) \Big(\frac{x+y}{2} \Big) \, L_a^\kappa(y-x) \tag{4.14}$$

with new potentials and kernels

$$\Lambda_{a,\kappa} = \frac{1}{|\kappa|!}\, \partial^\kappa \Lambda_a \qquad \text{and} \qquad L_a^\kappa(y-x) := (y-x)^\kappa\, L_a(y-x)\,,$$

where κ runs over all multi-indices. The first expansion term with $\kappa = 0$ is exactly of the form (2.23) with $A_a = \partial \Lambda_a$. Combining a with κ to a new summation index, also the higher-order terms can be written in the form (2.23).

We finally point out that summing from 1 to N in (4.6) merely is a matter of convenience, which will later facilitate to get a connection between the dynamical potentials and the gauge phases. But at this stage, the number of summands in (4.6) could be chosen arbitrarily. Taking the limit where the number of summands tends to infinity, one can also describe situations where the sum becomes a momentum integral, giving rise to the continuous ansatz

$$(U\psi)(x) := \int \frac{d^4k}{(2\pi)^4} \int d^4y\, e^{i\Lambda(x,p)}\, e^{-ik(x-y)}\, \psi(y)\,,$$

involving a momentum-dependent phase function $\Lambda(x,p)$. This ansatz may be of advantage for explicit computations.

4.3. Dynamical Gauge Potentials Including Holographic Phases

The method for introducing gauge phases introduced in the previous section was too simple for two reasons. First, choosing the operators L_a as projection operators (4.5) is not compatible with the form of the operators L_a in the nonlocal potential $\mathcal{B}$ in Sect. 2.5 (see (2.23) and the discussion of the support of $\hat{L}_a$ shown in Fig. 1). Second, assuming that the images of the summands in (4.6) are orthogonal (4.7) seems too restrictive. In order to improve the situation, we need to drop conditions (4.5) and (4.7). Thus, we consider the operator U of the form (4.6) with general phase functions Λ_a and homogeneous scalar operators (4.4). Then, this operator is no longer unitary on $L^2(M,\mathbb{C})$, because in (4.8) and (4.9) summands with $a \neq b$ do not drop out. In more general terms, we must take into account that the different phase components interact with each other. In what follows, we will explain step by step how this can be done. This section is devoted to the first step, which consists in specifying the form of the dynamical gauge potential in the presence of holographic phases.

The simplest approach is to proceed similarly to (4.10) by introducing the holographic phases into the Dirac operator again by the transformation

$$i\partial\!\!\!/ \to U(i\partial\!\!\!/)U^*\,. \tag{4.15}$$

Let us analyze how a nonlocal potential of the form (2.23) and (2.32) transforms under the operator U of the form (4.6). Including the nonlocal potential in the transformation (4.15), we obtain

$$i\partial\!\!\!/ + \mathcal{B} \to U(i\partial\!\!\!/ + \mathcal{B})U^* = i\partial\!\!\!/ + \mathcal{B}_\Lambda + \mathcal{B}_{\text{dyn}}\,,$$

where $\mathcal{B}_\Lambda$ is again the holographic gauge potential (in the simplest case given again by (4.12)) and

$$\begin{aligned}\mathcal{B}_{\text{dyn}}(x,y) &= \big(U\mathcal{B}U^*\big)(x,y)\\ &= \sum_{a,b,c=1}^{N} \int_M d^4z_1 \int_M d^4z_2\\ &\quad\times e^{i\Lambda_a(x)}\, L_a(z_1-x)\, \not{A}_c\Big(\frac{z_1+z_2}{2}\Big)\, L_c(z_2-z_1)\, L_b(y-z_2)\, e^{-i\Lambda_b(y)}\,. \end{aligned} \tag{4.16}$$

This consideration shows that the dynamical potentials naturally contain different phase factors on the left and on the right. Moreover, the momentum change of the operator sandwiched between these phase factors is determined by the potential A_c. The potential (4.16) has the disadvantage that it is somewhat complicated. Therefore, we now generalize it using a compact notation. First, introducing in position space the notation

$$\big(\not{A} \triangleright L\big)(x,y) := \not{A}\Big(\frac{x+y}{2}\Big)\, L(y-x)$$

(and likewise in momentum space by taking the Fourier transform of this expression), the integral operator corresponding to the nonlocal potential (4.16) can be written as

$$\mathcal{B}_{\text{dyn}} = \sum_{a,b,c=1}^{N} e^{i\Lambda_a}\, L_a\, \big(\not{A}_c \triangleright L_c\big)\, L_b\, e^{-i\Lambda_b}\,. \tag{4.17}$$

In order to further simplify the notation, we take the ansatz

$$\mathcal{B}_{\text{dyn}} = \sum_{a,b,c=1}^{N} e^{i\Lambda_a}\, \big(\not{A}_c \triangleright L^c_{a,b}\big)\, e^{-i\Lambda_b}\,, \tag{4.18}$$

which includes potentials A_c as well as homogeneous operators L^c_{ab} with $a,b,c \in \{1,\ldots,N\}$. Clearly, the potential (4.18) should again be symmetric, meaning that

$$\overline{A^j_c(z)} = A^j_c(z) \qquad \text{and} \qquad \overline{L^c_{a,b}(\xi)} = L^c_{b,a}(-\xi)\,.$$

We conclude this section with a discussion of the new ansatz (4.18). We begin by pointing out that it is very general and covers a large class of nonlocal potentials. The following constructions will not depend on how precisely the potential $\mathcal{B}_{\text{dyn}}$ is chosen.

In order to related the new ansatz (4.18) to (4.17), one can choose $L^c_{b,a}$ as the product $L_a L_b L_c$. However, this does not quite give back (4.17), because in general

$$\not{A}_c \triangleright \big(L_a\, L_c\, L_b\big) \neq L_a\, \big(\not{A}_c \triangleright L_c\big)\, L_b\,. \tag{4.19}$$

Nevertheless, the mathematical structure of these expressions is quite similar, so much so that their difference will be of no relevance for what follows. Indeed, all our considerations and computations apply also to (4.17), in which case one can take the right side of (4.19) as the definition of the operation $\not{A}_c \triangleright L^c_{a,b}$.

Another class of potentials which seems sensible is to choose

$$L^c_{a,b} = d(c,a,b)\;\; L_a\, L_b$$

with real coefficients $d(c,a,b)$. This choice has the desirable property that for large momenta (i.e., if the frequency p in Fig. 1 is chosen as $p^0 \ll -\omega_{\min}$) the product $L_a L_b$ vanishes unless $a = b$, so that the potential simplifies considerably.

We finally note that, in the case without holographic phases, our general ansatz (4.18) reduces to

$$\mathcal{B}_{\rm dyn} = \sum_{c=1}^N \not{A}_c \triangleright L_c \qquad \text{with} \qquad L_c := \sum_{a,b=1}^N L^c_{a,b} \,,$$

giving back (2.23). With this in mind, we can regard (4.18) as a natural generalization of (2.23), obtained by suitably inserting the holographic phases.

4.4. Realizing the Canonical Commutation Relations

We now define the field operators in analogy to (3.35) by

$$\hat{\mathcal{B}}^j_q = \sum_{a,b,c=1}^N \hat{\mathcal{B}}^{j,c}_{a,b,q} \qquad \text{with} \qquad \hat{\mathcal{B}}^{j,c}_{a,b,q} := e^{i\Lambda_a} \left(A^j_{c,q} \triangleright L^c_{a,b}\right) e^{-i\Lambda_b} \,, \tag{4.20}$$

where $A^j_{c,q}$ is the plane-wave component, i.e.,

$$A^j_{c,q}(x) := \hat{A}^j_c(q)\, e^{-iqx} \,. \tag{4.21}$$

When taking products of these operators,

$$\hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{d,e,q'} = e^{i\Lambda_a} \left(A^j_{c,q} \triangleright L^c_{a,b}\right) e^{-i\Lambda_b + i\Lambda_d} \left(A^k_{f,q'} \triangleright L^f_{d,e}\right) e^{-i\Lambda_e} \,, \tag{4.22}$$

the intermediate phase factor $e^{-i\Lambda_b+i\Lambda_d}$ oscillates rapidly unless $b = d$. Therefore, we expect that the leading contributions are obtained in the case $b = d$, i.e.,

$$\hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{d,e,q'} \approx \delta_{b,d}\, \hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{b,e,q'} \,. \tag{4.23}$$

This approximation will be justified in Sect. 4.5 using a stationary phase analysis, which will also give a precise scaling of the error terms. In order not to distract from the main construction, we here restrict attention to the leading contributions, disregarding the error terms with the notation "≈". We thus obtain

$$\hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{d,e,q'} \approx \delta_{bd}\, e^{i\Lambda_a} \left(A^j_{c,q} \triangleright L^c_{a,b}\right) \left(A^k_{f,q'} \triangleright L^f_{b,e}\right) e^{-i\Lambda_e}$$

and

$$\begin{aligned} &[\hat{\mathcal{B}}^j_q,\, \hat{\mathcal{B}}^k_{q'}] \\ &\approx \sum_{a,b,c,e,f=1}^N e^{i\Lambda_a} \Big(\big(A^j_{c,q} \triangleright L^c_{a,b}\big) \big(A^k_{f,q'} \triangleright L^f_{b,e}\big) - \big(A^k_{c,q'} \triangleright L^c_{a,b}\big) \big(A^j_{f,q} \triangleright L^f_{b,e}\big) \Big) e^{-i\Lambda_e} \\ &= \sum_{a,b,c,e,f=1}^N \hat{A}^j_c(q)\, \hat{A}^k_f(q')\, e^{i\Lambda_a} \Big[(E_q \triangleright L^c_{a,b})\,(E_{q'} \triangleright L^f_{b,e}) - (E_{q'} \triangleright L^f_{a,b})\,(E_q \triangleright L^c_{b,e}) \Big] e^{-i\Lambda_e} \,, \end{aligned}$$

where we used (4.21) and defined E_q as the operator of multiplication by a plane wave of momentum q, i.e.,

$$(E_q \psi)(x) := e^{-iqx}\, \psi(x) \,. \tag{4.24}$$

In order to write these formulas in a more compact form, it is useful to introduce a matrix notation by setting

$$\mathbf{L}^c := (L^c_{a,b})_{a,b=1,\ldots,N} \qquad \text{and} \qquad \mathbf{D} := \left(e^{i\Lambda_a}\, \delta_{ab}\,\right)_{a,b=1,\ldots,N}\,, \tag{4.25}$$

making it possible to write the above commutator more compactly as

$$\big[\hat{\mathcal{B}}^j_q, \hat{\mathcal{B}}^k_{q'}\big] \approx \sum_{a,c,e,f=1}^{N} \hat{A}^j_c(q)\, \hat{A}^k_f(q') \Big(\mathbf{D}\,\big[\,(E_q \triangleright \mathbf{L}^c),\, (E_{q'} \triangleright \mathbf{L}^f)\big]\, \mathbf{D}^{-1}\Big)_{a,e}\,. \tag{4.26}$$

We remark that, at this point, one could expand the operators $E_q \triangleright \mathbf{L}^c$ and $E_{q'} \triangleright \mathbf{L}^f$ in a Taylor series in q using that $q\ell_{\min} \ll 1$. This would lead to a formalism similar to that used in Sect. 3. (See, for example, Lemma 3.5.) We prefer not to work with this expansion, also in order to clarify that such an expansion is not really essential for the CCR.

We are now ready to state and prove the main result of this section. We again assume that the potentials $\hat{A}^j_a(q)$ are Gaussian with mean zero and covariance given by (3.31).

Theorem 4.1. (Realizing the canonical commutation relations) *Considering a nonlocal potential involving holographic phases (4.18), the bosonic field operators defined by (4.20) can be arranged by a suitable choice of the operators $L^c_{a,b}$ in (4.17) and of the covariance in (3.31) to satisfy the CCR in the operator sense, i.e., (in analogy to (3.26))*

$$\ll\big[\hat{\mathcal{B}}^j_q, \hat{\mathcal{B}}^k_{q'}\big] \otimes \hat{\mathcal{B}}^l_r \otimes \hat{\mathcal{B}}^{l'}_{r'}\gg\; \approx (2\pi)^4\, \delta^4(q+q')\, \hat{K}^{jk}(q)\, \mathbb{1} \otimes \ll\hat{\mathcal{B}}^l_r \otimes \hat{\mathcal{B}}^{l'}_{r'}\gg\,, \tag{4.27}$$

where $\hat{K}^{ij}(q)$ is the causal fundamental solution of the Maxwell field. Here, "$\approx$" refers to the approximation (4.23). (The errors of this approximation will be specified in Theorems 4.3 and 5.1.)

In the example of the Feynman gauge, the causal fundamental solution takes the form

$$\hat{K}^{jk}(q) = g^{jk}\, \delta(q^2)\, \epsilon(q^0)\,.$$

Our methods apply similarly also to other choices of gauge. We note that, in view of the nonlocality of the potential $\mathcal{B}$, also the causal fundamental solution in the CCR will be regularized on the length scale $\ell_{\min}$. We do not need to make this explicit here, because the regularization can be absorbed into the error terms.

Proof of Theorem 4.1. Taking the statistical mean of (4.26), we obtain

$$\ll\!\big[\hat{\mathfrak{B}}^j_q,\, \hat{\mathfrak{B}}^k_{q'}\big]\!\gg \\ \approx (2\pi)^4\, \delta^4(q+q') \sum_{a,c,e,f=1}^{N} \hat{h}^{jk}_{c,f}(q) \Big(\mathbf{D}\,\big[\,(E_q \triangleright \mathbf{L}^c),\,(E_{-q} \triangleright \mathbf{L}^f)\big]\,\mathbf{D}^{-1}\Big)_{a,e}\,. \tag{4.28}$$

This formula can be further simplified. We recall that the covariance has the symmetry and positivity properties (3.32) and (3.33). The symmetry property (3.32) implies that the matrix $\hat{h}(q)$ is Hermitian. Therefore, it can be diagonalized, having positive eigenvalues λ_d and corresponding eigenvectors $\phi_d \in \mathbb{C}^{4N}$ with complex components, i.e.,

$$\hat{h}^{jk}_{a,b}(q) = \sum_{d=1}^{4N} \lambda_d(q)\, \big(\phi_d(q)\big)^j_a\, \overline{\big(\phi_d(q)\big)^k_b}\,.$$

Using this formula in (4.28), we obtain

$$\ll\!\big[\hat{\mathfrak{B}}^j_q,\, \hat{\mathfrak{B}}^k_{q'}\big]\!\gg\, \approx (2\pi)^4\, \delta^4(q+q') \sum_{a,e=1}^{N} \sum_{d=1}^{4N} \Big(\mathbf{D}\,\big[\mathbf{M}^j_d(q),\, \mathbf{M}^k_d(-q)\big]\,\mathbf{D}^{-1}\Big)_{a,e}$$

with

$$\mathbf{M}^j_d(q) := \sqrt{\lambda_d(q)}\, \sum_{c=1}^{N} \big(\phi_d(q)\big)^j_c\, (E_q \triangleright \mathbf{L}^c) \tag{4.29}$$

$$\mathbf{M}^k_d(-q) := \sqrt{\lambda_d(q)}\, \sum_{f=1}^{N} \overline{\big(\phi_d(q)\big)^k_f}\, (E_{-q} \triangleright \mathbf{L}^f)\,. \tag{4.30}$$

We note that, using again the symmetry properties (3.32), by suitably ordering the eigenvectors of $\hat{h}(q)$ and $\hat{h}(-q)$ we can arrange that

$$\big(\mathbf{M}^k_d(-q)\big)^* = \mathbf{M}^k_d(q) \tag{4.31}$$

(where the star is the Hermitian adjoint of an $N \times N$-matrix).

In this formulation, the stochastic field and its coupling to the fermions is described by the family of matrices $\mathbf{M}^j_d(q)$ with j a tensor index, $d \in \{1, \ldots, 4N\}$ and $q \in \hat{M}$. Apart from the symmetry condition (4.31), these matrices can be chosen arbitrarily. This brings us into the position to satisfy the CCR, which (in analogy to (3.5) and (3.7)) we write as (4.27). In order to satisfy the CCR in the statistical mean, we must satisfy the relation

$$\sum_{a,e=1}^{N} \sum_{d=1}^{4N} \Big(\mathbf{D}\,\big[\mathbf{M}^j_d(q),\, \mathbf{M}^k_d(-q)\big]\,\mathbf{D}^{-1}\Big)_{a,e} = \hat{K}^{jk}(q)\, \mathbb{1}_{\mathcal{K}}\,. \tag{4.32}$$

Moreover, in order to ensure that the outer parings are negligible (as defined and explained after (3.38)), it suffices to satisfy for all $q \neq -q'$ the conditions

$$\big[\mathbf{M}^j_d(q), \mathbf{M}^k_e(q')\big] \ll \big[\mathbf{M}^j_d(q), \mathbf{M}^k_d(-q)\big]. \tag{4.33}$$

In order to satisfy conditions (4.32) and (4.33), we proceed as follows. Noting that (4.33) involves the commutator of an operator with its own adjoint (see (4.31)), the commutator can be arranged to be large, even if the operators have a small rank. Using this fact, for every d and q one chooses the operators $\mathbf{M}^j_d(q)$ in such a way that (4.32) holds. This can be done in agreement with the concept described in Sect. 2.5 that the operators L_a are projection operators to subspaces of $\mathcal{K}$ (see also Fig. 1). Next, the condition (4.33) can be satisfied by choosing the covariance in such a way that the images of the operators $\mathbf{M}^j_d(q)$ and $\mathbf{M}^k_e(q')$ for $d \neq e$ or $q \neq q'$ are orthogonal, up to errors which can be made arbitrarily small by choosing N large.

We finally point out that the positivity condition (3.33) can be satisfied by adding to $\hat{h}^{jk}_{ab}$ a sufficiently large multiple of the identity matrix $\delta^{jk}\delta_{ab}$. This has no effect on the CCR, because the identity matrix drops out of the commutator. $\square$

Remark 4.2. The above arguments even show that *the CCR arise naturally*, as we now explain. It is easiest to argue directly with the operators $\mathbf{M}^j_d(q)$ introduced in (4.29), which we regard as a family of stochastic operators on the Krein space $(\mathcal{K}, <.|.>)$ (as introduced after (2.34)). It is important to observe that these operators act as matrices on the holographic components labeled by $a \in \{1, \ldots, N\}$ (as one sees from the matrix notation introduced in (4.25)). The typical situation is that a holographic component a is mapped to itself or to other holographic components in a commuting way (like the contributions obtained in (4.17) to zero order in an expansion in $q\ell_{\min}$). The non-commuting character of these operators can typically be regarded as a small correction (obtained either by higher orders in an expansion in $q\ell_{\min}$ or by a coupling of the holographic components described in the next paragraph below). Clearly, in the commutators in (4.32) and (4.33) only these small corrections contribute. The images of these corrections are typically low-dimensional subspaces of $\mathcal{K}$ (as can be understood from the factors $\mathbf{L}^c$ in (4.29); see again Fig. 1). Consequently, in a stochastic description, the images of two such operators will be orthogonal up to errors which tend to zero if N tends to infinity. The commutator on the right of (4.33), on the other hand, is typically *not* small because one operator is the adjoint of the other (see (4.31)), meaning that these two operators are not statistically independent. This explains (4.33). Moreover, it becomes clear why the left of (4.32) is typically *not* small. It remains to explain why the right side in (4.32) has this specific form. Here we can argue with causality and symmetries: Combining (3.31) and (4.32) with the fact that the propagator $\hat{K}^{jk}$ is causal, these conditions mean that these stochastic operators should satisfy causality in the sense that they are uncorrelated for points with spatial separation. Moreover, the precise form of the propagator in (4.32) follows from the assumption of local Lorentz covariance. Clearly, here the notions "causality" and "Lorentz covariance" are to be understood modulo the effects of the regularization, which break Lorentz symmetry on the length scale $\ell_{\min}$.

We finally explain how this qualitative picture can be understood from the constructions in [16]. In this description, the holographic components correspond to null directions along the mass cone (as depicted in Fig. 1). In [16, Section 9.3], the linearized field equations are analyzed using the so-called mass cone expansion. The zero order in this expansion is local in rays along the mass cone, meaning that the linearized field equations decouple into independent equations for each holographic component. This corresponds to the above-mentioned commuting contributions to the operators $\mathbf{M}^j_d(q)$. The higher orders in the mass cone expansion, however, describe a coupling of the holographic components, giving rise to non-commuting contributions to these operators. ♢

4.5. Stationary Phase Analysis of the Error Terms

In (4.23), we used the approximation where in the operator products (4.22) we disregarded the terms involving oscillatory factors $e^{i\Lambda_b - i\Lambda_a}$ with $a \neq b$. We now justify this approximation by working out the corresponding correction terms. To this end, we choose two compactly supported wave functions ψ, ϕ in spacetime and consider the Krein inner product

$$\begin{aligned} &<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{d,e,q'}\, \phi> \\ &\quad = \int_M d^4x \int_M d^4z \int_M d^4y \prec\!\psi(x) \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}(x,z)\, \hat{\mathfrak{B}}^{k,f}_{d,e,q'}(z,y)\, \phi(y)\!\succ . \end{aligned} \tag{4.34}$$

We are interested in the situation where the phase factors oscillate on a length scale which is much smaller than $\ell_{\min}$. In this limiting case, we can compute the operator products in the *stationary phase approximation*, giving the following result:

Theorem 4.3. (Error estimates, method I) *The approximation "≈" introduced in (4.23) holds up to relative errors with the scaling behavior (1.7).*

The remainder of this section is devoted to the proof of this theorem. Before entering the detailed computations, we determine the scaling behavior and the errors of this approximation. We begin with the case $b = d$ where the operator product involves *no* intermediate phase factors, giving rise to the integral

$$\int_M e^{-iq\frac{x+z}{2}}\, L^c_{a,b}(z-x)\, e^{-iq'\frac{z+y}{2}}\, L^f_{b,e}(y-z)\, d^4z \,.$$

In this case, the scaling of the integral is given simply by the value of the integrand times the volume of the integration domain, i.e.,

$$\sim L^c_{a,b}(y-x)\, L^f_{b,e}(y-x)\, \ell_{\min}^4 \,. \tag{4.35}$$

In the case $b \neq d$, however, intermediate phase factors arise, giving integrals of the form

$$\int_M e^{-iq\frac{x+z}{2}}\, L^c_{a,b}(z-x)\, e^{-i\lambda\left(\Lambda_b(z)-\Lambda_d(z)\right)}\, e^{-iq'\frac{z+y}{2}}\, L^f_{d,e}(y-z)\, d^4z \,. \tag{4.36}$$

Now, the main contribution to the integral comes from the critical points of the phase function $\Lambda_b - \Lambda_d$. In order to determine the scalings, let us assume that this function has an extremal point at z_0, i.e.,

$$\frac{\partial}{\partial x^j} \big(\Lambda_b - \Lambda_d \big)(z_0) = 0 \,.$$

Then, the phase is stationary provided that

$$\big| z^j - z_0^j \big| \lesssim \ell_\Lambda \,,$$

where ℓ_Λ is again the length scale on which the phase functions oscillate (4.13). Therefore, the integral scales like

$$\sim L^c_{a,b}(z_0 - x)\, L^f_{d,e}(y - z_0)\, \ell_\Lambda^4 \,. \tag{4.37}$$

Comparing (4.35) with (4.37), one sees that the dephasing decreases the size of the operator products for each stationary point by a scaling factor

$$\Big(\frac{\ell_\Lambda}{\ell_{\min}} \Big)^4 \,. \tag{4.38}$$

The errors of the stationary phase approximation can be quantified by expanding the other functions in (4.36) in a Taylor series about the stationary points, giving scaling factors $\ell_\Lambda / \ell_{\min}$. Therefore, the error of the stationary phase approximation is of the desired order (1.7). We conclude that, under the assumption $\ell_\Lambda \ll \ell_{\min}$, the stationary phase approximation is justified. The dephasing effect can be described by scaling factors (4.38), which are typically very small.

The analysis of the error terms becomes more involved because we also must take into account their combinatorics. In particular, the operator products for $b \neq d$ may still be relevant if the number of stationary points is very large. The remainder of this section is devoted to a detailed analysis of the resulting counting. Keeping in mind that the results of this analysis will not be used later in this paper, it may be skipped by the reader more interested in the improved scaling in Sect. 5.4. Before quantifying the counting, we collect the relevant formulas of the stationary phase approximation.

Lemma 4.4. *Evaluating the inner product (4.34) in the stationary phase approximation gives the following results, up to errors of the order (1.7). In the case $b = d$,*

$$\begin{aligned} <\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> &= c \int d^4z \sum_{k=1}^{K_2} \frac{1}{\sqrt{|\det D^2 \Lambda_a(x_k)|}}\, \frac{1}{\sqrt{|\det D^2 \Lambda_e(y_k)|}} \\ &\quad \times \prec\!\psi(x_k) \,|\, e^{i\Lambda_a(x_k)}\, \hat{A}^j_c(q)\, e^{-iq\frac{x_k+z}{2}}\, L^c_{a,b}(z - x_k) \\ &\quad \times\, \hat{A}^k_f(q')\, e^{-iq'\frac{z+y_k}{2}}\, L^f_{b,e}(y_k - z)\, e^{-i\Lambda_e(y_k)}\, \phi(y_k)\!\succ \end{aligned} \tag{4.39}$$

with a numerical constant c. Here (x_k, y_k) with $k \in \{1, \ldots, K_2\}$ denote the critical points of the phase functions determined by the equations

$$\frac{\partial}{\partial x^j} \bigg(\Lambda_a(x) - \frac{q}{2}\, x + \mathrm{Im} \log \Big(\overline{\psi(x)}\, L^c_{a,b}(z-x) \Big) \bigg) = 0 \tag{4.40}$$

$$\frac{\partial}{\partial y^j} \bigg(- \Lambda_e(y) - \frac{q'}{2}\, y + \mathrm{Im} \log \Big(L^f_{d,e}(y-z)\, \phi(y) \Big) \bigg) = 0\,. \tag{4.41}$$

Likewise, in the case $b \neq d$,

$$\begin{aligned}
&<\!\psi \,|\, \hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{b,e,q'}\, \phi\!> \\
&\quad = c \sum_{k=1}^{K_3} \frac{1}{\sqrt{|\det D^2 \Lambda_a(x_k)|}}\, \frac{1}{\sqrt{|\det D^2 (\Lambda_b - \Lambda_d)(z_k)|}}\, \frac{1}{\sqrt{|\det D^2 \Lambda_e(y_k)|}} \\
&\qquad \times \prec\!\psi(x_k) \,|\, e^{i\Lambda_a(x_k)}\, \hat{A}^j_c(q)\, e^{-iq \frac{x_k+z_k}{2}}\, L^c_{a,b}(z_k - x_k)\, e^{-i\left(\Lambda_b(z_k) - \Lambda_d(z_k)\right)} \\
&\qquad \times\, \hat{A}^k_f(q')\, e^{-iq' \frac{z_k+y_k}{2}}\, L^f_{d,e}(y_k - z_k)\, e^{-i\Lambda_e(y_k)}\, \phi(y_k)\!\succ ,
\end{aligned} \tag{4.42}$$

where the critical points (x_k, z_k, y_k) with $k \in \{1, \ldots, K_3\}$ are determined by the equations

$$\frac{\partial}{\partial x^j} \bigg(\Lambda_a(x) - \frac{q}{2}\, x + \mathrm{Im} \log \Big(\overline{\psi(x)}\, L^c_{a,b}(z-x) \Big) \bigg) = 0 \tag{4.43}$$

$$\frac{\partial}{\partial z^j} \bigg(- \Lambda_b(z) + \Lambda_d(z) - \frac{q+q'}{2}\, z + \mathrm{Im} \log \Big(L^c_{a,b}(z-x)\, L^f_{d,e}(y-z) \Big) \bigg) = 0 \tag{4.44}$$

$$\frac{\partial}{\partial y^j} \bigg(- \Lambda_e(y) - \frac{q'}{2}\, y + \mathrm{Im} \log \Big(L^f_{d,e}(y-z)\, \phi(y) \Big) \bigg) = 0\,. \tag{4.45}$$

Proof. Using (4.20) in (4.34), we obtain

$$\begin{aligned}
&<\!\psi \,|\, \hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{d,e,q'}\, \phi\!> \\
&\quad = \int_M d^4x \int_M d^4z \int_M d^4y \prec\!\psi(x) \,|\, e^{i\Lambda_a(x)}\, \hat{A}^j_c(q)\, e^{-iq \frac{x+z}{2}}\, L^c_{a,b}(z-x) \\
&\qquad \times e^{-i\left(\Lambda_b(z) - \Lambda_d(z)\right)}\, \hat{A}^k_f(q')\, e^{-iq' \frac{z+y}{2}}\, L^f_{d,e}(y-z)\, e^{-i\Lambda_e(y)}\, \phi(y)\!\succ .
\end{aligned}$$

In the case $b = d$, the intermediate phase factor vanishes,

$$\begin{aligned}
&<\!\psi \,|\, \hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{b,e,q'}\, \phi\!> \\
&\quad = \int_M d^4x \int_M d^4z \int_M d^4y \prec\!\psi(x) \,|\, e^{i\Lambda_a(x)}\, \hat{A}^j_c(q)\, e^{-iq \frac{x+z}{2}}\, L^c_{a,b}(z-x) \\
&\qquad \times \hat{A}^k_f(q')\, e^{-iq' \frac{z+y}{2}}\, L^f_{b,e}(y-z)\, e^{-i\Lambda_e(y)}\, \phi(y)\!\succ .
\end{aligned}$$

In the case $b \neq d$, we can also apply the stationary phase approximation to the z-integral. $\square$

We now describe the method for evaluating the sums over the stationary points. Here, we need to take into account that the formulas for the expectation values in Lemma 4.4 involve phase factors, which means that summing over the stationary points may lead to cancellations due to dephasing. This effect can be treated systematically as follows.

In preparation, we specify the number of stationary points further. Clearly, the two equations (4.40) and (4.41) (and similarly the three equations (4.43)–(4.45)) are coupled because both equations involve both x and y. However, they decouple when only the functions Λ_a and Λ_e are taken into account. Therefore, the solutions of these coupled equations can be obtained by first computing the stationary points of Λ_a and Λ_e and then treating the other functions in (4.40) and (4.41) perturbatively (giving again an expansion in powers of $\ell_\Lambda/\ell_{\min}$). We do not need to enter the details of this construction. This procedure shows that the total number of stationary points is given by

$$K_2 \simeq K^2 \qquad \text{and} \qquad K_3 \simeq K^3 \,,$$

where the parameter K denotes the average number of stationary points of the functions Λ_a.

We begin with the formula (4.39) in the case $b = d$. First, we recall that evaluating each stationary point gives a scaling factor (4.38). It is convenient to introduce the abbreviation

$$\alpha := \left(\frac{\ell_\Lambda}{\ell_{\min}} \right)^4 \ll 1 \,. \tag{4.46}$$

Therefore, each summand in (4.39) has the scaling

$$\sim \alpha^2 \left| L^\circ_{\circ,\circ} \right|^2 \ell_{\min}^{12}$$

(where for ease of notation, we leave out the factors ψ, ϕ, $\hat{A}^j_c(q)$ and $\hat{A}^k_f(q')$ and do not specify the arguments and indices of the operator $L^\circ_{\circ,\circ}$). Next, one must keep in mind that the product of the kernels $L^\circ_{\circ,\circ}(z - x_k)$ and $L^\circ_{\circ,\circ}(y_k - z)$ vanishes unless the points x_k and y_k are close on the scale $\ell_{\min}$ in the sense that

$$\left| x^j_k - y^j_k \right| \lesssim \ell_{\min} \,. \tag{4.47}$$

This reduces the number of stationary points to be taken into account by a factor which we denote by

$$\beta \ll 1 \,. \tag{4.48}$$

If the stationary points x_k and y_k are close even on the scale ℓ_Λ, i.e., if

$$\left| x^j_k - y^j_k \right| \lesssim \ell_\Lambda \,, \tag{4.49}$$

then the phase function $\Lambda_a(x_k) - \Lambda_a(y_k)$ is close to zero. As a consequence, there are no relative phases, and no dephasing occurs. Therefore, the total contribution of these stationary points to (4.39) is obtained simply by counting the number of such stationary points. For this counting, we must take into account that the condition (4.49) reduces the number of stationary points by an additional factor α (because, given x_k, the four-volume of the ball in which y_k may lie is reduced by a factor α). Moreover, we must keep in mind that, to leading order in $\ell_\Lambda/\ell_{\min}$, the equations in (4.40) and (4.41) have the same solutions, implying that there are at least K stationary points which satisfy (4.49). Therefore, the total number of stationary points satisfying (4.47) is given by $(1+\alpha\beta K)K$. We thus obtain the scaling

$$<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> \simeq (1+\alpha\beta K)\, K\, \alpha^2 \left|L^\circ_{\circ,\circ}\right|^2 \ell_{\min}^{12} \,. \tag{4.50}$$

Before going on, we point out that the number $\alpha\beta K$ tells us about the average number of stationary points inside a ball of radius ℓ_Λ. In order for the stationary phase approximation to be sensible, the distance of the stationary phases must be much larger than ℓ_Λ. This leads to the condition

$$\alpha\beta K \ll 1\,. \tag{4.51}$$

Therefore, we may leave out the summand $\alpha\beta K$ in (4.50) to obtain

$$<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> \simeq K\alpha^2 \left|L^\circ_{\circ,\circ}\right|^2 \ell_{\min}^{12} \qquad \text{without dephasing}\,. \tag{4.52}$$

If the stationary points do *not* satisfy (4.49), we can treat the factors $e^{i(\Lambda_a(x_k)-\Lambda_a(y_k)}$ stochastically as random phases. Therefore, the amplitude of the sum is obtained by multiplying the amplitude of one summand by the square root of the number of summands. This gives the scaling

$$<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> \simeq \sqrt{\beta K^2}\, \alpha^2 \left|L^\circ_{\circ,\circ}\right|^2 \ell_{\min}^{12} \qquad \text{with dephasing}\,. \tag{4.53}$$

Note that, in view of (4.48), this contribution is much *smaller* than (4.52).

In the case $b \neq d$, we can proceed similarly. Yet, we must keep in mind that also the z-integral is evaluated in the stationary phase approximation. Therefore, each summand in (4.42) has the scaling

$$\sim \alpha^3 \left|L^\circ_{\circ,\circ}\right|^2 \ell_{\min}^{12}.$$

(For ease of notation, we again leave out the factors ψ, ϕ, $\hat{A}^j_c(q)$ and $\hat{A}^k_f(q')$.) If all three points x_k, y_k and z_k are close to the scale ℓ_Λ, we have no dephasing. When counting the number of corresponding stationary points, we must keep in mind that, similar as explained above, there are at least K stationary points satisfying (4.49), and in this case there is also a solution z_k which is close to x_k and y_k on the scale ℓ_Λ. We thus obtain the scaling

$$<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> \simeq (1+\alpha^2\beta^2 K^2)\, K\, \alpha^3 \left|L^\circ_{\circ,\circ}\right|^2 \ell_{\min}^{12} \,.$$

In view of (4.51), we can leave out the summand $\alpha^2\beta^2K^2$, i.e.,

$$<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> \simeq K\alpha^3 \left|L^\circ_{\circ,\circ}\right|^2 \ell_{\min}^{12} \qquad \text{without dephasing}\,. \tag{4.54}$$

If none of the points x_k, y_k and z_k are close to the scale ℓ_Λ, we obtain

$$<\psi \,|\, \hat{\mathfrak{B}}^{j,c}_{a,b,q}\, \hat{\mathfrak{B}}^{k,f}_{b,e,q'}\, \phi> \simeq \sqrt{\beta^2 K^3}\, \alpha^3 \left|L^\circ_{\circ,\circ}\right| \ell_{\min}^{12} \qquad \text{with dephasing}\,. \tag{4.55}$$

Finally, there is an additional contribution if two of the points x_k, y_k and z_k are close to the scale ℓ_Λ, but not the third point. They have the scaling

$$<\!\psi \mid \hat{\mathcal{B}}^{j,c}_{a,b,q}\, \hat{\mathcal{B}}^{k,f}_{b,e,q'}\, \phi\!> \simeq \sqrt{\alpha\beta^2K^3}\, \alpha^3\, \big|L^\circ_{\circ,\circ}\big|^2\, \ell^{12}_{\min}\,.$$

Since this is much smaller than (4.55), we may disregard these contributions in what follows.

The precise scaling behavior of the above stationary phase contributions (4.52), (4.53) and (4.54), (4.55) will be important when working out corrections to the Fock space dynamics (see Section 8). For the purpose of the present paper, it suffices to note that, apart from the constraints in (4.46), (4.48) and (4.51), the parameters α, β and K can be chosen arbitrarily. Therefore, all the stationary phase contributions can be made arbitrarily small by choosing α sufficiently small. In this way, we can arrange that the error terms are indeed of the desired form (1.7).

5. The Dirac Dynamics with Holographic Mixing

5.1. Preparatory Considerations

We now briefly outline the path taken by the authors to arrive at the formulation of the dynamical equations with holographic mixing. Although not quite straightforward, these considerations may nevertheless explain and motivate the subsequent construction of the holographic Green's operator (in Section 5.2). At the beginning of Sect. 4.3, we argued that the conditions (4.5) and (4.7) are too restrictive and must be relaxed. If this is done, the operator U defined by (4.6) is no longer unitary. Introducing the holographic phases again by (4.15) has the disadvantage that, rewriting the Dirac operator similar to (4.11) as the sum $i\partial\!\!\!/ + \mathcal{B}_\Lambda$ of the vacuum Dirac operator and a perturbation, the operator $\mathcal{B}_\Lambda$ involves a differential operator whose coefficients are convolution operators including the holographic phases. As a consequence, the operator $\mathcal{B}_\Lambda$ can no longer be regarded as a small perturbation, and it cannot be treated perturbatively in a straightforward way. In order to bypass this difficulty, one can try to construct a *unitary* operator V on the Krein space $(\mathcal{K}, <\!.|.\!>)$ (introduced after (2.34)) which again involves the holographic phases. Then, the procedure

$$\begin{aligned} i\partial\!\!\!/ \rightarrow V(i\partial\!\!\!/)V^* &= \frac{1}{2}\,(i\partial\!\!\!/)\,VV^* - \frac{1}{2}\,\big[i\partial\!\!\!/, V\big]\,V^* + \frac{1}{2}\,VV^*\,(i\partial\!\!\!/) + \frac{1}{2}\,V\,\big[i\partial\!\!\!/, V^*\big] \\ &= i\partial\!\!\!/ + \mathcal{B}_\Lambda \end{aligned}$$

with

$$\mathcal{B}_\Lambda := -\frac{1}{2}\,\big[i\partial\!\!\!/, V\big]\,V^* + \frac{1}{2}\,V\,\big[i\partial\!\!\!/, V^*\big]\,, \tag{5.1}$$

gives rise to a potential $\mathcal{B}_\Lambda$ which, due to its commutator structure, is no longer a differential operator. We refer to V as the *unitary holographic mixing operator*. The obvious question is how to choose or construct this operator.

One strategy is to begin with a potential of the form (4.14); in the simplest case

$$\mathcal{B}_\Lambda(x,y) = \sum_{a=1}^{N} (\not{\partial}\Lambda_a)\Big(\frac{x+y}{2}\Big)\, L_a(y-x)\,, \tag{5.2}$$

and to construct a unitary operator V with the property that

$$V(i\not{\partial})V^* = i\not{\partial} + \mathcal{B}_\Lambda\,.$$

A perturbative treatment in powers of $\mathcal{B}_\Lambda$ is not suitable because the oscillatory phase factors must be treated non-perturbatively. But, as is worked out in detail in Appendix A, a resummation technique makes it possible to compute V in an

$$\text{expansion in powers of } \ell_{\min}/\ell_\Lambda\,. \tag{5.3}$$

Here, all phase factors $e^{\pm i\Lambda_a}$ are taken into account non-perturbatively. However, there remains the limitation that the length scale ℓ_Λ on which the phase factors oscillate (see (4.13)) must be much larger than $\ell_{\min}$. At present, it is not clear whether this condition is satisfied in physically realistic situations. For this reason and for the sake of mathematical generality, it seems preferable not to rely on an expansion of the form (5.3).

In order to avoid the expansion (5.3), one can construct the operator V non-perturbatively, for example, by taking the polar decomposition of the operator U. To this end, one merely needs to assume that

$$U \;:\; L^2(M,\mathbb{C}) \rightarrow L^2(M,\mathbb{C}) \quad \text{is bounded and has a bounded inverse.}$$

(Note that, for technical simplicity, we restrict attention to mappings which act trivially on the spinor indices, making it possible to work in the Hilbert space $L^2(M,\mathbb{C})$ rather than the Krein space $(\mathcal{K}, <.|.>)$ introduced after (2.34).) Then, the operator V defined by

$$V := U\,\big(U^*U\big)^{-\frac{1}{2}} \;:\; L^2(M,\mathbb{C}) \rightarrow L^2(M,\mathbb{C}) \tag{5.4}$$

is unitary, as desired. This abstract method has the disadvantage that the operator V can no longer be given explicitly. Yet, as illustrated in Appendix B, it can still be computed using microlocal techniques, but again under the assumption $\ell_\Lambda \gg \ell_{\min}$.

Above we outlined various methods for defining and analyzing a unitary holographic mixing operator V. It turns out that these methods are not yet sufficient, because there is the general problem that the fermionic Green's operator is incompatible with the bosonic commutation relations. We now explain the basic difficulty, which will be resolved by the constructions in Sect. 5.2. We write the Dirac equation as

$$0 = \big(i\not{\partial} + \mathcal{B} - m\big)\tilde{\psi} = V(i\not{\partial} - m)V^*\tilde{\psi} + \mathcal{B}_{\text{dyn}}\tilde{\psi}\,, \tag{5.5}$$

where V is a unitary operator on the Krein space $(\mathcal{K}, <.|.>)$ (for example, the operator (5.4)), and $\mathcal{B}_{\text{dyn}}$ is the dynamical gauge potential (for example, of the form (4.18)). Considering the operator $V(i\not{\partial}-m)V^*\tilde{\psi}$ as the unperturbed operator, the corresponding Green's operator is given explicitly by $V s_m V^*$, where s_m is a Dirac Green's operator (like, for example, the retarded Green's operator as given in momentum space by (3.2)). Now, we can treat $\mathcal{B}_{\text{dyn}}$ perturbatively, giving rise to the perturbation series

$$\begin{aligned}\tilde{\psi} &= \sum_{n=0}^{\infty} \Big(-V s_m V^* \mathcal{B}_{\text{dyn}} \Big)^n V\,\psi \\ &= V\psi - \big(V s_m\, V^*\big)\mathcal{B}_{\text{dyn}}\, V\psi + \big(V s_m V^*\big)\mathcal{B}_{\text{dyn}}\big(V s_m\, V^*\big)\mathcal{B}_{\text{dyn}}\, V\psi - \cdots\,.\end{aligned} \tag{5.6}$$

This perturbation series describes the dynamics of the system completely. Therefore, it should correspond to the Dyson series in perturbative QFT.

It is useful to consider $V s_m\, V^*$ as the effective Green's operator; we also refer to it as the *holographic Green's operator* and denote it by

$$s^{\text{hol}} := V s_m\, V^*\,.$$

In Sects. 4.4 and 4.5 we showed that the potential $\mathcal{B}_{\text{dyn}}$ can be regarded as being composed of the bosonic field operators, which act on the Krein space $(\mathcal{K}, <.|.>)$ and satisfy the CCR. In order to make use of these commutation relations in (5.6), the bosonic field operators must also be commuted with the holographic Green's operators. In the setting of QFT, this is unproblematic, because the bosonic and fermionic field operators act on different spaces (the bosonic and fermionic Fock space, respectively), and therefore they commute by construction. In our setting, however, all field operators act on the same space: the Krein space $(\mathcal{K}, <.|.>)$ formed of one-particle wave functions in Minkowski space. Therefore, the bosonic field operators will in general *not* commute with the holographic Green's operator. Arranging such commutation relations poses non-trivial conditions for the choice of the unitary holographic mixing operator.

In order to satisfy these conditions, we need more freedom for the choice of the holographic Green's operator. A first step in this direction is to drop the condition that V be unitary. Thus, we let V be an operator on the Krein space $(\mathcal{K}, <.|.>)$ which is merely assumed to be invertible, i.e.,

$$V, V^{-1} \,:\, \mathcal{K} \to \mathcal{K}.$$

(More precisely, we assume that V and V^{-1} are continuous with respect to the Krein topology.) Writing the Dirac equation again in the form (5.5), the only difference compared to the above setting is that we need to carefully distinguish between V^* and V^{-1} as well as between V and $(V^{-1})^*$. In particular, the holographic Green's operator now becomes

$$s^{\text{hol}} := \big(V^{-1}\big)^*\, s_m\, V^{-1}\,. \tag{5.7}$$

Likewise, in the perturbation expansion (5.6) we need to replace the factors $V s_m V^*$ by this holographic Green's operator.

The next and final step is to observe that it is not crucial for our constructions that the Dirac operator and the Green's operator are of the form (5.5) and (5.7). Instead, we can work more generally with a pair of operators $\mathcal{D}^{\text{hol}}$

and s^{hol} with the following properties:

$$\big(\mathcal{D}^{\text{hol}}\big)^* = \mathcal{D}^{\text{hol}} \quad \text{(symmetry of the Dirac operator)} \tag{5.8}$$

$$\big(\mathcal{D}^{\text{hol}} - m\big)\, s^{\text{hol}} = \mathbb{1} \quad \text{(defining property of the Green's operator)}\,. \tag{5.9}$$

Here the symmetry is needed in order to obtain a conserved inner product which generalizes (2.26)–(2.28). More precisely, this inner product can be constructed as follows. In case that the operator $\mathcal{D}^{\text{hol}} - m$ can be written as an integral operator with a locally integrable kernel, denoting this kernel by $Q(x,y)$ one can work directly with (2.20). In case that the integral kernel is a singular distribution on the diagonal $x = y$, one needs to first regularize, then compute the surface layer integral (2.20) and finally remove the regularization. For example, for the singular kernel $Q(x,y) = (i\partial\!\!\!/_x - m)\, \delta^4(y-x)$, this procedure gives back, up to a prefactor, the usual Dirac scalar product (2.26).[1]

Of course, working in the general setting (5.8) and (5.9), one faces the computational difficulty that the solutions of the holographic Dirac equation $(\mathcal{D}^{\text{hol}} - m)\psi = 0$ can no longer be obtained from standard Dirac solutions by a perturbative treatment. From the conceptual point of view, however, the holographic Dirac equation provides a clean and simple way of describing the dynamics including holographic phases. This will be worked out in more detail in the next section.

5.2. The Holographic Green's and Dirac Operators

We now return to our ansatz for the dynamical potential involving holographic phases (4.18). We have the situation in mind that the considered wave function ψ can be decomposed into a sum of holographic components

$$\psi = \sum_{a=1}^N e^{i\Lambda_a}\, \psi_a \tag{5.10}$$

with wave functions $\psi_1, \ldots, \psi_N$ whose momenta are much smaller than $1/\ell_\Lambda$. Then, using again the approximation (4.23) of disregarding dephased contributions, the potential (4.18) acts componentwise, i.e.,

$$\mathcal{B}_{\text{dyn}}\psi \approx \sum_{a,b,c=1}^N e^{i\Lambda_a}\, \big(A\!\!\!/_c \triangleright L^c_{a,b}\big)\, \psi_b\,.$$

[1] More precisely, choosing a test function $\eta \in C^\infty_0(M, \mathbb{R}^+_0)$ which is symmetric (i.e., $\eta(-\xi) = \eta(\xi)$ for all $\xi \in M$), setting

$$Q^\delta(x,y) := \frac{1}{\delta^4}\, (i\partial\!\!\!/_x - m)\, \eta\Big(\frac{x-y}{\delta}\Big)$$

and choosing Ω as the past of the Cauchy surface at time t, a direct computation using integration by parts yields for any smooth ψ and ϕ

$$\begin{aligned} &-2i \lim_{\delta \searrow 0} \bigg(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x) \int_\Omega d\rho(y) \bigg) \prec\!\psi(x) \,|\, Q^\delta(x,y)\, \phi(y)\!\succ_x \\ &= c \int \prec\!\psi \,|\, \gamma^0\, \phi\!\succ_{(t,\vec{x})}\, d^3x \end{aligned}$$

with c a nonzero constant.

This corresponds to the concept that, similar to classical gauge phases, the holographic phase factors cannot be observed but merely describe a mixing of internal degrees of freedom. Following this concept, the Green's operator should also act on each holographic component irrespective of the holographic phases, i.e.,

$$s^{\mathrm{hol}}\psi \approx \sum_{a=1}^{N} e^{i\Lambda_a} \left(s_m \psi_a\right), \tag{5.11}$$

where s_m is again the vacuum Dirac Green's operator. Following up on the explanation in the last paragraph of Remark 4.2, we note that the relation (5.11) is also motivated by the constructions in [16] which show that the Green's operator s^{hol} should act on each holographic component like the vacuum Dirac Green's operator.

A simple and direct way of implementing the relation (5.11) is to define the holographic Green's operator by

$$s^{\mathrm{hol}} := \sum_{a=1}^{N} e^{i\Lambda_a}\; s_m\, e^{-i\Lambda_a}\;. \tag{5.12}$$

This Green's operator can be arranged to be *advanced* or *retarded* simply by choosing $s_m = s_m^\vee$ or $s_m = s_m^\wedge$, respectively. It acts on wave functions by

$$s^{\mathrm{hol}}\,\psi = \sum_{a=1}^{N} e^{i\Lambda_a} \left(s_m \psi_a\right) + \sum_{a,b=1,\ldots,N \text{ with } a\neq b} e^{i\Lambda_a}\; s_m \left(e^{-i\Lambda_a + i\Lambda_b}\; \psi_b\right).$$

The summands for $a \neq b$ are small for the following reason. The phase factor typically gives rise to high-frequency wave functions. Due to the decay of the Green's operator for large momenta, these high-frequency contributions become very small when acted upon by the Green's operator. More precisely, they give rise to errors of the order

$$\times\Big(1 + \mathcal{O}\big(\ell_{\mathrm{macro}}/\ell_\lambda\big)\Big)\,.$$

This can be made precise with a stationary phase analysis as worked out in Sect. 4.5. We conclude that the holographic Green's operator (5.12) indeed satisfies the condition (5.11).

Having introduced the holographic Green's operator, we can arrange (5.9) by defining the *holographic Dirac operator* by

$$\mathcal{D}^{\mathrm{hol}} := \left(s^{\mathrm{hol}}\right)^{-1} + m.$$

(Here, we need to assume that the inverse exists.) Using (5.12), this operator is again symmetric (5.8). This symmetry property is needed in order for current conservation to hold, and it is indeed the only assumption needed for this conservation law. In order to see this, we write the holographic Dirac equation as

$$\int_M Q^{\mathrm{hol}}(x,y)\,\psi(y)\,d^4y = 0\,,$$

where Q^{hol} is a distributional kernel which is symmetric in the sense that

$$Q^{\mathrm{hol}}(x,y)^* = Q^{\mathrm{hol}}(y,x)$$

(where the star is the adjoint with respect to the spin inner product). Then, the commutator inner product (2.26)–(2.28) can be generalized and written again in the form (2.20) with the kernel Q^{dyn} replaced by $Q^{\mathrm{hol}}(x,y)$ (as was explained in more detail after (5.9)). Although this conservation law will not enter the subsequent constructions, it is important conceptually because it yields a scalar product on the physical wave functions even in the interacting regime.

5.3. The Holographic Perturbation Expansion

We now insert the dynamical gauge potentials (4.18) into the holographic Dirac equation,

$$\big(\mathcal{D}^{\mathrm{hol}} + \mathcal{B}_{\mathrm{dyn}} - m\big)\,\psi = 0\,.$$

The corresponding perturbation expansion becomes

$$\tilde{\psi} = \sum_{n=0}^\infty \big(- s^{\mathrm{hol}}\, \mathcal{B}_{\mathrm{dyn}} \big)^n\, \psi\,. \tag{5.13}$$

Using (5.10), (4.18) and (5.12), we can proceed as in Sect. 4.4 and work with the approximation (4.23), leaving out all terms involving rapidly oscillating phase factors. We again use a matrix notation in the components (4.25). Moreover, we use a vector notation for the fermionic waves

$$|\psi> := (\psi)_{a=1,\ldots,N}$$

and similarly with the tilde. We thus obtain the *holographic perturbation expansion*

$$|\tilde{\psi}> \approx \sum_{n=0}^\infty \Big[-s_m \sum_{b=1}^N \not{A}_b \triangleright \mathbf{L}^b \Big]^n\, |\psi>\,. \tag{5.14}$$

The error of this approximation will be worked out in detail in the next section.

5.4. Improved Scaling of the Error Terms

We now adapt the stationary phase analysis of the error terms in Sect. 4.5 to the perturbation expansion (5.13). Our main finding will be that, as a consequence of the intermediate Green's operators, the scaling behavior of the error terms improves considerably:

Theorem 5.1. (Error estimates, method II) *The approximation "≈" introduced in (4.23) holds up to relative errors with the scaling behavior (1.8).*

The major improvement compared to (1.7) is that it becomes unnecessary to assume that $\ell_\Lambda \ll \ell_{\min}$. Instead, we can work with the opposite limiting case $\ell_{\min} \ll \ell_\Lambda$, which also makes it possible to apply the perturbative and microlocal techniques outlined in Appendices A and B.

The remainder of this section is devoted to the proof of this theorem. Our starting point is the expectation value of an operator product of the general form

$$<\psi \,|\, e^{i\Phi_0}\, K_1\, e^{i\Phi_1}\, K_2\, e^{i\Phi_2} \cdots e^{i\Phi_{p-1}}\, K_p\, e^{i\Phi_p}\, \phi> , \tag{5.15}$$

where the factors K_ℓ are homogeneous operators (either s_m or $L^\circ_{\circ,\circ}$), and $e^{i\Phi_\ell}$ are products of two adjacent phase factors, which we write as

$$\Phi_\ell := -\Lambda_{a_\ell} + \Lambda_{b_\ell} \tag{5.16}$$

with parameters $a_\ell, b_\ell \in \{0, \ldots, p\}$. We write the operator products as multiple integrals in position space,

$$\begin{aligned}
&<\psi \,|\, e^{i\Phi_0}\, K_1\, e^{i\Phi_1}\, K_2 \cdots e^{i\Phi_{p-1}}\, K_p\, e^{i\Phi_p}\, \phi> \\
&\quad = \int_M d^4x_0 \cdots \int_M d^4x_p \prec\!\psi(x_0) \,|\, e^{i\Phi_0(x_0)}\, K_1(x_1 - x_0)\, e^{i\Phi_1(x_1)}\, K_2(x_2 - x_1) \\
&\qquad \cdots e^{i\Phi_{p-1}(x_{p-1})}\, K_p(x_p - x_{p-1})\, e^{i\Phi_p(x_p)}\, \phi(x_p)\!\succ .
\end{aligned}$$

We now transform to the new integration variables

$$x_0 \qquad \text{and} \qquad \xi_\ell := x_\ell - x_{\ell-1} \text{ with } \ell \in \{1, \ldots, p\}$$

to obtain

$$\begin{aligned}
&<\psi \,|\, e^{i\Phi_0}\, K_1\, e^{i\Phi_1}\, K_2 \cdots e^{i\Phi_{p-1}}\, K_p\, e^{i\Phi_p}\, \phi> \\
&\quad = \int_M d^4x_0 \int_M d^4\xi_1 \cdots \int_M d^4\xi_p \prec\!\psi(x_0) \,|\, e^{i\Phi_0(x_0)}\, K_1(\xi_1)\, e^{i\Phi_1(x_1)}\, K_2(\xi_2) \\
&\qquad\qquad \cdots e^{i\Phi_{p-1}(x_{p-1})}\, K_p(\xi_p)\, e^{i\Phi_p(x_p)}\, \phi(x_p)\!\succ ,
\end{aligned}$$

where the variables $x_1, \ldots, x_p$ are expressed in terms of x_0 and $\xi_1, \ldots, \xi_p$ by

$$x_\ell = x_0 + \xi_1 + \cdots + \xi_\ell .$$

We now carry out the integrals successively. The x_0-integral involves all the phase factors and the wave functions ψ and ϕ but not the kernels K_ℓ. (Note that these kernels do not depend on x_0.) Therefore, we get a contribution only if all the phases cancel. Using the form of the phase factors (5.16), we conclude that there must be a permutation $\sigma \in S_p$ with

$$a_\ell = b_{\sigma(\ell)} \qquad \text{for all } \ell = 0, \ldots, p . \tag{5.17}$$

The error of this approximation can again be specified with the stationary phase analysis. Similar to (4.38) the error terms involve scaling factors

$$\Big(\frac{\ell_\Lambda}{\ell_{\text{macro}}} \Big)^4 .$$

We next carry out the integrals over ξ_p, ξ_{p-1}, $\ldots$, ξ_1 in this order. The integral over ξ_p takes the form

$$\int_M K_p(\xi_p)\, e^{i\Phi_p(\xi_p + \zeta_p)}\, \phi(\xi_p + \zeta_p)\, d^4\xi_p$$

with $\zeta_p := x_0 + \xi_1 + \cdots + \xi_{p-1}$. We need to consider the cases $K_p = s_m$ and $K_p = L^\circ_{\circ,\circ}$ separately. In the first case, we integrate the Dirac matrices by parts,

$$\begin{aligned}
&\int_M s^\wedge_m(\xi)\, e^{i\Phi_p(\xi+\zeta)}\, \phi(\xi+\zeta)\, d^4\xi \\
&\qquad = \int_M S^\wedge_{m^2}(\xi)\, (i\partial\!\!\!/_\xi + m)\big(e^{i\Phi_p(\xi+\zeta)}\, \phi(\xi+\zeta)\big)\, d^4\xi\,,
\end{aligned}$$

where the factors $S^\wedge_{m^2}$ are the Klein–Gordon Green's operators, which in position space take the form (see, for example, [11, eq. (2.2.6)]),

$$\begin{aligned}
S^\wedge_{m^2}(x,y) &= -\frac{1}{2\pi}\, \delta\big((y-x)^2\big)\, \Theta\big(x^0-y^0\big) \\
&\qquad + \frac{m^2}{4\pi}\, \frac{J_1\Big(\sqrt{m^2\,(y-x)^2}\Big)}{\sqrt{m^2\,(y-x)^2}}\, \Theta\big((y-x)^2\big)\, \Theta\big(x^0-y^0\big)\,.
\end{aligned}$$

Being smooth, the Bessel function can be treated as in Sect. 4.5. In the contribution involving the δ-distribution, however, we can carry out the time integral to obtain

$$\begin{aligned}
&\int_M S^\wedge_0(\xi)\, (i\partial\!\!\!/_\xi + m)\, e^{i\Phi_p(\xi+\zeta)}\, \phi(\xi+\zeta)\, d^4\xi \\
&\qquad = -\frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{1}{2\,|\vec{\xi}|}\, (i\partial\!\!\!/_\xi + m)\big(e^{i\Phi_p(\xi+\zeta)}\, \phi(\xi+\zeta)\big)\Big|_{\xi^0=-|\vec{\xi}|}\, d^3\xi\,.
\end{aligned}$$

Now, we can again use the stationary phase method, but in three dimensions. Every stationary point gives rise to a scaling factor

$$\Big(\frac{\ell_\Lambda}{\ell_{\text{macro}}}\Big)^3\,.$$

Moreover there may be a stationary point near the origin. Here the factor $1/|\vec{\xi}|$ effectively reduces the dimension by one (as is obvious in polar coordinates). We thus get the scaling factor

$$\Big(\frac{\ell_\Lambda}{\ell_{\text{macro}}}\Big)^2\,.$$

But the stationary points near the origin are combinatorially suppressed because there is at most one of them. One should also keep in mind that the derivative $i\partial\!\!\!/_\xi$ gives scaling factor ℓ_Λ^{-1}. Nevertheless, at least one factor ℓ_Λ remains, giving rise to error terms of the form

$$\times \Big(1 + \mathcal{O}\Big(\frac{\ell_\Lambda}{\ell_{\text{macro}}}\Big)\Big)\,. \tag{5.18}$$

We next consider the case $K_p = L^\circ_{\circ,\circ}$. In this case, an explicit analysis is possible in two complementary limiting cases. If $\ell_\Lambda \ll \ell_{\min}$, we can proceed with the stationary phase analysis of Section 4.5. In the opposite limiting case $\ell_{\min} \ll \ell_\Lambda$, on the other hand, we may expand the phase factor in a Taylor series. As a consequence, we do not get dephasing effects. In general (in particular, if we are in none of the above limiting cases), no computational tools are available for analyzing the integral explicitly.

The other integrals over $\xi_{p-1}, \ldots, \xi_1$ can be carried out similarly. The result on the phase factors $e^{i\Phi_\ell}$ with phases Φ_ℓ given by (5.16) can be described systematically as follows. We first note that, writing the perturbation expansion (5.13) in the form (5.15), the factors K_ℓ are alternating Green's operators and factors $L^\circ_{\circ,\circ}$. More precisely,

$$K_\ell = \begin{cases} s_m & \text{if } \ell \text{ is odd} \\ L^\circ_{\circ,\circ} & \text{if } \ell \text{ is even}\,. \end{cases}$$

Moreover, the total number p of such factors is odd. As explained above, the stationary phase method applies to all factors s_m, i.e., to all the integrals

$$\xi_1, \xi_3, \ldots, \xi_p$$

(with odd subscripts). Carrying out the last integral over ξ_p gives (as shown in detail above) the condition that the phase factor Φ_p is trivial,

$$e^{i\Phi_p} = 1\,. \tag{5.19}$$

According to (5.16), this means that

$$a_p = b_p\,. \tag{5.20}$$

Next, integrating over ξ_{p-2} gives the condition that

$$e^{i\Phi_{p-2}+i\Phi_{p-1}+i\Phi_p} = 1\,.$$

Using (5.19) and again (5.16), we conclude that

$$\Lambda_{a_{p-2}} - \Lambda_{b_{p-2}} + \Lambda_{a_{p-1}} - \Lambda_{b_{p-1}} = 0\,.$$

This condition can be satisfied in two ways, namely

$$\begin{cases} \text{either } a_{p-2} = b_{p-2}, & a_{p-1} = b_{p-1} \\ \quad\text{or} \quad a_{p-2} = b_{p-1}, & a_{p-1} = b_{p-2}\,. \end{cases}$$

Proceeding inductively, one obtains for each ℓ the alternative conditions

$$\begin{cases} \text{either } a_{\ell-2} = b_{\ell-2}, & a_{\ell-1} = b_{\ell-1} \\ \quad\text{or} \quad a_{\ell-2} = b_{\ell-1}, & a_{\ell-1} = b_{\ell-2} \end{cases} \qquad \text{for all } \ell = 3, 5, \ldots, p\,. \tag{5.21}$$

Finally, using all these relations, the condition (5.17) obtained by carrying out the x_0-integral simplifies to the condition

$$a_0 = b_0\,. \tag{5.22}$$

We point out that, at this stage, we applied the stationary phase method only to the Green's operators s_m. Therefore, the error terms scale like (5.18). We also note that, so far, the parameter $\ell_{\min}$ has not yet come into play.

We next write out parts of the operator products in the perturbation expansion (5.13) in detail

$$\sum_{a,\ldots,f} \cdots e^{i\Lambda_a}\, s_m\, e^{-i\Lambda_a}\, e^{i\Lambda_e} \left(\not{A}_c \triangleright L^c_{e,f}\right) e^{-i\Lambda_f}\, e^{i\Lambda_b}\, s_m\, e^{-i\Lambda_b} \cdots\,.$$

In the first case in (5.21), this simplifies to

$$\sum_{a,b,c} \cdots e^{i\Lambda_a}\, s_m \left(\not{A}_c \triangleright L^c_{a,b} \right) s_m\, e^{-i\Lambda_b} \cdots \,. \tag{5.23}$$

This gives back precisely the approximation used in (5.14). The interesting point is that now we also have the second case in (5.21), which gives

$$\sum_{a,b,c} \cdots e^{i\Lambda_a}\, s_m\, e^{-i(\Lambda_a-\Lambda_b)} \left(\not{A}_c \triangleright L^c_{b,b} \right) e^{-i(\Lambda_b-\Lambda_a)}\, s_m\, e^{-i\Lambda_a} \cdots \,.$$

Due to the additional phase factors $e^{\pm i(\Lambda_a-\Lambda_b)}$, this operator product can in general not be simplified further. But it simplifies in the limiting case $\ell_\Lambda \gg \ell_{\min}$. Namely, in this limiting case, the operator $L^c_{a,b}$ acts on the phase factors like a multiplication operator, so that we obtain

$$\sum_{a,b,c} \cdots e^{i\Lambda_a}\, s_m \left(\not{A}_c \triangleright L^c_{b,b} \right) s_m\, e^{-i\Lambda_a} \cdots \left(1 + \mathcal{O}\Big(\frac{\ell_\Lambda}{\ell_{\min}} \Big) \right) . \tag{5.24}$$

Writing both contributions (5.23) and (5.24) in the matrix notation (4.25), they can be combined to

$$\sum_c \cdots s_m \left(\not{A}_c \triangleright \mathbf{L}^c + \not{A}_c \triangleright \mathrm{Tr}\big(\mathbf{L}^c\big)\, \mathbb{1} \right) s_m \cdots \left(1 + \mathcal{O}\Big(\frac{\ell_\Lambda}{\ell_{\min}} \Big) \right) . \tag{5.25}$$

After these preparations, we are ready to analyze the commutation relations, following the procedure in Sect. 4.4. The difference of (5.25) compared to the formalism in Sect. 4.4 is the additional term involving the trace in (5.25). This difference can be implemented by modifying the matrices $\mathbf{L}^c$ by a multiple of the identity matrix, as is made precise by the replacement rule

$$\mathbf{L}^c \to \mathbf{L}^c + \mathrm{Tr}\big(\mathbf{L}^c\big)\, \mathbb{1} \,.$$

Since the identity matrix drops out of all commutators, all the methods and results in Sect. 4.4 remain valid. In particular, we can satisfy the conditions (4.32) and (4.33). In this way, we have realized the CCR (4.27), up to errors of the form (1.8). We finally remark that the combinatorics of the stationary points could be analyzed similarly to what described in Sect. 4.5.

6. Description with Bosonic and Fermionic Fock Spaces

6.1. Separating the Fermionic and Bosonic Degrees of Freedom

In Sect. 4 it was shown that decomposing the nonlocal perturbation operator $\mathcal{B}_{\rm dyn}$ of the form (4.18) according to their momentum transfer (4.20), the resulting operators $\hat{\mathcal{B}}^j_q$ can be identified with the bosonic field operators which satisfy the commutation relations (4.27). Clearly, starting from the bosonic field operators, we obtain the dynamical gauge potential by integrating over q,

$$\mathcal{B}_{\rm dyn} = \int \frac{d^4q}{(2\pi)^4}\, \gamma_j \hat{\mathcal{B}}^j_q \,.$$

The goal of this section is to study the bosonic field operators in the perturbation expansion (5.13). Before entering the construction, we point out that the operator $\mathcal{B}_{\mathrm{dyn}}$ in (5.13) has two tasks. On the one hand, in view of the CCR (4.27), it plays the role of the bosonic field operators. On the other hand, in the operator products in (5.13) it changes the momenta of the fermionic wave functions. In order to get into the standard formalism, we need to disentangle these two tasks such as to obtain separate operators acting on the bosonic and fermionic degrees of freedom.

Using again the approximation of dropping all rapidly oscillating contributions (5.14), we obtain

$$|\tilde{\psi}> \approx \sum_{n=0}^{\infty}(-1)^n \int_{\mathbb{R}^4} \frac{d^4q_1}{(2\pi)^4} \cdots \int_{\mathbb{R}^4} \frac{d^4q_n}{(2\pi)^4}$$
$$\times \sum_{b_1,\ldots,b_n=1}^{N} s_m \left(\hat{\not{A}}_{b_1}(q_1)\, E_{q_1} \triangleright \mathbf{L}^{b_1}\right) \cdots s_m \left(\hat{\not{A}}_{b_n}(q_n)\, E_{q_n} \triangleright \mathbf{L}^{b_n}\right) |\psi> ,$$

where E_q again denotes the operator of multiplication by a plane wave (4.24). Here, the symbol "$\approx$" again means that we allow for error terms of the form (1.7) or (1.8). Working with this approximation gives us the conservation of energy-momentum within the perturbation expansion. This fact is crucial for the following construction steps. For clarity, we write out the momenta at each operator,

$$|\tilde{\psi}(x)> \approx \sum_{n=0}^{\infty}(-1)^n \int_{\mathbb{R}^4} \frac{d^4q_1}{(2\pi)^4} \cdots \int_{\mathbb{R}^4} \frac{d^4q_n}{(2\pi)^4} \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}$$
$$\times \sum_{b_1,\ldots,b_n=1}^{N} e^{-i(q_1+\cdots+q_n+p)\,x}\, \hat{\psi}_{b_1,\ldots,b_n}(q_1,\ldots,q_n,p)$$

with

$$\hat{\psi}_{b_1,\ldots,b_n}(q_1,\ldots,q_n,p) := s_m\big|_{p+q_1+\cdots+q_n} \hat{\not{A}}_{b_1}(q_1)\, (E_{q_1} \triangleright \mathbf{L}^{b_1})\big|_{p+q_2+\cdots+q_n}$$
$$\times \cdots s_m\big|_{p+q_n} \hat{\not{A}}_{b_n}(q_n)\, (E_{q_n} \triangleright \mathbf{L}^{b_n})\big|_p\, |\hat{\psi}(p)> .$$

Using that the kernels $E_{q_\ell} \triangleright \mathbf{L}^{b_\ell}(p + q_\ell + \cdots + q_n, p + q_{\ell+1} + \cdots + q_n)$ are complex-valued and thus act trivially on the spinors, we may factor them out to obtain

$$\hat{\psi}_{b_1,\ldots,b_n}(q_1,\ldots,q_n,p) = s_m\big|_{p+q_1+\cdots+q_n} \gamma^{i_1}\, s_m\big|_{p+q_2+\cdots+q_n} \gamma^{i_2} \cdots s_m\big|_{p+q_n} \gamma^{i_n} \tag{6.1}$$
$$\times \left(\hat{A}^{i_1}_{b_1}(q_1)\, E_{q_1} \triangleright \mathbf{L}^{b_1}\right)\big|_{p+q_2+\cdots+q_n} \cdots \left(\hat{A}^{i_n}_{b_n}(q_n)\, E_{q_n} \triangleright \mathbf{L}^{b_n}\right)\big|_p\, |\hat{\psi}(p)> . \tag{6.2}$$

Now, (6.1) corresponds to the usual fermionic tree diagram involving the Dirac matrices and the momentum transfer q_ℓ at each leg. The factors in (6.2), on the other hand, can be written as a bosonic operator product acting on $|\psi>$,

$$\left(\hat{A}^{i_1}_{b_1}(q_1)\, E_{q_1} \triangleright \mathbf{L}^{b_1}\right) \left(\hat{A}^{i_2}_{b_2}(q_2)\, E_{q_2} \triangleright \mathbf{L}^{b_2}\right) \cdots \left(\hat{A}^{i_n}_{b_n}(q_n)\, E_{q_n} \triangleright \mathbf{L}^{b_n}\right) |\psi> . \tag{6.3}$$

Now, we can commute the bosonic operators using the commutation relations as worked out in Sect. 4.4, keeping the fermionic diagram (6.1) and the momenta therein fixed. In this way, we have separated the bosonic and fermionic degrees of freedom in the desired way.

The factorization (6.1) and (6.2) also give a direct understanding of how the fermionic and bosonic degrees of freedom are encoded in the causal fermion system: The fermionic degrees of freedom describe the low-frequency behavior of the wave function (modulo gauge phases); this is why in (6.1) the momenta are shifted by $q_1, \ldots, q_n$. The high-frequency behavior of the wave function, however, encodes the bosonic degrees of freedom, as is obvious from the fact that the bosonic operators in (6.2) act on the components $a \in \{1, \ldots, N\}$ of the holographic components.

In order to formulate these considerations in a mathematically clean way, it is helpful to let the operators in the above perturbation series act on a pair of functions, one being a standard spinorial wave function, and the other consisting of N complex-valued functions in spacetime,

$$|\psi \otimes \phi> \in \mathcal{K} \otimes C^\infty(M, \mathbb{C})^N \, . \tag{6.4}$$

We then replace the operator $\hat{\not{A}}_b(q)\, E_q \triangleright \mathbf{L}^b$ by

$$\gamma_j E_q \otimes \big(\hat{A}^j_b(q)\, E_q \triangleright \mathbf{L}^b \big) \, . \tag{6.5}$$

Now, the Dirac matrices appear in the first factor, acting on the spinorial wave function. The second factor, on the other hand, contains a matrix acting on $\mathbb{C}^N$. Note that the operator E_q describing the momentum shift acts both on the first and the second factor. This has the effect that if ψ and ϕ have the same momentum, then this property is preserved, in agreement with the momenta in (6.1) and (6.2). But now the operator (6.5) may act more generally on a tensor product $\psi \otimes \phi$ where ψ and ϕ have different momenta. This is needed once the bosonic operators in (6.3) are commuted.

Finally, we simplify the notation by setting

$$\mathcal{A}^j(q) := \sum_{b=1}^N \hat{A}^j_b(q)\, E_q \triangleright \mathbf{L}^b$$
$$\gamma_j E \otimes \mathcal{A}^j := \int \frac{d^4q}{(2\pi)^4}\, \gamma_j E_q \otimes \mathcal{A}^j(q) \, .$$

Using the above results, the operators $\mathcal{A}^j$ can be identified with the usual bosonic field operators.

6.2. Incorporating the Fermionic Fock Space

In order to get into the setting of QFT, the remaining task is to also rewrite the spinorial wave functions and Green's operators in terms of field operators acting on a fermionic Fock space. Moreover, we want to formulate the whole dynamics in the standard formalism of QED. To this end, we again drop all rapidly oscillating contributions and begin with the perturbation series (5.14), again allowing for error terms of the form (1.7) or (1.8). Acting with the Dirac operator and working again with pairs of functions (6.4), we obtain the Dirac equation

$$(i\partial\!\!\!/ \otimes \mathbb{1} + \gamma_j E \otimes \mathcal{A}^j - m)\,|\psi \otimes \phi\!> = 0\,.$$

This Dirac equation can be written in the Hamiltonian form as

$$\big(i\partial_t - H\big)|\psi \otimes \phi\!> = 0\,, \tag{6.6}$$

where the Hamiltonian is of the form

$$H = H_0 + V$$

with the standard Dirac Hamiltonian $H_0 := -i\gamma^0 \vec{\gamma}\vec{\nabla} + m\gamma^0$ and the nonlocal operator V given by

$$V = -\gamma^0\left(\gamma_j E \otimes \mathcal{A}^j\right) = -\big(\alpha^j E \otimes \mathcal{A}_j\big)\,,$$

where we used the standard notation $\alpha^j := \gamma^0\gamma^j$ (see, for example, [48]). The operator H acts on the fermionic component as a purely spatial operator. (Indeed, H_0 is the usual Dirac Hamiltonian, whereas $\gamma_j E$ is a multiplication operator in position space.) Therefore, we can consider the Hamiltonian equation (6.6) as an evolution equation on the function space

$$|\psi \otimes \phi\!> \in L^2(\mathbb{R}^3, \mathbb{C}^4) \otimes C^\infty(M, \mathbb{C})^N\,. \tag{6.7}$$

Next, we want to rewrite the dynamics in terms of fermionic Fock spaces. For clarity, we proceed in two steps. The first step is to rewrite (6.6) equivalently as the evolution equation for a one-fermion state on the Fock space. In the second step we will move on to many-fermion states. In the first step we can proceed in the standard way. Working for convenience in position space, we introduce the fermionic field operators Ψ and $\Psi^\dagger$ by their equal-time anti-commutation relations

$$\big\{\Psi^\alpha(\vec{x}), \Psi^\beta(\vec{y})^\dagger\big\} = \delta^{\alpha\beta}\,\delta^3\big(\vec{x} - \vec{y}\big) \qquad \text{and} \qquad \big\{\Psi^\alpha(\vec{x}), \Psi^\beta(\vec{y})\big\} = 0\,. \tag{6.8}$$

Now, we replace the function $|\psi \otimes \phi\!>$ in (6.7) by the Fock vector

$$|\Psi\!> := \Psi^\dagger(\psi)\,|0\!> \otimes\, \phi\,,$$

where $|0\!>$ denotes the fermionic vacuum vector and

$$\Psi^\dagger(\psi) := \int_{\mathbb{R}^3} \Psi^\dagger(\vec{x})\,\psi(\vec{x})\,d^3x\,.$$

Then, the dynamics of $|\Psi\!>$ is described by the Schrödinger equation

$$i\partial_t|\Psi\!> = H|\Psi\!> \tag{6.9}$$

with the "second-quantized" Hamiltonian computed by

$$\begin{aligned} H &= \int_{\mathbb{R}^3} d^3x \left(\Psi(t,\vec{x})^\dagger H_0 \Psi(t,\vec{x}) + \int \frac{d^4q}{(2\pi)^4} \Psi(t,\vec{x})^\dagger\,\alpha_j\,e^{-iqx}\Psi(t,\vec{x}) \otimes \mathcal{A}^j(q)\right) \\ &= \int_{\mathbb{R}^3} \Psi(t,\vec{x})^\dagger\,(H_0 - \alpha^j \mathcal{A}_j(t,\vec{x}))\Psi(t,\vec{x})\,d^3x\,. \end{aligned}$$

This is the standard Hamiltonian of QED.

Before moving on to many-fermion states, it is convenient to describe the fermions instead of equal-time anti-commutation relations by covariant CAR for general spacetime points $x = (t, \vec{x})$ and $y = (t', \vec{y})$ as

$$\begin{aligned} \{\Psi^\alpha(x), \Psi^\beta(y)^\dagger\} &= 2\pi \left(k_m(x,y)\, \gamma^0\right)^\alpha_\beta \\ \{\Psi^\alpha(x), \Psi^\beta(y)\} &= 0 = \{\Psi^\alpha(x)^\dagger, \Psi^\beta(y)^\dagger\}\,, \end{aligned} \tag{6.10}$$

where k_m is the causal fundamental solution of the vacuum Dirac equation (defined as the difference of the advanced and retarded Dirac Green's operators divided by $2\pi i$). This has the advantage that the bosonic CCR and the fermionic CAR are described in a similar formalism (cf. (4.27) and (6.10)). In more physical terms, the free fields are described in the Heisenberg picture (where the field operators are time-dependent and satisfy the homogeneous field equations). Accordingly, in the Hamiltonian we need to omit the free Hamiltonians (of both the bosonic and fermionic fields), giving rise to the Hamiltonian of QED in the interaction picture

$$H = -\int_{\mathbb{R}^3} \Psi(t,\vec{x})^\dagger\, \alpha^j \mathcal{A}_j(t,x)\, \Psi(t,\vec{x})\, d^3x\,. \tag{6.11}$$

We now come to the question of how a many-fermion state is to be described in our setting. This question is rather subtle, as we now explain in detail. We saw that the nonlocal Dirac equation (1.1) can be written in the Hamiltonian form (6.6). In this description, the microscopic structure of the physical wave function is described by the function $\phi \in C^\infty(M, \mathbb{C})^N$, whereas its macroscopic form is described by a usual Dirac wave function ψ. Clearly, the causal fermion system is composed of many physical wave functions. The naive idea is to describe each of them as in (6.6) and to take their anti-symmetrized product, i.e.,

$$|\Psi\!> := |\psi_1 \otimes \phi_1\!> \wedge \cdots \wedge |\psi_L \otimes \phi_L\!> \tag{6.12}$$

(Here, L denotes the total number of physical wave functions, including those describing the Dirac sea; here for ease in presentation we choose L to be finite.) Considering the anti-symmetrized product (6.12) as the quantum state of the system is problematic, as we now explain. Suppose we want to let a fermionic field operator $\Psi^\dagger(\psi)$ act on $|\Psi\!>$. Then, ψ is a fermionic wave function. The naive ansatz

$$\Psi^\dagger(\psi)\, |\Psi\!> = |\psi\!> \;\wedge\; |\psi_1 \otimes \phi_1\!> \wedge \cdots \wedge |\psi_L \otimes \phi_L\!>$$

does not work, because the first factor does not have a bosonic component, so that anti-symmetrization is ill-defined. Building in a bosonic component ϕ makes mathematical sense,

$$\Psi^\dagger(\psi)\, |\Psi\!> = |\psi \otimes \phi\!> \wedge |\psi_1 \otimes \phi_1\!> \wedge \cdots \wedge |\psi_L \otimes \phi_L\!>\,, \tag{6.13}$$

but this raises the question how ϕ is to be chosen. As the vacuum? Or as the physical bosonic state? This is unclear, showing that also the ansatz (6.13) is not a sensible concept. More generally, it is essential to anti-symmetrize only the fermionic degrees of freedom. The resulting totally anti-symmetric fermionic wave function should then be tensored with the bosonic degrees of freedom.

In order to implement this concept in our construction, we need a simplifying assumption, which we refer to as the *bosonic Fock space approximation*. It states that the bosonic field operator can be approximated by a matrix acting on the index $a = 1, \ldots, N$ (similar as in the matrix notation (4.25)) which does not change the spacetime dependence of the bosonic component ϕ. Thus, using again the notation (6.7), we assume that

$$\mathcal{A}_j \, |\psi \otimes \phi> = |\psi \otimes \mathcal{A}_j \phi> \qquad \text{and} \qquad (\mathcal{A}_j \phi)^a(x) = \sum_{b=1}^{N} \left(\mathcal{A}_j\right)^a_b \, \phi^b(x) \,. \tag{6.14}$$

Under this assumption, the spacetime dependence of the bosonic component is irrelevant and can be dropped. Then, the bosonic component is described only by the index $a \in \{1, \ldots, N\}$, making it possible to write

$$|\psi \otimes \phi >= \sum_{a=1}^{N} |\psi^a \otimes \phi_a> \,,$$

where $(\phi_a)_{a=1,\ldots,N}$ can be identified with the canonical basis of $\mathbb{C}^N$. Now, we define the Fock state by

$$|\Psi> := \sum_{a=1}^{N} \psi_1^a \wedge \cdots \wedge \psi_L^a \;\otimes\; \phi_a \,. \tag{6.15}$$

We point out that the fermionic wave functions all carry the same bosonic index a. Therefore, as desired, the anti-symmetrization takes into account only the fermionic degrees of freedom. The fermionic field operator act on this Fock state in an obvious way. The action of the bosonic field operators, on the other hand, is defined by

$$\mathcal{A}_j |\Psi> := \sum_{a,b=1}^{N} \psi_1^a \wedge \cdots \wedge \psi_L^a \;\otimes\; (\mathcal{A}_j)^b_a \, \phi_b \,.$$

This definition has the advantage that the commutation relations are, respected, because, for example,

$$\mathcal{A}_k \mathcal{A}_j |\Psi> := \sum_{a,b,c=1}^{N} \psi_1^a \wedge \cdots \wedge \psi_L^a \;\otimes\; (\mathcal{A}_k)^c_b (\mathcal{A}_j)^b_a \, \phi_c \,.$$

Having specified how the fermionic and bosonic field operators act on the state $|\Psi>$ defined by (6.15), it is also clear how the Hamiltonian H of QED given by (6.11) acts on $|\Psi>$.

Likewise, the Hamiltonian acts on the Fock space according to

$$H|\Psi> := \sum_{a,b=1}^{N} \Big(\big(\alpha^j E \psi_1^a\big) \wedge \cdots \wedge \psi_L^a + \cdots + \psi_1^a \wedge \cdots \wedge \big(\alpha^j E \psi_L^a\big) \Big) \;\otimes\; (\mathcal{A}_j)^b_a \, \phi_b \,.$$

We note that, as a consequence of the bosonic Fock space approximation, the product structure in (6.15) is not preserved by the time evolution. Instead, the state is described by a general vector in $\mathcal{F}^{\rm f} \otimes \mathcal{F}^{\rm b}$, where $\mathcal{F}^{\rm b}$ effectively describes all the "bosonic" factors $\phi_\ell \in \mathbb{C}^N$ in (6.15). In this way, we also obtain fermionic entanglement.

With the Schrödinger equation (6.9) with the Hamiltonian (6.11), we have rewritten the dynamics in the notions of standard QED. For clarity and completeness, we now outline the construction steps needed for the perturbative description and point out how the standard constructions need to be modified in each step.

(i) First, one needs to choose the quantum state ω at an initial time. One way of doing so is to specify ω as a positive linear functional on the algebra generated by the bosonic and fermionic field operators. More concretely, one can make use of the fact that we introduced the field operators as acting on tensor products of wave functions, giving a natural representation on a Fock space $\mathcal{F}$. Then, ω can be represented by a density operator on $\mathcal{F}$. In the simplest case of a pure state, ω can be written as $\omega = |\Psi\!>\!<\!\Psi|$ with a unit vector $\Psi \in \mathcal{F}$.
When describing a scattering process, the situation is particularly simple because we may assume that the system is non-interacting initially. Then, the causal fermion system is described by solutions of the vacuum Dirac equation. In non-technical terms, the initial fermionic state is chosen as the Hartree–Fock state formed of all physical wave functions, i.e.,

$$\Psi^{\rm f} := \psi_1 \wedge \cdots \wedge \psi_f$$

where $(\psi_\ell)_{\ell=1,\ldots,f}$ is an orthonormal basis of $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$. This equation becomes mathematically meaningful in the infinite-dimensional case $f = \infty$ by choosing ω as a regularized quasi-free Hadamard state.

(ii) The Schrödinger equation (6.9) can be solved perturbatively with the Dyson series. The only modification in our setting is that the bosonic field operators $\mathcal{A}^j$ contained in the potential V are nonlocal in space and time on the scale $\ell_{\min}$. The reader interested in more details on this Dyson series is referred to [29].

(iii) Next, one needs to specify the initial state $|\Psi(t_0)\!>$. The easiest choice is the vacuum $|0\!>$ determined as usual by the condition that it vanishes when acted upon by the annihilation operators. In preparation, for the bosonic operators, one implements the usual frequency splitting by writing $\mathcal{A}^j = a^j + (a^j)^\dagger$, where the operators

$$a_j := \int \frac{d^4q}{(2\pi)^4}\, \Theta(q^0)\, \hat{\mathcal{A}}^j_q \qquad \text{and} \qquad (a^j)^\dagger := \int \frac{d^4q}{(2\pi)^4}\, \Theta(-q^0)\, \hat{\mathcal{A}}^j_q$$

are composed of all positive and negative frequencies, respectively. (Note that the dagger can be regarded as the adjoint on the Krein space $(\mathcal{K}, <\!.|.\!>)$.) Then, the vacuum is characterized by the conditions

$$\Psi^\alpha(x)\, |0\!> = 0 = a^j(x)\, |0\!> \qquad \text{for all } \alpha, j \text{ and } x \in M\,. \tag{6.16}$$

Clearly, instead of the vacuum one can also choose an initial state $|\Psi(t_0)>$ involving particles and anti-particles and/or photons. This state is obtained by acting on the vacuum $|0>$ with a finite number of creation operators.

(iv) Expectation values of the state $|\Psi(t)>$ at a later time t can be computed to every order in perturbation theory by expanding the time-ordered exponential in the Dyson series and using the canonical commutation and anti-commutation relations together with the relations (6.16) characterizing the vacuum. This gives the standard *Feynman diagrams*, involving both fermionic and bosonic loops. The time ordering in the Dyson series has the effect that the diagrams are formed of the *Feynman propagators*, characterized by the property that positive frequencies propagate to the future, whereas negative frequencies propagate to the past.

(v) In view of the ultraviolet cutoff on the scale ε, all obtained Feynman diagrams are well defined and finite. Using the standard renormalization techniques, one could analyze the limiting case $\varepsilon \searrow 0$ when the ultraviolet regularization is removed.

We conclude with a remark concerning *Haag's theorem* [39,41,46], which states that that an interacting quantum field theory in the interaction picture cannot be unitarily equivalent to a free field theory. The critical reader may wonder whether our description is meaningful in view of this no-go theorem. The point is that, in our setting, the interaction potential is nonlocal, thus breaking Poincaré invariance. Therefore, we are not abiding by the hypotheses of this theorem and, more generally, of the local quantum physics approach.

7. The Dynamics of the Quantum State

In [20,21,23], an interacting causal fermion system was described at any time t by a quantum state ω^t. We now explain how the above description of the dynamics on Fock spaces can be related to the dynamics of this quantum state. Before beginning, we point out that the construction of the quantum state applies in a more general setting than the constructions in the present paper. In particular, the separation of the fermionic and bosonic degrees of freedom in Sect. 6.1 is possible only under additional assumptions and involves error terms. In case that these assumptions are not satisfied, the constructions in [21,23] still give a dynamics of a quantum state. However, it will differ from the standard unitary time evolution on Fock spaces.

We recall that, in the setting of algebraic QFT, a *quantum state* is defined as a positive linear mapping from the algebra of observables $\mathcal{A}$ (defined from linear fields in the vacuum) to the complex numbers. For convenience we choose a state which can be written as the expectation value of a density operator σ^t acting on the Fock space $\mathcal{F}$,

$$\omega^t(A) = \mathrm{tr}_{\mathcal{F}}(\sigma^t A) \qquad \text{for all } A \in \mathcal{A}\,.$$

The density operator, in turn, can be written in *bra/ket* notation as

$$\sigma^t = \sum_{a \in \mathfrak{S}} c_a \,\Big|\sum_{\alpha \in \mathfrak{T}_a} \Psi^{\mathcal{F}}_{a\alpha}\Big\rangle\Big\langle \sum_{\beta \in \mathfrak{T}_a} \Psi^{\mathcal{F}}_{a\beta}\Big|$$

with $\mathfrak{T}_a$ an index set depending on a. In the case of a *pure state*, the set $\mathfrak{S}$ has only one element, so that the density operator can be written as

$$\sigma^t = \Big|\sum_{\alpha=1}^{L} \Psi^{\mathcal{F}}_{\alpha}\Big\rangle\Big\langle \sum_{\beta=1}^{L} \Psi^{\mathcal{F}}_{\beta}\Big| \,,$$

where L denotes the number of Fock components.

The methods employed for the construction of the quantum state in [21, 23] are quite different from the techniques used here. The general idea behind the construction is to "compare" the interacting measure $\tilde{\rho}$ at a given time t with the vacuum measure. The freedom in identifying the Hilbert spaces of the interacting and the vacuum causal fermion systems is described by a unitary operator $\mathcal{U} \in \mathcal{G}$, where $\mathcal{G}$ is a compact group $\mathcal{G} \simeq \mathrm{U}(N)$ of unitary operators on the Hilbert space $\mathcal{H}$. In order to obtain information independent of this unitary freedom, we integrate over the unitary group. More precisely, the *refined partition function* is defined by

$$Z^t_V\big(\alpha, \beta, \tilde{\rho}\big) = \fint_{\mathcal{G}} d\mu_{\mathcal{G}}\big(\mathcal{U}_<\big) \fint_{\mathcal{G}} d\mu_{\mathcal{G}}\big(\mathcal{U}_>\big)\; e^{\alpha N \mathcal{T}^t_V \left(\tilde{\rho}, T_{\mathcal{U}_<, \mathcal{U}_>} \rho\right)} \,,$$

where V is the spacetime region under consideration, and $\mathcal{T}_V$ is a certain nonlinear surface layer integral. The *refined state* is introduced by

$$\omega^t_V\big(\cdots\big) = \frac{1}{Z^t_V\big(\alpha, \beta, \tilde{\rho}\big)} \fint_{\mathcal{G}} d\mu_{\mathcal{G}}\big(\mathcal{U}_<\big) \fint_{\mathcal{G}} d\mu_{\mathcal{G}}\big(\mathcal{U}_>\big)\; e^{\alpha N \mathcal{T}^t_V \left(\tilde{\rho}, T_{\mathcal{U}_<, \mathcal{U}_>} \rho\right)} \big(\cdots\big) \,, \tag{7.1}$$

where the dots on the left stand for an operator in the observable algebra $\mathcal{A}$ formed of the linearized fields in the vacuum spacetime, whereas the dots on the right stand for suitable surface layer integrals which again involve the linearized fields in the vacuum.

This construction yields a quantum space at any time t. However, it is a shortcoming of this approach that it does not tell us about the time evolution of the quantum state. This shortcoming is overcome with the constructions of the present paper, which make it possible to rewrite the dynamics in terms of a unitary time evolution on Fock spaces. Nevertheless, the present results do not make the constructions in [21, 23] obsolete. On the contrary, it seems that these constructions complement those in the present paper, giving a different perspective and thereby giving a more complete picture of the quantum dynamics.

On a technical level, at present it is unclear how to relate the refined state (7.1) to the constructions in the present paper. But on a conceptual level, the constructions fit together nicely, as we now briefly explain. The unitary operators $\mathcal{U}_<$ and $\mathcal{U}_>$ in (7.1) may compensate for the dephasing of the wave functions in the interacting spacetime relative to the vacuum. More concretely, with specific choices of these unitary operators one may "detect" the different summands of the dephasing operator U in (4.6) and similarly of the unitary holographic mixing operator V (as defined by (5.4)). Likewise, the entanglement as found in [23, Section 6] by evaluating the low-energy saddle points should be related to and arise dynamically as a consequence of the bosonic Fock space approximation introduced in (6.14). Making these relations and correspondences mathematically precise seems an interesting topic for future research.

8. Remarks and Outlook

In this paper, QED was derived in a well-defined limiting case from a more fundamental physical theory. *Corrections* to this limiting case are physical predictions by the theory of causal fermion systems to be tested in future experiments. Therefore, it is of utmost importance to work out and analyze different correction terms, which at present are subsumed in the error terms (1.7) or (1.8). We now discuss a few of these corrections, leaving the detailed analysis as future research projects.

Before beginning, we point out that the description of the dynamics by a nonlocal operator (2.23) in the Dirac equation involves many unknowns. It leaves us with the freedom in choosing the operators L_a as well as the covariances of the corresponding Gaussian stochastic fields B_a. This freedom was partially removed by the requirement of satisfying the linearized field equations and of getting covariant commutation relations on scales much larger than $\ell_{\min}$. But even then, we are left with a many freedoms in choosing the unknowns. This situation also made our analysis rather involved, because we had to carefully analyze to what extent these unknowns affect our end results. In order to improve the situation, one needs to derive stronger structural results which would impose further constraints on the form of the nonlocal potential. As long as such stronger results are not available, one should consider (2.23) as a suitable ansatz for describing the microscopic structure of spacetime. Our lack of knowledge on the microstructure of spacetime is reflected in the freedom in choosing the operators L_a and the stochastic fields B_a. Taking the statistical mean amounts to taking averages of the unknown microstructure. Our analysis shows that, doing so in a suitable way, does indeed give rise to an effective description by bosonic quantum fields. The fact that this limiting case comes with error terms gives the hope that, by quantifying the errors, one can get corrections to standard quantum field theory. But clearly, these corrections will again depend on the above-mentioned unknowns. The goal is to find corrections which, in our stochastic description, take a simple and quantifiable form. One strategy is to consider the high-precision measurements of QFT (like the Lamb shift or the spectrum of the hydrogen atom) and try to work out corresponding corrections. Alternatively, one could consider cosmological phenomena. Going into details seems premature and should better be the objective of future research.

Clearly, the first question in this context concerns the values of the parameters ℓ_Λ and $\ell_{\min}$. Thus, how large are the corrections to be expected? At present, not much is known about these parameters, except that $\ell_{\min}$ lies between the Planck scale and the length scale of macroscopic physics (1.3). This lack of knowledge is also the reason why we kept our analysis as general as possible by allowing for error terms of the form (1.7) or (1.8). Thinking of the phase factors $e^{i\Lambda_a}$ as being generated by the stochastic potentials A_a, the assumption $\ell_\Lambda \gg \ell_{\min}$ seems favorable. But at present, even this is not clear. The number N of the stochastic potentials is determined by $\ell_{\min}$ (see (1.5)). Another unknown is the strength of the stochastic fields as described by the covariances (3.31). Clearly, these covariances are partly determined by the CCR.

A promising approach to reduce the number of unknowns is to make use of the fact that the parameter $\ell_{\min}$ as well as the covariances also arise in the collapse phenomena as worked out in the non-relativistic limit in [29]. In this case, the nonlocality of the potential in time gives rise to non-symmetric potentials in the non-relativistic limit, which are crucial for the reduction of the wave function in the measurement process. These connections give the hope that the effective parameters in the resulting collapse model will give information on $\ell_{\min}$ and the strength of the stochastic fields, which will in turn pose constraints or even give predictions for the corrections to QED.

In addition to the corrections presently contained in our error terms, there are also corrections to QED of a different nature due to the *nonlocality in time* and the *nonlinearity* of the EL equations. These effects are closely related to the collapse phenomena as analyzed and discussed in [29]. Finally, corrections arise when going beyond the *bosonic Fock space approximation* introduced in (6.14). The exact dynamics cannot be formulated on a tensor product of a bosonic and a fermionic Fock space. Working out what this means quantitatively is a challenging problem for the future.

Apart from these corrections to QED, it is also an important open problem to extend our methods to *non-abelian gauge fields*. Generalizing the methods to the gravitational field would lead to a mathematically precise formulation of *quantum gravity*.

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Declarations

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Appendix A. A Perturbation Expansion in Powers of $\ell_{\min}/\ell_\Lambda$

This section provides a detailed construction of the unitary holographic mixing operator. Our starting point is the Dirac operator including holographic gauge potentials of the form (5.2), i.e.,

$$\mathcal{D} := i\partial\!\!\!/ + \mathcal{B} \qquad \text{with} \qquad \mathcal{B}(x,y) = \sum_{a=1}^{L} (\partial\!\!\!/ \Lambda_a)\Big(\frac{x+y}{2}\Big)\, L_a(y-x)\,. \tag{A.1}$$

Our goal is to construct a unitary operator on the Krein space,

$$V \,:\, \mathcal{K} \to \mathcal{K} \qquad \text{unitary on } (\mathcal{K}, <.|.>)\,, \tag{A.2}$$

which has the property that the Green's operator $\tilde{s}_m$ of as defined by

$$(\mathcal{D} - m)\, \tilde{s}_m = \mathbb{1}$$

has the representation

$$\tilde{s}_m = V s_m V^* \tag{A.3}$$

(where the star denotes the adjoint with respect to the Krein inner product $<.|.>$ in spacetime).

Our method is to apply perturbative techniques as developed in [6, 17, 35] (see also [11, Section 2.1] or [25, Chapter 18]). Before stating our result, we must introduce the spectral decomposition of the Dirac operator in the vacuum. Since we want to compute the Green's operators, we must take into account all the eigenspaces of the Dirac operator, including the imaginary eigenvalues (as first done in [5, Section 2.1]).

Definition A.1. For $a \in \mathbb{R}$ and $m \in \mathbb{R} \cup i\mathbb{R}$, $m \neq 0$, we define the following tempered distributions in momentum space,

$$P_a(k) := \delta(k^2 - a) \tag{A.4}$$

$$p_m(k) := \frac{|m|}{m}\, (\not{k} + m)\, \delta(k^2 - m^2)\,. \tag{A.5}$$

Likewise, for $m = 0$ we set

$$p_0(k) := \not{k}\, \delta(k^2)\,.$$

We also regard these distributions as multiplication operators in momentum space.

By direct computation, one verifies that these distributions are solutions of the Klein–Gordon and Dirac equations, respectively. More precisely,

$$(k^2 - a)\, P_a(k) = 0 \qquad \text{and} \qquad (\not{k} - m)\, p_m(k) = 0\,. \tag{A.6}$$

Products of the above operators are well defined if we work with a δ-normalization in the mass. Indeed, by direct computation one finds that

$$P_a\, P_b = \delta(k^2 - a)\, \delta(k^2 - b) \;=\; \delta(a - b)\, P_a \tag{A.7}$$

$$\begin{aligned}
p_m\, p_{m'} &= \frac{|mm'|}{mm'}\, (\not{k} + m)(\not{k} + m')\, \delta(k^2 - m^2)\, \delta\big(k^2 - (m')^2\big) \\
&= \delta\big(m^2 - (m')^2\big)\, \frac{|mm'|}{mm'}\, \big(k^2 + (m + m')\,\not{k} + mm'\big)\, \delta\big(k^2 - (m')^2\big) \\
&= \delta\big(m^2 - (m')^2\big)\, \frac{|mm'|}{mm'}\, (m + m')\, \big(\not{k} + m'\big)\, \delta\big(k^2 - (m')^2\big) \\
&= \frac{1}{2|m|}\, \delta(m - m')\, \frac{|mm'|}{mm'}\, (m + m')\, \big(\not{k} + m'\big)\, \delta\big(k^2 - (m')^2\big) \\
&= \delta(m - m')\, \frac{|m'|}{m'}\, \big(\not{k} + m'\big)\, \delta\big(k^2 - (m')^2\big) = \delta\big(m - m'\big)\, p_m\,.
\end{aligned} \tag{A.8}$$

Moreover, a straightforward computation using the symmetry properties under reflections $k \to -k$ yields the following completeness relations,

$$\int_{-\infty}^{\infty} P_a\, da = \int_{-\infty}^{\infty} \delta(k^2 - a)\, da = \mathbb{1}$$

$$\int_{\mathbb{R} \cup i\mathbb{R}} p_m\, dm = \int_{\mathbb{R}^+ \cup i\mathbb{R}^+} \frac{|m|}{m}\, 2m\, \delta(k^2 - m^2)\, dm$$

$$= \int_{-\infty}^{\infty} \delta(k^2 - m^2)\, d(m^2) = \mathbb{1}\ ,$$

where dm denotes the Lebesgue measure on $\mathbb{R} \cup i\mathbb{R}$. In view of (A.6) and the completeness relations, the distributions P_a and p_m can be viewed as the spectral projection operators of the Klein–Gordon and Dirac equations, respectively. Note that, in contrast to the situation for symmetric operators on Hilbert spaces, the spectrum of the Dirac operator is complex and the corresponding spectral projections are not symmetric but

$$p_m^* = p_{\overline{m}}$$

(as is obvious from (A.5)). The corresponding functional calculus is obtained by integrating over the spectral parameter. For example,

$$\int_{-\infty}^{\infty} a\ P_a\, da = \int_{-\infty}^{\infty} a\ \delta(k^2 - a)\, da = k^2$$

$$\int_{\mathbb{R}\cup i\mathbb{R}} m\ p_m\, dm = \int_{\mathbb{R}^+\cup i\mathbb{R}^+} 2|m|\ \not{k}\ \delta(k^2 - m^2)\, dm = \not{k} = i\not{\partial}_x\ ,$$

and using that multiplication in momentum space corresponds to differentiation in position space, one recovers the Klein–Gordon and Dirac operators. Moreover, the *symmetric Dirac Green's operator* can be defined by

$$s_m = \int_{\mathbb{R}\cup i\mathbb{R}} \frac{\mathrm{PP}}{\mu - m}\, p_\mu\, d\mu\ . \tag{A.9}$$

Here, "symmetry" refers to the fact that

$$s_m^* = s_{\overline{m}}\ .$$

Before going on, we make two remarks. We first point out that the above spectral theorem does not have an abstract underpinning, because there is no general spectral theorem for symmetric operators on Krein spaces. Moreover, we note that in the case that m is real, the symmetric Green's operator as defined by (A.9) coincides with the mean of the advanced and retarded Green's operators. In the case that m is imaginary, however, the connection to the causal Green's operators is not clear. The advanced and retarded Green's operator can still be constructed in position space using the theory of linear symmetric hyperbolic systems (as explained, for example, in [25, Chapter 13]). However, these solutions increase exponentially in time. Consequently, they are ill-defined as tempered distributions, making it impossible to take their Fourier transforms. In what follows, we bypass these issues by restricting attention to the symmetric Green's operators.

After these definitions, we are in the position to state the main result of this appendix.

Theorem A.2. *Let* Γ *be the integral operator on the Krein space* $(\mathcal{K}, <.|.>)$ *with integral kernel*

$$\Gamma(x,y) := \sum_{a=1}^{L} \Lambda_a\Big(\frac{x+y}{2}\Big)\, L_a(y-x)\ . \tag{A.10}$$

We introduce the following operator involving the double commutator $[[\not{\partial}, \Gamma], \Gamma]$,

$$\mathcal{C} := i \int_0^1 s\, e^{-is\Gamma}\, \big[[\not{\partial}, \Gamma], \Gamma\big]\, e^{is\Gamma}\, ds\,. \tag{A.11}$$

Then, the unitary holographic mixing operator V satisfying (A.2) and (A.3) can be written as

$$V = \int_{\mathbb{R} \cup i\mathbb{R}} V_m\, dm$$

where the operators V_m are defined in terms of the following perturbation series in $\mathcal{C}$,

$$V_m := e^{i\Gamma} \sum_{n=0}^\infty \big(-s_m\, \mathcal{C}\big)^n\, p_m \bigg(\mathbb{1} + \sum_{p=1}^\infty (-1)^p\, \frac{(2p-1)!!}{p!\, 2^p}\, \big(B_m\big)^p \bigg)$$

$$B_m := \pi^2\, \epsilon(m^2) \sum_{n,n'=0}^\infty \big(-\mathcal{C}\, s_m\big)^{n'}\, \mathcal{C}\, p_m\, \mathcal{C}\, \big(-s_m\, \mathcal{C}\big)^n\, p_m$$

(where $\epsilon(m^2)$ denotes the sign function).

We note for clarity that the operator $e^{i\Gamma}$ is defined via a spectral calculus on $(\mathcal{K}, <.|.>)$ or alternatively by the power series

$$e^{i\Gamma} := \sum_{k=0}^\infty \frac{(i\Gamma)^k}{k!}$$

(Thus, one takes powers of the nonlocal operator, not the exponential of the integral kernel.) Using that Γ is symmetric on $(\mathcal{K}, <.|.>)$, it follows that $(e^{i\Gamma})^* = e^{-i\Gamma}$, showing that the operator $e^{i\Gamma}$ is unitary on the Krein space $(\mathcal{K}, <.|.>)$.

The remainder of this appendix is devoted to the proof of this theorem. Having Green's operators to our disposal, the potential $\mathcal{B}$ in (A.1) can be treated perturbatively. However, using the specific structure of this potential, we can even treat generalized phase factors non-perturbative, as we now explain. By direct computation, one verifies that $\mathcal{B}$ can be written as a commutator,

$$\mathcal{B} = [\not{\partial}, \Gamma]\,,$$

where Γ is the operator defined in (A.10). Moreover, we set

$$\check{s}_m = e^{i\Gamma}\, s_m\, e^{-i\Gamma}\,. \tag{A.12}$$

This operator is an approximate Green's operator of the Dirac operator with nonlocal gauge potentials, as is made precise in the next lemma.

Lemma A.3. *For any $m \in \mathbb{R} \cup i\mathbb{R}$,*

$$(i\not{\partial} + \mathcal{B})\, \check{s}_m = E\, \check{s}_m\,, \tag{A.13}$$

where the operator E is given by

$$E = i \int_0^1 (1-s)\, e^{is\Gamma}\, \big[[\not{\partial}, \Gamma], \Gamma\big]\, e^{-is\Gamma}\, ds\,.$$

Proof. Using (A.9), we can rewrite (A.12) as

$$\check{s}_m = \int_{\mathbb{R}\cup i\mathbb{R}} \frac{\mathrm{PP}}{\mu - m}\, e^{i\Gamma}\, p_\mu\, e^{-i\Gamma}\, .$$

Therefore, it suffices to prove that

$$(i\partial\!\!\!/ + \mathcal{B} - \mu)\,\big(e^{i\Gamma}\, p_\mu\big) = E\,\big(e^{i\Gamma}\, p_\mu\big)\, .$$

In order to derive this relation, we make use of the well-known formula for the commutator with an exponential[2]

$$\big[\partial\!\!\!/, e^{i\Gamma}\big] = \int_0^1 e^{i\tau\Gamma}\, [\partial\!\!\!/, i\Gamma]\, e^{i(1-\tau)\Gamma}\, d\tau\, . \tag{A.14}$$

We thus obtain

$$\begin{aligned}(i\partial\!\!\!/ - \mu)\big(e^{i\Gamma}\, p_\mu\big) &= \big[i\partial\!\!\!/, e^{i\Gamma}\big]\, p_\mu = -\int_0^1 e^{i\tau\Gamma}\, [\partial\!\!\!/, \Gamma]\, e^{i(1-\tau)\Gamma}\, p_\mu\, d\tau \\ &= -[\partial\!\!\!/, \Gamma]\, e^{i\Gamma}\, p_\mu + \int_0^1 \big[[\partial\!\!\!/, \Gamma], e^{i\tau\Gamma}\big]\, e^{i(1-\tau)\Gamma}\, p_\mu\, d\tau\, ,\end{aligned}$$

showing that

$$(i\partial\!\!\!/ + \mathcal{B} - \mu)\big(e^{i\Gamma}\, p_\mu\big) = \int_0^1 \big[[\partial\!\!\!/, \Gamma], e^{i\tau\Gamma}\big]\, e^{-i\tau\Gamma}\, \big(e^{i\Gamma}\, p_\mu\big)\, d\tau\, .$$

Again using (A.14), the last integral can be simplified as follows,

$$\begin{aligned}&\int_0^1 \big[[\partial\!\!\!/, \Gamma], e^{i\tau\Gamma}\big]\, e^{-i\tau\Gamma}\, d\tau = \int_0^1 d\tau \int_0^1 d\tau'\, e^{i\tau'\tau\Gamma}\big[[\partial\!\!\!/, \Gamma], i\tau\Gamma\big]\, e^{-i\tau'\tau\Gamma} \\ &\quad= \left\{ \begin{array}{c} \tau\tau' =: s \\ \tau\, d\tau' = ds \end{array} \right\} = i\int_0^1 d\tau \int_0^\tau ds\, e^{is\Gamma}\big[[\partial\!\!\!/, \Gamma], \Gamma\big]\, e^{-is\Gamma} \\ &\quad= i\int_0^1 ds \int_s^1 d\tau\, e^{is\Gamma}\big[[\partial\!\!\!/, \Gamma], \Gamma\big]\, e^{-is\Gamma} = i\int_0^1 (1-s)\, e^{is\Gamma}\big[[\partial\!\!\!/, \Gamma], \Gamma\big]\, e^{-is\Gamma}\, ds\, .\end{aligned}$$

This concludes the proof. □

Now, we can treat the operator E in (A.13) perturbatively, giving the following result.

Lemma A.4. *For any $m \in \mathbb{R} \cup i\mathbb{R}$, the Green's operator $\tilde{s}_m$ corresponding to the operator $(i\partial\!\!\!/ + \mathcal{B})$ has the form*

$$\tilde{s}_m = \sum_{n=0}^\infty \big(-\check{s}_m\, E\big)^n\, \check{s}_m\, .$$

[2] For the proof, one uses that, for any $N \in \mathbb{N}$,

$$\big[\partial\!\!\!/, e^{i\Gamma}\big] = \big[\partial\!\!\!/, \big(e^{i\Gamma/N}\big)^N\big] = \sum_{k=0}^{N-1} e^{i\Gamma\, \frac{k}{N}}\, \big[\partial\!\!\!/, e^{i\Gamma/N}\big]\, e^{i\Gamma\, \frac{N-k-1}{N}}\, .$$

Expanding for large N, the commutator on the right becomes $[\partial\!\!\!/, i\Gamma]/N + \mathcal{O}(N^{-2})$. Viewing the sum as a Riemann sum and taking the limit $N \to \infty$ gives (A.14).

Proof. A direct computation using (A.13) gives

$$(i\partial\!\!\!/ + \mathcal{B})\big(- \check{s}_m \, E\big)^n \, \check{s}_m = E \left(- \check{s}_m \, E\right)^n \check{s}_m + \begin{cases} \mathbb{1} & \text{if } n = 0 \\ -E \left(- \check{s}_m \, E\right)^{n-1} \check{s}_m & \text{if } n > 0 \,. \end{cases}$$

Summing over n gives the result. □

The following lemma generalizes formulas from [17, Lemma 2.1] to the case of imaginary mass parameters.

Lemma A.5. *For any $m, m' \in \mathbb{R} \cup i\mathbb{R}$,*

$$p_m \, s_{m'} = s_{m'} \, p_m = \frac{PP}{m - m'} \, p_m \tag{A.15}$$

$$s_m \, s_{m'} = \frac{PP}{m - m'} \, (s_m - s_{m'}) + \pi^2 \, \epsilon(m^2) \, \delta(m - m') \, p_m \,. \tag{A.16}$$

Proof. The relation (A.15) follows immediately from (A.9) and (A.7). The proof of (A.16) is more subtle. In the case $m, m' \in \mathbb{R}$, one can argue with the support properties of the causal Green's operators. (For details, see [17, proof of Lemma 2.1].) Here, instead we carefully analyze regularized distributions in the complex plane. If m is real, we write the principal part in (A.9) as

$$s_m = \frac{1}{2} \lim_{\varepsilon \searrow 0} \sum_{s=\pm} \int_{\mathbb{R} \cup i\mathbb{R}} \frac{1}{\mu - m + is\varepsilon} \, p_\mu \, d\mu \,.$$

Now, the two summands can be treated individually by using the distributional relation (for details, see, for example, [11, eqs (1.2.33), (1.2.33) and Exercises 1.10-–1.12])

$$\lim_{\varepsilon \searrow 0} \frac{1}{x \pm i\varepsilon} = \frac{\mathrm{PP}}{x} \mp i\pi \, \delta(x). \tag{A.17}$$

In order to extend this method to the case where μ and m can be both either real or purely imaginary, we must regularize such as to avoid poles on both the real and the imaginary axes. This can be achieved, for example, by setting

$$s_m = \frac{1}{2} \lim_{\varepsilon \searrow 0} \sum_{s=\pm} \int_{\mathbb{R}_\varepsilon \cup i\mathbb{R}_\varepsilon} \frac{1}{\mu - m + (1-i)s\varepsilon} \, p_\mu \, d\mu \,,$$

where we used the abbreviation

$$R_\varepsilon := \mathbb{R} \setminus (-2\varepsilon, 2\varepsilon).$$

(Cutting out a neighborhood of the origin has the purpose of avoiding poles if m and μ are close to zero.) If m is real, we can again use (A.17). If m is imaginary, however, one must take into account that the Lebesgue measure dm differs from the contour integral dz along the imaginary axis by a factor i. We thus obtain the distributional relation

$$\lim_{\varepsilon \searrow 0} \frac{1}{\mu - m \pm (1-i)\varepsilon} = \frac{\mathrm{PP}}{\mu - m} \mp i\pi \, \sigma(m) \, \delta(\mu - m) \tag{A.18}$$

with

$$\sigma(m) := \begin{cases} 1 \text{ if } m \in \mathbb{R} \\ i \text{ if } m \in i\mathbb{R} \,. \end{cases}$$

Using (A.7), we write the left side of (A.16) as

$$s_m\, s_{m'} = \frac{1}{4} \lim_{\varepsilon,\varepsilon' \searrow 0} \sum_{s,s'=\pm} \int_{\mathbb{R}_\varepsilon \cup i\mathbb{R}_\varepsilon} \frac{1}{\mu - m + (1-i)s\varepsilon}\; \frac{1}{\mu - m' + (1-i)s'\varepsilon'}\; p_\mu\, d\mu \tag{A.19}$$

Next, we use the partial sum decomposition

$$\begin{aligned} &\frac{1}{\mu - m + (1-i)s\varepsilon}\; \frac{1}{\mu - m' + (1-i)s'\varepsilon'} \\ &\quad = \left(\frac{1}{\mu - m + (1-i)s\varepsilon} - \frac{1}{\mu - m' + (1-i)s'\varepsilon'} \right) \frac{1}{m - m' - (1-i)(s\varepsilon - s'\varepsilon')} \,. \end{aligned} \tag{A.20}$$

Now, we first take the limit $\varepsilon \searrow 0$. (Of course, taking the limits in another order gives the same end result.) Then, in the last factor in (A.20) we can leave out ε. Applying (A.18) gives

$$\begin{aligned} &\frac{1}{2} \lim_{\varepsilon \searrow 0} \sum_{s=\pm} \frac{1}{\mu - m + (1-i)s\varepsilon}\; \frac{1}{\mu - m' + (1-i)s'\varepsilon'} \\ &\quad = \left(\frac{\mathrm{PP}}{\mu - m} - \frac{1}{\mu - m' + (1-i)s'\varepsilon'} \right) \frac{1}{m - m' + (1-i)s'\varepsilon'} \,. \end{aligned}$$

Next, we take the limit $\varepsilon' \searrow 0$ and apply again (A.18). We thus obtain

$$\begin{aligned} &\frac{1}{4} \lim_{\varepsilon' \searrow 0} \lim_{\varepsilon \searrow 0} \sum_{s,s'=\pm} \frac{1}{\mu - m + (1-i)s\varepsilon}\; \frac{1}{\mu - m' + (1-i)s'\varepsilon'} \\ &\quad = \frac{\mathrm{PP}}{\mu - m}\; \frac{\mathrm{PP}}{m - m'} - \frac{1}{2} \sum_{s'=\pm} \left(\frac{\mathrm{PP}}{\mu - m'} - i\pi s'\, \sigma(m')\, \delta\big(\mu - m'\big) \right) \\ &\qquad \left(\frac{\mathrm{PP}}{m - m'} - i\pi s'\, \sigma(m)\, \delta\big(m - m'\big) \right) \\ &\quad = \frac{\mathrm{PP}}{\mu - m}\; \frac{\mathrm{PP}}{m - m'} - \frac{\mathrm{PP}}{\mu - m'}\; \frac{\mathrm{PP}}{m - m'} + \pi^2\, \sigma(m)^2\, \delta\big(\mu - m'\big)\, \delta\big(m - m'\big) \,. \end{aligned}$$

Using this identity in (A.19) gives (A.16), concluding the proof. □

We next want to compute an operator which maps the unperturbed to the perturbed solutions. Our first ansatz is

$$U_m := \sum_{n=0}^{\infty} \big(-\check{s}_m\, E \big)^n\, e^{i\Gamma}\, p_m \,.$$

Exactly as in the proof of Lemma A.4, one verifies that this operator maps to solutions, i.e.,

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, U_m = 0 \,.$$

However, the normalization is not correct, because (for more details, see similar computations in [17])

$$\begin{aligned}
(U_{\overline{m'}})^*\, U_m &= \sum_{n,n'=0}^{\infty} p_{m'}\, e^{-i\Gamma}\, \big(-E\, \check{s}_{m'}\big)^{n'} \big(-\check{s}_m\, E\big)^n\, e^{i\Gamma}\, p_m \\
&= \delta(m-m') \\
\Big(p_m + \pi^2\, \epsilon(m^2) &\sum_{n,n'=0}^{\infty} p_m\, e^{-i\Gamma}\, \big(-E\, \check{s}_m\big)^{n'}\, E\, \check{p}_m\, E\, \big(-\check{s}_m\, E\big)^n\, e^{i\Gamma}\, p_m\Big)
\end{aligned}$$

with

$$\check{p}_m := e^{i\Gamma} p_m e^{-i\Gamma}\,. \tag{A.21}$$

We write this result in the short form

$$(U_{\overline{m}})^*\, U_m = \delta(m-m')\, p_m \big(1 + A_m\big)$$

with

$$A_m := \pi^2\, \epsilon(m^2) \sum_{n,n'=0}^{\infty} e^{-i\Gamma}\, \big(-E\, \check{s}_m\big)^{n'}\, E\, \check{p}_m\, E\, \big(-\check{s}_m\, E\big)^n\, e^{i\Gamma}\, p_m\,.$$

Following the rescaling procedure in [17], we set

$$V_m := U_m\, \big(\mathbb{1} + A_m\big)^{-\frac{1}{2}}\,,$$

where the last factor is defined by the perturbation series given by the Taylor series, i.e.,

$$\big(\mathbb{1} + A_m\big)^{-\frac{1}{2}} := \mathbb{1} + \sum_{n=1}^{\infty} (-1)^n\, \frac{(2n-1)!!}{n!\, 2^n}\, \big(A_m\big)^n\,.$$

Then,

$$\begin{aligned}
(V_{\overline{m'}})^*\, V_m &= \delta(m-m')\, \big(1 + (A_{\overline{m}})^*\big)^{-\frac{1}{2}}\, p_m \big(1 + A_m\big)\, \big(1 + A_m\big)^{-\frac{1}{2}} \\
&= \delta(m-m')\, p_m\,,
\end{aligned}$$

as desired. As a consequence, the operator

$$V := \int_{\mathbb{R}\cup i\mathbb{R}} V_m\, dm$$

is unitary, because

$$\begin{aligned}
V^*V = \int_{\mathbb{R}\cup i\mathbb{R}} dm' &\int_{\mathbb{R}\cup i\mathbb{R}} dm\, (V_{\overline{m}})^*\, V_m \\
&= \int_{\mathbb{R}\cup i\mathbb{R}} p_m\, dm = \mathbb{1}\,.
\end{aligned}$$

Therefore, the perturbed spectral projectors can be written as

$$\tilde{p}_m = V p_m V^*\,,$$

and the corresponding symmetric Green's operators are given in analogy to (A.9) by

$$\tilde{s}_m = \int_{\mathbb{R}\cup i\mathbb{R}} \frac{\mathrm{PP}}{\mu - m}\, \tilde{p}_m\, d\mu = \int_{\mathbb{R}\cup i\mathbb{R}} \frac{\mathrm{PP}}{\mu - m}\, V p_m V^*\, d\mu = V s_m V^* \,.$$

The statement of Theorem A.2 is obtained by rewriting $\check{s}_m$ and $\check{p}_m$ in terms of s_m and p_m using (A.12) and A.21, and noting that

$$e^{-i\Gamma}\, E\, e^{i\Gamma} = i \int_0^1 (1-s)\, e^{-i(1-s)\Gamma}\, \big[[\partial\!\!\!/, \Gamma], \Gamma\big]\, e^{i(1-s)\Gamma}\, ds = \mathcal{C}\,,$$

where in the last step we transformed the integration variable according to $s \to 1-s$ and used (A.11).

Appendix B. Microlocal Expansion of the Unitary Holographic Mixing Operator

In this appendix, we explain how the unitary holographic mixing operator V defined by (5.4) can be expanded in powers of $\ell_{\min}/\ell_\Lambda$. Clearly, this expansion is sensible only if $\ell_\Lambda \gg \ell_{\min}$, as is made precise in (4.13). Our method is inspired by the pseudo-differential calculus and microlocal analysis; for the general context see, for example, [37]. In preparation, we expand the phase factors in (4.6) in a Taylor expansion about the arithmetic mean of the left and right arguments of the kernel. Thus, setting

$$\xi := y - x \qquad \text{and} \qquad \zeta := \frac{y+x}{2}\,,$$

we expand

$$\begin{aligned} U(x,y) &= \sum_{a=1}^N e^{i\Lambda_a(x)}\, L_a(x,y) = \sum_{a=1}^N e^{i\Lambda_a(\zeta-\xi/2)}\, L_a(x,y) \\ &= \sum_\kappa \sum_{a=1}^N \frac{1}{|\kappa|!} \left(\partial_\kappa e^{i\Lambda_a(\zeta)}\right) \Big(-\frac{\xi}{2}\Big)^\kappa L_a(x,y) \end{aligned}$$

(where κ is a multi-index). We also write this expansion as

$$U = \sum_{p=0}^\infty U^{(p)}$$

with

$$U^{(p)}(x,y) := \sum_{\kappa \text{with } |\kappa|=p} \sum_{a=1}^N \frac{1}{p!} \left(\partial_\kappa e^{i\Lambda_a(\zeta)}\right) \Big(-\frac{\xi}{2}\Big)^\kappa L_a(x,y)\,. \tag{B.1}$$

This has the advantage that the index p tells us about the scaling power in $\ell_\Lambda/\ell_{\min}$. Indeed, counting orders starting from L_a with the order zero, we have

$$U^{(p)} = \mathcal{O}\Big(\big(\ell_{\min}/\ell_\Lambda\big)^p \Big)\,.$$

For what follows, it is crucial that these operator commute to leading order in $\ell_\Lambda/\ell_{\min}$, i.e.,

$$\big[U^{(p)}, U^{(q)}\big] = \mathcal{O}\Big(\big(\ell_{\min}/\ell_\Lambda\big)^{p+q+1} \Big) \tag{B.2}$$

and similarly for U^* and mixed commutators. In order to see how this comes about, one should keep in mind that if the kernels in (B.1) do not depend on ζ, then the operators $U^{(p)}$ and $U^{(q)}$ are multiplication operators in momentum space, which clearly commute. Therefore, the commutator in (B.2) is nonzero as a consequence of a linear expansion of the ζ-dependent factors in (B.1), giving an additional scaling factor $\ell_{\min}/\ell_\Lambda$.

Clearly, the above formalism also applies to U^* or to composite operators. In general terms, the contributions to the resulting expansions can be written as

$$A(x,y) = f(\zeta)\, K(\xi)\,,$$

where the kernel $K(\xi)$ decays on the scale $\ell_{\min}$, and the derivatives of $f(\zeta)$ scale in powers of $1/\ell_\Lambda$. Under these assumptions, one can introduce a an approximate spectral calculus and one can compute approximate inverses, where "approximate" means that we allow for errors of the order $\ell_{\min}/\ell_\Lambda$. This is made precise in the next lemma.

Lemma B.1. (microlocal spectral calculus) *Given a function $g \in C^1(\mathbb{C},\mathbb{C})$ with $g \circ f \in C^1(M,\mathbb{C})$ and $g \circ \hat{K} \in L^1(\hat{M},\mathbb{C})$, the operator $g_{ml}(A)$ defined by*

$$\big(g_{ml}(A)\big)(x,y) := (g \circ f)(\zeta) \int \frac{d^4k}{(2\pi)^4}\, g\big(\hat{K}(k)\big)\, e^{-ik\xi}$$

satisfies the approximate spectral calculus

$$g_{ml}(A)\, \tilde{g}_{ml}(A) = \big(g\tilde{g}\big)_{ml}(A) + \mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)\,.$$

The microlocal inverse

$$A^{-1}_{ml}(x,y) := \frac{1}{f(\zeta)} \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{\hat{K}(k)}\, e^{-ik\xi}$$

is an approximate inverse even up to second-order corrections, i.e.,

$$A^{-1}_{ml}\, A = \mathbb{1} + \mathcal{O}\Big(\big(\ell_{\min}/\ell_\Lambda\big)^2 \Big)\,.$$

Proof. We write the operator product with integral kernels,

$$\begin{aligned}&\big(g_{\rm ml}(A)\, \tilde{g}_{\rm ml}(A)\big)(x,y) \\ &= \int_M (g\circ f)\Big(\frac{x+z}{2}\Big)\, (\tilde{g}\circ f)\Big(\frac{z+y}{2}\Big)\, \big(g\circ \hat{K}\big)(k)\, \big(\tilde{g}\circ\hat{K}\big)(k)\, d^4z\,.\end{aligned}$$

Replacing the arguments of $g\circ f$ and $\tilde{g}\circ f$ by $\zeta := (x+y)/2$ gives an error of order $\ell_{\min}/\ell_\Lambda$. Then, we can carry out the z-integral, giving the result.

For the inverse, we obtain the formula

$$\big(A^{-1}_{\rm ml}\, A\big)(x,y) = \int_M \frac{f((z+y)/2)}{f(x+z)/2}\, K^{-1}(x,z)\, K(z,y)\, d^4z\,, \tag{B.3}$$

where $K^{-1}(x,y)$ is the Fourier transform of $1/\hat{K}(k)$. We now expand the quotient in the integrand about x,

$$\begin{aligned}\frac{f((z+y)/2)}{f(x+z)/2} &= 1+\frac{\partial_j f(x)}{f(x)}\left(\frac{z+y-2x}{2}-\frac{x+z-2x}{2}\right)+\mathcal{O}\Big(\big(\ell_{\min}/\ell_\Lambda\big)^2\Big)\\ &= 1+\frac{\partial_j f(x)}{2f(x)}\,\xi^j+\mathcal{O}\Big(\big(\ell_{\min}/\ell_\Lambda\big)^2\Big)\,.\end{aligned}$$

Since this expansion does not involve z, we can carry out the z-integral in (B.3). Using the relation

$$\int_M K^{-1}(x,z)\,K(z,y)\,d^4z=\delta^4(x-y)\,,$$

, the factor ξ in the linear expansion term drops out, giving the result. □

Using the above formalism, we can compute V and the potential $\mathcal{B}_\Lambda$ in (5.1) order by order. We now illustrate this expansion by a computation to zeroth order. We begin with the operator U defined in (4.6),

$$\begin{aligned}U(x,y) &= \sum_{a=1}^N e^{i\Lambda_a(\zeta)}\,L_a(x,y)+\mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)\\ (U^*U)(x,y) &= \sum_{a,b=1}^N e^{-i\Lambda_a(\zeta)+i\Lambda_b(\zeta)}\,\big(L_a\,L_b\big)(x,y)+\mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)\\ \big((U^*U)^{-\frac{1}{2}}\big)(x,y) &= \int\frac{d^4k}{(2\pi)^4}\left(\sum_{a,b=1}^N e^{-i\Lambda_a(\zeta)+i\Lambda_b(\zeta)}\,\hat{L}_a(k)\,\hat{L}_b(k)\right)^{-\frac{1}{2}}e^{-ik(x-y)}\\ &\quad+\mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big).\end{aligned}$$

Consequently, the operator V given by (5.4) takes the form

$$\begin{aligned}V(x,y) &= \int\frac{d^4k}{(2\pi)^4}\left(\sum_{d=1}^N e^{i\Lambda_d(\zeta)}\,\hat{L}_d(k)\right)\\ &\qquad\times\left(\sum_{a,b=1}^N e^{-i\Lambda_a(\zeta)+i\Lambda_b(\zeta)}\,\hat{L}_a(k)\,\hat{L}_b(k)\right)^{-\frac{1}{2}}e^{-ik(x-y)}+\mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)\\ V^*(x,y) &= \int\frac{d^4k}{(2\pi)^4}\left(\sum_{d=1}^N e^{-i\Lambda_d(\zeta)}\,\hat{L}_d(k)\right)\\ &\qquad\times\left(\sum_{a,b=1}^N e^{i\Lambda_a(\zeta)-i\Lambda_b(\zeta)}\,\hat{L}_a(k)\,\hat{L}_b(k)\right)^{-\frac{1}{2}}e^{-ik(x-y)}+\mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)\,.\end{aligned}$$

Using these formulas in (5.1), we obtain

$$\begin{aligned}&\big[i\partial\!\!\!/,V^*\big](x,y)\\ &\quad=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(\sum_{d=1}^N e^{-i\Lambda_d(\zeta)}\,\hat{L}_d(k)\right)\left(\sum_{a,b=1}^N e^{i\Lambda_a(\zeta)-i\Lambda_b(\zeta)}\,\hat{L}_a(k)\,\hat{L}_b(k)\right)^{-\frac{3}{2}}\end{aligned}$$

$$\times \Bigg(\sum_{a,b=1}^N \Big(-(\partial\!\!\!/ \Lambda_a)(\zeta) + (\partial\!\!\!/ \Lambda_b)(\zeta) \Big)\, e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)\, e^{-ik(x-y)}$$

$$+ \int \frac{d^4k}{(2\pi)^4} \Bigg(\sum_{d=1}^N (\partial\!\!\!/ \Lambda_d)(\zeta)\, e^{-i\Lambda_d(\zeta)}\, \hat{L}_d(k) \Bigg)$$

$$\times \Bigg(\sum_{a,b=1}^N e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)^{-\frac{1}{2}} e^{-ik(x-y)} + \mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)$$

$$\mathcal{B}_{\rm dyn}(x,y) = V\, \big[i\partial\!\!\!/, V^* \big](x,y)$$

$$= -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Bigg(\sum_{a,b=1}^N e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)^{-1}$$

$$\times \Bigg(\sum_{a,b=1}^N \Big(-(\partial\!\!\!/ \Lambda_a)(\zeta) + (\partial\!\!\!/ \Lambda_b)(\zeta) \Big)\, e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)\, e^{-ik(x-y)}$$

$$+ \int \frac{d^4k}{(2\pi)^4} \Bigg(\sum_{d,e=1}^N (\partial\!\!\!/ \Lambda_d)(\zeta)\, e^{-i\Lambda_d(\zeta) + i\Lambda_d(\zeta)}\, \hat{L}_d(k)\, \hat{L}_e(k) \Bigg)$$

$$\times \Bigg(\sum_{a,b=1}^N e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)^{-1} e^{-ik(x-y)} + \mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Bigg(\sum_{a,b=1}^N \Big((\partial\!\!\!/ \Lambda_a)(\zeta) + (\partial\!\!\!/ \Lambda_b)(\zeta) \Big)\, e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)$$

$$\times \Bigg(\sum_{a,b=1}^N e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k) \Bigg)^{-1} e^{-ik(x-y)} + \mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4}\, \frac{\sum_{a,b=1}^N \Big((\partial\!\!\!/ \Lambda_a)(\zeta) + (\partial\!\!\!/ \Lambda_b)(\zeta) \Big)\, e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k)}{\sum_{a,b=1}^N e^{i\Lambda_a(\zeta) - i\Lambda_b(\zeta)}\, \hat{L}_a(k)\, \hat{L}_b(k)}$$

$$e^{-ik(x-y)} + \mathcal{O}\big(\ell_{\min}/\ell_\Lambda\big)\,.$$

If we expand this potential to first order in the phase functions Λ_a, assuming that

$$\sum_{a=1}^n L_a = \mathbb{1}\,,$$

we recover the potential (5.2), giving a connection to the perturbative treatment in Appendix A. However, we point out that the above formula for $\mathcal{B}_{\rm dyn}(x,y)$ is much more general due to the additional phase factors evaluated at ζ.

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