

On the Growth of Koszul Homology  
and Estimates of Betti Numbers  
of Semi-Perverse Sheaves  
on Pro- $\ell$  Towers



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# Zusammenfassung

In dieser Arbeit wird eine allgemeine Version eines Verschwindungssatzes von P. Scholze und H. Esnault zur Kohomologie von pro- $\ell$ -Türmen bewiesen, um diese, mittels der generalisierten Fourier-Mellin Transformation, wie von A. A. Beilinson angeregt, mit klassischen Wachstumsabschätzungen der Betti-Zahlen von semi-perversen Garben auf pro- $\ell$  Türmen in Verbindung zu bringen. Dadurch motiviert wird das Wachstum der Koszul-Homologie für Potenzen von Sequenzen und dessen Verknüpfungen mit der Dimensionstheorie untersucht. Dabei werden einige Aspekte behandelt, die in einem weit gefassten Kontext mannigfaltig mit Fragen im Zusammenhang stehen, die historisch stark mit dem Satz von Bézout verknüpft sind.

## Summary

In this thesis we will prove a general version of a vanishing result of P. Scholze and H. Esnault on the cohomology of pro- $\ell$  towers and relate it, via the generalized Fourier-Mellin transform, to classical growth estimates of Betti numbers of semi-perverse sheaves on pro- $\ell$  towers, as posed by A. A. Beilinson. Motivated by this, we will investigate the growth of Koszul homology for powers of sequences, and we will study its interplay with dimension theory. In doing so, we will examine several aspects that, in a bigger picture, are related to questions historically associated with Bézout's theorem.



*Für meine Schwester*



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# Conventions and Preliminaries

In this chapter, we establish some conventions and preliminaries, primarily for Part I. Additional conventions relevant to Part II will be introduced later.

All rings in this thesis are assumed to be commutative rings with unity. If not said otherwise, all rings are assumed to be noetherian. By default, we will denote such a ring by  $R$ .

In general, we consider 0 to be a natural number; however, if it is dictated by the context that 0 has to be excluded, we will not. A real number  $r$  is considered positive if  $r > 0$  and it is considered negative if  $r < 0$ . We apply these conventions specifically to integer indices. In a general partially ordered abelian group, however, we call an element  $a$  positive if  $a \geq 0$ . The slight ambiguity in this usage of language will be resolved by the context.

**Conventions on Indices, Rings and Modules I.** Let  $R$  be an arbitrary commutative ring, and let  $s$  be a natural number. Typically, we will denote a sequence  $(x_1, \dots, x_s)$  of elements in  $R$  by  $\mathbf{x} = x_1, \dots, x_s$  and we will typically denote an  $s$ -tuple  $(n_1, \dots, n_s)$  of real numbers by  $\underline{n} = n_1, \dots, n_s$ . In the following, let  $\mathbf{x} = x_1, \dots, x_s$  denote an arbitrary sequence of elements in  $R$ , and let  $\underline{n}$  denote an arbitrary  $s$ -tuple of real numbers. The entries of  $\underline{n}$  are referred to as  $n_1, \dots, n_s$  by default, and we put  $|\underline{n}| := n_1 + \dots + n_s$ . We set  $\mathbf{x}^{\underline{n}} := x_1^{n_1}, \dots, x_s^{n_s}$ , and for every natural number  $n$  we set  $\mathbf{x}^n := x_1^n, \dots, x_s^n$ . We apply the same convention to variables. We denote the ideal generated by  $x_1, \dots, x_s$  by  $(\mathbf{x})$  and for an  $R$ -module  $M$  we set  $M/\mathbf{x}M := M/(\mathbf{x})M$ . For every index  $1 \leq r \leq s$  we set  $(x_1, \dots, \widehat{x}_r, \dots, x_s) := (x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_s)$ . Let  $I$  be a subset of  $\{1, \dots, s\}$ . We define  $\mathbf{x}|_I := (x_i)_{i \in I}$  and  $\underline{n}|_I := (n_i)_{i \in I}$ . If the set  $I$  is of the form  $\{a, \dots, b\}$  with natural numbers  $a \leq b$  in  $\{1, \dots, s\}$ , then we occasionally write  $\mathbf{x}_{a, \dots, b}$  instead of  $\mathbf{x}|_I$  and  $\underline{n}|_{a, \dots, b}$  instead of  $\underline{n}|_I$ . Furthermore, we use “implicit type casting”. For example, if  $\mathbf{x} = (x_1, \dots, x_s)$  and  $\mathbf{y} = (x_1, \dots, x_s)$  are sequences in  $R$ , we denote the concatenated sequence  $(x_1, \dots, x_s, y_1, \dots, y_t)$  by  $(\mathbf{x}, \mathbf{y})$ . Similarly,  $\underline{0}$  (or  $\underline{1}$ ) denotes a tuple consisting only of zeroes (or ones, respectively) with its length determined by the context.

Let  $M$  be a finitely generated module over a noetherian ring. We denote the set comprising the minimal prime ideals in the support of  $M$  by  $\text{minSupp}(M)$ . Further, we denote the dual module  $\text{Hom}_R(M, R)$  by  $M^\vee$ . We define the zero module to have dimension  $-1$ .

## Conventions and Preliminaries

Given a pair  $\underline{n}, \underline{\beta}$  of  $s$ -tuples of natural numbers, we say that  $\underline{n}$  is larger than  $\underline{\beta}$ , in symbols  $\underline{n} > \underline{\beta}$ , if  $n_i > \beta_i$  for all  $i = 1, \dots, s$ . We say that a mathematical statement  $\mathbf{P}(\underline{n})$  depending on an  $s$ -tuple of elements of natural numbers holds true for all sufficiently large values of  $\underline{n}$ , in symbols  $\underline{n} \gg 0$ , if there exists an  $s$ -tuple  $\underline{\beta}$  in  $\mathbb{N}^s$  such that  $\mathbf{P}(\underline{n})$  holds true whenever  $\underline{n} > \underline{\beta}$ .

**Convention on Categories II.** Let  $R$  be a commutative ring. We denote the category of  $R$ -modules by  $\mathbf{R} - \mathbf{Mod}$ , and we denote the category of finitely generated  $R$ -modules by  $\mathbf{R} - \mathbf{mod}$ . Further,  $\mathbf{R} - \mathbf{Alg}$  denotes the category of  $R$ -algebras. We let  $\mathbf{POrd} - \mathbf{Ab}$  denote the category of partially ordered abelian groups and we let  $\mathbf{CMon}$  denote the category of commutative cancellation monoids. Let  $\ell$  be a prime number, let  $v$  be a positive number, and let  $R$  denote any of the rings  $\mathbb{F}_{\ell^v}, \mathbb{Z}_{\ell}, \mathbb{Q}_{\ell}, \overline{\mathbb{Q}}_{\ell}$ . We denote the category of middle perverse sheaves, as introduced in [10, 2.2] by  ${}^p\mathcal{D}_c(X_{\acute{e}t}, R)$ . In particular we use the definition [10, 2.2.16] for  $R = \mathbb{Z}_{\ell}$ .

Let  $\mathcal{C}$  be a category, let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor, and let  $\eta$  be a natural transformation from an endofunctors  $G$  of  $\mathcal{C}$  to an endofunctor  $H$  of  $\mathcal{C}$ . Then we denote the induced natural transformation from  $F \circ G$  to  $F \circ H$  by  $F \circ \eta$ .

Let  $S$  be an  $R$ -algebra. For every sequence  $\mathbf{x} = x_1, \dots, x_s$  in  $R$  we let  $\mathbf{S} - \mathbf{mod}_{\mathbf{x}}$  denote the full exact abelian subcategory of  $\mathbf{S} - \mathbf{mod}$  consisting of all finitely generated  $S$ -modules  $N$  that are annihilated by a power of  $(\mathbf{x})$ .

We denote the derived category  $\mathcal{D}(\mathbf{R} - \mathbf{Mod})$  by  $\mathcal{D}(R)$ , and we denote the full subcategory of  $\mathcal{D}^b(R)$  consisting of all objects with finitely generated cohomology by  $\mathcal{D}_{ft}^b(R)$ .

Given  $M$  in  $\mathcal{D}^b(R)$ , we call a sequence  $\mathbf{x}$  of  $R$  a *multiplicity system* of  $M$  if  $\ell(H^i(M) \otimes R/\mathbf{x}R)$  is finite for every integer  $i$ .

**Conventions on Partially Ordered Groups III.** Let  $(G, \leq)$  be a partially ordered abelian group. We call  $G^+ := \{a \in G \mid 0 \leq a\}$  the *positive cone* of  $G$ . Next, we define the partially ordered semigroup  $G_{\infty}$  as follows: As a set we let  $G_{\infty} = G \cup \{\infty\}$  and we extend the addition on  $G$  via  $g + \infty = \infty$  and  $\infty + g = \infty$  for all elements  $g$  in  $G$ , and  $\infty + \infty = \infty$ . We extend the partial order on  $G$  by setting  $g \leq \infty$  for all elements  $g$  in  $G_{\infty}$ , and we extend the scalar multiplication of  $\mathbb{Z}$  on  $G$  via  $z \cdot \infty := \infty$  for all  $z$  in  $\mathbb{Z}$ . In particular,  $0 \cdot \infty = \infty$  and  $-\infty = \infty$ .<sup>1</sup> We call an element  $g$  in  $G_{\infty}$  *finite* if  $g \neq \infty$ .

<sup>1</sup>The element  $\infty$  is a zero element for the addition in  $G_{\infty}$ . Hence, the group completion of  $G_{\infty}$  is the trivial group. Therefore, we have to work with the semigroup  $G_{\infty}$ .

Let  $\mathcal{C}$  denote a subclass of the class of objects of an exact category  $\mathcal{D}$ , and let  $\mu : \mathcal{C} \rightarrow G_\infty$  be a function.<sup>2</sup> If one has  $\mu(A) = \mu(B) + \mu(C)$  whenever there is an exact sequence

$$0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$$

in  $\mathcal{D}$  with objects  $A, B, C$  in  $\mathcal{C}$ , then we say that  $\mu$  is an *additive function*. This terminology is consistent with [107, 6.1.2.]. We say that  $\mu$  is *faithful* if  $\mu(M) = 0$  holds true if and only if  $M$  is the zero module. We say that  $\mu$  is *positiv* if  $\mu(A) \geq 0$  for every object  $A$  in  $\mathcal{C}$ . If  $\mu(A)$  is finite for all objects  $A$  in  $\mathcal{C}$ , then we say that  $\mu$  is *finite*.

**Conventions on Functions IV.** Let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be a function. We set

$$\Delta_r(f)(N_1, \dots, N_s) := f(N_1, \dots, N_{r-1}, N_r + 1, N_{r+1}, \dots, N_s) - f(N_1, \dots, N_s)$$

for every index  $1 \leq r \leq s$ .

We call  $f$  *polynomial-like* if there exists a polynomial  $P$  in  $\mathbb{R}[X_1, \dots, X_s]$  such that  $f(\underline{n})$  and  $P(\underline{n})$  agree for all sufficiently large  $s$ -tuples of natural numbers. This terminology is consistent with [91, B. §2].

For every multivariate polynomial we refer to its *total degree* simply as its *degree*. If  $f$  is polynomial-like with associated polynomial  $P$ , we define the *degree* of  $f$  to be the degree of  $P$ . We define the 0-polynomial to have degree  $-1$ . Given two polynomials  $P, Q$  in  $\mathbb{R}[X_1, \dots, X_s]$ , we put  $P \leq Q$  if  $P(\mathbf{a}) \leq Q(\mathbf{a})$  for all  $\mathbf{a}$  in  $\mathbb{R}^s$ . Dually we define  $P \geq Q$ .

**Preliminaries on Koszul Homology V.** Let  $R$  be a noetherian ring, and let  $x$  be a non-zero divisor of  $R$ . We denote by  $\mathcal{K}(x)$  the following complex:

$$\begin{aligned} \mathcal{K}_i(x) &= 0 \quad \text{for all } i \neq 0, 1; \\ \mathcal{K}_1(x) &= R; \\ \mathcal{K}_0(x) &= R; \end{aligned}$$

and the derivation  $\mathcal{K}_1(x) \rightarrow \mathcal{K}_0(x)$  is given by multiplication with  $x$ . We put

$$\mathcal{K}(\mathbf{x}) := \mathcal{K}(x_1) \otimes_R^L \mathcal{K}(x_2) \otimes_R^L \dots \otimes_R^L \mathcal{K}(x_s)$$

for every sequence  $\mathbf{x} = x_1, \dots, x_s$  of non-zero divisors of  $R$ . Let  $M$  be an object of  $\mathcal{D}^b(R)$ . We define the  $i$ -th *Koszul homology module* of  $M$  to be

$$H_i(\mathbf{x}; M) := H^{-i}(M \otimes_R^L \mathcal{K}(\mathbf{x})).$$

For the so defined *Koszul homology*, the Tor base change spectral sequence [93, Tag 0662] give rise to

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<sup>2</sup>That is to say,  $\mu$  assigns to every isomorphism class  $[A]$  of an object  $A$  in  $\mathcal{C}$  an element  $\mu([A])$  of  $G_\infty$ . Instead of  $\mu([A])$ , we will simply write  $\mu(A)$ .

## Conventions and Preliminaries

**Theorem 0.1** (Universal Coefficient Theorem). *Let  $R$  be a noetherian ring, let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of non-zero divisors of  $R$ , and let  $M$  be an object of  $\mathcal{D}^b(R)$ . Then we have a short exact sequence*

$$0 \rightarrow H_i(\mathbf{x}|_{1, \dots, s-1}, M) \otimes R/x_s R \rightarrow H_i(\mathbf{x}, M) \rightarrow H_{i-1}(\mathbf{x}|_{1, \dots, s-1}, M)[x_s] \rightarrow 0$$

for every index  $i$ .

Given  $M$  in  $\mathcal{D}^b(R)$ , we call a sequence  $\mathbf{x}$  of  $R$  a *multiplicity system* of  $M$  if  $\ell(H^i(M) \otimes R/\mathbf{x}R)$  is finite for every integer  $i$ . In this case, the Tor base change spectral sequence [93, Tag 0662] implies that the Koszul homology module  $H_i(\mathbf{x}, M)$  has finite length for every integer  $i$ .

Let  $M$  be a finitely generated  $R$ -module. Then for every sequence  $\mathbf{x}$  of non-zero divisors of  $R$ , the above implicitly defines Koszul homology modules  $H_i(\mathbf{x}; M)$  and we will use this terminology henceforth. We remark that this terminology can conflict with the classical one given in [91] if  $\mathbf{x}$  is not a regular sequence of  $R$ . This poses no issue, however, due to.

**Lemma 0.2** ([22, Exercise 17.10]). *Let  $R$  be a ring, let  $M$  be an  $R$ -module, and let  $x_1, \dots, x_s$  be a sequence of elements of  $R$ . Let  $S$  a ring (for example  $R[[Y_1, \dots, Y_s]]$ ), let  $y_1, \dots, y_s$  be a regular sequence of  $S$ , and let  $S \rightarrow R$  be a ring homomorphism that maps  $y_i$  to  $x_i$ . Then*

$$\mathrm{Tor}_i^S(S/(y_1, \dots, y_s)S, M_S) \cong H_i(\mathbf{x}; M)$$

in the classical sense.

The motivation for the above convention on Koszul homology is as follows: When working with Koszul homology, we primarily rely on the Universal Coefficient Theorem 0.1, and the chosen language is the most consistent way to ensure its validity in several relevant cases.

Let  $M$  be a finitely generated  $R$ -module, let  $\mathbf{x}$  be a sequence of elements of  $R$ , and let  $\mu : \mathbf{R}\text{-mod} \rightarrow \mathbb{N}_\infty$  be an additive function. For every natural number  $k$  we let

$$\chi_{\mu, k}(\mathbf{x}; M) := \sum_{i=k}^s (-1)^{i-k} \mu(H_i(\mathbf{x}; M))$$

denote the  $k$ -th *partial Euler-Poincaré characteristic* of the Koszul complex of  $M$  with coefficients  $\mathbf{x}$  with respect to  $\mu$ . We denote  $\chi_{0, \mu}(\mathbf{x}; M)$  also by  $\chi_\mu(\mathbf{x}; M)$  or  $e_\mu(\mathbf{x}, M)$ , depending on context. If  $\mu$  is the length function, we typically drop  $\mu$  from the notation. If  $\mathbf{x}$  is a system of parameters of  $M$ , then

$$\chi_{\ell_R}(\mathbf{x}; M) = e_0((\mathbf{x}); M),$$

where  $e_0(\mathbf{x}; M)$  denotes the leading binomial coefficient of the *Hilbert-polynomial* as was proven in [6, Thm. 4.1] based on mimeographed, non-published notes of J.-P. Serre. For that reason  $e_\ell(\mathbf{x}; M)$  is referred to as the *multiplicity* of  $M$  (with coefficients  $\mathbf{x}$ ).



# Introduction

The original motivation for this thesis stems from P. Scholze's and H. Esnault's result on the cohomological dimension of pro- $\ell$  towers and its connection to the growth of Betti numbers of semi-perverse sheaves, as questioned by A. A. Beilinson. We explore this connection in Part II, where we study the

## Étale Cohomology of Pro- $\ell$ Galois Towers

In the following, let  $k$  be an algebraically closed field, let  $\ell$  be a prime number different from the characteristic of  $k$ , and let  $\ell^v > 1$  be a power of  $\ell$ .

Let us briefly introduce the main notation. A *tower* of schemes is a collection  $\mathbb{I} = (\mathbb{I}\phi^{n+1} : I_{n+1} \rightarrow I_n)_{n \in \mathbb{N}}$  of morphisms of schemes.<sup>3</sup> For all pairs of natural numbers  $s, r$  with  $s > r$  we put  $\mathbb{I}\phi_r^s := \mathbb{I}\phi^{r+1} \circ \mathbb{I}\phi^{r+2} \circ \dots \circ \mathbb{I}\phi^s$ . If all the morphisms  $\mathbb{I}\phi_n^{n+1}$  are finite étale, we say that  $\mathbb{I}$  is *finite étale*. We say that  $\mathbb{I}$  is a *pro- $\ell$  Galois tower* if  $I_0$  is separated and of finite type over  $k$  and there exists a natural number  $t$  such that  $\mathbb{I}\phi_0^r$  is a connected Galois covering with Galois group  $(\mathbb{Z}/\ell^r\mathbb{Z})^t$  for every natural numbers  $r$ . In this case, we call  $t$  the type of  $\mathbb{X}$ . A morphism of towers

$$(\mathbb{X}\phi^{n+1} : X_{n+1} \rightarrow X_n)_{n \in \mathbb{N}} \rightarrow (\mathbb{Y}\phi^{n+1} : Y_{n+1} \rightarrow Y_n)_{n \in \mathbb{N}}$$

is a collection of morphisms  $f_n : Y_n \rightarrow X_n$  such that  $f_n \circ \mathbb{Y}\phi^{n+1} = \mathbb{X}\phi^{n+1} \circ f_{n+1}$  on every level  $n$ .

In astérisque 100 [10] the theory of (middle) perverse sheaves was developed by A. A. Beilinson, J. Bernstein and P. Deligne. In [19] P. Deligne introduced the theory of weights that is central for the proof of the *general Weil Conjectures*. In astérisque 100 the theory of global weights and the notation of purity of sheaves are placed into the framework of the category of perverse sheaves and it is only in this generality that the theory of weights obtains its final form. In this course, the following proposition is central, as it is used in [10] to prove via [10, Cor. 4.5.5] the purity theorem for the intermediate perverse extension [10, Cor. 5.3.3].

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<sup>3</sup>If we refer to a tower symbolically by  $\mathbb{I}$ , we refer to the schemes and morphisms that are part of its datum as  $I_n$  and  $\mathbb{I}\phi^{n+1}$  by default.

## Introduction

**Proposition 0.3** ([10, 4.5.1]). *Let  $(X_a)_{1 \leq a \leq n}$  be smooth projective curves over  $k$ , let  $X$  denote their product and let  $K$  be an object of  ${}^p\mathcal{D}_c^{\leq 0}(X, \overline{\mathbb{Q}}_\ell)$ . Then there exists a constant  $C$  such that, given connected étale coverings  $\tilde{X}_a$  of  $X_a$  of degree  $d_a$  for every  $1 \leq a \leq n$ , one has*

- i.)  $\dim_{\overline{\mathbb{Q}}_\ell} H^i(\tilde{X}, K) \leq C \cdot \prod_{a=1}^n d_a$  for every integer  $i$ ;
- ii.)  $\dim_{\overline{\mathbb{Q}}_\ell} H^i(\tilde{X}, K) \leq C \cdot \sup(\prod_{a \in A} d_a \mid |A| \subseteq \{1, \dots, n\} \text{ with } |A| = n - i)$  for every integer  $0 < i \leq n$ ;
- iii.)  $H^i(\tilde{X}, K) = 0$  for every  $i > n$ ,

where  $\tilde{X}$  denotes the product of the  $\tilde{X}_a$ .

The primary objective of Part II is to prove the following variant of Proposition 0.3 using new methods.

**Theorem 0.4** (11.1). *Let  $\mathbb{X}$  be a pro- $\ell$  Galois tower, let  $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_s$  be type one pro- $\ell$  Galois towers of curves, let  $\mathbb{C} = \mathbb{C}_1 \times_k \dots \times_k \mathbb{C}_s$  denote their product<sup>4</sup>, and assume that there exists a level-wise quasi-finite separated morphism  $f : \mathbb{X} \rightarrow \mathbb{C}$ . Let  $R$  denote any of the rings  $\mathbb{F}_{\ell^v}, \mathbb{Z}_\ell, \mathbb{Q}_\ell, \overline{\mathbb{Q}}_\ell$ . For every object  $K$  of  ${}^p\mathcal{D}_c^{\geq 0}(X_{\text{ét}}, R)$  one has*

$$\text{rank}_R(H_c^i(X_n, (\mathbb{X}\phi_0^n)^* K)) \in \begin{cases} \mathcal{O}(\ell^{ns}) & \text{for all } i \in \mathbb{Z}; \\ \mathcal{O}(\ell^{n(s-|i|)}) & \text{for all } -s < i < 0, \end{cases}$$

and  $H^i(X_n, (\mathbb{X}\phi_0^n)^* K)$  vanishes for all  $n$  and all  $i \leq -s$ . Dually, for every object  $K$  of  ${}^p\mathcal{D}_c^{\leq 0}(X_{\text{ét}}, R)$  one has

$$\text{rank}_R(H^i(X_n, (\mathbb{X}\phi_0^n)^* K)) \in \begin{cases} \mathcal{O}(\ell^{ns}) & \text{for all } i \in \mathbb{Z}; \\ \mathcal{O}(\ell^{n(s-i)}) & \text{for all } s > i > 0, \end{cases}$$

and  $H^i(X_n, (\mathbb{X}\phi_0^n)^* K)$  vanishes for all  $n$  and all  $i \geq s$ .

We note that Theorem 0.3 suffices for the prove of [10, Cor. 5.3.3]. A. A. Beilinson remarked, as documented in [24], that [10, 4.5.1] has some resemblance to

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<sup>4</sup>If the expression is unclear; see 8.1.

**Theorem 0.5** (H. Esnault, [24]). *Let  $X \subseteq \mathbb{P}^n$  be a projective scheme of dimension  $d$  over  $k$ . Choose coordinates  $(x_0 : \dots : x_n)$  on  $\mathbb{P}^n$  and define the covers*

$$\begin{aligned} \phi^r : \mathbb{P}^n &\rightarrow \mathbb{P}^n \\ (x_0 : \dots : x_n) &\mapsto (x_0^{\ell^r} : \dots : x_n^{\ell^r}). \end{aligned}$$

*Let  $X_r$  be the inverse image of  $X$  by  $\phi^r$ , and let  $\mathcal{F}$  be a constructible étale sheaf of  $\mathbb{F}_\ell$ -vector spaces on  $\mathbb{P}^n$  with support of dimension  $d$ . Then*

$$\operatorname{colim}_r H^i(X_r, (\phi^r)^* \mathcal{F}) = 0$$

*for all  $i > d$ .*

Theorem 0.5 was first obtained by P. Scholze in the case  $\operatorname{char}(k) = 0$  and  $\mathcal{F} = \mathbb{F}_\ell$  in [90] as a corollary of his theory of perfectoid spaces.

Our primary insight is that the cohomology modules in Theorem 0.5 and those in Proposition 0.4 are connected by the (*generalized*) *Fourier-Mellin transform*, as studied in [25]. This connection enables us to use a variant of Theorem 0.5 in order to prove Theorem 0.4. This method is new, as it differs from the approach in [10]. Let us provide a brief outline of the individual steps.

*Step 1:* In a first step, we will study the étale cohomology of finite étale towers in *Chapter 8*. Given a finite étale tower  $\mathbb{I}$ , an object  $K$  in  $\mathcal{D}_{\operatorname{tor}}^+(I_{0\text{ét}})$ , and an index  $k$ , we define

$$H^k(\mathbb{I}, K) := \operatorname{colim}_r H^k(I_{r, \mathbb{I}} \phi_0^{r*} K).$$

If  $I_0$  is separated and of finite type over an algebraic separably closed field, we define

$$H_c^k(\mathbb{I}, K) := \operatorname{colim}_r H_c^k(I_{r, \mathbb{I}} \phi_0^{r*} K).$$

In both cases the transition morphisms are induced by the restriction; see [93, Tag 03SH]. We will demonstrate that the so defined étale cohomology of finite étale towers satisfies fundamental theorems, including excision 8.12, the existence of a Mayer-Vietoris sequence 8.14, and the existence of Leray spectral sequences 8.15, 8.16.

*Step 2:* By utilizing the étale cohomology of finite étale towers, in *Chapter 9* we will prove the following theorem along the lines of [24]:

**Theorem 0.6** (9.12). *Let  $\mathbb{C}_1, \dots, \mathbb{C}_n$  be type one pro- $\ell$  Galois towers of curves over  $k$ , let  $f : \mathbb{X} \rightarrow \mathbb{C}_1 \times_k \dots \times_k \mathbb{C}_n$  be a morphism of towers that is level-wise quasi-finite and separated over  $k$ , and let  $K$  be an object of  ${}^p\mathcal{D}_c^{\leq 0}(X_{\text{ét}}, \mathbb{F}_{\ell^v})$ . Then  $H_c^i(\mathbb{X}, K)$  vanishes for every  $i > 0$ .*

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*Step 3:* In *Chapter 10* we will study the (generalized) Fourier-Mellin transform. In the following, let  $\mathbb{X}$  be a pro- $\ell$  Galois tower of type  $s$ . The (generalized) Fourier-Mellin transform of  $\mathbb{X}$  is a functor

$$\mathrm{FM}_{\mathbb{X}} : \mathcal{D}_c^b(X, \mathbb{F}_{\ell^v}) \rightarrow \mathcal{D}_{perf}^b(\mathbb{F}_{\ell^v}[[T_1, \dots, T_s]])$$

coherently interpolating the cohomology on  $\mathbb{X}$ , in the sense that

$$H_i(T_1^{\ell r}, \dots, T_s^{\ell r}; \mathrm{FM}_{\mathbb{X}}(D(K))) \cong H_c^i(X_r, (\mathbb{X}\phi_0^n)^*K)^\vee \quad (1)$$

and

$$\lim_r H^{-i}(X_r, (\mathbb{X}\phi_0^r)^*D(K)) \cong H^{-i}(\mathrm{FM}_{\mathbb{X}}(D(K))) \cong H_c^i(\mathbb{X}, K)^\vee \quad (2)$$

for all objects  $K$  of  $\mathcal{D}_c^b(X, \mathbb{F}_{\ell^v})$  and every index  $i$ . In (1) the homology on the left is Koszul homology<sup>5</sup>. In (2) the transition morphisms on the left-hand side are induced by the trace, while those on the right-hand side are induced by the restriction; see [93, Tag 03SH].

Let  $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_s$  be type one pro- $\ell$  Galois towers of curves<sup>6</sup>, let  $\mathbb{C}$  denote their product, assume that there exists a level-wise quasi-finite separated morphism  $f : \mathbb{X} \rightarrow \mathbb{C}$  over  $k$ , and let  $K$  be an object of  ${}^p\mathcal{D}_c^{\geq 0}(X_{et}, \mathbb{F}_{\ell^v})$ . Then Theorem 0.6 and (2) imply that  $\mathrm{FM}_{\mathbb{X}}(D(K))$  is an object of  $\mathcal{D}_{perf}^{\leq 0}(\mathbb{F}_{\ell^v}[[X_1, \dots, X_s]])$  for all objects  $K$  in  ${}^p\mathcal{D}_c^{\geq 0}(X_{et}, \mathbb{F}_{\ell^v})$ . The latter fact allows us, making use of (1) and the Tor Base change spectral sequence, to deduce Theorem 0.4 from classical growth estimates of Koszul homology due to D. Kirby [56]. This will be accomplished in *Chapter 11*.

Motivated by this new point of view, more generally, we question:

**Question 0.7.** Let  $R$  be a noetherian ring, let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of non-zero divisors of  $R$ , and let  $M$  be an object of  $\mathcal{D}_{ft}^b(R)$ . What can be said about the growth of the Koszul homology groups  $H_i(x_1^{n_1}, \dots, x_s^{n_s}; M)$  in the natural numbers  $n_1, \dots, n_s$ ?<sup>7</sup>

In particular, one may question whether it is possible to derive polynomial(-like)<sup>8</sup> growth in the context of Theorem 0.4. The primary focus of Part I is to answer Question 0.7.

<sup>5</sup>See Preliminaries on Koszul Homology V.

<sup>6</sup>That is to say, the underlying schemes of their datum are curves.

<sup>7</sup>Here, *Koszul homology* is to be understood in a derived sense; see the Preliminaries on Koszul Homology V.

<sup>8</sup>For the definition of polynomial-linkeness; see the Conventions on Functions IV.

## Growth of Koszul Homology

In the following, if not said otherwise, let  $R$  be a noetherian local ring, let  $M$  be a finitely generate  $R$ -module of dimension  $d$ , and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system<sup>9</sup> of  $M$ . Let  $\chi_{M,i}^{\mathbf{x}}(n) := \ell_R(\text{Tor}_i(R/(\mathbf{x})^{n+1}, M))$ . Going back to D. Hilbert and P. Samuel, the growth of the function  $\chi_{M,0}^{\mathbf{x}}(n)$ , which we nowadays refer to as the *Hilbert-Samuel function*, has been well studied as an integral part of *dimension theory*; see [91]. As an application of the Hilbert-Serre Theorem [78, Ch. 7 Thm. 19], one derives that the function  $\chi_{M,0}^{\mathbf{x}}(n)$  agrees with a polynomial  $P$  of degree  $d$  for all sufficiently large values of  $n$ . In this context, the polynomial  $P$  is referred to as the *Hilbert-Samuel polynomial*. Under certain mild modifications of the prerequisites, one can observe polynomial-like growth in higher degrees as well; see [59, Thm. 2]. Exploiting the inclusions

$$(\mathbf{x})^{sn} \subseteq (x_1^n, \dots, x_s^n) \subseteq (\mathbf{x})^n,$$

it follows that the function  $\ell_R(M/(x_1^n, \dots, x_s^n)M)$  is both upper and lower bounded by polynomials of degree  $d$ . Moreover,  $\ell_R(M/(x_1^{n_1}, \dots, x_s^{n_s})M)$  is polynomial-like in every single variable. Hence, it arises as a natural question whether the functions

$$\ell_R(M/(x_1^n, \dots, x_s^n)M) \quad \text{and} \quad \ell_R(M/(x_1^{n_1}, \dots, x_s^{n_s})M)$$

themselves are polynomial(-like)? In this strong form the answer to the question is negative in general. J.-L. García Roig and D. Kirby provided a counterexample in [33].<sup>10</sup> In *Chapter 1* we will additionally provide the following example, which can be aligned with the situation of the above Step 3.

**Example 0.8** (1.7, 1.9). Let  $K$  be a field, let  $a, b$  be elements of  $K^\times$ , and let  $u, i$  and  $j$  be positive natural numbers. The function

$$\ell(K[X, Y]/(aX^i + bX^jY^u, X^n, Y^m))$$

is polynomial-like if and only if  $1 \leq i \leq j$ .

Although the growth of Koszul homology for powers of a sequences fails to be polynomial-like in general, it can nonetheless be described “very tightly” by a uniquely determined polynomial that encodes structural information of the module. In *Chapter 6* we will present the following new

<sup>9</sup>For the definition see the Preliminaries on Koszul Homology V.

<sup>10</sup>We remark that  $\ell_R(M/(x_1^n, \dots, x_s^n)M)$  may be polynomial-like, while  $\ell_R(M/(x_1^{n_1}, \dots, x_s^{n_s})M)$  is not polynomial-like; see [71].

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**Theorem 0.9** (6.11). *Let  $R$  be a noetherian ring, let  $M$  be a bounded object of  $\mathcal{D}_{ft}^{\geq 0}(R)$ , and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $M$  consisting of non-zero divisors. For every index  $k$  let*

$$\Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M) := \sum_{I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \cdot \prod_{i \in I} N_i.^{11}$$

*Then for all  $s$ -tuples  $\underline{n} = n_1, \dots, n_s$  of positive natural numbers one has the estimate*

$$\Upsilon_{\mathbf{x},k}(\underline{n}) \leq \ell_R(H_k(\mathbf{x}^{\underline{n}}; M)).$$

*If  $P$  is a polynomial in  $\mathbb{R}[X_1, \dots, X_s]$  such that*

$$\Upsilon_{\mathbf{x},k}(\underline{n}; M) \leq P(\underline{n}) \leq \ell_R(H_k(\mathbf{x}^{\underline{n}}; M))$$

*for all positive  $\underline{n} = n_1, \dots, n_s$ , then  $P = \Upsilon_{\mathbf{x},k}$ .*

We will prove that all of the coefficients of the polynomials  $\Upsilon_{\mathbf{x},k}(\underline{N}; M)$  are non-negative. If  $R$  is a noetherian local ring,  $M$  is a finitely generated  $R$ -module, and  $\mathbf{x} = x_1, \dots, x_s$  is a system of parameters of  $M$ <sup>12</sup>, then employing results of Nguyen Tu Cuong and Vu The Khoi from [76], we will derive that the function  $\ell_R(H_k(x_1^{n_1}, \dots, x_s^{n_s}; M))$  gets bounded above by a polynomial of total degree  $\deg(\Upsilon_{\mathbf{x},k}(\underline{N}; M))$ . In fact, the total degree  $\deg(\Upsilon_{\mathbf{x},k}(\underline{N}; M))$  is independent of the system of parameters and we denote the corresponding numerical invariant of  $M$  by  $\rho_k(M)$ . It is natural to question what the numerical invariants  $\rho_k(M)$  encode. Our principal new result is

**Theorem 0.10** (6.19). *Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module of dimension  $d$ , and let  $-1 \leq n \leq d - 1$  be a natural number. Then  $\rho_1(M) = n$  if and only if*

- i.) The module  $M/\mathbf{x}|_{1, \dots, d-n-1}M$  has a submodule of dimension  $n$ ;*
- ii.) For every index  $s = 1, \dots, d - n - 2$  the quotient module  $M/(\mathbf{x}|_{1, \dots, s})M$  has no submodule of dimension  $m$  with  $n < m < d - s$ .*

The numerical invariant  $\rho_1(M)$  was studied before in [17], where it was shown that  $\rho_1(M)$  equals the dimension of the non-Cohen-Macaulay locus of  $M$ , provided that  $R$  admits a dualizing complex.

<sup>11</sup>Here  $e(-, -)$  denote the multiplicity symbol; see Preliminaries on Koszul Homology V.

<sup>12</sup>Via the canonical map  $\mathbf{R} - \mathbf{mod} \rightarrow \text{Comp}(R) \rightarrow \mathcal{D}(R)$  we regard  $M$  as an element of  $\mathcal{D}(R)$ .

In *Chapter 7* we will proceed to study those sequences for which equality holds true in Theorem 0.9 in degree 0.

In the following, we provide a brief overview of the *Chapters 2, 3, 4* and *5*.

*Chapter 2:* As a preparation for *Chapter 6*, we study functions  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  that exhibit polynomial-likeness in the single variables for any fixed value of the other variables.

*Chapter 3:* When studying the growth of Koszul homology, a priori it seems advantageous to take the “non-artinian case” into consideration. Instead of working with the length, one may use any suitable additive function  $\mathbf{R} - \mathbf{mod}_x \rightarrow G^+$ <sup>13</sup> where  $G^+$  is the positive cone of a partially ordered abelian group  $G$ . We will demonstrate that there is a universal way for such functions to arise. The key new result is as follows:

**Theorem 0.11** (3.3,3.10). *Let  $R$  be a noetherian local ring, and let  $\mathbb{Z}[\min\text{Spec}(R)]$  denote the free abelian group generated by the minimal prime ideals of  $R$ . The group  $\mathbb{Z}[\min\text{Spec}(R)]$  is a direct summand of the Grothendieck group  $G_0(R)$ <sup>14</sup> and the projection corresponds to the morphism*

$$G_0(R) \rightarrow \mathbb{Z}[\min\text{Spec}(R)]$$

*given on generators via*

$$[M] \mapsto \sum_{\mathfrak{p} \in \min\text{Spec}(R)} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot \mathfrak{p}.$$

*If  $\varphi : \mathbf{R} - \mathbf{mod} \rightarrow G^+$  is an additive function<sup>15</sup> to a positive cone  $G^+$  of a partially ordered abelian group  $G$ , then  $\varphi$  uniquely factors over*

$$\mathbf{R} - \mathbf{mod} \rightarrow G_0(R) \rightarrow \mathbb{Z}[\min\text{Spec}(R)].$$

A posteriori Theorem 0.11 indicates that, when studying the growth of sequences of modules, one should always use the length function. Anyways, Theorem 0.11 has the following application which is “valuable” in its own right.

**Corollary 0.12** (3.12). *Let  $R$  be a noetherian local ring, and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow G^+$  be an additive function to the positive cone  $G^+$  of a partially ordered abelian group  $G$ . Then  $\mu(R/\mathfrak{q})$  is zero whenever  $\mathfrak{q}$  is not a minimal prime ideal of  $R$ .*

<sup>13</sup>The category  $\mathbf{R} - \mathbf{mod}_x$  is defined in Conventions on Partially Ordered Groups III.

<sup>14</sup>For a definition; see [107, Def. 6.2].

<sup>15</sup>For a definition; see Conventions on Partially Ordered Groups III.

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Corollary 0.12 complements results of G. Krause; see [60]. For a noetherian local integral domain and  $G = \mathbb{R}$  the corollary is well-known; see [79, Thm. 2].

*Chapter 4:* We will very briefly study logical dependences between fundamental properties of intersection multiplicities, as Corollary 0.12 offers direct “minor” applications.

*Chapter 5:* We will study generalization of the multiplicity symbol  $e(-, -)$ , as it plays an important role in Theorem 0.9, and we will establish preparatory results to be used in *Chapter 6*. Using the results of *Chapter 3*, we will construct a “dimension sensitive general multiplicity symbol”.

Historically, the growth of Koszul homology as studied in Part I is “deeply” connected with the theory of *generalized Cohen-Macaulay* modules. We will recall the precise definition of generalized Cohen-Macaulay modules below. Briefly explained, the connection is as follows: Let  $R$  be a noetherian local ring, and let  $M$  be a finitely generated  $R$ -module. Then  $M$  is Cohen-Macaulay if and only if  $\rho_1(M) = -1$  and it is generalized Cohen-Macaulay if and only if  $\rho_1(M) = 0$ . The results of *Chapter 6* provide new perspectives on the theory of generalized Cohen-Macaulay modules. They may serve as a new foundation for a more general theory, as they pave the way for studying “general Cohen-Macaulay complexes”. A vast part of properties of Cohen-Macaulay modules continue to hold for generalized Cohen-Macaulay modules with an “exception in a single degree”. For example, Theorem 0.10 is a new appearance of this phenomena. The author conjectures that a significant portion of statements regarding (generalized) Cohen-Macaulay modules should extend naturally, with the coefficients of the polynomials  $\Upsilon_{\mathbf{x},k}(\underline{N}; M)$  determining the specific range in which these statements continue to hold true; see Outlook 6.22. In the following, we provide a historical overview of generalized Cohen-Macaulay modules and related theories, while highlighting where these theories had valuable applications. A more detailed account can be found in [101] and a comprehensive overview is provided in [102].

## Historical Background

In [15] D.A. Buchsbaum questioned the following: Let  $R$  be a noetherian local ring. Does there exist an invariant  $I(R)$  such that

$$\chi_1(\mathbf{x}; R) := \ell_R(R/\mathbf{x}R) - e_0(\mathbf{x}; R) = I(R)$$

for all system of parameters  $\mathbf{x}$  of  $R$ ? Here  $e_0(\mathbf{x}; R)$  denotes the leading binomial coefficient of the Hilbert-Samuel polynomial associated to the pair  $(R, \mathbf{x})$ . We note that  $e_0(\mathbf{x}; R) = e(\mathbf{x}; R)$  after J.-P. Serre.

As is documented in [101], this question was motivated by the work on

**Bézout’s Theorem.** Let  $X, Y$  be unmixed (i.e each component of  $X$  and  $Y$  has the dimension of  $X$  or  $Y$ , resp.) varieties of the projective  $n$ -space  $\mathbb{P}_k^n$  over an algebraically closed field  $k$  such that  $\dim(X \cap Y) = \dim(X) + \dim(Y) - n$ . Letting  $C$  run over all proper components of the intersection  $X \cap Y$  (i.e (irreducible) components with dimension equal to  $\dim(X \cap Y)$ ), we get that there exists an “intersection multiplicity”, say  $i(X, Y, C)$ , of  $X$  and  $Y$  along  $C$  such that the following “*Anzahl-Formel*” is true:

$$\deg(X) \cdot \deg(Y) = \sum_C i(X, Y, C) \cdot \deg(C).$$

The development of Bézout’s Theorem in its full generality has spanned a century or two, requiring considerable effort and extensive work to reach its current form; see Kleiman [57, 58]. At the beginning of the twentieth century one investigated the notion of length of a primary ideal in order to define an intersection multiplicity  $\mu(X, Y; C)$  with the hope that it provides the correct multiplicity for Bézout’s Theorem in  $\mathbb{P}^n$ . In 1928 B. L. van der Waerden has show that this idealtheoretic intersection multiplicity does not yield the correct multiplicity for Bézout’s Theorem to be valid in projective spaces  $\mathbb{P}^n$  with  $n \geq 4$ ; see [103]. Nowadays, it is of course known that in general the “length”  $\mu(X, Y; C)$  gives the “correct” multiplicity  $i(X, Y; C)$  if and only if the local rings  $A(X, C)$  of  $X$  at  $C$  and  $A(Y, C)$  of  $Y$  at  $C$  are Cohen-Macaulay for all proper components  $C$  of  $X \cap Y$  where  $\dim(X \cap Y) = \dim(X) + \dim(Y) - n$ ; see [91, V] and the introduction of [101]. In order to prove Bézout’s Theorem, one can assume without loss of generality that the intersection variety  $X$  is a complete intersection; see [101, Thm. 1]. Then

$$\mu(X, Y; C) \geq i(X, Y; C)$$

for each proper component  $C$ , and from the historical perspective it was natural to ask whether the difference  $\mu(X, Y; C) - i(X, Y; C)$  is independent of  $X$ . From the

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viewpoint of local algebra, this is equivalent to the question of D. A. Buchsbaum stated above. Certainly, based on the preceding discussion, we observe that the answer to the question is negative. If such an invariant  $I(R)$  exists for a given noetherian local ring  $R$ , then  $\ell_R(R/\mathbf{x}^n R)$  is a polynomial function for all systems of parameters  $\mathbf{x}$  of  $R$ , which is not the case in general.<sup>16</sup> Historically, first negative answers to the problem of D. A. Buchsbaum were given by J. Stückrad and W. Vogel in [97] and, eventually the theory of *Buchsbaum* and *generalized Cohen-Macaulay* modules arose from this [99, 98, 68, 70].

Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module, let  $P(M)$  denote the set of systems of parameters of  $M$ , and for every function  $f : P(M) \rightarrow \mathbb{Z}$  let  $\text{Spec}(f)$  denote the image of  $f$  in  $\mathbb{Z}$ . The module  $M$  is called *Buchsbaum* if  $|\text{Spec}(\chi_1(-; M))| = 1$  and it is called *generalized Cohen-Macaulay* if  $|\text{Spec}(\chi_1(-; M))|$  is finite. We recall that  $M$  is Cohen-Macaulay if and only if  $0$  is contained in  $\text{Spec}(\chi_1(-; M))$ , equivalently  $\text{Spec}(\chi_1(-; M)) = \{0\}$ ; see [91, Ch. IV Thm. 3; Appendix II]. We note that Buchsbaum and generalized Cohen-Macaulay modules can be characterized by the cardinality of  $\text{Spec}(f)$  for various maps  $P(M) \rightarrow \mathbb{Z}$  where Cohen-Macaulay modules are characterized by  $0 \in \text{Spec}(f)$  or  $\text{Spec}(f) = \{0\}$ , thereby offering a suitable generalization of the theory. This is the case for  $\ell(H_1(-; M))$  by [76] and it is true for the first Hilbert coefficient  $e_1(-; M)$  provided that the ring  $R$  is unmixed; see [34, 40].

We remark that the theory of generalized Cohen-Macaulay modules is necessary for the classification of algebraic curves in  $\mathbb{P}^3$ , and it was applied in the studies of singularities of algebraic varieties. It was shown that interesting and extensive classes of Buchsbaum and generalized Cohen-Macaulay modules exist.

Our principal results of *Chapter 6* were previously known only in the highly specific case of systems of parameters that are *dd-sequences*. They are characterized as follows: Let  $R$  be a noetherian local ring, and let  $M$  be a finitely generated  $R$ -module. A system of parameters  $\mathbf{x} = x_1, \dots, x_s$  of  $M$  is a *dd-sequence* if and only if there exist natural numbers  $\lambda_0, \dots, \lambda_s$  such that

$$\ell(M/(x_1^{n_1}, \dots, x_s^{n_s})M) = \sum_{r=0}^s \lambda_r \prod_{i=1}^r n_i$$

for all  $n_1, \dots, n_s > 0$ . For the general definition of a *dd-sequences*; see [2]. Equivalent characterization of *dd-sequences* and explicit formulas for the coefficients  $\Lambda_r$  are provided in [1]. We remark that from our new point of view the formulas for the coefficients  $\Lambda_r$  can be directly derived from Theorem 0.9; see Corollary 6.6. Further, we note that *Chapter 6* is quite “technical” and it contains several involved

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<sup>16</sup>We remark that the two statements are in-fact equivalent due to [71, Cor. 7].

calculations. In the aftermath, however, we derive general results that unify the calculations in the literature; e.g. see [1, 2]. This allows to reduce redundancies. It was proven in [75, Thm. 1.2, Thm. 1.3] that a local ring is a homomorphic image of a Cohen-Macaulay ring if and only if every finitely generated module has a system of parameters which is a  $dd$ -sequence. We remark that the existence of  $dd$ -sequences alone, however, does not suffice for the proof of Theorem 0.10. We further remark that  $dd$ -sequences are a stronger form of  $d$ -sequences introduced by C. Huneke in [52]. Systems of parameters that are  $d$ -sequences constitute the only class of systems of parameters for which the Hilbert coefficients are “well” understood; see [40, 2]. It is natural to ask, whether Theorem 0.10 offers new perspectives; see Outlook 8.14.

Systems of parameters which are  $dd$ -sequence have several important applications. They are at the heart of the theory of Buchsbaum and generalized Cohen-Macaulay modules [88, 68, 89], and they are crucial in the theory of sequential Cohen-Macaulay and sequential generalized Cohen-Macaulay modules [73, 74]. Additionally, they played a pivotal role in G. Falting’s Macaulayfication program; see [26, 53, 55, 75], which was recently completed by K. Česnavičius in [106]. Relying on the work of T. Kawasaki, the key point in the work of K. Česnavič is to use  $dd$ -sequences<sup>17</sup> to determine the “correct centers for blowing ups” such that locally the invariants  $\rho_1(-)$  can be successively decreased. In [54] T. Kawasaki utilized the theory of  $dd$ -sequences to establish the arithmetic Cohen-Macaulayfication of noetherian local rings and subsequently he gave an affirmative answer to the *Sharp conjecture* concerning the existence of dualizing complexes as posed in [92].

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<sup>17</sup>The notion of T. Kawasaki’s  $p$ -standard sequences (of type 0) [55, Def. 2.2] is equivalent to the notion of  $dd$ -sequences by [2, Prop. 3.4].



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# Part I

## Growth of Koszul Homology



# Chapter 1

## A Case Study in Dimension 2

In the following, let  $K$  be a field. In this chapter, we will study the growth of  $\ell(K[X, Y]/(P, X^n, Y^m))$  where  $P$  is a polynomial in  $K[X, Y]$ . We will provide several examples that demonstrate whether  $\ell(K[X, Y]/(P, X^n, Y^m))$  is polynomial-like or not.

Let  $M$  be a finitely generated module over a noetherian ring  $R$ , and let  $x, y$  be elements in  $R$ . Then we have

**Lemma 1.1.** *Let  $M$  be a noetherian module over a commutative ring  $R$ . Then we have an exact sequence*

$$0 \rightarrow M[x] \rightarrow M[xy]M \rightarrow M[y] \rightarrow M/xM \rightarrow M/(xy)M \rightarrow M/yM \rightarrow 0,$$

for all elements  $x, y$  in  $R$ .

*Proof:* Alluding to Lemma 0.2, we can assume without loss of generality that  $x$  and  $y$  are regular elements of  $R$ . The image of  $R/xR$  in  $R/(xy)R$  under the multiplication-by- $y$ -map is given by the quotient  $yR/xyR$ , whence the exact sequence

$$0 \rightarrow R/xR \xrightarrow{y} R/(xy)R \rightarrow R/yR \rightarrow 0.$$

Tensoring with  $M$  yields the proclaimed exact sequence. ■

Assume additionally that  $M/(x, y)M$  is artinian. Then there are two exact sequences

$$0 \rightarrow M/x^n M[y] \rightarrow M/x^n M[y^m] \rightarrow M/x^n M[y^{m-1}] \rightarrow$$

$$M \otimes R/(x^n, y)R \rightarrow M \otimes R/(x^n, y^m)R \rightarrow M \otimes R/(x^n, y^{m-1})R \rightarrow 0$$

and

$$0 \rightarrow M/y^m M[x] \rightarrow M/y^m M[x^n] \rightarrow M/y^m M[x^{n-1}] \rightarrow$$

$$M \otimes R/(y^m, x)R \rightarrow M \otimes R/(y^m, x^n)R \rightarrow M \otimes R/(y^m, x^{n-1})R \rightarrow 0$$

for all pairs of natural numbers  $n, m$  and  $\ell(M/(x^n, y^m)M)$  is polynomial-like if and only if there exists a constant  $\beta_1$  that is independent of  $n$  and a constant  $\beta_2$  that is independent of  $m$  such that

$$M/x^n M[y^m] = M/x^n M[y^\infty],$$

for all  $m \geq \beta_1$ , and

$$M/y^m M[x^n] = M/y^m M[x^\infty],$$

for all  $n \geq \beta_2$ .<sup>1</sup> This leads to the guiding

**Question 1.2.** What can be said about the existence of such constants?

In order to precisely track the stabilization of principal torsion, we make use of

**Definition 1.3.** Given a noetherian module  $M$  over a commutative ring  $R$ , and given an element  $a$  of  $R$ , we let  $\tau(M, a)$  denote the smallest natural number  $n$  satisfying the identity  $M[a^n] = M[a^{n+1}]$ .

Let us present a first

**Example 1.4.** Let  $x, y$  be regular sequence of a noetherian local ring  $R$  of dimension 2. Then the function  $\ell(\mathfrak{a}/(x^n, y^m)\mathfrak{a})$  is polynomial-like for every ideal  $\mathfrak{a}$  of  $R$ . Indeed, for every natural number  $n$  we have an exact sequence

$$0 \rightarrow R/\mathfrak{a}R[x^n] \rightarrow \mathfrak{a}/x^n \mathfrak{a} \rightarrow R/x^n R \rightarrow R/(\mathfrak{a}, x^n)R \rightarrow 0,$$

subsequently

$$R/\mathfrak{a}R[x^n, y^m] \cong \mathfrak{a}/x^n \mathfrak{a}[y^m]$$

for every natural number  $m$ , and the roles of  $n$  and  $m$  can be interchanged.

In the situation of Example 1.4, there are instances where  $\ell(R/(\mathfrak{a}, x^n, y^m)R)$  fails to be polynomial-like, due to an obstruction arising from the “telescopic sum phenomena”

**Lemma 1.5.** Let  $a, b, p, q$  be elements of a commutative ring  $R$ , and let  $t \leq s$ ,  $0 < r$  and  $0 \leq i < r$  be natural numbers. Then

$$(bp^t q - ap^s) \cdot \left( \sum_{j=0}^i p^{(s-t)(r-j-1)} q^j a^{i-j} b^j \right) = -a^{i+1} p^{(s-t)r+t} + b^{i+1} p^{(s-t)(r-i-1)+t} q^{i+1}.$$

---

<sup>1</sup>A detailed general prove of this fact is given in *Chapter 7*. We also note that Lemma 7.8 will establish the equivalence between the existence of  $\beta_1$  and  $\beta_2$ .

*Proof:* We prove the lemma by an ascending induction on  $i$ . The induction start  $i = 0$  is given by

$$(bp^tq - ap^s) \cdot p^{(s-t)(r-1)} = -ap^{(s-t)r+t} + bp^{(s-t)(r-1)+t}q.$$

Let  $i > 0$ , and assume that the stated equation holds true for  $i - 1$ . Then

$$\begin{aligned} & (bp^tq - ap^s) \cdot \left( \sum_{j=0}^i p^{(s-t)(r-j-1)} q^j a^{i-j} b^j \right) \\ &= (bp^tq - ap^s) \cdot \left( a \sum_{j=0}^{i-1} p^{(s-t)(r-j-1)} q^j a^{i-1-j} b^j \right) + (bp^tq - ap^s) \cdot p^{(s-t)(r-i-1)} q^i b^i \\ &= a \cdot \left( -a^i p^{(s-t)r+t} + b^i p^{(s-t)(r-i)+t} q^i \right) + (bp^tq - ap^s) \cdot p^{(s-t)(r-i-1)} q^i b^i \\ &= -a^{i+1} p^{(s-t)r+t} + ab^i p^{(s-t)(r-i)+t} q^i - ab^i p^{(s-t)(r-i-1)+s} q^i + b^{i+1} p^{(s-t)(r-i-1)+t} q^{i+1} \end{aligned}$$

and employing

$$(s-t) \cdot (r-i) + t = (s-t) \cdot (r-i) - (s-t) + s = (s-t) \cdot (r-i-1) + s,$$

we derive the induction step. ■

**Corollary 1.6.** *Let  $a, b, p$  and  $q$  be non-zero elements of an integral domain  $R$ , and let  $t < s$  and  $1 < r$  be natural numbers where  $p$  is a prime element not dividing  $bq$ , and the image of  $b \cdot a^{r-1}$  in  $R/qR$  is not divisible by the residue class of  $p$  in  $R/qR$ . Then*

$$\tau \left( R / (bp^tq - ap^s, p^{(s-t)r+t}) R, q \right) \geq r - 1.$$

*Proof:* Let  $\mathfrak{a}$  denote the ideal  $(bp^tq - ap^s, p^{(s-t)r+t})$ , and let  $i$  denote  $r - 2$ . Alluding to Lemma 1.5, it suffices to prove that  $b^{i+1} p^{(s-t)(r-i-1)+t} q^i$  is not contained in  $\mathfrak{a}$ .

Let  $\tilde{r}$  denote  $r - 1$ . Then

$$(bp^tq - ap^s) \cdot \left( \sum_{j=0}^{i-1} p^{(s-t)(\tilde{r}-j-1)} q^j a^{i-1-j} b^j \right) = -a^i p^{(s-t) \cdot \tilde{r} + t} + b^i p^{(s-t)(\tilde{r}-i)+t} q^i$$

by Lemma 1.5. Consequently,  $b^{i+1} p^{(s-t)(r-i-1)+t} q^i$  is an element of  $\mathfrak{a}$  if and only if  $-a^i p^{(s-t)(r-1)+t} \cdot b$  is an element of  $\mathfrak{a}$ . For the sake of contradiction, assume that there are elements  $c$  and  $d$  in  $R$  satisfying the equation

$$cp^t (bq - ap^{s-t}) = p^{(s-t)r+t} d - p^{(s-t)(r-1)+t} a^i \cdot b.$$

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Then

$$c(bq - ap^{s-t}) = p^{(s-t)(r-1)}(p^{s-t}d - a^i \cdot b).$$

Since  $p$  is a prime element not dividing  $bq$ , there exists an element  $c'$  in  $R$  such that  $c' \cdot p^{(s-t)(r-1)} = c$ . Then

$$c'(bq - ap^{s-t}) = p^{s-t}d - a^i \cdot b$$

that is

$$0 = p^{s-t} \left( c'a + d \right) - ba^i - bc'q.$$

Then  $p$  divides the residue class of  $b \cdot a^i$  in  $R/qR$ , contradicting the assumption. ■

Let  $P$  be a polynomial in  $K[X, Y]$ . Let  $i$  and  $j$  be positive natural numbers and assume that  $P$  can be written in the form

$$P = P'X^i + P''X^jY + c$$

where  $c$  is an element of  $K$ ,  $P'$  is a non-zero polynomial in  $K[X]$  that can not be divided by  $X$  and  $P''$  is a non-zero polynomial in  $K[X, Y]$  that can not be divided by  $X$ .

If  $c$  is non-zero, then  $X, Y$  is a regular sequence of  $K[X, Y]/(P)$ , and accordingly  $\ell(K[X, Y]/(P, X^n, Y^m))$  is a polynomial function. If  $c = 0$  and  $j < i$ , and  $P' \cdot P''$  is not divisible by  $\bar{X}$  in  $K[X, Y]/Y$ , then Corollary 1.6 implies that  $\tau(K[X, Y]/(P, X^n), Y)$  is unbounded in  $n$ . We derive

**Example 1.7.** Let  $a, b$  be elements in  $K^\times$ , and let  $u, i$  and  $j$  be positive natural numbers. The function  $\ell(K[X, Y]/(aX^i + bX^jY^u, X^n, Y^m))$  is not polynomial-like whenever  $1 \leq j < i$ .

The following lemma demonstrates that, in the situation of Example 1.7, the failure of the function  $\ell(K[X, Y]/(P, X^n, Y^m))$  to be polynomial-like can be attributed solely to the “telescopic sum obstruction” due to Lemma 1.5.

**Lemma 1.8.** Let  $\mathfrak{a}$  be an ideal of a noetherian ring  $R$ , and let  $x$  and  $y$  be two elements of  $R$ . Let  $\alpha$  denote  $\tau(R/\mathfrak{a}, x)$ , and let  $n > \alpha + 1$ . If

$$\tau(R/(\mathfrak{a}, x^n), y) > \tau(R/(\mathfrak{a}, x^{n-1}), y),$$

then there exists a non-trivial  $y$ -torsion element in  $(x^\alpha, \mathfrak{a})/(x^{\alpha+1}, \mathfrak{a})$ .

*Proof:* Assume that

$$\tau(R/(\mathfrak{a}, x^n), y) > \tau(R/(\mathfrak{a}, x^{n-1}), y).$$

First we prove that there exists an element  $g$  in  $\mathfrak{a} + (x^{n-1}) \setminus \mathfrak{a} + (x^n)$  such that  $yg$  is contained in  $\mathfrak{a} + (x^n)$ . Therefore, let  $u$  denote  $\tau(R/(\mathfrak{a}, x^{n-1}), y)$ , let  $v$  denote  $\tau(R/(\mathfrak{a}, x^n), y)$ , and let  $g'$  be an element in  $R$  such that  $y^v g'$  is contained in  $\mathfrak{a} + (x^n)$  and  $y^{v-1} g'$  is not contained in  $\mathfrak{a} + (x^n)$ . Then  $y^u g'$  is contained in  $\mathfrak{a} + (x^{n-1})$ , since  $v > u$ . Hence,  $g := y^{v-1} g'$  does the job.

We can write  $g$  in the form  $g = a + zx^{n-1}$  with  $a$  in  $\mathfrak{a}$  and  $z$  in  $R$ . Further, we find elements  $a'$  in  $\mathfrak{a}$  and  $z'$  in  $R$  with

$$yg = y(a + zx^{n-1}) = a' + z'x^n.$$

Then  $x^{n-1}(yz - xz')$  is contained in  $\mathfrak{a}$ , and we conclude that  $x^\alpha(yz - xz')$  is contained in  $\mathfrak{a}$ , since  $n-1 \geq \alpha$ . Thus,  $yx^\alpha z$  is contained in  $\mathfrak{a} + (x^{\alpha+1})$ . Making use of the fact that  $g$  is not contained in  $\mathfrak{a} + x^n$ , we conclude that  $x^\alpha z$  is not contained in  $\mathfrak{a} + (x^{\alpha+1})$ . Consequently, the residue class of  $x^\alpha z$  in  $(\mathfrak{a} + (x^\alpha))/(\mathfrak{a} + (x^{\alpha+1}))$  is nontrivial and gets annihilated by  $y$ , whence the lemma.  $\blacksquare$

If  $c = 0$  and  $j \geq i$ , in the above situation, then  $\tau(K[X, Y]/(P), X) = i$  and  $(X^i, P)/(X^{i+1}, P)$  has no non-trivial  $Y$ -torsion. Then Lemma 1.8 implies that  $\tau(K[X, Y]/(P, X^n), Y)$  is bounded in  $n$ , and we derive

**Example 1.9.** Let  $a, b$  be elements in  $K^\times$ , and let  $u, i$  and  $j$  be positive natural numbers. The function  $\ell(K[X, Y]/(aX^i + bX^j Y^u, X^n, Y^m))$  is polynomial-like whenever  $1 \leq i \leq j$ .

**Outlook 1.10.** With a little bit more effort one should be able to classify all polynomials  $P$  in  $K[X_1, \dots, X_m]$  such that  $\ell(K[\underline{X}]/(P, \underline{X}^n))$  is a polynomial(-like) function in  $\underline{n}$ .



# Chapter 2

## Optimal Polynomial Estimates

As a preparation for *Chapter 6*, in this chapter we study real valued functions with domains of the form  $\mathbb{N}^s$  that exhibit polynomial-likeness in the single variables for any fixed value of the other variables.

**Definition 2.1.** *Let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be a function, and let  $1 \leq r \leq s$  be a natural number. We say that  $f$  is a polynomial in the variables  $N_1, \dots, N_r$  if there exists a polynomial  $P$  in  $\mathbb{R}[N_1, \dots, N_r]$ , such that  $P(N_1, \dots, N_r)$  equals  $f(N_1, \dots, N_r, x_{r+1}, \dots, x_s)$  for all natural numbers  $x_{r+1}, \dots, x_s$ . Given a subtuple  $I$  of  $(1, \dots, s)$ , we say that  $f$  is a polynomial in the variables  $N_i$  with  $i$  in  $I$  if there exists a permutation  $\sigma$  of  $(1, \dots, s)$  such that  $\sigma(I) = (1, \dots, |I|)$ , and the function*

$$x_1, \dots, x_s \mapsto f(x_{\sigma(1)}, \dots, x_{\sigma(s)})$$

*is a polynomial in the variables  $N_1, \dots, N_{|I|}$ . Here  $|I|$  denotes the number of entries of  $I$ .*

*Let  $C$  be a subset of  $\mathbb{R}$ , and let  $P$  be a polynomial in  $\mathbb{R}[X_1, \dots, X_s]$  with coefficients in  $C$ . We say that  $P$  is lower polynomial estimate of  $f$  over  $C$  if  $P(\mathbf{a}) \leq f(\mathbf{a})$  for all  $\mathbf{a}$  in  $\mathbb{N}^s$ . We say that  $P$  is an upper polynomial estimate of  $f$  over  $C$  if  $-P$  is a lower polynomial estimate of  $-f$  over  $C$ .*

*If we do not specify the coefficients  $C$ , then we intend them to be the real numbers.*

**Definition 2.2.** *Let  $s$  be a natural number, and let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be a function. We set*

$$f_{\mathbf{a},r}(X) := f(a_1, \dots, a_{r-1}, X, a_{r+1}, \dots, a_s)$$

*for every index  $1 \leq r \leq s$  and every  $\mathbf{a} = (a_1, \dots, a_s)$  in  $\mathbb{N}^s$ . These functions are termed the component functions of  $f$ .*

*We say that  $f$  is a separated polynomial if all its component functions are polynomials. In this case, we refer to the functions  $f_{\mathbf{a},r}$  as the component polynomials of  $f$ .*

*We say that  $f$  is separated polynomial-like if all of its component functions are polynomial-like functions. In this case, we denote the polynomial that agrees with*

the component function  $f_{\mathbf{a},r}(X)$  for all large values of  $X$  by  $f_{\mathbf{a},r}^\infty(X)$ , and we refer to the functions  $f_{\mathbf{a},r}^\infty(X)$  as the component polynomials of  $f$ .

**Example 2.3.** Let  $f(X_1, \dots, X_s)$  and  $g(X_1, \dots, X_s)$  be two separated polynomial-like functions from  $\mathbb{N}^s$  to  $\mathbb{R}$ . Then the minimum and maximum functions

$$\min(f(X_1, \dots, X_s), g(X_1, \dots, X_s)) \quad \text{and} \quad \max(f(X_1, \dots, X_s), g(X_1, \dots, X_s))$$

are separated polynomial-like.

In the context of our study, we encounter functions defined on domains of the form  $\mathbb{N}^s$  with codomain  $\mathbb{N}$ . Let us, however, make the following

**Historical Remark 2.4.** In [16] F. W. Carroll proved that every separated polynomial function  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is actually a polynomial. If we replace  $\mathbb{R}$  with an infinite countable field this result, however, is no longer true. This insight is due to R. S. Palais who showed that over an arbitrary field  $K$  a necessary and sufficient condition for every separated polynomial function <sup>1</sup> from  $K \times K$  to  $K$  to be a polynomial is that  $K$  is either finite or uncountable; see [80]. For infinite countable fields *loc. cit.* provides an explicit counterexample that has the peculiarity that the set of the occurring degrees of the component polynomials is infinite. Counterexamples of that form for infinite fields are the only exceptions, due to a classical linear algebra argument.

**Proposition 2.5.** Let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be a separated polynomial <sup>2</sup>. Then  $f$  is a polynomial with coefficients in  $K$  if and only if the set of the occurring degrees of the component polynomials of  $f$  is finite.

*Proof:* The only-if part is clear.

We prove the if-part by an ascending induction on  $s$ . The base case  $s = 1$  is immediate. Assume that the statement holds true for  $s = t$  with  $t \geq 1$ . Let  $s = t + 1$ , and let  $\beta$  denote a natural number that bounds the degrees of the component polynomials of  $f$ .

Given natural numbers  $n_1, \dots, n_t$ , let  $a_0(n_1, \dots, n_t), \dots, a_\beta(n_1, \dots, n_t)$  denote the coefficients of the component polynomial  $X \mapsto f(n_1, \dots, n_t, X)$ . Let  $x_0, \dots, x_\beta$  be pairwise distinct natural numbers. The equation

$$f(n_1, \dots, n_t, X) = \sum_{i=0}^{\beta} a_i(n_1, \dots, n_t) X^i$$

---

<sup>1</sup>Adapt Definition 2.2 to this context.

<sup>2</sup>Here  $\mathbb{R}$  can be replaced by an arbitrary infinite field  $K$  and then  $\mathbb{N}$  can be replaced by an arbitrary infinite subset of  $K$ .

gives rise to the system of equations

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^\beta \\ 1 & x_1 & x_1^2 & \cdots & x_1^\beta \\ 1 & x_2 & x_2^2 & \cdots & x_2^\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_\beta & x_\beta^2 & \cdots & x_\beta^\beta \end{pmatrix} \cdot \begin{pmatrix} a_0(n_1, \dots, n_t) \\ \vdots \\ a_2(n_1, \dots, n_t) \\ \vdots \\ a_\beta(n_1, \dots, n_t) \end{pmatrix} = \begin{pmatrix} f(n_1, \dots, n_t, x_0) \\ \vdots \\ f(n_1, \dots, n_t, x_2) \\ \vdots \\ f(n_1, \dots, n_t, x_\beta) \end{pmatrix}$$

that is solvable, since the Vandermonde matrix on the left is regular. Thus, the coefficient  $a_i(n_1, \dots, n_{t-1})$  can be expressed as linear combinations of the values  $f(n_1, \dots, n_{t-1}, x_j)$  for all  $i = 0, \dots, \beta$ . This completes the induction step.  $\blacksquare$

As observed previously in *Chapter 1*, a separated linear polynomial-like function does not necessarily need to be polynomial-like in general. Furthermore, a separated polynomial-like function  $f$  in general may not even be bounded by polynomials as gets illustrated by the pathological example

$$\begin{aligned} \mathbb{N}^2 &\rightarrow \mathbb{N} \\ (x, y) &\mapsto \begin{cases} 2^{xy} & \text{if } x = y; \\ 2xy & \text{else.} \end{cases} \end{aligned}$$

The remainder of this section is devoted to providing conditions that prohibit the aforementioned phenomena and that give rise to a class of separated polynomial-like functions that can be bounded by polynomials. These conditions are rather “technical” in nature, and the resulting statements may not seem immediately applicable. Nevertheless, they are applicable in the context of our studies, and, as a pleasant surplus, the polynomial bounds obtained under these conditions turn out to be “optimal” in a precise sense of the word.

**Definition 2.6.** *Let  $C$  be subsets of  $\mathbb{R}$ , let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be a function, and let  $P$  be a polynomial in  $\mathbb{R}[X_1, \dots, X_s]$ .*

*We say that  $P$  is an optimal polynomial lower estimate of  $f$  over  $C$  if  $P$  is a lower polynomial estimate of  $f$  over  $C$  such that for every real polynomial  $P'$  in  $\mathbb{R}[X_1, \dots, X_s]$  with coefficients in  $C$  subject to the constraint  $P \leq P' \leq f$ <sup>3</sup> we find  $P' = P$ . Likewise, we say that  $P$  is an optimal polynomial upper estimate of  $f$  over  $C$  if  $-P$  is an optimal polynomial lower estimate of  $-f$  over  $C$ .*

*If we do not specify  $C$ , we intend  $C$  to be the real numbers.*

<sup>3</sup>That is  $P(\underline{n}) \leq P'(\underline{n}) \leq f(\underline{n})$  for all  $(\underline{n})$  in  $\mathbb{N}^s$ .

**Lemma 2.7.** *Let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be a function admitting an optimal polynomial lower estimate  $\Upsilon^-$ . Then the function  $\Upsilon^- - P$  is an optimal polynomial lower estimate of  $f - P$  for every polynomial  $P$  in  $\mathbb{R}[X_1, \dots, X_s]$ .*

*Proof:* Let  $Q$  be a polynomial in  $\mathbb{R}[X_1, \dots, X_s]$  with  $\Upsilon^- - P \leq Q \leq f - P$ . Then equivalently  $\Upsilon^- \leq P + Q \leq f$ , resulting in  $P + Q = \Upsilon^-$ . Thus  $Q = \Upsilon^- - P$ . ■

**Lemma 2.8.** *Let  $R$  be an integral domain, and let  $P$  and  $Q$  be two multivariate polynomials in  $R[X_1, \dots, X_s]$ . Assume that there exists subsets  $S_1, \dots, S_s$  in  $R$  with cardinality greater than the maximum of the total degrees of  $P$  and  $Q$  such that  $P$  and  $Q$  agree on  $S_1 \times \dots \times S_s$ . Then  $P$  and  $Q$  are identical.*

*Proof:* We can assume without loss of generality that  $R = K$  is a field. We prove the assertion by an ascending induction on  $s$ ; the induction start  $s = 1$  being immanent. So let  $s > 1$ , and assume that the assertion holds true for  $s - 1$ . Let  $y_1, \dots, y_n$  be pairwise different elements in  $S_s$  where  $n$  is exceeding the maximum of the total degrees of  $P$  and  $Q$ . By the induction assumption we find

$$P(X_1, \dots, X_{s-1}, y_i) = Q(X_1, \dots, X_{s-1}, y_i)$$

for every index  $1 \leq i \leq n$ . Let  $x_1, \dots, x_s$  be arbitrary elements in  $K$ . Then

$$P(x_1, \dots, x_{s-1}, y_i) = Q(x_1, \dots, x_{s-1}, y_i)$$

for every index  $1 \leq i \leq n$ . Thus, the induction assumption implies that the polynomials

$$P(x_1, \dots, x_{s-1}, X_s) = Q(x_1, \dots, x_{s-1}, X_s)$$

agree, whence  $P = Q$ . ■

**Proposition 2.9.** *Let  $f : \mathbb{N}^s \rightarrow \mathbb{N}$  be a function that can be estimated above and below by polynomials with coefficients in  $\mathbb{N}$ . Then  $f$  admits an optimal polynomial upper and an optimal polynomial lower estimate over  $\mathbb{N}$ .*

*Proof:* We will prove that  $f$  admits an optimal polynomial lower estimate over  $\mathbb{N}$ ; the proof of the existence of an optimal polynomial upper estimate over  $\mathbb{N}$  follows by changing signs.

Let  $Q_u$  be an upper polynomial estimate of  $f$ , and let  $Q_l$  be a lower polynomial estimate of  $f$  over  $\mathbb{N}$ . Further, let  $d$  denote the total degree of  $Q_u$ , let  $t$  denote  $(d + 1)^s$ , and let  $y_1, \dots, y_t$  denote the elements of  $\mathbb{N}_{\leq d}^s$ . For every index  $0 \leq i \leq t$  we define the set  $S_i$  iteratively as follows:

We let  $S_0$  denote the set of all lower polynomial estimates of  $f$  over  $\mathbb{N}$ . Let  $i > 0$ , and assume that  $S_{i-1}$  has been defined. Then there exists a polynomial  $\tilde{P}$  in  $S_{i-1}$

such that  $P(y_i) \leq \tilde{P}(y_i)$  for all polynomials  $P$  in  $S_{i-1}$ . We define  $S_i$  to be set comprising all polynomials  $P$  in  $S_{i-1}$  such that  $P(y_i) = \tilde{P}(y_i)$ .

By construction, there exists a polynomial  $\Upsilon^-$  in  $S_t$ . Let  $P$  be an arbitrary polynomial in  $\mathbb{N}[X_1, \dots, X_s]$  subject to the constraint  $\Upsilon^- \leq P \leq f$ . Then, by the construction of  $\Upsilon^-$ , we find that  $\Upsilon^-$  and  $P$  agree on all points  $y_1, \dots, y_t$ . Hence, the polynomials  $\Upsilon^-$  and  $P$  must coincide by Lemma 2.8.  $\blacksquare$

In the following, let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be an arbitrary separated polynomial-like function. For every  $s$ -tuple  $\mathbf{a}$  in  $\mathbb{N}^s$  and every admissible index  $k$ , we define

$$f_{\mathbf{a},k}^-(N_k) := f_{\mathbf{a},k}^\infty(N_k) + \inf_{a_k \in \mathbb{N}} \{f_{\mathbf{a},k}(a_k) - f_{\mathbf{a},k}^\infty(a_k)\}$$

so that

$$f_{\mathbf{a},k}^-(N_k) \leq f_{\mathbf{a},k}(N_k).$$

Moreover, we define the function

$$\begin{aligned} f_k^- : \mathbb{N}^s &\rightarrow \mathbb{R}_\infty \\ (n_1, \dots, n_s) &\mapsto f_{(n_1, \dots, n_s), k}^-(n_k). \end{aligned}$$

Let  $n_1, \dots, n_{s-1}$  be fixed elements in  $\mathbb{N}$ . Then every polynomial  $Q(N_s)$  subject to the constraint

$$f_s^-(n_1, \dots, n_{s-1}, N_s) \leq Q(N_s) \leq f(n_1, \dots, n_{s-1}, N_s)$$

necessarily is equal to  $f_s^-(n_1, \dots, n_{s-1}, N_s)$ . Now we formally write down the conditions that allow us to generalize this property to the multivariate case by the virtue of an induction principle.

Let  $\mathfrak{Pol}_0$  be the set comprising all separated polynomial-like functions  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  such that

- i.) The function  $f$  can be estimated both above and below by polynomials;
- ii.) The set of the occurring degrees of the component functions of  $f$  is bounded by a natural number  $d(f)$ .

For every index  $r$  with  $1 \leq r \leq s$  let

$$\mathfrak{Pol}_r := \left\{ \begin{array}{l} \text{separated polynomial-like functions } f : \mathbb{N}^s \rightarrow \mathbb{R} \text{ in } \mathfrak{Pol}_0 \\ \text{being a polynomial in the variables } N_1, \dots, N_r. \end{array} \right\}$$

and let  $\widetilde{\mathfrak{Pol}}_r$  denote the subset of  $\mathfrak{Pol}_r$ , comprising all functions of the form

$$f(N_1, \dots, N_s) = \sum_{\underline{i} \in \{0, \dots, d(f)\}^r} a_{\underline{i}}(N_{r+1}, \dots, N_s) \prod_{k=1}^r N_k^{i_k}$$

in  $\mathfrak{Pol}_r$ , where all of the functions  $(a_{\underline{i}})_{r+1}^-$  are separated polynomial-like.<sup>4</sup>

For every index  $r$  with  $0 \leq r < s$  we define a transformation

$$\mathfrak{T}_{r,r+1} : \widetilde{\mathfrak{Pol}}_r \rightarrow \mathfrak{Pol}_{r+1}$$

as follows: Let

$$f(N_1, \dots, N_s) = \sum_{\underline{i} \in \{0, \dots, d(f)\}^r} a_{\underline{i}}(N_{r+1}, \dots, N_s) \prod_{k=1}^r N_k^{i_k}$$

be an arbitrary function in  $\widetilde{\mathfrak{Pol}}_r$ . Then the function  $(a_{\underline{i}})_{r+1}^-$  is a polynomial in the variable  $N_{r+1}$  for every index  $\underline{i}$  in  $\{0, \dots, d(f)\}^r$ , and it is separated polynomial-like, so that we have an equation of the form

$$(a_{\underline{i}})_{r+1}^-(N_{r+1}, \dots, N_s) = \sum_{j=0}^{d(f)} b_{\underline{i},j}(N_{r+2}, \dots, N_s) N_{r+1}^j.$$

Let  $x_0, \dots, x_{d(f)}$  be pairwise different elements in  $D$ . Then the system of equations

$$V(x_0, \dots, x_{d(f)}) \cdot (b_{\underline{i},j})_{0 \leq j \leq d(f)} = ((a_{\underline{i}})_{r+1}^-(x_j, N_{r+2}, \dots, N_s))_{0 \leq j \leq d(f)}$$

where  $V(x_0, \dots, x_{d(f)})$  denotes the Vandemonde matrix of  $x_0, \dots, x_{d(f)}$  is solvable. Hence, all of the functions  $b_{\underline{i},j}$  are separated polynomial-like. We set

$$a_{(\underline{i},j)}(N_{r+2}, \dots, N_s) := b_{\underline{i},j}(N_{r+2}, \dots, N_s)$$

and

$$\mathfrak{T}_{r,r+1}(f) := \sum_{\underline{i} \in \{0, \dots, d(f)\}^{r+1}} a_{\underline{i}}(N_{r+2}, \dots, N_s) \prod_{k=1}^{r+1} N_k^{i_k}.$$

Next, we iteratively define a subclass  $\mathfrak{E}_i^-$  of  $\widetilde{\mathfrak{Pol}}_i$  for all indices  $i$  with  $0 \leq i \leq s$ . Therefore, let  $\mathfrak{E}_0^- := \widetilde{\mathfrak{Pol}}_0$ , and assuming  $\mathfrak{E}_{i-1}^-$  is defined for some  $i > 0$ , we set

$$\mathfrak{E}_i^- := \left\{ \begin{array}{l} \text{separated polynomial-like functions } f : \mathbb{N}^s \rightarrow \mathbb{R} \text{ in } \mathfrak{E}_{i-1}^- \\ \text{such that } \mathfrak{T}_{i-1,i}(f) \text{ is an element of } \widetilde{\mathfrak{Pol}}_i \end{array} \right\}.$$

Finally, we set  $\mathfrak{E}^- := \mathfrak{E}_s^-$ , and for every function  $f$  in  $\mathfrak{E}^-$  we set

$$\Upsilon_f^- := \mathfrak{T}_{s-1,s}(\mathfrak{T}_{s-2,s-1}(\dots(\mathfrak{T}_{0,1}(f)))) \dots.$$

<sup>4</sup>Formally, it is preferable to denote  $(a_{\underline{i}})_1^-$  rather than  $(a_{\underline{i}})_{r+1}^-$ . We choose, however, to perceive  $a_{\underline{i}}$  as a function in  $s$  many variables.

**Theorem 2.10.** *The function  $\Upsilon_f^-$  is an optimal polynomial lower estimate for every function  $f$  in  $\mathfrak{E}^-$ .*

*Proof:* Let  $f : \mathbb{N}^s \rightarrow \mathbb{R}$  be an arbitrary function in  $\mathfrak{E}^-$ . We prove the theorem by an ascending induction on  $s$ . The base case  $s = 1$  is clear. So let  $s > 1$ , and assume that the assertion holds true for  $s - 1$ .

It is readily apparent from the construction that  $\Upsilon_f^-$  is a lower polynomial estimate of  $f$ . Let  $Q(N_1, \dots, N_s)$  be a real multivariate polynomial such that

$$\Upsilon_f^- \leq Q \leq f.$$

Let  $\beta$  be a natural number that bounds the degrees of the component functions of  $f$ . We can express  $Q$  as

$$Q(N_1, \dots, N_s) = \sum_{i \in \{0, \dots, \beta\}} b_i(N_2, \dots, N_s) \cdot N_1^i$$

with polynomials  $b_i(N_2, \dots, N_s)$ . Similarly, we can express  $\mathfrak{T}_{0,1}(f)$  as

$$\mathfrak{T}_{0,1}(f) = \sum_{i \in \{0, \dots, \beta\}} a_i(N_2, \dots, N_s) \cdot N_1^i$$

with separated polynomial-like functions  $a_i(N_2, \dots, N_s)$ . Then

$$\sum_{i \in \{0, \dots, \beta\}} \Upsilon_{a_i}^- \cdot N_1^i = \Upsilon_f^-.$$

We assert that the equality

$$\Upsilon_{a_i}^- = b_i$$

holds true for all indices  $i = 0, \dots, \beta + 1$ , where we stipulate that  $b_{\beta+1} = \Upsilon_{a_{\beta+1}}^- = 0$ . We prove the assertion by a descending induction on  $i$ ; the induction start  $i = \beta + 1$  being immanent. For the induction step, let  $j \leq \beta$ , and assume that the assertion holds true for  $i = j + 1$ .

The induction assumption implies  $\Upsilon_{a_j}^- \leq b_j$ , and according to the induction assumption of the induction running on  $s$ , it suffices to prove  $\Upsilon_{a_j}^- \leq b_j \leq a_j$ . Let

$$Q' := Q - \sum_{k=j+1}^{\beta} \Upsilon_{a_k}^- \cdot N_1^k = \sum_{i \in \{0, \dots, j\}} b_i(N_2, \dots, N_s) \cdot N_1^i$$

and

$$f' := f - \sum_{k=j+1}^{\beta} \Upsilon_{a_k}^- \cdot N_1^k.$$

Then Lemma 2.7 implies

$$\sum_{i \in \{0, \dots, j\}} \Upsilon_{a_i}^- \cdot N_1^i = \Upsilon_{f'}^-.$$

Further

$$\mathfrak{T}_{0,1}(f') = \sum_{i \in \{0, \dots, j\}} a_i(N_2, \dots, N_s) \cdot N_1^i.$$

For the sake of contradiction, suppose that there exists natural numbers  $n_2, \dots, n_s$  such that  $a_j(n_2, \dots, n_s) < b_j(n_2, \dots, n_s)$ . Then

$$\mathfrak{T}_{0,1}(f')(n_1, \dots, n_s) < Q'(n_1, \dots, n_s) \leq f'(n_1, \dots, n_s)$$

for all sufficiently large values of  $n_1$ . Further, the functions  $\mathfrak{T}_{0,1}(f')(N_1, n_2, \dots, n_s)$  and  $f'(N_1, n_2, \dots, n_s)$  differ by a constant for all sufficiently large values of  $n_1$ . By the above there exists a positive constant  $C > 0$  such that

$$Q'(N_1, n_2, \dots, n_s) = \mathfrak{T}_{0,1}(f')(N_1, n_2, \dots, n_s) + C;$$

contradiction.

We have now completed the induction step, showing that  $\Upsilon_{a_i} = b_i$  holds true for all indices  $i$  in  $\{0, \dots, \beta + 1\}$ . Consequently, we have shown that  $\Upsilon_f^- = Q$ , which completes the induction step of the induction on  $s$ , and the theorem is proven. ■

Let  $R$  be a noetherian ring, let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of non-zero divisors of  $R$ , and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_\infty$  be an additive function. The (0-th) Euler-Poincaré characteristic  $\chi_\mu(\mathbf{x}; -)$  constitutes an additive function from  $\mathbf{R} - \mathbf{mod}$  to  $\mathbb{N}_\infty$ , and we recall the *recursion formula* from [6, Thm. 3.3 ]: Let  $N$  be a finitely generated  $R$ -module. If  $\mathbf{x}$  is a multiplicity system of  $N$ , then

$$\chi_\ell(x_1, \dots, x_s; N) = \chi_\ell(x_1, \dots, x_{s-1}; N/x_s N) - \chi_\ell(x_1, \dots, x_{s-1}; N[x_s]).$$

More generally, given  $M$  in  $\mathcal{D}_{ft}^b(R)$ , the UCT 0.1 implies

$$\begin{aligned} & \chi_\mu(x_1, \dots, x_{s-1}; H_i(x_s; M)) \\ &= \chi_\mu(x_1, \dots, x_{s-1}; H_i(M) \otimes_R R/x_s R) + \chi_\mu(x_1, \dots, x_{s-1}; H_{i-1}(M)[x_s]) \\ &= \chi_\mu(x_1, \dots, x_s; H_i(M)) + \chi_\mu(x_1, \dots, x_{s-1}; H_i(M)[x_s]) \\ & \quad + \chi_\mu(x_1, \dots, x_{s-1}; H_{i-1}(M)[x_s]). \end{aligned}$$

In the virtue of Lemma 1.1, we have

$$\chi_\mu(x_1^{n_1}, \dots, x_s^{n_s}; N) = n_1 \cdot \dots \cdot n_s \cdot \chi_\mu(\mathbf{x}; N),$$

for all natural numbers  $n_1, \dots, n_s$ . Employing these fundamental formal properties, we derive

**Theorem 2.11.** *Let  $R$  be a noetherian ring, let  $M$  be an object of  $\mathcal{D}_{ft}^b(R)$ , let  $\mathbf{x}$  be a multiplicity system of  $M$  consisting of non-zero divisors of  $R$ , and let  $i$  be an integer. The function*

$$\begin{aligned} \varphi : \mathbb{N}^s &\rightarrow \mathbb{R} \\ (n_1, \dots, n_s) &\mapsto \ell(H_i(x_1^{n_1}, \dots, x_s^{n_s}; M)) \end{aligned}$$

belongs to  $\mathfrak{E}^-$ , and for every polynomial  $P$  in  $\mathbb{R}_{\geq 0}[X_1, \dots, X_s]$  with  $P \leq \varphi$  one has  $P \leq \Upsilon_{\varphi}^-$ .

*Proof:* Let  $\mathcal{C}$  denote the full subcategory of  $\mathcal{D}_{ft}^b(R)$  comprising all objects  $M$  such that  $\mathbf{x}$  is a multiplicity system of  $M$ . First, we prove by a nested induction on  $(r, s)$  that the function

$$\begin{aligned} \varphi_r : \mathbb{N}^s &\rightarrow \mathbb{R} \\ (n_1, \dots, n_s) &\mapsto \chi_{\mu}(x_1, \dots, x_r; H_i(x_{r+1}^{n_{r+1}}, \dots, x_s^{n_s}; M)) \end{aligned}$$

belongs to  $\mathfrak{E}^-$  for every additive function  $\mu : \mathcal{C} \rightarrow \mathbb{N}$  and for all indices  $0 \leq r \leq s$ .

The first induction is running on  $s$ . The induction start  $s = 1$  follows from the UCT 0.1. So let  $s > 1$ , and assume that the assertion holds true for  $s - 1$ . In order to prove that the assertion holds true for  $s$ , we prove by a descending induction on  $r$  that the function

$$\begin{aligned} \varphi_r : \mathbb{N}^s &\rightarrow \mathbb{R} \\ (n_1, \dots, n_s) &\mapsto \chi_{\mu}(x_1, \dots, x_r, H_i(x_{r+1}^{n_{r+1}}, \dots, x_s^{n_s}; M)) \end{aligned}$$

belongs to  $\mathfrak{E}^-$  for all  $M$  in  $\mathcal{D}_{ft}^b(R)$ ; the induction start  $r = s$  being immanent. So let  $r < s$ , and assume that the assertion holds true for  $r - 1$ .

Employing the formal properties of the Euler-Poincaré characteristic, we compute

$$\begin{aligned} &\chi_{\mu}(x_1, \dots, x_r; H_i(x_{r+1}^{n_{r+1}}, \dots, x_s^{n_s}; M)) \\ &= \chi_{\mu}(x_1, \dots, x_r, x_{r+1}^{n_{r+1}}; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)) \\ &\quad + \chi_{\mu}(x_1, \dots, x_r; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)[x_{r+1}^{n_{r+1}}]) \\ &\quad + \chi_{\mu}(x_1, \dots, x_r; H_{i-1}(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)[x_{r+1}^{n_{r+1}}]) \\ &= n_{r+1} \cdot \chi_{\mu}(x_1, \dots, x_r, x_{r+1}; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)) \\ &\quad + \chi_{\mu}(x_1, \dots, x_r; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)[x_{r+1}^{n_{r+1}}]) \\ &\quad + \chi_{\mu}(x_1, \dots, x_r; H_{i-1}(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)[x_{r+1}^{n_{r+1}}]) \end{aligned}$$

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for all positive natural numbers  $n_{r+1}, \dots, n_s$ . Thus,

$$\begin{aligned}
& \mathfrak{T}_{0,1}(\varphi_r)(n_{r+1}, \dots, n_s) \\
&= n_{r+1} \cdot \chi_\mu(x_1, \dots, x_r, x_{r+1}; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)) \\
&\quad + \chi_\mu(x_1, \dots, x_r; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)[x_{r+1}]) \\
&\quad + \chi_\mu(x_1, \dots, x_r; H_{i-1}(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)[x_{r+1}]) \\
&= (n_{r+1} - 1) \cdot \chi_\mu(x_1, \dots, x_r, x_{r+1}; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)) \\
&\quad + \chi_\mu(x_1, \dots, x_r; H_i(x_{r+1}, x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)) \\
&= (n_{r+1} - 1) \cdot \chi_\mu(x_1, \dots, x_r, x_{r+1}; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M)) \\
&\quad + \chi_\mu(x_1, \dots, x_r; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M \otimes_R^L \mathcal{K}(x_{r+1}))).
\end{aligned}$$

According to the induction assumption on  $r$ , the function

$$\chi_\mu(x_1, \dots, x_{r+1}; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M))$$

belongs to  $\mathfrak{E}^-$ , and according to the induction assumption on  $s$ , the function

$$\chi_\mu(x_1, \dots, x_r; H_i(x_{r+2}^{n_{r+2}}, \dots, x_s^{n_s}; M \otimes_R^L \mathcal{K}(x_{r+1})))$$

belongs to  $\mathfrak{E}^-$ . A fortiori,  $\varphi_r$  belongs to  $\mathfrak{E}^-$ . This settles the induction step for the induction running on  $r$ , immediately resulting in the induction step for the induction running on  $s$ .

Let  $P$  be a polynomial in  $\mathbb{R}_{\geq 0}[X_1, \dots, X_s]$  subject to the constraint  $P \leq \varphi$ . We wish to prove  $P \leq \Upsilon_\varphi^-$ . We prove the assertion by an ascending induction on  $s$ ; the induction start  $s = 0$  being immanent. So let  $s > 0$ , and assume that the assertion holds true for  $s - 1$ . We can express  $P$  in the form

$$P(N_1, \dots, N_s) = (N_1 - 1) \cdot a_1(N_2, \dots, N_s) + a_0(N_2, \dots, N_s)$$

with multivariate linear polynomials  $a_0$  and  $a_1$ . The UCT 0.1 and Lemma 1.1 indicate that

$$\Delta_1(\varphi)(n_1, \dots, n_s) = \chi_\mu(x_1; H_i(x_2^{n_2}, \dots, x_s^{n_s}; M)),$$

whenever

$$n_1 \geq \max(\tau(H_i(x_2^{n_2}, \dots, x_s^{n_s}; M), x_1), \tau(H_{i-1}(x_2^{n_2}, \dots, x_s^{n_s}; M), x_1)).$$

A fortiori,

$$a_1(N_2, \dots, N_s) \leq \chi_\mu(x_1; H_i(x_2^{n_2}, \dots, x_s^{n_s}; M)).$$

Likewise,  $P(1, N_2, \dots, N_s) \leq \varphi(1, N_2, \dots, N_s)$  implies

$$a_0(N_2, \dots, N_s) \leq \mu(H_i(x_2^{n_2}, \dots, x_s^{n_s}; M \otimes^L R/x_1R)).$$

The function  $\chi_\mu$  is additive and the induction assumption implies

$$a_1(N_2, \dots, N_s) \leq \mathfrak{T}_{2,s}(\chi_\mu(x_1; H_i(x_2^{n_2}, \dots, x_s^{n_s}; M)))$$

and

$$a_0(N_2, \dots, N_s) \leq \mathfrak{T}_{2,s}(\mu(H_i(x_2^{n_2}, \dots, x_s^{n_s}; M \otimes^L R/x_1R))).$$

Arguing as in the above, we find

$$\begin{aligned} & \Upsilon_\varphi^-(N_1, \dots, N_s) \\ &= \mathfrak{T}_{2,s}(\mathfrak{T}_{0,1}(\varphi)(N_1, \dots, N_s)) \\ &= (N_1 - 1) \cdot \mathfrak{T}_{2,s}(\chi_\mu(x_1; H_i(x_2^{n_2}, \dots, x_s^{n_s}; M))) \\ &+ \mathfrak{T}_{2,s}(\mu(H_i(x_2^{n_2}, \dots, x_s^{n_s}; M \otimes^L \mathcal{K}(x_1)))). \end{aligned}$$

This yields the inequality  $P \leq \Upsilon_\varphi^-$ , settling the induction step. ■

In the setting of Theorem 2.11, we especially derive that  $\Upsilon_\varphi^-$  is an optimal polynomial lower estimate of  $\varphi$  over  $\mathbb{R}$ . In the above, we implicitly gave an algorithm for the computation of  $\Upsilon_\varphi^-$ . Making use of this, we wish to give an explicit formula for  $\Upsilon_\varphi^-$ . This will be achieved in *Chapter 6*.

We close with

**Question 2.12.** Given two functions  $f, g : \mathbb{N}^s \rightarrow \mathbb{R}$  in  $\mathfrak{E}^-$  with  $g \leq f$ , is  $\Upsilon_g^- \leq \Upsilon_f^-$ ?



# Chapter 3

## Ordered G-Theory

In [91, Appendix II] J.-P. Serre defines an element of the Grothendieck group  $K_0(\mathcal{C})$  of an essentially small abelian category  $\mathcal{C}$  to be positive if it is of the form  $[K]$  with  $K$  in  $\mathcal{C}$ . This notion of positivity, however, in general cannot be extended to a partial order on  $K_0(\mathcal{C})$ , making this ad-hoc definition of positivity somewhat “informal” and “badly behaved”. The objective of this chapter is to formalize and make this notion precise. Our interest in this stems from the following: In order to study the growth of a series of finitely generated modules, instead of working with the length, one may use any suitable additive function  $\mathbf{R} - \mathbf{mod}_x \rightarrow G^+$  where  $G^+$  is the positive cone of a partially ordered abelian group  $G$  and, not being restricted to finite length, one may hope to gain greater flexibility. In the following, we will demonstrate how such maps arise universally.

Every commutative monoid  $(M, +)$  admits a *group completion*. That is to say, the inclusion functor

$$\mathbf{Ab} \hookrightarrow \mathbf{CMon}$$

from the category of abelian groups to the category of commutative monoids admits a left adjoint  $G$ . The monoid  $M$  injects into  $G(M)$  if and only if  $M$  is a cancellation monoid. Partial orders come into play because, although there are advantages of replacing a monoid by its group completion, in some cases one needs to keep the original monoid in mind. Any commutative monoid  $M$  carries a preorder “ $\leq$ ” where “ $n \leq m$ ” if there exists an element  $x$  in  $M$  such that  $n + x = m$ . We have a bijection between partially ordered group structures and positive cones in a group. If for all elements  $n$  and  $m$  in  $M$  with  $n + m = 0$  it follows that  $n = m = 0$ , then  $M$  is a positive cone of  $G(M)$ , and subsequently  $G(M)$  naturally carries the structure of a partially ordered group, where the elements of  $M$  are exactly the positive ones; see [66]. In this fashion,  $K_0$  of a ring is a partially ordered group whenever all finitely generated projective modules are *directly finite*, i.e two finitely generated projective modules  $A$  and  $B$  can satisfy  $A \oplus B \cong A$  only if  $B$  is the zero module, and in this case the partial order on  $K_0$  was used in order to study the initial ring or algebra. We note that the *directly finiteness* condition is not particularly restrictive<sup>1</sup>. With all of this in mind, the

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<sup>1</sup>This holds true for  $R$ -modules  $A, B$  if  $R$  is a commutative ring, or if  $R$  is a directed union

idea behind what we refer to as *Ordered G-Theory* is to complete the addition of  $M$  and its preorder in order to derive an initial morphism  $M \rightarrow G$  to a partially ordered abelian group that maps  $M$  to the positive cone of  $G$ . But language first:

**Ordered Group Completion.** Let  $(M, +)$  be a commutative monoid. We denote the quotient group of the group completion  $G(M)$  by the subgroup generated by all equivalence classes  $[m]$  with  $m$  in  $M$  such that there exists an element  $n$  in  $M$  with  $[m] + [n] = [0]$  by  $\mathfrak{G}(M)$ . For every element  $m$  in  $M$  we denote the residue class of  $m$  in  $\mathfrak{G}(M)$  by  $\llbracket m \rrbracket$ .

We define the relation “ $\leq$ ” on  $\mathfrak{G}(M)$  by setting “ $n \leq m$ ” if there exists an object  $x$  in  $M$  such that  $n + \llbracket x \rrbracket = m$ . This constitutes a translation invariant partial order on  $\mathfrak{G}(M)$ . In order to see that  $\leq$  is indeed antisymmetric, assume that  $n \leq m$  and  $m \leq n$ . Then there exists objects  $x$  and  $y$  in  $M$  such that  $n + \llbracket x \rrbracket + \llbracket y \rrbracket = m + \llbracket y \rrbracket = n$ , resulting in  $\llbracket x + y \rrbracket = \llbracket x \rrbracket + \llbracket y \rrbracket = 0$ . Consequently, there exists an element  $z$  in  $M$  such that  $[x + y] + [z] = [x] + [z + y] = 0$ . Hence,  $\llbracket x \rrbracket = \llbracket y \rrbracket = 0$ .

We call  $(\mathfrak{G}(M), \leq)$  the *ordered group completion* of  $M$ . The ordered group completion  $(\mathfrak{G}(M), \leq)$  of a commutative monoid  $M$  together with the canonical morphism  $M \rightarrow \mathfrak{G}(M)$  has the following universal property:

Every monoid homomorphism  $\varphi : M \rightarrow G^+$  from  $M$  to the positive cone  $G^+$  of a partially ordered abelian group  $G$  factors uniquely as  $M \rightarrow \mathfrak{G}(M) \rightarrow G$ , where  $\mathfrak{G}(M) \rightarrow G$  is a morphism of partially ordered abelian groups.

Indeed, let  $G$  be a partially ordered abelian group, and let  $\varphi : M \rightarrow G$  be a monoid homomorphism that maps  $M$  to the positive cone  $G^+$ . Let  $n, m$  be elements in  $M$  such that  $[n] + [m] = 0$ . Then the universal property of the group completion  $G(M)$  implies that  $\varphi(n) + \varphi(m) = 0$ . We assert that in effect  $\varphi(n) = \varphi(m) = 0$ . Indeed, we have  $-\varphi(n) \leq 0$ , resulting in  $\varphi(m) \leq 0$ . Thus  $\varphi(m) = 0$  and  $\varphi(n) = 0$ . Therefore, the universal property of the group completion  $G(M)$  and the universal property of the quotient group  $\mathfrak{G}(M)$  imply that there exists a unique group homomorphism  $\tilde{\varphi} : \mathfrak{G}(M) \rightarrow G$ , such that  $\tilde{\varphi}(\llbracket m \rrbracket) = \varphi(m)$  for all elements  $m$  in  $M$ .

Furthermore,  $\tilde{\varphi}$  respects the partial orders: If  $x, y$ , are elements in  $\mathfrak{G}(M)$ , and  $z$  is an element of  $M$  with  $x + \llbracket z \rrbracket = y$ , then  $\tilde{\varphi}(x) + \tilde{\varphi}(\llbracket z \rrbracket) = \tilde{\varphi}(y)$ , equivalently  $\tilde{\varphi}(x) - \tilde{\varphi}(y) \leq -\varphi(z) \leq 0$ . Hence  $\tilde{\varphi}(x) \leq \tilde{\varphi}(y)$ .

Given a partially ordered abelian group  $G$ , its positive cone  $G^+$  is a commutative

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of finite-dimensional algebras, or if  $R$  is noetherian on either side, see [36, Prologue], or  $R$  is unit-regular; see [46, Thm. 2;9, Thm. 4.5].

cancelation monoid. This gives rise to the positive cone functor

$$(-)^+ : \mathbf{POrd} - \mathbf{Ab} \rightarrow \mathbf{CMon}.$$

**Proposition 3.1.** *The ordered group completion  $\mathfrak{G}(-)$  is left adjoint to the positive cone functor  $(-)^+$ .*

*Proof:* Let  $G$  be a partially ordered abelian group, and let  $M$  be a commutative monoid. Then every monoid homomorphism  $M \rightarrow G^+$  renders into a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & G^+ \\ \downarrow & & \downarrow \\ \mathfrak{G}(M) & \cdots\cdots\cdots & G \end{array}$$

with a unique dotted arrow. This constitutes an injective map

$$\varphi : \mathrm{Hom}_{\mathbf{CMon}}(M, G^+) \rightarrow \mathrm{Hom}_{\mathbf{POrd} - \mathbf{Ab}}(\mathfrak{G}(M), G).$$

Furthermore, every morphism  $\mathfrak{G}(M) \rightarrow G$  of partially ordered abelian groups renders into a commutative diagram

$$\begin{array}{ccc} \mathfrak{G}(M)^+ & \cdots\cdots\cdots & G^+ \\ \downarrow & & \downarrow \\ \mathfrak{G}(M) & \longrightarrow & G \end{array}$$

with a unique dotted arrow. This constitutes a map

$$\mathrm{Hom}_{\mathbf{POrd} - \mathbf{Ab}}(\mathfrak{G}(M), G) \rightarrow \mathrm{Hom}_{\mathbf{CMon}}(M, G^+)$$

that is right-inverse to  $\varphi$ . Hence,  $\varphi$  is bijective and  $\mathfrak{G}(-)$  is left adjoint to  $(-)^+$ . ■

Two comments in order:

- 1.) If  $M$  is a commutative monoid, an additive map  $\varphi : M \rightarrow G$  to a partially ordered abelian group  $G$  maps  $M$  to the positive cone of  $G$  if and only if it respects the canonic preorder of  $M$ ; that is  $\varphi(n) \leq \varphi(m)$  if there exists an element  $x$  in  $M$  with  $n + x = m$ .

2.) If  $M$  is a group, then  $\mathfrak{G}(M) = 0$ .

We shall transfer the concept to abelian categories. We bring to mind

**Grothendieck Group of an Abelian Category.** Let  $\mathcal{C}$  be an essentially small abelian category. The *Grothendieck group*  $K_0(\mathcal{C})$  of  $\mathcal{C}$  is the abelian group generated by the isomorphism classes of objects in  $\mathcal{C}$ , subject to the relations  $[A] = [B] + [C]$  whenever there is a short exact sequence

$$0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$$

in  $\mathcal{C}$ . Here,  $[A]$  denotes the isomorphism class of the object  $A$  in  $K_0(\mathcal{C})$ . To be more precise, if  $S$  denotes the set of isomorphism classes of objects in  $\mathcal{C}$ , then  $K_0(\mathcal{C})$  is the quotient of  $\mathbb{Z}[S]$  modulo the free subgroup generated by the symbols  $[A] - [B] - [C]$  such that there exists an exact sequence

$$0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$$

in  $\mathcal{C}$ . By a slight abuse of notation, for every object  $A$  in  $\mathcal{C}$  we will denote by  $[A]$  its residue class in  $K_0(\mathcal{C})$ .

The Grothendieck group satisfies the following universal property: Given any additive function  $\varphi : \mathcal{C} \rightarrow G$  from  $\mathcal{C}$  to an abelian group  $G$ , there exists a unique group homomorphism  $\tilde{\varphi} : K_0(\mathcal{C}) \rightarrow G$  such that  $\tilde{\varphi}([A]) = \varphi(A)$  for every object  $A$  in  $\mathcal{C}$ .

If  $R$  is a noetherian ring, we write  $G_0(R)$  for  $K_0(\mathbf{R} - \mathbf{mod})$ .

Let us explain how

**Grothendieck Groups are Preordered.** Let  $\mathcal{C}$  be an essentially small abelian category. We introduce a relation “ $\leq$ ” on the Grothendieck group  $K_0(\mathcal{C})$  as follows: For every elements  $A$  and  $B$  in  $K_0(\mathcal{C})$ , we set  $A \leq B$  if there exists an object  $C$  in  $\mathcal{C}$  such that  $A + [C] = B$ . We assert that  $\leq$  constitutes a preorder on  $K_0(\mathcal{C})$  that is compatible with the group structure of  $K_0(\mathcal{C})$ .

First, the relation  $\leq$  is reflexive, since  $A + [0] = A$  for every element  $A$  in  $K_0(\mathcal{C})$ .

Second, it is transitive. If  $A \leq B$  and  $B \leq C$ , then there exist objects  $X$  and  $Y$  in  $\mathcal{C}$  such that  $A + [X] = B$  and  $B + [Y] = C$ , resulting in  $A + [X \oplus Y] = C$ .

Finally, the preorder  $\leq$  is compatible with the addition of the Grothendieck group. If  $A \leq B$ , then for every elements  $C$  and  $D$  in  $K_0(\mathcal{C})$  with  $C \leq D$  we find  $A + C \leq B + D$ .

**Definition 3.2.** Let  $\mathcal{C}$  be an essentially small abelian category. We denote the quotient group of  $K_0(\mathcal{C})$  by the group generated by all symbols  $[C]$  with  $C$  in  $K_0(\mathcal{C})$  such that there exists an object  $D$  in  $\mathcal{C}$  with  $[C] + [D] = [0]$  by  $\mathfrak{K}_0(\mathcal{C})$ . For every object  $A$  in  $\mathcal{C}$  we denote the residue class of  $A$  in  $\mathfrak{K}_0(\mathcal{C})$  by  $\llbracket A \rrbracket$ .

We define the relation “ $\leq$ ” on  $\mathfrak{K}_0(\mathcal{C})$  by setting  $A \leq B$  for every elements  $A, B$  in  $\mathfrak{K}_0(\mathcal{C})$  if there exists an object  $C$  in  $\mathcal{C}$  such that  $A + \llbracket C \rrbracket = B$ .

This constitutes a translation invariant partial order on  $\mathfrak{K}_0(\mathcal{C})$ , the proof being essentially the same as for the ordered group completion: Assume that  $A \leq B$  and  $B \leq A$ . Then there exists objects  $C$  and  $D$  in  $\mathcal{C}$  such that  $A + \llbracket C \rrbracket + \llbracket D \rrbracket$  equals  $B + \llbracket D \rrbracket = A$ , resulting in  $\llbracket C \oplus D \rrbracket = \llbracket C \rrbracket + \llbracket D \rrbracket = 0$ . Consequently,  $[C \oplus D]$  is contained in the kernel of  $K_0(R) \rightarrow \mathfrak{K}_0(R)$ . Equivalently,

$$[C \oplus D] = \sum_{i=1}^n [A_i] - \sum_{i=n}^m [A_i]$$

where the  $A_i$  are objects in  $\mathcal{C}$  that satisfy  $[A_i] + [B_i] = 0$  for some object  $B_i$  in  $\mathcal{C}$ . Then

$$[C] + [D] + \sum_{i=n}^m [A_i] + \sum_{i=1}^n [B_i] = 0.$$

Hence,  $\llbracket C \rrbracket = \llbracket D \rrbracket = 0$ .

We call  $(\mathfrak{K}_0(\mathcal{C}), \leq)$  the ordered Grothendieck group of  $\mathcal{C}$ . If  $R$  is a noetherian ring, we denote  $\mathfrak{K}_0(\mathbf{R} - \mathbf{mod})$  by  $\mathfrak{G}_0(R)$ .

We remark that for a noetherian ring  $R$  the element  $\llbracket R \rrbracket$  serves as an order unit of  $\mathfrak{G}_0(R)$ . That is to say, for every element  $A$  in  $\mathfrak{G}_0(R)$  there exists a natural number  $n$  with  $A \leq n \cdot \llbracket R \rrbracket$ .

**Proposition 3.3.** The ordered Grothendieck group  $(\mathfrak{K}_0(\mathcal{C}), \leq)$  of an essentially small abelian category  $\mathcal{C}$  together with the canonical morphism  $\mathcal{C} \rightarrow \mathfrak{K}_0(\mathcal{C})$  has the following universal property:

If  $\varphi : \mathcal{C} \rightarrow G^+$  is an additive function to a positive cone  $G^+$  of a partially ordered abelian group  $G$ , then there exist a unique morphism  $\tilde{\varphi} : \mathfrak{K}_0(\mathcal{C}) \rightarrow G$  of partially ordered abelian groups such that  $\tilde{\varphi}(\llbracket A \rrbracket) = \varphi(A)$  for all objects  $A$  in  $\mathcal{C}$ .

*Proof:* The proof is essentially the same as for the ordered group completion.

Let  $G$  be a partially ordered abelian group, and let  $\varphi : \mathcal{C} \rightarrow G^+$  be an additive function. Let  $C, D$  be objects in  $\mathcal{C}$  such that  $[C] + [D] = 0$ . Then the universal property of the Grothendieck group  $K_0(\mathcal{C})$  implies that  $\varphi(C) + \varphi(D) = 0$ . We

assert that in effect  $\varphi(C) = \varphi(D) = 0$ . Indeed, by the positivity of  $\varphi$  we have  $-\varphi(C) \leq 0$ , resulting in  $\varphi(D) \leq 0$ . Thus,  $\varphi(D) = 0$  and  $\varphi(C) = 0$ . Therefore, the universal property of the Grothendieck group  $K_0(\mathcal{C})$  and the universal property of the quotient group  $\mathfrak{K}_0(\mathcal{C})$  imply that there exists a unique group homomorphism  $\tilde{\varphi} : \mathfrak{G}_0(\mathcal{C}) \rightarrow G$  such that  $\tilde{\varphi}(\llbracket A \rrbracket) = \varphi(A)$  for all objects  $A$  in  $\mathcal{C}$ .

Furthermore,  $\tilde{\varphi}$  respects the partial orders: If  $A, B$  are objects in  $\mathfrak{K}_0(\mathcal{C})$  and  $C$  is an object of  $\mathcal{C}$  with  $A + \llbracket C \rrbracket = B$ , then  $\tilde{\varphi}(A) + \tilde{\varphi}(\llbracket C \rrbracket) = \tilde{\varphi}(B)$ , equivalently  $\tilde{\varphi}(A) - \tilde{\varphi}(B) \leq -\varphi(C) \leq 0$ . Hence  $\tilde{\varphi}(A) \leq \tilde{\varphi}(B)$ . ■

We note that  $\mathfrak{K}_0(\mathcal{C}) \neq K_0(\mathcal{C})$  in general. An examples is given as follows:

**Example 3.4.** Let  $R = k\langle x_0, \dots, x_n \rangle$  be the free algebra over a field  $k$  on  $n + 1$  (non-commuting) generators. Let  $\mathcal{C}$  denote the category of finitely presented left  $R$ -modules. Then  $K_0(\mathcal{C})$  is a non-trivial cyclic group generated by  $[R]$ , see [3, Prop. 3.2], and subsequently  $\mathfrak{K}_0(\mathcal{C})$  is the trivial group.

**Remark 3.5.** Let  $R$  be a noetherian ring. Then every epimorphism in  $\mathbf{R} - \mathbf{mod}$  that is an endomorphism is an isomorphism.<sup>2</sup> Therefore, we can define a partial order “ $\leq$ ” on the isomorphism classes of finitely generated  $R$ -modules by setting “ $N \leq M$ ” if there exists an epimorphism  $M \twoheadrightarrow N$ .

Accordingly, an additive function  $\mathbf{R} - \mathbf{mod} \rightarrow G$  to a partially ordered abelian group maps  $\mathbf{R} - \mathbf{mod}$  to the positive cone of  $G$  if and only if it is order preserving in the above sense.

**Lemma 3.6.** *Let  $\mathcal{C}$  be an essentially small abelian category. The Grothendieck group  $K_0(\mathcal{C})$  carries a partial order compatible with its group structure such that the natural morphism  $\mathcal{C} \rightarrow K_0(\mathcal{C})$  maps  $\mathcal{C}$  to the positive cone of  $K_0(\mathcal{C})$  if and only if the natural morphism  $K_0(\mathcal{C}) \rightarrow \mathfrak{K}_0(\mathcal{C})$  is a group isomorphism.*

*Proof:* The only-if part of the assertion is immanent. In order to prove the converse, assume that the Grothendieck group  $K_0(\mathcal{C})$  carries a partial order  $\leq$  compatible with addition such that the natural morphism  $\mathcal{C} \rightarrow K_0(\mathcal{C})$  maps  $\mathcal{C}$  to the positive cone of  $K_0(\mathcal{C})$ . Then, by the universal property of the ordered Grothendieck group, we derive a commutative diagram

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<sup>2</sup>In the noetherian case the proof is straightforward. We remark that it holds for commutative rings in general. This result was originally established by A. Grothendieck [44, Prop. 8.9.3] in the case where  $M$  is finitely presented. The general case is credited to J.R. Strooker [94] and W.V. Vasconcelos [104, Prop. 1.2.].

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{[-]} & K_0(\mathcal{C}) \\
\llbracket - \rrbracket \downarrow & \nearrow \exists! \varphi & \\
\mathfrak{K}_0(\mathcal{C}) & & 
\end{array}$$

Let  $A$  be an arbitrary element in  $K_0(\mathcal{C})$ . Then there exists objects  $A^+$  and  $A^-$  in  $\mathcal{C}$  such that  $A = [A^+] - [A^-]$ . Then  $\varphi(\llbracket A^+ \rrbracket - \llbracket A^- \rrbracket) = A$ . Thus,  $\varphi$  is onto.

Furthermore,  $\varphi$  is clearly injective. Indeed, let  $B$  be an arbitrary object in  $\mathfrak{K}_0(\mathcal{C})$  with  $\varphi(B) = 0$ . Let  $A$  be a preimage of  $B$  in  $K_0(\mathcal{C})$ . Then  $A$  can be written in the form  $A = [A^+] - [A^-]$  with objects  $A^+, A^-$  in  $\mathcal{C}$ . Hence,

$$0 = \varphi(B) = \varphi(\llbracket A^+ \rrbracket) - \varphi(\llbracket A^- \rrbracket) = [A^+] - [A^-] = A,$$

resulting in  $B = 0$ . ■

**Examples 3.7.** Lemma 3.6 applies in the following examples:

- i.) Let  $(R, \mathfrak{m})$  be an artinian local ring. Then  $G_0(R)$  is isomorphic to the free abelian group generated by  $[R/\mathfrak{m}]$ . That is to say, we have an isomorphism  $G_0(R) \cong \mathbb{Z}$  that identifies  $[M]$  with  $\ell(M)$ . If  $R$  is an arbitrary artinian ring, then  $R$  is a finite product of artinian local rings and we have an isomorphism

$$G_0(R) \cong \bigoplus_{\mathfrak{m} \in \mathfrak{mSpec}(R)} G_0(R_{\mathfrak{m}}) \cong \bigoplus_{\mathfrak{m} \in \mathfrak{mSpec}(R)} \mathbb{Z}.$$

Thus,  $G_0(R) \cong \mathfrak{G}_0(R)$ .

- ii.) Assume that  $R$  is a regular local ring. Since every finitely generated module over a regular local ring has a finite projective dimension, the Cartan homomorphism  $K_0(R) \cong G_0(R)$  is an isomorphism. Then  $K_0(R)$  carries a partial order by [36]. If we transfer the partial order to  $G_0(R)$ , then the canonical morphism  $\mathbf{R} - \mathbf{mod} \rightarrow G_0(R)$  maps  $\mathbf{R} - \mathbf{mod}$  to the positive cone  $G_0(R)$ . Therefore,  $\mathfrak{G}_0(R) \cong K_0(R)$ .
- iii.) Let  $R$  be a noetherian ring, and let  $\mathbf{x}$  be a sequence of elements in  $R$ . Then by devissage, see [108, 6.3.3], we have an isomorphism  $K_0((\mathbf{R} - \mathbf{mod})_{\mathbf{x}}) \cong K_0(\mathbf{R}/\mathbf{x} - \mathbf{mod})$ . This isomorphism descends to an isomorphism  $\mathfrak{K}_0((\mathbf{R} - \mathbf{mod})_{\mathbf{x}}) = \mathfrak{K}_0(\mathbf{R}/\mathbf{x}\mathbf{R} - \mathbf{mod})$ .

**Remark 3.8.** We note that we can provide a more nuanced and less redundant perspective on the theory developed so far. We conjecture the following:

There is an equivalence of categories

$$\left( \begin{array}{c} \text{injective maps from monoids} \\ \text{to abelian groups} \end{array} \right) \rightarrow \left( \begin{array}{c} \text{partially preordered} \\ \text{abelian groups} \end{array} \right)$$

which one objects can be described as follows: An injective map  $M \rightarrow G$  from a monoid  $M$  to an abelian group  $G$  give rise to a partial preorder on  $G$  where  $a \leq b$  in  $G$  if there exists an element  $c$  in  $M$  such that  $a + c = b$ .

Similarly, there is an equivalence of categories

$$\left( \begin{array}{c} \text{injective maps from semigroups} \\ \text{without identity to abelian groups} \end{array} \right) \rightarrow \left( \begin{array}{c} \text{partially ordered} \\ \text{abelian groups} \end{array} \right)$$

such that the forgetful functor from partially ordered abelian groups to partially preordered abelian groups corresponds to the functor

$$\left( \begin{array}{c} \text{injective maps from semigroups} \\ \text{without identity to abelian groups} \end{array} \right) \rightarrow \left( \begin{array}{c} \text{injective maps from monoids} \\ \text{to abelian groups} \end{array} \right)$$

that is adding an identity element on the domain. This functor admits a left adjoint  $F$  that is essentially given by modding out all elements in the domain that admit an inverse and removing the identity element.

Given a commutative monoid  $M$ , let  $N$  denote the image of  $M$  in the Grothendieck group  $G(M)$ . Then the injection  $N \rightarrow G(M)$  of commutative cancellation monoids corresponds to a partially preordered abelian group  $G$  and  $\mathfrak{G}(M)$  is equal to  $F(G)$ . Likewise, given an essentially small abelian category  $\mathcal{C}$ , let  $N'$  denote the image of  $\mathcal{C}$  in  $K_0(\mathcal{C})$ . Then the injection  $N' \rightarrow K_0(\mathcal{C})$  corresponds to a partially preordered abelian group  $G'$  and  $\mathfrak{K}_0(\mathcal{C})$  is equal to  $F(G')$ .

Our next objective is to provide an explicit characterization of the ordered Grothendieck group of a noetherian local ring.

**Theorem 3.9.** *Let  $R$  be a noetherian local ring. The ordered Grothendieck group  $\mathfrak{G}_0(R)$  gets generated by the residue classes  $\llbracket R/\mathfrak{p} \rrbracket$  where  $\mathfrak{p}$  is a minimal prime ideal of  $R$ . If  $\mathfrak{q}$  is a prime ideal in  $R$  that is not a minimal prime ideal, then  $\llbracket R/\mathfrak{q} \rrbracket$  is zero in  $\mathfrak{G}_0(R)$ .*

*Proof:* We prove the assertion by an ascending induction on the Krull dimension of  $R$ . The induction start where  $R$  is artinian is clear by Examples 3.7. Let  $R$  be a noetherian local ring of Krull dimension  $\dim(R) \geq 1$  and assume that the assertion of the theorem holds true for all noetherian local rings of Krull dimension  $\dim(R) - 1$ . Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  denote the minimal prime ideals of  $R$ .

According to [93, Tag 00L0], the ordered Grothendieck group  $\mathfrak{G}_0(R)$  gets generated by  $\{\llbracket R/\mathfrak{p} \rrbracket \mid \mathfrak{p} \in \text{Spec}(R)\}$ . Now let  $\mathfrak{q}$  be a prime ideal of  $R$  that is not a minimal prime ideal. Due to prime avoidance, there exists an element  $z$  in  $\mathfrak{q}$  such that  $z$  is not contained in any minimal prime ideal of  $R$ . Then  $\dim(R/zR) = \dim(R) - 1$ . Let  $n > \tau(R, z)$ , and let  $y$  denote  $z^n$ . Our next objective is to prove that the transfer morphism  $\mathfrak{G}_0(R/yR) \rightarrow \mathfrak{G}_0(R)$ , induced by restriction of scalars, is the zero morphism. Let  $N$  denote  $R/R[y] \otimes R/yR$ , and let  $\mathfrak{q}_1, \dots, \mathfrak{q}_m$  denote the minimal prime ideals over  $(y)$ . Krull's principal ideal theorem implies that the ideals  $\mathfrak{q}_1, \dots, \mathfrak{q}_m$  are height one prime ideals of  $R$ . There exists an exact sequence

$$0 \rightarrow R[y] \rightarrow R/yR \rightarrow N \rightarrow 0.$$

Then  $y$  is a system of parameters of  $R_{\mathfrak{q}_i}$ , equivalently

$$\ell(R/yR \otimes R_{\mathfrak{q}_i}) - \ell(R[y] \otimes R_{\mathfrak{q}_i}) > 0$$

for every index  $i = 1, \dots, m$ , and we derive that  $N$  is supported at the  $\mathfrak{q}_i$ <sup>3</sup>. Let  $\mathbf{z} = z_1, \dots, z_s$  be a system of parameters of  $R/yR$ . Then  $\mathbf{z}$  is a system of parameters of  $N$ . The universal property of the ordered Grothendieck group  $\mathfrak{G}_0(R/yR)$  implies that the function  $e(\mathbf{z}; -) : \mathbf{R}/\mathbf{yR} - \mathbf{mod} \rightarrow \mathbb{N}$ , mapping a finitely generated  $R/yR$  module  $M$  to  $\sum_{i=0}^s (-1)^i \ell_{R/yR}(H_i(\mathbf{z}; M))$ , factors over  $\mathfrak{G}_0(R/yR)$ . A fortiori,  $\llbracket N \rrbracket$  is non-zero in  $\mathfrak{G}_0(R/yR)$ . Then the induction assumption and [93, Tag 00L0Tag 00L7] imply that there are positive natural numbers  $c_1, \dots, c_m$  such that

$$\llbracket N \rrbracket = \sum_{i=1}^m c_i \cdot \llbracket R/\mathfrak{q}_i \rrbracket \quad \text{in} \quad \mathfrak{G}_0(R/yR).$$

There are two exact sequences

$$0 \rightarrow yR \rightarrow R \rightarrow R/yR \rightarrow 0$$

and

$$0 \rightarrow R[y] \rightarrow R \rightarrow yR \rightarrow 0.$$

Hence,  $\llbracket N \rrbracket = 0$  in  $\mathfrak{G}_0(R)$ , resulting in  $\llbracket R/\mathfrak{q}_i \rrbracket = 0$  for all  $i = 1, \dots, m$ . In particular,  $\llbracket R/\mathfrak{q} \rrbracket = 0$  in  $\mathfrak{G}_0(R)$ , whence the induction step.  $\blacksquare$

Subsequently, given a noetherian local ring  $R$ , up to canonical isomorphism we get the following explicit description of the ordered Grothendieck group  $\mathfrak{G}_0(R)$ :

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<sup>3</sup>I.e. the  $\mathfrak{q}_i$  are contained in the support of  $N$ .

**Corollary 3.10.** *Let  $\mathbb{Z}[\min\text{Spec}(R)]$  denote the free abelian group generated by the minimal prime ideals of  $R$ . Then  $\mathbb{Z}[\min\text{Spec}(R)]$  is partially ordered via the product order and the morphism*

$$\mathfrak{G}_0(R) \rightarrow \mathbb{Z}[\min\text{Spec}(R)]$$

*given on generators via*

$$[[M]] \mapsto \sum_{\mathfrak{p} \in \min\text{Spec}(R)} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot \mathfrak{p}$$

*is an isomorphism of partially ordered abelian groups.*

*Proof:* Let  $\varphi$  denote the map

$$\begin{aligned} \mathbf{R} - \mathbf{mod} &\rightarrow \mathbb{Z}[\min\text{Spec}(R)] \\ [M] &\mapsto \sum_{\mathfrak{p} \in \min\text{Spec}(R)} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}). \end{aligned}$$

Then  $\varphi(R/\mathfrak{p}) = 1$  for every minimal prime ideal  $\mathfrak{p}$  of  $R$  and  $\varphi$  uniquely factors over the ordered Grothendieck group  $\mathfrak{G}_0(R)$ . Theorem 3.9 in conjugation with [93, Tag 00L0, Tag 00L7] implies the theorem.  $\blacksquare$

As a direct consequence of the splitting Lemma we derive

**Corollary 3.11.** *Let  $R$  be a noetherian local ring. Then the ordered Grothendieck group  $\mathfrak{G}_0(R)$  is canonical isomorphic to a direct summand of the Grothendieck group  $G_0(R)$  and under this identification the canonical morphism  $G_0(R) \rightarrow \mathfrak{G}_0(R)$  corresponds to the projection.*

Furthermore, the universal property of the ordered Grothendieck group 3.3, implies

**Corollary 3.12.** *Let  $R$  be a noetherian local ring, and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow G^+$  be an additive function mapping  $\mathbf{R} - \mathbf{mod}$  to the positive cone  $G^+$  of a partially ordered abelian group  $G$ . Then  $\mu(R/\mathfrak{q})$  vanishes whenever  $\mathfrak{q}$  is not a minimal prime ideal of  $R$ .*

A posteriori, Corollary 3.12 indicates that, in our context, working with Grothendieck groups provides no advantage in the “non-artinian case”. Consequently, we will assume artinianess in both *Chapter 6* and *Chapter 7*.

**Remark 3.13.** For a noetherian local domain Corollary 3.12 is well known in the case  $G = \mathbb{R}$ ; see [79, Thm. 2]. Furthermore, we remark that in [60] G. Krause proved that for a right noetherian ring  $R$  there is a finite number of, what he terms atomic rank functions  $\rho^1, \rho^2, \dots, \rho^n$ , where  $n$  is the number of minimal prime ideals of  $R$  and where the  $\rho^i$  arises from the minimal prime ideals such that every additive rank function, i.e a function  $\mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}$ , is of the form  $k_1\rho^1 + k_2\rho^2 + \dots + k_n\rho^n$  with non-negative integers  $k_i$ . Nevertheless, a priori, it is not very clear that Krause's point of view is equivalent to the one given above. In particular, it is not clear, that Krause's atomic rank functions are uniquely determined.



# Chapter 4

## Fundamental Properties of Intersection Multiplicities

In this chapter, we briefly diverge from our primary goal of addressing Question 0.7, as Corollary 3.12 offers immediate applications to logical implications among fundamental properties of intersection multiplicities.

In the following, let  $R$  be an equidimensional, universal catenary, noetherian local ring, and let

$$\tilde{\chi} : \mathbf{R} - \mathbf{mod} \times \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{R}_\infty$$

be a biadditive function such that  $\tilde{\chi}(M, N)$  is finite whenever  $M \otimes N$  has finite length, and  $M$  or  $N$  has finite projective dimension. The primary example of such a function is the intersection multiplicity of two modules after J.-P. Serre.

**Definition 4.1.** *Let  $M$  and  $N$  be a pair of finitely generated modules over a noetherian local ring  $R$  such that  $N$  has finite projective dimension and  $M \otimes N$  has finite length. Then the module  $\text{Tor}_i(M, N)$  has finite length, for every index  $i$ , and the intersection multiplicity of  $M$  and  $N$  is defined to be*

$$\chi_R(M, N) := \sum_{i=0}^{\infty} (-1)^i \cdot \ell_R(\text{Tor}_i(M, N)).$$

For  $\tilde{\chi}$  to be considered a “good” intersection multiplicity, it is crucial that it possesses certain classical properties. The most essential among these are the following:

We say that  $R$  possesses

**(HT)**: If  $\dim(M) + \dim(N) \leq \dim(R)$  for all finitely generated  $R$ -modules  $M, N$  where  $M \otimes_R N$  has finite length;

**(HT\*)**: If  $\dim(M) + \dim(N) \leq \dim(R)$  for all finitely generated  $R$ -modules  $M, N$  where  $M \otimes_R N$  has finite length, and  $N$  or  $M$  has finite projective dimension.

We say that  $(R, \tilde{\chi})$  possesses

**(NN)**: If  $\tilde{\chi}(M, N) \geq 0$  for all finitely generated  $R$ -modules  $M, N$  where  $M \otimes_R N$  has finite length, and  $N$  or  $M$  has finite projective dimension;

**(V)**: If  $\tilde{\chi}(M, N) = 0$  for all finitely generated  $R$ -modules  $M, N$  where  $M \otimes_R N$  has finite length,  $M$  or  $N$  has finite projective dimension, and  $\dim(M) + \dim(N) < \dim(R)$ ;

**(P)** If  $\tilde{\chi}(M, N) > 0$  for all finitely generated  $R$ -modules  $M, N$  where  $M \otimes_R N$  has finite length,  $M$  or  $N$  has finite projective dimension, and  $\dim(M) + \dim(N) = \dim(R)$ .

We say that a noetherian local ring  $R$  possesses  $(\star)$  with  $\star$  in  $\{\mathbf{P}, \mathbf{V}, \mathbf{NN}\}$  if the pair  $(R, \chi_R)$  possesses  $(\star)$ .

J.-P. Serre demonstrated that every regular local ring  $R$  possesses **(HT)**, and he conjectured that every regular local ring  $R$  possesses **(P)** and **(V)**. These conjecture are known as the *multiplicity conjectures*. He successfully proved the multiplicity conjectures for regular local rings of equal characteristic and for regular local rings of mixed characteristic in the unramified case; see [91].

In around 1985, P. C. Roberts and independently H. Gillet and C. Soulé proved the vanishing property **(V)** for regular local rings. Both proofs work for a complete intersection ring, provided that in **(V)** both modules  $M$  and  $N$  are required to have finite projective dimension. In around 1995, Gabber established nonnegativity **(NN)** for regular local rings, in the course of which he also gave a new proof for **(V)** in the regular case; see [35, 86, 85]. The *positivity conjecture*, which asserts that every regular local ring possesses **(P)**, remains open.

Historically, the question whether the multiplicity conjectures still hold when the regularity condition is dropped, naturally arose during the investigation of the original conjectures of J.-P. Serre. Whether there exists a noetherian local ring, or more specifically a complete intersection ring  $R$  that does not possess **(P)** or **(V)** was an open problem for some time. This problem was eventually resolved in [21] by S. P. Dutta, M. Hochster and J.E. McLaughlin, who constructed a module  $M$  of finite length and finite projective dimension over

$$R = K[X_1, X_2, X_3, X_4]_{\mathfrak{p}} / (X_1X_4 - X_2X_3),$$

where  $K$  is a field and  $\mathfrak{p}$  is the ideal  $(X_1, X_2, X_3, X_4)$  such that

$$\chi(M, R/(X_1, X_2)) = -1.$$

The ring  $R$  is also an example of a ring not possessing **(HT)**, since

$$\dim(K[X_1, X_2, X_3, X_4]/(X_1, X_2)) + \dim(K[X_1, X_2, X_3, X_4]/(X_3, X_4)) = 4.$$

Whether there are noetherian local rings not possessing **(HT\*)**, is very much an open problem. This was questioned by C. Peskine and L. Szpiro in [81, 82]. In [81] C. Peskine and L. Szpiro even raised the following question: Let  $R$  be a noetherian local ring, and let  $M$  and  $N$  be finitely generated  $R$ -modules such that  $M \otimes_R N$  has finite length, and  $M$  has finite projective dimension. Let  $I$  denote the annihilator of  $M$ . Must it be true that  $\dim(N) \leq \text{depth}_I(R)$ , (which is always  $\leq \dim(R) - \dim(M)$ )?

In the following, making use of Corollary 3.12, we will prove logical implications among the properties **(NN)**, **(V)** **(P)**, **(HT\*)** and **(HT)**. First we will prove some preliminary Lemmas.

**Lemma 4.2.** *Let  $\mathfrak{a}, \mathfrak{b}$  be ideals of a noetherian local ring  $R$  such that  $R/(\mathfrak{a} + \mathfrak{b})$  has finite length. Then  $\mathfrak{b}$  contains a system of parameters of the  $R$ -module  $R/\mathfrak{a}$ .*

*Proof:* Let  $\mathbf{x} = x_1, \dots, x_s$  in  $R/\mathfrak{a}$  denote a system of parameters of the ring  $R/\mathfrak{a}$ . The ring  $R/(\mathfrak{a} + \mathfrak{b})$  is the quotient of  $R/\mathfrak{a}$  by the ideal  $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$  and by the prerequisites it is of finite length. Thus, there exists a natural number  $n$  such that the elements  $x_1^n, \dots, x_s^n$  are contained in  $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$ . For every index  $i = 1, \dots, s$  let  $y_i$  in  $\mathfrak{b}$  be a representative of the residue class of  $x_i^n$ . Then  $y_1, \dots, y_s$  is a system of parameters of the  $R$ -module  $R/\mathfrak{a}$ , whence the Lemma.  $\blacksquare$

**Proposition 4.3.** *Let  $R$  be a noetherian local ring, let  $\mathfrak{b}$  be an ideal of  $R$ , and let  $N$  be a finitely generate  $R$ -module such that  $R/\mathfrak{b} \otimes_R N$  has finite length. Then  $\mathfrak{b}$  contains a system of parameters of  $N$ .*

*Proof:* We can assume without loss of generality that  $N$  is not artinian. Let  $\mathfrak{q}_1, \dots, \mathfrak{q}_n$  denote the minimal prime ideals in the support of  $N$ . Then [93, Tag 00L0, Tag 00L7] implies that  $R/\mathfrak{q}_i \otimes R/\mathfrak{b}$  has finite length for every  $i = 1, \dots, n$ . Let  $\mathfrak{q}$  be a minimal prime ideal in the support of  $N$  such that  $\dim(R/\mathfrak{q}) = \dim(N)$ . Let  $\mathfrak{a}$  denote the annihilator of  $N$ . Then, by prime avoidance, there exists elements  $b_1, \dots, b_m$  in  $\mathfrak{b} \cdot \mathfrak{q}$  such that  $\mathfrak{q}$  is the unique minimal prime ideal over  $(\mathfrak{a}, b_1, \dots, b_m)$ . Then there exists a system of parameters  $\mathbf{x} = x_1, \dots, x_s$  of  $R/\mathfrak{b}$  that is contained in  $\mathfrak{q}$  by Lemma 4.2. Further, there exists a natural number  $n$  such that  $x_1^n, \dots, x_s^n$  is contained in  $\mathfrak{a} + (b_1, \dots, b_m)$ . Hence,  $R/(\mathfrak{a} + \mathfrak{b})$  has finite length, and the proposition follows from Lemma 4.2.  $\blacksquare$

**Lemma 4.4.** *Let  $R$  be a noetherian local ring possessing **(HT\*)**, let  $N$  be a finitely generated  $R$ -module of finite projective dimension, and let  $\mathbf{x} = x_1, \dots, x_s$  be a system of parameters of  $N$ . Then  $\mathbf{x}$  is part of a system of parameters of  $R$ .*

*Proof:* Making use of **(HT\*)**, we derive

$$\dim(N) + \dim(R/\mathbf{x}R) \leq \dim(R).$$

Thus,  $\dim(R/\mathbf{x}R) \leq \dim(R) - s$ , resulting in  $\dim(R/\mathbf{x}R) = \dim(R) - s$ . ■

**Theorem 4.5.** *If  $(R, \tilde{\chi})$  possesses **(NN)** and  $R$  possesses **(HT\*)**, then  $(R, \tilde{\chi})$  possesses **(V)**.*

*Proof:* Let  $\mathfrak{p}$  be a prime ideal of  $R$ , and let  $N$  be a finitely generated  $R$ -module of finite projective dimension such that  $R/\mathfrak{p} \otimes N$  has finite length. Then, by Lemma 4.2, there exists a system of parameters  $\mathbf{x} = x_1, \dots, x_s$  of  $N$  that is contained in  $\mathfrak{p}$ , and Lemma 4.4 says that  $\mathbf{x}$  is part of a system of parameters of  $R$ . Assume that  $\dim(R/\mathfrak{p}) + \dim(N) < \dim(R)$ , equivalently  $\text{ht}(\mathfrak{p}) > s$ . Then, according to Krull's height theorem,  $\mathfrak{p}$  is not a minimal prime ideal over  $(\mathbf{x})$ .

The function  $\chi(-, N)$  gives rise to an additive function from  $\mathbf{R}/\mathbf{x}\mathbf{R} - \mathbf{mod}$  to  $\mathbb{N}$ , and Corollary 3.12 implies that  $\chi(R/\mathfrak{p}, N) = 0$ . The additivity of  $\chi(-, N)$  and [93, Tag 00L0, Tag 00L7] imply that  $(R, \tilde{\chi})$  possesses **(V)**. ■

**Lemma 4.6.** *If  $(R, \tilde{\chi})$  possesses **(P)**, then  $(R, \tilde{\chi})$  possesses **(NN)**.*

*Proof:* Let  $M, N$  be a pair of finitely generated  $R$ -modules, where  $N$  has finite projective dimension,  $\dim(N) + \dim(M) < \dim(R)$ , and  $M \otimes N$  has finite length. For the sake of contradiction, assume that  $\chi(M, N)$  is negative. Let  $\mathbf{x} = x_1, \dots, x_s$  be a system of parameters of  $N$ . Then we conclude from Krull's height theorem that there exists a prime ideal  $\mathfrak{p}$  over  $(\mathbf{x})$  of height  $s$ . Let  $M' := R/\mathfrak{p}$ . Then  $\chi(M', N) > 0$ . Let

$$Q = \bigoplus_{i=1}^{\chi(M', N)} M \bigoplus_{i=1}^{-\chi(M, N)} M'.$$

Then  $\dim(Q) + \dim(N) = \dim(R)$  and  $\chi(Q, N) = 0$ ; contradiction. ■

In order to prove **(HT)** for regular local rings, J.-P. Serre used Cohen's structure theorem in order to apply the validity of the multiplicity conjectures for formal power series rings over a complete discrete valuation ring. More generally, we have

**Theorem 4.7.** *If  $(R, \tilde{\chi})$  possesses **(P)**, then  $R$  possesses **(HT\*)**.*

*Proof:* First, we note that  $(R, \tilde{\chi})$  possesses **(NN)** by Lemma 4.6. Let  $M, N$  be finitely generated  $R$ -modules, where  $N$  has finite projective dimension. Let  $\mathfrak{p}$  be

a minimal prime ideal in the support of  $M$  such that  $\dim(M) = \dim(R/\mathfrak{p})$ . Then, [93, Tag 00L0, Tag 00L7] implies that  $R/\mathfrak{p} \otimes N$  has finite length. So there exists a system of parameters  $\mathbf{x} = x_1, \dots, x_s$  of  $N$  that is contained in  $\mathfrak{p}$  by Proposition 4.3.

For the sake of contradiction, assume that  $\dim(M) + \dim(N) > \dim(R)$ . Then  $\text{ht}(\mathfrak{p}) < s$  and there exists a prime ideal  $\mathfrak{q}$  of  $R$  over  $\mathfrak{p}$  of height  $s$ . Then  $\dim(R/\mathfrak{q}) + \dim(N) = \dim(R)$ . Hence,  $\chi(R/\mathfrak{q}, N) > 0$ .

The function  $\chi(-, N)$  gives rise to an additive function from  $\mathbf{R}/\mathbf{xR} - \mathbf{mod}$  to  $\mathbb{N}$  and Corollary 3.12 implies that  $\mathfrak{q}$  is a minimal prime ideal over  $(\mathbf{x})$ . Thus  $\mathfrak{p} = \mathfrak{q}$ ; contradiction. ■



# Chapter 5

## General Multiplicities

In this chapter, we will establish some standard properties of generalized multiplicities. It serves as a preparation for *Chapter 6*. As is foreshadowed by the proof of Theorem 2.11, such a multiplicity theory can serve as a “great” computational tool in our context. In fact, our further studies of Question 0.7 heavily relies on the syzygy of the UCT 0.1 and properties of the J.-P. Serre’s multiplicity symbol, joint by Lemma 1.1. In the remainder of this chapter if not said otherwise let  $R$  denote a noetherian ring.

**Definition 5.1.** *Let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements in  $R$ . For every  $s$ -tuple  $\underline{i}$  with entries in  $\{0, 1\}$ , we recursively define the module  $M\langle\langle\mathbf{x}\rangle\rangle_{\underline{i}}$ . We put  $M\langle\langle\mathbf{x}\rangle\rangle_{\emptyset} := M$  in the case  $s = 0$ , and we put*

$$M\langle\langle x \rangle\rangle_0 := M/xM \quad \text{and} \quad M\langle\langle x \rangle\rangle_1 := M[x]$$

*in the case  $s = 1$ . Now let  $s \geq 2$ , and assume that  $M\langle\langle\mathbf{x}\rangle\rangle_{\underline{j}}$  has been defined for  $\underline{j} = \underline{i}|_{1, \dots, s-1}$ . Then we set*

$$M\langle\langle x_1, \dots, x_s \rangle\rangle_{\underline{i}} := (M\langle\langle x_1, \dots, x_{s-1} \rangle\rangle_{(i_1, \dots, i_{s-1})})\langle\langle x_s \rangle\rangle_{i_s}.$$

**Definition 5.2.** *Let  $R$  be a noetherian ring, and let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements of  $R$ . Let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_{\infty}$  be an additive function. For every finitely generated  $R$ -module  $M$  we define*

$$e_{\mu}(\mathbf{x}; M) := \sum_{\underline{i} \in \{0, 1\}^s} (-1)^{|\underline{i}|} \mu(M\langle\langle\mathbf{x}\rangle\rangle_{\underline{i}})$$

*to be the multiplicity of  $M$  with coefficients  $\mathbf{x}$  with respect to  $\mu$ .*

*We will occasionally omit  $\mu$  from the notation when the specific additive function is clear from the context.*

In the following, let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements of  $R$ , and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_{\infty}$  be an additive function.

**Lemma 5.3.** *The function  $e_\mu(\mathbf{x}; -)$  is additive.*

*Proof:* Let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

be an exact sequence of finitely generated  $R$ -modules. We prove the assertion by an ascending induction on  $s$ ; the induction start  $s = 0$  being immanent. So assume that the assertion holds true for  $s - 1$ . The induction step follows immediately from the exact sequence

$$0 \rightarrow M'[x_1] \rightarrow M[x_1] \rightarrow M''[x_1] \rightarrow M'/x_1M' \rightarrow M/x_1M \rightarrow M''/x_1M'' \rightarrow 0$$

and the additivity of  $e_\mu(x_2, \dots, x_s; -)$ . ■

**Lemma 5.4.** *Let  $M$  be a finitely generated  $R$ -module. Then*

$$e_\mu(\mathbf{x}; M) = e_\mu(\mathbf{x}|_{2, \dots, s}; M/x_1M) - e_\mu(\mathbf{x}|_{2, \dots, s}; M[x_1]).$$

*Proof:* We calculate

$$\begin{aligned} e_\mu(\mathbf{x}; M) &= \sum_{\underline{i} \in \{0,1\}^s} (-1)^{|\underline{i}|} \mu(M \langle \mathbf{x} \rangle_{\underline{i}}) \\ &= \sum_{\underline{i} \in \{0,1\}^{s-1}} (-1)^{|\underline{i}|} \mu(M/x_1M \langle \mathbf{x} \rangle_{\underline{i}}) + \sum_{\underline{i} \in \{0,1\}^{s-1}} (-1)^{|\underline{i}|} \mu(M[x_1] \langle \mathbf{x} \rangle_{\underline{i}}) \\ &= e_\mu(\mathbf{x}|_{2, \dots, s}; M/x_1M) - e_\mu(\mathbf{x}|_{2, \dots, s}; M[x_1]), \end{aligned}$$

whence the lemma. ■

In the following, we derive various standard properties of multiplicity à la D. J. Wright [110] from Lemma 5.4.

**Proposition 5.5.** *Let  $M, N$  be a pair of finitely generated  $R$ -modules, where  $M$  has finite projective dimension. Then*

$$\sum_{\underline{i} \in \{0,1\}^s} \sum_{k=0}^{\infty} (-1)^{|\underline{i}|+k} \mu(\text{Tor}_k(N \langle \mathbf{x} \rangle_{\underline{i}}; M)) = \sum_{\underline{i} \in \{0,1\}^s} \sum_{k=0}^{\infty} (-1)^{|\underline{i}|+k} \mu(\text{Tor}_k(N; M) \langle \mathbf{x} \rangle_{\underline{i}}).$$

*Proof:* We prove a more general assertion. Namely, for every index  $1 \leq r \leq s$  we find

$$\begin{aligned} &\sum_{\substack{\underline{i} \in \{0,1\}^r \\ \underline{j} \in \{0,1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{|\underline{i}|+|\underline{j}|+k} \mu(\text{Tor}_k(N \langle \mathbf{x}|_{\{1, \dots, r\}} \rangle_{\underline{i}}; M) \langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle_{\underline{j}}) \\ &= \sum_{\underline{i} \in \{0,1\}^s} \sum_{k=0}^{\infty} (-1)^{|\underline{i}|+k} \mu(\text{Tor}_k(N; M) \langle \mathbf{x} \rangle_{\underline{i}}). \end{aligned}$$

We prove the assertion by an ascending induction on  $r$ ; the induction start  $r = 0$  being immanent. So let  $r > 0$ , and assume that the assertion holds true for  $r - 1$ . Let  $\underline{i} \in \{0, 1\}^{r-1}$  be arbitrary. By the UCT 0.1, there exists a short exact sequence

$$0 \rightarrow \text{Tor}_k(M, N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}}) \otimes R/x_r R \rightarrow Q_{\underline{i}, k} \rightarrow \text{Tor}_{k-1}(M, N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}})[x_r] \rightarrow 0,$$

for every index  $k$ , and a long exact sequence

$$\cdots \rightarrow \text{Tor}_{k-1}(N\langle\langle \mathbf{x} \rangle\rangle_{\underline{i}, 1}, M) \rightarrow Q_{\underline{i}, k} \rightarrow \text{Tor}_k(N\langle\langle \mathbf{x} \rangle\rangle_{\underline{i}, 0}, M) \rightarrow \cdots.$$

Then

$$\begin{aligned} & \sum_{\substack{\underline{i} \in \{0, 1\}^r \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(Q_{\underline{i}, k} \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}) \\ &= \sum_{\substack{\underline{i} \in \{0, 1\}^{r-1} \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu((\text{Tor}_k(M, N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}}) \otimes R/x_r R) \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}) \\ &+ \sum_{\substack{\underline{i} \in \{0, 1\}^{r-1} \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu((\text{Tor}_{k-1}(M, N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}})[x_r]) \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}) \\ &= \sum_{\substack{\underline{i} \in \{0, 1\}^{r-1} \\ \underline{j} \in \{0, 1\}^{s-(r-1)}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(\text{Tor}_k(M, N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}}) \langle\langle \mathbf{x}|_{\{r, \dots, s\}} \rangle\rangle_{\underline{j}}) \end{aligned}$$

and

$$\begin{aligned} & \sum_{\substack{\underline{i} \in \{0, 1\}^r \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(Q_{\underline{i}, k} \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}) \\ &= \sum_{\substack{\underline{i} \in \{0, 1\}^{r-1} \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(\text{Tor}_k(N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}, 0}) \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}) \\ &+ \sum_{\substack{\underline{i} \in \{0, 1\}^{r-1} \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(\text{Tor}_{k-1}(N\langle\langle \mathbf{x}|_{1, \dots, r-1} \rangle\rangle_{\underline{i}, 1}, M) \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}) \\ &= \sum_{\substack{\underline{i} \in \{0, 1\}^r \\ \underline{j} \in \{0, 1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(\text{Tor}_k(N\langle\langle \mathbf{x}|_{1, \dots, r} \rangle\rangle_{\underline{i}}, M) \langle\langle \mathbf{x}|_{\{r+1, \dots, s\}} \rangle\rangle_{\underline{j}}). \end{aligned}$$

Making use of the induction assumption, we conclude

$$\begin{aligned} & \sum_{\substack{\underline{i} \in \{0,1\}^r \\ \underline{j} \in \{0,1\}^{s-r}}} \sum_{k=0}^{\infty} (-1)^{k+|\underline{j}|+|\underline{i}|} \mu(\text{Tor}_k(N \langle \mathbf{x}|_{1,\dots,r} \rangle_{\underline{i}}, M) \langle \mathbf{x}|_{\{r+1,\dots,s\}} \rangle_{\underline{j}}) \\ &= \sum_{\underline{i} \in \{0,1\}^s} \sum_{k=0}^{\infty} (-1)^{|\underline{i}|+k} \mu(\text{Tor}_k(N; M) \langle \mathbf{x} \rangle_{\underline{i}}), \end{aligned}$$

that is the induction step. ■

**Corollary 5.6.** *Let  $M$  be a finitely generated  $R$ -module. Then*

$$e_{\mu}(\mathbf{x}; M) = \sum_{i=0}^s (-1)^i \mu(H_i(\mathbf{x}; M)).$$

*Proof:* Use Lemma 0.2 in order to apply Proposition 5.5. ■

**Corollary 5.7.** *One has  $e_{\mu}(\mathbf{x}; M) \leq \mu(M/\mathbf{x}M)$  for every finitely generated  $R$ -module  $M$ .*

*Proof:* Follows immediately from Corollary 5.6 and [91, Appendix II]. ■

**Lemma 5.8.** *Let  $M$  be a finitely generated  $R$ -module, and let  $\underline{n}$  be an  $s$ -tuple of natural numbers. Then*

$$e_{\mu}(\mathbf{x}^{\underline{n}}; M) = n_1 \cdot \dots \cdot n_s \cdot e_{\mu}(\mathbf{x}; M).$$

*Proof:* In order to prove the assertion, it suffices to prove that

$$e_{\mu}(x_1^n, \mathbf{x}|_{\{2,\dots,s\}}; M) = n \cdot e_{\mu}(\mathbf{x}; M),$$

for all natural numbers  $n$ . Applying Corollary 5.6 and Lemma 0.2, we can assume without loss of generality that  $x_1$  is a non-zero divisor on  $R$ . Then we have a short exact sequence

$$0 \rightarrow R/x_1 R \rightarrow R/x_1^m R \rightarrow R/x_1^{m-1} R \rightarrow 0,$$

for all natural numbers  $m$ , and the case  $s = 1$  follows from that. The general case follows by making use of Proposition 5.5. ■

**Lemma 5.9.** *Let  $M$  be a finitely generated  $R$ -module, let  $i$  be a natural number, and let  $\mathbf{y} = y_1, \dots, y_t$  be sequences of elements of  $R$ . Then*

*i.) We find*

$$\begin{aligned} & e_\mu(\mathbf{x}|_{2,\dots,s}, H_i(x_1, \mathbf{y}; M)) - e_\mu(\mathbf{x}, H_i(\mathbf{y}; M)) \\ &= e_\mu(\mathbf{x}|_{2,\dots,s}, H_i(\mathbf{y}; M)[x_1]) + e_\mu(\mathbf{x}|_{2,\dots,s}, H_{i-1}(\mathbf{y}; M)[x_1]); \end{aligned}$$

*ii.) We find*

$$\begin{aligned} & \mu(H_i(\mathbf{x}, \mathbf{y}; M)) - e_\mu(\mathbf{x}, H_i(\mathbf{y}; M)) \\ &= \sum_{r=1}^s e_\mu(\mathbf{x}|_{r+1,\dots,s}, H_i(\mathbf{x}|_{1,\dots,r-1}, \mathbf{y}; M)[x_r]) \\ & \quad + \sum_{r=1}^s e_\mu(\mathbf{x}|_{r+1,\dots,s}, H_{i-1}(\mathbf{x}|_{1,\dots,r-1}, \mathbf{y}; M)[x_r]). \end{aligned}$$

*Proof:* Let  $1 \leq i \leq s$  be an arbitrary index. Then the UCT 0.1 implies

$$\begin{aligned} & e_\mu(\mathbf{x}|_{2,\dots,s}, H_i(x_1, \mathbf{y}; M)) \\ &= e_\mu(\mathbf{x}|_{2,\dots,s}, H_i(\mathbf{y}; M) \otimes R/x_1R) + e_\mu(\mathbf{x}|_{2,\dots,s}, H_{i-1}(\mathbf{y}; M)[x_1]) \end{aligned}$$

and Lemma 5.4 says that

$$\begin{aligned} & e_\mu(\mathbf{x}|_{1,\dots,s}, H_i(\mathbf{y}; M)) \\ &= e_\mu(\mathbf{x}|_{2,\dots,s}, H_i(\mathbf{y}; M) \otimes R/x_1R) - e_\mu(\mathbf{x}|_{2,\dots,s}, H_i(\mathbf{y}; M)[x_1]). \end{aligned}$$

We conclude *i.)* and *ii.)* is an immediate consequence. ■

**Historical Remark 5.10.** The Euler-Poincaré characteristic of Koszul complexes is dimension-sensitive. That is to say, given a noetherian semi-local ring  $R$ , a finitely generated  $R$ -module  $M$  and a multiplicity system  $\mathbf{x} = x_1, \dots, x_s$  of  $M$ , the Euler-Poincaré characteristic  $\chi_{\ell_R}(x_1, \dots, x_s; M)$  vanishes if and only if  $s$  is exceeding the dimension of  $M$  by [6, Thm. 4.1.]. Building upon J.-P. Serre's work, M. Auslander and D. A. Buchsbaum in [6] rigorously employed the methods of homological algebra to investigate multiplicities axiomatically. This development has opened the subject to a much simpler treatment, and an example of this may be found in D.J. Wright's paper [110] and more comprehensively in D.G. Northcott's book [78]. Here, a recursive definition, similar to Definition 5.2, of the so-called general multiplicity symbol is provided, as suggested by Auslander-Buchsbaum's recursion formula for the Euler-Poincaré characteristic [6, Thm. 3.3].

From a categorical point of view it seems natural to define a “general multiplicity symbol” such that the multiplicity symbols factors through the Grothendieck group; the “universal receiver” of generalized Euler characteristics. Approaches into this direction were taken by K. Blackburn [13], M. Fraser [27], J. R. Strooker [95], J. Stückrad [96] and K. Nishida [77].

Given a noetherian local ring  $R$ , for all multiplicity theories in existence, the multiplicity symbol “ $e(\mathbf{x}; -)$ ” essentially is an additive function from  $\mathbf{R} - \mathbf{mod}$  to  $G_0(R/\mathbf{x}R)$  for all sequences  $\mathbf{x} = x_1, \dots, x_s$  of  $R$  which is defined as an Euler-Poincaré characteristic of Koszul homology. This approach has some drawbacks. One of them is that it is not clear whether these multiplicity symbols are *dimension-sensitive*.<sup>1</sup> In other words: Given a sequence  $\mathbf{x}$  of a noetherian local ring  $R$  and a finitely generated  $R$ -module  $M$ , is “ $e(\mathbf{x}; M)$ ” nontrivial in, say  $G_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}})$  where  $S = R/\text{Ann}(M)$  if and only if  $\mathbf{x}$  is part of a system of parameters of  $M$ ? This may not hold true, because “ $e(\mathbf{x}^n; M)$ ” could vanish for some power  $n$ , whereas “ $e(\mathbf{x}; M)$ ” does not. This can not occur in  $\mathfrak{K}_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}})$  and in the remainder of this chapter we will prove that dimension sensitivity of the Euler-Poincaré characteristic of Koszul homology in  $\mathfrak{K}_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}})$  indeed holds true.

**Lemma 5.11.** *Let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements of  $R$ . Then the  $i$ 's partial Euler-Poincaré characteristic*

$$\chi_i(\mathbf{x}; M) := \sum_{j=i}^s (-1)^{j-i} \llbracket H_j(\mathbf{x}; M) \rrbracket$$

*is positive in  $\mathfrak{K}_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}})$  for every index  $0 \leq i \leq s$  where  $S$  denotes the ring  $R/\text{Ann}(M)R$ .*

*Proof:* By Serre [91, Appendix II], there exists a finitely generated  $S/\mathbf{x}S$ -module  $N$  such that the equation

$$[N] = \sum_{j=i}^s (-1)^{j-i} [H_j(\mathbf{x}; M)]$$

holds true in  $G_0(R/\text{Ann}(M) + \mathbf{x}R)$ , whence the Lemma. ■

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<sup>1</sup>Although, many results of the existing theories point into this direction; for example Fraser [27, Cor. 2.9, Thm. 1.25, Cor. 2.9].

**Lemma 5.12.** *Let  $M$  be a finitely generated  $R$ -module, let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements of  $R$ , let  $S$  denote  $R/\text{Ann}(M)$ , and let  $\mu : \mathbf{S} - \mathbf{mod}_{\mathbf{x}} \rightarrow \mathfrak{G}_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}})$  denote the canonical morphism where  $S := R/\text{Ann}(M)$ . For all sufficiently large natural numbers  $n_1, \dots, n_s$  there exists a finitely generated  $R/\text{Ann}(M) + (\mathbf{x}^n)R$ -module  $N$  such that  $e_\mu(\mathbf{x}^n; M) = \llbracket N \rrbracket$ . If  $R$  is local, and  $\mathbf{y} = y_1, \dots, y_t$  is a system of parameters of  $N$ , then  $\mathbf{y}$  is a system of parameters of  $R/\text{Ann}(M) + (\mathbf{x}^n)R$ .*

*Proof:* We prove the assertion by an ascending induction on  $s$ ; the induction base  $s = 0$  being immanent. So let  $s > 0$  and assume that the assertion holds true for  $s - 1$ .

Let  $\mathfrak{a}$  denote the annihilator of  $M$ . By the induction assumption, for all sufficiently large  $n_1, \dots, n_{s-1}$  there exists a finitely generated  $R/\mathfrak{a} + (x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}})R$ -module  $N'$  such that  $e(x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}}; M) = \llbracket N' \rrbracket$  and if  $R$  is local and  $\mathbf{y} = y_1, \dots, y_t$  is a system of parameters of  $N'$ , then  $\mathbf{y}$  is a system of parameters of  $R/\mathfrak{a} + (x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}})R$ .

Let  $n_s > \tau(N', x_s)$ . There exists an exact sequence

$$0 \rightarrow N'[x_s^\infty] \rightarrow N'/x_s^{n_s}N' \rightarrow N'/N'[x_s^\infty] \otimes R/x_s^{n_s}R \rightarrow 0.$$

Thus,

$$e(\mathbf{x}^n; M) = e_\mu(x_s^{n_s}; N') = \llbracket N'/N'[x_s^\infty] \otimes R/x_s^{n_s}R \rrbracket.$$

Let  $N := N'/N'[x_s^\infty] \otimes R/x_s^{n_s}R$ . Assume that  $R$  is local, and let  $\mathbf{y} = y_1, \dots, y_t$  be a system of parameters of  $N$ . Then

$$e(\mathbf{y}; N) = e(\mathbf{y}, x_s^{n_s}; N') \neq 0$$

in  $\mathbb{Z}$ . Therefore, the induction assumption implies that  $\mathbf{y}, x_s^{n_s}$  is a system of parameters of  $R/\mathfrak{a} + (x_1, \dots, x_{s-1})R$ , settling the induction step.  $\blacksquare$

**Theorem 5.13** (Dimension-Sensitivity). *Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module, let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements of  $R$ , and let  $\mu : \mathbf{S} - \mathbf{mod}_{\mathbf{x}} \rightarrow \mathfrak{G}_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}})$  denote the canonical morphisms where  $S := R/\text{Ann}(M)$ . Then  $e_\mu(\mathbf{x}; M) \neq 0$  if and only if  $\mathbf{x}$  is part of a system of parameters of  $M$ .*

*Proof:* Assume that  $\mathbf{x}$  is part of a system of parameters of  $M$ . Then Lemma 5.8, Lemma 5.12 and Corollary 3.10 imply  $e(\mathbf{x}; M) > 0$ .

Vice versa, assume that  $e(\mathbf{x}; M) > 0$ . According to Proposition 5.8 and Lemma 5.8, we can assume without loss of generality that there exist a finitely generated

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$R/\text{Ann}(M) + (\mathbf{x})R$ -module  $N$  such that  $e(\mathbf{x}; M) = \llbracket N \rrbracket$  and every system of parameters  $\mathbf{y}$  of  $N$  is a system of parameters of  $R/\text{Ann}(M) + (\mathbf{x})$ . Then, by the universal property of the ordered Grothendieck group, we derive a morphism

$$\mathfrak{K}_0(\mathbf{S} - \mathbf{mod}_{\mathbf{x}}) \rightarrow \mathbb{Z}$$

mapping  $\llbracket A \rrbracket$  to  $e(\mathbf{y}; A)$ . Then

$$e(\mathbf{y}, N) = e(\mathbf{x}, \mathbf{y}; M) \neq 0$$

in  $\mathbb{Z}$ . Hence,  $\mathbf{x}, \mathbf{y}$  is a system of parameters of  $M$ . ■

# Chapter 6

## Optimal Polynomial Lower Estimates for Koszul Homology

In this chapter, if not said otherwise, let  $R$  be a noetherian ring, let  $M$  be a bounded object of  $\mathcal{D}_{ft}^{\geq 0}(R)$ , and let  $\mathbf{x} = x_1, \dots, x_s$  denote a multiplicity system of  $M$  consisting of non-zero divisors of  $R$ . We will regard any finitely generated  $R$ -module  $M$  naturally as an element of  $\mathcal{D}_{ft}^{\leq 0}(R) \cap \mathcal{D}_{ft}^{\geq 0}(R)$ .

According to Theorem 2.11, the function

$$\begin{aligned} \varphi_k &: \mathbb{N}^s \rightarrow \mathbb{N} \\ (n_1, \dots, n_s) &\mapsto \ell(H_k(x_1^{n_1}, \dots, x_s^{n_s}, M)) \end{aligned}$$

admits an optimal polynomial lower estimate for every index  $k$ . Having established the necessary tools, we can now provide precise formulas. Additionally, we will present various results dealing with the vanishing of their coefficients.

**Definition 6.1.** *Let  $M$  be an object of  $\mathcal{D}_{ft}^b(R)$ , let  $\mathbf{y} = y_1, \dots, y_t$  be a sequence of non-zero divisors of  $R$ , and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_\infty$  be an additive function such that  $e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M))$  is finite for every index  $k$  and every subset  $J$  of  $\{1, \dots, t\}$ .*

*For every subset  $I$  of  $\{1, \dots, t\}$  and every natural number  $k$  we set*

$$\lambda_{\mathbf{y},k,\mu}(I; M) := \sum_{\substack{J \subseteq \{1, \dots, t\} \\ I \subseteq J}} (-1)^{|J|-|I|} e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M))$$

*and*

$$\Upsilon_{\mathbf{y},k,\mu}(N_1, \dots, N_t; M) := \sum_{I \subseteq \{1, \dots, t\}} \lambda_{\mathbf{y},k}(I; M) \cdot \prod_{i \in I} N_i.$$

*In order to simplify the notation, we shall omit the symbols  $M$ ,  $\mathbf{y}$  and  $\mu$  whenever their meanings are clear from the context. Specifically, if  $\mathbf{y}$  is a multiplicity system of  $M$  and  $\mu$  represents  $\ell_R$ , we will omit  $\mu$  from the notation.*

**Lemma 6.2.** *Let  $R$  be an arbitrary commutative ring, and let*

$$P(X_1, \dots, X_s) = \sum_{I \subseteq \{1, \dots, s\}} \lambda_I \cdot \prod_{i \in I} X_i$$

*be a multivariate linear polynomial in the variables  $X_1, \dots, X_s$  with coefficients in  $R$ . Then*

*i.) For any elements  $a_1, \dots, a_s$  in  $R$  we find*

$$P(X_1 + a_1, \dots, X_s + a_s) = \sum_{I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \left( \prod_{j \in J \setminus I} a_j \right) \cdot \lambda_J \cdot \prod_{i \in I} X_i;$$

*ii.) For any index  $1 \leq r \leq s$  we find*

$$\Delta_r P(X_1, \dots, X_s) = \sum_{\{r\} \in I \subseteq \{1, \dots, s\}} \lambda_I \prod_{i \in I \setminus \{r\}} X_i.$$

*Proof:* We prove i.) by an ascending induction on  $s$ ; the induction start  $s = 0$  being immanent. So let  $s > 0$ , and assume that the assertion holds true for  $s - 1$ . Then

$$\begin{aligned} & P(X_1 + a_1, \dots, X_s + a_s) \\ &= \sum_{I \subseteq \{1, \dots, s\}} \lambda_I \cdot \prod_{i \in I} (X_i + a_i) \\ &= \sum_{\substack{I \subseteq \{1, \dots, s\} \\ s \in I}} \lambda_I \cdot \prod_{i \in I} (X_i + a_i) + \sum_{I \subseteq \{1, \dots, s-1\}} \lambda_I \cdot \prod_{i \in I} (X_i + a_i). \end{aligned}$$

Employing the induction assumption, we derive

$$\begin{aligned} & \sum_{\substack{I \subseteq \{1, \dots, s\} \\ s \in I}} \lambda_I \cdot \prod_{i \in I} (X_i + a_i) \\ &= (X_s + a_s) \cdot \sum_{\substack{I \subseteq \{1, \dots, s\} \\ s \in I}} \lambda_I \cdot \prod_{i \in I \setminus \{s\}} (X_i + a_i) \\ &= (X_s + a_s) \cdot \sum_{I \subseteq \{1, \dots, s-1\}} \lambda_{I \cup \{s\}} \cdot \prod_{i \in I} (X_i + a_i) \\ &= (X_s + a_s) \cdot \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s-1\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_{J \cup \{s\}} \cdot \prod_{i \in I} X_i \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{I \subseteq \{1, \dots, s\} \\ s \in I}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i \\
&+ \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s-1\} \\ I \subseteq J}} a_s \cdot \prod_{j \in J \setminus I} a_j \cdot \lambda_{J \cup \{s\}} \cdot \prod_{i \in I} X_i \\
&= \sum_{\substack{I \subseteq \{1, \dots, s\} \\ s \in I}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i + \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \cup \{s\} \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{I \subseteq \{1, \dots, s-1\}} \lambda_I \cdot \prod_{i \in I} (X_i + a_i) \\
&= \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s-1\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i. \\
&= \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i - \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \cup \{s\} \in J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i.
\end{aligned}$$

To conclude,

$$\begin{aligned}
&P(X_1 + a_1, \dots, X_s + a_s) \\
&= \sum_{\substack{I \subseteq \{1, \dots, s\} \\ s \in I}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i + \sum_{I \subseteq \{1, \dots, s-1\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i \\
&= \sum_{I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} \prod_{j \in J \setminus I} a_j \cdot \lambda_J \cdot \prod_{i \in I} X_i,
\end{aligned}$$

settling the induction step.

In order to prove ii.), we can assume without loss of generality that  $r = s$ . We compute

$$\begin{aligned}
&\Delta_s(P(X_1, \dots, X_s)) \\
&= \sum_{I \subseteq \{1, \dots, s-1\}} \lambda_I \cdot \prod_{i \in I} X_i + (X_s + 1) \cdot \sum_{\substack{I \subseteq \{1, \dots, s\} \\ \{s\} \in I}} \lambda_I \cdot \prod_{i \in I \setminus \{s\}} X_i \\
&- \sum_{I \subseteq \{1, \dots, s-1\}} \lambda_I \cdot \prod_{i \in I} X_i - X_s \cdot \sum_{\substack{I \subseteq \{1, \dots, s\} \\ \{s\} \in I}} \lambda_I \cdot \prod_{i \in I \setminus \{s\}} X_i \\
&= \sum_{\substack{I \subseteq \{1, \dots, s\} \\ \{s\} \in I}} \lambda_I \cdot \prod_{i \in I \setminus \{s\}} X_i,
\end{aligned}$$

whence the result. ■

**Lemma 6.3.** *Let  $G$  be an abelian group, let  $S$  be a finite set, and let  $I$  be subset. For every subset  $J$  of  $S$  containing  $I$  let  $g_J$  be an element of  $G$ . Then*

$$\sum_{I \subseteq J \subseteq S} g_J = g_I + \sum_{u=1}^{|S|-|I|} (-1)^{u+1} \sum_{\substack{K \subseteq I^c \\ |K|=u}} \sum_{I \cup K \subseteq J} g_J.$$

*Proof:* We prove the assertion by an ascending induction on  $|S|$ ; the induction start  $|S| = 0$  being immanent. If  $S = I$ , then the assertion is true. Accordingly, we assume that there exists an element  $s \in S \setminus I$  such that  $S \setminus \{s\}$  is non-empty, and the assertion holds true for the pair  $(I, S \setminus \{s\})$ . We shall show that the assertion holds true for the pair  $(S, I)$ . We have

$$\sum_{I \subseteq J \subseteq S} g_J = \sum_{I \subseteq J \subseteq S \setminus \{s\}} g_J + \sum_{I \cup \{s\} \subseteq J \subseteq S} g_J.$$

By the induction assumption we find

$$\sum_{I \subseteq J \subseteq S \setminus \{s\}} g_J = g_I + \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S \setminus \{s\}} g_J.$$

Further,

$$\begin{aligned} & \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S \setminus \{s\}} g_J \\ = & \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S} g_J \\ & - \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup \{s\} \cup K \subseteq J \subseteq S} g_J \\ = & \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus I \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S} g_J \\ & - \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup \{s\} \cup K \subseteq J \subseteq S} g_J \end{aligned}$$

$$\begin{aligned}
& - \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus I \\ |K|=u, s \in K}} \sum_{I \cup K \subseteq J \subseteq S} g_J \\
& = \sum_{u=1}^{|S|-|I|} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus I \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S} g_J - \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup \{s\} \cup K \subseteq J \subseteq S} g_J \\
& \quad - \sum_{u=1}^{|S|-|I|} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u-1}} \sum_{I \cup \{s\} \cup K \subseteq J \subseteq S} g_J \\
& = \sum_{u=1}^{|S|-|I|} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus I \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S} g_J - \sum_{u=1}^{|S|-|I|-1} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup \{s\} \cup K \subseteq J \subseteq S} g_J \\
& \quad - \sum_{u=0}^{|S|-|I|-1} (-1)^u \sum_{\substack{K \subseteq S \setminus (I \cup \{s\}) \\ |K|=u}} \sum_{I \cup \{s\} \cup K \subseteq J \subseteq S} g_J \\
& = \sum_{u=1}^{|S|-|I|} (-1)^{u+1} \sum_{\substack{K \subseteq S \setminus I \\ |K|=u}} \sum_{I \cup K \subseteq J \subseteq S} g_J - \sum_{I \cup \{s\} \subseteq J \subseteq S} g_J,
\end{aligned}$$

whence the induction step. ■

**Corollary 6.4.** *For every admissible index  $k$  and every subset  $I$  of  $\{1, \dots, s\}$  one has*

$$\lambda_{\mathbf{x},k}(I; M) = e(\mathbf{x}|_I; H_k(\mathbf{x}|_{I^c}; M)) - \sum_{\emptyset \neq K \subseteq I^c} \lambda_{\mathbf{x},k}(I \cup K; M).$$

*Proof:* Lemma 6.3 implies that

$$\begin{aligned}
& \lambda_{\mathbf{x},k}(I; M) - e(\mathbf{x}|_I; H_k(\mathbf{x}|_{I^c}; M)) \\
& = \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \\
& = \sum_{u=1}^{s-|I|} \sum_{\substack{K \subseteq I^c \\ |K|=u}} \sum_{I \cup K \subseteq J} (-1)^{|J|-|I|+u+1} e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)).
\end{aligned}$$

and clearly

$$(-1)^{|J|-|I|+u+1} \cdot (-1)^{|J|-|I|-u} = (-1)^{2|J|-2|I|+1} = -1.$$

Hence,

$$(-1)^{|J|-|I|-u} = -(-1)^{|J|-|I|+u+1}$$

implying

$$\begin{aligned} & \lambda_{\mathbf{x},k}(I; M) - e(\mathbf{x}|_I; H_k(\mathbf{x}|_{I^c}; M)) \\ &= - \sum_{u=1}^{s-|I|} \sum_{\substack{K \subseteq I^c \\ |K|=u}} \sum_{I \cup K \subseteq J} (-1)^{|J|-|I \cup K|} e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \\ &= - \sum_{\emptyset \neq K \subseteq I^c} \lambda_{\mathbf{x},k}(I \cup K; M), \end{aligned}$$

whence the corollary. ■

**Corollary 6.5.** *Let  $r \leq s$  be positive natural numbers, and let  $M$  be a finitely generated  $R$ -module. The following are equivalent:*

- i.) *For all proper subsets  $I$  of  $\{1, \dots, s\}$  containing  $\{1, \dots, r\}$  we find  $\lambda_{\mathbf{x},0}(I; M) = 0$ ;*
- ii.) *For all subsets  $I$  of  $\{1, \dots, s\}$  containing  $\{1, \dots, r\}$  we find*

$$e(\mathbf{x}|_{1, \dots, s}; M) = e(\mathbf{x}|_I; H_0(\mathbf{x}|_{I^c}; M)).$$

*Proof:* That i.) implies ii.) is immediate by Corollary 6.4.

Vice versa, assume that ii.) holds true. We prove by a descending induction on the cardinality of  $I$  that  $\lambda_{\mathbf{x},0}(I; M)$  vanishes whenever  $I$  is a proper subset of  $\{1, \dots, s\}$  containing  $\{1, \dots, r\}$ .

The induction start  $|I| = s - 1$  is immanent by Corollary 6.4. Accordingly, assume that  $|I| < s - 1$  and that the assertion holds true for all proper subsets  $I'$  of  $\{1, \dots, s\}$  containing  $\{1, \dots, r\}$  with  $|I'| > |I|$ . Then Corollary 6.4 implies that  $\lambda_{\mathbf{x},k}(I; M) = 0$ , whence the result. ■

**Corollary 6.6.** *Let  $k$  be a natural number. If  $\Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M)$  is of the form*

$$\Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M) = \sum_{r=0}^s \lambda_r \prod_{i=1}^r N_r$$

*with coefficients  $\lambda_r$  in  $\mathbb{Z}$ , then*

$$\lambda_r = e(\mathbf{x}|_{1, \dots, r}; H_k(\mathbf{x}|_{r+1, \dots, s}; M)) - e(\mathbf{x}|_{1, \dots, r+1}; H_k(\mathbf{x}|_{r+2, \dots, s}; M))$$

*for all  $r = 1, \dots, s - 1$ .*

*Proof:* We prove the corollary by a descending induction on  $r$ ; the induction start  $r = s - 1$  being true by Corollary 6.4. For the induction step let  $r < s - 1$ , and assume that the assertion holds true for the index  $r + 1$ . Then Corollary 6.4 and the induction assumption imply

$$\begin{aligned}
\lambda_r &= e(\mathbf{x}|_{1,\dots,r}; H_k(\mathbf{x}|_{r+1,\dots,s}; M)) - \sum_{i=1}^{s-r} \lambda_{r+i} \\
&= e(\mathbf{x}|_{1,\dots,r}; H_k(\mathbf{x}|_{r+1,\dots,s}; M)) - \sum_{i=1}^{s-r} [e(\mathbf{x}|_{1,\dots,r+i}; H_k(\mathbf{x}|_{r+i+1,\dots,s}; M)) \\
&\quad - e(\mathbf{x}|_{1,\dots,r+i+1}; H_k(\mathbf{x}|_{r+i+2,\dots,s}; M))] \\
&= e(\mathbf{x}|_{1,\dots,r}; H_k(\mathbf{x}|_{r+1,\dots,s}; M)) - e(\mathbf{x}|_{1,\dots,r+1}; H_k(\mathbf{x}|_{r+2,\dots,s}; M)),
\end{aligned}$$

whence the result. ■

**Lemma 6.7.** *For every admissible index  $k$  and every subset  $K$  of  $\{1, \dots, s\}$  one has*

$$\begin{aligned}
&\Upsilon_{\mathbf{x}|_{K^c,k}}((N_1, \dots, N_s)|_{K^c}; \mathcal{K}(\mathbf{x}|_K) \otimes^L M) \\
&= \Upsilon_{\mathbf{x},k}((1 - \xi_K(1)) \cdot N_1 + \xi_K(1), \dots, (1 - \xi_K(s)) \cdot N_s + \xi_K(s); M)
\end{aligned}$$

where  $\xi_K$  denotes the indicator function of  $K$ .

*Proof:* In order to prove the Lemma, we can assume without loss of generality that  $K = \{1\}$ . We have

$$\Upsilon_{\mathbf{x},k}(1, N_2, \dots, N_s; M) = \sum_{I \subseteq \{2, \dots, s\}} [\lambda_{\mathbf{x},k}(I; M) + \lambda_{\mathbf{x},k}(I \cup \{1\}; M)] \cdot \prod_{i \in I} N_i$$

and for every subset  $I$  of  $\{2, \dots, s\}$  we find

$$\begin{aligned}
&\lambda_{\mathbf{x},k}(I; M) + \lambda_{\mathbf{x},k}(I \cup \{1\}; M) \\
&= \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} \cdot e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \\
&\quad - \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \cup \{1\} \subseteq J}} (-1)^{|J|-|I|} \cdot e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \\
&= \sum_{\substack{J \subseteq \{2, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} \cdot e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)),
\end{aligned}$$

whence the lemma. ■

**Lemma 6.8.** *Let  $M$  be an object of  $\mathcal{D}_{ft}^b(R)$ , let  $\mathbf{y} = y_1, \dots, y_t$  be a sequence of non-zero divisors of  $R$ , and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_\infty$  be an additive function such that  $e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M))$  is finite for every index  $k$  and every subset  $J$  of  $\{1, \dots, t\}$ . Let  $I$  be a subset of  $\{2, \dots, t\}$ , and let  $\mathbf{y}' := (y_2, \dots, y_t)$ . Then*

$$\lambda_{\mathbf{y},k}(I; M) = \lambda_{\mathbf{y}',k}(I; \mathcal{K}(y_1) \otimes^L M) + \lambda_{\mathbf{y}',k,e_\mu(y_1,-)}(I; M)$$

for every index  $k$ .

*Proof:* We calculate

$$\begin{aligned} \lambda_{\mathbf{y},k}(I; M) &= \sum_{\substack{J \subseteq \{1, \dots, t\} \\ I \subseteq J}} (-1)^{|J|-|I|} e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M)) \\ &= \sum_{\substack{J \subseteq \{2, \dots, t\} \\ I \subseteq J}} (-1)^{|J|-|I|} e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M)) \\ &\quad + \sum_{\substack{J \subseteq \{1, \dots, t\} \\ I \cup \{1\} \subseteq J}} (-1)^{|J|-|I|} e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M)) \\ &= \lambda_{\mathbf{y}',k,\mu}(I; \mathcal{K}(y_1) \otimes^L M) + \lambda_{e_\mu(y_1,-),\mathbf{y}',k}(I; M), \end{aligned}$$

whence the lemma. ■

**Proposition 6.9.** *All of the coefficients of  $\Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M)$  are non-negative for every index  $k$ .*

*Proof:* For every sequence  $\mathbf{y} = y_1, \dots, y_t$  of non-zero divisors of  $R$ , every natural number  $0 \leq r \leq t$ , every object  $M$  of  $\mathcal{D}_{ft}^b(R)$ , and every additive function  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_\infty$  such that  $e_\mu(\mathbf{y}|_J; H_k(\mathbf{y}|_{J^c}; M))$  is finite for every index  $k$ , and every subset  $J$  of  $\{1, \dots, t\}$ , let

$$\Upsilon_{\mathbf{y},k,\mu}^r(N_1, \dots, N_t; M) := \sum_{\substack{I \subseteq \{1, \dots, t\} \\ \{1, \dots, r\} \subseteq I}} \lambda_{\mathbf{y},k,\mu}(I) \prod_{i \in I \setminus \{1, \dots, r\}} N_i.$$

We prove by an ascending induction on  $t - r$  that all of the coefficients of  $\Upsilon_{\mathbf{y},k}^r(N_1, \dots, N_t; M)$  are non-negative; the induction start  $t - r = 0$  being immanent by Lemma 5.11. Accordingly, we assume that  $t - r > 0$  and that the assertion holds true for all sequences of non-zero divisors  $y'_1, \dots, y'_{t'}$  of  $R$  and all natural numbers  $0 \leq r' \leq t'$  with  $t' - r' < t - r$ . We have

$$\begin{aligned} &\Upsilon_{\mathbf{y},k,\mu}^r(N_1, \dots, N_t; M) \\ &= N_{r+1} \cdot \Delta_{r+1} \Upsilon_{\mathbf{y},k,\mu}^r(N_1, \dots, N_t; M) + \Upsilon_{\mathbf{y},k,\mu}^r(N_1, \dots, N_r, 0, N_{r+2}, \dots, N_t; M). \end{aligned}$$

Let  $\mathbf{y}'$  denote  $(y_1, \dots, \widehat{y_{r+1}}, \dots, y_t)$ . Lemma 6.8 and Lemma 6.2 together with the induction assumptions imply that all of the coefficients of

$$\begin{aligned}
& \Upsilon_{\mathbf{y},k,\mu}^r(N_1, \dots, N_r, 0, N_{r+2}, \dots, N_t; M) \\
&= \sum_{\substack{\{1, \dots, r\} \subseteq I \subseteq \{1, \dots, t\} \\ r+1 \notin I}} \lambda_{\mathbf{y},k,\mu}(I) \cdot \prod_{i \in I \setminus \{1, \dots, r\}} N_i \\
&= \sum_{\{1, \dots, r\} \subseteq I \subseteq \{1, \dots, \widehat{r+1}, \dots, t\}} \lambda_{\mathbf{y},k,\mu}(I) \cdot \prod_{i \in I \setminus \{1, \dots, r\}} N_i \\
&= \sum_{\{1, \dots, r\} \subseteq I \subseteq \{1, \dots, \widehat{r+1}, \dots, t\}} \lambda_{\mathbf{y}',k,\mu}(I; \mathcal{K}(x_{r+1}) \otimes^L M) \cdot \prod_{i \in I \setminus \{1, \dots, r\}} N_i \\
&\quad + \sum_{\{1, \dots, r\} \subseteq I \subseteq \{1, \dots, \widehat{r+1}, \dots, t\}} \lambda_{\mathbf{y}',k,e_\mu(x_{r+1},-)}(I; M) \cdot \prod_{i \in I \setminus \{1, \dots, r\}} N_i \\
&= \Upsilon_{\mathbf{y}',k,\mu}^r(N_1, \dots, \widehat{N_{r+1}}, \dots, N_t; \mathcal{K}(x_{r+1}) \otimes^L M) \\
&\quad + \Upsilon_{\mathbf{y}',k,e_\mu(x_{r+1},-)}^r(N_1, \dots, \widehat{N_{r+1}}, \dots, N_t; M)
\end{aligned}$$

and

$$\begin{aligned}
& \Delta_{r+1} \Upsilon_{\mathbf{y},k,\mu}^r(N_1, \dots, N_t; R, M) \\
&= \sum_{\{1, \dots, r+1\} \in I \subseteq \{1, \dots, t\}} \lambda_{\mathbf{y},k,\mu}(I) \prod_{i \in I \setminus \{1, \dots, r+1\}} N_i \\
&= \Upsilon_{\mathbf{y},k,\mu}^{r+1}(N_1, \dots, N_t; R, M)
\end{aligned}$$

are non-negative, whence the induction step.  $\blacksquare$

**Remark 6.10.** We conjecture that analogously to Proposition 6.9 one can prove that all of the coefficients of  $\sum_{i=k}^s (-1)^{i-k} \Upsilon_{\mathbf{x},\mu,i}(N_1, \dots, N_s; M)$  are non-negative for every index  $k$ .

**Theorem 6.11.** *Let  $i$  be a natural number. Then*

$$\sum_{k=i}^v (-1)^{k-i} \Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M) \leq \sum_{k=i}^v (-1)^{k-i} \ell(H_k(x_1^{N_1}, \dots, x_s^{N_s}; M))$$

*is an optimal polynomial lower estimate over  $\mathbb{R}$  whenever  $v = i$  or  $v = \infty$ <sup>1</sup>.*

<sup>1</sup>If the cohomology of  $M$  is vanishing in degree larger than  $a$ , then  $\infty$  can be replaced by  $a + s$ . This is clear by the Tor base change spectral sequence. See also the upcoming Lemma 6.15.

*Proof:* First we prove the following assertion: Let  $M$  be a bounded object of  $\mathcal{D}_{ft}^{\geq 0}(R)$ , let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of non-zero divisors of  $R$ , and let  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_\infty$  be an additive function such that  $e_\mu(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M))$  is finite for every index  $k$  and every subset  $J$  of  $\{1, \dots, s\}$ .

For all natural numbers  $0 \leq r \leq s$ , all  $s$ -tuples  $\underline{n}$  of natural numbers and all subsets  $I$  of  $\{r+1, \dots, s\}$  one has

$$\begin{aligned} & \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r\}}, \mathbf{x}|_{I^c}; M)) \\ & \geq \sum_{k=u}^v (-1)^{k-u} \cdot \sum_{J \subseteq \{1, \dots, r\}} e_\mu(\mathbf{x}|_{I \cup J}; H_k(\mathbf{x}|_{(I \cup J)^c}; M)) \cdot \prod_{j \in J} n_j \end{aligned}$$

where  $(I \cup J)^c$  denotes  $\{1, \dots, s\} \setminus \{I \cup J\}$ , in the two cases  $v = s$  and  $v = \infty$ .

We prove the assertion by an ascending induction on  $r$ ; the induction start  $r = 0$  being immanent. So let  $r > 0$  and assume that the assertion holds true for  $r - 1$  and let  $I$  be a subset of  $\{r+1, \dots, s\}$ .

For every natural number  $k > 0$  and every  $s$ -tuple  $\underline{n}$  of natural numbers the UCT 0.1 yields an exact sequence

$$\begin{aligned} 0 & \rightarrow H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\} \cup I^c}; M) \otimes R/x_r^{n_r+1} R \\ & \rightarrow H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r\} \cup I^c}; M) \\ & \rightarrow H_{k-1}(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\} \cup I^c}; M) [x_r^{n_r+1}] \rightarrow 0. \end{aligned}$$

Furthermore, Lemma 1.1 implies that for all natural numbers  $m$  we have

$$\begin{aligned} & \Delta_r(e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r^m R)) \\ & = \Delta_r(e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r^m])) \\ & \quad + e_\mu(\mathbf{x}|_{I \cup \{r\}}; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M)), \end{aligned}$$

implying

$$\begin{aligned} & e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r^{n_r+1} R) \\ & = e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r R) \\ & \quad + \sum_{m=1}^{n_r} \Delta_r(e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r^m R)) \\ & = e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r R) \\ & \quad + e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r^{n_r+1}]) \end{aligned}$$

$$\begin{aligned}
& - e_\mu (\mathbf{x}|_I; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r]) \\
& + n_r \cdot e_\mu (\mathbf{x}|_{I \cup \{r\}}; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M)) \\
\geq & n_r \cdot e_\mu (\mathbf{x}|_{I \cup \{r\}}; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M)) \\
& + e_\mu (\mathbf{x}|_I; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r R).
\end{aligned}$$

The UCT 0.1 yields

$$\begin{aligned}
& \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu (\mathbf{x}|_I; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r\}}, \mathbf{x}|_{I^c}; M)) \\
= & \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu (\mathbf{x}|_I; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r^{n_r+1} R) \\
& + \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu (\mathbf{x}|_I; H_{k-1} (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r^{n_r+1}]) \\
= & (-1)^{v-u} \cdot e_\mu (\mathbf{x}|_I; H_v (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r^{n_r+1} R) \\
& + e_\mu (\mathbf{x}|_I; H_{u-1} (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r^{n_r+1}]) \\
& + (n_r + 1) \cdot \sum_{k=u}^{v-1} (-1)^{k-u} e_\mu (\mathbf{x}|_{I \cup \{r\}}; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M)).
\end{aligned}$$

for all  $s$ -tuples  $\underline{n}$  of natural numbers.

If  $v$  is sufficiently large, then in fact

$$H_v (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) = 0.$$

In either case we can estimate

$$\begin{aligned}
& \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu (\mathbf{x}|_I; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r\}}, \mathbf{x}|_{I^c}; M)) \\
\geq & n_r \cdot e_\mu (\mathbf{x}|_{I \cup \{r\}}; H_v (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M)) \\
& + e_\mu (\mathbf{x}|_I; H_v (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r R) \\
& + e_\mu (\mathbf{x}|_I; H_{u-1} (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r]) \\
& + (n_r + 1) \cdot \sum_{k=u}^{v-1} (-1)^{k-u} e_\mu (\mathbf{x}|_{I \cup \{r\}}; H_k (\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M)).
\end{aligned}$$

Then by the above we derive that

$$\begin{aligned}
& e_\mu(\mathbf{x}|_I; H_v(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) \otimes R/x_r R) \\
& + e_\mu(\mathbf{x}|_I; H_{u-1}(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M) [x_r]) \\
& = \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r\}}, \mathbf{x}|_{I^c}; M)) \\
& \quad - \sum_{k=u}^{v-1} (-1)^{k-u} e_\mu(\mathbf{x}|_{I \cup \{r\}}; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{I^c}; M))
\end{aligned}$$

and we can further estimate

$$\begin{aligned}
& \sum_{k=u}^v (-1)^{k-u} \cdot e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r\}}, \mathbf{x}|_{I^c}; M)) \\
& \geq n_r \cdot \sum_{k=u}^v (-1)^{k-u} e_\mu(\mathbf{x}|_{I \cup \{r\}}; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{(I \cup \{r\})^c}; M)) \\
& \quad + \sum_{k=u}^v (-1)^{k-u} e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{\{1, \dots, r\} \setminus I}; M)),
\end{aligned}$$

where  $(I \cup \{r\})^c$  denotes  $\{1, \dots, r\} \setminus (I \cup \{r\})$ .

By the induction assumption we have

$$\begin{aligned}
& n_r \cdot \sum_{k=u}^v (-1)^{k-u} e_\mu(\mathbf{x}|_{I \cup \{r\}}; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{(I \cup \{r\})^c}; M)) \\
& \geq n_r \cdot \sum_{k=u}^v (-1)^{k-u} \sum_{J \subseteq \{1, \dots, r-1\}} e_\mu(\mathbf{x}|_{I \cup \{r\} \cup J}; H_k(\mathbf{x}|_{(I \cup \{r\} \cup J)^c}; M)) \cdot \prod_{j \in J} n_j \\
& = \sum_{k=u}^v (-1)^{k-u} \sum_{\substack{J \subseteq \{1, \dots, r\} \\ r \in J}} e_\mu(\mathbf{x}|_{I \cup J}; H_k(\mathbf{x}|_{(I \cup J)^c}; M)) \cdot \prod_{j \in J} n_j.
\end{aligned}$$

The set  $I$  is in particular a subset of  $\{r, \dots, s\}$ . Hence, applying the induction assumption we derive

$$\begin{aligned}
& \sum_{k=u}^v (-1)^{k-u} e_\mu(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r-1\}}, \mathbf{x}|_{\{1, \dots, r\} \setminus I}; M)) \\
& \geq \sum_{k=u}^v (-1)^{k-u} \sum_{J \subseteq \{1, \dots, r-1\}} e_\mu(\mathbf{x}|_{I \cup J}; H_k(\mathbf{x}|_{(I \cup J)^c}; M)) \cdot \prod_{j \in J} n_j
\end{aligned}$$

$$\geq \sum_{k=u}^v (-1)^{k-u} \sum_{\substack{J \subseteq \{1, \dots, r\} \\ r \notin J}} e_{\mu}(\mathbf{x}|_{I \cup J}; H_k(\mathbf{x}|_{(I \cup J)^c}; M)) \cdot \prod_{j \in J} n_j.$$

Thus,

$$\begin{aligned} & \sum_{k=u}^v (-1)^{k-u} e_{\mu}(\mathbf{x}|_I; H_k(\mathbf{x}^{n+1}|_{\{1, \dots, r\}}, \mathbf{x}|_{I^c}; M)) \\ & \geq \sum_{k=u}^v (-1)^{k-u} \sum_{J \subseteq \{1, \dots, r\}} e_{\mu}(\mathbf{x}|_{I \cup J}; H_k(\mathbf{x}|_{(I \cup J)^c}; M)) \cdot \prod_{j \in J} n_j, \end{aligned}$$

whence the induction step. In particular, if  $\mathbf{x}$  is a multiplicity system of  $M$  employing Lemma 6.2 in the case  $r = s$  we derive

$$\begin{aligned} & \sum_{k=u}^v (-1)^{k-u} \ell(H_k(\mathbf{x}^{\underline{n}}; M)) \\ & \geq \sum_{k=u}^v (-1)^{k-u} \sum_{J \subseteq \{1, \dots, s\}} \prod_{j \in J} (n_j - 1) \cdot e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \\ & = \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k}(\underline{n}). \end{aligned}$$

Our next objective is to establish the optimality of the estimate. Therefore we prove by an ascending induction on  $s$  that for every bounded object  $M$  of  $\mathcal{D}_{\text{ft}}^{\geq 0}(R)$ , every sequence  $\mathbf{x} = x_1, \dots, x_s$  of non-zero divisors of  $R$ , every additive function  $\mu : \mathbf{R} - \mathbf{mod} \rightarrow \mathbb{N}_{\infty}$  such that  $e_{\mu}(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M))$  is finite for every index  $k$  and every subset  $J$  of  $\{1, \dots, s\}$  and every polynomial  $P(N_1, \dots, N_s)$  in  $\mathbb{R}[N_1, \dots, N_s]$  satisfying

$$\sum_{k=u}^v (-1)^{k-u} \mu(H_k(\mathbf{x}^{\underline{N}}; M)) \geq P(\underline{N}) \geq \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(\underline{N}),$$

it follows that  $P(\underline{N}) = \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(\underline{N})$  in the cases  $v = \infty$  and  $v = u$ .

The induction start  $s = 0$  is immanent. So let  $s \geq 1$ , and assume that the assertion holds for  $s - 1$ . Lemma 6.2 implies

$$\sum_{k=u}^v (-1)^{k-u} \mu(H_k(\mathbf{x}; M)) = \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(\underline{N})(\underline{1}) = P(\underline{1}),$$

and  $P$  necessarily is linear in each variable. Thus

$$P(N_1, \dots, N_s) = \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(N_1, \dots, N_s),$$

if and only if

$$\Delta_r(P(N_1, \dots, N_s)) = \Delta_r \left( \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(N_1, \dots, N_s) \right)$$

for all  $1 \leq r \leq s$ . We can assume without loss of generality that  $r = s$ . Let  $\underline{n}$  be an arbitrary  $s$ -tuple of natural numbers. Once again the UCT 0.1 implies

$$\begin{aligned} & \Delta_s \left( \sum_{k=u}^v (-1)^{k-u} \mu(H_k(\mathbf{x}^{\underline{n}}; M)) \right) \\ &= \sum_{k=u}^v (-1)^{k-u} e_{\mu}(x_s, H_k(\mathbf{x}^{\underline{n}}|_{1, \dots, s-1}; M)) \\ & \quad + \sum_{k=u}^v (-1)^{k-u} \Delta_s(\mu(H_{k-1}(\mathbf{x}^{\underline{n}}|_{1, \dots, s-1}; M)[x_s^{n_s}])) \end{aligned}$$

and

$$\begin{aligned} & \Delta_s \left( \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(n_1, \dots, n_s) \right) \\ &= \sum_{k=u}^v (-1)^{k-u} \sum_{\{s\} \in I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} \cdot e_{\mu}(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \prod_{i \in I \setminus \{s\}} n_i \end{aligned}$$

by Lemma 6.2. From the inequality

$$P(\underline{n}) \geq \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(\underline{n})$$

we infer

$$\Delta_s(P(n_1, \dots, n_s)) \geq \Delta_s \left( \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu}(n_1, \dots, n_s) \right).$$

We assert that

$$\sum_{k=u}^v (-1)^{k-u} e_{\mu}(x_s, H_k(\mathbf{x}^{\underline{n}}|_{1, \dots, s-1}; M)) \geq \Delta_s(P(n_1, \dots, n_s)).$$

Indeed, fix arbitrary natural numbers  $n_1, \dots, n_{s-1}$ . Then we conclude from the inequality

$$\mu \left( \sum_{k=u}^v (-1)^{k-u} H_k(\mathbf{x}^{\underline{n}}|_{\{1, \dots, s-1\}}, x_s^{N_s}; M) \right) \geq P(n_1, \dots, n_{s-1}, N_s).$$

that there exists an infinite subset  $S \subset \mathbb{N}$  such that for all  $n_s$  in  $S$  we derive

$$\Delta_s \left( \sum_{k=u}^v (-1)^{k-u} \mu \left( H_k \left( \mathbf{x}^n |_{\{1, \dots, s-1\}}, x_s^{n_s}; M \right) \right) \right) \geq \Delta_s (P(n_1, \dots, n_s)).$$

Then for sufficiently large values of  $n_s$  we find that

$$\begin{aligned} & \Delta_s \left( \sum_{k=u}^v (-1)^{k-u} \mu \left( H_k \left( \mathbf{x}^n |_{1, \dots, s-1}, x_s^{n_s}; M \right) \right) \right) \\ &= \sum_{k=u}^v (-1)^{k-u} e \left( x_s, H_k \left( \mathbf{x}^n |_{1, \dots, s-1}; M \right) \right) \end{aligned}$$

and therefore the inequality

$$\sum_{k=u}^v (-1)^{k-u} e \left( x_s, H_k \left( \mathbf{x}^n |_{1, \dots, s-1}; M \right) \right) \geq \Delta_s (P(n_1, \dots, n_s))$$

holds true for all natural numbers  $n_s$ .

Let  $\tilde{\mu}$  denote the additive function

$$\begin{aligned} \tilde{\mu} : \mathbf{R} - \mathbf{mod} &\rightarrow \mathbb{N}_\infty \\ M &\mapsto e(x_s, M). \end{aligned}$$

Then

$$\begin{aligned} & \sum_{k=u}^v (-1)^{k-u} \tilde{\mu} \left( H_k \left( \mathbf{x}^N |_{1, \dots, s-1}; M \right) \right) \\ &= \sum_{k=u}^v (-1)^{k-u} e \left( x_s, H_k \left( \mathbf{x}^N |_{1, \dots, s-1}; M \right) \right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\tilde{\mu}, \mathbf{x}, k} (N_1, \dots, N_{s-1}) \\ &= \sum_{k=u}^v (-1)^{k-u} \sum_{\{s\} \in I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} \cdot e(\mathbf{x}|_J; H_k(\mathbf{x}|_{J^c}; M)) \prod_{i \in I \setminus \{s\}} N_i. \end{aligned}$$

The induction assumption implies that

$$\Delta_s (P(N_1, \dots, N_s)) = \Delta_s \left( \sum_{k=u}^v (-1)^{k-u} \Upsilon_{\mathbf{x}, k, \mu} (N_1, \dots, N_s) \right),$$

settling the induction step. ■

**Corollary 6.12.** *One has*

$$\sum_{k=0}^{\infty} (-1)^k \Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M) = e(\mathbf{x}; M) \cdot \prod_{i=1}^s N_i.$$

**Corollary 6.13.** *If  $i$  is an even natural number, then*

$$\sum_{k=0}^i (-1)^{k-i} \Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M) \leq \sum_{k=0}^i (-1)^{k-i} \ell(H_k(x_1^{N_1}, \dots, x_s^{N_s}; M))$$

*is an optimal polynomial lower estimate over  $\mathbb{R}$  and if  $i$  is odd, then*

$$\sum_{k=0}^i (-1)^{k-i} \Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M) \geq \sum_{k=0}^i (-1)^{k-i} \ell(H_k(x_1^{N_1}, \dots, x_s^{N_s}; M))$$

*is a optimal polynomial upper estimate over  $\mathbb{R}$ .*

*Proof:* Clear by Lemma 2.7 and Corollary 6.12. ■

In the following, let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $M$ . For every index  $k$  we define

- i.)  $\rho_k^{\leq}(\mathbf{x}; M)$  to be the least total degree of a polynomial in  $\mathbb{Q}[N_1, \dots, N_s]$  bounding  $\ell(H_k(\mathbf{x}^N; M))$  from above;
- ii.)  $\rho_k^{\geq}(\mathbf{x}; M)$  to be the greatest total degree of a polynomial in  $\mathbb{Q}[N_1, \dots, N_s]$  bounding  $\ell(H_k(\mathbf{x}^N; M))$  from below;
- iii.)  $\Delta\rho_k^{\leq}(\mathbf{x}; M)$  to be the least degree of a polynomial in  $\mathbb{Q}[N]$  bounding  $\ell(H_k(\mathbf{x}^N; M))$  from above;
- iv.)  $\Delta\rho_k^{\geq}(\mathbf{x}; M)$  to be the greatest degree of a polynomial in  $\mathbb{Q}[N]$  bounding  $\ell(H_k(\mathbf{x}^N; M))$  from below;

Then by [32] we find that  $\Delta\rho_k^{\leq}(\mathbf{x}; M)$  and  $\Delta\rho_k^{\geq}(\mathbf{x}; M)$  only depend on  $s$ ,  $M$  and  $k$ . In particular, from the existence of  $dd$ -sequence for finitely generated modules over complete noetherian local rings [75, Thm. 1.2, Thm. 1.3] we derive that

$$\Delta\rho_k^{\leq}(\mathbf{x}; M) = \Delta\rho_k^{\geq}(\mathbf{x}; M)$$

whenever  $\mathbf{x}$  is a system of parameters of  $M$ . Furthermore,  $\rho_k^{\leq}(\mathbf{x}; M) = \rho_k^{\leq}(\mathbf{y}; M)$  whenever  $\mathbf{x}$  and  $\mathbf{y}$  are systems of parameters of  $M$  see [76, Cor. 3.5]. It follows

from [76, Cor. 4.6, Thm. 4.2]<sup>2</sup> (and the fact that  $q_k(M) \geq q_{k+1}(M)$  in the notation of [76]), Lemma 5.9, Corollary 6.6 and Lemma 6.7 that

$$\Delta\rho_k^{\leq}(\mathbf{x}; M) = \rho_k^{\leq}(\mathbf{x}; M) = \deg(\Upsilon_{\mathbf{x},k}(N_1, \dots, N_s; M))$$

and subsequently

$$\Delta\rho_k^{\leq}(\mathbf{x}; M) = \Delta\rho_k^{\geq}(\mathbf{x}; M) = \rho_k^{\leq}(\mathbf{x}; M) = \rho_k^{\geq}(\mathbf{x}; M) = \deg(\Upsilon_{\mathbf{x},k})$$

for every system of parameters  $\mathbf{x}$  of  $M$ , and this total degree is independent of the system of parameters. We denote it by  $\rho_k(M)$ . We remark that  $\rho_{k+1}(M) \leq \rho_k(M)$  and  $\rho_k(M)$  is the least total degree of a polynomial in  $\mathbb{Q}[N_1, \dots, N_s]$  bounding  $\chi_k(\mathbf{x}^N; M)$  from above whenever  $\mathbf{x} = x_1, \dots, x_s$  is a system of parameters of  $M$ ; see [76].

We have the following principal estimates for the total degrees  $\rho_k^{\leq}(\mathbf{x}; M)$ :

**Proposition 6.14.** <sup>3</sup> *Let  $R$  be a noetherian ring, let  $M$  be a finitely generated  $R$ -modules, and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $M$ . Let  $0 \leq k \leq s$ , and let  $\alpha$  denote  $\min(\dim(M), s - k)$ . Then there are integers  $a_I$  for every subset  $I$  of  $\{1, \dots, s\}$  of cardinality  $\alpha$  such that*

$$\ell(H_k(x_1^{n_1}, \dots, x_s^{n_s}, M)) \leq \sum_{\substack{I \subseteq \{1, \dots, s\} \\ |I| = \alpha}} a_I \cdot \prod_{i \in I} n_i$$

for all sufficiently large  $s$ -tuples  $\underline{n}$  of natural numbers.

*Proof:* We prove the assertion by an ascending induction on  $s$ . The induction start  $s = 1$  follows from Lemma 1.1. For the induction step let  $s > 1$  and assume that the assertion holds true for  $s - 1$ . Let  $0 \leq k \leq s$ . The exact sequence

$$0 \rightarrow M[x_1^\infty] \rightarrow M \rightarrow M/M[x_1^\infty] \rightarrow 0,$$

gives rise to the estimate

$$\ell(H_k(\mathbf{x}^n, M)) \leq \ell(H_k(\mathbf{x}^n, M[x_1^\infty])) + \ell(H_k(\mathbf{x}^n, M/M[x_1^\infty])).$$

Furthermore, for  $n_1 \geq \tau(M, x_1)$  we can estimate

$$\ell(H_k(\mathbf{x}^n, M[x_1^\infty])) \leq \ell(H_k(\mathbf{x}^n|_{\{2, \dots, s\}}, M[x_1^\infty])) + \ell(H_{k-1}(\mathbf{x}^n|_{\{2, \dots, s\}}, M[x_1^\infty]))$$

<sup>2</sup>In order to meet the prerequisites of [76], use [93, Tag 0BFR].

<sup>3</sup>The proposition was proven in the case  $n_1 = n_2 = \dots = n_s$  by J-L. García Roig in [32].

due to the UCT 0.1. Since  $x_1$  is not a zero divisor on  $M/M[x_1^\infty]$ , we estimate

$$\ell(H_k(\mathbf{x}^n, M/M[x_1^\infty])) \leq n_1 \cdot \ell(H_k(x_1, \mathbf{x}^n|_{\{2, \dots, s\}}, M/M[x_1^\infty])).$$

The base change spectral sequence

$$H_p^{R/x_1R}(\bar{x}_1, \bar{\mathbf{x}}^n|_{\{2, \dots, s\}}, H_q^R(x_1, M/M[x_1^\infty])) \Rightarrow H_{p+q}^R(x_1, \mathbf{x}^n|_{\{2, \dots, s\}}, M/M[x_1^\infty])$$

gives rise to an isomorphism

$$H_k^R(x_1, \mathbf{x}^n|_{\{2, \dots, s\}}, M/M[x_1^\infty]) \cong H_k^{R/x_1R}(\bar{\mathbf{x}}^n|_{\{2, \dots, s\}}, M/M[x_1^\infty] \otimes R/x_1R).$$

and

$$\dim(M/M[x_1^\infty] \otimes R/x_1R) + 1 = \dim(M/M[x_1^\infty]) \leq \dim(M)$$

due to the fact that  $x_1$  is not a zero divisor of  $M/M[x_1^\infty]$ . Applying the induction assumption to the functions

$$\ell(H_k(\mathbf{x}^n|_{\{2, \dots, s\}}, M[x_1^\infty])), \ell(H_{k-1}(\mathbf{x}^n|_{\{2, \dots, s\}}, M[x_1^\infty]))$$

and

$$\ell\left(H_k^{R/x_1R}(\bar{\mathbf{x}}^n|_{\{2, \dots, s\}}, M/M[x_1^\infty] \otimes R/x_1R)\right),$$

we derive the induction step. ■

**Lemma 6.15.** *Let  $M$  be a bounded object of  $\mathcal{D}_{ft}^{\geq 0}(R)$ , and let  $\mathbf{x}$  be a multiplicity system of  $M$  consisting of non-zero divisors of  $R$ . Then one has the estimate*

$$\ell(H_k(\mathbf{x}, M)) \leq \sum_{\substack{i, j \geq 0 \\ j-i=k}} \ell(H_j(\mathbf{x}, H^i(M)))$$

for all  $k \geq 0$ . Dually, if  $M$  is a bounded object of  $\mathcal{D}_{ft}^{\leq 0}(R)$  and  $\mathbf{x}$  is a multiplicity system of  $M$ , then one has the estimate

$$\ell(H_k(\mathbf{x}, M)) \leq \sum_{\substack{i, j \leq 0 \\ i-j=k}} \ell(H_j(\mathbf{x}, H^i(M)))$$

for all  $k \leq 0$ .

*Proof:* The statement follows immediately from the Tor base change spectral sequence

$$E_2^{p,q} = H^p(\mathcal{K}(\mathbf{x}) \otimes_R^L H^q(M)) \Rightarrow H^{p+q}(\mathcal{K}(\mathbf{x}) \otimes_R^L M)$$

see [93, Tag 0662]. ■

Let  $M$  be a bounded object of  $\mathcal{D}_{ft}^{\geq 0}(R)$ , and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $M$  consisting of non-zero divisors of  $R$ . Proposition 6.14 and Lemma 6.15 imply that the function  $\ell(H_k(\mathbf{x}^N; M))$  gets bounded above by a polynomial of total degree at most  $\min s - k, -1$  for every index  $k$ .

**Question 6.16.** From the above point of view it is natural to ask the following:

Let  $k \geq 0$  and let  $M$  be a bounded object of  $\mathcal{D}_{ft}^{\geq 0}(R)$  and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $M$  consisting of non-zero divisors of  $R$ . Assume that

$$\sum_{i=0}^{\infty} (-1)^i \cdot \ell(H_i(\mathbf{x}; M)) \neq 0.$$

Is the function  $\ell(H_k(\mathbf{x}^N; M))$  bounded from above by a polynomial whose total degree matches the total degree of  $\Upsilon_{\mathbf{x},k}(\underline{N}; M)$ ? Likewise, are the functions  $\ell(H_k(\mathbf{x}^N; M))$  bounded above and below by a polynomial of the same degree?

Our next objective is to understand what the numerical invariant  $\rho_1(M)$  encode.

**Lemma 6.17.** *Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a system of parameters of  $M$ . Then*

$$\rho_1(M) - s \leq \rho_1(M/(\mathbf{x}|_{1,\dots,s})M) \leq \rho_1(M).$$

*Proof:* We can assume without loss of generality that  $R$  is complete. It suffices to prove the lemma in the case  $s = 1$ . Making use of Cohen's structure theorem, we can apply [75, Thm. 1.2, Thm. 1.3]. Hence, there exists a  $dd$ -sequence  $\mathbf{y} = y_1, \dots, y_{s-1}$  of  $M/x_s M$ . Then

$$\Upsilon_{\mathbf{y},0}(N_1, \dots, N_{s-1}; M/x_s M) = \sum_{r=0}^{s-1} \lambda_r \prod_{i=1}^r N_r$$

and

$$\Upsilon_{(\mathbf{y}, x_s),0}(N_1, \dots, N_s; M) = N_s \cdot \sum_{r=0}^{s-1} \lambda'_r \prod_{i=1}^r N_r + \sum_{r=0}^{s-1} \lambda''_r \prod_{i=1}^r N_r$$

where  $\lambda_r = \lambda'_r + \lambda''_r$  with coefficients  $\lambda_r, \lambda'_r, \lambda''_r$  in  $\mathbb{N}$  due to Lemma 6.7 and Proposition 6.9. The lemma follows by making use of Corollary 6.12.  $\blacksquare$

**Lemma 6.18.** *Let  $R$  be a noetherian local ring, and let  $M$  be a finitely generated  $R$ -module of dimension  $d$ . Then  $M$  has no submodule of dimension  $m$  with  $\rho_1(M) < m < d$ .*

*Proof:* Let  $N$  be a submodule of  $M$  of dimension at most  $d - 1$ , and let  $\mathbf{x} = x_1, \dots, x_d$  be a system of parameters of  $M$ .<sup>4</sup> We can estimate

$$\Upsilon_{\mathbf{x},0}(N_1, \dots, N_d; M/N) \leq \ell(M/N \otimes R/(\mathbf{x}^N R)) \leq \ell(M \otimes R/(\mathbf{x}^N R)).$$

Thus

$$\Upsilon_{\mathbf{x},0}(N_1, \dots, N_d; M/N) \leq \Upsilon_{\mathbf{x},0}(N_1, \dots, N_d; M)$$

by Theorem 2.11. Then  $e(\mathbf{x}; M) = e(\mathbf{x}; M/N)$ , and Proposition 6.9 and Corollary 6.12 imply that  $\rho_1(M/N) \leq \rho_1(M)$ . The exact sequence

$$H_1(\mathbf{x}^N; M/N) \rightarrow N \otimes R/(\mathbf{x}^N R) \rightarrow M \otimes R/(\mathbf{x}^N R) \rightarrow M/N \otimes R/(\mathbf{x}^N R) \rightarrow 0$$

yields the estimate

$$\ell(N \otimes R/(\mathbf{x}^N)) \leq \ell(M \otimes R/(\mathbf{x}^N R)) - \Upsilon_{\mathbf{x},0}(N; M/N) + \ell(H_1(\mathbf{x}^N; M/N)).$$

Let  $P(\underline{n})$  be a polynomial of total degree  $\rho_1(M/N)$ , bounding  $\ell(H_1(\mathbf{x}^N; M/N))$  from above. Then

$$\ell(N \otimes R/(\mathbf{x}^N)) \leq \Upsilon_{\mathbf{x},0}(N; M) - \Upsilon_{\mathbf{x},0}(N; M/N) + P(\underline{n})$$

by Theorem 2.11. Then  $\Upsilon_{\mathbf{x},0}(N; M) - \Upsilon_{\mathbf{x},0}(N; M/N)$  is a polynomial of total degree at most  $\rho_1(M)$ . Subsequently, the Hilbert-Serre theorem implies that the dimension of  $N$  is at most  $\rho_1(M)$ .  $\blacksquare$

Now for the numerical invariant  $\rho_1(M)$ , we derive the following

**Theorem 6.19.** *Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module of dimension  $d$ , and let  $-1 \leq n \leq d - 1$  be a natural number. Then  $\rho_1(M) = n$  if and only if for every system of parameters  $\mathbf{x} = x_1, \dots, x_d$  of  $M$  we find*

- i.) *The module  $M/\mathbf{x}_{1,\dots,d-n-1}M$  has a submodule of dimension  $n$ ;*
- ii.) *For every index  $s = 1, \dots, d - n - 2$  we find that  $M/(\mathbf{x}_{1,\dots,s})M$  has no submodule of dimension  $m$  with  $n < m < d - s$ .*

---

<sup>4</sup>We can assume without loss of generality that  $R$  is complete, allowing us to choose  $\mathbf{x}$  to be a  $dd$ -sequence of  $M$ , which simplifies some arguments. We avoided this to demonstrate that, in fact, the existence of  $dd$ -sequences is not required for the proof of the upcoming Theorem 6.19.

*Proof:* Assume that i.) and ii.) holds true. Let  $\mathbf{x}$  be an arbitrary system of parameters of  $M$ . Then i.) yields

$$e(\mathbf{x}_{s+2, \dots, d}; M/(\mathbf{x}_{1, \dots, s})M[x_{s+1}]) = 0$$

for every index  $s = 1, \dots, d - n - 2$ , and Lemma 6.18 and Lemma 6.17 imply that

$$e(\mathbf{x}_{d-n+1, \dots, d}; M/(\mathbf{x}_{1, \dots, d-n-1})M[x_{d-n}]) \neq 0.$$

Then Proposition 6.5 implies that  $p_1(M) = n$ .

Vice versa assume that  $p_1(M) = n$ . Then i.) follows from Proposition 6.5, and ii.) follows from Lemma 6.18 and Lemma 6.17.  $\blacksquare$

**Question 6.20.** The numerical invariant  $\rho_1(M)$  was studied before in [17], where it was shown that  $\rho_1(M)$  equals the dimension of the non-Cohen-Macaulay locus of  $M$ , provided that  $R$  admits a dualizing complex. In the virtue of Theorem 6.19 we question the following: Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module with  $\dim(M) \geq 1$ , and let  $n$  be a natural number. Is  $\rho_1(M)$  equal to  $n$  if and only if

- i.) For every prime ideal  $\mathfrak{p}$  of  $R$  with  $\text{ht}(\mathfrak{p}) < \dim(M) - n$  the module  $M_{\mathfrak{p}}$  is Cohen-Macaulay;
- ii.) There exists a prime ideal  $\mathfrak{p}$  of  $R$  with  $\text{ht}(\mathfrak{p}) = \dim(M) - n$  such that  $M_{\mathfrak{p}}$  is not Cohen-Macaulay ?

In the case  $n = 1$  this is true by [70, Satz 2.5].

**Remark 6.21.** We remark that Theorem 6.19 is clear in the Cohen-Macaulay case: Let  $M$  be a finitely generated Cohen-Macaulay module over a noetherian local ring  $R$ , and let  $\mathbf{x} = x_1, \dots, x_d$  be a system of parameters of  $M$ . If  $N$  is a submodule of dimension at most  $d - 1$ , then there exists an element  $x$  in  $R$  annihilating  $N$  that is a regular element of  $M$ , whence  $N = 0$ . Moreover, if every nontrivial submodule in  $M/\mathbf{x}_{1, \dots, s}M$  has dimension  $d - s$ , then in particular  $M/\mathbf{x}_{1, \dots, s}M[x_{s+1}] = 0$ .

**Outlook 6.22.** We define a natural number  $1 \leq n < s$  to be a *vanishing degree* of  $M$  if  $\lambda_{\mathbf{x}, k}(I; M) = 0$  for all subsets  $I \subseteq \{1, \dots, s\}$  of cardinality  $n$ . We call  $1 \leq n < s$  a *non-vanishing degree* of  $M$  if  $n$  is not a vanishing degree of  $M$ . We question, whether a more refined version of Theorem 6.19 exists, wherein the possible dimensions of submodules align exactly with the non-vanishing degrees? There should also be a parameter free formulation of the theorem.

By the classical work of G. Pólya, see [83], we can associate to the *numerical polynomials*  $\Upsilon_{\mathbf{x},k}$  uniquely determined *binomial coefficients*. It is natural to question what information these coefficients encode. The author conjecture that there is a strong connection between these coefficients and those of the Hilbert polynomial, which are notoriously difficult to understand. The intuition behind the conjecture is reinforced by [34, 40, 40, 2] and [68, §4]. We recall that, based on the work of O. Gabber, P. C. Roberts reduced Serre’s positivity conjecture to the question of whether a specific coefficient of a certain Hilbert polynomial is positive; see [86]. This is one of the motivations to study the (vanishing) of the coefficients of Hilbert polynomials.

The class of Cohen-Macaulay rings form an exceptionally well behaved class: local cohomology vanishes when it can, and duality works out beautifully. Further, Cohen-Macaulay modules have many “good” properties and a vast part of those properties continue to hold for generalized Cohen-Macaulay modules with an “exception in a single degree”. For example, Theorem 6.19 is a new appearance of this phenomena. The author conjectures that a significant portion of statements regarding (generalized) Cohen-Macaulay modules should extend naturally, with the coefficients of the polynomial  $\Upsilon_{\mathbf{x},k}$  determining the specific range in which these statements continue to hold true. Here are

**Examples 6.23.** 1.) The *surjectivity criterion* for Buchsbaum modules, see [100], states: Let  $(R, \mathfrak{m})$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module, and let  $H_{\mathfrak{m}}^i(M)$  denote the  $i$ -th local cohomology of  $M$  with respect to the maximal ideal  $\mathfrak{m}$ . If the canonical morphism

$$\varphi_M^i : \text{Ext}_R^i(R/\mathfrak{m}; M) \rightarrow H_{\mathfrak{m}}^i(M)$$

is surjective for all  $i \neq \dim_R(M)$ , then  $M$  is a Buchsbaum  $R$ -module. The converse is true if the base ring  $R$  is regular.

2.) Let  $(R, \mathfrak{m})$  be a noetherian local ring. The theory of Buchsbaum rings and modules is closely related to that of Cohen-Macaulayness in blowing-ups; see [43, 69, 42, 105, 41]. For example, S. Goto proved in [39] that  $R/H_{\mathfrak{m}}^0(R)$  is a Buchsbaum ring if and only if the scheme  $\text{Proj } R(\mathfrak{q})$  is locally Cohen-Macaulay for every parameter ideal  $\mathfrak{q}$  in  $R$ .

In both examples, we ask whether there are more refined statements of this form that relate to the numerical invariants  $\rho_k(M)$  on  $M$ ?

# Chapter 7

## Polynomial-Like Sequences

In this chapter, we characterize those sequences for which equality holds in Theorem 6.11 in degree 0. Unless stated otherwise, let  $R$  be a noetherian ring.

First, we prove some principal properties of the stabilization of principal torsion. We recall that for every finitely generated  $R$ -module  $M$  and every element  $a$  in  $R$  we defined  $\tau(M, a)$  to be the smallest natural number  $n$  satisfying the identity  $M[a^n] = M[a^{n+1}]$ .

**Lemma 7.1.** *Let  $M$  be a finitely generated  $R$ -module, and let  $x$  be an element of  $R$ . Then*

- i.) For all  $n \geq \tau(M, a)$  we have  $M[a^n] = M[a^{\tau(M, a)}]$ ;*
- ii.) If  $M[x^\infty]$  has finite length, then  $\tau(M, x) \leq \ell(M[x^\infty]) - \ell(M[x])$ ;*
- iii.) If  $M \cong N \oplus O$  with  $R$ -modules  $N$  and  $O$ , then  $\tau(M, x)$  equals  $\max(\tau(N, x), \tau(O, x))$ ;*
- iv.) If  $f : R \rightarrow S$  is a faithfully flat morphism of noetherian rings, then  $\tau(M \otimes S, x) = \tau(M, x)$ .*

*Proof:* All clear. ■

**Lemma 7.2.** *Let  $M$  be a finitely generated  $R$ -module, and let  $a, b$  be elements of  $R$ . Then*

- i.)  $\tau(M/M[b], a) \leq \tau(M, a)$ ;*
- ii.)  $\tau(M/M[a^n], a) \leq \max(\tau(M, a) - n, 0)$ ;*
- iii.)  $\tau(M, a \cdot b) \leq \max(\tau(M, a), \tau(M, b))$ .*

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*Proof:* Let  $\alpha$  denote  $\tau(M/M[b], a)$ , and let  $m$  be an element in  $M$  such that

$$[m] \in (M/M[b])[a^\alpha] \setminus (M/M[b])[a^{\alpha-1}].$$

Then

$$bm \in M[a^\alpha] \setminus M[a^{\alpha-1}],$$

proving i.).

Now let  $\alpha$  denote  $\tau(M/M[a^n], a)$ , and let  $\beta$  denote  $\tau(M, a)$ , and assume that  $\alpha \neq 0$  and  $\beta < \alpha + n$ . Let  $m$  be an element in  $M$  such that

$$[m] \in (M/M[a^n])[a^\alpha] \setminus (M/M[a^n])[a^{\alpha-1}].$$

Then  $a^{\alpha+n}m = 0$ , that is  $a^{\alpha+n-1}m = 0$ . Thus,  $[m]$  lies in  $(M/M[a])[a^{\alpha-1}]$ , contradicting the choice of  $m$ , whence ii.).

Now we prove iii.). Let  $\alpha$  denote  $\tau(M, a \cdot b)$ . Then there exists an element  $m$  in  $M$  such that  $(ab)^\alpha m = 0$  and  $(ab)^{\alpha-1}m \neq 0$ . Assume that  $\tau(M, a) < \alpha$ . Let  $m'$  denote  $a^\alpha m$ . Then  $b^{\alpha-1}m' \neq 0$  and  $b^\alpha m' = 0$ , that is  $\tau(M, a \cdot b) \leq \tau(M, b)$ , whence the assertion. ■

**Lemma 7.3.** *Let*

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

*be an exact sequence of finitely generated  $R$ -modules. Then for all elements  $a$  of  $R$  we find*

- i.)  $\tau(M', a) \leq \tau(M, a)$ ;
- ii.)  $\tau(M, a) \leq \tau(M', a) + \tau(M'', a)$ .

*Proof:* The first assertion is immanent. In order to prove the second one, let  $\alpha$  denote  $\tau(M, a)$ , let  $\beta$  denote  $\tau(M'', a)$ , and let  $\varphi$  denote the given morphism from  $M$  onto  $M''$ . Assume that  $\alpha > \beta$ . Let  $m$  be an element in  $M[a^\alpha] \setminus M[a^{\alpha-1}]$ . Then  $a^\beta \varphi(m) = 0$ . Thus,  $a^\beta m$  lies in  $M'$ , and  $a^\beta m$  is contained in  $M'[a^{\alpha-\beta}] \setminus M'[a^{\alpha-\beta-1}]$ . Hence,  $\tau(M', a) \geq \alpha - \beta$ . Consequently,

$$\tau(M, a) = (\alpha - \beta) + \beta \leq \tau(M', a) + \tau(M'', a).$$

■

**Definition 7.4.** *Let  $M$  be a finitely generated  $R$ -module. For every sequence  $\mathbf{x} = x_1, \dots, x_s$  of elements of  $R$  we define*

$$\underline{\beta}(\mathbf{x}) := \left( \sup_{\mathbf{n} \in \mathbb{N}^s} \tau(M \otimes R / (\mathbf{x}^{\mathbf{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}, x_r) R) \right)_{r \in \{1, \dots, s\}}$$

in  $\mathbb{N}_\infty^s$ . We say that  $\mathbf{x}$  is a polynomial-like-sequence (or a p.l.-sequence) of  $M$  if all of the entries of  $\underline{\beta}(\mathbf{x})$  are natural numbers. If  $\beta(\mathbf{x}) \leq \underline{1}$ , then we say that  $\mathbf{x}$  is a polynomial-sequence (or p-sequence) of  $M$ .

In particular, if  $\mathbf{x} = x_1, \dots, x_s$  is a p.l.-sequence, then  $\mathbf{x}^{\underline{\beta}(\mathbf{x})}$  is a p-sequence, and  $\underline{\beta}(\mathbf{x})$  is the smallest  $s$ -tuple of natural numbers with this property. Further,  $\underline{\beta}(\mathbf{x}) = \underline{0}$  if and only if  $\mathbf{x}$  is a regular sequence of  $M$ .

**Historical Remark 7.5.** Polynomial-like sequences were initially examined in [71] where they were subjected to more stringent conditions.

**Lemma 7.6.** *Let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a sequence of elements of  $R$ . Then*

- i.) *The sequence  $\mathbf{x}$  is a p-sequence of  $M$  if and only if  $\mathbf{x}$  is a p-sequence of  $M_{\mathfrak{p}}$  for every prime ideal  $\mathfrak{p}$  of  $R$  if and only if  $\mathbf{x}$  is a p-sequence of  $M_{\mathfrak{m}}$  for every maximal ideal  $\mathfrak{m}$  of  $R$ ;*
- ii.) *If  $\mathbf{x}$  is a p-sequence of  $M$ , then  $\mathbf{x}|_I$  is a p-sequence of  $M$  for every subset  $I$  of  $\{1, \dots, s\}$ .*

*Proof:* In order to prove i.), let  $1 \leq r \leq s$  be an arbitrary index. Then the cokernel of

$$M/\mathbf{x}|_{1, \dots, \hat{r}, \dots, s} M[x_r] \rightarrow M/\mathbf{x}|_{1, \dots, \hat{r}, \dots, s} M[x_r^2]$$

vanishes if and only if

$$\text{coker} (M_{\mathfrak{p}}/\mathbf{x}|_{1, \dots, \hat{r}, \dots, s} M_{\mathfrak{p}}[x_r] \rightarrow M_{\mathfrak{p}}/\mathbf{x}|_{1, \dots, \hat{r}, \dots, s} M_{\mathfrak{p}}[x_r^2])$$

vanishes for every prime ideal  $\mathfrak{p}$  if and only if

$$\text{coker} (M_{\mathfrak{m}}/\mathbf{x}|_{1, \dots, \hat{r}, \dots, s} M_{\mathfrak{m}}[x_r] \rightarrow M_{\mathfrak{m}}/\mathbf{x}|_{1, \dots, \hat{r}, \dots, s} M_{\mathfrak{m}}[x_r^2])$$

vanishes for every maximal ideal  $\mathfrak{m}$ . That is i.).

In particular, in order to prove ii.), we can assume without loss of generality that  $R$  is a local ring. Assume that  $\mathbf{x}$  is a p-sequence, and let  $i$  be an arbitrary index in  $I$ . Let  $\underline{n}$  be an arbitrary  $s$ -tuple of natural numbers, and let  $m$  be an element of  $M/\mathbf{x}^{\underline{n}}|_{I \setminus \{i\}} M[x_i^2]$ . Let  $I$  denote the ideal  $(\mathbf{x}|_{1, \dots, \hat{i}, \dots, s})^s$ . Then for every natural number we find that

$$I^n \subseteq (\mathbf{x}^{\cdot n}|_{1, \dots, \hat{i}, \dots, s}).$$

Then  $x_i \cdot m$  is contained in

$$I^n \cdot (M/\mathbf{x}^{\underline{n}}|_{I \setminus \{i\}} M)$$

for every natural number  $n$ , and thus Krull's intersection theorem [93, Tag 00IP] implies  $x_i \cdot m = 0$ , whence the lemma. ■

**Proposition 7.7.** *Let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $M$ . Then the function*

$$\begin{aligned} \mathbb{N}^s &\rightarrow \mathbb{N} \\ \underline{n} &\mapsto \ell(M/\mathbf{x}^{\underline{n}}M) \end{aligned}$$

*is a multivariate polynomial function if and only if  $\mathbf{x}$  is a p-sequence of  $M$  and in this case we find*

$$\ell(M/\mathbf{x}^{\underline{n}}M) = \Upsilon_{\mathbf{x},0}(n_1, \dots, n_s; M).$$

*Likewise, the above function is a multivariate polynomial-like function if and only if  $\mathbf{x}$  is a p.l.-sequence of  $M$ , and in this case we find*

$$\ell(M/\mathbf{x}^{\underline{n}}M) = \sum_{I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J|-|I|} e_{\mu}(\mathbf{x}|_J; M/(\mathbf{x}^{\beta(\mathbf{x})}|_J)M) \cdot \prod_{i \in I} n_i$$

*for all  $\underline{n} \geq \underline{\beta}(\mathbf{x})$ .*

*Proof:* If  $\mathbf{x}$  is a p-sequence of  $M$ , then for every  $s$ -tuple  $\underline{n}$  of natural numbers and every index  $1 \leq r \leq s$  we find that

$$\Delta_r(\ell(M/\mathbf{x}^{\underline{n}}M)) = e(x_r; M/\mathbf{x}^{\underline{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}M)$$

by Lemma 1.1 and accordingly  $\ell(M/\mathbf{x}^{\underline{n}}M)$  is a multivariate polynomial by Proposition 2.5. Vice versa, if  $\ell(M/\mathbf{x}^{\underline{n}}M)$  is a multivariate polynomial, then

$$\begin{aligned} &\Delta_r(\mu(M/\mathbf{x}^{\underline{n}}M)) \\ &= e(x_r; M/\mathbf{x}^{\underline{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}M) + \Delta_r(\ell(M/\mathbf{x}^{\underline{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}M[x_r^n])) \end{aligned}$$

is a constant in  $\underline{n}$  for every index  $1 \leq r \leq s$ . That is

$$\ell((M/\mathbf{x}^{\underline{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}M)[x_r^n]) - \ell((M/\mathbf{x}^{\underline{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}M)[x_r^{n-1}]) = 0$$

for all  $\underline{n} > \underline{1}$  and  $1 \leq r \leq s$ , proving that  $\mathbf{x}$  is a p-sequence.

The remainder of the statement follows from Theorem 6.11 and Lemma 2.8. ■

**Corollary 7.8.** *Let  $M$  be a finitely generated  $R$ -module, and let  $\mathbf{x} = x_1, \dots, x_s$  be a multiplicity system of  $R$ . Then  $\mathbf{x}$  is a p.l.-sequence of  $M$  if and only if*

$$\tau(M \otimes R/(\mathbf{x}^{\underline{n}}|_{\{1, \dots, \hat{r}, \dots, s\}}), x_r)$$

*is bounded in  $\underline{n}$  for all  $r = 2, \dots, s$ .*

*Proof:* For all  $r = 2, \dots, s$  let

$$\beta_r := \sup_{\underline{z} \in \mathbb{N}^s} \tau \left( M \otimes R / (\mathbf{x}^{\underline{z}} |_{\{1, \dots, \hat{r}, \dots, s\}}), x_r \right),$$

let  $\underline{\beta}' = (\beta_2, \dots, \beta_s)$ , and let

$$\beta_1 := \max_{J \subseteq \{2, \dots, s\}} \tau(M / \mathbf{x}^{\underline{\beta}'} |_{J^c} M, x_1).$$

Then let  $\underline{\beta} = (\beta_1, \dots, \beta_s)$ . Proposition 7.7 implies that

$$\begin{aligned} & \ell(M / (\mathbf{x}^{\underline{N}})M) \\ &= \sum_{I \subseteq \{2, \dots, s\}} \sum_{\substack{J \subseteq \{2, \dots, s\} \\ I \subseteq J}} (-1)^{|J| - |I|} e(\mathbf{x}|_J; M / (x_1^{N_1}, \mathbf{x}^{\underline{\beta}} |_{J^c})M) \cdot \prod_{i \in I} N_i \\ &= \sum_{I \subseteq \{1, \dots, s\}} \sum_{\substack{J \subseteq \{1, \dots, s\} \\ I \subseteq J}} (-1)^{|J| - |I|} e(\mathbf{x}|_J; M / (\mathbf{x}^{\underline{\beta}} |_{J^c})M) \cdot \prod_{i \in I} N_i. \end{aligned}$$

for all  $\underline{N} > \underline{\beta}$ , whence the corollary. ■

**Corollary 7.9.** *Let  $R$  be a noetherian local ring, let  $M$  be a finitely generated  $R$ -module of dimension  $s$ , and let  $P(M)$  denote the set comprising the system of parameters of  $M$ . Let  $\varphi$  denote the map*

$$\begin{aligned} \varphi : P(M) &\rightarrow \mathbb{Z} \\ \mathbf{x} = x_1, \dots, x_s &\mapsto \sum_{i=1}^s \ell(M / (x_{\tau_i(1)}, \dots, x_{\tau_i(s-1)})M[x_{\tau_i(s)}]) \end{aligned}$$

where  $\tau_i$  denotes the transposition of the indices  $i$  and  $s$ . Let  $\text{Spec}(\varphi)$  denote the image of  $\varphi$ . Then

- i.)  $M$  is Cohen-Macaulay if and only if  $\text{Spec}(\varphi) = \{0\}$ ;
- ii.)  $M$  is Buchsbaum if and only if  $|\text{Spec}(\varphi)| = 1$ ;
- iii.)  $M$  is generalized Cohen-Macaulay if and only if  $|\text{Spec}(\varphi)| < \infty$ .

*Proof:* First assume that  $|\text{Spec}(\varphi)| \leq \infty$ . Then every system of parameters of  $M$  is a polynomial-like sequence, and  $M$  is generalized Cohen-Macaulay by [71, Cor. 7]. Vice versa, let  $M$  be a generalized Cohen-Macaulay module. Then Theorem 6.11, Corollary 6.6, and Corollary 6.12 imply that

$$\prod_{i=1}^s N_i \cdot e(\mathbf{x}; M) + \ell(M / (x_1, \dots, x_{s-1})M[x_s])$$

is the optimal polynomial lower estimate of  $\ell(M/\mathbf{x}^N M)$  for every system of parameters  $\mathbf{x}$  of  $M$ . Hence, for every system of parameters  $\mathbf{x}$  of  $M$  there exists an  $s$ -tuple  $\underline{\beta} = \beta_1, \dots, \beta_s$  of natural numbers such that

$$\ell(M/(x_1, \dots, x_{s-1})M[x_s]) \leq \ell\left(M/(x_1^{\beta_1}, \dots, x_{s-1}^{\beta_{s-1}})M[x_s^{\beta_s}]\right) = \chi_1(\mathbf{x}^{\underline{\beta}}; M)$$

by Proposition 7.7 and Corollary 6.12, whence  $|\text{Spec}(\varphi)| \leq \infty$ .

If  $|\text{Spec}(\varphi)| = 1$ , then the above implies that  $M$  is generalized Cohen-Macaulay and

$$\ell(M/(x_1, \dots, x_{s-1})M[x_s]) = \ell(M/(x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}})M[x_s^{n_s}])$$

for all  $n_1, \dots, n_s$  and every system of parameters  $\mathbf{x}$  of  $M$ . A fortiori, every system of parameters of  $M$  is a  $p$ -sequence of  $M$ , and we derive that

$$\ell(M/(x_1, \dots, x_{s-1})M[x_s]) = \chi_1(\mathbf{x}, M)$$

for every system of parameters  $\mathbf{x}$  of  $M$  and subsequently  $M$  is Buchsbaum.

If  $M$  is Buchsbaum, then Theorem 6.11 and Corollary 6.6 imply that

$$\ell(M/(x_1, \dots, x_{s-1})M[x_s]) = \chi_1(\mathbf{x}; M)$$

for every system of parameters  $\mathbf{x}$  of  $M$ , whence  $|\text{Spec}(\varphi)| = 1$ .

Finally, i.) follows from the above. ■

**Remark 7.10.** In the situation of Corollary 7.9, it is natural to ask whether  $M$  is Cohen-Macaulay if  $0$  is an element of  $\text{Spec}(\varphi)$ . We conjecture that this is not true in general. Nevertheless, we conjecture the following: Let

$$\begin{aligned} \varphi' : P(M) &\rightarrow \mathbb{Z} \\ \mathbf{x} = x_1, \dots, x_s &\mapsto \sum_{\underline{n} \in \{1,2\}^s} \sum_{i=1}^s \ell(M/(x_{\tau_i(1)}^{n_1}, \dots, x_{\tau_i(s-1)}^{n_{s-1}})M[x_{\tau_i(s)}^{n_s}]) \end{aligned}$$

Then  $M$  is Cohen-Macaulay if  $0$  is contained in  $\text{Spec}(\varphi)$ .

## Part II

# Cohomology of Pro- $\ell$ Towers



# Conventions

Throughout this part, by “sheaf” or “étale sheaf” we mean a sheaf on the small étale site. Let  $R$  be a commutative rings, and let  $X$  be a scheme. We denote the category of  $R$ -module presheaves on the small étale site  $X_{\acute{e}t}$  by  $\mathbf{PSh}(X_{\acute{e}t}, R)$  and we denote the category of  $R$ -modules sheaves on  $X_{\acute{e}t}$  by  $\mathbf{Sh}(X_{\acute{e}t}, R)$ . We write  $\mathbf{PSh}(X_{\acute{e}t})$  and  $\mathbf{Sh}(X_{\acute{e}t})$  instead of  $\mathbf{PSh}(X_{\acute{e}t}, \mathbb{Z})$  and  $\mathbf{Sh}(X_{\acute{e}t}, \mathbb{Z})$ . For every morphism  $f : X \rightarrow Y$  we denote the inverse image functor on presheaves by  $f^p$  and we denote the direct image functor on presheaves by  $f_p$ . We denote the corresponding derived categories of  $R$ -modules sheaves by  $\mathcal{D}(X_{\acute{e}t}, R)$ , and we denote  $\mathcal{D}(X_{\acute{e}t}, \mathbb{Z})$  by  $\mathcal{D}(X_{\acute{e}t})$ . By  $\mathcal{D}_{tor}(X_{\acute{e}t})$  we denote the subcategories of objects in  $\mathcal{D}(X_{\acute{e}t})$  whose cohomology sheaves are torsion sheaves. Such complexes will be briefly called torsion complexes. We recall from [93, Tag 0DD7] that  $\mathcal{D}_{tor}(X_{\acute{e}t})$  is the derived category of the category  $\mathbf{Sh}_{tor}(X_{\acute{e}t})$  of torsion sheaves on  $X$ . In this context, the subscript  $c$  indicates that the objects are assumed to be representable by a complex whose cohomology sheaves are constructible sheaves of sets. Such complexes will be briefly called constructible complexes.

Let  $\ell$  be a prime number. We say that a sheaf  $\mathcal{F}$  is a  $\ell$ -primary torsion sheaf if  $\mathcal{F}$  gets annihilated by a power of  $\ell$ .

Let  $E$  be a set, and let  $X$  be a scheme. We let  $\underline{E}$  denote the constant sheaf that is the sheafification of the constant presheaf with value  $E$  on  $X_{\acute{e}t}$ . If  $E$  is an abelian group, ring, module, etc, then  $\underline{E}$  is a sheaf of abelian groups, rings, modules, etc.

If  $U$  is an open subset of  $X$ , then we denote the corresponding open immersion by  $j_U$  or  ${}_U j$  by default. If  $Z$  is a closed subset of  $X$ , then we endow  $Z$  with the induced reduced subscheme structure and we denote the corresponding closed immersion by  $i_Z$  or  ${}_Z i$  by default.



# Chapter 8

## Étale Cohomology of Finite Étale Towers

This chapter serves as a preparation. To prove Theorem 0.6 in *Chapter 9*, we will follow the approach outlined in [24]. Here, we will develop the étale cohomology theory of finite étale towers and we will establish the fundamental theorem such as the Mayer-Vietoris sequence and The Leray spectral sequence (with proper support). In the aftermath, we do not have to deal with “technical” difficulties in the chapters to follow.

### 8.1 Basic Constructions

**Definition 8.1.** *We define a tower  $\mathbb{X}$  of schemes to be a collection  $(\phi^{n+1} : X_{n+1} \rightarrow X_n)_{n \in \mathbb{N}}$  of morphisms of schemes. We call  $X_n$  the  $n$ -th level of the tower  $\mathbb{X}$  and we call  $\phi^n$  the  $n$ -th transition morphism of  $\mathbb{X}$ . We refer to  $X_0$  as the base of  $\mathbb{X}$ . For all pairs of natural numbers  $m, n$  with  $m \geq n + 1$  we put  $\phi_n^m := \phi^{n+1} \circ \phi^{n+2} \circ \dots \circ \phi^m$ , and we set  $\phi_n^n := \text{id}$  for  $m = n$ . Given a tower  $\mathbb{X}$ , we refer to the so defined morphisms by  ${}_{\mathbb{X}}\phi_t^s$  by default. Further, we will refer to the base  $X_0$  of  $\mathbb{X}$  by  $X$  by default. If all the levels  $X_n$  of  $\mathbb{X}$  belong to a class  $\mathcal{C}$  of schemes, then we say that  $\mathbb{X}$  is a  $\mathcal{C}$  tower. If the base of the tower  $X$  is an  $S$ -scheme, then we say that  $\mathbb{X}$  is a tower over  $S$ . If the base  $X$  is an  $S$ -scheme such that the structure morphism  $X \rightarrow S$  is separated and of finite type, then we say that the tower  $\mathbb{X}$  is separated and of finite type over  $S$ .*

*Given two towers  $\mathbb{X}, \mathbb{Y}$ , we define a morphism  $f : \mathbb{Y} \rightarrow \mathbb{X}$  of towers to be a collection of morphisms  $(f_n : Y_n \rightarrow X_n)_{n \in \mathbb{N}}$  such that  $f_{n-1} \circ {}_{\mathbb{Y}}\phi^n = {}_{\mathbb{X}}\phi^n \circ f_n$  on every level  $n$ . We call  $f_r$  the  $r$ -th level morphism of  $f$ . If  $f : \mathbb{Y} \rightarrow \mathbb{X}$  is a morphism of towers where all level morphisms  $f_n$  belong to a class  $\mathcal{C}$  of morphisms, then we say that  $f$  is a  $\mathcal{C}$  morphism.*

*Given an open immersion  $f : \mathbb{Y} \rightarrow \mathbb{X}$ , we say that  $\mathbb{Y}$  is an open subtower of  $\mathbb{X}$ . If  $f$  is a closed immersion, then we say that  $\mathbb{Y}$  is a closed subtower of  $\mathbb{X}$ , respectively.*

We call a morphism  $f : \mathbb{Y} \rightarrow \mathbb{X}$  cartesian if  $f$  exhibits  $Y_n$  as  $Y_{n-1} \times_{X_{n-1}} X_n$  for every admissible index  $n$ . Given two towers  $\mathbb{X}$  and  $\mathbb{Y}$  over a scheme  $S$ , we denote by  $\mathbb{X} \times_s \mathbb{Y}$  the tower whose  $n$ -th level is given by  $X_n \times_S Y_n$  and whose transition morphisms are derived by the universal property of the fibre product. Given a tower  $\mathbb{X}$  and a morphism  $f : Y \rightarrow X$ , we denote by  $\mathbb{Y}$  the tower, whose  $n$ -th level is given by  $Y \times_X X_n$ , and where the transition morphisms are induced by the universal property of the fibre product. We call  $\mathbb{Y}$  the induced tower over  $Y$ . If  $U$  and  $V$  are open subscheme of  $X$ , then we denote the tower derived in this fashion from the open immersion  $U \cap V \rightarrow X$  by  $\mathbb{U} \cap \mathbb{V}$ .

We recall

**Cohomology with compact support** from [5, XVII 3-6], [18, Arcate IV 5,6]. Let  $X$  and  $Y$  be  $S$ -schemes. An  $S$ -morphism  $f : X \rightarrow Y$  is called  *$S$ -compactifiable* if  $Y$  is quasi-compact and quasi-separated and there exists a proper  $S$ -scheme  $P$  such that  $f$  is the composition of a separated quasi-finite morphism  $X \rightarrow Y \times_S P$  and the projection  $p_1 : Y \times_S P \rightarrow Y$ . If  $f$  is compactifiable, then  $f$  is necessarily a quasi-compact separated morphism, and  $X$  is necessarily a quasi-compact and quasi-separated scheme. We say that an  $S$ -scheme  $X$  is *compactifiable* if the structure morphism  $X \rightarrow S$  is  $S$ -compactifiable. We recall that any separated morphism of finite type between noetherian schemes is compactifiable by a theorem of Nagata and every scheme of finite type over a field is noetherian by [93, Tag 01T6].

Let  $f : X \rightarrow Y$  be  $S$ -compactifiable. Then  $f$  factors as a composition

$$X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Y$$

of an open immersion  $j$  and a proper  $S$ -compactifiable morphism  $\bar{f}$  due to Zariski's Main Theorem [20, IV 18.12.13]. We call a factorisation of the latter form a *compactification* of  $f$ .

As a matter of fact, given an  $S$ -compactification

$$X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Y$$

of an  $S$ -morphism  $f : X \rightarrow Y$ , the isomorphism class of the functor  $\mathcal{R}\bar{f}_*j_!$  is independent of the choice of the compactification. We call

$$\mathcal{R}f_! := \mathcal{R}\bar{f}_*j_! : \mathcal{D}_{tor}^+(X_{ét}) \rightarrow \mathcal{D}_{tor}^+(Y_{ét})$$

the *extension by zero functor*. According to [29, Rem. 7.4.6],[93, Tag 07K7], and [93, Tag 09T5], this construction gives rise to a functor

$$\mathcal{R}f_! := \mathcal{R}\bar{f}_*j_! : \mathcal{D}_{tor}(X_{ét}) \rightarrow \mathcal{D}_{tor}(Y_{ét}).$$

For every  $K$  in  $\mathcal{D}_{tor}^+(X_{\acute{e}t})$  we set  $\mathcal{R}^q f_! K = H^q(\mathcal{R} f_! K)$ . Suppose that  $S = \text{Spec } k$  is the spectrum of an algebraic separably closed field  $k$ . Then in the above setting we define

$$\mathcal{R}\Gamma_c(X, K) := \Gamma(\text{Spec}(k), \mathcal{R} f_! K) \cong \mathcal{R}\Gamma(\bar{X}, j_! K)$$

and

$$H_c^q(X, K) = H^q(\mathcal{R}\Gamma_c(X, K)) \cong H^q(\bar{X}, j_! K)$$

for every  $K$  in  $\mathcal{D}_{tor}^+(X_{\acute{e}t})$ .

*Caveat:* In general  $\mathcal{R} f_!$  is not the right derived functor of  $\mathcal{R}^0 f_!$ .

Let  $X, Y$  be separated schemes of finite type over an algebraic separably closed field  $k$ , and let  $\varphi : X \rightarrow Y$  be a proper  $k$ -morphism. Let  $f$  denote the structure morphism of  $X$ , and let  $g$  denote the structure morphism of  $Y$ . By the above  $\mathcal{R}\varphi_! = \mathcal{R}\varphi_*$ . Utilizing [29, Lem. 7.4.4], we derive that the unit  $1 \mapsto \mathcal{R}\varphi_*\varphi^*$  of the adjunction  $(\mathcal{R}\varphi^* \dashv \mathcal{R}\varphi_*)$  induces a natural transformation

$$\mathcal{R}g_! \rightarrow \mathcal{R}g_! \circ \mathcal{R}\varphi_! \circ \varphi^* = \mathcal{R}(g_! \circ \varphi_!) \varphi^* = \mathcal{R}f_! \varphi^*,$$

giving rise to the canonical morphism

$$H_c^\bullet(Y, K) \rightarrow H_c^\bullet(X, \varphi^* K)$$

in cohomology for every  $K$  in  $\mathcal{D}_{tor}^+(Y_{\acute{e}t})$ . We assert that this construction is compatible with the composition of morphisms: Say  $\psi : Y \rightarrow Z$  is another proper  $k$ -morphism where  $Z$  is separated and of finite type over  $k$ . Then we have the natural transformation

$$\mathcal{R}h_! \rightarrow \mathcal{R}h_! \circ \mathcal{R}\psi_! \circ \psi^* = \mathcal{R}(h_! \circ \psi_!) \psi^* = \mathcal{R}g_! \psi^*,$$

where  $h$  denotes the structure morphism of  $Z$ . Then  $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ <sup>1</sup> and consequently  $\mathcal{R}(\psi \circ \varphi)_* = \mathcal{R}\psi_* \mathcal{R}\varphi_*$  by [93, Tag 09T5] and [63, Ch. IV.8 Thm. 1], and we derive the asserted compatibility.

**Remark 8.2.** Let  $\mathbb{X}$  and  $\mathbb{Y}$  be towers that are separated and of finite type over an algebraic separably closed field  $k$ , and let  $f : \mathbb{Y} \rightarrow \mathbb{X}$  be a separated morphism of finite type over  $k$ . Then  $f$  factors as  $f : \mathbb{Y} \rightarrow \mathbb{Z} \rightarrow \mathbb{X}$ , where  $\mathbb{Y} \rightarrow \mathbb{Z}$  is a cartesian open immersion and  $\mathbb{Z} \rightarrow \mathbb{X}$  is a cartesian proper morphism.

**Recollection 8.3.** If  $f : Y \rightarrow X$  is a finite morphism of schemes, then  $f_*$  is exact by [29, Prop. 5.3.7]. Employing [29, Prop. 5.7.4], we derive that the unit morphism  $1 \mapsto f_* f^*$  of the adjunction  $(f^* \dashv f_*)$  induces a canonical morphism

$$H^\bullet(X, K) \rightarrow H^\bullet(X, f_* f^* K) = H^\bullet(Y, f^* K)$$

for every  $K$  in  $\mathcal{D}_{tor}^+(X_{\acute{e}t})$ .

<sup>1</sup>One may derive this from [63, Ch. IV.8 Thm. 1] and the fact that  $(\psi \circ \varphi)_* = \varphi_* \circ \psi_*$ .

**Definition 8.4.** Let  $\mathbb{X}$  be a finite étale tower of schemes. For every  $K$  in  $\mathcal{D}_{\text{tor}}^+(X_{\text{ét}})$  we define

$$H^k(\mathbb{X}, K) := \text{colim}_r H^k(X_r, {}_{\mathbb{X}}\phi_0^{r*} K),$$

and we call  $H^k(\mathbb{X}, K)$  the  $k$ -th cohomology group of  $\mathbb{X}$  with coefficients  $K$ . If  $X$  is separated and of finite type over an algebraic separably closed field, then we define

$$H_c^k(\mathbb{X}, K) := \text{colim}_r H_c^k(X_r, {}_X\phi_0^{r*} K),$$

and we call  $H_c^k(\mathbb{X}, K)$  the  $k$ -th cohomology group with compact support of  $\mathbb{X}$  with coefficients  $K$ .

## 8.2 Transformation of Adjoints

In this section, we gather together a set of “technical” lemmas which are related to the transformation of adjoints within our framework.

**Lemma 8.5** (Variations of proper base change). *Let*

$$\begin{array}{ccc} Z & \xrightarrow{g'} & Y \\ f' \downarrow & & \downarrow f \\ X & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes of finite type over an algebraic separably closed field  $k$ , where  $f$  is finite étale. Then  $\mathcal{R}g'_* \circ f'^* \cong f^* \mathcal{R}g_*$  as functors from  $\mathcal{D}_{\text{tor}}(X_{\text{ét}})$  to  $\mathcal{D}_{\text{tor}}(Y_{\text{ét}})$ , where the isomorphism gets canonically induced by adjunctions. If  $g$  is  $k$ -compactifiable, then  $\mathcal{R}g'_! \circ f'^* \cong f^* \mathcal{R}g_!$ .

*Proof:* Let  $h : Z \rightarrow Z'$  be an arbitrary morphism of schemes of finite type over an algebraic separably closed field. Then  $\mathcal{R}h_*$  has finite cohomological dimension on the category of torsion sheaves on  $Z$ , see [29, Lem. 7.5.6], and therefore  $\mathcal{R}h_*$  is defined on  $\mathcal{D}_{\text{tor}}(Z_{\text{ét}})$ ; see [29, Rem. 7.4.6]. Then [93, Tag 0DDD] and [93, Tag 09T5] imply that the functor  $\mathcal{R}h_* : \mathcal{D}_{\text{tor}}(Z_{\text{ét}}) \rightarrow \mathcal{D}_{\text{tor}}(Z'_{\text{ét}})$  is right adjoint to  $h^*$ .

Assume that  $f$  is finite étale. Employing [93, Tag 03S7], we derive the natural equivalences

$$\text{Hom}(g^* f_*(-), -) \cong \text{Hom}(-, f^* \mathcal{R}g_*(-))$$

and

$$\text{Hom}(f'_* g'^*(-), -) \cong \text{Hom}(-, \mathcal{R}g'_* f'^*(-)).$$

Making use of the proper base change theorem [29, Lem. 7.3.1] and the explicit description of the base change isomorphism [93, Tag 07A7] we derive a natural equivalence

$$g^* f_* \cong f'_* g'^*$$

induced by adjunctions and subsequently the Yoneda lemma implies

$$\mathcal{R}g'_* f'^* \cong f^* \mathcal{R}g_*,$$

whence the first part.

In order to prove the second part, we can assume without loss of generality that  $g$  is an open immersion. Then we have the natural equivalences

$$\mathrm{Hom}(-, f'_* g'^*(-)) \cong \mathrm{Hom}(g'_! f'^*(-), -)$$

and

$$\mathrm{Hom}(-, g^* f_*(-)) \cong \mathrm{Hom}(f^* g_!(-), -).$$

Then the proper base change theorem and the Yoneda lemma imply  $g'_! \circ f'^* \cong f^* \circ g_!$ .  $\blacksquare$

**Recollection 8.6** ([64, Ch. II §2, §3]). Let  $f : Y \rightarrow X$  be a morphism of schemes, let  $\mathcal{F}$  be an étale presheaf on  $X$ , and let

$$\alpha : \mathrm{Hom}_{\mathbf{PSh}(Y_{\acute{e}t})}(f^p \mathcal{F}, f^p \mathcal{F}) \rightarrow \mathrm{Hom}_{\mathbf{PSh}(X_{\acute{e}t})}(\mathcal{F}, f_* f^p \mathcal{F})$$

denote the isomorphism induced by the adjunction  $(f^p \dashv f_*)$ .

We recall the explicit presentation of  $\alpha$  and its inverse. First, we recall that to give a morphism in  $\mathcal{F} \rightarrow f_* f^p \mathcal{F}$  is the same as to give maps  $\mathcal{F}(U) \rightarrow f^p \mathcal{F}(U \times_X Y)$  for all étale morphisms  $U \rightarrow X$  which are compatible with the restriction morphisms. Furthermore, to give a morphism  $f^p \mathcal{F} \rightarrow f^p \mathcal{F}$  is the same as to give maps  $\mathcal{F}(U) \rightarrow f^p \mathcal{F}(V)$  for all commutative squares

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where  $U \rightarrow X$  and  $V \rightarrow Y$  are étale morphisms which are compatible with the restriction morphisms. Thus, given a morphism  $f^p \mathcal{F} \rightarrow f^p \mathcal{F}$ , we canonically derive a morphism  $\mathcal{F} \rightarrow f_* f^p \mathcal{F}$  that actually is the image of  $f^p \mathcal{F} \rightarrow f^p \mathcal{F}$  under the isomorphism  $\alpha$ . In particular,  $\alpha(\mathrm{id})$  is the cocone morphism  $\mathcal{F}(U) \rightarrow f^p \mathcal{F}(U \times_X Y)$  that is part of the datum of the colimit presentation of  $f^p \mathcal{F}(U \times_X Y)$ .

In particular, if  $f$  is an étale morphism, then for every  $U \rightarrow X$  étale we find that  $f_* f^p \mathcal{F}(U) \cong \mathcal{F}(U \times_X Y)$  by cofinality and the morphism  $\mathcal{F} \rightarrow f_* f^p \mathcal{F}$  associated to the identity on  $f^p \mathcal{F}$  via  $\alpha$  evaluated on  $U$  is given by the restriction  $\mathcal{F}(U) \rightarrow \mathcal{F}(U \times_X Y)$ .

For that reason, the unit  $\text{id} \rightarrow f_* f^p = f_* f^*$  is called *restriction* for every étale morphism  $f$ . See [93, Tag 03SH] and [64, Rem. 2.8]. The counit is called *trace*.

Vice versa, let  $\varphi$  be an arbitrary morphism in  $\text{Hom}_{\mathbf{PSh}(X_{\text{ét}})}(\mathcal{F}, f_* f^p \mathcal{F})$  given by the collected datum  $\varphi_U : \mathcal{F}(U) \rightarrow f^p \mathcal{F}(U \times_X Y)$  where  $U \rightarrow X$  is étale. Given any commutative square

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

the morphism  $V \rightarrow U$  factors uniquely as  $V \rightarrow U \times_X Y \rightarrow U$  and  $\alpha^{-1}(\varphi)$  is the morphism given by the datum

$$\mathcal{F}(U) \xrightarrow{\varphi_U} f^p \mathcal{F}(U \times_X Y) \xrightarrow{\text{res}_V^{U \times_X Y}} f^p \mathcal{F}(V)$$

for all  $U \rightarrow X$  and  $V \rightarrow Y$  as above.

**Recollement of Sheaves 8.7** ([64, Ch. II. §3] , [29, Ch. 5.4]) Let  $X$  be a scheme, let  $j : U \hookrightarrow X$  be an open subscheme, and let  $i : Z \rightarrow X$  be a closed subscheme of  $X$ , whose underlying topological space is the complementary closed set  $Z = X \setminus U$ . The category  $\mathbf{T}(X, i, j)$  is defined to be the category whose objects are triples  $(\mathcal{F}_1, \mathcal{F}_2, \varphi)$  where  $\mathcal{F}_1$  is an object of  $\mathbf{Sh}(Z_{\text{ét}})$ ,  $\mathcal{F}_2$  is an object of  $\mathbf{Sh}(U_{\text{ét}})$ , and  $\varphi$  is a morphism from  $\mathcal{F}_1$  to  $i^* j_* \mathcal{F}_2$ . A morphism  $(\mathcal{F}_1, \mathcal{F}_2, \varphi) \rightarrow (\mathcal{F}'_1, \mathcal{F}'_2, \varphi')$  is a pair  $(\psi_1, \psi_2)$  where  $\psi_1$  is a morphism  $\mathcal{F}_1 \rightarrow \mathcal{F}'_1$  and  $\psi_2$  is a morphism  $\mathcal{F}_2 \rightarrow \mathcal{F}'_2$ , and the two are compatible with  $\varphi, \varphi'$  in the sense that the diagram

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{\varphi} & i^* j_* \mathcal{F}_2 \\ \psi_1 \downarrow & & \downarrow i^* j_* (\psi_2) \\ \mathcal{F}'_1 & \xrightarrow{\varphi'} & i^* j_* \mathcal{F}'_2 \end{array}$$

commutes. Sending an étale sheaf  $\mathcal{F}$  in  $\mathbf{Sh}(X_{\text{ét}})$  to the triple  $(i^* \mathcal{F}, j^* \mathcal{F}, \varphi)$  where  $\varphi : i^* \mathcal{F} \rightarrow i^* j_* j^* \mathcal{F}$  gets induced by the unit  $1 \rightarrow j_* j^*$  of the adjunction  $(j^* \dashv j_*)$  gives rise to an equivalence between the categories  $\mathbf{Sh}(X_{\text{ét}})$  and  $\mathbf{T}(X, i, j)$ . Here a morphism  $\varphi$  in  $\mathbf{Sh}(X_{\text{ét}})$  gets mapped to the morphism  $(i^* \varphi, j^* \varphi)$ .

**Lemma 8.8.** *Let  $f : Y \rightarrow X$  be a finite étale morphism of quasi-compact quasi-separated schemes, and let  $(\mathcal{F}_i)_{i \in I}$  be a directed system of abelian torsion étale sheaves on  $X$  all of whose transition morphisms are injective. The canonical diagram*

$$\begin{array}{ccc} \operatorname{colim}_i H^k(X, \mathcal{F}_i) & \longrightarrow & \operatorname{colim}_i H^k(Y, f^* \mathcal{F}_i) \\ \operatorname{id} \downarrow = & & = \downarrow \operatorname{id} \\ H^k(X, \operatorname{colim}_i \mathcal{F}_i) & \longrightarrow & H^k(Y, f^* \operatorname{colim}_i \mathcal{F}_i), \end{array}$$

*commutes for every natural number  $k$ .*

*Proof:* Making use of the fact that  $f^*$  and  $f_*$  commute with colimits by [93, Tag 03Q5] and [93, Tag 00DA], we derive that it suffices to prove that the canonical diagram

$$\begin{array}{ccc} H^k(X, \mathcal{F}_i) & \longrightarrow & H^k(Y, f^* \mathcal{F}_i) \\ \downarrow & & \downarrow \\ H^k(X, \operatorname{colim}_i \mathcal{F}_i) & \longrightarrow & H^k(Y, \operatorname{colim}_i f^* \mathcal{F}_i), \end{array}$$

commutes. We find

$$\operatorname{colim}_i (\mathcal{F}_i(X)) = (\operatorname{colim}_i \mathcal{F}_i)(X) \quad \text{and} \quad \operatorname{colim}_i (f^* \mathcal{F}_i(Y)) = (\operatorname{colim}_i f^* \mathcal{F}_i)(Y)$$

by [93, Tag 0738]. In the virtue of Recollection 8.6, let

$$\varphi_i : \mathcal{F}_i(X) \rightarrow f_* f^* \mathcal{F}_i(X) = \mathcal{F}_i(Y)$$

denote the restriction morphism. Then the induced morphism

$$\varphi : (\operatorname{colim}_i \mathcal{F}_i)(X) \rightarrow (\operatorname{colim}_i f^* \mathcal{F}_i)(Y) = (\operatorname{colim}_i \mathcal{F}_i)(Y)$$

is the restriction morphism. This can be concluded from the prove of [93, Tag 0738] and the commutative diagram in the introduction of [93, Tag 007X]. Consequently, we derive the needed commutativity.  $\blacksquare$

**Corollary 8.9.** *Let  $\mathbb{X}$  be a finite étale tower of quasi-compact quasi-separated schemes, and let  $(\mathcal{F}_i)_{i \in I}$  be a directed system of étale sheaves on  $X$  all of whose transition morphisms are injective. Then*

$$H^k(\mathbb{X}, \operatorname{colim}_i \mathcal{F}_i) = \operatorname{colim}_i H^k(\mathbb{X}, \mathcal{F}_i),$$

*for every index  $k \geq 0$ .*

**Lemma 8.10.** *Let*

$$\begin{array}{ccc} Z & \xrightarrow{h'} & Y \\ g \downarrow & & \downarrow f \\ X & \xrightarrow{h} & S \end{array}$$

be a cartesian diagram of schemes where  $f$  is finite étale. Let  $\eta$  denote the unit, and let  $\epsilon$  denote the counit of the (derived) adjunction  $(g^* \dashv g_*)$ . Further, let  $\eta'$  denote the unit, and let  $\epsilon'$  denote the counit of the (derived) adjunction  $(f^* \dashv f_*)$ . Then

- i.) One has  $\mathcal{R}h_*\eta = \eta'\mathcal{R}h_*$  as a functor from  $\mathcal{D}^+(X_{\acute{e}t})$  to  $\mathcal{D}^+(S_{\acute{e}t})$ ;
- ii.) One has  $\mathcal{R}h_*\epsilon' = \epsilon\mathcal{R}h_*$  as a functor from  $\mathcal{D}^+(X_{\acute{e}t})$  to  $\mathcal{D}^+(S_{\acute{e}t})$ ;
- iii.) Provided that  $h$  is an open immersion, one has  $h_!\eta = \eta'h_!$  as a functor from  $\mathcal{D}^+(X_{\acute{e}t})$  to  $\mathcal{D}^+(S_{\acute{e}t})$ .

*Proof:* In order to prove the first part of the Lemma, we note that

$$\mathcal{R}h_* \circ g_* = \mathcal{R}(h_* \circ g_*) = \mathcal{R}(f_* \circ h'_*) = f_* \circ \mathcal{R}h'_*$$

Hence, employing Lemma 8.5, we derive a canonical diagram

$$\begin{array}{ccc} \mathrm{Hom}(g^*K, g^*K) & \xrightarrow{\alpha} & \mathrm{Hom}(K, g_*g^*K) \\ \downarrow \epsilon & & \downarrow \gamma \\ \mathrm{Hom}(\mathcal{R}h'_*g^*K, \mathcal{R}h'_*g^*K) & & \mathrm{Hom}(\mathcal{R}h_*K, \mathcal{R}h_*g_*g^*K) \\ = \downarrow \zeta & & = \downarrow \delta \\ \mathrm{Hom}(f^*\mathcal{R}h_*K, f^*\mathcal{R}h_*K) & \xrightarrow{\beta} & \mathrm{Hom}(\mathcal{R}h_*K, f_*f^*\mathcal{R}h_*K) \end{array}$$

for every  $K$  in  $\mathcal{D}_{\mathrm{tor}}^+(X_{\acute{e}t})$ . According to [63, Ch. IV.7 Prop. 1], we need to prove that  $\delta \circ \gamma \circ \alpha(\mathrm{id}) = \beta(\mathrm{id})$ . We can assume without loss of generality that  $K = \mathcal{F}$  is a sheaf. In the realm of Recollection 8.6, we find that  $\beta(\mathrm{id})(U)$  is given by the restriction morphism

$$h_*\mathcal{F}(U) \rightarrow h_*\mathcal{F}(U \times_S Y)$$

for every étale morphism  $U \rightarrow S$ . Furthermore,  $\alpha(\mathrm{id})(V)$  is given by the restriction morphism

$$\mathcal{F}(V) \rightarrow \mathcal{F}(V \times_X Z)$$

for every étale morphism  $V \rightarrow X$  and consequently  $\gamma \circ \alpha(\mathrm{id})(U)$  is given by the restriction morphism

$$h_*\mathcal{F}(U) \rightarrow h_*\mathcal{F}(U \times_S Y)$$

## 8.2. Transformation of Adjoints

for every étale morphism  $U \rightarrow S$ . Thus  $\delta \circ \gamma \circ \alpha(\text{id}) = \beta(\text{id})$  and we conclude the first part of the Lemma.

If  $h$  is an open immersion, then, making use of the representation of  $h_!$  under the category equivalence discussed in 8.7, see [64, Ch. II. §3]), we derive that  $h_!\eta \cong \eta' h_!$ . ■

**Lemma 8.11.** *Let  $C$  and  $D$  be categories, and let  $F : D \rightarrow C$  and  $G : C \rightarrow D$  be functors, where  $F$  is left adjoint to  $G$ . Let  $\eta$  denote the unit, and let  $\varepsilon$  denote the counit of the adjunction  $(F \dashv G)$ . Let  $D_C : C \rightarrow C$  and  $D_D : D \rightarrow D$  be a pair of contravariant functors with  $D_C \circ D_C = \text{id}_C$  and  $D_D \circ D_D = \text{id}_D$ . Assume that the diagram*

$$\begin{array}{ccccc} D & \xrightarrow{F} & C & \xrightarrow{G} & D \\ \downarrow D_D & & \downarrow D_C & & \downarrow D_D \\ D & \xrightarrow{F} & C & \xrightarrow{G} & D \end{array}$$

*commutes. Then we have natural isomorphisms  $D_D \circ \eta \cong \varepsilon \circ D_D$ .*

*Proof:* Let  $d$  in  $D$  and  $c$  in  $C$  be arbitrary objects, and let  $f : D_D(d) \rightarrow G(D_C(c))$  be an arbitrary morphism. According to the universal property of  $\eta$ , there exists a unique morphism  $\varphi : F(D_D(d)) \rightarrow D_C(c)$  such that

$$\begin{array}{ccc} D_D(d) & \xrightarrow{\eta_{D_D(d)}} & GF(D_D(d)) \\ & \searrow f & \downarrow G(\varphi) \\ & & G(D_C(c)) \end{array}$$

commutes. Applying  $D_D$ , we find that  $D_C(\varphi)$  is the unique morphism such that

$$\begin{array}{ccc} d & \xleftarrow{D_D \eta_{D_D(d)}} & GF(d) \\ & \swarrow D_D(f) & \uparrow G(D_C(\varphi)) \\ & & G(c) \end{array}$$

commutes. Thus  $(D_D \eta_{D_D(d)})_{d \in \mathbf{Ob}(D)}$  has the universal property of  $(\varepsilon_d)_{d \in \mathbf{Ob}(D)}$  and we derive the proclaimed natural isomorphism. ■

### 8.3 Fundamental Theorems

In this section we extend some of the fundamental theorems of étale cohomology to the étale cohomology on finite étale towers.

**Theorem 8.12.** *Let  $\mathbb{X}$  be a finite étale tower that is separated and of finite type over an algebraic separably closed field  $k$ . Let  $j : U \hookrightarrow X$  be an open immersion with closed complement  $i : Z \rightarrow X$ . We have a long exact sequence*

$$\dots \rightarrow H_c^i(\mathbb{U}, K|_U) \rightarrow H_c^i(\mathbb{X}, K) \rightarrow H_c^i(\mathbb{Z}, K|_Z) \rightarrow \dots$$

for every  $K$  in  $\mathcal{D}_{\text{tor}}^+(X_{\text{ét}})$ .

*Proof:* We denote the open immersion  $\mathbb{U} \rightarrow \mathbb{X}$  by  $j$ , and we denote the closed immersion  $\mathbb{Z} \rightarrow \mathbb{X}$  by  $i$ . Let  $s, t$  be a pair of natural numbers with  $t \geq s$ , and let  $K' := (\phi_0^s)^* K$ . We recall that we have a distinguished triangle

$$(j_{s!} j_s^* K', K', i_{s*} i_s^* K');$$

see [93, Tag 014Z] and [29, Thm. 7.4.4]. Applying [93, Tag 05SC], we derive a morphism of distinguished triangles from

$$(j_{s!} j_s^*(K'), (K'), i_{s*} i_s^*(K'))$$

to

$$(\mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^* j_{s!} j_s^*(K'), \mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^*(K'), \mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^* i_{s*} i_s^*(K'))$$

induced by the unit  $1 \mapsto \mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^*$ . Due to Lemma 8.10, this morphism gives rise to a morphism from

$$(j_{s!} j_s^*(K'), (K'), i_{s*} i_s^*(K'))$$

to

$$(j_!(\mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^* j_{s!} j_s^* K'), \mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^* K', i_*(\mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^* i_{s*} i_s^* K'))$$

being induced by the units  $1 \mapsto \mathcal{R}(\phi_s^t)_* \circ (\phi_s^t)^*$  with  $W$  in  $\{X, U, Z\}$ . Employing [29, Thm. 7.4.4], we derive the Theorem.  $\blacksquare$

**Lemma 8.13.** *Let  $X$  be a scheme, let  $K$  be a sheaf complex on  $X$ , and let  $U \rightarrow X$  and  $V \rightarrow X$  be open immersions such that  $X = U \cup V$ . Then there exists an exact sequence*

$$0 \rightarrow K \rightarrow U j_! U j^* K \oplus V j_! V j^* K \rightarrow U \cap V j_! U \cap V j^* K \rightarrow 0.$$

*Proof:* First let  $\mathcal{F}$  be an étale sheaf on  $X$ . Let  $W$  stand either for  $U$  or  $V$ . First we note that

$${}_{U \cap V}j_! {}_{U \cap V}j^* \mathcal{F} \cong {}_{U \cap V}j_! {}_{U \cap V}j^* {}_Wj_! {}_Wj^* \mathcal{F}.$$

Indeed, by [64, Ch. II §3], we have an exact sequence

$$0 \rightarrow {}_Wj_! {}_Wj^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow {}_{X/W}i_* {}_{X/W}i^* \mathcal{F} \rightarrow 0,$$

where  ${}_{X/W}i$  denotes the closed immersion associated to  $X/W$  with the induced reduced closed subscheme structure [93, Tag 01J4]. Then

$${}_{U \cap V}j_! {}_{U \cap V}j^* {}_{X/W}i_* {}_{X/W}i^* \mathcal{F} = 0,$$

as, making use the category equivalence described in 8.7, can be easily seen on stalks; see [64, Ch. II §3]. Then  ${}_{U}j_!$  is left adjoint to  ${}_{U}j_*$  and we derive the canonical morphisms

$${}_U\psi : {}_Uj_! {}_Uj^* \mathcal{F} \rightarrow {}_{U \cap V}j_! {}_{U \cap V}j^* \mathcal{F} \quad \text{and} \quad {}_V\psi : {}_Vj_! {}_Vj^* \mathcal{F} \rightarrow {}_{U \cap V}j_! {}_{U \cap V}j^* \mathcal{F}.$$

Let  $\psi$  denote the morphism

$${}_Uj_! {}_Uj^* \mathcal{F} \oplus {}_Vj_! {}_Vj^* \mathcal{F} \rightarrow {}_{U \cap V}j_! {}_{U \cap V}j^* \mathcal{F}$$

derived from  $({}_U\psi, -{}_V\psi)$  by the universal property of the coproduct. We have canonical morphisms

$${}_U\varphi : \mathcal{F} \rightarrow {}_Uj_! {}_Uj^* \mathcal{F} \quad \text{and} \quad {}_V\varphi : \mathcal{F} \rightarrow {}_Vj_! {}_Vj^* \mathcal{F},$$

and we let  $\varphi$  denote the morphism

$$\mathcal{F} \rightarrow {}_Uj_! {}_Uj^* \mathcal{F} \oplus {}_Vj_! {}_Vj^* \mathcal{F}$$

derived from  $({}_U\varphi, {}_V\varphi)$  by the universal property of the product. Let  $\bar{x}$  be a geometric point of  $X$ . If  $\bar{x}$  is not contained in  $U \cap V$ , then  ${}_{U \cap V}j_! {}_{U \cap V}j^* \mathcal{F} = 0$  and  ${}_Uj_! {}_Uj^* \mathcal{F} \oplus {}_Vj_! {}_Vj^* \mathcal{F} \cong \mathcal{F}_{\bar{x}}$ , where the isomorphism is given by  $\varphi_{\bar{x}}$ . If  $\bar{x}$  is contained in  $U \cap V$ , then  $\varphi_{\bar{x}}$  identifies with the diagonal morphism

$$\mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}} \oplus \mathcal{F}_{\bar{x}}$$

and

$$\psi_{\bar{x}} : \mathcal{F}_{\bar{x}} \oplus \mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}}$$

maps an element  $(a, b)$  to  $a - b$ , whence the exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow {}_Uj_! {}_Uj^* \mathcal{F} \oplus {}_Vj_! {}_Vj^* \mathcal{F} \rightarrow {}_{U \cap V}j_! {}_{U \cap V}j^* \mathcal{F} \rightarrow 0.$$

This construction gives rise to the proclaimed exact sequence of complexes. ■

**Theorem 8.14** (Mayer-Vietoris sequence). *Let  $\mathbb{X}$  be a finite étale tower that is separated and of finite type over an algebraic separably closed field  $k$ . Suppose that  $X = U \cup V$  is the union of two open subschemes. Then there exists a long exact sequence*

$$\dots \rightarrow H_c^i(\mathbb{X}, K) \rightarrow H_c^i(\mathbb{U}, K|_U) \oplus H_c^i(\mathbb{V}, K|_V) \rightarrow H_c^i(\mathbb{U} \cap \mathbb{V}, K|_{U \cap V}) \rightarrow \dots$$

for every object  $K$  of  $\mathcal{D}_{tor}^+(X_{ét})$ .

*Proof:* Let  $f : X \rightarrow \text{Spec}(k)$  denote the structure morphism of  $X$ , and let  $s, t$  be a pair of natural numbers with  $t \geq s$ . Let  $K' := (\phi_0^s)^* K$ . Lemma 8.13 and [93, Tag 014Z] yield a distinguished triangle

$$\left( K', {}_U j_{s!} {}_U j_s^* K' \oplus {}_V j_{s!} {}_V j_s^* K', {}_{U \cap V} j_{s!} {}_{U \cap V} j_s^* K' \right).$$

Alluding to Lemma 8.10, we derive a morphism of distinguished triangles from

$$(\mathcal{R}f_! K', \mathcal{R}f_! {}_U j_{s!} K' |_U \oplus \mathcal{R}f_! {}_V j_{s!} K' |_V, \mathcal{R}f_! {}_{U \cap V} j_{s!} K' |_{U \cap V})$$

to

$$(H_1, H_2, H_3)$$

with

$$\begin{aligned} H_1 &= \mathcal{R}f_!(\phi_s^t)_*(\phi_s^t)^* K'; \\ H_2 &= \mathcal{R}f_! {}_U j_{s!} ({}_U \phi_s^t)_* ({}_U \phi_s^t)^* K' |_U \oplus \mathcal{R}f_! {}_V j_{s!} ({}_V \phi_s^t)_* ({}_V \phi_s^t)^* K' |_V; \\ H_3 &= \mathcal{R}f_!(\phi_s^t)_*(\phi_s^t)^* K', \end{aligned}$$

being induced by the units  $1 \mapsto \mathcal{R}({}_W \phi_s^t)_* \circ ({}_W \phi_s^t)^*$  with  $W$  in  $\{X, U, V, U \cap V\}$ , whence the theorem.  $\blacksquare$

**Theorem 8.15** (Leray spectral sequence). *Let  $f : \mathbb{Y} \rightarrow \mathbb{X}$  be a cartesian morphism of finite étale towers of schemes. Then we have a biregular Leray spectral sequence*

$$E_2^{pq} : H^p(\mathbb{X}, \mathcal{R}^q f_* K) \Rightarrow H^{p+q}(\mathbb{Y}, K)$$

for all  $K$  in  $\mathcal{D}_{tor}^+(Y_{ét})$ .

*Proof:* Let  $F$  denote the functor that associated to an abelian torsion étale sheaf  $\mathcal{F}$  on  $Y$  the abelian group

$$\text{colim}_s \Gamma(X_s, ((\phi_0^s)^* \mathcal{F})).$$

Then  $(\phi_0^s)_* = (\phi_0^s)_!$  is an exact left adjoint of  $(\phi_0^s)^*$  and subsequently  $(\phi_0^s)^*$  preserves injective objects. Then  $f_*$  maps injective objects to  $F$ -acyclic objects, and utilizing Proposition 8.5, we obtain the stated spectral sequence formally as a Grothendieck spectral sequence; see [93, Tag 015N].  $\blacksquare$

**Theorem 8.16** (Leray spectral sequence with compact support).

*Let  $f : \mathbb{Y} \rightarrow \mathbb{X}$  be a cartesian morphism of finite étale towers of schemes that are separated and of finite type over an algebraic separably closed field  $k$ . Then we have a biregular Leray spectral sequence*

$$E_2^{pq} : H_c^p(\mathbb{X}, \mathcal{R}^q f_! K) \Rightarrow H_c^{p+q}(\mathbb{Y}, K)$$

for all  $K$  in  $\mathcal{D}^+(Y_{\text{ét}})$ .

*Proof:* Lemma 8.5 implies that

$$H_c^i(\mathbb{Y}, K) = H_c^i(\mathbb{X}, \mathcal{R}f_! K).$$

for every admissible index  $i$  and the theorem follows from [29, Thm. 6.3.5]. ■



# Chapter 9

## Cohomological Dimension in Pro- $\ell$ Towers after Esnault

In this section, we will prove Theorem 0.6. If not said otherwise, let  $k$  be an algebraic closed field, let  $\ell$  be a prime number different from the characteristic of  $k$ , and let  $\ell^v > 1$  be a power of  $\ell$ .

**Definition 9.1.** *Let  $\mathbb{X}$  be a tower of schemes that is separated and of finite type over  $k$ . If there exists a positive natural number  $t$  such that the transition morphism  ${}_{\mathbb{X}}\varphi_0^r$  is a connected Galois covering with Galois group  $(\mathbb{Z}/\ell^r\mathbb{Z})^t$  for all natural numbers  $r$ , then we say that  $\mathbb{X}$  is a pro- $\ell$  (Galois) tower over  $k$ . In this case we call  $t$  the type of  $\mathbb{X}$ . If, in addition, all levels of  $\mathbb{X}$  are curves, then we say that  $\mathbb{X}$  is a pro- $\ell$  (Galois) tower of curves.*

**Lemma 9.2.** *Let  $S$  be a scheme, let  $X$  and  $X'$  be  $S$ -schemes, and let  $Y \rightarrow X$  and  $Y' \rightarrow X'$  be a pair of connected Galois coverings with Galois groups  $G_1$  and  $G_2$ . Assume that the fibre products  $X \times_S X'$  and  $Y \times_S Y'$  are connected. Then  $Y' \times_S Y \rightarrow X \times_S X'$  is a connected Galois covering with Galois group  $G_1 \times G_2$ .*

*Proof:* Finite étale morphisms are finite locally free. Therefore, the degree of  $Y' \times_S Y \rightarrow X \times_S X'$  equals the order of  $G_1 \times G_2$ . Furthermore, the universal property of the fibre product gives rise to a group homomorphism

$$\mathrm{Aut}_X(Y) \times \mathrm{Aut}_{X'}(Y') \hookrightarrow \mathrm{Aut}_{X \times_S X'}(Y \times_S Y'),$$

that is injective, as can be seen on fibers. We conclude from [64, Ch. I 3.13 and p.40 ] that the order of  $\mathrm{Aut}_{X \times_S X'}(Y \times_S Y')$  is smaller, hence equal to the order of  $G_1 \times G_2$ . ■

If the scheme  $S$  is the spectrum of an algebraic separably closed field in the context of Lemma 9.2, then the connectedness of  $X \times_S X'$  and  $Y \times_S Y'$  follows directly from [93, Tag 0385].

**Lemma 9.3.** *Let  $Z \rightarrow X$  be a connected Galois covering with Galois group  $G$ , and let  $Y \rightarrow X$  be a morphism of schemes, where  $Y$  and  $Z \times_X Y$  are connected. Then  $Z \times_X Y \rightarrow Y$  is a connected Galois covering with Galois group  $G$ .*

*Proof:* The proof is similar to the proof of Lemma 9.2. The universal property of the fibre product gives rise to an injective morphism  $G \rightarrow \text{Aut}_Y(Z \times_X Y)$  and the order of  $\text{Aut}_Y(Z \times_X Y)$  equals the order of  $G$ . ■

**Definition 9.4.** *Let  $\mathbb{X}$  be a finite étale tower that is separated and of finite type over  $k$ . We say that  $\mathbb{X}$  possesses **SEV** if  $H_c^i(\mathbb{X}, \mathcal{F})$  vanishes for all  $\ell$ -primary torsion étale sheaves  $\mathcal{F}$  on the base  $X$  of  $\mathbb{X}$  and all  $i > \dim(X)$ .*

**Remark 9.5.** Let  $\mathbb{X}$  be a finite étale tower that is separated and of finite type over  $k$  possessing **SEV**, and let  $\mathcal{F}$  be an  $\ell$ -primary torsion étale sheaf on  $X$ . Then

$$H_c^i(\mathbb{X}, \mathcal{F}) = 0 \quad \text{for all } i > \dim(\overline{\text{supp}(\mathcal{F})})$$

due to Theorem 8.12.

**Lemma 9.6.** *Let  $\mathbb{X}$  be a finite étale tower that is separated and of finite type over  $k$ . Then  $\mathbb{X}$  possesses **SEV** if the induced tower over the reduction of  $X$  possesses **SEV**.*

*Proof:* Let  $f : X_{red} \rightarrow X$  denote the reduction of  $X$ ; see [93, Tag 01IZ]. Let  $\mathbb{X}_{red}$  denote the tower over  $X_{red}$  that gets induced by  $f$ . Then  $\dim(X_{red})$  equals  $\dim(X)$ , and the induced morphism  $\mathbb{X}_{red} \rightarrow \mathbb{X}$  is finite, surjective and radiciel. Subsequently, the Lemma follows from [29, Cor. 5.3.10 and Cor. 5.7.1]. ■

**Lemma 9.7.** *Let  $\mathbb{X}$  be a finite étale tower that is separated and of finite type over  $k$ . Then  $\mathbb{X}$  possesses **SEV** if the induced towers over the irreducible components of  $X$  possess **SEV**.*

*Proof:* We prove the assertion by an ascending induction on the number of irreducible components of  $X$ . The induction start where  $X$  is irreducible is clear. Let  $X_1, \dots, X_s$  denote the pairwise different irreducible components of  $X$ , and assume that the assertion of the lemma holds true whenever the base of the tower can be covered by  $s - 1$  many irreducible components.

Let  $Z := X_2 \cup \dots \cup X_s$ , and let  $U := X \setminus Z$ . Then  $U$  is a dense open irreducible subset of  $X_1$  and  $X_2, \dots, X_s$  are the irreducible components of  $Z$  by [93, Tag 0G2Y]. The induction step follows from Theorem 8.12. ■

**Topology of a one-dimensional scheme 9.8** In the following, let  $X$  be an irreducible one-dimensional scheme. The underlying topological space of  $X$  is Kolmogorov. Hence, with the exception of the generic point all points of  $X$  are closed. Assume in addition that  $X$  is noetherian. Then every proper closed subsets  $Z$  of  $X$  consists solely of closed points, all of them being irreducible components of  $Z$ . In effect  $Z$  is finite. If  $X$  is of finite type over an algebraically closed field, then [29, Cor. 5.7.3], [38, Prop. 3.33] and Theorem 8.12 imply that  $X$  has étale cohomological dimension 0. Assume that  $X$  is noetherian and that the underlying topological space of  $X$  is not finite. Let  $\mathcal{F}$  be a constructible sheaf on  $X$ . Then there exists a non-empty open subscheme  $U$  of  $X$  such that  $\mathcal{F}|_U$  is finite locally constant.

**Proposition 9.9** (The Curve Case). *Every type one pro- $\ell$  Galois tower of curves  $\mathbb{C}$  over  $k$  possesses SEV.*

*Proof:* Let  $\mathbb{C}$  be a pro- $\ell$  Galois tower of curves, and let  $\mathcal{F}$  be an  $\ell$ -primary torsion étale sheaf on  $X$ . Without loss of generality it suffices to prove that the canonical morphism

$$H_c^2(C_0, \mathcal{F}) \rightarrow H_c^2(C_r, (\mathbb{C}\phi_0^r)^* \mathcal{F}).$$

vanishes for all sufficiently large  $r$ . The base  $C$  of  $\mathbb{C}$  is geometrically integral; see [47, Ch. II Exc. 3.15]. Applying Theorem 8.12, 9.8, Lemma 9.3 and [93, Tag 056V], we can assume without loss of generality that the base  $C$  of  $\mathbb{C}$  is a smooth affine curve and that the underlying topological space of  $C$  is not finite. Making use of [29, Thm. 5.8.8] and Corollary 8.9, we can assume without loss of generality that  $\mathcal{F}$  is constructible, and alluding to Theorem 8.12, 9.8 and Lemma 9.3 we can assume without loss of generality that  $\mathcal{F}$  is finite locally constant.

Let  $f : C \rightarrow k$  denote the structure morphism of  $C$ , let  $r$  be a natural number, and let  $n$  be a power of  $\ell$  annihilating  $\mathcal{F}$ . The ring  $\mathbb{Z}/n\mathbb{Z}$  is self-injective, so is any Tate twist, and [93, Tag 03PC] implies that the constant sheaf  $\mathbb{Z}/n\mathbb{Z}$  on  $\text{Spec}(k)$  is injective. Then

$$H^2(\mathcal{R}\Gamma_k \circ D_k \circ \mathcal{R}f_!(\mathcal{F})) \cong \text{Hom}(H_c^2(C, \mathcal{F}), \mathbb{Z}/n\mathbb{Z}).$$

Further,

$$D_k \circ \mathcal{R}f_! = \mathcal{R}f_* \circ D_C$$

by Poincaré duality; see [18]. Subsequently, for every locally constant étale  $\mathbb{Z}/n\mathbb{Z}$ -

sheaf  $\mathcal{G}$  on  $C$  we derive

$$\begin{aligned}
 H^2(\mathcal{R}\Gamma_k \circ D_k \circ \mathcal{R}f_!(\mathcal{G})) &= H^0(\mathcal{R}\Gamma_k \circ D_k \circ \mathcal{R}f_!(\mathcal{G}[2])) \\
 &= H^0(\mathcal{R}\Gamma_C \circ D_C(\mathcal{G}[2])) \\
 &= H^0(\mathcal{R}\Gamma_C(\mathcal{R}\mathcal{H}om(\mathcal{G}[2], f^!\mathbb{Z}/n\mathbb{Z})) \\
 &= H^0(\mathcal{R}\Gamma_C(\mathcal{R}\mathcal{H}om(\mathcal{G}[2], \mathbb{Z}/n\mathbb{Z}(1)[2])) \\
 &= H^0(C, \mathcal{H}om(\mathcal{G}, \mathbb{Z}/n\mathbb{Z}(1)))
 \end{aligned}$$

and  $\mathcal{H}om(\mathcal{G}, \mathbb{Z}/n\mathbb{Z}(1))$  is finite locally constant by [93, Tag 093T].<sup>1</sup> Let  $\eta : \text{id} \rightarrow (\mathbb{C}\phi_0^r)_*(\mathbb{C}\phi_0^r)^*$  denote the restriction, and let  $\varepsilon : (\mathbb{C}\phi_0^r)_*(\mathbb{C}\phi_0^r)^* \rightarrow \text{id}$  denote the trace. Poincaré duality allows to apply Lemma 8.11 in order to derive

$$\mathcal{R}\Gamma_C \circ \varepsilon \circ D_C = \mathcal{R}\Gamma_C \circ D_C \circ \eta = \mathcal{R}\Gamma_k \circ D_k \circ \mathcal{R}f_! \circ \eta.$$

Subsequently, it suffices to prove that, given a finite locally constant étale  $\mathbb{Z}/n\mathbb{Z}$ -sheaf  $\mathcal{G}$  on  $C$ , the morphism

$$H^0(C, (\mathbb{C}\phi_0^r)_*(\mathbb{C}\phi_0^r)^*\mathcal{G}) \rightarrow H^0(C, \mathcal{G})$$

induced by the trace vanishes for all sufficiently large natural numbers  $r$ . The morphism  $\mathbb{C}\phi_0^r$  is finite étale. Hence,

$$\Gamma(C, (\mathbb{C}\phi_0^r)_*(\mathbb{C}\phi_0^r)^*\mathcal{G}) = \Gamma(C_r, \mathcal{G}).$$

According to [64, Ch. V, Proof of Lem. 1.12 ], we find

$$\Gamma(C, \mathcal{G}) \cong \Gamma(C_r, \mathcal{G})^{G_r}$$

where  $G_r$  denotes the Galois group of  $\mathbb{C}\phi_0^r$  and taking sections over  $C$ , the trace map identifies with

$$\begin{aligned}
 \Gamma(C_r, \mathcal{G}) &\rightarrow \Gamma(C_r, \mathcal{G})^{G_r} \\
 s &\mapsto \sum_{\sigma \in G_r} \sigma \cdot s
 \end{aligned}$$

Let  $V_r$  denote the  $\mathbb{Z}/n\mathbb{Z}$ -module  $\Gamma(C_r, \mathcal{G})$ , let  $\bar{x}$  be a geometric point of  $C$ , and let  $\bar{x}_r$  be a geometric point of  $C_r$  lying over  $\bar{x}$ . Further, let  $V$  denote  $\mathcal{G}_{\bar{x}}$ . Then  $V_r \cong V^{\pi(C_r, \bar{x}_r)}$ , as documented in [65, Prop. 29.9] and the action of  $\pi(C_{r+1}, \bar{x}_{r+1})$  on  $V$

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<sup>1</sup>For the final equality, we employed standard arguments analogous to those given in [109, III.2 Proof of Prop. 2.1] for  $\overline{\mathbb{Q}}_\ell$ -coefficients.

factors over  $\pi(C_r, \bar{x}_r)$  for every natural number  $r$ . Utilizing that  $V$  is artinian, we derive that there exists a natural number  $\beta$  such that  $V_r \cong V_\beta$  for all  $r > \beta$ . Let

$$\psi_r : G_r \rightarrow \text{Aut}(V_r)$$

denote the  $G_r$ -representation of  $V_r$ . There exists a natural number  $c$  such that there exists no element of order greater than  $c$  in  $V_\beta$ . Subsequently, for sufficiently large  $r$  the representation  $\psi_r$  factor as

$$\mathbb{Z}/\ell^r\mathbb{Z} \rightarrow \mathbb{Z}/\ell^c\mathbb{Z} \xrightarrow{\psi'_r} \text{Aut}(V_r)$$

and for every element  $v$  in  $V_r$  we find

$$\sum_{\sigma \in \mathbb{Z}/\ell^r\mathbb{Z}} \psi_r(\sigma) \cdot v = \sum_{\sigma \in \mathbb{Z}/\ell^c\mathbb{Z}} \ell^{r-c} \psi'_r(\sigma) \cdot v = 0,$$

whence the proposition. ■

**Remark 9.10.** In the proof of Proposition 9.9, instead of requiring that  $\mathbb{C}$  has type 1, it suffices to require that there exists a positive natural number  $t$ , such that  ${}_{\mathbb{C}}\varphi_r^s$  is a connected Galois covering with Galois group  $(\mathbb{Z}/\ell^{s-r}\mathbb{Z})^t$  for all natural numbers  $s, r$  with  $s \geq r$ .

Now we can deduce a variant of the vanishing result 0.5 attributed to P. Scholze and H. Esnault.

**Theorem 9.11** (The General Case). *Let  $\mathbb{C}_1, \dots, \mathbb{C}_n$  be type one pro- $\ell$  Galois towers of curves over  $k$ , and let  $f : \mathbb{X} \rightarrow \mathbb{C}_1 \times_k \dots \times_k \mathbb{C}_n$  be a quasi-finite separated morphism over  $k$ . Then  $\mathbb{X}$  possesses **SEV**.*

*Proof:* Let  $\mathcal{F}$  be an  $\ell$ -primary torsion étale sheaf on the base  $X$  of  $\mathbb{X}$ , let  $i > \dim(X)$ , and let  $\mathbb{C}$  denote the product  $\mathbb{C}_1 \times_k \dots \times_k \mathbb{C}_n$ . Without loss of generality it suffices to prove that the morphism

$$H_c^i(X_0, \mathcal{F}) \rightarrow H_c^i(X_r, ({}_{\mathbb{X}}\phi_0^r)^* \mathcal{F})$$

induced by the restriction vanishes for sufficiently large values of  $r$ . The morphism  $f$  factors as

$$\mathbb{X} \xrightarrow{g} \mathbb{X}' \xrightarrow{h} \mathbb{C},$$

where  $h$  is a cartesian quasi-finite separated morphism over  $k$ , and  $g$  is the identity on the base. Employing the cancelation property of finite morphisms, we derive that the above morphism factors as

$$H_c^i(X_0, \mathcal{F}) \rightarrow H_c^i(X'_r, ({}_{\mathbb{X}'}\phi_0^r)^* \mathcal{F}) \rightarrow H_c^i(X_r, ({}_{\mathbb{X}}\phi_0^r)^* \mathcal{F})$$

and we can assume without loss of generality that  $f$  is cartesian. Applying Corollary 8.9 and [29, Thm. 5.8.8], we can assume without loss of generality that  $\mathcal{F}$  is constructible. According to Lemma 9.6 and Lemma 9.7, we can assume without loss of generality that  $X$  is integral.

We prove the assertion by an ascending induction on the dimension of  $X$ . If  $X$  is zero-dimensional, then  $X$  is finite discrete and the assertion of the theorem is immanent by Theorem 8.12 and 9.8.

So we assume that the dimension of  $X$  is greater than 0 and that the assertion of the theorem holds true for all quasi-finite separated morphisms  $f : \mathbb{Y} \rightarrow \mathbb{C}$  of towers of finite type over  $k$ , where the dimension of  $Y$  is smaller than the dimension of  $X$ . Fix some  $i > \dim(X)$ . Let

$$\rho : C_1 \times_k \dots \times_k C_n \rightarrow C_1$$

denote the projection to the first factor, where  $C_j$  denotes the base of  $\mathbb{C}_j$ , and let  $\psi$  denote the composition  $\rho \circ f_0$ . Applying 9.8 and [93, Tag 01JT], we can assume without loss of generality that  $\psi(X)$  is not a closed point. The Leray spectral sequence with compact support 8.16 and Proposition 9.9 yield a short exact sequence

$$0 \rightarrow H_c^1(\mathbb{C}_1, \mathcal{R}^{i-1}\psi_!\mathcal{F}) \rightarrow H_c^i(\mathbb{X}, \mathcal{F}) \rightarrow H_c^0(\mathbb{C}_1, \mathcal{R}^i\psi_!\mathcal{F}) \rightarrow 0.$$

Thus, it remains to show that both  $H_c^1(\mathbb{C}_1, \mathcal{R}^{i-1}\psi_!\mathcal{F})$  and  $H_c^0(\mathbb{C}_1, \mathcal{R}^i\psi_!\mathcal{F})$  vanish. Alluding to Lemma 8.5, it suffices to prove without loss of generality that the morphism

$$\mathcal{R}^j\psi_!\mathcal{F} \rightarrow ({}_{C_1}\phi_0^r)_* ({}_{C_1}\phi_0^r)^* \mathcal{R}^j\psi_!\mathcal{F}$$

vanishes for  $j = i - 1, i$  and all sufficiently large  $r$ . Let  $j \in \{i - 1, i\}$ . The sheaf  $\mathcal{R}^j\psi_!\mathcal{F}$  is constructible by [18, (finitude) Thm. 1.1]. Subsequently, there exists an open subscheme  $U$  of  $C_1$  such that  $\mathcal{R}^j\psi_!\mathcal{F}|_U$  is finite locally constant. The complement of the image of  $U$  in  $C_1$  consists solely of finitely many closed points  $x_1, \dots, x_m$ . Let  $\{U_w\}_{w=m+1, \dots, k}$  be a finite étale covering of  $U$  such that  $\mathcal{F}|_{U_w}$  is finite constant for all  $w = m + 1, \dots, k$ . Then  $U_w$  contains a closed point  $x_w$  for every  $w = m + 1, \dots, k$ . Let  $x$  denote any of the points  $x_1, \dots, x_m, x_{m+1}, \dots, x_k$ . We denote the transition morphisms from level  $s$  to level  $t$  of the induced tower over  $\psi^{-1}(x)$  by  $\phi_s^t$ . Lemma 8.10 and [29, Thm. 7.4.4] yield an isomorphism

$$(\mathcal{R}^j\psi_!\mathcal{F})_x \cong H^j(\psi^{-1}(x), \mathcal{F}|_{\psi^{-1}(x)})$$

such that for every  $r$  the canonical map

$$(\mathcal{R}^j\psi_!\mathcal{F})_x \rightarrow ({}_{C_1}\phi_0^r)_* ({}_{C_1}\phi_0^r)^* (\mathcal{R}^j\psi_!\mathcal{F})_x$$

identifies with the canonical map

$$H^j(\psi_0^{-1}(x), \mathcal{F}|_{\psi^{-1}(x)}) \rightarrow H^j(\psi_r^{-1}(x), (\phi_0^r)^* \mathcal{F}|_{\psi^{-1}(x)}).$$

Then the geometric fibre  $\psi^{-1}(x)$  is a proper closed subset of  $X$  and subsequently the dimension of the fiber  $\psi^{-1}(x)$  is smaller than the dimension of  $X$ . The induction assumption implies that there exists a natural number  $N_w$  for every index  $w = 1, \dots, k$  such that

$$H^j(\psi_0^{-1}(x_w), \mathcal{F}|_{\psi_r^{-1}(x_w)}) \rightarrow H^j(\psi_r^{-1}(x_w), (\phi_0^r)^* \mathcal{F}|_{\psi^{-1}(x_w)})$$

is the zero map for every index  $r$  exceeding  $N_w$ . Put  $N := \max(N_0, \dots, N_k)$ . Then the map

$$\mathcal{R}^j \psi_! \mathcal{F} \rightarrow (C_1 \phi_0^r)_* (C_1 \phi_0^r)^* \mathcal{R}^j \psi_! \mathcal{F}$$

vanishes for all  $r > N$ , whence the induction step. ■

**Corollary 9.12.** *Let  $\mathbb{C}_1, \dots, \mathbb{C}_n$  be type one pro- $\ell$  Galois towers of curves. Let  $f : \mathbb{X} \rightarrow \mathbb{C}_1 \times_k \dots \times_k \mathbb{C}_n$  be a quasi-finite separated morphism of towers over  $k$ . Then  $H_c^i(\mathbb{X}, K)$  vanishes for every  $K$  in  ${}^p\mathcal{D}_c^{\leq 0}(X_{\text{ét}}, \mathbb{F}_{\ell^v})$  and all  $i > 0$ .*

*Proof:* We prove the corollary by an ascending induction on the dimension of  $X$ . We can prove, analogously to Lemma 9.7 that we can assume without loss of generality that  $X$  is irreducible. If  $X$  is zero-dimensional, then the underlying topological space of  $X$  consists of a point and the induction start follows from [29, Prop. 6.3.5] and Theorem 9.11.

So let  $X$  be at least one-dimensional. By [109, Ch. III.2]<sup>2</sup> there exists an open dense essentially smooth subscheme  $j : U \hookrightarrow X$ , such that the restriction of  $K$  to  $U$  is smooth. Let  $i : Z \rightarrow X$  denote the complementary closed subset endowed with the reduced induced scheme structure. Then  $H_c^i(\mathbb{U}, j^*(K))$  vanishes for all  $i > 0$  by [29, Prop. 6.3.5] and Theorem 9.11. Furthermore,  $H_c^i(\mathbb{Z}, i^*(K))$  vanishes for all  $i > 0$  by the induction assumption and Theorem 8.12 implies that  $H_c^i(\mathbb{X}, K)$  vanishes for all  $i > 0$ . ■

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<sup>2</sup>The formulation in [109, III.2] is with  $\overline{\mathbb{Q}}_\ell$ -coefficients. However,  $\mathbb{F}_{\ell^v}$ -coefficients work equally well. See [10]. Alternatively, one may also prove the corollary directly by using the standard truncations of  $K$  and applying the associated spectral sequence.



# Chapter 10

## Fourier-Mellin Transform

In this chapter, we will study the (*generalized*) *Fourier-Mellin transform* for pro- $\ell$  towers, as introduced by M. Kerz and H. Esnault in [25]. For tori the Fourier-Mellin transform is called Mellin transform in [31, Prop. 3.1.3] and for complex abelian varieties a version of it was studied in [12].

If not said otherwise, let  $k$  denote an algebraically closed field, let  $\ell$  be a prime number different from the characteristic of  $k$ , and let  $\ell^v > 1$  denote a power of  $\ell$ .

First we need to recall some standard facts.

**Group Algebras 10.1.** Let  $R$  denote an arbitrary ring. The *group algebra* of a group  $G$  over  $R$  is the associative algebra whose elements are formal linear combinations over  $R$  of the elements of  $G$ . The addition of algebra elements is given by the componentwise addition of coefficients and the multiplication is given by the convolution product. There is an adjunction

$$(R[-] \dashv (-)^\times) : \mathbf{R - Alg} \rightleftarrows \mathbf{Grp}$$

between the category of associative algebras over  $R$  and that of groups, where  $R[-]$  forms the group algebra over  $R$  and where  $(-)^\times$  assigns to an  $R$ -algebra its group of units. Given finite groups  $H$  and  $G$ , one has a canonical isomorphism  $R[G] \otimes_R R[H] \cong R[G \oplus H]$ .

We call an  $R$ -module  $M$  a  $G$ -*module* if it is endowed with a  $G$ -action that is compatible with the group structure of  $M$ . We recall that we have a category equivalence

$$(R[G] - \text{modules}) \longleftrightarrow (G - \text{modules}).$$

In the virtue of the category equivalence, one has restriction and extension of scalars functors, which we denote by  $\text{Res}_H^G$  and  $\text{Ind}_H^G$  for every subgroup  $H$  of  $G$ .

**Example 10.2.** Let  $\ell^v$  be a power of  $\ell$ . We recall that we have isomorphisms

$$\mathbb{F}_{\ell^v}[T]/(T^{\ell^n}) \cong \mathbb{F}_{\ell^v}[T]/(T^{\ell^n} - 1) \cong \mathbb{F}_{\ell^v}[\mathbb{Z}/\ell^n\mathbb{Z}]$$

for every natural number  $n$ , as can be proven by the Yoneda lemma: Let  $A$  be an  $\mathbb{F}_{\ell^v}$ -algebra. There are natural bijections

$$\mathrm{Hom}_{\mathbb{F}_{\ell^v}\text{-Alg}}(\mathbb{F}_{\ell^v}[T]/(T^{\ell^n}), A) \cong \{ a \in A \mid a^{\ell^n} = 0 \}$$

and

$$\mathrm{Hom}_{\mathbb{F}_{\ell^v}\text{-Alg}}(\mathbb{F}_{\ell^v}[T]/(T^{\ell^n} - 1), A) \cong \{ a \in A \mid a^{\ell^n} = 1 \}.$$

By the universal property of the group ring there is a natural bijection

$$\mathrm{Hom}_{\mathbb{F}_{\ell^v}\text{-Alg}}(\mathbb{F}_{\ell^v}[\mathbb{Z}/\ell^n\mathbb{Z}], A) \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}/\ell^n\mathbb{Z}, A^\times)$$

given by restriction along the canonical injection  $\mathbb{Z}/\ell^n\mathbb{Z} \rightarrow \mathbb{F}_{\ell^v}[\mathbb{Z}/\ell^n\mathbb{Z}]$  and we have a canonical isomorphism

$$\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}/\ell^n\mathbb{Z}, A^\times) \cong \{ a \in A \mid a^{\ell^n} = 1 \}.$$

Finally,

$$\begin{aligned} \{ a \in A \mid a^{\ell^n} = 1 \} &\rightarrow \{ a \in A \mid a^{\ell^n} = 0 \} \\ a &\mapsto a - 1 \end{aligned}$$

is a natural bijection, since  $A$  has characteristic  $\ell$ . The Yoneda lemma implies the stated isomorphisms. Explicitly, the isomorphism

$$\mathbb{F}_{\ell^v}[\mathbb{Z}/\ell^n\mathbb{Z}] \cong \mathbb{F}_{\ell^v}[T]/(T^{\ell^n})$$

gets induced by mapping 1 in  $\mathbb{Z}/\ell^n\mathbb{Z}$  to  $\bar{T}$  in  $\mathbb{F}_{\ell^v}[T]/(T^{\ell^n})$ .

**Lemma 10.3.** *Let  $R$  be a commutative ring, let  $G$  be a group, and let  $H$  be a normal subgroup. Then for every  $G$ -module  $M$  one has*

$$\mathrm{Ind}_H^G \mathrm{Res}_H^G M \cong R[G/H] \otimes_R M.$$

*Proof:* This is principally due to the fact that  $G$  is the disjoint union of the cosets of  $H$  in  $G$ ; see [14, III.5]. In [14] the case  $R = \mathbb{Z}$  is covered. The arguments, however, extend to the general case. ■

**Étale Fundamental Group** [45, V 1.]. Let  $X$  be a connected scheme, and let  $\bar{x}$  be a geometric point of  $X$ . The fundamental group

$$\pi_1(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$$

of  $X$  with base point  $\bar{x}$  is the automorphism group of the fibre functor  $F_{\bar{x}} : \mathbf{F\acute{E}t}_X \rightarrow \mathbf{Sets}$  endowed with its canonical profinite topology. For every

finite étale  $X$ -scheme  $Y$ , the  $X$ -automorphism group  $\text{Aut}_X(Y)$  acts on the fibre  $F_{\bar{x}}(Y)$  and if  $Y$  is connected this action is faithful by [64, Cor. 3.13]. If  $Y$  is connected and  $\text{Aut}_X(Y)$  acts transitively on the fibre  $F_{\bar{x}}(Y)$ , equivalently if the above injection is an isomorphism  $\text{Aut}_X(Y) \cong F_{\bar{x}}(Y)$ , then  $Y$  is a connected Galois covering over  $X$ . By [62, Thm. 3.5, 3.15] we have an isomorphism

$$\pi_1(X, \bar{x}) = \lim_{Gal} \text{Aut}_{\mathbf{F}\acute{\text{E}}\mathbf{t}_X}(Y, \bar{y})$$

where  $Gal$  is the collection of all connected Galois coverings  $Y$  over  $X$  mapping  $\bar{y}$  to  $\bar{x}$ . The isomorphism is canonically determined up to an inner automorphism of  $\text{Aut}(F_{\bar{x}})$ .

The fibre functor defines an equivalence of categories

$$F_{\bar{x}} : \mathbf{F}\acute{\text{E}}\mathbf{t}_X \rightarrow \text{Finite} - \pi_1(X, \bar{x}) - \text{Sets}$$

and under this equivalence the connected Galois coverings of  $(X, \bar{x})$  identify with the finite  $\pi_1(X, \bar{x})$ -Sets of the form  $G = \pi_1(X, \bar{x})/N$ , where  $N$  is an open normal subgroup. Equivalently, if  $G$  is a finite group and  $\pi_1(X, \bar{x}) \rightarrow G$  is a continuous surjection, then  $G$  viewed as a  $\pi_1(X, \bar{x})$ -set corresponds to a Galois covering; see [62, Construction 3.18] and [93, Tag 03SF]. Then, by [93, Tag 03RV], a sheaf of sets on  $X_{\acute{e}t}$  is finite locally constant if and only if it can be represented by a finite étale morphism and the Yoneda lemma implies the category equivalence

$$\left( \begin{array}{c} \text{finite locally constant} \\ \text{sheaves of sets on } X_{\acute{e}t} \end{array} \right) \longleftrightarrow (\text{finite } \pi_1(X, \bar{x}) - \text{sets})$$

where a sheaf  $\mathcal{L}$  on the left gets identified with the stalk  $\mathcal{L}_{\bar{x}}$  on the right. Pull-backs and pushforwards along finite étale morphisms identify with restriction and extension of scalars, respectively; see also [93, Tag 095B]. Endowing the underlying (sheaves of) sets with additional structure, we derive the category equivalences

$$\left( \begin{array}{c} \text{finite locally constant} \\ \text{sheaves of abelian groups on } X_{\acute{e}t} \end{array} \right) \longleftrightarrow (\text{finite } \pi_1(X, \bar{x}) - \text{modules})$$

and for every finite ring  $\Lambda$  we have the category equivalences

$$\left( \begin{array}{c} \text{finite locally constant} \\ \text{sheaves of } \Lambda\text{-modules on } X_{\acute{e}t} \end{array} \right) \longleftrightarrow \left( \begin{array}{c} \text{finite } \pi_1(X, \bar{x}) - \text{modules} \\ \text{endowed with a commuting} \\ \Lambda - \text{module structure} \end{array} \right).$$

Assume that  $(X, \bar{x})$  is a connected pointed scheme over an algebraic separably closed field  $k$ . Let  $\mathcal{F}$  be a locally constant étale sheaf. Then  $\mathcal{F}$  is a constant

sheaf with stalk  $M$  at  $\bar{x}$  if and only if  $\mathcal{F}$  is equal to the pullback of  $\underline{M}$  on  $\text{Spec}(k)$  to  $X$ . Since  $\pi_1(\text{Spec}(K), \text{Spec}(K))$  is given by the trivial absolute Galois group of  $k$ , this is the same as to say that the action of  $\pi_1(X, \bar{x})$  on  $\mathcal{F}_{\bar{x}}$  is trivial.

We recall the following standard

**Lemma 10.4.** *Let  $f : Y \rightarrow X$  be a connected Galois covering with Galois group  $G$ , mapping a geometric point  $\bar{y}$  of  $Y$  to a geometric point  $\bar{x}$  of  $X$ . Then  $\pi_1(Y, \bar{y})$  is a normal open subgroup of  $\pi_1(X, \bar{x})$  and we have an exact sequence*

$$1 \rightarrow \pi_1(Y, \bar{y}) \xrightarrow{\pi_1(f)} \pi_1(X, \bar{x}) \rightarrow G \rightarrow 1.$$

*Proof:* Let  $Z_1$  be a connected Galois covering over  $X$  with Galois group  $G_1$ , let  $Z_2'$  be a connected Galois covering over  $Y$ , and let  $Z_2$  be a connected Galois covering over  $X$  dominating the finite étale morphism  $Z_2' \rightarrow Y \rightarrow X$ . According to [93, Tag 03SF], we have continuous surjections  $\pi_1(X, \bar{x}) \rightarrow G_1$  and  $\pi_1(X, \bar{x}) \rightarrow G_2$ . Let  $G$  denote the image of  $\pi_1(X, \bar{x})$  under the corresponding continuous homomorphisms  $\pi_1(X, \bar{x}) \rightarrow G_1 \times G_2$ . Then  $G$  corresponds to a connected Galois covering  $Z \rightarrow X$  via the category equivalence [93, Tag 0BND]. Then  $Z/X$  is dominating  $Z_1/X$  and  $Z/Y$  is a connected Galois covering dominating  $Z_2'/Y$ . Then  $\text{Aut}_Y(Z)$  is a normal subgroup of  $\text{Aut}_X(Z)$ , and we have an exact sequence

$$0 \rightarrow \text{Aut}_Y(Z) \rightarrow \text{Aut}_X(Z) \rightarrow G.$$

Then finally, the universal property of the limit and the fact that taking limits is left exact implies the result.  $\blacksquare$

**Remark 10.5.** In the setting of Lemma 10.4 assume that the morphism  $Y \rightarrow X$  factors as  $Y \rightarrow Z \rightarrow X$  where  $Z \rightarrow X$  is a connected Galois covering, and let  $\bar{z}$  be a geometric point of  $Z$  that gets mapped to  $\bar{y}$ . Then  $Y/Z$  is a connected Galois covering with Galois group  $H \trianglelefteq G$  and the Galois group of  $Z/X$  is isomorphic to  $G/H$ . Then Lemma 10.4 gives rise to the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(Y, \bar{y}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(Z, \bar{z}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & G/H \longrightarrow 1 \end{array}$$

In the following, let  $\mathbb{X}$  be a pro- $\ell$  Galois tower over  $k$ . For every pair of indices  $r \leq s$ , we denote the transition morphism  $X_s \rightarrow X_r$  by  $\phi_r^s$  and we put  $G_i := \text{Gal}(X_i/X)$ . We let  $G$  denote  $\lim_i G_i$ , and we set  $\mathbb{F}_{\ell^v}[G] := \lim_i \mathbb{F}_{\ell^v}[G_i]$ . Example 10.2 implies

$$\mathbb{F}_{\ell^v}[G] \cong \mathbb{F}_{\ell}[[T_1, \dots, T_n]] \quad \text{and} \quad \mathbb{F}_{\ell^v}[G_i] \cong \mathbb{F}_{\ell^v}[[T_1, \dots, T_n]]/(T_1^{\ell^i}, \dots, T_n^{\ell^i}).$$

**Definition 10.6.** For every natural number  $r$  we let  $\mathcal{L}_r$  denote  $\phi_{0*}^r \underline{\mathbb{F}_{\ell^v}}$  in  $\mathcal{D}_c^b(X_{\acute{e}t}, \mathbb{F}_{\ell^v})$ .

**Lemma 10.7.** For all pairs of natural numbers  $r \leq s$  we find

$$\mathcal{L}_s \otimes_{\mathbb{F}_{\ell^v}[G]} \underline{\mathbb{F}_{\ell^v}[G_r]} \cong \mathcal{L}_r$$

and

$$\mathcal{L}_s \otimes_{\mathbb{F}_{\ell^v}[G_s]} \underline{\mathbb{F}_{\ell^v}[G_r]} \cong \mathcal{L}_r.$$

*Proof:* Let  $\bar{x}$  denote a geometric point of  $X$ . We deduce from Lemma 10.3, Lemma 10.4 and Remark 10.5 that the locally constant sheaf  $\mathcal{L}_t$  corresponds to the finite  $\pi_1(X, \bar{x})$ -module  $\mathbb{F}_{\ell^v}[G_t]$  with its canonical action for every index  $t$ . Let  $r \leq s$ . Then the canonical morphism

$$\mathbb{F}_{\ell^v}[\pi_1(X, \bar{x})] \rightarrow \mathbb{F}_{\ell^v}[G_r]$$

factors as

$$\mathbb{F}_{\ell^v}[\pi_1(X, \bar{x})] \rightarrow \mathbb{F}_{\ell^v}[G_s] \rightarrow \mathbb{F}_{\ell^v}[G_r],$$

where the morphism on the right gets induced from the canonical surjection  $G_s \rightarrow G_r$ . The constant sheaf  $\underline{\mathbb{F}_{\ell^v}[G_r]}$  corresponds to the  $\pi_1(X, \bar{x})$ -module

$$\mathbb{F}_{\ell^v}[\pi_1(X, \bar{x})] \xrightarrow{\varphi} \mathbb{F}_{\ell^v}[G_r]$$

where  $\varphi$  gets induced from the zero morphism  $\pi_1(X, \bar{x}) \rightarrow G_r$ .

Then

$$\mathbb{F}_{\ell^v}[G_r] \otimes_{\mathbb{F}_{\ell^v}[G]} \mathbb{F}_{\ell^v}[G_s] \cong \mathbb{F}_{\ell^v}[G_r],$$

where  $a \otimes b$  gets mapped to  $a \cdot \bar{b}$  and  $\bar{b}$  is the residue class of  $b$  in  $\mathbb{F}_{\ell^v}[G_r]$ . Likewise, we have an isomorphism

$$\mathbb{F}_{\ell^v}[G_s] \otimes_{\mathbb{F}_{\ell^v}[G_s]} \mathbb{F}_{\ell^v}[G_r] \cong \mathbb{F}_{\ell^v}[G_r].$$

and we deduce the Lemma. ■

**Remark 10.8.** One has  $\mathcal{H}om(\mathcal{L}_r, \mathbb{F}_{\ell^v}) \cong (-1)^* \mathcal{L}_r$  for every natural number  $r$ .

Indeed, in the proof of Lemma 10.7 we constituted that  $\mathcal{L}_r$  corresponds to the finite  $\pi_1(X, \bar{x})$ -module  $\mathbb{F}_{\ell^v}[G_r]$  with its canonical action. Then the action of  $\pi_1(X, \bar{x})$  on  $\text{Hom}(\mathbb{F}_{\ell^v}[G_r], \mathbb{F}_{\ell^v})$  is the contragredient one. Namely,  $(g \cdot \lambda)(m) = \lambda(g^{-1} \cdot m)$ . See [93, Tag 0A2H]. Then  $\text{Hom}(\mathbb{F}_{\ell^v}[G_r], \mathbb{F}_{\ell^v}) \cong \mathbb{F}_{\ell^v}[G_r]$  and under this identification  $\pi_1(X, \bar{x})$  acts on  $\mathbb{F}_{\ell^v}[G_r]$  via  $g \cdot \lambda := g^{-1} \cdot \lambda$ . Let  $-1$  be the multiplication by  $-1$  map on  $X$ . Then the induced automorphism

$$[-1]_* : \pi^1(X) \rightarrow \pi^1(X)$$

acts as  $-1$  on  $\pi^1(X)$ , whence the result.

**Lemma 10.9.** *Let  $X$  be a scheme that is separated and of finite type over  $k$ . Then one has*

$$H^{-i}(X, K) \cong H_c^i(X, D(K))^\vee$$

for every object  $K$  in  $\mathcal{D}_c^b(X_{\acute{e}t}, \mathbb{F}_{\ell^v})$  and every index  $i$ .

*Proof:* Let  $f : X \rightarrow k$  denote the structure morphism of  $X$ . Then [93, Tag 03PC] implies that  $\underline{\mathbb{F}}_{\ell^v}$  is injective on  $k$  and Poincaré duality implies

$$\mathcal{R}f_* D_X(K) \cong D_{\text{Spec}(k)}(\mathcal{R}f_!(K)) = \mathcal{R}\mathcal{H}om(\mathcal{R}f_!(K), \underline{\mathbb{F}}_{\ell^v}).$$

Taking global sections on  $\text{Spec}(k)$  is exact, whence the Lemma. ■

Let  $r$  denote an arbitrary index, let  $f : X \rightarrow k$  denote the structure morphism of  $X$ , let  $g_r$  denote  $f \circ \phi_0^r$ , and let  $M$  be an object of  $\mathcal{D}_c^b(X, \mathbb{F}_{\ell^v})$ . Then making use of [109, Thm. D.1], we can apply the projection formula [28, Ch. I Prop. 8.14] in order to deduce

$$\begin{aligned} & \mathcal{R}f_*(M \otimes (\phi_0^r)_* \underline{\mathbb{F}}_{\ell^v}) \\ & \cong \mathcal{R}g_{r*}((\phi_0^r)^* M \otimes \underline{\mathbb{F}}_{\ell^v}) \\ & \cong \mathcal{R}g_{r*}((\phi_0^r)^* M \otimes (g_r)^* \underline{\mathbb{F}}_{\ell^v}) \\ & \cong \mathcal{R}g_{r*}((\phi_0^r)^* M \otimes \underline{\mathbb{F}}_{\ell^v}). \end{aligned}$$

Thus,

$$\mathcal{R}\Gamma(X_0, M \otimes \mathcal{L}_r) \cong \mathcal{R}\Gamma(X_r, \phi_0^{r*} M)$$

and by the very construction of the projection formula, see [28, Ch. I Prop. 8.14], varying  $r$ , we derive an isomorphism of inverse systems, where the transition morphisms are induced by the trace.

**Definition 10.10.** *Let  $M$  be an object of  $\mathcal{D}_c^b(X, \mathbb{F}_{\ell^v})$ . We define the (general) Fourier-Mellin transform of  $M$  to be*

$$\text{FM}_{\mathbb{X}}(M) := \mathcal{R}\lim_r \mathcal{R}\Gamma(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_r).$$

**Proposition 10.11** (Basic Properties). *For every object  $M$  in  $\mathcal{D}_c^b(X, \mathbb{F}_{\ell^v})$  and every natural number  $r$  we find*

- i.) *Finiteness:*  $\text{FM}_{\mathbb{X}}(M)$  is an element of  $\mathcal{D}_{\text{per}}^b(\mathbb{F}_{\ell^v}[G])$ ;
- ii.)  $\text{FM}_{\mathbb{X}}(M) \otimes_{\mathbb{F}_{\ell^v}[G]}^L \mathbb{F}_{\ell^v}[G_r] \cong \mathcal{R}\Gamma(X_0, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_r)$ ;
- iii.) *Limit Property:*  $H^j(\text{FM}_{\mathbb{X}}(M)) \cong \lim_i H^j(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)$ ;

iv.) *Duality:*  $H^{-i}(\mathrm{FM}_{\mathbb{X}}(M)) \cong H_c^i(\mathbb{X}, D(M))^\vee$ .

*Proof:* First we note that  $M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i$  is an element of  $\mathcal{D}_c^-(X, \mathbb{F}_{\ell^v})$  by [93, Tag 0961]. In fact, making use of the tor base change spectral sequence<sup>1</sup>, we derive that there are integers  $a, b$  such that  $M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i$  is an element of  $\mathcal{D}_c^{[a,b]}(X, \mathbb{F}_{\ell^v})$  for every index  $i$ . Let  $f : X \rightarrow k$  denote the structure morphism of  $X$ . Then the Leray spectral sequence 8.15

$$E_2^{p,q} = H^p(k, \mathcal{R}^q f_*(M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)) \Rightarrow H^{p+q}(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)$$

yields an isomorphism

$$H^0(k, \mathcal{R}^q f_*(M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)) \cong H^q(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i).$$

Let  $\tilde{p}, \tilde{q}$  be two indices. Alluding to [18, (finitude) Thm. 1.1], all of the sheaves  $\mathcal{R}^{\tilde{q}} f_*(H^{\tilde{p}}(M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i))$  are finite constant. Applying [29, Prop. 6.3.5], we deduce that the modules

$$H^q(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)$$

are finitely generated  $\mathbb{F}_{\ell^v}$ -modules. Additionally employing [109, Thm. D.1], we derive that

$$\mathcal{R}\Gamma(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)$$

is bounded. Hence, [93, Tag 066U] implies that  $\mathcal{R}\Gamma(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_i)$  is a perfect complex. Employing [93, Tag 0F10], we deduce that  $X$  has finite cohomological dimension and we can apply [93, Tag 0F12] in order to conclude

$$\mathcal{R}\Gamma(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_r) \cong \mathcal{R}\Gamma(X, M \otimes_{\mathbb{F}_{\ell^v}}^L \mathcal{L}_s) \otimes_{\mathbb{F}_{\ell^v}[G_s]}^L \mathbb{F}_{\ell^v}[G_r]$$

from Lemma 10.7. Then  $\mathrm{FM}_{\mathbb{X}}(M)$  is a perfect object of  $\mathcal{D}(\mathbb{F}_{\ell^v}[G])$  alluding to [11, Lem. 4.2]. Further, [11, Lem. 4.2] implies ii.). Assertion iii.) follows from the above and [93, Tag 0CQE, Tag 091D]. In particular, we derive boundedness in i.). Utilizing Lemma 8.11, Lemma 10.9 and iii.), we derive

$$\begin{aligned} & H_c^i(\mathbb{X}, D(M))^\vee \\ & \cong (\mathrm{colim}_r H_c^i(X_r, \phi_0^{r*} D(M)))^\vee \\ & \cong \lim_r (H_c^i(X_r, D(\phi_0^{r*} M))^\vee) \\ & \cong \lim_r H^{-i}(X_r, \phi_0^{r*} M) \\ & \cong H^{-i}(\mathrm{FM}_{\mathbb{X}}(M)), \end{aligned}$$

whence iv.). ■

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<sup>1</sup>Use the analogue of [93, Tag 0662] for  $\mathcal{D}_c^-(X, \mathbb{F}_{\ell^v})$ . This can be derived from the standard spectral sequences for the total complex.



# Chapter 11

## Betti Numbers of Semi-Perverse Sheaves Revisited

In this chapter, we finally prove Theorem 0.4. Let  $k$  be an algebraically closed field, let  $\ell$  be a prime number different from the characteristic of  $k$ , and let  $\ell^v > 1$  be a power of  $\ell$ . Let  $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_s$  be type one pro- $\ell$  Galois towers of curves over  $k$ , and let  $\mathbb{X}$  be a pro- $\ell$  Galois tower. Let  $\mathbb{C}$  denote  $\mathbb{C}_1 \times_k \mathbb{C}_2 \times_k \dots \times_k \mathbb{C}_s$ , and let  $f : \mathbb{X} \rightarrow \mathbb{C}$  be a quasi-finite separated morphism over  $k$ .

**Theorem 11.1.** *Let  $R$  denote any of the rings  $\mathbb{F}_{\ell^v}, \mathbb{Z}_{\ell}, \mathbb{Q}_{\ell}, \overline{\mathbb{Q}}_{\ell}$ . For every object  $K$  of  ${}^p\mathcal{D}_c^{\geq 0}(X_{et}, R)$  one has*

$$\text{rank}_R(H_c^i(X_n, (\mathbb{X}\phi_0^n)^* K)) \in \begin{cases} \mathcal{O}(\ell^{ns}) & \text{for all } i \in \mathbb{Z}; \\ \mathcal{O}(\ell^{n(s-|i|)}) & \text{for all } -s < i < 0, \end{cases}$$

and  $H^i(X_n, (\mathbb{X}\phi_0^n)^* K)$  vanishes for all  $n$  and all  $i \leq -s$ . Dually, for every object  $K$  of  ${}^p\mathcal{D}_c^{\leq 0}(X_{\acute{e}t}, R)$  one has

$$\text{rank}_R(H^i(X_n, (\mathbb{X}\phi_0^n)^* K)) \in \begin{cases} \mathcal{O}(\ell^{ns}) & \text{for all } i \in \mathbb{Z}; \\ \mathcal{O}(\ell^{n(s-i)}) & \text{for all } s > i > 0, \end{cases}$$

and  $H^i(X_n, (\mathbb{X}\phi_0^n)^* K)$  vanishes for all  $n$  and all  $i \geq s$ .

*Proof:* According to Lemma 10.9, it suffices to prove the first part of the theorem. In order to prove the theorem for  $R = \overline{\mathbb{Q}}_{\ell}$ , it suffices to prove it for  $R = \mathbb{Q}_{\ell}$  and we can immediately reduce this case to the case  $R = \mathbb{Z}_{\ell}$ . For any scheme  $Y = X_n$  with  $n$  in  $\mathbb{N}$  and any object  $K'$  of  ${}^p\mathcal{D}_c^{\geq 0}(Y_{et}, \mathbb{Z}_{\ell})$  the Tor base change spectral gives rise to the short exact sequence

$$0 \rightarrow H^i(K') \otimes_{\mathbb{Z}_{\ell}} \mathbb{Z}/\ell\mathbb{Z} \rightarrow H^i(K' \otimes_{\mathbb{Z}_{\ell}}^L \mathbb{Z}/\ell\mathbb{Z}) \rightarrow \text{Tor}_1(H^{i+1}(K'), \mathbb{Z}/\ell\mathbb{Z}) \rightarrow 0$$

and  $K' \otimes_{\mathbb{Z}_{\ell}}^L \mathbb{Z}/\ell\mathbb{Z}$  is an element of  ${}^p\mathcal{D}_c^{\geq 0}(Y_{\acute{e}t}, \mathbb{F}_{\ell})$  due to [10, 2.2.16] and the problem reduces to the case  $R = \mathbb{F}_{\ell^v}$ .

Let  $K$  be an object of  ${}^p\mathcal{D}_c^{\geq 0}(X_{et}, \mathbb{F}_{\ell^v})$ . Then

$$H^{-i}(\mathrm{FM}_{\mathbb{X}}(D(M))) \cong H_c^i(\mathbb{X}, M)^\vee$$

by Proposition 10.11 and  $D(K)$  is an element of  ${}^p\mathcal{D}_c^{\leq 0}(X_{et}, \mathbb{F}_{\ell^v})$ . Then Example 10.2, Proposition 10.11 and Corollary 9.12 imply that  $\mathrm{FM}_{\mathbb{X}}(D(M))$  can be treated as a bounded object of  $\mathcal{D}_{perf}^{\leq 0}(\mathbb{F}_{\ell^v}[[X_1, \dots, X_s]])$ . Proposition 10.11 and Lemma 10.9 imply

$$\begin{aligned} & H_i(\underline{T}^{\ell n}; \mathrm{FM}_{\mathbb{X}}(D(M))) \\ &= H^{-i}(\mathrm{FM}_{\mathbb{X}}(D(M)) \otimes_{\mathbb{F}_{\ell^v}[\underline{T}]}^L \mathbb{F}_{\ell^v}[\underline{T}] / (\underline{T}^{\ell n})) \\ &\cong H^{-i}(X_n, (\mathbb{X}\phi_0^n)^* D(M)) \\ &\cong H_c^i(X_n, (\mathbb{X}\phi_0^n)^* M)^\vee. \end{aligned}$$

The theorem follows from Lemma 6.15 and Proposition 6.14. ■

We demonstrated in Part I that we can not expect polynomial-likeness in the context of Theorem 11.1. In Theorem 11.1 we can also work with separated variables and then we can apply a dual version of Theorem 6.11 in order to obtain optimal polynomial lower estimates. Whether these polynomials give rise to interesting numerical invariant with a geometric interpretation remains an open question ...

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