Geometric and Arithmetic Volumes on Quasi-Projective Varieties

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Chapter 1

Introduction

1.1 Brief History of Arakelov geometry

Intersection theory is one of the classical tools studied in algebraic geometry. Given a projective variety X over a field k of dimension d, there exists a symmetric, multilinear pairing $\mathrm{Div}(X)^d \to \mathbb{R}$ where $\mathrm{Div}(X)$ denotes the group of Cartier divisors on X. Since the the associated divisor of rational function has degree 0, the intersection pairing factors modulo the class of a divisor modulo rational equivalence to give a pairing $\mathrm{Pic}(X)^d \to \mathbb{R}$ where $\mathrm{Pic}(X)$ denotes the Picard group of isomorphism classes of line-bundles on X.

It was pointed out by Arakelov that in Diophantine geometry, the theory of heights can be developed using an extension of intersection theory in the arithmetic setting. More precisely, we consider $\mathcal{X}$ to be a flat, integral, projective scheme over $\mathrm{Spec}(\mathbb{Z})$ with non-singular generic fiber X to be the analogue projective varieties from algebraic geometry. Moreover the analogue of divisors are tuples $(\mathcal{D}, g)$ where $\mathcal{D} \in \mathrm{Div}(\mathcal{X})$ and g to be a *Green function* of the divisor $\mathcal{D}$. A Green function g of a Cartier divisor $\mathcal{D}$ is a continuous function $g\colon \mathcal{X}(\mathbb{C})\backslash\mathcal{D}(\mathbb{C}) \to \mathbb{R}$ such that for any Zariski open subset U of $\mathcal{X}$ such that $\mathcal{D}|_U = \mathrm{div}(f)$ for a rational function f, the function $g + \log|f|$ extends to a continuous function $\tilde{g}$ on $U(\mathbb{C})$. Such a tuple $(\mathcal{D}, g)$ corresponds to a tuple $(O_X(D), \|\ \|)$ where $O_X(D)$ is the line bundle induces on the geenric fiber by $\mathcal{D}$ and $||\cdot||$ is *continuous metric* which is a data of continuously varying norms on the one dimensional fibers $O_X(D) \otimes_{O_{X,x}} \kappa(x)$ parametrised by the points x of the complex manifold $\mathcal{X}(\mathbb{C})$. Indeed suppose the Cartier divisor $D = \mathrm{div}(f)|_U$ locally on some Zariski open subset U and a rational function f. Then $s_i := \frac{1}{f}$ is a local frame of $O_X(D)$ on U and we set the metric of s_i to be $\|s_i(x)\| := \frac{1}{|f(x)|} \cdot e^{-g(x)}$. This gives a continuous function on U since $g + \log|f|$ extends to a continuous function on U and hence induces a metric on $O_X(D)$. We denote the tuples $(\mathcal{D}, g)$ where $\mathcal{D} \in \mathrm{Div}(\mathcal{X})$ and g is a Green function of $\mathcal{D}$, called *arithmetic divisors*, by $\widehat{\mathrm{Div}}(\mathcal{X})$ and we denote tuples $(\mathcal{L}, \|\ \|)$ where $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ and $\|\ \|$ is a continuous metric on the generic fiber L, called *metrised line bundles*, by $\widehat{\mathrm{Pic}}(\mathcal{X})$. In their seminal work [GS90], Gillet and Soule were able to construct a symmetric,multilinear *arithmetic intersection pairing* $\widehat{\mathrm{Div}}(\mathcal{X})^{d+1} \to \mathbb{R}$.

Zhang noted that the choice of a Cartier divisor $\mathcal{D} \in \mathrm{Div}(\mathcal{X})$ already induces a metric at non-Archimedean places which led to his notion of *adelic divisors*. This allows us to treat both the Archimedean and non-Archimedean places in the same footing. Suppose we have a projective variety X over K where K is either a number field with ring of integers O_K or a function field of a smooth projective curve C. An *arithmetic divisor* on X can thus be defined as tuple (D, g) where D is a Cartier divisor on X and g is a family of Green function indexed by all the places of K with a coherence condition. More precisely, we require that there is an open dense subset U of $\mathrm{Spec}(O_K)$ or C such that there is a model $\mathcal{X}$ over U with generic fiber X and the Green functions at all places $v \in U$ are induced by a model $\mathcal{D} \in \mathrm{Div}(\mathcal{X})$. In the case when K is a number field, we further require that the Green functions at the Archimedean places are smooth. Finally an *adelic divisor* can be defined as those tuples (D, g) where each Green function can be obtained as uniform limit of Green functions of arithmetic divisors. The intersection pairing of Gillet and Soule then extends to adelic divisors. We can pass from arithmetic divisors and adelic divisors to *Hermitian line bundles* and *adelic line bundles* via a process dividing by principal divisors in similarity from algebraic geometry. The notion of adelic divisor was introduced by Zhang to include the crucial example of even ample divisors on abelian varieties with their induced canonical Green functions coming from the structure of a dynamical system.

1.2 Generalisation to Quasi-Projective varieties

In their recent manuscript [YZ24], Yuan and Zhang develop a systematic generalisation of arithmetic intersection theory over quasi-projective varieties. It allows them to define arithmetic intersection numbers and develop a height theory and volume theory over quasi-projective varieties. In this thesis, we are primarily interested in the theory of volumes and how they relate to arithmetic intersection numbers under certain positivity assumptions. In view of that we give a short description of the theory of adelic line bundles, their space of sections and arithmetic (and geometric) volumes along the lines of [YZ24].

We recall the definitions of adelic divisors on quasi-projective varieties over Dedekind bases. We follow the treatment along [YZ24, Section 2.7]. We begin by considering a tuple $\overline{B} := (B, \Sigma)$ where B is either the ring of integers of a number field K, a smooth projective curve B with function field K or any field and $\Sigma \subseteq \mathrm{Hom}(K, \mathbb{C})$. Furthermore when B is a smooth projective curve or a field we require Σ to be empty. We do this to include the geometric case in our uniform terminology. By a variety over k we mean a flat, integral, separated, finite-type scheme $\mathcal{X}$ over B. Furthermore we say it is quasi-projective (projective) if $\mathcal{X}$ is quasi-projective (projective) over B. For a quasi-projective variety $\mathcal{U}$ over k, a *projective model* of $\mathcal{U}$ means a projective variety $\mathcal{X}$ over B together with an open dense immersion $\mathcal{U} \hookrightarrow \mathcal{X}$.

We want to include a slightly more general class of varieties that we want to work with. In order to do so, we recall the notion of *pro-open immersions*. A morphism between integral schemes is said to be *pro-open* if the underlying map of topological spaces is injective and it induces an isomorphism of local rings at each point (see [YZ24, Section 2.3.1]). By an *essentially quasi-projective variety* over B, we mean an integral, flat, finite-type scheme X over B together with a projective variety $\mathcal{X}$ over B such that there is a pro-open immersion $X \hookrightarrow \mathcal{X}$. By a *(quasi-) projective* model of X over B, we mean a (quasi-) projective variety $\mathcal{X}$ over B together with a pro-open immersion $X \hookrightarrow \mathcal{X}$. Suppose $\mathcal{X}$ is a projective arithmetic variety over B. Then we set $\mathcal{X}_\Sigma := \coprod_{\sigma\in\Sigma} \mathcal{X}_\sigma$ where $\mathcal{X}_\sigma$ denotes the base change of $\mathcal{X}$ to $\mathbb{C}$ via σ. By an *arithmetic divisor* over $\mathcal{X}$, we mean a tuple $(\mathcal{D}, g_\mathcal{D})$ where $\mathcal{D}$ is a divisor and $g_\mathcal{D}$ is a Green function $\mathcal{X}_\Sigma(\mathbb{C})\backslash|\mathcal{D}(\mathbb{C})| \to \mathbb{R}$ of continuous type which is invariant under the action F_∞ of complex conjugation. We denote the group of arithmetic divisors on $\mathcal{X}$ by $\widehat{\mathrm{Div}}(\mathcal{X})$. For further notions we refer to [YZ24, Section 2.7].

Given an open subscheme of $\mathcal{X}$, we denote the *objects of mixed coefficients* as

$$\widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U}) = \widehat{\mathrm{Div}}(\mathcal{X})_\mathbb{Q} \oplus_{\mathrm{Div}(\mathcal{U})_\mathbb{Q}} \mathrm{Div}(\mathcal{U})$$

Given a fixed quasi-projective arithmetic variety $\mathcal{U}$ on k, we can define the group of *model adelic divisors* as

$$\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} := \lim_{\mathcal{X}} \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$$

where we take the filtered colimit by varying $\mathcal{X}$ across all projective models of $\mathcal{U}$ and viewing the objects of mixed coefficients as a filtered system via birational pull-backs. We can extend the notions of effectivity to the group of model adelic divisors by passing to filtered colimits. By a *boundary divisor* on $\mathcal{U}$, we mean a model divisor $\overline{\mathcal{D}}_0 \in \widehat{\mathrm{Div}}(\mathcal{X}, \mathcal{U})$ such that $\mathrm{Supp}(\mathcal{D}_0) = \mathcal{X}\backslash\mathcal{U}$. Given the choice of such a boundary divisor we can endow the group of model divisors by a *boundary norm* as

$$||\cdot||_{\overline{\mathcal{D}}_0} : \widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}} \to \mathbb{R} \cup \{\infty\}$$

$$||\mathcal{D}||_{\overline{\mathcal{D}}_0} := \inf\{q \in \mathbb{Q} \mid -q\overline{\mathcal{D}}_0 \leq \overline{\mathcal{D}} \leq q\overline{\mathcal{D}}_0\}$$

We can define the group of *adelic divisors* on $\mathcal{U}$, denoted by $\widehat{\mathrm{Div}}(\mathcal{U}, \overline{B})$, as the Cauchy completion of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$ with respect to the topology induced by $||\cdot||_{\overline{\mathcal{D}}_0}$ for some boundary divisor $\overline{\mathcal{D}}_0$. Finally given an essentially quasi-projective variety U over k, we define the group of adelic divisors on U, denoted by $\widehat{\mathrm{Div}}(U, \overline{B})$ as

$$\widehat{\mathrm{Div}}(U, \overline{B}) := \lim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U}, \overline{B})$$

where we take the filtered colimit by varying $\mathcal{U}$ across all quasi-projective models of U over $\overline{B}$ and using birational pull-backs as transition maps. We further have the notions of *strongly nef*, *nef* and *integrable* adelic divisors denoted by $\widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{snef}}$, $\widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{nef}}$ and $\widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{int}}$ respectively and we refer to [YZ24, Section 2.4] for further definitions and details.
Note that the notions of effectivity of divisors on $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathrm{mod}}$ induce notions of effectivity on $\widehat{\mathrm{Div}}(\mathcal{U}, \overline{B})$ and $\widehat{\mathrm{Div}}(U, \overline{B})$ by passing to completions and filtered colimits. In other words, an adelic divisor in $\widehat{\mathrm{Div}}(\mathcal{U}, \overline{B})$ is effective if it can be represented by a Cauchy sequence of effective model divisors and over an essentially quasi-projective variety an adelic divisor is effective if it is represented by an effective adelic divisor on some quasi-projective model of it. Given $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$, Yuan and Zhang introduce the notion of *small sections* as

$$H^0(U, \overline{D}) := \{f \in k(U)^\times \mid \widehat{\mathrm{div}}(f) + \overline{D} \geq 0\}$$

where $k(U)$ denotes the function field of U. They are further able to show that these spaces are finite dimensional over the base field in the geometric case and are of finite cardinality in the arithmetic case (see [YZ24, Lemma 5.1.6])) which are expected in analogy to the classical projective case. This allows them to define the notions of geometric and arithmetic volumes $\widehat{\mathrm{vol}}(\overline{D})$ which will be our main objects of investigation in this thesis. We say that $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$ is *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$. We refer to [YZ24, Sections 5.1-5.2] for details and various properties of volumes that are obtained.

1.3 Volumes: Geometric Case

The study of volume functions is central in both algebraic and arithmetic geometry. In this section we focus on the algebro-geometric version of volumes on quasi-projective varieties defined by Yuan and Zhang and we go on to introduce Newton Okounkov bodies to study such volume functions. We furthermore introduce and study the notion of restricted volumes along a closed-subvariety. The results of this section appeared first in the article [Bis24a].
We introduce the notions of volumes and arithmetic volumes which is central to thesis. Let L be a line bundle on a projective variety X over an algebraically closed field K. We can define the *volume* of L as

$$\mathrm{vol}(L) := \limsup_{m \to \infty} \frac{\dim_K H^0(X, mL)}{m^{d+1}/d!}$$

We call a line bundle L big if $\mathrm{vol}(L) > 0$ and this quantity measures the asymptotic rate of growth of the space of global sections. In [LM09b], Lazarsfeld and Mustata associate convex geometric arguments to study volumes of line bundles. More precisely, given a big line bundle L, they construct a compact convex subset $\Delta(L) \subseteq \mathbb{R}^d$ called the Newton-Okounkov body of L. The Euclidean volume of this compact convex subset remarkably encodes the geometric volume in the sense that $\mathrm{vol}(L) = d! \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(L))$. The construction requires the choice of closed smooth point in $X(K)$ and one crucial fact about the construction in [LM09b] is that the construction makes sense for arbitary graded linear subseries of a line bundle even if the variety is not projective. Using this observation, we can associate Newton Okounkov bodies in the quasi-projective geometric setting of Yuan and Zhang. Let $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$ where $\overline{B} = (k, \emptyset)$ for some algebraically closed field k. Then as the first main result of this thesis, we show the following theorem in Chapter 2:

Theorem 1.3.1. *Suppose we have a big adelic divisor $\overline{D}$ on a normal quasi-projective variety U and suppose $\Delta(\overline{D})$ is the Okounkov body associated to $\overline{D}$. Furthermore suppose $\widehat{\text{vol}}(\overline{D})$ be the adelic volume defined in Theorem 5.2.1 of [YZ24]. Then we have*

$$\text{vol}_{\mathbb{R}^d}(\Delta(\overline{D})) = \lim_{m\to\infty} \frac{\dim_K(H^0(U, m\overline{D}))}{m^d} = \frac{1}{d!} \cdot \widehat{\text{vol}}(\overline{D})$$

In [LM09b], there is a further study of how the associated Okounkov bodies vary as we vary our divisor inside the vector space of all Cartier divisors. They are able to deduce properties of the volume such as continuity, log-concavity etc from convex geometric arguments as a corollary of this variational study. As our second main theorem of this thesis, we show that such an analogous study is possible in the quasi-projective setting. In particular we show the following theorem in Chapter 2:

Theorem 1.3.2. *Suppose $\overline{D}$ and $\overline{E}$ be adelic divisors on a normal quasi-projective variety U such that $\overline{D}$ is big. Then there exists a convex body $\Delta(U) = \Delta(U, \overline{D}, \overline{E}) \subset \mathbb{R}^{d+2}$ with the property that for any $\vec{a} = (a_1, a_2) \in \mathbb{Q}^2$ with $a_1\overline{D} + a_2\overline{E}$ big, we have*

$$\Delta(a_1\overline{D} + a_2\overline{E}) = \Delta(U) \cap (\mathbb{R}^d \times \{\vec{a}\})$$

We point out a difference that whereas in [LM09b] the global body is only depends on the variety X, in our quasi-projective analogue $\Delta(U)$ independent on the chosen directions $\overline{D}$ and $\overline{E}$. This difference is due to the fact the Neron-Severi space of a smooth projective variety is finite dimensional and can be generated by classes of ample divisors. No such analogue is know in the quasi-projective setting of Yuan and Zhang.
We continue along the lines of [LM09b] to study *restricted volumes* and *augmented base loci* of divisors. Given a divisor D on a projective variety X, we can define its *stable base locus* as

$$\text{SB}(D) := \cap_{m\in\mathbb{N}}\text{Bs}(mD)$$

where $\text{Bs}(mD)$ denotes the base locus of the divisor mD. We can then define the *augmented base locus* of D as

$$B_+(D) := \cap_{m\in\mathbb{N}}\text{SB}(mD - A)$$

where A is any arbitary ample divisor on X. Serre's theorem on vanishing cohomology of ample bundles easily shows that the above definition of $B_+(D)$ is independent of the chosen ample divisor A. Now given a closed subvariety E of X, we can define its *restricted volume* as the rate of growth of those section on $O(D)|_E$ which are in the image of the restriction map $O(D) \to O(D)|_E$. In [LM09b] it is shown that a similar study of restricted volumes using Newton-Okounkov bodies is possible when the closed subvariety E is not contained in $B_+(D)$. In particular they show that one can attach Newton-Okounkov bodies which encodes the restricted volume of the divisor along a closed subvariety in analogy with the projective case. In this thesis, we extend the notions of augmented base loci and restricted volumes to geometric adelic divisors on quasi-projective varieties and we are able to show the following theorem in Chapter 2:

Theorem 1.3.3. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K. Furthermore suppose E is a closed irreducible sub-variety of U not contained in the augmented base locus of $\overline{D}$. Then we have*

$$\text{vol}_{\mathbb{R}^d}(\Delta_{U|E}(\overline{D})) = \lim_{m\to\infty} \frac{\dim_K(H^0(U|E, O_E(m\overline{D})))}{m^d} = \frac{1}{d!} \cdot \widehat{\text{vol}}_{U|E}(\overline{D})$$

where $\dim(E) = d$

In order to make sense of the above, we have to give a suitable definition of augmented base locus for divisors in the quasi-projective setting. Since there is no known version of Serre's vanishing theorem or ample divisors in the quasi-projective setting, it needs some work to make sense of $B_+(D)$ and its independence from the chosen "ample" divisor A. We end this gap using the main result on augmented base locus from [Bir17]. We furthermore show that we can study variations of the restricted volumes using a global body as was the case for volumes.

1.4 Positive intersection products and Differentiability of Volumes

In this section, we turn our attention to both geometric and arithmetic volumes. In the classical projective case, both geometric and arithmetic volumes are known to be differentiable at big divisors. We go on to show a similar result in the quasi-projective setting of Yuan and Zhang. The results of this section appear in the article [Bis24b].

There is a natural analogue of global sections in Arakelov geometry which are termed as *small sections*. Suppose $\mathcal{X}$ is an arithmetic variety over $\mathrm{Spec}(\mathbb{Z})$ and $\overline{D}$ is an arithmetic divisor on X. Then we have defined the arithmetic volumes $\widehat{\mathrm{vol}}(\overline{D})$ which measures the rate of growth of small sections. In [BFJ09a], Boucksom, Favre and Jonsson show that the geometric volume function $\mathrm{vol}(L)$ is differentiable along any arbitary direction and the derivatives are given by *positive intgersection products* which are a slight variant of ordinary intersection products. In [Che11], Chen proves the arithmetic analogue of differentiability. In particular he shows that the arithmetic volume function $\widehat{\mathrm{vol}}(\overline{D})$ is differentiable on the big cone and the derivative is given by the arithmetic analogue of positive intersection products. In this thesis, we extend the results of [BFJ09a] and [Che11] to prove that the volume functions defined in [YZ24] on quasi-projective varieties are differentiable along integrable adelic divisors. More precisely given a big adelic divisor $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$ for some essentially quasi-projective variety U over $\overline{B}$ and integrable adelic divisor $\overline{E} \in \widehat{\mathrm{Div}}(U, \overline{B})$, we define positive intersection products $\langle \overline{D}^d \rangle \cdot \overline{E}$ as

$$\langle \overline{D}^d \rangle \cdot \overline{E} \coloneqq \sup_{X', \overline{A}} \overline{A}^d \cdot \pi^* \overline{E}$$

where $\overline{E}$ is nef and the tuples $(\pi, X', \overline{A})$ varies across all tuples such that $\pi\colon U' \to U$ is a birational modification of U, X' is a projective model of U' and $\overline{A}$ is a nef $\mathbb{Q}$-divisor on X' such that $\pi^*\overline{D} - \overline{A} \geq 0$ in $\widehat{\mathrm{Div}}(U', \overline{B})$. We can then extend the definition to integrable $\overline{E}$ by linearity and we show the following theorem in Chapter 3:

Theorem 1.4.1. *Suppose $\overline{D}$ is a big adelic divisor and $\overline{E}$ is an integrable adelic divisor on a normal essentially quasi-projective variety U over $\overline{B}$ of dimension d. Then the function $t \mapsto \widehat{\mathrm{vol}}(\overline{D} + t\overline{E})$ is differentiable at $t = 0$ with derivative given by*

$$\frac{d}{dt} \widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) \mid_{t=0} = (d+1) \cdot \langle \overline{D}^d \rangle \cdot \overline{E}$$

Note that the above theorem is for any arbitary base $\overline{B}$ and hence it simultaneously covers the arithmetic case when the base is a number field or function field as well as the geometric case when the base is any arbitary field.

Along similar lines of positive intersection products, in this thesis we go on to define a variant of *asymptotic intersection numbers* of an adelic divisor against a closed sub-variety inspired from the projective case defined in [Ein+09a]. We go on to show that restricted volumes also satisfy a Fujita approximation type result when the sub-variety is not contained in the augmented base locus using our definition above and this can be thought of as a quasi-projective analogue of [Ein+09a, Theorem 2.13]. More precisely, suppose E is a closed sub-variety of a smooth quasi-projective variety U over $B = (k, \emptyset)$ for $k = \mathbb{C}$ with $\dim E = d$. Furthermore we define

$$\langle D^d \rangle \cdot E \coloneqq \sup_{(X', A)} A^d \cdot \tilde{E}$$

where the supremum varies over all tuples (π, X', A) such that $\pi\colon U' \to U$ is a birational modification such that π is an isomoprhism over the generic point of E, X' is a projective model of U', A is a nef $\mathbb{Q}$-divisor on X' such that $\pi^*\overline{D} - \overline{A} \geq 0$ in $\widehat{\mathrm{Div}}(U', \overline{B})$ and $\tilde{E} = \pi^{-1}(E)$. Then we deduce the following theorem in Chapter 3:

Theorem 1.4.2. *Suppose D is an adelic divisor on U and E is a closed-subvariety of dimension d such that $E \nsubseteq B_+(D)$. Then we have*

$$\widehat{\mathrm{vol}}_{U|E}(D) = \langle D^d \rangle \cdot E$$

The above theorem can be thought of as a version of Fujita approximation for restricted volumes on quasi-projective varieties.

1.5 Equidistribution at Big Adelic Divisors

In the arithmetic setting, the variational approach allows us to deduce certain equidistribution results from the differentiability of arithmetic volumes which we are furthermore able to generalise in the quasi-projective setting in the article [Bis24b].

Equidistribution is a central phenomenon studied in Arakelov geometry. The variational approach to prove equidistribution was used by Szpiro, Ullmo and Zhang in [SUZ97] which in turn was used to prove the Bogomolov conjecture for Jacobians of curves by Ullmo in [Ull98] and for arbiatry abelian varieties by Zhang in [Zha98]. Suppose U is a quasi-projective variety over $\overline{B}$ and suppose we fix a place v on the field of fractions K of B and suppose K_v denotes the completion of K at v. Then there is an action of the Galois group $\mathrm{Gal}(\overline{K_v}/K_v)$ over a geometric point $x \in U(\overline{K})$. Suppose we denote by η_x the probability measure of the Galois orbit of x under $\mathrm{Gal}(\overline{K_v}/K_v)$. The phenomenon of equidistribution deals with conditions on a sequence of geometric points $\{x_m\}$ such that the sequence of measures η_{x_m} converge weakly to some probability measure.

We recall some notions related to the statement of equidistribution. A sequence of geometric points $\{x_m\}$ in $U(\overline{K})$ is called *generic* if it is Zariski dense in U. Furthermore suppose $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$ is an adelic divisor. Given $\overline{D}$ and $x \in U(\overline{K})$, we can define the *height* of x with respect to $\overline{D}$ as

$$h_{\overline{D}}(x) \coloneqq \sum_{v \in \Sigma_K} \frac{n_v}{\#O(x)_v} \sum_{y \in O(x)_v} g_{\overline{D}+\widehat{\mathrm{div}}(f),v}(y)$$

where f is a rational function on U such that $x \notin \mathrm{Supp}(D + \mathrm{div}(f))$, $\widehat{\mathrm{div}}(f)$ is a principal adelic divisor associated to f, $g_{\overline{D}+\widehat{\mathrm{div}}(f),v}$ denotes the Green function of the adelic divisor $\overline{D} + \widehat{\mathrm{div}}(f)$ at the place v, $O(x)$ denotes the Galois orbit of closed points in U_{K_v} by the action of $\mathrm{Gal}(\overline{K_v}/K_v)$ and n_v denotes the usual local factor at the place $v \in \Sigma_K$ on K. Recall that the local factor is given by $n_v = \frac{[K_v:\mathbb{Q}_p]}{[K:\mathbb{Q}]}$ when K is a number field an p is the restriction of the place v to $\mathbb{Q}$ and $n_v = [k(v) : k]$ when K is a smooth projective curve over a field k and v is identified with a closed point of the curve. We say that a sequence $\{x_m\}$ in $U(\overline{K})$ is *small* with respect to $\overline{D}$ is $h_{\overline{D}}(x_m) \to \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)}$ as $m \to \infty$. Note that this definition of $\overline{D}$-small sequences varies from the classical definition where we require the heights of the points to converge to the normalised height of U. When U is projective, K is a number field and $\overline{D}$ is an arithmetic divisor, Chen showed that the sequence of measures $\{\eta_{x_m}\}$ weakly converges to the measure given by $f \mapsto \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(f)}{\mathrm{vol}(D)}$ in [Che11]. In this thesis we generalise the above theorem to quasi-projective varieties. In particular we prove the following theorem in Chapter 3:

Theorem 1.5.1. *Suppose U is an essentially quasi-projective variety over K of dimension d and suppose $\overline{D}$ is a big arithmetic adelic divisor in $\widehat{\mathrm{Div}}(U, \overline{B})$. Furthermore suppose $\{x_m\}$ is a generic sequence of geometric points in $U(\overline{K})$ which is small with respect to $\overline{D}$. Then for any place v on K and for any $g \in C_c^0(U_v)$, we have*

$$\lim_{m \to \infty} \int_{U_v} g \; d\eta_{x_m} = \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g)}{n_v \cdot \mathrm{vol}(D)}$$

In particular, the sequence of Radon measures $\{\eta_{x_m}\}$ converge weakly to the Radon measure given by

$$g \in C_c^0(U_v) \mapsto \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g)}{n_v \cdot \mathrm{vol}(D)}$$

We go on to show that the above theorem implies an equidistribution theorem of Yuan and Zhang in the quasi-projective setting ([YZ24, Theorem 5.4.3]) where the adelic divisor $\overline{D}$ is assumed to be nef.

1.6 Concave transforms of compactified divisors

We continue our study of arithmetic volume functions in the quasi-projective setting. We use the machinery of concave transforms induced by filtrations to study the arithmetic volume function. We furthermore go on to show that the extended definition of arithmetic intersection numbers as introduced in [BK24] and [CG24] can be realised as the integral of a concave function on the Okounkov body of the generic fiber. The results of this chapter will appear in an ongoing work along with Yulin Cai.
In [BC11], Boucksom and Chen introduced the idea of using filtrations on the space of global sections to construct concave functions which can be used to study arithmetic volume functions. They can be thought of as the non-toric analogue of "global roof functios" introduced in [Bur+16]. In [CM20], Chen and Moriwaki generalise the idea of Boucksom and Chen on projective varieties defined over arbitary adelic curves and adelic line bundles defined over them using Harder-Narasimhan filtrations. In this thesis, we extend these results for compactified arithmetic divisors of [CG24] using the construction of [CM20].
For the remainder of this chapter, we always assume K to be a perfect field for technical reasons. In [CG24], the authors generalised the compactified arithmetic divisors of Yuan and Zhang from the number field case to the case of a general adelic curve S on K. The group of "compactified arithmetic divisors", denoted by $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, is defined over any quasi-projective variety U over an adelic curve S of dimension d. The compactified divisors are defined as a certain topological completion of the space of adelic divisors on projective varieties defined in [CM20] over arbitary adelic curves. Given a compactified divisor $\overline{D}$ defined over a normal quasi-projective variety U over an adelic curve S with underlying field K, we can consider the Okounkov bodies $\Delta(D)$ of the generic fiber D constructed in Chapter 1. In this thesis we associate a "graded algebra of adelic vector bundles" $\{H^0_+(U, m\overline{D})\}_{m\in\mathbb{N}}$ by considering those global sections which have finite sup-norm. Note that these spaces of global sections can differ from the space of all global sections since the sup.norm is not necessarily finite due to U being quasi-projective. Using Harder-Narasimhan filtrations on $\{H^0_+(U, m\overline{D})\}_{m\in\mathbb{N}}$, we can then construct a concave function $G_{\overline{D}}\colon \Delta(D)^\circ \to \mathbb{R}\cup\{-\infty\}$. In sub-section 4.4.2, we go on to define the arithmetic volume and arithmetic χ-volume of a compactified divisor as the asymptotic rate of growth of positive arithmetic degrees and arithmetic degrees of $\{H^0_+(U, m\overline{D})\}_{m\in\mathbb{N}}$ respectively. Our first main result in this chapter is the following analogue of [CM20, Theorem 6.4.6]

Theorem 1.6.1. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, and $\overline{V_\bullet} = \{\overline{V_m}\}_{m\in\mathbb{N}}$ its associated graded K-algebra of adelic vector bundles defined in Definition 4.4.2. Assume that $\mathbb{N}(V_\bullet) \neq 0$. Then the associated Borel probability measures $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ in 4.4.8 converge weakly to the probability measure $\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta_+(\overline{D}))} G_{\overline{D},*}(d\lambda)$, where $d\lambda$ is the standard Lebesgue measure on $\Delta(D) \subseteq \mathbb{R}^\kappa$ and κ is the Kodaira dimension $V_\bullet$ ($\kappa = d$ if D is big). In particular, if $\overline{D}$ is big, then D is big. Moreover, in the case where D is big (note that $\Delta(D)^\circ = \Delta_+(\overline{D})^\circ$ by Lemma 4.4.5), we have that*

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda. \tag{1.1}$$

and

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = \limsup_{m\to\infty} \frac{\widehat{\deg}(\overline{V_m})}{m^{d+1}/(d+1)!} \leq (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\, d\lambda. \tag{1.2}$$

with equality if $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) > -\infty$ (in this case the supremum limit in (1.2) is a limit).

We explain a few terms appearing in the above theorem. We denote by $\mathbb{N}(V_\bullet)$ the positive generator of the sub-group spanned by all integers m such that $V_m \neq \{0\}$ and this condition is always satisfied when D is big. The term $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D})$ is the asymptotic rate of growth of "minimal slopes" of the graded algebra of adelic vector bundles $\{H^0_+(U, m\overline{D})\}_{m\in\mathbb{N}}$ which arise from their associated Harder-Narasimhan filtrations. We refer to sub-section 4.4.2 for detailed definitions.
We go on to study further properties of the concave transform and arithmetic volumes. In particular we show that the arithmetic volumes vary continuously with respect to the boundary topology and using that we show that the concave transforms satisfy a pointwise convergence property on the Okounkov bodies. Note that in [BK24] and [CG24], the authors extend the intersection pairing defined by Yuan and Zhang. Such an extension is required since the intersection pairing of Yuan and Zhang is only defined for integrable adelic line-bundles whereas the line-bundle of Siegel-Jacobi modular forms along with the Petersson metric is not integrable in the sense of Yuan and Zang. We adapt the terminology of [CG24] where an *arithmetically nef* compactified divisor is Cauchy limit of model adelic divisors which are *nef* in the sense of [CM22, Definition 5.3.5]. In the case of number field or function fields as the base adelic curves, these are the *nef adelic divisors* of Yuan and Zhang introduced before. On the other hand we denote by *relatively nef* compactified divisors which are Cauchy limits of model divisors which are *relatively nef* in the sense of [CM22, Definition 5.1.14]. Then suppose we denote by $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ the monoid of relatively nef compactified divisors in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ such that there is an arithmetically nef reference metric on D. Then in [BK24] and more generally in [CG24], the authors construct an intersection pairing $(\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}})^{d+1} \to \mathbb{R}$ which extends the intersection pairing defined by Yuan and Zhang. As the second main result of this chapter, we show that this extended intersection number can be realised as the integral of the concave function that we defined.

Theorem 1.6.2. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ with D big. Then*

$$(\overline{D}^{d+1} \mid U)_S = (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}\, d\lambda.$$

In particular, if $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) > -\infty$, then by Theorem 4.4.9, we have that

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = (\overline{D}^{d+1} \mid U)_S.$$

The above theorem can be viewed as a non-toric analogue of [Bur+16, Theorem 5.6] for singular metrics over general adelic curves. The results of this chapter are from ongoing work with Yulin Cai.

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Chapter 2

Geometric Volumes on Quasi-Projective Varieties

2.1 Introduction

The theory of Okounkov bodies to study linear systems of line bundles on a projective variety was introduced by the Russian mathematician Andrei Okounkov in his articles [Oko96b] and [Oko03]. Given a linear series of an ample line bundle on a projective variety, he introduced certain convex bodies, which later came to be known as *Okounkov bodies* whose convex geometric properties encode interesting invariants of the graded series. In their article [LM09b] Robert Lazarsfeld and Mircea Mustață noticed that the constructions of Okounkov generalise from ample line bundles to arbitary big line bundles on projective varieties. In their paper [LM09b] they develop a systematic study of Okounkov bodies for big line bundles and prove various properties of volumes such as continuity, Fujita approximation and others. They also consider the notion of *restricted volumes* along a closed sub-variety and prove properties analogous to those of ordinary volumes.
A crucial feature of the approach in [LM09b] is that the construction of Okounkov bodies makes sense even when the variety is not projective as long as we have a graded series of the space of global sections on our given line bundle. In this article we use this observation to construct Okounkov bodies for "compactified" line-bundles on quasi-projective varieties. In their recent pre-print [YZ24] Xinyi Yuan and Shou-Wu Zhang introduced the notion of *adelic divisors* on a quasi-projective variety U over a field. They manage to put a topology on the space of all divisors which come from projective models X_i of U and consider all divisors which are "compactified" with this topology. In other words an *adelic divisor* on a normal quasi projective variety U is given by the data $\{X_i, D_i\}$ and a sequence of positive rationals q_i converging to 0 where X_i are projective models of U, D_i are $\mathbb{Q}$-divisors on X_i with $D_i|_U = D_j|_U$ for all i, j such that the following "Cauchy condition" holds.

$$-q_j D_0 \leq D_i - D_j \leq q_j D_0 \ \forall \ i \geq j$$

Here inequalities signify effectivity relations holding in a common projective model (see section 2.4 of [YZ24] for details). As a result of their consideration, given any divisor D on U and an adelic compactification on D denoted by $\overline{D}$, we get a space of adelic global sections $H^0(U, \overline{D})$ which is a **finite dimensional** sub-space of all global sections $H^0(U, O(D))$. Hence we can consider the notions of volumes similarly to the projective case and it is shown in [YZ24] that these volume functions shows properties analogous to the classical projective volumes (see [YZ24], section 5). However following the approach in [LM09b] in this article we construct **Okounkov bodies** $\Delta(\overline{D})$ for the graded series $\{H^0(U, m\overline{D})\}_{m\in\mathbb{N}}$. The construction is essentially a special case of the construction sketched in Definition 1.16 of [LM09b] where we take $W_m = H^0(U, m\overline{D}) \subseteq H^0(U, O(mD))$. If the divisor $\overline{D}$ is big *i.e* it has positive volume as defined in [YZ24], we show that the Lebesgue volume of the body is essentially the same as the algebraic volume upto scaling. The first main theorem of our article is as follows

Theorem 2.1.1. *Suppose we have a big adelic divisor $\overline{D}$ on a normal quasi-projective variety U and suppose $\Delta(\overline{D})$ is the Okounkov body associated to $\overline{D}$. Furthermore suppose $\widehat{\mathrm{vol}}(\overline{D})$ be the adelic volume defined in Section 5.2.2 of [YZ24]. Then we have*

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(\overline{D})) = \lim_{m\to\infty} \frac{\dim_K(H^0(U, m\overline{D}))}{m^d} = \frac{1}{d!}\cdot \widehat{\mathrm{vol}}(\overline{D})$$

Continuing with our analogy of the approach in the article [LM09b] we construct global bodies for adelic Okounkov bodies to study the variation of these bodies. Although we do not have finiteness of the Néron–Severi space associated to adelic divisors, it turns out there exist a canonical global convex body whose fibers give Okounkov bodies even if this global body depends on some choices of divisors in contrast to in [LM09b]. This is the content of our next theorem

Theorem 2.1.2. *Suppose $\overline{D}$ and $\overline{E}$ be adelic divisors on a normal quasi-projective variety U such that $\overline{D}$ is big. Then there exists a convex body $\Delta(U) = \Delta(U, \overline{D}, \overline{E}) \subset \mathbb{R}^{d+2}$ with the property that for any $\vec{a} = (a_1, a_2) \in \mathbb{Q}^2$ with $a_1\overline{D} + a_2\overline{E}$ big, we have*

$$\Delta(a_1\overline{D} + a_2\overline{E}) = \Delta(U) \cap (\mathbb{R}^d \times \{\vec{a}\})$$

The above two theorems combine to prove Theorem 5.2.1 of [YZ24] for big adelic divisors using convex geometric methods and Okounkov bodies. This volume essentially measures the asymptotic growth of the global sections which arise as restrictions of global sections from the bigger variety just as in the classical projective case (Furthermore they show that not only the volume of the big adelic divisor but also its Okounkov body constructed in this article is approximated (in terms of Hausdorff metric) by the corresponding Okounkov bodies of the projective models defining the divisor.
Next we go on to define the notions of restricted volumes of adelic divisors along a closed sub-variety E of U using Okounkov bodies. The restricted volume essentially measures the asymptotic growth of global sections of $\overline{D}|_E$ which arise as restrictions of sections of $\overline{D}$ over U to E analogously to the classical projective setting (see [Ein+09a] for more details). Analogously to the projective case, we can form the convex geometric objects $\Gamma_{U|E}(\overline{D})$, $\Delta_{U|E}(\overline{D})$ and the algebraic objects $H^0(U|E, \overline{D})$, $\widehat{\mathrm{vol}}_{U|E}(\overline{D})$ for a given adelic divisor $\overline{D}$. In order to have relations analogous to that of the adelic volume, we introduce the notion of augmented base locus of an adelic divisor in analogy with projective augmented base locus (see section 2.4, [LM09b]). Our definition, although being very similar to the projective case, requires some work to be shown well-defined. Since we do not have Serre finiteness on quasi-projective varieties, we have to use the main result of [Bir17] to show the well-definedness. We go on to show that when E is not contained in the augmented base locus, the expected properties hold which is our next theorem

Theorem 2.1.3. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K. Furthermore suppose E is a closed irreducible sub-variety of U not contained in the augmented base locus of $\overline{D}$. Then we have*

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta_{U|E}(\overline{D})) = \lim_{m\to\infty} \frac{\dim_K(H^0(U|E, O_E(m\overline{D})))}{m^d} = \frac{1}{d!}\cdot \widehat{\mathrm{vol}}_{U|E}(\overline{D})$$

where $\dim(E) = d$

We go on to show the existence of a global body even for restricted volumes and hence the variation of these restricted Okounkov bodies also has desirable satisfying properties like continuity etc.

The organisation of this chapter is as follows. In the first three sub-sections of the first section we review the notions of adelic divisors and their space of global sections following sub-section 2.4 in [YZ24]. In the fourth sub-section we construct the Okounkov bodies for adelic divisors and show some preleminary properties of them. In the fifth sub-section we prove our first main theorem relating the algebraic adelic volumes with Euclidean volumes of their Okounkov bodies. In the next two sub-sections we construct the global bodies and show that their fibers essentially gives the variation of Okounkov bodies in fixed directions. We also deduce certain corollaries of the existence of global bodies. In the first sub-section of the second section we define augmented base locus of an adelic divisor. We go on to define the restricted volume of an adelic divisor along a closed sub-variety in the next sub-section. We relate them to Euclidean volumes of restricted Okounkov bodies and show the existence of global bodies in analogy to the adelic volume in the next two sub-sections. We end the section by obtaining certain corollaries of restricted volumes similar to those of ordinary volumes in chapter 1. The results of this chapter are from [Bis24a].

2.2 Adelic divisors

We begin by giving a short review of adelic divisors which are our main objects of interest in this article. We fix a quasi-projective varity U over any field k. By a *projective model* of U, we mean a projective variety X over k which contains U as an open dense subset via an open immersion $U \hookrightarrow X$. Given a projective model X of U, we have the group of Cartier $\mathbb{Q}$-divisors denoted by $\mathrm{Div}(X)_{\mathbb{Q}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then we consider the group of $(\mathbb{Q}, \mathbb{Z})$-divisors $\mathrm{Div}(X, U)$ as follows

$$\mathrm{Div}(X, U) = \{(D, \mathcal{D}) \in \mathrm{Div}(U) \oplus \mathrm{Div}(X)_{\mathbb{Q}} \mid \mathcal{D}|_U = D \text{ in } \mathrm{Div}(U)_{\mathbb{Q}}\}$$

where $\mathcal{D}|_U$ denotes the image of $\mathcal{D}$ under the pull-back morphism $\mathrm{Div}(X)_{\mathbb{Q}} \to \mathrm{Div}(U)_{\mathbb{Q}}$.

Note that the set of *all* projective models of a given U form an inverse system which in turn makes the set of $(\mathbb{Q}, \mathbb{Z})$-divisors into a directed system via pull-backs. Then we can form the direct limit to define the group of *model divisors* as follows

$$\mathrm{Div}(U/k)_{\mathrm{mod}} = \lim_{X} \mathrm{Div}(X, U)$$

where above the direct limit is taken as X varies over all projective models of U. Next note that there is a notion of effectivity in both the groups $\mathrm{Div}(X)_{\mathbb{Q}}$ and $\mathrm{Div}(U)$ which induces a partial order on $\mathrm{Div}(X, U)$ where $(D, \mathcal{D}) \leq (D', \mathcal{D}')$ if and only if both $D' - D$ and $\mathcal{D}' - \mathcal{D}$ are effective in $\mathrm{Div}(U)$ and $\mathrm{Div}(X)_{\mathbb{Q}}$ respectively. This partial order induces a partial order in $\mathrm{Div}(U/k)_{\mathrm{mod}}$ by passing to direct limits.

By a *boundary divisor* of U over k, we mean a tuple (X_0, D_0) where X_0 is a projective model of U and D_0 is an effective Cartier divisor on X_0 with $\mathrm{Supp}(D_0) = X_0 - U$. Note that such a boundary divisor always exists which can be seen by choosing any projective model X_0' of U and blowing-up X_0' along the reduced center $X_0 - U$. Then note for any non-zero rational $r \in \mathbb{Q}$ we can view rD_0 as an element of $\mathrm{Div}(X_0, U)$ and hence as en element of $\mathrm{Div}(U/k)$ by setting the component on $\mathrm{Div}(U)$ to be 0.

We can finally put a norm, denoted by a *boundary norm* on $\mathrm{Div}(U/k)_{\mathrm{mod}}$ as follows

$$|| \cdot ||_{D_0} \colon \mathrm{Div}(U/k)_{\mathrm{mod}} \to [0, \infty]$$

$$||\overline{D}||_{D_0} = \inf\{q \in \mathbb{Q}_{>0} \mid -qD_0 \leq \overline{D} \leq qD_0 \text{ in } \mathrm{Div}(U/k)_{\mathrm{mod}}\}$$

It is shown Lemma 2.4.1 of [YZ24] that $|| \cdot ||_{D_0}$ is actually a norm and the topology induced by it on $\mathrm{Div}(U/k)_{\mathrm{mod}}$ is independent of the chosen boundary divisor (X_0, D_0). Hence we can talk about the *boundary topology* on $\mathrm{Div}(U/k)_{\mathrm{mod}}$ as the topology induced by a boundary norm coming from *any* boundary divisor. We finally define the *adelic divisors*, denoted by $\widehat{\mathrm{Div}}(U, K)$, as the completion of the topological space $\mathrm{Div}(U/k)_{\mathrm{mod}}$ with respect to the boundary topology described above. Note that then by an adelic divisor we mean the data $\{X_i, D_i\}$ where X_i are projective models and $D_i \in \mathrm{Div}(X_i, U)$ and a sequence of positive rationals $\{q_i\}$ converging to 0 satisfying the effectivity relations

$$-q_i D_0 \leq D_i - D_j \leq q_i D_0 \text{ in } \mathrm{Div}(U/k)_{\mathrm{mod}} \text{ for all } j \geq i$$

Remark 1. If we assume U to be normal, we can choose the models X_i to be normal and further embedding the group of Cartier divisors into Weil divisors we can look at D_i just as elements of $\mathrm{Div}(X_i)_{\mathbb{Q}}$ and the effectivity relation to be holding in just $\mathrm{Div}(X_i)_{\mathbb{Q}}$ instead of $\mathrm{Div}(U/k)_{\mathrm{mod}}$. This is due to the fact that group of Weil divisors on U has no torsion.

2.3 Space of global sections of an adelic divisor

We fix an algebraically closed field K and a normal quasi-projective variety U over K. As described in the previous section, we have the notion of the group of *adelic divisors* $\widehat{\mathrm{Div}}(U, K)$ ([YZ24] sub-section 2.4.1 for more details) which are given by a compatible sequence of models $\{X_i, D_i\}$ such that D_i are Cartier $\mathbb{Q}$-divisors on the projective models X_i such that D_i restrict to a Cartier divisor D on U and they satisfy the Cauchy condition with respect to a boundary divisor D_0 defined over a projective model X_0 *i.e* there exists a sequence of positive rational numbers $\{q_j\}$ converging to 0 such that

$$D_j - q_j D_0 \leq D_i \leq D_j + q_j D_0 \text{ for all } i \geq j\,. \tag{2.1}$$

where D_0 is an effective Cartier divisor on X_0 with support exactly equal to the complement of U in X_0 and the above effectivity relations are considered in a common model(for details see [YZ24, Section 2]). Note that the definition of adelic divisor does not depend on the particular choice of the boundary divisor D_0, as shown in [YZ24, Lemma 2.4.1]. We denote this data by $\overline{D}$. Given such an adelic divisor, we introduce the space of global sections

$$H^0(U, \overline{D}) = H^0(U, O(\overline{D})) = \{f \in \kappa(U)^{\times} \mid \mathrm{div}(f) + \overline{D} \geq 0\} \cup \{0\}$$

following [YZ24, section 5.1.2]. In the above definition, $\mathrm{div}(f)$ is the adelic divisor obtained by picking the divisor corresponding to $f \in \kappa(U)^{\times} = \kappa(X)^{\times}$ on any projective model X of U, and $\mathrm{div}(f) + \overline{D} \geq 0$ means that the left hand side can be represented by a sequence of effective divisors on the corresponding models.

Remark 2. It is shown in [YZ24, Lemma 5.1.7(2)] that this space is always finite dimensional. This will be our analogue for the usual space of global sections on which we construct Okounkov bodies. For this purpose, note that by restricting the effectivity relation $\mathrm{div}(f) + \overline{D} \geq 0$ to U, we can identify $H^0(U, \overline{D})$ with a finite dimensional vector sub-space of the space of all sections $H^0(U, O(D))$(which in general is very large and infinite dimensional). This will always be our way of viewing the vector spaces $H^0(U, \overline{D})$.

2.4 Different notions of effective sections

Note that D_i can be viewed as a (model) adelic divisor $\overline{D_i}$ in $\widehat{\mathrm{Div}}(U, K)$ and consequently we have the space of global sections $H^0(U, \overline{D_i})$ as before, where we put the overline to emphasize it is viewed as a model adelic divisor. However viewing D_i as a $\mathbb{Q}$-divisor on the projective variety X_i we can also define the space of global sections as before

$$H^0(X_i, D_i)' = \{f \in \kappa(X_i)^{\times} \mid \mathrm{div}(f) + D_i \geq 0 \text{ in } \mathrm{Div}(X_i)_{\mathbb{Q}}\} \cup \{0\}\,.$$

only by restricting our attention to the projective model X_i. These two notions of effective sections can be different a-priori. However if we consider U to be normal, then by [YZ24, Lemma 5.1.5 and Remark 5.1.6] they are canonically identified and we get that both these notions are the same. Next we will obtain some inclusions.

Lemma 2.4.1. *We have the sequence of inclusions*

$$H^0(X_j, k(D_j - q_j D_0))' \hookrightarrow\; H^0(U, O(k\overline{D})) \hookrightarrow H^0(X_j, k(D_j + q_j D_0))'$$

for all $k \in \mathbb{N}$ and for all j.

Proof. Note that by our discussion above the two extremes of the sequence can be replaced by

$$H^0(U, k(\overline{D_j} - q_j\overline{D_0})) \quad \text{and} \quad H^0(U, k(\overline{D_j} + q_j\overline{D_0}))$$

respectively as U is assumed to be normal. Therefore, the statement is equivalent to the chain of inequalities

$$\overline{D}_j - q_j\overline{D}_0 \leq \overline{D} \leq \overline{D}_j + q_j\overline{D}_0\,,$$

which is an immediate consequence of (2.1). ☐

Next we define the volume of an adelic line bundle following [YZ24, sub-section 5.2.2].

Definition 2.4.2. *Given an adelic line bunde $\overline{D}$ on a quasi-projective variety U as above, we define the* volume of $\overline{D}$ *as*

$$\widehat{\text{vol}}(\overline{D}) = \limsup_{m\to\infty} \frac{\dim_K(H^0(U, m\overline{D}))}{m^d/d!}\,,$$

where d is the dimension of U. We call an adelic divisor big *if $\widehat{\text{vol}}(\overline{D}) > 0$.*

We will primarily be interested in the Okounkov bodies of the big adelic divisors.

Remark 3. It is shown in [YZ24, Theorem 5.2.1(1)] that the lim sup in Definition 2.4.2 is actually a limit by using the fact that the volume is actually a limit of the volumes of the projective $\mathbb{Q}$-volumes of thze models. However we will not assume that here and we will use the theory of Okounkov bodies to independently show that this volume is given by a limit.

2.5 Okounkov bodies for adelic divisors

We recall the valuation function crucial in the definition of Okounkov bodies. Note that as we remarked at the end of section 1.1, every element of $H^0(U, O(\overline{D}))$ can be identified as a global section of $O(D)$ on U by restricting the effectivity relation $\text{div}(f)+\overline{D} \geq 0$ to U. Now we fix a closed regular point $x \in U(K)$ and consider any local trivialisation s_0 of $O(D)$ around x. Then every element $s \in H^0(U, \overline{D}) \subseteq H^0(U, O(D))$ induces a regular function by $f = \frac{s}{s_0}$ around x and hence an element in the completion $\widehat{O_{U,x}} \cong K[[x_1 \dots x_d]]$ where d is the dimension of U and the second congruence follows from the regularity of x. Then we define a valuation like function denoted by ord as follows:

$$\nu_x(f) = \min\{\alpha \in \mathbb{N}^d \mid f = \sum a_\alpha \mathrm{x}^\alpha \text{ in } \widehat{O_{U,x}},\ a_\alpha \neq 0\}$$

where the minimum is taken with respect to the lexicographic order on the variables $x_1 \dots x_d$ and this function is independent of the choice of s_0. Now the choice of a flag $x = Y_0 \subset Y_1 \subset \dots Y_d = U$ centered at x gives a choice of variables $x_1 \dots x_d$ as above and hence yields a valuation function ν_x on $H^0(U, \overline{D})$. Note that the sub-spaces $H^0(U, m\overline{D})$ are finite dimensional and induces a graded linear series $\{V_m \subseteq H(U, mD)\}$ in the sense of section 1.3 of [LM09b]. Hence we can define the semi-groups and convex bodies similarly

Definition 2.5.1. *Suppose we have the adelic divisor $\overline{D}$. Then we can define the semi-group*

$$\Gamma(\overline{D}) = \{(\alpha, m) \in \mathbb{N}^{d+1} \mid \alpha = \nu_x(s) \textit{ for some } s \in H^0(U, m\overline{D})\}$$

We further define $\Gamma(\overline{D})_m = \Gamma(\overline{D}) \cap (\mathbb{N}^d \times \{m\})$. Finally we define the associated Okounkov body *of $\overline{D}$ as*

$$\Delta(\overline{D}) = \textit{closed convex hull}(\cup_m \frac{1}{m} \cdot \Gamma(\overline{D})_m) = \Sigma(\Gamma(\overline{D})) \cap (\mathbb{R}^d \times \{1\})$$

where $\Sigma(\cdot)$ denotes taking the closed convex cone in the ambient Euclidean space.

We are going to derive required properties of $\Gamma(\overline{D})$ and $\Delta(\overline{D})$ with the goal of relating its volume to the volume of adelic line bundles. We begin by showing that eventually the models are big when perturbed a little by the boundary divisor D_0 provided $\overline{D}$ is big. This is immediate if we assume Proposition 5.2.1 of [YZ24]. However even without the full strength of the result, we have the following lemma:

Lemma 2.5.2. *Suppose $\overline{D}$ is a big adelic divisor given by models $\{X_i, D_i\}$ as above with boundary divisor D_0. Then for $j >> 0$, $D_j - q_j D_0$ (and hence D_j) is a big $\mathbb{Q}$-divisor on X_j. In particular, we deduce that there exists a r_0 such that $H^0(U, r\overline{D}) \neq \{0\}$ for all $r > r_0$.*

Proof. We are going to use Fujita approximation ([Fuj94]) for $\mathbb{Q}$-divisors on projective models. Note that the RHS of the inclusions in Lemma 2.4.1 gives us that $\mathrm{vol}(D_j + q_j D_0) \geq \widehat{\mathrm{vol}}(\overline{D})$. Hence for $\epsilon_i > 0$, we can find by Fujita approximation an ample $\mathbb{Q}$-divisor A_j on a birational modification $\pi \colon X_j' \to X_j$ such that $\pi^*(D_j + q_j D_0) \geq A_j$ and $\mathrm{vol}(A_j) \geq \mathrm{vol}(D_j + q_j D_0) - \epsilon_j$. Then consider the $\mathbb{Q}$-divisor $A_j - 2q_j D_0 \leq D_j - q_j D_0$ where we consider this effectivity relation in X_j' by pulling back both D_0 and D_j to X_j' and we omit the notaions of pull-backs. Write $D_0 = A - B$ where A and B are nef effective $\mathbb{Q}$-divisors in X_0. Then we have

$$\mathrm{vol}(D_j - q_j D_0) \geq \mathrm{vol}(A_j - 2q_j D_0) = \mathrm{vol}(A_j + 2q_j B - 2q_j A)$$

$$\geq (A_j + 2q_j B)^d - 2dq_j (A_j + 2q_j B)^{d-1} \cdot A \geq \; A_j^d - 2dq_j (A_j + 2q_j B)^{d-1} A \geq$$

$$\mathrm{vol}(D_j + q_j D_0) - \epsilon_j - 2dq_j (A_j + 2q_j B)^{d-1} \cdot A$$

Here in the second inequality we have used Siu's inequality to the nef divisors $A_j + 2q_j B$ and $2q_j A$ since both A and B were nef in X_0 and nefness is preserved under bi-rational pull-backs whereas A_j is ample in X_j', in the third inequality we have used that A_j is nef and B is nef and effective and in the last one we have used $A_j^d = \mathrm{vol}(A_j) \geq \mathrm{vol}(D_j + q_j D_0) - \epsilon_j$. Now choosing $\epsilon_j \to 0$ as $j \to \infty$ and suppose we can choose a nef model divisor N such that $A_j + 2q_j \pi_j^* B \leq \pi_j^* N$ for all j. Then we get that

$$\mathrm{vol}(D_j - q_j D_0) \geq \mathrm{vol}(D_j + q_j D_0) - 2dq_j M - \epsilon_j$$

where $M = N^{d-1} A$ is a fixed number independent of j. Noting that both ϵ_j and q_j go to 0 as $j \to \infty$ and noting that $\mathrm{vol}(D_j + q_j D_0) \geq \widehat{\mathrm{vol}}(\overline{D}) > 0$ is bounded from below independently of j, the above inequality shows that for large enough j, $\mathrm{vol}(D_j - q_j D_0) > 0$ which finishes the claim.
Hence we are reduced to showing that there exists a model nef divisor N in $\mathrm{Div}(U, k)_{\mathrm{mod}}$ such that $\pi_j^* N \geq A_j + 2q_j \pi_j^* B$ for all j. To this end choose a positive integer r such that $r > q_j$ for all j. Then consider the divisor $D_1 + 2rD_0 + 2rB$ in X_0. By Serre's finiteness there is a nef divisor N on X_0 such that $N \geq D_1 + 2rD_0 + 2rB$. Then since $r > q_1 > 0$ and D_0 is effective, we get that $N \geq D_1 + q_1 D_0 + q_j D_0 + 2q_j B$ for all j. But note that we have the effectivity relation $D_1 + q_1 D_0 \geq D_j$ and hence we conclude $N \geq D_j + q_j D_0 + 2q_j B$. Since we have the effectivity $\pi_j^*(D_j + q_j D_0) \geq A_j$, pulling back by π_j we deduce $\pi_j^* N \geq \pi_j^*(D_j + q_j D_0) + 2q_j \pi_j^* B \geq A_j + 2q_j \pi_j^* B$ as required. □

From now on onwards thanks to the previous lemma, we fix once and for all a j such that $D_k - q_k D_0$ is big for all $k \geq j$. The first result we want to state is the boundedness of $\Delta(\overline{D})$ where we use the similar result for integral divisors on projective varieties from [LM09b] to obtain our claim.
We start with a sequence of inclusions.

Lemma 2.5.3. *We have a sequence of inclusions*

$$\Gamma(D_j - q_j D_0)_k \subseteq \Gamma(\overline{D})_k \subseteq \Gamma(D_j + q_j D_0)_k$$

for all positive integers k and hence as a consequence

$$\Delta(D_j - q_j D_0) \subseteq \Delta(\overline{D}) \subseteq \Delta(D_j + q_j D_0)$$

Proof. First note that it makes sense to have $\Gamma(\cdot)$ and $\Delta(\cdot)$ in the right and left extremities above even though the arguments are $\mathbb{Q}$-divisors by just viewing them as model adelic divisors in $\mathrm{Div}(U,k)_{\mathrm{mod}}$. The first sequence of inclusions then follow easily from the set of injective maps in 2.4.1 and noting that the construction of ν_x is local. The second set of inclusions then easily follows from definition of a closed convex hull generated by subsets. □

Finally we can state the boundedness result that we wanted to obtain.

Lemma 2.5.4. *The subset $\Delta(\overline{D})$ is compact convex subset of $\mathbb{R}^d$.*

Proof. The said subset is already closed and convex. Hence it is enough to prove that it is bounded. Note that $R = D_j + q_jD_0$ is a $\mathbb{Q}$-divisor on X_i and hence there is an integer t such that tR is an integral Cartier divisor. Note that from the RHS of the set of inclusions 2.4.1 we conclude that for any section $s \in H^0(U, kt\overline{D})$ induces a section $s' \in H^0(X_i, ktR)' = H^0(U, kt\overline{R})$ and both of these have the same valuation vector. Hence we get that $\Gamma(t\overline{D}) \subseteq \Gamma(tR)$ where the RHS is well defined as tR is an integral Cartier divisor which in turn yields by construction that $\Delta(t\overline{D}) \subseteq \Delta(tR)$. On the other hand we have $\Gamma(\overline{D}) \subseteq \frac{1}{t}\cdot\Gamma(t\overline{D})$ and hence by construction we get $\Delta(\overline{D}) \subseteq \frac{1}{t}\cdot\Delta(t\overline{D})$. This readily gives the boundedness as $\Delta(tR)$ is bounded by Lemma 1.10 of [LM09b] as tR is integral divisor and X_i is projective. □

Remark 4. The proof of boundedness for the projective case in [LM09b] is based on intersecting ample divisors with the flag which gives us the Okounkov construction. It might be interesting to try to give a proof using intersection theory as there is a new intersection theory now with adelic line bundles on quasi-projective varieties. However the notion of "adelic ample divisors" are "positive" is not immediate to formulate since pull back of ample bundles by birational morphisms is not necessarily ample again and this might arise as a problem.

2.6 Volumes of Okounkov bodies

We want to relate the volume of the Okounkov body $\Delta(\overline{D})$ with the volume of the adelic divisor $\overline{D}$ as defined in [YZ24]. It will turn out that they are equal (upto scaling) analogous to the projective case. We start with a lemma listing the properties of the $\Gamma(\overline{D})$ which are sufficient to assert the volume equality. We fix a j as in the previous section once again.
We begin by recording a result which relates the dimension of the space of global sections with the cardinality of slices of $\Gamma(\overline{D})$. We denote by $\Gamma(\overline{D})_m = \Gamma(\overline{D}) \cap (\mathbb{N}^d \times \{m\})$. Then we have

Lemma 2.6.1. *We have $\#\Gamma_m = \dim_K(H^0(U, m\overline{D}))$*

Proof. The claim immediately follows from Lemma 1.4 of [LM09b] by taking $W = H^0(U, m\overline{D})$ and noting that W is finite dimensional from Lemma 5.1.7 in [YZ24]. □

Next we want to naturally extend the notion of Okounkov bodies to $\mathbb{Q}$-adelic line bundles. One necessary property is to show that the construction of $\Delta(\cdot)$ behaves well with taking integral multiples of adelic divisors which is the content of our next lemma. Note that if we can show $\mathrm{vol}_{\mathbb{R}}(\Delta(\overline{D})) = \lim_{m\to\infty}\frac{\#\Gamma_m}{m^d}$, then with the Lemma 2.6.1 we have that the Euclidean volume of $\Delta(\overline{D})$ is the same as the volume of $\overline{D}$ as defined in Definition 2.4.2 upto scaling by $d!$. It turns out that for the above equality to be true, it is enough for $\Gamma(\overline{D})$ to satisfy certain properties which are purely Euclidean geometric in nature. We wish to state and prove them in our main lemma of this section. Before that we prove a property necessary in our next lemma.

Lemma 2.6.2. *Suppose $\overline{D}$ is a big adelic divisor on a normal quasi-projective variety U given by the sequence of models $\{X_i, D_i\}$ and rationals $\{q_i \to 0\}$ as usual. Then there is a model X_j such that for all ample divisors $\overline{A}$ on X_j, there exists a non-zero section $s_0 \in H^0(U, m\overline{D} - \overline{A})$ whenever m is a sufficiently large positive integer.*

Proof. The idea is to use Kodaira lemma (Proposition 2.2.6 [LM09b]) in the projective case on models approximating $\overline{D}$ from below. More preciesely suppose $D'_j = D_j - q_j D_0$. Then as $\{D'_j\}$ is a sequence also representing the big divisor $\overline{D}$, by Lemma 2.5.2 we can find a j such that D'_j is a big divisor. Now applying the Kodaira lemma on the big divisor D'_j on the projective variety X_j, we conclude that for all sufficiently large m, there exists a non-zero section of $O(mD'_j - \overline{A})$ on X_j and restricting to U, we get a non-zero section $s_0 \in H^0(U, mD'_j - \overline{A})$ for all sufficiently large m. Now the claim follows from noting that the effectivity relation $\overline{D} \geq D'_j$ implies that $H^0(U, mD'_j - \overline{A}) \subseteq H^0(U, m\overline{D} - \overline{A})$. ☐

Lemma 2.6.3. *Suppose $\overline{D}$ is a big adelic divisor on a normal quasi-projective variety U over K. Then the convex body $\Gamma(\overline{D})$ satisfies the following properties:*

1. $\Gamma_0 = \{0\}$
2. *There exist finitely many vectors $(v_i, 1)$ spanning a semi-group $B \subseteq \mathbb{N}^{d+1}$ such that $\Gamma(\overline{D}) \subseteq B$.*
3. *$\Gamma(\overline{D})$ generates $\mathbb{Z}^{d+1}$ as a group.*

Proof. The first point is trivial. For the second point we follow the proof of Lemma 2.2 in [LM09b]. Denote by $v_i(s)$ the i-th co-ordinate in the valuation vector of a section s. Note then $v_i(s) \leq mb$ for some large constant b and for all non-zero $s \in H^0(U, m\overline{D})$ due to the fact that $\Delta(\overline{D})$ is bounded (Lemma 2.5.4) as $\Delta(\overline{D})$ contains $\frac{1}{m} \cdot \Gamma(\overline{D})_m$ for each $m \in \mathbb{N}$. Now a basic algebraic calculation easily shows that $\Gamma(\overline{D})$ is contained in the semi-group generated by the finite set of integer vectors $\{(a_i) \mid 0 \leq a_i \leq b\}$ which shows the second point. Hence it is enough to prove the third point.
To this end, choose a model X_j which satisfies the condition of Lemma 2.6.2. Then choose a very ample divisor $\overline{A}$ on X_j such that there exists sections $\overline{s_i}$ of $O(\overline{A})$ for $i = 0, 1, \ldots d$ with $v(\overline{s_i}) = e_i$ where v is the valuation vector with respect to the chosen flag and $\{e_i\}$ is the standard basis of $\mathbb{R}^d$ for $i = 1, \ldots d$ and e_0 is the zero vector, as suggested in the beginning of the proof of Lemma 2.2 in [LM09b]. Restricting these sections give sections $s_i \in H^0(U, \overline{A})$ with $v(s_i) = e_i$. Now thanks to Lemma 2.6.2 and our choice of X_j, we can find non-zero sections $t_i \in H^0(U, (m_0 + i)\overline{D} - \overline{A})$ for $i = 0, 1$ with valuation vectors $v(s_i) = f_i$. Then clearly we find non-zero sections $s'_i = s_i \otimes t_0 \in H^0(U, m_0\overline{D})$ and $s''_0 = s_0 \otimes t_1 \in H^0((m+1)\overline{D})$ with valuation vectors $v(s'_i) = (f_0 + e_i)$ for $i = 0, \ldots d$ and $v(s''_i) = f_1$. Hence $\Gamma(\overline{D})$ contains the vectors (f_0, m_0), $(f_0 + e_i, m_0)$ for $i = 1 \ldots d$ and $(f_1, m_0 + 1)$. Then it clearly shows that $\Gamma(\overline{D})$ generated $\mathbb{Z}^d$ as a group and finishes the proof. ☐

We are ready to state the first main theorem of this chapter.

Theorem 2.6.4. *Suppose we have a big adelic divisor $\overline{D}$ on a normal quasi-projective variety U and suppose $\Delta(\overline{D})$ is the Okounkov body associated to $\overline{D}$ as constructed above. Furthermore suppose $\widehat{\mathrm{vol}}(\overline{D})$ be the adelic volume defined in section 5 of [YZ24]. Then we have*

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(\overline{D})) = \lim_{m\to\infty} \frac{\#\Gamma_m}{m^d} = \lim_{m\to\infty} \frac{\dim_K(H^0(U, m\overline{D}))}{m^d} = \frac{1}{d!} \cdot \widehat{\mathrm{vol}}(\overline{D})$$

Proof. With Lemma 2.6.3 and by basic arguments of euclidean and convex geometry as indicated in the proof of Proposition 2.1 of [LM09b], we get that

$$\mathrm{vol}_{\mathbb{R}}(\Delta(\overline{D})) = \lim_{m\to\infty} \frac{\#\Gamma_m}{m^d} = \lim_{m\to\infty} \frac{\dim_K(H^0(U, m\overline{D})}{m^d} \tag{2.2}$$

exists which clearly gives the claim. ☐

Remark 5. Note that the above theorem also proves that the lim sup in the definition of $\widehat{\mathrm{vol}}(\overline{D})$ is actually given by a limit directly from convex geometric properties of the Okounkov bodies which is essentially the content of the first part of Theorem 5.2.1 of [YZ24].

We end this section by showing that the construction of Okounkov body is homogenous with respect to scaling.

Lemma 2.6.5. *Suppose $\overline{D}$ is a big adelic divisor on a normal quasi-projective variety U. Then*

$$\Delta(t\overline{D}) = t \cdot \Delta(\overline{D})$$

for all positive integers t. Hence we can naturally extend the construction of $\Delta(\cdot)$ to big adelic $\mathbb{Q}$-divisors.

Proof. We choose an integer r_0 such that $H^0(U, r\overline{D}) \neq \{0\}$ for all $r > r_0$. We can always do this as we assumed $\overline{D}$ is big (as explained in the proof of Lemma 2.6.3). Next choose $q_0 > 0$ such that $q_0 t - (t + r_0) > r_0$ for all t. Then for all $r_0 + 1 \leq r \leq r_0 + t$ we can find non-zero sections $s_r \in H^0(U, r\overline{D})$ and $t_r \in H^0(U, (q_0 t - r)\overline{D})$ which gives inclusions

$$H^0(U, mt\overline{D}) \xhookrightarrow{\otimes s_r} H^0(U, (mt + r)\overline{D}) \xhookrightarrow{\otimes t_r} H^0(U, (m + q_0)t\overline{D})$$

which gives the corresponding inclusion of the graded semi-groups

$$\Gamma(t\overline{D})_m + e_r + f_r \subseteq \Gamma(\overline{D})_{mt+r} + f_r \subseteq \Gamma(t\overline{D})_{m+q_0}$$

where $e_r = v(s_r)$ and $f_r = v(t_r)$. Now recalling the construction of $\Delta(\cdot)$ and letting $m \to \infty$ we get

$$\Delta(t\overline{D}) \subseteq t \cdot \Delta(\overline{D}) \subseteq \Delta(t\overline{D})$$

which clearly finishes our proof. □

Remark 6. Note that this homogeneity allows us to define Okounkov bodies for adelic $\mathbb{Q}$-divisors by passing to integral multiples and hence conclude that adelic volumes are homogenous for big divisors as in the projective case.

2.7 Variation of Okounkov bodies

We fix a normal quasi-projective variety U over K and a big adelic divisor $\overline{D}$ on it. Furthermore suppose $\overline{E}$ is any adelic divisor on U. We will construct a global convex body $\Delta(U) = \Delta(U, \overline{D}, \overline{E}) \subseteq \mathbb{R}^d \times \mathbb{R}^2$ such that the fiber of this body over a vector $(a_1, a_2) \in \mathbb{Q}^2$ under the projection to $\mathbb{R}^2$ will give us the Okounkov body of the adelic $\mathbb{Q}$-divisor $a_1\overline{D} + a_2\overline{E}$ provided it is big. Furthermore we fix a flag $Y_d \subset \ldots \subset Y_0$ as before. We are going to follow closely the arguments in Section 4 of [LM09b]. All constructions are dependent on the choice of the divisor $\overline{D}$ and $\overline{E}$ but we fix them for this section and we omit them in the notation. We start by defining the semi-group associated to these two adelic divisors.

Definition 2.7.1. *Suppose $\overline{D}$ and $\overline{E}$ are as before. We define the graded semi-group $\Gamma(U)$ as*

$$\Gamma(U) = \{((v(s), a_1, a_2) \mid a_i \in \mathbb{Z}, s \neq 0 \in H^0(U, a_1\overline{D} + a_2\overline{E})\}$$

where $v(\cdot)$ is the valuation corresponding to the chosen flag. Further more we define the global Okounkov body $\Delta(U)$ as

$$\Delta(U) = \textit{closed convex cone}(\Gamma(U))$$

which is a closed convex subset of $\mathbb{R}^d \times \mathbb{R}^2$.

As in the case with one bundle, we will deduce the properties needed from general properties of convex bodies and graded semi-groups. Before doing that we define certain terms necessary.

Definition 2.7.2. *Suppose we have an additive semi-group* Γ *in* $\mathbb{R}^d \times \mathbb{R}^2$*. Denote by* P *the projection from* $\mathbb{R}^d \times \mathbb{R}^2$ *to* $\mathbb{R}^2$ *and* $\Delta = \Sigma(\Gamma)$ *is the closed convex cone generated by* Γ*. We define the* support *of* Δ*, denoted as Supp*(Δ) *to be its image under* P*. It coincides with the closed convex cone in* $\mathbb{R}^2$ *generated by the image of* Γ *under* P*. Finally given a vector* $\vec{a} = (a_1, a_2) \in \mathbb{Z}^2$ *we denote*

$$\Gamma_{\mathbb{N}\vec{a}} = \Gamma \cap (\mathbb{N}^r \times \mathbb{N}\vec{a})$$

$$\Delta_{\mathbb{R}\vec{a}} = \Delta \cap (\mathbb{R}^d \times \mathbb{R}\vec{a})$$

Furthermore we denote $\Gamma_{\mathbb{N}\vec{a}}$ *as a semi-group inside* $\mathbb{N}^d \times \mathbb{N}\vec{a} \cong \mathbb{N}^{d+1}$ *and denote the closed convex cone generated by it in* $\mathbb{R}^{d+1}$ *as* $\Sigma(\Gamma_{\mathbb{N}\vec{a}})$*.*

With the above definitions we can state our next lemma.

Lemma 2.7.3. *Suppose the semi-group* Γ *generates a sub-group of finite index in* $\mathbb{Z}^{d+2}$ *and suppose* $\vec{a} \in \mathbb{N}^2$ *such that* $\vec{a} \in \operatorname{int}(\operatorname{Supp}(\Delta))$*. Then we have*

$$\Delta_{\mathbb{R}\vec{a}} = \Sigma(\Gamma_{\mathbb{N}\vec{a}})$$

Proof. The statement and the proof of the Lemma is identical as in Proposition 4.9 of [LM09b]. ☐

Next we want to show that the vectors which gives rise to big combinations of the bundles $\overline{D}$ and $\overline{E}$ in fact belong to the interior $\operatorname{int}(\operatorname{Supp}(\Delta))$ which is the content of the next lemma. Note that by passing to rational multiples just as in the projective case, we can similarly define $\mathbb{Q}$-adelic divisors. Furthermore by the remark at the end of the previous section, we can also define Okounkov bodies for $\mathbb{Q}$-adelic divisors which behave homogenously.

Lemma 2.7.4. *Suppose* $\vec{a} \in \mathbb{Q}^2$ *such that* $a_1\overline{D} + a_2\overline{E}$ *is a big adelic divisor. Then* $\vec{a} \in \operatorname{int}(\operatorname{Supp}(\Delta))$*.*

Proof. We assume that $\dim(U) > 0$ as the 0-dimensional case is degenerate. Clearly it is enough to show the case when $a_i \in \mathbb{Z}$ because $\operatorname{Supp}(\Delta)$ is a cone and scaling sends open sets to open sets. We can assume that both $\overline{D}$ and $\overline{E}$ are given by models D_i and E_i on projective models X_i of U respectively along with a boundary divisor D_0 and rationals $q_i \to 0$ as in our usual notation. We first prove that for any rational $q \in \mathbb{Q}$ such that $\overline{D} + q\overline{E}$ is big, there is an $\epsilon > 0$ such that $(1, x)$ is in $\operatorname{Supp}(\Delta)$ for all $x \in (q-\epsilon, q+\epsilon)$. Suppose first that $q > 0$. Then note that the sequence of models $S_j = (D_j - q_jD_0) + q(E_j - q_jD_0)$ gives a Cauchy sequence defining $\overline{D} + q\overline{E}$ and hence by Lemma 2.5.2 we get that S_j is big for large enough j. Now due to the continuity of the volume function in the projective setting, we can find a rational $0 < q < p$ such that $(D_j - q_jD_0) + p(E_j - q_jD_0)$ is big. Now due to the effectivity relation

$$(D_j - q_jD_0) + p(E_j - q_jE_0) \leq \overline{D} + p\overline{E}$$

we deduce that the right hand side above is big. Hence we get that for some positive integer p_0, $p_0 \cdot (1, p) \in P(\Gamma)$ where $P\colon \mathbb{R}^d \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection and $\Gamma = \Gamma(\overline{D})$. As $\overline{D}$ is assumed to be big, we also obtain that $r_0 \cdot (1, 0) \in P(\Gamma)$ as well for some large positive integer r_0. As p_0 and r_0 are positive, it is enough to find an $\epsilon > 0$ such that $(1, x)$ is in the convex cone generated by $(1, 0)$ and $(1, p)$ for all $x \in (q-\epsilon, q+\epsilon)$ because $\operatorname{Supp}(\Delta)$ is exactly the convex cone generated by $P(\Gamma)$. But clearly $(1, x)$ is in the convex cone generated by $(1, 0)$ and $(1, p)$ for all $0 < x < p$ which clearly yields the existence of one such ϵ because $0 < q < p$. For the case when $q < 0$ we do a similar calculation but with $E_j - q_jD_0$ being replaced by $E_j + q_jD_0$. Finally for the case $q = 0$ using similar arguments we can find a rational number q_0 such that all the three vectors $p_0 \cdot (1, 0)$, $p_0 \cdot (1, -q_0)$ and $p_0 \cdot (1, q_0)$ are in $P(\Gamma)$ for some large positive integer p_0. Hence by the above arguments we get that $(1, x)$ is in $\operatorname{Supp}(\Delta)$ for $x \in (-q_0, q_0)$.
Next we take any $\vec{a} = (a_1, a_2)$ such that $a_1\overline{D} + a_2\overline{E}$ is big. First suppose $a_1 \leq 0$. It is easy to see that the sum of two big adelic divisors is again big. Hence adding $(-a_1)\overline{D}$ we conclude that $a_2\overline{E}$ is big. Since the trivial adelic divisor is not big, we conclude that $a_2 \neq 0$. Then adding the big adelic divisor

$-a_1\overline{D}$ we deduce that $\overline{E}$ (resp. -$\overline{E}$) is big if $a_2 > 0$ (resp. $a_2 < 0$). Hence in these two cases replacing $\overline{D}$ by $\overline{E}$ or $-\overline{E}$ we are reduced to the case when $a_1 > 0$ and hence we can assume WLOG that $a_1 > 0$. In that case scaling by a_1 we obtain that $\overline{D} + q\overline{E}$ is big for $q = \frac{a_2}{a_1}$ and by our considerations before we deduce that for some $\epsilon > 0$, $(1, x)$ is in the convex cone generated by $P(\Gamma)$ for all $x \in (q-\epsilon, q+\epsilon)$. We assume that $a_2 \geq 0$ and the argument for $a_2 < 0$ will just be the analogue by changing signs. Then for any $\kappa > 0$ we have

$$\frac{a_2 - \kappa}{a_1 + \kappa} \leq \frac{a_2 + t_2}{a_1 + t_1} \leq \frac{a_2 + \kappa}{a_1 - \kappa}$$

for all $t_1, t_2 \in (-\kappa, +\kappa)$. Choose $\kappa > 0$ so small that

$$(\frac{a_2 - \kappa}{a_1 + \kappa}, \frac{a_2 + \kappa}{a_1 - \kappa}) \subset (q - \epsilon, q + \epsilon)$$

and $a_1 \pm \kappa > 0$ which we can do as $q = \frac{a_2}{a_1}$ and $a_1 > 0$. Hence by the choice of ϵ for any $t_1, t_2 \in (-\kappa, \kappa)$, the vector $(1, \frac{a_2+t_2}{a_1+t_1})$ and hence $(a_1 + t_1, a_2 + t_2)$ is in the convex cone generated by $P(\Sigma)$ and hence in $\mathrm{Supp}(\Delta)$ as $a_1 + t_1 > 0$. This clearly shows that $(a_1, a_2) \in \mathrm{int}(\mathrm{Supp}(\Delta))$ and finishes the proof. □

Next to use Lemma 2.7.3 we have to prove that $\Gamma(U)$ generates a sub-group of finite index in $\mathbb{Z}^{d+2}$ which in particular guarantees that $\mathrm{int}(\mathrm{Supp}(\Delta(U))$ is non-empty. This is going to be the content of our next Lemma.

Lemma 2.7.5. *The multi-graded semi-group $\Gamma(U)$ constructed in Definition 2.7.1 generates $\mathbb{Z}^{d+2}$ as a group.*

Proof. Arguing similarly as in the proof of Lemma 2.7.4, as $\overline{D}$ is big, we can find a positive integer m such that $m\overline{D} - \overline{E}$ is big. On the other hand we already know that $\overline{D}$ is big. Note that the semi-groups $\Gamma(m\overline{D} - \overline{E})$ and $\Gamma(\overline{D})$ sit naturally as sub-semigroups of $\Gamma(U)$. Moreover from Lemma 2.6.3 we deduce that $\Gamma(\overline{D})$ and $\Gamma(m\overline{D} - \overline{E})$ generate the sub-groups $\mathbb{Z}^d \times \mathbb{Z} \cdot (1, 0)$ and $\mathbb{Z}^d \times \mathbb{Z} \cdot (m, -1)$. But the vectors $(1, 0)$ and $(m, -1)$ generate $\mathbb{Z}^2$ which clearly shows that $\Gamma(U)$ generates $\mathbb{Z}^{d+2}$ as a group. □

Finally we are ready to state and prove the main theorem of this section.

Theorem 2.7.6. *Suppose $\overline{D}$ and $\overline{E}$ be adelic divisors on a normal quasi-projective variety U such that $\overline{D}$ is big. Then there exists a convex body $\Delta(U) = \Delta(U, \overline{D}, \overline{E}) \subset \mathbb{R}^{d+2}$ with the property that for any $\vec{a} = (a_1, a_2) \in \mathbb{Q}^2$ with $a_1\overline{D} + a_2\overline{E}$ big, we have*

$$\Delta(a_1\overline{D} + a_2\overline{E}) = \Delta(U) \cap (\mathbb{R}^d \times \{\vec{a}\})$$

where $\Delta(a_1\overline{D} + a_2\overline{E})$ is the Okounkov body of $a_1\overline{D} + a_2\overline{E}$ as constructed in Definition 2.5.1.

Proof. Clearly it is enough to show when $\vec{a} \in \mathbb{Z}^2$ by homogeneity of Okounkov bodies (Lemma 2.6.5). Note that the semi-group $\Gamma(a_1\overline{D} + a_2\overline{E})$ sits naturally in $\mathbb{N}^d \times \mathbb{N} \cdot \vec{a} \cong \mathbb{N}^{d+1}$ and by construction of $\Delta(\cdot)$ as in Definition 2.5.1, we deduce that $\Delta(a_1\overline{D} + a_2\overline{E}) = \Sigma(\Gamma(U)_{\mathbb{N}\vec{a}}) \cap (\mathbb{R}^d \times \{\vec{a}\})$. By Lemma 2.7.4 we get that $\vec{a} \in \mathrm{int}(\mathrm{Supp}(\Delta(U))$ and hence by Lemma 2.7.3 we have $\Delta(U)_{\mathbb{R}\vec{a}} = \Sigma(\Gamma(U)_{\mathbb{N}\vec{a}})$. Hence we deduce that

$$\Delta(a_1\overline{D} + a_2\overline{E}) = \Sigma(\Gamma(U)_{\mathbb{N}\vec{a}}) \cap (\mathbb{R}^d \times \{\vec{a}\}) = \Delta(U)_{\mathbb{R}\vec{a}} \times (\mathbb{R}^d \times \{\vec{a}\}) = \Delta(U) \cap (\mathbb{R}^d \times \{\vec{a}\})$$

concluding the proof. □

Remark 7. The construction of the Global body $\Delta(U, \overline{D}, \overline{E})$ is done here by mimicking the constructions in section 4 of [LM09b]. However one stark difference is that the Global body constructed in [LM09b] is independent of the chosen basis of the Neron-Severi group because they work modulo numerical equivalences. However even if there can be a notion of "numerical equivalence" in the adelic setting,it is certainly not known if the corresponding Neron-Severi space is finitely generated and hence such a "canonical global body" cannot be constructed using similar methods and our $\Delta(U, \overline{D}, \overline{E})$ is dependent on the chosen divisors $\overline{D}$ and $\overline{E}$. However our version still gives some interesting corollaries which we shall see next.

2.8 Corollaries : Continuity, Fujita approximation and more

Before going to state our first corollary, we introduce the notion of *Hausdorff distance* which will be the correct metric under which we want to show the convergence of bodies.

Definition 2.8.1. *Let $(V, \|\cdot\|)$ be a normed real vector space. The* Hausdorff distance *between two closed compact subsets C_1 and C_2 in V is defined as*

$$d_H(C_1, C_2) = inf\{\epsilon > 0 \mid C_1 \subseteq C_2 + \epsilon\mathbb{B}, C_2 \subseteq C_1 + \epsilon\mathbb{B}\}$$

where $\mathbb{B}$ is the unit ball in V with respect to $||\cdot||$.

Now we can state our first main corollary.

Corollary 2.8.2. *Suppose $\overline{D}$ is a big adelic divisor on a normal quasi-projective variety U given by models $\{X_i, D_i\}$ in our usual notation. Then*

$$\lim_{j\to\infty} d_H(\Delta(\overline{D}), \Delta(\overline{D_j})) = 0$$

where $\overline{D_j}$ is just D_j looked at as a model divisor in $\mathrm{Div}(U,k)_{\mathrm{mod}}$. In particular, we have

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{j\to\infty} \mathrm{vol}(D_j)$$

where $\mathrm{vol}(\cdot)$ is the classical projective volume considering D_j as a $\mathbb{Q}$-divisor in X_j.

Proof. We prove the first claim at first. Begin by noting that the sequence of inclusions

$$\Delta(\overline{D} - q_j\overline{D_0}) \subseteq \Delta(\overline{D_j}) \subseteq \Delta(\overline{D} + q_j\overline{D_0})$$

implies that it is enough to show that $d_H(\Delta(\overline{D} - q_jD_0), \Delta(\overline{D} + q_jD_0)) \to 0$ as $j \to \infty$. But this immediately follows from Theorem 2.7.6 taking $\overline{E} = \overline{D_0}$ and Theorem 13 in [Kho12] noting that $q_j \to 0$ as $j \to \infty$. Now the second claim follows readily from Theorem 7 in [SW65a] and the first claim noting that $\mathrm{vol}(D_j) = \widehat{\mathrm{vol}}(\overline{D_j}) = d! \cdot \mathrm{vol}(\Delta(\overline{D_j}))$ and $\mathrm{vol}(\overline{D}) = d! \cdot \mathrm{vol}(\Delta(\overline{D}))$. □

Remark 8. Note that Corollary 2.8.2 and Theorem 2.6.4 prove Theorem 5.2.1 of [YZ24] for big adelic divisors independently using convex geometric methods and hence we can deduce all the corollaries of section 5 of [YZ24] coming from Theorem 5.2.1 for big divisors which we list next.

Corollary 2.8.3 (log-concavity). *Suppose $\overline{D}_1$ and $\overline{D}_2$ are two effective adelic divisors on a normal quasi-projective variety U. Then we have*

$$\widehat{\mathrm{vol}}(\overline{D}_1 + \overline{D}_2)^{\frac{1}{d}} \geq \widehat{\mathrm{vol}}(\overline{D}_1)^{\frac{1}{d}} + \widehat{\mathrm{vol}}(\overline{D}_2)^{\frac{1}{d}}$$

where $d = \dim(U)$.

Proof. The statement is trivial if one of the divisors is not big. When both of them are big, applying Corollary 2.8.2 the problem gets converted into the projective case which is proved in Corollary 4.12 in [LM09b]. □

Corollary 2.8.4 (Fujita approximation). *Suppose $\overline{D}$ is a big adelic $\mathbb{Q}$-divisor on a normal quasi-projective variety U. Then for any $\epsilon > 0$ there exists a normal quasi-projective variety U', a birational morphism $\pi\colon U' \to U$, a projective model X' of U' and an ample $\mathbb{Q}$-divisor A' on X' such that $\pi^*\overline{D} - A' \geq 0$ in $\widehat{\mathrm{Div}}(U,K)$ and*

$$\mathrm{vol}(A') \geq \widehat{\mathrm{vol}}(\overline{D}) - \epsilon$$

where $\mathrm{vol}(A')$ is the volume of A' as a divisor on X'.

Proof. Using the fact that the adelic volume is the limit of its models in Corollary 2.8.2, the claim gets reduced to the original Fujita approximation which was proved in [Fuj94]. □

Next we come to the final corollary of this section which shows the continuity of the volume function.

Corollary 2.8.5 (continuity)**.** *Suppose $\overline{D}, \overline{M}_1, \dots \overline{M}_r$ are adelic $\mathbb{Q}$-divisors on a normal quasi-projective variety U. Then we have*

$$\lim_{t_1,t_2\dots t_r\to 0} \widehat{\mathrm{vol}}(\overline{D} + t_1\overline{M}_1 + \dots t_r\overline{M}_r) = \widehat{\mathrm{vol}}(\overline{D})$$

where $t_1 \dots t_r$ are rational numbers converging to 0. Furthermore we have $\widehat{\mathrm{vol}}(\overline{D}) = \lim_{j\to\infty} \mathrm{vol}(D_j)$ for a sequence of model D_j representing $\overline{D}$.

Proof. As in the proof of Theorem 5.2.8 in [YZ24], we choose nef model adelic divisors $\overline{M}'_i$ such that $\overline{M}'_i \pm \overline{M}_i \geq 0$ and we set $\overline{M} = \overline{M}'_1 + \dots \overline{M}'_r$. Then it is enough to show that

$$\lim_{t\to 0} \widehat{\mathrm{vol}}(\overline{D} + t\overline{M}) = \widehat{\mathrm{vol}}(\overline{D})$$

as t converges to 0 over the rationals. First assume that $\overline{D}$ is big. Then from Theorem 13 of [Kho12] we get

$$\lim_{t\to 0} d_H(\Delta(\overline{D} + t\overline{M}), \Delta(\overline{D})) = 0$$

by taking $\overline{E} = \overline{M}$ in Theorem 2.7.6 since we saw in the proof of Lemma 2.7.4 that $\overline{D} + t\overline{M}$ is big for small enough t whenever $\overline{D}$ is big. Now the claim follows from Theorem 7 of [SW65a]. The second claim is also true when $\overline{D}$ is big thanks to Corollary 2.8.2. Hence we can assume that $\overline{D}$ is not big. Now suppose the claim does not hold. Then there is a $c > 0$ and a sequence of rationals $t_i \to 0$ such that $\widehat{\mathrm{vol}}(\overline{D} + t_i\overline{M}) > c$ for all t_i. By Corollary 2.8.4 we can choose an ample $\mathbb{Q}$-divisor A_{t_i} on a projective model X' of a birational modification $\pi\colon U' \to U$ of U such that $\pi^*(\overline{D} + t_i\overline{M}) - A_{t_i} \geq 0$ and $\mathrm{vol}(A_{t_i}) > c/2$. Then clearly

$$\widehat{\mathrm{vol}}(\overline{D}) \geq \mathrm{vol}(A_{t_i} - t_i\overline{M}) \geq A_{t_i}^d - dt_iA_{t_i}^{d-1}\overline{M}$$

where in the second inequality we used the Siu's criterion for model nef divisors A_{ti} and $\overline{M}$. We can bound the intersection number $A_{t_i}^{d-1}\overline{M}$ as in the proof of Theorem 5.2.8 in [YZ24] to conclude that

$$\widehat{\mathrm{vol}}(\overline{D}) \geq A_{t_i}^d - O(t_i) > c/2 - O(t_i) \text{ as } t_i \to 0$$

which clearly contradicts the hypothesis $\widehat{\mathrm{vol}}(\overline{D}) = 0$ and finishes the proof of the first claim. Furthermore the effectivity relation $\overline{D}_j \leq \overline{D} + q_j\overline{D}_0$ shows that $\mathrm{vol}(D_j) \leq \widehat{\mathrm{vol}}(\overline{D} + q_j\overline{D}_0)$. Now as $j \to \infty$ we know that $q_j \to 0$ and hence by the first claim $\lim_{j\to\infty} \widehat{\mathrm{vol}}(\overline{D} + q_j\overline{D}_0) = 0$ which clearly shows the second claim. □

2.9 Augmented base loci and Restricted volumes

In this section, recall the concepts of base loci and stable base loci of a graded linear series of an adelic line bundle. Using these concepts we introduce the notion of the *augmented base locus* of an adelic divisor $\overline{D}$ in analogy to the projective setting (see [LM09b] section 2.4). In the projective setting, it is shown that the definition of augmented base locus is independent of the choice of the ample divisor using Serre's finiteness. However as in our setting, model divisors are only defined upto bi-rational pull-backs and ampleness is not preserved under such pull-backs, Serre's finiteness does not work. It turns out that this gap can be fixed using the main theorem due to [Bir17] and provides us with a similar independence of choice which will be the main result of this section.

Definition 2.9.1. *Suppose U is a normal quasi-projective variety over an algebraically closed field K and suppose D is a divisor. Furthermore suppose $W \subseteq H^0(U, O(D)) = H^0(U, D)$ is a finite dimensional sub-space of the space of global sections of $O(D)$. Then we define the* base locus

$$\mathrm{Bs}(W) = \{p \in U \mid s(p) = 0 \text{ in } \kappa(p) = O_{U,p}/m_{U,p} \text{ for all } s \in W\}$$

Now suppose we have a graded linear series $W = \{W_m\}$ of $O(D)$. We define the stable base locus *as*

$$\mathrm{SB}(W) = \cap_{m \in \mathbb{N}} \mathrm{Bs}(W_m)$$

Finally suppose $\overline{D}$ is an adelic divisor on U. Then it determines graded linear series $W = \{W_m = H^0(U, m\overline{D})\}$ as explained in the beginning of chapter 1. Then we define the base locus *and* stable base locus *of $\overline{D}$ as*

$$\mathrm{Bs}(\overline{D}) = \mathrm{Bs}(W_1) \text{ and } \mathrm{SB}(\overline{D}) = \mathrm{SB}(W)$$

Remark 9. Note that it is easy to check that the stable base locus $\mathrm{SB}(\overline{D})$ is indeed eventually stable *i.e* there exists an integer p_0 such that $\mathrm{SB}(\overline{D}) = \mathrm{Bs}(p_0\overline{D})$ by using noetherianity of U just like in the projective case.

As discussed above we want to show that this above notion is invariant under passing to other model ample divisors. Our next lemma is the main ingredient to show that

Lemma 2.9.2. *Suppose X_1 and X_2 are two normal projective models of a normal quasi-projective variety U over K, $f\colon X_1 \to X_2$ a birational morphism which is an isomorphism over U and $\overline{A}_1$, $\overline{A}_2$ ample divisors on X_1 and X_2 respectively. Furthermore suppose $\overline{D}$ is an adelic divisor on U. Then for any closed irreducible sub-variety E of U, $E \nsubseteq \mathrm{Bs}(m_0\overline{D} - \overline{A}_2)$ for some positive integer m_0 if and only if $E \nsubseteq \mathrm{Bs}(n_0\overline{D} - \overline{A}_1)$ for some positive integer n_0.*

Proof. We first suppose that $E \nsubseteq \mathrm{Bs}(m_0\overline{D} - \overline{A}_2)$. We denote $f^*\overline{A}_2 = \overline{A}_2'$ which is a big nef divisor on X_1 as $\overline{A}_2$ was big and nef(being ample) and this notions are invariant under bi-rational pull-backs. Let $\overline{E}$ be the Zariski closure of E in X_1. Then clearly $\overline{A}_2'|_{\overline{E}}$ is big and as $\overline{A}_2'$ is also nef, we can deduce from Theorem 1.4 of [Bir17] that for large enough integer s_0, $\overline{E}$ is not contained in the (projective) stable base locus of $s_0\overline{A}_2' - \overline{A}_1$ since $\overline{A}_1$ is ample in X_1. Restricting everything to U, we can find a positive integer p_0 and section $s' \in H^0(U, s_0p\overline{A}_2 - p\overline{A}_1)$ such that s' does not vanish along E whenever $p_0 \mid p$. Tensoring by a section of $H^0(U, (p-1)\overline{A}_1)$ non-vanishing on E, which we can find as $\overline{A}_1$ is ample, we produce a section $s \in H^0(s_0p\overline{A}_2 - \overline{A}_1)$ non-vanishing on E whenever $p_0 \mid p$. By hypothesis we can find a section $s_0 \in H^0(U, m_0s_0p_0\overline{D} - s_0p_0\overline{A}_2)$ non-vanishing along E. Hence picking $p = p_0$ and tensoring s and s_0 we produce a section in $H^0(U, m_0s_0p_0\overline{D} - \overline{A}_1)$ which does not vanish identically on E and hence $E \nsubseteq \mathrm{Bs}(n_0\overline{D} - \overline{A}_1)$ and finishes one direction of the claim with $n_0 = m_0s_0p_0$.
For the other side, suppose $E \nsubseteq \mathrm{Bs}(n_0\overline{D} - \overline{A}_1)$. Hence for every positive integer p we can find a section $s_0 \in H^0(U, n_0p\overline{D} - p\overline{A}_1)$ which does not vanish identically on E. Now chose p large enough such that $p\overline{A}_1 - \overline{A}_2'$ is very ample which we can do by Serre's finiteness theorem on projective varieties because $\overline{A}_1$ is ample on X_1. Then chosing a section of $p\overline{A}_1 - \overline{A}_2'$ on X_1 not vanishing identically on $\overline{E}$ and restricting to U, we obtain a section $s_0 \in H^0(U, p\overline{A}_1 - \overline{A}_2)$ not vanishing identically on E for large enough p. Once again tensoring s and s_0 we obtain that $E \nsubseteq \mathrm{Bs}(m_0\overline{D} - \overline{A}_2)$ with $m_0 = n_0p$ for large enough p and finishes the proof. □

Remark 10. The proof of the above lemma follows along similar lines as the independence of the augmented base locus on the choice of the ample divisor is shown in the projective case. However it uses Serre's finiteness theorem which has no known versions in the adelic setting due to non-invariance of ampleness under birational pull-backs. However it turns out the gap in one direction of the proof can be bridged by the main result due to [Bir17] as we have shown above and in the other direction we already have Serre finiteness.

Finally we can deduce the the desired invariance under pull-backs of model ample divisors as a direct corollary of Lemma 2.9.2 which we do next.

Corollary 2.9.3. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K and suppose X_1 and X_2 are two projective models of E with ample divisors $\overline{A}_1$ and $\overline{A}_2$ respectively on them. Then for any closed irreducible sub-variety E of U, we have that $E \nsubseteq \mathrm{Bs}(m_0\overline{D} - \overline{A}_2)$ for some positive integer m_0 if and only if $E \nsubseteq \mathrm{Bs}(n_0\overline{D} - \overline{A}_1)$ for some positive integer n_0. In particular the set $B_+(\overline{D}, \overline{A}) = \cap_{m\in\mathbb{N}}\mathrm{Bs}(m\overline{D} - \overline{A})$ is independent of the chosen model ample divisor $(X, \overline{A})$.*

Proof. Clearly the the second claim follows from the first and the first claim follows directly from Lemma 2.9.2 by noting that we can always find a projective model X of U dominating both X_1 and X_2 via a birational morphism over U and an ample divisor on X. □

The above corollary clearly shows what should be the definition of our augmented base locus which we record in the next definition.

Definition 2.9.4. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi projective variety U over K. We define the* augmented base locus *of $\overline{D}$ as $B_+(\overline{D}) = \cap_{m\in\mathbb{N}}\mathrm{Bs}(m\overline{D} - \overline{A})$ for any ample divisor $\overline{A}$ on a projective model of U.*

Remark 11. Note that the above definition makes sense thanks to Corollary 2.9.3. It is easy to check that $B_+(m_0\overline{D}) = B_+(\overline{D})$ for any positive integer m_0 and hence we can define an augmented base locus of an adelic $\mathbb{Q}$-divisor by passing to integral multiples.

We end this section with a lemma which will be necessary later to show that the Okounkov bodies of restricted linear series behave nicely when the sub-variety is not contained in the augmented base locus.

Corollary 2.9.5. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K and suppose E is a closed irreducible sub-variety with $E \nsubseteq B_+(\overline{D})$. Then there exist a projective model X such that for any ample divisor $\overline{A}$ on X, there exists sections $s_i \in H^0(U, (m_0 + i)\overline{D} - p_i\overline{A})$ not vanishing identically on E for some positive integers m_0, p_0, p_1 and $i = 0, 1$.*

Proof. Suppose $\overline{D}$ is given a sequence of models $\{X_i, D_i\}$ and rationals $q_i \to 0$ as usual and let $X = X_1$. Then as $E \nsubseteq B_+(\overline{D})$, for any ample divisor $\overline{A}$ on X_1 we can assume that $E \nsubseteq \mathrm{Bs}(n_0\overline{D} - \overline{A})$ for some $n_0 \in \mathbb{N}$ and hence we can produce a section $s_0 \in H^0(U, 2n_0p\overline{D} - 2p\overline{A})$ not vanishing identically on E for every positive integer p. Choose p so large that $D_1' + p\overline{A}$ is very ample where $D_1' = D_1 - q_1D_0$ and choose a section $s' \in H^0(U, D_1' + p\overline{A})$ which does not vanish identically on E. Then tensoring s_0 and s' we get a section $s_1 \in H^0(U, 2n_0p\overline{D} + D_1' - p\overline{A}) \subseteq H^0(U, (2n_0p + 1)\overline{D} - p\overline{A})$ where the inclusion follows from the effectivity relation $D_1' \leq \overline{D}$. Clearly s_0 and s_1 satisfy the claim with $m_0 = 2n_0p$, $p_1 = 2p$ and $p_2 = p$. □

2.10 Restricted volumes

In this section, we define the restricted volume of an adelic divisor along a closed sub-variety E of U in analogy to the projective setting. Then we go on to show that if E is such an irreducible closed sub-variety with $E \nsubseteq B_+(\overline{D})$, then this restricted volume can be realised as the volume of an Okounkov body calculated with respect to a suitable flag dominated by E. Much in the spirit of Theorem 2.6.4 we deduce that the lim sup defining the restricted volume is actually a limit.

Suppose we have an irreducible closed sub-variety $E \overset{i}{\hookrightarrow} U$ embedding in U via the closed immersion i. Then as explained in sub-section 5.2.2 of [YZ24] we can consider the pullback of the adelic line bundle $O(\overline{D})$ by i which we denote as the *restriction* of $O(\overline{D})$ to E and denote as $O(\overline{D})|_E$. We recall that this line bundle is given by the datum $\{E_i, O(D_i)|_{E_i}\}$ where D_i are the ,models defining $\overline{D}$ and E_i are the Zariski closures of E in the projective models X_i of U. Then there is a restriction map of vector spaces on the space of global sections

$$H^0(U, O(\overline{D})) \xrightarrow{\text{restr}} H^0(E, O(\overline{D})|_E)$$

and we denote the image of this map by $H^0(U|E, O_E(\overline{D}))$ obtained by just restriction maps on sections model wise. This lets us define the notion of restricted volume.

Definition 2.10.1. *Suppose E is a closed irreducible sub-variety of a normal quasi-projective variety U over K and $\overline{D}$ be an adelic divisor on U. Then we define the* restricted volume *of $\overline{D}$ along E as*

$$\widehat{\mathrm{vol}}_{U|E}(\overline{D}) = \limsup_{m\to\infty} \frac{dim_K(H^0(U|E, O_E(m\overline{D}))}{m^d/d!}$$

where $d = dim(E)$.

We can view the finite-dimensional vector spaces $W_m = H^0(U|E, O_E(m\overline{D}))$ as a graded linear subseries of $H^0(E, O(m\overline{D})|_E) \subseteq H^0(E, O(mD)|_E)$. And hence if we can fix a flag in E, we can construct an Okounkov body corresponding to $\{W_m\}$ as indicated in section 1 of [LM09b].
Now given a closed sub-variety E in U, we fix a flag $Y_0 \subset Y_1 \ldots \subset Y_d = E$ in E where $\dim(E) = d$. Note that in any projective model of U this flag induces a canonical partial flag contained in the closure E_j of E in U by taking closures we obtain a flag in the model such that the (partial) valuation of a global section of some bundle with respect to this on the model is the same after restricting to U and evaluating w.r.t to the flag $Y_0 \subset Y_1 \ldots \subset Y_d$ and we always take this flag to calculate valuation vectors in the projective models. We fix this flag to calculate the Okounkov body of the linear series $\{W_m\}$. Then we have the notions of the graded semi-group $\Gamma_{U|E}(\overline{D}) \subseteq \mathbb{N}^{d+1}$ and the Okounkov body $\Delta_{U|E}(\overline{D}) \subseteq \mathbb{R}^d$. As in chapter 1, we also define $\Gamma_{U|E}(\overline{D})_m$ to be the fiber of the graded semi-group over the positive integer m. Next we show that when $E \nsubseteq B_+(\overline{D})$, then the Okounkov body behaves nicely in the sense of satisfying properties analogous to Lemma 2.6.3.

Lemma 2.10.2. *Suppose $\overline{D}$ is a adelic divisor on a normal quasi-projective variety U over K. Furthermore suppose E is a closed irreducible sub-variety of U such that $E \nsubseteq B_+(\overline{D})$. Then the graded semi-group $\Gamma_{U|E}(\overline{D})$ satisfies the following properties*

1. $\Gamma_{U|E}(\overline{D})_0 = \{0\}$
2. *There exists finitely many vectors $(v_i, 1)$ spanning a semi-group $B \subseteq \mathbb{N}^{d+1}$ such that $\Gamma_{U|E}(\overline{D}) \subseteq B$.*
3. $\Gamma_{U|E}(\overline{D})$ *generates $\mathbb{Z}^{d+1}$ as a group.*

Remark 12. Note that in analogy to Lemma 2.6.3 it is desirable that $\overline{D}$ is big in the above lemma. However since we assume that $E \nsubseteq B_+(\overline{D})$, by Definition 2.9.4 we already have a non-zero section $s \in H^0(m\overline{D} - \overline{A})$ for some model ample divisor $\overline{A}$ on a projective model X of U. Hence we have the inclusion

$$H^0(U, n\overline{A}) \xhookrightarrow{s^{\otimes n}} H^0(U, mn\overline{D})$$

for all positive integers n which shows that $\widehat{\mathrm{vol}}(m\overline{D}) \geq \widehat{\mathrm{vol}}(\overline{A}) > 0$ and hence $\overline{D}$ is big. In other words the assumption $E \nsubseteq B_+(\overline{D})$ already implies that $\overline{D}$ is big.

Proof. Suppose $\overline{D}$ is given by the sequence of models $\{X_i, D_i\}$ and rationals $q_i \to 0$ as usual. Note that as in the proof of Lemma 2.6.3, the first point is trivial and the second point can be deduced once we know that the vectors of $\Gamma_{U|E}(\overline{D})_m$ are bounded by mb for some large positive constant b as explained in the proof of Lemma 2.2 in [LM09b]. In other words we need to show that restricted graded series satisifes the condition (A) as defined in Definition 2.4 of [LM09b]. Note that the effectivity relation $\overline{D} \leq D_j + q_j D_0$ implies the inclusion $\Gamma_{U|E}(\overline{D})_m \subset \Gamma_{U|E}(\overline{D_j + q_j D_0})_m$ for all positive integers m. The right hand side is the same as the graded semi-group of $D_j + q_j D_0$ viewed as a $\mathbb{Q}$-divisor on the projective variety X_j calculated with closures of our flag on E and hence by the footnote on page 803 of [LM09b], we conclude that $\Gamma_{U|E}(\overline{D_j + q_j D_0})_m$ satisifes condition (A) which clearly shows the second point as $\Gamma_{U|E}(\overline{D})_m$ is a subset. Hence we just need to show the third point.

We argue as in the proof of Lemma 2.6.3. Choose a model very ample divisor $\overline{A}$ on X_j such that it has sections $\overline{s_i}$ on X_j with $v(\overline{s_i}) = (e_i)$ for $i = 0 \ldots d$ where e_0 is the zero vector, $\{e_i\}$ is the standard basis of $\mathbb{R}^d$ for $i = 1, \ldots d$ and $v(\cdot)$ is the valuation corresponding to the closures in X_j of the chosen flag in E. We can always do this as $\overline{A}$ is chosen very ample and hence the restriction $\overline{A}|_{E_j}$ is very ample where E_j is the closure of E in X_j, as explained in proof of Lemma 2.2 in [LM09b]. Restricting to U gives sections $s_i \in H^0(U, \overline{A})$ with the same valuation vectors. Note that then for all positive integers p, by appropriately tensoring these sections we can also find sections $s_{ip} = s_0^{\otimes p-1} \otimes s_i \in H^0(U, p\overline{A})$ such that $v(s_{ip}) = (e_i)$. Then by restricting to E, we get non-zero sections $s_{ip}|_E \in H^0(U|E, O_E(p\overline{A}))$ with $v(s_{ip}|_E) = e_i$. Now using Corollary 2.9.5 we can find positive integers m_0, p_0, p_1 and sections t_0, t_1 (recalling them t_i for notational convenience) satisfying the properties stated in the corollary. Restricting t_i's to E we get non-zero sections $t_i|_E \in H^0(U|E, O_E((m_0 + i)\overline{D} - p_i\overline{A}))$ and suppose $v(t_i|_E) = f_i$ for $i = 0, 1$. Then arguing like in the proof Lemma 2.6.3 by tensoring $s_{ip}|_E$'s with $t_i|_E$'s we conclude that the vectors (f_0, m_0), $(f_0 + e_i, m_0)$ and $(f_1, m_0 + 1)$ all belong to $\Gamma_{U|E}(\overline{D})$ which clearly completes the proof. □

Then arguing just like in chapter 1, we deduce the main theorem of this section which we state next.

Theorem 2.10.3. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K. Furthermore suppose E is a closed irreducible sub-variety of U such that $E \nsubseteq B_+(\overline{D})$. Then we have*

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta_{U|E}(\overline{D})) = \lim_{m\to\infty} \frac{\#\Gamma_{U|E}(\overline{D})_m}{m^d} = \lim_{m\to\infty} \frac{\dim_K(H^0(U|E, O_E(m\overline{D})))}{m^d} = \frac{1}{d!} \cdot \widehat{\mathrm{vol}}_{U|E}(\overline{D})$$

where $\dim(U) = d$

We end this section with a homogeneity property analogous to Lemma 2.6.5. Before going to that we obtain a crucial property needed for the homogeneity.

Lemma 2.10.4. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K and E is a closed sub-variety with $E \nsubseteq B_+(\overline{D})$. Then there exists an integer r_0 such that $H^0(U|E, O_E(r\overline{D})) \neq \{0\}$ for all positive integers $r > r_0$.*

Proof. By Corollary 2.9.5 there exist non-zero sections $s_i \in H^0(U, (m + i)\overline{D})$ which do not vanish identically on E for $i = 0, 1$ by tensoring with sections of $p_i\overline{A}$ which do not vanish identically on E which exists as $\overline{A}$ can be assumed very ample. Then for all $r \geq m^2$, write it as $r = a_r m + b_r$ for non-negative integers $a_r \geq m$ and $0 \leq b_r \leq m - 1 < a_r$. Then note that $s_0^{\otimes a_r - b_r} \otimes s_1^{b_r}$ is a section of $H^0(U, r\overline{D})$ which does not vanish identically on E which clearly finishes the claim with $r_0 = m^2 - 1$ □

Lemma 2.10.5. *Suppose $\overline{D}$ is an adelic divisor on a normal quasi-projective variety U over K. Then for any closed irreducible sub-variety E of U with $E \nsubseteq B_+(\overline{D})$, we have*

$$\Delta_{U|E}(t\overline{D}) = t \cdot \Delta_{U|E}(\overline{D})$$

for all positive integers t. Hence we can naturally extend the construction of $\Delta_{U|E}(\cdot)$ to big adelic $\mathbb{Q}$-divisors.

Proof. We choose an integer r_0 such that $H^0(U|E, O_E(r\overline{D})) \neq \{0\}$ for all $r > r_0$ thanks to Lemma 2.10.4. Next choose $q_0 > 0$ such that $q_0 t - (t + r_0) > r_0$ for all positive integers t. Then for all $r_0 + 1 \leq r \leq r_0 + t$ we can find non-zero sections $s_r \in H^0(U|E, O_E(r\overline{D}))$ and $t_r \in H^0(U|E, O_E((q_0 t - r)\overline{D}))$ which gives inclusions

$$H^0(U|E, O_E(mt\overline{D})) \xhookrightarrow{\otimes s_r} H^0(U|E, O_E((mt + r)\overline{D})) \xhookrightarrow{\otimes t_r} H^0(U|E, O_E((m + q_0)t\overline{D}))$$

which gives the corresponding inclusion of the graded semi-groups

$$\Gamma_{U|E}(t\overline{D})_m + e_r + f_r \subseteq \Gamma_{U|E}(\overline{D})_{mt+r} + f_r \subseteq \Gamma_{U|E}(t\overline{D})_{m+q_0}$$

where $e_r = v(s_r)$ and $f_r = v(t_r)$. Now recalling the construction of $\Delta_{U|E}(\cdot)$ and letting $m \to \infty$ we get

$$\Delta_{U|E}(t\overline{D}) \subseteq t \cdot \Delta_{U|E}(\overline{D}) \subseteq \Delta_{U|E}(t\overline{D})$$

which clearly finishes our proof. □

2.11 Variation of bodies for restricted volumes

In this section, we construct global bodies whose fibers give the Okounkov bodies $\Delta_{U|F}(\cdot)$ for a "sufficiently general" closed sub-variety F of U much in analogy with Theorem 2.7.6. Most of the constructions follow analogously as in the global case. The crucial point that we need to show is that given a fixed irreducible sub-variety F, the set of divisors $\overline{D}$ with $F \nsubseteq B_+(\overline{D})$ is in the interior of the support of the global body as was shown in Lemma 2.7.4. Most of the other arguments will follow identically as in section 5 of chapter 1. However for sake of clarity we will anyway repeat some constructions. We fix a flag $Y_0 \subset \ldots Y_d = F$ as explained in the previous section and all calculations of Okounkov bodies is with respect to this flag.

Definition 2.11.1. *Suppose $\overline{D}$ and $\overline{E}$ are two adelic divisors on a normal quasi-projective variety U over K. Given a closed irreducible sub-variety F of U with $F \nsubseteq B_+(\overline{D})$, we define the graded semi-group $\Gamma_{U|F}(F)$ as*

$$\Gamma_{U|F}(F) = \{((v(s), a_1, a_2) \mid a_i \in \mathbb{Z}, s \neq 0 \in H^0(U|F, O_F(a_1\overline{D} + a_2\overline{E}))\}$$

where $v(\cdot)$ is the valuation corresponding to the chosen flag. Further more we define the global Okounkov body $\Delta(U)$ as

$$\Delta_{U|F}(F) = \textit{closed convex cone}(\Gamma_{U|F}(F)) = \Sigma(\Gamma_{U|F}(F))$$

which is a closed convex subset of $\mathbb{R}^d \times \mathbb{R}^2$.

Definition 2.11.2. *Suppose we have an additive semi-group Γ in $\mathbb{R}^d \times \mathbb{R}^2$. Denote by P the projection from $\mathbb{R}^d \times \mathbb{R}^2$ to $\mathbb{R}^2$ and $\Delta = \Sigma(\Gamma)$ is the closed convex cone generated by Γ. We define the* support *of Δ, denoted as Supp(Δ) to be its image under P. It coincides with the closed convex cone in $\mathbb{R}^2$ generated by the image of Γ under P. Finally given a vector $\vec{a} = (a_1, a_2) \in \mathbb{Z}^2$ we denote*

$$\Gamma_{\mathbb{N}\vec{a}} = \Gamma \cap (\mathbb{N}^r \times \mathbb{N}\vec{a})$$

$$\Delta_{\mathbb{R}\vec{a}} = \Delta \cap (\mathbb{R}^d \times \mathbb{R}\vec{a})$$

Furthermore we denote $\Gamma_{\mathbb{N}\vec{a}}$ as a semi-group inside $\mathbb{N}^d \times \mathbb{N}\vec{a} = \mathbb{N}^{d+1}$ and denote the closed convex cone generated by it in $\mathbb{R}^{d+1}$ as $\Sigma(\Gamma_{\mathbb{N}\vec{a}})$.

We begin by showing the crucial property of the "good" divisors being open.

Lemma 2.11.3. *Suppose $\overline{D}$ and $\overline{E}$ are two adelic divisors such that $F \nsubseteq B_+(\overline{D})$ and $F \nsubseteq B_+(\overline{D}+q\overline{E})$ for some $q \in \mathbb{Q}$. Then there is an $\epsilon > 0$ such that $(1, x) \in \mathrm{Supp}(\Delta_{U|F}(F))$ for all $x \in (q-\epsilon, q+\epsilon)$.*

Proof. Suppose both $\overline{D}$ and $\overline{E}$ are given by models D_i, E_i on projective models X_i and rationals $\{q_i \to 0\}$ as usual. Further more we denote $E_1' = D_1 - q_1 D_0$ and $E_1'' = E_1 + q_1 E_0$. We first consider the case when $q \neq 0$. Then by hypothesis there exists an integer m_0 depending on $\overline{A}$ such that $F \nsubseteq \mathrm{Bs}(m_0 p\overline{D} + m_0 qp\overline{E} - p\overline{A})$ for some very ample divisor $\overline{A}$ on X_1 and for all sufficiently divisible integers p. Choose p so large that that $E_1' + p\overline{A}$ (resp $-E_1'' + p\overline{A}$) is very ample when $q > 0$(resp. $q < 0$). Then choosing sections in

$$H^0(U, 2m_0 p\overline{D} + 2m_0 qp\overline{E} - 2p\overline{A}) \text{ and } H^0(U, p\overline{A} + E_1') \text{ (resp } H^0(U, p\overline{A} - E_1''))$$

which do not vanish identically on F and finally tensoring them, we get sections in $H^0(U, 2m_0p\overline{D} + 2m_0qp\overline{E} + E_1' - p\overline{A})$ (resp $H^0(U, 2m_0p\overline{D} + 2m_0qp\overline{E} - E_1'' - p\overline{A})$) which do not vanish identically on F. Now the effectivity relation $E_1' \leq \overline{E}$ (resp $E_1'' \geq \overline{E}$) induces the inclusion

$$H^0(U, 2m_0p\overline{D} + 2m_0qp\overline{E} + E_1' - p\overline{A}) \subseteq H^0(U, 2m_0p\overline{D} + (2m_0qp + 1)\overline{E} - p\overline{A})$$

$$(\text{resp. } H^0(U, 2m_0p\overline{D} + 2m_0qp\overline{E} - E_1'' - p\overline{A}) \subseteq H^0(U, 2m_0p\overline{D} + (2m_0qp - 1)\overline{E} - p\overline{A}))$$

when $q > 0$ (resp $q < 0$). Hence noting the remark at the end of Definition 2.9.4, we conclude that $F \nsubseteq B_+(2m_0p\overline{D} + (2m_0qp+1)\overline{D}) = B_+(\overline{D} + r\overline{E})$ (resp. $F \nsubseteq B_+(2m_0p\overline{D} + (2m_0qp-1)\overline{D}) = B_+(\overline{D} + r\overline{E})$) where $r = q + \frac{1}{2m_0p} > q$ (resp $r = q - \frac{1}{2m_0p} < q$) when $q > 0$ (resp $q < 0$). Note that then thanks to Lemma 2.10.4 we conclude that for some large integer p_0 the points $p_0(1, r),\ p_0(1, 0) \in \mathrm{Supp}(\Delta_{U|F}(F))$ as $F \nsubseteq B_+(\overline{D} + r\overline{E})$ and $F \nsubseteq B_+(\overline{D})$. Hence arguing as in the proof of Lemma 2.7.4 we obtain the claim. Finally for the case $q = 0$ we repeat the arguments above in both positive and negative directions with E_1' and E_1'' to obtain such an ϵ. □

As a corollary of the above, we obtain the necessary property which we record next.

Corollary 2.11.4. *Suppose $\overline{D}$ and $\overline{E}$ are adelic divisors on a normal quasi-projective variety U over K and let F be a closed sub-variety of U with $F \nsubseteq B_+(\overline{D})$. Then for any $\vec{a} = (a_1, a_2) \in \mathbb{Q}^2$ such that $a_1\overline{D} + a_2\overline{E}$ satisfies $F \nsubseteq B_+(a_1\overline{D} + a_2\overline{E})$ we have $\vec{a} \in \mathrm{int}(\mathrm{Supp}(\Delta_{U|F}(F)))$*

Proof. Due to the homogeneity property in Lemma 2.10.5 it is enough to show the claim for a_i integers. First note that if $\overline{D}_1$ and $\overline{D}_2$ are two adelic divisors with $F \nsubseteq B_+(\overline{D}_i)$ for $i = 1, 2$, then $F \nsubseteq B_+(\overline{D}_1 + \overline{D}_2)$. To see this pick a positive integer m such that $F \nsubseteq \mathrm{Bs}(m\overline{D}_i - \overline{A})$ for $i = 1, 2$ and some ample divisor $\overline{A}$ on some projective model X. Then choosing sections on each of the bundles non-vanishing on F and tensoring them, we produce a section in $H^0(U, m(\overline{D}_1 + \overline{D}_2) - 2\overline{A})$ which does not vanish identically on E which clearly shows that $F \nsubseteq B_+(\overline{D}_1 + \overline{D}_2)$ by definition of the augmented base locus.
Now first suppose that $a_1 \leq 0$. Then as $F \nsubseteq B_+(\overline{D})$ by hypothesis, by adding $(-a_1)\overline{D}$ we deduce using our discussion above that $F \nsubseteq B_+(a_2\overline{E}) = B_+(\overline{E})($ resp. $B_+(-\overline{E}))$ if $a_2 > 0($ resp. $a_2 < 0)$ by the remark at the end of Definition 2.9.4 and clearly $a_2 \neq 0$. Then switching $\overline{D}$ with $\overline{E}($ resp. $-\overline{E})$ we can assume that $a_1 > 0$. Then once again by the remark, we conclude that $F \nsubseteq B_+(a_1\overline{D} + a_2\overline{E}) = B_+(\overline{D} + q\overline{E})$ for $q = \frac{a_2}{a_1}$. Once we obtain this then thanks to Lemma 2.11.3 we can argue exactly as in the end of the proof of Lemma 2.7.4 to obtain the claim. □

Next we show that the interior of the the support is actually non-empty- To show this we show that the graded semi-group generates the whole $\mathbb{Z}^{d+2}$ in our next Lemma.

Lemma 2.11.5. *Suppose $\overline{D}$ and $\overline{E}$ are adelic divisors on a normal quasi-projective variety U over K and F a closed irreducible sub-variety of U with $F \nsubseteq B_+(\overline{D})$. Then $\Gamma_{U|F}(F)$ generates $\mathbb{Z}^{d+2}$ as a group.*

Proof. The proof is almost identical to the proof of Lemma 2.7.5. We just need to note that by the proof of Lemma 2.11.3, when $q = 0$ we can find a positive integer n such that $F \nsubseteq B_+(\overline{D} - \frac{1}{n}\overline{E}) = B_+(n\overline{D} - \overline{E})$. The rest of the argument is identical to Lemma 2.7.5 thanks to the third property in Lemma 2.10.2 and as $(1, 0)$ and $(n, -1)$ generate $\mathbb{Z}^2$ as a group. □

Finally we are ready to state and prove the main theorem of this section.

Theorem 2.11.6. *Suppose $\overline{D}$ and $\overline{E}$ be adelic divisors on a normal quasi-projective variety U over K and let F be an irreducible closed sub-variety such that $F \nsubseteq B_+(\overline{D})$. Then there exists a convex body $\Delta_{U|F}(F) = \Delta_{U|F}(F, \overline{D}, \overline{E}) \subset \mathbb{R}^{d+2}$ with the property that for any $\vec{a} = (a_1, a_2) \in \mathbb{Q}^2$ with $F \nsubseteq B_+(a_1\overline{D} + a_2\overline{E})$, we have*

$$\Delta_{U|F}(a_1\overline{D} + a_2\overline{E}) = \Delta_{U|F}(F) \cap (\mathbb{R}^d \times \{\vec{a}\})$$

where $\Delta_{U|F}(a_1\overline{D}+a_2\overline{E})$ *is the restricted Okounkov body of* $a_1\overline{D}+a_2\overline{E}$ *as constructed in section 2.*

Proof. Clearly it is enough to show when $\vec{a}\in\mathbb{Z}^2$ by homogeneity of Okounkov bodies(Lemma 2.10.5). Note that the semi-group $\Gamma_{U|F}(a_1\overline{D}+a_2\overline{E})$ sits naturally in $\mathbb{N}^d\times\mathbb{N}\cdot\vec{a}\cong\mathbb{N}^{d+1}$ and by construction of $\Delta_{U|F}(\cdot)$, we deduce that $\Delta_{U|F}(a_1\overline{D}+a_2\overline{E})=\Sigma(\Gamma_{U|F}(a_1\overline{D}+a_2\overline{E})_{\mathbb{N}\vec{a}})\cap(\mathbb{R}^d\times\{\vec{a}\})$. By Lemma 2.11.3 we get that $\vec{a}\in\operatorname{int}(\operatorname{Supp}(\Delta(U))$ and hence by Lemma 2.7.3 we have $\Delta_{U|F}(F)_{\mathbb{R}\vec{a}}=\Sigma(\Gamma_{U|F}(a_1\overline{D}+a_2\overline{E})_{\mathbb{N}\vec{a}})$. Hence we deduce that

$$\Delta_{U|F}(a_1\overline{D}+a_2\overline{E})=\Sigma(\Gamma_{U|F}(a_1\overline{D}+a_2\overline{E})_{\mathbb{N}\vec{a}})\cap(\mathbb{R}^d\times\{\vec{a}\})=\Delta_{U|F}(F)_{\mathbb{R}\vec{a}}\cap(\mathbb{R}^d\times\{\vec{a}\})=\Delta_{U|F}(F)\cap(\mathbb{R}^d\times\{\vec{a}\})$$

concluding the proof. □

2.12 Corollaries

In this section we deduce some corollaries which are direct from the existence of global bodies for restricted volumes as shown in Theorem 2.11.6. Note that we already have the notion of restricted volume of a line bundle L along the closed sub-variety E of a projective variety X defined similarly as defined before Lemma 2.16 in [LM09b] which we denote by *projective restricted volume* in the next corollary.

Corollary 2.12.1. *Suppose* $\overline{D}$ *is an adelic divisor on a normal quasi-projective variety* U *over* K *and suppose* F *is a closed irreducible sub-variety of* U *with* $F\nsubseteq B_+(\overline{D})$*. Furthermore suppose* $\overline{D}$ *is given by a Cauchy sequence of models* $\{X_i,D_i\}$ *and let* F_j *be the Zariski closure of* F *in* X_j*. Then we have*

$$\lim_{i\to\infty}d_H(\Delta_{U|F}(\overline{D}),\Delta_{U|F}(\overline{D_i}))=0$$

where $d_H(\cdot,\cdot)$ *is the Hausdorff metric and* $\overline{D_i}$ *is* D_i *considered as a model adelic divisor. In particular, we have*

$$\widehat{\operatorname{vol}}_{U|F}(\overline{D})=\lim_{i\to\infty}\operatorname{vol}_{X_i|F_i}(O(D_i))$$

where $\operatorname{vol}_{X_i|F_i}(O(D_i))$ *is the projective restricted volume of the line-bundle* $O(D_i)$ *with respect to* F_i*.*

Proof. The proof is very similar to that of the proof of Lemma 2.8.2. We begin by noting the set of inclusions

$$\Delta_{U|F}(\overline{D}-q_j\overline{D_0})\subseteq\Delta_{U|F}(\overline{D_j})\subseteq\Delta_{U|F}(\overline{D}+q_j\overline{D_0})$$

where we put overlines to emphasize that they are looked as model divisors. Now the first claim follows once again noting that the two extremities of the above inclusions converge under the Hausdorff metric thanks to Theorem 2.11.6 and Theorem 13 of [Kho12] when q_j is small enough. Then note that from Lemma 2.11.3, as $F\nsubseteq B_+(\overline{D})$ we conclude that $F\nsubseteq B_+(\overline{D}-q_j\overline{D_0})\supseteq B_+(\overline{D_j})$ for large enough j as $q_j\to0$. Hence for large enough j we have $F\nsubseteq B_+(\overline{D_j})$ which implies $\operatorname{vol}(\Delta_{U|F}(\overline{D_j}))=\frac{1}{k!}\widehat{\operatorname{vol}}_{U|F}(\overline{D_j})=\frac{1}{k!}\operatorname{vol}_{X_j|F_j}(O(D_j))$ thanks to Theorem 2.10.3 which now clearly gives the second claim together with the first claim. □

Corollary 2.12.2 (log-concavity)**.** *Suppose* $\overline{D_i}$ *are two adelic divisors on a normal quasi-projective variety* U *over* K *for* $i=1,2$*. Furthermore suppose* F *is a closed irreducible sub-variety of* U *with* $F\nsubseteq\operatorname{Bs}(\overline{D_i})$ *for* $i=1,2$*. Then we have*

$$\widehat{\operatorname{vol}}_{U|F}(\overline{D_1}+\overline{D_2})^{\frac{1}{k}}\geq\widehat{\operatorname{vol}}_{U|F}(\overline{D_1})^{\frac{1}{k}}+\widehat{\operatorname{vol}}_{U|F}(\overline{D_2})^{\frac{1}{k}}$$

where $\dim(E)=k$*.*

Proof. When $F\nsubseteq B_+(\overline{D_i})$ for both i, so is their sum and hence passing to models, we are reduced to the claim in the projective setting thanks to Corollary 2.12.1. The projective case can be deduced from the existence of global bodies as indicated in Example 4.22 of [LM09b]. □

Chapter 3

Differentiability of Volumes on Quasi-Projective Varieties

3.1 Introduction

Let k be a field and X be a projective variety over k. If L is a line bundle on X then we define

$$\mathrm{vol}(L) := \limsup_{m\to\infty} \frac{d! \cdot \dim_k(H^0(X, L^{\otimes m}))}{m^d}$$

where $d = \dim(X)$. We say L is big if $\mathrm{vol}(L) > 0$. Boucksom, Favre and Jonsson showed that the function $\mathrm{vol}(L)$ is differentiable on the big cone in [BFJ09b]. Lazarsfeld and Mustață independently showed the differentiability result by using Okounkov bodies in [LM09b]. The differential of the volume function in a suitable direction involves the definition of *positive intersection products* which were first introduced in [Bou+13].

In Arakelov geometry, there is a similar notion of arithmetic volumes for Hermitian line bundles which measures the asymptotic growth of *small global sections* of a given Hermitian line bundle. More precisely if $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ is a projective arithmetic variety of dimension d with generic fiber X and $\overline{\mathcal{L}}$ is a Hermitian line bundle with generic fiber L, then there is an induced Hermitian metric $|\cdot|$ on the complexified line bundle $L_{\mathbb{C}} := L \otimes_{\mathbb{Q}} \mathbb{C}$. In this article, we use the alternative notion *arithmetic divisors* which means the data $(\mathcal{D}, g)$ where $\mathcal{D}$ is a Cartier Divisor on $\mathcal{X}$ and g is a Green function for $\mathcal{D}_{\mathbb{C}}$. Given a rational section s of a Hermitian line-bundle $\mathcal{L}$, we can associate an arithmetic divisor $(\mathrm{div}(s), -\log\|s\|)$.

A Hermitian line-bundle $\mathcal{L}$ induces a sup-norm $||\cdot||$ corresponding to every global section $H^0(X, L)$ and we define the space of *small sections* as

$$\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}) := \{s \in H^0(\mathcal{X}, \mathcal{L}) \mid \|s\| \le 1\} \text{ and } \widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}) := \log \# \hat{H}^0(\mathcal{X}, \overline{\mathcal{L}})$$

Then we can finally define the *arithmetic volume* of $\overline{\mathcal{L}}$ as

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) := \limsup_{m\to\infty} \frac{d! \cdot \widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes m})}{m^d}$$

where d is the Krull dimension of $\mathcal{X}$. We call $\overline{\mathcal{L}}$ to be *big* if $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) > 0$. Analogously to the classical volume function in algebraic geometry, Chen showed that the arithmetic volume function is differentiable in the big cone. In [Che11], he introduces an arithmetic analogue of the positive intersection products in [Bou+13] and shows that the derivative at the big point is given by the arithmetic positive intersection product with a suitable direction analogously.

In their recent work [YZ24], Yuan and Zhang introduce the notion of *adelic divisors* on a quasi-projective (arithmetic) variety $\mathcal{U}$ as a general framework to have an arithmetic intersection theory with a very general class of singularities (in fact they work with the much more general notion of essentially quasi-projective varieties). The principal motivation of their theory is to include the Hodge bundle with Petersson metric which has singularities of log-log type along the boundary under a framework of arithmetic intersection theory analogous to the classical projective case.
Let $\overline{B}$ be either the tuple $(\mathrm{Spec}(O_K), \mathrm{Hom}(K, \mathbb{C}))$ or $(B, \emptyset)$ where O_K is the ring of integers of a number field K or B is a smooth projective curve over some field with fraction field K. By a (quasi)-projective arithmetic variety over $\overline{B}$, we mean a (quasi)-projective integral, flat, finite type scheme over $\mathrm{Spec}(O_K)$ or B. The idea of Yuan and Zhang is to consider all arithmetic divisors (with smooth Green functions) coming from some projective model of the fixed quasi-projective variety $\mathcal{U}$ and put a *boundary topology* on them which measures their differences along the boundary outside $\mathcal{U}$. Then they define the *adelic divisors* as those divisors which are "compactified" with respect to the boundary topology. More precisely if we fix a boundary divisor $\overline{D_0} := (D_0, g_0)$ *i.e* an arithmetic divisor $\overline{D_0}$ on some projective compactification X_0 of $\mathcal{U}$ such that $\mathrm{Supp}(D_0) = X_0 \backslash \mathcal{U}$, then an *adelic divisor* is given by sequence $\{X_i, \overline{D_i}\}$ where X_i are projective compactifications of $\mathcal{U}$ and $\overline{D}_i$ are arithmetic divisors on X_i satisfying a Cauchy condition *i.e* there is a sequence of positive rationals $\{q_i\}$ converging to 0 such that

$$-q_i\overline{D_0} \leq \overline{D_j} - \overline{D_i} \leq q_i\overline{D_0} \text{ for all } j \geq i$$

where the effectivity relations are understood to hold after passing to a common projective model dominating both X_i and X_j via birational pull-backs. The notion of adelic divisors are closely related to the notion of *b-divisors* introduced in [Sho03] (see [BG22] for a nice review on b-divisors). It is worth mentioning that the construction of Yuan and Zhang has been recently extended to quasi-projective varieties over arbitary *adelic curves* by Cai and Gubler in [CG24].
Yuan and Zhang also introduce the notions of volumes and arithmetic volumes $\widehat{\mathrm{vol}}(\overline{D})$ for an adelic divisor $\overline{D}$ on a quasi-projective variety and show that they satisfy many properties of the classical volumes on projective varieties like continuity, log-concavity etc (see [YZ24, Chapter 5]). Following [BFJ09b] in the geometric case and [Che11] in the arithmetic case, we define *positive intersection products* of adelic divisors $\overline{D}$ against integrable directions $\overline{E}$ denoted by $\langle \overline{D}^d \rangle \cdot \overline{E}$ (see Definition 3.2.4 in section 3) where $\dim(\mathcal{U}) = d + 1$. In this chapter, we work with the slightly more general notion of *essentially quasi-projective* varieties which are defined later. Our first main result in this chapter is to show that arithmetic volume functions of adelic divisors are differentiable. Suppose $\overline{B}$ be as above and suppose U is an essentially quasi-projective variety over $\overline{B}$. Then in Theorem 3.2.15 we show the following differentiability of adelic volumes

Theorem 3.1.1. *Suppose $\overline{D}$ is a big adelic divisor and $\overline{E}$ is an integrable adelic divisor on a normal essentially quasi-projective variety U over $\overline{B}$ of dimension d. Then the function $t \mapsto \widehat{\mathrm{vol}}(\overline{D} + t\overline{E})$ is differentiable at $t = 0$ with derivative given by*

$$\frac{d}{dt}\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) \mid_{t=0} = (d+1) \cdot \langle \overline{D}^d \rangle \cdot \overline{E}$$

The arguments to deduce the differentiability of both geometric and arithmetic adelic volumes are very similar to the ones used by Boucksom, Favre and Jonsson in [BFJ09b] to deduce differentiability of geometric volumes (see the proof of 3.2.15). The crucial point is using the continuity of positive intersection products and adelic volumes to obtain a limit version of the bounds in [BFJ09b] while perturbing in small directions of the boundary divisor. We also use Fujita approximation [YZ24, Theorem 5.2.8] and Siu's inequality [YZ24, Theorem 5.2.2(2)] for adelic volumes.
Let us now assume that $k = \mathbb{C}$. Given a closed subvariety of a projective variety over k, there is an invariant called *asymptotic intersection number* define along the closed subvariety for a line bundle which is constructed in [Ein+09b]. The definition of this intersection number is closely related to the definition of positive intersection products. In [Ein+09b] it is showed that these intersection numbers

are related to the *restricted volumes* of the line bundle along the closed subvariety provided that the closed subvariety is "general enough" with respect to the bundle. Given a closed subvariety E of a quasi-projective variety and a (geometric) adelic divisor D on U, we have introduced the notions of the restricted volumes denoted by $\widehat{\mathrm{vol}}_{U|E}(D)$ and *augmented base locus* of D denoted by $B_+(D)$ in Chapter 2. In this article we introduce an invariant similar to the asymptotic intersection numbers for adelic divisors which we denote by $\langle D^d\rangle \cdot E$ and show that they are equal to the restricted volume when the closed sub-variety is not contained in the augmented base locus. This allows us to obtain a version of Fujita approximation for restricted volumes. Now suppose U is a quasi-projective variety over k and E is a closed subvariety of dimension d in U. In Theorem 3.3.8, we show the following

Theorem 3.1.2. *Suppose D is a big adelic divisor and suppose E is a closed subvariety of U such that $E \not\subseteq B_+(D)$. Then we have*

$$\widehat{\mathrm{vol}}_{U|E}(D) = \langle D^d\rangle \cdot E$$

where $d = \dim(E)$

Next as an application of our differentiability results we consider the variational principle useful in proving equidistribution conjectures. We work uniformly over the base K which can either be a number field or a function field of one variable. We furthermore set $\overline{B}$ as before. The differentiability of both the geometric and arithmetic adelic volumes allows us to mimick the arguments in [Che11, Section 5.2] to obtain an equdistribution for big arithmetic adelic divisors, Suppose U is now a quasi-projective variety over $K = \mathrm{Frac}(B)$ where B is either the ring of integers of a number field or a smooth projective curve. We fix a place v on K and we denote by U_v the K_v-analytic space associated to U where K_v is the completion of K at v. We furthermore dneote by $\widehat{\mathrm{Div}}(U, \overline{B})$ to be the group of adelic divisors on the essentially quasi-projective variety U over $\overline{B}$. In Theorem 3.5.15, we show the following

Theorem 3.1.3. *Suppose U is a quasi-projective variety over K of dimension d and suppose $\overline{D}$ is a big arithmetic adelic divisor in $\widehat{\mathrm{Div}}(U, \overline{B})$ with generic fiber D. Furthermore suppose $\{x_m\}$ is a generic sequence of geometric points in $U(\overline{K})$ which is small with respect to $\overline{D}$. Then for any place v on K and for any compactly supported continuous function $g \in C_c^0(U_v)$, we have*

$$\lim_{m\to\infty} \int_{U_v} g \; d\eta_{x_m} = \frac{\langle \overline{D}^d\rangle \cdot \mathcal{O}(g)}{\mathrm{vol}(D)}$$

In particular, the sequence of Radon measures $\{\eta_{x_m}\}$ converge weakly to the Radon measure given by

$$g \in C_c^0(U_v) \mapsto \frac{\langle \overline{D}^d\rangle \cdot \mathcal{O}(g)}{\mathrm{vol}(D)}$$

We go on to show that this strengthens the equidistribution theorem on quasi-projective varieties proved by Yuan and Zhang in [YZ24, Theorem 5.4.3] by being able to relax the positivity hypothesis of arithmetic nefness to arithmetic bigness. The above theorem can be thought of as an analogue of the equidistribution theorem obtained by Berman and Boucksom in [BB10b] for the the quasi-projective case where we require an extra positivity assumption for the arithmetic adelic divisor to be **arithmetically** big. The results of this chapter are from [Bis24b].

3.2 Positive Intersection products

In this section, we define the positive intersection product of (arithmetic and geometric) adelic divisors with an integrable adelic divisor inspired by positive intersection products defined in [BFJ09b] in the geometric case and in [Che11] in the arithmetic case. We go on to show that at big adelic divisors, this product behaves continuously. Then using arguments like in [BFJ09b] we show that the function $t \mapsto \widehat{\mathrm{vol}}(\overline{D}+t\overline{E})$ is differentiable at $t = 0$ and the derivative is given by the defined positive intersection product when $\overline{E}$ is an integrable adelic divisor.

3.2.1. We are going to treat the arithmetic and geometric cases parallely as in [YZ24]. Throughout this section $\overline{B}$ will denote either $(\mathrm{Spec}(O_K), \mathrm{Hom}(K,\mathbb{C}))$, $(B,\emptyset)$ or $(k,\emptyset)$ as in Section 2. We will denote by U an essentially quasi-projective variety over $\overline{B}$ and by $\mathcal{U}$ any quasi-projective model of U over $\overline{B}$. Given two such models of U, we say one dominates the other if there is a morphism over U between them in the appropriate direction. Note that given two models $\mathcal{U}_1, \mathcal{U}_2$ of U such that $\mathcal{U}_1$ dominates $\mathcal{U}_2$, $\mathcal{U}_1$ is necessarily a birational modification of $\mathcal{U}_2$ which is an isomorphism over U.

3.2.2. We have defined adelic divisors on U in the previous section and we give a further description here on a quasi-projective model which will be useful for our computations later. An adelic divisor $\overline{D}$ on a quasi-projective variety $\mathcal{U}$ over $\overline{B}$ is given by the data $\{\mathcal{X}_i, \overline{D_i}, q_i\}$ where $\mathcal{X}_i$ are projective varieties over $\overline{B}$ containing $\mathcal{U}$ as a dense open subset, $\overline{D_i} \in \widehat{\mathrm{Div}}(\mathcal{X}_i, \mathcal{U})$ and $\{q_i\}$ is a sequence of positive rationals converging to 0 such that they satisfy the "Cauchy condition"

$$-q_j\overline{D_0} \le \overline{D_i} - \overline{D_j} \le q_j\overline{D_0} \text{ in } \mathrm{Div}(\mathcal{U})_{\mathrm{mod}} \text{ for all } i \ge j$$

for a boundary divisor $\overline{D_0}$. Finally we define the group of adelic divisors, nef adelic divisors and integrable adelic divisors on an essentially quasi-projective variety U over $\overline{B}$ to be

$$\widehat{\mathrm{Div}}(U,\overline{B}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U},\overline{B}) \quad \widehat{\mathrm{Div}}(U,\overline{B})_{\mathrm{nef}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U},\overline{B})_{\mathrm{nef}} \quad \widehat{\mathrm{Div}}(U,\overline{B})_{\mathrm{int}} := \varinjlim_{\mathcal{U}} \widehat{\mathrm{Div}}(\mathcal{U},\overline{B})_{\mathrm{int}}$$

where we vary $\mathcal{U}$ over all the quasi-projective models of U and we take the filtered colimit by viewing the collections of $\widehat{\mathrm{Div}}(\mathcal{U},\overline{B})$ as a filtered system via birational pull-backs. Note that it is crucial here that the notions of nefness and integrability are preserved under birational pull-backs.

Next we give the main definition of *positive intersection product* of an adelic divisor $\overline{D}$ with a nef adelic divisor $\overline{E}$. Note that Yuan and Zhang has defined top absolute intersection numbers in [YZ24, Prop 4.1.1]. Our strategy will be to define the positive intersection first for quasi-projective models and then pass to the essentially quasi-projective case. Hence we first give the definition for quasi-projective varieties over $\overline{B}$. We first start with the definition of "free divisors".

Definition 3.2.3. *Suppose $\mathcal{X}$ is a projective variety over $\overline{B}$ and $\overline{A}$ an arithmetic divisor on $\mathcal{X}$. We say that $\overline{A}$ is free if*

- *the underlying line bundle $\mathcal{O}(A)$ is semi-ample i.e some positive tensor power of it is generated by global sections when $\overline{B} = (B,\emptyset)$ or $(k,\emptyset)$.*
- *the associated curvature current of $\overline{A}$ on $\mathcal{X}(\mathbb{C})$ is semi-positive and some positive tensor power of the underlying line bundle $\mathcal{O}(A)$ is generated by small sections when $\overline{B} = (\mathrm{Spec}(O_K), Hom(K,\mathbb{C}))$.*

Remark 13. First we remark that it is easy to check that the definition above can be easily extended to the case of $\mathbb{Q}$-divisors. Furthermore for any open subset $\mathcal{U}$ of $\mathcal{X}$, $\overline{A}$ is model nef on $\mathcal{U}$. This follows from [Che11, Prop 2.3] when $\overline{B} = (\mathrm{Spec}(O_K), \mathrm{Hom}(K,\mathbb{C}))$ and follows from classical geometric intersection theory for the case when $\overline{B} = (B,\emptyset)$ or $\overline{B} = (k,\emptyset)$. Furthermore note that if $\overline{A}$ is arithmetically ample then it is free. Moreover note that since the notion of freenes is invariant under birational pull-backs, we can talk about free model $\mathbb{Q}$-divisors.

Definition 3.2.4. *Suppose $\mathcal{U}$ is a quasi-projective variety of dimension d over $\overline{B}$ and $\overline{D}, \overline{E}$ adelic divisors on $\mathcal{U}$ such that $\overline{E}$ is nef. Then we define the positive intersection product of $\overline{D}$ with $\overline{E}$, denoted $\langle \overline{D}^{d-1} \rangle \cdot \overline{E}$, as*

$$\langle \overline{D}^{d-1} \rangle \cdot \overline{E} = \sup_{X', \overline{A}} \overline{A}^{d-1} \cdot (\pi^*\overline{E})$$

where $(X', \overline{A})$ runs over all tuples such that X' is a projective model of a birational modification $\pi\colon \mathcal{U}' \to \mathcal{U}$ of $\mathcal{U}$ and $\overline{A}$ is free model $\mathbb{Q}$-divisor on X' such that $\pi^*\overline{D} - \overline{A} \ge 0$ in $\widehat{\mathrm{Div}}(\mathcal{U},\overline{B})$. We denote any such above tuple $(X', \overline{A})$ as an admissible approximation *of $\overline{D}$ on $\mathcal{U}$.*

Remark 14. Note that by [YZ24, Proposition 4.1.1], intersection numbers $\overline{A}^{d-1} \cdot \pi^*\overline{E}$ are defined as $\overline{E}$ (and hence $\pi^*\overline{E}$) is nef and in particular integrable and hence the definition makes sense. Also note that as the notions of intersection numbers and effectivity can be extended to $\mathbb{Q}$-adelic divisors, the same can be done for positive intersection products and it is furthermore easy to check that positive intersection products are homogeneous with respect to scaling by positive rationals. Note that for an arbitary $\overline{D}$, the set of its admissible approximations might be empty and in that case we set the positive intersection products to be $-\infty$ as a matter of set theoretic convention. However we almost always exclusively work with a big $\overline{D}$ in which case there always exists an admissible approximation of $\overline{D}$. Moreover by choosing model nef divisors $\overline{B} \geq \overline{D}$ it is also easy to see that the intersection numbers $\overline{A}^{d-1} \cdot \pi^*\overline{E}$ are bounded from above uniformly as we vary our admissible approximations and hence the positive intersection is a finite number whenever $\overline{D}$ is big.

We want to extend the above definition to essentially quasi-projective varieties over $\overline{B}$. The ordinary intersection products are defined by choosing any quasi-projective model and calculating intersection numbers over there and it is well-defined because of projection formula (since any two quasi-projective models of U are birational). We will do something similar for positive intersection products which is the content of our next lemma.

Lemma 3.2.5. *Suppose $\pi_0\colon \mathcal{U}_1 \to \mathcal{U}_2$ are two quasi-projective models with dimension d of an essentially quasi-projective variety U over $\overline{B}$ and with π_0 a birational morphism. Suppose $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{U}_2, \overline{B})$ and $\overline{E} \in \widehat{\mathrm{Div}}(\mathcal{U}_2, \overline{B})_{\mathrm{nef}}$. Then we have*

$$\langle \pi_0^*\overline{D}^{d-1} \rangle \cdot \pi_0^*\overline{E} = \langle \overline{D}^{d-1} \rangle \cdot \overline{E}$$

Proof. We first suppose that $(\mathcal{U}_2', X_2', \overline{A}, \pi)$ is an admissible approximation of $\overline{D}$ on $\mathcal{U}_2$. Since $\mathcal{U}_1$ itself is a birational modification of $\mathcal{U}_2$, we get that the fiber product $\mathcal{U}_1' := \mathcal{U}_2' \times_{\mathcal{U}_2} \mathcal{U}_1$ is a birational modification of $\mathcal{U}_1$ and we have a pull-back square

$$\begin{array}{ccc} \mathcal{U}_1' & \xrightarrow{\pi_0'} & \mathcal{U}_2' \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{U}_1 & \xrightarrow{\pi_0} & \mathcal{U}_2 \end{array}$$

Denote by $\pi'\colon \mathcal{U}_1' \to \mathcal{U}_1$ be the canonical birational morphism. Since $\pi^*\overline{D} \geq \overline{A}$ in $\widehat{\mathrm{Div}}(\mathcal{U}_2', \overline{B})$, pulling back by π_0' we get that $\pi'^*(\pi_0^*\overline{D}) = (\pi_0')^*(\pi^*\overline{A}) \geq \pi_0'^*\overline{A}$ where $\pi_0'\colon \mathcal{U}_1' \to \mathcal{U}_2'$ is the canonical morphism and hence $(\mathcal{U}_1', (\pi_0')^*X_2', (\pi_0')^*\overline{A}, \pi')$ is an admissible approximation of $\pi_0^*\overline{D}$ on $\mathcal{U}_1$ since $\pi_0'^*\overline{A}$ is free. The projection formula clearly yields $\pi_0'^*\overline{A} \cdot \pi'^*(\pi_0^*\overline{E}) = \overline{A} \cdot \pi^*\overline{E}$ since all morphisms involved are birational. We easily deduce from the definition of positive intersection products that

$$\langle \pi_0^*\overline{D}^{d-1} \rangle \cdot \pi_0^*\overline{E} \geq \langle \overline{D}^{d-1} \rangle \cdot \overline{E} \tag{3.1}$$

On the other hand, suppose $(\mathcal{U}_1', X_1', \overline{A}, \pi)$ is an admissible approximation of $\pi_0^*\overline{D}$ on $\mathcal{U}_1$. Then clearly $\mathcal{U}_1'$ is a birational modification of $\mathcal{U}_2$ as well. Moreover since $\pi^*(\pi_0^*\overline{D}) \geq \overline{A}$, we clearly get that $(\mathcal{U}_1', X_1', \overline{A}, \pi_0 \circ \pi)$ is an admissible approximation of $\overline{D}$ on $\mathcal{U}_2$. This clearly yields

$$\langle \overline{D}^{d-1} \rangle \cdot \overline{E} \geq \langle \pi_0^*\overline{D}^{d-1} \rangle \cdot \pi_0^*\overline{E} \tag{3.2}$$

Equations (3.1) and (3.2) together clearly finishes the proof of the claim. ☐

With the above lemma which shows invariance of positive intersection products with respect to birational pull-backs, we can now extend the definition of our positive intersection products to essentially quasi-projective varieties as follows

Definition 3.2.6. *Suppose U is an essentially quasi-projective variety over $\overline{B}$ such that a quasi-projective model of U has dimension d and $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B}), \overline{E} \in \widehat{\mathrm{Div}}(U, \overline{B})_{nef}$. Furthermore suppose $\overline{D}$ arises as an arithmetic adelic divisor $\overline{\mathcal{D}}$ on a quasi-projective model $\mathcal{U}$ of U. Then we define the positive intersection product of $\overline{D}$ against $\overline{E}$, denoted by $\langle \overline{D}^{d-1} \rangle \cdot \overline{E}$ as*

$$\langle \overline{D}^{d-1} \rangle \cdot \overline{E} := \langle \overline{\mathcal{D}}^{d-1} \rangle \cdot \overline{E}$$

where the right hand side is as defined in Definition 3.2.4. Note that the definition does not depend on the choice of $\mathcal{U}$ due to Lemma 3.2.5.

We begin by recording some easy positivity properties of nef adelic divisors and their intersection numbers.

Lemma 3.2.7. *Suppose $\overline{E}_1, \ldots \overline{E}_d$ be nef adelic divisors and suppose $\overline{B}$ is a model nef adelic divisor on an essentially quasi-projective variety U over $\overline{B}$ whose quasi-projective models have dimension d and $\overline{B} \geq \overline{E}_i$ for all i and suppose any quasi-projective model of U has dimension d. Then*

$$0 \leq \prod_{i=1}^{d} \overline{E}_i \leq \overline{B}^d$$

Proof. We assume that all the divisors live in a common quasi-projective model $\mathcal{U}$ of U since intersection numbers are birational invariants. Note that it is easy to check that the absolute intersection numbers defined in [YZ24, Prop 4.1.1] is continuous in the sense that for all rational t and integrable adelic divisors $\overline{E}_i, \overline{M}$, $\lim_{t\to 0} \prod_i (\overline{E}_i + t\overline{M}) = \prod_i \overline{E}_i$ and hence it is enough to check the above inequalities for strongly nef $\overline{E}_i$'s. Note that the left hand side of the inequality is already obtained in Propositon 4.1.1 and hence it is enough to show the right hand side. Suppose $\overline{E}_i$ is given by the models $\{X_j, \overline{E}_{ij}\}$ and rationals $\{q_j \to 0\}$ such that each $\overline{E}_{ij}$ is nef which we can always assume by passing to finer models. Then note that we have the effectivity relation $\overline{B} + q_j \overline{D}_0 \geq \overline{E}_{ij}$ by the Cauchy condition $\overline{E}_i \geq \overline{E}_{ij} - q_j \overline{D}_0$. Moreover we can choose a model ample divisor $\overline{D}_0' \geq \overline{D}_0$ and shrinking $\mathcal{U}$ if necessary, we can assume that $\overline{D}_0'$ has support $X_0 \backslash \mathcal{U}$. Hence for the proof we can assume that the boundary divisor is nef since intersection numbers do not change if we shrink U (and $\mathcal{U}$) to a dense open subset. Then clearly we have $(\overline{B} + q_j \overline{D}_0)^d \geq \prod_{i=1}^d \overline{E}_{ij}$ for all j since each $\overline{E}_{ij}$, $\overline{B}$ and $\overline{D_0}$ is nef. Now as indicated in the proof of Proposition 4.1.1 the intersection number $\prod_{i=1}^d \overline{E}_i$ is just given by the limit of the products $\prod_{i=1}^d \overline{E}_{ij}$ as j goes to infinity and hence we conclude

$$\overline{B}^d = \lim_{j\to\infty} (\overline{B} + q_j \overline{D}_0)^d \geq \lim_{j\to\infty} \prod_{i=1}^{d} \overline{E}_{ij} = \prod_{i=1}^{d} \overline{E}_i$$

where the first equality follows since $q_j \to 0$ by continuity of intersection numbers and this finishes the proof of the lemma. □

Remark 15. Note that the above lemma can be used easily to show that the positive intersection product is not $+\infty$ by bounding our adelic divisor by a large nef model divisor. In case that the set of admissible approximations is empty, we set the positive intersection product to be $-\infty$ as a matter of set theoretic convention. However we will later see that it is never the case when $\overline{D}$ is big.

Next we want to show that the positive intersection product defined above is continuous at big divisors.

Lemma 3.2.8. *Suppose $\overline{D}$ is a big adelic divisor, $\overline{E}$ is a nef adelic divisor and $\overline{F}$ is any adelic divisor on an essentially quasi-projective variety U over $\overline{B}$ such that any quasi-projective model of it has dimension d. Then there is a positive integer m depending only on $\overline{D}, \overline{E}$ and $\overline{F}$ such that*

$$(1-mt)^{d-1} \langle \overline{D}^{d-1} \rangle \cdot \overline{E} \leq \langle (\overline{D} + t\overline{F})^{d-1} \rangle \cdot \overline{E} \leq (1+mt)^{d-1} \langle \overline{D}^{d-1} \rangle \cdot \overline{E}$$

for all rational $\frac{1}{m} \geq t \geq 0$*. In particular for* t *rational*

$$\lim_{t \to 0} \langle (\overline{D} + t\overline{F})^{d-1} \rangle \cdot \overline{E} = \langle \overline{D}^{d-1} \rangle \cdot \overline{E}$$

Proof. As usual we assume that all the divisors live in a common quasi-projective model $\mathcal{U}$ of U. We begin by noting that there is a positive integer m such that $m\overline{D} \pm \overline{F} \geq 0$. Indeed if $\overline{D}, \overline{F}$ are represented by a sequence $\{\overline{D_j}, \overline{F_j}\}$ on some common models X_j and rationals $q_j \to 0$ as usual, then as $\overline{D}$ is big, thanks to [YZ24, Theorem 5.2.1(2)] we can find a j such that $\overline{D_j} - q_j\overline{D_0}$ is big since $\{\overline{D_j} - q_j\overline{D_0}\}$ is also a sequence of models representing $\overline{D}$. Then by Kodaira's lemma ([Laz04, Prop 2.2.6)]) for the geometric case and [Yua08a, Corollary 2.4(3)] for the arithmetic case, we can find a positive integer m such that $m(\overline{D_j} - q_j\overline{D_0}) - (\overline{F_j} + q_j\overline{D_0}) \geq 0$ and $m(\overline{D_j} - q_j\overline{D_0}) + (\overline{F_j} - q_j\overline{D_0}) \geq 0$. Now the usual Cauchy effectivity relations $\overline{D} \geq \overline{D_j} - q_j\overline{D_0}$ and $\overline{F_j} - q_j\overline{D_0} \leq \overline{F} \leq \overline{F_j} + q_j\overline{D_0}$ yield $m\overline{D} - \overline{F} \geq 0$ and $m\overline{D} + \overline{F} \geq 0$ respectively.
Next note it is easy to check from the definition of positive intersection products and Lemma 3.2.7 that if $\overline{D}_1 \leq \overline{D}_2$ are two adelic divisors and $\overline{E}$ any nef adelic divisor, then

$$\langle \overline{D}_1^{d-1} \rangle \cdot \overline{E} \leq \langle \overline{D}_2^{d-1} \rangle \cdot \overline{E}$$

since $\overline{E}$ is nef. Clearly by the choice of m we have the effectivity relations

$$(1 - mt)\overline{D} \leq \overline{D} + t\overline{F} \leq (1 + mt)\overline{D}$$

when $0 < t < \frac{1}{m}$. Then the L.H.S of the inequality claimed in the lemma easily follows for all rational $0 < t < \frac{1}{m}$ and the right hand follows for all rational t by noticing that the positive intersection product is homogeneous with respect to positive scaling and using the above effectivity relations. □

As a consequence of the continuity of positive intersection products, we derive two corollaries which will be useful for us. The first one gives an alternate description of the positive intersection products for big divisors in terms of nef divisors instead of free ones.

Corollary 3.2.9. *Suppose* $\overline{D}$ *is a big adelic divisor and* $\overline{E}$ *is a nef adelic divisor on a quasi-projective variety* $\mathcal{U}$ *over* $\overline{B}$ *of dimension* d*. Then we have*

$$\langle \overline{D}^{d-1} \rangle \cdot \overline{E} = \sup_{(X', \overline{A})} \overline{A}^{d-1} \cdot \overline{E}$$

where the supremum is taken as $(X', \overline{A})$ *varies over all tuples such that* X' *is a projective model of a birational modification* $\pi \colon \mathcal{U}' \to \mathcal{U}$ *and* $\overline{A}$ *is a nef* $\mathbb{Q}$*-divisor on* X' *such that* $\pi^*\overline{D} \geq \overline{A}$ *in* $\widehat{\mathrm{Div}}(\mathcal{U}, \overline{B})$*.*

Proof. Note that since all free $\mathbb{Q}$-divisors are automatically nef by our remark earlier, we immediately get that $\langle \overline{D}^{d-1} \rangle \cdot \overline{E} \leq \sup_{(X', \overline{A})} \overline{A}^{d-1} \cdot \overline{E}$ where $(X', \overline{A})$ are tuples of the form in the statement of the lemma. For the converse inequality, choose any arithmetically ample (or ample in the geometric case) divisor $\overline{B}$ on X'. Then since $\overline{A}$ is nef, we get that $\overline{A} + \epsilon \cdot \overline{B}$ is ample and hence free for any positive rational ϵ. Note that then $(X', \overline{A} + \epsilon \cdot \overline{B})$ is an admissible approximation of the big adelic divisor $\overline{D} + \epsilon \cdot \overline{B}$. Thus we get by the definition of positive intersection products that

$$(\overline{A} + \epsilon \cdot \overline{B})^{d-1} \cdot \overline{E} \leq \langle (\overline{D} + \epsilon \cdot \overline{B})^{d-1} \rangle \cdot \overline{E}$$

Then letting $\epsilon \to 0$ and noting that positive intersection products are continuous at $\overline{D}$ since it is big (Lemma 3.2.8), we deduce

$$\overline{A}^{d-1} \cdot \overline{E} \leq \langle \overline{D}^{d-1} \rangle \cdot \overline{E}$$

Since $(X', \overline{A})$ was an arbitary admissible approximation, taking supremum of the left hand side above across all such approximations easily gives the desired converse inequality and finishes the proof. □

Remark 16. Note that the above lemma shows that we can choose our approximations such that $\overline{A}$ is just nef instead of being free. Hence for the rest of the article, we will freely choose approximations to be either nef or free depending on what is suitable for our argument and this in particular shows almost by definition that for model nef divisors $\overline{D}$, the positive intersection products are the same as usual intersection products.

As a second corollary of the continuity of positive intersections, we deduce that for nef and big arithmetic adelic divisors, the positive intersection products are the same as usual absolute intersection numbers. For the rest of this section, we always assume that quasi-projective models of an essentially quasi-projective variety has Krull dimension d.

Corollary 3.2.10. *Suppose $\overline{D}$ is a big and nef arithmetic adelic divisor on U whose quasi-projective models are of dimension d and $\overline{E}$ be any nef arithmetic adelic divisor. Then we have*

$$\langle \overline{D}^{d-1} \rangle \cdot \overline{E} = \overline{D}^{d-1} \cdot \overline{E}$$

Proof. Note that after choosing a common quasi-projective model of U, we can assume that $\overline{D}$ is given by a Cauchy sequence $\{\overline{D}_i, q_i\}$ as usual where each $\overline{D}_i$ is model nef. Then we have the effectivity relation

$$\overline{D} - q_i\overline{D}_0 \leq \overline{D}_i \leq \overline{D} + q_i\overline{D}_0 \text{ for all } i$$

for a sequence of positive rational numbers q_i converging to 0. Then we have the inequality of positive intersection products

$$\langle (\overline{D} - q_i\overline{D}_0)^{d-1} \rangle \cdot \overline{E} \leq \langle \overline{D}_i^{d-1} \rangle \cdot \overline{E} \leq \langle (\overline{D} + q_i\overline{D}_0)^{d-1} \rangle \cdot \overline{E}$$

Hence we can deduce from Lemma 3.2.8 that $\lim_{i\to\infty} \langle \overline{D}_i^{d-1} \rangle \cdot \overline{E} = \langle \overline{D}^{d-1} \rangle \cdot \overline{E}$ since $\overline{D}$ is big. However since $\{\overline{D}_i\}$ is a sequence of model nef divisors converging to $\overline{D}$ and $\overline{E}$ is nef we have that $\lim_{i\to\infty} \overline{D}_i^{d-1} \cdot \overline{E} = \overline{D}^{d-1} \cdot \overline{E}$ by how absolute intersection numbers for integrable adelic divisors are constructed in [YZ24, Prop 4.1.1]. Now since $\overline{D}_i$ are model nef, by Remark 16 we have that $\langle \overline{D}_i^{d-1} \rangle \cdot \overline{E} = \overline{D}_i^{d-1} \cdot \overline{E}$. Putting everything together we conclude

$$\langle \overline{D}^{d-1} \rangle \cdot \overline{E} = \lim_{i\to\infty} \langle \overline{D}_i^{d-1} \rangle \cdot \overline{E} = \lim_{i\to\infty} \overline{D}_i^{d-1} \cdot \overline{E} = \overline{D}^{d-1} \cdot \overline{E}$$

which finishes the proof. □

We want to show that the positive intersection products are additive in $\overline{E}$ nef so that we can extend the product to integrable $\overline{E}$. For that we first need to show that the family of admissible approximations are filtered under the natural relation of dominance. We essentially use an argument of Chen in [Che11] but since we work more generally over function fields, we restate it for clarity.

Lemma 3.2.11. *Suppose X is a projective variety over $\overline{B}$. Suppose $\overline{D}$ is an arithmetic divisor on X and $\overline{A}_1, \overline{A}_2$ are two free $\mathbb{Q}$-arithmetic divisors on X such that $\overline{D} \geq \overline{A}_i$ for $i = 1, 2$. Then there is a birational modification $\pi\colon X' \to X$ and a free arithmetic $\mathbb{Q}$-divisor $\overline{A}$ on X' such that $\overline{A} \geq \pi^*\overline{A}_i$ for $i = 1, 2$ and $\pi^*\overline{D} \geq \overline{A}$.*

Proof. Note that the arithmetic case when $\overline{B} = (\mathrm{Spec}(O_K), \mathrm{Hom}(K, \mathbb{C}))$ is exactly the content of [Che11, Proposition 3.1] and the geometric case follows by the same construction by ignoring the part about Hermitian metrics. □

Lemma 3.2.12. *For nef adelic divisors $\overline{E}_i$, $1 \leq i \leq n$ and any big adelic divisor $\overline{D}$ living in a common quasi-projective model of U, we can find a sequence of positive rational numbers q_m converging to 0, a boundary divisor $\overline{D_0}$ and a sequence $(X'_m, \overline{A_m})$ of admissible approximations of $\overline{D} + q_m\overline{D_0}$ such that the following conditions hold*

1. $\lim_{m\to\infty} \overline{A_m}^d = \widehat{\mathrm{vol}}(\overline{D})$

2. $\lim_{m\to\infty} \overline{A_m}^{d-1} \cdot \overline{E_i} = \langle \overline{D}^{d-1}\rangle \cdot \overline{E_i}$ *for all* $1 \leq i \leq n$.

In particular the positive intersection products are linear i.e

$$\langle \overline{D}^{d-1}\rangle \cdot (\sum_{i=1}^n \overline{E}_i) = \sum_{i=1}^n \langle \overline{D}^{d-1}\rangle \cdot \overline{E}_i$$

Proof. We begin by noting that it suffices to show the existence of such a sequence. Indeed consider the $n+1$ nef adelic divisors $\overline{E}_i$ for $1 \leq i \leq n$ and $\sum_{i=1}^n \overline{E}_i$. Choose a sequence of model free divisors such that it satisfies the two conditions for the above $n+1$ nef divisors. Clearly the linearity is true for ordinary intersection products of $\overline{A_m}$ against the $\overline{E}_i$. Then we can conclude by taking the limit as $m \to \infty$ and using the property (ii) of the sequence.
To prove the existence of such a sequence, first assume $(X_j, \overline{D_j}, q_j)$ is a Cauchy sequence of model divisors converging to $\overline{D}$ with respect to the boundary divisor of $\overline{D_0}$. Now suppose $(X'_{ij}, \overline{A}_{ij})$ are sequences of admissible approximations of $\overline{D}$ for $1 \leq i \leq n$ such that

$$\lim_{j\to\infty} \overline{A}_{ij}^{d-1} \cdot \overline{E}_i = \langle \overline{D}^{d-1}\rangle \cdot \overline{E_i}$$

for each i and suppose $(X'_{0j}, \overline{A}_{0j})$ is a sequence of admissibile approximations such that

$$\overline{A_{0j}}^d \geq \widehat{\mathrm{vol}}(\overline{D}) - \frac{1}{j}$$

Note that we can find such sequences for $1 \leq i \leq n$ merely by the definition of positive intersection products and for $i = 0$ due to Fujita approximation (see [YZ24, Theorem 5.2.8]) since we assumed $\overline{D}$ is big and ample divisors are free. Furthermore by passing to finer models we can assume that $X'_{ij} = X'_j$ for $0 \leq i \leq n$ and that there are birational morphisms $\pi_j \colon X'_j \to X_j$. Then note that for $0 \leq i \leq n$, $\overline{A}_{ij} \leq \pi_j^*(\overline{D_j} + q_j\overline{D_0})$. By applying Lemma 3.2.11 repeatedly we can find a birational $\pi'_j \colon X''_j \to X'_j$ and a free $\mathbb{Q}$-divisor $\overline{A_j} \geq \pi'^*_j \overline{A}_{ij}$ on X''_j for $0 \leq i \leq n$ such that $(\pi_j \circ \pi'_j)^*(\overline{D_j} + q_j\overline{D_0}) - \overline{A_j} \geq 0$. Then the effectivity relations $\overline{D_j} + q_j\overline{D_0} \leq \overline{D} + 2q_j\overline{D_0}$ imply that $(X''_j, \overline{A_j})$ is an admissible approximation of $\overline{D} + 2q_j\overline{D_0}$ over U. Since $\overline{A}, \overline{A}_i, \overline{E}_i$ are all nef, from Lemma 3.2.7 we have

$$\overline{A_j}^d \geq \overline{A_{0j}}^d \geq \widehat{\mathrm{vol}}(\overline{D}) - \frac{1}{j} \text{ and } \overline{A_j}^{d-1} \cdot \overline{E}_i \geq \overline{A}_{ij}^{d-1} \cdot \overline{E}_i$$

for $1 \leq i \leq n$. Hence we deduce

$$\widehat{\mathrm{vol}}(\overline{D} + 2q_j\overline{D_0}) \geq \overline{A_j}^d \geq \widehat{\mathrm{vol}}(\overline{D}) - \frac{1}{j} \text{ and } \langle (\overline{D} + 2q_j\overline{D_0})^{d-1}\rangle \cdot \overline{E}_i \geq \overline{A_j}^{d-1} \cdot \overline{E}_i \geq \overline{A}_{ij}^{d-1} \cdot \overline{E}_i$$

for $1 \leq i \leq n$. Now using the continuity of volume functions (see [YZ24, Theorem 5.2.9]) and positive intersection products (see Lemma 3.2.8), taking limits of the above inequality as $j \to \infty$ we obtain

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{j\to\infty} \overline{A_j}^d \text{ and } \langle \overline{D}^{d-1}\rangle \cdot \overline{E}_i \geq \lim_{j\to\infty} \overline{A_j}^{d-1} \cdot \overline{E_i} \geq \lim_{j\to\infty} \overline{A_{ij}}^{d-1} \cdot \overline{E_i} = \langle \overline{D}^{d-1}\rangle \cdot \overline{E_i}$$

$1 \leq i \leq n$ since $q_j \to 0$ as $j \to \infty$. Taking limits of the two above inequalities as $j \to \infty$, we see that the sequence of model free $\mathbb{Q}$-divisors $\{(X''_j, \overline{A_j})\}$ and positive rationals $\{2q_j\}$ does the job. □

Remark 17. Note that the linearity described above allows us to extend the definition of positive intersection product of any big adelic divisor $\overline{D}$ with an integrable adelic divisor $\overline{E}$ which we also denote as $\langle \overline{D}^{d-1}\rangle \cdot \overline{E}$.

We want to obtain the main inequality required for the differentiability where we estimate terms of order at least 2. However for later purpose we want to keep track of the constants involved in the estimates and for that we need an easy lemma which we record next.

Lemma 3.2.13. *Suppose $\overline{E}$ is integrable. Then we can write $\overline{E} = \overline{E}_1 - \overline{E}_2$ where $\overline{E}_i$ is nef and effective for $i = 1, 2$.*

Proof. As usual we choose a common quasi-projective model $\mathcal{U}$ of U where all the divisors live. By definition of integrable divisors we can choose $\overline{E}_i'$ nef such that $\overline{E} = \overline{E}_1' - \overline{E}_2'$. Choose models $\overline{E_1'}$ and $\overline{E_2'}$ on a common projective model X of U such that $\overline{E}_i' \geq \overline{E_i'}$. Then we can choose an ample divisor $\overline{A}_0$ on X such that $\overline{E_i'} + \overline{A}_0$ is effective for $i = 1, 2$ by Serre's theorem (and its arithmetic version). Then the effectivity relation shows $\overline{E}_i' + \overline{A}_0 \geq 0$ and nef and hence we deduce the claim for $\overline{E_i} = \overline{E_i'} + \overline{A}_0$. □

Next we state the main inequality required for our differentiability.

Lemma 3.2.14. *Suppose $\overline{A}$ is a free model $\mathbb{Q}$-divisor on U and suppose $\overline{E} = \overline{E}_1 - \overline{E}_2$ is an integrable adelic divisors with $\overline{E}_i$ nef. Then we have*

$$\widehat{\mathrm{vol}}(\overline{A} + t\overline{E}) \geq \overline{A}^d + (d\overline{A}^{d-1}\overline{E}) \cdot t - Ct^2$$

for all rational $0 \leq t \leq 1$ where $C = 2^d d^2 \overline{B}^d$ for any model nef divisor $\overline{B}$ such that $\overline{B} \geq \overline{A}$ and $\overline{B} \geq \overline{E}_i$ for $i = 1, 2$.

Proof. Choose a model nef divisor $\overline{B}$ such that $\overline{B} \geq \overline{A}$ and $\overline{B} \geq \overline{E}_i$ for $i = 1, 2$. First note that by Siu's inequality ([YZ24, Theorem 5.2.2(2)]) we obtain

$$\widehat{\mathrm{vol}}(\overline{A}+t\overline{E}) = \widehat{\mathrm{vol}}(\overline{A}+t\overline{E}_1-t\overline{E}_2) \geq (\overline{A}+t\overline{E}_1)^d - d(\overline{A}+t\overline{E}_1)^{d-1}\cdot(t\overline{E}_2) \text{ for all positive rational } t \tag{3.3}$$

noting that $\overline{A} + t\overline{E}_1$ and $t\overline{E}_2$ are nef when $t > 0$. Next we have the binomial expansion of intersection numbers

$$(\overline{A} + t\overline{E}_1)^d - d(\overline{A} + t\overline{E}_1)^{d-1} \cdot (t\overline{E}_2) \tag{3.4}$$

$$= \overline{A}^d + (d\overline{A}^{d-1} \cdot \overline{E}) \cdot t - \sum_{k=1}^{d-1} B_k(d\overline{A}^{d-1-k}\overline{E}_1^k\overline{E}_2)t^{k+1} + \sum_{k\geq 2} C_k(\overline{A}^{d-k}\overline{E}_1^k)t^k \tag{3.5}$$

where B_k and C_k are appropriate binomial coefficients all bounded by 2^d. Since by choice we have $\overline{B} \geq \overline{A}$ and $\overline{B} \geq \overline{E}_i$ for $i = 1, 2$, by Lemma 3.2.7 we can bound each coefficient of t^k in the above sum for $k \geq 2$ by $2^d\overline{B}^d$. Since all the terms of the last summand above are positive, ignoring them we have the lower bound

$$(\overline{A} + t\overline{E}_1)^d - d(\overline{A} + t\overline{E}_1)^{d-1} \cdot (t\overline{E}_2) \geq \overline{A}^d + (d\overline{A}^{d-1} \cdot \overline{E}) \cdot t - Ct^2 \text{ for all } 0 \leq t \leq 1 \tag{3.6}$$

by counting the number of such higher order terms in (3.4), (3.5) which together with (3.3) completes the proof.

□

We are ready to state and prove the main theorem of differentiability in this section

Theorem 3.2.15. *Suppose $\overline{D}$ is a big adelic divisor and $\overline{E}$ is an integrable adelic divisor on a normal essentially quasi-projective variety U over $\overline{B}$ whose quasi-projective models are of dimension d. Then the function $t \mapsto \widehat{\mathrm{vol}}(\overline{D} + t\overline{E})$ is differentiable at $t = 0$ with derivative given by*

$$\frac{d}{dt}\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) \mid_{t=0} = d \cdot \langle \overline{D}^{d-1} \rangle \cdot \overline{E}$$

Proof. Since volumes and intersection numbers are birational invariants, we harmlessly omit the birational pull-backs by abuse of notation. We begin by choosing a sequence of free model $\mathbb{Q}$-divisors $\{\overline{A_m}\}$ and positive rationals q_m converging to 0 which satisfies the conditions of Lemma 3.2.12 for the big divisor $\overline{D}$ and nef effective divisors $\overline{E_i}$ for $i = 1, 2$ such that $\overline{E} = \overline{E_1} - \overline{E_2}$ and we assume $q_m < A$ for all m and for some positive rational A. We choose a model nef divisor $\overline{B} \geq \overline{D} + A\overline{D}_0$ and $\overline{B} \geq \overline{E_i}$ for $i = 1, 2$. We claim that the following inequality holds:

$$\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) - \widehat{\mathrm{vol}}(\overline{D}) \geq (d\langle \overline{D}^{d-1}\rangle \cdot \overline{E}) \cdot t - Ct^2 \tag{3.7}$$

where $C = 2^d d^2 \overline{B}^d$. Indeed since $\overline{B} \geq \overline{D} + A\overline{D}_0 \geq \overline{D} + q_m D_0 \geq \overline{A_m}$ and $\overline{B} \geq \overline{E_i}$, thanks to Lemma 3.2.14 we have the inequality

$$\widehat{\mathrm{vol}}(\overline{D} + q_m\overline{D_0} + t\overline{E}) \geq \widehat{\mathrm{vol}}(\overline{A_m} + t\overline{E}) \geq \overline{A_m}^d + d(\overline{A_m}^{d-1} \cdot \overline{E_1} - \overline{A_m}^{d-1} \cdot \overline{E_2}) - Ct^2$$

Now taking limits of the above inequality as $m \to \infty$, we obtain the inequality (3.7) noting the property of the sequence $\overline{A_m}$ and linearity of positive intersection products (Lemma 3.2.12). Replacing $\overline{E}$ by $-\overline{E}$, we obtain the inequality

$$\widehat{\mathrm{vol}}(\overline{D}) - \widehat{\mathrm{vol}}(\overline{D} - t\overline{E}) \leq (d\langle \overline{D}^{d-1}\rangle \cdot \overline{E} + Ct^2 \tag{3.8}$$

We want to apply the inequality (3.7) with the flips $\overline{D} \longleftrightarrow \overline{D} + t\overline{E}$ and $\overline{E} \longleftrightarrow -\overline{E}$ and the inequality (3.8) with the flips $\overline{D} \longleftrightarrow \overline{D} - t\overline{E}$ and $\overline{E} \longleftrightarrow -\overline{E}$ for $0 < t \ll 1$ small enough positive rational such that $\widehat{\mathrm{vol}}(\overline{D} \pm t\overline{E}) > 0$ which can be obtained by continuity of adelic volumes ([YZ24], Theorem 5.2.9). Note that $2\overline{B} \geq \overline{D} + t\overline{E}_1 \geq \overline{D} + t\overline{E}$ for $0 \leq t \leq 1$ since $\overline{E}_i$ are effective. Replacing $\overline{E}$ by $-\overline{E}$ we also deduce $2\overline{B} \geq \overline{D} - t\overline{E}$ for $0 \leq t \leq 1$. Hence we change the constant C to $2^d d^2 (2\overline{B})^d = 2^d \cdot C$. In other words we have

$$\widehat{\mathrm{vol}}(\overline{D}) - \widehat{\mathrm{vol}}(\overline{D} - t\overline{E}) \geq (d\langle(\overline{D} - t\overline{E})^{d-1}\rangle \cdot \overline{E}) \cdot t - 2^d Ct^2 \quad 0 \leq t \ll 1 \tag{3.9}$$

$$\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) - \widehat{\mathrm{vol}}(\overline{D}) \leq (d\langle(\overline{D} + t\overline{E})^{d-1}\rangle \cdot \overline{E}) \cdot t + 2^d Ct^2 \quad 0 \leq t \ll 1 \tag{3.10}$$

for $C = 2^d d^2 \overline{B}^d$. Finally the claim follows from the equations (3.7), (3.8), (3.9) and (3.10) together with continuity of positive intersections described in Lemma 3.2.8. $\square$

We end the section by deducing the *isoperimetric inequality* for adelic divsors which is the analogue of [Che11, Proposition 4.5]).

Corollary 3.2.16. *Suppose $\overline{D}$ and $\overline{E}$ are adelic divisors such that $\overline{D}$ is big and $\overline{E}$ is integrable and effective. Then we have*

$$\langle \overline{D}^{d-1}\rangle \cdot \overline{E} \geq \widehat{\mathrm{vol}}(\overline{D})^{\frac{d-1}{d}} \widehat{\mathrm{vol}} \cdot (\overline{E})^{\frac{1}{d}}$$

where d is the dimension of any quasi-projective model of U.

Proof. By Theorem 3.2.15 we have

$$\lim_{t \to 0} \frac{\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) - \widehat{\mathrm{vol}}(\overline{D})}{t} = d \cdot \langle \overline{D}^{d-1}\rangle \cdot \overline{E} \tag{3.11}$$

Moreover since $\overline{E}$ is effective, by [YZ24, Theorem 5.2.6] we have

$$\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) \geq (\widehat{\mathrm{vol}}(\overline{D})^{\frac{1}{d}} + t\widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d}})^d \text{ for all positive rational } t$$

Using the binomial expansion in the above inequality we get

$$\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) - \widehat{\mathrm{vol}}(\overline{D}) \geq (d \cdot \widehat{\mathrm{vol}}(\overline{D})^{\frac{d-1}{d}} \cdot \widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d}}) \cdot t + O(t^2)$$

which together with the equation (3.11) yields the claim by taking the limit as $t \to 0$.

$\square$

3.3 Relation with restricted volumes in the geometric case

In this section we define a slight variant of the positive intersection product that we defined in the last chapter inspired by the *asymptotic intersection numbers* defined in [Ein+09b]. We show using very similar arguments as in the previous chapter that these intersection numbers are continuous as well on big divisors. As a consequence we obtain a "generalized Fujita approximation" like theorem for restricted volumes in the case that the sub-variety is not contained in the augmented base locus as a quasi-projective version of [Ein+09b, Theorem 2.13]. In order to be able to use the results from [Ein+09b], we work over smooth complex varieties in this section.

3.3.1. We begin by recalling relevant definitions and the set up for this section. In this section we always have $\overline{B} = (k, \emptyset)$ where $k = \mathbb{C}$. We furthermore denote by U a smooth quasi-projective variety over $\overline{B}$ and by abuse of notation we write U is a quasi-projective variety over k and we denote the group of adelic divisors by $\widehat{\mathrm{Div}}(U, k)$. Furthermore since we always work in the geometric case, we omit the bars from the notation to denote divisors, both adelic and model. We recollect two crucial defeinitions from Chapter 2 that will be required for this section.

- Suppose $D \in \widehat{\mathrm{Div}}(U, k)$. The *augmented base locus* of D is defined as

$$B_+(D) \coloneqq \cap_{m\in\mathbb{N}} \mathrm{Bs}(mD - A)$$

  for any model divisor A such that A is ample on some projective model X of U. It is shown in Lemma 2.9.2 and Corollary 2.9.3 that the above definition is well defined.

- Suppose now E is a closed subvariety of U of dimension d. The *restricted volume* of $\widehat{\mathrm{vol}}_{U|E}(D)$ is defined as

$$\widehat{\mathrm{vol}}_{U|E}(D) \coloneqq \frac{d! \cdot (\dim_k(\mathrm{Im}(H^0(U, mD) \to H^0(E, mD|E))}{m^d}$$

  where $H^0(U, mD) \to H^0(E, mD|E)$ denotes the restriction map. We refer to Chapter 2 for further details and properties of the restricted volume.

3.3.2. We have showed that in Corollary 2.12.1 that if $E \not\subseteq B_+(D)$ and D is given by models $\{X_i, D_i\}$ as usual, then we have $\widehat{\mathrm{vol}}_{U|E}(D) = \lim_{i\to\infty} \mathrm{vol}_{X_i|E_i}(D_i)$ where E_i is the Zariski closure of E in X_i and the terms in the right hand side of the equality denote the classical projective restricted volumes as defined in [Ein+09b, Definition 2.1] viewing D_i as a $\mathbb{Q}$-divisor in X_i. It is easy to check using the Zariski density of U in each X_i that $\widehat{\mathrm{vol}}_{U|E}(D_i) = \mathrm{vol}_{X_i|E_i}(D_i)$ where in the left we view D_i as model adelic divisors. We next define a positive intersection product of a divisor D along E.

Definition 3.3.3. *Suppose D and E are as described above. Then we define an E-*admissible approximation *of D with respect to U by the data (U', X', A, π) where $\pi\colon U' \to U$ is a birational morphism such that it is an isomorphism over the generic point of E, X' is a projective model of U' and A is a nef $\mathbb{Q}$-divisor on X' such that $\pi^*D - A \geq 0$. We generally abbreviate the whole data by (X', A).*

Definition 3.3.4. *Suppose D and E are as described above. We define*

$$\langle D^d \rangle \cdot E = \sup_{(X',A)} A^d \cdot \tilde{E}$$

where $(\pi\colon X' \to X, A)$ varies over all E-admissible approximations of D on U and $\tilde{E} = \pi^{-1}(E)$.

Remark 18. Note that since we assume $\pi\colon U' \to U$ to be an isomorphism over the generic point of E, $\tilde{E}$ always have the same Krull dimension as E and hence the intersection numbers appearing in the definition actually make sense. Note further that it is easy to check that the above product is homogenous with respect to scaling by positive rationals in D just as for positive intersection products defined above. Hence we can extend the notion of the above product also to $\mathbb{Q}$-divisors.

We begin by showing that this product is continuous in the sense of Lemma 3.2.8. The proof will be very similar and hence we mention only the crucial points.

Lemma 3.3.5. *Suppose D is a big adelic divisor and F is any adelic divisor on U. Then there is a positive integer m depending only on D and F such that*

$$(1-mt)^d\langle D^d\rangle \cdot E \leq \langle (D+tF)^d\rangle \cdot E \leq (1+mt)^d \langle D^d\rangle \cdot \ E$$

for all rational $\frac{1}{m} > t \geq 0$. In particular for t rational

$$\lim_{t\to 0}\langle (D+tF)^d\rangle \cdot E = \langle D^d\rangle \cdot E$$

Proof. Note first that since D is big, using the same argument as in the beginning of the proof of Lemma 3.3.5 we can find a positive integer m depending on D and F such that $mD \pm F \geq 0$. Next note that it is easy to check from the definition of the positive intersection that if $D_1 \leq D_2$ are two adelic divisors then $\langle D_1^d\rangle \cdot E \leq \langle D_2^d\rangle \cdot E$. Now noting the sequence of effectivity relations

$$(1-mt)D \leq D+tF \leq (1+mt)D \text{ for all positive rational } t$$

and noting the homogneity of positive intersections with positive scaling we obtain the claim readily for all rational $0 \leq t < \frac{1}{m}$.

□

3.3.6. Suppose D is a $\mathbb{Q}$-divisor on a projective model X of U and suppose $\overline{E}$ is the Zariski closure of E in X. Then we can consider the *asymptotic intersection number* $||D^d \cdot \overline{E}||$ as defined in [Ein+09a, Definition 2.6]. Furthermore using the alternate description of this product when $\overline{E} \nsubseteq \overline{B_+}(D)$ as described in [Ein+09a, Prop 2.11] and our definition in 3.3.4 we have $\langle D^d\rangle \cdot E = ||D^d \cdot \overline{E}||$ provided $\overline{E} \nsubseteq \overline{B_+}(D)$. Here in the left we consider D as an model adelic $\mathbb{Q}$-divisor and $\overline{B_+}(D)$ denotes the classical augmented base locus as defined in [Ein+09a, Section 1]. With this observation and using the continuity in Lemma 3.3.5 we show that positive intersection at a big adelic divisor is given by limit of positive intersections of models after perturbing them a little by the boundary divisor.

Lemma 3.3.7. *Suppose D is a big adelic divisor given by a sequence $\{X_i, D_i, q_i\}$ as usual and furthermore suppose that $E \nsubseteq B_+(D)$. Then we have*

$$\langle D^d\rangle \cdot E = \lim_{i\to\infty}\langle (D_i+q_iD_0)^d\rangle \cdot E = \lim_{i\to\infty} ||(D_i+q_iD_0)^d \cdot \overline{E}_i||$$

where $\overline{E}_i$ is the Zariski closure of E in X_i.

Proof. Note first that since $D \leq D_i + q_iD_0$ we have that $B_+(D_i+q_iD_0) \subseteq B_+(D)$ which in particular implies that $E \nsubseteq B_+(D_i+q_iD_0)$. Now using Zariski density of U in each X_i we conclude that $\overline{E}_i \nsubseteq \overline{B_+}(D_i+q_iD_0)$ now looking at the models as $\mathbb{Q}$-divisors on projective varieties. Hence we conclude from 3.3.6 that $\langle (D_i+q_iD_0)^d\rangle \cdot E = ||(D_i+q_iD_0)^d \cdot \overline{E}_i||$ for all i. Thus we just need to prove the first equality in the above claim. Note that the effectivity relations $D \leq D_j + q_jD_0 \leq D + 2q_jD_0$ yields the inequalities

$$\langle D^d\rangle \cdot E \leq \langle (D_i+q_iD_0)^d\rangle \cdot E \leq \langle (D+2q_jD_0)^d\rangle \cdot E$$

This then evidently yields the claim by the continuity of the products (Lemma 3.3.5) and as $q_j \to 0$ by noting that D is assumed big. □

Finally we can state the main theorem of this section which is a variant of Fujita approximation for restricted volumes.

Theorem 3.3.8. *Suppose D is a big adelic divisor and suppose E is a closed sub-variety of U such that $E \nsubseteq B_+(D)$. Then we have*

$$\widehat{\mathrm{vol}}_{U|E}(D) = \langle D^d\rangle \cdot E$$

Proof. We noted in the proof of the previous lemma that $\overline{E}_i \nsubseteq \overline{B_+}(D_i + q_iD_0)$ for all i. Hence by [Ein+09b, Theorem 2.13] we conclude that $\mathrm{vol}_{X_i|\overline{E}_i}(D_i + q_iD_0) = ||(D_i + q_iD_0)^d \cdot \overline{E}_i||$. Furthermore since $E \nsubseteq B_+(D)$ we have shown in Corollary 2.12.1 that $\mathrm{vol}_{U|E}(D) = \lim_{i\to\infty} \mathrm{vol}_{X_i|\overline{E}_i}(D_i + q_iD_0)$ since $\{D_i + q_iD_0\}$ is a Cauchy sequence of models representing D. Thus combing the above with Lemma 3.3.7 we deduce the claim of the theorem. □

Remark 19. Note that when D is a model $\mathbb{Q}$-divisor which is ample on some projective model of U the restricted volume is the same as the intersection number *i.e* we have $D^d \cdot E = \mathrm{vol}_{U|E}(D)$ and hence the above theorem states that for general enough subvarieties E with respect to D *i.e* $E \nsubseteq B_+(D)$, the restricted volume is "approximated" by restricted volume of nef divisors dominated by it which is analogous to the usual Fujita approximation for volumes. A point to note here is that just like we assume the divisor to be big for usual Fujita approximation *i.e* we require $\mathrm{vol}(D) > 0$, we require $E \nsubseteq B_+(D)$ for restricted Fujita approximation to hold above and that readily implies $\widehat{\mathrm{vol}}_{U|E}(D) > 0$. A natural question to ask would be whether the converse is true *i.e* whether $\widehat{\mathrm{vol}}_{U|E}(D) > 0 \Rightarrow E \nsubseteq B_+(D)$. This is always the case in the projective setting which is obtained as the main theorem in [Ein+09b, Theorem (C)]. Unfortunately the proof heavily uses cohomological methods via separation of jets and it is not clear how to extend these methods in the quasi-projective setting.

Remark 20. We end this section by remarking how the results of this section could be generalised to varieties over arbitary fields of characteristic 0. For this we will need all varieties and subvarieties to be geometrically integral. Suppose now U is any quasi-projective variety over an arbitary field of characteristic 0 and D is an adelic divisor on U. Then using the Lefschetz principle the projective models of U and D descend to U_0 and D_0 over a finitely generated subfield of k. As a consequence we have a diagram

$$\begin{array}{ccc} H^0(U_0, mD_0) \otimes_{k_0} k & \longrightarrow & H^0(E_0, mD \mid E_0) \otimes_{k_0} k \\ \downarrow & & \downarrow \\ H^0(U_0, mD_0) & \longrightarrow & H^0(E_0, mD_0 \mid E_0) \end{array}$$

where $H^0(U_0, mD_0) \otimes_{k_0} k \cong H^0(U, mD)$, $H^0(E_0, mD \mid E_0) \otimes_{k_0} k \cong H^0(E, mD \mid E)$ where the left and right vertical arrows are isomorphisms. This readily shows that the notions of restricted volumes remain invariant under base change. Moreover the hypothesis $E \nsubseteq B_+(D)$ also remains invariant under base change which gives an idea how to translate the results from base field $\mathbb{C}$ to arbitary base fields.

3.4 Fundamental inequality in the Function Field case

In this section, we derive a version of the fundamental inequality for arithmetic adelic divisors analogous to [YZ24, Lemma 5.3.4] for function fields. The result is only stated in the case K is a number field in [YZ24] and the proof uses Minkowski's techniques from geometry of lattices. Since over a function field of a curve there are no Archimedean places these methods can not be applied. Due to a lack of a proper reference in earlier literature (at least in the quasi-projective case), we add a proof of this fact in the function field case.

3.4.1. We will use the more general theory of adelic curves as in [CM20]. Note that given a number field K with ring of integers O_K or a function field K of a smooth projective curve B over an arbitary field k, we can impose the structure of an adelic curve on it. More precisely, we set Σ to be the set of all places of K and we put the structure of a discrete σ-algebra on Σ. Furthermore we declare the measure of a singleton $\{v\}$ for any place $v \in \Sigma$ to be $n_v := \frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}$ when K is a number field and $n_v := [k(v) : k]$ when K is the function field of B. We refer to [CM20, Chapter 3] for further details on adelic curves.

- A *normed family* is the data $(V, \| \|_v)$ where V is a finite dimensional vector space over K and $\| \|_v$ is a norm on $V \otimes_K K_v$ for each place $v \in \Sigma$ and we abbreviate the whole data by $\overline{V}$. Such a normed family is called an *adelic vector bundle* if both the functions $\sum_{v\in\Sigma}\log\|s\|_v<\infty$ and $\sum_{v\in\Sigma}\log\|s'\|_v^\vee<\infty$ where s is an arbitary element of V, s' is an arbitary element of the dual $V^\vee$ and $\|\ \|_v^\vee$ is the dual norm of $\|\ \|_v$ on $V^\vee$. Note that this definition is the special case of the more general definition given in [CM20, Definition 4.1.28].

- Given an adelic vector bundle $\overline{V}$ such that $\dim_K(V)=1$, we define its *Arakelov degree* as

$$\widehat{\deg}(\overline{V}) \coloneqq \sum_{v\in\Sigma} -n_v \log\|s\|_v$$

Furthermore given a general adelic vector bundle $\overline{V}$, we denote it Arakelov degree by

$$\widehat{\deg}(\overline{V}) \coloneqq \widehat{\deg}(\overline{\det V})$$

where $\overline{\det V}$ denotes the *determinant adelic vector bundle* associated to $\overline{V}$ by setting the underlying vector space to be $\det V$ and the norm at v to be the determinant norm $\det\|\ \|_v$. We furthermore define the *positive degree* of $\overline{V}$ to be

$$\widehat{\deg}_+(\overline{V}) \coloneqq \sup_{\overline{W}\neq 0} \widehat{\deg}(\overline{W})$$

where $\overline{W}$ varies over all non-zero adelic vector bundles usch that W is a K-subspace of V and the norms of $\overline{W}$ are obtained by restriction from $\overline{V}$. We refer to [CM20, Section 4.3] for details.

- Given an adelic vector bundle $\overline{V}$, we denote its *slope* to be

$$\widehat{\mu}(\overline{V}) \coloneqq \frac{\widehat{\deg}(\overline{V})}{\dim_K V}$$

as defined in [CM20, Section 4, 3, 7]. Furthermore we define the *maximal slope* to be

$$\widehat{\mu}_{\max}(\overline{V}) \coloneqq \sup_{\overline{W}\neq 0} \widehat{\mu}(\overline{W})$$

where $\overline{W}$ varies over all non-zero adelic vector bundles such that W is a K-subspace of V and the norms of $\overline{W}$ are obtained by restriction from $\overline{V}$.

- Suppose $\overline{V_m}$ is a family of adelic vector bundles indexed by integers $m\in\mathbb{N}$. We denote the *asymptotic maximal slope* of $\overline{V_m}$ as

$$\widehat{\mu}^{\text{asy}}_{\max}(\overline{V_m}) \coloneqq \limsup_{m\to\infty} \frac{\widehat{\mu}_{\max}(\overline{V_m})}{m}$$

3.4.2. We recall the notions of *adelic vector spaces* relevant from [Bur+16] which are very closely related to adelic vector bundles.

- An *adelic vector space* is the data $(V,\|\ \|_v)$ where V is a finite dimensional vector space over K and $\|\ \|_v$ is a norm on $V\otimes_K K_v$ for each place $v\in\Sigma$ which saitisfies the condition that for any $s\in V$

$$\|s\|_v \le 1 \text{ for almost all } v\in\Sigma$$

For each non-Archimedean place $v\in\Sigma$, we set

$$V_v^\circ \coloneqq \{s\in V\otimes_K K_v \mid \|s\|_v\le 1\}$$

- An adelic vector space is called *pure* if $V \otimes_K K_v$ has an orthonormal basis with respect to $\| \ \|_v$ for each non-Archimedean place $v \in \Sigma$. Given an adelic vector space $(V, \| \ \|_v)$, we denote its *purification* to be the adelic vector space $(V, \| \ \|_{v,\mathrm{pur}})$ where we set $\| \ \|_{v,\mathrm{pur}}$ to be the lattice norm with respect to the K_v°-module V_v° when $v \in \Sigma$ is non-Archimedean and setting $\| \ \|_v = \| \ \|_{v,\mathrm{pur}}$ when $v \in \Sigma$ is Archimedean.

- An adelic vector space is called *generically trivial* if there is a basis $\mathbf{e}$ of V which is an orthonormal basis of $V \otimes_K K_v$ with respect to $\| \ \|_v$ for almost all $v \in \Sigma$.

- A *normed vector bundle* is the data of a locally free O_B-module together with a norm $\| \ \|_v$ on $\mathcal{V} \otimes_{O_B} K_v$ for all Archimedean places v when $B = \mathrm{Spec}(O_K)$ and just a locally free O_B-module when B is a smooth projective curve over some field k with function field K.

We refer to [Bur+16, Definition 2.10] for further details.

3.4.3. We set $\overline{B} = (\mathrm{Spec}(O_K), \mathrm{Hom}(K, \mathbb{C}))$ or $\overline{B} = (B, \emptyset)$ where K is any number field, O_K is the ring of integers of K and B is a smooth projective curve over some field k with function field K. We furthermore set Σ to be the set of all places over K. Suppose U is an essentially quasi-projective variety over $\overline{B}$ and further $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(X, U)$ is a model adelic divisor for some projective model X of U over $\overline{B}$. Then $\overline{\mathcal{D}}$ induces a metric $\| \ \|_v$ on the generic fiber $\mathcal{O}(D)$ for each place $v \in \Sigma$ and corresponding to $\| \ \|_v$ we have a sup norm $\| \ \|_{v,\sup}$ on $\mathcal{O}(D) \times_K K_v$ which by restriction to U_v gives a norm on $H^0(U, D) \times_K K_v$. We argue that the vector space $V = H^0(U, D) = H^0(X, D)$ along with the norms $\| \ \|_{v,\sup}$ constitute an adelic vector space which is furthermore generically trivial. Indeed we begin by noting that for each place $v \in \Sigma$, $\mathcal{O}(\mathcal{D})_v \coloneqq \mathcal{O}(\mathcal{D}) \otimes_{O_B} O_{B,v}$ is a model of the generic fiber $\mathcal{O}(D) \times_K K_v$ for each non-Archimedean place $v \in \Sigma$. Since the generic fiber of X is geometrically integral, by Lemma 37.26.4 there is a Zariski open subset V of B such that $\pi_* \mathcal{O}(\mathcal{D})|_V$ is a free O_U-module and the special fiber $X_{v,s}$ over v is geometrically reduced where $\pi \colon X \to B$ is the structure morphism. If we choose an O_V-basis $\mathbf{e}$ of $\pi_* \mathcal{O}(\mathcal{D})|_V$, then by [BE21b, Lemma 6.3(iii)] we can conclude that $\| \ \|_{v,\sup}$ coincides with the lattice norm induced by $\mathcal{O}(\mathcal{D})_v$. Since $\pi_* \mathcal{O}(\mathcal{D})|_V$ is free over O_U, we can conclude that $\mathbf{e}$ is an $O_{B,v}$-basis of $\mathcal{O}(\mathcal{D})_v$ for all $v \in U$. It is then immediate to see that $\mathbf{e}$ is an orthonormal basis of $\mathcal{O}(D) \otimes_K K_v$ for all $v \in V$. We can then conclude that $(\mathcal{O}(D), \| \ \|_{v,\sup})$ is a generically trivial adelic vector space by definition. We furthermore mention that for an open Zariski subset V as above, the norms $\| \ \|_{v,\sup}$ are pure for all $v \in V$.

3.4.4. Recall that given $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$, there is an associated height function $h_{\overline{D}}(\cdot) \colon U(\overline{K}) \to \mathbb{R}$. More precisely, it is given as

$$h_{\overline{D}}(x) = \sum_{v \in \Sigma} -n_v g_{\overline{D}(f)}(x)$$

where f is rational function on U such that $x \notin \mathrm{Supp}(D + \mathrm{div}(f))$ is a geometric point of U and $\overline{D}(f) = (D + \mathrm{div}(f), g_{\overline{D}} - \log|f|)$. Furthermore we can define the *essential minima* of $\overline{D}$ to be

$$\widehat{\mu}^{\mathrm{ess}}(\overline{D}) \coloneqq \sup_{V \subset U} \inf_{x \in V(\overline{K})} h_{\overline{D}}(x)$$

It is easy to note the furthermore that for a projective model X of U and $\overline{D} \in \widehat{\mathrm{Div}}(X, U)$, the above definition of essential minima conicides with the usual definition in the projective case.

Lemma 3.4.5. *Suppose $\overline{D} \in \widehat{\mathrm{Div}}(X, U)$ is a model adelic divisor with a big generic fiber D. Furthermore we assume here that $\overline{B} = (B, \emptyset)$ for a smooth projective curve B over a field k. Then we have*

$$\widehat{\mu}_{\mathrm{ess}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)}$$

Proof. We use the notation of 3.4.3 and we set $V_m = H^0(X, \mathcal{O}(mD))$ and we denote by $\overline{V_m}$ to be the adelic vector space $(V_m, \| \ \|_{m,v,\sup})$ as explained before. Then we know that $\overline{V_m}$ is a generically trivial adelic vector space. Suppose $\overline{V}_{m,\mathrm{pur}} \coloneqq (V_m, \| \ \|_{m,v,\mathrm{pur}})$ denote the purification of $\overline{V_m}$ and let U be a Zariski open dense subset as constructed in 3.4.3. We know from [BE21b, Lemma 6.3(iii)] and [BE21b, Lemma 6.4] that there is a constant $C > 0$ such that

$$\| \ \|_{m,v,\sup} \leq \| \ \|_{m,v,\mathrm{pur}} \leq C\| \ \|_{m,v,\sup} \text{ for all } v \in \Sigma \text{ and for all } m \in \mathbb{N} \tag{3.12}$$

and furthermore we have that $\| \ \|_{m,v,\sup} = \| \ \|_{m,v,\mathrm{pur}}$ for all $v \notin V$.
Next we use the notaion of 3.4.3 and choose an open subset V as explained there. Now note that as explained in 3.4.3, there is a basis $\mathbf{e}$ of V_m which induces the norm $\| \ \|_{v,\mathrm{pur}}$ at every place $v \in V$ in the sense of [CM20, Example 4.1.5]. This in turn implies from [CM20, Example 4.1.5] that each $\overline{V}_{m,\mathrm{pur}}$ is an adelic vector bundle. Then applying [CM20, Proposition 4.3.31] to (3.12), we get

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_{m,\mathrm{pur}}) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D})$$

where $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D})$ denotes the asymptotic maximal slope of the model divisor $\overline{D}$ as defined in [CM20, Section 6.4.1]. Note then since $\overline{V}_{m,\mathrm{pur}}$ is a generically trivial adelic vector space, we can deduce from [Bur+16, Proposition 2.13] that there is a normed vector bundle $\overline{\mathcal{V}_m}$ which induces the norms of $\overline{V}_{m,\mathrm{pur}}$ in the sense of [Bur+16, Example 2.12]. Using the bounds in [CM20, Theorem 4.3.23] and [CM20, Theorem 4.3.34], we have the inequality

$$|\widehat{h^0}(m\overline{D}) - \widehat{\deg_+}(\overline{V}_{m,\mathrm{pur}})| \leq C \cdot h^0(V_m) \tag{3.13}$$

for some constant $C > 0$ where $h^0(V_m)$ denotes the K-dimension of V_m. Note that here we have used the fact that after passing to purification, the space of small sections does not change, as indicated in [Bur+16, Proposition 2.23]. Since the norm $\| \ \|_{v,\sup}$ is ultrametric for each place v, we can deduce from [CM20, Proposition 4.3.44] that

$$\widehat{\deg_+}(\overline{V}_{m,\mathrm{pur}}) = \sum_{i=1}^{h^0(V_m)} \widehat{\mu}_i(\overline{V}_{m,\mathrm{pur}}) \leq h^0(V_m) \cdot \widehat{\mu}_1(\overline{V}_{m,\mathrm{pur}}) \tag{3.14}$$

where $\widehat{\mu}_i(\overline{V}_m)$ denotes the successive slopes as defined in [CM20, Definition 4.3.39]. Moreover it is easy to deudce from [CM20, Proposition 4.3.37] that $\widehat{\mu}_1(\overline{V}_{m,\mathrm{pur}}) = \widehat{\mu}_{\max}(\overline{V}_{m,\mathrm{pur}})$ from the definitions. Equations (3.13) and (3.14) yield

$$\frac{\widehat{h^0}(m\overline{D})}{m^{d+1}} \leq \frac{h^0(V_m)}{m^d} \cdot \frac{\widehat{\mu}_1(\overline{V}_{m,\mathrm{pur}})}{m}$$

where we have used in (3.13) that $h^0(V_m) = O(m^d)$. Then from the definitions of volumes and asymptotic maximal slopes, we can conclude that

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)}$$

Finally from [CM20, Proposition 6.4.4] we can conclude that

$$\widehat{\mu}_{\mathrm{ess}}(\overline{D}) \geq \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)}$$

which finishes the proof of the claim. □

In light of the previous lemma, we can easily generalise the above fundamental inequality to the case of quasi-projective varieties.

Theorem 3.4.6. *Suppose U is an essentially quasi-projective variety over $\overline{B}$ such that every quasi-projective model of U has dimension d and suppose $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$ with big generic fiber D. We furthermore assume that $\overline{B} = (B, \emptyset)$ for a smooth projective curve B over a field k. Then we have*

$$\widehat{\mu}_{\mathrm{ess}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)}$$

Proof. Suppose $\overline{D}$ is given by a Cauchy sequence of model divisors $\overline{D_i}$ in $\widehat{\mathrm{Div}}(X, U)$. Then from 3.4.5 we have the inequality

$$\widehat{\mu}_{\mathrm{ess}}(\overline{D_i}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D_i})}{(d+1)\mathrm{vol}(D_i)}$$

for each i. Furthermore from [BS24, Lemma 11.4] we have that

$$\lim_{i \to \infty} \widehat{\mu}_{\mathrm{ess}}(\overline{D_i}) = \widehat{\mu}_{\mathrm{ess}}(\overline{D})$$

Then the proof is finished by noting that

$$\lim_{i \to \infty} \widehat{\mathrm{vol}}(\overline{D_i}) = \widehat{\mathrm{vol}}(\overline{D})$$

$$\lim_{i \to \infty} \widehat{\mathrm{vol}}(D_i) = \widehat{\mathrm{vol}}(D)$$

from [YZ24, Theorem 5.2.1(2)]. □

3.5 Application to Equidistribution

In this section, we simultaneously consider the arithmetic and geometric case to obtain a differentiability result for *asymptotic slopes* following the arguments given by Chen for arithmetic divisors on a projective arithmetic variety (see [Che11, Proposition 5.1]). As a principal application of it we deduce an equidistribution result for big divisors and we show that it generalises the equidistribution result obtained by Yuan and Zhang in [YZ24, Theorem 5.4.3].

3.5.1. For this section we set $\overline{B} := (\mathrm{Spec}(O_K), \mathrm{Hom}(K, \mathbb{C}))$ or $\overline{B} := (B, \emptyset)$ where B is a smooth projective curve over some field k with function field K. Suppose U is an essentially quasi-projective variety over $\overline{B}$. We denote the group of "arithmetic" adelic divisors on U by $\widehat{\mathrm{Div}}(U, \overline{B})$ and the group of "geometric" adelic divisors by $\widehat{\mathrm{Div}}(U, K)$. Unless otherwise stated, d will always denote the dimension of any quasi-projective model of U. Furthermore for a place v of K, we set $n_v := \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}$ when K is a number field and we set $n_v := [k(v) \colon k]$ when K is a function field. Furthermore for this section we assume that U is of Krull dimension d so that any quasi-projective model of U over $\overline{B}$ will have Krull dimension $d + 1$.

Remark 21. Suppose $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$ is a big arithmetic adelic divisor. Then from [YZ24, Theorem 5.2.1(2)] we can deduce that there is a projective model $\mathcal{X}$ of U and an arithmetic divisor $\overline{D}_1$ such that $\overline{D}_1 \leq \overline{D}$ in $\widehat{\mathrm{Div}}(U, \overline{B})$ and $\overline{D}_1$ is big. Then from [CM20, Prop 6.4.18] we can deduce that the generic fiber D_1 of $\overline{D}_1$ is big and $D_1 \leq D$ in $\widehat{\mathrm{Div}}(U, K)$ where D is the generic fiber of $\overline{D}$. Hence we deduce that the generic fiber D is always big whenever $\overline{D}$ is arithmetically big.

For any big arithmetic adelic divisor $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})$, we can then define the "asymptotic positive slope" $\hat{\mu}_+^{\pi}(\overline{D})$ as

$$\hat{\mu}_+^{\pi}(\overline{D}) = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)}$$

It is defined in the projective setting by Chen (see [Che11, Section 5]) working over number fields. We derive as a corollary of our differentiability results before that this slope function is differentiable when $\overline{D}$ is big.

Corollary 3.5.2. *Suppose $\overline{D}$ is a big arithmetic adelic divisor and $\overline{M}$ an integrable arithmetic adelic divisor in $\widehat{\mathrm{Div}}(U,k)$. Then we have*

$$\lim_{t\to 0}\frac{\hat{\mu}^{\pi}_{+}(\overline{D}+t\overline{M})-\hat{\mu}^{\pi}_{+}(\overline{D})}{t}=\frac{\langle \overline{D}^{d}\rangle\cdot\overline{M}}{\mathrm{vol}(D)}-\frac{d\langle D^{d-1}\rangle\cdot M}{\mathrm{vol}(D)}\hat{\mu}^{\pi}_{+}(\overline{D})$$

Proof. The proof is quite straightforward by applying the following elementary principle from Analysis 1. If $f(t),g(t)$ are two differentiable functions on $\mathbb{R}$, then the quotient $\frac{f(t)}{g(t)}$ is also differentiable at a point t_0 such that $g(t_0)\neq 0$ and the derivative at t_0 is given by $\frac{g(t_0)f'(t_0)-f(t_0)g'(t_0)}{g(t_0)^2}$. We apply the above to the functions $f(t)=\widehat{\mathrm{vol}}(\overline{D}+t\overline{M})$ and $g(t)=\widehat{\mathrm{vol}}(D+tM)$ which are both differentiable by Theorem 3.2.15 at $t=0$ to deduce the claim since $\mu^{\pi}_{+}(\overline{D}+t\overline{M})=\frac{1}{(d+1)}\frac{f(t)}{g(t)}$. □

3.5.3. We recall here quickly the variational principle used frequently to obtain equidistribution results. We want to use the differentiability of the asymptotic positive slope to study the well known "variational principle". We fix a place v over K and we denote by U_v the K_v-analytic space associated to $U\times_K K_v$ where K_v is the completion of K at v. Note that as explained in [Ber90b, Chapter 1], they are the the usual complex analytic spaces associated to $U\times_K K_v$ when v is complex Archimedean and the quotient of the usual complex analytic spaces of $U\times_K K_v$ by complex conjugation when v is real Archimedean and the Berkovich spaces studied in [Ber90b] when v is non-Archimedean. For a geometric point $x\in U(\overline{K})$, we consider the discrete measure

$$\eta_x := \frac{1}{\#O(\tilde{x})}\sum_{y\in O(\tilde{x})}\delta_y$$

on U_v where $\tilde{x}$ denotes the induced closed point on $U\times_K\overline{K_v}$ from x, $O(\tilde{x})$ denotes the Galois orbit of x under the action of $\mathrm{Gal}(\overline{K_v}/K_v)$ and δ_y is the Dirac measure concentrated at y.
Now we consider a sequence of points $\{x_m\}$ of geometric points in $U(\overline{K})$ and we recall two definitions of sequence of points crucial in stating equidistribution conjectures. Recall that given $\overline{D}\in\widehat{\mathrm{Div}}(U,\overline{B})$ on U, there is real valued height function $h_{\overline{D}}(\cdot)\colon U(\overline{K})\to\mathbb{R}$ as explained in Section 5.

Definition 3.5.4. *We call a sequence $\{x_m\}$ in $U(\overline{K})$* generic *if the sequence is dense in the Zariski topology of U* i.e *for all non-empty Zariski open subset V of U, $x_m\in V(\overline{K})$ for large enough m. Given an arithmetic adelic divisor $\overline{D}$, we call $\{x_m\}$ is* small *w.r.t $\overline{D}$ if $\lim_{m\to\infty}h_{\overline{D}}(x_m)=\mu^{\pi}_{+}(\overline{D})$*

Remark 22. Note that the definition of a small sequence above is slightly different from the usual definition of a small sequence in literature (or *directionally small* as termed by Yuan and Zhang, see [YZ24, Section 5.4.1]). However when $\overline{D}$ is nef then our definition agrees with the classical one since then $\mu^{\pi}_{+}(\overline{D})=h_{\overline{D}}(U)$.

The equidistribution theorems and conjectures deal with the problem of determining conditions on the sequence $\{x_m\}$ and $\overline{D}$ such that the sequence of integrals $\int_{U_v}g\,d\eta_{x_m}$ converges to some real number in $\mathbb{R}$ as $m\to\infty$ for every continuous function g on U_v with compact support.
In [YZ24, Theorem 3.6.4], Yuan and Zhang introduce the notion of "compactified functions" as

$$C^0_{\mathrm{cptf}}(U_v):=\{f\in C^0(U_v)\mid \frac{f}{g_{0,v}}\to 0 \text{ along the boundary}\}$$

where $g_{0,v}$ is the Green function of any boundary divisor at v. Theorem [YZ24, Theorem 3.6.4] tells us that for any $g\in C^0_{\mathrm{cptf}}(U_v)$ there is an adelic divisor $\mathcal{O}(g)$ in $\widehat{\mathrm{Div}}(U,\overline{B})$. Furthermore construction identifies $C^0_{\mathrm{cptf}}(U_v)$ with a subgroup of $\widehat{\mathrm{Div}}(U,\overline{B})$ and contains the subgroup of compactly supported continuous functions $C^0_c(U_v)$ on U_v. The following description of the height function allows us to relate the measures η_{x_m} with the integrals $\int_{U_v}g\,d\eta_{x_m}$

$$h_{\mathcal{O}(g)}(x_m)=n_v\int_{U_v}g\,d\eta_{x_m} \tag{3.15}$$

We want to use the variational principle and obtain our equidistribution from differentiability in the direction of the divisor $\mathcal{O}(g)$ but $\mathcal{O}(g)$ is not integrable in the sense of Yuan and Zhang for any arbitary g. However there are a particular class of functions for which the associated divisor is integrable and they approximate any arbitary function with compact support under the uniform topology which we introduce next.

Definition 3.5.5. *Suppose $g \in C^0(U_v)$ is a continuous function on U_v. We say that g is a* model function *in either of the two following cases*

1. *v is Archimedean and g is the restriction of a smooth function on some projective model of U_v.*
2. *v is non-Archimedean and there is a projective model $\mathcal{X}$ of $U \times_B \operatorname{Spec}(O_{B,v})$ over the local ring $O_{B,v}$ of B at v and a vertical Cartier divisor $\mathcal{A}$ on $\mathcal{X}$ such that a positive multiple of g is the function induced by $\mathcal{A}$ on U_v via restriction.*

We denote the group of model functions on U_v by $C_{mod}(U_v)$.

Note that any model function f is bounded and hence in particular $C^0_{\text{mod}}(U_v) \subseteq C^0_{\text{cptf}}(U_v)$. We go on to show that the adelic divisors $\mathcal{O}(g)$ are integrable in the sense of Yuan and Zhang when g is model.

Lemma 3.5.6. *Suppose $g \in C^0(U_v)$ is a model function. Then the associated arithmetic adelic divisor $\mathcal{O}(g)$ is integrable.*

Proof. We first assume the case when v is non-archimedean. Then since g is model, by definition there is a projective model $\mathcal{X}$ of U over $\overline{B}$ with generic fiber X and a vertical divisor $\mathcal{A}$ on $\mathcal{X}$ such that g is induced by the model function of $\mathcal{A}$. Then using Serre's theorem we can write $\mathcal{A} = \mathcal{A}_1 - \mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ are ample divisors on $\mathcal{X}$. Then clearly writing $\mathcal{O}(g) = \mathcal{A}_1 - \mathcal{A}_2$ we conclude that $\mathcal{O}(g)$ is integrable.
For Archimedean g, we give a sketch of the argument. Like before we choose a projective model $\mathcal{X}$ of U with generic fiber X. Since g is smooth, $\mathcal{O}(g)$ induces a smooth Chern form on X_v. Choose an ample divisor $\overline{A}$ on X with usual Fubiny-Study form which has positive curvature and adding high enough powers of it to $\mathcal{O}(g)$, we get that $\mathcal{O}(g) + m\overline{A}$ also has positive curvature. Then writing $\mathcal{O}(g) = \mathcal{O}(g) + m\overline{A} - m\overline{A}$ we deduce the claim. We refer to [Gub03, Prop 10.4] and [YZ24, Section 2.1.1] for more general versions and details. ☐

Remark 23. To highlight the importance of the model functions, we remark here that these functions approximate arbitary continuous functions on U_v with compact support (looking at them as continuous functions on the analytification of some projective model of U) under the uniform topology. Indeed this is the famous Weierstrass approximation theorem in the Archimedean case and was proved by Gubler in [Gub98b, Theorem 7.12] in the non-archimedean case over algebraically closed fields. Gubler's arguments were adapted by Yuan [Yua08a, Section 3] for the non-algebraically closed case. In particular this shows that the functions considered in Lemma 3.5.6 approximate arbitary continuous functions uniformly in both the cases of number fields and function fields and this fact will be important to us to reduce the case of weak convergence to model situations. Furthermore note that the space of model functions with compact support form a subgroup of $\widehat{\operatorname{Div}}(U, \overline{B})$ under the inclusion into $C^0_{\text{cptf}}(U_v)$ since model functions are always bounded and we denote this subgroup of $\widehat{\operatorname{Div}}(U, \overline{B})$ by $C^0_{\text{mod}}(U_v)$.

3.5.7. Our strategy to attack the equidistribution problem is identical to the strategy in [Che11, Section 5.1] but we still repeat the arguments for clarity. Given a sequence of geometric points $\{x\} := \{x_m\}_{m \in \mathbb{N}}$ in $U(\overline{K})$, we begin by defining the following function

$$\phi_{\{x\}} \colon \widehat{\operatorname{Div}}(U, \overline{B}) \to \mathbb{R} \cup \{\pm\infty\}$$

$$\overline{D} \mapsto \liminf_{m \to \infty} h_{\overline{D}}(x_m)$$

The additivity of $h_{\overline{D}}(\cdot)$ on geometric points easily shows that $\phi_{\{x\}}(\cdot)$ is a super-additive function on $\widehat{\mathrm{Div}}(U,\overline{B})$. Next we record two important properties of $\phi_{\{x\}}(\cdot)$ that will be relevant in our future considerations. Recall that a sequence of Radon measures $\{d\eta_m\}$ on U_v is said to *weakly converge* to another Radon measure $d\eta$ if $\lim_{m\to\infty}\int_{U_v} g\ d\eta_m = \int_{U_v} g\ d\eta$ for all $g \in C_c^0(U_v)$. If the limit measure is not specified and the integrals converge to some real number, then we say the sequence $\{d\eta_m\}$ converges weakly. Given a big arithmetic adelic divisor $\overline{D}$, we can define the sub-semigroup

$$C(U,\overline{D}) := \{m\overline{D} + \mathcal{O}(f) \mid m \in \mathbb{N},\ m\overline{D} + \mathcal{O}(f) \text{ is big and } f \in C^0_{\mathrm{mod}}(U_v)\}$$

of $\widehat{\mathrm{Div}}(\mathcal{U},k)$.

3.5.8. Recall that given a sub-semigroup C and a subgroup H of an abelian group G, we say that C is *open with respect to* H if for all $g \in C$ and $h \in H$, we have that $mg + h \in C$ for all large enough positive integers m. It is easy to see that $C(U,\overline{D})$ is open with respect to the sub-group $C^0_{\mathrm{mod}}(U_v)$ which will be crucial to use [Che11, Proposition 5.4]. Indeed given $g \in C^0_{\mathrm{mod}}(U_v)$ and $f \in C^0_{\mathrm{mod}}(U_v)$, $m \in \mathbb{N}$ such that $m\overline{D} + \mathcal{O}(f) \in C(U,\overline{D})$, for large enough positive integer n we have $n(m\overline{D}+\mathcal{O}(f))+\mathcal{O}(g) = mn\overline{D}+\mathcal{O}(nf+g) \in C(U,\overline{D})$ since $nf+g \in C^0_{\mathrm{mod}}(U_v)$ for all positive integers n, the adelic volume function $\widehat{\mathrm{vol}}(\cdot)$ is positive homogeneous and continuous and $m\overline{D}+\mathcal{O}(f)$ itself is big. Next we record two lemmas which will be useful later.

Lemma 3.5.9. *Suppose $\overline{D} \in \widehat{\mathrm{Div}}(U,\overline{B})$ and $\{x\} := \{x_m\}$ be a sequence of geometric points in $U(\overline{K})$ such that the sequence $\{h_{\overline{D}}(x_m)\}$ is convergent. For any $f \in C^0_{\mathrm{cptf}}(U_v)$, the following are equivalent*

- *The sequence $\{\int_{U_v} f\ d\eta_{x_m})\}$ converges as $m \to \infty$.*
- *The function $\phi_{\{x\}}(\cdot)$ is differentiable at $\overline{D}$ in the direction $\mathcal{O}(f)$.*

Furthermore if one of the conditions hold, then

$$\lim_{m\to\infty}\int_{U_v} f\ d\eta_{x_m} = \frac{\phi'_{\{x\},\overline{D}}(f)}{n_v}$$

where $\phi'_{\{x\},\overline{D}}(f)$ denotes the derivative of $\phi_{\{x\}}$ at $\overline{D}$ calculated at $\mathcal{O}(f)$.

Proof. The proof of the equivalence is identical to the proof of [Che11, Theorem 5.3] but we still sketch a proof for clarity. The series of equalities that is crucial for the proof is

$$\phi_{\{x\}}(n\overline{D}+\mathcal{O}(f)) = \liminf_{m\to\infty}(nh_{\overline{D}}(x_m) + n_v\int_{U_v} f\ d\eta_{x_m}) = n\phi_{\{x\}}(\overline{D}) + \phi_{\{x\}}(\mathcal{O}(f)) \tag{3.16}$$

for all positive integers n and $f \in C^0_{\mathrm{cptf}}(U_v)$. The first equality above easily follows from the additivity of the height function on the geometric points. Now suppose that the first assertion holds. Then the second equality easily follows since the second summand in the middle term of equality (3.16) is convergent as $m \to \infty$. Replacing f by $-f$ in the above equality clearly shows that the function $\phi_{\{x\}}$ is differentiable at $\overline{D}$ along the direction of f. In fact it shows that it is linear in the direction of f. Conversely if we assume that the second assertion holds, then also the second equality of the equation (3.16) holds but this time since the **first** summand of the middle term in equation (3.16) is convergent. The differentiability of $\phi_{\{x\}}$ then allows us to deduce by taking limits as $n \to \infty$ that

$$\phi'_{\{x\},\overline{D}}(\mathcal{O}(f)) = \phi_{\{x\}}(\mathcal{O}(f))$$

where the left hand-side above denotes the differential of $\phi_{\{x\}}$ at $\overline{D}$ in the direction of $\mathcal{O}(f)$. Since differentials are linear maps, replacing $\mathcal{O}(f)$ by $\mathcal{O}(-f)$ we deduce from the above equality that

$$\phi_{\{x\}}(\mathcal{O}(-f)) = -\phi_{\{x\}}(\mathcal{O}(f))$$

Recalling the definition of the function $\phi_{\{x\}}$ we deduce that

$$\limsup_{m\to\infty} \int_{U_v} f \; d\eta_{x_m} = \liminf_{m\to\infty} \int_{U_v} f \; d\eta_{x_m} = \frac{\phi'_{\{x\},\overline{D}}(\mathcal{O}(f))}{n_v}$$

which proves the first assumption. Note that equation (3.16) also shows the last additional claim. □

Lemma 3.5.10. *If $\{x\}$ is a generic sequence in $U(\overline{K})$, then we have*

$$\phi_{\{x\}}(\overline{D}) \geq \mu_+^{\pi}(\overline{D})$$

for big arithmetic adelic divisors $\overline{D}$ in $\widehat{\mathrm{Div}}(U, \overline{B})$.

Proof. We recall the definition of the essential minima of an arithmetic adelic divisor, denoted by $\widehat{\mu}_{\mathrm{ess}}(\cdot)$ as

$$\widehat{\mu}_{\mathrm{ess}}(\overline{D}) := \sup_{V} \inf_{x\in V(\overline{K})} h_{\overline{D}}(x)$$

where V ranges over all proper non-empty Zariski open subsets of U. Then the definition of $\phi_{\{x\}}$ readily shows that for any Zariski open $V \subset U$, since $x_m \in V(\overline{K})$ for large enough m we have that $\phi_{\{x\}} \geq \widehat{\mu}_{\mathrm{ess}}$ on $\widehat{\mathrm{Div}}(U, \overline{B})$. Then we can conclude the claim of the lemma since $\widehat{\mu}_{\mathrm{ess}} \geq \mu_+^{\pi}$ from [YZ24, Lemma 5.3.4] in the number field case and Theorem 3.4.6 in the function field case. □

We are going to use the above Lemma together with our differentiability result to obtain an equidistribution theorem. As the above lemma suggests our strategy will be to deduce the differentiability of $\phi_{\{x\}}$ at big adelic divisors from the differentiability of μ_+^{π}. We will need a result for that on differentiability of super-additive functions as done in [Che11] which we restate for the sake of completeness.

Lemma 3.5.11. *[Che11, Prop 5.4] Let G be a group, H be a sub-group, C be a sub semi-group which is open with respect to H and $x \in C$. Furthermore let $f, t\colon C \to \mathbb{R}$ be two positively homogeneous functions satisfying*

1. *For all $a, b \in C$, we have $f(a + b) \geq f(a) + f(b)$*
2. *$f \geq t$, $f(x) = t(x)$*
3. *t is differentiable at x along the directions in H.*

Then the function f is differentiable as well and we have $D_x f = D_x t$.

Next we show that the positive intersections respect effectivity relations in terms of the direction $\overline{E}$ which helps us to control positive intersection products against generically trivial divisors with constant metrics.

Lemma 3.5.12. *Suppose $\overline{D}$ is an arithmetic adelic divisor in $\widehat{\mathrm{Div}}(U, \overline{B})$ with generic fiber D and suppose $\overline{E}_1, \overline{E}_2 \in \widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{int}}$ such that $\overline{E}_1 \leq \overline{E}_2$. Then we have*

$$\langle \overline{D}^d \rangle \cdot \overline{E}_1 \leq \langle \overline{D}^d \rangle \cdot \overline{E}_2$$

Furthermore for any $r > 0 \in \mathbb{R}$, we have

$$\langle \overline{D}^d \rangle \cdot \mathcal{O}(r) \leq r \cdot \mathrm{vol}(D)$$

Proof. We begin with proof of the first statement. As usual we choose a common quasi-projective model of U and suppose $(X, \overline{A})$ is an admissible approximation and suppose A is the generic fiber.

Furthermore we write $\overline{E_1} = \overline{E_{11}} - \overline{E_{12}}$ and $\overline{E_2} = \overline{E_{21}} - \overline{E_{22}}$ where $\overline{E_{11}}$, $\overline{E_{12}}$, $\overline{E_{21}}$ and $\overline{E_{22}}$ are elements of $\widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{nef}}$. Then from the given effectivity relation we can deduce

$$\overline{E_{11}} + \overline{E_{22}} \leq \overline{E_{21}} + \overline{E_{12}} \Rightarrow \overline{A}^d \cdot (\overline{E_{11}} + \overline{E_{22}}) \leq \overline{A}^d (\cdot \overline{E_{21}} + \overline{E_{12}})$$

$$\Rightarrow \langle \overline{D}^d \rangle \cdot (\overline{E_{11}} + \overline{E_{22}}) \leq \langle \overline{D}^d \rangle \cdot (\overline{E_{21}} + \overline{E_{12}})$$

which then allows us to deduce the first inequality by linearity of positive intersection products from Lemma3.2.12. Since the positive intersection products are defined by taking the supremum of all such products appearing as we vary $(X', \overline{A})$ across all admissible approximations in the above display we can deduce the claim.
For the second claim, we note that $\mathcal{O}(r)$ is nef and hence in particular integrable for all constants $r > 0$ and so it makes sense to take positive intersection products against it. As in the first paragraph we take an admissible approximation $(X', \overline{A})$ of $\overline{D}$ on a common quasi-projective model. Then we have

$$\overline{A}^d \cdot \mathcal{O}(r) = r \cdot A^d \leq r \cdot \mathrm{vol}(D) \text{ since } r \geq 0 \tag{3.17}$$

To see the last inequality above, we note that $A \leq D$ in the level of generic fibers and hence $\mathrm{vol}(A) \leq \mathrm{vol}(D)$. However since $\overline{A}$ was arithmetically nef (as it was an admissible approximation of $\overline{D}$), we deduce that A is nef and hence $\mathrm{vol}(A) = A^d \leq \mathrm{vol}(D)$ and hence we deduce the inequality in (3.17). Finally to obtain our second claim, we note that positive intersection product $\langle \overline{D}^d \rangle \cdot \mathcal{O}(r)$ is defined by taking the supremum of the first term in (3.17) as we vary admissible approximations $(X', \overline{A})$ which gives us the desired inequality. □

Next we extend the definition of positive intersection products against generically trivial adelic line bundles $\mathcal{O}(g)$ where $g \in C_{\mathrm{cptf}}(U_v)$ is arbitary. Using the lemma above, we show that we can extend positive intersection products to uniform limits of model functions. Recall from Definition 3.5.5 that we say $g \in C_c^0(U_v)$ is a *model function* if g is smooth when v is Archimedean and if g is induced by a model when v is non-Archimedean.

Lemma 3.5.13. *Suppose g_n is a sequence of uniformly convergent model functions in $C_{\mathrm{mod}}(U_v)$ converging uniformly to an arbitary continuous function $g \in C_{\mathrm{cptf}}(U_v)$ and $\overline{D} \in \widehat{\mathrm{Div}}(U, k)$ is big. Then we have that the sequence*

$$\{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g_n)\}$$

converges in $\mathbb{R}$. Furthermore the limit is independent of the sequence of functions g_n.

Proof. We only prove the first assertion as the second one can be deduced very similarly. As we have remarked before, since the functions g_n are model we know that each $\mathcal{O}(g_n)$ is integrable and hence the terms $\langle \overline{D}^d \rangle \cdot \mathcal{O}(g_n)$ make sense. Since we have that $\{g_m\}$ is uniformly convergent, we can find a sequence of positive rational numbers r_m converging to 0 such that $-r_m \leq g_n - g_m \leq r_m$ for all $n \geq m$. In terms of Hermitian line bundles, we have then the effectivity relations

$$-\mathcal{O}(r_m) \leq \mathcal{O}(g_n) - \mathcal{O}(g_m) \leq \mathcal{O}(r_m) \tag{3.18}$$

Using the linearity of positive intersection products in Lemma 3.2.12 and the first assertion of Lemma 3.5.12, we deduce

$$-\langle \overline{D}^d \rangle \cdot \mathcal{O}(r_m) \leq \langle \overline{D}^d \rangle \cdot \mathcal{O}(g_n) - \langle \overline{D}^d \rangle \cdot \mathcal{O}(g_m) \leq \langle \overline{D}^d \rangle \cdot \mathcal{O}(r_m)$$

Estimating the two extremities of the above inequality using the second assertion of Lemma 3.5.12, we get

$$-r_m \cdot \mathrm{vol}(D) \leq \langle \overline{D}^d \rangle \cdot \mathcal{O}(g_n) - \langle \overline{D}^d \rangle \cdot \mathcal{O}(g_m) \leq r_m \cdot \mathrm{vol}(D) \text{ for all } n \geq m$$

This gives us the desired claim since $r_m \to 0$ as $m \to \infty$. □

We can then finally extend our positive intersection products against $\mathcal{O}(g)$ where g is an arbitary continuous function with compact support.

Definition 3.5.14. *Suppose $\overline{D}$ is any big arithmetic adelic divisor in $\widehat{\mathrm{Div}}(U,\overline{B})$ and suppose $g \in C_c^0(U_v)$. Then we define*

$$\langle \overline{D}^d \rangle \cdot \mathcal{O}(g) := \lim_{m\to\infty} \langle \overline{D}^d \rangle \cdot \mathcal{O}(g_m)$$

where $\{g_m\}$ is any sequence of model functions on U_v converging uniformly to g.

Remark 24. We remark that our definition above makes sense. Indeed choosing a projective model X of U and viewing g as a continuous function on X_v by extending it to 0 outside the support, we can use the Weierstrass approximation theorem in the Archimedean case and [Gub98b, Theorem 7.12] in the algebraically closed non-Archimedean case (adapted by Yuan in [Yua08a] in the arbitary non-Archimedean case) to deduce that there is indeed such a sequence $\{g_m\}$ approximating g uniformly as above. Furthermore in the second assertion of Lemma 3.5.13 we also see that the limit is independent of the sequence $\{g_m\}$.

We can finally state our main result on equidistribution of small and generic points with respect to a big adelic divisor.

Theorem 3.5.15. *Suppose U is an essentially quasi-projective variety over K of dimension d and suppose $\overline{D}$ is a big arithmetic adelic divisor in $\widehat{\mathrm{Div}}(U,\overline{B})$. Furthermore suppose $\{x_m\}$ is a generic sequence of geometric points in $U(\overline{K})$ which is small with respect to $\overline{D}$. Then for any place v on K and for any $g \in C_c^0(U_v)$, we have*

$$\lim_{m\to\infty} \int_{U_v} g \; d\eta_{x_m} = \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g)}{n_v \cdot \mathrm{vol}(D)}$$

In particular, the sequence of Radon measures $\{\eta_{x_m}\}$ converge weakly to the Radon measure given by

$$g \in C_c^0(U_v) \mapsto \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g)}{n_v \cdot \mathrm{vol}(D)}$$

Proof. We begin by noting that we can reduce to the case when g is a model function. Indeed as we have remarked before, any continuous function with compact support on U_v can be approximated by model functions in the uniform topology. Note that then for such a sequence of model functions approximating our arbitary continuous function, the L.H.S will converge to the desired integral and the R.H.S will also converge to the desired quantity simply by how we define the positive intersection products against $\mathcal{O}(g)$ (see Definition 3.5.14 and Lemma 3.5.13). Hence we can safely assume that g is a model function.
Note that now since we have assumed g to be model, from Lemma 3.5.6 we know that each $\mathcal{O}(g) \in \widehat{\mathrm{Div}}(U,\overline{B})_{\mathrm{int}}$. The proof is then a consequence of all the results that we obtained in this section so far. We begin by considering Theorem 3.5.11 and suppose we set $G = \widehat{\mathrm{Div}}(U,\overline{B})$, $H = C^0_{\mathrm{mod}}(U_v)$, $C = C(U,\overline{D})$. We set $f = \phi_{\{x\}}$ and $t = \mu_+^{\pi}$. We check that these functions satisfy the hypotheses of Theorem 3.5.11. Indeed note that we get from positive homogeneity of heights and volumes that both f and t are positively homogeneous functions. Moreover

$$\phi_{\{x\}}(\overline{D}_1 + \overline{D}_2) = \liminf_{m\to\infty} h_{\overline{D}_1+\overline{D}_2}(x_m) = \liminf_{m\to\infty} (h_{\overline{D}_1}(x_m) + h_{\overline{D}_2}(x_m)) \geq \phi_{\{x\}}(\overline{D}_1) + \phi_{\{x\}}(\overline{D}_2)$$

so $f = \mu_+^{\pi}$ satisfies super-additivity. From Lemma 3.5.10 we have that $f \geq t$ on $C(U,\overline{D})$ since all its elements are big by definition. Furthermore since the sequence $\{x\}$ is small with respect to $\overline{D}$, from the definitions of $\phi_{\{x\}}$ and μ_+^{π} we have that $f(x) = t(x)$ and hence also the second hypothesis of Theorem 3.5.11 is satisfied. Finally from Corollary 3.5.2 and the assumed bigness of $\overline{D}$, we deduce that t is differentiable at x in the direction of $\mathcal{O}(g)$ since $\mathcal{O}(g)$ is integrable as mentioned in the first paragraph. Furthermore we also deduce from Corollary 3.5.2 that the derivative is given by

$$D_x t(g) = \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g)}{\mathrm{vol}(D)}$$

since the underlying generic fiber of $\mathcal{O}(g)$ is trivial. Hence from Theorem 3.5.11 we deduce that $f = \phi_{\{x\}}$ is differentiable at $x = \overline{D}$ with the differential given by the derivative of t at x. Then from Lemma 3.5.9 we deduce that the sequence of Radon measures $\{\eta_{x_m}\}$ converges weakly to the Radon measure

$$g \in C_c^0(U_v) \mapsto \lim_{m\to\infty} \int_{U_v} g \; d\eta_{x_m} = \frac{D_x f(g)}{n_v} = \frac{D_x t(g)}{n_v} = \frac{\langle \overline{D}^d \rangle \cdot \mathcal{O}(g)}{n_v \mathrm{vol}(D)}$$

which is what we wanted to prove. □

3.6 Relation to other Equidistribution Results

In this section, we relate our equidistribution result obtained in Theorem 3.5.15 to two other equidistribution results already known. We recall that we work over an essentially quasi-projective variety U over K where K is either a number field or a function field of a smooth projective curve over a field. We denote by $\overline{B} = (\mathrm{Spec}(O_K), \mathrm{Hom}(K, \mathbb{C}))$ or $\overline{B} = (B, \emptyset)$. The first equidistribution result is due to Yuan and Zhang obtained for nef adelic line bundles on quasi-projective varieties as in [YZ24, Theorem 5.4.3] and we show it can be deduced from our result. We end by remarking the relation of our result with a previously obtained equidistribution of Berman and Boucksom.

3.6.1. Recall that given an integrable arithmetic adelic divisor $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{int}}$, there is a Radon measure on U_v, denoted as the *Chambert-Loir measure* by Yuan and Zhang, corresponding to $\overline{D}$ as explained in [YZ24, Section 3.6.6]. We denote this Radon measure by $c_1(\overline{D}_v)^{\wedge d}$. They are constructed as weak limits of the already existing measures for the projective models approximating $\overline{D}$. The property crucial for us is that when $\overline{D}$ is nef, we have for any $g \in C_c^0(U_v)$ the equality

$$\overline{D}^d \cdot \mathcal{O}(g) = n_v \cdot \int_{U_v} g \; c_1(\overline{D}_v)^{\wedge d} \tag{3.19}$$

Indeed note that a similar equality holds in the projective setting by the definition of arithmetic intersections (see [GS90]). Then the above equality can be deduced by noting that both the left and right hand sides are constructed by taking limits along projective models approximating $\overline{D}$ as in [YZ24, Prop 4.1.1] and [YZ24, section 3.6.6] respectively. Then we can deduce Yuan and Zhang's equidistribution in the next theorem.

Theorem 3.6.2. *[YZ24, Theorem 5.4.3] Suppose $\overline{D} \in \widehat{\mathrm{Div}}(U, \overline{B})_{\mathrm{nef}}$ is a nef arithmetic adelic divisor such that $\mathrm{vol}(D) = D^d > 0$ and suppose $\{x_m\}$ is a generic sequence in $U(\overline{K})$ which is small with respect to $\overline{D}$. Then the sequence of Radon measure $\{d\eta_{x_m}\}$ converge weakly to the Radon measures $\frac{c_1(\overline{D}_v)^{\wedge d}}{D^d}$.*

Proof. We have to show that for all $g \in C_c^0(U_v)$, we have

$$\lim_{m\to\infty} \int_{U_v} g \; d\eta_{x_m} \mapsto \frac{1}{D^d} \cdot \int_{U_v} g \; c_1(\overline{D})^{\wedge d}$$

The trick is to deduce it from Theorem 3.5.15 by twisting $\overline{D}$. Note that for any $r > 0$, we have that the r-twist $\overline{D}(r)$ is again nef. Moreover since it is nef, we have $\widehat{\mathrm{vol}}(\overline{D}(r)) = \overline{D}(r)^{d+1} = \overline{D}^{d+1} + (d+1)rD^d > 0$ from Theorem [YZ24, Theorem 5.2.2(1)] and since $D^d > 0$ by hypothesis. Hence we deduce that the r-twist $\overline{D}(r)$ is big for all $r > 0$. From the previous equality we can also deduce $\widehat{\mathrm{vol}}(\overline{D}(r)) =$

$\widehat{\mathrm{vol}}(\overline{D}) + (d+1)r \cdot \mathrm{vol}(D) \Rightarrow \hat{\mu}^{\pi}_{+}(\overline{D}(r)) = \hat{\mu}^{\pi}_{+}(\overline{D}) + r$. Moreover we have $h_{\overline{D}(r)}(x_m) = h_{\overline{D}}(x_m) + r$ and hence we deduce that the sequence $\{x_m\}$ is small with respect to $\overline{D}(r)$ as well. Then we can apply Theorem 3.5.15 and deduce that the sequence of measures $\{d\eta_{x_m}\}$ converge weakly to the measure given by

$$g \in C^0_c(U_v) \mapsto \frac{\langle \overline{D}(r)^d \rangle \cdot \mathcal{O}(g)}{n_v \cdot D^d}$$

However note that since $\overline{D}(r)$ is already nef, by Corollary 3.2.10 we have that in this case the positive intersection products are the same as usual intersection products and hence

$$\frac{\langle \overline{D}(r)^d \rangle \cdot \mathcal{O}(g)}{n_v \cdot D^d} = \frac{\overline{D}(r)^d \cdot \mathcal{O}(g)}{n_v \cdot D^d}$$

However note that since $\mathcal{O}(g)$ is generically trivial and $\mathcal{O}(r)$ is also generically trivial with constant metric, we have that $\overline{D}(r)^d \cdot \mathcal{O}(g) = \overline{D}^d \cdot \mathcal{O}(g)$ which finishes the proof together with Equation (3.19). □

Remark 25. We end this section by relating our equidistribution with an equidistribution result obtained by Berman and Boucksom. In [BB10b, Theorem D] Berman and Boucksom show that for a smooth projective variety over $\mathbb{C}$ and a big line bundle on it endowed with a continuous metric denoted by $\overline{L}$, the Galois orbits of a generic sequence $\{x_m\}$ equidistribute to the *equilibrium measure* whenever the height of the points with respect to $\overline{L}$ converge to the *adelic energy at equilibrium*. They obtain it as a corollary of differentiability of *energy at equilibrium* as in [BB10b, Theorem A]. Our equidistribution can be thought of as a generalisation of Berman-Boucksom equidistribution in the quasi-projective setting. However as in Chen's equidistribution (see [Che11, Corollary 5.5]) we need an additional assumption of the divisor $\overline{D}$ to be arithmetically big compared to the more relaxed positivity assumptions of Berman-Boucksom.

Chapter 4

Concave Transforms on Okounkov Bodies

4.1 Introduction

4.1.1 Okounkov bodies in algebraic geometry

The notion of Okounkov bodies is introduced by Okounkov [Oko96a] to study the volume of any ample line bundle on a projective varietiy which is generalized by Lazarsfeld-Mustaţă [LM09a] to arbitrary *big* line bundles. Let X be a d-dimension integral projective variety over a field K, L a big line bundle on X and $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ a *graded linear series* of L i.e. a graded sub-algebra of $\{H^0(X, mL)\}_{m\in\mathbb{N}_{\geq 1}}$. If $V_\bullet$ *contains an ample series* (see 4.3.12), we can associate a concave set $\Delta(V_\bullet) \subset \mathbb{R}^d$ to $V_\bullet$. The crucial property is that

$$\lim_{m\to\infty} \frac{\dim_K(V_m)}{m^d/d!} = d! \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(V_\bullet)). \tag{4.1}$$

where $\mathrm{vol}_{\mathbb{R}^d}(\Delta)$ denotes the Lebesgue measure of any measurable subset $\Delta \subset \mathbb{R}^d$. The left-hand side of (4.1) is denoted by $\mathrm{vol}(V_\bullet)$. If $V_m = H^0(X, L^{\otimes m})$, we set $\mathrm{vol}(L) \coloneqq \mathrm{vol}(\{H^0(X, mL)\}_{m\in\mathbb{N}})$, called the *volume* of L. We have the classical formula

$$\mathrm{vol}(L) = L^d \tag{4.2}$$

which relates volume with the auto-intersection number of L.

4.1.2 Concave transforms in local case for projective varieties

Let K be a valued field complete with respect to a norm $|\ |_v$, X a projective variety over K and L a big line bundle on X. We denote by X^{an} the Berkovich analytification of X, see [Ber90a, §3.4, §3.5]. Let $\|\ \|_\varphi$ be a continuous metric on L. Then for any $m \in \mathbb{N}$, $\|\ \|_{m\varphi} \coloneqq \|\ \|_\varphi^{\otimes m}$ is a metric on $L^{\otimes m}$ and we have the *sup-norm* $\|\ \|_{m\varphi,\sup}$ on $H^0(X, L^{\otimes m})$:

$$\|\ \|_{m\varphi,\sup} \coloneqq \sup_{x\in X^{\mathrm{an}}} \{\|s\|_{m\varphi}(x)\}. \tag{4.3}$$

When $K = \mathbb{C}$, for any $m \in \mathbb{N}$, we set

$$\mathcal{B}(m\varphi) \coloneqq \{s \in H^0(X, L^{\otimes m}) \mid \|s\|_{m\varphi,\sup} \leq 1\} \quad \text{and} \quad d_m \coloneqq \dim_{\mathbb{C}}(H^0(X, L^{\otimes m})).$$

Nyström [Nys14] associated a convex function $c[\varphi]\colon \Delta(L)^\circ \to \mathbb{R}$ to φ. Given another metric ψ of L, he was able to show that

$$\lim_{m\to\infty} \frac{1}{m^{d+1}/(d+1)!} \log\left(\frac{\mathrm{vol}_{\mathbb{R}^{d_m}}(\mathcal{B}^\infty(m\psi))}{\mathrm{vol}_{\mathbb{R}^{d_m}}(\mathcal{B}^\infty(m\varphi))}\right) = 2(d+1)! \int_{\Delta(L)^\circ} (c[\varphi](\lambda) - c[\psi](\lambda))\, d\lambda \tag{4.4}$$

For any arbitrary complete valued field K, using the method developed in [BC11], by considering a superaddtive function associated to (L, φ) in [CM15, §4.2], Chen-Maclean constructed a concave transform $G_{(L,\varphi)}\colon \Delta(L)^\circ \to \mathbb{R}$ in [CM15, Remark 4.4]. Given another metric ψ of L, by [CM15, Theorem 4.3], we have that

$$\lim_{m\to\infty} \frac{1}{m^{d+1}/(d+1)!} \log \left(\| \ \|_{m\psi,\det}/\| \ \|_{m\varphi,\det}\right) = (d+1)! \int_{\Delta(L)^\circ} (G_{(L,\varphi)}(\lambda) - G_{(L,\psi)}(\lambda))\, d\lambda, \quad (4.5)$$

see also [Séd23, §2.4]. We denote by $\mathrm{vol}(L, \varphi, \psi)$ the value of the left-hand side of (4.5). If the metrics φ, ψ are *semi-positive* in the sense of [Zha95] when v is non-trivial, and are uniform limits of *Fubini-Study metrics* when v is trivial, then from the Bedford–Taylor theory [BT76] in the archimedean case, and from [CD12, §5.6] (or [Cha06, Proposition 2.7] when v is non-trivial and K has a countable dense subfield) in the non-Archimedean case, for any $0 \leq j \leq d$ we can associate a mixed Monge-Ampére measure

$$c_1(L, \varphi)^j \wedge c_1(L, \psi)^{d-j}$$

on X^{an}. The *relative Monge-Ampère energy* is defined as

$$E(\varphi, \psi) \coloneqq \frac{1}{d+1} \sum_{j=0}^{d} \int_{X^{\mathrm{an}}} \log \left(\| \ \|_{\psi}/\| \ \|_{\varphi}\right) c_1(L, \varphi)^j \wedge c_1(L, \psi)^{d-j}.$$

Then we have the equality

$$\mathrm{vol}(L, \varphi, \psi) = (d+1) \cdot E(\varphi, \psi) \quad (4.6)$$

which was proved in [BB10a, Theorem A] for the Archimedean case (notice that if X is not smooth, we can pass to its resolution of singularity) and in [BGM21, Theorem 1.1] for the non-Archimedean case (or [BE21a, Theorem 9.15] if L is semi-ample). Notice that the above results in [BGM21] can be applied in the trivially valued case. The formulas (4.4), (4.5) are local analogues of (4.1), and (4.6) is a local analogue of (4.2).

4.1.3 Concave transforms in Arakelov theory for projective varieties

The framework of the construction of concave transforms was given by Boucksom-Chen [BC11] which is also used in the local case above. Let X be a d-dimensional projective variety over a field K and L be a big line bundle on X. Let $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ be a graded sub-algebra of $\{H^0(X, mL)\}_{m\in\mathbb{N}}$ *containing an ample series* (see [BC11, Definition 1.1]) and $\mathcal{F}$ be a *multiplicative* filtration (see [BC11, Definition 1.3]) of $V_\bullet$. Boucksom-Chen [BC11] associated a concave transform $G_{\mathcal{F}}\colon \Delta(V_\bullet)^\circ \to \mathbb{R}$ satisfying certain measure convergence property if $\mathcal{F}$ is furthermore *pointwise bounded below* and *linearly bounded above* in the sense of [BC11, Definition 1.3]. When K is a number field with the ring of adéles $\mathbb{A}_K$ and $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ is equipped with a multiplicative norm $\| \ \|_m$ on each V_m such that $\overline{V_m} \coloneqq (V_m, \| \ \|_m)$ is an *adelically normed space* in the sense of [BC11, p. 2.1]. Set

$$\widehat{V}_m \coloneqq \{s \in V_m \mid \|s\|_{m,\omega} \leq 1 \text{ for any } \omega \in \Omega\},$$

$$\widehat{\dim}_K(\widehat{V}_m) \coloneqq \frac{1}{[K:\mathbb{Q}]} \log \#\widehat{V}_m, \quad (4.7)$$

$$\chi(\overline{V_m}) \coloneqq \frac{1}{[K:\mathbb{Q}]} \log \mathrm{vol}(\widehat{V}_m),$$

where $\mathrm{vol}(\cdot)$ is the Haar measure of $V_{m,\mathbb{A}_K} \coloneqq V_m \otimes_K \mathbb{A}_K$ normalized by $\mathrm{vol}(V_{m,\mathbb{A}_K}/V_m) = 1$ and $\widehat{V}_m$ is viewed as a subset of $V_{m,\mathbb{A}_K}$. Notice that the norms $\| \ \|_m$ give a multiplicative filtration of $V_\bullet$, namely the *filtration by minima* $\mathcal{F}_{\min}$ (see [BC11, p. 2.5]). Hence we can associate a concave transform $G_{\mathcal{F}_{\min}}\colon \Delta(V_\bullet)^\circ \to \mathbb{R}$ to $\overline{V_\bullet} = \{(V_m, \|\ \|_m)\}_{m\in\mathbb{N}}$. If $\mathcal{F}_{\min}$ is linearly bounded above, then the measure convergence property implies that

$$\lim_{m\to\infty} \frac{\widehat{\dim}_K(\widehat{V}_m)}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(V_\bullet)^\circ} \max\{0, G_{\mathcal{F}_{\min}}(\lambda)\}\, d\lambda. \tag{4.8}$$

If $\mathcal{F}_{\min}$ is furthermore *linearly bounded below* (in particular, $\inf\limits_{\lambda\in\Delta(V_\bullet)^\circ} G_{\mathcal{F}_{\min}}(\lambda) > -\infty$), then

$$\lim_{m\to\infty} \frac{\chi(\widehat{V}_m)}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(V_\bullet)^\circ} G_{\mathcal{F}_{\min}}(\lambda)\, d\lambda. \tag{4.9}$$

These two formulas (4.8), (4.9) are the arithmetic version of (4.1),(4.4), respectively. In particular, if L is endowed with a *model metric* $\|\ \|_\varphi = (\|\ \|_{\varphi,\omega})_{\omega\in M_K}$ (M_K is the set of places of K) i.e. $\|\ \|_\varphi$ induced by a *metrized line bundle* on a projective $\mathcal{O}_K$-model of X (see [Zha95]) and $V_m = H^0(X, L^{\otimes m})$ with $\|\ \|_m$ given by the sup-norms defined in (4.3), then the left-hand sides of (4.8), (4.9), denoted by $\widehat{\mathrm{vol}}(\overline{L})$ and $\widehat{\mathrm{vol}}_\chi(\overline{L})$, are called the *arithmetic volume* and the *arithmetic χ-volume* of $\overline{L} \coloneqq (L, \|\ \|_\varphi)$, respectively. Moreover, if $\|\ \|_{\varphi,\omega}$ is induced by a *semi-positive* metrized line bundle on some projective $\mathcal{O}_K$-model of X with underlying line bundle ample (in this case, the corresponding filtration $\mathcal{F}_{\min}$ satisfies all properties above), we have the arithmetic Hilbert-Samuel formula (see [AB95, Théorèm Principal] or [Mor14, Theorem 5.36]):

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = \overline{L}^{d+1}, \tag{4.10}$$

where $\overline{L}^{d+1}$ is the arithmetic intersection number given by [GS90].

In [CM20], Chen-Moriwaki proposed a new framework of Arakelov theory. They consider projective varieties over an *adelic curve* which is much more general than the classical cases of number fields or function fields. An *adelic curve* consisted of a field F and absolute values on F which are parameterized by a measure space $(\Omega, \mathcal{A}, \nu)$ similar to the notion of M-fields considered by Gubler in [Gub97]. Throughout this paper, the adelic curve is assumed to be *proper* which means that the following product formula holds: for any $a \in F^\times$,

$$\int_\Omega \log|a|_\omega\, \nu(d\omega) = 0.$$

More examples of adelic curves are given in [CM20, §3.2.4-6]. If K is endowed with a structure of adelic curve $S = (K, \Omega, \mathcal{A}, \nu)$ with Ω is discrete or K countable, let $\|\ \|_\varphi = (\|\ \|_{\varphi,\omega})_{\omega\in\Omega}$ be a family of metrics on L which is *dominated* and *measurable* in the sense of [CM20], then for any $m \in \mathbb{N}$, $H^0(X, L^{\otimes m})$ together with the sup-norms $\|\ \|_{m\varphi,\sup}$ is an *adelic vector bundle* over S, so the *Arakelov degree* $\widehat{\deg}(H^0(X, L^{\otimes m}), \|\ \|_{m\varphi,\sup})$ and the *positive Arakelov degree* $\widehat{\deg}_+(H^0(X, L^{\otimes m}), \|\ \|_{m\varphi,\sup})$ are well-defined [CM20, Definition 4.3.7, §4.3.4]. Write $\overline{L} \coloneqq (L, \|\ \|_\varphi)$. After generalizing [BC11, Theorem A] to the superadditive filtrations (see [CM20, Theorem 6.3.20]), Chen-Moriwaki [CM20, Theorem 6.4.6] associated a concave transform $G_{\overline{L}}\colon \Delta(L)^\circ \to \mathbb{R}$ satisfying

$$\lim_{m\to\infty} \frac{\widehat{\deg}_+(H^0(X, L^{\otimes m}), \|\ \|_{m\varphi,\sup})}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(L)^\circ} \max\{0, G_{\overline{L}}(\lambda)\}\, d\lambda. \tag{4.11}$$

If $\inf\limits_{\lambda\in\Delta*(L)} G_{(L,\varphi)}(\lambda) < \infty$, then

$$\lim_{m\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes m}), \|\ \|_{m\varphi,\sup})}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(L)^\circ} G_{\overline{L}}(\lambda)\, d\lambda. \tag{4.12}$$

As the classical case, the left-hand sides of (4.11), (4.12) are denoted by $\widehat{\mathrm{vol}}(\overline{L})$, $\widehat{\mathrm{vol}}_\chi(\overline{L})$, called the *arithmetic volume*, the *arithmetic χ-volume* of $\overline{L}$, respectively. In their subsequent works [CM21] [CM22], if $\overline{L}$ is *relatively ample* in the sense of [CM22, Definition 6.3.2], they defined the arithmetic intersection number $(\overline{L}^{d+1} \mid X)_S$ as integrals of local intersection numbers [CM21, Definition 4.4.3] and prove a Hilbert-Samuel type formula [CM22, Theorem B]:

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = (\overline{L}^{d+1} \mid X)_S \tag{4.13}$$

generalizing (4.10) to the adelic curve case.

4.1.4 Arakelov theory for line bundles with singular metrics

The formulas we presented in the local and global case are for continuous metrics of line bundles over projective varieties. In arithmetic geometry, there are singular metrics appearing naturally, e.g. the Petersson metric of the Hodge bundle on a compactification of the moduli space of principally polarized abelian varieties of dimension $g \geq 1$. In [BKK07], Burgos-Kramer-Kühn developed a systematic approach to arithmetic intersection theory for singular metrics with log-log singularities along the boundary.

Another approach was given recently by Yuan-Zhang [YZ24]. Let K be a number field or the function field of a curve over a field, U a quasi-projective variety. Let X be a projective compactification of U such that $B = X \setminus U$ is a Cartier divisor. We take a positive *arithmetic model divisor* $\overline{\mathcal{B}}$ for B, then $\overline{\mathcal{B}}$ induces a *model metrized divisor* $\overline{B} \coloneqq (B, g_B)$, called a *boundary divisor* of U. A *compactified metrized line bundle* $\overline{L}$ is given by a line bundle L_U on U and a sequence of line bundles L_j on projective compactifications X_j equipped with model metrics $\|\ \|_j$ satisfying the following conditions:

- $L_j|_U = L_U$ for any $j \in \mathbb{N}_{\geq 1}$;
- let s be a rational section of L_U, and s_j the extension of s to L_j, then for any $\varepsilon \in \mathbb{Q}_{>0}$, there is $j_0 \in \mathbb{N}_{\geq 1}$ such that for all $i, j \geq j_0$,

  $$-\varepsilon B \leq \mathrm{div}(s_i) - \mathrm{div}(s_j) \leq \varepsilon B \quad \text{and} \quad -\varepsilon g_B \leq \log\|s_j\|_j - \log\|s_i\|_i \leq \varepsilon g_B,$$

  where the inequalities take place after pulling back to a joint projective compactification of U.

Yuan-Zhang then defined the arithmetic intersection number for *arithmetically nef* (called *nef* in [YZ24]) compactified line bundles. Using the energy approach, Burgos-Kramer [BK24] could extend the arithmetic intersection number to *relatively nef* compactified line bundles which allowed them to define the arithmetic intersection number of the line bundle of Siegel–Jacobi forms with canonical metrics, and their extension covers the arithmetic intersection number defined in [BKK07]. In [CG24], Gubler and the second author generalized the arithmetic intersection numbers of compactified metrized line bundles in case where K is any field endowed with a structure of adelic curve. They also used the energy approach of Burgos and Kramer to extend the arithmetic intersection number to more singular metrics.

Yuan-Zhang worked on the geometric case in [YZ24] at the same time, and defined similarly the *compactified line bundles* and corresponding intersection theory on a quasi-projective variety over an arbitrary field. Let L be a compactified line bundle on U. We can define the space of global sections $H^0(U, L)$ and the volume $\mathrm{vol}(L)$ of L:

$$\mathrm{vol}(L) \coloneqq \limsup_{m\to\infty} \frac{\dim_K(H^0(U, L^{\otimes m}))}{m^d/d!}$$

which is in fact a limit, see [YZ24, Theorem 5.2.1 (1)]. Using the continuity of $\mathrm{vol}(\cdot)$ [YZ24, Theorem 5.2.1 (2)], if L is *nef* in the sense of [YZ24], then we have a Hilbert-Samuel type formula [YZ24, Theorem 5.2.2 (1)]:

$$\mathrm{vol}(L) = L^d,$$

where L^d is the auto-intersection number of L given by [YZ24, Proposition 4.1.1]. In Chapter 2, we followed the method in [LM09a] and defined the Okounkov body $\Delta(L)$ associated to L with the property

$$\mathrm{vol}(L) = d! \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(L)).$$

For the arithmetic case, let $\overline{L}$ be a compactified metrized line bundle on U with underlying compactified line bundle L. As the projective case, using (4.7), we can define the *arithmetic volume* $\widehat{\mathrm{vol}}(\overline{L})$ of $\overline{L}$, see [YZ24, Definition 5.1.3]. Using the continuity of $\widehat{\mathrm{vol}}(\cdot)$ in [YZ24, Theorem 5.2.1 (2)], if $\overline{L}$ is arithmetically nef, then we have the following arithmetic Hilbert-Samuel formula [YZ24, Theorem 5.2.2 (1)]:

$$\widehat{\mathrm{vol}}(\overline{L}) = \overline{L}^{d+1},$$

where $\overline{L}^{d+1}$ is the arithmetic intersection number given by [YZ24, Proposition 4.1.1].

In fact, before Yuan-Zhang's theory, Berman-Freixas [BF14, Theorem 1.1] established an arithmetic Hilbert Samuel formula for semi-positive metrics with log-log singularity. Instead of considering sup-norms, they consider the L^2-norms at archimedean places.

4.1.5 Main results

The goal of this paper is to use the theory of concave transforms along the lines of [BC11] and [CM20, Chapter 6] to study the arithmetic volumes and arithmetic χ-volumes of compactified metrized divisors on quasi-projective varieties.

Let $S = (K, \Omega, \mathcal{A}, \nu)$ be a proper adelic curve such that $\mathcal{A}$ is discrete or that K is countable, and let U be a d-dimensional normal quasi-projective variety over K.Let $\overline{D} = (D, g)$ be a *compactified S-metrized divisor* of U in the sense of 4.3.8. In Section 4.4, we associate a concave transform $G_{\overline{D}} \colon \Delta(D)^\circ \to \mathbb{R}$ to $\overline{D}$ if D is big, where $\Delta(D)$ be the Okounkov body of D. The first main result is given as follows:

Theorem 1 (Theorem 4.4.9). Let $\overline{V_\bullet} = \{\overline{V_m}\}_{m\in\mathbb{N}}$ be the graded K-algebra of adelic vector bundles associated to $\overline{D}$ defined in Definition 4.4.6. If D is big, then

$$\lim_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda, \tag{4.14}$$

and

$$\limsup_{m\to\infty} \frac{\widehat{\deg}(\overline{V_m})}{m^{d+1}/(d+1)!} \leq (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\, d\lambda \tag{4.15}$$

with equality if $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) > -\infty$, where $d\lambda$ is the standard Lebesgue measure on $\Delta(D) \subseteq \mathbb{R}^d$.

The left-hand sides of (4.14), (4.15) are denoted by $\widehat{\mathrm{vol}}(\overline{D})$, $\widehat{\mathrm{vol}}_\chi(\overline{D})$, called the *arithmetic volume*, the *arithmetic χ-volume* of $\overline{D}$, respectively (we will show in Proposition 4.6.5 that the notion arithmetic volume coincide with the one defined in [YZ24, Definition 5.1.3] if S is given by a number field). We denote $\widehat{\mathrm{vol}}^{\mathrm{num}}_\chi(\overline{D}) \coloneqq (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\, d\lambda$ which plays an important role in this paper. We also give applications of the concave transform $G_{\overline{D}}$, for example, we show following fundamental inequality in Theorem 4.4.21:

$$\sup_{V \subset U \text{ open}} \inf_{x\in V(\overline{K})} h_{\overline{D}}(x) \geq \frac{\widehat{\mathrm{vol}}^{\mathrm{num}}_\chi(\overline{D})}{(d+1)\cdot \mathrm{vol}(D)},$$

and generalize the height inequality [YZ24, Theorem 5.3.5 (3)] to the adelic curve case in Theorem 4.4.22.

Now we assume that the base field K we work with is perfect. We consider the set $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ of *relatively nef compactified* YZ-*divisors* $\overline{E}$ (see 4.6.2) such that there is *arithmetically nef compactified* YZ-*divisor* whose underlying compactified divisor is the same as the one of $\overline{E}$. The arithmetic intersection number $(\overline{E}^{d+1} \mid U)_S$ of an element $\overline{E}$ in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ is well-defined, see [CG24, Theorem 11.3]. We then obtain the following Hilbert-Samuel type formula.

Theorem 2 (Theorem 4.5.5). If $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ and D is big. Then

$$\widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S.$$

If moreover, $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) > -\infty$, then

$$\widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S.$$

An obvious consequence is that $(\overline{D}^{d+1} \mid U)_S > 0$ implies that $\overline{D}$ is *big* in the sense of Definition 4.4.6, see Corollary 4.5.6. Another application of concave transforms is the following equidistribution theorem generalizing [YZ24, Theorem 5.4.3] and Theorem 3.5.15.

Theorem 3 (Theorem 4.6.11). Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ with D big. Let $(x_m)_{m\in I} \subset U(\overline{K})$ be a generic net of points which is small with respect to $\overline{D}$. Then for any $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $\overline{D}(c)$ is big, for any $v \in \Omega$ and for any $f_v \in C_c(U_v^{\mathrm{an}})$, we have that

$$\lim_{m\in I} \frac{1}{\# O(x_m)} \sum_{\sigma \in \mathrm{Gal}(\overline{K}/K)} f_v(\sigma(x_m)) = \frac{\langle \overline{D}(c)^d \rangle \cdot (0, f_v)}{\mathrm{vol}(D)}. \tag{4.16}$$

(here $O(x_m)$ is the Galois orbits of x_m and $\langle \overline{D}(c)^d \rangle \cdot (0, f_v)$ is the positive intersection number defined in Definition 3.2.6, see also 4.6.8.)

Notice that we don't require $\overline{D}$ to be arithmetically integrable which is a rather strong condition, see [CG24, Example 7.13]. This is ongoing work with Yulin Cai.

Notation

For any scheme X, we denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X. An algebraic variety X over a field k is defined as a geometrically integral separated scheme of finite type over k, and we denote by $k(X)$ its function field.

Let $S = (K, \Omega, \mathcal{A}, \nu)$ be an adelic curve, see Definition 4.2.1. We denote by $\mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ the space of integrable functions $\phi : \Omega \to \mathbb{R}$. For any $\omega \in K$, we denote $|\ |_\omega$ the absolute value corresponding to ω and K_ω the completion of K with respect to $|\ |_\omega$.

For any vector space V over K, we denote $V_\omega \coloneqq V \otimes_K K_\omega$. Similarly, for any algebraic variety X over K and any ($\mathbb{Q}$-)Cartier divisor D on X, we denote by X_ω, D_ω the base change of X, D respectively. For a measurable subset Δ of $\mathbb{R}^d$, Δ° denotes the topological interior of Δ in $\mathbb{R}^d$, and $\mathrm{vol}_{\mathbb{R}^d}(\Delta)$ denotes the Lebesgue measure of Δ.

4.2 Adelic curves and adelic vector bundles

4.2.1 Adelic Curves

We begin by introducing the notion of *adelic curves* in this subsection. Our main reference is [CM20].

Definition 4.2.1 ([CM20] § 3.1). *Let K be a field and M_K the set of all absolute values on K. An* adelic structure *on K is a measure space $(\Omega, \mathcal{A}, \nu)$ equipped with a map $\phi : \Omega \to M_K$, $\omega \mapsto |\cdot|_\omega$ such that for any $a \in K^\times$, the function $\log|a|_\omega : \Omega \to \mathbb{R}$ is measurable. The data $(K, (\Omega, \mathcal{A}, \nu), \phi)$ (or simply $(K, \Omega, \mathcal{A}, \nu)$) is called an* adelic curve*. Furthermore, an adelic curve $(K, (\Omega, \mathcal{A}, \nu), \phi)$ is said to be* proper *if the* product formula *holds: for any $a \in K^\times$,*

$$\int_\Omega \log|a|_\omega \, \nu(d\omega) = 0.$$

Remark 26. Given an adelic curve $(K, (\Omega, \mathcal{A}, \nu), \phi)$, we fix the following notation throughout the paper:

$$\Omega_\infty := \{\omega \in \Omega \mid \phi(\omega) \text{ archimedean}\},$$

$$\Omega_{\text{fin}} := \{\omega \in \Omega \mid \phi(\omega) \text{ non-archimedean}\},$$
$$\Omega_0 := \{\omega \in \Omega \mid \phi(\omega) \text{ trivial}\} \subset \Omega_{\text{fin}}.$$

4.2.2 Adelic vector bundles

Throughout the rest of this section, we fix a proper adelic curve $S = (K, \Omega, \mathcal{A}, \nu)$. Recall that given a function $\psi\colon \Omega \to \mathbb{R}$, we define its *upper integral* as

$$\overline{\int_\Omega} \psi(\omega)\,\nu(d\omega) := \inf\left\{\int_\Omega \phi(\omega)\,\nu(d\omega) \mid \phi \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu),\ \phi \geq \psi \text{ almost everywhere on } \Omega\right\}$$

and its *lower integral* as

$$\underline{\int_\Omega} \psi(\omega)\,\nu(d\omega) := \sup\left\{\int_\Omega \phi(\omega)\,\nu(d\omega) \mid \phi \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu),\ \phi \leq \psi \text{ almost everywhere on } \Omega\right\}.$$

We say such a function ψ is *upper dominated* (resp. *lower dominated*) if ψ has a finite upper integral (resp. lower integral).

4.2.2. By an *S-normed vector space* over K, we mean a finite dimensional vector space V over K equipped with a family of norms $\|\ \| = (\|\ \|_\omega)_{\omega\in\Omega}$ denoted by $\overline{V} := (V, \|\ \|) = (V, (\|\ \|_\omega)_{\omega\in\Omega})$ where $\|\ \|_\omega$ is a norm on $V_\omega := V \otimes_K K_\omega$. Recall that given an S-normed vector space $\overline{V}$ over K, we consider the dual vector space $V^\vee$ along with dual norms $\|\ \|_\omega^\vee$ of $\|\ \|_\omega$ at each $\omega \in \Omega$, see [CM20, §1.1.5]. This gives us an S-normed vector space which we call the *dual S-normed vector space* of $\overline{V}$ over K and denote it by $\overline{V}^\vee := (V^\vee, \|\ \|^\vee)$.

Let $\overline{V}$ be an S-normed vector space over K. We say that $\overline{V}$ is an *adelic vector bundle* on S if the following conditions hold:

1. it is *measurable*: for any $s \in V$, the function

   $$\omega(\in \Omega) \mapsto \|s\|_\omega$$

   is measurable.

2. it is *dominated*: for any $s \in V$ and $t \in V^\vee$ both the functions

   $$\omega(\in \Omega) \mapsto \log\|s\|_\omega \quad\text{and}\quad \omega(\in \Omega) \mapsto \log\|t\|_\omega^\vee$$

   are upper dominated.

4.2.3. Let V be a vector space over K and $\|\ \| = (\|\ \|_\omega)_{\omega\in\Omega}$, $\|\ \|' = (\|\ \|'_\omega)_{\omega\in\Omega}$ be families of norms on V. For any $\omega \in \Omega$, set

$$d_\omega(\|\ \|, \|\ \|') := \sup_{s\in V_\omega\setminus\{0\}} |\log\|s\|_\omega - \log\|s\|_{\mathbf{e},\omega}|$$

called the *local distance* of $\|\ \|$ and $\|\ \|'$ at ω. The *global distance* of $\|\ \|, \|\ \|'$ is defined as

$$d(\|\ \|, \|\ \|') := \overline{\int_\Omega} d_\omega(\|\ \|, \|\ \|')\,\nu(d\omega).$$

4.2.4. Let $\overline{V} = (V, \|\ \|)$ be an adelic vector bundle on S. We can construct the following spaces.

- The *determinant* $\det(\overline{V})$ of $\overline{V}$ is given by fixing the determinant $\det(V)$ of V to be the underlying vector space and the norm to be the determinant norm at each $\omega \in \Omega$ (see [CM20, Definition 1.1.65]). It is proven in [CM20, Proposition 4.1.32] that $\det(\overline{V})$ is an adelic vector bundle of dimension 1 over K.
- For a vector subspace W of V, we can restrict the norms of $\overline{V}$ at each $\omega \in \Omega$ to obtain a normed family with underlying vector space W which is called the *restricted norm*. The subspace W with the restricted norm is an adelic vector bundle by [CM20, Proposition 4.1.32 (2)].
- For a quotient vector space of G of V, we can consider G along with the quotient norms at every $\omega \in \Omega$, see [CM20, §1.1.3], to construct a family of norms, called the *quotient norm*, whose underlying space is G. The quotient space with the quotient norm is an adelic vector bundle by [CM20, Proposition 4.1.32 (2)].

Using the notation above, we are able to define the following invariants of $\overline{V}$.

1. The *Arakelov degree* of $\overline{V}$ is defined as

$$\widehat{\deg}(\overline{V}) = \widehat{\deg}(\det(\overline{V})) := \int_{\Omega} -\log \|s\|_{\det,\omega} \, \nu(d\omega),$$

   where s is any non-zero section of $\det(V)$ and it is well-defined since S is proper and $\dim_K \det(V) = 1$.

2. The *positive Arakelov degree* of $\overline{V}$ is defined as

$$\widehat{\deg}_+(\overline{V}) := \sup_{\overline{W}} \{\widehat{\deg}(\overline{W})\},$$

   where $\overline{W}$ varies over all subspaces of $\overline{V}$ equipped with restriction norms.

3. The *slope* of $\overline{V}$ is defined as

$$\widehat{\mu}(\overline{V}) := \frac{\widehat{\deg}(\overline{V})}{\dim_K(V)}.$$

4. The *maximal slope* and *minimal slope* of $\overline{V}$ are defined as

$$\widehat{\mu}_{\max}(\overline{V}) := \sup_{\overline{W}} \{\widehat{\mu}(\overline{W})\} \text{ and } \widehat{\mu}_{\min}(\overline{V}) := \inf_{\overline{G}} \{\widehat{\mu}(\overline{G})\},$$

   respectively, where $\overline{W}$ varies over all subspaces of $\overline{V}$ with the restricted norms and $\overline{G}$ varies over all quotient spaces of $\overline{V}$ with quotient norms.

Let $\overline{M} = (M, \| \ \|_M)$, $\overline{N} = (N, \| \ \|_N)$ be two adelic vector bundles, and $f \colon M \to N$ a K-linear map. For any $\omega \in \Omega$, there is an induced map of normed vector spaces

$$f_\omega \colon (M_\omega, \| \ \|_{M,\omega}) \to (N_\omega, \| \ \|_{N,\omega})$$

and we can consider its *operator norm* $\|f\|_\omega := \|f_\omega\|$. Then we define the *height* of f as

$$h(f) := \underline{\int}_{\Omega} \log \|f\|_\omega \, \nu(d\omega).$$

4.2.3 Purification

For our later application in the case of the classical global fields, we consider pure norms in order to relate our arithmetic volumes with the classical ones.

4.2.5. Let $\overline{V} = (V, \| \ \|)$ be an S-normed vector space. For any $\omega \in \Omega \setminus \Omega_0$, we denote

$$\widehat{V}_\omega := \{s \in V_\omega \mid \|s\|_\omega \leq 1\}.$$

The *purification* of $\overline{V}$, denoted by $\overline{V}_{\mathrm{pur}}$, is the same underlying vector space V equipped with a family of norms $\| \ \|_{\mathrm{pur}} := \{\| \ \|_{\mathrm{pur},\omega}\}_{\omega\in\Omega}$ defined as follows:

- for any $\omega \in \Omega \setminus (\Omega_0 \cup \Omega_\infty)$ and $0 \neq s \in V_\omega$,

$$\|s\|_{\mathrm{pur},\omega} := \inf\{|\alpha| \mid \alpha \in K_\omega^\times,\ s \in \alpha\widehat{V}_\omega\};$$

- for any $\omega \in \Omega_\infty$ and $0 \neq s \in V_\omega$, we set

$$\|s\|_{\mathrm{pur},\omega} = \|s\|_\omega;$$

- for any $\omega \in \Omega_0$ and $0 \neq s \in V_\omega$, $\|s\|_\omega = 1$.

Notice that $\| \ \|_\omega \leq \| \ \|_{\mathrm{pur},\omega}$ for any $\omega \in \Omega \setminus \Omega_\infty$. Moreover it is not always true that $\| \ \|_\omega = \| \ \|_{\mathrm{pur},\omega}$ in the case when $\omega \in \Omega_{\mathrm{fin}}$ which leads to the following definition of purity.

Definition 4.2.6. *For $\omega \in \Omega$, we say that $\overline{V}$ is* pure at ω *if $\| \ \|_\omega = \| \ \|_{\mathrm{pur},\omega}$. We say that $\overline{V}$ is* pure *if it is pure at each $\omega \in \Omega$.*

Remark 27. Let $V = (V, \| \ \|)$ be an S-normed vector space such that $\| \ \|_\omega$ is ultrametric for any $\omega \in \Omega_{\mathrm{fin}}$. By [CM20, Proposition 1.1.30 (3), Proposition 1.1.32 (2)], $\overline{V}$ is pure at $\omega \in \Omega$ if and only if $(V_\omega, \| \ \|_\omega)$ is pure in the sense of [CM20, Definition 1.1.29]. In particular, if $\omega \in \Omega_{\mathrm{fin}}$ gives a discrete absolute value $| \ |_\omega$ on K, then the following statements are equivalent:

- $\overline{V}$ is pure at ω;
- $\|V\|_\omega = |K|_\omega$, see also [Bur+16, Proposition 2.9];
- $\| \ \|_\omega$ is induced by some *lattice* $\mathcal{V}_\omega$ of V_ω, see [CM20, Definition 1.1.29, Proposition 1.1.30 (2)].

Here, by a lattice of V_ω, we mean a sub-K_ω°-module $\mathcal{V}_\omega$ of V_ω generating V_ω as a vector space over K_ω with $\mathcal{V}_\omega$ bounded in V_ω for some norm of V_ω, see [CM20, Definition 1.1.23].

For further application, we give the following lemma.

Lemma 4.2.7. *Let $\overline{V}$ be an S-normed vector bundle. If $\overline{V}$ is dominated, then for any $s' \in V^\vee$, the function $\omega(\in \Omega) \mapsto \log \|s'\|_{\mathrm{pur},\omega}^\vee$ is upper dominated where $(V^\vee, \| \ \|_{\mathrm{pur}}^\vee)$ is the dual of $\overline{V}_{\mathrm{pur}}$.*

Proof. Since $\| \ \|_\omega \leq \| \ \|_{\mathrm{pur},\omega}$, the identity on V induces a morphism of normed vector spaces

$$f_\omega \colon (V_\omega, \| \ \|_{\mathrm{pur},\omega}) \to (V_\omega, \| \ \|_\omega)$$

for any $\omega \in \Omega$ and clearly the operator norm $\|f\|_\omega \leq 1$. Then passing to the dual and using [CM20, Proposition 1.1.22], for any $s' \in V^\vee$ and $\omega \in \Omega$

$$\|s'\|_{\mathrm{pur},\omega}^\vee \leq \|s'\|_\omega^\vee$$

Since $\overline{V}$ is dominated, we have that $\log \|s'\|_\omega^\vee$ is upper dominated, then so is $\log \|s'\|_{\mathrm{pur},\omega}^\vee$ by the above inequality using [CM20, Proposition A.4.2 (2)]. □

4.2.4 Harder-Narasimhan Filtration

4.2.8. Let $\overline{V}$ be an adelic vector bundle on S. The *Harder-Narasimhan filtration* of V is an $\mathbb{R}$-indexed non-increasing filtration $\mathcal{H}$ on V defined as

$$\mathcal{H}^t(V) := \mathrm{Vect}_K(\{W \subseteq V \mid \widehat{\mu}_{\min}(\overline{W}) \geq t\})$$

for $t \in \mathbb{R}$, where $\mathrm{Vect}_K(\cdot)$ denotes the vector space generated over K by the subset, see also [CM20, Proposition 4.3.46]. The *jumping numbers* of the Harder-Narasimhan filtration is defined as usual by

$$\widehat{\mu}_i(\overline{V}) := \sup\{t \in \mathbb{R} \mid \dim_K(\mathcal{H}^t(V)) \geq i\}$$

for every integer $i \in \mathbb{N}$. By [CM20, Proposition 4.3.46], the last slope of $\overline{V}$ is exactly the minimal slope $\widehat{\mu}_{\min}(\overline{V})$ defined in 4.2.4(4). See [CM20, Proposition 4.3.50, Proposition 4.3.51, Corollary 4.3.52] for the relation between the jumping numbers and the Arakelov degree of $\overline{V}$.

4.2.5 Graded algebra of adelic vector bundles

In this subsection, we introduce graded algebras where each of the graded pieces are adelic vector bundles. Later on we will introduce arithmetic volumes as the rate of asymptotic growth of positive degrees of such algebras. In order to have a satisfying theory of filtration, we need to introduce a technical condition on our base adelic curve. Note that given two adelic vector bundles $\overline{E} = (E, \|\ \|_E)$ and $\overline{F} = (F, \|\ \|_F)$ over an adelic curve S, we can define their (ϵ, π)-*tensor product* by setting the norm at every $\omega \in \Omega$ to be the (ϵ, π)-tensor product $\|\ \|_{E,\omega} \otimes_{\epsilon,\pi} \|\ \|_{F,\omega}$ (see [CM20, Chapter 4] for more details). We denote this adelic vector bundle by $\overline{E} \otimes_{\epsilon,\pi} \overline{F}$.

4.2.9 ([CM20] Definition 4.3.73, Definition 6.3.23). Let $C \in \mathbb{R}_{\geq 0}$. We say that S has

- the *tensorial minimal slope property of level* $\geq C$ if for any two adelic vector bundles $\overline{E}$ and $\overline{F}$ on S, we have

$$\widehat{\mu}_{\min}(\overline{E} \otimes_{\varepsilon,\pi} \overline{F}) \geq \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}) - C \log(\dim_K(E) \dim_K(F));$$

- the *strong tensorial minimal slope property of level* $\geq C$ if for any $n \in \mathbb{N}_{\geq 2}$ and any family $\{\overline{E_i}\}_{1 \leq i \leq n}$ of adelic vector bundles on S, we have

$$\widehat{\mu}_{\min}(\overline{E_1} \otimes_{\varepsilon,\pi} \cdots \otimes_{\varepsilon,\pi} \overline{E_n}) \geq \sum_{i=1}^{n} \widehat{\mu}_{\min}(\overline{E_i}) - C \log(\dim_K(E_i));$$

- the *Minkowski property of level* $\geq C$ if for any adelic vector bundles $\overline{E}$ on S, we have

$$\widehat{\nu}_1(\overline{E}) \geq \widehat{\mu}_{\max}(\overline{E}) - C \log(\dim_K(E)).$$

Note that Chen-Moriwaki showed in [CM20, Corollary 5.6.2] that S has the tensorial minimal slope property of level $\geq \frac{3}{2}\nu(\Omega_\infty)$ when the underlying field K is perfect. In particular number fields and function fields of characteristic 0 are included in this formalism. When S is given by a regular projective curve over some field k, then S has the Minkowski property of level $\geq C$ for some $C \in \mathbb{R}_{\geq 0}$, see [CM20, Remark 4.3.74], which implies that S has the tensorial minimal slope property of level $\geq C$ by [CM20, Corollary 4.3.76].

Remark 28. Note that Chen-Moriwaki showed in [CM20, Corollary 5.6.2] that S has the tensorial minimal slope property of level $\geq \frac{3}{2}\nu(\Omega_\infty)$ when the underlying field K is perfect.

Definition 4.2.10 ([CM20] Definition 6.3.24)**.** *Let $R := \mathrm{Frac}(\overline{K}[[z_1, \ldots z_d]])$ be the field of formal power series. We call a* graded K-algebra of S-normed vector spaces *(*with respect to R*) to be any family $\overline{E_\bullet} := \{\overline{E_m}\}_{m \in \mathbb{N}} = \{(E_m, \|\ \|_m)\}_{m \in \mathbb{N}}$ of S-normed vector spaces satisfying*

1. $\bigoplus_{m=0}^{\infty} E_{\overline{K},m}T^m$ *is contained in a* $\overline{K}$*-subalgebra of finite type inside* $R[T]$*, where* $E_{\overline{K},m} := E_m \otimes_K \overline{K}$*;*

2. *for any* $m \in \mathbb{N}$*,* $\|\ \|_{m,\omega}$ *is ultrametric for any* $\omega \in \Omega_{\mathrm{fin}}$*;*

3. *for any* $m_1, m_2 \in \mathbb{N}_{\geq 1}$*,* $s_1 \in E_m$*,* $s_2 \in E_n$ *and* $\omega \in \Omega$*, we have*

$$\|s_1 \cdot s_2\|_{m_1+m_2,\omega} \leq \|s_1\|_{m_1,\omega} \cdot \|s_2\|_{m_2,\omega}$$

A graded K-algebra of adelic vector bundles *is a graded* K*-algebra of* S*-normed vector spaces* $\overline{E_\bullet}$ *such that each* $\overline{E_m}$ *is an adelic bundle bundle.*

Remark 29. Our definition here is slightly more general than [CM20, Definition 6.3.24], with this definition, we are able to remove the assumption that X has a rational regular point $p \in X(K)$ in [CM20, Theorem 6.4.6] as pointed out in [CM20, Remark 6.3.29]. In this paper, we will prove a similar result for quasi-projective varieties, but without this assumption, see Theorem 4.4.9.

Remark 30. Recall from [CM20, Proposition 6.3.18] that for any graded K-algebra $E_\bullet = \{E_m\}_{m\in\mathbb{N}}$ such that $\bigoplus_{m=0}^{\infty} E_{\overline{K},m}T^m$ is $\mathrm{Frac}(\overline{K}[[z_1,\ldots z_d]])$ where $E_{\overline{K},m} := E_m \otimes_K \overline{K}$, there is a minimal non-negative integer $0 \leq \kappa \leq d$ such that $\dim_K(E_m) = \dim_K(E_{m,\overline{K}}) = O(m^\kappa)$, i.e. there is $C \in \mathbb{R}_{>0}$ such that $|\dim_K(E_m)/m^\kappa| < C$ for any $m \in \mathbb{N}$, see [CM20, Definition 6.3.9, Proposition 6.3.18], and we call this integer the *Kodaira dimension* of $E_\bullet$.

We next introduce the *arithmetic volume* of a graded K-algebra of adelic vector bundles.

Definition 4.2.11. *Let* $\overline{E_\bullet}$ *be a graded* K*-algebra of adelic vector bundles with respect to* $R := \mathrm{Frac}(\overline{K}[[z_1,\ldots z_d]])$*. We define the* arithmetic volume *and* arithmetic χ-volume *of* $\overline{E_\bullet}$ *as*

$$\widehat{\mathrm{vol}}(\overline{E_\bullet}) := \limsup_{m\to\infty} \frac{\widehat{\deg}_+(\overline{E_m})}{m^{d+1}/(d+1)!}$$

and

$$\widehat{\mathrm{vol}}_\chi(\overline{E_\bullet}) := \limsup_{m\to\infty} \frac{\widehat{\deg}(\overline{E_m})}{m^{d+1}/(d+1)!},$$

respectively.

Definition 4.2.12. *Let* $\overline{E_\bullet}$ *be a graded* K*-algebra of adelic vector bundles. We define the* asymptotic maximal slope *and* asymptotic minimal slope *of* $\overline{E_\bullet}$ *as*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{E_\bullet}) := \limsup_{m\to\infty} \frac{\widehat{\mu}_{\max}(\overline{E_m})}{m}$$

and

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{E_\bullet}) := \liminf_{m\to\infty} \frac{\widehat{\mu}_{\min}(\overline{E_m})}{m},$$

respectively.

Remark 31. Note that these quantities are not necessarily bounded. However we will show that for compactified divisors the asymptotic maximal slope is always finite and it turns out that under suitable positivity assumptions on the geometric divisor, the asymptotic minimal slope is also bounded.

4.2.6 Over global fields

We close this section by considering the adelic vector bundles over a global field.

4.2.13. Let C be the spectrum of the ring of integers of a number field K or a smooth projective curve over some field k. Let K be the function field of C. We can define an adelic curve, denoted by S, as follows. Set Ω_{fin} the set of closed points on C and

$$\Omega_\infty := \begin{cases} \mathrm{Hom}(K, \mathbb{C}), & \text{if } K \text{ is a number field;} \\ \emptyset & \text{if } K \text{ is the function field of a smooth curve over } k. \end{cases}$$

We equip $\Omega := \Omega_\infty \coprod \Omega_{\text{fin}}$ with the discrete σ-algebra $\mathcal{A}$. Then every $\omega \in \Omega$ determines an absolute value on K in an obvious way. Let ν be the measure on $(\Omega, \mathcal{A})$ such that for any $\omega \in \Omega$,

$$\nu(\omega) = \nu(\{\omega\}) := \begin{cases} \frac{[K_\omega : \mathbb{Q}_\omega]}{[K : \mathbb{Q}]} & \text{if } K \text{ is a number field;} \\ [k(\omega) : k] & \text{if } K \text{ is the function field of a smooth curve over } k. \end{cases}$$

Then S is proper.

In the following, we fix C, the spectrum of the ring of integers of a number field K or a smooth projective curve over some field k with function field K, and let $S = (K, \Omega, \mathcal{A}, \nu)$ be the adelic curve associated to C given in 4.2.13.

Classically, the *coherent* S-normed vector spaces over K as defined in Definition 4.2.15 are frequently studied. For a graded K-algebra of coherent adelic vector bundles $\overline{E_\bullet}$, the arithmetic volume $\widehat{\mathrm{vol}}(\overline{E_\bullet})$ can be defined in two ways, either using positive degree as we did in Definition 4.2.11 or using small sections in Definition 4.2.19. We will show these two definitions of arithmetic volume coincide if $\overline{E_\bullet}$ is *asymptotically pure*.

Recall from [CM20, §1.2.3] that for an S-normed space $\overline{V} = (V, \|\ \|)$, a basis $\mathbf{e} = \{e_1, \cdots, e_r\}$ of V is a *orthonormal basis* of V_ω if

- $\|e_i\|_\omega = 1$ for any $i = 1, \cdots, r$;
- $\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega \geq \max_{1 \leq i \leq r} \{|\lambda_i|_\omega\}$ for any $(\lambda_1, \cdots, \lambda_r) \in K^r$.

In particular, if $\omega \in \Omega_{\text{fin}}$ and $\|\ \|_\omega$ is ultrametric, then the second condition is equivalent to that $\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega = \max_{1 \leq i \leq r} \{|\lambda_i|_\omega\}$ for any $(\lambda_1, \cdots, \lambda_r) \in K^r$.

Example 4.2.14 ([CM20] Example 4.1.5). Let V be a vector space over K and $\mathbf{e} = \{e_1, \cdots, e_r\}$ a basis of V. We can associate $\mathbf{e}$ an S-norm $\|\ \|_{\mathbf{e}} = \{\|\ \|_{\mathbf{e},\omega}\}_{\omega \in \Omega}$ as follows: for any $s = \sum_{i=1}^{r} \lambda_i e_i \in V_\omega = V \otimes_K K_\omega$, we set

$$\|s\|_{\mathbf{e},\omega} := \begin{cases} \sum_{i=1}^{r} |\lambda_i|_\omega & \text{if } \omega \in \Omega_\infty; \\ \max_{1 \leq i \leq r} \{|\lambda_i|_\omega\} & \text{if } \omega \in \Omega_{\text{fin}}. \end{cases}$$

Then by [CM20, Example 4.1.5], the basis $\mathbf{e}$ as above gives a structure of adelic vector bundle which we denote by $\overline{V}_{\mathbf{e}}$. Moreover, $\overline{V}_{\mathbf{e}}$ is pure by [Bur+16, Proposition 2.9].

Definition 4.2.15. *Let $\overline{V} = (V, \|\ \|)$ be an S-normed vector space such that $\|\ \|_\omega$ is ultrametric for any $\omega \in \Omega_{\text{fin}}$. We say that $\overline{V}$*

1. *([Bur+16, Definition 2.10]) is* generically trivial *if there is a basis $\mathbf{e} := \{e_1, \cdots, e_r\}$ of V such that $\mathbf{e}$ forms an orthonormal basis of V_ω over K_ω for all but finitely many $\omega \in \Omega$;*
2. *([Bur+16, Definition 4.4.4]) is* coherent *if for any $s \in V$, we have that $\|s\|_\omega \leq 1$ for all but finitely many $\omega \in \Omega$.*

The generically trivial S-normed vector spaces and adelic vector bundles are closely related.

Lemma 4.2.16. *Let $\overline{V} = (V, \|\ \|)$ be an S-normed vector space such that $\|\ \|_\omega$ is ultrametric for any $\omega \in \Omega_{\mathrm{fin}}$.*

1. *We have that $\overline{V}$ is generically trivial if and only if there is a basis $\mathbf{e}$ of V such that $\|\ \|_\omega = \|\ \|_{\mathbf{e},\omega}$ for all but finitely many $\omega \in \Omega$. In particular, if $\overline{V}$ is a generically trivial, then it is a coherent adelic vector bundle.*

2. *If $\overline{V}$ is pure at all but finitely many $\omega \in \Omega$, then $\overline{V}$ is an adelic vector bundle if and only if $\overline{V}$ is generically trivial. In particular, in this case, $\overline{V}$ is coherent by 1.*

3. *If $\overline{V}$ is a coherent adelic vector bundle, then the purification $\overline{V}_{\mathrm{pur}}$ is a pure generically trivial adelic vector bundle.*

Proof. 1 The first statement is from the definition. If $\overline{V}$ is a generically trivial, by [CM20, Corollary 4.1.10], then $\overline{V}$ is an adelic vector bundle. On the other hand, notice that $\overline{V}_{\mathbf{e}}$ is coherent for any basis $\mathbf{e}$, so $\overline{V}$ is coherent by the first statement.

2 If $\overline{V}$ is generically trivial, then it is an adelic vector bundle by 1. For the converse statement, assume that $\overline{V}$ is an adelic vector bundle. Let $\mathbf{e}$ be a basis of V, and $\overline{V}_{\mathbf{e}} = (V, \|\ \|_{\mathbf{e}})$ the associated adelic vector bundle. Since $\overline{V}$ and $\overline{V}_{\mathbf{e}}$ are adelic vector bundles, by [CM20, Corollary 4.1.8] we have that

$$d(\|\ \|, \|\ \|_{\mathbf{e}}) = \sum_{\omega \in \Omega} \sup_{s \in V_\omega \setminus \{0\}} |\log \|s\|_\omega - \log \|s\|_{\mathbf{e},\omega}| < \infty$$

Now observing that all pure norms have discrete value groups in the non-archimedean places by [Bur+16, Proposition 2.9], we deduce that $\|\ \|_\omega$ and $\|\ \|_{\mathbf{e},\omega}$ must agree at all but finitely many $\omega \in \Omega$. This implies that $\overline{V}$ is generically trivial by 1.

3 Let $s \in V$. By the definition of the purification, for $\omega \in \Omega$, if $\|s\|_\omega \leq 1$, then $\|s\|_{\mathrm{pur},\omega} \leq 1$. Since $\overline{V}$ is coherent, we have that $\|s\|_\omega \leq 1$ for almost all $\omega \in \Omega$. So $\|s\|_{\mathrm{pur},\omega} \leq 1$ for almost all $\omega \in \Omega$. Hence

$$\overline{\int_\Omega} \log \|s\|_{\mathrm{pur},\omega}\, \nu(d\omega) < \infty.$$

Now noting that the σ-algebra $\mathcal{A}$ is discrete, by Lemma 4.2.7 we can easily conclude that $\overline{V}_{\mathrm{pur}}$ is an adelic vector bundle. Moreover, $\overline{V}_{\mathrm{pur}}$ is generically trivial by 2. □

Definition 4.2.17. *Let $\overline{V} = (V, \|\ \|)$ be an S-norm vector space such that $\|\ \|_\omega$ is ultrametric for any $\omega \in \Omega_{\mathrm{fin}}$. For any $\omega \in \Omega$, notice that $\|\ \|_\omega \leq \|\ \|_{\mathrm{pur},\omega}$. We set*

$$\sigma_\omega(\overline{V}) \coloneqq d_\omega(\|\ \|, \|\ \|_{\mathrm{pur}}) = \sup_{s \in V_\omega \setminus \{0\}} (\log \|s\|_\omega - \log \|s\|_{\mathrm{pur},\omega})$$

The impurity *of $\overline{V}$ is defined as*

$$\sigma(\overline{V}) \coloneqq d(\|\ \|, \|\ \|_{\mathrm{pur}}) = \sum_{\omega \in \Omega} \sigma_\omega(\overline{V}) \in [0, +\infty].$$

From the definition, $\overline{V}$ is pure if and only $\sigma(\overline{V}) = 0$. By Lemma 4.2.16(3) and [CM20, Corollary 4.1.8], if $\overline{V}$ is a coherent adelic vector bundle, then $\sigma(\overline{V}) < +\infty$.

Remark 32. Keep the notation in Definition 4.2.17. If K is a number field and $\overline{V}$ is coherent, then the impurity $\sigma(\overline{V})$ is exactly the one defined in [CM20, §4.4.3]. We explain the reason as follows. Set

$$\mathcal{V} \coloneqq \{s \in V \mid \|s\|_\omega \leq 1 \text{ for all } \omega \in \Omega_{\mathrm{fin}}\}.$$

Since $\overline{V}$ is coherent, by [CM20, Proposition 4.4.2 (2), Remark 4.4.1 (2)], for any $\omega \in \Omega_{\text{fin}}$, we have that

$$\mathcal{V} \otimes_{\mathcal{O}_K} K_\omega^\circ = \{x \in V_\omega \mid \|s\|_\omega \leq 1\}.$$

By definition of purification, the norm $\| \ \|_{\text{pur},\omega}$ is exactly the one arising from $\mathcal{V} \otimes_{\mathcal{O}_K} K_\omega^\circ$ in [CM20, Definition 1.1.23]. This implies our claim by definitions of impurity.

As [Bur+16, Definition 2.14, Definition 2.18], we have the following definition.

Definition 4.2.18. *Let $\overline{V} = (V, \| \ \|)$ be a coherent S-normed vector space. We define the space of its* small sections *as*

$$\widehat{V} \coloneqq \{s \in V \mid \|s\|_\omega \leq 1 \text{ for all } \omega \in \Omega\}$$

and define

$$\widehat{h}^0(\overline{V}) \coloneqq \begin{cases} \log(\#\widehat{V}) & \text{if } K \text{ is a number field} \\ \dim_k(\widehat{V}) & \text{if } K \text{ is the function field of a curve of a field } k \end{cases}$$

We further assume that $\overline{V}$ is generically trivial.

- *Assume that K is a function field of curve over a field k. Let $\mathbf{b}$ be a basis of V over K. For any $\omega \in \Omega$, we choose a basis $\mathbf{b}_\omega$ of $\widehat{V}_\omega \coloneqq \{s \in V_\omega \mid \|s\|_\omega \leq 1\}$ over K_ω°. The* Euler characteristic *of $\overline{V}$ is defined as*

$$\chi(\overline{V}) \coloneqq \sum_{\omega \in \Omega} \nu(\omega) \log |\det(\mathbf{b}_\omega/\mathbf{b})|_\omega, \tag{4.17}$$

  where $\mathbf{b}_\omega/\mathbf{b}$ denotes the matrix of $\mathbf{b}_\omega$ with respect to the basis $\mathbf{b}$. This quantity does not depend on the choice of bases.

- *Assume that K is a number field. Let $\mathbb{A}_K$ be its ring of adeles and $V_{\mathbb{A}_K} \coloneqq V \otimes_K \mathbb{A}_K$. The* Euler characteristic *of $\overline{V}$ is defined as*

$$\chi(\overline{V}) \coloneqq \frac{1}{[K:\mathbb{Q}]} \log \operatorname{vol}(\widehat{V}), \tag{4.18}$$

  where $\widehat{V} \subseteq V_{\mathbb{A}_K}$ is viewed as the adelic unit ball with respect to the norms $\| \ \|_\omega$ and $\operatorname{vol}(\cdot)$ denotes the Haar measure on $V_{\mathbb{A}_K}$ normalized by $\operatorname{vol}(V_{\mathbb{A}_K}/V) = 1$.

Remark 33. Let $\overline{V}$ be a generically trivial adelic vector bundle. If K is a function field, by [CM20, Proposition 4.3.18], we have that $\widehat{\deg}(\overline{V}) = \chi(\overline{V})$. If K is a number field, the expression (4.18) for $\chi(\overline{V})$ can have a similar formula as (4.17), see [Bur+16, Remark 2.20]. Unlike the function field case, $\widehat{\deg}(\overline{V}) \neq \chi(\overline{V})$ in general. However, we have that

$$\widehat{\deg}(\overline{V}) = \chi(\overline{V}) + O(\dim_K(V)\log(\dim_K(V)))$$

see [BC11, (3.5)].

Remark 34. Let $\overline{V} = (V, \| \ \|)$ be a coherent adelic vector bundle. By the definition of purification, for any $s \in V$ and $\omega \in \Omega$, we have that $\|s\|_\omega \leq 1$ if and only if $\|s\|_{\text{pur},\omega} \leq 1$. Hence

$$\widehat{h}^0(\overline{V}) = \widehat{h}^0(\overline{V}_{\text{pur}}).$$

If furthermore, $\overline{V}$ is generically trivial, by [Bur+16, Proposition 2.23], then

$$\chi(\overline{V}) = \chi(\overline{V}_{\text{pur}}).$$

Next, we consider the classical arithmetic volume of a graded K-algebra of coherent S-normed vector spaces.

Definition 4.2.19. *Let $\overline{E_\bullet}$ be a graded K-algebra of coherent S-normed vector spaces (i.e. $\overline{E_m}$ is coherent for each $m \in \mathbb{N}$). We define the* classical arithmetic volume

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{E_\bullet}) := \limsup_{m\to\infty} \frac{\widehat{h}^0(\overline{E_m})}{m^{d+1}/(d+1)!},$$

where d is the Kodaira dimension of the graded K-algebra $E_\bullet$.

We further assume that $\overline{E_m}$ is generically trivial for any $m \in \mathbb{N}$. We define the classical arithmetic χ-volume

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{E_\bullet}) := \limsup_{m\to\infty} \frac{\chi(\overline{E_m})}{m^{d+1}/(d+1)!}.$$

Definition 4.2.20 ([CM20] Definition 7.5.1)**.** *Let $\overline{E_\bullet}$ be a graded K-algebra of coherent S-normed vector spaces. We say that $\overline{E_\bullet}$ is* asymptotically pure *if*

$$\limsup_{m\to\infty} \frac{\sigma(\overline{E_m})}{m} = 0.$$

Proposition 4.2.21*.* Let $\overline{E_\bullet}$ be a graded K-algebra of coherent (resp. generically trivial) adelic vector bundles. One has the following inequality:

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{E_\bullet}) \leq \widehat{\mathrm{vol}}(\overline{E_\bullet}) \text{ (resp. } \widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{E_\bullet}) \leq \widehat{\mathrm{vol}}_\chi(\overline{E_\bullet})),$$

the equality holds if $\overline{E_\bullet}$ is asymptotically pure.

Proof. We first consider the arithmetic volume. Let $\overline{E_{\mathrm{pur},m}} = (E_m, \|\ \|_{\mathrm{pur},m})$ be the purification of $\overline{E_m} = (E_m, \|\ \|_m)$. Since each $\overline{E_m}$ is a coherent adelic vector bundle, by Lemma 4.2.16(3), $\overline{E_{\mathrm{pur},\bullet}} := \bigoplus_{m=0}^{\infty} \overline{E_{\mathrm{pur},m}}$ is a graded K-algebra of pure generically trivial adelic vector spaces. By Remark 33, $\widehat{h}^0(\overline{E_{\mathrm{pur},m}}) = \widehat{h}^0(\overline{E_m})$ and hence $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{E_{\mathrm{pur},\bullet}}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{E_\bullet})$. On the other hand, by [Bur+16, Proposition 2.13], $\overline{E_{\mathrm{pur},m}}$ is the adelic vector bundle associated to some Hermitian vector bundle $\overline{\mathcal{E}_m}$ on C if K is a number field or to some vector bundle $\mathcal{E}_m$ on C if K is the function field of a curve. By [CM20, Proposition 4.3.23, Proposition 4.3.24], we have that

$$|\widehat{h}^0(\overline{E_{\mathrm{pur},m}}) - \widehat{\deg}_+(\overline{E_{\mathrm{pur},m}})| \leq C_1 \cdot \dim_K(E_m)\log(\dim_K(E_m)) \tag{4.19}$$

for some constant $C_1 > 0$ dependent only on K. Combine (4.19) and the fact that $\frac{\dim_K(E_m)}{m^d}$ is bounded when $m \to \infty$, we have that $\widehat{\mathrm{vol}}(\overline{E_{\mathrm{pur},\bullet}}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{E_{\mathrm{pur},\bullet}})$. It remains to compare $\widehat{\mathrm{vol}}(\overline{E_{\mathrm{pur},\bullet}})$ and $\widehat{\mathrm{vol}}(\overline{E_\bullet})$. Notice that

$$e^{-\sigma_\omega(\overline{E_m})}\|\ \|_{\mathrm{pur},m,\omega} \leq \|\ \|_{m,\omega} \leq \|\ \|_{\mathrm{pur},m,\omega} \tag{4.20}$$

for any $\omega \in \Omega$. By [CM20, Proposition 4.3.21], we have that

$$\begin{aligned}\widehat{\deg}_+(\overline{E_{\mathrm{pur},m}}) \leq \widehat{\deg}_+(\overline{E_m}) &\leq \widehat{\deg}_+(E_m, (e^{-\sigma_\omega(\overline{E_m})}\|\ \|_{\mathrm{pur},m,\omega})_{\omega\in\Omega}) \\ &\leq \widehat{\deg}_+(\overline{E_{\mathrm{pur},m}}) + \dim_K(E_m)\sigma(\overline{E_m}).\end{aligned}$$

Hence $\widehat{\mathrm{vol}}(\overline{E_{\mathrm{pur},\bullet}}) \leq \widehat{\mathrm{vol}}(\overline{E_\bullet})$ and

$$\begin{aligned}\widehat{\mathrm{vol}}(\overline{E_\bullet}) &\leq \limsup_{m\to\infty} \frac{\widehat{\deg}_+(\overline{E_{\mathrm{pur},m}}) + \dim_K(E_m)\sigma(\overline{E_m})}{m^{d+1}/(d+1)!} \\ &\leq \widehat{\mathrm{vol}}(\overline{E_{\mathrm{pur},\bullet}}) + (d+1)!\limsup_{m\to\infty} \frac{\dim_K(E_m)\sigma(\overline{E_m})}{m^{d+1}}.\end{aligned}$$

Again, notice that $\frac{\dim_K(E_m)}{m^d}$ is bounded when $m \to \infty$, if $\overline{E_\bullet}$ is asymptotically pure, i.e. $\limsup_{m\to\infty} \frac{\sigma(\overline{E_m})}{m} = 0$, then $\widehat{\mathrm{vol}}(\overline{E_\bullet}) \leq \widehat{\mathrm{vol}}(\overline{E_{\mathrm{pur},\bullet}})$. This completes the proof for arithmetic volume.

For the arithmetic χ-volume, the proof is similar. Assume that each $\overline{E_m}$ is generically trivial, we keep the notation as before. By [Bur+16, Proposition 2.23], we have that $\chi(\overline{E_{\mathrm{pur},m}}) = \chi(\overline{E_m})$ and hence $\widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{E_{\mathrm{pur},\bullet}}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{E_\bullet})$. By Remark 33, we have that

$$|\chi(\overline{E_{\mathrm{pur},m}}) - \widehat{\deg}(\overline{E_{\mathrm{pur},m}})| \leq C_2 \cdot \dim_K(E_m) \log(\dim_K(E_m)) \tag{4.21}$$

for some constant $C_2 > 0$ dependent only on K. This implies that $\mathrm{vol}_\chi(E_{\mathrm{pur},\bullet}) = \mathrm{vol}_\chi^{\mathrm{YZ}}(E_{\mathrm{pur},\bullet})$. It remains to compare $\widehat{\mathrm{vol}}_\chi(\overline{E_{\mathrm{pur},\bullet}})$ and $\widehat{\mathrm{vol}}_\chi(\overline{E_\bullet})$. By (4.20), we have

$$\widehat{\deg}(\overline{E_{\mathrm{pur},m}}) \leq \widehat{\deg}(\overline{E_m}) \leq \widehat{\deg}(\overline{E_{\mathrm{pur},m}}) + \dim_K(E_m)\sigma(\overline{E_m}).$$

As above, this implies that $\widehat{\mathrm{vol}}_\chi(\overline{E_{\mathrm{pur},\bullet}}) \leq \widehat{\mathrm{vol}}_\chi(\overline{E_\bullet})$ and the equality holds if $\overline{E_\bullet}$ is asymptotically pure, which completes the proof. □

4.3 Review of compactified S-metrized divisors

In this section, we recall the notion of compactified S-metrized divisors on quasi-projective varieties over adelic curves defined in [CG24]. Recall that given a $\mathbb{Q}$-vector space M, and a subset N, the the closure $\overline{N}$ of N with respect to the *finite subspace topology* is given by $\overline{N} = \varinjlim_E \overline{(N \cap E)}$, where E runs through all finite dimensional subspaces of M and $\overline{(N \cap E)}$ is the closure of $N \cap E$ in E with respect to the canonical euclidean topology.

4.3.1 Compactified (geometric) divisors

4.3.1. Let X be a projective variety over a field K. We set $N_{\mathrm{gm}}(X)$ the set of nef divisors in $\mathrm{Div}(X)$, and $N_{\mathrm{gm},\mathbb{Q}}(X)$ the cone in $\mathrm{Div}_\mathbb{Q}(X) \coloneqq \mathrm{Div}(X) \otimes_\mathbb{Z} \mathbb{Q}$ generated by $N_{\mathrm{gm}}(X)$. For a quasi-projective variety U over K, a (*geometric*) *boundary divisor* of U is a pair (X_0, B) consisting of a projective K-model $U \hookrightarrow X_0$ and an effective divisor $B \in \mathrm{Div}_\mathbb{Q}(X_0)$ such that $|B| = X_0 \setminus U$. We set

$$\mathrm{Div}_\mathbb{Q}(U)_{\mathrm{mo}} \coloneqq \varinjlim_X \mathrm{Div}_\mathbb{Q}(X), \quad N_{\mathrm{gm},\mathbb{Q}}(U) \coloneqq \varinjlim_X N_{\mathrm{gm},\mathbb{Q}}(X), \tag{4.22}$$

where X ranges over all projective K-models of U. Let (X_0, B) be a boundary divisor of U. The (B-)*boundary topology* on $\mathrm{Div}_\mathbb{Q}(U)_{\mathrm{mo}}$ is defined such that a basis of neighborhoods of a divisor D is given by

$$B(r, D) \coloneqq \{E \in \mathrm{Div}_\mathbb{Q}(U) \mid -rB \leq E - D \leq rB\}, \quad r \in \mathbb{Q}_{>0}.$$

Notice that the boundary topology is independent of the choice of (X_0, B). We write $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{cpt}}$ (resp. $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{snef}}$) the completion of $\mathrm{Div}_\mathbb{Q}(U)_{\mathrm{mo}}$ (resp. $N_{\mathrm{gm},\mathbb{Q}}(U)$) with respect to the boundary topology and $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{int}} \coloneqq \widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{snef}} - \widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{snef}}$. Moreover, write $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{nef}}$ for the closure of $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{snef}}$ in $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{int}}$ with respect to the finite subspace topology. An element in $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{cpt}}$ is called a *compactified* (*geometric*) *divisor*, we have a natural map

$$\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{cpt}} \to \mathrm{Div}_\mathbb{Q}(U), \ D \mapsto D|_U.$$

For $D \in \widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{cpt}}$, we say D is *strongly nef* (resp. *nef*, *integrable*) if $D \in \widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{snef}}$ (resp. $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{nef}}$, $\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{int}}$). We have a symmetric multilinear map

$$\underbrace{\widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{int}} \times \cdots \times \widetilde{\mathrm{Div}}_\mathbb{Q}(U)_{\mathrm{int}}}_{d\text{-times}} \to \mathbb{R}, \quad (D_1, \cdots, D_d) \mapsto D_1 \cdots D_d,$$

see [YZ24, Proposition 4.1.1]. For simplicity, for $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{int}}$ we denote $\deg_D(U) := D^d$.

Remark 35. In [CG24, Definition 3.4], the space $\mathrm{Div}_{\mathbb{Q}}(U)_{\mathrm{mo}}$ defined to be $\varinjlim_X \mathrm{Div}_{\mathbb{Q}}(X)$, where X ranges over all proper K-models of U (although U is quasi-projective), so the completion $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ in [CG24] may be larger than the one defined above. In the local and global cases below, the same situation happens. The reason why we only consider projective K-models is that the ampleness condition in Definition 4.3.12 is crucial in our paper and that the results in [YZ24] are induced from the ones in projective case.

4.3.2. Let U be a d-dimensional normal quasi-projective variety over a field K and $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$. Notice that an element f in the function field $K(U)$ of U can determine a compactified divisor denoted by $\mathrm{div}(f)$. We can introduce the space of *global sections* of $\overline{D}$ as

$$H^0(U, D) := \{f \in K(U)^\times \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

which is a K-vector space since U is normal by [CM20, Proposition 2.4.13 (6)] (on projective K-models of U). The set $H^0(U, D)$ is not necessarily stable under addition if U is not normal, see [CM20, Example 2.4.15]. We have a natural injective map $H^0(U, D) \hookrightarrow H^0(U, D|_U)$. If D is induced by a divisor on some projective model X of U which still denoted by D, then $H^0(X, D) \subset H^0(U, D)$, the equality holds if X is integrally closed in U by [YZ24, Lemma 2.3.7]. We have the following properties:

1. ([YZ24, Lemma 5.1.6, Theorem 5.2.1 (1)]) The K-vector space $H^0(U, D)$ has finite dimension and the limit
$$\mathrm{vol}(D) := \lim_{m\to\infty} \frac{\dim_K(H^0(U, mD))}{m^d/d!}$$
exists;

2. ([YZ24, Theorem 5.2.1 (2), Theorem 5.2.9]) If D_n converges to D in $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ with respect to the boundary topology, then $\lim_{n\to\infty} \mathrm{vol}(D_n) = \mathrm{vol}(D)$.

We call $\mathrm{vol}(D)$ the *volume* of D. We say that $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ is *big* if $\mathrm{vol}(D) > 0$. We refer to [YZ24, §5.1–5.2] for details and various properties of volumes.

4.3.2 Local theory: mixed Monge-Ampère measures

In this subsection, we fix a field K with an absolute value $|\ |_v$ and a d-dimensional quasi-projective variety U over K. We denote by $U^{\mathrm{an}} = U_v^{\mathrm{an}}$ the Berkovich analytification of U with respect to the place v, see [Ber90a, §3.4–3.5].

4.3.3. For a $\mathbb{Q}$-Cartier divisor $D \in \mathrm{Div}_{\mathbb{Q}}(U) := \mathrm{Div}(U) \otimes_{\mathbb{Z}} \mathbb{Q}$, D can be written by local equations $f \in \mathcal{O}_U(V)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. Then we get a well-defined continuous real function $\log|f|$ on V^{an} for such open subsets V. A *Green function* for D is a continuous real function g_D such that on $(U \setminus |D|)^{\mathrm{an}}$ such that for any local equation f of D on an open subset V of U, we have that $g_D + \log|f|$ is a continuous function on V^{an}.

We denote by $\widehat{\mathrm{Div}}(U)$ (resp. $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)$) the group of pairs (D, g) with $D \in \mathrm{Div}(U)$ (resp. $D \in \mathrm{Div}_{\mathbb{Q}}(U)$) and g a Green function for D. Obviously, we have that $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U) \simeq \widehat{\mathrm{Div}}(U) \otimes_{\mathbb{Z}} \mathbb{Q}$.

4.3.4. For a projective variety X over K, we denote $\widehat{\mathrm{Div}}_{\mathbb{Q}}(X)_{\mathrm{mo}}$ the group of $\mathbb{Q}$-Cartier divisors with *model* Green functions, $N_{\mathrm{mo},\mathbb{Q}}(X)$ (resp. $N_{\mathrm{tFS},\mathbb{Q}}(X)$) the cone in $\widehat{\mathrm{Div}}_{\mathbb{Q}}(X)$ generated by the Cartier divisors with *semi-positive* model Green functions (resp. tFS-Green functions) and $\widehat{\mathrm{Div}}_{\mathbb{Q}}(X)_{\mathrm{tFS}} := N_{\mathrm{tFS},\mathbb{Q}}(X) - N_{\mathrm{tFS},\mathbb{Q}}(X)$.

4.3.5. Let U be a quasi-projective variety over K. Assume that v is non-trivial. A *boundary divisor* of U is a pair $(X_0, \overline{B})$ consisting of a projective K-model $U \hookrightarrow X_0$ and an effective divisor $\overline{B} = (B, g_B) \in \widehat{\mathrm{Div}}_{\mathbb{Q}}(X_0)_{\mathrm{mo}}$ such that $|B| = X_0 \setminus U$. We set

$$\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{mo}} := \varinjlim_X \widehat{\mathrm{Div}}_{\mathbb{Q}}(X)_{\mathrm{mo}}, \quad N_{\mathrm{mo},\mathbb{Q}}(U) := \varinjlim_X N_{\mathrm{mo},\mathbb{Q}}(X) \tag{4.23}$$

where X ranges over all projective K-models of U. Then a boundary divisor (X_0, B) gives a *boundary topology* on $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{mo}}$ as 4.3.1 which is independent of the choice of (X_0, B). We denote by $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ (resp. $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{snef}}$) the completion of $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{mo}}$ (resp. $N_{\mathrm{mo},\mathbb{Q}}(U)$) with respect to the boundary topology and $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{int}} := \widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{snef}} - \widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{snef}}$. Moreover, we denote by $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{nef}}$ the closure of $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{snef}}$ in $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{int}}$ with respect to the finite subspace topology. When v is trivial, we can do a similar procedure on $(\widehat{\mathrm{Div}}_{\mathrm{tFS},\mathbb{Q}}, N_{\mathrm{tFS}})$, the corresponding spaces are also denoted similarly. An element $\overline{D}$ in $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)$ is called a *compactified metrized divisor* on U, it is called *strongly nef* (resp. *nef*) if $\overline{D} \in \widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{snef}}$ (resp. $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{nef}}$). We have a homomorphism

$$\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}} \to \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$$

and an injective homomorphism

$$\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}} \hookrightarrow \widehat{\mathrm{Div}}_{\mathbb{Q}}(U),$$

see [CG24, p. 4.20]. So an element in $\widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ can be uniquely written as (D, g) with $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ and g a Green function for $D|_U$.

There is a unique way to associate to any d-dimension quasi-projective U over K and to a family $\overline{D_1}, \cdots, \overline{D_d} \in \widehat{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{nef}}$ a positive Radon measure $c_1(\overline{D_1}) \wedge \cdots \wedge c_1(\overline{D_d})$ on U^{an} satisfying multilinearity, symmetry, continuity (respect to boundary topology) and other properties, see [CG24, Proposition 4.45].

Remark 36. If we have finitely many nef compactified metrized divisors $\overline{D_j} = (D_j, g_j)$ for $j = 0, \ldots, k$, then there is an open subset $U' \subset U$ such that the restriction of $\overline{D_j}$ on U' is strongly nef, see [CG24, Remark 4.34].

4.3.3 Global theory: Compactified (model) divisors

In the rest of this section, we fix an adelic curve $S = (K, \Omega, \mathcal{A}, \nu)$ satisfying the following conditions:

1. $\nu(\Omega_\infty) < \infty$;
2. The set $\nu(\mathcal{A}) \not\subset \{0, +\infty\}$;
3. either the σ-algebra $\mathcal{A}$ is discrete, or that K is countable.

We also fix a d-dimensional quasi-projective variety U over K and set $U_{\overline{K}} := U \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$.

4.3.6. An *S-Green function* for a divisor $D \in \mathrm{Div}_{\mathbb{Q}}(U)$ is a family of Green functions $g_{D,\omega}$ for the base change D_ω of D to U_ω with ω running over Ω. We denote $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)$ the group of pairs (D, g_D) with D a $\mathbb{Q}$-Cartier divisor and g_D an *S-measurable*, *locally S-bounded* S-Green function for D, see [CG24, Definition 6.3, Definition 6.9]. Given $\overline{D} := (D, g_D) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)$ and $x \in U(\overline{K})$, we denote its *height function* as

$$h_{\overline{D}}(x) := \int_{\Omega_{\overline{K}}} g_\omega(x)\ \nu_{\overline{K}}(d\omega),$$

where $S_{\overline{K}} = (\overline{K}, \Omega_{\overline{K}}, \mathcal{A}_{\overline{K}}, \nu_{\overline{K}})$ denotes the canonical extension of the adelic curve structure on K to the algebraic closure $\overline{K}$ as explained in [CM20, §3.4.2].

If $U = X$ is projective, a pair $(D, g_D) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$ if and only if (D, g_D) is an *adelic $\mathbb{Q}$-divisor* in the sense of Chen and Moriwaki, [CM20, §6.2.3], see [CG24, Remark 6.21]. Note that in this case, it also ocincides with the definition of height function given in [CM20, Definition 6.2.1].

4.3.7. Let X be a d-dimensional projective variety over K. We consider the submonoid $N_{S,\mathbb{Q}}(X)$ of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$ consisting of $(D, g_D) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$ with $g_{D,\omega}$ a uniform limit of semipositive model (resp. tFS-)Green functions for D_ω on X_ω^{an} if $\omega \in \Omega \setminus \Omega_0$ (resp. $\omega \in \Omega_0$). Write $M_{S,\mathbb{Q}}(X) := N_{S,\mathbb{Q}}(X) -$

$N_{S,\mathbb{Q}}(X)$ which is an ordered $\mathbb{Q}$-vector space: $(D, g_D) \geq 0$ if and only if $D \geq 0$ and $g_{D,\omega} \geq 0$ for any $\omega \in \Omega$. We have a symmetric multilinear map

$$M_{S,\mathbb{Q}}(X)^{d+1} \to \mathbb{R}, \quad (D_0, \cdots, D_d) \mapsto (D_0 \cdots D_d \mid X)_S. \tag{4.24}$$

For any $\overline{D} \in M_{S,\mathbb{Q}}(X)$ and integral closed subscheme Z of X, set $h_{\overline{D}}(Z) := (\overline{D}^{\dim(Z)+1} \mid Z)_S$ (the arithmetic intersection number is well-defined without assuming that X is geometrically integral, see [CM21]). Notice that the restriction of $\overline{D}$ on Z may not be defined, but the restriction of the corresponding *S-metrized* ($\mathbb{Q}$-)*line bundle* is defined (see [CG24, §7.2]), so $h_{\overline{D}}(Z)$ is well-defined.

We say that $\overline{D} \in N_{S,\mathbb{Q}}(X)$ is *S-ample* if D is ample and if there is $\varepsilon > 0$ such that for every integral closed subscheme Z of X, we have

$$h_{\overline{D}}(Z) \geq \varepsilon \deg_D(Z)(\dim(Z) + 1).$$

The S-ample divisors form a cone $N^+_{S,\mathbb{Q}}(X)$ of $M_{S,\mathbb{Q}}(X)$. We say that $\overline{D} \in N_{S,\mathbb{Q}}(X)$ is *S-nef* if $\overline{D}$ is in the closure of $N^+_{S,\mathbb{Q}}(X)$ in $M_{S,\mathbb{Q}}(X)$ with respect to the finite subspace topology. The S-nef divisors form a subcone $N'_{S,\mathbb{Q}}(X)$ of $N_{S,\mathbb{Q}}(X)$ which is preserved under pull-backs and tensor products.

4.3.8. Set

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}} := \varinjlim_X \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X), \quad N_{S,\mathbb{Q}}(U) := \varinjlim_X N_{S,\mathbb{Q}}(X), \quad N'_{S,\mathbb{Q}}(U) := \varinjlim_X N'_{S,\mathbb{Q}}(X),$$

where X runs through all projective K-model of U.

A *weak boundary divisor* (resp. a *boundary divisor*) of U is a pair $(X_0, \overline{B})$ consisting of a projective K-model $U \hookrightarrow X_0$ over K and $\overline{B} \in M_{S,\mathbb{Q}}(U)_{\geq 0}$ such that $|B| \subset X_0 \setminus U$ (resp. $|B| = X_0 \setminus U$). The weak boundary divisors form a directed subset T of $M_{S,\mathbb{Q}}(U)_{\geq 0}$. As in 4.3.1, a weak boundary divisor $(X_0, \overline{B})$ gives a topology, called *$\overline{B}$-boundary topology*, on $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$. We set $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}_{\mathrm{CM}}$ (resp. $N_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}$, resp. $N'_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}$) as the completion of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$ (resp. $N_{S,\mathbb{Q}}(U)$, resp. $N'_{S,\mathbb{Q}}(U)$) with respect to the $\overline{B}$-boundary topology. The space

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}} := \varinjlim_{\overline{B} \in T} \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}_{\mathrm{CM}}$$

is called the space of *compactified S-metrized* ($\mathbb{Q}$-)*divisors* of U. We also set

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}} := \varinjlim_{\overline{B} \in T} N_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}, \quad \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-snef}} := \varinjlim_{\overline{B} \in T} N'_{S,\mathbb{Q}}(U)^{d_{\overline{B}}},$$

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-int}} := \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}} - \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}},$$

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-int}} := \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-snef}} - \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-snef}},$$

and denote by $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$) the closure of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-snef}}$) in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-int}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-int}}$) with respect to the finite subspace topology. A compactified S-metrized divisors $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ is called *relatively integrable* (resp. *strongly relatively nef*, resp. *relatively nef*, resp. *arithmetically integrable*, resp. *strongly arithmetically nef*, resp. *arithmetically nef*) if $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-int}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}}$, resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}$, resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-int}}$, resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$).

We have injective homomorphisms

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-int}} \hookrightarrow \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-int}} \hookrightarrow \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}} \hookrightarrow \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U) \tag{4.25}$$

(see [CG24, Proposition 7.4] for the injectivity) and a forgetting homomorphism

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}} \to \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}.$$

By the injectivity of (4.25), throughout this paper, we write an element $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ as (D,g) with $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ and g an S-measurable, locally S-bounded S-Green function for $D|_U$. For a compactified divisor $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$, an *$S$-Green function for D* is an S-Green function g for $D|_U$ such that $(D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. Let $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ and $x \in U(\overline{K})$, we write $h_{\overline{D}}(x) \coloneqq \int_\Omega g_\omega(x)\,\nu(d\omega)$.

Remark 37. If $U = X$ is projective, by [CG24, Proposition 7.10], we have that

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{CM}} = \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{cpt}} = \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\text{rel-int}} = \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\text{ar-int}},$$

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\text{rel-nef}} = N_{S,\mathbb{Q}}(X) \quad \text{and} \quad \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\text{ar-nef}} = N'_{S,\mathbb{Q}}(X).$$

4.3.4 Extension of Yuan-Zhang's arithmetic intersection number

We have a symmetric bilinear map

$$\underbrace{\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-int}} \times \cdots \times \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-int}}}_{(d+1)\text{-times}} \to \mathbb{R}, \quad (\overline{D_0}, \cdots, \overline{D_d}) \mapsto (\overline{D_0} \cdots \overline{D_d} \mid U)_S,$$

see [CG24, Theorem 7.22]. We want to extend this map to relatively nef case, see [BK24, Definition 4.6] for archimedean case and [CG24, Theorem 11.3] for general case.

4.3.9. Let $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ and g_1, g_2 measurable S-Green functions for D, see 4.3.7. We say that g_1 is *more singular than* g_2, denoted by $[g_1] \le [g_2]$, if there is a ν-integrable function $C \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $g_{1,\omega} \le g_{2,\omega} + C(\omega)$ for any $\omega \in \Omega$. We say that g_1, g_2 *have equivalent singularities*, denoted by $[g_1] = [g_2]$, if $[g_2] \le [g_1]$ and $[g_1] \le [g_2]$.

Denote by $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ the monoid of relatively nef compactified S-metrized divisors $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}$ satisfying the following property: there is an S-Green function g' for D such that $(D,g') \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$.

4.3.10. Let $\overline{D_j} = (D_j, g_j) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}$ for $j = 0, \cdots, d$. Assume that there are $\overline{D'_j} = (D_j, g'_j) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$ with $[g_j] \le [g'_j]$ for $j = 0, \cdots, d$ (in particular, $\overline{D_j} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$). For any $\omega \in \Omega$, we set

$$E(\mathbf{g}'_\omega, \mathbf{g}_\omega) \coloneqq \sum_{j=0}^{d} \int_{U^{\mathrm{an}}_\omega} (g_{j,\omega} - g'_{j,\omega})\, c_1(\overline{D_{0,\omega}}) \wedge \cdots \wedge c_1(\overline{D_{j-1,\omega}}) \wedge c_1(\overline{D'_{j+1,\omega}}) \wedge \cdots \wedge c_1(\overline{D'_{d,\omega}}),$$

$$E(\mathbf{g}', \mathbf{g}) \coloneqq \int_\Omega E(\mathbf{g}'_\omega, \mathbf{g}_\omega)\,\nu(d\omega)$$

and define

$$(\overline{D_0} \cdots \overline{D_d} \mid U)_S \coloneqq (\overline{D'_0} \cdots \overline{D'_d} \mid U)_S + E(\mathbf{g}', \mathbf{g}).$$

By [CG24, Lemma 10.7], the map $\omega \mapsto E(\mathbf{g}'_\omega, \mathbf{g}_\omega)$ is ν-integrable. The value $(\overline{D_0} \cdots \overline{D_d} \mid U)_S$ is in $\mathbb{R} \cup \{-\infty\}$ and it is finite if $[g_j] = [g'_j]$ for any j. It is shown in [CG24, Proposition 10.10, Theorem 11.2] that $(\overline{D_0} \cdots \overline{D_d} \mid U)_S$ is independent of the choices of $\overline{D'_j}$. Notice that every $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ satisfies the assumption based on the following fact: if $\overline{D'} = (D,g') \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, then $h \coloneqq \max\{g, g'\}$ is an S-Green function for D with $(D,h) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$ and $[g] \le [h]$. Hence, we have a symmetric mulitlinear map

$$\underbrace{\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}} \times \cdots \times \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}}_{(d+1)\text{-times}} \to \mathbb{R} \cup \{-\infty\}, \quad (\overline{D_0}, \cdots, \overline{D_d}) \mapsto (\overline{D_0} \cdots \overline{D_d} \mid U)_S.$$

4.3.5 Graded linear series

4.3.11. An *admissible flag* of $U_{\overline{K}}$ is a flag

$$Y_\bullet \colon \{\mathrm{pt}\} = Y_d \subset Y_{d-1} \subset \cdots \subset Y_0 = U_{\overline{K}}$$

of irreducible subvarieties of $U_{\overline{K}}$ such that $\mathrm{codim}(Y_i) = i$ in $U_{\overline{K}}$ and that the closed point Y_d is regular in each Y_i. Giving an admissible flag $Y_\bullet$ is equivalent to choosing a closed regular point $p := Y_d$ in $U_{\overline{K}}$ and a system of parameters $\{z_1, \cdots, z_d\}$ such that $\widehat{\mathcal{O}_{U_{\overline{K}},p}} = \overline{K}[[z_1, \cdots, z_d]]$, where $\kappa(p)$ is the residue field of p, and $\widehat{\mathcal{O}_{U_{\overline{K}},p}}$ is the Krull completion of the local ring $\mathcal{O}_{U_{\overline{K}},p}$ with respect to its maximal ideal. Hence we have a valuation v_Y on $\widehat{\mathcal{O}_{U_{\overline{K}},p}}$ as follows: for any $f = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha \in \widehat{\mathcal{O}_{U_{\overline{K}},p}} = \overline{K}[[z_1, \cdots, z_d]]$,

$$v_Y(f) := \min\{\alpha \in \mathbb{N}^d \mid a_\alpha \neq 0\},$$

where the minimum is taken with respect to the lexicographic order on the variables $z_1, \cdots, z_d$.

In the rest of this section, we assume that U is normal and fix an admissible flag $Y_\bullet$ of $U_{\overline{K}}$. Then we have a valuation function v_Y on $H^0(U, D) \otimes_K \overline{K}$ for any $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$.

Next we introduce the notion of a graded linear series containing an ample series. It can be thought of as a generalization of the bigness property of D to arbitrary graded linear sub-series of D.

Definition 4.3.12. *A* graded linear series *of $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ is a graded sub-algebra $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ of $\{H^0(U, mD)\}_{m\in\mathbb{N}}$ with $\dim_K(V_m) < \infty$ for any $m \in \mathbb{N}$. We say a graded linear series $V_\bullet$ of D* contains an ample series *if the following two condition holds:*

1. *$V_m \neq \{0\}$ for large enough m;*
2. *there is a positive integer n, a projective model X of U and an ample divisor A on it such that $nD \geq A$ and the canonical injection $H^0(U, mA) \xhookrightarrow{\otimes s^{\otimes m}} H^0(U, mnD)$ factors through V_{mn}, i.e.*

$$H^0(U, mA) \xhookrightarrow{\otimes s^{\otimes m}} V_{mn} \hookrightarrow H^0(U, mnD),$$

for some non-zero $s \in H^0(U, nD - A)$ for all $m \in \mathbb{N}$.

4.3.13. Let $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ be a graded linear series of D. As 4.3.2(1), we can define the *algebraic volume* of $V_\bullet$ as

$$\mathrm{vol}(V_\bullet) := \limsup_{m\to\infty} \frac{\dim_K(V_m)}{m^d/d!}.$$

Let $D_{\overline{K}}$ be the pull-back of D to $U_{\overline{K}}$. Then the valuation v_Y is defined on $H^0(U_{\overline{K}}, D_{\overline{K}}) = H^0(U, D) \otimes_K \overline{K}$. We denote set the semi-group

$$\Gamma(V_\bullet) := \left\{(m, \gamma) \in \mathbb{N}_{\geq 1} \times \mathbb{Z}^d \mid \gamma = v_Y(s) \text{ for some } s \in V_m \otimes_K \overline{K}\right\}$$

and $\Gamma(V_\bullet)_m := \Gamma(V_\bullet) \cap (\{m\} \times \mathbb{Z}^d)$. The associated *Okounkov body of $V_\bullet$* is defined as

$$\Delta(V_\bullet) := \text{ the closure of } \bigcup_{m\in\mathbb{N}_{\geq 1}} \frac{1}{m}\Gamma(V_\bullet)_m \text{ in } \{1\} \times \mathbb{R}^d \ .$$

We set $\Gamma(D) := \Gamma\left(\{H^0(U, mD)\}_{m\in\mathbb{N}}\right)$ and $\Delta(D) := \Delta\left(\{H^0(U, mD)\}_{m\in\mathbb{N}}\right)$.

On the other hand, let $\Gamma(V_\bullet)_{\mathbb{Z}}$ be the subgroup of $\mathbb{Z}^{d+1}$ generated by $\Gamma(V_\bullet)$, and $\Gamma(V_\bullet)_{\mathbb{Z},0} := \Gamma(V_\bullet)_{\mathbb{Z}} \cap (\{0\} \times \mathbb{Z}^d)$. Notice that $V_{\overline{K},\bullet}$ is a $\overline{K}$-subalgebra of $\overline{K}[[z_1, \cdots, z_d]]$, we have the Kodaira dimension $\kappa = \dim(\Delta(V_\bullet)) \leq d$ of $V_\bullet$, see Remark 30. Then $\kappa = \mathrm{rk}_{\mathbb{Z}}(\Gamma(V_\bullet)_{\mathbb{Z},0})$ by [CM20, Proposition 6.3.6 (1)]. Moreover, by [CM20, Proposition 6.3.18],

$$\lim_{m\in\mathbb{N}(V_\bullet), m\to\infty} \frac{\dim_K(V_m)}{m^\kappa} = \mathrm{vol}_{\mathbb{R}^\kappa}(\Delta(V_\bullet)) > 0,$$

where $\mathbb{N}(V_\bullet) \coloneqq \{m \in \mathbb{N} \mid \dim_K(V_m) \neq 0\}$. In particular, for $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$, D is big if and only if the Kodaira dimension of $\{H^0(U, mD)\}_{m\in\mathbb{N}}$ is d.

We state our next lemma which will be repeatedly used throughout this paper.

Lemma 4.3.14. *Let $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ be a graded linear series of $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$. Suppose $V_\bullet$ contains an ample series. Then $\Gamma(V_\bullet)_{\mathbb{Z}} = \mathbb{Z}^{d+1}$. In particular, we have*

$$\mathrm{vol}(V_\bullet) = \lim_{m\to\infty} \frac{\dim_K(V_m)}{m^d/d!} = d! \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(V_\bullet)) = d! \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(V_\bullet)^\circ) \tag{4.26}$$

where S° denotes the topological interior of any subset $S \subseteq \mathbb{R}^d$.

Proof. We choose a projective K-model X of U and an ample divisor A as in Definition 4.3.12 which can be furthermore chosen to be very ample. As argued in the proof of [LM09a, Lemma 2.2], there exist sections $s_0, s_1, \cdots, s_d$ in $H^0(X, A)$, hence in $H^0(U, A)$, with $v_Y(s_0) = 0$ and $v_Y(s_i) = e_i$ for $1 \leq i \leq d$, where $\{e_i\}_{1\leq i\leq d}$ is the standard basis of $\mathbb{R}^d$ and 0 is the zero vector. Choose m such that there is a sequence of inclusions

$$H^0(U, A) \xhookrightarrow{\otimes t_0} V_m \hookrightarrow H^0(U, mD)$$

for a non-zero $t_0 \in H^0(U, mD - A)$. Write $f_0 \coloneqq v_Y(t_0)$. Since $V_m \neq \{0\}$ for large enough m, we can choose m above such that there is a non-zero section $t_1 \in V_{m+1}$. Write $f_1 \coloneqq v(t_1)$. Then clearly the vectors (m, f_0), $(m, f_0 + e_i)$ and $(m + 1, f_1)$ are all in $\Gamma(V_\bullet)$. Hence $\Gamma(V_\bullet)_{\mathbb{Z}} = \mathbb{Z}^{d+1}$, $\Gamma(V_\bullet)_{\mathbb{Z},0} \simeq \mathbb{Z}^d$. This implies that the Kodaira dimension of $V_\bullet$ is d, so

$$\lim_{m\to\infty} \frac{\dim_K(V_m)}{m^d} = \lim_{m\in\mathbb{N}(V_\bullet), m\to\infty} \frac{\dim_K(V_m)}{m^d} = \mathrm{vol}_{\mathbb{R}^d}(\Delta(V_\bullet)),$$

the first equality uses Definition 4.3.12(1). Hence (4.26) holds. □

We end this subsection by noting a result on the behavior of the Okounkov bodies under perturbation.

4.3.15. A *convex body* in $\mathbb{R}^d$ is a compact concave subset in $\mathbb{R}^d$ with non-empty interior. For example, by 4.3.13, for $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)$, D is big if and only if $\Delta(D)$ is a convex body in $\mathbb{R}^d$ (i.e. the corresponding Kodaira dimension is d). We denote by $\mathscr{K}^d$ the set of concave bodies in $\mathbb{R}^d$.

Let Δ_1, Δ_2 be compact concave subsets in $\mathbb{R}^d$ (not necessarily with interior points). We define the *Hausdorff distance* between Δ_1, Δ_2 as

$$d_H(\Delta_1, \Delta_2) \coloneqq \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid \Delta_1 \subset \Delta_2 + \varepsilon\mathbb{B}, \Delta_2 \subset \Delta_1 + \varepsilon\mathbb{B}\}$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^d$. We also set

$$d_S(\Delta_1, \Delta_2) \coloneqq \mathrm{vol}_{\mathbb{R}^d}((\Delta_1 \cup \Delta_2) \setminus (\Delta_1 \cap \Delta_2)).$$

Then d_H and d_S are metrics, called the *Hausdorff metric* and the *symmetric difference metric*, respectively, on $\mathscr{K}^d$. Let $(\Delta_j)_{j\in\mathbb{N}_{\geq 1}} \subset \mathscr{K}^d$ and $\Delta \in \mathscr{K}^d$. By [SW65b, Theorem 7], $\lim\limits_{j\to\infty} d_H(\Delta_j, \Delta) = 0$ if and only if $\lim\limits_{j\to\infty} d_S(\Delta_j, \Delta) = 0$.

Lemma 4.3.16. *Let $(\Delta_j)_{j\in\mathbb{N}_{\geq 1}}$ be a sequence of compact concave bodies in $\mathbb{R}^d$ converging to Δ, i.e. $\lim\limits_{j\to\infty} d_H(\Delta_j, \Delta) = 0$. Then for any $\lambda \in \Delta^\circ$, we have $\lambda \in \Delta_j^\circ$ when j is large enough.*

Proof. Let $\lambda \in \Delta^\circ$ and $r \in \mathbb{R}_{>0}$ such that $B := \{x \in \mathbb{R}^d \mid |x-\lambda| \leq r\} \subset \Delta$. We consider $B_j := B \cap \Delta_j$. Since $B \setminus B_j \subset (\Delta \cup \Delta_j) \setminus (\Delta \cap \Delta_j)$ and $\lim_{j\to\infty} d_S(\Delta_j, \Delta) = 0$ by 4.3.15, we have that $\lim_{j\to\infty} d_S(B_j, B) = 0$, i.e. B_j converge to B. Since $B_j \subset B$, we have that

$$\lim_{j\to\infty} \mathrm{vol}_{\mathbb{R}^d}(B_j) = \mathrm{vol}_{\mathbb{R}^d}(B) \tag{4.27}$$

and B_j is a convex body for j large enough. To show $\lambda \in \Delta_j^\circ$ for j large, it suffices to show that $\lambda \in B_j^\circ$ for j large. If $\lambda \notin B_j^\circ$, since B_j is a convex body,

- there is a hyperplane in $\mathbb{R}^d$ separating λ and B_j when $\lambda \notin B_j$;
- there is a supporting hyperplane of B_j containing λ when λ is in the boundary of B_j by [ROC70, Theorem 11.6].

In either case, B_j is contained in a semi-ball of B. By (4.27), we know that $\lambda \in B_j^\circ$ for j large enough. This completed the proof of the lemma. □

Lemma 4.3.17. *Let $(D_j)_{j\in\mathbb{N}_{\geq 1}}$ be a Cauchy sequence in $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ converging to D. Assume that D is big. Then*

$$\lim_{j\to\infty} d_H(\Delta(D_j), \Delta(D)) = 0,$$

where $d_H(\cdot,\cdot)$ is the usual Hausdorff metric on the space of compact convex bodies. In particular, if $(D_j)_{j\in\mathbb{N}_{\geq 1}}$ is decreasing (resp. increasing) defining D, then

$$\Delta(D) = \bigcap_{j=1}^{\infty} \Delta(D_j)$$

$$\textit{(resp. } \Delta(D)^\circ = \bigcup_{j=1}^{\infty} \Delta(D_j)^\circ\textit{).}$$

Proof. The first claim is a consequence of Corollary 2.8.2. For the second claim, if D_j is decreasing, then $\Delta(D) \subset \bigcap_{j=1}^{\infty} \Delta(D_j)$. By 4.3.15 $\lim_{j\to\infty} d_H(\Delta(D_j), \Delta(D)) = 0$ implies that

$$\lim_{j\to\infty} d_S(\Delta(D_j), \Delta(D)) = \lim_{j\to\infty} \mathrm{vol}_{\mathbb{R}^d}(\Delta(D_j)) - \mathrm{vol}_{\mathbb{R}^d}(\Delta(D)) = 0,$$

so $\mathrm{vol}_{\mathbb{R}^d}\left(\bigcap_{j=1}^{\infty} \Delta(D_j)\right) = \mathrm{vol}_{\mathbb{R}^d}(\Delta(D))$. This implies that $\Delta(D) = \bigcap_{j=1}^{\infty} \Delta(D_j)$ since they are both concave bodies. If D_j is increasing, then $\bigcup_{j=1}^{\infty} \Delta(D_j)^\circ \subset \Delta(D)^\circ$. By Lemma 4.3.16, the equality holds. □

4.4 Arithmetic volumes and concave transforms of compactified S-metrized divisors

In this section, we will define the arithmetic volume, arithmetic χ-volume and concave transform of a compactified S-metrized divisor, and use these notion to study essential minimal and give a height inequality. Throughout this section, we fix an adelic curve $S = (K, \Omega, \mathcal{A}, \nu)$ satisfying the condition 1 2 3 in §4.3.3. We also fix a d-dimensional normal quasi-projective variety U over K. We fix an admissible flag $Y_\bullet$ of $U_{\overline{K}}$ with $Y_d = p$, see 4.3.11 for the definition of admissible flag, then we have the algebra $\widehat{\mathcal{O}_{U_{\overline{K}},p}} = \overline{K}[[z_1, \cdots, z_d]]$. By a graded K-algebra of adelic vector bundle, we mean a graded K-algebra of adelic vector bundle with respect to $R := \mathrm{Frac}(\overline{K}[[z_1, \cdots, z_d]])$ in the sense of Definition 4.2.10.

4.4.1 Auxiliary graded linear series for compactified S-metrized divisors

In this subsection, we associate a graded K-algebra of adelic vector bundles to any compactified S-metrized divisor. We need to show that this definition actually gives us a graded K-algebra of adelic vector bundles. We derive the necessary results from the corresponding facts in the projective case.

4.4.1. For any compactified S-metrized divisor $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ introduced in 4.3.8 and any $\omega \in \Omega$, we can define the *sup-norm* on $H^0(U_\omega, D_\omega)$ as follows: for any $s \in H^0(U_\omega, D_\omega)$, set

$$\log \|s\|_{\sup,\omega} := \sup\{(-g_\omega + \log |s|_\omega)(x) \mid x \in U_\omega^{\mathrm{an}}\}.$$

Note that by the above definition, $\| \ \|_{\sup,\omega}$ can take ∞ as a value and hence it is not an honest norm. Our idea will be to look at those sections in $H^0(U, D)$ which satisfies an appropriate dominancy.

Definition 4.4.2. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. The space of* auxiliary global sections *as*

$$H^0_+(U, \overline{D}) := \{s \in H^0(U, D) \mid \|s\|_{\sup,\omega} < \infty \text{ for any } \omega \in \Omega, \overline{\int_\Omega} \log \|s\|_{\sup,\omega} \, \nu(d\omega) < \infty\}.$$

With the above definition we associate an auxiliary graded K-algebra *to $\overline{D}$ as $\{H^0_+(U, m\overline{D}) \otimes_K \overline{K}\}_{m\in\mathbb{N}}$ which is a $\overline{K}$-subalgebra of $R[T]$, see 4.3.11. Each of the graded pieces $H^0_+(U, m\overline{D})$ equipped with the sup-norm $\| \ \|_{\sup,m,\omega}$ for each $\omega \in \Omega$ becomes an S-normed K-vector space.*

Remark 38. Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$, and X a projective K-model of U such that $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$. If X is integrally closed in U, then $H^0_+(U, \overline{D}) = H^0(U, D) = H^0(X, D)$. Indeed, the second equality is from 4.3.2. For the first equality, notice that for any $\omega \in \Omega$, U_ω^{an} is dense in X_ω^{an}, then the sup-norm on $H^0(U, D)$ is exactly the one on $H^0(X, D)$ via the identity, so $\|s\|_{\sup,\omega} < \infty$ for any $s \in H^0(U, D)$. Moreover, by [CM20, Theorem 6.1.13], we have that $\overline{\int}_\Omega \log \|s\|_{\sup,\omega} \, \nu(d\omega) < \infty$ for any $s \in H^0(X, D)$. This proves our claim.

We want to show next that $\{(H^0_+(U, m\overline{D}), \| \ \|_{\sup,m})\}_{m\in\mathbb{N}}$ is a graded K-algebra of adelic vector bundles, so that we can study their arithmetic volumes. The main task is to prove that $(H^0_+(U, m\overline{D}), \| \ \|_{\sup,m})$ is an adelic vector bundle (see 4.2.2) for each $m \in \mathbb{N}$ and we will use approximating projective models to deduce that. We need the following lemma.

Lemma 4.4.3. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, and $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}} \subset \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ a sequence decreasingly converging to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B}$ of U. Then for every $s \in H^0_+(U, \overline{D}) \subset H^0_+(U, \overline{D_j})$ and $\omega \in \Omega$, we have that*

$$\lim_{j\to\infty} \log \|s\|_{j,\sup,\omega} = \log \|s\|_{\sup,\omega},$$

where $\| \ \|_{\sup,\omega}$ (resp. $\| \ \|_{j,\sup,\omega}$) is the sup-norm on $H^0_+(U, \overline{D})$ (resp. $H^0_+(U, \overline{D_j})$) at ω.

Proof. Write $\overline{D} = (D, g)$ and $\overline{D_j} = (D_j, g_j)$ with $D, D_j \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$. Let $s \in H^0_+(U, \overline{D}) \subseteq H^0_+(U, \overline{D_j})$ and $\omega \in \Omega$. The effectivity relation $\overline{D_j} \geq \overline{D}$ implies that for any $x \in U_\omega^{\mathrm{an}}$, we have

$$-g_{j,\omega}(x) + \log |s(x)|_\omega \leq -g_\omega(x) + \log |s(x)|_\omega,$$

which implies $\log \|s\|_{j,\sup,\omega} \leq \log \|s\|_{\sup,\omega}$. Similarly, the monotonicity of the Cauchy sequence clearly implies that $\log \|s\|_{j,\sup,\omega}$ is increasing. Then $\lim_{j\to\infty} \log \|s\|_{j,\sup,\omega}$ exists and

$$\lim_{j\to\infty} \log \|s\|_{j,\sup,\omega} \leq \log \|s\|_{\sup,\omega}. \tag{4.28}$$

On the other hand, let $\varepsilon \in \mathbb{R}_{>0}$. By definition, there is a point $x \in U_\omega^{\mathrm{an}}$ such that

$$-g_\omega(x) + \log |s(x)|_\omega \geq \log \|s\|_{\sup,\omega} - \varepsilon.$$

Since $\overline{D_j}$ converges to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B}$ of U, we have that

$$\begin{aligned}\lim_{j\to\infty} \log \|s\|_{j,\sup,\omega} &\geq \lim_{j\to\infty} (-g_{j,\omega}(x) + \log |s(x)|_\omega) \\ &= -g_\omega(x) + \log |s(x)|_\omega \\ &\geq \log \|s\|_{\sup,\omega} - \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have that $\lim_{j\to\infty} \log \|s\|_{j,\sup,\omega} \geq \log \|s\|_{\sup,\omega}$. Combining this with (4.28), we complete the proof. □

Proposition 4.4.4. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. Then $(H^0_+(U,\overline{D}), \|\ \|_{\sup})$ is an adelic vector bundle. Moreover, the graded K-algebra $\{(H^0_+(U, m\overline{D}), \|\ \|_{\sup,m})\}_{m\in\mathbb{N}}$ equipped with the sup-norms is a graded K-algebra of adelic vector bundles in the sense of Definition 4.2.10.*

Proof. We prove the proposition in the following steps.

Step 1: The S-normed space $(H^0_+(U,\overline{D}), \|\ \|_{\sup})$ is measurable.

Let $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}}$ be a sequence in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$ decreasingly converging to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B}$ of U. Assume that $\overline{D_j} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X_j)$ for some projective K-model of U which is integrally closed in U. Then $H^0_+(U,\overline{D}) = H^0(U,\overline{D_j}) = H^0(X_j, D_j)$ by Remark 38. Let $\|\ \|_{j,\sup,\omega}$ the sup-norm on $H^0(X, D_j)$ at $\omega \in \Omega$. By [CM20, Proposition 6.1.26], for any $j \in \mathbb{N}_{\geq 1}$ and $s \in H^0_+(U,\overline{D}) \subset H^0(X, D_j)$, the function $\log \|s\|_{j,\sup,\omega}$ is $\mathcal{A}$-measurable. By Lemma 4.4.3, this implies that $\log |s|_{\sup,\omega}$ is $\mathcal{A}$-measurable. Hence $(H^0_+(U,\overline{D}), \|\ \|_{\sup})$ is measurable.

Step 2: The S-normed space $(H^0_+(U,\overline{D}), \|\ \|_{\sup})$ is dominated.

Let $\overline{V}^\vee = (V^\vee, \|\ \|^\vee)$ be the dual of $(H^0_+(U,\overline{D}), \|\ \|_{\sup})$. By definition of $H^0_+(U,\overline{D})$ in Definition 4.4.2, we have that

$$\overline{\int_\Omega} \log \|s\|_{\sup,\omega}\, \nu(d\omega) < \infty$$

for any $s \in H^0_+(U,\overline{D})$. It remains to show that

$$\overline{\int_\Omega} \log \|t\|^\vee_\omega\, \nu(d\omega) < \infty$$

for any $t \in V^\vee$. Let $\overline{D'} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$ for some projective K-model X of U such that X is integrally closed in U and $\overline{D'} \geq \overline{D}$ in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$. By Remark 38, we have that $H^0_+(U,\overline{D'}) = H^0(X, D')$. We denote $\overline{V'}^\vee = (V'^\vee, \|\ \|'^\vee)$ the dual of $H^0(X, D')$ equipped with the sup-norm $\|\ \|'_{\sup}$. Since $\overline{D} \leq \overline{D'}$, we have a canonical injective map of K-vector spaces

$$f\colon H^0_+(U,\overline{D}) \to H^0_+(U,\overline{D'}) = H^0(X, D')$$

which induces a contractive map

$$f_\omega\colon (H^0_+(U,\overline{D}) \otimes_K K_\omega, \|\ \|_{\sup,\omega}) \to (H^0(X, D') \otimes_K K_\omega, \|\ \|'_{\sup,\omega}),$$

i.e. the operator norm $\|f_\omega\| \leq 1$, for any $\omega \in \Omega$. For the dual map

$$f^\vee_\omega\colon (V'^\vee \otimes_K K_\omega, \|\ \|'^\vee_\omega) \to (V^\vee \otimes_K K_\omega, \|\ \|^\vee_\omega)$$

which is surjective by [CM20, Proposition 1.1.22], we have that $\|f^\vee_\omega\| \leq \|f_\omega\| \leq 1$. Then for any $t \in V^\vee$, there is $t' \in V'^\vee$ such that $t = f^\vee(t')$, and we have that

$$\log \|t\|^\vee_\omega = \log \|f^\vee(t')\|^\vee_\omega \leq \log \|t'\|'^\vee_\omega$$

for any $\omega \in \Omega$. Note that $(H^0(X, D'), \| \|'_{\sup})$ is dominated by [CM20, Theorem 6.1.13], then

$$\overline{\int}_{\Omega} \log \|t\|_{\omega}^{\vee} \, \nu(d\omega) \leq \overline{\int}_{\Omega} \log \|t'\|'^{\vee}_{\omega} \, \nu(d\omega) < \infty.$$

This completes the proof of Step 2.

Step 3: The graded K-algebra $\{(H^0_+(U, m\overline{D}), \| \|_{\sup,m})\}_{m\in\mathbb{N}}$ is a graded K-algebra of adelic vector bundle.

For any $m \in \mathbb{N}$, by Step 1 and Step 2 the S-normed space $(H^0_+(U, m\overline{D}), \| \|_{\sup,m})$ is an adelic vector bundle on S. It remains to verify the three conditions in Definition 4.2.10. To verify condition 1, note that we can choose a projective model X of U and a very ample divisor A in X such that X is integrally closed in U and $A \geq D$ in $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$, then there are inclusions

$$H^0_+(U, m\overline{D}) \hookrightarrow H^0(U, mD) \hookrightarrow S_m := H^0(U, mA) = H^0(X, mA).$$

Since A is very ample on X, Serre's theorem implies that $\bigoplus_{m\in\mathbb{N}} S_m T^m$ is a K-algebra of finite type which clearly shows condition 1 ($\bigoplus_{m\in\mathbb{N}} (S_m \otimes_K \overline{K})T^m$ is $\overline{K}$-subalgebra of $R[T]$ by restricting at p). Condition 2 is clear and condition 3 is clearly satisfied for the sup-norms $\| \|_{\sup,\omega}$ due to how tensor product of two metrics are defined. This finishes the proof of Step 3. □

4.4.2 Arithmetic volumes of compactified S-metrized divisors

In this subsection, we define the arithmetic volumes for compactified S-metrized divisors. We first consider the geometric volume. For $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, we use the graded linear series $\{H^0(U, mD)\}_{m\in\mathbb{N}}$ to define the geometric volume $\mathrm{vol}(D)$ and the Okounkov body $\Delta(D)$ in 4.3.13 (see also Definition 4.4.6). On the other hand, we have the graded linear series $\{H^0_+(U, m\overline{D})\}_{m\in\mathbb{N}}$ defined in Definition 4.4.2, we denote the corresponding volume and Okounkov body by $\mathrm{vol}_+(\overline{D})$ and $\Delta_+(\overline{D})$ (see 4.3.13), respectively. We have the following result.

Lemma 4.4.5. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with D big. Then the graded linear series $\{H^0_+(U, m\overline{D})\}_{m\in\mathbb{N}}$ contains an ample series (see Definition 4.3.12). Furthermore, we have that*

$$\mathrm{vol}_+(\overline{D}) = \mathrm{vol}(D)$$

In particular, we have that

$$\Delta_+(\overline{D})^\circ = \Delta(D)^\circ.$$

Proof. Let $\overline{B} = (B, g_B)$ be a weak boundary divisor of U, $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}} = (D_j, g_j)_{j\in\mathbb{N}_{\geq 1}} \subset \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$ and $(\varepsilon_j)_{j\in\mathbb{N}_{\geq 1}} \subset \mathbb{Q}_{>0}$ such that $\lim_{j\to\infty} \varepsilon_j = 0$ and

$$-\varepsilon_j \overline{B} \leq \overline{D} - \overline{D_j} \leq \varepsilon_j \overline{B}.$$

For any $j \in \mathbb{N}_{\geq 1}$, let X_j be a projective model of U such that $\overline{D_j} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X_j)$. We claim that there is an inclusion

$$H^0(U, mD'_j) \hookrightarrow H^0_+(U, m\overline{D}) \subseteq H^0(U, mD) \tag{4.29}$$

for all j and m where $\overline{D'_j} = (D'_j, g'_j) := \overline{D_j} - \varepsilon_j \overline{B}$. If the claim holds, then by [YZ24, Theorem 5.2.1] we can deduce that D'_j is big for large enough j since D is assumed to be big. We fix such a j_0. Since D'_{j_0} is big, we have that $H^0(X_{j_0}, mD'_{j_0}) \neq 0$ for large m, so $H^0(U, m\overline{D}) \neq 0$. By Kodaira lemma

([Laz04, Proposition 2.2.6]), we can find a positive integer n and an ample divisor A on X_{j_0} such that $nD'_{j_0} - A \geq 0$ which in turn will induce injections

$$H^0(U, mA) \xhookrightarrow{\otimes s^{\otimes m}} H^0(U, mnD'_{j_0}) \hookrightarrow H^0_+(U, mn\overline{D})$$

for a non-zero section $s \in H^0(U, nD'_{j_0} - A)$. This clearly shows that $\{H^0_+(U, n\overline{D})\}_{n\in\mathbb{N}}$ contains an ample series. Moreover from (4.29), we have inequalities

$$\mathrm{vol}(D'_j) \leq \mathrm{vol}_+(\overline{D}) \leq \mathrm{vol}(D)$$

which also shows the equality of volumes by 4.3.2(2). Then we can deduce the claim about the interiors by noting that two compact convex subsets of $\mathbb{R}^d$, one contained in another, can have the same volume if and only if their difference has empty interior. Hence it is enough to show our claim. To show the claim, for any $j \in \mathbb{N}_{\geq 1}$, note that the effectivity $\overline{D'_j} \leq \overline{D}$ induces an inequality

$$-g_\omega + \log |s|_\omega \leq -g'_{j,\omega} + \log |s|_\omega \tag{4.30}$$

on U^{an}_ω for all $s \in H^0(U, D'_j)$ and $\omega \in \Omega$. Furthermore since $\overline{D'_j} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X_j)$, by [CM20, Theorem 6.1.13] we have that

$$\log \|s\|'_{j,\sup,\omega} := \sup\{-g'_{j,\omega}(x) + \log |s(x)|_\omega \mid x \in X^{\mathrm{an}}_{j,\omega}\} < \infty,$$

$$\overline{\int_\Omega} \log \|s\|'_{j,\sup,\omega}\, \nu(d\omega) < \infty$$

for all $s \in H^0(U, D'_j)$. Now we can easily deduce the claim noting the inequality (4.30) and [CM20, Proposition A.4.2 (2)]. ☐

Recall the arithmetic χ-volume $\widehat{\mathrm{vol}}_\chi(\overline{E_\bullet})$ and arithmetic volume $\widehat{\mathrm{vol}}(\overline{E_\bullet})$ of a graded algebra of adelic vector bundles $\overline{E_\bullet} = \{\overline{E_m}\}_{m\in\mathbb{N}}$ in Definition 4.2.11.

Definition 4.4.6. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. We call*

$$\overline{V_{\overline{D},\bullet}} = \left\{\overline{V_{\overline{D},m}}\right\}_{m\in\mathbb{N}} := \left\{(H^0_+(U, m\overline{D}), \|\ \|_{\sup,m})\right\}_{m\in\mathbb{N}}$$

the graded K-algebra of adelic vector bundles associated to $\overline{D}$ *and define the* arithmetic volume *and* arithmetic χ-volume *of $\overline{D}$ as*

$$\widehat{\mathrm{vol}}(\overline{D}) := \widehat{\mathrm{vol}}(\overline{V_{\overline{D},\bullet}}) = \limsup_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_{\overline{D},m}})}{m^{d+1}/(d+1)!}$$

and

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) := \widehat{\mathrm{vol}}_\chi(\overline{V_{\overline{D},\bullet}}) = \limsup_{m\to\infty} \frac{\widehat{\deg}(\overline{V_{\overline{D},m}})}{m^{d+1}/(d+1)!},$$

respectively. We furthermore set

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) := \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V_{\overline{D},\bullet}}) \quad \textit{and} \quad \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) := \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{V_{\overline{D},\bullet}}).$$

We say that $\overline{D}$ is big *if $\widehat{\mathrm{vol}}(\overline{D}) > 0$.*

Remark 39. Even if we can define the arithmetic volume of any compactified S-metrized divisor with any assumption on the generic fiber thanks to the formalism of graded K-algebras, we will mostly stick to the case where D is big. In that case Lemma 4.4.5 shows that the graded K-algebra $\{H^0_+(U,\overline{D})\}_{m\in\mathbb{N}}$ has Kodaira dimension d. Furthermore the construction of Okounkov bodies in Definition 2.5.1 of Chapter 2 coincides with the more general construction sketched in [CM20, Section 6.3.2]. The main difference is that in the big case, all the Okounkov bodies live in the same ambient Euclidean space $\mathbb{R}^d$ even if we perturb it in small enough directions whereas the Okounkov bodies in [CM20] may live in different ambient spaces of different dimension. Hence we almost always stick to the big case to be able to use properties shown in Chapter 2.

Remark 40. When $U = X$ is projective, and $\overline{D} = (D, g) \in N_{S,\mathbb{Q}}(X)$ a relatively nef element, recall $N_{S,\mathbb{Q}}(X)$ in 4.3.7, then Chen-Moriwaki also define the asymptotic minimal slope for $\overline{D}$ in [CM22, Definition 6.4.2]. Since X is normal, if K is perfect, D is ample and with $\mathbb{Z}$-coefficients, then our definition above is the same as their definition in [CM22, Definition 6.4.2].

Lemma 4.4.7. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. Then*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) < \infty.$$

Proof. The idea will be to deduce it from the projective case as usual. Let $\overline{D'} = (D', g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$ such that $\overline{D'} \geq \overline{D}$. For any $m \in \mathbb{N}$, write $\overline{V_m} := (H^0(U, mD), \| \ \|_{m,\sup})$ (resp. $\overline{V'_m} := (H^0(U, mD'), \| \ \|'_{m,\sup})$), where $\| \ \|_{m,\sup}$ (resp. $\| \ \|'_{m,\sup}$) is the corresponding sup-norm. Assume that $\overline{D'} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$ for some projective K-model of U which is integrally closed in U. Then $H^0_+(U, m\overline{D'}) = H^0(X, mD')$ for any $m \in \mathbb{N}$ by Remark 38. Moreover, [CM20, Proposition 6.4.4] and [CM20, Proposition 6.2.7] together show that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D'}) < \infty$. Furthermore the effectivity relation $\overline{D'} \geq \overline{D}$ implies that there is an injective K-linear map

$$f_m \colon H^0_+(U, m\overline{D}) \hookrightarrow H^0_+(U, m\overline{D'})$$

which is furthermore norm-contractive at every place ω, i.e. $\|f_m\|_\omega \leq 1$ for all $\omega \in \Omega$ and $m \in \mathbb{N}$. Consequently we have that $h(f_m) \leq 0$ for all $m \in \mathbb{N}$. Then [CM20, Proposition 4.3.31 (1)] shows that

$$\widehat{\mu}_{\max}(\overline{V_m}) \leq \widehat{\mu}_{\max}(\overline{V'_m})$$

for any $m \in \mathbb{N}$ which clearly implies

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) \leq \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D'})$$

and finishes the proof. □

4.4.3 Concave transforms of compactified S-metrized divisors

4.4.8. Let $\overline{V_\bullet} = \{\overline{V_m}\}_{m\in\mathbb{N}} = \{(V_m, \| \ \|_m)\}_{m\in\mathbb{N}}$ be a graded K-algebra of adelic vector bundles over S, and $\mathcal{H}_\bullet = \{\mathcal{H}_m\}_{m\in\mathbb{N}}$ be the the Harder-Narasimhan filtration of $V_\bullet = \{V_m\}_{m\in\mathbb{N}}$ defined in 4.2.8. We assume that

$$\mathbb{N}(V_\bullet) := \{n \in \mathbb{N} \mid V_n \neq 0\} \neq \{0\}.$$

Write $V_{\overline{K},\bullet} = \{V_{\overline{K},m}\}_{m\in\mathbb{N}} := \{V_m \otimes_K \overline{K}\}_{m\in\mathbb{N}}$ $\mathcal{H}_{\overline{K},\bullet} := \{\mathcal{H}_m \otimes_K \overline{K}\}_{m\in\mathbb{N}}$, notice that $\mathcal{H}_{\overline{K},\bullet}$ may not be the Harder-Narasimhan filtration of $V_{\overline{K},\bullet}$. However, by [CM20, Proposition 6.3.25](notice that the canonical extension $S_{\overline{K}}$ of S to $\overline{K}$ (see [CM20, §3.4.2]) satisfies the tensorial minimal slope property in sense of 4.2.9), $\mathcal{H}_{\overline{K},\bullet}$ is a δ-superadditive $\mathbb{R}$-filtration on $V_{\overline{K},\bullet}$ in the sense of [CM20, Definition 6.3.19 (c)], where $\delta\colon \mathbb{N} \mapsto \mathbb{R}_{\geq 0}$, $\ n \mapsto C\log(\dim_K(V_n))$ (although $\overline{V_\bullet}$ may not be a graded K-algebra of adelic vector bundles in the sense of [CM20, Definition 6.4.24], but the proof of [CM20, Proposition 6.3.25] works without the assumption that $V_\bullet$ is a K-subalgebra of $\mathrm{Frac}(K[[T_1, \cdots, T_d]])[T]$).

Notice that $\dim_K(V_n) \leq C_1 n^\kappa$ for some $C_1 \in \mathbb{R}_{>0}$, where κ is the Kodaira dimension of $V_\bullet$, we can replace δ by the map $n \mapsto C_0 \log n$ for some suitable $C_0 \in \mathbb{R}_{>0}$. It is obvious that δ is increasing, and

$$\sum_{a\in\mathbb{N}} \frac{\delta(2^a)}{2^a} = C_0 \log 2 \cdot \sum_{a\in\mathbb{N}} \frac{a}{2^a} = C_0 \log 2 < \infty,$$

i.e. the condition in [CM20, Theorem 6.3.20] holds (similarly, the condition for $\dim_K(W_n)$ in [CM20, Proposition 6.3.28] automatically holds). Then we can construct a *concave transform* $G_{\mathcal{H}_\bullet}\colon \Delta(V_\bullet)^\circ \to \mathbb{R} \cup \{+\infty\}$ associated to $\mathcal{H}_\bullet$ as follows (see 4.3.13 for the definitions of $\Gamma(V_\bullet)$ and $\Delta(V_\bullet)$). We set

$$\begin{aligned}\rho\colon \Gamma(V_\bullet) &\to \mathbb{R} \cup \{\infty\},\\ (m,\gamma) &\mapsto \sup\{t \in \mathbb{R} \mid \text{there is } x \in \mathcal{H}^t_{\overline{K},m}(V_{\overline{K},m}) \text{ such that } v_Y(x) = \gamma\},\end{aligned}$$

and construct the auxiliary function $\widetilde{\rho}(m,\gamma) \coloneqq \limsup_{n\to\infty} \frac{\rho(nm,n\gamma)}{n}$, where v_Y is the valuation defined by the admissible flag $Y_\bullet$, see 4.3.11. We furthermore define

$$\Gamma^t(V_\bullet) \coloneqq \{(m,\gamma) \in \Gamma(V_\bullet) \mid \widetilde{\rho}(m,\gamma) \geq mt\},$$

$$\Gamma^t(V_\bullet)_m \coloneqq \Gamma^t(V_\bullet) \cap (\{m\} \times \mathbb{Z}^d),$$

$$\Delta(\Gamma^t(V_\bullet)) \coloneqq \text{ the closure of } \bigcup_{m\in\mathbb{N}_{\geq 1}} \frac{1}{m}\Gamma^t(V_\bullet)_m \text{ in } \{1\} \times \mathbb{R}^d\ .$$

For any $\lambda \in \Delta(V_\bullet)^\circ$,

$$G_{\mathcal{H}_\bullet}(\lambda) \coloneqq \sup\{t \in \mathbb{R} \mid \lambda \in \Delta(\Gamma^t(V_\bullet))\}.$$

Since $G_{\mathcal{H}_\bullet}$ is convex, we have that $G_{\mathcal{H}_\bullet} \equiv +\infty$ or $G_{\mathcal{H}_\bullet}(\Delta(V_\bullet)^\circ) \subset \mathbb{R}$. On the other hand, for $m \in \mathbb{N}(V_\bullet)$, let $\|\ \|_{\mathcal{H}_m}$ (resp. $\|\ \|_{\mathcal{H}_{\overline{K},m}}$) be the norm on V_m (resp. $V_{\overline{K},m}$) associated to $\mathcal{H}_m$ (resp. $\mathcal{H}_{\overline{K},m}$) in [CM20, Remark 1.1.40], by [CM20, Remark 4.3.63], we have that

$$\widehat{\mu}_i(V_{\overline{K},m}, \|\ \|_{\mathcal{H}_{\overline{K},m}}) = \widehat{\mu}_i(V_m, \|\ \|_{\mathcal{H}_m}) = \widehat{\mu}_i(\overline{V_m}), \tag{4.31}$$

where $\widehat{\mu}_i$ is the i-th jumping number of the Harder-Narasimhan filtration defined in 4.2.8 (we view $(V_m, \|\ \|_{\mathcal{H}_m})$ (resp. $(V_{\overline{K},m}, \|\ \|_{\mathcal{H}_{\overline{K},m}})$) as adelic vector bundle over K (resp. $\overline{K}$) with the trivial absolute value). Write ν_m the Borel probability measure on $\mathbb{R}$ such that for any Borel function f on $\mathbb{R}$, we have that

$$\int_{\mathbb{R}} f(t)\ \nu_m(dt) \coloneqq \frac{1}{\dim_K(V_m)} \sum_{i=1}^{\dim_K(V_m)} f\left(\frac{1}{m}\widehat{\mu}_i(\overline{V_m})\right).$$

From our discussion above and apply [CM20, Theorem 6.3.20] to $\mathcal{H}_{\overline{K},\bullet}$, we have that $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ vaguely converge to the zero measure or to the pushforward $\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta(V_\bullet))} G_{\mathcal{H}_\bullet,*}(d\lambda)$, where $d\lambda$ is the standard Lebesgue measure on $\Delta(V_\bullet)^\circ \subset \mathbb{R}^\kappa$. More precisely, from Step 6 of the proof of [CM20, Theorem 6.3.12], if $G_{\mathcal{H}_\bullet} \equiv +\infty$, then $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ vaguely converge to the zero measure; if $G_{\mathcal{H}_\bullet}(\Delta(V_\bullet)^\circ) \subset \mathbb{R}$, then $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ weakly converge to $\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta(V_\bullet))} G_{\mathcal{H}_\bullet,*}(d\lambda)$. By [CM20, Remark 4.3.48, Remark 6.3.21] and (4.31), if $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V_\bullet}) < +\infty$, then

$$\sup_{\lambda\in\Delta(V_\bullet)^\circ} G_{\mathcal{H}_\bullet}(\lambda) = \lim_{m\in\mathbb{N}(V_\bullet), m\to+\infty} \frac{1}{n}\widehat{\mu}_1(\overline{V_m}) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V_\bullet}) < +\infty, \tag{4.32}$$

hence in this case, $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ converge weakly to $\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta(V_\bullet))} G_{\mathcal{H}_\bullet,*}(d\lambda)$.

Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ such that there is $n \in \mathbb{N}_{\geq 1}$ with $\dim_K(H^0_+(U, m\overline{D})) > 0$. We consider the case where $\overline{V_\bullet}$ is the graded K-algebra of adelic vector bundles associated to D defined in Definition 4.4.2, then all above assumptions for $\overline{V_\bullet}$ are satisfied (notice that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) < \infty$ by Lemma 4.4.7). The corresponding concave transform $G_{\overline{D}}\colon \Delta_+(\overline{D})^\circ \to \mathbb{R}$ is called the *concave transform* of $\overline{D}$.

We have the following theorem which generalizes a result of Chen-Moriwaki [CM20, Theorem 6.4.6].

Theorem 4.4.9. *Let* $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$*, and* $\overline{V_\bullet} = \{\overline{V_m}\}_{m\in\mathbb{N}}$ *its associated graded* K*-algebra of adelic vector bundles defined in Definition 4.4.2. Assume that* $\mathbb{N}(V_\bullet) \neq 0$*. Then the associated Borel probability measures* $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ *in 4.4.8 converge weakly to the probability measure* $\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta_+(\overline{D}))} G_{\overline{D},*}(d\lambda)$*, where* $d\lambda$ *is the standard Lebesgue measure on* $\Delta(D) \subseteq \mathbb{R}^\kappa$ *and* κ *is the Kodaira dimension* $V_\bullet$ *(*$\kappa = d$ *if* D *is big). In particular, if* $\overline{D}$ *is big, then* D *is big. Moreover, in the case where* D *is big (note that* $\Delta(D)^\circ = \Delta_+(\overline{D})^\circ$ *by Lemma 4.4.5), we have that*

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m^{d+1}/(d+1)!} = (d+1)! \int_{\Delta(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda. \tag{4.33}$$

and

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = \limsup_{m\to\infty} \frac{\widehat{\deg}(\overline{V_m})}{m^{d+1}/(d+1)!} \leq (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\, d\lambda. \tag{4.34}$$

with equality if $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) > -\infty$ *(in this case the supremum limit in (4.34) is a limit).*

Proof. By our discussion in 4.4.8, $(\nu_m)_{m\in\mathbb{N}(V_\bullet)}$ converge weakly to the probability measure $\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta_+(\overline{D}))} G_{\overline{D},*}(d\lambda)$. In particular, by [CM20, Remark 6.3.27],

$$\begin{aligned}\frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta_+(\overline{D}))} \int_{\mathbb{R}} \max\{x, 0\}\, ((G_{\overline{D}})_* d\lambda)(dx) =& \frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta_+(\overline{D}))} \int_{\Delta_+(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda \\ =& \lim_{m\in\mathbb{N}(V_\bullet), m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m \dim_K(V_m)}.\end{aligned} \tag{4.35}$$

Since $\mathbb{N}(V_\bullet) \neq 0$, there are infinitely many $m \in \mathbb{N}$ such that $V_m \neq 0$. So

$$\mathrm{vol}_+(\overline{D}) = \limsup_{m\to\infty} \frac{\dim_K(V_m)}{m^d/d!} = \limsup_{m\in\mathbb{N}(V_\bullet), m\to\infty} \frac{\dim_K(V_m)}{m^d/d!}, \tag{4.36}$$

$$\begin{aligned}\widehat{\mathrm{vol}}(\overline{D}) =& \limsup_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m^{d+1}/(d+1)!} = \limsup_{m\in\mathbb{N}(V_\bullet), m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m^{d+1}/(d+1)!} \\ =& (d+1) \cdot \limsup_{m\in\mathbb{N}(V_\bullet), m\to\infty} \left(\frac{\dim_K(V_m)}{m^d/d!} \cdot \frac{\widehat{\deg}_+(\overline{V_m})}{m \dim_K(V_m)} \right)\end{aligned} \tag{4.37}$$

Combine (4.37) (4.35) (4.36), we have that

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1) \cdot \mathrm{vol}_+(\overline{D}) \cdot \frac{1}{\mathrm{vol}_{\mathbb{R}^\kappa}(\Delta_+(\overline{D}))} \int_{\Delta_+(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda. \tag{4.38}$$

If $\overline{D}$ is big, i.e. $\widehat{\mathrm{vol}}(\overline{D}) > 0$, then by (4.38), $\mathrm{vol}_+(\overline{D}) > 0$ which implies that D is big since $\mathrm{vol}(D) \geq \mathrm{vol}_+(\overline{D}) > 0$.

In the case where D is big, by Lemma 4.4.5, $V_\bullet$ contains an ample series, $\mathrm{vol}_+(\overline{D}) = \mathrm{vol}(D)$, $\Delta_+(\overline{D})^\circ = \Delta(D)^\circ$ and $\kappa = d$ (see 4.3.13). In particular, $V_m \neq 0$ for m large enough, so (4.35) becomes

$$\begin{aligned}\frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D}))} \int_{\Delta_+(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda =& \lim_{m\in\mathbb{N}(V_\bullet), m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m \dim_K(V_m)} \\ =& \lim_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m \dim_K(V_m)}.\end{aligned} \tag{4.39}$$

We can deduce from Lemma 4.3.14 that

$$\mathrm{vol}_+(\overline{D}) = \lim_{m\to\infty} \frac{\dim_K(V_m)}{m^d/d!} = d! \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D})). \tag{4.40}$$

Combine (4.38) and (4.40), we have that

$$\begin{aligned}\widehat{\mathrm{vol}}(\overline{D}) =&(d+1)\cdot \mathrm{vol}_+(\overline{D}) \cdot \frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D}))} \int_{\Delta_+(\overline{D})^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda \\ =&(d+1)! \int_{\Delta(D)^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\, d\lambda\end{aligned}$$

On the other hand, combine (4.38) (4.39) (4.40), we have that

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\cdot \lim_{m\to\infty} \frac{\dim_K(V_m)}{m^d/d!} \cdot \lim_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m \dim_K(V_m)} = \lim_{m\to\infty} \frac{\widehat{\deg}_+(\overline{V_m})}{m^{d+1}/(d+1)!}, \tag{4.41}$$

i.e. the supremum limit in the definition of $\widehat{\mathrm{vol}}(\overline{D})$ is actually a limit when D is big. Hence we have completed the proof of (4.33).

It remains to consider (4.34). For any large $m \in \mathbb{N}$ (notice that $m \in \mathbb{N}(V_\bullet)$), $\omega \in \Omega$, we recall the invariant $\delta_\omega(\overline{V_m})$ of $\overline{V_m}$ defined in [CM20, Definition 4.3.9] and set

$$\delta(\overline{V_m}) := \int_{\omega\in\Omega} \log \delta_\omega(\overline{V_m})\, \nu(d\omega)$$

as in [CM20, Definition 4.3.12]. Then by [CM20, Proposition 4.3.10]

$$0 \le \delta(\overline{V_m}) \le \frac{1}{2} \dim_K(V_m) \log(\dim_K(V_m))\nu(\Omega_\infty),$$

i.e.

$$0 \le \frac{\delta(\overline{V_m})}{m \dim_K(V_m)} \le \frac{1}{2} \cdot \frac{\log(\dim_K(V_m))}{m} \cdot \nu(\Omega_\infty).$$

By Remark 30, we have that $\log(\dim_K(V_m)) \le C' \log(m)$ for some $C' \in \mathbb{R}_{>0}$, then

$$0 \le \frac{\delta(\overline{V_m})}{m \dim_K(V_m)} \le \frac{C'}{2} \cdot \frac{\log(m)}{m} \cdot \nu(\Omega_\infty). \tag{4.42}$$

By our assumption that $\nu(\Omega_\infty) < \infty$, after taking the limit on (4.42), we have that

$$\lim_{m\to\infty} \frac{\delta(\overline{V_m})}{m \dim_K(V_m)} = 0. \tag{4.43}$$

By [CM20, Proposition 4.3.50, Proposition 4.3.51],

$$\sum_{i=1}^{\dim_K(V_m)} \widehat{\mu}_i(\overline{V_m}) \le \widehat{\deg}(\overline{V_m}) \le \sum_{i=1}^{\dim_K(V_m)} \widehat{\mu}_i(\overline{V_m}) + \delta(\overline{V_m}).$$

i.e.

$$\frac{\sum_{i=1}^{\dim_K(V_m)} \widehat{\mu}_i(\overline{V_m})}{m \dim_K(V_m)} \le \frac{\widehat{\deg}(\overline{V_m})}{m \dim_K(V_m)} \le \frac{\sum_{i=1}^{\dim_K(V_m)} \widehat{\mu}_i(\overline{V_m}) + \delta(\overline{V_m})}{m \dim_K(V_m)}. \tag{4.44}$$

After taking the supremum limit on (4.44), by Definition 4.4.6 and (4.43), we have that

$$\limsup_{m\to\infty}\frac{\widehat{\deg}(\overline{V_m})}{m\dim_K(V_m)}=\limsup_{m\to\infty}\frac{\sum_{i=1}^{\dim_K(V_m)}\widehat{\mu}_i(\overline{V_m})}{m\dim_K(V_m)}. \tag{4.45}$$

Then

$$\begin{aligned}\widehat{\mathrm{vol}}_\chi(\overline{D})=&\limsup_{m\to\infty}\frac{\widehat{\deg}(\overline{V_m})}{m^{d+1}/(d+1)!}\\ =&(d+1)\cdot\limsup_{m\to\infty}\left(\frac{\dim_K(V_m)}{m^d/d!}\cdot\frac{\widehat{\deg}(\overline{V_m})}{m\dim_K(V_m)}\right)\\ =&(d+1)\cdot\mathrm{vol}_+(\overline{D})\cdot\limsup_{m\to\infty}\frac{\sum_{i=1}^{\dim_K(V_m)}\widehat{\mu}_i(\overline{V_m})}{m\dim_K(V_m)}.\end{aligned} \tag{4.46}$$

where the last equality is from (4.40) and from (4.45), notice that the supremum limit in the definition of $\mathrm{vol}_+(\overline{D})$ (see (4.36)) is actually a limit in (4.40) when D is big. On the other hand, notice that $\sup_{\lambda\in\Delta(D)^\circ}\{G_{\overline{D}}(\lambda)\}=\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D})=\limsup_{m\to\infty}\frac{\widehat{\mu}_{\max}(V_m)}{m}<\infty$, so there exists $c>\max\{0,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D})\}$ such that $\frac{\widehat{\mu}_i(V_m)}{m}\leq\frac{\widehat{\mu}_{\max}(V_m)}{m}\leq c$ for any $m\in\mathbb{N}(V_\bullet)$ and $1\leq i\leq\dim_K(V_m)$. Hence for any $\alpha\in\mathbb{R}_{>0}$,

$$\begin{aligned}\frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D}))}\int_{-\alpha}^{\infty}x\,((G_{\overline{D}})_*d\lambda)(dx)=&\frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D}))}\int_{-\alpha}^{c}x\,((G_{\overline{D}})_*d\lambda)(dx)\\ =&\lim_{m\to\infty}\int_{-\alpha}^{c}x\,\nu_m(dx)\\ =&\lim_{m\to\infty}\int_{-\alpha}^{\infty}x\,\nu_m(dx)\\ \geq&\limsup_{m\to\infty}\int_{-\infty}^{\infty}x\,\nu_m(dx)\\ =&\limsup_{m\to\infty}\frac{\sum_{i=1}^{\dim_K(V_m)}\widehat{\mu}_i(\overline{V_m})}{m\dim_K(V_m)}.\end{aligned}$$

Let $\alpha\to\infty$ in the above inequality, we get

$$\frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D}))}\int_{-\infty}^{\infty}x\,((G_{\overline{D}})_*d\lambda)(dx)=\frac{d!}{\mathrm{vol}_+(\overline{D})}\int_{\Delta(D)^\circ}G_{\overline{D}}(\lambda)\,d\lambda\geq\limsup_{m\to\infty}\frac{\sum_{i=1}^{\dim_K(V_m)}\widehat{\mu}_i(\overline{V_m})}{m\dim_K(V_m)}. \tag{4.47}$$

Combine (4.46) and (4.47), we have that

$$\widehat{\mathrm{vol}}_\chi(\overline{D})\leq(d+1)\cdot\mathrm{vol}_+(\overline{D})\cdot\frac{d!}{\mathrm{vol}_+(\overline{D})}\int_{\Delta(D)^\circ}G_{\overline{D}}(\lambda)\,d\lambda=(d+1)!\cdot\int_{\Delta(D)^\circ}G_{\overline{D}}(\lambda)\,d\lambda.$$

If $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D})=\liminf_{m\to\infty}\frac{\widehat{\mu}_{\min}(m\overline{D})}{m}>-\infty$, by [CM20, Remark 6.3.27], we have that

$$\frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta_+(\overline{D}))}\int_{-\infty}^{\infty}x\,((G_{\overline{D}})_*d\lambda)(dx)=\frac{d!}{\mathrm{vol}_+(\overline{D})}\int_{\Delta(D)^\circ}G_{\overline{D}}(\lambda)\,d\lambda=\lim_{m\to\infty}\frac{\sum_{i=1}^{\dim_K(V_m)}\widehat{\mu}_i(\overline{V_m})}{m\dim_K(V_m)},$$

in particular, the supremum limits in (4.46) are limits. Combine (4.46) and (4.47), we have that

$$\widehat{\mathrm{vol}}_\chi(\overline{D})=(d+1)\cdot\mathrm{vol}_+(\overline{D})\cdot\frac{d!}{\mathrm{vol}_+(\overline{D})}\int_{\Delta(D)^\circ}G_{\overline{D}}(\lambda)\,d\lambda=(d+1)!\cdot\int_{\Delta(D)^\circ}G_{\overline{D}}(\lambda)\,d\lambda.$$

This completed the proof of (4.34). □

For any $c = (c(\omega))_{\omega\in\Omega} \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$, we denote $(0, c) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ the compactified S-metrized divisor such that the underlying compactified divisor of $(0, c)$ on a projective model of U is 0, and the Green functions at $\omega \in \Omega$ are $c(\omega)$. For any $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, we write $\overline{D}(c) := \overline{D} + (0, c)$.

We list the following properties of the concave transforms.

Proposition 4.4.10. *Let $\overline{D} = (D, g), \overline{D'} = (D', g') \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with D, D' big. Then the following statements hold.*

1. *For any $\alpha \in \mathbb{Q}_{>0}$, we have that $\Delta(\alpha\overline{D}) = \alpha\Delta(\overline{D})$ and*
$$G_{\alpha\overline{D}}(\alpha\lambda) = \alpha \cdot G_{\overline{D}}(\lambda)$$
for any $\lambda \in \Delta(\overline{D})^\circ$.

2. *Let $c = (c(\omega))_{\omega\in\Omega} \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$. Then*
$$G_{\overline{D}(c)} = G_{\overline{D}} + \int_\Omega c(\omega)\,\nu(d\omega)$$
on $\Delta(\overline{D}(c))^\circ = \Delta(D)^\circ$.

3. *If $\overline{D} \le \overline{D'}$, then $G_{\overline{D}} \le G_{\overline{D'}}$ on $\Delta(D)^\circ \subseteq \Delta(D')^\circ$.*

4. *We have that*
$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) \le G_{\overline{D}} \le \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}).$$
Moreover, $\sup\limits_{\lambda\in\Delta(D)^\circ} G_{\overline{D}}(\lambda) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D})$.

5. *For any $\lambda \in \Delta(\overline{D})^\circ$ and $\lambda' \in \Delta(\overline{D'})^\circ$, we have that $\lambda + \lambda' \in \Delta(\overline{D} + \overline{D'})^\circ$ and*
$$G_{\overline{D}+\overline{D}'}(\lambda + \lambda') \ge G_{\overline{D}}(\lambda) + G_{\overline{D}'}(\lambda').$$

6. *Let $\varphi\colon V \to U$ a birational morphism of normal quasi-projective varieties such that $\varphi^*Y_\bullet$ is an admissible flag of V. Then $\widehat{\mathrm{vol}}(\varphi^*\overline{D}) = \widehat{\mathrm{vol}}(\overline{D})$, $\Delta(\varphi^*D)^\circ = \Delta(D)^\circ$ and $G_{\varphi^*\overline{D}} = G_{\overline{D}}$ on $\Delta(D)^\circ$.*

Proof. Let $\overline{V}_\bullet = \{\overline{V}_m\}_{m\in\mathbb{N}} := \{(H^0_+(U, m\overline{D}), \|\ \|_{m,\sup})\}_{m\in\mathbb{N}}$ (resp. $\overline{V'_\bullet} = \{\overline{V'_m}\}_{m\in\mathbb{N}} := \{(H^0_+(U, m\overline{D'}), \|\ \|'_{m,\sup})\}_{m\in\mathbb{N}}$) be the graded K-algebra of adelic vector bundles corresponding to $\overline{D}$ (resp. $\overline{D'}$), $\mathcal{H}$ the Harder-Narasimhan filtration on $V_\bullet$ (resp. $V'_\bullet$) induced by $\overline{D}$ (resp. $\overline{D}'$) and $\rho, \widetilde{\rho}, \Gamma^t(V_\bullet)$ (resp. $\rho', \widetilde{\rho}', \Gamma^t(V'_\bullet)$) the corresponding functions and semi-group given in 4.4.8.

1 This is from [CM20, Remark 6.3.21 (2)].

2 For any $m \in \mathbb{N}_{\ge 1}$, let $\|\ \|_{m,c,\sup}$ be the sup-norm on $H^0(U, m\overline{D}(c))$. Then
$$\|\ \|_{m,c,\sup,\omega} = e^{-c(\omega)}\|\ \|_{m,\sup,\omega}$$
on $H^0_+(U, m\overline{D}) = H^0_+(U, m\overline{D}(c))$. This equality above of sup-norms now easily implies 2 by the construction of concave transforms in 4.4.8.

3 The effectivity relation $\overline{D} \le \overline{D'}$ implies that there is an injective K-linear map
$$f_m\colon V_m \hookrightarrow V'_m$$
for every $m \in \mathbb{N}$ which is furthermore norm-contractive at every $\omega \in \Omega$, i.e. $\|f_m\|_\omega \le 1$ for all $\omega \in \Omega$ (see 4.2.4 for the notation). This implies that $h(f_m) \le 0$. By [CM20, Proposition 4.3.49] we have that $\mathcal{H}^t(V_m) \subseteq \mathcal{H}^t(V'_m)$ since $h(f_m) \le 0$ and the filtrations are non-increasing. This implies that $\rho \le \rho', \widetilde{\rho} \le \widetilde{\rho}'$, hence $\Gamma^t(V_\bullet) \subset \Gamma^t(V'_\bullet)$, for any t. By the construction of $G_{\overline{D}}, G_{\overline{D'}}$, we have $G_{\overline{D}} \le G_{\overline{D'}}$, this completes the proof of 3.

4 From the discussion in 4.4.8, we have that $\sup_{\lambda\in\Delta(D)^\circ} G_{\overline{D}}(\lambda) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D})$. It remains to show that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) \leq G_{\overline{D}}$. Notice that $\rho(m,\gamma) \geq \widehat{\mu}_{\min}(\overline{V_m})$ for any $(m,\gamma) \in \Gamma(V_\bullet)$ (see 4.4.8). For any $t < \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D})$, by definition we have that $\widehat{\mu}_{\min}(\overline{V_m}) > mt$ for all large enough m. Then for any large enough m with $(m,\gamma) \in \Gamma(V_\bullet)$, we conclude that $\rho(m,\gamma) \geq \widehat{\mu}_{\min}(\overline{V_m}) > mt$ which in turn implies $\widetilde{\rho}(m,\gamma) > mt$ for all $(m,\gamma) \in \Gamma(V_\bullet)$ by definition. In particular, we conclude that for any $t < \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D})$, $\Gamma^t(V_\bullet) = \Gamma(V_\bullet)$, $\Delta(\Gamma^t(V_\bullet)) = \Delta(V_\bullet)$ which clearly shows that $G_{\overline{D}}(\lambda) \geq t$ for all $\lambda \in \Delta(D)^\circ$. As $t < \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D})$ was arbitrary we deduce that $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) \leq G_{\overline{D}}$.

5 This is from [CM20, Proposition 6.3.28], notice that the condition for $\dim_K(W_n)$ in [CM20, Proposition 6.3.28] holds, see 4.4.8.

6 Since the sup-norm is stable under birational morphism, it suffices to show that $H^0(V,\varphi^*D) = H^0(U,D)$ which will imply the corresponding graded algebras of adelic vector bundles associated to $\varphi^*\overline{D}, \overline{D}$, respectively, are isomorphic, hence proves 6. We have a homomorphism $H^0(U,D) \to H^0(V,\varphi^*D)$ which is injective since φ is birational, it remains to show that if $f \in K(V) \simeq K(U)$ such that $\widehat{\mathrm{div}}(f) + \varphi^*\overline{D} \geq 0$, then $\widehat{\mathrm{div}}(f) + D \geq 0$. It suffices to show that $\varphi^*\overline{D} \geq 0$ implies $\overline{D} \geq 0$. If $\varphi^*\overline{D} \geq 0$, let $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}}$ be a sequence of S-metrized divisors in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$ decreasingly converging to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B}$ of U. Let X_j be a normal projective K-model of U such that $\overline{D_j} = (D_j, g_j) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X_j)$. Let X'_j be a projective K-model of V such that the extension $\varphi_j \colon X'_j \to X_j$ of φ exists. Moreover, since $\varphi^*\overline{D_j} \geq \varphi^*\overline{D} \geq 0$ in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, we can assume that $\varphi_j^*\overline{D_j} \geq 0$ in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X'_j)$. This implies that $g_j \circ \varphi_j \geq 0$ and the Weil divisor corresponding to $\varphi_j^* D_j$ is non-negative, so $g_j \geq 0$ and the Weil divisor corresponding to $\overline{D_j}$ is non-negative. Since X_j is normal, so $\overline{D_j} \geq 0$. This completes the proof of 6. □

Corollary 4.4.11. *Let $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. Then $\overline{D}$ is big if and only if D is big and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) > 0$.*

Proof. As argued in the proof of [CM20, Proposition 6.4.18], the corollary is from Proposition 4.4.10(4) and Theorem 4.4.9. □

For further application, we give the following definition.

Definition 4.4.12. *Let $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with D big and $G_{\overline{D}}$ the concave transform of $\overline{D}$ given in 4.4.8. We define the* numerical χ-volume *of $\overline{D}$ as*

$$\widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}) \coloneqq (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\ d\lambda.$$

By Theorem 4.4.9, we have that $\widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}) \geq \widehat{\mathrm{vol}}_{\chi}(\overline{D})$ with equality if $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{D}) > -\infty$.

Next we record an easy corollary of the previous theorem which will be useful later.

Corollary 4.4.13. *Let $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with D big. If $G_{\overline{D}} \geq 0$ on $\Delta(D)^\circ$, then $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D})$. More generally, if $\inf_{\lambda\in\Delta(D)^\circ} G_{\overline{D}}(\lambda) > -\infty$, then there exists $c \in \mathscr{L}^1(\Omega,\mathcal{A},\nu)$ such that $\widehat{\mathrm{vol}}(\overline{D}(c)) = \widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}(c))$.*

Proof. The proof is immediate from the two statements of Theorem 4.4.9 and Proposition 4.4.10(2). □

4.4.4 Continuity of arithmetic volumes

Suppose $(K,(\Omega,\mathcal{A},\nu),\phi)$ be an adelic curve. Furthermore recall that U is a normal quasi-projective variety over K of dimension d. We first note that by assumption we have $\nu(\mathcal{A}) \not\subseteq \{0,\infty\}$. Thus we can always choose a measurable subset $B \in \mathcal{A}$ such that $0 < \nu(B) < \infty$.

Lemma 4.4.14. *Suppose $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ is a compactified divisor with D big. Then there is a non-negative integrable function $f\colon \Omega \to \mathbb{R}_{\geq 0}$ such that the twist $\overline{D}(f) := (D, g_\omega + f(\omega))$ is arithmetically big.*

Proof. Suppose $\alpha \in \Delta(D)^\circ$ and let $r := G_{\overline{D}}(\alpha) > -\infty$. Furthermore suppose $B \in \mathcal{A}$ such that $0 < \nu(B) < \infty$. Then we choose $f(\omega) = 0$ if $\omega \notin B$ and $\kappa := f(\omega) > \max\{-\frac{r}{\nu(B)}, 0\}$ for all $\omega \in B$. Then clearly from Proposition 4.4.10(2) we have $G_{\overline{D(f)}}(\alpha) = G_{\overline{D}}(\alpha) + \int_\Omega f(\omega)\ d\nu > 0 = G_{\overline{D}}(\alpha) + f(\omega) \cdot \nu(B) > 0$ and thus we have $\widehat{\mathrm{vol}}(\overline{D}(f)) > 0$ by Theorem 4.4.9. □

Next we show that if $\overline{D}$ is arithmetically big, then the generic fiber D is big.

Lemma 4.4.15. *Suppose $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ is arithmetically big. Then the generic fiber D is big.*

Proof. We first begin by noting that since $\widehat{\mathrm{vol}}(\overline{D}) > 0$, we have that for big enough m, $\widehat{\deg}_+(\overline{V_m}) > 0$ where $\overline{V_m}$ is adelic vector bundle associated to $m\overline{D}$. Then by [CM20, Remark 4.3.53] we have that

$$\widehat{\deg}_+(\overline{V_m}) \leq \dim_K(V_m) \cdot \widehat{\mu_1}(\overline{V_m}).$$

Then we can see that

$$0 < \widehat{\mathrm{vol}}(\overline{D}) = \limsup_{m\to\infty} \frac{\deg_+(\overline{V_m})}{m^{d+1}} \leq \limsup_{m\to\infty} \frac{\dim_K(V_m)}{m^d} \cdot \frac{\widehat{\mu_1}(\overline{V_m})}{m} = \mathrm{vol}(D) \cdot \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}).$$

This then clearly implies that $\mathrm{vol}(D) > 0$ which finishes the proof of the lemma. □

Next we show that the sum of two arithmetically big compactified divisors is again arithmetically big.

Lemma 4.4.16. *Suppose $\overline{D}$ and $\overline{E}$ are two arithmetically big divisors. Then we have*

$$\widehat{\mathrm{vol}}(\overline{D} + \overline{E})^{\frac{1}{d+1}} \geq \widehat{\mathrm{vol}}(\overline{D})^{\frac{1}{d+1}} + \widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}}.$$

In particular, $\overline{D} + \overline{E}$ is arithmetically big.

Proof. The proof is essentially the same as in [CM20, Theorem 6.4.7]. First note that since $\overline{D}$ and $\overline{E}$ is big, we get from Lemma 4.4.15 that D, E and hence $D+E$ is big. Now note that if $\overline{W}_\bullet$, $\overline{U}_\bullet$ and $\overline{V}_\bullet$ are the associated graded algebra of adelic vector bundles of $\overline{D}+\overline{E}$, $\overline{D}$ and $\overline{E}$ respectively, then it satisfies the hypothesis of [CM20, Proposition 6.3.28]. Indeed since $D+E$ are big and $\mathrm{vol}(W_\bullet) = \mathrm{vol}(D+E) > 0$, we deduce that $\dim_K(W_m) \leq Cm^d$ for some $C > 0$. Thus clearly $\log \dim_K(W_{2^a q}) \leq \log C + ad \log 2 + d \log q$ for some fixed q and hence $\sum_{a \in N, 2^a q \in \mathbb{N}(W_\bullet)} \frac{\log(\dim_K(W_{2^a q})}{2^a} < \infty$ as $\sum_{a\in\mathbb{N}} \frac{a}{2^a} < \infty$. In other words, the linear series $W_\bullet$ satisfies condition (6.37) in [CM20, Proposition 6.3.28]. Thus for $(x,y) \in \Delta(D)^\circ \times \Delta(E)^\circ$, we have $G_{\overline{D}+\overline{E}}(x+y) \geq G_{\overline{D}}(x) + G_{\overline{E}}(y)$. Then arguing as in the proof of [CM20, Theorem 6.4.7] we easily deduce the claim using Theorem 4.4.9 and the classical Brunn-Minkowski inequality. □

We next show that the big cone is open along arbitary directions. This is the quasi-projective analogue of [CM20, Proposition 6.4.23].

Lemma 4.4.17. *Suppose $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ be arithmetically big and $\overline{A} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ be any compactified divisor. Then there is a positive integer n_0 such that $n\overline{D} + \overline{A}$ is big for all $n \geq n_0$.*

Proof. By bigness of $\overline{D}$, we know that the geometric divisor D is big from Lemma 4.4.15. Hence by continuity of geometric volumes, there is an integer m such that $mD + A$ is big. Let us denote by $S_{\overline{D}} := \max\{G_{\overline{D}}, 0\}$. Then by Proposition 4.4.10(2) for any integrable function $f\colon \Omega \to \mathbb{R}_{\geq 0}$ we have

$$G_{\overline{D(f)}} = G_{\overline{D}} + \int_\Omega f\ d\nu,$$

which clearly implies

$$S_{\overline{D}}(\alpha) \leq S_{\overline{D}(f)}(\alpha) \leq S_{\overline{D}}(\alpha) + \int_{\Omega} f \ d\nu,$$

for all $\alpha \in \Delta(D)^{\circ}$. Similarly we have

$$S_{\overline{D}}(\alpha) \geq S_{\overline{D}(-f)}(\alpha) \geq S_{\overline{D}}(\alpha) - \int_{\Omega} f \ d\nu,$$

for all $\alpha \in \Delta(D)^{\circ}$. By Theorem 4.4.9 the above then clearly implies

$$\widehat{\mathrm{vol}}(\overline{D}) \leq \widehat{\mathrm{vol}}(\overline{D}(f)) \leq \widehat{\mathrm{vol}}(\overline{D}) + (d+1)\mathrm{vol}(D) \cdot \int_{\Omega} f \ d\nu. \tag{4.48}$$

$$\widehat{\mathrm{vol}}(\overline{D}) \geq \widehat{\mathrm{vol}}(\overline{D}(-f)) \geq \widehat{\mathrm{vol}}(\overline{D}) - (d+1)\mathrm{vol}(D) \cdot \int_{\Omega} f \ d\nu. \tag{4.49}$$

By Lemma 4.4.14 there is a non-negative integrable function f on Ω such that $(m\overline{D} + \overline{A})(f)$ is arithmetically big. Applying our observation in (4.49) before to the non-negative integrable function $\frac{f}{a}$ for any positive integer a, we get

$$\widehat{\mathrm{vol}}(D, g - \frac{f}{a}) \geq \widehat{\mathrm{vol}}((D, g)) - \frac{d + 1(\mathrm{vol}(D))}{a} \int_{\Omega} f \ d\nu.$$

Since $\widehat{\mathrm{vol}}((D, g)) = \widehat{\mathrm{vol}}(\overline{D}) > 0$ and f is an integrable function, we can choose a large integer a such that $\widehat{\mathrm{vol}}((D, g - \frac{f}{a}) > 0$. Now we write

$$(m + a)\overline{D} + \overline{A} = (m\overline{D} + \overline{A})(f) + a(D, g - \frac{f}{a})$$

which is clearly big by choice of f and a and Lemma 4.4.16. Now clearly $n_0 = m + a$ satisfies the claim again by Lemma 4.4.16. □

We now state the main theorem which is the quasi-projective analogue of [CM20, Theorem 6.4.24].

Theorem 4.4.18. *Suppose $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with D big and $\overline{B}$ be any compactified divisor. Then*

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{t \to 0} \widehat{\mathrm{vol}}(\overline{D} + t\overline{B}).$$

In particular, if $\{\overline{D_j}\}$ is sequence of compactfied divisors converging to $\overline{D}$ in some $\overline{B}$-topology for some effective divisor $\overline{B}$, then

$$\lim_{i \to \infty} \widehat{\mathrm{vol}}(\overline{D_i}) = \widehat{\mathrm{vol}}(\overline{D}).$$

Proof. We begin by choosing a non-negative integrable function $f \colon \Omega \to \mathbb{R}$ such that $\overline{D}(f) = (D, g + f)$ is arithmetically big by Lemma 4.4.14. Then by Lemma 4.4.17 we can choose a positive integer a such that $a\overline{D}(\phi) \pm \overline{B}$ is arithmetically big and let $t > 0$. Then note that by Lemma 4.4.16 we have

$$\widehat{\mathrm{vol}}(\overline{D} - at \cdot \overline{D}(f)) \leq \widehat{\mathrm{vol}}(\overline{D} - t\overline{B}) \leq \widehat{\mathrm{vol}}(\overline{D} + at \cdot \overline{D}(f))$$

and

$$\widehat{\mathrm{vol}}(\overline{D} - at \cdot \overline{D}(f)) \leq \widehat{\mathrm{vol}}(\overline{D} + t\overline{B}) \leq \widehat{\mathrm{vol}}(\overline{D} + at \cdot \overline{D}(f)),$$

since $at\overline{D}(f) \pm t\overline{B}$ is arithmetically big. But

$$\overline{D} - at \cdot \overline{D}(f) = (1 - at)(D, g) - at \cdot (0, f) = (1 - at) \cdot ((D, g) - \frac{at}{1 - at}(0, f)).$$

Similarly

$$\overline{D} + at \cdot \overline{D}(f) = (1+at) \cdot ((D,g) + \frac{at}{1+at}(0,f)).$$

Thus it reduces to showing

$$\lim_{t\to 0} \widehat{\mathrm{vol}}((1-at)\cdot((D,g) - \frac{at}{1-at}(0,f))) = \lim_{t\to 0} \widehat{\mathrm{vol}}((1+at)\cdot((D,g) + \frac{at}{1+at}(0,f))) = \mathrm{vol}(\overline{D})$$

Using homegeneity of arithmetic volumes and the fact that $1 \pm at \to 1$ as $t \to 0$ it is enough to show that

$$\lim_{t\to 0} \widehat{\mathrm{vol}}((D,g) - \frac{at}{1-at}(0,f)) = \lim_{t\to 0} \widehat{\mathrm{vol}}((D,g) + \frac{at}{1-at}(0,f)) = \widehat{\mathrm{vol}}(D,g)$$

Since $\frac{at}{1-at} \to 0$ as $t \to 0$ it amounts showing that for any non-negative integrable function $f\colon \Omega \to \mathbb{R}$, we have

$$\lim_{t\to 0} \widehat{\mathrm{vol}}((D,g) + t(0,f)) = \widehat{\mathrm{vol}}(D,g)).$$

However from (4.48), we have that for $t > 0$

$$\widehat{\mathrm{vol}}(\overline{D}) \leq \widehat{\mathrm{vol}}(\overline{D}(tf)) \leq \widehat{\mathrm{vol}}(\overline{D}) + t(d+1)\mathrm{vol}(D) \cdot \int_\Omega f \ d\nu.$$

Similarly from (4.49) on the other side we have

$$\widehat{\mathrm{vol}}(\overline{D}) \geq \widehat{\mathrm{vol}}(\overline{D}(-tf)) \geq \widehat{\mathrm{vol}}(\overline{D}) - t(d+1)\mathrm{vol}(D) \cdot \int_\Omega f \ d\nu.$$

The claim now clearly follows by letting $t \to 0$ as f is an integrable function.
For the second claim note that since $\{\overline{D_i}\}$ is a Cauchy sequence converging to $\overline{D}$, we have the effectivity relations

$$\overline{D} - t_i\overline{B} \leq \overline{D_i} \leq \overline{D} + t_i\overline{B},$$

for all i and $t_i > 0$ rational such that $t_i \to 0$ as $i \to \infty$. Hence there exists norm-contractive injective linear maps

$$H^0_+(U, m(\overline{D} - t\overline{B})) \hookrightarrow H^0_+(U, m\overline{D}) \hookrightarrow H^0_+(U, m(\overline{D} + t\overline{B})).$$

Taking positive arithmetic degrees above, it follows that

$$\widehat{\mathrm{vol}}(\overline{D} - t\overline{B}) \leq \widehat{\mathrm{vol}}(\overline{D_i}) \leq \widehat{\mathrm{vol}}(\overline{D} + t\overline{B}).$$

The second claim now easily follows from the first since $t_i \to 0$. □

4.4.5 Essential minimum

Definition 4.4.19. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. We define the* essential minimum *of $\overline{D}$ as*

$$\zeta_{\mathrm{ess}}(\overline{D}) \coloneqq \sup_{V \,\subset\, U \ open} \ \inf_{x \in V(\overline{K})} h_{\overline{D}}(x).$$

We next show Theorem 4.4.21 which gives an analogue for compactified divisors of the fundamental inequality of Zhang. To prove this theorem, we need the following definition from [CM20, §1.1.10, Remark 1.1.67].

4.4.20. Let $\overline{V} = (V, \|\ \|_V), \overline{W} = (W, \|\ \|_W)$ be S-norm vector spaces and $\psi\colon [0,1] \to [0,1]$ such that $\max\{t, 1-t\} \leq \psi(t)$ for any $t \in [0,1]$. For any $(x,y) \in V_\omega \oplus W_\omega$, set

$$\|(x,y)\|_{\psi,\omega} \coloneqq (\|x\|_{V,\omega} + \|y\|_{W,\omega}) \cdot \psi\left(\frac{\|x\|_{V,\omega}}{\|x\|_{V,\omega} + \|y\|_{W,\omega}}\right).$$

We call $\| \ \|_\psi = \{\| \ \|_{\psi,\omega}\}_{\omega\in\Omega}$ the *ψ-direct sum* of $\| \ \|_V$ and $\| \ \|_W$.

A basis $\{e_1, \cdots, e_r\} \subset V$ is a *Hadamard basis* if

$$\|e_1 \wedge \cdots \wedge e_r\|_{V,\det,\omega} = \|e_1\|_{V,\omega} \cdots \|e_r\|_{V,\omega},$$

for every $\omega \in \Omega$ where $\| \ \|_{V,\det} = \{\| \ \|_{V,\det,\omega}\}_{\omega\in\Omega}$ is the determinant norm of $\| \ \|_V$ defined in 4.2.4.

Theorem 4.4.21. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. Then*

$$\zeta_{\mathrm{ess}}(\overline{D}) \geq \widehat{\mu}_{\max}(\overline{D}) \quad \textit{and} \quad \zeta_{\mathrm{ess}}(\overline{D}) \geq \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}). \tag{4.50}$$

In particular, if D is big, then

$$\zeta_{\mathrm{ess}}(\overline{D}) \geq \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}^{\mathrm{num}}_\chi(\overline{D})}{(d+1)\cdot \mathrm{vol}(D)} \geq \frac{\widehat{\mathrm{vol}}_\chi(\overline{D})}{(d+1)\cdot \mathrm{vol}(D)}. \tag{4.51}$$

Proof. Notice that $\zeta_{\mathrm{ess}}(m\overline{D}) = m\zeta_{\mathrm{ess}}(\overline{D})$, then the second inequality of (4.50) is a consequence of the first inequality of (4.50). If D is big, let $\Delta(D)$ be the Okounkov body of D defined in 4.3.13. By Proposition 4.4.10(4) and Lemma 4.3.14, we have that

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) \geq \frac{1}{\mathrm{vol}_{\mathbb{R}^d}(\Delta(D))}\int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\, d\lambda = \frac{d!}{\mathrm{vol}(D)} \cdot \frac{\widehat{\mathrm{vol}}^{\mathrm{num}}_\chi(\overline{D})}{(d+1)!} = \frac{\widehat{\mathrm{vol}}^{\mathrm{num}}_\chi(\overline{D})}{(d+1)\cdot \mathrm{vol}(D)}.$$

This shows the second inequality of (4.51). Hence it suffices to show the first inequality of (4.50).

We follow the idea of the proof of [CM20, Proposition 6.4.4] to show that $\zeta_{\mathrm{ess}}(\overline{D}) \geq \widehat{\mu}_{\max}(\overline{D})$. Let $S_{\overline{K}} = (\overline{K}, \Omega_{\overline{K}}, \mathcal{A}_{\overline{K}}, \nu_{\overline{K}})$ be the adelic curve associated to the algebraic closure $\overline{K}$ induced by S and let $\overline{V} = (V, \| \ \|) = (H^0_+(U, \overline{D}), \| \ \|_{\sup})$ be the adelic vector bundle of auxiliary global section associated to $\overline{D}$ defined in Definition 4.4.2. Let $t \in \mathbb{R}$ such that $\zeta_{\mathrm{ess}}(\overline{D}) < t$. Then there is an infinite subset Λ of $U(\overline{K})$ such that Λ is dense in U and $h_{\overline{D}}(x) < t$ for all $x \in \Lambda$. Let $W \subset V$ be an arbitrary subspace. Notice that V is a subspace of $H^0(U, D|_U)$ and hence a subspace of $K(U)$ since U is integral. We consider the evaluation map

$$W \otimes_K \overline{K} \to \prod_{x\in\Lambda} \kappa(x)$$

which is injective since Λ is dense in U, where $\kappa(x)$ is the residue field of $x \in \Lambda$. Then there are points $x_1, \cdots, x_{\dim_K(W)} \in \Lambda$ such that the evaluation map $\varphi\colon W \otimes_K \overline{K} \to \bigoplus_{i=1}^{\dim_K(W)} \kappa(x_i)$ is bijective. For any $\omega \in \Omega_{\overline{K}}$, let $\| \ \|_{i,\omega}$ be a norm of $\kappa(x_i)_\omega \coloneqq \kappa(x_i) \otimes_{\overline{K}} \overline{K}_\omega$ given by $\|1\|_{i,\omega} \coloneqq \exp(g_{\pi(\omega)}(\sigma(x_i)))$ where $\pi\colon \Omega_{\overline{K}} \to \Omega_K$ is the canonical map and $\sigma\colon (U_{\overline{K}})^{\mathrm{an}}_\omega \to U^{\mathrm{an}}_{\pi(\omega)}$ is the canonical morphism as analytic spaces. We equip $\bigoplus_{i=1}^{\dim_K(W)} \kappa(x_i)$ with the ψ-direct sum $\| \ \|_\psi = \{\| \ \|_{\psi,\omega}\}_{\omega\in\Omega}$, where $\psi\colon [0,1] \to [0,1], \quad t \mapsto \max\{t, 1-t\}$, i.e. for any $(s_1, \cdots, s_{\dim_K(W)}) \in \bigoplus_{i=1}^{\dim_K(W)} \kappa(x_i)_\omega$, we have that

$$\|(s_1, \cdots, s_{\dim_K(W)})\|_{\psi,\omega} = \max\{\|s_1\|_{1,\omega}, \cdots, \|s_{\dim_K(W)}\|_{\dim_K(W),\omega}\}.$$

If we denote $\{e_i\}_{i=1}^{\dim_K(W)}$ a basis of $\bigoplus_{i=1}^{\dim_K(W)} \kappa(x_i)$ such that $e_i \in \kappa(x_i)$, then this basis is orthogonal with respect to $\| \ \|_{\psi,\omega}$ for any $\omega \in \Omega_{\overline{K}}$. By [CM20, Proposition 1.2.23], $\{e_i\}_{i=1}^{\dim_K(W)}$ is a Hadamard basis for $\| \ \|_{\psi,\omega}$ for any $\omega \in \Omega_{\overline{K}}$. In particular,

$$\widehat{\deg}\left(\bigoplus_{i=1}^{\dim_K(W)} \left(\kappa(x_i), \| \ \|_\psi\right)\right) = \sum_{i=1}^{\dim_K(W)} h_{\overline{D}}(x_i) \leq \dim_K(W)\cdot t.$$

Moreover, for any $\omega \in \Omega_{\overline{K}}$ the operator norm $\|\varphi\|_\omega \leq 1$. By [CM20, Proposition 4.3.18], one has

$$\widehat{\mu}(\overline{W}) = \widehat{\mu}(\overline{W} \otimes_K \overline{K}) \leq \frac{1}{\dim_K(W)} \widehat{\deg}\left(\bigoplus_{i=1}^{\dim_K(W)} \left(\kappa(x_i), \|\ \|_\psi\right)\right) \leq t.$$

This completes the proof of (4.50). □

4.4.6 Height Inequality

We end this section by obtaining a height inequality which is a generalization of the height inequalities obtained in [YZ24, Theorem 5.3.5 (1), (3)]. Note that this height inequality has been used later as one of the tools to obtain a uniform Bogomolov type result, see [Yua21, §4.5]. We generalize this in the setting of adelic curves.

Theorem 4.4.22. *Let $\overline{D} = (D, g), \overline{M} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$.*

1. *If $\overline{D}$ is big, then there exists $\epsilon > 0$ and a non-empty Zariski open subset V of U such that*

$$h_{\overline{D}}(x) \geq \epsilon \cdot h_{\overline{M}}(x).$$

2. *If D is big, then there exists $\epsilon > 0$, $c \in \mathbb{R}$ and a non-empty Zariski open subset V of U such that*

$$h_{\overline{D}}(x) \geq \epsilon \cdot h_{\overline{M}}(x) - c \text{ for all } x \in V(\overline{K}).$$

.

Proof. We see that 1 implies that 2. In fact, if $(\overline{D}, D)$ is as in Item 2, there exists $c \in \mathbb{R}$ and $\lambda \in \Delta(D)^\circ$ such that $G_{\overline{D}}(\lambda) > -c$. We take a function $\widetilde{c} \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ with $c = \int_\Omega \widetilde{c}(\omega)\, \nu(d\omega)$. By Proposition 4.4.10(2), (4),

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}(\widetilde{c})) \geq G_{\overline{D}(\widetilde{c})}(\lambda) = G_{\overline{D}}(\lambda) + c > 0.$$

Then $\overline{D}(\widetilde{c})$ is big by Corollary 4.4.11. We can apply 1 to $\overline{D}(\widetilde{c})$. This gives 2 by the simple relation

$$h_{\overline{D}(\widetilde{c})}(x) = h_{\overline{D}}(x) + \int_\Omega \widetilde{c}(\omega)\, \nu(d\omega) = h_{\overline{D}}(x) + c.$$

It remains to show 1. By Theorem 4.4.18 and Lemma 4.4.15, there is $\epsilon \in \mathbb{Q}_{>0}$ such that $\widehat{\mathrm{vol}}(\overline{D} - \epsilon\overline{M}) > 0$, i.e. $\overline{D} - \epsilon\overline{M}$ is big. By Theorem 4.4.21 and Corollary 4.4.11, we then have that

$$\zeta_{\mathrm{ess}}(\overline{D} - \epsilon \cdot \overline{M}) \geq \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D} - \epsilon \cdot \overline{M}) > 0.$$

By definition of the essential minima, we then have a non-empty Zariski open subset V of U such that

$$h_{\overline{D} - \epsilon\cdot\overline{M}}(x) = h_{\overline{D}}(x) - \epsilon \cdot h_{\overline{M}}(x) > 0 \text{ for all } x \in V(\overline{K})$$

which readily proves 1. □

4.5 Arithmetic Hilbert-Samuel formula

In this section, we will build the relation between the numerical χ-volume and arithmetic auto-intersection number of a given relatively nef compactified S-metrized divisor, i.e. the arithmetic Hilbert-Samuel formula. We fix an adelic curve $S = (K, \Omega, \mathcal{A}, \nu)$ satisfying the condition 1 2 3 in §4.3.3, and assume that S has either the strong tensorial minimal slope property of level $\geq C$ or the Minkowski property of level $\geq C$ for some $C \in \mathbb{R}_{\geq 0}$ (see 4.2.9). We also fix a d-dimensional normal quasi-projective variety U over K and an admissible flag $Y_\bullet$ of $U_{\overline{K}}$, see 4.3.11.

4.5.1 Behaviour of the concave transform under perturbations

Recall that for $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ big, we denote by $\Delta(D)^\circ$ the topological interior of the Okounkov body of D calculated with respect to the fixed flag $Y_\bullet$ in U, see 4.3.13. Furthermore we have the concave transform $G_{\overline{D}}$ on $\Delta(D)^\circ$ constructed in 4.4.8 corresponding to $\overline{D}$.

4.5.1. Let $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}} = (D_j, g_j)_{j\in\mathbb{N}_{\geq 1}}$ be a Cauchy sequence in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ converging to $\overline{D} = (D, g)$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$. If D is big, by the continuity of (geometric) volume, see 4.3.2(2), D_j is big for j large enough. Hence $G_{\overline{D_j}}$ is well-defined for j large enough. By Lemma 4.3.16 and the Hausdorff convergence of Okounkov bodies in Lemma 4.3.17, we can conclude that every $\lambda \in \Delta(D)^\circ$ is in $\Delta(D_j)^\circ$ for j large enough. Hence for any $\lambda \in \Delta(D)^\circ$, it makes sense to consider the limits $\limsup_{j\to\infty} G_{\overline{D_j}}(\lambda)$ and $\liminf_{j\to\infty} G_{\overline{D_j}}(\lambda)$.

Theorem 4.5.2. *Let $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}}$ be a sequence in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ converging to $\overline{D} = (D, g)$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$. If D is big, then*

$$\lim_{j\to\infty} G_{\overline{D_j}} = G_{\overline{D}} \tag{4.52}$$

pointwise on $\Delta(D)^\circ$ (as we discussed in 4.5.1, the limit makes sense for j large enough).

Proof. We prove the theorem in the following steps.

Step 1: If the theorem holds when $\overline{D_j}$ is decreasing or $\overline{D_j}$ is increasing, then it holds in general.

Let $\overline{B}$ be a weak boundary divisor in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$, and $(\varepsilon_j)_{j\in\mathbb{N}_{\geq 1}}$ a sequence in $\mathbb{Q}_{>0}$ such that $\overline{D} - \varepsilon_j\overline{B} \leq \overline{D_j} \leq \overline{D} + \varepsilon_j\overline{B}$ for all $j \in \mathbb{N}_{\geq 1}$ and $\lim_{j\to\infty} \varepsilon_j = 0$. Let $\lambda \in \Delta(D)^\circ$. By Proposition 4.4.10(3), we have that

$$G_{\overline{D}-\varepsilon_j\overline{B}}(\lambda) \leq G_{\overline{D_j}}(\lambda) \leq G_{\overline{D}+\varepsilon_j\overline{B}}(\lambda). \tag{4.53}$$

It suffices to show that

$$G_{\overline{D}}(\lambda) = \limsup_{j\to\infty} G_{\overline{D}+\varepsilon_j\overline{B}}(\lambda) = \liminf_{j\to\infty} G_{\overline{D}-\varepsilon_j\overline{B}}(\lambda). \tag{4.54}$$

We only consider the the sequence $(\overline{D} + \varepsilon_j\overline{B})_{j\in\mathbb{N}_{\geq 1}}$, it is similar for $(\overline{D} - \varepsilon_j\overline{B})_{j\in\mathbb{N}_{\geq 1}}$. Let $(\varepsilon_{j_i})_{i\in\mathbb{N}_{\geq 1}}$ be a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}_{\geq 1}}$ such that

$$\lim_{i\to\infty} G_{\overline{D}+\varepsilon_{j_i}\overline{B}}(\lambda) = \limsup_{j\to\infty} G_{\overline{D}+\varepsilon_j\overline{B}}(\lambda).$$

Furthermore, we can assume that $\varepsilon_{j_i} \leq 1/i$ for any $i \in \mathbb{N}_{\geq 1}$. Since the theorem holds for $(\overline{D} + \frac{1}{i}\overline{B})_{j\in\mathbb{N}_{\geq 1}}$ by our assumption, from Proposition 4.4.10(3), we have that

$$G_{\overline{D}}(\lambda) \leq \limsup_{j\to\infty} G_{\overline{D}+\varepsilon_j\overline{B}}(\lambda) = \lim_{i\to\infty} G_{\overline{D}+\varepsilon_{j_i}\overline{B}}(\lambda) \leq \lim_{i\to\infty} G_{\overline{D}+\frac{1}{i}\cdot\overline{B}}(\lambda) = G_{\overline{D}}(\lambda).$$

This shows the first equality of (4.54). The proof for the second equality of (4.54) is similar. Since $\lambda \in \Delta(D)^\circ$ is arbitrary, we have that Step 1 holds.

In the following, we fix an integrable function $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $\int_\Omega c(\omega)\,\nu(d\omega) = 1$. For any $\alpha > 0$, we set

$$\Delta(D)^\alpha := \{\lambda \in \Delta(D) \mid G_{\overline{D}}(\lambda) \geq -\alpha\},$$

and denote $\overline{D}(\alpha) := \overline{D} + (0, \alpha c)$, similarly for $\overline{D_j}(\alpha)$.

Step 2: If $\overline{D_j}$ is decreasing, then the theorem holds.

If $\overline{D_j}$ is decreasing, then for any $\lambda \in \Delta(D)^\circ$, $G_{\overline{D_j}}(\lambda) \geq G_{\overline{D}}(\lambda)$ and $\lim_{j\to\infty} G_{\overline{D_j}}(\lambda)$ exists by Proposition 4.4.10(3). So it suffices to show that $\lim_{j\to\infty} G_{\overline{D_j}} \leq G_{\overline{D}}$ almost everywhere on $\Delta(D)^\circ$. For any $r \in \mathbb{Q}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$, we set

$$K^+_{\mathrm{Sing},r} \coloneqq \{\lambda \in \Delta(D)^\circ \mid \lim_{j\to\infty} G_{\overline{D_j}}(\lambda) > G_{\overline{D}}(\lambda) + r\},$$

$$\Delta(D)^{\alpha,r}_+ \coloneqq \Delta(D)^\alpha \cap K^+_{\mathrm{Sing},r} \subset \Delta(D)^\circ.$$

We claim that the measure $\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)^{\alpha,r}_+)$ of $\Delta(D)^{\alpha,r}_+$ is 0 for any $r \in \mathbb{Q}_{>0}, \alpha \in \mathbb{R}_{>0}$. Indeed, given $r \in \mathbb{Q}_{>0}, \alpha \in \mathbb{R}_{>0}$, for any $\lambda \in \Delta(D)^{\alpha,r}_+$ we have

$$G_{\overline{D_j}(\alpha)}(\lambda) \geq \lim_{i\to\infty} G_{\overline{D_i}(\alpha)}(\lambda) > G_{\overline{D}(\alpha)}(\lambda) + r \geq r > 0 \tag{4.55}$$

thanks to Proposition 4.4.10(2). Write $f_{\overline{D}(\alpha)} \coloneqq \max\{G_{\overline{D}(\alpha)}, 0\}$ on $\Delta(D)$ (resp. $f_{\overline{D}_j(\alpha)} \coloneqq \max\{G_{\overline{D_j}(\alpha)}, 0\}$ on $\Delta(D_j)$), then $f_{\overline{D_j}(\alpha)} \geq f_{\overline{D}(\alpha)}$ on $\Delta(D)^\circ$. For all $\lambda \in \Delta(D)^{\alpha,r}_+ \subseteq \Delta(D)^\circ$ and $j \in \mathbb{N}_{\geq 1}$, by (4.55) we have

$$f_{\overline{D_j}(\alpha)}(\lambda) = G_{\overline{D_j}(\alpha)}(\lambda), \quad f_{\overline{D}(\alpha)}(\lambda) = G_{\overline{D}(\alpha)}(\lambda),$$

then

$$f_{\overline{D_j}(\alpha)}(\lambda) - f_{\overline{D}(\alpha)}(\lambda) = G_{\overline{D_j}(\alpha)}(\lambda) - G_{\overline{D}(\alpha)}(\lambda) = G_{\overline{D_j}}(\lambda) - G_{\overline{D}}(\lambda) > r, \tag{4.56}$$

the last inequality holds since $\lambda \in K^+_{\mathrm{Sing},r}$. Using the first claim of Theorem 4.4.9, we have

$$\begin{aligned}
&\frac{1}{(d+1)!}(\widehat{\mathrm{vol}}(\overline{D_j}(\alpha)) - \widehat{\mathrm{vol}}(\overline{D}(\alpha))) \\
=&\int_{\Delta(D)^\circ} \left(f_{\overline{D_j}(\alpha)}(\lambda) - f_{\overline{D}(\alpha)}(\lambda)\right) d\lambda + \int_{\Delta(D_j)^\circ \setminus \Delta(D)^\circ} f_{\overline{D_j}(\alpha)}(\lambda)\, d\lambda \\
\geq&\int_{\Delta(D)^{\alpha,r}_+} \left(f_{\overline{D_j}(\alpha)}(\lambda) - f_{\overline{D}(\alpha)}(\lambda)\right) d\lambda \\
\geq& r \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(D)^{\alpha,r}_+) \geq 0.
\end{aligned}$$

By the continuity of arithmetic volumes, see Theorem 4.4.18, we have that the left-hand side of the equation above goes to 0 as $j \to \infty$. Hence the $\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)^{\alpha,r}_+) = 0$. This proves our claim. So $K^+_{\mathrm{Sing},r} = \bigcup_{\alpha\in\mathbb{Q}_{>0}} \Delta(D)^{\alpha,r}_+$ has measure 0. Note the set of points where the mentioned pointwise convergence fails to happen is exactly given by $\bigcup_{r\in\mathbb{Q}} K^+_{\mathrm{Sing},r}$. So (4.52) holds almost everywhere on $\Delta(D)^\circ$. Since $\lim_{j\to\infty} G_{\overline{D_j}}$ and $G_{\overline{D}}$ are concave, hence continuous on $\Delta(D)^\circ$, we have that (4.52) holds pointwise on $\Delta(D)^\circ$, which proves Step 2.

Step 3: If $\overline{D_j}$ is increasing, then the theorem holds.

If $\overline{D_j}$ is increasing, as before, for any $\lambda \in \Delta(D)^\circ$, the limit $\lim_{j\to\infty} G_{\overline{D_j}}(\lambda)$ exists. For any $r \in \mathbb{Q}_{>0}, \alpha \in \mathbb{R}_{>0}$ and $j \in \mathbb{N}_{\geq 1}$, we set

$$K^-_{\mathrm{Sing},r} \coloneqq \{\lambda \in \Delta(D)^\circ \mid \lim_{j\to\infty} G_{\overline{D_j}}(\lambda) < G_{\overline{D}}(\lambda) - r\},$$

$$\Delta(D)^{\alpha,r}_- \coloneqq \Delta(D)^\alpha \cap K^-_{\mathrm{Sing},r} \subset \Delta(D)^\circ,$$

$$\Delta(D_j)^{\alpha,r}_- \coloneqq \Delta(D)^{\alpha,r}_- \cap \Delta(D_j)^\circ \subset \Delta(D_j)^\circ.$$

As before, to show the theorem in this case, it suffices to show that the measure $\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)_-^{\alpha,r}) = 0$ for any $r \in \mathbb{Q}_{>0}, \alpha \in \mathbb{R}_{>0}$. Given $r \in \mathbb{Q}_{>0}, \alpha \in \mathbb{R}_{>0}$, we can assume that $\alpha \geq r$ (notice that $\Delta(D)_-^{\alpha,r}$ is increasing at α). By Lemma 4.3.17 we have that $\Delta(D)_-^{\alpha,r} = \bigcup_{j=1}^{\infty} \Delta(D_j)_-^{\alpha,r}$. Notice that $\Delta(D_i)^{\alpha,r} \subset \Delta(D_j)^{\alpha,r}$ if $i \leq j$, then

$$\lim_{j\to\infty} \mathrm{vol}_{\mathbb{R}^d}(\Delta(D_j)_-^{\alpha,r}) = \mathrm{vol}_{\mathbb{R}^d}(\Delta(D)_-^{\alpha,r}). \tag{4.57}$$

For any $\lambda \in \Delta(D_j)_-^{\alpha,r}$ we have

$$G_{\overline{D_j}(2\alpha)}(\lambda) \leq \lim_{i\to\infty} G_{\overline{D_i}(2\alpha)}(\lambda) < G_{\overline{D}(2\alpha)}(\lambda) - r, \tag{4.58}$$

$$G_{\overline{D}(2\alpha)}(\lambda) - r \geq \alpha - r \geq 0 \tag{4.59}$$

thanks to Proposition 4.4.10(2). Write $f_{\overline{D}(2\alpha)} := \max\{G_{\overline{D}(2\alpha)}, 0\}$ on $\Delta(D)$ (resp. $f_{\overline{D_j}(2\alpha)} := \max\{G_{\overline{D_j}(2\alpha)}, 0\}$ on $\Delta(D_j)$), then $f_{\overline{D_j}(2\alpha)} \leq f_{\overline{D}(2\alpha)}$ on $\Delta(D_j)^\circ$. For all $\lambda \in \Delta(D_j)_-^{\alpha,r}$ and $j \in \mathbb{N}_{\geq 1}$, by (4.58) and (4.59), we have

$$f_{\overline{D_j}(2\alpha)}(\lambda) = \max\{G_{\overline{D_j}(2\alpha)}(\lambda), 0\} \leq G_{\overline{D}(2\alpha)}(\lambda) - r \leq f_{\overline{D}(2\alpha)}(\lambda) - r \tag{4.60}$$

Now we will apply a similar argument as the decreasing case, more precisely using Theorem 4.4.9, we have

$$\begin{aligned}
&\frac{1}{(d+1)!}(\widehat{\mathrm{vol}}(\overline{D_j}(2\alpha)) - \widehat{\mathrm{vol}}(\overline{D}(2\alpha))) \\
&= \int_{\Delta(D_j)^\circ} (f_{\overline{D_j}(2\alpha)}(\lambda) - f_{\overline{D}(2\alpha)}(\lambda))\, d\lambda - \int_{\Delta(D)^\circ\setminus\Delta(D_j)^\circ} f_{\overline{D}(2\alpha)}(\lambda)\, d\lambda \\
&\leq \int_{\Delta(D_j)^\circ} (f_{\overline{D_j}(2\alpha)}(\lambda) - f_{\overline{D}(2\alpha)}(\lambda))\, d\lambda \\
&\leq \int_{\Delta(D_j)_-^{\alpha,r}} (f_{\overline{D_j}(2\alpha)}(\lambda) - f_{\overline{D}(2\alpha)}(\lambda))\, d\lambda \\
&\leq -r \cdot \mathrm{vol}_{\mathbb{R}^d}(\Delta(D_j)_-^{\alpha,r}) \leq 0.
\end{aligned}$$

By the continuity of arithmetic volumes, see Theorem 4.4.18, and (4.57), taking $j \to \infty$, we have that $\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)_-^{\alpha,r}) = 0$. This implies Step 3. Hence we complete the proof. ☐

The following proposition is based on the fact that the underlying field of K is perfect.

Proposition 4.5.3. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$ with D big. We further assume that the underlying field K is perfect. Then the following statements hold:*

1. *we have that*
$$\widehat{\mathrm{vol}}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S.$$
2. *$G_{\overline{D}} \geq 0$ on $\Delta(D)^\circ$.*

Proof. We prove the proposition in the following steps.

Step 1: If the underlying field of S is perfect, $U = X$ is projective and $\overline{D}$ is S-ample, then the Proposition holds.

It follows directly from [CM22, Proposition 9.1.2].

Step 2: If $U = X$ is projective and $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\text{ar-nef}}$, then $\widehat{\mathrm{vol}}(\overline{D}) = (\overline{D}^{d+1} \mid X)_S$. If, furthermore, D is big, then $G_{\overline{D}} \geq 0$.

We first consider the case where D is not big. Then $\overline{D}$ is not big, i.e. $\widehat{\mathrm{vol}}(\overline{D}) = 0$, by [CM20, Proposition 6.4.18]. On the other hand, by [CG24, Proposition 7.10 (v), Proposition 6.34 (1)], we have that $(\overline{D}^{d+1} \mid X)_S \geq 0$. Hence we have the inequalities

$$0 = \widehat{\mathrm{vol}}(\overline{D}) \geq \widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (\overline{D}^{d+1} \mid X)_S \geq 0$$

which implies that $\overline{D} = (\overline{D}^{d+1} \mid X)_S = 0$.

Assume that D is big. Let $\overline{A} \in N^{+}_{S,\mathbb{Q}}(X)$ be an S-ample divisor on X such that the point corresponding to the admissible flag $Y_\bullet$ is not in the support of A. Then for any $n \in \mathbb{N}_{\geq 1}$ large enough, $\overline{D} + \frac{1}{n}\overline{A}$ is S-ample by definition of nefness in [CM22, Definition 9.1.5]. This implies that $G_{\overline{D}+\frac{1}{n}\overline{A}} \geq 0$ by Step 1 and Step 2 using Proposition 4.4.10(4). Let $V = X \setminus |A|$, and $i \colon V \to X$ the corresponding open immersion. Notice that $i^*\left(\overline{D} + \frac{1}{n}\overline{A}\right)$ converges to $\overline{D}$ in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(V)_{\mathrm{cpt}}$ with respect to the $\overline{A}$-boundary topology, by Proposition 4.4.10(6) and Theorem 4.5.2, we have that

$$G_{\overline{D}} = G_{i^*\overline{D}} = \lim_{n\to\infty} G_{i^*\left(\overline{D}+\frac{1}{n}\overline{A}\right)} = \lim_{n\to\infty} G_{\overline{D}+\frac{1}{n}\overline{A}} \geq 0$$

almost everywhere on $\Delta(D)^\circ$. Since $G_{\overline{D}}$ is continuous on $\Delta(D)^\circ$, so the statement holds in this case.

Step 3: If U is quasi-projective and $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, then $\widehat{\mathrm{vol}}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S$. If, furthermore, D is big, then $G_{\overline{D}} \geq 0$.

For the general case where U is quasi-projective and $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, by Remark 46 and Proposition 4.4.10(6), after shrinking U, we can assume that $\overline{D}$ is the limit of a sequence $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}} \subset N'_{S,\mathbb{Q}}(U)$ with respect to the $\overline{B}$-boundary topology for some weakly boundary divisor $\overline{B} \in N'_{S,\mathbb{Q}}(U)$. Since $\overline{D_j} \in N'_{S,\mathbb{Q}}(U)$, we can deduce from Step 3 Proposition 4.4.10(6) that $\widehat{\mathrm{vol}}(\overline{D_j}) = (\overline{D_j}^{d+1} \mid U)_S$ and $G_{\overline{D_j}} \geq 0$ on $\Delta(D_j)^\circ$. By Theorem 4.4.18 and continuity of intersection numbers as in [CG24, Theorem 7.22(v)], we have that

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{j\to\infty} \widehat{\mathrm{vol}}(\overline{D_j}) = \lim_{j\to\infty} (\overline{D_j}^{d+1} \mid U)_S = (\overline{D}^{d+1} \mid U)_S.$$

By Theorem 4.5.2, we have that $G_{\overline{D}} \geq 0$ on $\Delta(D)^\circ$. □

4.5.2 Arithmetic Hilbert-Samuel formula

Proposition 4.5.4. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ and $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}} = (D_j, g_j)_{j\in\mathbb{N}_{\geq 1}}$ a sequence in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ converging to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{CM}}$ of U. Assume that $D_j, D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ are big and one of the following condition holds:*

1. *there is $c_0 \in \mathbb{R}$ such that $\inf\limits_{\lambda\in\Delta(D_j)^\circ} G_{\overline{D_j}}(\lambda)$, $\inf\limits_{\lambda\in\Delta(D)^\circ} G_{\overline{D}}(\lambda) > c_0$ for any $j \in \mathbb{N}_{\geq 1}$;*

2. *$\overline{D_j}$ is decreasing and $D_j = D$ in $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ for arbitrary large $j \in \mathbb{N}_{\geq 1}$.*

Then

$$\lim_{j\to\infty} \widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D_j}) = \widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}).$$

Proof. 1 Let $c_0 \in \mathbb{R}$ such that $\inf\limits_{\lambda\in\Delta(D_j)^\circ} G_{\overline{D_j}}(\lambda)$, $\inf\limits_{\lambda\in\Delta(D)^\circ} G_{\overline{D}}(\lambda) > c_0$ for any $n \geq 1$. We fix an integrable function $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $c_0 = \int_\Omega c(\omega)\, \nu(d\omega)$. By Proposition 4.4.10(2), we have that $G_{\overline{D_j}(-c)} > 0$ on $\Delta(D_j)^\circ$ (resp. $G_{\overline{D}(-c)} > 0$ on $\Delta(D)^\circ$). By Corollary 4.4.13 and Lemma 4.3.14, we have that

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_j}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_j}(-c)) + c_0(d+1)\mathrm{vol}(D_j) = \widehat{\mathrm{vol}}(\overline{D_j}(-c)) + c_0(d+1)\mathrm{vol}(D_j).$$

Similarly, we have $\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{D}(-c)) + c_0(d+1)\mathrm{vol}(D)$. By 4.3.2(2) and Theorem 4.4.18, we have that

$$\begin{aligned}\lim_{j\to\infty}\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_j}) &= \lim_{j\to\infty}\Big(\widehat{\mathrm{vol}}(\overline{D_j}(-c)) + c_0(d+1)\mathrm{vol}(D_j)\Big)\\ &= \widehat{\mathrm{vol}}(\overline{D}(-c)) + c_0(d+1)\mathrm{vol}(D)\\ &= \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}).\end{aligned}$$

2 By Theorem 4.5.2, the concave transforms $G_{\overline{D_j}}$ decreasingly converges to $G_{\overline{D}}$ pointwise on $\Delta(D)^\circ$. Hence, by the monotone convergence theorem, we have that

$$\lim_{n\to\infty}\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_j}) = \lim_{n\to\infty}(d+1)!\int_{\Delta(D)^\circ} G_{\overline{D_j}}(\lambda)\,d\lambda = (d+1)!\int_{\Delta(D)^\circ} G_{\overline{D}}(\lambda)\,d\lambda = \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}).$$

□

We are ready to prove the main result of this section.

Theorem 4.5.5. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ with D big and assume the underlying field of the adelic curve S is perfect. Then*

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S.$$

In particular, if $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{D}) > -\infty$, then by Theorem 4.4.9, we have that

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = (\overline{D}^{d+1} \mid U)_S.$$

Proof. We prove the theorem in the following steps.

Step 1: If $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, then $\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S$.

By Proposition 4.5.3(2) and Theorem 4.4.9, we have that

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}) = (d+1)!\int_{\Delta(\overline{D})^\circ} G_{\overline{D}}(\lambda)\,d\lambda = (d+1)!\int_{\Delta(\overline{D})^\circ} \max\{G_{\overline{D}}(\lambda), 0\}\,d\lambda = \widehat{\mathrm{vol}}(\overline{D}).$$

By Proposition 4.5.3(1), we have that $\widehat{\mathrm{vol}}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S$. This proves Step 1.

Step 2: If there is $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $\overline{D}(c) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, then $\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}) = (\overline{D}^{d+1} \mid U)_S$.

Let $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ be the one in the assumption of Step 2. By Proposition 4.4.10(2) and Lemma 4.3.14, we have that

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}(c)) = \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}) + (d+1)\int_\Omega c(\omega)\,\nu(d\omega)\cdot\mathrm{vol}(D).$$

On the other hand, by the properties of arithmetic intersection number on quasi-projective varieties (see [CG24, Theorem 7.22]), we can induce that

$$(\overline{D}(c)^{d+1} \mid U)_S = (\overline{D}^{d+1} \mid U)_S + (d+1)\int_\Omega c(\omega)\,\nu(d\omega)\cdot D^d.$$

By [YZ24, Theorem 5.2.2], we have $\mathrm{vol}(D) = D^d$. Combining these with Step 1, we easily see that Step 2 holds.

Step 3: Theorem holds.

Let $\overline{D'} = (D, g') \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$ such that $g_\omega = g'_\omega$ for all but except ω in a finite subset T of Ω. By Proposition 4.4.10(6), we can shrink U and assume that there is a $\overline{B} \in N_{S,\mathbb{Q}}(U)_{\text{mo}}$ such that $\overline{D} \in N_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}$ (resp. $\overline{D'} \in N'_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}$). We can assume that $g' \geq g$, otherwise, by [CG24, Lemma 10.3 (ii)], we can replace g' by $\max\{g', g\}$. By [CG24, Lemma 10.6], there is an increasing sequence of ν-integrable functions $(c_m)_{m\in\mathbb{N}_{\geq 1}} \subset \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $\overline{D_m} = (D, g'_m) \coloneqq (D, \max\{g, g' - c_m\}) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}}$ and $\overline{D_m}$ converges decreasingly to $\overline{D}$ with respect to the $\overline{B}$-boundary topology. By [CG24, Theorem 10.9 (v)], we have that

$$\lim_{m\to\infty} (\overline{D_m}^{d+1} \mid U)_S = (\overline{D}^{d+1} \mid U)_S. \tag{4.61}$$

On the other hand, by Proposition 4.5.4(2), we have that

$$\lim_{m\to\infty} \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_m}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}). \tag{4.62}$$

Notice that $\overline{D_m}(c_m) \geq \overline{D'}$ and $\overline{D'} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$, so $\overline{D_m}(c_m) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$ by [CG24, Lemma 10.3]. By Step 2, we have that

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_m}) = (\overline{D_m}^{d+1} \mid U)_S.$$

Take $m \to \infty$ on both sides and by (4.61), (4.62), we prove the theorem. $\square$

Remark 41. Theorem 4.5.5 gives another definition of arithmetic auto-intersection of relatively nef compactified line bundle in [CG24] when the underlying geometry line bundles are big by realising it as the integral of a concave function in analogy to the toric setting. In particular, the Hodge bundles on Shimura varieties and the Jacobi line bundles on the universal elliptic surface are *relatively nef compactified line bundles.*

The following corollary follows directly.

Corollary 4.5.6. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ with D big. Then*

$$\widehat{\mathrm{vol}}(\overline{D}) \geq (\overline{D}^{d+1} \mid U)_S.$$

In particular, if $\overline{D}^{d+1} > 0$, then $\overline{D}$ is big.

We have the following height inequality generalizing [YZ24, Theorem 5.3.5 (2)].

Proposition 4.5.7. *Let $\overline{D} = (D, g), \overline{M} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{cpt}}$. If $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef}}_{\text{rel-nef}}$ and $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\text{cpt}}$ is big, then for any $c > 0$, there is $\epsilon \in \mathbb{Q}_{>0}$ and a Zariski open, dense subset V of U such that*

$$h_{\overline{D}}(x) \geq \epsilon h_{\overline{M}}(x) + \frac{(\overline{D}^{d+1} \mid U)_S}{(d+1)D^d} - c, \quad \textit{for any } x \in V(\overline{K}).$$

Proof. We can assume that $(\overline{D}^{d+1} \mid U)_S > -\infty$. Let c be an arbitrary positive number and $\widetilde{c} \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that

$$\int_\Omega \widetilde{c}(\omega)\, \nu(d\omega) = -\frac{(\overline{D}^{d+1} \mid U)_S}{(d+1)D^d} + c.$$

Since D is nef and big, we have that $\mathrm{vol}(D) = D^d > 0$ by [YZ24, Theorem 5.2.2] and

$$(\overline{D}(\widetilde{c})^{d+1} \mid U)_S = (\overline{D}^{d+1} \mid U)_S + (d+1) \cdot \int_\Omega \widetilde{c}(\omega)\, \nu(d\omega) \cdot D^d > 0.$$

Then $\overline{D}(\widetilde{c})$ is big by Corollary 4.5.6. By Theorem 4.4.22(1), there is $\epsilon > 0$ and a Zariski open, dense subvariety V of U such that

$$h_{\overline{D}(\widetilde{c})} = h_{\overline{D}}(x) + \int_{\Omega} \widetilde{c}(\omega)\, \nu(d\omega) \geq \epsilon h_{\overline{M}}(x), \quad \text{for any } x \in V(\overline{K}).$$

This completes the proof. □

4.6 Equidistribution on quasi-projective varieties over classical adelic curves and comparison with classical volumes

In our final section, we obtain an equidistribution result for compactified metrized divisors. We compare our definition of arithmetic volumes with the definition of Yuan and Zhang in [YZ24, Section 5] and show that they agree. Throughout this section, we fix the adelic curve $S = (K, \Omega, \mathcal{A}, \nu)$ given by a number field or a smooth projective curve over some field. We denote by C either the spectrum of the ring of integers of the number field whose function field is K or the smooth projective curve whose function field is K. We also fix a d-dimensional normal quasi-projective variety U over K and an admissible flag $Y_\bullet$ of $U_{\overline{K}}$.

4.6.1 Compactified S-metrized YZ-divisors

Recall the space of compactified S-metrized divisors $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, the relatively nef cone $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}$ and the arithmetically nef cone $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-nef}}$ defined in 4.3.8.

4.6.1. Let X be a projective variety over K. A *projective C-model* of X is a flat, integral scheme $\mathcal{X}$ projective over C with generic fiber X. We say $(D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$ is a *model S-metrized* ($\mathbb{Q}$-)*divisor* if it is induced by a pairing $(\mathcal{D}, g_\infty)$ on a projective C-model $\mathcal{X}$ of X where $\mathcal{D}$ is a $\mathbb{Q}$-Cartier divisor on $\mathcal{X}$ and $g_\infty = \{g_\omega\}_{\omega\in\Omega_\infty}$ is a family of smooth Green functions for D on X_ω^{an} for each $\omega \in \Omega_\infty$. Set $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$ to be the subspace of model S-metrized divisors in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)$. We denote

$$N_{S,\mathbb{Q}}(X)_{\mathrm{mo}} \coloneqq \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}} \cap N_{S,\mathbb{Q}}(X), \quad N'_{S,\mathbb{Q}}(X)_{\mathrm{mo}} \coloneqq \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}} \cap N'_{S,\mathbb{Q}}(X)$$

(see 4.3.7 for $N_{S,\mathbb{Q}}(X), N'_{S,\mathbb{Q}}(X)$).

4.6.2. The space of *model S-metrized* ($\mathbb{Q}$-)*divisors* on U is defined as the limit

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}} \coloneqq \varinjlim_{X} \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}},$$

where X runs through all projective K-models of U. Similarly, we also denote

$$N_{S,\mathbb{Q}}(U)_{\mathrm{mo}} \coloneqq \varinjlim_{X} N_{S,\mathbb{Q}}(X)_{\mathrm{mo}}, \quad N'_{S,\mathbb{Q}}(U)_{\mathrm{mo}} \coloneqq \varinjlim_{X} N'_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$$

Let T be the subset of weak boundary divisors in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$. For a weak boundary divisor $\overline{B} \in T$, we have a $\overline{B}$*-boundary topology* on $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$, see 4.3.8, and denote by $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}^{d_{\overline{B}}}$ (resp. $N_{S,\mathbb{Q}}(U)_{\mathrm{mo}}^{d_{\overline{B}}}$, resp. $N'_{S,\mathbb{Q}}(U)_{\mathrm{mo}}^{d_{\overline{B}}}$) the completion of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$ (resp. $N_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$, resp. $N'_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$) with respect to the $\overline{B}$-boundary topology. The space of *compactified S-metrized* YZ*-divisors* is defined as

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}^{\mathrm{YZ}} \coloneqq \varinjlim_{\overline{B}\in T} \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}^{d_{\overline{B}}}.$$

We also set

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}}^{\mathrm{YZ}} \coloneqq \varinjlim_{\overline{B}\in T} N_{S,\mathbb{Q}}(U)_{\mathrm{mo}}^{d_{\overline{B}}}, \quad \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{ar-snef}}^{\mathrm{YZ}} \coloneqq \varinjlim_{\overline{B}\in T} N'_{S,\mathbb{Q}}(U)_{\mathrm{mo}}^{d_{\overline{B}}}$$

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-int}} \coloneqq \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-snef}} - \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-snef}},$$
$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-int}} \coloneqq \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-snef}} - \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-snef}},$$

and denote by $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-nef}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-nef}}$) the closure of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-snef}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-snef}}$) in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-int}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-int}}$) with respect to the finite subspace topology. The elements of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-int}}$ (resp. $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-snef}}$, $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-nef}}$, $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-int}}$, $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-snef}}$, $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-nef}}$) are called *relatively integrable* (resp. *strongly relatively nef*, *relatively nef*, *arithmetically integrable*, *strongly arithmetically nef*, *arithmetically nef*) compactified S-metrized YZ-divisors of U. We have that

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}} \subset \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}} \subset \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)$$

and a forgetting homomorphism

$$\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}} \to \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}},$$

where $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ is the space of compactified (geometric) divisors of U defined in 4.3.1. As 4.3.8, from now on we write an element $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ as (D, g) with $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ and g an S-measurable, locally S-bounded S-Green function for $D|_U$.

Remark 42. Our definition above is slightly different from Yuan-Zhang's definition given in [YZ24, §2.4, §2.5.3] when K is a number field, their space of adelic (compactified) divisors on U is a subspace of $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$, see [CG24, Section 8]. However, it is not hard to show that the results for (classical) arithmetic volumes (see Definition 4.6.3 below) in [YZ24, Section 5] can be generalized to $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$.

Remark 43. Obviously, we have that $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-nef}} \subset \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}} \cap \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}$, it is natural to ask if the converse holds or not, i.e. if $\overline{D}$ is a relative nef compactified S-metrized divisor which is also a compactified S-metrized YZ-divisor, is $\overline{D}$ a relatively nef compactified S-metrized YZ-divisor?

4.6.2 Classical arithmetic volume

In this subsection, we compare the arithmetic volume of a compactified S-metrized YZ-divisor $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)$ defined in Definition 4.4.6 and the classical one defined in [YZ24, Definition 5.1.3]. The comparison relies on the results given in §4.2.6. Let us recall the definition of purification in 4.2.5, and the notion of generically trivial S-normed vector spaces, coherent S-normed vector space in Definition 4.2.15. The space of global sections of an adelic line bundle over a projective variety with sup-norms in [CM20] is not necessarily generically trivial. It is similar for quasi-projective case. Recall the space of auxiliary sections $H^0_+(U, \overline{D})$ of a compactified S-metrized divisor defined in Definition 4.4.6, it is an adelic vector bundle on S by Proposition 4.4.4. We introduce a subspace of $H^0_+(U, \overline{D})$ which will naturally give rise to coherent adelic vector bundles.

Definition 4.6.3. *Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. We define*

$$H^0_{\mathrm{YZ}}(U, \overline{D}) \coloneqq \{s \in H^0_+(U, \overline{D}) \mid \|s\|_{\sup,\omega} \leq 1 \textit{ for almost all } \omega \in \Omega\},$$

where $\|\ \|_{\sup} = (\|\ \|_{\sup,\omega})_{\omega\in\Omega}$ is the family of sup-norms on $H^0_+(U, \overline{D})$, see 4.4.1. Since $(H^0_+(U, \overline{D}), \|\ \|_{\sup})$ is an adelic vector bundle, we have that $(H^0_{\mathrm{YZ}}(U, \overline{D}), \|\ \|_{\sup})$ is a coherent adelic vector bundle. Set $\overline{V_{\mathrm{YZ},m}} \coloneqq (H^0_{\mathrm{YZ}}(U, m\overline{D}), \|\ \|_{\sup,m})$, where $\|\ \|_{\sup,m}$ is the family of sup-norms on $H^0_{\mathrm{YZ}}(U, m\overline{D})$. We get a graded K-algebra of coherent adelic vector bundles $\overline{V_{\mathrm{YZ},\bullet}} = \{\overline{V_{\mathrm{YZ},m}}\}_{m\in\mathbb{N}}$. We define the classical arithmetic volume *of $\overline{D}$ as*

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) \coloneqq \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\mathrm{YZ},\bullet}}) = \limsup_{m\to\infty} \frac{\widehat{h}^0(\overline{V_{m,\mathrm{YZ}}})}{m^{d+1}/(d+1)!}$$

(see Definition 4.2.19 for the notion $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\cdot)$*). By Lemma 4.2.16(3), the purification* $\overline{V_{\mathrm{YZ,pur},m}}$ *of* $\overline{V_{\mathrm{YZ},m}}$ *is a pure generically trivial adelic vector bundle. Set* $\overline{V_{\mathrm{YZ,pur},\bullet}} \coloneqq \{\overline{V_{\mathrm{YZ,pur},m}}\}_{m\in\mathbb{N}}$*. We define the* classical arithmetic χ-volume *of* $\overline{D}$ *as*

$$\widehat{\mathrm{vol}}_{\chi}^{\mathrm{YZ}}(\overline{D}) \coloneqq \limsup_{m\to\infty} \frac{\chi(\overline{V_{m,\mathrm{YZ,pur}}})}{m^{d+1}/(d+1)!}$$

(see Definition 4.2.19 for the notion $\widehat{\mathrm{vol}}_{\chi}^{\mathrm{YZ}}(\cdot)$*). Notice that* $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\mathrm{YZ,pur},\bullet}})$ *by Remark 34.*

Remark 44. Let $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$. Notice that our definition of $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D})$ coincide with the one in [YZ24, Definition 5.1.3 (2)]. Following a similar statement as [YZ24, Theorem 5.2.1], we know that the limit $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) = \lim\limits_{m\to\infty} \frac{\widehat{h}^0(\overline{V_{m,\mathrm{YZ}}})}{m^{d+1}/(d+1)!}$ exists and if a sequence of model S-metrized divisor $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}} \subset \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ converges to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B}$ in $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$, then

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) = \lim_{j\to\infty} \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D_j}).$$

Example 4.6.4. Assume that K is a number field. Let X be a projective variety over K, $\mathcal{X}$ a projective C-model of X with the structure morphism $\varphi\colon \mathcal{X} \to C$, and $\overline{\mathcal{D}} = (\mathcal{D}, g_\infty)$ a metrized divisor on $\mathcal{X}$ corresponding to a line bundle with smooth metrics. We can assume that $\mathcal{X}$ is integrally closed in X. Let $H^0(\mathcal{X},\mathcal{D})$ be the space of global sections of $\mathcal{L} \coloneqq \mathcal{O}_{\mathcal{X}}(\mathcal{D})$ which is an $\mathcal{O}_K$-module. Then $H^0(X,D) = H^0(\mathcal{X},\mathcal{D}) \otimes_{\mathcal{O}_K} K$. We consider an S-normed $\|\ \|_{\overline{\mathcal{D}}}$ on $H^0(X,D)$ defined as follows:

- For any $\omega \in \Omega_\infty$ and $s \in H^0(X,D) \otimes_K K_\omega$, we set

$$\|s\|_{\overline{\mathcal{D}},\omega} \coloneqq \|s\|_{\sup,\omega} = \sup_{x\in X_\omega^{\mathrm{an}}} \|s(x)\|_\omega.$$

- For any $\omega \in \Omega_{\mathrm{fin}}$ and $s \in H^0(X,D) \otimes_K K_\omega$, we set

$$\|s\|_{\overline{\mathcal{D}},\omega} \coloneqq \inf\{|\alpha|_\omega \mid \alpha \in K_\omega^\times, s \in \alpha(H^0(\mathcal{X},\mathcal{D}) \otimes_{\mathcal{O}_K} K_\omega^\circ)\}.$$

On the other hand, $\overline{\mathcal{D}}$ determines an element $\overline{D} = (D,g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$, and we have a family of sup-norms $\|\ \|_{\sup} = \{\|\ \|_{\sup,\omega}\}_{\omega\in\Omega}$ on $H^0(X,D)$, see 4.4.1. Then the following statements hold:

1. The S-norm vector space $(H^0(X,D), \|\ \|_{\overline{\mathcal{D}}})$ is a pure generically trivial adelic vector bundle.
2. The purification of $\|\ \|_{\sup}$ is $\|\ \|_{\overline{\mathcal{D}}}$, and $\|\ \|_{\sup,\omega} = \|\ \|_{\overline{\mathcal{D}},\omega}$ for all but finitely many $\omega \in \Omega$ (in fact, $\|\ \|_{\sup,\omega} = \|\ \|_{\overline{\mathcal{D}},\omega}$ if the special fiber of $\mathcal{X}$ over $\omega \in \Omega_{\mathrm{fin}}$ is reduced). In particular, $H^0_{\mathrm{YZ}}(X,\overline{D}) = H^0_+(X,\overline{D}) = H^0(X,D)$, and $(H^0(X,D), \|\ \|_{\sup})$ is a generically trivial adelic vector bundle.
3. Recall the arithmetic volume $\widehat{\mathrm{vol}}(\overline{D})$ and arithmetic χ-volume $\widehat{\mathrm{vol}}_\chi(\overline{D})$ of $\overline{D}$ defined in Definition 4.2.11. Write $\overline{V_{\overline{\mathcal{D}},\bullet}} = \{(H^0(X,mD), \|\ \|_{m\overline{\mathcal{D}}})\}_{m\in\mathbb{N}}$. Then

$$\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}}) = \widehat{\mathrm{vol}}(\overline{V_{\overline{\mathcal{D}},\bullet}}),$$

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{D}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}}) = \widehat{\mathrm{vol}}_\chi(\overline{V_{\overline{\mathcal{D}},\bullet}}).$$

Notice that $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}})$ (resp. $\widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}})$) is exactly the arithmetic volume (resp. arithmetic χ-volume) of $\overline{\mathcal{D}}$ defined [Mor08, Section 4] (resp. [Yua09, §1.3]), see also [Bur+16, Definition 3.14].

When K is a function field of a curve over a field k, with the same notation, the module $H^0(\mathcal{X},\mathcal{D})$ is replaced by $\varphi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ on C and we can define $\|\ \|_{\overline{\mathcal{D}}}$ on $H^0(X,D)$ similarly. Then the statements 1, 2, 3 also hold.

We will show 1, 2, 3. For 1, notice that $H^0(\mathcal{X},\mathcal{D})\otimes_{\mathcal{O}_K} K_\omega^\circ$ is a lattice of $H^0(X,D)\otimes_K K_\omega$ in the sense of [CM20, Definition 1.1.23], then $(H^0(X,D),\|\ \|_{\overline{\mathcal{D}}})$ is pure by Remark 27. Write $\varphi\colon \mathcal{X}\to C$ the structure morphism. Since $\varphi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ is free locally around the generic point of C, by choosing an open subset $W\subset C$ and an $\mathcal{O}_W$-basis $\mathbf{e}$ of $\varphi_*\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ on W, it can be seen easily that $(H^0(X,D),\|\ \|_{\overline{\mathcal{D}}})$ is generically trivial given by the basis $\mathbf{e}$.

For 2, by [BE21a, Proposition 6.3], for any $\omega\in\Omega$, if the special fiber of $\mathcal{X}$ over $\omega\in\Omega_{\mathrm{fin}}$ is reduced, then $\|\ \|_{\overline{\mathcal{D}},\omega}=\|\ \|_{\mathrm{sup},\omega}$. Hence $\|\ \|_{\overline{\mathcal{D}},\omega}=\|\ \|_{\mathrm{sup},\omega}$ for all but finitely many $\omega\in\Omega$, this implies that $H^0_{\mathrm{YZ}}(X,\overline{D})=H^0_+(X,\overline{D})=H^0(X,D)$, and $(H^0(X,D),\|\ \|_{\mathrm{sup}})$ is a generically trivial adelic vector bundle. Moreover, since $\mathcal{X}$ is assumed to be integrally closed in X, by [YZ24, Lemma 3.3.3], we have that

$$H^0(\mathcal{X},\mathcal{D})=\{s\in H^0(X,D)\mid \|s\|_{\mathrm{sup},\omega}\le 1 \text{ for any } \omega\in\Omega_{\mathrm{fin}}\}.$$

Since $(H^0(X,D),\|\ \|_{\mathrm{sup}})$ is coherent, by [CM20, Proposition 4.4.2], for any $\omega\in\Omega_{\mathrm{fin}}$, we have that

$$H^0(\mathcal{X},\mathcal{D})\otimes_{\mathcal{O}_K} K_\omega^\circ=\{s\in H^0(X,D)\otimes_K K_\omega\mid \|s\|_{\mathrm{sup},\omega}\le 1\},$$

this implies that the purification of $\|\ \|_{\mathrm{sup}}$ is $\|\ \|_{\overline{\mathcal{D}}}$ by the definitions.

For 3, by 2 and Remark 34, we have that

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D})=\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}}),\quad \widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{D})=\widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}}).$$

By 1 and Proposition 4.2.21, we have that

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}})=\widehat{\mathrm{vol}}(\overline{V_{\overline{\mathcal{D}},\bullet}}),\quad \widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{V_{\overline{\mathcal{D}},\bullet}})=\widehat{\mathrm{vol}}_\chi(\overline{V_{\overline{\mathcal{D}},\bullet}}).$$

By [CM20, Theorem 7.5.9], the graded K-algebra of generically trivial adelic vector bundles $\{(H^0(X,m\overline{D}),\|\ \|_{m,\mathrm{sup}})\}_{m\in\mathbb{N}}$ is asymptotically pure, where $\|\ \|_{m,\mathrm{sup}}$ is the sup-norm on $H^0(X,m\overline{D})$. By 2 and Proposition 4.2.21 again, we have that

$$\widehat{\mathrm{vol}}(\overline{D})=\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}),\quad \widehat{\mathrm{vol}}_\chi(\overline{D})=\widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{D}).$$

This completes the proof of 3.

Finally we can state the main result of this subsection which shows that the arithmetic volumes and the classical arithmetic volumes coincide.

Proposition 4.6.5. *Let $\overline{D}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$. Then*

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D})\le\widehat{\mathrm{vol}}(\overline{D}).$$

The equality holds if $\overline{D}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$.

Proof. Let $\overline{V_\bullet}=\{\overline{V_m}\}_{m\in\mathbb{N}}:=\{(H^0_+(U,\overline{D}),\|\ \|_{\mathrm{sup},m})\}_{m\in\mathbb{N}}$ be the graded algebra of adelic vector bundles associated to $\overline{D}$ viewed as a compactified divisor. Similarly we denote by $\overline{V_{\mathrm{YZ},\bullet}}=\{\overline{V_{\mathrm{YZ},m}}\}_{m\in\mathbb{N}}:=H^0_{\mathrm{YZ}}(U,\overline{D}),\|\ \|_{\mathrm{sup},m})\}_{m\in\mathbb{N}})$, where $\|\ \|_{\mathrm{sup},m}$ is the family of sup-norms on $H^0_+(U,m\overline{D})$. Let $\overline{V_{\mathrm{YZ},\mathrm{pur},m}}$ be the purification of $\overline{V_{\mathrm{YZ},m}}$, and $\overline{V_{\mathrm{YZ},\mathrm{pur},\bullet}}:=\{\overline{V_{\mathrm{YZ},\mathrm{pur},m}}\}_{m\in\mathbb{N}}$. By Remark 34 and Proposition 4.2.21, we have that

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D})=\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{V_{\mathrm{YZ},\mathrm{pur},\bullet}})=\widehat{\mathrm{vol}}(\overline{V_{\mathrm{YZ},\mathrm{pur},\bullet}}).$$

Note that there is a canonical norm-contractive injection of adelic vector bundles

$$f_m\colon \overline{V_{\mathrm{YZ},\mathrm{pur},m}}\hookrightarrow\overline{V_m},$$

i.e. the operator norms $\|f_m\|_\omega \leq 1$ at all $\omega \in \Omega$ since $\| \ \|_\omega \leq \| \ \|_{\omega,\mathrm{pur}}$. It is easy to deduce then from [CM20, Proposition 4.3.18] that

$$\widehat{\deg}_+(\overline{V_{\mathrm{YZ,pur},m}}) \leq \widehat{\deg}_+(\overline{V_m})$$

which clearly shows that $\widehat{\mathrm{vol}}(\overline{V_{\mathrm{YZ,pur},\bullet}}) \leq \widehat{\mathrm{vol}}(\overline{D})$. This proves the inequality.

If $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$, we choose any model S-metrized divisor $\overline{D'}$ on some projective model of U with $\overline{D} \leq \overline{D'}$. Then by Example 4.6.4(3), we have that $\widehat{\mathrm{vol}}(\overline{D'}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D'})$. However the effectivity relation $\overline{D} \leq \overline{D'}$ shows by arguments just like before that

$$\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) \leq \widehat{\mathrm{vol}}(\overline{D}) \leq \widehat{\mathrm{vol}}(\overline{D'}) = \widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D'}).$$

Now choosing a sequence of such model S-metrized divisors $\overline{D'}$ converging to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$, by the generalization of [YZ24, Theorem 5.2.1], see Remark 44, one has $\widehat{\mathrm{vol}}^{\mathrm{YZ}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{D})$. □

Remark 45. Similarly, for $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$, we have that $\widehat{\mathrm{vol}}_\chi^{\mathrm{YZ}}(\overline{D}) \leq \widehat{\mathrm{vol}}_\chi(\overline{D})$. We may expect that the equality holds if $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$, but it is unknown.

Proposition 4.6.5 allows us to translate the results for in [YZ24, § 5.2] in the language of our paper (although $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ is slightly large than the group of adelic divisors defined in [YZ24]).

4.6.3 Suitably approximating sequences

In this subsection, we construct suitable Cauchy sequences of compactified S-metrized divisors which will be crucial for obtaining arithmetic Hilbert-Samuel formula later.

Lemma 4.6.6. *Let $\overline{D} = (D, g), \overline{D'} = (D', g') \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{rel\text{-}snef}}$ be given as a limit of a sequence in $N_{S,\mathbb{Q}}(U)$ with respect to the $\overline{B}$-boundary topology for some boundary divisor $\overline{B} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)$. Assume that $D \geq D'$ in $\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$ and write $h := \max\{g, g'\}$.*

1. *If we can choose $\overline{B}$ in $N_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$, then h is an S-Green function for $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$, and $(D, h) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{rel\text{-}snef}}$.*
2. *If we can choose $\overline{B}$ in $N_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$ and if additionally $(D, g) \in N'_{S,\mathbb{Q}}(U)^{d_{\overline{B}}}_{\mathrm{mo}}$, then $(D, h) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{ar\text{-}snef}}$.*

Proof. The proof is similar as the one of [CG24, Lemma 10.3]. By [CG24, Lemma 10.3], h is a S-Green function for $D \in \widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$.

Step 1: *If $U = X$ is projective and $\overline{D}, \overline{D'} \in N_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$, let $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $c_\omega = 0$ for any $\omega \in \Omega_{\mathrm{fin}}$ and $c_\omega > 0$ for any $\omega \in \Omega_\infty$, then (D, h) is the limit of a decreasing sequence (D, h_n) in $N_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$ with respect to $(0, c)$-boundary topology with $h_{n,\omega} = h_\omega$ for any $\omega \in \Omega_{\mathrm{fin}}$.*

Let $\overline{\mathcal{D}} = (\mathcal{D}, g_\infty), \overline{\mathcal{D}'} = (\mathcal{D}', g'_\infty)$ be pairs of Cartier divisors with Green functions on a projective C-model $\mathcal{X}$ of X inducing $\overline{D}, \overline{D'}$, respectively. Since $D \geq D'$, there is a finite subset T of C such that $\mathcal{D}|_{C\setminus T} \geq \mathcal{D}'|_{C\setminus T}$, where $\mathcal{D}|_{C\setminus T}$ (resp. $\mathcal{D}|_{C\setminus T}$) is the restriction of $\mathcal{D}$ (resp. $\mathcal{D}'$) on the $\mathcal{X} \times_C (C \setminus T)$. Then when $\omega \notin T$, we have that $h_\omega = g_\omega$ and when $\omega \in T \cap \Omega_{\mathrm{fin}}$, we have that $(D_\omega, h_\omega) \in N_{\mathrm{mo},\mathbb{Q}}(X_\omega)$ (see proof of [CG24, Lemma 9.3]). Then after replacing $\mathcal{X}$ by its blowing up along a subscheme of $\coprod_{\omega \in T\setminus\Omega_\infty} \mathcal{X}_\omega$ where $\mathcal{X}_\omega$ is the special fiber of $\mathcal{X}$ over $\omega \in T \setminus \Omega_\infty$, we can find a divisor $\mathcal{D}''$ on $\mathcal{X}$ inducing (D_ω, h_ω) for any $\omega \in \Omega_{\mathrm{fin}}$. When $\omega \in T \cap \Omega_\infty$, we have that $(D_\omega, g_\omega) \in \widehat{\mathrm{Div}}_{\mathbb{Q}}(X_\omega)_{\mathrm{nef}}$ by [CG24, Lemma 9.3]. Moreover, it is the limit of a decreasing sequence in $N_{\mathrm{mo},\mathbb{Q}}(X_\omega)$. Then (D, h) is the limit of some decreasing sequence in $N_{S,\mathbb{Q}}(X)_{\rm mo}$ with respect to $(0,c)$-boundary topology. This proves Step 1.

Step 2: *Under the assumptions from Step 1 and if $(D,g) \in N'_{S,\mathbb{Q}}(X)_{\rm mo}$, then $(D,h) \in \widehat{\rm Div}_{S,\mathbb{Q}}(X)^{\rm YZ}_{\text{ar-nef}}$.*

By [CG24, Lemma 10.3, Proposition 7.10 (iv)], we have that $(D,h) \in N'_{S,\mathbb{Q}}(X)$, i.e. S-nef on X. From Step 1, (D,h) is the limit of a decreasing sequence $(D,h_j)_{j\in\mathbb{N}_{\geq 1}}$ in $N_{S,\mathbb{Q}}(X)_{\rm mo}$. Since $(D,h_j) \geq (D,h)$ and $(D,h) \in N'_{S,\mathbb{Q}}(X)$, we have that $(D,h_j) \in N'_{S,\mathbb{Q}}(X)_{\rm mo}$. This proves Step 2.

Step 3: *1 holds.*

Let $\overline{B} = (B, g_B) \in N_{S,\mathbb{Q}}(U)$ be a weak boundary divisor, and $(\overline{D_j})_{j\in\mathbb{N}_{\geq 1}}, (\overline{D'_j})_{j\in\mathbb{N}_{\geq 1}}$ sequences in $N_{S,\mathbb{Q}}(U)_{\rm mo}$ converging to $\overline{D}$, $\overline{D'}$ with respect to the $\overline{B}$-boundary topology, respectively. We can assume that there is a positive constant c_0 such that $g_{B,\omega} > c_0$ for any $\omega \in \Omega_\infty$. Let X_j be a projective model of U such that $\overline{D_j} = (D_j, g_j), \overline{D'_j} = (D'_j, g'_j) \in N_{S,\mathbb{Q}}(X_j)_{\rm mo}$. Since $D \geq D'$, after adding small multiples of $\overline{B}$ to $\overline{D}_j$, we can assume that $D_j \geq D'_j$. By Step 1, we have that $(D_j, \max\{g_j, g'_j\}) \in \widehat{\rm Div}_{S,\mathbb{Q}}(X_j)^{\rm YZ}_{\text{rel-nef}}$ and $(D_j, \max\{g_j, g'_j\})$ converges to (D,h) in $\widehat{\rm Div}_{S,\mathbb{Q}}(U)_{\rm cpt}$ with respect to $\overline{B}$-boundary topology. By Step 1, we can choose an element $\overline{D'_j}$ in $N_{S,\mathbb{Q}}(X_j)_{\rm mo}$ closed to $(D_j, \max\{g_j, g'_j\})$ with respect to the $\overline{B}$-boundary topology (notice that $g_{B,\omega} > c_0 > 0$ on $U^{\rm an}_\omega$ for any $\omega \in \Omega_\infty$). Then $\overline{D'_j}$ converges to (D,h) in $\widehat{\rm Div}_{S,\mathbb{Q}}(U)_{\rm cpt}$ with respect to $\overline{B}$-boundary topology. This proves Step 3.

Step 4: *2 holds .*

We can twist a boundary divisor $\overline{B} \in N_{S,\mathbb{Q}}(U)_{\rm mo}$ with a positive constant at some place $\omega \in \Omega$, and assume that $\overline{B} \in N'_{S,\mathbb{Q}}(U)$. Then Step 4 follows from the same proof as in Step 3 relying now on Step 2 instead of Step 1. □

Remark 46. We can obtain a weak boundary divisor $\overline{B} \in N_{S,\mathbb{Q}}(U)_{\rm mo}$ by shrinking U. The shrinking procedure is useful in the following situation: as Remark 36, by shrinking U we may assume that finitely many given relatively nef (resp. arithmetically nef) YZ-divisors become strongly relatively nef (resp. strongly arithmetically nef).

Lemma 4.6.7. *Let $\overline{D_0} = (D_0, g_0), \ldots, \overline{D_d} = (D_d, g_d) \in \widehat{\rm Div}_{S,\mathbb{Q}}(U)_{\text{rel-snef}}$ and $\overline{D'_0} = (D_0, g'_0), \ldots, \overline{D'_d} = (D_d, g'_d) \in \widehat{\rm Div}_{S,\mathbb{Q}}(U)^{\rm YZ}_{\text{ar-snef}}$. Assume that for any $j = 0, \ldots, d$,*

- *there is a boundary divisor $\overline{B} \in N_{S,\mathbb{Q}}(U)_{\rm mo}$ and a sequence in $N_{S,\mathbb{Q}}(U)_{\rm mo}$ (resp. $N'_{S,\mathbb{Q}}(U)_{\rm mo}$) converging to $\overline{D_j}$ (resp. $\overline{D'_j}$);*
- *we have that $g_{j,\omega} = g'_{j,\omega}$ for all but finitely many $\omega \in \Omega$.*

For any $m \in \mathbb{N}_{\geq 1}$, set

$$h_{j,m} \coloneqq \max\{g_j, g'_j - m\}.$$

Then the following statements hold.

1. *For each $j = 0, \ldots, d$, the function $h_{j,m}$ is an S-Green function for $D_j \in \widetilde{\rm Div}_{\mathbb{Q}}(U)_{\rm cpt}$ and $\overline{D_{j,m}} \coloneqq (D_j, h_{j,m}) \in \widehat{\rm Div}_{S,\mathbb{Q}}(U)^{\rm YZ}_{\text{rel-snef}}$.*
2. *For each $j = 0, \ldots, d$, the sequence $(\overline{D_{j,m}})_{m\in\mathbb{N}_{\geq 1}}$ decreasingly converges to $\overline{D_j}$ with respect to the $\overline{B}$-boundary topology for some weak boundary divisor $\overline{B} \in N_{S,\mathbb{Q}}(U)_{\rm mo}$.*
3. *We have that*
$$\lim_{m\to\infty} (\overline{D_{0,m}} \cdots \overline{D_{d,m}} \mid U)_S = (\overline{D_0} \cdots \overline{D_d} \mid U)_S.$$

4. *For any* $\overline{E}=(E,f)\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-int}}$ *with underlying compactified (geometric) divisor* $E=0\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$*, we have that*

$$\lim_{m\to\infty}(\overline{D_{1,m}}\cdots\overline{D_{d,m}}\cdot\overline{E}\mid U)_S=\sum_{\omega\in\Omega}\nu(\omega)\int_{U_\omega}f_\omega\, c_1(\overline{D_{1,\omega}})\wedge\cdots\wedge c_1(\overline{D_{d,\omega}}). \tag{4.63}$$

Proof. 1 Let $T\subset\Omega$ be a finite subset such that for any $1\le j\le d$ and $\omega\in\Omega$, $g_{j,\omega}\neq g_{j,\omega}$ implies that $\omega\in T$. Then for any $m\in\mathbb{N}_{\ge1}$ and $\omega\notin\Omega\setminus T$, we have that $g_{j,\omega}\ge g'_{j,\omega}-m$, and $h_{j,m,\omega}=g_{j,\omega}$. By Lemma 4.6.6(1), we know that $h_{j,m}$ is an S-Green function for $D_j\in\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{cpt}}$, and $\overline{D_{j,m}}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{rel-snef}}$.

2 Obviously, we have that $(\overline{D_{j,m}})_{m\in\mathbb{N}_{\ge1}}$ is decreasing, as the proof of [CG24, Lemma 10.6], it is not hard to show 2 holds similarly.

3 Recall the definition of $E(\mathbf{g}',\mathbf{g})$ in 4.3.10. By 2 and [CG24, Theorem 10.9 (v)], we have that

$$\lim_{m\to\infty}E(\mathbf{g}',\mathbf{h}_m)=E(\mathbf{g}',\mathbf{g}).$$

Hence

$$\lim_{m\to\infty}(\overline{D_{0,m}}\cdots\overline{D_{d,m}}\mid U)_S=(\overline{D_0}\cdots\overline{D_d}\mid U)_S.$$

This proves 3.

4 Write $\overline{E}=\overline{E_1}-\overline{E_2}$ with $\overline{E_1}=(E_1,f_1),\overline{E_2}=(E_2,f_2)\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-nef}}$. Since $E=0$, we have that $E_1=E_2\in\widetilde{\mathrm{Div}}_{\mathbb{Q}}(U)_{\mathrm{nef}}$. We may assume that $[f_1]\le[f_2]$ in the sense of **??**. Otherwise, we consider $f_3:=\max\{f_1,f_2\}$ which is still a Green function for $E_1=E_2$ and $(E_1,f_3)\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-nef}}$ (see Lemma 4.6.6(2), $[f_1]\le[f_3],[f_2]\le[f_3]$, then (4.63) for $\overline{E_1}-\overline{E_3}$ and for $\overline{E_2}-\overline{E_3}$ implies (4.63) for $\overline{E}=\overline{E_1}-\overline{E_2}$, i.e. 4 holds. After assigning $\overline{D_0}=\overline{E_1}$ and $\overline{D_0}=\overline{E_2}$, notice that $f_1=\max\{f_1,f_1-m\},f_2=\max\{f_2,f_2-m\}$, by 3, we have that

$$\lim_{m\to\infty}(\overline{D_{1,m}}\cdots\overline{D_{d,m}}\cdot\overline{E_1}\mid U)_S=(\overline{D_1}\cdots\overline{D_d}\cdot\overline{E_1}\mid U)_S. \tag{4.64}$$

By 4.3.10

$$(\overline{D_1}\cdots\overline{D_d}\cdot\overline{E_1}\mid U)_S=(\overline{D'_1}\cdots\overline{D'_d}\cdot\overline{E_1}\mid U)_S+E((g'_1,\cdots,g'_d,f_1),(g_1,\cdots,g_d,f_1)). \tag{4.65}$$

Similarly, we have that

$$\lim_{m\to\infty}(\overline{D_{1,m}}\cdots\overline{D_{d,m}}\cdot\overline{E_2}\mid U)_S=(\overline{D_1}\cdots\overline{D_d}\cdot\overline{E_2}\mid U)_S. \tag{4.66}$$

$$(\overline{D_1}\cdots\overline{D_d}\cdot\overline{E_2}\mid U)_S=(\overline{D'_1}\cdots\overline{D'_d}\cdot\overline{E_2}\mid U)_S+E((g'_1,\cdots,g'_d,f_2),(g_1,\cdots,g_d,f_2)). \tag{4.67}$$

We take (4.65)−(4.67) and substitute it into (4.64)−(4.66), then

$$\begin{aligned}&\lim_{m\to\infty}(\overline{D_{1,m}}\cdots\overline{D_{d,m}}\cdot\overline{E}\mid U)_S\\ =&(\overline{D_1}\cdots\overline{D_d}\cdot\overline{E_1}\mid U)_S-(\overline{D_1}\cdots\overline{D_d}\cdot\overline{E_2}\mid U)_S\\ =&(\overline{D'_1}\cdots\overline{D'_d}\cdot\overline{E_1}\mid U)_S-(\overline{D'_1}\cdots\overline{D'_d}\cdot\overline{E_2}\mid U)_S+E((g'_1,\cdots,g'_d,f_1),(g_1,\cdots,g_d,f_1))\\ &-E((g'_1,\cdots,g'_d,f_2),(g_1,\cdots,g_d,f_2)).\end{aligned}$$

By [CG24, Theorem 11.2] and our assumption that $[f_1]\le[f_2]$, we have that

$$(\overline{D'_1}\cdots\overline{D'_d}\cdot\overline{E_1}\mid U)_S-(\overline{D'_1}\cdots\overline{D'_d}\cdot\overline{E_2}\mid U)_S+E((g'_1,\cdots,g'_d,f_2),(g'_1,\cdots,g'_d,f_1)).$$

Then

$$\begin{aligned}
&\lim_{m\to\infty}(\overline{D_{1,m}}\cdots\overline{D_{d,m}}\cdot\overline{E}\mid U)_S\\
=&E((g_1',\cdots,g_d',f_2),(g_1',\cdots,g_d',f_1))+E((g_1',\cdots,g_d',f_1),(g_1,\cdots,g_d,f_1))\\
&-E((g_1',\cdots,g_d',f_2),(g_1,\cdots,g_d,f_2))\\
=&E((g_1',\cdots,g_d',f_2),(g_1,\cdots,g_d,f_1))-E((g_1',\cdots,g_d',f_2),(g_1,\cdots,g_d,f_2))\\
=&E((g_1,\cdots,g_d,f_2),(g_1,\cdots,g_d,f_1))\\
=&\sum_{\omega\in\Omega}\nu(\omega)\int_{U_\omega}f_\omega\,c_1(\overline{D_{1,\omega}})\wedge\cdots\wedge c_1(\overline{D_{d,\omega}}),
\end{aligned}$$

the second and third equality are from [CG24, Proposition 10.10]. This completed the proof of 4. ☐

4.6.4 Positive intersection product

We recall the *positive intersection product* defined in Definition 3.2.4.

4.6.8. Let X be a projective variety over K. We say a model S-metrized divisor $\overline{A}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$ is *free* if $\overline{A}\in N_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$ (see 4.6.1) and there is $n\in\mathbb{N}_{\geq 1}$ such that $n\overline{A}$ is with $\mathbb{Z}$-coefficients induced by some semiample divisor $\mathcal{A}$ on some projective C-model of X.

Let $\overline{D}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ and $\overline{E}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-nef}}$. If $\overline{D}$ is big, the *positive intersection product* of $\overline{D}$ and $\overline{E}$ is defined as

$$\langle\overline{D}^d\rangle\cdot\overline{E}\coloneqq\sup_{(\pi,\overline{A})}\{\overline{A}^d\cdot\pi^*\overline{E}\},$$

where $(\pi,\overline{A})$ runs over all tuples such that $\pi\colon U'\to U$ is a birational morphism, and $\overline{A}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U')_{\mathrm{mo}}$ is induced by a free model S-metrized divisor on some projective model of U' such that $\pi^*\overline{D}-\overline{A}\geq 0$. By linearity, if $\overline{E}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-int}}$, $\langle\overline{D}^d\rangle\cdot\overline{E}$ is well-defined, see Lemma 3.2.12. Notice that the positive intersection product is stable under birational pull-back, see Lemma 3.2.5.

Lemma 4.6.9. *Let $\overline{D}=(D,g)\in\widehat{\mathrm{Div}}_S(U)^{\text{ar-nef},\mathrm{YZ}}_{\text{rel-nef}}$ with D is big, and $(0,f)\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{mo}}$. If $\inf\limits_{\lambda\in\Delta(D)^\circ}G_{\overline{D}}(\lambda)>0$, then*

$$\langle\overline{D}^d\rangle\cdot(0,f)=\sum_{\omega\in\Omega}\nu(\omega)\int_{U_\omega^{\mathrm{an}}}f_\omega\,c_1(\overline{D}_\omega)^d. \tag{4.68}$$

Proof. Write $\overline{E}=(0,f)$. As the proof of Step 3 in the proof of Theorem 4.5.5, after shrinking U (notice that the both sides of (4.68) are stable under birational base change), there is a sequence $(\overline{D_m})_{m\in\mathbb{N}_{\geq1}}\subset\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-snef}}$ and a sequence $(c_m)_{m\in\mathbb{N}_{\geq1}}\subset\mathscr{L}^1(\Omega,\mathcal{A},\nu)_{\geq0}$ satisfying the following properties:

- $(\overline{D_m})_{m\in\mathbb{N}_{\geq1}}$ is decreasing converging to $\overline{D}$ with respect to the $\overline{B}$-boundary topology for some boundary divisor $\overline{B}\in N_{S,\mathbb{Q}}(U)_{\mathrm{mo}}$ and

$$\lim_{m\to\infty}(\overline{D_m}^d\cdot\overline{E}\mid U)_S=\sum_{\omega\in\Omega}\nu(\omega)\int_{U_\omega^{\mathrm{an}}}f_\omega\,c_1(\overline{D})^d \tag{4.69}$$

  by Lemma 4.6.7;
- for any $m\in\mathbb{N}_{\geq1}$, $c_m(\omega)=0$ for all but finitely many $\omega\in\Omega$;
- $\overline{D_m}(c_m)\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-snef}}$, then

$$\langle\overline{D_m}(c_m)^d\rangle\cdot\overline{E}=(\overline{D_m}(c_m)^d\cdot\overline{E}\mid U)_S$$

  by the linearity and Corollary 3.2.10.

We claim that
$$\langle \overline{D_m}(c_m)^d \rangle \cdot \overline{E} = \langle \overline{D_m}^d \rangle \cdot \overline{E}.$$
If this holds, notice that
$$(\overline{D_m}(c_m)^d \cdot \overline{E} \mid U)_S = (\overline{D_m}^d \cdot \overline{E} \mid U)_S$$
by Lemma 4.6.7(4) since $c_1(\overline{D_m}(c_m)_\omega)^{\wedge d} = c_1((\overline{D_m})_\omega)^{\wedge d}$ noting that $\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{mo}} \subseteq \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\text{ar-int}}$. Then we have
$$\langle \overline{D_m}^d \rangle \cdot \overline{E} = \langle \overline{D_m}(c_m)^d \rangle \cdot \overline{E} = (\overline{D_m}(c_m)^d \cdot \overline{E} \mid U)_S = (\overline{D_m}^d \cdot \overline{E} \mid U)_S.$$
Taking $m \to \infty$ on both sides, by Lemma 3.2.8 and (4.69), the lemma holds. It remains to show the claim. To show the claim, we consider the functions φ, ψ on $\mathbb{R}$ as follows:
$$\varphi(t) := \widehat{\mathrm{vol}}(\overline{D_m}(c_m) + t\overline{E}), \quad \psi(t) := \widehat{\mathrm{vol}}(\overline{D_m} + t\overline{E}).$$
Since $\overline{D_m}$ is decreasingly converging to $\overline{D}$, then
$$G_{\overline{D_m}(c_m)} \geq G_{\overline{D_m}} \geq G_{\overline{D}} \geq \inf_{\lambda \in \Delta(D)^\circ} G_{\overline{D}}(\lambda) > 0.$$
Notice that since $\overline{E} = (0, f) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{mo}}$, there is a $b \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)_{\geq 0}$ such that $b(\omega) = 0$ for all but finitely many $\omega \in \Omega$ and $\sup_{x \in U_\omega^{\mathrm{an}}} \{|f_\omega(x)|\} \leq b(\omega)$ for any $\omega \in \Omega$. Then for any $t \in \mathbb{R}$,
$$\overline{D_m} - |t|(0, b) \leq \overline{D_m} + t\overline{E} \leq \overline{D_m} + |t|(0, b),$$
Hence for small enough $|t|$ using Proposition 4.4.10(3), we have
$$G_{\overline{D_m}(c_m) + t\overline{E}} \geq G_{\overline{D_m} + t\overline{E}} \geq G_{\overline{D_m} - |t|(0,b)} = G_{\overline{D_m}} - |t| \int_\Omega b(\omega)\, \nu(d\omega) > 0.$$
Thus using Theorem 4.4.9, we have
$$\widehat{\mathrm{vol}}(\overline{D_m}(c_m) + t\overline{E}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_m}(c_m) + t\overline{E}), \quad \widehat{\mathrm{vol}}(\overline{D_m} + t\overline{E}) = \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_m} + t\overline{E}),$$
for small enough $|t|$. Then by Theorem 4.4.9, when $|t|$ is small enough, we have that
$$\begin{aligned}\varphi(t) =& \widehat{\mathrm{vol}}(\overline{D_m}(c_m) + t\overline{E}) \\ =& \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_m}(c_m) + t\overline{E}) \\ =& \widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D_m} + t\overline{E}) + (d+1) \sum_{\omega \in \Omega} c_m(\omega)\nu(\omega) \cdot D_m^d \\ =& \psi(t) + (d+1) \sum_{\omega \in \Omega} c_m(\omega)\nu(\omega) \cdot D_m^d,\end{aligned}$$
where in the third equality we have used Proposition 4.4.10(2), the definition of numerical χ-volume and the fact that $\mathrm{vol}(\Delta(D)^\circ) = \frac{\mathrm{vol}(D)}{d!} = \frac{D^d}{d!}$ since D is nef. Taking the derivatives on both sides of $\varphi(t) = \psi(t) + \sum_{\omega \in \Omega} c_m(\omega)\nu(\omega) \cdot D_m^d$, then Theorem 3.2.15 implies our claim. This finishes the proof of the lemma. □

4.6.5 Equidistribution

We quickly recall the setting in which we prove the equidistribution.

4.6.10. Let $(x_m)_{m \in I} \subset U(\overline{K})$ a net of geometric points.

- We say that $(x_m)_{m\in I} \subset U(\overline{K})$ is *generic* if it is Zariski dense in the Zariski topology of U.
- We say that $(x_m)_{m\in I} \subset U(\overline{K})$ is *small with respect to* $\overline{D}$ if

$$\lim_{m\in I} h_{\overline{D}}(x_m) = \frac{\widehat{\mathrm{vol}}_{\chi}^{\mathrm{num}}(\overline{D})}{(d+1)\mathrm{vol}(D)}.$$

- Let $v \in \Omega$, and μ a measure on U_v^{an}. We say that the Galois orbit of $(x_m)_{m\in I}$ *equidistributes in* U_v^{an} *with respect to* μ if

$$\lim_{m\in I} \frac{1}{\#O(x_m)} \sum_{\sigma\in\mathrm{Gal}(\overline{K}/K)} f(\sigma(x_m)) = \int_{U_v^{\mathrm{an}}} f\,d\mu$$

for every compactly supported function f on U_v^{an}, where $O(x_m)$ denotes the Galois orbit of x_m under the action of $\mathrm{Gal}(\overline{K}/K)$ and U_v^{an} denotes the Berkovich analytification of $U \times_K K_v$. See [Bur+19, Section 2] for more details.

Remark 47. Let $(x_m)_{m\in I} \subset U(\overline{K})$ a net of geometric points, and $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\mathrm{cpt}}$ with D big. By Theorem 4.4.21, we have that

$$\limsup_{m\in I} h_{\overline{D}}(x_m) \geq \zeta_{\mathrm{ess}}(\overline{D}) \geq \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}_{\chi}^{\mathrm{num}}(\overline{D})}{(d+1)\mathrm{vol}(D)}.$$

If $(x_m)_{m\in I}$ is small with respect to $\overline{D}$, then $G_{\overline{D}}(\lambda) = \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D})$ on $\Delta(D)^\circ$ since we always have $G_{\overline{D}}(\lambda) \leq \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D})$ from Proposition 4.4.10(4), Indeed $\{x_m\}$ is small with respect to $\overline{D}$ if and only if the series of inequalities above is an equality which implies

$$\widehat{\mathrm{vol}}_{\chi}^{\mathrm{num}}(\overline{D}) = (d+1)! \int_{\Delta(D)^\circ} G_{\overline{D}}\,d\lambda = (d+1)\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D}) \cdot \mathrm{vol}(D) = (d+1)! \int_{\Delta(D)^\circ} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D})\,d\lambda.$$

The above along with $G_{\overline{D}}(\lambda) \leq \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D})$ clearly implies that $G_{\overline{D}}(\lambda) = \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D})$ on $\Delta(D)^\circ$. In particular, in the above case, $\inf_{\lambda\in\Delta(D)^\circ} G(\lambda) = \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D}) > -\infty$. Furthermore, let us consider the following statements which often appear as hypothesis in equidistribution theorem:

1. $(x_m)_{m\in I}$ is small with respect to $\overline{D}$;
2. $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{D}) > -\infty$ and
$$\lim_{m\in I} h_{\overline{D}}(x_m) = \frac{\widehat{\mathrm{vol}}_{\chi}(\overline{D})}{(d+1)\mathrm{vol}(D)};$$
3. $\overline{D}$ is big and
$$\lim_{m\in I} h_{\overline{D}}(x_m) = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)};$$
4. $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)_{\text{rel-nef}}^{\text{ar-nef,YZ}}$ and
$$\lim_{m\in I} h_{\overline{D}}(x_m) = \frac{(\overline{D}^{d+1} \mid U)_S^{d+1}}{(d+1)D^d}.$$

If $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{D}) > -\infty$, by Theorem 4.4.9, then 1 and 2 are equivalent. If $\overline{D}$ is big, by Corollary 4.4.11, then $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D}) > 0$. By Lemma 4.3.14, Proposition 4.4.10(4) and Theorem 4.4.9, we have that

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D}) = \frac{d!}{\mathrm{vol}(D)} \int_{\Delta(D)^\circ} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{D})\,d\lambda \geq \frac{d!}{\mathrm{vol}(D)} \int_{\Delta(D)^\circ} \max\{G_{\overline{D}}, 0\}\,d\lambda = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\mathrm{vol}(D)},$$

and the equality holds if and only if $G_{\overline{D}}(\lambda) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) > 0$ on $\Delta(D)^\circ$. Hence in this case by Theorem 4.4.9, $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D})$, then 1 and 3 are equivalent. If $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef,YZ}}_{\text{rel-nef}}$ and the underlying field K is perfect, then by Theorem 4.5.5, then 1 and 4 are equivalent.

For any $v \in \Omega$, and $f_v \in C_c(U_v^{\mathrm{an}})$, we simply denote by $(0, f_v)$ the compactified metrized divisor $\overline{D} = (D, g)$ of U such that D is trivial, $g_v = f_v$, and $g_\omega = 0$ if $\omega \in \Omega \setminus v$.

Theorem 4.6.11. *Let $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\mathrm{YZ}}_{\mathrm{cpt}}$ with D big. Let $(x_m)_{m\in I}$ be a generic net of points which is small with respect to $\overline{D}$. Then for any $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $\overline{D}(c)$ is big, for any $v \in \Omega$ and for any $f_v \in C_c(U_v^{\mathrm{an}})$, we have that*

$$\lim_{m\in I} \frac{\nu(v)}{\#O(x_m)} \sum_{\sigma\in Gal(\overline{K}/K)} f_v(\sigma(x_m)) = \frac{\langle \overline{D}(c)^d \rangle \cdot (0, f_v)}{\mathrm{vol}(D)}. \tag{4.70}$$

In particular, if $\overline{D} \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef,YZ}}_{\text{rel-nef}}$ and K is perfect, then the Galois orbit of $(x_m)_{m\in I}$ equidistributes in U_v^{an} with respect to $\frac{1}{D^d} \cdot c_1(\overline{D}_v)^d$.

Proof. By Remark 47, we have that $G_{\overline{D}}(\lambda) \equiv \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{D}) > -\infty$ on $\Delta(D)^\circ$. Let $c \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$ such that $\overline{D}(c)$ is big. Then $\inf\limits_{\lambda\in\Delta(D)^\circ} G_{\overline{D}(c)}(\lambda) > 0$.

Let X be a projective K-model of U. Recall the space $\widehat{\mathrm{Div}}_{\mathbb{Q}}(X_v)_{\mathrm{mo}}$ in 4.3.4. We first assume that $\overline{E} := (0, f_v) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$. We consider functions on $\mathbb{R}$: for any $m \in I$,

$$\varphi_m(t) := h_{\overline{D}(c)+t\overline{E}}(x_m) = h_{\overline{D}(c)}(x_m) + \frac{\nu(v)\cdot t}{\#O(x_m)} \sum_{\sigma\in \mathrm{Gal}(\overline{K}/K)} f_v(\sigma(x_m)), \tag{4.71}$$

$$\psi(t) := \frac{\widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}(c) + t\overline{E})}{(d+1)\cdot \mathrm{vol}(D)}$$

(the last equality of (4.71) is from [Bur+19, Proposition 2.3]). We prove the theorem in the following steps.

Step 1: *The following statements hold:*

1. *$\varphi_m(t)$ is concave and differentiable with derivative $\frac{\nu(v)}{\#O(x_m)} \sum\limits_{\sigma\in Gal(\overline{K}/K)} f_v(\sigma(x_m))$ at $t = 0$;*
2. *$\liminf\limits_{m\in I} \varphi_m(t) \geq \psi(t)$;*
3. *$\lim\limits_{m\in I} \varphi_m(0) = \psi(0)$.*

1 is from linearity of height of closed points. For 2, since $(x_m)_{m\in I}$ is generic, we have that

$$\liminf_{m\in I} h_{\overline{D}(c)+t\overline{E}}(x_m) \geq \zeta_{\mathrm{ess}}(\overline{D}(c) + t\overline{E}) \geq \frac{\widehat{\mathrm{vol}}^{\mathrm{num}}_{\chi}(\overline{D}(c) + t\overline{E})}{(d+1)\cdot \mathrm{vol}(D)}$$

which proves 2, here the last inequality is from Theorem 4.4.21. 3 is from Proposition 4.4.10(2) and the assumption that $(x_m)_{m\in I}$ is small.

Step 2: *If $(0, f_v) \in \widehat{\mathrm{Div}}_{S,\mathbb{Q}}(X)_{\mathrm{mo}}$ (i.e. locally $(0, f_v) \in \widehat{\mathrm{Div}}_{\mathbb{Q}}(X_v)_{\mathrm{mo}}$), then $\psi(t)$ is differentiable at $t = 0$ with derivative given by*

$$\frac{d\psi}{dt}(0) = \frac{\langle \overline{D}(c)^d \rangle \cdot (0, f_v)}{\mathrm{vol}(D)}.$$

Hence, by Step 2 and [BB10a, Lemma 7.6], we have that

$$\lim_{m\in I}\frac{\nu(v)}{\#O(x_m)}\sum_{\sigma\in Gal(\overline{K}/K)} f_v(\sigma(x_m)) = \frac{\langle \overline{D}(c)^d\rangle\cdot(0,f_v)}{\mathrm{vol}(D)}. \tag{4.72}$$

Let $b_v \coloneqq \sup_{x\in X_v^{\mathrm{an}}}\{|f_v(x)|\} < \infty$. For any $\omega\in\Omega\setminus\{v\}$, set $b_\omega = 0$. Write $b = (b_\omega)_{\omega\in\Omega}$. Let $t\in\mathbb{R}$. Then

$$\overline{D}(c) + t\overline{E} \geq \overline{D}(c) - |t|(0,b)$$

which implies that

$$G_{\overline{D}(c)+t\overline{E}} \geq G_{\overline{D}(c)} - |t|b_v \geq \inf_{\lambda\in\Delta(D)^\circ} G_{\overline{D}(c)}(\lambda) - |t|b_v > 0,$$

for small enough $|t|$ by Proposition 4.4.10(3). By Theorem 4.4.9, we have that

$$\widehat{\mathrm{vol}}_\chi^{\mathrm{num}}(\overline{D}(c)+t\overline{E}) = \widehat{\mathrm{vol}}(\overline{D}(c)+t\overline{E})$$

when $|t|$ is small enough. By Theorem 3.2.15, we have that

$$\frac{d\psi}{dt}(0) = \frac{\langle\overline{D}(c)^d\rangle\cdot\overline{E}}{\mathrm{vol}(D)}.$$

Step 3: (4.70) *holds for any* $f_v\in C_c(U_v^{\mathrm{an}})$.

To show the theorem is equivalent to show (4.72) holds for any $f_v\in C_c(U_v^{\mathrm{an}})$. Notice that $C_c(U_v^{\mathrm{an}})\subset C(X_v^{\mathrm{an}})$. By [Gub98a, Theorem 7.12] and [Yua08b, Lemma 3.5], the set

$$\{f_v\in C(X_v^{\mathrm{an}}) \mid (0,f_v)\in\widehat{\mathrm{Div}}_{\mathbb{Q}}(X_v)_{\mathrm{mo}}\}$$

is dense in $C(X_v^{\mathrm{an}})$ under the topology of uniform convergence. Hence it suffices to show (4.72) for $f_v\in C(X_v^{\mathrm{an}})$ such that $(0,f_v)\in\widehat{\mathrm{Div}}_{\mathbb{Q}}(X_v)_{\mathrm{mo}}$. This is Step 2, so the theorem holds.

Step 4: *If* $\overline{D}\in\widehat{\mathrm{Div}}_{S,\mathbb{Q}}(U)^{\text{ar-nef,YZ}}_{\text{rel-nef}}$*, then the Galois orbit of* $(x_m)_{m\in I}$ *equidistributes in* U_v^{an} *with respect to* $\frac{1}{D^d}\cdot c_1(\overline{D}_v)^d$.

This is from (4.70) and Lemma 4.6.9. Notice that $c_1(\overline{D}(c)_v)^d = c_1(\overline{D}_v)^d$. □

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