Bounding torsion homology growth: algebraic cheap rebuilding, inner amenability and dynamics

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Summary

Let Γ be a residually finite group, and Λ_* be a residual chain in Γ. Let $j \in \mathbb{N}$ and $\mathbb{F}$ be a field. We define the *Betti number gradient* resp. the *torsion homology gradient* by

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) := \limsup_{i\to\infty} \frac{b_j(\Lambda_i; \mathbb{F})}{[\Gamma : \Lambda_i]} \quad \text{and} \quad \widehat{t}_j(\Gamma, \Lambda_*) := \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_j(\Lambda_i; \mathbb{Z})}{[\Gamma : \Lambda_i]}.$$

This thesis is mainly concerned with vanishing results for these two invariants.

Inspired by inheritance results for classical homological properties, we consider the notion of an equivariantly bootstrappable property. Roughly speaking, equivariantly bootstrappable properties are classes of chain complexes over group rings satisfying inheritance axioms for the degree, suspensions, mapping cones, and inductions. From these axioms, we obtain the Bootstrapping Theorem: We can demonstrate that a group has a given bootstrappable property by constructing an action on a CW-complex with suitable stabilisers.

As a main example of a bootstrappable property, we consider the algebraic cheap rebuilding property. It is inspired by Abért–Bergeron–Frączyk–Gaboriau's (geometric) cheap rebuilding property [ABFG24]. Like geometric cheap rebuilding, algebraic cheap rebuilding implies the vanishing of Betti number gradients and torsion homology gradients. We show that all infinite amenable groups have the algebraic cheap weak rebuilding property, thus recovering a result by Kar–Kropholler–Nikolov [KKN17, Corollary 2] that the torsion homology gradients of amenable groups vanish.

In degree 1, we extend vanishing results to the class of inner-amenable groups. We use a structure theorem of Tucker-Drob [Tuc20] to obtain suitable subgroups, to which we can apply the Bootstrapping Theorem.

In Chapter 4, we consider the two dynamical constants *measured embedding dimension* and *measured embedding volume*, which provide upper bounds on the Betti number gradient resp. the torsion homology gradient. The main result of that section is the monotonicity of measured embedding dimension and measured embedding volume under weak containment. This theorem provides upper bounds on (torsion) homology growth that are independent of the fixed residual chain.

Contents

Introduction

In algebraic topology, one of the most fundamental invariants of topological spaces are the Betti numbers, named after the Italian mathematician Enrico Betti. Intuitively, the j-th Betti number measures the number of 'j-dimensional holes' in a topological space. More formally, we define the *j-th Betti number* of a topological space X as the dimension of its j-th homology group, i.e.

$$b_j(X;\mathbb{F}) \coloneqq \dim_{\mathbb{F}} H_j(X;\mathbb{F}) \in \mathbb{N} \cup \{\infty\},$$

where $\mathbb{F}$ is a field. The number can depend on the chosen field; often we choose $\mathbb{F} = \mathbb{Q}$. For CW-complexes and simplicial complexes, the Betti numbers are algorithmically computable. Moreover, they have nice algebraic properties, e.g. for products, they satisfy the Künneth formula [Hat02, Corollary 3B.7], they are invariant under homotopy equivalence and there is a Mayer–Vietoris sequence.

This thesis is mainly concerned with vanishing of Betti number gradients and torsion homology gradients of groups. We will explain these notions in the first part of the introduction. The main theorems will be stated starting from Section 4.

Recall that a *(model for a) classifying space* for a group Γ is a path-connected CW-complex X with fundamental group $\pi_1(X) \cong \Gamma$, whose universal covering $\widetilde{X}$ is contractible. For each group, classifying spaces exist and are unique up to homotopy equivalence [Hat02, Theorem 1B.8]. We thus refer to its homotopy type as the *classifying space* $B\Gamma$. Thus, for a group Γ, we define its *Betti numbers* by $b_j(\Gamma;\mathbb{F}) \coloneqq b_j(B\Gamma;\mathbb{F})$. In the following, we often assume property F_j, i.e. that there exists a classifying space that consists of finitely many cells up to some dimension $j \in \mathbb{N}$. This implies in particular property FP_j (see Section 1.6).

Convention. The natural numbers $\mathbb{N}$ contain 0.

1 ℓ^2-Betti numbers

Betti numbers can behave quite erratically under (even finite-index) subgroups. For example, the group $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ satisfies $b_1(\Gamma;\mathbb{Q}) = 0$. It contains the free group F_2 of rank 2 as a subgroup of index 6 and $b_1(F_2;\mathbb{Q}) = 2$. Hence, in general $b_1(\,\cdot\,;\mathbb{Q})$ does *not* scale with the index, nor does it satisfy any other meaningful relation.

Instead of considering Betti numbers, it can therefore be beneficial to consider the following variant, originally introduced by Atiyah [Ati76] in an effort to study solutions of elliptic differential equations. Lück defines the j-th *ℓ^2-Betti number* of a group Γ as

$$b_j^{(2)}(\Gamma) \coloneqq \dim_{\mathcal{N}\Gamma} H_j(B\Gamma; \ell^2\Gamma) \in [0, \infty],$$

where $\dim_{\mathcal{N}\Gamma}$ denotes the von Neumann dimension and $H_j(B\Gamma; \ell^2\Gamma)$ denotes the *ℓ^2-homology.* For an introduction to these notions, we refer to the books by Lück [Lüc02] and Kammeyer [Kam19].

Substantial work has been done by Cheeger–Gromov. In particular, all ℓ^2-Betti numbers of infinite amenable groups vanish [CG86, Theorem 0.2].

Definition 1 (amenable group)**.** A group Γ is called *amenable* if there exists a left-invariant mean on Γ, i.e. a function $\mu\colon \mathcal{P}(\Gamma) \to [0,1]$ that is finitely additive, satisfies $\mu(\Gamma) = 1$, and $\mu(\gamma A) = \mu(A)$ for all $\gamma \in \Gamma$ and $A \subseteq \Gamma$.

Cheeger–Gromov's work shows that in the theory of ℓ^2-Betti numbers, amenable groups are 'small'.

2 (Torsion) homology growth

An interesting viewpoint is opened up by Lück's approximation theorem [Lüc94, Theorem 0.1]: A countable group Γ is *residually finite*, if there exists a chain $\Lambda_* = (\Lambda_i)_{i\in\mathbb{N}}$

$$\Gamma = \Lambda_0 \supseteq \Lambda_1 \supseteq \dots$$

of finite-index, normal subgroups satisfying $\bigcap_{i\in\mathbb{N}} \Lambda_i = \{1\}$. We call such a sequence a *residual chain* of Γ.

Theorem 2 (Lück's approximation theorem, [Kam19, Theorem 5.3])**.** *Let Γ be a residually finite group, and suppose that there is a model for $B\Gamma$ of finite type (i.e. in every dimension, there are only finitely many cells). Then, for all $j \in \mathbb{N}$ and all residual chains Λ_* of Γ, we have*

$$b_j^{(2)}(\Gamma) = \lim_{i\to\infty} \frac{b_n(\Lambda_i; \mathbb{Q})}{[\Gamma : \Lambda_i]}.$$

In particular, the sequence on the right-hand side is convergent and the limit is independent of the residual chain.

For residually finite groups, we can consider the right hand side of Lück's approximation theorem as a definition of the *Betti number gradient.*

This viewpoint of 'stabilising' Betti numbers under finite, regular coverings has the advantage that it is easy to generalise: For instance, we can replace the field $\mathbb{Q}$ by a general field $\mathbb{F}$. We define

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) \coloneqq \limsup_{i\to\infty} \frac{b_j(\Lambda_i; \mathbb{F})}{[\Gamma : \Lambda_i]}.$$

Note that in this case, in general, it is unknown whether the lim sup is a proper limit, and whether the limit depends on the chosen residual chain. Note that if Γ

is of type F_j, i.e. admits a classifying space with finite j-skeleton, then $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F})$ is finite by the following argument: Let C be the number of j-cells in a model for $B\Gamma$. Then, $B\Lambda_i$ admits a model, where the number of j-cells is $C \cdot [\Gamma : \Lambda_i]$. Thus, $b_j(\Lambda_i; \mathbb{F}) \leq C \cdot [\Gamma : \Lambda_i]$.

When considering homology with integer coefficients, we can also focus on the torsion part: By $\operatorname{tors} A$, denote the cardinality of the torsion subgroup of an abelian group A. Then, we define

$$\widehat{t}_j(\Gamma, \Lambda_*) \coloneqq \limsup_{i \to \infty} \frac{\log \operatorname{tors} H_j(\Lambda_i; \mathbb{Z})}{[\Gamma : \Lambda_i]}.$$

We use the convention $\log \infty \coloneqq \infty$.

Remark 3 (directed systems)**.** We can generalise the definitions of $\widehat{b}_n$ and $\widehat{t}_n$ to directed systems of subgroups (instead of a chain of subgroups).

We additionally define

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{Z}) \coloneqq \limsup_{i \to \infty} \frac{\operatorname{rk}_{\mathbb{Z}} H_j(\Lambda_i; \mathbb{Z})}{[\Gamma : \Lambda_i]}.$$

Note that we have $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{Z}) = \widehat{b}_j(\Gamma, \Lambda_*; \mathbb{Q})$.

There are more invariants defined in a similar way by 'stabilisation' along a residual chain, e.g. the rank gradient [Lac05] or stable integral simplicial volume [Löh18].

If Γ is finitely presented, a calculation shows that in degree 1, $\operatorname{tors} H_1(\Lambda_i; \mathbb{Z})$ admits an upper bound that is exponential in $[\Gamma : \Lambda_i]$ [Ber18, Section 2]. Thus, $\widehat{t}_1(\Gamma, \Lambda_*)$ is finite. On the other hand, without the assumption on finite presentation, in general, there is no bound on torsion: More precisely, for every function $f\colon \mathbb{N} \to \mathbb{N}$, there exists a finitely generated, residually finite group Γ and a residual chain Λ_* of Γ such that

$$\log \operatorname{tors} H_1(\Lambda_i; \mathbb{Z}) > f([\Gamma : \Lambda_i])$$

for all $i \in \mathbb{N}$ [KKN17, Theorem 3].

Surprisingly though, no example of a finitely presented group Γ that satisfies $\widehat{t}_1(\Gamma, \Lambda_*) > 0$ is known [ABFG24, Introduction]. At the same time, it is a widely-accepted conjecture that for a closed hyperbolic 3-manifold M, $\widehat{t}_1(\pi_1(M), \Lambda_*)$ is proportional to the hyperbolic volume of M and thus positive. Lê proved that $\widehat{t}_1\big(\pi_1(M), \Lambda_*\big) \leq \frac{\operatorname{vol}(M)}{6\pi}$ [Lê18, Theorem 1.1].

Conjecture 4 ([Ber18, Conjecture 6.1] [Lê18, Conjecture 1.3])**.** Let M be a closed hyperbolic 3-manifold and Λ_* be a residual chain of $\pi_1(M)$. Then,

$$\widehat{t}_1\big(\pi_1(M), \Lambda_*\big) = \frac{\operatorname{vol}(M)}{6\pi}.$$

The torsion homology growth detects the difference in $\widehat{b}_j(\Gamma, \Lambda_i; \mathbb{F})$ for different fields $\mathbb{F}$: Avramidi–Okun–Schreve show that for a flag triangulation L of $\mathbb{R}P^2$, the corresponding right-angled Artin group (RAAG) A_L satisfies

$$\widehat{b}_3(A_L, \Lambda_*; \mathbb{Q}) = 0 \neq 1 = \widehat{b}_3(A_L, \Lambda_*, \mathbb{F}_2)$$

for every residual chain Λ_* of A_L [AOS21, Corollary 2]. Note that by the universal coefficient theorem, $H_3(\Lambda_i; \mathbb{F}_2)$ is determined by $H_3(\Lambda_i; \mathbb{Q})$ and the $\mathbb{F}_2$-summands in $H_3(\Lambda_i; \mathbb{Z})$ and $H_2(\Lambda_i; \mathbb{Z})$. Because $H_3(\Lambda_i; \mathbb{Q})$ grows sublinearly in $[\Gamma : \Lambda_i]$ and $H_3(\Lambda_i; \mathbb{Z})$ is torsion-free [AOS21, p. 713], we obtain that $H_2(\Lambda_i; \mathbb{Z})$ contains many $\mathbb{F}_2$-factors, implying that

$$\widehat{t}_2(A_L, \Lambda_*) > 0.$$

In number theory, torsion homology growth of certain arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z}[i])$ parametrises field extensions over quadratic number fields [Ber18, Section 4].

Since Lück's approximation theorem (Theorem 2) yields an analytic description of $\widehat{b}_n(\Gamma, \Lambda_*; \mathbb{Q})$ by $b_n^{(2)}(\Gamma)$, it is natural to ask the following question.

Question 5. Is there an analytic description of $\widehat{b}_n(\Gamma, \Lambda_*; \mathbb{F})$ for finite fields $\mathbb{F}$? Can we describe $\widehat{t}_n(\Gamma, \Lambda_*)$ analytically?

In the latter case, the following is conjectured.

Conjecture 6 ([KKL23, Conjecture 6.11])**.** Let M be an aspherical (i.e. the universal covering $\widetilde{M}$ of M is contractible), closed manifold of odd dimension $2m+1$. Assume that $\pi_1(M)$ is residually finite and let Λ_* be a residual chain of $\pi_1(M)$. Then,

$$\sum_{j=0}^{2m+1} (-1)^j \cdot \widehat{t}_j(\pi_1(M), \Lambda_*) = \rho^{(2)}(\pi_1(M) \curvearrowright \widetilde{M}),$$

where the right-hand side denotes the ℓ^2-*torsion* of M (for a definition, we refer to the books by Lück [Lüc02, Chapter 3] and Kammeyer [Kam19, Chapter 6]).

3 Vanishing (torsion) homology gradients

In order to show vanishing of the Betti number gradients, it is sufficient to find suitable bounds on the number of cells of classifying spaces. Let Γ be a residually finite group with residual chain Λ_* and $i \in \mathbb{N}$. Suppose that there exists a model for $B\Lambda_i$ with C_i cells of dimension $j \in \mathbb{N}$. Then, for all fields $\mathbb{F}$, we have

$$\begin{aligned}\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) &= \limsup_{i\to\infty} \frac{b_j(B\Lambda_i; \mathbb{F})}{[\Gamma : \Lambda_i]} \\ &\leq \limsup_{i\to\infty} \frac{C_i}{[\Gamma : \Lambda_i]}\end{aligned}$$

Thus, if we can find suitable classifying spaces with $\limsup_{i\to\infty} \frac{C_i}{[\Gamma:\Lambda_i]} = 0$, we can show that $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0$.

For the vanishing of the torsion homology gradient, we do not only need to control the number of cells but also the boundary maps. Essentially all known vanishing results for torsion homology growth rely on Gabber's estimate (compare e.g. [BGS20]).

Lemma 7 (Gabber)**.** *Let (X, ∂) be a chain complex consisting of finitely generated based free $\mathbb{Z}$-modules. Then,*

$$\log \operatorname{tors} H_j(X; \mathbb{Z}) \leq \operatorname{rk}(X_j) \cdot \log_+ \|\partial_{j+1}\|,$$

where $\|\partial_{j+1}\|$ denotes the operator-norm of the differential $\partial_{j+1}\colon X_{j+1} \to X_j$ with respect to the ℓ^1-norms on X_{j+1} and X_j induced by the bases, and we set $\log_+ := \max\{\log, 0\}$.

Note that the original estimate was phrased for ℓ^2-norms [Sou99]. We will instead work with the ℓ^1-norm.

A sophisticated way of showing vanishing of Betti number and torsion homology gradients was recently introduced by Abért–Bergeron–Frączyk–Gaboriau through the (geometric) cheap rebuilding property [ABFG24]. In this thesis, we add the word "geometric" to distinguish it from the algebraic cheap rebuilding property, which we will present later. Roughly speaking, for $n \in \mathbb{N}$, a residually finite group Γ has the *(geometric) cheap n-rebuilding property*, if for all Farber sequences $(\Lambda_i)_{i\in\mathbb{N}}$ (a notion generalising residual chains, see Definition 2.1.4), the following holds in a uniform way: If i is large enough, we can find a CW-model of $B\Lambda_i$ with few cells up to dimension n, maintaining tame norms on the cellular boundary operators and homotopies to the canonical covering of $B\Gamma$, corresponding to the subgroup $\Lambda_i \subseteq \Gamma$. For the precise definition, see Definition 2.1.6. The geometric cheap n-rebuilding property implies vanishing of the Betti number gradient up to dimension n. This is because

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) \leq \limsup_{i\to\infty} \frac{C_i}{[\Gamma : \Lambda_i]},$$

where C_i denotes the minimal number of j-cells in a model for $B\Lambda_i$ and the conditions in the definition of geometric cheap n-rebuilding assure that the right hand side converges to zero. Moreover, the torsion homology gradients vanish up to degree $n-1$. This is a consequence of a similar argument, using bounds on the number of cells and boundary operators that are made such that Gabber's estimate (Lemma 7) yields vanishing of the torsion homology gradients.

4 Bootstrappable properties

Abért–Bergeron–Frączyk–Gaboriau prove a robust 'bootstrapping' criterion for the geometric cheap n-rebuilding property [ABFG24, Theorem F] (see Theorem 2.1.8), which has been used for proving the vanishing of the torsion homology gradients e.g. for $\mathrm{SL}_d(\mathbb{Z})$ (where $d \geq 3$) [ABFG24, Theorem B], polynomially growing mapping tori [AGHK23], and outer automorphisms of free Coxeter groups [GGH22, Theorem 1]. Inspired by these applications and classical results, we introduce the notion of an *equivariantly bootstrappable property* in a joint paper with Li–Löh–Moraschini–Sauer [LLMSU24]. An *equivariantly bootstrappable property* over a ring R is a family $\mathsf{B}_*^{\diamondsuit}$, that for every $n \in \mathbb{Z}$ and for every group $\Gamma \in \diamondsuit$ contains a class B_n^{Γ} of chain complexes over the group ring $R\Gamma$, that satisfy inheritance axioms for the degree, suspensions, mapping cones, and inductions of chain complexes (see Definition 1.2.1 and Definition 1.4.1). We may restrict the potential groups to a suitable *is-class* of groups, denoted by $\diamondsuit$, i.e. a class that is closed under isomorphisms and subgroups (e.g. the class of residually finite groups). From an equivariantly bootstrappable property $\mathsf{B}_*^{\diamondsuit} = (\mathsf{B}_n^{\Gamma})_{n\in\mathbb{Z},\Gamma\in\diamondsuit}$, we obtain a family $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ of classes of groups as follows: For all $n \in \mathbb{Z}$, define

$$\mathsf{B}_n \coloneqq \{\Gamma \in \diamondsuit \mid \exists \text{ a projective } R\Gamma\text{-resolution } X \text{ of } R \text{ with } X \in \mathsf{B}_n^{\Gamma}\}.$$

Classical examples for equivariantly bootstrappable properties come from algebraic finiteness properties (see Section 1.6)

The main motivation for introducing the notion of equivariantly bootstrappable properties is the following Bootstrapping Theorem.

Theorem A (Theorem 1.5.3)**.** *Let $\mathsf{B}_*^{\diamondsuit}$ be an equivariantly bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R (see Definition 1.4.1). Let $\Gamma \in \diamondsuit, n \in \mathbb{N}$ and Ω be a Γ-CW-complex. If all of the following conditions hold, then $\Gamma \in \mathsf{B}_n$.*

(i) The complex Ω is $(n-1)$-acyclic over R;

(ii) The quotient $\Gamma\backslash\Omega^{(n)}$ is compact;

(iii) For every cell σ of Ω with $\dim(\sigma) \leq n$, the stabiliser Γ_σ is in $\mathsf{B}_{n-\dim(\sigma)}$.

Starting from the group of integers, we can use the Bootstrapping Theorem to establish bootstrappable properties for fundamental groups of graphs of groups, elementary amenable groups, and chordal RAAGs (see Section 1.7). It is natural to ask if the results for elementary amenable groups extend to all amenable groups.

Question 8 (Question 1.7.4)**.** Let $\mathsf{B}_*^{\diamondsuit}$ be an equivariantly bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R and $n \in \mathbb{N}$. Suppose that $\mathbb{Z} \in \mathsf{B}_j$ for all $j \leq n$. Let $\Gamma \in \diamondsuit$ be an infinite amenable group of type FP_∞. Is $\Gamma \in \mathsf{B}_n$?

5 Algebraic cheap rebuilding

While the geometric cheap rebuilding property satisfies the Bootstrapping Theorem (Theorem 2.1.8), we do *not* expect it to be induced by an equivariantly bootstrappable property in our sense. This is our main motivation for considering the following equivariantly bootstrappable properties.

The main examples for equivariantly bootstrappable properties in a joint paper with Li–Löh–Moraschini–Sauer [LLMSU24] are the algebraic cheap rebuilding properties, which we introduce in the three variants $\mathsf{CR}_*^\diamondsuit$ (*cheap rebuilding*), $\mathsf{CWR}_*^\diamondsuit$ (*cheap weak rebuilding*), and $\mathsf{CD}_*^\diamondsuit$ (*cheap domination*). Let Γ be a residually finite group. Roughly speaking, an $R\Gamma$-chain complex X lies in $\mathsf{CR}_*^\diamondsuit$, if for every residual chain Λ_* of Γ, the following holds: Whenever i is large enough, we can write $\mathbb{Z} \otimes_{\mathbb{Z}\Lambda_i} X$ as a retract of a 'small' complex Y_i. Here, 'small' means that the ranks of the chain modules of Y_i, the operator norms of the differentials in Y_i as well as in the homotopy retraction are small (in comparison to X). For $\mathsf{CWR}_*^\diamondsuit$ and $\mathsf{CD}_*^\diamondsuit$, we demand fewer conditions (for the precise definitions, see Section 2.4).

Theorem B (Theorem 2.4.6)**.** *The classes* $\mathsf{CD}_*^\diamondsuit$ *and* $\mathsf{CR}_*^\diamondsuit$ *define equivariantly bootstrappable properties.*

The class $\mathsf{CWR}_*^\diamondsuit$ only partially satisfies the axioms of an equivariantly bootstrappable property. Yet, these properties are good enough to obtain a modified Bootstrapping Theorem (Theorem 2.4.10).

Algebraic cheap rebuilding is an algebraic analogue of Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding, and also implies vanishing of (torsion) homology gradients.

Theorem C (Theorem 2.5.1)**.** *Let* $n \in \mathbb{N}$ *and* Γ *be a residually finite group.*

(i) If $\Gamma \in \mathsf{CD}_n$*, then, for* $j \leq n$*, all coefficient fields* $\mathbb{F}$ *and residual chains* Λ_**, we have*

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0.$$

(ii) If $\Gamma \in \mathsf{CWR}_n$*, then, for* $j \leq n-1$ *and residual chains* Λ_**, we have*

$$\widehat{t}_j(\Gamma, \Lambda_*) = 0.$$

(iii) If $\Gamma \in \mathsf{CWR}_n$ *is of type* FP_{n+1}*, then, for all residual chains* Λ_**, we have*

$$\widehat{t}_n(\Gamma, \Lambda_*) = 0.$$

Many amenable groups satisfy the property CWR_n for every $n \in \mathbb{N}$.

Theorem D (Theorem 2.8.5)**.** *Let* $n \in \mathbb{N}$ *and* Γ *be a residually finite infinite amenable group of type* FP_n*. Then,* Γ *satisfies* CWR_n*.*

As a corollary, we obtain that torsion homology and Betti number gradients of infinite amenable groups vanish, recovering a result by Kar–Kropholler–Nikolov [KKN17, Corollary 2].

6 Inner-amenable groups

There is another class of groups where the bootstrapping principle works particularly well, namely the class of *inner-amenable* groups.

Definition 9 (inner-amenability, [Eff75; Tuc20])**.** A group Γ is called *inner-amenable* if there exists an atomless conjugation-invariant mean, i.e. a function $\mu\colon \mathcal{P}(\Gamma) \to [0,1]$ that is finitely additive, satisfies $\mu(\Gamma) = 1$, $\mu(\{\gamma\}) = 0$ and $\mu(A^\gamma) = \mu(A)$ for all $\gamma \in \Gamma$ and $A \subseteq \Gamma$. Here, $A^\gamma \coloneqq \{\gamma^{-1} \cdot A \cdot \gamma \mid x \in A\}$.

All infinite amenable groups are inner-amenable (Example 3.1.5(i)). From a structure theorem for inner-amenable groups, we obtain the following theorem.

Theorem E (Theorem 3.3.3)**.** *Let $\mathsf{B}_*^\diamondsuit$ be an equivariantly bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R. Suppose that the following conditions are satisfied:*

(i) All infinite groups in $\diamondsuit$ satisfy B_0.

(ii) The integers $\mathbb{Z}$ satisfy B_1.

Then, all finitely generated, virtually torsion-free inner-amenable, non-amenable groups in $\diamondsuit$ satisfy B_1.

By applying this theorem to $\mathsf{B}_*^\diamondsuit \coloneqq \mathsf{CR}_*^\diamondsuit$, we obtain the following result.

Theorem F (Corollary 3.3.4)**.** *Let Γ be a finitely generated, residually finite, virtually torsion-free, inner-amenable group. Then, for all fields $\mathbb{F}$ and residual chains Λ_*, we have*

$$\widehat{b}_1(\Gamma, \Lambda_*; \mathbb{F}) = 0.$$

If, additionally, Γ is of type FP_2, we have

$$\widehat{t}_1(\Gamma, \Lambda_*) = 0.$$

The first part was already known by work of Chifan–Sinclair–Udrea [CSU16, Corollary D] and later reproved by Tucker-Drob [Tuc20, Theorem 5]. It is natural to ask if the results can be generalised to higher degrees.

Question 10 (Question 3.3.5)**.** Let $\mathsf{B}_*^\diamondsuit$ be an equivariantly bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R and $n \in \mathbb{N}_{\geq 2}$. Suppose that the two conditions of Theorem E are satisfied. Do all virtually torsion-free, inner-amenable, non-amenable groups in $\diamondsuit$ (satisfying suitable finiteness properties) satisfy B_n ?

If the answer is affirmative, this would in particular apply to cheap rebuilding, thus Theorem F would generalise to higher degrees.

7 A dynamical approach

Inspired by the geometric and algebraic cheap rebuilding properties, we aim to introduce a dynamical analogon. The basic principle is the following: The asymptotic behaviour along finite-index subgroups is encoded in the dynamical system given by the profinite completion. Recall that the profinite completion is defined as follows.

Definition 11 (profinite completion)**.** Let Γ be a residually finite group and let $\Lambda_* = (\Lambda_i)_{i\in I}$ be a residual chain (indexed over $I = \mathbb{N}$) or the system of all finite-index normal subgroups in Γ. For each $i \in I$, we equip Γ/Λ_i with the normalised counting measure and the action of Γ by left translation. If $\Lambda_i \subseteq \Lambda_j$, we have the canonical projection $\Gamma/\Lambda_i \to \Gamma/\Lambda_j$. By taking the (inverse) limit over the system Λ_* with inclusions, we obtain a standard probability space $\widehat{\Lambda}_*$, called the *profinite completion* of Γ w.r.t. Λ_*, together with a free action $\Gamma \curvearrowright \widehat{\Lambda}_*$. The probability measure is given by the normalised *Haar measure* on $\widehat{\Lambda}_*$ [Wei40, §7, Appendice II].

Previously, it was established that $\mathbb{Q}$-Betti number gradients (see [Gab02] in combination with Theorem 2), rank gradients [AN12] and stable integral simplicial volume [LP16] are encoded in the profinite completion. In a joint project with Li–Löh–Moraschini–Sauer [LLMSU25], we introduce the *measured embedding dimension* medim_n as an upper bound to $\widehat{b}_n$ and the *measured embedding volume* mevol_n as an upper bound to $\widehat{t}_n$ (see Section 4.3).

Theorem G (Theorem 4.3.4)**.** *Let $n \in \mathbb{N}$ and Γ be a residually finite group of type FP_{n+1}. Let Λ_* be a residual chain or the system of all finite-index normal subgroups of Γ. Let Z denote the integers or a finite field. Then,*

$$\widehat{b}_n(\Gamma, \Lambda_*; Z) \leq \operatorname{medim}_n^Z(\Gamma \curvearrowright \widehat{\Lambda}_*)$$
$$\widehat{t}_n(\Gamma, \Lambda_*) \leq \operatorname{mevol}_n(\Gamma \curvearrowright \widehat{\Lambda}_*).$$

The basic principle for defining the measured embedding dimension resp. volume is the following: Given a free probability measure preserving action $\alpha\colon \Gamma \curvearrowright (X, \mu)$ of a group Γ on a standard probability space (X, μ), we consider the *crossed product ring* $L^\infty(X) *_\alpha \Gamma$ and an $L^\infty(X) *_\alpha \Gamma$-resolution C_* of $L^\infty(X)$. We say that α is 'small' if there exists an $L^\infty(X) *_\alpha \Gamma$-chain map $C_* \to D_*$ extending the identity on $L^\infty(X)$ to an $L^\infty(X) *_\alpha \Gamma$-chain complex D_* that is 'small'. Here, 'small' refers to small dimension of the chain modules (in the case of medim_n) or small lognorm (in the case of mevol_n), a quantity that we consider as a refinement of the expression "$\dim D_n \cdot \log_+ \|\partial_{n+1}^D\|$", mimicking the bound in Gabber's estimate (Lemma 7). For the precise definitions, see Section 4.3.

It is a common theme for dynamical quantities to be monotone under weak containment (see cost [Kec10, Corollary 10.14] and integral foliated simplicial volume [FLPS16, Theorem 1.5]). We prove that the same holds for medim_n^Z and mevol_n, providing evidence that these are indeed dynamical quantities.

Theorem H (Theorem 4.7.1)**.** *Let $n \in \mathbb{N}$ and Γ be a countable group of type FP_{n+1} and let $\alpha\colon \Gamma \curvearrowright (X, \mu)$, $\beta\colon \Gamma \curvearrowright (Y, \nu)$ be free probability measure preserving actions of Γ on standard probability spaces. Let α be weakly contained in β. Let Z denote the integers or a finite field. Then, we have*

$$\begin{aligned} \operatorname{medim}_n^Z(\beta) &\leq \operatorname{medim}_n^Z(\alpha), \\ \operatorname{mevol}_n(\beta) &\leq \operatorname{mevol}_n(\alpha). \end{aligned}$$

In measured group theory, monotonicity under weak containment is a useful property as it sometimes allows to replace an action by a different action that is potentially better-understood. As an application, we obtain upper bounds that are independent of the action (see Section 4.8). Thus, this result is related to the question if (torsion) homology growth is independent of the fixed residual chain.

On the other hand, we can ask if theses dynamical quantities are orbit equivalence invariants.

Question 12. Let Γ and Λ be orbit equivalent groups (see [Fur11, Section 2.2]). Let Z denote the integers or a finite field. Is

$$\begin{aligned} \operatorname{medim}_n^Z(\Gamma \curvearrowright \widehat{\Gamma}) &= \operatorname{medim}_n^Z(\Lambda \curvearrowright \widehat{\Lambda}) \\ \operatorname{mevol}_n(\Gamma \curvearrowright \widehat{\Gamma}) &= \operatorname{mevol}_n(\Lambda \curvearrowright \widehat{\Lambda}) \end{aligned}$$

for all $n \in \mathbb{N}$?

8 Organisation of the Thesis

In Chapter 1, we consider the notion of an (equivariantly) bootstrappable property. The key feature is the Bootstrapping Theorem 1.5.3.

Chapter 2 is dedicated to the prime example of an equivariantly bootstrappable property, called *algebraic cheap rebuilding*. It is largely inspired by Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding, which we review in Section 2.1. We show that algebraic cheap rebuilding implies the vanishing of homology gradient invariants (Section 2.5) and discuss the example of the group of integers (Section 2.7) and amenable groups (Section 2.8).

In Chapter 3, we deduce the results about equivariantly bootstrappable properties of inner-amenable groups.

Chapter 4 considers a dynamical viewpoint. We define the quantities medim and mevol, which are upper bounds to (torsion) homology growth. A large part of the chapter is dedicated to proving monotonicity of these invariants under weak containment (see Section 4.7).

9 Contributions

I will briefly outline my contributions and new results in this thesis. The main results are the following:

- Theorem E and Theorem F already appeared in a published paper of mine [Usc24]. In this thesis, the proofs are presented in Chapter 3 and adapted to the setting in Chapters 1 and 2. In particular, I introduce the notion of a bootstrappable action property in order to incorporate Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding into the setting.
- Theorem H, presented in Chapter 4, is my own project.
- Theorem A, Theorem B, Theorem C and Theorem D are contained in a joint project with Li–Löh–Moraschini–Sauer [LLMSU24]. In this project, I was mainly involved with the Novikov–Shubin invariants, the choice of the 1-norm instead of the 2-norm, the vanishing of torsion homology growth in top degree, and amenability.
- Theorem G is contained in another joint project with Li–Löh–Moraschini–Sauer [LLMSU25] and is a prerequisite for the relevance of Theorem H. I was particularly involved with the approximation of lognorm with dense subalgebras.

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1 The bootstrapping principle

Many homological properties of groups satisfy inheritance results. For a (bi-infinite) sequence $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ of classes of groups, the following properties are desirable:

(i) Let Γ be the fundamental group of a finite graph of groups (e.g., an amalgamated product or an HNN-extension) and $n \in \mathbb{N}$. If all the vertex groups lie in B_n and all edge groups lie in B_{n-1}, then $\Gamma \in \mathsf{B}_n$.

(ii) Let $1 \to N \to \Gamma \to Q \to 1$ be a group extension and $n \in \mathbb{N}$. If $N \in \mathsf{B}_m$ for all $m \leq n$ and Q is of type F_n, then $\Gamma \in \mathsf{B}_n$. (Recall that a group Q is of type F_n if there exists a model for the classifying space $K(\Gamma, 1)$ with finite n-skeleton.)

A classical example is given by defining B_n to be the class of groups satisfying FP_n (see Section 1.6).

In the following, we will define stronger conditions on the classes B_n, of which the two aforementioned conditions are special cases. We first consider chain complexes over a fixed ring (e.g. the group ring of a fixed group) in Section 1.2. We generalise to the setting of equivariantly bootstrappable properties, which are families $\mathsf{B}_*^{\diamondsuit}$, where $*$ ranges over the integers and $\diamondsuit$ ranges over a class of groups (see Section 1.4). Finally, this yields classes of groups $(\mathsf{B}_n)_{n\in\mathbb{Z}}$ (see Definition 1.4.2).

The following definitions emerged in a joint project with Li–Löh–Moraschini–Sauer [LLMSU24] in an effort to axiomatise the concept of (geometric) cheap rebuilding introduced by Abért–Bergeron–Frączyk–Gaboriau (see Section 2.1).

1.1 Preliminaries on chain complexes

We recall some basics on chain complexes, suspensions, cones and homotopy commutative cubes. In this section, we omit proofs that are straightforward computations. They can be found in the paper [LLMSU24].

Definition 1.1.1 (Suspension)**.** Let X be a chain complex and $k \in \mathbb{Z}$. The *k-fold suspension* of X is the chain complex $\Sigma^k X$ with chain modules $(\Sigma^k X)_j \coloneqq X_{j-k}$ and differentials $\partial_j^{\Sigma^k X} \coloneqq (-1)^k \partial_{j-k}^X$. Similarly, if $f\colon X \to Y$ is a chain map, we define $\Sigma^k f\colon \Sigma^k X \to \Sigma^k Y$ by $(\Sigma^k f)_j \coloneqq f_{j-k}$.

For $k = 1$, we write Σ instead of Σ^1.

Definition 1.1.2 (Cone)**.** Let $f\colon X \to Y$ be a chain map between chain complexes. Then, the *cone* of f is defined to be the chain complex $\mathrm{Cone}(f)$ with chain modules $(\mathrm{Cone}(f))_j \coloneqq X_{j-1} \oplus Y_j$ and differentials $\partial_j\colon \mathrm{Cone}(f)_j \to \mathrm{Cone}(f)_{j-1}$ given by

$$\partial_j(x,y) = (-\partial^X_{j-1}(x), \partial^Y_j(y) + f_{j-1}(x)).$$

Example 1.1.3. Let $f\colon X \to Y$ be a chain map. Then, the inclusion defines a chain map $Y \to \operatorname{Cone}(f)$.

In analogy to skeleta of CW-complexes, we define the following notion.

Definition 1.1.4 (skeleton of a chain complex)**.** Let X be a chain complex and $n \in \mathbb{Z}$. Then, the *n-skeleton* of X is defined as the chain complex $X^{(n)}$ with chain modules $(X^{(n)})_j := X_j$ if $j \leq n$ and $(X^{(n)})_j := 0$ otherwise. The differentials are given by $\partial_j^{X^{(n)}} := \partial_j^X$ if $j \leq n$ and zero otherwise. Similarly, we define $X^{(>n)}$ by replacing all modules in degrees $\leq n$ with zero modules.

An important observation is that every chain complex is isomorphic to a mapping cone in the following ways.

Example 1.1.5 (Splitting a chain complex)**.** Let X be a chain complex and $n \in \mathbb{N}$. Let $f\colon \Sigma^{-1}(X^{(>n)}) \to X^{(n)}$ be the chain map given by $f_n := \partial^X_{n+1}$ and zero in all other degrees. Then, $\operatorname{Cone}(f)$ is isomorphic to X.

Definition 1.1.6 (homotopy commutative square)**.** A *homotopy commutative square* consists of chain maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & \circlearrowleft_H & \downarrow{\scriptstyle b} \\ Z & \xrightarrow[g]{} & W \end{array} \tag{1.1.1}$$

together with a chain homotopy $H\colon g \circ a \simeq b \circ f$, i.e. a family of maps $H_j\colon X_j \to W_{j+1}$ such that for all $j \in \mathbb{Z}$

$$\partial^W_{j+1} \circ H_j + H_{j-1} \circ \partial^X_j = g \circ a - b \circ f. \tag{1.1.2}$$

Note that in the case $H = 0$, the square is *strictly commutative*.

We often employ homotopy commutative squares to obtain maps between mapping cones [Wei95, Section 1.5].

Lemma 1.1.7 (induced map on mapping cones)**.** *Let a homotopy commutative square as in Equation* (1.1.1) *be given. Then, we obtain an induced chain map*

$$(a,b;H)\colon \operatorname{Cone}(f) \to \operatorname{Cone}(g)$$

given by

$$(a,b;H)_j(x,y) := \big(a_{j-1}(x), b_j(y) - H_{j-1}(x)\big).$$

in degree $j \in \mathbb{Z}$.

Lemma 1.1.8. *Let $f\colon X \to Y$ be a chain map. Then, there exists a short exact sequence of chain complexes*

$$0 \to Y \xrightarrow{\iota} \mathrm{Cone}(f) \xrightarrow{\pi} \Sigma X \to 0,$$

which splits degreewise. Consequently, there exists a natural long exact sequence of homology groups

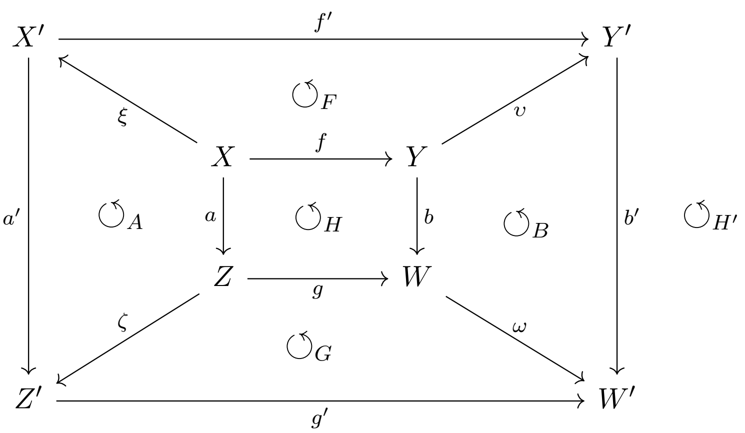
$$\cdots \to H_j(X) \to H_j(Y) \to H_j(\mathrm{Cone}(f)) \to H_{j-1}(X) \to H_{j-1}(Y) \to \cdots .$$

Lemma 1.1.9. *Suppose we have a homotopy commutative square as in Definition 1.1.6. We have the following diagram:*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{Cone}(f) \\ {\scriptstyle a}\big\downarrow & \circlearrowleft_H & \big\downarrow{\scriptstyle b} & \circlearrowleft_0 & \big\downarrow{\scriptstyle (a,b;H)} \\ Z & \xrightarrow[g]{} & W & \longrightarrow & \mathrm{Cone}(g) \end{array}$$

If the chain maps a and b are weak equivalences, then so is $(a, b; H)$.

Ultimately, we will need a "higher" functoriality of the mapping cone with respect to cubes that are coherently homotopy commutative. We introduce the following definition: A *homotopy commutative cube* consists of six homotopy commutative squares



(1.1.3)

where

$$\begin{aligned} H&\colon g \circ a \simeq b \circ f; \\ H'&\colon g' \circ a' \simeq b' \circ f'; \\ A&\colon \zeta \circ a \simeq a' \circ \xi; \\ B&\colon \omega \circ b \simeq b' \circ \upsilon; \\ F&\colon f' \circ \xi \simeq \upsilon \circ f; \\ G&\colon g' \circ \zeta \simeq \omega \circ g; \end{aligned}$$

together with a map $\Phi\colon X_* \to W'_{*+2}$, called the *filler* of the cube, such that

$$\begin{aligned} &\partial^{W'}_{j+2} \circ \Phi_j - \Phi_{j-1} \circ \partial^X_j = \\ &\qquad \omega_{j+1} \circ H_j - H'_j \circ \xi_j + B_j \circ f_j - g'_{j+1} \circ A_j + G_j \circ a_j - b'_{j+1} \circ F_j. \end{aligned} \tag{1.1.4}$$

The right hand side in Equation (1.1.4) is the sum of all six maps $X_* \to W'_{*+1}$ in Diagram (1.1.3), where opposite faces of the cube contribute with opposite signs.

Lemma 1.1.10 ([LLMSU24, Lemma 2.3])**.** *Let a homotopy commutative cube as in Diagram* (1.1.3) *be given. Then, the following square is homotopy commutative*

$$\begin{array}{ccc} \mathrm{Cone}(f) & \xrightarrow{(\xi,\upsilon;F)} & \mathrm{Cone}(f') \\ {\scriptstyle (a,b;H)}\downarrow & \circlearrowleft_{\Psi} & \downarrow{\scriptstyle (a',b';H')} \\ \mathrm{Cone}(g) & \xrightarrow[(\zeta,\omega;G)]{} & \mathrm{Cone}(g') \end{array}$$

where $\Psi\colon \mathrm{Cone}(f)_* \to \mathrm{Cone}(g')_{*+1}$ *is defined as*

$$\Psi_j(x, y) \coloneqq \big(-A_{j-1}(x), B_j(y) - \Phi_{j-1}(x)\big).$$

1.2 Bootstrappable properties for chain complexes

We first introduce bootstrappable properties of chain complexes over a fixed ring.

Definition 1.2.1 (Bootstrappable property)**.** Let R be a ring. A sequence $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ of classes of R-chain complexes is a *bootstrappable property of R-chain complexes* if it is closed under isomorphisms and satisfies the following axioms: Let X be an R-chain complex and $n \in \mathbb{Z}$.

(B-deg) *Degree.* If X is concentrated in degrees ≥ 0, then $X \in \mathsf{B}_m$ for all $m < 0$;

(B-susp) *Suspension.* We have $X \in \mathsf{B}_n$ if and only if $\Sigma X \in \mathsf{B}_{n+1}$;

(B-cone) *Mapping cone.* Let $f\colon X \to Y$ be an R-chain map. If $X \in \mathsf{B}_{n-1}$ and $Y \in \mathsf{B}_n$, then $\mathrm{Cone}(f) \in \mathsf{B}_n$.

An easy example is given by interpreting B_n as 'finitely generated in degrees $\leq n$', which obviously satisfies all three axioms. We will give more complex examples in Section 1.8 and Chapter 2. We first record some immediate consequences of the definition.

Lemma 1.2.2. *Let B_* be a bootstrappable property of R-chain complexes.*

(i) If $n \in \mathbb{Z}$ and X is concentrated in degrees $> n$, then $X \in \mathsf{B}_n$.

(ii) If $X, Y \in \mathsf{B}_n$, then $X \oplus Y \in \mathsf{B}_n$.

Proof.

(i) We obtain that $\Sigma^{-(n+1)}X$ is concentrated in degrees ≥ 0, thus $\Sigma^{-(n+1)} \in \mathsf{B}_{-1}$ by Axiom (B-deg). Then, a repeated application of Axiom (B-susp) yields that $X = \Sigma^{n+1}\Sigma^{-(n+1)}X \in \mathsf{B}_n$.

(ii) Since $X \oplus Y = \mathrm{Cone}(0\colon \Sigma^{-1}X \to Y)$, we obtain $X \oplus Y \in \mathsf{B}_n$ from Axiom (B-susp) and Axiom (B-cone). ☐

1.3 Projective replacements

In this section, we recall the technique of projective replacements. Starting with a resolution of a module M and projective resolutions of the modules occurring in the resolution, we can build a new resolution of M consisting of projective modules that satisfies additional properties.

Let R be a ring and M be an R-module. Recall that a *projective resolution* of M is an chain complex X consisting of projective R-modules that is concentrated in non-negative degrees, together with an *augmentation map* $\epsilon\colon X_0 \to M$ such that the augmented complex $X_* \to M$ is exact. A *weak equivalence* is a chain map that induces isomorphisms on homology in all degrees.

Proposition 1.3.1 (Projective replacement [Bro82b, Lemma 1.5] [LLMSU24, Proposition 2.12])**.** *Let X be a chain complex that is concentrated in degrees ≥ 0. For each $j \geq 0$, let $P^j = (P^j_i)_{i\geq 0}$ be a projective resolution of the module X_j. Then, there exists a projective chain complex $\widehat{X}$ together with a filtration $(\widehat{X}^{[k]})_{k\geq 0}$ and a chain map $q\colon \widehat{X} \to X$ such that for every $k \geq 0$, the following hold:*

(i) We have $\widehat{X}^{[0]} = P^0$.

(ii) $\widehat{X}^{[k]}$ is the mapping cone of a chain map $\Sigma^{k-1}P^k \to \widehat{X}^{[k-1]}$.

(iii) The restriction $q^k\colon \widehat{X}^{[k]} \to X^{(k)}$ is a weak equivalence.

In particular, the chain modules of $\widehat{X}$ are of the form

$$\widehat{X}_n = \bigoplus_{j+i=n} P^j_i$$

and $q\colon \widehat{X} \to X$ is a weak equivalence.

Proof. We prove the claim by induction over $k \in \mathbb{N}$. We set $\widehat{X}^{[0]} := P^0$ and $q^0\colon P^0 \to X^{(0)}$ defined by the augmentation $P^0_0 \to X_0$ in degree 0 and zero in all other degrees. Since P^0 is a resolution of X_0, the chain map q^0 is a weak equivalence.

For the induction step, assume that the chain complex $\widehat{X}^{[k-1]}$ and a weak equivalence $q^{k-1}\colon \widehat{X}^{[k-1]} \to X^{(k-1)}$ have been constructed. We consider the following diagram:

$$\begin{array}{ccc} \Sigma^{k-1}P^k & & \widehat{X}^{[k-1]} \\ {\scriptstyle \Sigma^{k-1}\varepsilon^k}\big\downarrow & & \big\downarrow{\scriptstyle q^{k-1}} \\ \Sigma^{k-1}X_k & \xrightarrow{\partial_k^X} & X^{(k-1)} \end{array}$$

where $\varepsilon^k \colon P^k \to X_k$ is given by the augmentation $P_0^k \to X_k$ in degree 0 (and zero in all other degrees). Note that in all degrees, $\Sigma^{k-1}P^k$ consists of projective modules and q^{k-1} is a weak equivalence. Thus, there exists a chain map $\widehat{\partial_k^X} \colon \Sigma^{k-1}P^k \to \widehat{X}^{[k-1]}$ making the square homotopy commutative [Bro82b, Lemma 1.1]. We define $\widehat{X}^{[k]} \coloneqq \mathrm{Cone}(\widehat{\partial_k^X})$ and $q^k \colon \widehat{X}^{[k]} \to X^{(k)} = \mathrm{Cone}(\partial_k^X)$ to be the induced chain map on mapping cones. We thus have a (homotopy commutative) diagram of mapping cone sequences as follows.

$$\begin{array}{ccccc} \Sigma^{k-1}P^k & \xrightarrow{\widehat{\partial_k^X}} & \widehat{X}^{[k-1]} & \longrightarrow & \widehat{X}^{[k]} \\ {\scriptstyle \Sigma^{k-1}\varepsilon^k}\big\downarrow & & \big\downarrow{\scriptstyle q^{k-1}} & & \big\downarrow{\scriptstyle q^{k}} \\ \Sigma^{k-1}X_k & \xrightarrow{\partial_k^X} & X^{(k-1)} & \longrightarrow & X^{(k)} \end{array}$$

By Lemma 1.1.9, because $\Sigma^{k-1}\epsilon^k$ and q^{k-1} are weak equivalences, so is the chain map q^k.

Finally, we take the colimit $\widehat{X} \coloneqq \mathrm{colim}_k \widehat{X}^{[k]}$ and $q \coloneqq \mathrm{colim}_k q^k$. By construction, we have

$$\widehat{X}_n^{[k]} = \bigoplus_{\substack{j+i=n,\\ 0\le j\le k}} P_i^j,$$

and thus $\widehat{X}_n^{[k]} = \widehat{X}_n$ for $k \geq n$. Consequently, we also have $\widehat{X}_n = \bigoplus_{j+i=n} P_i^j$. Since homology commutes with directed colimits [Sta25, Lemma 00DB] and all maps $(q^k)_k$ are weak equivalences, the map q is a weak equivalence. □

Proposition 1.3.2 (bootstrapping for chain complexes)**.** *Let B_* be a bootstrappable property of R-chain complexes and $n \in \mathbb{N}$. Let X be an R-chain complex that is concentrated in degrees ≥ 0. Suppose that for all $j \leq n$, the R-module X_j admits a projective resolution P^j lying in B_{n-j}. Then, there exists a projective R-chain complex $\overline{X} \in \mathsf{B}_n$ that is weakly equivalent to X.*

Proof. **Case 1:** Suppose that X is of finite length, i.e. there is $k \in \mathbb{N}$ such that X is concentrated in degrees $\leq k$. By Proposition 1.3.1, there exists a projective replacement $\overline{X}$ that is weakly equivalent to X. Moreover, we have a filtration

$$P^0 = \widehat{X}^{[0]} \subseteq \widehat{X}^{[1]} \subseteq \cdots \subseteq \widehat{X}^{[k]} = \widehat{X},$$

where for all $j \leq n$, the complex $\widehat{X}^{[j]}$ is the mapping cone of a map $\Sigma^{j-1}P^j \to \widehat{X}^{[j-1]}$.

We prove the claim by induction: We have $P^0 \in \mathsf{B}_{n-0}$ by assumption. For the induction step, assume $\widehat{X}^{[j-1]} \in \mathsf{B}_n$. By assumption, we have $P^j \in \mathsf{B}_{n-j}$, thus by

Axiom (B-susp), we obtain $\Sigma^{j-1}P^j \in \mathsf{B}_{n-1}$. Thus, $\widehat{X}^{[j]} \in \mathsf{B}_n$. Finally, we obtain that $\widehat{X} = \widehat{X}^{[k]} \in \mathsf{B}_n$.

Case 2: Let X be arbitrary (possibly of infinite length). For $j > n$, let P^j denote an arbitrary projective resolution of X_j. By Example 1.1.5, X is isomorphic to $\mathrm{Cone}(\partial^X_{n+1})\colon \Sigma^{-1}(X^{(>n)}) \to X^{(n)}$, where the chain map is given by ∂^X_{n+1} in degree n. By Proposition 1.3.1, there are projective replacements $q^{(n)}\colon \widehat{X^{(n)}} \to X^{(n)}$ and $q^{(>n)}\colon \widehat{X^{(>n)}} \to X^{(>n)}$ with respect to the resolutions $(P^j)_j$. We consider the following diagram:

$$\begin{array}{ccc} \Sigma^{-1}(\widehat{X^{(>n)}}) & & \widehat{X^{(n)}} \\ \Sigma^{-1}(q^{(>n)})\big\downarrow & & \big\downarrow q^{(n)} \\ \Sigma^{-1}(X^{(>n)}) & \xrightarrow{\partial^X_{n+1}} & X^{(n)} \end{array}$$

The R-chain complex $\Sigma^{-1}(\widehat{X^{(>n)}})$ is projective and the R-chain map $q^{(n)}$ is a weak equivalence, thus we can lift ∂^X_{n+1} up to homotopy to $\widehat{\partial^X_{n+1}}\colon \Sigma^{-1}(\widehat{X^{(>n)}}) \to \widehat{X^{(n)}}$ that is an R-chain map [Bro82b, Lemma 1.1]. We now define $\overline{X} \coloneqq \mathrm{Cone}(\widehat{\partial^X_{n+1}})$ and $q\colon \overline{X} \to X$ to be the induced map on mapping cones (see Lemma 1.1.7). We thus have the following diagram of mapping cone sequences.

$$\begin{array}{ccccc} \Sigma^{-1}(\widehat{X^{(>n)}}) & \xrightarrow{\widehat{\partial^X_{n+1}}} & \widehat{X^{(n)}} & \longrightarrow & \overline{X} \\ \Sigma^{-1}(q^{(>n)})\big\downarrow & & \big\downarrow q^{(n)} & & \big\downarrow q \\ \Sigma^{-1}(X^{(>n)}) & \xrightarrow{\partial^X_{n+1}} & X^{(n)} & \longrightarrow & X \end{array}$$

As a cone over projective R-chain complexes, $\overline{X}$ is projective. Lemma 1.1.9 yields that q is a weak equivalence. By Case (1), we obtain that $\widehat{X^{(n)}}$ lies in B_n. The construction in Proposition 1.3.1 yields that $\Sigma^{-1}(\widehat{X^{(>n)}})$ is concentrated in degrees $\geq n$, thus lies in B_{n-1} by Lemma 1.2.2. By Axiom (B-cone), we conclude that $\overline{X} = \mathrm{Cone}(\widehat{\partial^X_{n+1}}) \in \mathsf{B}_n$. □

1.4 Bootstrappable properties for groups

We will now define a notion of an equivariantly bootstrappable property for groups. In order to relate chain complexes over group rings for different groups, we demand compatibility with the induction functor. We define an *is-class* of groups as a class of groups that is closed both under isomorphisms and taking arbitrary subgroups. We use the symbol $\diamondsuit$ to denote such a class of groups. The most common examples in this thesis are the class of all groups and the class of residually finite groups.

Definition 1.4.1 (Equivariantly bootstrappable property)**.** Let R be a ring and let $\diamondsuit$ be an is-class of groups. An *equivariantly bootstrappable property of chain* *complexes over R* is a family $\mathsf{B}_*^{\diamondsuit} = (\mathsf{B}_n^{\Gamma})_{\Gamma\in\diamondsuit, n\in\mathbb{Z}}$, where B_n^{Γ} is a class of $R\Gamma$-chain complexes for all $\Gamma \in \diamondsuit$ and $n \in \mathbb{Z}$ such that B_*^{Γ} is a bootstrappable property of $R\Gamma$-chain complexes, and, moreover, for all $n \in \mathbb{Z}$, the following holds:

(B-ind) *Induction.* Let $\Gamma \in \diamondsuit$, let Δ be a subgroup of Γ, and let X be an $R\Delta$-chain complex. If $X \in \mathsf{B}_n^{\Delta}$, then $\operatorname{ind}_{\Delta}^{\Gamma} X := R\Gamma \otimes_{R\Delta} X \in \mathsf{B}_n^{\Gamma}$.

From an equivariantly bootstrappable property, we obtain a bootstrappable property of groups.

Definition 1.4.2 (Bootstrappable property of groups)**.** Let $\mathsf{B}_*^{\diamondsuit}$ be an equivariantly bootstrappable property of chain complexes over R. We define the associated class B_n as follows:

$$\mathsf{B}_n := \{\Gamma \in \diamondsuit \mid \exists \text{ a projective } R\Gamma\text{-resolution } X \text{ of } R \text{ with } X \in \mathsf{B}_n^{\Gamma}\}.$$

We call a family of classes of groups a *bootstrappable property of the is-class* $\diamondsuit$ *of groups* whenever it arises in this way from some equivariantly bootstrappable property of chain complexes over R. We denote the intersection $\mathsf{B}_\infty := \bigcap_{n\in\mathbb{Z}} \mathsf{B}_n$.

Many bootstrappable properties additionally satisfy the following restriction axiom.

(B-res) *Restriction to finite index subgroups.* Let $\Gamma \in \diamondsuit$, let Δ be a finite index subgroup of Γ, and let X be an $R\Gamma$-chain complex. If $X \in \mathsf{B}_n^{\Gamma}$, then we have $\operatorname{res}_{\Delta}^{\Gamma} X \in \mathsf{B}_n^{\Delta}$.

1.5 The Bootstrapping Theorem

The main motivation for defining bootstrappable properties is the following bootstrapping theorem. It allows to “bootstrap” a property of a group from properties of its stabilisers. In typical applications, we will start with a “small” group (typically $\mathbb{Z}$) where we can show a certain property by hand, and then work our way up using the Bootstrapping Theorem. For the latter, we have to construct actions that have stabilisers that we have already verified the desired bootstrappable property for.

Definition 1.5.1 (bootstrappable action property)**.** Let $\diamondsuit$ be an is-class of groups and $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ be a sequence of group properties for groups in $\diamondsuit$. We say that B_* is a *bootstrappable action property* over a ring R if for all groups $\Gamma \in \diamondsuit, n \in \mathbb{N}$ and all Γ-CW-complexes Ω, we have that if the following conditions hold, then $\Gamma \in \mathsf{B}_n$.

(i) Ω is $(n-1)$-acyclic over R (i.e., $H_j(\Omega; R) \cong H_j(\mathrm{pt}; R)$ for all $j \leq n-1$);

(ii) $\Gamma\backslash\Omega^{(n)}$ is compact; and

(iii) For every cell σ of Ω with $\dim(\sigma) \leq n$, the stabiliser Γ_σ lies in $\mathsf{B}_{n-\dim(\sigma)}$;

The main class of examples are bootstrappable properties of groups. We first start with an algebraic observation.

Theorem 1.5.2 (Bootstrapping theorem, algebraic version)**.** *Let $\mathsf{B}_*^{\diamondsuit}$ be an equivariantly bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R. Let $\Gamma \in \diamondsuit$ and $n \in \mathbb{N}$. Let X be an $R\Gamma$-resolution of the trivial $R\Gamma$-module R. Suppose that for all $j \leq n$, the $R\Gamma$-module X_j is isomorphic to a finite direct sum $\bigoplus_{\sigma \in S_j} R[\Gamma/\Gamma_\sigma]$, where Γ_σ is a subgroup of Γ lying in B_{n-j}. Then, $\Gamma \in \mathsf{B}_n$.*

Proof. For each $j \leq n$ and $\sigma \in S_j$, because $\Gamma_\sigma \in \mathsf{B}_{n-j}$, there exists a projective $R[\Gamma_\sigma]$-resolution P^σ of R lying in B_{n-j}^Γ. By axiom (B-ind), we obtain that the projective $R[\Gamma]$-resolution

$$P^j := \bigoplus_{\sigma \in S_j} \operatorname{ind}_{\Gamma_\sigma}^\Gamma P^\sigma$$

of X_j lies in B_{n-j}^Γ. For $j > n$, we pick any resolution P^j. In this case, we have $P^j \in \mathsf{B}_{n-j}^\Gamma$ by Axiom (B-deg). We apply Proposition 1.3.1 to X (without the augmentation map) and $(P^j)_{j \in \mathbb{N}}$. We obtain a projective chain complex $\widehat{X}$ with a filtration $(\widehat{X}^{[k]})_{k \in \mathbb{N}}$ and a chain map $q \colon \widehat{X} \to X$, which is a weak equivalence. Since X is a resolution of R, and $\widehat{X}$ is projective, also $\widehat{X}$ is a projective resolution of R. We will show that $\widehat{X} \in \mathsf{B}_n^\Gamma$. By Proposition 1.3.1, we have $\widehat{X}^{[0]} = P^0 \in \mathsf{B}_n^\Gamma$. Moreover, for all $k \in \mathbb{N}$, the chain complex $\widehat{X}^{[k]}$ is the mapping cone of a chain map $\Sigma^{k-1} P^k \to \widehat{X}[k-1]$. Since $P^k \in \mathsf{B}_{n-k}^\Gamma$, repeated application of Axiom (B-susp) yields that $\Sigma^{k-1} P^k \in \mathsf{B}_{n-1}^\Gamma$. Axiom (B-cone) yields that $\widehat{X}^{[k]} \in \mathsf{B}_n^\Gamma$. We obtain by induction that $\widehat{X}^{[n]} \in \mathsf{B}_n^\Gamma$. Proposition 1.3.1 also yields that $\widehat{X}$ is the mapping cone of a chain map $Y \to X^{[n]}$, where Y is a chain complex concentrated in degrees $\geq n$. Thus, by Lemma 1.2.2, we have $Y \in \mathsf{B}_{n-1}^\Gamma$ and Axiom (B-cone) yields that $\widehat{X} \in \mathsf{B}_n^\Gamma$. Since $\widehat{X}$ is a projective resolution of R, this shows that $\Gamma \in \mathsf{B}_n$. □

This implies the following topological version.

Theorem 1.5.3 (Bootstrapping Theorem, [LLMSU24, Theorem 3.6])**.** *Bootstrappable properties have the bootstrappable action property.*

Proof. Let $\mathsf{B}_*^{\diamondsuit}$ be an equivariantly bootstrappable property of chain complexes over R that induces B_*. Let $\Gamma \in \diamondsuit$, $n \in \mathbb{N}$ and Ω be a Γ-CW-complex satisfying the conditions (i)–(iii) in Definition 1.5.1. We can attach cells in dimensions $> n$ and assume that Ω is acyclic. Thus, the associated cellular $R\Gamma$-chain complex X of Ω, together with the augmentation, is exact in all degrees. Moreover, for all $j \leq n$, the $R\Gamma$-module is isomorphic to a finite direct sum $\bigoplus_{\sigma \in S_j} R[\Gamma/\Gamma_\sigma]$, where Γ_σ is the stabiliser of a cell σ (running over all orbits in Ω of a given dimension) lying in B_{n-j}. We can thus apply Theorem 1.5.2 to conclude. □

Note that properties can satisfy the bootstrapping theorem property without even being associated to a property for chain complexes. This will be very handy when dealing with geometric cheap rebuilding (see Theorem 2.1.8).

Remark 1.5.4. The construction in the proof of Theorem 1.5.2 is an algebraic version of a topological 'blow-up' construction by Lück [Lüc00; MS20] and Geoghegan [Geo08], given as follows: Let Ω be a Γ-CW-complex and let classifying spaces $(B\Gamma_\sigma)_\sigma$ for its cell stabilisers be given. Then, there exists a *free* Γ-CW-complex $\widehat{\Omega}$ that is (non-equivariantly) homotopy equivalent to Ω, and, roughly speaking, obtained from Ω by replacing each Γ-orbit of cells $\Gamma/\Gamma_\sigma \times \sigma$ with the free Γ-CW-complex $\operatorname{ind}_{\Gamma_\sigma}^\Gamma \widetilde{B\Gamma_\sigma} \times \sigma$. Thus, if Ω is (non-equivariantly) contractible, we obtain that $\widehat{\Omega}/\Gamma$ is a classifying space for Γ.

1.6 Classical examples: Algebraic finiteness properties

Classical examples of an equivariantly bootstrappable property are given by algebraic finiteness properties of chain complexes. For an introduction, we refer to Brown's book [Bro82a, Chapter VIII]. We work over an arbitrary ring R. Whenever we drop the ring from the notation, it is understood that $R = \mathbb{Z}$.

Definition 1.6.1. Let Γ be a group and $n \in \mathbb{Z}$. The class $\mathsf{FG}_n^\Gamma(R)$ consists of all $R\Gamma$-chain complexes X satisfying for all $j \leq n$ that the $R\Gamma$-module X_j is finitely generated.

Definition 1.6.2. Let $n \in \mathbb{Z}$. A group Γ is of type $\mathsf{FP}_n(R)$ if there exists a projective $R\Gamma$-resolution P of the trivial $R\Gamma$-module R satisfying $P \in \mathsf{FG}_n^\Gamma(R)$.

By definition, a group lies in $\mathsf{FG}_n(R)$ if and only if it is of type $\mathsf{FP}_n(R)$. In particular, it is easy to see that the group of integers $\mathbb{Z}$ and all finite groups lie in $\mathsf{FG}_\infty(R)$.

Lemma 1.6.3. *Let $\diamondsuit$ denote the is-class of all groups. The family $\mathsf{FG}_*^\diamondsuit(R)$ is an equivariantly bootstrappable property of chain complexes over R that additionally satisfies Axiom (B-res).*

Proof. It is straightforward to verify that Axioms (B-deg), (B-susp) and (B-cone) hold. For Axiom (B-ind), let $\Delta \subseteq \Gamma$ be a subgroup and M be a finitely generated $R\Delta$-module, generated by $x_1, \dots, x_k$. Then, $\operatorname{ind}_\Delta^\Gamma M = R\Gamma \otimes_{R\Delta} M$ is generated by $1 \otimes x_1, \dots, 1 \otimes x_k$. In particular, it is finitely generated. Applying this fact degreewise yields Axiom (B-ind). In order to verify Axiom (B-res), note that for a finite index subgroup $\Delta \subseteq \Gamma$, we have $\operatorname{res}_\Delta^\Gamma R\Gamma \cong (R\Delta)^{[\Gamma:\Delta]}$. Thus restrictions of finitely generated modules are again finitely generated. Applying this fact degreewise yields Axiom (B-res). □

Remark 1.6.4. Note that, instead of demanding finite generation in all degrees $\leq n$, we could have only demanded finite generation in degree n and we could have still obtained an equivariantly bootstrappable property satisfying Axiom (B-res). However, the above setup has the advantage that $\mathsf{FG}_{n+1}^\Gamma(R) \subseteq \mathsf{FG}_n^\Gamma(R)$, which fits better with the examples we will develop in this thesis (e.g. algebraic cheap rebuilding, see Chapter 2).

We obtain the Bootstrapping Theorem 1.5.3 for $\mathsf{FG}_*(R)$, which is a classical theorem [Bro87, Proposition 1.1]. We give another example related to cohomological dimension.

Definition 1.6.5. Let Γ be a group and $n \in \mathbb{Z}$. The class $\mathsf{Z}_n^\Gamma(R)$ consists of all $R\Gamma$-chain complexes X satisfying $X_n = 0$.

The class $\mathsf{Z}_*^\diamondsuit$ relates to cohomological dimension in the following way.

Lemma 1.6.6. *Let Γ be a group and $n \in \mathbb{N}$. Then, Γ lies in the associated class of groups $\mathsf{Z}_{n+1}(R)$ if and only if $\operatorname{cd}_R(\Gamma) \leq n$. Here, $\operatorname{cd}_R(\Gamma)$ denotes the* cohomological dimension *of Γ over the ring R.*

Proof. We use the result that $\operatorname{cd}_R(\Gamma) \leq n$ if and only if there exists a projective $R\Gamma$-resolution of R of length at most n [Bro82a, Lemma VIII.2.1]. Thus, if $\operatorname{cd}_R(\Gamma) \leq n$, there exists a projective $R\Gamma$-resolution X of R with $X_{n+1} = 0$, thus $\Gamma \in \mathsf{Z}_{n+1}(R)$. Conversely, if $\Gamma \in \mathsf{Z}_{n+1}(R)$, i.e. there is a projective $R\Gamma$-resolution X of R with $X_{n+1} = 0$, we can set $X_k := 0$ for $k > n + 1$. Thus, $\operatorname{cd}_R(\Gamma) \leq n$. □

Lemma 1.6.7. *Let $\diamondsuit$ denote the is-class of all groups. The family $\mathsf{Z}_*^\diamondsuit$ is an equivariantly bootstrappable property of chain complexes over R that additionally satisfies Axiom (B-res).*

Proof. All the axioms are straightforward to verify. □

We thus obtain the Bootstrapping Theorem 1.5.3 for $\mathsf{Z}_*^\diamondsuit$. This is a weakening of the following classical theorem, which doesn't assume co-compactness of the action.

Theorem 1.6.8 ([Bro82a, Exercise VIII.2.4])**.** *Let $\Gamma \curvearrowright X$ be an action of a group on an acyclic CW-complex. Then,*

$$\operatorname{cd}\Gamma \leq \sup_\sigma \{\operatorname{cd}\Gamma_\sigma + \dim\sigma\},$$

where σ ranges of all cells of X.

1.7 Consequences of the Bootstrapping Theorem

We record immediate consequences of the Bootstrapping Theorem 1.5.3 for a general property B_*. Recall that the finiteness property $\mathsf{FP}_*^\diamondsuit$ was defined in Definition 1.6.2.

Proposition 1.7.1. *Let B_* be a bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R. For all $n \in \mathbb{Z}$, we have:*

(i) (Graph of groups). *Let $\Gamma \in \diamondsuit$ be the fundamental group of a finite (connected) graph of groups. (For an introduction to this subject, see [Ser80, Section I.5].) If all vertex groups lie in B_n and all edge groups lie in B_{n-1}, then $\Gamma \in \mathsf{B}_n$.*

(ii) (Extensions). *Let $\Gamma \in \diamondsuit$ be a group containing a normal subgroup N such that the quotient Γ/N is of type $\mathsf{FP}_n(R)$. (This is the case, e.g., if Γ is of type $\mathsf{FP}_n(R)$ and N is of type $\mathsf{FP}_{n-1}(R)$.) If $N \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$.*

(iii) (Finite extensions). *Let $\Gamma \in \diamondsuit$ and let N be a finite index normal subgroup of Γ. If $N \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$.*

(iv) (Finitely generated infinite abelian groups). *Let $\Gamma \in \diamondsuit$ be a be a finitely generated, infinite abelian group. If $\mathbb{Z} \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$*

Proof. (i) The claim follows by application of the Bootstrapping Theorem 1.5.3 to the Bass-Serre tree [Ser80, Section I.5].

(ii) Write $Q := \Gamma/N$ for the quotient, which by assumption is of type $\mathsf{FP}_n(R)$. Hence, there exists an RQ-resolution P of R such that P_j is finitely generated, free for all $j \leq n$ [Bro82a, Proposition VIII.4.3]. We then consider the $R\Gamma$-resolution given by the restriction $\operatorname{res}_{\Gamma \twoheadrightarrow Q} P$ of R and apply Theorem 1.5.2.

(iii) This follows from Part (ii) because finite groups are of type F_∞, thus in particular of type $\mathsf{FP}_\infty(R)$.

(iv) From Part (ii), we obtain that $\mathbb{Z}^m \in \mathsf{B}_n$ for all $m \in \mathbb{N}_{>0}$. As Γ contains some $\mathbb{Z}^m$ as a finite index normal subgroup, by Part (iii), we obtain that $\Gamma \in \mathsf{B}_n$. □

We spell out consequences of this for some classes of groups. The underlying principle remains the same: From the group $\mathbb{Z}$, we 'bootstrap' our way up using the inheritance properties in Proposition 1.7.1. We will also see the same principle later (see e.g. Theorem 3.3.3 for an application to inner-amenable groups).

In the following examples, let B_* be a bootstrappable property of an is-class $\diamondsuit$ of groups over a ring R and $n \in \mathbb{N}$.

Example 1.7.2 (Polycyclic-by-finite groups)**.** Let $\Gamma \in \diamondsuit$ be an infinite polycyclic-by-finite group. If $\mathbb{Z} \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$.

Proof. The group Γ contains a finite-index normal subgroup Λ that is poly-infinite-cyclic [CD21, Lemma 5.11], i.e., Λ admits a subnormal series whose factors are infinite cyclic. By Proposition 1.7.1 (iii), it suffices to show that $\Lambda \in \mathsf{B}_j$ for all $j \leq n$. Let

$$1 = \Lambda_0 \triangleleft \Lambda_1 \triangleleft \cdots \triangleleft \Lambda_k = \Lambda$$

be a subnormal series of Λ with infinite cyclic quotients. Since $\Lambda_1 \cong \mathbb{Z}$, we have $\Lambda_1 \in \mathsf{B}_j$ for all $j \leq n$. By induction on the length k of the subnormal series, assume that $\Lambda_{k-1} \in \mathsf{B}_j$ for all $j \leq n$. By Proposition 1.7.1 (ii), we obtain that $\Lambda = \Lambda_k \in \mathsf{B}_j$ for all $j \leq n$. □

We can even bootstrap properties to infinite, elementary amenable groups of type FP_∞. Recall that the class of *elementary amenable* groups is the smallest class that contains all finite groups, all abelian groups, and that is closed under taking subgroups, quotients, extensions and directed unions [Day57, Section 4].

Example 1.7.3 (elementary amenable groups)**.** Let $\Gamma \in \diamondsuit$ be an infinite elementary amenable group of type FP_∞. If $\mathbb{Z} \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$.

Proof. We use a characterisation of Kropholler–Martínez–Pérez–Nucinkis [KMN09, Theorem 1.1]: We have either of the following:

(i) Γ is infinite polycyclic-by-finite, or

(ii) Γ contains a normal subgroup N such that
   - N is a strictly ascending HNN-extension $H*_{H,t}$ over a finitely generated, virtually nilpotent group, and
   - Γ/N is a Euclidean crystallographic group.

In the first case, we have $\Gamma \in \mathsf{B}_n$ by Example 1.7.2. In the second case, Euclidean crystallographic groups are virtually $\mathbb{Z}^k$ for some $k \in \mathbb{N}$ [Bie12], thus of type F_∞, thus by Proposition 1.7.1 (ii), it suffices to show that $N \in \mathsf{B}_j$ for all $j \leq n$. The HNN-extension $N = H*_{H,t}$ is strictly ascending, so H must be infinite. The infinite finitely generated virtually nilpotent group H is infinite polycyclic-by-finite and hence lies in B_j for all $j \leq n$ by Example 1.7.2. HNN-extensions are a particular example of fundamental groups of graphs of groups, thus Proposition 1.7.1 (i) yields the claim. ☐

Question 1.7.4. Does the statement of the above Example 1.7.3 generalise to the class of amenable groups?

For the definition of amenability, see Definition 1 in the Introduction. We will obtain a positive answer for this question for the property CWR_* in Theorem 2.8.5.

We can also bootstrap properties of $\mathbb{Z}$ to certain right-angled Artin groups (abbreviated RAAGs). Recall that a finite simplicial graph $\mathcal{G}$ induces a presentation of a RAAG as follows: For every vertex, we have a generator. We add the relation that two generators commute if their corresponding vertices are connected by an edge in $\mathcal{G}$. A graph is called *chordal* if it does not contain cycles of length ≥ 4 as full subgraphs.

Example 1.7.5 (Chordal RAAGs)**.** Let $\mathcal{G}$ be a non-empty connected finite simplicial graph that is chordal. Suppose that the associated right-angled Artin group $A_{\mathcal{G}}$ is contained in $\diamondsuit$. If $\mathbb{Z} \in \mathsf{B}_j$ for all $j \leq n$, then $A_{\mathcal{G}} \in \mathsf{B}_n$.

Proof. It is a classical result [Dir61, Theorem 1] about chordal graphs that either

(i) $\mathcal{G}$ is complete, or

(ii) there exist full proper subgraphs $\mathcal{G}_1, \mathcal{G}_2$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ and the intersection $\mathcal{G}_0 \coloneqq \mathcal{G}_1 \cap \mathcal{G}_2$ is complete.

In the first case, $A_{\mathcal{G}}$ is an infinite, finitely generated, free abelian group, thus in B_n by Proposition 1.7.1 (iv). In the latter case, we have that $A_{\mathcal{G}}$ splits as $A_{\mathcal{G}_1} *_{A_{\mathcal{G}_0}} A_{\mathcal{G}_2}$. In particular, it is the fundamental group of a graph of groups, consisting of two vertex groups and one edge group. By induction, we can assume that $A_{\mathcal{G}_1}, A_{\mathcal{G}_2} \in \mathsf{B}_n$. Finally $A_{\mathcal{G}_0}$ is a free abelian group. Because $\mathcal{G}$ is non-empty and connected, $A_{\mathcal{G}_0}$ must be non-trivial. Thus, $A_{\mathcal{G}_0} \in B_{n-1}$ by Proposition 1.7.1 (iv). Hence, $A_{\mathcal{G}} \in B_n$ by Proposition 1.7.1 (i). □

In Section 3.4, we will bootstrap the property B_1 to general Artin groups. In the following, we investigate a few consequences of the additional axiom (B-res).

Proposition 1.7.6. *Let R be a ring and $\diamondsuit$ be an is-class of groups. Let $\mathsf{B}_*^{\diamondsuit}$ be an equivariantly bootstrappable property of chain complexes over R that additionally satisfies (B-res). For all $n \in \mathbb{Z}$, the following hold:*

(i) (Finite index subgroups). *Let $\Gamma \in \diamondsuit$ and let Δ be a finite index subgroup of Γ. If $\Gamma \in \mathsf{B}_n$, then $\Delta \in \mathsf{B}_n$.*

(ii) (Finite index overgroups). *Let $\Gamma \in \diamondsuit$ and let Δ be a finite index subgroup of Γ. If $\Delta \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$.*

(iii) (Commensurated subgroups). *Let $\Gamma \in \diamondsuit$ and let Λ be a commensurated subgroup of Γ (i.e., for all $\gamma \in \Gamma$, the intersection $\Lambda \cap \gamma^{-1}\Lambda\gamma$ has finite index both in Λ and in $\gamma^{-1}\Lambda\gamma$). Furthermore, let Γ be of type F_n and Λ be of type F_{n-1}. If $\Lambda \in \mathsf{B}_j$ for all $j \leq n$, then $\Gamma \in \mathsf{B}_n$.*

Proof. (i) If Γ lies in B_n, by definition, there exists a projective $R\Gamma$-resolution P of R lying in B_n^{Γ}. By Axiom (B-res), the restriction $\operatorname{res}_{\Delta}^{\Gamma} P$ lies in B_n^{Δ}. It is a projective $R\Delta$-resolution of R, witnessing that $\Delta \in \mathsf{B}_n$.

(ii) Suppose that $\Delta \in \mathsf{B}_j$ for all $j \leq n$. The normal core $\operatorname{Core}_{\Gamma}(\Delta) \coloneqq \bigcap_{\gamma \in \Gamma} \Delta^{\gamma}$ has finite index in Δ. Here, we write $\Delta^{\gamma} \coloneqq \gamma^{-1}\Delta\gamma$. Thus, $\operatorname{Core}_{\Gamma}(\Delta) \in \mathsf{B}_j$ for all $j \leq n$ by part (i). Then, Proposition 1.7.1 (iii) yields that $\Gamma \in \mathsf{B}_n$.

(iii) We consider the *Schlichting completion* G of Γ relative to Λ, i.e., the closure of the image of the translation action $\tau \colon \Gamma \to \operatorname{Sym}(\Gamma/\Lambda)$, where we equip $\operatorname{Sym}(\Gamma/\Lambda)$ with the topology of pointwise convergence. For details on the Schlichting completion, see [SW13, Section 3], where the Schlichting completion is called the *relative profinite completion*. We have that G is a locally compact totally disconnected group and the closure U of $\tau(\Lambda)$ is a compact-open subgroup. A result of Bonn-Sauer yields that $U \cap \tau(\Gamma) = \tau(\Lambda)$ [BS24, Section 2]. Since Γ is of type F_n and Λ is of type F_{n-1}, there is a contractible G-CW complex Ω with compact-open stabilisers such that the n-skeleton is co-compact. Bonn–Sauer say that G has type F_n in this case

[BS24, Definition 3.1, Theorem 1.2]. Every compact-open subgroup K of G is commensurable with U (i.e. $H \cap K$ has finite index in both H and K). Thus, the stabilisers of the Γ-CW-complex $\mathrm{res}_\tau \Omega$ are commensurable with Λ and hence lie in B_j for all $j \leq n$ by Parts (i) and (ii). We can apply the Bootstrapping Theorem 1.5.3 to conclude that $\Gamma \in \mathsf{B}_n$. □

1.8 ℓ^2-invariants

We survey some examples presented in an article with Li–Löh–Moraschini–Sauer. More precisely, we show that vanishing of ℓ^2-homology, of ℓ^2-Betti numbers and of Novikov-Shubin invariants are equivariantly bootstrappable properties. In this thesis, we state the results. The proofs can be found in the article [LLMSU24, Section 4.2]. For background on ℓ^2-invariants, we refer to Lück's book [Lüc02]. We work over the ring $R = \mathbb{Z}$ and denote by $\mathcal{N}\Gamma$ the group von Neumann algebra.

ℓ^2-invisibility

Definition 1.8.1 (ℓ^2-invisibility)**.** Let Γ be a group and $n \in \mathbb{Z}$. The class I_n^Γ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all $j \leq n$ that

$$H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X) = 0.$$

A group Γ thus lies in I_n if and only if $H_j(\Gamma; \mathcal{N}\Gamma) = 0$ for all $j \leq n$. Groups that lie in I_∞ are said to be *ℓ^2-invisible*. For an overview on ℓ^2-invisibility and its relevance for the Zero-in-the-spectrum Conjecture, we refer to Lück's book [Lüc02, Chapter 12]. We point out a few remarkable facts: For a group Γ of type F_∞, we obtain that it lies in I_∞ if and only if $H_j(\Gamma; \ell^2\Gamma) = 0$ for all $j \in \mathbb{Z}$ [Lüc02, Lemma 12.3]. The class I_0 is the class of non-amenable groups [Lüc02, Lemma 12.11 (4)]. There is a product formula: If $\Gamma_1 \in \mathsf{I}_{n_1}$ an $\Gamma_2 \in \mathsf{I}_{n_2}$, then $\Gamma_1 \times \Gamma_2 \in \mathsf{I}_{n_1+n_2+1}$ [Lüc02, Lemma 12.11 (3)].

Proposition 1.8.2 ([LLMSU24, Proposition 4.5])**.** *Let $\diamondsuit$ denote the is-class of all groups. The family $\mathsf{I}_*^\diamondsuit$ is an equivariantly bootstrappable property of chain complexes over $\mathbb{Z}$ that additionally satisfies Axiom (B-res).*

As a consequence, we obtain the Bootstrapping Theorem 1.5.3 for I_*, which is implicit in the work of Sauer–Thumann [ST14, Proof of Theorem 1.1]. This result can be used to provide examples of ℓ^2-invisible groups of type F_∞, which are given by certain local similarity groups [ST14]. We do not know whether there exists an ℓ^2-invisible group of type F.

ℓ^2-acyclicity Similarly, we obtain results for vanishing of ℓ^2-Betti numbers.

Definition 1.8.3 (ℓ^2-acyclicity)**.** Let Γ be a group and $n \in \mathbb{Z}$. The class A_n^Γ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all $j \leq n$ that

$$\dim_{\mathcal{N}\Gamma} H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X) = 0.$$

Here, by $\dim_{\mathcal{N}\Gamma}$, we denote the von Neumann dimension over $\mathcal{N}\Gamma$.

Then, a group Γ lies in A_n if and only if its ℓ^2-Betti numbers $b_j^{(2)}(\Gamma) = 0$ for all $j \leq n$. Groups that lie in A_∞ are called *ℓ^2-acyclic*. For an overview on ℓ^2-acyclicity, we refer to Lück's book [Lüc02, Section 7.1]. We point out some facts: The class A_0 is the class of infinite groups [Lüc02, Theorem 6.54 (8)]. All infinite amenable groups lie in A_∞ [Lüc02, Corollary 6.75]. Clearly, I_n is contained in A_n and this inclusion is strict for each $n \in \mathbb{N}$, as witnessed by infinite amenable groups.

Proposition 1.8.4 ([LLMSU24, Proposition 4.7])**.** *Let $\diamondsuit$ denote the is-class of all groups. The family $\mathsf{A}_*^\diamondsuit$ is an equivariantly bootstrappable property of chain complexes over $\mathbb{Z}$ that additionally satisfies Axiom (B-res).*

Consequently, we obtain the Bootstrapping Theorem 1.5.3 for A_*, which was proved (in a slightly weaker form) by Jo [Jo07, Theorem 3.5].

Novikov-Shubin invariants and capacity The Novikov-Shubin invariants are spectral invariants measuring the difference between ℓ^2-acyclicity and ℓ^2-invisibility. We refer to Lück's book [Lüc02, Chapter 2] for an introduction. Following the setup developed by Lück–Reich–Schick [LRS99], we state the results in terms of the *capacity*, which is reciprocal to the Novikov-Shubin invariants. Capacity takes values in the extended range of numbers

$$[\![0,\infty]\!] \coloneqq \{0^-\} \sqcup [0,\infty].$$

We extend the usual ordering on $[0,\infty]$ by $0^- < 0$ and the arithmetic operations as expected.

Definition 1.8.5. Let Γ be a group, $n \in \mathbb{Z}$ and $\kappa \in [\![0,\infty]\!]$.

- The class CM_n^Γ consists of all $\mathbb{Z}\Gamma$ chain complexes X satisfying for all $j \leq n$ that the $\mathcal{N}\Gamma$-module $H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X)$ is cofinal-measurable [LRS99, Definition 2.1].
- The class $\mathsf{C}(\leq \kappa)_n^\Gamma$ consists of all $\mathbb{Z}\Gamma$-chain complexes $X \in \mathsf{CM}_n^\Gamma$ satisfying for all $j \leq n$ that
$$c_{\mathcal{N}\Gamma}\big(H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X)\big) \leq \kappa.$$
Here, $c_{\mathcal{N}\Gamma}$ is the capacity of $\mathcal{N}\Gamma$-modules [LRS99, Section 2], i.e. the reciprocal of the Novikov-Shubin invariant.
- Similarly, we define the class $\mathsf{C}(< \kappa)_n^\Gamma$ by replacing the inequality with a strict equality.

All cofinal-measurable modules have von Neumann dimension 0. In particular, this implies that CM_n is a subclass of A_n. Note that we have $\mathsf{I}_n^\Gamma = \mathsf{C}(\leq 0^-)_n^\Gamma$.

Proposition 1.8.6 ([LLMSU24, Proposition 4.9])**.** *Let $\diamondsuit$ denote the is-class of all groups. The families $\mathsf{CM}_*^{\diamondsuit}, C(\leq 0^-)_*^{\diamondsuit}, C(\leq 0)_*^{\diamondsuit}$, and $C(< \infty)_*^{\diamondsuit}$ are equivariantly bootstrappable properties of chain complexes over $\mathbb{Z}$ that additionally satisfy Axiom (B-res).*

Consequently, we obtain the corresponding versions of the Bootstrapping Theorem 1.5.3.

Example 1.8.7. The group of integers $\mathbb{Z}$ lies in A_∞ as it is infinite and amenable, and for the same reason not in I_n for any $n \in \mathbb{N}$. Moreover, we have that $H_j(\mathbb{Z}; \mathcal{N}\mathbb{Z})$ is cofinal-measurable for all $j \in \mathbb{Z}$, and $c_0(\mathbb{Z}) = 1$ and $c_j(\mathbb{Z}) = 0^-$ for $j \geq 1$ [LRS99, Theorem 3.7 (4)].

2 Algebraic cheap rebuilding

We introduce a notion called the *algebraic cheap rebuilding property.* It is heavily inspired by Abért–Bergeron–Frączyk–Gaboriau's (geometric) cheap rebuilding property [ABFG24], which was introduced to show the vanishing of (torsion) homology growth. Roughly, the geometric cheap rebuilding property is satisfied for a residually finite group Γ if for every residual chain Λ_*, we can find an 'efficient' model for $B\Lambda_i$, once the index $[\Gamma : \Lambda_i]$ is large enough. In this context, 'efficient' is quantified in terms of a small number of cells, operator norms of the boundary maps and homotopies involved. In the following, we coin an algebraic version of this definition, by demanding certain bounds on norms and ranks of projective resolutions.

We first recall Abért–Bergeron–Frączyk–Gaboriau's definition of geometric cheap rebuilding (Section 2.1) before stating the relevant definitions (until Section 2.4). We show that algebraic cheap rebuilding implies the vanishing of (torsion) homology growth (Section 2.5). Moreover, we characterise algebraic cheap rebuilding in degree 0 (Section 2.6) and state the example of the group of integers (Section 2.7). Finally, we show that amenable groups satisfy the algebraic weak cheap rebuilding property (Section 2.8).

The following notions, except for Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding, emerged in an article of the author together with Li–Löh–Moraschini–Sauer [LLMSU24, Section 4.4]. I was particularly involved in working out the details why in our setting, it is better to work with the 1-norm instead of the 2-norm (esp. Section 2.7). Moreover, I contributed particularly to the proof that in the top degree, the logarithmic torsion homology growth vanishes in presence of CWR_n (Section 2.5) and to making the topological argument for amenable groups algebraically precise (Section 2.8).

2.1 Geometric cheap rebuilding

The *(geometric) cheap rebuilding property* was introduced by Abért–Bergeron–Frączyk–Gaboriau in order to show vanishing results for Betti numbers and torsion gradients. The majority of their results builds on the fact that this notion is a bootstrappable action property. In the following, we recall the definition of the cheap rebuilding property, following the article by Abért–Bergeron–Frączyk–Gaboriau [ABFG24].

First, we define the notion of a rebuilding of a CW-complex and its quality.

Definition 2.1.1 (rebuilding [ABFG24, Definition 1]). Let $n \in \mathbb{N}$ and let Y be a CW-complex with finite n-skeleton. An *n-rebuilding* of Y is a tuple $(Y, Y', \mathbf{g}, \mathbf{h}, \mathbf{P})$, consisting of the following data:

(i) Y' is a CW-complex with finite n-skeleton,

(ii) $\mathbf{g}\colon Y^{(n)} \to Y'^{(n)}$ and $\mathbf{h}\colon Y'^{(n)} \to Y^{(n)}$ are cellular maps that are homotopy inverse to each other up to dimension $n-1$, i.e., $\mathbf{h} \circ \mathbf{g}_{\upharpoonright Y^{(n-1)}} \simeq \mathrm{id}_{\upharpoonright Y^{(n-1)}}$ within $Y^{(n)}$ and $\mathbf{g} \circ \mathbf{h}_{\upharpoonright Y'^{(n-1)}} \simeq \mathrm{id}_{\upharpoonright Y'^{(n-1)}}$ within $Y'^{(n)}$, and

(iii) a cellular homotopy $\mathbf{P}\colon [0,1] \times Y^{(n-1)} \to Y^{(n)}$ between the identity and $\mathbf{h} \circ \mathbf{g}$, i.e., $\mathbf{P}(0, \cdot) = \mathrm{id}_{\upharpoonright Y^{(n-1)}}$ and $\mathbf{P}(1, \cdot) = \mathbf{h} \circ \mathbf{g}_{\upharpoonright Y^{(n-1)}}$.

We often write (Y, Y'), leaving the maps implicit.

Definition 2.1.2 (quality of a rebuilding, [ABFG24, Definition 2]). Given real numbers $T \geq 1, \kappa \geq 1$, we say that an n-rebuilding $(Y, Y', \mathbf{g}, \mathbf{h}, \mathbf{P})$ is of *quality* (T, κ) if we have for all $j \leq n$

$$|X'^{(j)}| \leq \kappa T^{-1} |X^{(j)}| \qquad \text{(cells bound)}$$

$$\max\bigl\{\log \|g_j\|, \log \|h_j\|, \log \|\rho_{j-1}\|, \log \|\partial'_j\|\bigr\} \leq \kappa(1 + \log T) \qquad \text{(norms bound)}$$

where $|\cdot|$ denotes the number of cells and the norms $\|\cdot\|$ are the operator norms associated to the canonical ℓ^2-norms on the cellular chain complexes given by the basis of open cells. Moreover, ∂' is the cellular boundary map on Y', and g and h are the chain maps respectively associated to $\mathbf{g}$ and $\mathbf{h}$, and $\rho\colon C_\bullet(Y) \to C_{\bullet+1}(Y)$ is the chain homotopy induced by $\mathbf{P}$ in the cellular chain complexes:

$$\begin{array}{ccccccc}
C_n(Y) & \xrightarrow{\partial_n} & \cdots & \quad & \cdots \longrightarrow C_1(Y) & \xrightarrow{\partial_1} & C_0(Y) \\
g_n \downarrow \uparrow h_n & \overset{\rho_{n-1}}{\swarrow} & & & \overset{\rho_1}{\swarrow} \quad g_1 \downarrow \uparrow h_1 & \overset{\rho_0}{\swarrow} & g_0 \downarrow \uparrow h_0 \\
C_n(Y') & \xrightarrow{\partial'_n} & \cdots & \quad & \cdots \longrightarrow C_1(Y') & \xrightarrow{\partial'_1} & C_0(Y').
\end{array}$$

In the words of Abért–Bergeron–Frączyk–Gaboriau, the above definition captures “an intrinsic tension between ‘having few cells’ and ‘maintaining tame norms’ ” [ABFG24, p. 7].

Abért–Bergeron–Frączyk–Gaboriau eventually apply this notion to classifying spaces of subgroups. In order to obtain a notion for groups, we demand a uniform rebuilding along subgroups stemming from Farber sequences. We first recall the definition of a Farber sequence.

Definition 2.1.3 ([ABFG24, Section 10.1]). Let Γ be a countable group and let $\mathrm{Sub}^{\mathrm{fi}}_\Gamma$ denote the space of finite index subgroups of Γ with the topology induced from the topology of pointwise convergence on $\{0,1\}^\Gamma$. For $\gamma \in \Gamma$, we consider the following function.

$$\mathrm{fx}_{\Gamma,\gamma} : \mathrm{Sub}^{\mathrm{fi}}_\Gamma \to [0,1], \quad \Gamma' \mapsto \frac{|\{g\Gamma' \mid \gamma g \Gamma' = g\Gamma'\}|}{[\Gamma : \Gamma']}$$

Definition 2.1.4 (Farber sequence [ABFG24, Definition 10.1])**.** A sequence $(\Gamma_i)_{i\in\mathbb{N}}$ of subgroups of Γ is a *Farber sequence* if it consists of finite index subgroups and for every $\gamma \in \Gamma \setminus \{e\}$, we have $\lim_{i\to\infty} \mathrm{fx}_{\Gamma,\gamma}(\Gamma_i) = 0$.

Note that Farber sequences in Γ exist if and only if Γ is residually finite. The most common example of Farber sequences are *residual chains*, i.e., nested sequences of finite index normal subgroups whose intersection is trivial.

Definition 2.1.5 (Farber neighbourhood [ABFG24, Definition 10.2])**.** Let Γ be a residually finite group. An open subset $U \subseteq \mathrm{Sub}_\Gamma^{\mathrm{fi}}$ is a Γ-*Farber neighbourhood* if it is invariant by the conjugation action of Γ on $\mathrm{Sub}_\Gamma^{\mathrm{fi}}$ and every Farber sequence in $\mathrm{Sub}_\Gamma^{\mathrm{fi}}$ eventually belongs to U.

Finally, we can define the cheap n-rebuilding property.

Definition 2.1.6 ([ABFG24, Definition 10.5])**.** Let Γ be a countable group and $n \in \mathbb{N}$. Then, Γ has the *(geometric) cheap n-rebuilding property* if it is residually finite and there is a $K(\Gamma,1)$-space X with finite n-skeleton and a constant $\kappa_X \geq 1$ such that the following holds: For every real number $T \geq 1$, there exists a Farber neighbourhood $U = U(X,T) \subseteq \mathrm{Sub}_\Gamma^{\mathrm{fi}}$ such that for every finite covering $Y \to X$ with $\pi_1(Y) \in U$, there is an n-rebuilding (Y,Y') of quality (T,κ_X).

We denote by GCR_n the class of residually finite groups satisfying the geometric cheap n-rebuilding property.

The main motivation for this definition are its consequences on vanishing of Betti number and torsion gradients.

Theorem 2.1.7 ([ABFG24, Theorem 10.20])**.** *Let $n \in \mathbb{N}$ and Γ be a residually finite group that has the geometric cheap n-rebuilding property. Then, for every Farber sequence $(\Lambda_i)_{i\in\mathbb{N}}$, coefficient field $\mathbb{F}$ and $0 \leq j \leq n$, we have*

$$\widehat{b}_j(\Gamma,\Lambda_*;\mathbb{F}) = 0$$

and for $0 \leq j \leq n-1$, we have

$$\widehat{t}_j(\Gamma,\Lambda_*) = 0.$$

If, additionally, Γ is of type F_{n+1}, we have

$$\widehat{t}_n(\Gamma,\Lambda_*) = 0.$$

Abért–Bergeron–Frączyk–Gaboriau state the following version of a bootstrapping theorem.

Theorem 2.1.8 ([ABFG24, Theorem 10.9])**.** *The class $\mathsf{GCR} = (\mathsf{GCR}_n)_{n\in\mathbb{N}}$ has the bootstrappable action property (see Definition 1.5.1).*

Together with the fact that $\mathbb{Z} \in \mathsf{GCR}_\infty$ [ABFG24, Lemma 10.10], Abért–Bergeron–Frączyk–Gaboriau establish vanishing (torsion) homology groups for certain examples. We mention two of them.

Theorem 2.1.9 ([ABFG24, Theorem A and C]). *Let $d \geq 3$. Then $\Gamma \coloneqq \mathrm{SL}_d(\mathbb{Z})$ has the geometric cheap $(d-2)$-rebuilding property.*

In particular, let $(\Lambda_i)_{i\in\mathbb{N}}$ be a sequence of pairwise distinct finite index subgroups of Γ. Then, using a strengthening of Theorem 2.1.7 in this case, for every field $\mathbb{F}$ and $j \leq d-2$, we have

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0 \quad \textit{and} \quad \widehat{t}_j(\Gamma, \Lambda_*) = 0.$$

Theorem 2.1.10 ([ABFG24, Theorem D]). *Let S be an orientable surface of genus $g > 0$ with b boundary components. Then, $\Gamma \coloneqq \mathcal{MCG}(S)$, i.e. the mapping class group of S, has the geometric cheap $\alpha(g,b)$-rebuilding property, where*

$$\alpha(g,b) = \begin{cases} 2g - -2 & \textit{if } b = 0 \\ 2g - 3 + b & \textit{if } b > 0 \end{cases}.$$

In particular, Theorem 2.1.7 yields that for all Farber chains $(\Lambda_i)_{i\in\mathbb{N}}$ in Γ, for every field $\mathbb{F}$ and $j \leq \alpha(g,b)$, we have

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0 \quad \textit{and} \quad \widehat{t}_j(\Gamma, \Lambda_*) = 0.$$

2.2 Quantitative homotopy retracts of chain complexes

In order to work towards the definition of algebraic cheap rebuilding, we aim to quantify the size of homotopy retracts. We will work over the ring $\mathbb{Z}$. A free $\mathbb{Z}$-module endowed with a $\mathbb{Z}$-basis is called *based free*. The basis of such a $\mathbb{Z}$-module induces an ℓ^1-norm. In the following, norms $|\cdot|_1$ of elements will always refer to this ℓ^1-norm, unless otherwise stated. For a linear map $f\colon M \to L$ between based free $\mathbb{Z}$-modules, we denote by $\|f\|$ the *operator norm* with respect to the ℓ^1-norms on M and L. A $\mathbb{Z}$-chain complex is *based free* if every chain module is based free.

A *homotopy retract* of a chain complex X is a tuple (X, X', ξ, ξ', Ξ) consisting of chain maps $\xi\colon X \to X'$, $\xi'\colon X' \to X$ and a chain homotopy $\Xi\colon \mathrm{id}_X \simeq \xi' \circ \xi$. When leaving the maps implicit, we just write (X, X') for a homotopy retract.

Definition 2.2.1 (Rebuildings). Let $n \in \mathbb{Z}$ and X, X' be based free $\mathbb{Z}$-chain complexes such that X_j and X'_j are finitely generated for all $j \leq n$. Let $T, \kappa \in \mathbb{R}_{\geq 1}$. We say that a homotopy retract (X, X', ξ, ξ', Ξ) is

- an *n-domination* of X of quality (T, κ) if for all $j \leq n$

$$\mathrm{rk}_{\mathbb{Z}}(X'_j) \leq \kappa T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j);$$

- a *weak n-rebuilding* of X of quality (T, κ) if for all $j \leq n$

$$\mathrm{rk}_{\mathbb{Z}}(X'_j) \leq \kappa T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j);$$
$$\max\bigl\{ \|\partial_j^{X'}\|, \|\xi_j\| \bigr\} \leq \exp(\kappa) T^{\kappa};$$

- an *n-rebuilding* of X of quality (T, κ) if for all $j \leq n$

$$\operatorname{rk}_{\mathbb{Z}}(X'_j) \leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j);$$
$$\max\{\|\partial_j^{X'}\|, \|\xi_j\|, \|\xi'_j\|, \|\Xi_j\|\} \leq \exp(\kappa) T^{\kappa};$$

By definition, an n-rebuilding is in particular a weak n-rebuilding of the same quality. A weak n-rebuilding is also an n-domination of the same quality. For $T' \leq T$, an n-domination of quality (T, κ) is in particular of quality (T', κ). However, the analogous statements for weak n-rebuildings and n-rebuildings do not hold.

Remark 2.2.2. The above definition of n-rebuildings for chain complexes is an algebraic version of Abért–Bergeron–Frączyk–Gaboriau's geometric definition of n-rebuildings for CW-complexes (see Definition 2.1.1 and Definition 2.1.2). We point out the main differences:

- We require a homotopy retract in all degrees, not just a truncated homotopy equivalence.
- We work with the ℓ^1-norm on based free $\mathbb{Z}$-modules, not with the ℓ^2-norm. This simplifies some calculations. We could also work with the ℓ^2-norm and all results, especially on vanishing of homology gradients (Proposition 2.2.3), would still hold, with potentially different constants.
- We ask for control on the norm of the homotopy Ξ_j in degrees $j \leq n$ (as opposed to degrees $j \leq n-1$). This is needed in the proof of Proposition 2.3.2.

There is no obvious implication between the algebraic and geometric notions. Still, we consider the geometric one, introduced by Abért–Bergeron–Frączyk–Gaboriau, to be more general. The definitions above are mainly chosen so to cover the main example of the circle (Example 2.2.8) and to induce equivariantly bootstrappable properties (see Theorem 2.4.6).

The definition is designed in such a way such that the following estimate on (torsion) homology holds.

Proposition 2.2.3.

(i) Let (X, X', ξ, ξ', Ξ) be an n-domination of quality (T, κ). For $j \leq n$, we have

$$\operatorname{rk}_{\mathbb{Z}} H_j(X) \leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j).$$

(ii) Let (X, X', ξ, ξ', Ξ) be a weak n-rebuilding of quality (T, κ). For $j \leq n-1$, we have

$$\log \operatorname{tors} H_j(X) \leq \kappa^2 T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j)(1 + \log T)$$

Proof. In both cases, since (X, X', ξ, ξ', Ξ) is a homotopy retract, we have $\mathrm{id}_X \simeq \xi' \circ \xi$. In particular, ξ induces an inclusion $H_*(X) \hookrightarrow H_*(X')$.

(i) We thus have

$$\begin{aligned}\operatorname{rk}_{\mathbb{Z}} H_j(X) &\leq \operatorname{rk}_{\mathbb{Z}} H_j(X') \\ &\leq \operatorname{rk}_{\mathbb{Z}}(X'_j) && \text{(trivial bound)} \\ &\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j). && \text{(n-domination)}\end{aligned}$$

(ii) For $j \leq n-1$, using Gabber's estimate (Lemma 7), we obtain

$$\begin{aligned}\log \operatorname{tors} H_j(X) &\leq \log \operatorname{tors} H_j(X') \\ &\leq \operatorname{rk}_{\mathbb{Z}}(X'_j) \log_+ \|\partial^{X'}_{j+1}\| && \text{(Gabber's estimate)} \\ &\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j)\kappa(1+\log T). && \text{(weak $n-$rebuilding)} \ \square\end{aligned}$$

Example 2.2.4. Suppose that X is a $\mathbb{Z}$-chain complex and that there exists a uniform $\kappa \in \mathbb{R}_{\geq 1}$ such that for all $T \in \mathbb{R}_{\geq 1}$, the complex X admits a weak n-rebuilding of quality (T, κ). Then, Proposition 2.2.3 shows that $H_j(X) = 0$ for $j \leq n$ (as $\kappa T^{-1} \to 0$ for $T \to \infty$) and $\log \operatorname{tors} H_j(X) = 0$ for $j \leq n-1$ (as $T^{-1} \log T \to 0$ for $T \to \infty$) . Later, we will make use of an asymptotic version of this argument (Theorem 2.5.1).

Remark 2.2.5. It is worth pointing out that while we obtain bounds in the rank of homology up to degree n, we only obtain bounds for torsion homology up to degree $n-1$. This is because Gabber's estimate requires control on the norm of the differential $\partial^{X'}_{j+1}$ in degree $j+1$. However, by imposing additional finiteness requirements, it is often possible to obtain control on torsion homology in degree n as well (see e.g. Theorem 2.5.1 (iii)).

Remark 2.2.6. The additional conditions in the definition of an n-rebuilding (Definition 2.2.1) ensure controlled norms when taking mapping cones (see Proposition 2.3.2). This will be the essential point why rebuildings define an equivariantly bootstrappable property while weak rebuildings do not seem so (see Remark 2.4.8).

The following are algebraic versions of (topological) homotopy retracts of circles [ABFG24, Proof of Lemma 10.10].

Example 2.2.7. For $d \in \mathbb{N}$, let $S^{[0,d]}$ be the chain complex with chain modules

$$S^{[0,d]}_j = \begin{cases} \bigoplus_{i=0}^{d-1} \mathbb{Z}\langle v_i \rangle & \text{for } j = 0; \\ \bigoplus_{i=0}^{d-1} \mathbb{Z}\langle e_i \rangle & \text{for } j = 1; \\ 0 & \text{otherwise;} \end{cases}$$

and differential $\partial_1(e_i) = v_{i+1} - v_i$ for $i \in \{0, \ldots, d-1\}$ considered modulo d.

We construct an n-rebuilding $(S^{[0,d]}, S^{[0,1]})$ of quality $(d, 1)$. There is a homotopy retract $(S^{[0,d]}, S^{[0,1]}, \xi, \xi', \Xi)$, where the chain maps $\xi\colon S^{[0,d]} \to S^{[0,1]}$ and $\xi'\colon S^{[0,1]} \to$

$S^{[0,d]}$ are given by

$$\begin{aligned}
\xi_0(v_i) &= v_0 \quad \text{for all } i;\\
\xi_1(e_i) &= \begin{cases} e_0 & \text{if } i = 0;\\ 0 & \text{otherwise;}\end{cases}\\
\xi'_0(v_0) &= v_0;\\
\xi'_1(e_0) &= \sum_{i=0}^{d-1} e_i;
\end{aligned}$$

and the chain homotopy $\Xi\colon \mathrm{id}_{S^{[0,d]}} \simeq \xi' \circ \xi$ is given by

$$\Xi_0(v_i) = \begin{cases} 0 & \text{if } i = 0;\\ -e_i - \cdots - e_{d-1} & \text{if } i \in \{1, \dots, d-1\}.\end{cases}$$

For all $j \in \mathbb{Z}$, we have

$$\mathrm{rk}_{\mathbb{Z}}(S^{[0,1]}_j) \le d^{-1}\,\mathrm{rk}_{\mathbb{Z}}(S^{[0,d]}_j);$$

$$\|\partial^{S^{[0,1]}}_j\| \le 0; \quad \|\xi_j\| \le 1; \quad \|\xi'_j\| \le d; \quad \|\Xi_j\| \le d.$$

Hence, for all $n \in \mathbb{Z}$, the homotopy retract $(S^{[0,d]}, S^{[0,1]})$ is an n-rebuilding of quality $(d, 1)$. For $T \le d$, the pair $(S^{[0,d]}, S^{[0,1]})$ provides a weak n-rebuilding of quality $(T, 1)$, but in general not an n-rebuilding of quality $(T, 1)$.

Example 2.2.8. For $d \in \mathbb{N}$ and $T \in \mathbb{R}_{\ge 1}$ with $T \le d$, we construct an n-rebuilding of $S^{[0,d]}$ of quality $(T, 2)$. Choose a sequence of integers $0 = a_0 < a_1 < \cdots < a_m = d$ with

$$T/2 \le a_{k+1} - a_k \le T$$

for all $k \in \{0, \dots, m-1\}$. There is a homotopy retract $(S^{[0,d]}, S^{[0,m]}, \xi, \xi', \Xi)$, where the chain maps $\xi\colon S^{[0,d]} \to S^{[0,m]}$ and $\xi'\colon S^{[0,m]} \to S^{[0,d]}$ are given by

$$\begin{aligned}
\xi_0(v_i) &= v_k \quad \text{if } i \in \{a_{k-1}+1, \dots, a_k\};\\
\xi_1(e_i) &= \begin{cases} e_k & \text{if } i = a_k \text{ for some } k;\\ 0 & \text{otherwise;}\end{cases}\\
\xi'_0(v_i) &= v_{a_i};\\
\xi'_1(e_i) &= e_{a_i} + \cdots + e_{a_{i+1}-1};
\end{aligned}$$

and the chain homotopy $\Xi\colon \mathrm{id}_{S^{[0,d]}} \simeq \xi' \circ \xi$ is given by

$$\Xi_0(v_i) = \begin{cases} 0 & \text{if } i = a_k \text{ for some } k;\\ -e_i - \cdots - e_{a_k - 1} & \text{if } i \in \{a_{k-1}+1, \dots, a_k - 1\}.\end{cases}$$

For all $j \in \mathbb{Z}$, we have

$$\mathrm{rk}_{\mathbb{Z}}(S^{[0,m]}_j) \le 2T^{-1}\,\mathrm{rk}_{\mathbb{Z}}(S^{[0,d]}_j);$$

$$\|\partial_j^{S^{[0,m]}}\| \leq 2; \quad \|\xi_j\| \leq 1; \quad \|\xi'_j\| \leq T; \quad \|\Xi_j\| \leq T.$$

Hence, for all $n \in \mathbb{Z}$, the homotopy retract $(S^{[0,d]}, S^{[0,m]})$ is an n-rebuilding of quality $(T, 2)$.

2.3 Rebuilding mapping cones

The main goal of this section is the rebuilding of mapping cones of chain complexes. In the case of direct sums, we obtain slightly better bounds for the quality of the rebuildings involved. We spell out this case.

Lemma 2.3.1. *Let* $\mathbf{X} = (X, X', \xi, \xi', \Xi)$ *and* $\mathbf{Y} = (Y, Y', \upsilon, \upsilon', \Upsilon)$ *be homotopy retracts of* $\mathbb{Z}$*-chain complexes. Define*

$$\mathbf{X} \oplus \mathbf{Y} := (X \oplus Y, X' \oplus Y', \xi \oplus \upsilon, \xi' \oplus \upsilon', \Xi \oplus \Upsilon).$$

Let $n \in \mathbb{N}$ *and let* $T, \kappa_{\mathbf{X}}, \kappa_{\mathbf{Y}} \in \mathbb{R}_{\geq 1}$. *Set* $\kappa := \max\{\kappa_{\mathbf{X}}, \kappa_{\mathbf{Y}}\}$.

(i) The tuple $\mathbf{X} \oplus \mathbf{Y}$ *is a homotopy retract;*

(ii) If $\mathbf{X}$ *is an* n*-domination of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-domination of quality* $(T, \kappa_{\mathbf{Y}})$*, then,* $\mathbf{X} \oplus \mathbf{Y}$ *is an* n*-domination of quality* (T, κ).

(iii) If $\mathbf{X}$ *is a weak* n*-rebuilding of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is a weak* n*-rebuilding of quality* $(T, \kappa_{\mathbf{Y}})$*, then,* $\mathbf{X} \oplus \mathbf{Y}$ *is a weak* n*-rebuilding of quality* (T, κ).

(iv) If $\mathbf{X}$ *is an* n*-rebuilding of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-rebuilding of quality* $(T, \kappa_{\mathbf{Y}})$*, then,* $\mathbf{X} \oplus \mathbf{Y}$ *is an* n*-rebuilding of quality* (T, κ).

Proof. (i) This is a straightforward calculation.

(ii) Let $\mathbf{X}, \mathbf{Y}$ be n-dominations of the claimed qualities. For all $j \leq n$, we have

$$\begin{aligned} \operatorname{rk}_{\mathbb{Z}}(X'_j \oplus Y'_j) &= \operatorname{rk}_{\mathbb{Z}}(X'_j) + \operatorname{rk}_{\mathbb{Z}}(Y'_j) \\ &\leq \kappa_{\mathbf{X}} T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j) + \kappa_{\mathbf{Y}} T^{-1} \operatorname{rk}_{\mathbb{Z}}(Y_j) \\ &\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j \oplus Y_j). \end{aligned}$$

(iii) Suppose that $\mathbf{X}, \mathbf{Y}$ are weak n-rebuildings. In addition to the results of Part (ii), we have for all $j \leq n$ (because we're dealing with the ℓ^1-norms):

$$\begin{aligned} \|\partial_j^{X'} \oplus \partial_j^{Y'}\| &\leq \max\bigl\{\|\partial_j^{X'}\|, \|\partial_j^{Y'}\|\bigr\} \\ &\leq \max\bigl\{\exp(\kappa_{\mathbf{X}}) T^{\kappa_{\mathbf{X}}}, \exp(\kappa_{\mathbf{Y}}) T^{\kappa_{\mathbf{Y}}}\bigr\} \\ &\leq \exp(\kappa) T^{\kappa}. \end{aligned}$$

We also have $\|\xi_j \oplus \upsilon_j\| \leq \exp(\kappa) T^{\kappa}$.

(iv) Similar to the computations in Part (iii), we obtain

$$\max\{\|\xi' \oplus \upsilon'\|, \|\Xi_j \oplus \Upsilon_j\|\} \leq \exp(\kappa)T^{\kappa}$$

for all $j \leq n$. □

Proposition 2.3.2 (Rebuilding of mapping cones)**.** *Let* $\mathbf{X} = (X, X', \xi, \xi', \Xi)$ *and* $\mathbf{Y} = (Y, Y', \upsilon, \upsilon', \Upsilon)$ *be homotopy retracts of* $\mathbb{Z}$*-chain complexes. Let* $f\colon X \to Y$ *be a chain map and define the chain map* $f' \coloneqq \upsilon \circ f \circ \xi'\colon X' \to Y'$*. Consider the tuple*

$$\mathbf{C}(f) \coloneqq \big(\mathrm{Cone}(f), \mathrm{Cone}(f'), (\xi, \upsilon; -\upsilon \circ f \circ \Xi), (\xi', \upsilon'; \Upsilon \circ f \circ \xi'), L\big),$$

where $L\colon \mathrm{Cone}(f)_* \to \mathrm{Cone}(f)_{*+1}$ *is defined as*

$$L_j(x, y) \coloneqq \big(-\Xi_{j-1}(x), \Upsilon_j(y) + \Upsilon_j \circ f_j \circ \Xi_{j-1}(x)\big).$$

Let $n \in \mathbb{N}$*, let* $T, \kappa^{\mathbf{X}}, \kappa^{\mathbf{Y}} \in \mathbb{R}_{\geq 1}$*, and set*

$$\kappa \coloneqq \kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log 3 + \max\{\log_+ \|f_j\| \mid j \leq n\}.$$

Then the following hold:

(i) The tuple $\mathbf{C}(f)$ *is a homotopy retract;*

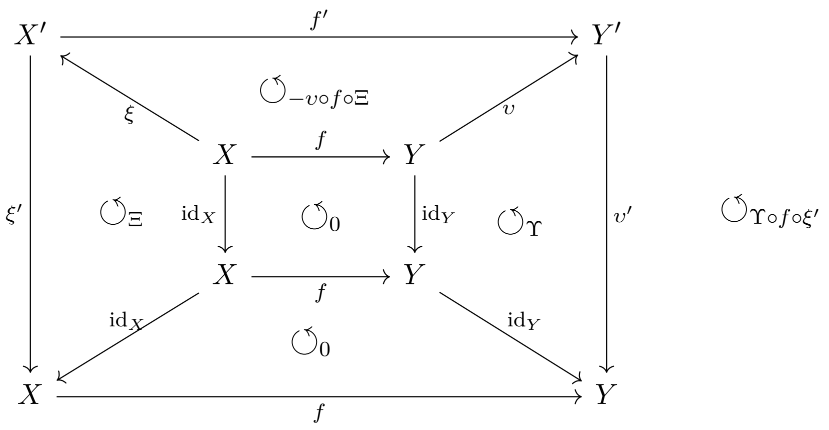
(ii) If $\mathbf{X}$ *is a* $(n-1)$*-domination of quality* $(T, \kappa^{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-domination of quality* $(T, \kappa^{\mathbf{Y}})$*, then* $\mathbf{C}(f)$ *is an* n*-domination of quality* (T, κ)*;*

(iii) If $\mathbf{X}$ *is a* $(n-1)$*-rebuilding of quality* $(T, \kappa^{\mathbf{X}})$ *and* $\mathbf{Y}$ *is a weak* n*-rebuilding of quality* $(T, \kappa^{\mathbf{Y}})$*, then* $\mathbf{C}(f)$ *is a weak* n*-rebuilding of quality* (T, κ)*;*

(iv) If $\mathbf{X}$ *is a* $(n-1)$*-rebuilding of quality* $(T, \kappa^{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-rebuilding of quality* $(T, \kappa^{\mathbf{Y}})$*, then* $\mathbf{C}(f)$ *is an* n*-rebuilding of quality* (T, κ)*.*

We point out that in Part (iii), $\mathbf{X}$ is assumed to be an $(n-1)$-rebuilding, and not only a weak $(n-1)$-rebuilding. This slight modification will be responsible for why only a weaker bootstrapping theorem for CWR holds (see Theorem 2.4.10).

Proof. (i) The following cube is homotopy commutative:



(2.3.1)

with $K_* := -\Upsilon_{*+1} \circ f_{*+1} \circ \Xi_* \colon X_* \to Y_{*+2}$. Indeed, one easily checks that the upper and outer squares are homotopy commutative. The other four squares are clearly homotopy commutative. We check that K is a filler of the cube:

$$\begin{aligned}
&0 - \Upsilon_j \circ f_j \circ \xi'_j \circ \xi_j + \Upsilon_j \circ f_j - f_{j+1} \circ \Xi_j + 0 + \upsilon'_{j+1} \circ \upsilon_{j+1} \circ f_{j+1} \circ \Xi_j \\
&= \Upsilon_j \circ f_j \circ (\mathrm{id}_{X_j} - \xi'_j \circ \xi_j) - (\mathrm{id}_{Y_{j+1}} - \upsilon'_{j+1} \circ \upsilon_{j+1}) \circ f_{j+1} \circ \Xi_j \\
&= \Upsilon_j \circ f_j \circ (\partial^X_{j+1} \circ \Xi_j + \Xi_{j-1} \circ \partial^X_j) - (\partial^Y_{j+2} \circ \Upsilon_{j+1} + \Upsilon_j \circ \partial^Y_{j+1}) \circ f_{j+1} \circ \Xi_j \\
&= \Upsilon_j \circ f_j \circ \Xi_{j-1} \circ \partial^X_j - \partial^Y_{j+2} \circ \Upsilon_{j+1} \circ f_{j+1} \circ \Xi_j \\
&= \partial^Y_{j+2} \circ K_j - K_{j-1} \circ \partial^X_j .
\end{aligned}$$

By Lemma 1.1.10, the homotopy commutative cube (2.3.1) induces a homotopy commutative square of mapping cones

$$\begin{array}{ccc}
\mathrm{Cone}(f) & \xrightarrow{(\xi,\upsilon;-\upsilon\circ f\circ\Xi)} & \mathrm{Cone}(f') \\
{\scriptstyle (\mathrm{id}_X,\mathrm{id}_Y;0)}\downarrow & \circlearrowleft_L & \downarrow{\scriptstyle (\xi',\upsilon';\Upsilon\circ f\circ\xi')} \\
\mathrm{Cone}(f) & \xrightarrow[(\mathrm{id}_X,\mathrm{id}_Y;0)]{} & \mathrm{Cone}(f)
\end{array}$$

where

$$L_j(x,y) = \big(-\Xi_{j-1}(x), \Upsilon_j(y) - K_{j-1}(x)\big).$$

Hence $\mathbf{C}(f)$ is a homotopy retract of chain complexes.

(ii) Suppose that $\mathbf{X}$ is an $(n-1)$-domination of quality $(T, \kappa^{\mathbf{X}})$ and $\mathbf{Y}$ is an n-domination of quality $(T, \kappa^{\mathbf{Y}})$. Then we have for all $j \le n$

$$\begin{aligned}
\mathrm{rk}_{\mathbb{Z}}(\mathrm{Cone}(f')_j) &= \mathrm{rk}_{\mathbb{Z}}(X'_{j-1}) + \mathrm{rk}_{\mathbb{Z}}(Y'_j) \\
&\le \kappa^{\mathbf{X}} T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_{j-1}) + \kappa^{\mathbf{Y}} T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(Y_j) \\
&\le \kappa T^{-1} \big(\mathrm{rk}_{\mathbb{Z}}(X_{j-1}) + \mathrm{rk}_{\mathbb{Z}}(Y_j)\big) \\
&= \kappa T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(\mathrm{Cone}(f)_j).
\end{aligned}$$

(iii) Suppose that $\mathbf{X}$ is an $(n-1)$-rebuilding of quality $(T, \kappa^{\mathbf{X}})$ and $\mathbf{Y}$ is an n-weak rebuilding of quality $(T, \kappa^{\mathbf{Y}})$. Additionally to Part (ii), we have for all $j \le n$

$$\begin{aligned}
\|\partial_j^{\mathrm{Cone}(f')}\| &\le \|\partial^{X'}_{j-1}\| + \|\partial^{Y'}_j\| + \|\upsilon_{j-1}\| \cdot \|f_{j-1}\| \cdot \|\xi'_{j-1}\| \\
&\le \exp(\kappa^{\mathbf{X}}) T^{\kappa^{\mathbf{X}}} + \exp(\kappa^{\mathbf{Y}}) T^{\kappa^{\mathbf{Y}}} + \\
&\qquad \exp(\kappa^{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \kappa^{\mathbf{X}}) T^{\kappa^{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \kappa^{\mathbf{X}}} \\
&\le 3 \exp(\kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log_+ \|f_{j-1}\|) T^{\kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log_+ \|f_{j-1}\|} \\
&\le \exp(\kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \log 3) T^{\kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \log 3} \\
&\le \exp(\kappa) T^{\kappa}.
\end{aligned}$$

Note that for this inequality, we needed to have control on $\|\xi'_{j-1}\|$. We do so if $\mathbf{X}$ is an $(n-1)$-rebuilding. In general, we would *not* have this control if $\mathbf{X}$ were just a weak $(n-1)$-rebuilding. Similarly, we obtain

$$\|(\xi, \upsilon; -\upsilon \circ f \circ \Xi)_j\| \leq \|\xi_{j-1}\| + \|\upsilon_j\| + \|\upsilon_j\| \cdot \|f_j\| \cdot \|\Xi_{j-1}\| \leq \exp(\kappa) T^\kappa.$$

(iv) Suppose that $\mathbf{X}$ is an $(n-1)$-rebuilding of quality $(T, \kappa^{\mathbf{X}})$ and $\mathbf{Y}$ is an n-rebuilding of quality $(T, \kappa^{\mathbf{Y}})$. Additionally to Part (iii), we have for all $j \leq n$

$$\begin{aligned} \|(\xi, \upsilon; -\upsilon \circ f \circ \Xi)_j\| &\leq \|\xi_{j-1}\| + \|\upsilon_j\| + \|\upsilon_j\| \cdot \|f_j\| \cdot \|\Xi_{j-1}\| \leq \exp(\kappa) T^\kappa; \\ \|(\xi', \upsilon'; \Upsilon \circ f \circ \xi')_j\| &\leq \|\xi'_{j-1}\| + \|\upsilon'_j\| + \|\Upsilon_{j-1}\| \cdot \|f_{j-1}\| \cdot \|\xi'_{j-1}\| \leq \exp(\kappa) T^\kappa; \\ \|L_j\| &\leq \|\Xi_{j-1}\| + \|\Upsilon_j\| + \|\Upsilon_j\| \cdot \|f_j\| \cdot \|\Xi_{j-1}\| \leq \exp(\kappa) T^\kappa; \end{aligned}$$

by similar computations. □

We finish this section with the result that homotopy retracts, dominations and weak rebuildings are closed under compositions. A similar statement (involving a degree shift) is true for rebuildings, though we will not require it here. The following is the algebraic version of [ABFG24, Lemma 6.3].

Lemma 2.3.3. *Let $\mathbf{X} = (X, X', \xi, \xi', \Xi)$ and $\mathbf{Y} = (X', X'', \upsilon, \upsilon', \Upsilon)$ be homotopy retracts of $\mathbb{Z}$-chain complexes. Consider the tuple*

$$\mathbf{Y} \circ \mathbf{X} \coloneqq (X, X'', \upsilon \circ \xi, \xi' \circ \upsilon', \Xi + \xi' \circ \Upsilon \circ \xi).$$

Let $n \in \mathbb{N}$, let $T, S, \kappa^{\mathbf{X}}, \kappa^{\mathbf{Y}} \in \mathbb{R}_{\geq 1}$, and set $\kappa \coloneqq 2\kappa^{\mathbf{Y}}\kappa^{\mathbf{X}}$. Then the following hold:

(i) The tuple $\mathbf{Y} \circ \mathbf{X}$ is a homotopy retract;

(ii) If $\mathbf{X}$ is an n-domination of quality $(T, \kappa^{\mathbf{X}})$ and $\mathbf{Y}$ is an n-domination of quality $(S, \kappa^{\mathbf{Y}})$, then $\mathbf{Y} \circ \mathbf{X}$ is an n-domination of quality (ST, κ);

(iii) If $\mathbf{X}$ is a weak n-rebuilding of quality $(T, \kappa^{\mathbf{X}})$ and $\mathbf{Y}$ is a weak n-rebuilding of quality $(S, \kappa^{\mathbf{Y}})$, then $\mathbf{Y} \circ \mathbf{X}$ is a weak n-rebuilding of quality (ST, κ).

Proof.

(i) A straight-forward calculation shows that $\Xi + \xi' \circ \Upsilon \circ \xi$ indeed provides a chain homotopy between the chain maps id_X and $\xi' \circ \upsilon' \circ \upsilon \circ \xi$.

(ii) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are n-dominations of quality $(T, \kappa^{\mathbf{X}})$ and $(T, \kappa^{\mathbf{Y}})$, respectively. For all $j \leq n$, we have

$$\mathrm{rk}_{\mathbb{Z}}(X''_j) \leq \kappa^{\mathbf{Y}} S^{-1} \mathrm{rk}_{\mathbb{Z}}(X'_j) \leq \kappa^{\mathbf{Y}} S^{-1} \kappa^{\mathbf{X}} T^{-1} \mathrm{rk}_{\mathbb{Z}}(X_j) \leq \kappa (ST)^{-1} \mathrm{rk}_{\mathbb{Z}}(X_j).$$

(iii) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are weak n-rebuildings of quality $(T, \kappa^{\mathbf{X}})$ and $(T, \kappa^{\mathbf{Y}})$, respectively. Additionally to Part (ii), we have for all $j \leq n$

$$\begin{aligned} \|\partial_j^{X''}\| &\leq \exp(\kappa^{\mathbf{Y}}) T^{\kappa^{\mathbf{Y}}} \leq \exp(\kappa) T^\kappa; \\ \|\upsilon_j \circ \xi_j\| &\leq \|\upsilon_j\| \cdot \|\xi_j\| \leq \exp(\kappa^{\mathbf{Y}}) S^{\kappa^{\mathbf{Y}}} \exp(\kappa^{\mathbf{X}}) T^{\kappa^{\mathbf{X}}} \leq \exp(\kappa)(ST)^\kappa. \end{aligned}$$

Hence $\mathbf{Y} \circ \mathbf{X}$ is a weak n-rebuilding of quality (ST, κ). □

2.4 Algebraic cheap rebuilding

We always work over the ring $R = \mathbb{Z}$. Let Γ be a residually finite group. For a subgroup Λ of Γ, and a $\mathbb{Z}\Gamma$-chain complex X, we write $X_\Lambda \coloneqq \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \operatorname{res}^\Gamma_\Lambda X$ for the $\mathbb{Z}$-chain complex of Λ-*coinvariants*. Similarly, for a $\mathbb{Z}\Gamma$-module M, we write $M_\Lambda \coloneqq \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \operatorname{res}^\Gamma_\Lambda M$. We consider the is-class $\diamondsuit$ of residually finite groups.

We record the following basic properties:

Lemma 2.4.1. *Let Γ be a group and Λ be a finite-index subgroup of Γ. Let M be a free $\mathbb{Z}\Gamma$-module of finite rank. Then, the coinvariants M_Λ are a free $\mathbb{Z}$-module of rank*

$$\operatorname{rk}_\mathbb{Z} M_\Lambda = [\Gamma : \Lambda] \cdot \operatorname{rk}_{\mathbb{Z}\Gamma} M.$$

Proof. Because both restriction and tensor products are compatible with direct sums, it suffices to show the claim for $M = \mathbb{Z}\Gamma$. As $\mathbb{Z}\Lambda$-modules, we have $\operatorname{res}^\Gamma_\Lambda \mathbb{Z}\Gamma \cong \bigoplus_{[\Gamma:\Lambda]} \mathbb{Z}\Lambda$, thus

$$M_\Lambda \cong \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \bigoplus_{[\Gamma:\Lambda]} \mathbb{Z}\Lambda \cong \bigoplus_{[\Gamma:\Lambda]} \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \mathbb{Z}\Lambda \cong \bigoplus_{[\Gamma:\Lambda]} \mathbb{Z},$$

implying the claim. □

Lemma 2.4.2. *Let Γ be a group. The following hold:*

(i) Let Λ be a finite index normal subgroup of Γ. Let Δ be a subgroup of Γ and let M be a $\mathbb{Z}\Delta$-module. Then there is an isomorphism of $\mathbb{Z}$-modules

$$(\operatorname{ind}^\Gamma_\Delta M)_\Lambda \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} M_{\Lambda\cap\Delta}$$

that is natural in M;

(ii) Let Λ be a finite index normal subgroup of Γ. Let Δ be a subgroup of Γ and let X be a $\mathbb{Z}\Delta$-chain complex. Then there is a natural isomorphism of $\mathbb{Z}$-chain complexes

$$(\operatorname{ind}^\Gamma_\Delta X)_\Lambda \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} X_{\Lambda\cap\Delta};$$

(iii) Let $f\colon M \to L$ be a $\mathbb{Z}\Gamma$-chain map between based free $\mathbb{Z}\Gamma$-modules. Let Λ be a subgroup of Γ. Then the induced map $f_\Lambda\colon M_\Lambda \to L_\Lambda$ of based free $\mathbb{Z}$-modules satisfies $\|f_\Lambda\| \le \|f\|$.

Proof. (i) We have natural isomorphisms of $\mathbb{Z}$-modules

$$\begin{aligned}(\operatorname{ind}^\Gamma_\Delta M)_\Lambda &\cong \mathbb{Z}[\Lambda \setminus \Gamma] \otimes_{\mathbb{Z}\Gamma} (\operatorname{ind}^\Gamma_\Delta M) \cong \mathbb{Z}[\Lambda \setminus \Gamma] \otimes_{\mathbb{Z}\Gamma} (\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} M) \\ &\cong \mathbb{Z}[\Lambda \setminus \Gamma] \otimes_{\mathbb{Z}\Delta} M \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} \mathbb{Z}[\Lambda \cap \Delta \setminus \Delta] \otimes_{\mathbb{Z}\Delta} M \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} M_{\Lambda\cap\Delta}.\end{aligned}$$

For the second to last isomorphism, it is crucial that Λ is normal in Γ.

(ii) This follows directly from the first part.

(iii) Consider the diagram of based free $\mathbb{Z}$-modules

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_L} \\ M_\Lambda & \xrightarrow[f_\Lambda]{} & L_\Lambda \end{array}$$

where p_M and p_L are the obvious projections. For $m \in M_\Lambda$, there exists $\widetilde{m} \in M$ with $p_M(\widetilde{m}) = m$ and $|\widetilde{m}|_1 = |m|_1$. Then we have

$$|f_\Lambda(m)|_1 = |f_\Lambda \circ p_M(\widetilde{m})|_1 = |p_L \circ f(\widetilde{m})|_1 \leq \|p_L\| \cdot \|f\| \cdot |\widetilde{m}|_1 \leq \|f\| \cdot |m|_1$$

and hence $\|f_\Lambda\| \leq \|f\|$.

□

Lemma 2.4.3. *Let Γ be a group, $f\colon X \to Y$ be a linear map between $\mathbb{Z}\Gamma$-chain complexes and Λ be a subgroup of Γ. Then, there is a natural isometric isomorphism of normed chain complexes*

$$\operatorname{Cone}(f_\Lambda) \cong \operatorname{Cone}(f)_\Lambda$$

Proof. It is straightforward to check that the map defined by $\big(k \otimes x, l \otimes y\big) \mapsto kl \otimes (x, y)$ for $k, l \in \mathbb{Z}, x \in X_{j-1}, y \in Y_j$ for all $j \in \mathbb{Z}$ induces an isomorphism. □

Inspired by the definition of Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding property (Definition 2.1.6), we make the following definition:

Definition 2.4.4 (Algebraic cheap rebuilding)**.** Let Γ be a residually finite group and let $n \in \mathbb{Z}$. The class CR_n^Γ (resp. CWR_n^Γ, CD_n^Γ) consists of all based *free* $\mathbb{Z}\Gamma$-chain complexes X lying in $\mathsf{FG}_n^\Gamma(\mathbb{Z})$ satisfying the following: there exists $\kappa \in \mathbb{R}_{\geq 1}$ such that for all $T \in \mathbb{R}_{\geq 1}$ and all residual chains Λ_* in Γ, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits an n-rebuilding (resp. weak n-rebuilding, n-domination) of quality (T, κ). We repeat the just stated (for CR_n^Γ) in symbols:

$$\exists_{\kappa \geq 1}\ \forall_{T \geq 1}\ \forall_{\Lambda_* \text{res. chain in } \Gamma}\ \exists_{i_0 \in \mathbb{N}}\ \forall_{i \geq i_0}\ \exists_{n\text{-rebuilding of } X_{\Lambda_i} \text{ of quality } (T,\kappa)}.$$

Recall that we obtain associated classes of residually finite groups as follows: A group Γ lies in CR_n (resp. CWR_n, CD_n) if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a based free resolution X lying in CR_n^Γ (resp. CWR_n^Γ, CD_n^Γ). In this case, we say that Γ satisfies the *algebraic cheap n-rebuilding property* (resp. *algebraic cheap weak n-rebuilding property*, *algebraic cheap n-domination property*. A group lies in CR_∞ (resp. CWR_∞, CD_∞) if it lies in CR_n (resp. CWR_n, CD_n) for all $n \in \mathbb{Z}$ (compare Definition 1.4.2).

Remark 2.4.5. Compared to Abért–Bergeron–Frączyk–Gaboriau's notion of geometric rebuilding, the main difference apart from those pointed out in Remark 2.2.2 is that we only work with residual chains, not with general Farber chains. This is conceptually easier, and provides a more 'hands-on' approach. We suspect that similar statements to those presented in this thesis hold when working with Farber chains. We expect this to be technically more involved, but might come at the advantage of having Axiom (B-res) satisfied. For residual chains, we couldn't verify Axiom (B-res) since normal subgroups of finite-index subgroups do not have to be normal in the ambient group.

Theorem 2.4.6. *The classes $\mathsf{CD}_*^\diamondsuit$ and $\mathsf{CR}_*^\diamondsuit$ define equivariantly bootstrappable properties. The class $\mathsf{CWR}_*^\diamondsuit$ satisfies the Axioms (B-deg), (B-susp) and (B-ind).*

Proof. The Axioms (B-deg) and (B-susp) are straightforward to verify. For Axiom (B-cone), suppose that Γ is a residually finite group, $f\colon X \to Y$ is a $\mathbb{Z}\Gamma$-chain map between $\mathbb{Z}\Gamma$-chain complexes. Suppose that $X \in \mathsf{CD}_{n-1}^\Gamma$ (resp. CR_{n-1}^Γ), witnessed by the constant $\kappa^{\mathbf{X}}$ and $Y \in \mathsf{CD}_n^\Gamma$ (resp. CR_n^Γ), witnessed by $\kappa^{\mathbf{Y}}$. Let $T \geq 1$ and Λ_* be a residual chain in Γ. Then, by hypothesis, there exists $i^{\mathbf{X}} \in \mathbb{N}$ such that for all $i \geq i^{\mathbf{X}}$, the complex X_{Λ_i} admits an $(n-1)$-domination (resp. -rebuilding) of quality $(T, \kappa^{\mathbf{X}})$. Similarly, there exists $i^{\mathbf{Y}} \in \mathbb{N}$ such that for all $i \geq i^{\mathbf{Y}}$, the complex Y_{Λ_i} admits an n-domination (resp. -rebuilding) of quality $(T, \kappa^{\mathbf{Y}})$. We set $i_0 \coloneqq \max\{i^{\mathbf{X}}, i^{\mathbf{Y}}\}$. For $i \geq i_0$, we consider the $\mathbb{Z}$-chain map $f_{\Lambda_i} : X_{\Lambda_i} \to Y_{\Lambda_i}$ between $\mathbb{Z}$-chain complexes. Note that $\operatorname{Cone}(f)_{\Lambda_i} \cong \operatorname{Cone}(f_{\Lambda_i})$ (see Lemma 2.4.3). From Proposition 2.3.2, we obtain that $\operatorname{Cone}(f_{\Lambda_i})$ admits an n-domination (resp. -rebuilding) of quality (T, κ_i), where

$$\kappa_i \coloneqq \kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log 3 + \max\bigl\{\log_+ \|(f_j)_{\Lambda_i}\| \mid j \leq n\bigr\}.$$

Note that, because $\|(f_j)_{\Lambda_i}\| \leq \|f_j\|$ (see Lemma 2.4.2 (iii)), we have

$$\kappa_i \leq \kappa \coloneqq \kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log 3 + \max\bigl\{\log_+ \|f_j\| \mid j \leq n\bigr\}.$$

Thus, we obtain an n-domination (resp. -rebuilding) of quality (T, κ).

It remains to show Axiom (B-ind). Let Γ be a residually finite group, $\Delta \subseteq \Gamma$ be a subgroup and X be a $\mathbb{Z}\Delta$-chain complex with $X \in \mathsf{CD}_n^\Delta$ (resp. $X \in \mathsf{CWR}_n^\Delta$, $X \in \mathsf{CR}_n^\Delta$), witnessed by a constant κ. We need to show that $\operatorname{ind}_\Delta^\Gamma X \in \mathsf{CD}_n^\Gamma$ (resp. CWR_n^Γ, CR_n^Γ). Let $T \geq 1$ and Λ_* be a residual chain in Γ. Then, $\Delta_i \coloneqq \Delta \cap \Lambda_i$ is a residual chain in Δ. Thus, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Δ_i} admits an n-domination (resp. weak n-rebuilding, n-rebuilding) of quality (T, κ). Note that by Lemma 2.4.2 (i), we have an isomorphism of $\mathbb{Z}$-chain complexes

$$(\operatorname{ind}_\Delta^\Gamma X)_{\Lambda_i} \cong \bigoplus_{\frac{[\Gamma:\Lambda_i]}{[\Delta:\Delta_i]}} X_{\Delta_i}.$$

Thus, by Lemma 2.3.1, $(\operatorname{ind}_\Delta^\Gamma X)_{\Lambda_i}$ admits an n-domination (resp. weak n-rebuilding, n-rebuilding) of quality (T, κ). Thus, $\operatorname{ind}_\Delta^\Gamma X \in \mathsf{CD}_n^\Gamma$ (resp. CWR_n^Γ, CR_n^Γ). $\square$

As a corollary, we obtain the Bootstrapping Theorem 1.5.3 for CD and CR, which we spell out in the following.

Corollary 2.4.7 (Bootstrapping theorem for CD, CR)**.** *Let Γ be a residually finite group, $n \in \mathbb{N}$ and Ω be a Γ-CW-complex, satisfying the following conditions:*

(i) Ω is $(n-1)$-acyclic over $\mathbb{Z}$ (i.e., $H_j(\Omega;\mathbb{Z}) \cong H_j(pt;\mathbb{Z})$ for all $j \leq n-1$);

(ii) $\Gamma\backslash\Omega^{(n)}$ is compact;

(iii) For every cell σ of Ω with $\dim(\sigma) \leq n$, the stabiliser Γ_σ lies in $\mathsf{CD}_{n-\dim(\sigma)}$ (resp. $\mathsf{CR}_{n-\dim(\sigma)}$).

Then, $\Gamma \in \mathsf{CD}_n$ (resp. CR_n).

We will mainly build on this corollary to establish the cheap rebuilding property, starting with the group of integers (see Proposition 2.7.1).

Remark 2.4.8. Note that for $\mathsf{CWR}_n^{\diamondsuit}$, we could not show the Axiom (B-cone), because in Proposition 2.3.2, we only have a weaker inheritance result for weak rebuildings of mapping cones. We spell out the weaker version for $\mathsf{CWR}_n^{\diamondsuit}$ as follows.

Theorem 2.4.9. *Let $f\colon X \to Y$ be a $\mathbb{Z}\Gamma$-chain map. If $X \in \mathsf{CR}_{n-1}^{\Gamma}$ and $Y \in \mathsf{CWR}_n^{\Gamma}$, then $\mathrm{Cone}(f) \in \mathsf{CWR}_n^{\Gamma}$.*

This works analogously to the proof of Theorem 2.4.6. We spell out the argument for the convenience of the reader.

Proof. Let $\kappa^{\mathbf{X}} \geq 1$ be a constant witnessing that $X \in \mathsf{CR}_{n-1}^{\Gamma}$ and $\kappa^{\mathbf{Y}}$ be a constant witnessing that $Y \in \mathsf{CWR}_n^{\Gamma}$. Let $T \geq 1$ and Λ_* be a residual chain in Γ. Then, by hypothesis, there exists $i^{\mathbf{X}} \in \mathbb{N}$ such that for all $i \geq i^{\mathbf{X}}$, the complex X_{Λ_i} admits an $(n-1)$-rebuilding of quality $(T, \kappa^{\mathbf{X}})$. Similarly, there exists $i^{\mathbf{Y}} \in \mathbb{N}$ such that for all $i \geq i^{\mathbf{Y}}$, the complex Y_{Λ_i} admits a weak n-rebuilding of quality $(T, \kappa^{\mathbf{Y}})$. We set $i_0 \coloneqq \max\{i^{\mathbf{X}}, i^{\mathbf{Y}}\}$. For $i \geq i_0$, we consider the $\mathbb{Z}$-chain map $f_{\Lambda_i} : X_{\Lambda_i} \to Y_{\Lambda_i}$ between $\mathbb{Z}$-chain complexes. Note that $\mathrm{Cone}(f)_{\Lambda_i} \cong \mathrm{Cone}(f_{\Lambda_i})$ (see Lemma 2.4.3). From Proposition 2.3.2(iii), we obtain that $\mathrm{Cone}(f_{\Lambda_i})$ admits a weak n-rebuilding of quality (T, κ_i), where

$$\kappa_i \coloneqq \kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log 3 + \max\bigl\{\log_+ \|(f_j)_{\Lambda_i}\| \mid j \leq n\bigr\}.$$

Note that, because $\|(f_j)_{\Lambda_i}\| \leq \|f_j\|$ (see Lemma 2.4.2 (iii)), we have

$$\kappa_i \leq \kappa \coloneqq \kappa^{\mathbf{X}} + \kappa^{\mathbf{Y}} + \log 3 + \max\bigl\{\log_+ \|f_j\| \mid j \leq n\bigr\}.$$

Thus, we obtain a weak n-rebuilding of quality (T, κ). □

In spite of the fact that cheap weak rebuilding does not define a bootstrappable property, we still obtain a weaker version of the bootstrapping theorem.

Theorem 2.4.10 (modified bootstrapping theorem for CWR)**.** *Let Γ be a residually finite group, $n \in \mathbb{N}$ and Ω be a Γ-CW-complex, satisfying the following conditions:*

(i) Ω is $(n-1)$-acyclic over $\mathbb{Z}$ (i.e., $H_j(\Omega;\mathbb{Z}) \cong H_j(pt;\mathbb{Z})$ for all $j \leq n-1$);

(ii) $\Gamma\backslash\Omega^{(n)}$ is compact;

(iii) For every vertex v of Ω, the stabiliser Γ_v lies in CWR_n;

(iv) For every cell σ of Ω with $0 < \dim(\sigma) \leq n$, the stabiliser Γ_σ lies in $\mathsf{CR}_{n-\dim(\sigma)}$.

Then, $\Gamma \in \mathsf{CWR}_n$.

The argument works in analogy with the proof of the Bootstrapping Theorem 1.5.3. For the convenience of the reader, we spell out the crucial points of the argument, highlighting the differences.

Proof. As in the proof of the Bootstrapping Theorem 1.5.3, we can assume that Ω is acyclic. We obtain that the associated $\mathbb{Z}\Gamma$-chain complex X consists in degrees $\leq n$ of finite direct sums $\sum_{\sigma\in S_j} \mathbb{Z}[\Gamma/\Gamma_\sigma]$, where $\Gamma_\sigma \in \mathsf{CR}_{n-j}$ for $\sigma \in S_j, 0 < j \leq n$ and $\Gamma_\sigma \in \mathsf{CWR}_n$ for $\sigma \in S_0$.

We now need an analogue of the algebraic bootstrapping theorem (Theorem 1.5.2). Note that CWR_* (which is implied by CR_*) satisfies the Axioms (B-deg), (B-susp) and (B-ind) (see Theorem 2.4.6). We have to replace the use of Axiom (B-cone).

The argument goes as follows: For each $0 < j \leq n$ and $\sigma \in S_j$, because $\Gamma_\sigma \in \mathsf{CR}_{n-j}$, there exists a projective $\mathbb{Z}[\Gamma_\sigma]$-resolution P^σ of $\mathbb{Z}$ lying in CR^Γ_{n-j}. For $j = 0$ and $\sigma \in S_0$, there exists a projective $\mathbb{Z}[\Gamma_\sigma]$-resolution P^σ of $\mathbb{Z}$ lying in CWR^Γ_n. By axiom (B-ind), we obtain that the projective $R[\Gamma]$-resolution

$$P^j := \bigoplus_{\sigma\in S_j} \operatorname{ind}^\Gamma_{\Gamma_\sigma} P^\sigma$$

of X_j lies in CR^Γ_{n-j} for $0 < j \leq n$ and in CWR^Γ_n for $j = 0$. For $j > n$, we pick any resolution P^j. In this case, we have $P^j \in \mathsf{CR}_{n-j}$ by Axiom (B-deg). We apply Proposition 1.3.1 to X (without the augmentation map) and $(P^j)_{j\in\mathbb{N}}$. We obtain a projective chain complex $\widehat{X}$ with a filtration $(\widehat{X}^{[k]})_{k\in\mathbb{N}}$ and a chain map $q\colon \widehat{X} \to X$, which is a weak equivalence. Since X is a resolution of $\mathbb{Z}$, and $\widehat{X}$ is projective, also $\widehat{X}$ is a projective resolution of $\mathbb{Z}$. We will show that $\widehat{X} \in \mathsf{CWR}^\Gamma_n$. By Proposition 1.3.1, $\widehat{X}^{[0]} = P^0 \in \mathsf{CWR}^\Gamma_n$. Moreover, for all $k \in \mathbb{N}$, the chain complex $\widehat{X}^{[k]}$ is the mapping cone of a chain map $\Sigma^{k-1}P^k \to \widehat{X}[k-1]$. Since $P^k \in \mathsf{CR}^\Gamma_{n-k}$, repeated application of Axiom (B-susp) yields that $\Sigma^{k-1}P^k \in \mathsf{CR}^\Gamma_{n-1}$. We can thus apply Theorem 2.4.9 to obtain that $\widehat{X}^{[k]} \in \mathsf{CWR}^\Gamma_n$. By induction, we obtain that $\widehat{X}^{[n]} \in \mathsf{CWR}^\Gamma_n$. Proposition 1.3.1 also yields that $\widehat{X}$ is the mapping cone of a chain map $Y \to X^{[n]}$, where Y is a chain complex concentrated in degrees $\geq n$. Thus, by Lemma 1.2.2, we have $Y \in \mathsf{CR}^\Gamma_{n-1}$. Thus, Theorem 2.4.9 yields that $\widehat{X} \in \mathsf{CWR}^\Gamma_n$. Since $\widehat{X}$ is a projective resolution of $\mathbb{Z}$, this shows that $\Gamma \in \mathsf{CWR}_n$. ☐

We close this section with a remark on equivariant cheap rebuildings.

Remark 2.4.11 (Equivariant rebuilding)**.** Let Γ be a group and let (X, X', ξ, ξ', Ξ) be a $\mathbb{Z}\Gamma$-homotopy retract of based free $\mathbb{Z}\Gamma$-chain complexes lying in $\mathsf{FG}_n^\Gamma(\mathbb{Z})$. Inspired by the definition of an n-rebuilding (see Definition 2.2.1), we suppose that there exist real numbers $T, \kappa \geq 1$ such that for all $j \leq n$, we have

$$\operatorname{rk}_{\mathbb{Z}\Gamma}(X'_j) \leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j);$$
$$\max\bigl\{\|\partial_j^{X'}\|, \|\xi_j\|, \|\xi'_j\|, \|\Xi_j\|\bigr\} \leq \exp(\kappa) T^\kappa.$$

Let Λ be a finite index subgroup of Γ. The $\mathbb{Z}\Gamma$-homotopy retract (X, X') descends to a homotopy retract (X_Λ, X'_Λ). Moreover, for all $j \leq n$, we have

$$\begin{aligned} \operatorname{rk}_{\mathbb{Z}}\bigl((X'_\Lambda)_j\bigr) &= [\Gamma : \Lambda] \operatorname{rk}_{\mathbb{Z}\Gamma}(X'_j) && \text{(Lemma 2.4.1)} \\ &\leq [\Gamma : \Lambda] \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j) \\ &= \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}\bigl((X_\Lambda)_j\bigr). && \text{(Lemma 2.4.1)} \end{aligned}$$

Furthermore, the functor $(-)_\Lambda$ does not increase the norm of maps (Lemma 2.4.2 (iii)), and we obtain

$$\max\bigl\{\|(\partial_\Lambda^{X'})_j\|, \|(\xi_\Lambda)_j\|, \|(\xi'_\Lambda)_j\|, \|(\Xi_\Lambda)_j\|\bigr\} \leq \exp(\kappa) T^\kappa.$$

Thus, (X_Λ, X'_Λ) is an n-rebuilding of quality (T, κ). Similar statements hold for weak rebuildings and dominations.

2.5 Vanishing of homology gradient invariants

Similar to geometric rebuilding (cf. Theorem 2.1.7), the properties CD_n and CWR_n are designed to yield vanishing (torsion) homology gradients.

The finiteness properties FG and FP were introduced in Section 1.6.

Theorem 2.5.1. *Let $n \in \mathbb{N}$ and Γ be a residually finite group.*

(i) If Γ satisfies CD_n, then, for $j \leq n$, all coefficient fields $\mathbb{F}$ and residual chains $(\Lambda_i)_{i \in \mathbb{N}}$, we have

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0.$$

(ii) If $\Gamma \in \mathsf{CWR}_n$, then, for $j \leq n-1$ and residual chains $(\Lambda_i)_{i \in \mathbb{N}}$, we have

$$\widehat{t}_j(\Gamma, \Lambda_*) = 0.$$

(iii) Let X be a $\mathbb{Z}\Gamma$-chain complex with $X \in \mathsf{CWR}_n^\Gamma$ and $X \in \mathsf{FG}_{n+1}^\Gamma(\mathbb{Z})$. Then, for all residual chains $(\Lambda_i)_{i \in \mathbb{N}}$, we have

$$\widehat{t}_n(X, \Lambda_*) = 0.$$

(iv) If $\Gamma \in \mathsf{CWR}_n$ *is of type* FP_{n+1}*, then for all residual chains* $(\Lambda_i)_{i\in\mathbb{N}}$*, we have*

$$\widehat{t}_n(\Gamma, \Lambda_*) = 0.$$

Proof. (i) The fact that $\Gamma \in \mathsf{CD}_n$ is witnessed by a based free resolution X of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ and some $\kappa \geq 1$. Then, for all $T \geq 1$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, there exists an n-domination X'_{Λ_i} of X_{Λ_i} of quality (T, κ). By Proposition 2.2.3 (i), we have for $j \leq n$

$$\begin{aligned}
\dim_{\mathbb{F}} H_j(\Lambda_i; \mathbb{F}) &\leq \dim_{\mathbb{F}} H_j(X'_{\Lambda_i}; \mathbb{F}) \\
&\leq \dim_{\mathbb{F}} \mathbb{F} \otimes_{\mathbb{Z}} X'_{\Lambda_i} \\
&= \operatorname{rk}_{\mathbb{Z}} X'_{\Lambda_i} \\
&\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_{\Lambda_i}) \\
&\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X) \cdot [\Gamma : \Lambda_i] \qquad \text{(Lemma 2.4.1)}
\end{aligned}$$

Thus,

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}} H_j(\Lambda_i; \mathbb{F})}{[\Gamma : \Lambda_i]} \leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X).$$

As $\operatorname{rk}_{\mathbb{Z}\Gamma}(X) < \infty$ and this inequality holds for all $T \geq 1$, we obtain that $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0$.

(ii) Let $\Gamma \in \mathsf{CWR}_n$, witnessed by a based free resolution X and constant $\kappa \geq 1$. Let $j \leq n-1$ and $(\Lambda_i)_{i\in\mathbb{N}}$ be a residual chain in Γ. Let $T \geq 1$. By definition of CWR_n, there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$, the based free $\mathbb{Z}$-complex X_{Λ_i} admits a cheap weak n-rebuilding of quality (T, κ). Thus, by Proposition 2.2.3 (ii), we have

$$\begin{aligned}
\log \operatorname{tors} H_j(X_{\Lambda_i}; \mathbb{Z}) &\leq \kappa^2 T^{-1} \operatorname{rk}_{\mathbb{Z}}\big((X_{\Lambda_i})_j\big)(1 + \log T) \\
&= \kappa^2 T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j) \cdot [\Gamma : \Lambda_i] \cdot (1 + \log T) \quad \text{(Lemma 2.4.1)}
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{t}_j(\Gamma, \Lambda_*) &= \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_j(\Lambda_i; \mathbb{Z})}{[\Gamma : \Lambda_i]} \\
&= \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_j(X_{\Lambda_i}; \mathbb{Z})}{[\Gamma : \Lambda_i]} \\
&\leq \kappa^2 T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j) \cdot (1 + \log T).
\end{aligned}$$

As $\operatorname{rk}_{\mathbb{Z}\Gamma}(X_j) < \infty$ and $T^{-1} \log T \to 0$ for $T \to \infty$, we obtain $\widehat{t}_j(\Gamma, \Lambda_*) = 0$.

(iii) The crucial point for requiring that $j \leq n-1$ in the second part is that in Proposition 2.2.3 (ii), we require control on the boundary operator in degree $j + 1 \leq n$. We do not have this control here. However, we can use the following trick: Let $X \in \mathsf{CWR}_n^\Gamma$ be witnessed by the constant κ and $X \in \mathsf{FG}_{n+1}^{\mathbb{Z}}$. Let Λ_* be a residual chain. By definition, for all $T \geq 1$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits a weak n-rebuilding $(X_{\Lambda_i}, X'_{\Lambda_i}, \xi, \xi', \Xi)$ of quality (T, κ). We have $\|\partial_{n+1}^{X_{\Lambda_i}}\| \leq \|\partial_{n+1}^X\|$ by Lemma 2.4.2 (iii) and the latter quantity is finite because X_{n+1} is finitely generated over $\mathbb{Z}\Gamma$. We define another $\mathbb{Z}$-chain complex X''_{Λ_i} with the chain modules

$$(X''_{\Lambda_i})_j := \begin{cases} 0 & \text{if } j \geq n+2; \\ (X_{\Lambda_i})_{n+1} & \text{if } j = n+1; \\ (X'_{\Lambda_i})_j & \text{if } j \leq n; \end{cases}$$

and differentials

$$\partial_j^{X''_{\Lambda_i}} := \begin{cases} 0 & \text{if } j \geq n+2; \\ \xi_n \circ \partial_{n+1}^{X_{\Lambda_i}} & \text{if } j = n+1; \\ \partial_j^{X'_{\Lambda_i}} & \text{if } j \leq n. \end{cases}$$

Consider the partial chain maps $\overline{\xi}\colon X_{\Lambda_i} \to X''_{\Lambda_i}$ and $\overline{\xi'}\colon X''_{\Lambda_i} \to X_{\Lambda_i}$ defined up to degree $n+1$ as follows:

$$\overline{\xi}_j := \begin{cases} \mathrm{id}_{(X_{\Lambda_i})_{n+1}} & \text{if } j = n+1; \\ \xi_j & \text{if } j \leq n; \end{cases}$$

$$\overline{\xi'}_j := \begin{cases} \mathrm{id}_{(X_{\Lambda_i})_{n+1}} - \Xi_n \circ \partial_{n+1}^{X_{\Lambda_i}} & \text{if } j = n+1; \\ \xi'_j & \text{if } j \leq n. \end{cases}$$

We have constructed

$$\begin{array}{ccccccc}
(X_{\Lambda_i})_{n+1} & \xrightarrow{\partial_{n+1}^{X_{\Lambda_i}}} & (X_{\Lambda_i})_n & \xrightarrow{\partial_n^{X_{\Lambda_i}}} & (X_{\Lambda_i})_{n-1} & \longrightarrow & \cdots \\
{\scriptstyle \mathrm{id} - \Xi_n \circ \partial_{n+1}^{X_{\Lambda_i}}} \uparrow \; \downarrow {\scriptstyle \mathrm{id}} & & {\scriptstyle \xi'_n} \uparrow \; \downarrow {\scriptstyle \xi_n} & & {\scriptstyle \xi'_{n-1}} \uparrow \; \downarrow {\scriptstyle \xi_{n-1}} & & \\
(X_{\Lambda_i})_{n+1} & \xrightarrow[\xi_n \circ \partial_{n+1}^{X_{\Lambda_i}}]{} & (X'_{\Lambda_i})_n & \xrightarrow[\partial_n^{X'_{\Lambda_i}}]{} & (X'_{\Lambda_i})_{n-1} & \longrightarrow & \cdots
\end{array}$$

The $\mathbb{Z}$-chain homotopy Ξ provides a partial chain homotopy between $\mathrm{id}_{X_{\Lambda_i}}$ and $\overline{\xi'} \circ \overline{\xi}$ up to degree n. In particular, $H_n(X_{\Lambda_i}; \mathbb{Z})$ is a retract of $H_n(X''_{\Lambda_i}; \mathbb{Z})$ and hence

$$\log \operatorname{tors} H_n(X_{\Lambda_i}; \mathbb{Z}) \leq \log \operatorname{tors} H_n(X''_{\Lambda_i}; \mathbb{Z}).$$

We have

$$\begin{aligned}
\operatorname{rk}_{\mathbb{Z}}\bigl((X''_{\Lambda_i})_n\bigr) &= \operatorname{rk}_{\mathbb{Z}}\bigl((X'_{\Lambda_i})_n\bigr) \\
&\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}\bigl((X_{\Lambda_i})_n\bigr) \\
&= \kappa T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_n) && \text{(Lemma 2.4.1)}
\end{aligned}$$

and

$$\begin{aligned}\log_+ \|\partial_{n+1}^{X''_{\Lambda_i}}\| &= \log_+ \|\xi_n \circ \partial_{n+1}^{X_{\Lambda_i}}\| \\ &\le \log_+ \|\xi_n\| + \log_+ \|\partial_{n+1}^{X_{\Lambda_i}}\| \\ &\le \kappa(1+\log T) + \log_+ \|\partial_{n+1}^{X}\|.\end{aligned}$$

We can now use Gabber's estimate (Lemma 7) to conclude that

$$\begin{aligned}\widehat{t}(X,\Lambda_*) &\le \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_n(X''_{\Lambda_i})}{[\Gamma:\Lambda_i]} \\ &\le \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_n)\big(\kappa(1+\log T) + \log_+ \|\partial_{n+1}^{X}\|\big).\end{aligned}$$

Since $\|\partial_{n+1}^{X}\| < \infty$, taking $T \to \infty$ shows that $\widehat{t}(X,\Lambda_*) = 0$.

(iv) Since Γ is of type FP_{n+1}, there exists a based free $\mathbb{Z}\Gamma$-resolution Y of $\mathbb{Z}$ lying in $\mathsf{FG}^{\Gamma}_{n+1}(\mathbb{Z})$. It suffices to show that $Y \in \mathsf{CWR}^{\Gamma}_n$. Then, Part (iii) yields the claim.

By assumption, Γ lies in CWR_n, thus there exists a based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ lying in CWR^{Γ}_n witnessed by a constant κ. Fix a $\mathbb{Z}\Gamma$-chain homotopy equivalence between Y and X given by the Fundamental Lemma of homological algebra. Since Y and X lie in $\mathsf{FG}^{\Gamma}_n(\mathbb{Z})$, there exists $\mu \in \mathbb{R}_{\ge 1}$ such that for every finite index subgroup Λ of Γ, the $\mathbb{Z}\Gamma$-chain homotopy equivalence between Y and X descends to a weak n-rebuilding (Y_Λ, X_Λ) of quality $(1,\mu)$ by Remark 2.4.11.

Let $T \ge 1$ and Λ_* be a residual chain in Γ. By assumption $X \in \mathsf{CWR}^{\Gamma}_n$, thus there exists $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$, there exists a weak n-rebuilding $(X_{\Lambda_i}, X'_{\Lambda_i})$ of quality (T,κ). Lemma 2.3.3 (iii) yields that the composition of the weak n-rebuildings $(Y_{\Lambda_i}, X_{\Lambda_i})$ and $(X_{\Lambda_i}, X'_{\Lambda_i})$ is a weak n-rebuilding $(Y_{\Lambda_i}, X'_{\Lambda_i})$ of quality $(T, 2\kappa\mu)$. Hence, $Y \in \mathsf{CWR}^{\Gamma}_n$ is witnessed by the constant $2\kappa\mu$.

□

Remark 2.5.2. While CWR_* implies vanishing of logarithmic torsion homology growth, it does *not* seem to be a bootstrappable property. This was the main motivation for defining CR_*, as it is strong enough to be equivariantly bootstrappable and also implies CWR_*.

2.6 Algebraic cheap rebuilding in degree 0

Proposition 2.6.1. *We have*

$$\mathsf{CR}_0 = \mathsf{CWR}_0 = \mathsf{CD}_0 = \big\{\Gamma \mid \Gamma \textit{ res. fin. and infinite}\big\}.$$

Proof. It is obvious from the definition that $\mathsf{CR}_0 \subseteq \mathsf{CWR}_0 \subseteq \mathsf{CD}_0$.

Suppose that $\Gamma \in \mathsf{CD}_0$, witnessed by a free $\mathbb{Z}\Gamma$-resolution $X \in \mathsf{CD}_0^\Gamma$ of $\mathbb{Z}$ with constant κ as in Definition 2.4.4. We show that Γ is infinite. Let $T \geq 1$ and Λ_* be a residual chain in Γ. Thus, by definition, there exists $i_0 = i_0(T) \in \mathbb{N}$ such that for all $i \geq i_0$, there is a 0-domination X'_{Λ_i} of X_{Λ_i}, i.e.,

$$\begin{aligned} \operatorname{rk}_{\mathbb{Z}}\bigl((X'_{\Lambda_i})_0\bigr) &\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}\bigl((X_{\Lambda_i})_0\bigr) \\ &= \kappa T^{-1}[\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_0). \qquad \text{(Lemma 2.4.1)} \end{aligned}$$

Since $H_0(X) \cong \mathbb{Z}$, we also have $H_0(X_{\Lambda_i}) \cong \mathbb{Z}$, which by the property of being a homotopy retract is contained in $H_0(X'_{\Lambda_i})$. Thus, $\operatorname{rk}_{\mathbb{Z}}\bigl((X'_{\Lambda_i})_0\bigr) \geq 1$, implying that

$$1 \leq \operatorname{rk}_{\mathbb{Z}}\bigl((X'_{\Lambda_i})_0\bigr) \leq \kappa T^{-1}[\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_0),$$

hence

$$[\Gamma : \Lambda_i] \geq \frac{T}{\kappa \cdot \operatorname{rk}_{\mathbb{Z}\Gamma}(X_0)}.$$

Since for every $T \geq 1$, we can find a suitable $i_0 \in \mathbb{N}$, we obtain that $[\Gamma : \Lambda_i] \to \infty$ as $i \to \infty$, i.e., Γ is infinite.

Now, suppose that Γ is an infinite, residually finite group. We want to show that $\Gamma \in \mathsf{CR}_0$. Fix a based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ with $X_0 = \mathbb{Z}\Gamma, \partial_0^X(1) = 1, X_1 = \bigoplus_\Gamma \mathbb{Z}\Gamma$, and $\partial_1^X(e_\gamma) = \gamma - 1$. Here $e_\gamma \in X_1$ denotes the $\mathbb{Z}\Gamma$-basis element corresponding to $\gamma \in \Gamma$. We will show for every finite index subgroup Λ of Γ, that X_Λ admits a 0-rebuilding of quality $(T, 1)$ for all $T \leq [\Gamma : \Lambda]$. Since Γ is infinite and residually finite, we can find finite-index subgroups of arbitrarily large index. Thus, this shows that X lies in CR_0^Γ witnessed by the constant $\kappa = 1$.

Now, let Λ be a finite index subgroup of Γ. Choose a set S of right-coset representatives for $\Lambda \setminus \Gamma$ with $1_\Gamma \in S$. Consider the isomorphism of $\mathbb{Z}\Lambda$-modules

$$\begin{aligned} \operatorname{res}_\Lambda^\Gamma \mathbb{Z}\Gamma &\cong \bigoplus_S \mathbb{Z}\Lambda \\ s &\leftarrow\!\shortmid e_s \\ \gamma &\mapsto \gamma t(\gamma)^{-1} e_{t(\gamma)} \end{aligned}$$

where e_s is the $\mathbb{Z}\Lambda$-basis element corresponding to $s \in S$ and $t(\gamma) \in S$ is such that $\Lambda\gamma = \Lambda t(\gamma)$. Under this isomorphism, the based free $\mathbb{Z}\Lambda$-resolution $\operatorname{res}_\Lambda^\Gamma X$ of $\mathbb{Z}$ is given in degrees 1 and 0 by

$$\cdots \to \bigoplus_{\Gamma \times S} \mathbb{Z}\Lambda \xrightarrow{\operatorname{res}_\Lambda^\Gamma \partial_1^X} \bigoplus_S \mathbb{Z}\Lambda,$$

where

$$\operatorname{res}_\Lambda^\Gamma \partial_1^X(e_{(\gamma,s)}) = s\gamma t(s\gamma)^{-1} e_{t(s\gamma)} - e_s.$$

Let Y be a based free $\mathbb{Z}\Lambda$-resolution of $\mathbb{Z}$ with $Y_0 = \mathbb{Z}\Lambda$. Then there exist mutually $\mathbb{Z}\Lambda$-chain homotopy inverse $\mathbb{Z}\Lambda$-chain maps $\xi\colon \operatorname{res}_\Lambda^\Gamma X \to Y$ with $\xi_0(e_s) = 1$ and

$\xi' \colon Y \to \operatorname{res}^\Gamma_\Lambda X$ with $\xi'_0(1) = e_{1_\Gamma}$. The $\mathbb{Z}\Lambda$-chain homotopy $\Xi \colon \mathrm{id}_{\operatorname{res}^\Gamma_\Lambda X} \simeq \xi' \circ \xi$ is given in degree 0 by $\Xi_0(e_s) = -e_{(s^{-1},s)}$. By construction, we have

$$\operatorname{rk}_{\mathbb{Z}\Lambda}(Y_0) \leq [\Gamma : \Lambda]^{-1} \operatorname{rk}_{\mathbb{Z}\Lambda}(\operatorname{res}^\Gamma_\Lambda X_0);$$
$$\max\bigl\{\|\partial_0^Y\|, \|\xi_0\|, \|\xi'_0\|, \|\Xi_0\|\bigr\} \leq 1.$$

Hence by Remark 2.4.11, the $\mathbb{Z}\Lambda$-homotopy equivalence between $\operatorname{res}^\Gamma_\Lambda X$ and Y descends to a 0-rebuilding (X_Λ, Y_Λ) of quality $(T, 1)$ for all $T \leq [\Gamma : \Lambda]$. □

2.7 The fundamental example: the group of integers

The fundamental example of a group satisfying cheap rebuilding in all degrees – and actually the main example where we can verify this property by hand – is the group of integers.

Proposition 2.7.1. *The group of integers $\mathbb{Z}$ lies in CR_∞, thus in CWR_∞ and CD_∞.*

Proof. Recall that the group ring of the integers is isomorphic to the ring of Laurent polynomials $\mathbb{Z}[\langle t\rangle] = \mathbb{Z}[t, t^{-1}]$. We need to provide a suitable free resolution of $\mathbb{Z}$ over this ring. We consider the following resolution X:

$$0 \longrightarrow X_1 \coloneqq \mathbb{Z}[\langle t\rangle] \xrightarrow{\partial_1} X_0 \coloneqq \mathbb{Z}[\langle t\rangle] \xrightarrow{\partial_0} \mathbb{Z} \longrightarrow 0,$$

where $\partial_1(t) = t - 1$ and $\partial_0(t) = 1$. Set $\kappa \coloneqq 2$. Let $T \geq 1$ and Λ_* be a residual chain in the group of integers. Fix $i_0 \in \mathbb{N}$ such that $[\Gamma : \Lambda_{i_0}] \geq T$. Let $i \geq i_0$. We have $\Lambda_i = d_i \cdot \mathbb{Z} = \langle t^{d_i}\rangle$ for $d_i \coloneqq [\Gamma : \Lambda_i] \geq T$. Thus, X_{Λ_i} is isomorphic to $S^{[0,d_i]}$ as defined in Example 2.2.7, which admits an n-rebuilding of quality $(T, 2)$ for every $n \in \mathbb{Z}$ by Example 2.2.8. □

2.8 Amenable groups

Inspired by the work of Kar–Kropholler–Nikolov [KKN17], we present a proof that amenable groups satisfy the algebraic cheap weak rebuilding property (Definition 2.4.4), thus implying the vanishing of the (torsion) homology gradients. This argument appeared in the article with Li–Löh–Moraschini–Sauer [LLMSU24, Section 5].

Recall that amenable groups admit left-invariant means (see Definition 1 in the Introduction). In this section, we use a characterisation using Følner sequences that are compatible with residual chains (Theorem 2.8.4) and work over the ring $R = \mathbb{Z}$.

Lemma 2.8.1. *Let $0 \to A \xrightarrow{f} X \xrightarrow{g} Y \to 0$ be a short exact sequence of free $\mathbb{Z}$-chain complexes. If the inclusion f is nullhomotopic, then there exists a $\mathbb{Z}$-chain map $h \colon Y \to X$ such that $\mathrm{id}_X \simeq h \circ g$.*

Proof. The obvious $\mathbb{Z}$-chain map $q\colon \mathrm{Cone}(f) \to Y$ makes the following diagram commute.

$$\begin{array}{ccc} X & \hookrightarrow & \mathrm{Cone}(f) \\ & \searrow^{g} & \downarrow q \\ & & Y \end{array}$$

The map q is a weak equivalence by the long exact sequences in homology and the Five Lemma. Since $\mathrm{Cone}(f)$ and Y are free $\mathbb{Z}$-chain complexes, the weak equivalence q is a chain homotopy equivalence. Let $r\colon Y \to \mathrm{Cone}(f)$ be a chain homotopy inverse of q. Then the composition $r \circ g$ is chain homotopic to the inclusion $X \hookrightarrow \mathrm{Cone}(f)$. Set $h\colon Y \to X$ to be the composition

$$h\colon Y \xrightarrow{r} \mathrm{Cone}(f) \xrightarrow{(\mathrm{id}_A, \mathrm{id}_X; H)} \mathrm{Cone}(0) \cong \Sigma A \oplus X \xrightarrow{p_2} X,$$

where H is a chain homotopy between the nullhomotopic inclusion f and the zero map 0, and p_2 is the projection to the second summand. By construction, the composition $h \circ g$ is homotopic to id_X. □

In the following, we will prove an 'augmented' version of Lemma 2.8.1. Recall that a *based short exact sequence* of based free $\mathbb{Z}$-modules is a short exact sequence

$$0 \to K \xrightarrow{f} M \xrightarrow{g} L \to 0$$

of based free $\mathbb{Z}$-modules with fixed bases I_K, I_M, and I_L, respectively, such that $f(I_K) \subseteq I_M$ and $g(I_M \setminus f(I_K)) = I_L$. We say that $f(K)$ is a *based submodule* of M. For based free $\mathbb{Z}$-chain complexes, we obtain corresponding degreewise notions of based subcomplexes and based short exact sequences.

Let X be a based free $\mathbb{Z}$-chain complex that is concentrated in degrees ≥ 0. Consider the *standard augmentation map* $\varepsilon\colon X_0 \to \mathbb{Z}$ that maps each $\mathbb{Z}$-basis element of X_0 to $1 \in \mathbb{Z}$. We say that X is *augmented over* $\mathbb{Z}$ if $\varepsilon \circ \partial_1^X = 0$. If X is augmented over $\mathbb{Z}$, we define the *augmented chain complex* X^ε *associated to* X as the based free $\mathbb{Z}$-chain complex with chain modules

$$X_j^\varepsilon := \begin{cases} X_j & \text{if } j \geq 0; \\ \mathbb{Z} & \text{if } j = -1; \end{cases}$$

and differentials

$$\partial_j^{X^\varepsilon} := \begin{cases} \partial_j^X & \text{if } j \geq 1; \\ \varepsilon & \text{if } j = 0. \end{cases}$$

If X is augmented over $\mathbb{Z}$, then so is every based subcomplex of X. We say that the inclusion $f\colon A \to X$ of a based subcomplex is *augmentedly nullhomotopic* if the $\mathbb{Z}$-chain map $f^\varepsilon\colon A^\varepsilon \to X^\varepsilon$ given by

$$f_j^\varepsilon := \begin{cases} f_j & \text{if } j \geq 0; \\ \mathrm{id}_{\mathbb{Z}} & \text{if } j = -1; \end{cases}$$

is nullhomotopic. In this case we also say that A is *augmentedly contractible in* X.

For a $\mathbb{Z}$-chain complex Y that is concentrated in degrees ≥ 0, we denote by Y^+ the $\mathbb{Z}$-chain complex with chain modules

$$Y_j^+ := \begin{cases} Y_j & \text{if } j \geq 1; \\ Y_0 \oplus \mathbb{Z} & \text{if } j = 0; \end{cases}$$

and differentials

$$\partial_j^{Y^+} := \begin{cases} \partial_j^Y & \text{if } j \geq 2; \\ (\partial_1^Y, 0) & \text{if } j = 1. \end{cases}$$

Lemma 2.8.2. *Let* $0 \to A \xrightarrow{f} X \xrightarrow{g} Y \to 0$ *be a based short exact sequence of based free* $\mathbb{Z}$*-chain complexes that are concentrated in degrees* ≥ 0*. If* X *is augmented over* $\mathbb{Z}$ *and the inclusion* f *is augmentedly nullhomotopic, then there exist* $\mathbb{Z}$*-chain maps* $g^+ \colon X \to Y^+$ *and* $h^+ \colon Y^+ \to X$ *such that* $\mathrm{id}_X \simeq h^+ \circ g^+$*.*

Moreover, let $n \in \mathbb{N}$ *and suppose that for all* $j \leq n$ *the* $\mathbb{Z}$*-module* X_j *is finitely generated. Denote*

$$T' := \min\left\{ \frac{\mathrm{rk}_{\mathbb{Z}}(X_j)}{\mathrm{rk}_{\mathbb{Z}}(Y_j^+)} \;\middle|\; 0 \leq j \leq n \right\}, \quad \kappa := \max\Bigl\{1, \max\bigl\{\log_+ \|\partial_j^X\| \bigm| 0 \leq j \leq n\bigr\}\Bigr\}.$$

Then, for all $T \in \mathbb{R}_{\geq 1}$ *with* $T \leq T'$*, there exists a weak* n*-rebuilding* (X, Y^+) *of quality* (T, κ)*.*

Proof. We have a short exact sequence of free $\mathbb{Z}$-chain complexes

$$0 \to A^\varepsilon \xrightarrow{f^\varepsilon} X^\varepsilon \xrightarrow{g} Y \to 0,$$

where, by abuse of notation, $g \colon X^\varepsilon \to Y$ is the $\mathbb{Z}$-chain map given by $g \colon X \to Y$ in degrees ≥ 0 and by zero in degree -1. Lemma 2.8.1 yields a $\mathbb{Z}$-chain map $h \colon Y \to X^\varepsilon$ such that there exists a chain homotopy $H \colon \mathrm{id}_{X^\varepsilon} \simeq h \circ g$. We define the chain maps $g^+ \colon X \to Y^+$ and $h^+ \colon Y^+ \to X$ by

$$g_j^+ := \begin{cases} g_j & \text{if } j \geq 1; \\ (g_0, \varepsilon) & \text{if } j = 0; \end{cases}$$

$$h_j^+ := \begin{cases} h_j & \text{if } j \geq 1; \\ h_0 \oplus H_{-1} & \text{if } j = 0. \end{cases}$$

We have constructed

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
X_2 & \xrightarrow{g_2} & Y_2 & \xrightarrow{h_2} & X_2 \\
\partial_2^X \downarrow & & \downarrow \partial_2^Y & & \downarrow \partial_2^X \\
X_1 & \xrightarrow{g_1} & Y_1 & \xrightarrow{h_1} & X_1 \\
\partial_1^X \downarrow & & \downarrow (\partial_1^Y, 0) & & \downarrow \partial_1^X \\
X_0 & \xrightarrow{(g_0, \varepsilon)} & Y_0 \oplus \mathbb{Z} & \xrightarrow{h_0 \oplus H_{-1}} & X_0
\end{array}$$

Then one easily checks that $H_{*\geq 0}$ provides a chain homotopy $\mathrm{id}_X \simeq h^+ \circ g^+$.

Now, suppose that X_j is finitely generated for all $j \leq n$. For all $j \leq n$, we have

$$\begin{aligned} \mathrm{rk}_{\mathbb{Z}}(Y_j^+) &\leq T'^{-1}\,\mathrm{rk}_{\mathbb{Z}}(X_j) \leq T^{-1}\,\mathrm{rk}_{\mathbb{Z}}(X_j); \\ \|\partial_j^{Y^+}\| &\leq \|\partial_j^X\| \leq \exp(\kappa); \\ \|g_j^+\| &\leq 2. \end{aligned}$$

Hence (X, Y^+) is a weak n-rebuilding of quality (T, κ). $\square$

In the following, when we write $\lim_{i\to\infty} \frac{a_i}{b_i}$ for $a_i, b_i \in \mathbb{Z}$, we assume implicitly $b_i \neq 0$ for all large enough $i \in \mathbb{N}$.

Proposition 2.8.3. *Let Γ be an infinite residually finite group and let $n \in \mathbb{N}$. Let X be a based free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$ (with the standard augmentation) such that the $\mathbb{Z}\Gamma$-module X_j is finitely generated for all $j \leq n$. Suppose that for all residual chains Λ_* in Γ, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the based free $\mathbb{Z}$-chain complex $X^i := X_{\Lambda_i} = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \mathrm{res}^{\Gamma}_{\Lambda_i} X$ admits a based subcomplex A^i that is augmentedly contractible in X^i such that for all $j \in \{0, \dots, n\}$, we have*

$$\lim_{i\to\infty} \frac{\mathrm{rk}_{\mathbb{Z}}(X_j^i) - \mathrm{rk}_{\mathbb{Z}}(A_j^i)}{\mathrm{rk}_{\mathbb{Z}}(X_j^i)} = 0. \tag{2.8.1}$$

Then $\Gamma \in \mathsf{CWR}_n$.

Proof. Set $\kappa := \max\{1, \max\{\log_+ \|\partial_j^X\| \mid j \leq n\}\}$. Let $T \in \mathbb{R}_{\geq 1}$ and let Λ_* be a residual chain in Γ. For $i \geq i_0$, we denote by $Y^i := X^i/A^i$ the quotient chain complex. For all $j \in \{0, \dots, n\}$, we have

$$\lim_{i\to\infty} \frac{\mathrm{rk}_{\mathbb{Z}}(Y_j^i)}{\mathrm{rk}_{\mathbb{Z}}(X_j^i)} = 0$$

by assumption. Moreover, in degree 0 we have

$$\lim_{i\to\infty} \frac{\mathrm{rk}_{\mathbb{Z}}(Y_0^i) + 1}{\mathrm{rk}_{\mathbb{Z}}(X_0^i)} = 0$$

using that $\lim_{i\to\infty} \mathrm{rk}_{\mathbb{Z}}(X_0^i) = \infty$ because Γ is infinite. Together, we have

$$\lim_{i\to\infty} \frac{\mathrm{rk}_{\mathbb{Z}}(X_j^i)}{\mathrm{rk}_{\mathbb{Z}}\big((Y^i)_j^+\big)} = \infty.$$

Hence there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$

$$\min\left\{ \frac{\mathrm{rk}_{\mathbb{Z}}(X_j^i)}{\mathrm{rk}_{\mathbb{Z}}\big((Y^i)_j^+\big)} \;\middle|\; 0 \leq j \leq n \right\} \geq T.$$

By Lemma 2.4.2 (iii), we have

$$\|\partial_j^{X^i}\| \leq \|\partial_j^X\| \leq \exp(\kappa).$$

The based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ is in particular augmented over $\mathbb{Z}$ and thus, so is the based free $\mathbb{Z}$-chain complex X^i. Hence Lemma 2.8.2 applies and yields a weak n-rebuilding $(X^i, (Y^i)^+)$ of quality (T, κ). $\square$

Condition (2.8.1) means that asymptotically the subcomplex A^i is 'large' in X^i. We will apply Proposition 2.8.3 in the context of amenable groups to appropriate subcomplexes coming from Følner sequences. We first recall the notion of a Følner sequence.

Let Γ be a finitely generated group and let S be a finite generating set of Γ. For a subset F of Γ, we write $\partial_S F := \{\gamma \in F \mid \gamma \cdot s \in \Gamma \setminus F \text{ for some } s \in S\}$. We also write $\operatorname{int}_S(F) := F \setminus \partial_S F$. A *Følner sequence* in Γ with respect to S is a sequence $F_* = (F_i)_{i\in\mathbb{N}}$ of non-empty finite subsets of Γ satisfying

$$\lim_{i\to\infty} \frac{\#\partial_S F_i}{\#F_i} = 0.$$

A finitely generated group is *amenable* if and only if it admits a Følner sequence with respect to one (hence all) finite generating sets [BHV08, Theorem G.5.1]. Residually finite amenable groups admit Følner sequences satisfying additional properties:

Theorem 2.8.4 (Weiss [Wei01][KKN17, Theorem 7])**.** *Let Γ be a finitely generated amenable group that is residually finite and infinite. Let S be a finite generating set of Γ and let Λ_* be a residual chain in Γ. Then there exists a Følner sequence F_* in Γ with respect to S such that for all $i \in \mathbb{N}$, the set F_i is a set of right-coset representatives for $\Lambda_i \backslash \Gamma$.*

We say that a Følner sequence F_* as in Theorem 2.8.4 is *compatible* with the residual chain Λ_*.

Theorem 2.8.5. *Let $n \in \mathbb{N}$ and let Γ be an infinite residually finite and amenable group of type FP_n. Then $\Gamma \in \mathsf{CWR}_n$.*

Proof. Since Γ is of type FP_n, there exists a based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ such that the free $\mathbb{Z}\Gamma$-module X_j is finitely generated for all $j \leq n$ [Bro82a, Proposition VIII.4.3]. The differentials $\partial_1, \dots, \partial_n$ are given by (right-)multiplication with matrices $M_1, \dots, M_n$, respectively, whose entries are in $\mathbb{Z}\Gamma$. Let S be a finite generating set of Γ containing all group elements appearing in the entries of the matrices $M_1, \dots, M_n$. For $j \geq 0$, denote by S^j the finite set of all words with letters in S of length $\leq j$. Let Λ_* be a residual chain in Γ. By Theorem 2.8.4, there exists a Følner sequence F_* in Γ with respect to the finite generating set S^{n+1} that is compatible with the residual chain Λ_*. For all $i \in \mathbb{N}$, we consider the based free $\mathbb{Z}$-chain complex $X^i := X_{\Lambda_i} = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \operatorname{res}^\Gamma_{\Lambda_i} X$. For i large enough, we will construct a based subcomplex A^i of X^i that is augmentedly contractible in X^i such that for all $j \in \{0, \dots, n\}$, we have

$$\lim_{i\to\infty} \frac{\mathrm{rk}_{\mathbb{Z}}(X^i_j) - \mathrm{rk}_{\mathbb{Z}}(A^i_j)}{\mathrm{rk}_{\mathbb{Z}}(X^i_j)} = 0.$$

Then Proposition 2.8.3 yields that Γ lies in CWR_n.

Now, since the $\mathbb{Z}\Gamma$-resolution X is based free, for all $j \in \mathbb{N}$ we have an isomorphism of $\mathbb{Z}\Gamma$-modules $X_j \cong \bigoplus_{I_j} \mathbb{Z}\Gamma$, where I_j is the set of basis elements. The differential $\partial_j \colon X_j \to X_{j-1}$ is given by (right-)multiplication with the $(I_j \times I_{j-1})$-matrix $M_j = (m_j^{kl})_{k\in I_j, l\in I_{j-1}}$, where $m_j^{kl} = \sum_{s\in S} \lambda_j^{kl}(s)s \in \mathbb{Z}\Gamma$ with $\lambda_j^{kl}(s) \in \mathbb{Z}$. Consider the isomorphism of $\mathbb{Z}[\Lambda_i]$-modules

$$\begin{aligned} \mathrm{res}^{\Gamma}_{\Lambda_i} \mathbb{Z}\Gamma &\cong \bigoplus_{F_i} \mathbb{Z}[\Lambda_i] \\ f &\leftarrow\!\shortmid e_f \\ \gamma &\mapsto \gamma t_\gamma^{-1} e_{t_\gamma} \end{aligned}$$

where e_f is the $\mathbb{Z}[\Lambda_i]$-basis element corresponding to $f \in F_i$, and $t_\gamma \in S$ is such that $\Lambda_i\gamma = \Lambda_i t_\gamma$. We can identify the $\mathbb{Z}$-chain modules of X^i as

$$X^i_j = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \mathrm{res}^{\Gamma}_{\Lambda_i} X_j \cong \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \bigoplus_{I_j} \bigoplus_{F_i} \mathbb{Z}[\Lambda_i] \cong \bigoplus_{I_j\times F_i} \mathbb{Z}.$$

Under this identification, the differential $\partial^i_j \colon X^i_j \to X^i_{j-1}$ maps the $\mathbb{Z}$-basis element $e_{(k,f)} \in X^i_j$ for $(k, f) \in I_j \times F_i$ to

$$\sum_{l\in I_{j-1}} \sum_{s\in S} \lambda_j^{kl}(s) e_{(l,t_{fs}))} \in X^i_{j-1}.$$

Define the based subcomplex A^i of X^i by

$$A^i_j := \begin{cases} \bigoplus_{I_j\times \mathrm{int}_{S^{j+1}}(F_i)} \mathbb{Z} & \text{if } j \in \{0, \dots, n\}; \\ 0 & \text{if } j \geq n+1. \end{cases}$$

The chain complex A^i is well-defined, as the differential $\partial^i_j \colon X^i_j \to X^i_{j-1}$ restricts to a differential $A^i_j \to A^i_{j-1}$. For $j \leq n$, the index set I_j is finite by assumption. Hence, for all $j \leq n$, we have

$$\lim_{i\to\infty} \frac{\mathrm{rk}_{\mathbb{Z}}(X^i_j) - \mathrm{rk}_{\mathbb{Z}}(A^i_j)}{\mathrm{rk}_{\mathbb{Z}}(X^i_j)} = \lim_{i\to\infty} \frac{\#I_j \cdot \#\partial_{S^{j+1}}(F_i)}{\#I_j \cdot \#F_i} \leq \lim_{i\to\infty} \frac{\#\partial_{S^{n+1}}(F_i)}{\#F_i} = 0.$$

It remains to show that the inclusion $A^i \to X^i$ is augmentedly nullhomotopic. The inclusion $A^i \to X^i$ factors through the projection $X \to X^i$ via the $\mathbb{Z}$-chain map $A^i \to X$ that maps the $\mathbb{Z}$-basis element $e_{(k,f)} \in A^i_j \cong \bigoplus_{I_j \times \mathrm{int}_{S^{j+1}}(F_i)} \mathbb{Z}$ to $f \cdot e_k \in X_j \cong \bigoplus_{I_j} \mathbb{Z}\Gamma$. In particular, we obtain a factorisation of the augmented inclusion $(A^i)^\varepsilon \to (X^i)^\varepsilon$ through X^ε. Since X is a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$, the augmented $\mathbb{Z}$-chain complex X^ε is contractible. Hence the augmented inclusion $(A^i)^\varepsilon \to (X^i)^\varepsilon$ is nullhomotopic. □

By Theorem 2.5.1, we recover the following known result [CG86; LLS11; KKN17].

Corollary 2.8.6. *Let $n \in \mathbb{N}$ and let Γ be an infinite residually finite amenable group of type FP_n. Then Γ satisfies the following: For every residual chain Λ_* and every field $\mathbb{F}$, we have*

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0$$

for all $j \leq n$ and

$$\widehat{t}_j(\Gamma, \Lambda_*) = 0$$

for all $j \leq n-1$. If, additionally, Γ is of type FP_{n+1} we have

$$\widehat{t}_n(\Gamma, \Lambda_*) = 0.$$

for every field $\mathbb{F}$.

As a consequence of the modified Bootstrapping Theorem 2.4.10 for CWR_* and Example 1.7.3, we obtain:

Corollary 2.8.7. *Let Γ be a residually finite fundamental group of a finite graph of groups and let $n \in \mathbb{Z}$. If all vertex groups are infinite amenable of type FP_n and all edge groups are infinite elementary amenable of type FP_∞, then for every residual chain Λ_* of Γ and $j \leq n-1$, we have*

$$\widehat{t}_j(\Gamma, \Lambda_*) = 0.$$

Proof. Let Ω be the associated Bass-Serre tree [Ser80, Section I.5]. Then, Γ acts on Ω. As a (connected) tree, Ω is acyclic. Moreover, the action is cocompact. For every vertex of Ω, the stabiliser is the corresponding vertex group, thus infinite amenable of type FP_n by assumption and hence lies in CWR_n by Theorem 2.8.5. For every edge, the stabiliser is the corresponding edge group and therefore infinite elementary amenable of type FP_∞ by assumption, hence it lies in CR_{n-1} by Example 1.7.3. For this, we used that $\mathbb{Z} \in \mathsf{CR}_{n-1}$ (Proposition 2.7.1). Thus, the modified Bootstrapping Theorem for CWR (Theorem 2.4.10) yields that $\Gamma \in \mathsf{CWR}_n$. The claim then follows from Corollary 2.8.6. □

Remark 2.8.8 (ℓ^2-Torsion)**.** The above Corollary 2.8.7 has the following relevance: Li–Thom [LT14, Theorem 1.3] show that non-trivial amenable groups of type FL have vanishing ℓ^2-torsion. (A group Γ is *of type* FL if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a finite free resolution.) Hence the fundamental group of a finite graph of non-trivial amenable groups that are of type FL has vanishing ℓ^2-torsion [Lüc02, Theorem 3.93].

Let Γ be a residually finite fundamental group of a finite graph of groups with non-trivial amenable vertex groups of type FL and non-trivial elementary amenable edge groups of type FL. Then the ℓ^2-torsion $\rho^{(2)}(\Gamma)$ vanishes by the above, and Γ satisfies $\widehat{t}_j(\Gamma, \Lambda_*) = 0$ for every residual chain Λ_* and all $j \in \mathbb{Z}$ by Corollary 2.8.7. In particular, we have

$$\rho^{(2)}(\Gamma) = 0 = \sum_{j \geq 0} (-1)^j \cdot \widehat{t}_j(\Gamma, \Lambda_*).$$

This confirms Lück's approximation conjecture [Lüc13, Conjecture 1.11 (3)] for L^2-torsion for the group Γ.

3 Inner-amenable groups

A group Γ is called *amenable* if there exists a left-invariant mean (Definition 1 in the Introduction). The definition stems from an article by von Neumann [Neu29] in the context of paradoxical decompositions. Similarly, amenable actions of a group on a set were defined. These two definitions are compatible in the following sense: A group is amenable if and only if its action $\Gamma \curvearrowright \Gamma$ by left translation is. Instead of the left translation action, it is also interesting to consider the action $\Gamma \curvearrowright \Gamma$ by conjugation. This leads to the notion of inner-amenable groups. This class was originally defined by Effros [Eff75] in an attempt to characterise property Gamma. Effros showed that property Gamma of the group von Neumann algebra implies inner-amenability of the group and asked whether the converse also holds. This was answered negatively by Vaes [Vae12].

In this chapter, we give an introduction to the class of inner-amenable groups. We then prove a structure theorem for inner-amenable groups, allowing us to apply the Bootstrapping Theorem 1.5.3.

The material presented here was already published for Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding [Usc24]. In contrast to the published paper, we present everything in terms of the axiomatic approach provided by the bootstrapping principle that was developed in Chapter 1.

3.1 Definition and examples

We follow a definition of Tucker-Drob.

Definition 3.1.1 (Inner amenability [Tuc20, Definition 0.7])**.** A group Γ is *inner-amenable* if the conjugation action $\Gamma \curvearrowright \Gamma$ admits an atomless conjugation-invariant mean, i.e., if there exists a finitely additive probability measure $\mu\colon \mathcal{P}(\Gamma) \to [0,1]$ such that the following conditions hold:

(i) (atomlessness) For all $g \in \Gamma$, we have $\mu(\{g\}) = 0$.

(ii) (conjugation-invariance) For all subsets $A \subseteq \Gamma$ and $\gamma \in \Gamma$, we have

$$\mu(A^\gamma) = \mu(A).$$

Here, we use the notation $A^\gamma := \{\gamma^{-1} \cdot x \cdot \gamma \mid x \in A\}$.

Remark 3.1.2. It is important to point out that the definition of inner-amenability by Tucker-Drob, which we follow, is more restrictive than the original definition by Effros. Effros' approach does not demand atomlessness of the measure, but only that the measure is not concentrated on the identity element. As a consequence, fewer groups are inner-amenable in our setting, e.g., $\mathbb{Z}/2 \times F_2$ is inner-amenable in Effros' setting (as one can define a mean concentrating on the non-trivial element of the centre) but is not in our setting (by Proposition 3.1.3, this would imply inner-amenability of F_2, which contradicts Example 3.1.5 (xii)). One advantage of requiring atomlessness of the mean in the definition is that inner-amenability becomes a commensurability invariant. If we do not require atomlessness, the inheritance to finite-index subgroups requires the group to be of finite type and to have infinite conjugacy classes [GH91, Théorème 1(ii)]. Note that Duchesne–Tucker-Drob–Wesolek do not state these conditions because they also work in the atomless setting [DTW21, Proposition 2.7(1)].

For ICC groups (i.e., groups where every non-trivial conjugation class is infinite), the two notions coincide.

Note that our definition implies that inner-amenable groups are infinite.

Proposition 3.1.3 (Finite-index subgroups [DTW21, Proposition 2.7(1)])**.** *Let Γ be an inner-amenable group. Let $\Lambda \subseteq \Gamma$ be a finite-index subgroup. Then, Λ is also inner-amenable.*

It is worth noting that Duchesne–Tucker-Drob–Wesolek show stronger, more uniform results, namely that there exists an atomless conjugation-invariant mean μ on Γ such that for every finite-index subgroup Λ, we have $\mu(\Lambda) = 1$ [DTW21, Proposition 2.3].

Proof of Proposition 3.1.3. We give a simplified version of the proof of Duchesne–Tucker-Drob–Wesolek [DTW21, Section 2]. Let μ be an atomless conjugation-invariant mean on Γ. We define a new mean $\check{\mu} * \mu \colon \mathcal{P}(\Lambda) \to [0,1]$, called the *convolution*, by

$$\check{\mu} * \mu(A) := \int_{\gamma\in\Gamma} \mu(\gamma \cdot A)\, d\mu(\gamma).$$

Because $\gamma \mapsto \mu(\gamma \cdot A)$ is a non-negative bounded map, the integral is well-defined and converges [AB06, Chapter 11.2]. It is straightforward to verify that $\check{\mu} * \mu$ is again an atomless conjugation-invariant mean on Γ. For a subset $A \subseteq \Lambda$, we define

$$\mu_\Lambda(A) := \frac{\check{\mu} * \mu(A)}{\check{\mu} * \mu(\Lambda)}.$$

It is easy to verify that μ_Λ is an atomless conjugation-invariant mean on Λ. The only non-trivial part is its well-definedness, i.e., it remains to show that $\check{\mu}*\mu(\Lambda) > 0$.

Let $n \coloneqq [\Gamma : \Lambda] < \infty$ and pick left coset representatives $\gamma_1, \dots, \gamma_n$ of Λ in Γ. Then,

$$\begin{aligned}
\check{\mu} * \mu(\Lambda) &= \int_{\gamma \in \Gamma} \mu(\gamma \cdot \Lambda)\, d\mu(\gamma) \\
&= \sum_{i=1}^{n} \int_{\gamma \in \gamma_i \cdot \Lambda} \mu(\gamma \cdot \Lambda)\, d\mu(\gamma) \\
&= \sum_{i=1}^{n} \int_{\gamma \in \gamma_i \cdot \Lambda} \mu(\gamma_i \cdot \Lambda)\, d\mu(\gamma) && (\gamma \cdot \Lambda = \gamma_i \cdot \Lambda) \\
&= \sum_{i=1}^{n} \mu(\gamma_i \cdot \Lambda)^2,
\end{aligned}$$

which is positive, as the sum without the squares satisfies

$$\sum_{i=1}^{n} \mu(\gamma_i \cdot \Lambda) = \mu(\Gamma) = 1. \qquad \square$$

Moreover, also the converse holds.

Proposition 3.1.4 (Finite-index subgroups [DTW21, Proposition 2.7(1)])**.** *Let* Γ *be a group and* $\Lambda \subseteq \Gamma$ *be a finite-index subgroup that is inner-amenable. Then,* Γ *is inner-amenable.*

For the convenience of the reader, we give a simplified version of the proof by Duchesne–Tucker-Drob–Wesolek.

Proof of Proposition 3.1.4. By Proposition 3.1.3, the normal core $\operatorname{Core}_\Gamma(\Lambda)$ is inner-amenable. Since this is a finite-index normal subgroup of Γ, we can assume without loss of generality that Λ is a normal subgroup of Γ. Let μ be a conjugation-invariant mean for Λ and let $\gamma_1, \dots, \gamma_n \in \Gamma$ be a family of (left) coset representatives for Λ. Then, a conjugation-invariant mean for Λ is given by

$$\widetilde{\mu}(A) \coloneqq \frac{1}{n} \cdot \sum_{i=1}^{n} \mu\big(A^{\gamma_i} \cap \Lambda\big)$$

for all $A \subseteq \Gamma$. $\square$

We collect some examples and inheritance properties of inner-amenable groups that can be found in the literature.

Example 3.1.5.

(i) *Infinite* amenable groups are inner-amenable, because we can choose bi-invariant means [Gre69, Lemma 1.1.1 and Lemma 1.1.3].

(ii) Products $A \times \Gamma$, where A is inner-amenable, and Γ is an arbitrary group, are inner-amenable [BH86, Corollaire 2(iii)].

(iii) Γ is inner-amenable if it is an extension $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$, where Γ' is inner-amenable and Γ'' is amenable [BH86, Corollaire 2(iv)].

(iv) Direct limits of inner-amenable groups are inner-amenable [BH86, Corollaire 2(v)].

(v) All Baumslag-Solitar groups $\mathrm{BS}(m, n)$ (where $m, n \neq 0$) are inner-amenable [Sta06, Exemple 3.2].

(vi) For an abelian group $H = \Lambda$, every HNN-extension $\mathrm{HNN}(\Lambda, H, K, \phi)$ is inner-amenable [Sta06, Exemple 3.3].

(vii) There is a criterion for the inner-amenability of non-ascending HNN extensions [DTW21, Theorem 1.2]. Two specific examples are given by Kida and Ozawa, where the associated subgroups are cyclic [Kid14, Theorem 1.1 and 1.4].

(viii) There is a criterion for inner-amenability of wreath products [DTW21, Theorem 1.5].

(ix) Groups that are (JS-)stable, McDuff or have property Gamma, are inner-amenable [DV18, Figure 1]. Sufficient conditions for stability can be found in the article by Tucker-Drob [Tuc20, Theorem 18, Corollary 19].

(x) Thompson's group F is inner-amenable [Jol97]. In fact, it is even stable [Tuc20, Corollary 21].

(xi) Thompson's groups T and V are *not* inner-amenable [HO17, Theorem 4.4].

(xii) Nonabelian free groups are *not* inner-amenable [BH86, Corollaire 3(iii)].

(xiii) Discrete ICC groups having property (T) are *not* inner-amenable [BH86, Corollaire 3(i)].

Similar to amenability, there is a characterisation in terms of Følner-sequences.

Lemma 3.1.6 (inner-Følner sequence [BH86, Théorème 1(F)]). *A countable group Γ is inner-amenable if and only if it admits an* inner-Følner sequence*, i.e., a sequence $(F_n)_{n\in\mathbb{N}}$ of finite, nonempty subsets of Γ with $\lim_{n\to\infty} |F_n| = \infty$ such that for all $\gamma \in \Gamma$,*

$$\lim_{n\to\infty} \frac{|(F_n)^\gamma \triangle F_n|}{|F_n|} = 0.$$

Here, $\triangle$ denotes the symmetric difference*.*

Note, that the condition on atomlessness discussed in Remark 3.1.2 imposes the condition that $|F_n| \to \infty$.

Another large class of examples was pointed out to me by Francesco Fournier-Facio.

Example 3.1.7. Let Γ be a countably infinite group with commuting conjugates, i.e., for every finitely generated subgroup $H \leq \Gamma$, there is $f \in \Gamma$ such that H commutes with H^f. Then, Γ is inner-amenable.

Proof. We will show that Γ admits an inner-Følner-sequence (Lemma 3.1.6). Because Γ is countable, we may enumerate $\Gamma = \{\gamma_0, \gamma_1, \dots\}$. Since Γ has commuting conjugates, for all $n \in \mathbb{N}$, there is $f_n \in \Gamma$ such that $\{\gamma_0, \dots, \gamma_n\}$ commutes with $\{(\gamma_0)^{f_n}, \dots, (\gamma_n)^{f_n}\}$. In particular, this implies that $F_n \coloneqq \{(\gamma_0)^{f_n}, \dots, (\gamma_n)^{f_n}\}$ defines an inner-Følner sequence. □

Together with Example 3.1.5((iii)), we obtain that the following groups are inner-amenable.

Example 3.1.8. Let Γ be a countable group with commuting conjugates or a group extension of the form

$$1 \longrightarrow H \longrightarrow \Gamma \longrightarrow K \longrightarrow 1,$$

where H is countable with commuting conjugates and K is amenable. Then, Γ is inner-amenable.

In particular, this includes countable groups of piecewise linear or piecewise projective homomorphisms of the real line [FL23, Proof of Theorem 1.3] such as Thompson's group F. More examples of this type can be found in the work of Fournier-Facio and Lodha [FL23].

A natural question to ask is whether the result of Fournier-Facio and Lodha that second bounded cohomology vanishes for group extensions as in Example 3.1.8 [FL23, Theorem 1.2] extends to all inner-amenable groups. Already the group $\mathbb{Z} \times F_2$ shows that this is *not* the case, as the second bounded cohomology $H^2_b(\mathbb{Z} \times F_2)$ retracts onto $H^2_b(F_2) \ncong 0$ [Joh72, Proposition 2.8][Fri17, Section 2.6].

3.2 A structure theorem for inner-amenable groups

In the following, we will develop a structure theorem for inner-amenable groups, stating the existence of a chain of q-normal subgroups. We recall the definition of the latter notion.

Definition 3.2.1 (q-normality [Pop06, Definition 2.3])**.** Let Γ be a group. A subgroup $\Lambda \subseteq \Gamma$ is *q-normal* if there exists a generating set $S \subseteq \Gamma$ such that for all $s \in S$, the intersection $\Lambda \cap \Lambda^s$ is infinite. In this case, we will denote $\Lambda \leq_q \Gamma$ and we say that such a generating set S *witnesses* the q-normality of Λ in Γ.

Example 3.2.2. Let Γ be an infinite group and $\Lambda \subseteq \Gamma$ be a subgroup of finite index. Then, $\Lambda \leq_q \Gamma$. Indeed, for all $\gamma \in \Gamma$, the subgroup $\Lambda \cap \Lambda^\gamma$ is of finite index, thus infinite.

Example 3.2.3. Infinite, normal subgroups are q-normal as every generating set is a witness. However, in general, not every generating set is a witness. For example, in the semi-direct product $\mathbb{Z}^4 \rtimes S_4$, where the symmetric group acts by permutation of the coordinates, the subgroup $\Lambda := \mathbb{Z}^2 \times \{0\} \times \{0\}$ of $\mathbb{Z}^4$ is q-normal in $\mathbb{Z}^4 \rtimes S_4$, but not normal. More precisely, q-normality of Λ is witnessed by a generating set built from a generating set of $\mathbb{Z}^2$ and transpositions, but there are permutations σ such that $\Lambda \cap \Lambda^\sigma = \{1\}$.

Example 3.2.4. If $\Lambda \leq_q \Gamma$ and Γ is finitely generated, we can pick a *finite* generating set as a witness. Indeed, suppose that a (possibly infinite) set $S \subseteq \Gamma$ is a witness. Because Γ is finitely generated, there is a finite subset $S' \subseteq S$ that is a generating set. By definition, also S' then is a finite witness for the q-normality.

Remark 3.2.5. The notion of q-normality fits into a whole family of weak notions of normality, see e.g. wq-normality [Pop06, Definition 2.3] or (n-step) s-normality [BFS14, Definition 1.1].

The following is contained in an article by Tucker-Drob [Tuc20, Section 4]. In order to make the exposition clearer, we will spell out the arguments in an elementary way for our purposes. The main ingredient is the following classical theorem attributed to Rosenblatt, for which we first recall the notion of an amenable action.

Definition 3.2.6 (amenable action)**.** Let Γ be a group. An action of Γ on a set X is *amenable* if there exists an invariant mean, i.e. a finitely additive probability measure $\mu\colon \mathcal{P}(X) \to [0,1]$ such that for all $A \subseteq X$ and $\gamma \in \Gamma$, we have

$$\mu(\gamma \cdot A) = \mu(A).$$

Note that a group Γ is amenable if and only if the left translation action $\Gamma \curvearrowright \Gamma$ is.

Theorem 3.2.7 (Rosenblatt's theorem [Ros81, Proposition 3.5] [HO17, Proposition 4.2])**.** *Let Γ be a non-amenable group that acts amenably on X. Let μ be an atomless, invariant mean for this action. Then,*

$$\mu\big(\{x \in X \mid \Gamma_x \textit{ is not amenable}\}\big) = 1.$$

Here, Γ_x denotes the stabiliser of an element $x \in X$.

This has the following consequence for inner-amenable groups.

Corollary 3.2.8 ([Tuc20, Lemma 4.2])**.** *Let Γ be an inner-amenable, non-amenable group. Then, there exists an element of $\Gamma \setminus \{e\}$ with non-amenable centraliser.*

Proof. We apply Rosenblatt's theorem (Theorem 3.2.7) to the conjugation action $\Gamma \curvearrowright \Gamma$. This action is amenable because Γ is inner-amenable. Because there is an atomless mean for this action, the set $\{x \in \Gamma \mid \Gamma_x \text{ is not amenable}\}$ must contain a nontrivial element. Note that for $x \in \Gamma$, since we consider the conjugation action, we have

$$\Gamma_x = \{\gamma \in \Gamma \mid x^\gamma = x\} = \{\gamma \in \Gamma \mid x \cdot \gamma = \gamma \cdot x\} =: C_\Gamma(x),$$

which is the centraliser of x. □

The following theorem about non-amenable subgroups of inner-amenable groups was observed by Tucker-Drob.

Theorem 3.2.9 ([Tuc20, Theorem 4.3(i)])**.** *Let Γ be a torsion-free, inner-amenable group and $L \subseteq \Gamma$ be a non-amenable subgroup. Then, there exists a subgroup K such that*

$$L \leq_q K \leq_q \Gamma.$$

We give a direct proof of this fact, starting from Rosenblatt's theorem.

Proof. For all $\gamma \in \Gamma$, we denote its *centraliser* by $C_\Gamma(\gamma)$. We set

$$S_L \coloneqq \{\gamma \in \Gamma \mid L \cap C_\Gamma(\gamma) \text{ is non-amenable}\}$$

and $K \coloneqq \langle L \cup S_L \rangle_\Gamma$. The generating set $L \cup S_L$ witnesses that $L \leq_q K$.

It remains to show that $K \leq_q \Gamma$. Note that for $\gamma \in \Gamma$, the intersection $L \cap C_\Gamma(\gamma)$ is the stabiliser of γ of the conjugation action $L \curvearrowright \Gamma$. As the restriction of the conjugation action $\Gamma \curvearrowright \Gamma$, it is an amenable action. We can pick an atomless conjugation-invariant mean μ on Γ. Thus, Rosenblatt's theorem (Theorem 3.2.7) implies that $\mu(S_L) = 1$, and hence $\mu(K) = 1$. Let $\gamma \in \Gamma$. Then, by conjugation-invariance of μ, we have $\mu(K^\gamma) = \mu(K) = 1$. Because $\mu(\Gamma) = 1$, this implies that $\mu(K \cap K^\gamma) = 1$. In particular, $K \cap K^\gamma$ is infinite. ☐

In the presence of finiteness properties, we can impose the subgroups to be finitely generated.

Lemma 3.2.10. *Let L, K, Γ be groups, Γ and L be finitely generated and K be torsion-free. Assume that $L \leq_q K \leq_q \Gamma$. Then, there exists a* finitely generated *subgroup $K' \subseteq K$ such that*

$$L \leq_q K' \leq_q \Gamma.$$

Proof. Let S_Γ be a generating set of Γ witnessing that $K \leq_q \Gamma$, let S_K be a generating set of K witnessing that $L \leq_q K$ and let S_L be a finite generating set of L. Because Γ is finitely generated, we can assume that S_Γ is finite. Let $s \in S_\Gamma$. Because S_Γ witnesses the q-normality of K in Γ, the set $K \cap K^s$ is infinite. In particular, we can pick a nontrivial element in $k_s \in K \cap K^s \setminus \{e\}$. We can therefore pick words w_s and u_s in $S_K \cup S_K^{-1}$ such that in the group Γ, we have

$$k_s = w_s = s^{-1} \cdot u_s \cdot s. \tag{3.2.1}$$

Because we are considering only finitely many words, the set $\widetilde{S}$ of letters t for which there exists $s \in S_\Gamma$ such that t appears in w_s or u_s, is finite. We set $K' \coloneqq \langle \widetilde{S} \cup S_L \rangle_\Gamma$. By construction, K' is finitely generated. The generating set $\widetilde{S} \cup S_L$ witnesses that $L \leq_q K'$ (because $\widetilde{S}$ is a subset of the witness S_K). The generating set S_Γ witnesses that $K' \leq_q \Gamma$: Let $s \in S_\Gamma$. Then, Equation (3.2.1) shows that $K' \cap K'^s$ is nontrivial. Because $K' \cap K'^s$ is contained in K, which is torsion-free, the intersection must be infinite, as desired. ☐

Altogether, we obtain the following structure theorem. Recall that a group Γ is virtually torsion-free if there exists a finite-index subgroup $\Gamma' \subseteq \Gamma$ that is torsion-free.

Corollary 3.2.11 (structure theorem for inner-amenable groups)**.** *Let Γ be a finitely generated, virtually torsion-free, inner-amenable, non-amenable group. Then, there exist $g \in \Gamma \setminus \{e\}$ and finitely generated subgroups L, K, Γ' such that*

$$\mathbb{Z} \cong \langle g \rangle \leq_q L \leq_q K \leq_q \Gamma' \leq_q \Gamma.$$

Proof. Because Γ is virtually torsion-free, there is a finite-index subgroup $\Gamma' \subseteq \Gamma$ that is torsion-free. Example 3.2.2 yields that $\Gamma' \leq_q \Gamma$. Moreover, note that as a finite-index subgroup of a finitely generated group, Γ' itself is finitely generated. Then, Γ' is also inner-amenable by Proposition 3.1.3 and non-amenable. Thus, by Corollary 3.2.8, there exists an element $g \in \Gamma' \setminus \{e\}$ with non-amenable centraliser $C_{\Gamma'}(g)$. Since $C_{\Gamma'}(g)$ is non-amenable, there exists a finitely generated subgroup $L \subseteq C_{\Gamma'}(g)$ that is non-amenable [Day57, p. 517, Property (K)]. In particular, $\langle g \rangle \cong \mathbb{Z}$ is central, thus q-normal in L. Theorem 3.2.9 yields an intermediate subgroup K with $L \leq_q K \leq_q \Gamma'$. Finally, Lemma 3.2.10 yields that K can be chosen to be finitely generated. □

3.3 Bootstrapping for inner-amenable groups

The structure theorem for inner-amenable groups (Corollary 3.2.11) yields a strategy to prove that certain equivariantly bootstrappable properties hold for inner-amenable groups. We can show that the Bootstrapping Theorem 1.5.3 enables the transport through q-normality.

Lemma 3.3.1 (Transport through q-normality)**.** *Let B_* be a bootstrappable action property of an is-class $\diamondsuit$ of groups (see Definition 1.5.1). Let $\Gamma \in \diamondsuit$ be finitely generated and suppose that all infinite subgroups of Γ satisfy B_0. Let $\Lambda \leq_q \Gamma$ be a q-normal subgroup. If $\Lambda \in \mathsf{B}_1$, then $\Gamma \in \mathsf{B}_1$.*

Proof. It suffices to construct an action of Γ as in the Bootstrapping Theorem 1.5.3. Choose a generating set S of Γ witnessing the q-normality of Λ. Because Γ is finitely generated, we can choose S to be finite. We define a graph $\Omega = (V, E)$ with vertex set $V \coloneqq \Gamma/\Lambda$ and edge set

$$E \coloneqq \big\{\{g\Lambda, gs\Lambda\} \mid g \in \Gamma, s \in S\big\}.$$

Then, left-translation defines an action $\Gamma \curvearrowright \Omega$. Since S is a generating set, Ω is connected, i.e., 0-acyclic. Moreover, $\Gamma \backslash \Omega^{(1)}$ is compact, as it contains a single 0-cell and at most $|S|$ many 1-cells. Finally, we need to examine stabilisers. Vertex stabilisers are conjugate to Λ, thus satisfy B_1 by assumption. For edge stabilisers, fix an edge $\{g\Lambda, gs\Lambda\}$ for some $g \in \Lambda$ and $s \in S$. Its stabiliser contains

$$\Lambda^{g^{-1}} \cap \Lambda^{(gs)^{-1}} = \big(\Lambda^s \cap \Lambda\big)^{(gs)^{-1}},$$

which is infinite, because S witnesses the q-normality of Λ. As an infinite subgroup of Γ, the stabiliser of $\{g\Lambda, gs\Lambda\}$ thus satisfies B_0 by assumption. The Bootstrapping Theorem 1.5.3 yields that Γ satisfies B_1. □

Remark 3.3.2. There's also a more conceptual way of obtaining suitable actions from a 'blow-up' construction, based on a technique by Lück and Weiermann (see a previous paper by the author [Usc24, Section 5] for the proof in the setting of this thesis).

Combining Lemma 3.3.1 with the structure theorem for inner-amenable groups (Corollary 3.2.11), we obtain the following result.

Theorem 3.3.3 (Bootstrappable properties for inner-amenable groups)**.** *Let* B_* *be a bootstrappable action property of an is-class* $\diamondsuit$ *of groups. Suppose that the following conditions are satisfied:*

(i) All infinite groups in $\diamondsuit$ *satisfy* B_0*.*

(ii) The integers $\mathbb{Z}$ *satisfy* B_1*.*

Then, all finitely generated, virtually torsion-free, inner-amenable, non-amenable groups in $\diamondsuit$ *satisfy* B_1*.*

Proof. Let Γ be a finitely generated, virtually torsion-free, inner-amenable, non-amenable group. Then, by Corollary 3.2.11, there exist $g \in \Gamma \setminus \{e\}$ and finitely generated subgroups L, K, Γ' such that

$$\mathbb{Z} \cong \langle g \rangle \leq_q L \leq_q K \leq_q \Gamma' \leq_q \Gamma.$$

By assumption $\langle g \rangle \cong \mathbb{Z}$ satisfies B_1. By repeated application of Lemma 3.3.1, we obtain that $\Gamma \in \mathsf{B}_1$. □

By applying the theorem to $\mathsf{B}_* = \mathsf{CR}_*$, we obtain the following corollary.

Corollary 3.3.4. *Let* Γ *be a finitely generated, residually finite, virtually torsion-free, inner-amenable group. Then,* Γ *lies in* CWR_1 *and thus, for all fields* $\mathbb{F}$ *and residual chains* $(\Lambda_i)_{i\in\mathbb{N}}$*, we have*

$$\widehat{b}_1(\Gamma, \Lambda_*; \mathbb{F}) = 0.$$

If, additionally, Γ *is of type* FP_2*, we have*

$$\widehat{t}_1(\Gamma, \Lambda_*) = 0.$$

Proof. By Theorem 2.5.1, it suffices to show that Γ satisfies CWR_1. We consider the is-class $\diamondsuit$ of all residually finite groups. If Γ is amenable, then it satisfies CWR_1 by Theorem 2.8.5. If Γ is non-amenable, we apply Theorem 3.3.3 to the stronger, equivariantly bootstrappable property $\mathsf{B}_* = \mathsf{CR}_*$ (see Theorem 2.4.6). Indeed, all infinite, residually finite groups satisfy CR_0 (Proposition 2.6.1) and $\mathbb{Z}$ satisfies CR_1 (Proposition 2.7.1). □

Using ergodic-theoretic methods, the vanishing of the first ℓ^2-Betti number, which by Lück's Approximation Theorem 2 is equal to $\widehat{b}_1(\Gamma, \Lambda_*; \mathbb{Q})$, was already proved for all countable inner-amenable groups by Chifan–Sinclair–Udrea [CSU16, Corollary D]. Later, Tucker-Drob proved that the *cost* of these groups is equal to 1, thus also implying the vanishing of the first ℓ^2-Betti number [Tuc20, Theorem 5].

It is natural to ask if the result generalises to higher degrees.

Question 3.3.5. Let B_* be a bootstrappable action property of an is-class $\diamondsuit$ of groups and $n \in \mathbb{N}_{\geq 2}$. Suppose that the two conditions in Theorem 3.3.3 are satisfied. Do all virtually torsion-free, inner-amenable, non-amenable groups (satisfying suitable finiteness properties) in $\diamondsuit$ satisfy B_n ?

In order to demonstrate that this question has a positive answer, we would need new insights about the structure of inner-amenable groups. The following example shows that inheritance along q-normal subgroups does not hold for higher degrees.

Example 3.3.6. We consider the bootstrappable action property CWR_*. We have

$$\mathbb{Z} \leq_q \mathbb{Z} \times F_2 \leq_q F_2 \times F_2,$$

where the first inclusion is given by the inclusion in the first factor, and the second inclusion is induced by $\mathbb{Z} \hookrightarrow F_2$, given by mapping $1 \in \mathbb{Z}$ to one of two free generators of F_2. Then, $\mathbb{Z} \in \mathsf{CWR}_\infty$ (Proposition 2.7.1) and $\mathbb{Z} \times F_2 \in \mathsf{CWR}_\infty$ (apply Proposition 1.7.1(ii) to CR_*). However, $F_2 \times F_2 \notin \mathsf{CWR}_2$, as this would imply vanishing of the Betti number gradients in degree 2 (Theorem 2.5.1(i)), but we have $\widehat{b}_2(F_2 \times F_2, \Lambda_*; \mathbb{Q}) = 1 \neq 0$ for every residual chain Λ_* of $F_2 \times F_2$ [Kam19, Theorem 4.15(i)].

Note that this does *not* answer Question 3.3.5 in the negative, as $F_2 \times F_2$ is *not* inner-amenable [BH86, Théorème 5(f) and 5(c)].

Remark 3.3.7. Tracing back the proof of Theorem 3.3.3, we use multiple times that we can construct actions of the form $\Gamma \curvearrowright \Omega$, where Ω is a graph with vertex set Γ/Λ, $\Lambda \in \mathsf{B}_1$ and for a generating set S, we have for all $s \in S$ that $\Lambda \cap \Lambda^s$ is infinite, thus lies in B_0. It seems plausible that for higher degrees, we need to require a condition in the spirit of a result by Bader–Furman–Sauer [BFS14, Theorem 1.3], such as the following: For all $k \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_k \in \Gamma$, we have that

$$\Lambda \cap \Lambda^{\gamma_1} \cap \dots \cap \Lambda^{\gamma_k} \in \mathsf{B}_{n-k}.$$

It is not known whether we can always find a suitable chain of finite-index subgroups in inner-amenable groups.

On the other hand, for right-angled Artin groups, Question 3.3.5 admits a positive answer.

Proposition 3.3.8. *Let B_* be a bootstrappable action property of an is-class $\diamondsuit$ of groups and $n \in \mathbb{N}$ such that $\mathbb{Z} \in \mathsf{B}_n$. Let Γ be a non-trivial, right-angled Artin group that is inner-amenable. Then, Γ satisfies B_n.*

Proof. By a result of Duchesne–Tucker-Drob–Wesolek, Γ splits as a direct product with $\mathbb{Z}$ [DTW21, Corollary 4.21]. We can then conclude by Proposition 1.7.1(ii). □

3.4 More consequences of transport through q-normality

We end this chapter by recording a few more consequences of the transport through q-normality (Lemma 3.3.1) that are not related to inner-amenability.

Definition 3.4.1 (chain-commuting)**.** A group Γ is *chain-commuting* if there exists a finite generating set $\{\gamma_0, \dots, \gamma_k\}$ for Γ of elements of infinite order such that for all $i \in \{0, \dots, k-1\}$ we have that the commutator $[\gamma_i, \gamma_{i+1}]$ is trivial.

Application of transport through q-normality yields the following.

Proposition 3.4.2. *Let B_* be a bootstrappable action property of an is-class $\diamondsuit$ of groups such that the following conditions are satisfied:*

(i) All infinite groups in $\diamondsuit$ satisfy B_0.

(ii) The integers $\mathbb{Z}$ satisfy B_1.

Let $\Gamma \in \diamondsuit$ be a chain-commuting group. Then, Γ satisfies B_1.

Proof. Let $\{\gamma_0, \dots, \gamma_k\}$ be a generating set witnessing that Γ is chain-commuting. Set $\Gamma_i \coloneqq \langle \gamma_0, \dots, \gamma_i \rangle_\Gamma$. Because $\diamondsuit$ is closed under taking subgroups, we have that $\Gamma_i \in \diamondsuit$. It is straightforward to verify that $\Gamma_i \leq_q \Gamma_{i+1}$ for $i \in \{0, \dots, k-1\}$. Thus, we can apply Lemma 3.3.1 to the chain

$$\mathbb{Z} \cong \Gamma_0 \leq_q \Gamma_1 \leq_q \cdots \leq_q \Gamma_k = \Gamma,$$

and obtain that $\Gamma \in \mathsf{B}_1$. □

This allows us to apply this result to Artin groups with connected nerve.

Corollary 3.4.3. *Let B_* be a bootstrappable action property of an is-class $\diamondsuit$ of groups such that the following conditions are satisfied:*

(i) All infinite groups in $\diamondsuit$ satisfy B_0.

(ii) The integers $\mathbb{Z}$ satisfy B_1.

Let $\Gamma \in \diamondsuit$ be an Artin group with connected nerve. Then, Γ lies in B_1.

Proof. By Proposition 3.4.2, it suffices to show that Γ is chain-commuting. Since the nerve of Γ is connected, we can find a finite sequence of standard generators of Γ (possibly with repetitions) such that for every two subsequent generators s and t, the corresponding vertices in the nerve v_s, v_t are connected by an edge. Note that the generators s and t do not necessarily commute (unless the exponent of the edge is 2), but this means that the Artin group $\langle s, t \rangle$ is *spherical*. Thus, the centre of $\langle s, t \rangle$ is infinite cyclic [BS72, Satz 7.2]. Denote one generator of this centre by $\gamma_{s,t}$. Thus, the sequence $(s, \gamma_{s,t}, t)$ is chain-commuting. By concatenation of these sequences, we obtain a chain-commuting generating set for Γ. □

We also obtain the following corollary for products.

Corollary 3.4.4. *Let* B_* *be a bootstrappable action property of an is-class* $\diamondsuit$ *of groups such that the following conditions are satisfied:*

(i) All infinite groups in $\diamondsuit$ *satisfy* B_0*.*

(ii) The integers $\mathbb{Z}$ *satisfy* B_1*.*

Let Γ *and* Λ *be infinite groups such that* Γ *contains* $\mathbb{Z}$ *as a subgroup. Then,* $\Gamma \times \Lambda$ *lies in* B_1*.*

Proof. Let $H \subseteq \Gamma$ be isomorphic to $\mathbb{Z}$. Then, H is a normal subgroup of $H \times \Lambda$, thus q-normal by Example 3.2.3. Hence,

$$\mathbb{Z} \cong H \leq_q H \times \Lambda \leq_q \Gamma \times \Lambda,$$

where the latter is witnessed by any generating set of $\Gamma \times \Lambda$. Thus, twofold application of Lemma 3.3.1 yields that $\Gamma \times \Lambda \in \mathsf{B}_1$. $\square$

4 A dynamical upper bound and weak containment

We consider a dynamical viewpoint on (torsion) homology growth. The basic principle is the following: The asymptotic behaviour along finite-index subgroups is encoded in the dynamical system given by the profinite completion (see Definition 11 in the Introduction). We introduce the dynamical invariants *measured embedding dimension* (medim) and *measured embedding volume* (mevol) of a probability measure preserving (pmp) action on a standard probability space (see Section 4.3). Measured embedding dimension and volume are upper bounds to (torsion) homology growth (Theorem 4.3.4). The main part of this chapter is concerned with proving that medim and mevol are monotone under weak containment of actions (see Definition 4.4.1). The proof is concluded in Section 4.7 and some applications are given in Section 4.8. In preparation, we develop techniques regarding the translation of chain complexes and maps to a different action (see Section 4.5) and making almost chain complexes strict (see Section 4.6). We begin this chapter by fixing the algebraic notions (Section 4.1) and some explicit computations (Section 4.2).

Most of the content is based on a joint project with Kevin Li, Clara Löh, Marco Moraschini, and Roman Sauer [LLMSU25]. The theorem on monotonicity under weak containment (Theorem 4.7.1) is my own project.

For the whole section, we fix the following setup.

Setup 4.0.1. In this chapter, let Z denote the integers $\mathbb{Z}$ with the standard norm or a finite field with the trivial norm.

4.1 Decompositions, norms and almost equality

In this section, we consider the following situation.

Setup 4.1.1. Let Γ be a countably infinite group, $\alpha\colon \Gamma \curvearrowright (X,\mu)$ be an (essentially) free probability measure preserving action on a standard probability space. The latter implies that (X,μ) is (non-equivariantly) isomorphic to the interval $[0,1]$ with the Lebesgue measure.

4.1.1 Rings and modules

We write $L^\infty(X)$ for the ring of essentially bounded measurable functions $X \to Z$ considered up to equality μ-almost everywhere (with pointwise addition and multiplication). Note that because balls in Z are finite, essentially bounded functions are exactly the functions that can be represented by finite Z-linear combinations of characteristic functions on measurable subsets of X.

We define the *crossed product ring* $L^\infty(X) *_\alpha \Gamma$ as the free $L^\infty(X)$-module with basis Γ, endowed with a ring structure given by the following multiplication: For $\lambda \in L^\infty(X)$ and $\gamma \in \Gamma$, denote by (λ, γ) the inclusion of λ into the summand indexed by γ. For $\lambda, \lambda' \in L^\infty(X)$ and $\gamma, \gamma' \in \Gamma$, we define

$$(\lambda, \gamma) \cdot (\lambda', \gamma') \coloneqq \big(\lambda \cdot (\gamma \cdot \lambda'), \gamma \cdot \gamma'\big),$$

where the function $\gamma \cdot \lambda'$ is defined by $(\gamma \cdot \lambda')(x) \coloneqq \lambda'(\gamma^{-1}x)$. We spell out the following example, for it will appear often in this thesis.

Example 4.1.2. Let $\gamma, \eta \in \Gamma$ and $U, V \subseteq X$ be measurable subsets. We denote by $\chi_U \in L^\infty(X)$ the *characteristic function* of U. Then, in $L^\infty(X) *_\alpha \Gamma$, we have

$$(\chi_U, \gamma) \cdot (\chi_V, \eta) = (\chi_{U \cap \gamma V}, \gamma\eta).$$

4.1.2 Marked decompositions

Definition 4.1.3 (marked projective module)**.** A *marked projective* $L^\infty(X) *_\alpha \Gamma$-*module* is a triple $\big(M, (A_i)_{i\in I}, \varphi\big)$, consisting of

- an $L^\infty(X) *_\alpha \Gamma$-module M,
- a *finite* family $(A_i)_{i\in I}$ of measurable subsets of X, and
- an $L^\infty(X) *_\alpha \Gamma$-isomorphism $\varphi\colon M \to \bigoplus_{i\in I}(L^\infty(X) *_\alpha \Gamma) \cdot (\chi_{A_i}, 1)$.

In the following, we will abbreviate for $i \in I$:

$$\big\langle A_i \big\rangle_\alpha \coloneqq (L^\infty(X) *_\alpha \Gamma) \cdot (\chi_{A_i}, 1) \subseteq M.$$

We often just write $M = \bigoplus_{i\in I}\big\langle A_i\big\rangle_\alpha$ and leave the isomorphism φ implicit. Moreover, we define the *dimension* of the marked projective module $\big(M, (A_i)_{i\in I}, \varphi\big)$ over $L^\infty(X) *_\alpha \Gamma$ by

$$\dim_\alpha M \coloneqq \sum_{i\in I} \mu(A_i)$$

and the *rank* by

$$\operatorname{rk} M \coloneqq |I|.$$

Definition 4.1.4 (marked decomposition)**.** Let $M = \bigoplus_{i\in I}\big\langle A_i\big\rangle_\alpha$ be a marked projective $L^\infty(X) *_\alpha \Gamma$-module. A *marked decomposition* is the canonical $L^\infty(X) *_\alpha \Gamma$-isomorphism

$$M \cong \bigoplus_{i\in I}\big\langle A_i \setminus B_i\big\rangle_\alpha \oplus \bigoplus_{i\in I}\big\langle B_i\big\rangle_\alpha,$$

induced by a family $(B_i)_{i\in I}$ of (possibly empty) measurable subsets $B_i \subseteq A_i$. We call $\bigoplus_{i\in I}\big\langle A_i \setminus B_i\big\rangle_\alpha$ and $\bigoplus_{i\in I}\big\langle B_i\big\rangle_\alpha$ *marked summands* of M. There are canonical embeddings of the marked summands into M. We often denote a marked decomposition just by the family $(B_i)_{i\in I}$. Similarly, we can define marked decompositions into more than two summands.

Lemma 4.1.5 (maps between marked projective modules)**.** *Let*

$$M = \bigoplus_{i\in I} \langle A_i \rangle_\alpha \quad \text{and} \quad N = \bigoplus_{j\in J} \langle B_j \rangle_\alpha$$

be marked projective modules over $L^\infty(X) *_\alpha \Gamma$ and $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-linear map. Then, f is given by right-multiplication with a matrix $z \coloneqq (z_{i,j})_{i\in I, j\in J}$, where $z_{i,j} \in L^\infty(X) *_\alpha \Gamma$. Thus, there is a finite family $(U_k)_{k\in K}$ of pairwise disjoint, measurable subsets of X and a finite subset $F \subseteq \Gamma$ and $a_{i,j,k,\gamma} \in Z$ such that for all $i \in I, j \in J$, we have

$$z_{i,j} = \sum_{(k,\gamma)\in K\times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma). \tag{4.1.1}$$

In particular, if $i \in I, j \in J, k \in K, \gamma \in F$ such that $a_{i,j,k,\gamma} \neq 0$, we have $\gamma^{-1} A_i \cap U_k \subseteq B_j$.

Proof. Because f is a linear map between left-modules, it is given by right-multiplication with a matrix z over $L^\infty(X) *_\alpha \Gamma$. The coefficient $z_{i,j} \in L^\infty(X) *_\alpha \Gamma$ is a finite sum of elements of the form $a \cdot (\chi_U, \gamma)$, where $a \in Z, \gamma \in \Gamma$ and $U \subseteq X$ is a measurable subset of X. Since $U = \gamma \cdot (\gamma^{-1} U)$, we can write $z_{i,j}$ in the above form by potentially subdividing and regrouping the U_k to make them pairwise disjoint.

For the second claim, note that

$$(\chi_X, \gamma^{-1}) \cdot (\chi_{A_i}, 1) \cdot (\chi_{\gamma U_k}, \gamma) = \big(\chi_{\gamma^{-1} A_i \cap U_k)}, 1\big) \in \langle B_j \rangle_\alpha$$

if and only if $\gamma^{-1} A_i \cap U_k \subseteq B_j$ (up to a null-set) by Lemma 4.1.6 below. Note that the lemma below cites Equation (4.1.1), which we already proved. □

Lemma 4.1.6 (recognising marked summands)**.** *Let $M = \langle B \rangle_\alpha$ be a marked summand and $x \in M$. Let $\pi_1\colon L^\infty(X) *_\alpha \Gamma \to L^\infty(X)$ be the $L^\infty(X)$-linear projection to the summand corresponding to the identity element $1 \in \Gamma$. Assume that $a \in Z, a \neq 0$ and $A \subseteq X$ such that $\pi_1(x) = a \cdot \chi_A$. Then, $\langle A \rangle_\alpha \subseteq M$.*

Proof. It suffices to show that $A \setminus B$ is a null-set. Because $x \in M = (L^\infty(X) *_\alpha \Gamma) \cdot (\chi_B, 1)$, we have $x = \lambda \cdot (\chi_B, 1)$ for some

$$\lambda = \sum_{(k,\gamma)\in K\times F} a_{(k,\gamma)} \cdot (\chi_{\gamma U_k}, \gamma),$$

with K and $F \subseteq \Gamma$ finite sets, $a_{(k,\gamma)} \in Z$, and $U_k \subseteq X$, as in Equation (4.1.1).

Then, we have

$$\begin{aligned}
a \cdot \chi_{A\setminus B} &= \chi_{A\setminus B} \cdot a \cdot \chi_A && \text{(pointwise equality)}\\
&= \pi_1(\chi_{A\setminus B} \cdot x)\\
&= \pi_1\Bigg((\chi_{A\setminus B}, 1) \cdot \sum_{(k,\gamma)\in K\times F} a_{(k,\gamma)} \cdot (\chi_{\gamma U_k}, \gamma) \cdot (\chi_B, 1)\Bigg)\\
&= \sum_{k\in K} a_{(k,1)} \cdot \chi_{A\setminus B} \cdot \chi_{U_k} \cdot \chi_B && \text{(projection)}\\
&= 0 && ((A \setminus B) \cap B = \emptyset)
\end{aligned}$$

Thus, we have $a \cdot \chi_{A\setminus B} = 0$ almost everywhere with $a \neq 0$, which is only possible if $\mu(A \setminus B) = 0$. □

Remark 4.1.7. The presentation in Equation (4.1.1) is unique up to subdividing the partition $(U_k)_{k\in K}$ further and adding zeros. Thus, by subdividing further, we can assume that $U_k \subseteq B_j$ and $\gamma U_k \subseteq A_i$ if there are $i \in I$ and $\gamma \in F$ with $a_{i,j,k,\gamma} \neq 0$.

We record that whenever we have an $L^\infty(X) *_\alpha \Gamma$-linear map between marked projective modules, we are in the following setup.

Setup 4.1.8. Let

$$M = \bigoplus_{i\in I} \langle A_i \rangle_\alpha \quad \text{and} \quad N = \bigoplus_{j\in J} \langle B_j \rangle_\alpha$$

be marked projective modules over $L^\infty(X) *_\alpha \Gamma$ and $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism given by right-multiplication with a matrix $z \coloneqq (z_{i,j})_{i\in I, j\in J}$ that is given as follows: There is a finite family $(U_k)_{k\in K}$ of pairwise disjoint, measurable subsets of X and a finite subset $F \subseteq \Gamma$ and a family $(a_{i,j,k,\gamma})_{i\in I, j\in J, k\in K, \gamma\in F}$ over Z such that for all $i \in I, j \in J$, we have

$$z_{i,j} = \sum_{(k,\gamma)\in K\times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma).$$

Moreover, for all $i \in I, j \in J, k \in K, \gamma \in F$ with $a_{i,j,k,\gamma} \neq 0$, the following hold:

(i) $\gamma^{-1} A_i \cap U_k \subseteq B_j$;

(ii) $U_k \subseteq B_j$;

(iii) $\gamma U_k \subseteq A_i$.

Often, we will estimate norms by the following quantity.

Definition 4.1.9. Let $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Let P be a presentation of f as specified in Setup 4.1.8. We define

$$Q(f,P) \coloneqq \sum_{(i,j,k,\gamma)\in I\times J\times K\times F} |a_{i,j,k,\gamma}|$$

and

$$Q(f) \coloneqq \min_P Q(f,P),$$

where the minimum is taken over all possible presentations of f. Note that this is indeed a minimum, as norms of elements in Z lie in $\mathbb{N}$. If $z \in L^\infty(X) *_\alpha \Gamma$, define $Q(z) \coloneqq Q(f_z)$, where $f_z\colon L^\infty(X) *_\alpha \Gamma \to L^\infty(X) *_\alpha \Gamma$ is the map given by right-multiplication with z.

If $\eta\colon \langle A\rangle_\alpha \to L^\infty(X)$ is an $L^\infty(X) *_\alpha \Gamma$-linear map, we define

$$Q(\eta) \coloneqq Q\big(\iota(\eta(\chi_A, 1))\big),$$

where $\iota\colon L^\infty(X) \hookrightarrow L^\infty(X) *_\alpha \Gamma$ is the canonical inclusion into the summand indexed by $1 \in \Gamma$. Finally, if $\eta\colon M = \bigoplus_{i\in I}\langle A_i\rangle_\alpha \to L^\infty(X)$ is an $L^\infty(X) *_\alpha \Gamma$-linear map, we define

$$Q(\eta) \coloneqq \sum_{i\in I} Q\Big(\eta|_{\langle A_i\rangle_\alpha}\Big).$$

Lemma 4.1.10. *We record a few basic properties of this quantity.*

(i) *Let $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective modules over $L^\infty(X) *_\alpha \Gamma$ and $M = M_1 \oplus M_2$ be a marked decomposition. Then,*

$$Q(f) \le Q(f|_{M_1}) + Q(f|_{M_2}).$$

(ii) *Let $f\colon M \to N$ and $g\colon N \to P$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms between marked projective $L^\infty(X) *_\alpha \Gamma$- modules. Then,*

$$Q(g\circ f) \le Q(g)\cdot Q(f).$$

(iii) *Let $f_1\colon M \to N_1$ and $f_2\colon M \to N_2$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Then, $(f_1, f_2)\colon M \to N_1 \oplus N_2$ satisfies*

$$Q((f_1,f_2)) \le Q(f_1) + Q(f_2).$$

Proof. These properties follow from straightforward computations. □

4.1.3 Norms

We use the 1-norm to measure the size of elements in marked projective modules and consider the associated operator norm for homomorphisms between marked projective modules.

Definition 4.1.11. Let $A \subseteq X$ and $\lambda \in L^\infty(A)$. Then, we define the *norm* of λ by

$$|\lambda|_1 \coloneqq \int_A |\lambda| \, d\mu.$$

Definition 4.1.12. Let $A \subseteq X$ and $\lambda \in \langle A \rangle_\alpha \subseteq L^\infty(X) *_\alpha \Gamma$. Then, λ can be written as a finite sum $\lambda = \sum_{\gamma \in \Gamma} (\lambda_\gamma, \gamma)$, and we define its *norm* by

$$|\lambda|_1 \coloneqq \sum_{\gamma \in \Gamma} |\lambda_\gamma|_1.$$

We extend this norm to the ℓ^1-norm $|\cdot|_1$ on marked projective modules. We can thus define norms of homomorphisms as the operator norm (with respect to the ℓ^1-norms).

Definition 4.1.13 (operator norm)**.** Let $f \colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Then, the *norm* of f is the operator norm $\|f\|$, i.e. the least real number c such that

$$\forall_{x \in M} \ |f(x)|_1 \leq c \cdot |x|_1.$$

Remark 4.1.14. From the explicit description of the operator norm (Lemma 4.2.1), we obtain that the operator norm of $L^\infty(X) *_\alpha \Gamma$-homomorphisms between marked projective $L^\infty(X) *_\alpha \Gamma$-modules is always finite and in fact an integer.

4.1.4 Lognorm

Let $f \colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective modules over $L^\infty(X) *_\alpha \Gamma$. We consider the quantity $\operatorname{lognorm}_\alpha f$ to refine the expression "$\dim_\alpha N \cdot \log_+ \|\cdot\|$", which is the analogue of the expression in Gabber's estimate (Lemma 7 in the Introduction).

Definition 4.1.15 (lognorm)**.** Let $f \colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules.

- The *marked outer rank* of f is defined as

$$\operatorname{mrk}_\alpha f \coloneqq \inf\bigl\{\dim_\alpha N' \mid N' \subseteq N \text{ is a marked direct summand with } f(M) \subseteq N'\bigr\} \in [0, \dim_\alpha N]$$

- We define

$$\operatorname{lognorm}'_\alpha f \coloneqq \min\bigl\{\dim_\alpha M, \operatorname{mrk}_\alpha(f)\bigr\} \cdot \log_+ \|f\|.$$

  Here, $\log_+ \coloneqq \max\{\log, 0\}$ denotes the *truncated logarithm*.

- Let $D(M)$ denote the class of all finite marked decompositions of M. If $(M_i)_{i\in I} \in D(M)$, we set

$$\operatorname{lognorm}'_\alpha\bigl(f, (M_i)_{i\in I}\bigr) := \sum_{i\in I} \operatorname{lognorm}'_\alpha(f|_{M_i}\colon M_i \to N).$$

- Finally, we set

$$\operatorname{lognorm}_\alpha f := \inf_{M_*\in D(M)} \operatorname{lognorm}'_\alpha(f, M_*) \in \mathbb{R}_{\geq 0}.$$

Note that the marked outer rank is an approximation from above to "$\dim_\alpha f(M)$". The example of the diagonal map $\Delta\colon \langle X\rangle_\alpha \to \langle X\rangle_\alpha \oplus \langle X\rangle_\alpha$ shows that $f(M)$ is not necessarily a marked projective summand (see Lemma 4.1.6) and that the marked outer rank can be greater than the dimension of the domain: In this case, we have

$$\operatorname{mrk}_\alpha \Delta = 2 > 1 = \dim_\alpha \langle X\rangle_\alpha.$$

Note that we define $\operatorname{lognorm}_\alpha$ in terms of marked decompositions. Thus, the value $\operatorname{lognorm}_\alpha f$ might depend on the marked structure.

4.1.5 Almost equality

We introduce a measure for the difference between two morphisms between marked projective modules.

Definition 4.1.16 (almost equality)**.** Let $\delta \in \mathbb{R}_{>0}$ and $f, g\colon M \to N$ be maps between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. We write $f =_\delta g$ if there exists a marked decomposition $M \cong M_0 \oplus M_1$ with

$$f|_{M_0} = g|_{M_0} \quad \text{and} \quad \dim_\alpha M_1 \leq \delta.$$

In this case, we say that f and g are *almost equal* with error at most δ. Similarly, for $\lambda, \kappa \in L^\infty(X)$, we write $\lambda =_\delta \kappa$ if there exists a measurable subset $A \subseteq X$ with

$$\lambda|_A = \kappa|_A \quad \text{and} \quad \mu(A) \geq 1 - \delta.$$

We state the following elementary example.

Lemma 4.1.17. *Let $\gamma \in \Gamma$ and $U, V \subset X$ be measurable subsets. Let f_U be the $L^\infty(X) *_\alpha \Gamma$-linear map $L^\infty(X) *_\alpha \Gamma \to L^\infty(X) *_\alpha \Gamma$ given by right-multiplication with $(\chi_{\gamma U}, \gamma)$ and $f_V\colon L^\infty(X) *_\alpha \Gamma \to L^\infty(X) *_\alpha \Gamma$ be the $L^\infty(X) *_\alpha \Gamma$-linear map given by right-multiplication with $(\chi_{\gamma V}, \gamma)$. Set $\delta := \mu(U \triangle V)$, where $\triangle$ denotes the symmetric difference. Then, $f_U =_\delta f_V$.*

Proof. It suffices to show that the maps f_U and f_V coincide on

$$\big\langle X \setminus \gamma(U \triangle V)\big\rangle_\alpha \cong \big\langle \gamma(U \cap V)\big\rangle_\alpha \oplus \big\langle X \setminus \gamma(U \cup V)\big\rangle_\alpha,$$

where the isomorphism is to be understood in the category of $L^\infty(X) *_\alpha \Gamma$-modules. Indeed, we have

$$\begin{aligned}
(\chi_{\gamma(U\cap V)}, 1) \cdot (\chi_{\gamma U}, \gamma) &= (\chi_{\gamma(U\cap V)}, \gamma) \\
&= (\chi_{\gamma(U\cap V)}, 1) \cdot (\chi_{\gamma V}, \gamma), \\
(\chi_{X\setminus\gamma(U\cup V)}, 1) \cdot (\chi_{\gamma U}, \gamma) &= (\chi_{\emptyset}, \gamma) \\
&= (\chi_{X\setminus\gamma(U\cup V)}, 1) \cdot (\chi_{\gamma V}, \gamma).
\end{aligned}$$

□

We record three elementary inheritance properties for almost equality. The proofs are straightforward and therefore omitted.

Lemma 4.1.18. *Let $f_1, f_2, g_1, g_2 \colon M \to N$ be maps between marked projective modules over $L^\infty(X) *_\alpha \Gamma$. Let $\delta_1, \delta_2 \in \mathbb{R}_{>0}$ such that $f_1 =_{\delta_1} g_1$ and $f_2 =_{\delta_2} g_2$. Then,*

$$f_1 + f_2 =_{\delta_1+\delta_2} g_1 + g_2.$$

Lemma 4.1.19. *Let $f, g, h \colon M \to N$ be maps between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Let $\delta, \epsilon \in \mathbb{R}_{>0}$ such that $f =_\delta g$ and $g =_\epsilon h$. Then $f =_{\delta+\epsilon} h$.*

Lemma 4.1.20. *Let $M = \bigoplus_{i\in I}\langle A_i\rangle_\alpha$ and $N = \bigoplus_{j\in J}\langle B_j\rangle_\beta$ be marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Let $f, g \colon M \to N$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms. For $i \in I$ and $j \in J$, let $f_{i,j}, g_{i,j} \colon \langle A_i\rangle_\alpha \to \langle B_j\rangle_\alpha$ denote the restrictions to the specified summands in the domain and codomain. Let $\delta \in \mathbb{R}_{>0}$ such that for all $i \in I, j \in J$, we have $f_{i,j} =_\delta g_{i,j}$. Then, we have $f =_{|I|\cdot|J|\cdot\delta} g$.*

Almost equality does not behave well enough with operator norms. We introduce the following controlled modification.

Definition 4.1.21. Let M, N be marked projective $L^\infty(X) *_\alpha \Gamma$-modules, and let $f, f' \colon M \to N$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms and $\delta, K \in \mathbb{R}_{>0}$. We say that f is (δ, K)-*almost equal* to f' if there exists a marked decomposition $M \cong M_0 \oplus M_1$ such that

$$f|_{M_0} = f'|_{M_0} \quad \text{and} \quad \dim_\alpha M_1 \le \delta \quad \text{and} \quad \|f|_{M_1} - f'|_{M_1}\| \le K.$$

In this case, we write $f =_{\delta,K} f'$.

The quantity $\operatorname{lognorm}_\alpha$ behaves as follows under almost equality with this modification.

Lemma 4.1.22 (lognorm and almost equality, [LLMSU25, Proposition 6.4(iv)])**.** *Let f, $g \colon M \to N$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms between marked projective modules over $L^\infty(X) *_\alpha \Gamma$. Let $\delta, K \in \mathbb{R}_{>0}$. If $f =_{\delta,K} g$, then*

$$\operatorname{lognorm}_\alpha f \le \operatorname{lognorm}_\alpha g + \delta \cdot \log_+ K.$$

4.1.6 Almost chain maps and complexes

Almost equality allows us to define almost chain complexes.

Definition 4.1.23 (almost chain complex)**.** Let $n \in \mathbb{N}$ and $\delta \in \mathbb{R}_{>0}$. A *marked projective* (δ, n)*-almost* $L^\infty(X) *_\alpha \Gamma$*-chain complex* is a sequence (D_*, η) of the form

$$D_{n+1} \xrightarrow{\partial_{n+1}} D_n \longrightarrow \cdots \longrightarrow D_0 \xrightarrow{\partial_0 = \eta} L^\infty(X)$$

consisting of marked projective $L^\infty(X) *_\alpha \Gamma$-modules $D_0, \dots, D_{n+1}$ and $L^\infty(X) *_\alpha \Gamma$-homomorphisms $\eta =: \partial_0, \partial_1, \dots, \partial_{n+1}$ such that

$$\forall_{r \in \{0,\dots,n\}}\ \partial_r \circ \partial_{r+1} =_\delta 0$$

and such that η is δ-surjective, i.e. there exists $z \in D_0$ with $\eta(z) =_\delta 1$.

Note that in particular, an (ordinary) chain complex of marked projective modules over $L^\infty(X) *_\alpha \Gamma$ that augments to $L^\infty(X)$ is a (δ, n)-almost chain complex for all $\delta \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$. In the following, we will sometimes write *strict* chain complex for an ordinary chain complex to highlight the difference. All chain complexes over crossed product rings in this chapter are marked projective.

Definition 4.1.24 (almost chain map)**.** Let $\delta, \epsilon \in \mathbb{R}_{>0}$, let $n \in \mathbb{N}$ and let (C_*, ζ) and (D_*, η) be marked projective (δ, n)-almost $L^\infty(X) *_\alpha \Gamma$-chain complexes. An (ϵ, n)*-almost* $L^\infty(X) *_\alpha \Gamma$*-chain map* $C_* \to D_*$ *extending the identity on* $L^\infty(X)$ is a sequence $(f_r \colon C_r \to D_r)_{r \in \{0,\dots,n+1\}}$ of $L^\infty(X) *_\alpha \Gamma$-homomorphisms with

$$\eta \circ f_0 =_\epsilon \zeta \quad \text{and} \quad \forall_{r \in \{1,\dots,n+1\}}\ \partial_r^D \circ f_r =_\epsilon f_{r-1} \circ \partial_r^C.$$

Lemma 4.1.25 (compositions of almost chain maps)**.** *Let* $\delta, \epsilon, \lambda \in \mathbb{R}_{>0}$*,* $n \in \mathbb{N}$*, and let* $(C_*, \zeta), (D_*, \eta)$ *and* (E_*, θ) *be* (λ, n)*-almost chain complexes. Let* $f \colon C_* \to D_*$ *be a* (δ, n)*-almost chain map and* $g \colon D_* \to E_*$ *be an* (ϵ, n)*-almost chain map extending the identity on* $L^\infty(X)$*. Then, we have that the composition* $g \circ f \colon C_* \to E_*$ *is a* $(Q(f, n) \cdot \epsilon + \delta, n)$*-almost chain map extending the identity on* $L^\infty(X)$*. Here, we define* $Q(f, n) := \max\{Q(f_0), \dots Q(f_n)\}$*. (For the definition of* Q*, see Definition 4.1.9.)*

Proof. For $i \in \{0, \dots, n\}$, we have

$$\begin{aligned} \partial_i^E \circ (g_i \circ f_i) &=_{N_1(f_i)\cdot\epsilon} (g_{i-1} \circ \partial_i^D) \circ f_i && \text{[LLMSU25, Lemma 3.6(iii)]} \\ &=_\delta g_{i-1} \circ (f_{i-1} \circ \partial_i^C) && \text{[LLMSU25, Lemma 3.6(ii)]} \end{aligned}$$

Thus, by Lemma 4.1.19, the maps $\partial_i^E \circ (g_i \circ f_i)$ and $(g_{i-1} \circ f_{i-1}) \circ \partial_i^C$ almost agree, with error bounded by

$$N_1(f_i) \cdot \epsilon + \delta \leq Q(f, n) \cdot \epsilon + \delta.$$

Note that $N_1(f_i)$ is defined in [LLMSU25, Definition 2.27] and is bounded from above by $Q(f, n)$ (Remark 4.2.4). □

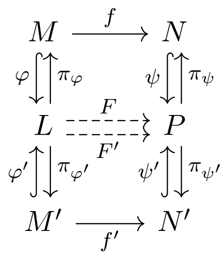
4.1.7 A Gromov-Hausdorff distance for chain complexes

Definition 4.1.26 (marked symmetric difference)**.** Let $M = \bigoplus_{i\in I}\langle M_i\rangle_\alpha$ be a marked projective $L^\infty(X) *_\alpha \Gamma$-module and let $N = \bigoplus_{i\in I}\langle A_i\rangle_\alpha$, $N' = \bigoplus_{i\in I}\langle A'_i\rangle_\alpha$ be marked projective summands of M. Then we define the *marked symmetric difference of N and N'* by

$$N \circledtriangle N' := \bigoplus_{i\in I}\langle A_i \triangle A'_i\rangle_\alpha.$$

Definition 4.1.27 (Gromov–Hausdorff distance, homomorphisms)**.** Let $f\colon M \to N$, $f'\colon M' \to N'$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms between marked projective modules over $L^\infty(X) *_\alpha \Gamma$, and let $\delta, K \in \mathbb{R}_{>0}$. We write $d^K_{\mathrm{GH}}(f, f') < \delta$ if there exist marked projective $L^\infty(X) *_\alpha \Gamma$-modules L, P and marked embeddings $\varphi\colon M \to L$, $\varphi'\colon M' \to L$, $\psi\colon N \to P$, $\psi'\colon N' \to P$ with the following properties:

- $\dim_\alpha\big(\varphi(M) \circledtriangle \varphi'(M')\big) < \delta$
- $\dim_\alpha\big(\varphi(N) \circledtriangle \varphi'(N')\big) < \delta$
- $F =_{\delta,K} F'$, where $F := \psi \circ f \circ \pi_\varphi$, $F' := \psi' \circ f' \circ \pi_{\varphi'}$ and π_φ, $\pi_{\varphi'}$ are the marked projections associated with the marked embeddings φ and φ', respectively.



We call $d^K_{\mathrm{GH}}(\cdot)$ the *K-Gromov–Hausdorff distance.* Note that for the triangle inequality, we need to sum the constants in the place of K [LLMSU25, Proposition 3.17].

Maps that are close in the Gromov–Hausdorff sense have similar dimensions and $\mathrm{lognorm}_\alpha$. We make this more precise in the following.

Lemma 4.1.28 ([LLMSU25, Proposition 3.14((i)) and 6.4((v))])**.** *Let $f\colon M \to N$ and $f'\colon M' \to N'$ be $L^\infty(X) *_\alpha \Gamma$-homomorphisms between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Let $\delta, K \in \mathbb{R}_{>0}$ such that $d^K_{\mathrm{GH}}(f, f') < \delta$. Then,*

$$|\dim_\alpha M - \dim_\alpha M'| < \delta \quad \textit{and} \quad \mathrm{lognorm}_\alpha f \le \mathrm{lognorm}_\alpha f' + \delta \cdot \log_+ K.$$

We extend the notion of Gromov–Hausdorff distance to chain complexes.

Definition 4.1.29 (Gromov–Hausdorff distance for chain complexes [LLMSU25, Definition 3.16])**.** Let $n \in \mathbb{N}$, let (D_*, η), (D'_*, η') be marked projective $L^\infty(X) *_\alpha \Gamma$-chain complexes (up to degree $n+1$), and let $\delta, K \in \mathbb{R}_{>0}$.

We then say that $d^K_{\mathrm{GH}}(D_*, D'_*, n) < \delta$ if there exist marked projective $L^\infty(X) *_\alpha \Gamma$-modules $P_0, \dots, P_{n+1}$ and marked embeddings $\varphi_r \colon D_r \to P_r$, $\varphi'_r \colon D'_r \to P_r$ for all $r \in \{0, \dots, n+1\}$ with the following properties:

- For all $r \in \{0, \dots, n+1\}$, we have
$$\dim_\alpha\bigl(\varphi_r(D_r) \circledcirc \varphi'_r(D'_r)\bigr) < \delta.$$
- For all $r \in \{0, \dots, n+1\}$, we have
$$F_r =_{\delta,K} F'_r,$$
where $F_r \coloneqq \varphi_{r-1} \circ \partial_r \circ \pi_{\varphi_r}$ and $F'_r \coloneqq \varphi'_{r-1} \circ \partial'_r \circ \pi_{\varphi'_r}$. Here, $P_{-1} \coloneqq L^\infty(X)$ and $\varphi_{-1} \coloneqq \mathrm{id}_{L^\infty(X)} \eqqcolon \varphi'_{-1}$.

$$\begin{array}{ccccccc}
D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \dots & D_0 & \xrightarrow{\partial_0} & L^\infty(X) \\
{\scriptstyle \varphi_{n+1}}\downarrow\uparrow{\scriptstyle \pi_{\varphi_{n+1}}} & & {\scriptstyle \varphi_n}\downarrow\uparrow{\scriptstyle \pi_{\varphi_n}} & & {\scriptstyle \varphi_0}\downarrow\uparrow{\scriptstyle \pi_{\varphi_0}} & & \| \\
P_{n+1} & \overset{F_{n+1}}{\underset{F'_{n+1}}{\dashrightarrow}} & P_n & \dots & P_0 & \overset{F_0}{\underset{F'_0}{\dashrightarrow}} & L^\infty(X) \\
{\scriptstyle \varphi'_{n+1}}\uparrow\downarrow{\scriptstyle \pi_{\varphi'_{n+1}}} & & {\scriptstyle \varphi'_n}\uparrow\downarrow{\scriptstyle \pi_{\varphi'_n}} & & {\scriptstyle \varphi'_0}\uparrow\downarrow{\scriptstyle \pi_{\varphi'_0}} & & \| \\
D'_{n+1} & \xrightarrow[\partial'_{n+1}]{} & D'_n & \dots & D'_0 & \xrightarrow[\partial'_0]{} & L^\infty(X)
\end{array}$$

In the same way, we will define the Gromov–Hausdorff distance for sequences of $L^\infty(X) *_\alpha \Gamma$-homomorphisms of marked projective $L^\infty(X) *_\alpha \Gamma$-modules ending with an $L^\infty(X) *_\alpha \Gamma$-homomorphism to $L^\infty(X)$. In particular, this includes the case of almost chain complexes.

4.2 Explicit descriptions

We describe the operator norm and the marked outer rank explicitly.

4.2.1 Operator norm

There is the following explicit description of the operator norm.

Lemma 4.2.1. *Let $f \colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Then, in the notation of Setup 4.1.8, we have*

$$\|f\| = \max_{i \in I} \max\Bigl\{ \sum_{j \in J, (k,\gamma) \in L} |a_{i,j,k,\gamma}| \Bigm| L \subseteq K \times F \textit{ with } \mu\bigl(\bigcap_{(k,\gamma) \in L} \gamma U_k \bigr) > 0 \Bigr\}.$$

In particular, we have $\|f\| \in \mathbb{N}$.

We first show that we can restrict to modules of the form $L^\infty(A)$.

Lemma 4.2.2. *Let $A \subseteq X$ and $f : \langle A \rangle_\alpha \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Then,*

$$\|f\| = \|f|_{L^\infty(A)}\|.$$

Here, $f|_{L^\infty(A)}$ denotes the precomposition of f with the canonical inclusion $L^\infty(A) \hookrightarrow L^\infty(X) *_\alpha \Gamma$ into the summand induced by the identity element.

Proof. For $\gamma \in \Gamma$, we define $f_\gamma \colon L^\infty(A) \to N$ by

$$f_\gamma(g) := f\big((\chi_X, \gamma) \cdot (g, 1)\big)$$

for all $g \in L^\infty(A)$. By definition of the ℓ^1-norm on $\langle A \rangle_\alpha$, it is clear that

$$\|f\| = \sup_{\gamma \in \Gamma} \|f_\gamma\|.$$

It thus suffices to show that for $\gamma \in \Gamma$, we have $\|f_\gamma\| \leq \|f_1\|$. Indeed, because f is $L^\infty(X) *_\alpha \Gamma$-linear, for all $g \in L^\infty(A)$, we have

$$\begin{aligned}
\|f_\gamma(g)\|_1 &= \big\|f\big((\chi_X, \gamma) \cdot (g, 1)\big)\big\|_1 \\
&= \big\|(\chi_X, \gamma) \cdot f((g, 1))\big\|_1 \\
&= \|f((g, 1))\|_1 \\
&\leq \|f_1\| \cdot |g|_1,
\end{aligned}$$

where in the penultimate line, we note that multiplication with (χ_X, γ) defines an isometry because the action of γ preserves the probability measure μ. □

For the proof of the explicit description of the operator norm, we need the following computation.

Lemma 4.2.3. *Let $f \colon \langle A \rangle_\alpha \to \langle B \rangle_\alpha$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism, given as in Setup 4.1.8, i.e. by right multiplication with*

$$z := \sum_{(k,\gamma) \in K \times F} a_{k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma),$$

where $a_{k,\gamma} \in Z$, and $F \subseteq \Gamma$ is a finite set, and $(U_k)_{k \in K}$ are pairwise disjoint measurable subsets of B and $\gamma U_k \subseteq A$ whenever $a_{k,\gamma} \neq 0$. For $L \subseteq K \times F$, we define

$$U(L) := \bigcap_{(k,\gamma) \in L} \gamma U_k \;\cap \bigcap_{(k,\gamma) \in (K \times F) \setminus L} A \setminus \gamma U_k.$$

Then, $(U(L))_{L \subseteq K \times F}$ is a family of pairwise disjoint subsets of A, and for $L \subseteq K \times F$,

$$f\big((\chi_{U(L)}, 1)\big) = \sum_{\gamma \in \Gamma} a^L_{*,\gamma} \cdot (\chi_{U(L)}, \gamma),$$

where

$$a^L_{*,\gamma} := \begin{cases} 0, & \text{if } U(L) = \emptyset; \\ a_{k,\gamma}, & \text{if there exists } k \in K \text{ with } (k,\gamma) \in L; \\ 0, & \text{else} \end{cases}$$

Note that because the U_k are pairwise disjoint, there is at most one $k \in K$ with $(k, \gamma) \in L$ unless $U(L) = \emptyset$.

Proof. It is straightforward to check that the $U(L)$ are pairwise disjoint. We note that for $(k, \gamma) \in K \times F$, we have

$$U(L) \cap \gamma U_k = \begin{cases} U(L) & \text{if } (k,\gamma) \in L \\ \emptyset & \text{if } (k,\gamma) \notin L \end{cases}. \tag{4.2.1}$$

Thus, we have

$$\begin{aligned} f\big((\chi_{U(L)}, 1)\big) &= (\chi_{U(L)}, 1) \cdot \sum_{(k,\gamma) \in K \times F} a_{k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma) \\ &= \sum_{(k,\gamma) \in K \times F} a_{k,\gamma} \cdot (\chi_{U(L) \cap \gamma U_k}, \gamma) \\ &= \sum_{(k,\gamma) \in L} a_{k,\gamma} \cdot (\chi_{U(L)}, \gamma) && \text{(Equation (4.2.1))} \\ &= \sum_{\gamma \in \Gamma} a^L_{*,\gamma} \cdot (\chi_{U(L)}, \gamma). \end{aligned}$$

□

We can now prove the explicit description of the operator norm.

Proof of Lemma 4.2.1. Because the operator norm is defined with respect to the ℓ^1-norm on M, we can assume without loss of generality that $M = \langle A \rangle_\alpha$. We will therefore drop the index i from the notation. Let m be the maximum on the right hand side of the claim, i.e.

$$m := \max\Big\{ \sum_{j \in J, (k,\gamma) \in L} |a_{j,k,\gamma}| \;\Big|\; L \subseteq K \times F \text{ s.t. } \mu\big(\bigcap_{(k,\gamma) \in L} \gamma U_k \big) > 0 \Big\}.$$

In order to prove that $\|f\| \geq m$, let $L_m \subseteq K \times F$ be a subset realising the maximum, i.e.

$$m = \sum_{j \in J, (k,\gamma) \in L_m} |a_{j,k,\gamma}| \quad \text{and} \quad \mu\Big(\bigcap_{(k,\gamma) \in L_m} \gamma U_k \Big) > 0.$$

If we assume maximality of L_m, we also have $\mu(U(L_m)) > 0$. We consider the characteristic function $x \coloneqq (\chi_{U(L_m)}, 1)$ as a witness and compute

$$\begin{aligned}
|f(x)|_1 &= \sum_{j\in J} |f_j(\chi_{U(L_m)})|_1 \\
&= \sum_{j\in J} \Big| \sum_{\gamma\in\Gamma} a^{L_m}_{j,*,\gamma} \cdot (\chi_{U(L_m)}, \gamma) \Big|_1 && \text{(Lemma 4.2.3)} \\
&= \sum_{j\in J} \sum_{\gamma\in\Gamma} |a^{L_m}_{j,*,\gamma}| \cdot \mu(U(L_m)) && \text{(Def } \ell^1\text{-norm)} \\
&= m \cdot |x|_1
\end{aligned}$$

Since $|x|_1 = \mu(U(L_m)) > 0$, this proves that $\|f\| \geq m$.

To show the converse inequality, note that there is a canonical isomorphism

$$\langle A \rangle_\alpha \cong \bigoplus_{L \subseteq K\times F} \langle U(L) \rangle_\alpha,$$

where $U(L)$ is defined as in Lemma 4.2.3. Equipping the right hand side with the ℓ^1-norms of the summands, the canonical isomorphism of $L^\infty(X) *_\alpha \Gamma$-modules is an isometry. Thus, it suffices to prove that $|f(x)|_1 \leq m \cdot |x|_1$ for $L \subseteq K \times F$ and $x \in \langle U(L) \rangle_\alpha$. By Lemma 4.2.2, it suffices to consider elements $g \in L^\infty(U(L)) \subseteq \langle U(L) \rangle_\alpha$. The calculation in Lemma 4.2.3 shows that

$$\begin{aligned}
|f(g)|_1 &= \sum_{j\in J} |f_j(g)|_1 \\
&= \sum_{j\in J} \Big| \sum_{\gamma\in\Gamma} a^{L}_{j,*,\gamma} \cdot g \Big|_1 \\
&\leq \Big(\sum_{j\in J} \sum_{\gamma\in\Gamma} |a^{L}_{j,*,\gamma}| \Big) \cdot |g|_1 \\
&\leq m \cdot |g|_1.
\end{aligned}$$

□

Remark 4.2.4. Inspired by this explicit description of the operator norm, we defined the quantity Q (see Definition 4.1.9). From Lemma 4.2.1 we obtain that $\|f\| \leq Q(f)$. Straightforward calculations show that Q also provides an upper bound to the ∞-norm $\|\cdot\|_\infty$, to N_1 and to N_2 [LLMSU25, Definition 2.27].

4.2.2 Marked outer rank

Lemma 4.2.5. *Let $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. Then, in the notation of Setup 4.1.8, we have*

$$\operatorname{mrk}_\alpha f = \sum_{j\in J} \mu\Big(\bigcup_{\substack{(i,k,\gamma)\in I\times K\times F,\\ a_{i,j,k,\gamma}\neq 0}} \gamma^{-1} A_i \cap U_k \Big).$$

Proof. Since the marked outer rank is defined via marked decompositions, we can assume without loss of generality that $N = \langle B \rangle_\alpha$. We will thus drop $j \in J$ from the notation. Let

$$B' := \bigcup_{\substack{(i,k,\gamma)\in I\times K\times F,\\ a_{i,k,\gamma}\neq 0}} \gamma^{-1}A_i \cap U_k.$$

It suffices to show that $\langle B' \rangle_\alpha$ is the smallest marked projective summand of N containing $f(M)$.

We first show that the image is contained in this summand. We denote by $\pi_{B\setminus B'}$ the canonical projection to the marked summand $\langle B \setminus B' \rangle_\alpha$. For all $i \in I$, we have that $\pi_{B\setminus B'} \circ f|_{\langle A_i \rangle_\alpha}$ is given by right-multiplication with the element

$$\begin{aligned}
&(\chi_{A_i}, 1) \cdot \sum_{\substack{(k,\gamma)\in K\times F,\\ a_{i,k,\gamma}\neq 0}} a_{i,k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma) \cdot (\chi_{B\setminus B'}, 1)\\
&= \sum_{\substack{(k,\gamma)\in K\times F,\\ a_{i,k,\gamma}\neq 0}} a_{i,k,\gamma} \cdot (\chi_{A_i \cap \gamma U_k \cap \gamma B \setminus \gamma B'}, \gamma)\\
&= 0,
\end{aligned}$$

because

$$\begin{aligned}
A_i \cap \gamma U_k \cap \gamma B \setminus \gamma B' &\subseteq \gamma(\gamma^{-1}A_i \cap U_k \setminus B')\\
&\subseteq \gamma\big(\gamma^{-1}A_i \cap U_k \setminus (\gamma^{-1}A_i \cap U_k)\big)\\
&= \emptyset.
\end{aligned}$$

This shows that $f(M) \subseteq \langle B' \rangle_\alpha$.

Conversely, let $(i_0, k_0, \gamma_0) \in I \times K \times F$ such that $a_{i_0,k_0,\gamma_0} \neq 0$. It suffices to show that $\langle \gamma_0^{-1} A_{i_0} \cap U_{k_0} \rangle_\alpha$ is contained in every marked projective summand containing $f(M)$. Indeed, let $\pi_1 \colon L^\infty(X) *_\alpha \Gamma \to L^\infty(X)$ be the $L^\infty(X)$-linear projection to the summand indexed by $1 \in \Gamma$. We have

$$\begin{aligned}
&\pi_1 \circ f\big((\chi_{U_{k_0}}, \gamma_0^{-1}) \cdot (\chi_{A_{i_0}}, 1)\big)\\
&= \pi_1 \circ f(\chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0}}, \gamma_0^{-1})\\
&= \pi_1\bigg((\chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0}}, \gamma_0^{-1}) \cdot \sum_{(k,\gamma)\in K\times F} a_{i_0,k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma)\bigg)\\
&= \pi_1\bigg(\sum_{(k,\gamma)\in K\times F} a_{i_0,k,\gamma} \cdot (\chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0} \cap \gamma_0^{-1}\gamma U_k}, \gamma_0^{-1}\gamma)\bigg)\\
&= \sum_{k\in K} a_{i_0,k,\gamma_0} \cdot \chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0} \cap U_k} && \text{(projection)}\\
&= a_{i_0,k_0,\gamma_0} \cdot \chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0}} && \text{($(U_k)_k$ pairwise disjoint)}
\end{aligned}$$

Since $a_{i_0,k_0,\gamma_0} \neq 0$, we obtain from Lemma 4.1.6 that $\langle U_{k_0} \cap \gamma_0^{-1} A_{i_0} \rangle_\alpha$ is contained in the marked direct summand generated by $f(M)$. □

4.3 Measured embedding dimension and volume

We consider the quantities medim_n^Z and mevol_n. We first fix the following setup.

Setup 4.3.1. Let Γ be a countable group, $\alpha\colon \Gamma \curvearrowright (X,\mu)$ be an essentially free probability measure preserving (pmp) action on a standard probability space.

Let $n \in \mathbb{N}$. We assume that the group Γ is of type FP_{n+1}. Thus, we can fix a free $Z\Gamma$-resolution $C_* \twoheadrightarrow Z$ of the trivial $Z\Gamma$-module Z with finitely generated free modules in degrees $\leq n+1$. By Schanuel's lemma [Bro82a, Proposition VIII.4.3], we can additionally assume that $C_0 = Z\Gamma$ and that the augmentation map $\eta\colon Z\Gamma \to Z$ is given by mapping all $\gamma \in \Gamma$ to $1 \in Z$.

We now tensor with $L^\infty(X) *_\alpha \Gamma$ and obtain a free $L^\infty(X) *_\alpha \Gamma$-resolution

$$C_*^\alpha \coloneqq (L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} C_* \twoheadrightarrow (L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} Z \cong L^\infty(X),$$

that is finitely generated in degrees $\leq n+1$. The isomorphism is an isomorphism in the category of $L^\infty(X) *_\alpha \Gamma$-modules. Recall that $Z\Gamma$ embeds into $L^\infty(X) *_\alpha \Gamma$ as an extension of the embedding $Z \hookrightarrow L^\infty(X)$ via constant functions. Note that $L^\infty(X) *_\alpha \Gamma$ is a flat $Z\Gamma$-module [LLMSU25, Proposition 2.10].

Note that by elementary homological algebra, the subsequent definitions of medim_n^Z and mevol_n will be independent of the choice of C_*.

Definition 4.3.2 (α-embedding)**.** An *α-embedding* is a pair consisting of a marked projective augmented $L^\infty(X) *_\alpha \Gamma$-chain complex $D_* \twoheadrightarrow L^\infty(X)$ and an $L^\infty(X) *_\alpha \Gamma$-chain map $C_*^\alpha \to D_*$ extending the identity on $L^\infty(X)$ in degree -1. We write $\mathsf{A}(\alpha)$ for the class of all marked projective augmented complexes arising in α-embeddings.

For every real-valued isomorphism invariant Δ of marked projective and augmented $L^\infty(X) *_\alpha \Gamma$-chain complexes, we define

$$\Delta(\alpha) \coloneqq \inf_{(D_* \twoheadrightarrow L^\infty(X)) \in \mathsf{A}(\alpha)} \Delta\big(D_* \twoheadrightarrow L^\infty(X)\big).$$

The two most important instances will be the following:

Definition 4.3.3 (medim, mevol [LLMSU25, Section 1.1])**.** In the situation of Setup 4.3.1, we define the following:

- For the *measured embedding dimension* over Z in degree n, we set

$$\Delta\big(D_* \twoheadrightarrow L^\infty(X)\big) \coloneqq \dim_\alpha D_n.$$

  and define $\operatorname{medim}_n^Z(\alpha) \coloneqq \Delta(\alpha)$.

- In the case that $Z = \mathbb{Z}$, for the *measured embedding volume* in degree n, we set
$$\Delta\big(D_* \twoheadrightarrow L^\infty(X)\big) := \operatorname{lognorm}_\alpha \partial_{n+1}^{D_*}.$$
and define $\operatorname{mevol}_n(\alpha) := \Delta(\alpha)$.

The main motivation for introducing medim and mevol is the bound to (torsion) homology growth they provide.

Theorem 4.3.4 ([LLMSU25, Theorem 1.2]). *Let $n \in \mathbb{N}$ and Γ be a residually finite group of type FP_{n+1}. Let Λ_* be a residual chain or the system of all finite-index normal subgroups of Γ. Then,*

$$\widehat{b}_n(\Gamma, \Lambda_*; Z) \le \operatorname{medim}_n^Z(\Gamma \curvearrowright \widehat{\Lambda}_*)$$
$$\widehat{t}_n(\Gamma, \Lambda_*) \le \operatorname{mevol}_n(\Gamma \curvearrowright \widehat{\Lambda}_*).$$

Note that the profinite completion $\widehat{\Lambda}_*$ was defined in the Introduction (Definition 11) for the system of all finite-index normal subgroups of Γ and is defined similarly for a single chain (see Example 4.4.3).

Remark 4.3.5. Instead of working over the crossed product ring $L^\infty(X) *_\alpha \Gamma$, we could also define medim and mevol over the equivalence relation ring, which is defined as follows: Let

$$\mathcal{R} := \mathcal{R}_{\Gamma \curvearrowright X} := \big\{(x, \gamma \cdot x) \bigm| x \in X, \gamma \in \Gamma\big\}$$

be the *orbit relation* of α. Note that the measurable structure on $\mathcal{R}$ is induced by the one on $X \times X$. We equip $\mathcal{R}$ with the measure ν that is defined as follows: For all $A \subseteq \mathcal{R}$, let
$$\nu(A) := \int_X \#(A \cap (\{x\} \times X)) \, d\mu(x).$$
Let $Z\mathcal{R}$ be the *equivalence relation ring* of $\mathcal{R}$ with coefficients in Z, i.e.,

$$\begin{aligned} Z\mathcal{R} := \big\{f \in L^\infty(\mathcal{R}; Z) \mid\ & \sup_{x\in X} \#\{y \mid f(x,y) \neq 0\} < \infty, \\ & \sup_{y \in X} \#\{x \mid f(x,y) \neq 0\} < \infty\big\} \end{aligned}$$

and this ring carries the convolution product defined by

$$(f \cdot g)(x, y) := \sum_{z \in [x]_\mathcal{R}} f(x, z) g(z, y)$$

for all $x, y \in X$ and the topology induced by the ∞-norm. If $\alpha\colon \Gamma \curvearrowright (X, \mu)$ is essentially free, there is a canonical inclusion $L^\infty(X) *_\alpha \Gamma \hookrightarrow Z\mathcal{R}$, defined by

$$(f, \gamma) \mapsto \big((\gamma^\alpha \cdot x, x) \mapsto f(\gamma^\alpha \cdot x)\big)$$

for all $f \in L^\infty(X)$ and $\gamma \in \Gamma$. Moreover, the image of this inclusion is ℓ^1-dense in $Z\mathcal{R}$. It can be shown by approximation techniques that we obtain the same notions of mevol and medim when replacing the crossed product ring $L^\infty(X) *_\alpha \Gamma$ with the equivalence relation ring $Z\mathcal{R}$ in Definition 4.3.3. In general, it could be advantageous to work with the equivalence relation ring to tackle problems related to orbit equivalence. However, in our setup, the crossed product ring simplifies the arguments for the monotonicity under weak containment (see Section 4.7).

4.4 Weak containment

We briefly recall the definition of weak containment, a few examples, and a useful characterisation. This section summarises results from the literature, and does *not* rely on the previous sections.

Definition 4.4.1 (weak containment [Kec10, p. 64])**.** Let Γ be a group and $\alpha\colon \Gamma \curvearrowright (X,\mu)$ and $\beta\colon \Gamma \curvearrowright (Y,\nu)$ be probability measure preserving actions of Γ on standard probability spaces. We say that α is *weakly contained* in β (in symbols $\alpha \prec \beta$) if for all $n \in \mathbb{N}$, measurable sets $A_1,\dots,A_n \subseteq X$, finite sets $F \subseteq \Gamma$ and $\epsilon > 0$, there are measurable sets $B_1,\dots,B_n \subseteq Y$ such that

$$\forall_{\gamma\in F}\ \forall_{i,j\in\{1,\dots,n\}}\ |\mu(\gamma^\alpha(A_i)\cap A_j) - \nu(\gamma^\beta(B_i)\cap B_j)| < \epsilon.$$

Example 4.4.2. Let Γ be a group and $\alpha\colon \Gamma \curvearrowright (X,\mu)$ and $\beta\colon \Gamma \curvearrowright (Y,\nu)$ be probability measure preserving actions of Γ on standard probability spaces. It is straightforward to check that $\alpha \prec \alpha\times\beta$, where $\alpha\times\beta\colon \Gamma \curvearrowright (X\times Y, \mu\otimes\nu)$ is the product action.

Example 4.4.3. Let Γ be a residually finite group and Λ_* be a residual chain of Γ. Let (X,μ) be the inverse limit of the system $(\Gamma/\Lambda_i)_{i\in\mathbb{N}}$, equipped with the Haar measure. Then, we have an action $\Gamma \curvearrowright X$ via left translation. This action is weakly contained in the profinite completion action $\Gamma \curvearrowright \widehat{\Gamma}$ (see Definition 11 in the Introduction).

For countably infinite groups, there is a smallest action with respect to weak containment.

Example 4.4.4 (Bernoulli shift)**.** Let (X,μ) be a non-trivial probability space (i.e. μ is *not* concentrated in one point) and Γ be a countably infinite group. The *Bernoulli shift* of Γ on X is the action of Γ on $\prod_\Gamma X$ (endowed with the product measure) via shifting of the factors. Abért and Weiss proved that every Bernoulli shift of Γ is weakly contained in every free probability measure preserving action of Γ [AW13, Theorem 1].

Example 4.4.5 (amenable groups)**.** Let Γ be an infinite amenable group and $\alpha\colon \Gamma \curvearrowright (X,\mu)$ and $\beta\colon \Gamma \curvearrowright (Y,\nu)$ be free probability measure preserving actions on standard probability spaces. Then, $\alpha \prec \beta$ [Kec10, p. 91].

Definition 4.4.6 (ergodic action)**.** An action $\Gamma \curvearrowright (X, \mu)$ is *ergodic* if for every measurable subset $A \subseteq X$ with $\Gamma \cdot A = A$, we have

$$\mu(A) = 0 \quad \text{or} \quad \mu(X \setminus A) = 0.$$

Definition 4.4.7 (EMD* [Kec12, Definition 4.4, Proposition 4.5])**.** An infinite countable residually finite group Γ satisfies EMD* if every ergodic standard probability action of Γ is weakly contained in the profinite completion action $\Gamma \curvearrowright \widehat{\Gamma}$.

For the examples below, note that Tucker-Drob proved that for all groups property EMD* is equivalent to a similarly defined property MD [Tuc15, Theorem 1.4].

Example 4.4.8 (EMD*)**.** The following groups satisfy EMD*:

- countable free groups [Kec12, Theorem 3.1, Proposition 4.5]
- residually finite infinite amenable groups [Kec10, Proposition 13.2]
- fundamental groups of closed surfaces [BT13, Theorem 1.4]
- fundamental groups of closed hyperbolic 3-manifolds [FLPS16, Corollary 3.11]

More examples can be found in the survey by Burton and Kechris [BK20, pp. 2698f].

Many dynamical invariants are monotone under weak containment (see cost [Kec10, Corollary 10.14] and integral foliated simplicial volume [FLPS16, Theorem 1.5]). In Theorem 4.7.1, we will prove monotonicity of measured embedding dimension and measured embedding volume under weak containment. We will work with a characterisation of weak containment using weak neighbourhoods.

Definition 4.4.9 (weak neighbourhoods, [Kec10, Section 1(B)])**.** Let Γ be a group and (X, μ) be a standard probability space. The *weak topology* on the space of probability measure preserving actions $\Gamma \curvearrowright (X, \mu)$ is defined by the following basic open neighbourhoods: Let $\alpha \colon \Gamma \curvearrowright X$, $F \subseteq \Gamma$ be finite and $n \in \mathbb{N}$, $A_1, \dots, A_n \subseteq X$ be measurable and $\varepsilon \in \mathbb{R}_{>0}$. Then,

$$\bigl\{\beta \colon \Gamma \curvearrowright X \bigm| \forall_{\gamma \in F}\ \forall_{i \in \{1,\dots,n\}}\ \mu\bigl((\gamma^\alpha A_i) \triangle (\gamma^\beta A_i)\bigr) < \varepsilon\bigr\}$$

is open in the weak topology.

Proposition 4.4.10 ([Kec10, Proposition 10.1])**.** *Let $\alpha, \beta \colon \Gamma \curvearrowright (X, \mu)$ be probability measure preserving actions. Then, α is weakly contained in β if and only if in every weak neighbourhood U of α, there is $\beta' \in U$ such that β' is isomorphic to β (as actions on standard probability spaces).*

4.5 Translating actions

Let α and β be probability measure preserving actions of a group Γ on a standard probability space (X,μ). In this section, we consider modules or maps defined over $L^\infty(X) *_\alpha \Gamma$, and we produce "corresponding" $L^\infty(X) *_\beta \Gamma$-modules or $L^\infty(X) *_\beta \Gamma$-chain maps.

4.5.1 Definition and direct consequences

Definition 4.5.1 (translation of modules)**.** Let $M = \bigoplus_{i\in I} \langle A_i \rangle_\alpha$ be a marked projective module over $L^\infty(X) *_\alpha \Gamma$. We define the *translation* of M to β as the marked projective module M_β over $L^\infty(X) *_\beta \Gamma$ via

$$M_\beta \coloneqq \bigoplus_{i\in I} \langle A_i \rangle_\beta.$$

Remark 4.5.2. It is immediate from the definition of the dimension (Definition 4.1.3) that $\dim_\beta M_\beta = \dim_\alpha M$.

We can also translate maps to the action β.

Definition 4.5.3. Let $f\colon (L^\infty(X) *_\alpha \Gamma)^m \to (L^\infty(X) *_\alpha \Gamma)^n$ be a linear map between marked free $L^\infty(X) *_\alpha \Gamma$-modules. Recall from Setup 4.1.8 that f is given by right multiplication with a matrix z over $L^\infty(X) *_\alpha \Gamma$, where

$$z_{i,j} = \sum_{(k,\gamma)\in K\times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma^\alpha U_k}, \gamma),$$

and $(U_k)_{k\in K}$ is a finite family of disjoint subsets of X, the set $F \subseteq \Gamma$ is finite, and $a_{i,j,k,\gamma} \in Z$. We define the *translation of f to β* to be the $L^\infty(X) *_\beta \Gamma$-linear map $f_\beta\colon (L^\infty(X) *_\beta \Gamma)^m \to (L^\infty(X) *_\beta \Gamma)^n$ defined by right multiplication with the matrix $z_\beta = ((z_\beta)_{i,j})_{i\in I, j\in J}$ that is defined by

$$(z_\beta)_{i,j} = \sum_{(k,\gamma)\in K\times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma^\beta U_k}, \gamma).$$

It is straightforward to show that f_β is well-defined and does not depend on the chosen presentation of z in Setup 4.1.8.

More generally, let $f\colon M = \bigoplus_{i\in I} \langle A_i \rangle_\alpha \to N = \bigoplus_{j\in J} \langle B_j \rangle_\alpha$ be a linear map between marked projective $L^\infty(X) *_\alpha \Gamma$-modules. We define its *translation to β* by

$$f_\beta \coloneqq \pi_{N_\beta} \circ (\iota_N \circ f \circ \pi_M)_\beta \circ \iota_{M_\beta} \colon M_\beta \to N_\beta,$$

where $\iota_N\colon N \to (L^\infty(X) *_\alpha \Gamma)^{|J|}$ and $\iota_{M_\beta}\colon M_\beta \to (L^\infty(X) *_\beta \Gamma)^{|I|}$ denote the canonical marked inclusions and the maps $\pi_{N_\beta}\colon (L^\infty(X) *_\beta \Gamma)^{|J|} \to N_\beta$ as well as $\pi_M\colon (L^\infty(X) *_\alpha \Gamma)^{|I|} \to M$ denote the canonical marked projections.

Remark 4.5.4. The action α is replaced by β in three spots:

- In the generation of modules: we generated an $L^\infty(X) *_\beta \Gamma$-module instead of one over $L^\infty(X) *_\alpha \Gamma$.
- In the multiplication of the matrix: we multiply over $L^\infty(X) *_\beta \Gamma$.
- In the coefficients: we multiply with $\chi_{\gamma^\beta U_k}$, where γ now acts via β on U_k. Previously, we considered the action via α.

Remark 4.5.5. From Definition 4.1.9, it follows that Q is monotone under translation, i.e. $Q(f_\beta) \leq Q(f)$.

For complexes obtained from $Z\Gamma$-chain complexes by tensoring, there is an easy description of the translation.

Lemma 4.5.6. *Let $C_* \twoheadrightarrow Z$ be a free $Z\Gamma$-chain complex. Then, there is a canonical isomorphism of $L^\infty(X) *_\beta \Gamma$-chain complexes*

$$\big((L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} C_*\big)_\beta \cong (L^\infty(X) *_\beta \Gamma) \otimes_{Z\Gamma} C_*.$$

If $C_0 = Z\Gamma$ and the augmentation map $\eta\colon Z\Gamma \to Z$ is given by sending all $\gamma \in \Gamma$ to $1 \in Z$, then we can define an augmentation map $(L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} C_0 \to L^\infty(X)$ and the isomorphism can be extended by the identity on $L^\infty(X)$ in degree -1.

Proof. The tensor product and the translation are compatible with direct sums, so we can work componentwise. Without loss of generality, we assume $C_j \cong_{Z\Gamma} Z\Gamma$. Then, we have isomorphisms of $L^\infty(X) *_\beta \Gamma$-modules

$$\begin{aligned}\big((L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} C_j\big)_\beta &\cong \big(L^\infty(X) *_\alpha \Gamma\big)_\beta \\ &= \big(\langle X \rangle_\alpha\big)_\beta \\ &= \langle X \rangle_\beta \\ &\cong (L^\infty(X) *_\beta \Gamma) \otimes_{Z\Gamma} C_j.\end{aligned}$$

For the boundary maps, because all of the above isomorphisms are compatible with direct sums, we can suppose that $\partial_{j+1}\colon Z\Gamma \to Z\Gamma$ is given by right-multiplication with

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma.$$

Then, $(L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} \partial_{j+1}$ is given by right multiplication with

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot (\chi_X, \gamma) = \sum_{\gamma \in \Gamma} a_\gamma \cdot (\chi_{\gamma^{\alpha} X}, \gamma).$$

Thus, its translation $\big((L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} \partial_{j+1}\big)_\beta$ to β is given by

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot (\chi_{\gamma^{\beta} X}, \gamma).$$

Because $\gamma^\beta X = \gamma^\alpha X$, this agrees with $(L^\infty(X) *_\beta \Gamma) \otimes_{Z\Gamma} \partial_{j+1}$. For the extension to degree -1, we define $\eta\colon (L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} C_0 \cong L^\infty(X) *_\alpha \Gamma \to L^\infty(X)$ as the $L^\infty(X)$-linear extension of $\eta((\lambda, \gamma)) := \gamma\lambda$ for all $\lambda \in L^\infty(X)$ and $\gamma \in \Gamma$. Note that this map is $(L^\infty(X) *_\alpha \Gamma)$-linear. $\square$

4.5.2 Almost functoriality of translation

Compositions behave well under translation in the following sense:

Lemma 4.5.7 (composition estimate)**.** *Let $f\colon M \to N$ and $g\colon N \to P$ be linear maps over $L^\infty(X) *_\alpha \Gamma$ and $\delta \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, we have*

$$(g \circ f)_\beta =_\delta g_\beta \circ f_\beta.$$

Proof. By Lemma 4.1.20 and Lemma 4.1.18, we can assume that $M = \langle A \rangle_\alpha$, $N = \langle B \rangle_\alpha$ and $P = \langle C \rangle_\alpha$. By Lemma 4.1.5, we are in the situation of Setup 4.1.8, i.e. f and g are given by multiplication with elements $z_f, z_g \in L^\infty(X) *_\alpha \Gamma$, respectively. Since sums behave well with almost equality (Lemma 4.1.18), we can assume without loss of generality that

$$z_f = a \cdot (\chi_{\gamma^\alpha W}, \gamma) \quad \text{and} \quad z_g = b \cdot (\chi_{\eta^\alpha V}, \eta).$$

where $a, b \in Z$, $\gamma, \eta \in \Gamma$ and $W \subseteq B$ and $V \subseteq C$ are measurable subsets. We define U to be the weak neighbourhood of α defined by $F := \{\eta^{-1}\}, n := 2, A_1 := W, A_2 := V$, and $\epsilon := \delta$ (in the notation used in Definition 4.4.9).

Then, $g \circ f$ is given by right-multiplication with

$$\begin{aligned} z_f \cdot z_g &= ab \cdot (\chi_{\gamma^\alpha W \cap \gamma^\alpha \eta^\alpha V}, \gamma\eta) \\ &= ab \cdot (\chi_{(\gamma\eta)^\alpha((\eta^{-1})^\alpha W \cap V)}, \gamma\eta) \end{aligned}$$

Thus, for all $\beta \in U$, the translation $(g \circ f)_\beta$ is given by right-multiplication with

$$ab \cdot (\chi_{(\gamma\eta)^\beta((\eta^{-1})^{\textcircled{\alpha}} W \cap V)}, \gamma\eta) \tag{4.5.1}$$

Similarly, $g_\beta \circ f_\beta$ is given by right-multiplication with

$$ab \cdot (\chi_{(\gamma\eta)^\beta((\eta^{-1})^{\textcircled{\beta}} W \cap V)}, \gamma\eta) \tag{4.5.2}$$

Note that the expressions in Equation (4.5.1) and Equation (4.5.2) differ only by an α resp. β in the exponent of η^{-1} (indicating via which action η^{-1} is acting on W; this is highlighted with a circle in the equation). Thus, by Lemma 4.1.17, $(g \circ f)_\beta$ and $g_\beta \circ f_\beta$ are almost equal with error at most

$$\begin{aligned} &\mu\big((\gamma\eta)^\beta((\eta^{-1})^\alpha W \cap V) \triangle (\gamma\eta)^\beta((\eta^{-1})^\beta W \cap V)\big) \\ &= \mu\big(((\eta^{-1})^\alpha W \cap V) \triangle ((\eta^{-1})^\beta W \cap V)\big) \\ &\leq \delta, \end{aligned}$$

where the last inequality is given by the choice of the weak neighbourhood U. $\square$

Corollary 4.5.8 (translation of chain complexes)**.** *Let $n \in \mathbb{N}$, $\delta \in \mathbb{R}_{>0}$ and (D_*, η) be a marked projective chain complex over $L^\infty(X) *_\alpha \Gamma$ with an augmentation map $\eta\colon D_0 \to L^\infty(X)$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, the translated sequence $((D_*)_\beta, \eta_\beta)$ is a marked projective (δ, n)-almost chain complex.*

Proof. We apply Lemma 4.5.7 multiple times. Define U to be the intersection of the neighbourhoods where the estimate holds for $\partial_r \circ \partial_{r+1}$ for $r \in \{0, \dots, n\}$. Moreover, because η is an augmentation, there exists $z \in D_0$ with $\eta(z) = 1$. Thus, in a suitable neighbourhood, we have $\eta_\beta(z_\beta) =_\delta 1_\beta = 1$. Note that z_β was also defined in Definition 4.5.3. □

Corollary 4.5.9 (translation of chain maps)**.** *Let $n \in \mathbb{N}$, $\delta \in \mathbb{R}_{>0}$ and $f_*\colon C_* \to D_*$ be an $L^\infty(X) *_\alpha \Gamma$-chain map between marked projective chain complexes (C_*, ζ) and (D_*, η) extending the identity on $L^\infty(X)$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, the translations of the chain complexes $(C_*)_\beta$ and $(D_*)_\beta$ are marked projective (δ, n)-almost chain complexes and moreover, the map $(f_*)_\beta\colon (C_*)_\beta \to (D_*)_\beta$ is a (δ, n)-almost chain map.*

Proof. We apply the above Corollary 4.5.8 to C_* and D_* and intersect the resulting neighbourhoods. Moreover, we apply Lemma 4.5.7 with error term $\frac{\delta}{2}$ to the compositions $\eta \circ f_0$, $\partial_r^D \circ f_r$ and $f_{r-1} \circ \partial_r^C$ for $r \in \{1, \dots, n+1\}$. We intersect all resulting neighbourhoods of α to obtain a neighbourhood U. In degree -1, for all $\beta \in U$, we have

$$\eta_\beta \circ (f_0)_\beta =_{\delta/2} (\eta \circ f_0)_\beta = \zeta_\beta.$$

Moreover, for $r \in \{1, \dots, n+1\}$, we have

$$\begin{aligned}(\partial_r^D)_\beta \circ (f_r)_\beta &=_{\delta/2} (\partial_r^D \circ f_r)_\beta \\ &= (f_{r-1} \circ \partial_r^C)_\beta \\ &=_{\delta/2} (f_{r-1})_\beta \circ (\partial_r^C)_\beta\end{aligned}$$

Thus by Lemma 4.1.19, we obtain that $(\partial_r^D)_\beta \circ (f_r)_\beta =_\delta (f_{r-1})_\beta \circ (\partial_r^C)_\beta$. □

4.5.3 Continuous change of invariants

We show that the marked outer rank, the norm and lognorm change continuously in the action in the following sense:

Lemma 4.5.10 (translation and marked outer rank)**.** *Let $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective modules and $\delta \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, we have $\operatorname{mrk}_\beta f_\beta \leq \operatorname{mrk}_\alpha f + \delta$.*

Proof. By Lemma 4.2.5, we have (in the notation of Setup 4.1.8)

$$\mathrm{mrk}_\alpha f = \sum_{j\in J} \mu\Big(\bigcup_{\substack{(i,k,\gamma)\in I\times K\times F,\\ a_{i,j,k,\gamma}\neq 0}} ((\gamma^{-1})^\alpha A_i \cap U_k)\Big).$$

Thus, for every action β, we have

$$\begin{aligned}\mathrm{mrk}_\beta f_\beta - \mathrm{mrk}_\alpha f &\leq \sum_{\substack{(i,j,k,\gamma)\in I\times J\times K\times F,\\ a_{i,j,k,\gamma}\neq 0}} \mu\big(((\gamma^{-1})^\alpha A_i \cap U_k) \triangle ((\gamma^{-1})^\beta A_i \cap U_k)\big)\\ &\leq \sum_{(i,j,k,\gamma)\in I\times J\times K\times F} \mu\big(((\gamma^{-1})^\alpha A_i \cap U_k) \triangle ((\gamma^{-1})^\beta A_i \cap U_k)\big)\\ &\leq \delta,\end{aligned}$$

where the last inequality holds in a suitable weak neighbourhood U that is defined as in Definition 4.4.9 with error term $\epsilon := \frac{\delta}{|I|\cdot|J|\cdot|K|\cdot|F|}$ (or $\epsilon := 1$ if the denominator is zero), and the $(A_i)_{i\in I}$ as the test sets. □

Lemma 4.5.11 (translation and norm)**.** *Let $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective $L^\infty(X) *_\alpha \Gamma$-modules and $\delta \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, the following holds: There is an $L^\infty(X) *_\beta \Gamma$-homomorphism $f'_\beta\colon M_\beta \to N_\beta$ such that*

$$f_\beta =_{\delta,Q(f)} f'_\beta \quad \textit{and} \quad \|f'_\beta\| \leq \|f\| \quad \textit{and} \quad f'_\beta(M_\beta) \subseteq f_\beta(M_\beta).$$

Proof. By an analogue of Lemma 4.1.20, we can assume that $M = \langle A \rangle_\alpha$. We will thus drop $i \in I$ from the notation. Let P be a presentation representing f as in Setup 4.1.8 such that $Q(f,P) = Q(f)$. Fix the notation of that setup. Pick a weak neighbourhood U of α where for all $\gamma \in F, k \in K$ and $\beta \in U$, we have

$$\mu\big(\gamma^\alpha U_k \triangle \gamma^\beta U_k\big) \leq \frac{\delta}{|K|\cdot|F|}.$$

Let $\beta \in U$. We define

$$A'' := \bigcup_{(k,\gamma)\in K\times F} \gamma^\alpha U_k \triangle \gamma^\beta U_k$$

and $A' := A \setminus A''$. We set $M' := \langle A' \rangle_\beta$ and $M'' := \langle A'' \cap A \rangle_\beta$. Hence, we have an isomorphism of $L^\infty(X) *_\beta \Gamma$-modules $M_\beta \cong M' \oplus M''$. By construction, we have that $\dim_\beta M'' \leq \mu(A'') \leq \delta$. We define $f'_\beta := f_\beta \circ \iota_{A'} \circ \pi_{A'}$, where $\pi_{A'}$ is the projection onto the marked summand M' and $\iota_{A'} : M' \to M$ is the canonical marked embedding. In particular,

$$f'_\beta|_{M'} = f_\beta|_{M'} \quad \text{and} \quad f'_\beta|_{M''} = 0$$

and $f'_\beta(M_\beta) \subseteq f_\beta(M_\beta)$. Moreover, note that for $L \subseteq K \times F$, we have

$$A' \cap \bigcap_{(k,\gamma)\in L} \gamma^{\beta} U_k \subseteq \bigcap_{(k,\gamma)\in L} \gamma^{\alpha} U_k. \tag{4.5.3}$$

Thus, by the explicit description of the operator norm (Lemma 4.2.1), we have

$$\begin{aligned} \|f'_\beta\| &= \max\Bigl\{ \sum_{j\in J,(k,\gamma)\in L} |a_{j,k,\gamma}| \Bigm| L \subseteq K \times F \text{ with } \mu\Bigl(A' \cap \bigcap_{(k,\gamma)\in L} \gamma^\beta U_k\Bigr) > 0\Bigr\} && \text{(Lemma 4.2.1)} \\ &\leq \max\Bigl\{ \sum_{j\in J,(k,\gamma)\in L} |a_{j,k,\gamma}| \Bigm| L \subseteq K \times F \text{ with } \mu\Bigl(\bigcap_{(k,\gamma)\in L} \gamma^\alpha U_k\Bigr) > 0\Bigr\} && \text{(Equation (4.5.3))} \\ &= \|f\|. && \text{(Lemma 4.2.1)} \end{aligned}$$

It remains to estimate $\|f_\beta|_{M''}\|$: we have

$$\begin{aligned} \|f_\beta|_{M''}\| &\leq \|f_\beta\| \\ &\leq Q(f_\beta) && \text{(Remark 4.2.4)} \\ &\leq Q(f). && \text{(Remark 4.5.5)} \end{aligned}$$

□

We use these two estimates to show that also lognorm is continuous in the action.

Lemma 4.5.12 (translation and lognorm). *Let $f\colon M \to N$ be an $L^\infty(X) *_\alpha \Gamma$-homomorphism between marked projective modules and $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, we have*

$$\operatorname{lognorm}_\beta f_\beta \leq \operatorname{lognorm}_\alpha f + \epsilon.$$

Proof. We set $\delta := \frac{\epsilon}{4\cdot\log_+ Q(f)} > 0$ (or $\delta := 1$ if $\log_+ Q(f) = 0$). By definition of $\operatorname{lognorm}_\alpha$, there exists a marked decomposition $(M_i)_{i\in I}$ of M over $L^\infty(X) *_\alpha \Gamma$ with

$$\sum_{i\in I} \operatorname{lognorm}'_\alpha(f|_{M_i}\colon M_i \to N) = \operatorname{lognorm}'_\alpha\bigl(f, (M_i)_{i\in I}\bigr) < \operatorname{lognorm}_\alpha f + \frac{\epsilon}{2}. \tag{4.5.4}$$

To simplify notation, we will assume that $M = M_i$ consists of a single summand and estimate the change of $\operatorname{lognorm}'_\alpha(f|_{M_i})$ under translation. (The general case will then follow by dividing the allowed error by $|I|$, which stays constant during this proof.)

By Lemma 4.5.10, there exists a weak neighbourhood U_1 of α, such that for all $\beta \in U_1$, we have $\operatorname{mrk}_\beta f_\beta \leq \operatorname{mrk}_\alpha f + \delta$. Moreover, by Lemma 4.5.11, there exists a weak neighbourhood U_2 of α, such that for $\beta \in U_2$, there exists $f'_\beta\colon M_\beta \to N_\beta$ with

$$f_\beta =_{\delta,Q(f)} f'_\beta \quad\text{and}\quad \|f'_\beta\| \leq \|f\| \quad\text{and}\quad f'_\beta(M_\beta) \subseteq f_\beta(M_\beta).$$

We define $U := U_1 \cap U_2$. Then, for all $\beta \in U$, we have

$$\begin{aligned}
&\operatorname{lognorm}_\beta f_\beta \\
\le\ &\operatorname{lognorm}_\beta(f'_\beta) + \delta \cdot \log_+ Q(f) && \text{(Lemma 4.1.22)} \\
\le\ &\operatorname{lognorm}'_\beta(f'_\beta) + \delta \cdot \log_+ Q(f) && \text{(Def. of lognorm)} \\
=\ &\min\bigl\{\dim_\beta M_\beta, \operatorname{mrk}_\beta(f'_\beta)\bigr\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) \\
=\ &\min\bigl\{\dim_\alpha M, \operatorname{mrk}_\beta(f'_\beta)\bigr\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) && \text{(Remark 4.5.2)} \\
\le\ &\min\bigl\{\dim_\alpha M, \operatorname{mrk}_\beta(f_\beta)\bigr\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) && (f'_\beta(M_\beta) \subseteq f_\beta(M_\beta)) \\
\le\ &\min\bigl\{\dim_\alpha M, \operatorname{mrk}_\alpha f + \delta\bigr\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) && (\beta \in U_1) \\
\le\ &\min\bigl\{\dim_\alpha M, \operatorname{mrk}_\alpha f + \delta\bigr\} \cdot \log_+ \|f\| + \delta \cdot \log_+ Q(f) && (\|f'_\beta\| \le \|f\|) \\
\le\ &\min\bigl\{\dim_\alpha M, \operatorname{mrk}_\alpha f\bigr\} \cdot \log_+ \|f\| + 2\delta \cdot \log_+ Q(f) && \text{(Remark 4.2.4)} \\
\le\ &\operatorname{lognorm}'_\alpha(f) + \frac{\epsilon}{2} && \text{(Def. of } \delta) \\
\le\ &\operatorname{lognorm}_\alpha f + \epsilon && \text{(Equation (4.5.4))} \ \square
\end{aligned}$$

4.6 Strictification

The two main results of this section are the following:

- Every almost chain complex is 'close' to a strict chain complex (Theorem 4.6.3).
- Every almost chain map is 'close' to a strict chain map (Theorem 4.6.4).

We call the construction of a strict chain complex (resp. map) 'strictification' of the almost chain complex (resp. map). The results were previously proved in the joint paper [LLMSU25], and we only deduce more abstract versions of these theorems here.

Let $\alpha\colon \Gamma \curvearrowright (X, \mu)$ be an essentially free probability measure preserving action on a standard probability space. We consider modules over the crossed product ring $L^\infty(X) *_\alpha \Gamma$.

4.6.1 Chain complexes

We bound the complexity of the input data by the following notion:

Definition 4.6.1 (translation-invariant constant)**.** Let (X, μ) be a standard probability space, Γ be a group and $n \in \mathbb{N}$. A *translation-invariant constant* is a family of maps $\kappa = (\kappa_\delta)_{\delta \in \mathbb{R}_{>0}}$ that, given $\delta \in \mathbb{R}_{>0}$, an essentially free probability measure preserving action $\alpha\colon \Gamma \curvearrowright (X, \mu)$ and an (augmented) marked projective (δ, n)-almost chain complex (D_*, η) of $L^\infty(X) *_\alpha \Gamma$-modules, assigns a positive real number $\kappa_\delta(D_*, \eta)$ such that the following holds:

Let $\delta \in \mathbb{R}_{>0}$, $\alpha\colon \Gamma \curvearrowright X$ and (D_*, η) be a marked projective $(\delta/2, n)$-almost chain complex over $L^\infty(X) *_\alpha \Gamma$. Then, there exists a neighbourhood U of α (see

Definition 4.4.9) such that for every $\beta \in U$, we have that $((D_*)_\beta, \eta_\beta)$ is a (δ, n)-almost chain complex and

$$\kappa_\delta((D_*)_\beta, \eta_\beta) \leq \kappa_{\delta/2}(D_*, \eta) + 1.$$

We often write $\kappa_\delta(D_*)$ instead of $\kappa_\delta(D_*, \eta)$.

In a similar fashion, we can define translation-invariant constants of almost chain maps between almost chain complexes.

Definition 4.6.2 (translation-invariant constant of chain complexes)**.** Let (X, μ) be a standard probability space, Γ be a group and $n \in \mathbb{N}$. A *translation-invariant constant* is a family of maps $(\kappa_\delta)_{\delta \in \mathbb{R}_{>0}}$ that, given $\delta \in \mathbb{R}_{>0}$, an essentially free probability measure preserving action $\alpha \colon \Gamma \curvearrowright (X, \mu)$ and a (δ, n)-almost chain map $f_* \colon C_* \to D_*$ between augmented marked projective (δ, n)-almost chain complexes, assigns a positive real number $\kappa_\delta(f_*)$ such that the following holds:
Let $\delta \in \mathbb{R}_{>0}$, $\alpha \colon \Gamma \curvearrowright X$ and $f_* \colon C_* \to D_*$ be a $(\delta/2, n)$-almost chain map between marked projective $(\delta/2, n)$-almost chain complexes. Then, there exists a neighbourhood U of α such that for every $\beta \in U$, we have that $(f_*)_\beta \colon (C_*)_\beta \to (D_*)_\beta$ is a (δ, n)-almost chain map between (δ, n)-almost chain complexes and

$$\kappa_\delta((f_*)_\beta) \leq \kappa_{\delta/2}(f_*) + 1.$$

Theorem 4.6.3. *Let (X, μ) be a standard probability space, Γ be a group, $n \in \mathbb{N}$. Then, there exist monotone increasing functions $K, p \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, and a translation-invariant constant $\kappa = (\kappa_\delta)_{\delta \in \mathbb{R}_{>0}}$ such that for every essentially free probability measure preserving action $\alpha \colon \Gamma \curvearrowright (X, \mu)$, $\delta \in \mathbb{R}_{>0}$ and every marked projective (δ, n)-almost $L^\infty(X) *_\alpha \Gamma$-chain complex (D_*, η), there exists a marked projective (strict) $L^\infty(X) *_\alpha \Gamma$-chain complex $(\widehat{D}_*, \widehat{\eta})$ (up to degree $n+1$) such that*

$$d_{\mathrm{GH}}^{K(\kappa_\delta(D_*))}(\widehat{D}_*, D_*, n) \leq K(\kappa_\delta(D_*)) \cdot \delta.$$

Moreover, $\widehat{D}_*$ can be chosen such that the following hold:

(i) For each $j \in \{0, \dots, n\}$, the module D_j is a submodule of $\widehat{D}_j$ and the inclusion map $D_* \hookrightarrow \widehat{D}_*$ is a $(K(\kappa_\delta(D_*)) \cdot \delta, n)$-almost chain map.

(ii) We have $\kappa_\delta(\widehat{D}_*) \leq p(\kappa_\delta(D_*))$.

Proof. We employ [LLMSU25, Theorem 4.8] to define K: For the fixed $n \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$, we define $K(x)$ to be one of the $K \in \mathbb{R}_{>0}$, for which [LLMSU25, Theorem 4.8] holds (with $\kappa \coloneqq x$). Without loss of generality, we can assume that the function K is monotone increasing. We define κ as follows: Let $\alpha \colon \Gamma \curvearrowright (X, \mu)$ be an essentially free probability measure preserving action, $\delta \in \mathbb{R}_{>0}$ and (D_*, η) be a marked projective (δ, n)-almost chain complex of $L^\infty(X) *_\alpha \Gamma$-modules. We define

$$\kappa'_\delta \coloneqq \inf_z Q(z),$$

where z ranges over all $z \in D_0$ with $\eta(z) =_\delta 1$. We then define

$$\kappa_\delta(D_*, \eta) \coloneqq \max\bigl\{\kappa'_\delta, Q(\eta), Q(\partial_1^D), \ldots, Q(\partial_{n+1}^D), \operatorname{rk} D_1, \ldots, \operatorname{rk} D_{n+1}\bigr\} + 1.$$

Note that Q is monotone (see Remark 4.5.5) and rk is invariant under translation. For the (almost-) invariance of κ', note that if $z \in D_0$ such that $\eta(z) =_{\delta/2} 1$, then the composition estimate (Lemma 4.5.7) shows that in a suitable neighbourhood U, we have for all $\beta \in U$ that $\eta_\beta(z_\beta) =_\delta 1$. Moreover, $Q(z_\beta) \le Q(z)$. Thus, we can find a neighbourhood as in the translation-invariance condition. Note that because κ' is defined as an infimum, we add the "+1" in the definition of a translation-invariant constant to obtain an open neighbourhood. Thus, κ defines a translation-invariant constant.

We show that the functions K and κ satisfy the desired conditions: Let $\alpha\colon \Gamma \curvearrowright (X, \mu)$, $\delta \in \mathbb{R}_{>0}$ and (D_*, η) be a marked projective (δ, n)-almost chain complex. We write $\kappa_D \coloneqq \kappa_\delta(D_*)$ and $K_D \coloneqq K(\kappa_\delta(D_*))$. We have (in the notation of [LLMSU25]) $\overline{\kappa}_n(D_*) < \kappa_D$, because $N_1(\cdot)$, $N_2(\cdot)$, $|\cdot|_\infty$ and $\|\cdot\|$ are bounded from above by $Q(\cdot)$ (see Remark 4.2.4). Thus, [LLMSU25, Theorem 4.8] yields a marked projective $L^\infty(X) *_\alpha \Gamma$-chain complex $(\widehat{D}_*, \widehat{\eta})$ (up to degree $n+1$) with

$$d_{\mathrm{GH}}^{K_D}(\widehat{D}_*, D_*, n) \le K_D \cdot \delta.$$

Moreover, [LLMSU25, Theorem 4.8] states that the inclusion map $D_* \hookrightarrow \widehat{D}_*$ is a $(K_D \cdot \delta, n)$-almost chain map. For the second statement, we have to dive into the details of the proof of [LLMSU25, Theorem 4.8] and proceed by induction over the degrees. We advise the reader to simultaneously have a look at [LLMSU25], as we are also using the notation from that paper. The proof of [LLMSU25, Lemma 4.10] yields that $\kappa'_\delta(\widehat{D}_*) \le \kappa'_\delta(D_*)$. Furthermore, that proof shows that $\widehat{D}_0 = D_0 \oplus \bigl\langle B \bigr\rangle_\alpha$ for some $B \subseteq X$ and thus,

$$\operatorname{rk} \widehat{D}_0 = \operatorname{rk} D_0 + 1.$$

Lemma 4.1.10 then shows that

$$\begin{aligned} Q(\widehat{\eta}) &\le Q(\eta) + Q\bigl(\widehat{\eta}|_{\langle B\rangle}\bigr) \\ &= Q(\eta) + Q(1 - \eta(z)) && \text{(Proof of [LLMSU25, Lemma 4.10])} \\ &\le Q(\eta) + (1 + Q(\eta) \cdot Q(z)) \\ &\le p_0(\kappa_\delta(D_*)) \end{aligned}$$

for the function $p_0\colon x \mapsto 1 + x + x^2$.

For the inductive step, assume that $\widehat{D}_{r-1}$ and $\partial_r^{\widehat{D}}$ have been constructed and satisfy the theorem with the function p_{r-1}. The proof of [LLMSU25, Lemma 4.10] constructs $\widehat{D}_r$ as $D_r \oplus E_r$, where $\operatorname{rk} E_r \le \operatorname{rk} D_r$, and $\dim_\alpha E_r$ is bounded by a function in $\kappa_\delta(\widehat{D}_*)$. Moreover, Lemma 4.1.10 yields that

$$\begin{aligned} Q(\widehat{\partial}_r) &\le Q(\widetilde{\partial}_r) + \operatorname{rk} E_r \cdot Q(\widetilde{\partial}_r) \cdot Q(\partial_{r+1}) \\ &\le Q(\widetilde{\partial}_r) \cdot (1 + \operatorname{rk} D_r \cdot Q(\partial_{r+1})) \\ &\le (Q(\partial_{r+1}) + \operatorname{rk} E_r) \cdot (1 + \operatorname{rk} D_r \cdot Q(\partial_{r+1})) \\ &\le (Q(\partial_{r+1}) + \operatorname{rk} D_r) \cdot (1 + \operatorname{rk} D_r \cdot Q(\partial_{r+1})), \end{aligned}$$

which is bounded from above by a function in $\kappa_\delta(\widehat{D}_*)$. Moreover, in degree $n+1$, we have

$$\begin{aligned} Q(\widehat{\partial}_{n+1}) &= Q(\widetilde{\partial}_{n+1}) \\ &\leq Q(\partial_{n+1}) + \operatorname{rk} E_n \\ &\leq Q(\partial_{n+1}) + \operatorname{rk} D_n, \end{aligned}$$

which is also bounded from above by $2 \cdot \kappa_\delta(\widehat{D}_*)$. Finally, by construction, we have $\operatorname{rk} \widehat{D}_r \leq 2 \cdot \operatorname{rk} D_r$. Altogether, we obtain that

$$\kappa_\delta(\widehat{D}_*) \leq p(\kappa_\delta(D_*)),$$

where p is defined to be the maximum of all the upper bounds encountered so far. □

4.6.2 Chain maps

We can also strictify chain maps between (strict) chain complexes. We fix a free $L^\infty(X) *_\alpha \Gamma$-resolution (C_*, ζ) of $L^\infty(X)$ as in Setup 4.3.1.

Theorem 4.6.4. *Let (X, μ) be a standard probability space, Γ be a group, $n \in \mathbb{N}$. Then, there exists a monotone increasing function $K \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and a translation-invariant constant κ such that for all $\alpha \colon \Gamma \curvearrowright X$, $\delta \in \mathbb{R}_{>0}$ and for every (δ, n)-almost chain map $f_* \colon (C_*, \zeta) \to (\widehat{D}_*, \widehat{\eta})$ between (strict) marked projective $L^\infty(X) *_\alpha \Gamma$-chain complexes, extending the identity on $L^\infty(X)$, there exists a strict marked projective $L^\infty(X) *_\alpha \Gamma$-chain complex $\widetilde{D}_*$ with*

$$d_{\mathrm{GH}}^{K(\kappa_\delta(f_*))}(\widetilde{D}_*, \widehat{D}_*, n) < K(\kappa_\delta(f_*)) \cdot \delta$$

that admits a chain map $\widetilde{f}_* \colon C_* \to \widetilde{D}_*$ extending the identity on $L^\infty(X)$.

Proof. Given $\alpha \colon \Gamma \curvearrowright (X, \mu)$, $\delta \in \mathbb{R}_{>0}$ and a (δ, n)-almost-chain map $f \colon C_* \to \widehat{D}_*$, we define

$$\begin{aligned} \kappa_\delta(f_*) \coloneqq \max\bigl\{ & Q(\zeta), Q(\partial_1^C), \ldots, Q(\partial_{n+1}^C), Q(\widehat{\eta}), Q(\partial_1^{\widehat{D}}), \ldots, Q(\partial_{n+1}^{\widehat{D}}), \\ & Q(f_0), \ldots, Q(f_n)\bigr\} + 1. \end{aligned}$$

Because Q is monotone under translation (Remark 4.5.5), κ is a translation-invariant constant. We employ [LLMSU25, Theorem 4.15] to define the map $K \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, which can be assumed to be monotone.

Because the norm is bounded by Q (Remark 4.2.4), in the notation of [LLMSU25], we have $\max\{\kappa_n(C_*), \nu_n(C_*)\} < \kappa_\delta(f_*)$ and $\kappa_n(\widehat{D}_*) < \kappa_\delta(f_*)$. Moreover, $\kappa_n(f_*) \leq \kappa_\delta(f_*)$. Thus, [LLMSU25, Theorem 4.15] yields a marked projective $L^\infty(X) *_\alpha \Gamma$-chain complex $(\widetilde{D}_*, \widetilde{\eta})$ with

$$d_{\mathrm{GH}}^{K(\kappa_\delta(f_*))}(\widetilde{D}_*, \widehat{D}_*, n) < K(\kappa_\delta(f_*)) \cdot \delta$$

that admits a chain map $\widetilde{f}_* \colon C_* \to \widetilde{D}_*$ extending the identity on $L^\infty(X)$. □

4.7 Monotonicity under weak containment

The main goal of this section is to prove the following monotonicity result for measured embedding dimension and volume under weak containment of actions.

Theorem 4.7.1 (weak containment)**.** *Let $n \in \mathbb{N}$ and Γ be a group of type FP_{n+1} and let $\alpha\colon \Gamma \curvearrowright (X,\mu)$, $\beta\colon \Gamma \curvearrowright (Y,\nu)$ be free probability measure preserving actions of Γ on standard probability spaces. Let $\alpha \prec \beta$. Let Z denote the integers (equipped with the standard norm) or a finite field (equipped with the trivial norm). Then, we have*

$$\operatorname{medim}_n^Z(\beta) \le \operatorname{medim}_n^Z(\alpha),$$
$$\operatorname{mevol}_n(\beta) \le \operatorname{mevol}_n(\alpha).$$

Remark 4.7.2. The proof of this theorem consists of several steps. We give a roadmap to the proof outlining the main ideas:

(i) In the setting of Setup 4.3.1, we fix an α-embedding from C_*^α to an augmented $L^\infty(X) *_\alpha \Gamma$-chain complex D_* with $\dim_\alpha D_n$ (resp. $\operatorname{lognorm}_\alpha \partial_{n+1}^D$) 'close' to $\operatorname{medim}_n^Z \alpha$ (resp. to $\operatorname{mevol}_n(\alpha)$).

(ii) Because α is weakly contained in β, the action β is (isomorphic to) an action $\beta'\colon \Gamma \curvearrowright X$ 'close' to α in the weak topology (see Proposition 4.4.10).

(iii) We can translate D_* from α to β' (see Section 4.5) and obtain $(D_*)_{\beta'}$. However, in general, $(D_*)_{\beta'}$ will no longer be a chain complex, but only an *almost* chain complex over $L^\infty(X) *_{\beta'} \Gamma$ and the α-embedding $C_*^\alpha \to D_*$ translates to an *almost* chain map $C_*^{\beta'} \to (D_*)_{\beta'}$ over $L^\infty(X) *_{\beta'} \Gamma$. The error depends on the previous distances.

(iv) We can strictify $(D_*)_{\beta'}$ to obtain a (strict) $L^\infty(X) *_{\beta'} \Gamma$-chain complex $\widehat{D}_*$. We obtain an almost β'-embedding to this complex.

(v) We can also strictify the almost β'-embedding to obtain a strict $L^\infty(X) *_{\beta'} \Gamma$-chain map $C_*^{\beta'} \to \widetilde{D}_*$ to a (different) strict chain complex $\widetilde{D}_*$.

(vi) If β' is 'close' enough to α, the strictified chain complex $\widetilde{D}_*$ is 'close' to D_*, thus $\dim_{\beta'} \widetilde{D}_n$ (resp. $\operatorname{lognorm}_\beta \partial_{n+1}^{\widetilde{D}}$) is 'close' to $\dim_\alpha D_n$ (resp. $\operatorname{lognorm}_\alpha \partial_{n+1}^D$).

(vii) We can make the error arbitrarily small, thus proving the claim.

The main difficulty is making the notions of closeness precise and controlling the distances. These distances depend on one another, thus some work needs to be done to obtain global control on the errors.

The key approximation is contained in the following lemma.

Lemma 4.7.3. *Let $n \in \mathbb{N}$ and Γ be a group of type FP_{n+1}, and let $\alpha\colon \Gamma \curvearrowright (X,\mu)$ be a free probability measure preserving standard action. Let $f_*\colon C_*^\alpha \to D_*$ be an α-embedding and $\epsilon > 0$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, there exists a β-embedding $C_*^\beta \to \widetilde{D}_*$ satisfying*

$$\dim_\beta \widetilde{D}_n \le \dim_\alpha D_n + \epsilon \quad \textit{and} \quad \operatorname{lognorm}_\beta \partial_{n+1}^{\widetilde{D}} \le \operatorname{lognorm}_\alpha \partial_{n+1}^{D} + \epsilon.$$

Proof. We fix a free $Z\Gamma$-resolution $C_* \twoheadrightarrow Z$ of the trivial $Z\Gamma$-module Z with finitely generated $Z\Gamma$-modules in degrees $\le n+1$ and $C_0 = Z\Gamma$ with $\eta\colon C_0 \to Z$ as the standard augmentation map (see Setup 4.3.1). As in the definition of medim_n^Z and mevol_n (Definition 4.3.3), we set $C_*^\alpha := (L^\infty(X) *_\alpha \Gamma) \otimes_{Z\Gamma} C_*$. Let $f_*\colon C_*^\alpha \to D_*$ be the given α-embedding, i.e. an $L^\infty(X) *_\alpha \Gamma$-chain map extending the identity on $L^\infty(X)$. By Lemma 4.5.12, we can pick a neighbourhood U such that for all $\beta \in U$, we have

$$\operatorname{lognorm}_\beta (\partial_{n+1}^D)_\beta \le \operatorname{lognorm}_\alpha \partial_{n+1}^D + \frac{\epsilon}{2}.$$

We fix a translation-invariant constant κ, and monotone increasing maps K and p as in Theorem 4.6.3. Moreover, we fix a translation-invariant constant and monotone-increasing constant as in Theorem 4.6.4. By taking the maximum, we can also denote the latter by κ resp. K. We restrict the neighbourhood U to a potentially smaller neighbourhood that additionally satisfies the translation-invariance condition in Definition 4.6.1 (for $\delta := \frac{\epsilon}{2}$). We define

$$M = \Big[K\big(\max\big\{\kappa_\delta(f_*), 2 \cdot p(\kappa_\delta(D_*))\big\}\big) \cdot Q(f,n) + 1\Big] \cdot K(\kappa_{\delta/2}(D_*) + 1).$$

and choose $\delta \in \mathbb{R}_{>0}$ such that

$$M \cdot \delta \le \epsilon \quad \text{and} \quad M \cdot \log_+(M \cdot \delta) \le \frac{\epsilon}{2} \quad \text{and} \quad \delta \le \epsilon.$$

Note that $Q(f,n)$ was defined in Lemma 4.1.25. By Corollary 4.5.9, we can restrict U to a neighbourhood of α such that for all $\beta \in U$ the translated chain complex $(D_*)_\beta$ is a (δ,n)-almost chain complex and $(f_*)_\beta\colon (C_*^\alpha)_\beta \to (D_*)_\beta$ is a (δ,n)-almost chain map. By Lemma 4.5.6, we have $(C_*^\alpha)_\beta \cong C_*^\beta$.

Let $\beta \in U$. We first strictify $(D_*)_\beta$. By Theorem 4.6.3, we obtain a strict marked projective $L^\infty(X) *_\beta \Gamma$-chain complex $\widehat{D}_*$ such that the inclusion $i_*\colon (D_*)_\beta \hookrightarrow \widehat{D}_*$ is a $(K(\kappa_\delta((D_*)_\beta)) \cdot \delta, n)$-almost chain map and

$$d_{\mathrm{GH}}^{K(\kappa_\delta((D_*)_\beta))}(\widehat{D}_*, (D_*)_\beta, n) \le K(\kappa_\delta((D_*)_\beta)) \cdot \delta.$$

By translation-invariance and because of our choice of U, we have

$$\kappa_\delta((D_*)_\beta) \le \kappa_{\delta/2}(D_*) + 1.$$

Because K is monotone-increasing, we can directly use $K(\kappa_{\delta/2}(D_*)+1)$ as an upper bound in the following. Then, by Lemma 4.1.25, we have that the composition

$i_* \circ (f_*)_\beta \colon C_*^\beta \to (\widehat{D}_*)_\beta$ is a $(Q(f_\beta, n) \cdot K(\kappa_{\delta/2}(D_*) + 1) \cdot \delta + \delta, n)$-almost chain map. Note that $Q(f_\beta, n) \leq Q(f, n)$.

Now, we can strictify the chain map: By Theorem 4.6.4, there exists a strict marked projective $L^\infty(X) *_\beta \Gamma$-chain complex $\widetilde{D}_*$ admitting a (strict) $L^\infty(X) *_\beta \Gamma$-chain map $C_*^\beta \to \widetilde{D}_*$ extending the identity on $L^\infty(X)$ with

$$d_{\mathrm{GH}}^{K(\kappa_\delta(i_* \circ (f_*)_\beta))}(\widetilde{D}_*, \widehat{D}_*, n) \leq K(\kappa_\delta(i_* \circ (f_*)_\beta)) \cdot Q(f, n) \cdot K(\kappa_{\delta/2}(D_*) + 1) \cdot \delta.$$

Thus, $C_*^\beta \to \widetilde{D}_*$ is a β-embedding. From the explicit descriptions of the translation-invariant constants (see the proofs of Theorem 4.6.3 and Theorem 4.6.4), we obtain that

$$\kappa_\delta(i_* \circ (f_*)_\beta) \leq \max\bigl\{\kappa_\delta(f_*), 2 \cdot \kappa_\delta(\widehat{D}_*)\bigr\} \leq \max\bigl\{\kappa_\delta(f_*), 2 \cdot p(\kappa_\delta(D_*))\bigr\}.$$

Thus, because the Gromov-Hausdorff distance satisfies a modified version of the triangle inequality [LLMSU25, Proposition 3.17], we obtain

$$d_{\mathrm{GH}}^M(\widetilde{D}_*, (D_*)_\beta, n) \leq M \cdot \delta$$

with

$$M = \Bigl[K\bigl(\max\bigl\{\kappa_\delta(f_*), 2 \cdot p(\kappa_\delta(D_*))\bigr\}\bigr) \cdot Q(f, n) + 1\Bigr] \cdot K(\kappa_{\delta/2}(D_*) + 1),$$

as defined at the beginning of this proof. For the dimension, we thus obtain that

$$\begin{aligned} \dim_\beta \widetilde{D}_n &\leq \dim_\beta (D_n)_\beta + M \cdot \delta && \text{(Lemma 4.1.28)} \\ &= \dim_\alpha D_n + M \cdot \delta && \text{(Remark 4.5.2)} \\ &\leq \dim_\alpha D_n + \epsilon. && \text{(choice of } \delta) \end{aligned}$$

For the lognorm, we have

$$\begin{aligned} \operatorname{lognorm}_\beta \partial_{n+1}^{\widetilde{D}} &\leq \operatorname{lognorm}_\beta (\partial_{n+1}^D)_\beta + M \cdot \log_+(M \cdot \delta) && \text{(Lemma 4.1.28)} \\ &\leq \operatorname{lognorm}_\beta (\partial_{n+1}^D)_\beta + \frac{\epsilon}{2} && \text{(choice of } \delta) \\ &\leq \operatorname{lognorm}_\alpha \partial_{n+1}^D + \epsilon. && \text{(choice of } U) \ \square \end{aligned}$$

We can now prove the theorem that measured embedding dimension and volume are monotone under weak embeddings.

Proof of Theorem 4.7.1. We show how to deduce the statement for mevol_n. The proof for $\mathrm{medim}_n^{\mathbb{Z}}$ works analogously by replacing every occurrence of '$\operatorname{lognorm}_* \partial_{n+1}$' by '$\dim_* D_n$'.

Note that property FP_{n+1} implies that $\mathrm{mevol}_n(\alpha) < \infty$. Let $\epsilon \in \mathbb{R}_{>0}$. By definition of mevol_n, there is an α-embedding $C_*^\alpha \to D_*$ with

$$\operatorname{lognorm}_\alpha \partial_{n+1}^D \leq \mathrm{mevol}_n(\alpha) + \epsilon.$$

Because $\alpha \prec \beta$, in every weak neighbourhood U of α, there is $\beta' \in U$ such that $\beta' \cong \beta$ (Proposition 4.4.10). Thus, Lemma 4.7.3 yields a weak neighbourhood U of α and $\beta' \in U$ with $\beta' \cong \beta$ such that there is a β'-embedding $C_*^{\beta'} \to \widetilde{D}_*$ with

$$\operatorname{lognorm}_{\beta'} \partial_{n+1}^{\widetilde{D}} \leq \operatorname{lognorm}_\alpha \partial_{n+1}^{D} + \epsilon.$$

As $\beta' \cong \beta$, this defines a β-embedding $C_*^{\beta} \to \widetilde{D}_*$. Thus,

$$\begin{aligned} \operatorname{mevol}_n(\beta) &\leq \operatorname{lognorm}_\beta \partial_{n+1}^{\widetilde{D}} \\ &\leq \operatorname{lognorm}_\alpha \partial_{n+1}^{D} + \epsilon \\ &\leq \operatorname{mevol}_n(\alpha) + 2 \cdot \epsilon. \end{aligned}$$

Taking the limit $\epsilon \to 0$ yields the claim. □

4.8 Applications

We collect direct applications of the inheritance of measured embedding dimension and volume under weak containment (Theorem 4.7.1).

Corollary 4.8.1. *Let* Γ *be a residually finite group of type* FP_{n+1} *and* Λ_* *be a residual chain of* Γ*. Let* $\beta\colon \Gamma \curvearrowright (X, \mu)$ *be the profinite completion of the residual chain* Λ_* *(see Example 4.4.3). Let* $\alpha\colon \Gamma \curvearrowright (Y, \nu)$ *be a probability measure preserving action such that* $\alpha \prec \beta$*. Then, for all* $n \in \mathbb{N}$*, we have*

$$\widehat{b}_n(\Gamma, \Lambda_*; Z) \leq \operatorname{medim}_n^Z(\alpha) \quad \textit{and} \quad \widehat{t}_n(\Gamma, \Lambda_*) \leq \operatorname{mevol}_n(\alpha).$$

Proof. We have

$$\begin{aligned} \widehat{b}_n(\Gamma, \Lambda_*; Z) &\leq \operatorname{medim}_n^Z(\beta) && \text{(Theorem 4.3.4)} \\ &\leq \operatorname{medim}_n^Z(\alpha) && \text{(Theorem 4.7.1)} \end{aligned}$$

and the analogous results hold for $\widehat{t}_n$ and mevol_n by the same theorems. □

In particular, the above corollary holds for α being a Bernoulli shift (Example 4.4.4), i.e. a Bernoulli shift provides an upper bound to the measured embedding dimension resp. volume of every action.

Corollary 4.8.2 (EMD*)**.** *Let* Γ *be an infinite, residually finite group of type* FP_{n+1} *that satisfies EMD* (see Definition 4.4.7; e.g. free groups, surface groups, fundamental groups of hyperbolic* 3*-manifolds). Let* $\alpha\colon \Gamma \curvearrowright X$ *be an essentially free ergodic standard probability measure preserving action. Then, for every residual chain* Λ_* *in* Γ *and* $n \in \mathbb{N}$*, we have*

$$\widehat{b}_n(\Gamma, \Lambda_*; Z) \leq \operatorname{medim}_n^Z(\alpha) \quad \textit{and} \quad \widehat{t}_n(\Gamma, \Lambda_*) \leq \operatorname{mevol}_n(\alpha).$$

In particular, if $\operatorname{mevol}_n(\alpha) = 0$*, then* $\widehat{t}_n(\Gamma, \Lambda_*) = 0$ *for every residual chain* Λ_**.*

Proof. Since Γ satisfies EMD*, the action α is weakly contained in the profinite completion $\Gamma \curvearrowright \widehat{\Gamma}$. Let Γ_* be the system of all finite-index normal subgroups of Γ. Then, since $\widehat{b}_n$ is defined via a lim sup, we have

$$\begin{aligned} \widehat{b}_n(\Gamma, \Lambda_*; Z) &\le \widehat{b}_n(\Gamma, \Gamma_*; Z) \\ &\le \operatorname{medim}_n^Z(\Gamma \curvearrowright \widehat{\Gamma}) && \text{(Theorem 4.3.4)} \\ &\le \operatorname{medim}_n^Z(\alpha) && \text{(Theorem 4.7.1)} \end{aligned}$$

and the analogous results hold for $\widehat{t}_n$ and mevol_n. □

Recall that if Γ is the fundamental group of a hyperbolic 3-manifold, we expect $t_1(\Gamma, \Lambda_*)$ to be positive (see Conjecture 4 in the Introduction).

Corollary 4.8.3 (amenable groups)**.** *Let* Γ *be an infinite amenable group of type* FP_{n+1} *and* $\alpha\colon \Gamma \curvearrowright (X, \mu)$ *and* $\beta\colon \Gamma \curvearrowright (Y, \nu)$ *be free probability measure preserving actions on standard probability spaces. Then,*

$$\operatorname{medim}_n^Z(\alpha) = \operatorname{medim}_n^Z(\beta) \quad \textit{and} \quad \operatorname{mevol}_n(\alpha) = \operatorname{mevol}_n(\beta).$$

Proof. By Example 4.4.5, we have $\alpha \prec \beta$ and $\beta \prec \alpha$. Then, Theorem 4.7.1 yields the claim. □

A separate argument [LLMSU25, Theorem 11.11] shows that we even have

$$\operatorname{medim}_n^Z(\alpha) = \operatorname{medim}_n^Z(\beta) = 0 \quad \text{and} \quad \operatorname{mevol}_n(\alpha) = \operatorname{mevol}_n(\beta) = 0.$$

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Index

M

N

O

P

Q

R

S

T

Index of Notation

Symbols

A

B

C

D

E

F

G

H

I

K

L