

Convergence of the Nonlocal Allen–Cahn Equation to Mean Curvature Flow

Asymptotic Analysis

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Abstract

We prove convergence of the nonlocal Allen–Cahn equation to mean curvature flow in the sharp interface limit, in the situation when the parameter corresponding to the kernel goes to zero fast enough with respect to the diffuse interface thickness. The analysis is done in the case of a $W^{1,1}$ -kernel, under periodic boundary conditions and in both two and three space dimensions. We use the approximate solution and spectral estimate from the local case, and combine the latter with an L^2 -estimate for the difference of the nonlocal operator and the negative Laplacian from Abels, Hurm Abels, H., & Hurm, C. (2024). *Journal of Differential Equations*, 402: 593–624. To this end, we prove a nonlocal Ehrling-type inequality to show uniform H^3 -estimates for the nonlocal solutions.

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Keywords

nonlocal Allen–Cahn equation, diffuse interface model, sharp interface limit, mean curvature flow

1 Introduction

Let $\mathbb{T}^n$, $n \in \{2, 3\}$, be the n -dimensional torus and $T > 0$ be fixed. We consider the following nonlocal Allen–Cahn equation,

$$\partial_t c_\varepsilon + \mathcal{L}_\eta c_\varepsilon + \frac{1}{\varepsilon^2} f'(c_\varepsilon) = 0 \quad \text{in } \mathbb{T}^n \times (0, T), \quad (1.1)$$

$$c_\varepsilon|_{t=0} = c_{0,\varepsilon} \quad \text{in } \mathbb{T}^n. \quad (1.2)$$

Here, $c_\varepsilon : \mathbb{T}^n \times (0, T) \rightarrow \mathbb{R}$ is an order parameter representing the relative difference of the two phases and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a double-well potential with wells of equal depth and minima at ± 1 . Moreover, the parameter $\varepsilon > 0$ is related to the thickness of the diffuse interface separating the two phases. Finally, for $\eta > 0$ the nonlocal operator is given by

$$\begin{aligned} \mathcal{L}_\eta \varphi(x) &:= \int_{\mathbb{T}^n} \tilde{J}_\eta(x-y) (\varphi(x) - \varphi(y)) \, dy \\ &= \int_{\mathbb{R}^n} J_\eta(x-y) (\varphi(x) - \varphi(y)) \, dy \quad \text{for all } x \in \mathbb{T}^n, \end{aligned} \quad (1.3)$$

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for integrable $\varphi : \mathbb{T}^n \rightarrow \mathbb{R}$ and suitable interaction kernels $J_\eta : \mathbb{R}^n \rightarrow \mathbb{R}$, cf. (A.2) below, which is compactly supported in $(-\pi, \pi)^n$. Here, $\tilde{J}_\eta : \mathbb{T}^n \rightarrow \mathbb{R}$ denotes a 2π -periodic extension of J_η . In this setting, the corresponding bilinear form

$$\mathcal{E}_\eta(\varphi) := \frac{1}{4} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \tilde{J}_\eta(x-y) |\varphi(x) - \varphi(y)|^2 dy dx \quad (1.4)$$

satisfies the following variational convergence

$$\mathcal{E}_\eta(\varphi) \rightarrow \frac{1}{2} \int_{\mathbb{T}^n} |\nabla \varphi|^2 dx \quad (1.5)$$

as $\eta \rightarrow 0$ for all $\varphi \in H^1(\mathbb{T}^n)$. For details, we refer to the results by Ponce (2004a, 2004b).

Based on (1.5), the nonlocal-to-local convergence $\mathcal{L}_\eta \varphi \rightarrow -\Delta \varphi$ as $\eta \rightarrow 0$ as well as corresponding results about the nonlocal-to-local convergence (without rates) of the Cahn–Hilliard equation have been shown in Davoli et al. (2020, 2021a, 2021b, 2023) and Elbar and Skrzeczkowski (2023). Recently, the authors in Abels and Hurm (2024) established rates of convergence for the nonlocal operator $\mathcal{L}_\eta$, see Theorem 2.4 below. This result has then been used in Abels and Hurm (2024) to prove the nonlocal-to-local limit with rates of convergence for the Cahn–Hilliard equation and the Allen–Cahn equation as well as in Hurm et al. (2024) and Hurm and Moser (2024) for coupled Cahn–Hilliard systems. Finally, we note that in Grasselli and Schimperna (2013) a nonlocal phase field model is studied, where a nonlocal Allen–Cahn equation as above is coupled to a heat equation.

In particular, solutions to the nonlocal Allen–Cahn equation converge to the solution to the corresponding local Allen–Cahn equation,

$$\partial_t c_\varepsilon - \Delta c_\varepsilon + \frac{1}{\varepsilon^2} f'(c_\varepsilon) = 0 \quad \text{in } \mathbb{T}^n \times (0, T), \quad (1.6)$$

$$c_\varepsilon|_{t=0} = c_{\varepsilon,0} \quad \text{in } \mathbb{T}^n. \quad (1.7)$$

It is well-known that in the sharp interface limit $\varepsilon \rightarrow 0$ the Allen–Cahn equation (1.6) and (1.7) converges toward mean curvature flow. This has first been analyzed in Chen (1992) and de Mottoni and Schatzman (1995) using the method of matched asymptotic expansions as long as there is a smooth solution of the mean curvature flow equation. Moreover, Ilmanen (1993) showed convergence globally in time to a Brakke solution of the mean curvature flow. On account of stability of the linearized Allen–Cahn operator, one may also derive rates of convergence. An alternative approach has been used by the authors in Fischer et al. (2020). Therein, the relative entropy method has been applied to prove the sharp interface asymptotics.

We are interested in the sharp interface limit for the nonlocal Allen–Cahn equation, that is, we intend to analyze the limit $\varepsilon \rightarrow 0$ for initial data close to a diffuse interface configuration. This is a very difficult task in general, therefore we restrict ourselves to the case when η is very small compared ε (see Theorem 1.1 below for a quantification of the latter). The idea is to enforce closeness to the local situation where the sharp interface limit is well-known. Indeed, we use the approximate solution and the spectral estimate from the local case, cf. Section 2.4 below, and combine the latter with an estimate for $\|\mathcal{L}_\eta c_\varepsilon + \Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}$ from Abels and Hurm (2024), cf. Theorem 2.4 below. In order to close the Gronwall argument, it turns out that we need to control the H^3 -norm of the nonlocal solutions uniformly. The latter is done in Section 3.2 with a novel nonlocal Ehrling inequality, cf. Theorem 2.5 below.

To the best of our knowledge this is the first rigorous sharp interface limit result in the nonlocal case. The main result of this contribution is the following:

Theorem 1.1 (Convergence). *Let $n \in \{2, 3\}$, $T_0 > 0$ and $(\Gamma_t)_{t \in [0, T_0]}$ with $\Gamma_t \subset (-\pi, \pi)^n$ for all $t \in [0, T_0]$ be a smoothly evolving compact closed hypersurface in $\mathbb{T}^n$ satisfying mean curvature flow, i.e. $V_{\Gamma_t} = H_{\Gamma_t}$ for all $t \in [0, T_0]$, where V_{Γ_t} is the normal velocity and H_{Γ_t} the mean curvature of Γ_t . Here, Γ_t separates $\mathbb{T}^n$ into two disjoint connected domains $\Omega_t^\pm$ for all $t \in [0, T_0]$. We set $\Gamma := \bigcup_{t \in [0, T_0]} \Gamma_t \times \{t\}$ and $\Omega^\pm := \bigcup_{t \in [0, T_0]} \Omega_t^\pm \times \{t\}$. Moreover, for sufficiently small $\delta > 0$, we denote by $\Gamma(\delta)$ the tubular neighborhood around Γ , and the tangential and normal derivatives by ∇_τ and ∂_n , respectively. Finally, let $M \in \mathbb{N}$.*

Then there exists an $\varepsilon_0 > 0$ and $c_\varepsilon^A : \mathbb{T}^n \times [0, T_0] \rightarrow \mathbb{R}$ smooth for $\varepsilon \in (0, \varepsilon_0]$ with $\lim_{\varepsilon \rightarrow 0} c_\varepsilon^A = \pm 1$ uniformly on compact subsets of $\Omega^\pm$ and the following holds:

1. Let $M \geq 3$ and consider initial values $c_{0,\varepsilon} \in H^3(\mathbb{T}^n)$, $\varepsilon \in (0, \varepsilon_0]$ for (1.1) and (1.2) with

$$\sup_{\varepsilon \in (0, \varepsilon_0]} \|c_{0,\varepsilon}\|_\infty < \infty \quad \text{and} \quad \|c_{0,\varepsilon} - c_\varepsilon^A\|_{L^2(\mathbb{T}^n)} \leq R\varepsilon^{M+\frac{1}{2}}, \quad R > 0 \text{ fixed},$$

$$\|c_{0,\varepsilon}\|_{H^1(\mathbb{T}^n)} \leq \frac{C}{\varepsilon^2}, \quad \|c_{0,\varepsilon}\|_{H^2(\mathbb{T}^n)} \leq \frac{C}{\varepsilon^4} \quad \text{and} \quad \|c_{0,\varepsilon}\|_{H^3(\mathbb{T}^n)} \leq \frac{C}{\varepsilon^{11}}.$$

Then there are constants $c, \varepsilon_1 > 0$ such that if $\eta = \eta(\varepsilon) \leq cR\varepsilon^{16+M+\frac{1}{2}}$, $\varepsilon \in (0, \varepsilon_1]$, then the solution c_ε of the nonlocal Allen–Cahn equation (1.1) and (1.2) satisfies for all $\varepsilon \in (0, \varepsilon_1]$ and $T \in (0, T_0]$, where $\bar{c}_\varepsilon := c_\varepsilon - c_\varepsilon^A$,

$$\sup_{t \in [0, T_0]} \|\bar{c}_\varepsilon(t)\|_{L^2(\mathbb{T}^n)} + \|\nabla \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0, T) \setminus \Gamma(\delta))} \leq C\varepsilon^{M+\frac{1}{2}},$$

$$\|\nabla_\tau \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0, T) \cap \Gamma(\delta))} + \varepsilon \|\partial_n \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0, T) \cap \Gamma(\delta))} \leq C\varepsilon^{M+\frac{1}{2}}.$$

2. Let $M = 2$. Then the same statement holds when time T is small.

Remark 1.2. 1. The c_ε^A in Theorem 1.1 is the approximate solution from the local case as given in Theorem 2.6 below. Here M determines the number of terms in the expansion, cf. Theorem 2.6 below for the details.

2. The condition $\eta = \eta(\varepsilon) \leq cR\varepsilon^{16+M+\frac{1}{2}}$ is needed technically in our proof. We expect that our result can be extended to a broader range of choices $\eta(\varepsilon)$ as long as at least $\eta = O(\varepsilon)$. Without this condition convergence and a possible limit system seem to be unclear.

The structure of this article is as follows. In Section 2, we introduce some notation, assumptions and preliminary results. Moreover, we prove a nonlocal version of the Ehrling inequality, which we need to derive uniform estimates of the solutions. In Section 3.1, we then show existence, uniqueness, higher regularity and uniform H^3 -bounds of solutions for the nonlocal Allen–Cahn equation (1.1) and (1.2). Section 4 contains the proof of Theorem 1.1 (Convergence). Therein, we show stability estimates, which then imply the convergence.

2 Preliminaries

2.1 Notation

Let $n \in \mathbb{N}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$, and $\mathbb{T}^n := \mathbb{R}^n / 2\pi\mathbb{Z}^n$. For any open set $\Omega \subseteq \mathbb{R}^n$ or $\Omega = \mathbb{T}^n$ and a Banach space X , we write $L^p(\Omega, X)$ and $W^{k,p}(\Omega, X)$ for the Lebesgue and Sobolev spaces of functions on Ω with values in X . The corresponding norms are denoted by $\|\cdot\|_{L^p(\Omega, X)}$ and $\|\cdot\|_{W^{k,p}(\Omega, X)}$, respectively. We also write $H^k(\Omega, X) := W^{k,2}(\Omega, X)$. If $X = \mathbb{R}$, we leave out the X in the notation. Moreover, $H^{-k}(\mathbb{T}^n)$ are the dual spaces corresponding to $H^k(\mathbb{T}^n)$. We note that we identify $f : \mathbb{T}^n \rightarrow \mathbb{C}$ with the corresponding function $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ with $\tilde{f}(x) = f(x + 2\pi\mathbb{Z}^n)$ for all $x \in \mathbb{R}^n$, which is 2π -periodic with respect to every component of $x \in \mathbb{R}^n$. Moreover, $f \in H^k(\mathbb{T}^n)$ if and only if $\tilde{f} \in H_{loc}^k(\mathbb{R}^n)$.

We recall that functions $f \in L^1(\mathbb{T}^n, \mathbb{C})$ can be represented by their Fourier expansion

$$f(x) = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \hat{f}_k e^{ik \cdot x},$$

where the corresponding Fourier coefficients are given by

$$\hat{f}_k = \hat{f}(k) := \int_{\mathbb{T}^n} e^{-ik \cdot x} f(x) dx \quad \text{for } k \in \mathbb{Z}^n.$$

We note that this (non-standard) definition of the Fourier coefficients is useful to simplify the relation between the Fourier expansion and Fourier transform. This will be used in the proof of Theorem 2.5 below. We remark that with this choice for the Fourier coefficients, the properties of the Fourier series hold true with different constants than usual.

Let us briefly recall some basic properties of Fourier series, which are needed throughout this contribution.

Lemma 2.1. (a) (Convolution Theorem) For $f, g \in L^1(\mathbb{T}^n, \mathbb{C})$ and $k \in \mathbb{Z}^n$, it holds

$$\widehat{f * g}(k) = \hat{f}(k) \hat{g}(k). \quad (2.1)$$

(b) (Parseval's relation) For $f, g \in L^2(\mathbb{T}^n, \mathbb{C})$, it holds

$$\int_{\mathbb{T}^n} f(x) \overline{g(x)} dx = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \hat{f}(k) \overline{\hat{g}(k)}. \quad (2.2)$$

(c) For $f \in L^2(\mathbb{T}^n, \mathbb{C})$, it holds

$$\|f\|_{L^2(\mathbb{T}^n)}^2 = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)|^2. \quad (2.3)$$

(d) The map $f \mapsto (\hat{f}(k))_{k \in \mathbb{Z}^n}$ is an isomorphism from $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$.

(e) (Riemann–Lebesgue Lemma) For $f \in L^1(\mathbb{T}^n, \mathbb{C})$, we have

$$|\hat{f}(k)| \rightarrow 0 \quad \text{as } |k| \rightarrow \infty. \quad (2.4)$$

For a proof, we refer to the book by Grafakos (2014). Moreover, $H_2^s(\mathbb{T}^n)$ for $s \in \mathbb{R}$ are the usual Bessel-Potential spaces endowed with the norm $\|f\|_{H_2^s(\mathbb{T}^n)}^2 := \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s |\hat{f}(k)|^2$. It is well-known that $H_2^k(\mathbb{T}^n)$ is isomorphic to $H^k(\mathbb{T}^n)$ for all $k \in \mathbb{Z}$.

For $f \in L^1(\mathbb{R}^n, \mathbb{C})$, we define the Fourier transform of f by $\mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx$ for $\xi \in \mathbb{R}^n$. Note that for clarity we use a different notation as for the torus. Note that if $\text{supp } f \subseteq [-\pi, \pi]^n$ and we roughly identify $\mathbb{T}^n \cong [-\pi, \pi]^n$, then $\hat{f}(k) = \mathcal{F}[f](k)$ for all $k \in \mathbb{Z}^n$. We recall that the Convolution Theorem as well as the Riemann–Lebesgue Theorem also hold true for the Fourier transform on $\mathbb{R}^n$. In this setting, Plancherel's Theorem states that $\mathcal{F} : L^2(\mathbb{R}^n, \mathbb{C}) \rightarrow L^2(\mathbb{R}^n, \mathbb{C})$ is an isomorphism.

Finally, we use the convention that constants can change from line to line, but are independent from the free variables, unless noted otherwise.

2.2 Assumptions

(A.1) Let $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, $n \in \{2, 3\}$, be the n -dimensional torus.

(A.2) Let $\eta > 0$ and $J_\eta : \mathbb{R}^n \rightarrow [0, \infty)$ be a non-negative function given by $J_\eta(x) = \rho_\eta(|x|)/|x|^2$ for all $x \in \mathbb{R}^n$ and $J_\eta \in W^{1,1}(\mathbb{R}^n)$, where $(\rho_\eta)_{\eta>0}$ is a family of mollifiers defined by $\rho_\eta(r) = \eta^{-n} \rho_1(\frac{r}{\eta})$ for all $r \in \mathbb{R}$ for some $\rho_1 : \mathbb{R} \rightarrow [0, \infty)$ such that $\rho_1(|\cdot|) \in L^1(\mathbb{R}^n)$ and $\rho_1(r) = \rho_1(-r)$ for all $r \in \mathbb{R}$. Moreover, we assume that ρ_1 be compactly supported in $(-\pi, \pi)$ and

$$\int_0^\infty \rho_1(r) r^{n-1} dr = \frac{2}{C_n},$$

where $C_n := \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 d\mathcal{H}^{n-1}(\sigma)$. Finally, let $\tilde{J}_\eta : \mathbb{T}^n \rightarrow [0, \infty)$ be the 2π -periodic extension of J_η .

(A.3) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth double-well potential with wells of equal depth, more precisely,

$$f \in C^\infty(\mathbb{R}), \quad f(\pm 1) = f'(\pm 1) = 0, \quad f''(\pm 1) > 0, \quad f > 0 \text{ in } (-1, 1)$$

and $f' < 0$ in $(-\infty, -R_0)$ as well as $f' > 0$ in (R_0, ∞) for some $R_0 \geq 1$. Finally, we assume

$$|f'(r)| \leq C(1 + |r|^3) \quad \text{for all } r \in \mathbb{R}$$

and $f'' \geq -\alpha$ for some $\alpha \geq 0$.

Remark 2.2. Note that a typical example for J_η is given through the choice

$$\rho_1(r) := C_{\beta,n,\tilde{\rho}} |r|^\beta \tilde{\rho}(r) \quad \text{for all } r \in \mathbb{R},$$

where $\beta \in (3 - n, 2)$ to ensure the integrability conditions and $\tilde{\rho} \in C_0^\infty((-\pi, \pi))$, $\tilde{\rho} \not\equiv 0$ is symmetric, non-negative and $C_{\beta,n,\tilde{\rho}} > 0$ is a normalization constant. Note that this choice corresponds to the $W^{1,1}$ -kernel setting in Davoli et al. (2021a).

Remark 2.3. Note that the assumptions on the double-well potential f in (A.3) are standard. Under these assumptions, we then define the function $\theta_0 \in C^2(\mathbb{R})$ to be the unique solution to the ordinary differential equation,

$$-\theta_0'' + f'(\theta_0) = 0 \quad \text{in } \mathbb{R}, \quad \lim_{\rho \rightarrow \pm\infty} \theta_0(\rho) = \pm 1, \quad \theta_0(0) = 0. \quad (2.5)$$

The function θ_0 is also referred to as the *optimal profile*.

2.3 Inequalities

Since we assume the interaction kernel to be of class $W^{1,1}$, we can use the result in Abels and Hurm (2024), where the authors provided concrete rates of convergence for the nonlocal operator $\mathcal{L}_\eta$. It is one of the central result for the main result. In fact, they proved the following assertion:

Theorem 2.4. *Let $n \in \{2, 3\}$. Moreover, let $\mathcal{L}_\eta$ be defined as in (1.3) and J_η satisfy (A.2) from Section 2.2. Then for all $c \in H^3(\mathbb{T}^n)$ it holds for a constant $K > 0$ independent of η*

$$\left\| \mathcal{L}_\eta c + \Delta c \right\|_{L^2(\mathbb{T}^n)} \leq K\eta \|c\|_{H^3(\mathbb{T}^n)}. \quad (2.6)$$

Proof. The proof of this theorem can be done analogously as in Abels and Hurm (2024, Lemma 3.1) by using Fourier series instead of Fourier transformation. $\square$

The following result has already appeared in Davoli et al. (2020, Lemma 3.4) and Davoli et al. (2021a, Lemma 4). However, these results do not apply in our case, since we explicitly need the dependence on the parameter R for the arguments in Section 3.2. It is important for the estimate of the error $u = c_\varepsilon - c_A$ with aid of the Gronwall Lemma.

Theorem 2.5 (Nonlocal Ehring Inequality). *Let $n \in \mathbb{N}$, $R \geq 1$, $0 < \eta \leq 1/R$ and $\mathcal{E}_\eta$ be defined as in (1.4) with J_η satisfying (A.2) from Section 2.2. Then for all $u \in L^2(\mathbb{T}^n)$ it holds with $C > 0$ independent of R, η, u ,*

$$\|u\|_{L^2(\mathbb{T}^n)}^2 \leq \frac{C}{R^2} \mathcal{E}_\eta(u) + CR^2 \|u\|_{H^{-1}(\mathbb{T}^n)}^2.$$

Proof. Let $R \geq 1$ and $0 < \eta \leq 1/R$ be arbitrary. Exploiting the symmetry of the interaction kernel J_η , it follows

$$2\mathcal{E}_\eta(u) = \int_{\mathbb{T}^n} \mathcal{L}_\eta u(x) u(x) \, dx.$$

Then, using Parseval's relation (2.2), the convolution theorem (2.1) and the definition of J_η , we obtain

$$\begin{aligned} \mathcal{E}_\eta(u) &= \frac{1}{2} \int_{\mathbb{T}^n} \mathcal{L}_\eta u(x) u(x) \, dx \\ &= \frac{1}{2(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \widehat{\mathcal{L}_\eta u}(k) \overline{\widehat{u}(k)} = \frac{1}{2(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \left(\widehat{J}_\eta(0) - \widehat{J}_\eta(k) \right) |\widehat{u}(k)|^2 \end{aligned}$$

for all $u \in L^2(\mathbb{T}^n)$. Let us note that $\widehat{J}_\eta$ is indeed well-defined, since the interaction kernel is of class $W^{1,1}$. Observe that the interaction kernel J_η is compactly supported in $(-\pi, \pi)^n$. Therefore,

$$\widehat{J}_\eta(k) = \int_{\mathbb{T}^n} \tilde{J}_\eta(z) e^{-ik \cdot z} \, dz = \int_{\mathbb{R}^n} J_\eta(z) e^{-ik \cdot z} \, dz = \mathcal{F}[J_\eta](k) \quad (2.7)$$

for all $k \in \mathbb{Z}^n$. For the following, we define the auxiliary function

$$\Psi : \mathbb{R}^n \rightarrow \mathbb{C} : \xi \mapsto \frac{1}{2(2\pi)^n} \left(\widehat{J}_1(0) - \mathcal{F}[J_1](\xi) \right).$$

Due to (2.7) and since $\mathcal{F}[J_1]$ is a function on $\mathbb{R}^n$, it follows by standard Fourier analysis that $\Psi \in C^\infty(\mathbb{R}^n)$ and Ψ is $\mathbb{R}$ -valued because J_1 is even. Moreover, a scaling argument yields

$$\frac{1}{2(2\pi)^n} \left(\widehat{J}_\eta(0) - \widehat{J}_\eta(k) \right) = \frac{1}{\eta^2} \Psi(k\eta) \quad \text{for all } k \in \mathbb{Z}^n, \quad 0 < \eta \leq \frac{1}{R}.$$

We need to derive more properties of Ψ . It holds $\Psi \geq 0$ because of

$$\left| \mathcal{F}[J_1](\xi) \right| \leq \int_{\mathbb{R}^n} J_1(x) dx = \widehat{J}_1(0) \quad \text{for all } \xi \in \mathbb{R}^n.$$

Moreover, the following contradiction argument shows $\Psi(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. We conversely assume that there exists some $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\Psi(\xi) = 0$. In particular, this also means that $\operatorname{Re}\Psi(\xi) = 0$. Thus, it holds

$$\begin{aligned} 0 = \operatorname{Re} \left(\widehat{J}_1(0) - \mathcal{F}[J_1](\xi) \right) &= \operatorname{Re} \int_{\mathbb{R}^n} J_1(x) (1 - e^{-i\xi \cdot x}) dx \\ &= \int_{\mathbb{R}^n} J_1(x) (1 - \cos(\xi \cdot x)) dx, \end{aligned}$$

which implies that $J_1(x)(1 - \cos(\xi \cdot x)) = 0$ for almost all $x \in \mathbb{R}^n$, since the integrand is non-negative. However, the properties of J_1 then yield $1 - \cos(\xi \cdot x) = 0$ for almost all $x \in \mathbb{R}^n$ and thus $\xi = 0$, which contradicts our choice of $\xi \in \mathbb{R}^n \setminus \{0\}$. This shows $\Psi(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. For the gradient we obtain

$$\nabla \Psi(\xi) = -i \frac{1}{2} \int_{\mathbb{R}^n} J_1(x) e^{-ix \cdot \xi} x dx.$$

In particular, this implies $\nabla \Psi(0) = 0$ since the interaction kernel J_1 is even. Moreover, we have

$$D^2 \Psi(\xi)_{k,j} = \frac{1}{2} \int_{\mathbb{R}^n} J_1(x) e^{-ix \cdot \xi} x_k x_j dx,$$

and therefore $D^2 \Psi(0) = \operatorname{Id}$ due to the properties of J_1 . More precisely, the radial symmetry and renormalization of ρ_1 imply

$$D^2 \Psi(0)_{k,j} = \frac{1}{2} \int_{\mathbb{R}^n} J_1(x) x_k x_j dx = 0 \quad \text{if } k \neq j$$

and if $k = j$,

$$D^2 \Psi(0)_{k,k} = \frac{1}{2} \int_{\mathbb{R}^n} J_1(x) x_k^2 dx = \frac{1}{2n} \int_{\mathbb{R}^n} J_1(x) |x|^2 dx = \frac{1}{2n} \int_{\mathbb{R}^n} \rho_1(|x|) dx = 1.$$

Hence, a Taylor expansion around 0 shows that Ψ has quadratic growth in a small neighborhood around 0. In particular, a compactness argument yields

$$\Psi(\xi) \geq c_0 |\xi|^2 \quad \text{for all } |\xi| \leq 1,$$

where $c_0 > 0$ is a constant independent of R, η, u . Finally, because of the Riemann–Lebesgue Lemma we have $|\mathcal{F}[J_1](\xi)| \rightarrow 0$ for $|\xi| \rightarrow \infty$ which implies that $\Psi(\xi) \rightarrow \frac{1}{2(2\pi)^n} \widehat{J}_1(0)$ for $|\xi| \rightarrow \infty$. In particular, by continuity it holds

$$\Psi(\xi) \geq c_1 \quad \text{for all } |\xi| \geq 1,$$

where $c_1 > 0$ is a constant independent of R, η, u .

Finally, we can prove the main assertion. Due to the arguments in the steps before, we have

$$\begin{aligned}
\mathcal{E}_\eta(u) &= \sum_{k \in \mathbb{Z}^n} \frac{\Psi(k\eta)}{\eta^2} |\hat{u}(k)|^2 \\
&\geq \sum_{k \in \mathbb{Z}^n, |k| \leq R} c_0 |k|^2 |\hat{u}(k)|^2 + \sum_{k \in \mathbb{Z}^n, |k| \geq \frac{1}{\eta}} \frac{c_1}{\eta^2} |\hat{u}(k)|^2 + \sum_{k \in \mathbb{Z}^n, \frac{1}{\eta} > |k| > R} c_0 |k|^2 |\hat{u}(k)|^2 \\
&\geq \sum_{k \in \mathbb{Z}^n, |k| \leq R} c_0 |k|^2 |\hat{u}(k)|^2 + c \sum_{k \in \mathbb{Z}^n, |k| > R} R^2 |\hat{u}(k)|^2 \\
&\geq c \sum_{k \in \mathbb{Z}^n, |k| > R} R^2 |\hat{u}(k)|^2 = c \sum_{k \in \mathbb{Z}^n} R^2 |\hat{u}(k)|^2 - c \sum_{k \in \mathbb{Z}^n, |k| \leq R} R^2 |\hat{u}(k)|^2,
\end{aligned}$$

where $c := \min\{c_0, c_1\} > 0$. Noting that $R^2 \leq 2R^4/(1 + |k|^2)$ for all $k \in \mathbb{Z}^n$ with $|k| \leq R$, we obtain

$$\begin{aligned}
\mathcal{E}_\eta(u) &\geq c \sum_{k \in \mathbb{Z}^n} R^2 |\hat{u}(k)|^2 - c \sum_{k \in \mathbb{Z}^n} \frac{2R^4}{1 + |k|^2} |\hat{u}(k)|^2 \\
&= (2\pi)^n c R^2 \|u\|_{L^2(\mathbb{T}^n)}^2 - 2c R^4 \|u\|_{H_2^{-1}(\mathbb{T}^n)}^2,
\end{aligned}$$

and thus the assertion follows. $\square$

2.4 Results for the Local Allen–Cahn Equation

First, we fix some notation concerning the evolving hypersurface.

Evolving hypersurfaces: Let $T_0 > 0$ and $(\Gamma_t)_{t \in [0, T_0]}$ with $\Gamma_t \subset (-\pi, \pi)^n$ for all $t \in [0, T_0]$ be a smoothly evolving compact hypersurface in $\mathbb{T}^n$. Moreover, let $\vec{n}_{\Gamma_t} = \vec{n}(\cdot, t)$ be the exterior normal of Γ_t for all $t \in [0, T_0]$ and $\Gamma := \bigcup_{t \in [0, T_0]} \Gamma_t \times \{t\}$. Here Γ_t separates $\mathbb{T}^n$ into two disjoint connected domains $\Omega_t^\pm$ for all $t \in [0, T_0]$, where $\vec{n}(\cdot, t)$ points into Ω_t^+ , and we set $\Omega^\pm := \bigcup_{t \in [0, T_0]} \Omega_t^\pm \times \{t\}$. In the following pr_t will denote the projection on the time variable, e.g., $\text{pr}_t(x, t) = t$ for $(x, t) \in \bar{\Omega} \times [0, T_0]$.

Parametrization: We consider a reference hypersurface Σ and a smooth parametrization $X_0 : \Sigma \times [0, T_0] \rightarrow \mathbb{T}^n$ such that $\bar{X}_0 := (X_0, \text{pr}_t) : \Sigma \times [0, T_0] \rightarrow \Gamma$ is a smooth diffeomorphism. Then we denote by

$$X : [-\delta, \delta] \times \Sigma \times [0, T_0] \rightarrow \mathbb{T}^n : (r, s, t) \mapsto X_0(s, t) + r\vec{n}(s, t)$$

for $\delta > 0$ small the standard tubular neighborhood coordinate system.

Tubular neighborhood: It is well known that, for $\delta_0 > 0$ small and all $\delta \in (0, \delta_0]$, $\bar{X} := (X, \text{pr}_t)$ is a smooth diffeomorphism onto the neighborhood $\bar{\Gamma}(\delta)$ of Γ , where $\bar{\Gamma}(\delta) := \bigcup_{t \in [0, T_0]} \Gamma_t(\delta) \times \{t\}$ and $\Gamma_t(\delta) := \{x \in \mathbb{T}^n : \text{dist}(x, \Gamma_t) < \delta\}$ is the tubular neighborhood of Γ_t for $t \in [0, T_0]$.

Finally, we set $(r, s, \text{pr}_t) := \bar{X}^{-1} : \bar{\Gamma}(\delta) \rightarrow [-\delta, \delta] \times \Sigma \times [0, T_0]$ and define the tangential and normal derivative as

$$\nabla_\tau \psi := (D_x s)^\top \left[\nabla_\Sigma(\psi \circ \bar{X}) \circ \bar{X}^{-1} \right] \quad \text{and} \quad \partial_n \psi := \partial_r(\psi \circ \bar{X}) \circ \bar{X}^{-1}$$

for sufficiently smooth $\psi : \bar{\Gamma}(\delta) \rightarrow \mathbb{R}$, $\delta \in (0, \delta_0]$. Here $D_x s(x) : \mathbb{R}^n \rightarrow T_{s(x)}\Sigma \subseteq \mathbb{R}^n$ is identified with an $n \times n$ matrix in the standard manner taking the standard basis on $\mathbb{R}^n$. Hence $D_x s(x)^T = (\partial_{x_j} s_k(x))_{j,k=1}^n$, cf. also Chen et al. (2010, equation (34)). Moreover, note that $|\nabla \psi|^2 = |\nabla_\tau \psi|^2 + |\partial_n \psi|^2$.

In the following theorem, we recall a standard result from asymptotic expansions for the local Allen–Cahn equation (1.6). This approximate solution is a central ingredient for comparison with the exact solution c_ε and to show convergence finally.

Theorem 2.6 (Approximate Solution for Local Allen–Cahn Equation). *Let the notation be as above, Γ be evolving according to mean curvature flow and let $M \in \mathbb{N} \cup \{0\}$.*

There are smooth $h_j : \Sigma \times [0, T_0] \rightarrow \mathbb{R}$, $u_{j+1} : \mathbb{R} \times \Sigma \times [0, T_0] \rightarrow \mathbb{R}$ for $j = 1, \dots, M$ with

$$\partial_\rho^j \partial_s^k \partial_t^l u_{j+1}(\rho, s, t) = \mathcal{O}(e^{-\beta|\rho|}) \quad \text{uniformly in } (\rho, s, t),$$

such that the following holds: defining for $\varepsilon > 0$, with the optimal profile θ_0 as in Remark 2.3,

$$\begin{aligned}\tilde{\rho}_\varepsilon(x, t) &:= \frac{r(x, t) - \varepsilon h_\varepsilon(s(x, t), t)}{\varepsilon}, \quad h_\varepsilon := \sum_{j=1}^M \varepsilon^{j-1} h_j, \\ u_\varepsilon^I(x, t) &:= \theta_0(\tilde{\rho}_\varepsilon(x, t)) + \sum_{k=2}^{M+1} \varepsilon^k u_{j+1}(\tilde{\rho}_\varepsilon(x, t), s(x, t), t) \quad \text{in } \Gamma(\delta)\end{aligned}$$

as well as with a cutoff $\eta : \mathbb{R} \rightarrow [0, 1]$ smooth with $\eta(r) = 1$ for $|r| \leq 1$, $\eta(r) = 0$ for $|r| \geq 2$,

$$u_\varepsilon^A := \begin{cases} \eta(\frac{r}{\delta_0})u_\varepsilon^I + (1 - \eta(\frac{r}{\delta_0}))\text{sign}(r) & \text{in } \Gamma(\delta_0), \\ \pm 1 & \text{in } \mathbb{T}^n \times (0, T_0) \setminus \Gamma(\delta_0), \end{cases}$$

then u_ε^A is smooth, uniformly bounded for small ε and for $r_\varepsilon^A := \partial_t u_\varepsilon^A - \Delta u_\varepsilon^A + \frac{1}{\varepsilon^2} f'(u_\varepsilon^A)$, the remainder of u_ε^A in the local Allen–Cahn equation (1.6), it holds

$$\begin{aligned}|r_\varepsilon^A| &\leq C(\varepsilon^M e^{-c|\tilde{\rho}_\varepsilon|} + \varepsilon^{M+1}) & \text{in } \Gamma(\delta_0), \\ r_\varepsilon^A &= 0 & \text{in } \mathbb{T}^n \times (0, T_0) \setminus \Gamma(\delta_0),\end{aligned}$$

where $c, C > 0$ are some constants and $\varepsilon > 0$ is small.

Proof. This essentially follows from the computations in Chen et al. (2010) and Abels and Liu (2018), see also the expansions in Moser (2023). $\square$

Remark 2.7. We note that

$$\lim_{\varepsilon \rightarrow 0} \tilde{\rho}_\varepsilon(x, t) = \pm\infty \quad \text{in } \Omega^\pm \cap \Gamma(2\delta)$$

and therefore

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon^A(x, t) = \pm 1 \quad \text{in } \Omega^\pm,$$

where $\Gamma_t = \partial\Omega_t^+$ evolves according to the mean curvature flow. This yields the convergence of c_ε to solutions of the mean curvature flow once the error $c_\varepsilon - c_A$ is estimated. Here $\tilde{\rho}_\varepsilon$ is called *stretched variable* and it resolves the behavior of c_ε close to the interface Γ at distance $O(\varepsilon)$. $\varepsilon h_\varepsilon$ can be considered as an approximation of the distance between Γ and the zero-level set $\{c_\varepsilon(x, t) = 0\}$.

Finally, we recall a classical spectral estimate for the linearized local Allen–Cahn operator.

Theorem 2.8 (Spectral Estimate for Local Allen–Cahn Operator). *Let $\delta \in (0, \delta_0/2]$ and $h_\varepsilon : \Sigma \times [0, T_0] \rightarrow \mathbb{R}$ be such that $h_\varepsilon(\cdot, t)$ is continuous for all $t \in [0, T_0]$ and $\|h_\varepsilon\|_\infty \leq C_0$, where $\varepsilon > 0$ is small and $C_0 > 0$ is a fixed constant. Moreover, we define the scaled variable*

$$\rho_\varepsilon := \frac{r - \varepsilon h_\varepsilon(s, t)}{\varepsilon} \quad \text{in } \overline{\Gamma(2\delta)}.$$

For $\varepsilon > 0$ small we consider $u_\varepsilon^A : \mathbb{T}^n \times [0, T_0] \rightarrow \mathbb{R}$ measurable with the property

$$u_\varepsilon^A = \begin{cases} \theta_0(\rho_\varepsilon) + \mathcal{O}(\varepsilon^2) & \text{in } \Gamma(\delta), \\ \pm 1 + \mathcal{O}(\varepsilon) & \text{in } [\mathbb{T}^n \times (0, T_0)] \setminus \Gamma(\delta), \end{cases}$$

where θ_0 is the optimal profile from (2.5).

Then there exist $\varepsilon_0, \bar{C} > 0$ independent of h_ε for fixed C_0 such that for all $\varepsilon \in (0, \varepsilon_0]$, $t \in [0, T_0]$ and all $\psi \in H^1(\mathbb{T}^n)$ it holds

$$\int_{\mathbb{T}^n} |\nabla \psi|^2 + \frac{1}{\varepsilon^2} f''(u_\varepsilon^A(\cdot, t)) \psi^2 dx \geq -\bar{C} \|\psi\|_{L^2(\mathbb{T}^n)}^2 + \|\nabla \psi\|_{L^2(\mathbb{T}^n \setminus \Gamma_t(\delta))}^2 + \|\nabla_\tau \psi\|_{L^2(\Gamma_t(\delta))}^2.$$

Proof. This goes back to Abels and Liu (2018) and Chen (1994, Theorem 2.13). $\square$

3 Properties of Solutions to the Nonlocal Allen–Cahn Equation

Let the assumptions (A.1)–(A.3) hold, $\varepsilon > 0$ and $\mathcal{L}_\eta$ be given as in (1.3). In this section, we consider the nonlocal Allen–Cahn equation (1.1) and (1.2) and show existence, uniqueness, higher regularity and boundedness of solutions in Section 3.1 and uniform H^3 -bounds for solutions for sufficiently small ε and $\eta \leq \eta(\varepsilon)$ in Section 3.2.

3.1 Existence, Uniqueness and Maximum Principle

Lemma 3.1 (Uniqueness). *Let the assumptions (A.1)–(A.3) hold, $T > 0$ and $\varepsilon > 0$. For initial values in $H^1(\mathbb{T}^n)$, solutions of (1.1) and (1.2) with regularity $H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^1(\mathbb{T}^n))$ are unique.*

Proof. Let $c_0 \in H^1(\mathbb{T}^n)$ and $c_1, c_2 \in H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^1(\mathbb{T}^n))$ be solutions of

$$\begin{aligned} \partial_t c + \mathcal{L}_\eta c + f'(c) &= 0 & \text{in } \mathbb{T}^n \times (0, T), \\ c|_{t=0} &= c_0 & \text{in } \mathbb{T}^n. \end{aligned}$$

Then, $\tilde{c} := c_1 - c_2$ solves

$$\begin{aligned} \partial_t \tilde{c} + \mathcal{L}_\eta \tilde{c} + f'(c_1) - f'(c_2) &= 0 & \text{in } \mathbb{T}^n \times (0, T), \\ \tilde{c}|_{t=0} &= 0 & \text{in } \mathbb{T}^n. \end{aligned}$$

Testing the latter by $\tilde{c}$ gives

$$\frac{1}{2} \frac{d}{dt} \|\tilde{c}\|_{L^2(\mathbb{T}^n)}^2 + 2\mathcal{E}_\eta(\tilde{c}) + \int_{\mathbb{T}^n} (f'(c_1) - f'(c_2)) \tilde{c} \, dx = 0,$$

where we note that $f'(c_i) \in L^2(0, T; L^2(\mathbb{T}^n))$ for $i = 1, 2$ because of (A.3) and embeddings. Moreover, the growth assumption of f from (A.3) yields

$$\int_{\mathbb{T}^n} (f'(c_1) - f'(c_2)) \tilde{c} \, dx \geq -\alpha \|\tilde{c}\|_{L^2(\mathbb{T}^n)}^2.$$

Therefore, it holds

$$\frac{1}{2} \frac{d}{dt} \|\tilde{c}\|_{L^2(\mathbb{T}^n)}^2 + 2\mathcal{E}_\eta(\tilde{c}) \leq \alpha \|\tilde{c}\|_{L^2(\mathbb{T}^n)}^2$$

and hence the Gronwall Lemma yields $c_1 = c_2$. □

Theorem 3.2 (Existence, Higher Regularity and Boundedness). *Let the assumptions (A.1)–(A.3) hold, $T > 0$ and $\varepsilon > 0$. Moreover, let $c_{0,\varepsilon} \in H^1(\mathbb{T}^n)$. Then, there exists a unique solution c_ε to (1.1) and (1.2) with the regularity*

$$c_\varepsilon \in H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^1(\mathbb{T}^n)) \cap C^0([0, T], L^p(\mathbb{T}^n))$$

for all $p \in [1, \infty)$ if $n = 2$ and $p \in [1, 6)$ if $n = 3$. Additionally, if $c_{0,\varepsilon} \in H^3(\mathbb{T}^n)$ and $|c_{0,\varepsilon}| \leq R_0$ with $R_0 \geq 1$ as in (A.3), then the solution c_ε has the regularity

$$c_\varepsilon \in H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^3(\mathbb{T}^n)) \cap C^1([0, T], C^0(\mathbb{T}^n)) \cap C^0([0, T], H^2(\mathbb{T}^n))$$

and $|c_\varepsilon|$ is uniformly bounded by R_0 .

Proof. For simplicity we set $\varepsilon = 1$ in the proof and omit the ε in the notation. All the arguments can be done analogously for the case of arbitrary $\varepsilon > 0$. Moreover, $\eta > 0$ is fixed in the following.

We prove this theorem using a Galerkin ansatz, in particular we project the equations and solution spaces onto finite dimensional subspaces of $H^1(\mathbb{T}^n)$. For the approximation scheme, we consider the finite-dimensional subspaces

$$V_N := \text{span}\{1, \cos(k \cdot \cdot), \sin(k \cdot \cdot) : k = (k_1, \dots, k_n) \in \mathbb{Z}^n, |k_1|, \dots, |k_n| \leq N\}, \quad N \in \mathbb{N},$$

generated by trigonometric functions. One can directly show that the Laplacian is diagonalizable on V_N and we denote the eigenfunctions by $w_1, \dots, w_{\dim V_N}$. Since $V_{N_1} \subseteq V_{N_2}$ for $N_1 \leq N_2$ this is consistent for all $N \in \mathbb{N}$ and it is well-known that

$(w_i)_{i \in \mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{T}^n)$. We denote with $(\lambda_i)_{i \in \mathbb{N}}$ the corresponding eigenvalues. Moreover, note that we have the following representation of the L^2 -projection for (real-valued) $f \in L^2(\mathbb{T}^n)$ and all $N \in \mathbb{N}$:

$$P_N f := P_{V_N}^{L^2} f = S_N f = D_N * f, \quad (3.1)$$

where

$$S_N f := \sum_{k \in \mathbb{Z}^n, |k_1|, \dots, |k_N| \leq N} \frac{1}{(2\pi)^n} \hat{f}_k e^{ik \cdot} \quad \text{and} \quad D_N := \sum_{k \in \mathbb{Z}^n, |k_1|, \dots, |k_N| \leq N} e^{ik \cdot}.$$

for all $N \in \mathbb{N}$ and $f \in L^1(\mathbb{T}^n)$. Properties of S_N , especially for convergence, are well-known from Fourier-Analysis, cf. Grafakos (2014). Recalling that convolution operators commute, we observe that $\mathcal{L}_\eta P_N u = P_N \mathcal{L}_\eta u$ for all $u \in L^2(\mathbb{T}^n)$. Moreover, we note that $\sup_{N \in \mathbb{N}} \|P_N\|_{\mathcal{L}(L^p(\mathbb{T}^n))} < \infty$, cf. Grafakos (2014, Theorem 4.1.8). In particular, $\text{span}\{w_i : i \in \mathbb{N}\}$ is dense in $H^k(\mathbb{T}^n)$ for all $k \in \mathbb{N}$. Finally, for vector-valued $f \in L^2(\mathbb{T}^n)^m$, $m \in \mathbb{N}$ equation (3.1) is understood component-wise.

1. Finite dimensional approximation. In this step, we determine approximate solutions c^N of (1.1) and (1.2) of the form

$$c^N = \sum_{i=1}^N g_i^N w_i,$$

where $\mathbf{g}^N := (g_i^N)_{i=1}^N : [0, T] \rightarrow \mathbb{R}$ will be continuously differentiable. In particular, it holds $c^N(t) \in V_N$ for all $t \in [0, T]$. We will construct $\mathbf{g}^N$ such that c^N is a solution of system

$$\int_{\mathbb{T}^n} \partial_t c^N(t) w_i \, dx + \int_{\mathbb{T}^n} \mathcal{L}_\eta c^N(t) w_i \, dx + \int_{\mathbb{T}^n} f'(c^N(t)) w_i \, dx = 0 \quad (3.2)$$

for all $t \in [0, T]$ and $i = 1, \dots, N$ together with the initial condition

$$c^N(0) = P_N c_0 = \sum_{i=1}^N (c_0, w_i)_{L^2(\mathbb{T}^n)} w_i. \quad (3.3)$$

Since the functions $(w_i)_{i \in \mathbb{N}}$ are an orthonormal basis of $L^2(\mathbb{T}^n)$, the approximate system (3.2) and (3.3) is equivalent to the following ordinary differential equation for $\mathbf{g}^N$:

$$\frac{d}{dt} g_i^N(t) = - \sum_{k=1}^N g_k^N(t) \int_{\mathbb{T}^n} \mathcal{L}_\eta w_k w_i \, dx - \int_{\mathbb{T}^n} f' \left(\sum_{k=1}^N g_k^N(t) w_k \right) w_i \, dx \quad (3.4)$$

together with the initial condition

$$g_i^N(0) = (c_0, w_i)_{L^2} \quad (3.5)$$

for all $i = 1, \dots, N$. Since the right-hand side in (3.4) depends continuously on $\mathbf{g}^N$, Peano's Theorem guarantees the existence of a local C^1 -solution of this initial value problem on a right-maximal interval $[0, T_N^*) \cap [0, T]$ for some $T_N^* > 0$. In the next two steps, we will derive an energy estimate and show global existence of the Galerkin approximation. Therefore we show uniform bounds on $[0, T_N]$ for $T_N \in [0, T_N^*) \cap [0, T]$ arbitrary.

2. Energy estimate. First of all, we define

$$E^N(t) := \frac{1}{4} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) |c^N(x, t) - c^N(y, t)|^2 \, dy \, dx + \int_{\mathbb{T}^n} f(c^N(t)) \, dx$$

for all $t \in [0, T_N]$. Then, we compute

$$\begin{aligned} \frac{d}{dt} E^N(t) &= \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) (c^N(x,t) - c^N(y,t)) (\partial_t c^N(x,t) - \partial_t c^N(y,t)) dy dx \\ &\quad + \int_{\mathbb{T}^n} f'(c^N(t)) \partial_t c^N(t) dx = - \int_{\mathbb{T}^n} |\partial_t c^N(t)|^2 dx \leq 0, \end{aligned}$$

where we used that $g_1^N, \dots, h_N^N$ are C^1 -functions and for the second equality we multiplied (3.2) by $(g_i^N)'(t)$ and summed over all $i = 1, \dots, N$. Integrating the inequality above with respect to time yields for all $s \in [0, T_N]$ and all $N \in \mathbb{N}$

$$E^N(s) + \int_0^s \int_{\mathbb{T}^n} |\partial_t c^N(x,t)|^2 dx dt \leq E^N(0). \quad (3.6)$$

In order to establish uniform bounds for our solutions, we need to verify that $E^N(0)$ can be bounded by some constant $C > 0$, which does not depend on N .

First of all, because of $c_0 \in H^1(\mathbb{T}^n)$, the initial condition (3.3) and the convolution representation in (3.1) it follows that the sequence $(c^N(0))_{N \in \mathbb{N}} \subseteq H^1(\mathbb{T}^n)$ is bounded. By Bourgain et al. (2001, Theorem 1), there exists a constant $C > 0$ such that

$$\frac{1}{4} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) |c^N(x,0) - c^N(y,0)|^2 dy dx = \mathcal{E}_\eta(c^N(0)) \leq C \|\nabla c^N(0)\|_{L^2(\mathbb{T}^n)}^2.$$

Moreover, the assumptions for f from (A.3) and the Sobolev embedding theorem imply that

$$\int_{\mathbb{T}^n} f(c^N(0)) dx \leq C \left(1 + \|c^N(0)\|_{L^4(\mathbb{T}^n)}^4 \right)$$

is bounded uniformly in $N \in \mathbb{N}$. Finally, we obtain the following energy estimate

$$E^N(s) + \int_0^s \int_{\mathbb{T}^n} |\partial_t c^N(x,t)|^2 dx dt \leq E^N(0) \leq C \quad (3.7)$$

for all $s \in [0, T_N]$ and all $N \in \mathbb{N}$, where C is independent of $T_N \in [0, T_N^*) \cap [0, T]$ and $N \in \mathbb{N}$.

3. Uniform estimates. From the energy inequality (3.7) we obtain the uniform estimate $\|\partial_t c^N\|_{L^2(0, T_N; L^2(\mathbb{T}^n))} \leq C$. In the next step we prove that $(c^N)_{N \in \mathbb{N}} \subseteq L^\infty(0, T_N; H^1(\mathbb{T}^n))$ is bounded. To this end, we first test (3.2) by Δc^N . More precisely, we multiply (3.2) by $\lambda_i g_i^N(t)$ and sum over all $i = 1, \dots, N$. This yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla c^N\|_{L^2(\mathbb{T}^n)}^2 + \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) |\nabla c^N(x,t) - \nabla c^N(y,t)|^2 dy dx \\ + \int_{\mathbb{T}^n} \nabla f'(c^N) \cdot \nabla c^N dx = 0. \end{aligned}$$

Due to the assumption on f'' from (A.3), it holds

$$\int_{\mathbb{T}^n} \nabla f'(c^N) \cdot \nabla c^N dx \geq -\alpha \|\nabla c^N\|_{L^2(\mathbb{T}^n)}^2.$$

Thus, we can apply Gronwall's inequality to conclude that $(\nabla c^N)_{N \in \mathbb{N}} \subseteq L^\infty(0, T_N; L^2(\mathbb{T}^n))$ is bounded. Next, we test (3.2) by c^N . On the Galerkin level, we multiply (3.2) by $g_i^N(t)$ and sum over all $i = 1, \dots, N$. This yields

$$\frac{1}{2} \frac{d}{dt} \|c^N\|_{L^2(\mathbb{T}^n)}^2 + \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) |c^N(x,t) - c^N(y,t)|^2 dy dx + \int_{\mathbb{T}^n} f'(c^N) c^N dx = 0.$$

In order to control the third term on the left-hand side, we use the growth assumption of f from (A.3), the Hölder inequality as well as the Gagliardo–Nirenberg inequality. Hence we obtain

$$\begin{aligned} \int_{\mathbb{T}^n} f'(c^N) c^N dx &\leq \|f'(c^N)\|_{L^2(\mathbb{T}^n)} \|c^N\|_{L^2(\mathbb{T}^n)} \\ &\leq C \|c^N\|_{L^2(\mathbb{T}^n)} + C \|c^N\|_{L^6(\mathbb{T}^n)}^3 \|c^N\|_{L^2(\mathbb{T}^n)} \\ &\leq C \|c^N\|_{L^2(\mathbb{T}^n)} + C \|\nabla c^N\|_{L^2(\mathbb{T}^n)}^n \|c^N\|_{L^2(\mathbb{T}^n)}^{4-n}. \end{aligned}$$

Since $(\nabla c^N)_{N \in \mathbb{N}}$ is bounded in $L^\infty(0, T_N; L^2(\mathbb{T}^n)^n)$, we apply Young's inequality and obtain from Gronwall that $(c^N)_{N \in \mathbb{N}} \subseteq L^\infty(0, T_N; L^2(\mathbb{T}^n))$ is bounded.

Altogether, the sequence $(c^N)_{N \in \mathbb{N}}$ is bounded in $C^0([0, T_N], H^1(\mathbb{T}^n)) \cap H^1(0, T_N; L^2(\mathbb{T}^n))$ uniformly for all $T_N \in [0, T_N^*) \cap [0, T]$ and $N \in \mathbb{N}$. In particular, this yields that the solution $\mathbf{g}^N$ is bounded on $[0, T_N]$ uniformly for all $T_N \in [0, T_N^*) \cap [0, T]$ and all $N \in \mathbb{N}$. An extension argument yields that $\mathbf{g}^N$ and hence c^N exist globally on $[0, T]$ and $T_N = T$ can be chosen above.

4. Passage to the limit. Due to the uniform estimates derived in the step before, there exists a subsequence, again denoted by $(c^N)_{N \in \mathbb{N}}$, such that for $N \rightarrow \infty$

$$\begin{aligned} c^N &\rightharpoonup c && \text{in } H^1(0, T; L^2(\mathbb{T}^n)), \\ c^N &\rightarrow c && \text{in } C^0([0, T]; L^p(\mathbb{T}^n)), \end{aligned}$$

for some $c \in H^1(0, T; L^2(\mathbb{T}^n)) \cap C^0([0, T]; L^p(\mathbb{T}^n))$ and for all $p \in [1, \infty)$ if $n = 2$ and all $p \in [1, 6)$ if $n = 3$, where the last convergence follows by the Aubin–Lions–Simon Lemma. We need to show equation (1.1) for c .

Let $\xi \in C_0^\infty(0, T)$. We multiply (3.2) by $\xi(t)$ and integrate with respect to time. Hence

$$\int_0^T \int_{\mathbb{T}^n} \partial_t c^N w_i \xi dx dt + \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\eta c^N w_i \xi dx dt + \int_0^T \int_{\mathbb{T}^n} f'(c^N) w_i \xi dx dt = 0.$$

For the first term on the left-hand side, we use the weak convergence $c^N \rightharpoonup c$ in $H^1(0, T; L^2(\mathbb{T}^n))$ for $N \rightarrow \infty$ to conclude that

$$\int_0^T \int_{\mathbb{T}^n} \partial_t c^N w_i \xi dx dt \rightarrow \int_0^T \int_{\mathbb{T}^n} \partial_t c w_i \xi dx dt.$$

For the second term we prove in the following that $\mathcal{L}_\eta c^N \rightarrow \mathcal{L}_\eta c$ in $L^2(0, T; L^2(\mathbb{T}^n))$ for $N \rightarrow \infty$. By definition and Young's convolution inequality, we have

$$\begin{aligned} \|\mathcal{L}_\eta c^N - \mathcal{L}_\eta c\|_{L^2(0, T; L^2(\mathbb{T}^n))}^2 &\leq \int_0^T \|J_\eta * (c^N(t) - c(t))\|_{L^2(\mathbb{T}^n)}^2 dt + \int_0^T \|(J_\eta * 1)(c^N(t) - c(t))\|_{L^2(\mathbb{T}^n)}^2 dt \\ &\leq \int_0^T \|J_\eta\|_{L^1(\mathbb{T}^n)}^2 \|c^N(t) - c(t)\|_{L^2(\mathbb{T}^n)}^2 dt + \int_0^T \|(J_\eta * 1)(c^N(t) - c(t))\|_{L^2(\mathbb{T}^n)}^2 dt. \end{aligned}$$

Since the term $J_\eta * 1$ is constant on the torus, the assertion follows from the strong convergence $c^N \rightarrow c$ in $C^0([0, T]; L^2(\mathbb{T}^n))$ for $N \rightarrow \infty$. Altogether, this implies for $N \rightarrow \infty$ that

$$\int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\eta c^N w_i \xi dx dt \rightarrow \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\eta c w_i \xi dx dt.$$

Finally, for the last term we apply the General Lebesgue Dominated Convergence Theorem. Due to the growth assumption of f' from (A.3), it holds

$$|f'(c)| \leq C|c|^3 + C.$$

Thus, the operator T defined by

$$T(x, c(x)) := f'(c(x))$$

is a Nemytskii operator and by standard theory it follows that T is a bounded and continuous operator from $L^4(\mathbb{T}^n)$ to $L^{4/3}(\mathbb{T}^n)$. Therefore, thanks to the strong convergence $c^N \rightarrow c$ in $C^0([0, T]; L^4(\mathbb{T}^n))$, we can pass to the limit and get

$$\int_0^T \int_{\mathbb{T}^n} f'(c^N) w_i \xi \, dx \, dt \rightarrow \int_0^T \int_{\mathbb{T}^n} f'(c) w_i \xi \, dx \, dt$$

for $N \rightarrow \infty$. Finally, we obtain

$$\int_0^T \int_{\mathbb{T}^n} \partial_t c w_i \xi \, dx \, dt + \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\eta c w_i \xi \, dx \, dt + \int_0^T \int_{\mathbb{T}^n} f'(c) w_i \xi \, dx \, dt = 0$$

for all $i \in \mathbb{N}$ and all $\xi \in C_0^\infty(0, T)$. Since $\text{span}\{w_i : i \in \mathbb{N}\}$ is dense in $H^1(\mathbb{T}^n)$, we obtain

$$\int_0^T \int_{\mathbb{T}^n} \partial_t c w \xi \, dx \, dt + \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\eta c w \xi \, dx \, dt + \int_0^T \int_{\mathbb{T}^n} f'(c) w \xi \, dx \, dt = 0$$

for all $w \in H^1(\mathbb{T}^n)$ and all $\xi \in C_0^\infty(0, T)$. Therefore, the Fundamental Lemma of Calculus of Variations implies that (1.1) holds for c .

Finally, we show that the initial condition holds. We have already seen that $c^N(0) \rightarrow c_0$ in $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ for $N \rightarrow \infty$. Since we also have the convergence $c^N \rightarrow c$ in $C^0([0, T]; L^2(\mathbb{T}^n))$, we conclude that $c(0) = c_0$ in $L^2(\mathbb{T}^n)$ for $N \rightarrow \infty$.

5. Higher order estimates. In this part we prove higher regularity for the solution obtained in the steps before provided that the initial value satisfies $c_0 \in H^3(\mathbb{T}^n)$ and $|c_0| \leq R_0$ with $R_0 \geq 1$ as in (A.3). In the end we prove that the solution is confined to $[-R_0, R_0]$ in this situation and hence we can change the potential outside of this interval suitably. Therefore from now on we assume without loss of generality that the potential f and its derivatives are uniformly bounded.

We first prove that $c \in L^\infty(0, T; H^2(\mathbb{T}^n))$. For the proof, we again use the Galerkin scheme introduced in step 1. We test equation (3.2) by $\Delta^2 c^N$, which means, on the Galerkin level, that we multiply (3.2) by $\lambda_i^2 g_i^N(t)$ and sum over all $i = 1, \dots, N$. This yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta c^N\|_{L^2(\mathbb{T}^n)}^2 + \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) \left| \Delta c^N(x, t) - \Delta c^N(y, t) \right|^2 dy \, dx \\ + \int_{\mathbb{T}^n} \Delta f'(c^N) \Delta c^N \, dx = 0. \end{aligned}$$

For the third term on the left-hand side, we observe

$$\begin{aligned} \int_{\mathbb{T}^n} \Delta f'(c^N) \Delta c^N \, dx &= \int_{\mathbb{T}^n} f'''(c^N) |\nabla c^N|^2 \Delta c^N \, dx + \int_{\mathbb{T}^n} f''(c^N) |\Delta c^N|^2 \, dx \\ &\geq \int_{\mathbb{T}^n} f'''(c^N) |\nabla c^N|^2 \Delta c^N \, dx - \alpha \|\Delta c^N\|_{L^2(\mathbb{T}^n)}^2. \end{aligned}$$

By our assumption on the potential f , it holds $\|f^{(k)}\|_{L^\infty(\mathbb{R})} \leq C$ for all $k \geq 3$. Thus, the remaining term can be controlled as follows:

$$\left| \int_{\mathbb{T}^n} f'''(c^N) |\nabla c^N|^2 \Delta c^N \, dx \right| \leq \|f'''\|_\infty \|\nabla c^N\|_{L^4(\mathbb{T}^n)^n}^2 \|\Delta c^N\|_{L^2(\mathbb{T}^n)}.$$

In order to control $\|\nabla c^N\|_{L^4(\mathbb{T}^n)^n}$, note that by (3.2)

$$0 = P_N(\partial_t c^N + \mathcal{L}_\eta c^N + f'(c^N)) = \partial_t c^N + \mathcal{L}_\eta c^N + P_N(f'(c^N)),$$

where we used that P_N and $\mathcal{L}_\eta$ commutes due to the representation (3.1) and properties of convolutions. We differentiate the latter equation with respect to the space variable and test by $P_N(|\nabla c^N|^{p-2}\nabla c^N)$, $p > 2$. This yields

$$\begin{aligned} & \int_{\mathbb{T}^n} \partial_t \nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N) dx + \int_{\mathbb{T}^n} \mathcal{L}_\eta \nabla c^N \cdot |\nabla c^N|^{p-2}\nabla c^N dx \\ & + \int_{\mathbb{T}^n} f''(c^N) \nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N) dx = 0, \end{aligned} \quad (3.8)$$

where we used $\mathcal{L}_\eta \nabla c^N = \mathcal{L}_\eta P_N \nabla c^N = P_N \mathcal{L}_\eta \nabla c^N$ for the second term. This follows from the fact that convolution operators commute. For the first term, we use $\partial_t \nabla c^N \in (V_N)^n$ to obtain

$$\int_{\mathbb{T}^n} \partial_t \nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N) dx = \frac{d}{dt} \int_{\mathbb{T}^n} \frac{1}{p} |\nabla c^N|^p dx.$$

For the term involving the nonlocal operator, we use a symmetry argument to conclude

$$\begin{aligned} & \int_{\mathbb{T}^n} \mathcal{L}_\eta \nabla c^N \cdot |\nabla c^N|^{p-2}\nabla c^N dx \\ & = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) (\nabla c^N(x) - \nabla c^N(y)) \cdot \nabla b(\nabla c^N(x)) dy dx \\ & = \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) (\nabla c^N(x) - \nabla c^N(y)) \cdot (\nabla b(\nabla c^N(x)) - \nabla b(\nabla c^N(y))) dy dx, \end{aligned}$$

where the function b is defined as $b(z) := |z|^p/p$ for all $z \in \mathbb{R}^n$. Since ∇b is monotone and since J_η is non-negative, this implies that

$$\int_{\mathbb{T}^n} \mathcal{L}_\eta \nabla c^N \cdot |\nabla c^N|^{p-2}\nabla c^N dx \geq 0.$$

Finally, for the last term in (3.8), we use Hölder's inequality to get

$$\left| \int_{\mathbb{T}^n} f''(c^N) \nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N) dx \right| \leq C \|\nabla c^N\|_{L^p(\mathbb{T}^n)^n} \|P_N(|\nabla c^N|^{p-2}\nabla c^N)\|_{L^{p'}(\mathbb{T}^n)^n}.$$

From Fourier Analysis and (3.1) it is well-known that $\sup_{N \in \mathbb{N}} \|P_N\|_{\mathcal{L}(L^{p'}(\mathbb{T}^n))} \leq K$ for every $1 < p < \infty$, where $K > 0$ is independent of $N \in \mathbb{N}$. Moreover, we observe that $\| |\nabla c^N|^{p-2}\nabla c^N \|_{L^{p'}(\mathbb{T}^n)^n} = \|\nabla c^N\|_{L^p(\mathbb{T}^n)^n}^{p-1}$. Hence (3.8) yields

$$\frac{d}{dt} \int_{\mathbb{T}^n} \frac{1}{p} |\nabla c^N|^p dx \leq C \|\nabla c^N\|_{L^p(\mathbb{T}^n)^n}^p.$$

Using the relation $c^N(0) = P_N c_0$ and (3.1), we conclude that the sequence $(c^N(0))_{N \in \mathbb{N}}$ is bounded in $H^3(\mathbb{T}^n)$ provided that $c_0 \in H^3(\mathbb{T}^n)$.

Thus by Gronwall's inequality the sequence $(\nabla c^N)_{N \in \mathbb{N}} \subseteq L^\infty(0, T; L^p(\mathbb{T}^n)^n)$ is bounded for all $p > 2$. Therefore it holds

$$\int_{\mathbb{T}^n} f'''(c^N) |\nabla c^N|^2 \Delta c^N dx \leq C(1 + \|\Delta c^N\|_{L^2(\mathbb{T}^n)}^2)$$

and thus, again by Gronwall's inequality, $(c^N)_{N \in \mathbb{N}} \subseteq L^\infty(0, T; H^2(\mathbb{T}^n))$ is bounded since the second order derivatives can be controlled with the Laplacian via integration by parts.

In the next step, we prove $c \in L^\infty(0, T; H^3(\mathbb{T}^n))$. To this end, we test equation (3.2) by $\Delta^3 c^N$. More precisely, we multiply (3.2) by $\lambda_i^3 g_i^N(t)$ and sum over all $i = 1, \dots, N$. This yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \Delta c^N\|_{L^2(\mathbb{T}^n)^n}^2 + \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\eta(x-y) \left| \nabla \Delta c^N(t, x) - \nabla \Delta c^N(t, y) \right|^2 dy dx \\ & + \int_{\mathbb{T}^n} \nabla \Delta f'(c^N) \cdot \nabla \Delta c^N dx = 0. \end{aligned}$$

We need to control the last term on the left-hand side. First of all, the chain rule gives

$$\begin{aligned}\nabla \Delta f'(c^N) &= f^{(4)}(c^N) |\nabla c^N|^2 \nabla c^N + 2f'''(c^N) D^2 c^N \nabla c^N \\ &\quad + f'''(c^N) \Delta c^N \nabla c^N + f''(c^N) \nabla \Delta c^N.\end{aligned}\tag{3.9}$$

In order to estimate the terms we use that we assumed without loss of generality that the derivatives of f are uniformly bounded. Moreover,

$$\begin{aligned}\| |\nabla c^N|^2 \nabla c^N \|_{L^2(\mathbb{T}^n)^n} &= \| \nabla c^N \|_{L^6(\mathbb{T}^n)^n}^3 \leq C, \\ \| D^2 c^N \nabla c^N \|_{L^2(\mathbb{T}^n)^n} &\leq \| D^2 c^N \|_{L^2(\mathbb{T}^n)^{n \times n}} \| \nabla c^N \|_{L^\infty(\mathbb{T}^n)^n} \leq C \| c^N \|_{H^2(\mathbb{T}^n)} \| c^N \|_{H^3(\mathbb{T}^n)} \\ &\leq C \| c^N \|_{H^2(\mathbb{T}^n)} (1 + \| \nabla \Delta c^N \|_{L^2(\mathbb{T}^n)^n}), \\ \| \Delta c^N \nabla c^N \|_{L^2(\mathbb{T}^n)^n} &\leq C \| c^N \|_{H^2(\mathbb{T}^n)} (1 + \| \nabla \Delta c^N \|_{L^2(\mathbb{T}^n)^n}).\end{aligned}$$

Therefore we can control the corresponding parts in the integral above using Young's inequality. For the last term, we use the growth condition of f from (A.3), which implies

$$\int_{\mathbb{T}^n} f''(c^N) |\nabla \Delta c^N|^2 dx \geq -\alpha \| \nabla \Delta c^N \|_{L^2(\mathbb{T}^n)^n}^2.$$

Altogether, Gronwall shows that $(c^N)_{N \in \mathbb{N}} \subseteq L^\infty(0, T; H^3(\mathbb{T}^n))$ is bounded.

From the previous estimates we obtain $c \in H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^3(\mathbb{T}^n))$. Now, due to the Aubin–Lions–Simon Lemma, we get

$$H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^3(\mathbb{T}^n)) \hookrightarrow C^0([0, T]; H^2(\mathbb{T}^n)),$$

which in particular yields $c \in C^0([0, T], C^0(\mathbb{T}^n))$. Finally, from the equation (1.1) for c we obtain $c \in C^1([0, T]; C^0(\mathbb{T}^n))$.

6. Boundedness. Finally, we prove that the solution $c \in C^1([0, T]; C^0(\mathbb{T}^n))$ of (1.1) and (1.2) is bounded by R_0 provided that $|c_0| \leq R_0$, where $R_0 \geq 1$ is from (A.3). Assume that there exists $(x_0, t_0) \in \mathbb{T}^n \times [0, T]$ such that c attains its maximum in (x_0, t_0) with $c(x_0, t_0) > R_0$. Then $\partial_t c(x_0, t_0) \geq 0$ and it holds

$$\partial_t c(x_0, t_0) + \mathcal{L}_\eta c(x_0, t_0) + f'(c(x_0, t_0)) > 0,$$

since $f'(c(x_0, t_0)) > 0$ if $c(x_0, t_0) > R_0$ and

$$\mathcal{L}_\eta c(x_0, t_0) = \int_{\mathbb{T}^n} J_\eta(x_0 - y)(c(x_0, t_0) - c(y, t_0)) dy \geq 0,$$

since $J_\eta \geq 0$. This is a contradiction since c solves (1.1). Analogously, one shows $c \geq -R_0$.

This concludes the proof of Theorem 3.2. □

3.2 Uniform H^3 -Bounds

In order to apply Theorem 2.4 and close the proof of the main result with a Gronwall argument, we need to show that the solutions obtained by Theorem 3.2 are bounded uniformly in η . Indeed, we have the following result:

Theorem 3.3 (Uniform H^3 -Bounds). *Let $T > 0$ be fixed. For any $\varepsilon \in (0, 1)$, let c_ε be the solution to (1.1) and (1.2) from Theorem 3.2 for initial values $c_{0,\varepsilon} \in H^3(\mathbb{T}^n)$ such that $|c_{0,\varepsilon}| \leq R_0$ with $R_0 \geq 1$ as in (A.3) and such that with some constant $K > 0$ independent of $\varepsilon \in (0, 1)$*

$$\|c_{0,\varepsilon}\|_{H^1(\mathbb{T}^n)} \leq \frac{K}{\varepsilon^2}, \quad \|c_{0,\varepsilon}\|_{H^2(\mathbb{T}^n)} \leq \frac{K}{\varepsilon^4} \quad \text{and} \quad \|c_{0,\varepsilon}\|_{H^3(\mathbb{T}^n)} \leq \frac{K}{\varepsilon^{11}}.$$

Then there exist constants $\varepsilon_0 \in (0, 1)$, $c, C > 0$ independent of $c_\varepsilon, \varepsilon, \eta$ such that for all $0 < \eta \leq c\varepsilon^3$, $\varepsilon \in (0, \varepsilon_0)$ it holds

$$\|c_\varepsilon\|_{L^\infty(0,T;H^3(\mathbb{T}^n))} \leq \frac{C}{\varepsilon^{11}}.$$

Proof. Let c_ε be the solution of (1.1) and (1.2) according to Theorem 3.2. We have to control the H^3 -norm of c_ε in dependence of ε . To this end, we proceed in several steps.

1. **H^1 -estimate.** We test (1.1) with $-\Delta c_\varepsilon$. After integration by parts, we obtain for all $t \in [0, T]$

$$\frac{1}{2} \|\nabla c_\varepsilon(t)\|_{L^2(\mathbb{T}^n)^n}^2 + \int_0^t \mathcal{E}_\eta(\nabla c_\varepsilon) \, ds \leq \frac{1}{2} \|\nabla c_{0,\varepsilon}\|_{L^2(\mathbb{T}^n)^n}^2 + \frac{\alpha}{\varepsilon^2} \int_0^t \|\nabla c_\varepsilon\|_{L^2(\mathbb{T}^n)^n}^2 \, ds$$

where we also used the condition for f'' from (A.3). Next, we apply Theorem 2.5 (Nonlocal Ehrling lemma) which yields for all $R \geq 1$, $0 < \eta \leq 1/R$ that

$$\begin{aligned} & \frac{1}{2} \|\nabla c_\varepsilon(t)\|_{L^2(\mathbb{T}^n)^n}^2 + \int_0^t \mathcal{E}_\eta(\nabla c_\varepsilon) \, ds \\ & \leq \frac{1}{2} \|\nabla c_{0,\varepsilon}\|_{L^2(\mathbb{T}^n)^n}^2 + \int_0^t \left[\frac{C\alpha}{\varepsilon^2 R^2} \mathcal{E}_\eta(\nabla c_\varepsilon) + \frac{C\alpha}{\varepsilon^2} R^2 \|c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \right] \, ds. \end{aligned}$$

We choose $R = 2(\sqrt{C\alpha})/\varepsilon$ and ε small such that $R \geq 1$. Thus using that $|c_\varepsilon|$ is bounded thanks to the maximum principle, we obtain

$$\|\nabla c_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{T}^n)^n)} \leq \frac{C}{\varepsilon^2}$$

for all $0 < \eta \leq (1/2\sqrt{C\alpha})\varepsilon$ and ε small.

2. **H^2 -estimate.** We test (1.1) with $\Delta^2 c_\varepsilon$. This yields for all $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \|\Delta c_\varepsilon(t)\|_{L^2(\mathbb{T}^n)}^2 + \int_0^t \mathcal{E}_\eta(\Delta c_\varepsilon) \, ds \\ & \leq \frac{1}{2} \|\Delta c_{0,\varepsilon}\|_{L^2(\mathbb{T}^n)}^2 + \frac{\alpha}{\varepsilon^2} \int_0^t \|\Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \, ds - \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^n} f'''(c_\varepsilon) |\nabla c_\varepsilon|^2 \Delta c_\varepsilon \, dx \, ds. \end{aligned}$$

Because of the maximum principle in Theorem 3.2, we have that $|c_\varepsilon|$ is uniformly bounded. Moreover, the Gagliardo–Nirenberg interpolation inequality gives for all $t \in [0, T]$

$$\|\nabla c_\varepsilon\|_{L^4(\mathbb{T}^n)^n} \leq C \|c_\varepsilon\|_{L^\infty(\mathbb{T}^n)}^{1/2} \|c_\varepsilon\|_{H^2(\mathbb{T}^n)}^{1/2} \leq C \|c_\varepsilon\|_{H^2(\mathbb{T}^n)}^{1/2}$$

for both $n \in \{2, 3\}$, where we omit the time-dependence for simplicity. Then by Young's inequality

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\mathbb{T}^n} f'''(c_\varepsilon) |\nabla c_\varepsilon|^2 \Delta c_\varepsilon \, dx & \leq \frac{C}{\varepsilon^2} \|\nabla c_\varepsilon\|_{L^4(\mathbb{T}^n)^n}^4 + \frac{C}{\varepsilon^2} \|\Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \\ & \leq \frac{C}{\varepsilon^2} \|c_\varepsilon\|_{H^2(\mathbb{T}^n)}^2 + \frac{C}{\varepsilon^2} \|\Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2. \end{aligned}$$

By definition of the H^2 -norm, the boundedness of c_ε and the estimate derived in step 1, we have

$$\|c_\varepsilon\|_{H^2(\mathbb{T}^n)}^2 \leq C \left(1 + \frac{1}{\varepsilon^4} + \|\Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \right).$$

This yields

$$\frac{1}{\varepsilon^2} \int_{\mathbb{T}^n} f'''(c_\varepsilon) |\nabla c_\varepsilon|^2 \Delta c_\varepsilon \, dx \leq \frac{C}{\varepsilon^6} + \frac{C}{\varepsilon^2} \|\Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2.$$

Together with Theorem 2.5 (Nonlocal Ehrling Inequality), we get for all $R \geq 1$, $0 < \eta \leq 1/R$ and all $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \|\Delta c_\epsilon(t)\|_{L^2(\mathbb{T}^n)}^2 + \int_0^t \mathcal{E}_\eta(\Delta c_\epsilon) \, ds \\ & \leq \frac{1}{2} \|\Delta c_{0,\epsilon}\|_{L^2(\mathbb{T}^n)}^2 + \frac{C}{\epsilon^6} + \frac{C}{\epsilon^2} \int_0^t \|\Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}^2 \, ds \\ & \leq \frac{1}{2} \|\Delta c_{0,\epsilon}\|_{L^2(\mathbb{T}^n)}^2 + \frac{C}{\epsilon^6} + \int_0^t \left[\frac{C}{\epsilon^2 R^2} \mathcal{E}_\eta(\Delta c_\epsilon) + \frac{C}{\epsilon^2} R^2 \|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)}^2 \right] \, ds. \end{aligned}$$

Choosing R proportional to $1/\epsilon$ to absorb the $\mathcal{E}_\eta$ -term, we get

$$\|\Delta c_\epsilon\|_{L^\infty(0,T;L^2(\mathbb{T}^n))} \leq \frac{C}{\epsilon^4}$$

for all $0 < \eta \leq c\epsilon$ and $\epsilon > 0$ small, where $c > 0$ is some fixed constant. This also controls the $L^2(\mathbb{T}^n)$ -norm of the second derivatives due to integration by parts.

3. H^3 -estimate. In the final step, we test (1.1) with $-\Delta^3 c_\epsilon$. Then we have for a.e. $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \|\nabla \Delta c_\epsilon(t)\|_{L^2(\mathbb{T}^n)}^2 - \frac{1}{2} \|\nabla \Delta c_{0,\epsilon}\|_{L^2(\mathbb{T}^n)}^2 \\ & + \int_0^t \mathcal{E}_\eta(\nabla \Delta c_\epsilon) \, ds + \frac{1}{\epsilon^2} \int_0^t \int_{\mathbb{T}^n} \nabla \Delta f'(c_\epsilon) \cdot \nabla \Delta c_\epsilon \, dx \, ds = 0. \end{aligned}$$

For simplicity, we often omit the time-dependence in the following. The estimates are uniform with respect to time. Using the chain rule as in (3.9) and that c_ϵ is uniformly bounded, we obtain

$$\left\| f^{(4)}(c_\epsilon) |\nabla c_\epsilon|^2 \nabla c_\epsilon \right\|_{L^2(\mathbb{T}^n)} \leq C \|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)}^2 \|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)} \leq C \|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)}^3.$$

In particular, this gives

$$\begin{aligned} \left| \frac{1}{\epsilon^2} \int_{\mathbb{T}^n} f^{(4)}(c_\epsilon) |\nabla c_\epsilon|^2 \nabla c_\epsilon \cdot \nabla \Delta c_\epsilon \, dx \right| & \leq \frac{C}{\epsilon^2} \|c_\epsilon\|_{H^2(\mathbb{T}^n)}^3 \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)} \\ & \leq \frac{C}{\epsilon^{22}} + \frac{C}{\epsilon^6} \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}^2. \end{aligned}$$

Here, we used the estimate derived in step 2 together with Young's inequality for the terms ϵ^{-11} and $\epsilon^{-3} \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}$. For the remaining terms, we have

$$\begin{aligned} \left\| f'''(c_\epsilon) D^2 c_\epsilon \nabla c_\epsilon \right\|_{L^2(\mathbb{T}^n)} & \leq C \|\Delta c_\epsilon\|_{L^2(\mathbb{T}^n)} \|\nabla c_\epsilon\|_{H^2(\mathbb{T}^n)} \\ & \leq \frac{C}{\epsilon^4} (\|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)} + \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}), \\ \left\| f'''(c_\epsilon) \Delta c_\epsilon \nabla c_\epsilon \right\|_{L^2(\mathbb{T}^n)} & \leq \frac{C}{\epsilon^4} (\|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)} + \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}) \end{aligned}$$

and therefore

$$\begin{aligned} & \left| \frac{1}{\epsilon^2} \int_{\mathbb{T}^n} (f'''(c_\epsilon) D^2 c_\epsilon \nabla c_\epsilon + f'''(c_\epsilon) \Delta c_\epsilon \nabla c_\epsilon) \cdot \nabla \Delta c_\epsilon \, dx \right| \\ & \leq \frac{C}{\epsilon^6} (\|\nabla c_\epsilon\|_{L^2(\mathbb{T}^n)} + \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}) \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)} \\ & \leq \frac{C}{\epsilon^6} \|\nabla \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)}^2 + \frac{C}{\epsilon^{10}}, \end{aligned}$$

where we applied Young's inequality in the last step. For the last term, (A.3) implies

$$\frac{1}{\varepsilon^2} \int_{\mathbb{T}^n} f''(c_\varepsilon) |\nabla \Delta c_\varepsilon|^2 dx \geq -\frac{\alpha}{\varepsilon^2} \|\nabla \Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2.$$

Combining these estimates with Theorem 2.5 (Nonlocal Ehrling Inequality), we obtain for all $R \geq 1$, $0 < \eta \leq \min\{c\varepsilon^2, 1/R\}$ and for a.e. $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} \|\nabla \Delta c_\varepsilon(t)\|_{L^2(\mathbb{T}^n)}^2 + \int_0^t \mathcal{E}_\eta(\nabla \Delta c_\varepsilon) ds \\ & \leq \frac{1}{2} \|\nabla \Delta c_{0,\varepsilon}\|_{L^2(\mathbb{T}^n)}^2 + \frac{C}{\varepsilon^{22}} + \frac{C}{\varepsilon^6} \int_0^t \|\nabla \Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 ds \\ & \leq \frac{1}{2} \|\nabla \Delta c_{0,\varepsilon}\|_{L^2(\mathbb{T}^n)}^2 + \frac{C}{\varepsilon^{22}} + \int_0^t \left[\frac{C}{\varepsilon^6 R^2} \mathcal{E}_\eta(\nabla \Delta c_\varepsilon) + \frac{C}{\varepsilon^6} R^2 \|\Delta c_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 \right] ds. \end{aligned}$$

Finally, we choose R proportional to $1/\varepsilon^3$ to absorb the $\mathcal{E}_\eta$ -term. Then we obtain for all $0 < \eta \leq c\varepsilon^3$ and $\varepsilon > 0$ small, where the constant c is possibly smaller than before, that

$$\|\nabla \Delta c_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{T}^n))} \leq \frac{C}{\varepsilon^{11}}.$$

The L^2 -norm of the third derivatives is then also controlled via integration by parts. □

4 Proof of Theorem 1.1 (Convergence)

We use the notation from Theorem 1.1. Moreover, let c_ε^A for $\varepsilon > 0$ small be the approximate solution from the local case, constructed for the evolving hypersurface $(\Gamma_t)_{t \in [0, T_0]}$ and the parameter $M \in \mathbb{N}$, cf. Theorem 2.6. Note that Γ has to evolve according to mean curvature flow in order to have the remainder estimates for c_ε^A in Theorem 2.6 for the local Allen–Cahn equation available. For the latter we set $r_\varepsilon^A := \partial_t c_\varepsilon^A - \Delta c_\varepsilon^A + 1/\varepsilon^2 f'(c_\varepsilon^A)$. Finally, let $\beta \geq 0$ be fixed (to be chosen later) and set $g_\beta(t) := e^{-\beta t}$ for all $t \geq 0$. We investigate the validity of the estimates

$$\begin{aligned} \sup_{t \in [0, T]} \|g_\beta \bar{c}_\varepsilon(t)\|_{L^2(\mathbb{T}^n)}^2 + \|g_\beta \nabla \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0, T) \setminus \Gamma(\delta))^n}^2 & \leq 2R^2 \varepsilon^{2M+1}, \\ \|g_\beta \nabla_\tau \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0, T) \cap \Gamma(\delta))^n}^2 + \varepsilon^2 \|g_\beta \partial_n \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0, T) \cap \Gamma(\delta))^n}^2 & \leq 2R^2 \varepsilon^{2M+1}, \end{aligned} \quad (4.1)$$

where $\varepsilon, R, M > 0$ and $T \in (0, T_0]$. Now let us define

$$T_{\varepsilon, \beta, R} := \sup\{T \in (0, T_0] : (4.1) \text{ holds for } \varepsilon, \beta, R\}.$$

For the different cases from Theorem 1.1 we have that $T_{\varepsilon, \beta, R}$ is well-defined and positive due to continuity. For the three cases we need to show the following:

1. Let $M \geq 3$. There are $\beta, \varepsilon_1, c > 0$ such that if $\eta = \eta(\varepsilon) \leq cR\varepsilon^{16+M+1/2}$, then it holds $T_{\varepsilon, \beta, R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$.
2. Let $M = 2$. The analogous statement as in the first case holds for $\beta = 0$ and small time T .

In the following we first carry out a general computation that works for every case. Taking the difference of the nonlocal Allen–Cahn equation (1.1) for c_ε and the local Allen–Cahn equation (1.6) for c_ε^A with remainder r_ε^A , we obtain

$$\partial_t \bar{c}_\varepsilon - \Delta \bar{c}_\varepsilon + f''(c_\varepsilon^A) \bar{c}_\varepsilon + \mathcal{L}_\eta c_\varepsilon + \Delta c_\varepsilon = -r_\varepsilon^A - r_\varepsilon(c_\varepsilon, c_\varepsilon^A), \quad (4.2)$$

where we have set $r_\epsilon(c_\epsilon, c_\epsilon^A) := (1/\epsilon^2)[f'(c_\epsilon) - f'(c_\epsilon^A) - f''(c_\epsilon^A)\bar{c}_\epsilon]$. Testing (4.2) with $g_\beta^2 \bar{c}_\epsilon$ and integrating over $\mathbb{T}^n \times (0, T)$ for $T \in (0, T_{\epsilon, \beta, R, M}]$ yields

$$\begin{aligned} & \frac{1}{2} g_\beta^2(T)^2 \|\bar{c}_\epsilon(T)\|_{L^2(\mathbb{T}^n)}^2 - \frac{1}{2} \|\bar{c}_\epsilon(0)\|_{L^2(\mathbb{T}^n)}^2 + \beta \int_0^T g_\beta^2 \|\bar{c}_\epsilon\|_{L^2(\mathbb{T}^n)}^2 dt \\ & + \int_0^T g_\beta^2 \int_{\mathbb{T}^n} |\nabla \bar{c}_\epsilon|^2 + \frac{1}{\epsilon^2} f''(c_\epsilon^A) |\bar{c}_\epsilon|^2 dx dt + \int_0^T g_\beta^2 \int_{\mathbb{T}^n} (\mathcal{L}_\eta c_\epsilon + \Delta c_\epsilon) \bar{c}_\epsilon dx dt \\ & = - \int_0^T g_\beta^2 \int_{\mathbb{T}^n} r_\epsilon^A \bar{c}_\epsilon + r_\epsilon(c_\epsilon, c_\epsilon^A) \bar{c}_\epsilon dx dt, \end{aligned}$$

where we used $(d/dt)g_\beta = -\beta g_\beta$. We have $1/2 \|\bar{c}_\epsilon(0)\|_{L^2(\mathbb{T}^n)}^2 \leq (1/2)R^2 \epsilon^{2M+1}$ due to the assumption in the theorem. Moreover, the spectral estimate for the local case from Theorem 2.8 yields

$$\begin{aligned} & \int_0^T g_\beta^2 \int_{\mathbb{T}^n} |\nabla \bar{c}_\epsilon|^2 + \frac{1}{\epsilon^2} f''(c_\epsilon^A) |\bar{c}_\epsilon|^2 dx dt \\ & \geq -\bar{C} \int_0^T g_\beta^2 \|\bar{c}_\epsilon\|_{L^2(\mathbb{T}^n)}^2 dt + \int_0^T g_\beta^2 \left(\|\nabla \bar{c}_\epsilon\|_{L^2(\mathbb{T}^n \setminus \Gamma(\delta))}^2 + \|\nabla_\tau \bar{c}_\epsilon\|_{L^2(\Gamma(\delta))}^2 \right) dt. \end{aligned}$$

Furthermore, we use the estimate $\|\mathcal{L}_\eta c_\epsilon + \Delta c_\epsilon\|_{L^2(\mathbb{T}^n)} \leq C\eta \|c_\epsilon\|_{H^3(\mathbb{T}^n)}$ from Theorem 2.4 and the uniform H^3 -estimate for c_ϵ from Theorem 3.3 for $0 < \eta \leq c\epsilon^4$. This yields

$$\begin{aligned} & \left| \int_0^T g_\beta^2 \int_{\mathbb{T}^n} (\mathcal{L}_\eta c_\epsilon + \Delta c_\epsilon) \bar{c}_\epsilon dx dt \right| \leq \int_0^T g_\beta^2 C\eta \epsilon^{-16} \|\bar{c}_\epsilon\|_{L^2(\mathbb{T}^n)} dt \\ & \leq \frac{\beta}{2} \int_0^T g_\beta^2 \|\bar{c}_\epsilon\|_{L^2(\mathbb{T}^n)}^2 dt + C \|g_\beta\|_{L^2(0,T)}^2 (\eta \epsilon^{-16})^2. \end{aligned}$$

Moreover, using the remainder estimate for r_ϵ^A from Theorem 2.6 and an integral transformation in tubular neighborhood coordinates yields

$$\left| \int_0^T g_\beta^2 \int_{\mathbb{T}^n} r_\epsilon^A \bar{c}_\epsilon dx dt \right| \leq C \int_0^T g_\beta^2 \epsilon^{M+\frac{1}{2}} \|\bar{c}_\epsilon\|_{L^2(\mathbb{T}^n)} \leq \bar{C}_1 \|g_\beta\|_{L^1(0,T)} R \epsilon^{2M+1},$$

where we used (4.1). Additionally, we use the uniform boundedness of c_ϵ and c_ϵ^A from Theorems 3.2 and 2.6 to estimate with Taylor

$$\left| \int_0^T g_\beta^2 r_\epsilon(c_\epsilon, c_\epsilon^A) \bar{c}_\epsilon dt \right| \leq \frac{C}{\epsilon^2} \int_0^T g_\beta^2 \|\bar{c}_\epsilon\|_{L^3(\mathbb{T}^n)}^3 dt.$$

The latter can be estimated by splitting $\mathbb{T}^n$ into $\mathbb{T}^n \setminus \Gamma(\delta)$ and $\Gamma(\delta)$, using tubular neighborhood coordinates for the latter set as well as Gagliardo–Nirenberg estimates, cf. Moser (2023), Lemma 6.6. This implies

$$\left| \int_0^T g_\beta^2 r_\epsilon(c_\epsilon, c_\epsilon^A) \bar{c}_\epsilon dt \right| \leq CR^3 \epsilon^{2M+1} \epsilon^{M-2} \|g_\beta^{-1}\|_{L^{\frac{4}{4-n}}(0,T)}.$$

Finally, we control the $\partial_n \bar{c}_\epsilon$ -term in (4.1) by the spectral term via

$$\begin{aligned} \epsilon^2 \|g_\beta \partial_n \bar{c}_\epsilon\|_{L^2(\mathbb{T}^n \times (0,T) \cap \Gamma(\delta))}^2 & \leq C \epsilon^2 \int_0^T g_\beta^2 \int_{\mathbb{T}^n} |\nabla \bar{c}_\epsilon|^2 + \frac{1}{\epsilon^2} f''(c_\epsilon^A) \bar{c}_\epsilon^2 dx dt \\ & + C \int_0^T g_\beta^2 \|\bar{c}_\epsilon\|_{L^2(\mathbb{T}^n)}^2 dt, \end{aligned}$$

where we used that c_ε^A is uniformly bounded with respect to small ε . The first term on the right-hand side is absorbed by half of the spectral term for ε small. Altogether this yields

$$\begin{aligned}
& \frac{1}{2} g_\beta(T)^2 \|\bar{c}_\varepsilon(T)\|_{L^2(\mathbb{T}^n)}^2 + \frac{1}{2} \|g_\beta \nabla \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0,T) \setminus \Gamma(\delta))^n}^2 \\
& + \frac{1}{2} \|g_\beta \nabla_\tau \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0,T) \cap \Gamma(\delta))^n}^2 + \frac{\varepsilon^2}{2} \|g_\beta \partial_n \bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n \times (0,T) \cap \Gamma(\delta))^n}^2 \\
& \leq \frac{R^2}{2} \varepsilon^{2M+1} + \int_0^T g_\beta^2 \left(-\frac{\beta}{2} + \bar{C}_0\right) \|\bar{c}_\varepsilon\|_{L^2(\mathbb{T}^n)}^2 dt + C \|g_\beta\|_{L^2(0,T)}^2 (\eta \varepsilon^{-16})^2 \\
& + \bar{C}_1 \|g_\beta\|_{L^1(0,T)} R \varepsilon^{2M+1} + C R^3 \varepsilon^{2M+1} \varepsilon^{M-2} \|g_\beta^{-1}\|_{L^{\frac{4}{4-n}}(0,T)}
\end{aligned} \tag{4.3}$$

for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \tilde{\varepsilon}_0]$, where $\tilde{\varepsilon}_0 > 0$ is small (independent of β, R, T).

Now we consider the distinct cases in the theorem.

Ad 1. Let $M \geq 3$. We choose $\beta \geq 2\bar{C}_0$ and large such that $\bar{C}_1/\beta \leq R/8$. We now roughly estimate $\|g_\beta\|_{L^2(0,T)}^2 \leq T_0^2$ for this case. Then if $\eta \leq cR\varepsilon^{16+M+1/2}$ for some $c > 0$ small, then the right-hand side in (4.3) is estimated by $(3/4)R^2\varepsilon^{2M+1}$ for all $T \in (0, T_{\varepsilon,\beta,R}]$ and $\varepsilon \in (0, \varepsilon_1]$, where $\varepsilon_1 > 0$ is small (depending on R, β). Finally, a contradiction and continuity argument shows $T_{\varepsilon,\beta,R} = T_0$ for all $\varepsilon \in (0, \varepsilon_1]$. More precisely, we conversely assume that $T_{\varepsilon,\beta,R} \neq T_0$. Then, by continuity of the left-hand side in (4.1), there exists some $\tilde{T} \in (T_{\varepsilon,\beta,R}, T_0)$ such that (4.1) still holds true for $\tilde{T}$. However, this contradicts the definition of $T_{\varepsilon,\beta,R}$.

Ad 2. Let $M = 2$. Then we set $\beta = 0$ and obtain for $\eta \leq cR\varepsilon^{16+M+1/2}$ with (4.1) that the right-hand side in (4.3) is estimated by

$$\left[\frac{R^2}{2} + \bar{C}_0 T + C R^2 T^2 + \bar{C}_1 T R + C R^3 T^{\frac{4-n}{4}} \right] \varepsilon^{2M+1}.$$

We choose $R > 0$ so small such that the right-hand side in (4.3) is estimated by $(3/4)R^2\varepsilon^{2M+1}$. Then, by continuity of the left-hand side in (4.1), there exist $\varepsilon_1 > 0$ and $T_1 > 0$ such that estimate (4.1) holds true for all $T \in (0, \min\{T_{\varepsilon,0,R}, T_1\})$ and $\varepsilon \in (0, \varepsilon_1]$. This implies $T_{\varepsilon,0,R} \geq T_1$ by a similar contradiction argument as before. Namely, if $T_{\varepsilon,0,R} < T_1$ for all $\varepsilon \in (0, \varepsilon_1]$, there exists some $\tilde{T} \in (T_{\varepsilon,0,R}, T_1)$ such that (4.1) still holds true for T_1 and all $\varepsilon \in (0, \varepsilon_1]$, which follows by continuity of the left-hand side in (4.1). However, this contradicts the definition of $T_{\varepsilon,0,R}$.

This shows Theorem 1.1. $\square$

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