Nonlocal-to-Local Limits for Phase-Field Models

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Abstract

In this thesis, we study systems of nonlinear partial differential equations that can be used to describe phase separation. This process can be modeled from both a macroscopic and microscopic point of view. Typically, the macroscopic model contains a local differential operator, which accounts for short-range interactions. In the microscopic model, however, this operator is then replaced by a nonlocal operator, which describes long-range interactions between the particles. This is done by convolution integrals weighted by suitable interaction kernels.

In the first part, we analyze the asymptotic behavior of the nonlocal operator in the case, when the corresponding interaction kernel concentrates around the origin suitably. We prove that the nonlocal operator converges to a local differential operator as the parameter in the interaction kernel is sent to zero. More precisely, we even provide concrete rates of convergence depending on this parameter. The analysis is done in the case of sufficiently smooth bounded domains and a large class of interaction kernels. Indeed, the kernel can be either anisotropic or isotropic as well as of $W^{1,1}$-regularity or even more singular. The convergence is shown with respect to the L^p-norm, where $p \in [1, \infty)$. The proof is based on localization and perturbation arguments. In the case of regular kernels, we also prove convergence on the torus with respect to the L^2-norm. This proof is based on methods from Fourier analysis.

In the second part, we focus on models, where two different phases are separated by a diffuse interface. We intend to apply the results from the first part to show that solutions to the nonlocal model converge to a solution to the local model as the parameter in the nonlocal operator is sent to zero. More precisely, we prove strong convergence for the sequence of solutions along with certain rates of convergence. This is done using an energy method. We consider the following models:

- Cahn–Hilliard equation
- Allen–Cahn equation
- Cahn–Hilliard model for tumor growth
- Cahn–Hilliard/Navier–Stokes system

Finally, we also investigate the case, where both the parameter in the nonlocal operator and the thickness of the diffuse interface are sent to zero suitably. The analysis is done for the nonlocal Allen–Cahn equation on the torus. We first prove a well-posedness result and suitable error estimates for the solutions. Then, we combine the convergence result from the first part together with existing results for the sharp interface limit of the local Allen–Cahn equation.

Zusammenfassung

In dieser Arbeit untersuchen wir Systeme nichtlinearer partieller Differentialgleichungen, die zur Beschreibung der Phasentrennung verwendet werden können. Dieser Prozess kann sowohl vom makroskopischen als auch vom mikroskopischen Standpunkt aus modelliert werden. Typischerweise enthält das makroskopische Modell einen lokalen Differentialoperator, der die Wechselwirkungen über kurze Distanzen berücksichtigt. Im mikroskopischen Modell wird dieser Operator dann jedoch durch einen nichtlokalen Operator ersetzt, der die Wechselwirkungen zwischen den Teilchen über große Entfernungen beschreibt. Dies geschieht durch Faltungsintegrale, die durch geeignete Wechselwirkungskerne gewichtet werden.

Im ersten Teil analysieren wir das asymptotische Verhalten des nichtlokalen Operators für den Fall, dass der entsprechende Wechselwirkungskern sich in geeigneter Weise um den Ursprung konzentriert. Wir beweisen, dass der nichtlokale Operator gegen einen lokalen Differentialoperator konvergiert, wenn sich der Parameter im Wechselwirkungskern der Null nähert. Insbesondere zeigen wir sogar konkrete Konvergenzraten in Abhängigkeit von diesem Parameter. Die Analyse wird für hinreichend glatte beschränkte Gebiete und einer großen Klasse von Wechselwirkungskernen durchgeführt. Der Kern kann entweder anisotrop oder isotrop sowie von $W^{1,1}$-Regularität oder noch singulärer sein. Die Konvergenz wird bezüglich der L^p-Norm gezeigt, wobei $p \in [1, \infty)$. Der Beweis basiert auf Lokalisierungs- und Störungsargumenten. Im Falle von regulären Kernen wird auch die Konvergenz auf dem Torus bezüglich der L^2-Norm bewiesen. Dieser Beweis basiert auf Methoden der Fourier-Analyse.

Im zweiten Teil konzentrieren wir uns auf Modelle, bei denen zwei verschiedene Phasen durch eine diffuse Grenzschicht getrennt sind. Wir wollen die Ergebnisse aus dem ersten Teil verwenden, um zu zeigen, dass die Lösungen des nichtlokalen Modells zu einer Lösung des lokalen Modells konvergieren, wenn sich der Parameter des nichtlokalen Operators der Null nähert. Genauer gesagt beweisen wir die starke Konvergenz für die Folge von Lösungen zusammen mit bestimmten Konvergenzraten. Dies geschieht mit Hilfe einer Energiemethode. Wir betrachten die folgenden Modelle:

- Cahn–Hilliard Gleichung
- Allen–Cahn Gleichung
- Cahn–Hilliard Modell für Tumorwachstum
- Cahn–Hilliard/Navier–Stokes System

Schließlich untersuchen wir auch den Fall, in dem sich sowohl der Parameter im nichtlokalen Operator als auch die Dicke der diffusen Grenzschicht in geeigneter Weise der Null nähern. Die Analyse wird für die nichtlokale Allen–Cahn Gleichung auf dem Torus durchgeführt. Wir beweisen zunächst ein Wohlgestelltheitsergebnis und geeignete Fehlerschätzungen für die Lösungen. Dann kombinieren wir das Konvergenzergebnis aus dem ersten Teil mit bestehenden Ergebnissen für den scharfen Grenzschichtlimes der lokalen Allen–Cahn Gleichung.

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Omnium enim rerum principia parva sunt
Cicero

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Chapter 1

Introduction

1.1. Phase-field models

In fluid dynamics, it is a challenging topic to investigate the mixture of two immiscible fluids. Some prominent examples are thermocapillary flows, spinodal decomposition, mixing and interfacial stretching, droplet breakup or moving contact lines, cf. [22] and the references therein. In order to describe the motion of such mixtures, it is crucial to understand the nature of the moving interface separating the different components. To this end, two important mathematical approaches have been developed: *sharp interface models* and *diffuse interface models*.

In the following, we briefly describe and compare the aforementioned approaches. However, we first introduce the general setting:
Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \in \{2, 3\}$, filled with a binary fluid consisting of its components A and B. We now denote the concentrations of the components, expressed as mass fractions, by $c_k : \Omega \to \mathbb{R}$ for $k \in \{A, B\}$. More precisely, the concentration is given by

$$c_k(x) := \frac{m_k(B_r(x))}{m(B_r(x))}, \quad k \in \{A, B\},$$

where m_k denotes the mass of the k-th component and m denotes the combined mass of the two components. Moreover, $B_r(x)$ denotes the ball with radius r around x. From this it follows that

$$c_A(x) + c_B(x) \equiv 1.$$

In order to investigate the motion of the mixture, one does not consider the concentrations c_A and c_B separately. Instead, one introduces a so-called *order parameter*

$$c : \Omega \to \mathbb{R}, \quad c(x) := c_B(x) - c_A(x),$$

which then determines the different components of the fluid.

Note that this approach can also be used in other disciplines like, e.g., materials science and its applications to biology or engineering. Depending on the application, one uses volume or mole fractions instead of mass fractions in the derivation. Typically, the order parameter is given by the density, the composition of two materials or some artificial variable, cf. [22, 102].

The classical approach in fluid dynamics are the so-called *sharp interface models*. There, we assume that the interface between the fluids is represented by a surface of thickness zero, i.e., a hypersurface or even more complicated objects (cf. [102]). See Figure 1.1 for a typical situation. In this case, the concentrations are given by characteristic functions in the respective components. As a consequence, the order parameter exhibits a jump from the value -1 to 1 at the interface. As we intend to capture the motion of the mixture, the interface is usually described by an evolving hypersurface. Thus, these models typically consist of an evolution law for the hypersurface, which is often coupled to equations in the bulk domains and relations on the interface. A famous example for a sharp interface model is the so-called *Stefan problem*, which describes, e.g., the melting of ice. For further examples, we refer to [111]. Typically, the interface between the fluids is unknown and also part of the problem. Such models are called *free boundary problems* and are very challenging from a mathematical point of view.

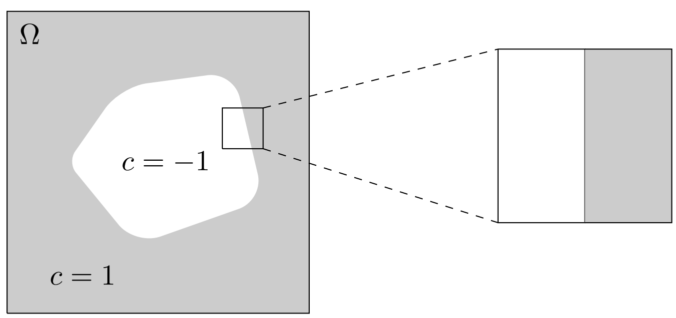

Figure 1.1: Sharp interface model.

The free boundary description turns out to be a successful model in many situations. However, these models break down, when the described phenomena act on length scales comparable to the interfacial thickness, e.g., when the topology changes or when the interfacial thickness diverges at a critical point, cf. [22].

Another approach are so-called *diffuse-interface models*, where we assume that the interface between the components is approximated by a thin tubular neighbourhood as shown in Figure 1.2. This interfacial region is then called the *diffuse interface*. It is fundamental in these models that the width of this region is proportional to a small parameter $\eta > 0$. Here, one typically introduces a *phase-field* c as order parameter to distinguish the different phases. It is assumed that the phase-field attains constant values close to -1 and 1 in the pure respective phase and experiences a rapid but smooth transition in the interfacial region.

The phase-field can be used to reformulate the free boundary problem. A major advantage of these models is the fact that quantities, which in the sharp interface model are localized to the interfacial surface, are usually distributed throughout the diffuse interface. Moreover, these models can be used to describe phenomena, which act on length scales comparable to the interfacial thickness. In this regard, they provide an appropriate alternative to sharp interface models.

Eminent examples for diffuse interface models are the Cahn–Hilliard equation and the Allen–Cahn equation, which model, e.g., spinodal decomposition or the evolution of antiphase boundaries in iron alloys, respectively, cf. [20, 30].

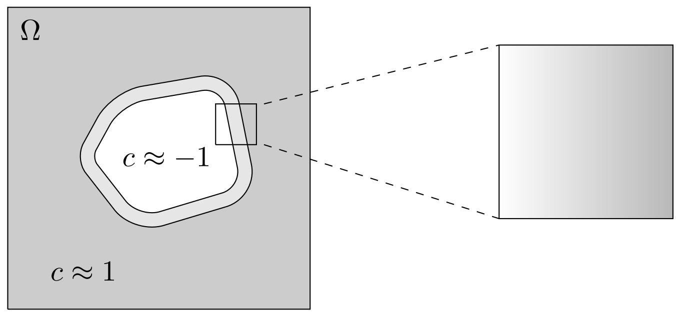

Figure 1.2: Diffuse interface model.

1.2. The Cahn–Hilliard and Allen–Cahn equations

Originally, the Cahn–Hilliard equation was introduced in [30] to model the phenomena of spinodal decomposition in binary alloys. Spinodal decomposition is a process of demixing, i.e., phase separation, that typically "*occurs when a uniform mixture of the alloy is quenched below a certain critical temperature underneath which the uniform mixture becomes unstable. As a result, a very fine microstructure of two spatially phases with different concentrations develops. In later stages of the evolution on a much larger time scale than the initial phase separation the structures become coarser: either by merging of particles or by the growth of bigger particles at cost of smaller ones*" (cf. [74]), as shown in Figure 1.3. For details, we refer to [74, 92, 106].

Meanwhile, the Cahn–Hilliard equation has been frequently used in a variety of different mathematical models describing phenomena such as population dynamics, image processing, two-phase flows and tumor growth, cf. [41, 43, 48, 90, 98] and the references therein. The system of equations reads as:

$$\partial_t c = \Delta\mu \qquad \text{in } \Omega_T, \tag{1.2.1a}$$
$$\mu = -\eta\Delta c + \tfrac{1}{\eta} f'(c) \qquad \text{in } \Omega_T, \tag{1.2.1b}$$
$$\partial_{\mathbf{n}} c = \partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \partial\Omega_T, \tag{1.2.1c}$$
$$c(0) = c_0 \qquad \text{in } \Omega. \tag{1.2.1d}$$

Here, $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$, whose outer unit normal vector field is denoted by $\mathbf{n}$. Furthermore, $T > 0$ is a prescribed final time and we set $\Omega_T := \Omega \times (0, T)$ as well as $\partial\Omega_T := \partial\Omega \times (0, T)$. The function $c : \Omega_T \to \mathbb{R}$ is the phase-field, $c_0 : \Omega \to \mathbb{R}$ a prescribed initial datum, and the function $\mu : \Omega_T \to \mathbb{R}$ denotes the chemical potential associated to c. The constant $\eta > 0$ is related to the thickness of the diffuse interface. [1]

[1] If η is not mentioned explicitly, we use the convention $\eta = 1$ throughout this thesis

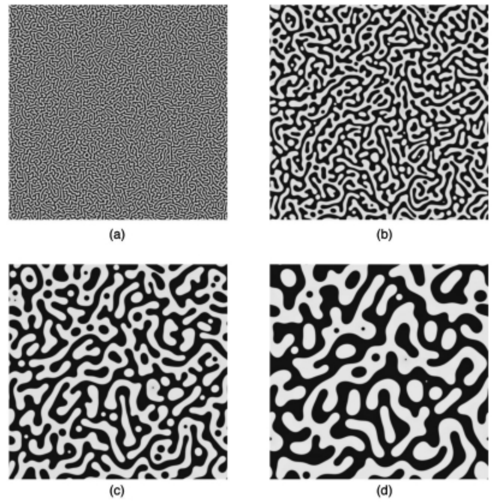

Figure 1.3: Patterns during spinodal decomposition and subsequent coarsening.
(Image taken from [31]).

The model (1.2.1) can be interpreted as a gradient flow (in the H^{-1}-metric) of the free energy functional

$$\mathcal{E}(c) := \int_\Omega \frac{\eta}{2}|\nabla c(x)|^2 + \frac{1}{\eta} f(c(x))\,\mathrm{d}x, \tag{1.2.2}$$

which is also known as the *Cahn–Hilliard energy* or also *Modica–Mortola energy*. The term $\frac{\eta}{2}|\nabla c|^2$ is often referred to as *Dirichlet energy density* and the function f is typically chosen as a double-well shaped function with wells of equal depth and minima at ± 1, see Figure 1.4. As already pointed out in [96], the concentration gradients were added to the free energy density, in order to regularize the problem. Otherwise, there is a spinodal region where the initial value problem is ill-posed due to the non-convex potential $f(c)$. Moreover, the gradient term also exhibits a physical nature. Jumps from -1 to 1 usually cost interfacial energy and are thus not really physical. Hence, these transitions need to be penalized, which happens on account of the gradient term.

A choice for a double-well potential is the so-called *Flory–Huggins potential* or *logarithmic potential* given by

$$f_{\log}(s) := \frac{\theta}{2}\left[(1+s)\ln(1+s) + (1-s)\ln(1-s)\right] - \frac{\theta_0}{2}s^2, \quad s \in [-1,1]. \tag{1.2.3}$$

Here, $\theta > 0$ denotes the temperature of the system and $\theta_0 > 0$ denotes the critical temperature, respectively. The temperature θ_0 is critical in the sense that phase separation takes place for temperatures $\theta < \theta_0$. We point out that this potential ensures the existence of *physical solutions*, i.e., solutions c lying in the physically consistent interval $[-1,1]$. The potential $f_{\log}$ is classified as a singular potential as its derivative tends to $\pm\infty$ as its argument approaches ± 1, i.e.,

$$f'_{\log}(s) = \frac{\theta}{2}[\ln(1+s) - \ln(1-s)] - \theta_c s \to \pm\infty \quad \text{as } s \to \pm 1.$$

Thus, the mathematical analysis turns out to be very challenging.

In the literature, one typically replaces the singular potential by suitable approximations by double-well shaped functions without any singularity at ± 1. A very common choice is the polynomial potential

$$f_{\text{reg}}(s) := \frac{1}{4}(1-s^2)^2, \quad s \in \mathbb{R}. \tag{1.2.4}$$

The main drawback of regular potentials, however, is that we can not ensure that the solution stays in the interval $[-1, 1]$.

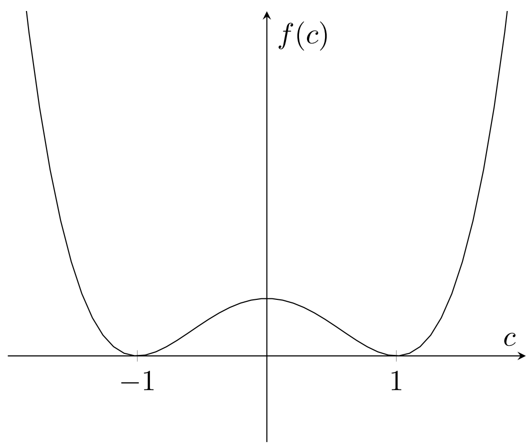

Figure 1.4: Typical graph of a double-well potential, $f_{\text{reg}}(c) = \frac{1}{4}(1-c^2)^2$.

The boundary condition $\partial_{\mathbf{n}}\mu = 0$, cf. (1.2.1c), implies that there is no mass flux across the boundary. It is also known as *no-flux boundary condition*. Indeed, sufficiently smooth solutions to the Cahn–Hilliard equation satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega c \,\mathrm{d}x = \int_\Omega \partial_t c \,\mathrm{d}x = 0,$$

which implies

$$\int_\Omega c(t) \,\mathrm{d}x = \int_\Omega c_0 \,\mathrm{d}x$$

for all $t \in [0, T]$. This means that the total mass of the mixture is conserved. On the other hand, the boundary conditions (1.2.1c) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(c) = -\int_\Omega |\nabla\mu|^2 \,\mathrm{d}x \leq 0$$

for all $t \in [0, T]$ entailing the *dissipation* of free energy, i.e., the energy $\mathcal{E}(c)$ is decreasing in time. For analytic results of the Cahn–Hilliard equation, we refer to [17, 19, 49, 50, 99, 100, 104, 125].

In the Cahn–Hilliard equation as introduced in (1.2.1), we assumed that the mobility of the mixture is constant. However, one can also study more general cases where the mobility is non-constant. In this context, equation (1.2.1a) reads as

$$\partial_t c = -\operatorname{div}(m(c)\nabla\mu) \quad \text{in } \Omega \times (0, T),$$

where $m : \mathbb{R} \to [0, \infty)$ denotes the mobility function of the mixture. In the literature, one usually assumes that the mobility is uniformly positive (or constant), since it simplifies the mathematical analysis. Instead, one can also assume that the mobility is degenerate at the pure phases, i.e., $m(s) = 0$ if $|s| = 1$. However, this complicates the mathematical analysis a lot.

The Ginzburg–Landau free energy as defined in (1.2.2) is *isotropic*, i.e., it is independent of the local orientation of the interface. In many applications, however, e.g., when crystalline materials are described, it is more realistic to consider *anisotropic* energies. In this case, the energy density depends on the local orientation of the interface. In this context, the energy is given by

$$\mathcal{E}(c) := \int_\Omega \eta\, \Phi(\nabla c(x)) + \frac{1}{\eta} f(c(x))\, \mathrm{d}x,$$

where Φ is given by

$$\Phi(p) := \tfrac{1}{2}\gamma(p)^2 \quad \text{for all } p \in \mathbb{R}^n.$$

Here, $\gamma : \mathbb{R}^n \to (0, \infty)$ is a convex and one-homogeneous function, cf. [69] for details. Then corresponding *anisotropic Cahn–Hilliard equation* then reads as

$$\partial_t c = \Delta\mu \qquad \text{in } \Omega_T, \tag{1.2.5a}$$

$$\mu = -\eta\, \mathrm{div}(\Phi'(\nabla c)) + \tfrac{1}{\eta} f'(c) \qquad \text{in } \Omega_T, \tag{1.2.5b}$$

$$\mathbf{n} \cdot \Phi'(\nabla c) = \partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \partial\Omega_T, \tag{1.2.5c}$$

$$c(0) = c_0 \qquad \text{in } \Omega. \tag{1.2.5d}$$

For analytic results of the anisotropic Cahn–Hilliard equation, we refer to [69] and the references therein.

As we pointed out before, the Cahn–Hilliard equation can be seen as a gradient flow of the Ginzburg–Landau free energy with respect to the H^{-1}-inner product. Considering the L^2-inner product instead, we arrive at

$$\partial_t c = -\mu \qquad \text{in } \Omega \times (0, T), \tag{1.2.6a}$$

$$\mu = -\Delta c + \frac{1}{\eta^2} f'(c) \qquad \text{in } \Omega \times (0, T), \tag{1.2.6b}$$

$$\partial_{\mathbf{n}} c = 0 \qquad \text{on } \partial\Omega \times (0, T) \tag{1.2.6c}$$

$$c(0) = c_0 \qquad \text{in } \Omega. \tag{1.2.6d}$$

This system is known as the *Allen–Cahn equation*. It was originally introduced in [20] to describe the evolution of antiphase boundaries in iron alloys. The notation is the same as before. In contrast to the Cahn–Hilliard equation, the total mass of the mixture is not conserved. This can be seen immediately by integrating (1.2.6a) over Ω. To overcome this problem, one studies the conserved Allen–Cahn equation, where (1.2.6a) is replaced by

$$\partial_t c = -(\mu - \overline{\mu}) \quad \text{in } \Omega \times (0, T).$$

Indeed, this guarantees that the total mass of the corresponding system is preserved. The energy identity in case of the Allen–Cahn equation reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(c) = -\int_\Omega |\mu|^2\, \mathrm{d}x \leq 0,$$

entailing the dissipation of the free energy. For analytic results of the Allen–Cahn equation, we refer to [102] and the references therein.

1.3. The nonlocal Cahn–Hilliard and Allen–Cahn equations

Phase separation is a process with many practical and theoretical applications. It can be observed in fields such as quantum physics, biological science and also cosmology, covering all relevant length and interaction scales, cf. [103]. As pointed out in [103], many of these systems are governed by long-range interactions between the particles. However, the Dirichlet energy density $\frac{\eta}{2}|\nabla c|^2$ in the Ginzburg–Landau energy only focuses on short-range interactions. For this reason, (1.2.1) is also called the *local* Cahn–Hilliard equation.

As observed by Giacomin and Lebowitz in [75, 76], there is no microscopic derivation of the (local) Cahn–Hilliard equation (1.2.1). Based on their observation, they rigorously derived a macroscopic equation, which describes phase separation in systems with long-range interaction. More precisely, they considered the hydrodynamic limit of a microscopic model describing an n-dimensional lattice gas evolving via the (Poisson) nearest neighbor exchange process. In their contribution, they derived a nonlocal free energy of the form

$$\mathcal{E}_{NL}(c) := \frac{\eta}{4}\int_\Omega\int_\Omega J(x-y)\big|c(x)-c(y)\big|^2 \,\mathrm{d}y\,\mathrm{d}x + \frac{1}{\eta}\int_\Omega f(c(x))\,\mathrm{d}x, \tag{1.3.1}$$

where $J:\mathbb{R}^n\to[0,\infty)$ is a sufficiently smooth and even interaction kernel and f a double-well shaped potential as in (1.2.2). In this case, the gradient term in (1.2.2) is replaced by a nonlocal operator which captures the long-range interactions.

The resulting *nonlocal* variant of the Cahn–Hilliard equation then reads as:

$$\begin{aligned}
\partial_t c &= \Delta\mu && \text{in } \Omega_T, && \text{(1.3.2a)}\\
\mu &= -\eta J * c_\Omega + \eta(J*\chi_\Omega)c + \tfrac{1}{\eta}f'(c) && \text{in } \Omega_T, && \text{(1.3.2b)}\\
\partial_{\mathbf{n}}\mu &= 0 && \text{on } \partial\Omega_T, && \text{(1.3.2c)}\\
c(0) &= c_0 && \text{in } \Omega. && \text{(1.3.2d)}
\end{aligned}$$

This system (1.3.2) can be seen as a gradient flow of the nonlocal free energy (1.3.1) with respect to the H^{-1}-inner product. In the following, we introduce the nonlocal operator

$$\begin{aligned}
\mathcal{L}c(x) := -(J*c_\Omega)(x) + (J*\chi_\Omega)c(x) &= \text{P.V.}\int_\Omega J(x-y)(c(x)-c(y))\,\mathrm{d}y\\
&:= \lim_{r\to 0}\int_{\Omega\cap B_r(x)} J(x-y)(c(x)-c(y))\,\mathrm{d}y,
\end{aligned} \tag{1.3.3}$$

for all $x\in\Omega$, where

$$c_\Omega(x) := \begin{cases} c(x) & \text{if } x\in\Omega,\\ 0 & \text{if } x\notin\Omega.\end{cases}$$

We observe that system (1.3.2) can be written as a second-order integrodifferential equation for the order parameter c. Formally, the definition of μ in (1.3.2a) yields

$$\partial_t c - \mathrm{div}\Big(-\eta\nabla J * c + \eta(\nabla J*\chi_\Omega)c + \eta(J*\chi_\Omega)\nabla c + \tfrac{1}{\eta}f''(c)\nabla c\Big) = 0.$$

Hence, we do not need to impose any boundary condition for the phase-field c. On the other hand, using the substitution

$$\tilde{f}(x, c(x)) := f(c(x)) - \tfrac{1}{2}a(x)c(x)^2, \tag{1.3.4}$$

where

$$a(x) := (J * \chi_\Omega)(x) \quad \text{for all } \ x \in \Omega,$$

we can rewrite the nonlocal free energy (1.3.1) as

$$\mathcal{E}_{NL}(c) := -\frac{\eta}{2}\int_\Omega\int_\Omega J(x-y)c(x)c(y)\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{\eta}\int_\Omega \tilde{f}(x, c(x))\,\mathrm{d}x. \tag{1.3.5}$$

This leads to the following variant of the nonlocal Cahn–Hilliard equation:

$$\begin{aligned}
\partial_t c &= \Delta\mu && \text{in } \Omega_T, && (1.3.6a)\\
\mu &= -\eta(J * c) + \tfrac{1}{\eta}\tilde{f}'(x, c) && \text{in } \Omega_T, && (1.3.6b)\\
\partial_{\mathbf{n}}\mu &= 0 && \text{on } \partial\Omega_T, && (1.3.6c)\\
c(0) &= c_0 && \text{in } \Omega. && (1.3.6d)
\end{aligned}$$

Notice that on account of the substitutions (1.3.4) and (1.3.5) as well as after some corrections, the systems (1.3.2) and (1.3.6) are equivalent.

As pointed out in [76], the first term in the nonlocal free energy (1.3.1) can be seen as an approximation of $\frac{\kappa}{2}|\nabla c|^2$ for some suitable $\kappa > 0$. Thus, the nonlocal Cahn–Hilliard equation (1.3.2) can be interpreted as an approximation of its local counterpart (1.2.1). If Ω is the tours $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, then the term $a(x) = (J * \chi_\Omega)(x)$ is constant and therefore, the form of $\mathcal{E}_{NL}$ is very close to the one of $\mathcal{E}$. Furthermore, the systems (1.2.1) and (1.3.6) can be related via their nonlocal-to-local limit (see Section 1.9).

Thanks to the no-flux boundary condition for μ, again the total mass of the mixture is conserved, i.e.,

$$\int_\Omega c(t)\,\mathrm{d}x = \int_\Omega c_0\,\mathrm{d}x,$$

for all $t \in [0, T]$. Furthermore, the boundary conditions (1.3.6c) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{NL}(c) = -\int_\Omega |\nabla\mu|^2\,\mathrm{d}x \leq 0,$$

entailing the dissipation of the nonlocal free energy. Let us also mention that if we assume f to be singular at the pure phases, we can deduce $c \in [-1, 1]$. Similar to the local system, we can also investigate more general situations where the mobility is either uniformly positive or degenerate at the pure phases. We call the nonlocal Cahn–Hilliard equation *isotropic* if the interaction kernel J is radially symmetric and even, and *anisotropic* if J is merely even.

In case of the Allen–Cahn equation, replacing the negative Laplacian $-\Delta$ by the nonlocal operator (1.3.3), the nonlocal Allen–Cahn equation reads as

$$\begin{aligned}
\partial_t c + \mathcal{L}c + \tfrac{1}{\eta^2}f'(c) &= 0 && \text{in } \Omega \times (0, T), && (1.3.7a)\\
c(0) &= c_0 && \text{in } \Omega. && (1.3.7b)
\end{aligned}$$

The notation is the same as before, i.e., c denotes the phase-field, c_0 a prescribed initial datum and f is a double-well shaped potential as in (1.2.2). We observe that the nonlocal Allen–Cahn equation behaves as a nonlocal ordinary differential equation for the order parameter c. Thus, we do not need to impose any boundary condition for c. Integrating (1.3.7a) over Ω, we deduce that the total mass of the mixture is not conserved. The formal energy identity of this system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{NL}(c) = -\int_\Omega |\mu|^2 \,\mathrm{d}x \leq 0.$$

For analytic results about the nonlocal Allen–Cahn equation, we refer to [12, 83] and the references therein.

1.4. A Cahn–Hilliard/Navier–Stokes model for two phase flow

In this subsection, we present one of the fundamental models to describe the motion of two incompressible, viscous, Newtonian fluids with *matched* densities, which is also called "*Model H*". It was first developed in [86], and later rigorously derived in [84]. One of the underlying assumptions of Model H is that the densities of both fluids are equal and hence the density of the mixture is constant. The system is given by an incompressible Navier–Stokes equation coupled to a convective Cahn–Hilliard system and reads as follows:

$$\rho(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \nu\Delta\mathbf{v} + \nabla p = \mu\nabla c \qquad \text{in } \Omega_T, \tag{1.4.1a}$$
$$\mathrm{div}(\mathbf{v}) = 0 \qquad \text{in } \Omega_T, \tag{1.4.1b}$$
$$\partial_t c + \mathbf{v}\cdot\nabla c = m\Delta\mu \qquad \text{in } \Omega_T, \tag{1.4.1c}$$
$$\mu = -\Delta c + f'(c) \qquad \text{in } \Omega_T, \tag{1.4.1d}$$
$$\mathbf{v}(0) = \mathbf{v}_0, \quad c(0) = c_0 \qquad \text{in } \Omega. \tag{1.4.1e}$$

In the case when Ω is a bounded domain, we further impose the standard boundary conditions

$$\mathbf{v} = \mathbf{0}, \quad \partial_\mathbf{n}\mu = 0, \quad \partial_\mathbf{n} c = 0 \quad \text{on } \partial\Omega_T, \tag{1.4.1f}$$

and if $\Omega = \mathbb{T}^n$, we assume periodic boundary conditions. Here, $\mathbf{v} : \Omega_T \to \mathbb{R}^n$ denotes the *velocity field* associated with the mixture of two fluids, $p : \Omega_T \to \mathbb{R}$ represents the corresponding *pressure*, $c : \Omega_T \to \mathbb{R}$ is the *phase-field* and $\mu : \Omega_T \to \mathbb{R}$ denotes the *chemical potential*. The quantities ρ, ν and m represent the *mass density* of the mixture, the *kinematic viscosity*, and the *mobility*, respectively, which are all assumed to be positive constants. The function f' is the derivative of a potential f, which is usually double-well shaped. A physically relevant choice is the logarithmic potential as in (1.2.3). As previously noted, this potential ensures the existence of physical solutions $c \in [-1, 1]$. In our mathematical analysis, we will even be able to handle a more general class of singular potentials that will be specified by the assumptions (S1)–(S3) in Section 2.5. For analytic results about Model H, we refer to [2, 26, 66, 80].

Observe that the total energy of the system (1.4.1a)–(1.4.1f) is the following:

$$E(c, \mathbf{v}) := \frac{\rho}{2}\int_\Omega |\mathbf{v}(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla c(x)|^2 \,\mathrm{d}x + \int_\Omega f(c(x)) \,\mathrm{d}x. \tag{1.4.2}$$

Here, the first summand in (1.4.2) denotes the kinetic energy, whereas the last two summands represent the free energy of the mixture, which is of Ginzburg–Landau type, cf. Section 1.2.

Note that the boundary conditions (1.4.1f) ensure that the total mass of the concentration c is conserved, i.e.,

$$\int_\Omega c(t)\,\mathrm{d}x = \int_\Omega c_0\,\mathrm{d}x \quad \text{for all } t \in [0,T]. \tag{1.4.3}$$

In the case $\Omega = \mathbb{T}^n$, we further have

$$\int_\Omega \mathbf{v}(t)\,\mathrm{d}x = \int_\Omega \mathbf{v}_0\,\mathrm{d}x \quad \text{for all } t \in [0,T]. \tag{1.4.4}$$

Sufficiently regular solutions to the Model H satisfy the energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}E\big(\mathbf{v}(t),c(t)\big) = -\nu \int_\Omega |\nabla \mathbf{v}(t)|^2\,\mathrm{d}x - m\int_\Omega |\nabla\mu|^2\,\mathrm{d}x \quad \text{for all } t \in [0,T]. \tag{1.4.5}$$

In particular, this means that the system (1.4.1) has a dissipative nature.

1.5. A nonlocal Cahn–Hilliard/Navier–Stokes model for two phase flow

In the local Model H as introduced in Section 1.4 only short-range interactions between the particles are taken into account. As already pointed out in Section 1.3, both the short and long-range interactions cannot be neglected in many applications. Thus, in order to also allow for long-range interactions, one replaces the gradient term in (1.4.2) by the nonlocal spatial interaction integral $\mathcal{E}_{NL}$ as in (1.3.1), namely

$$\begin{aligned} E(c,\mathbf{v}) := \frac{\rho}{2}\int_\Omega |\mathbf{v}(x)|^2\,\mathrm{d}x + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)|c(x)-c(y)|^2\,\mathrm{d}y\mathrm{d}x \\ + \int_\Omega f\big(c(x)\big)\,\mathrm{d}x. \end{aligned} \tag{1.5.1}$$

The first term in (1.5.1) represents the kinetic energy, whereas the last two terms represent the free energy of the mixture, which is now of Helmholtz type, cf. Section 1.3. In this context, the nonlocal variant of the Model H (1.4.1) then reads as follows:

$$\rho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \nu\Delta\mathbf{v} + \nabla p = \mu\nabla c \qquad \text{in } \Omega_T, \tag{1.5.2a}$$

$$\operatorname{div}(\mathbf{v}) = 0 \qquad \text{in } \Omega_T, \tag{1.5.2b}$$

$$\partial_t c + \mathbf{v}\cdot\nabla c = m\Delta\mu \qquad \text{in } \Omega_T, \tag{1.5.2c}$$

$$\mu = \mathcal{L}c + f'(c) \qquad \text{in } \Omega_T, \tag{1.5.2d}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad c|_{t=0} = c_0 \qquad \text{in } \Omega. \tag{1.5.2e}$$

In case Ω is a bounded domain, we further impose the standard boundary conditions

$$\mathbf{v} = \mathbf{0}, \quad \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \partial\Omega_T, \tag{1.5.2f}$$

and if $\Omega = \mathbb{T}^n$, we assume periodic boundary conditions. Observe that the nonlocal Cahn–Hilliard equation is a second-order differential equation in space. Thus, we do not need to impose any boundary condition for the phase-field variable c. Here, the nonlocal operator is given as in (1.3.3), i.e.,

$$\mathcal{L}c(x) := \text{P.V.} \int_\Omega J(x-y)\big(c(x) - c(y)\big)\,\mathrm{d}y \quad \text{for all } x \in \Omega.$$

Further, we use the same notation as for the local Model H, cf. Section 1.4.

From the mathematical point of view, the nonlocal Model H is very challenging, since the regularity of the phase-field variable c is lower than in the local Model H. As a consequence, the Korteweg force $\mu \nabla c$ acting on the fluid can have low regularity. Hence, the coupling with the Navier–Stokes equation leads to additional difficulties compared to the local model. Therefore, it is not easy to extend the results, which already hold for the local Model H, cf. [54] for details. Regarding the mathematical analysis of the nonlocal Model H, we refer to [34, 53–58, 65].

As for the local model, the boundary conditions (1.5.2f) ensure that the total mass is conserved, that is

$$\int_\Omega c(t)\,\mathrm{d}x = \int_\Omega c_0\,\mathrm{d}x \quad \text{for all } t \in [0,T]. \tag{1.5.3}$$

Again, if $\Omega = \mathbb{T}^n$, we have

$$\int_\Omega \mathbf{v}(t)\,\mathrm{d}x = \int_\Omega \mathbf{v}_0\,\mathrm{d}x \quad \text{for all } t \in [0,T]. \tag{1.5.4}$$

We also observe that sufficiently regular solutions to the system (1.5.2) satisfy the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} E\big(\mathbf{v}(t), c(t)\big) = -\nu \int_\Omega |\nabla \mathbf{v}(t)|^2\,\mathrm{d}x - m \int_\Omega |\nabla \mu|^2\,\mathrm{d}x \quad \text{for all } t \in [0,T], \tag{1.5.5}$$

entailing that the system (1.5.2) has a dissipative nature.

1.6. A Cahn–Hilliard tumor growth model

Here, we present an application of the phase-field approach that models tumor growth. In the last decades, many mathematical models have been proposed to describe the underlying biological and chemical phenomena, see [35–37, 59, 70–73, 89] and the references therein. In this section, we want to study the following variant of the Cahn–Hilliard model for tumor growth, which was originally derived in [72]:

$$\partial_t c = \Delta\mu + (\mathcal{P}\sigma - \mathcal{A})h(c) \quad \text{in } \Omega_T, \tag{1.6.1a}$$
$$\mu = -\Delta c + f'(c) \quad \text{in } \Omega_T, \tag{1.6.1b}$$
$$\partial_t \sigma = \Delta\sigma + \mathcal{B}(\sigma_S - \sigma) - \mathcal{C}\sigma h(c) \quad \text{in } \Omega_T, \tag{1.6.1c}$$
$$\partial_{\mathbf{n}} c = \partial_{\mathbf{n}} \mu = \partial_{\mathbf{n}} \sigma = 0 \quad \text{on } \partial\Omega_T, \tag{1.6.1d}$$
$$c(0) = c_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{1.6.1e}$$

Here, $c : \Omega_T \to \mathbb{R}$ is an *order parameter* distinguishing the healthy and tumor tissue. More precisely, the level sets $\{c = 1\}$ and $\{c = -1\}$ represent the pure tumorous and healthy phases, respectively, and the diffuse interface $\{|c| < 1 - \delta\}$ models the transition region separating them. The variable $\mu : \Omega_T \to \mathbb{R}$ denotes the *chemical potential* and $\sigma : \Omega_T \to \mathbb{R}$ is the *nutrient concentration*, e.g. oxygen or glucose. Moreover, $f : \mathbb{R} \to \mathbb{R}$ is a double well potential with wells of equal depth and minima at ± 1 and $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C} \geq 0$ are constants representing *tumor proliferation rate, tumor apoptosis rate, nutrient supply rate* and *nutrient consumption rate*, respectively. Additionally, $h : \mathbb{R} \to [0, 1]$ is an interpolation function only present in the tumor phase and the function $\sigma_S : \Omega_T \to \mathbb{R}$ is a *preexisting nutrient concentration.*

From a modeling point of view, the term $\mathcal{P}\sigma h(c)$ in (1.6.1a) denotes the tumor growth, whereas the term $\mathcal{A}h(c)$ models the death of tumor cells. Hence, on account of equation (1.6.1a), we observe that the tumor expands if $\mathcal{P}\sigma - \mathcal{A} > 0$ and shrinks if $\mathcal{P}\sigma - \mathcal{A} < 0$, respectively. In equation (1.6.1c), the term $\mathcal{C}\sigma h(c)$ represents the consumption of the nutrient by tumor cells and $\mathcal{B}(\sigma_S - \sigma)$ models the supply and transport of nutrients. More precisely, if $\sigma_S > \sigma$, nutrients are supplied from blood vessels and if $\sigma_S < \sigma$, the nutrients are transported away from the domain. For more modeling aspects of system (1.6.1a)–(1.6.1e), we refer to [89].

The system (1.6.1a)–(1.6.1e) is a special case of the models derived in Garcke, Lam, Sitka, Styles [72], in particular neglecting chemotaxis and active transport. Moreover, the system in Garcke, Lam, Rocca [73] reduces to our model when neglecting the control term. Indeed, this yields uniqueness and existence of strong solutions to our model, cf. Theorem 6.2.1 below. The interested reader may also consider the references in [72, 73] for other Cahn–Hilliard-type models for tumor growth, for example the Cahn–Hilliard–Darcy variant and optimal control problems. Let us just mention the results in [35–37, 59, 70, 71] for similar systems as (1.6.1a)–(1.6.1e).

The system (1.6.1a)–(1.6.1e) consists of a coupling of the Cahn–Hilliard equation together with a reaction-diffusion equation for the nutrient σ. The total energy of this system is given by

$$E(c, \sigma) := \frac{1}{2} \int_\Omega |\nabla c(x)|^2 \, \mathrm{d}x + \int_\Omega f(c(x)) \, \mathrm{d}x + \frac{1}{2} \int_\Omega |\sigma(x)|^2 \, \mathrm{d}x, \tag{1.6.2}$$

where the first summand is the Ginzburg–Landau free energy of the Cahn–Hilliard equation. In the context of tumor growth models, it describes the cell-to-cell adhesion. In particular, the gradient term penalizes too dispersed tumor patterns, and the term involving f represents the fact that tumor cells rather adhere to each other than to non-tumor cells. For a more detailed overview, we refer to [113].

1.7. A nonlocal Cahn–Hilliard tumor growth model

As already pointed out in the section before, in the context of tumor growth, the Ginzburg-Landau energy in (1.6.2) accounts for cell-to-cell adhesion. Due to the involving gradient, it only takes into account short-range interactions.

From a modeling perspective, however, the essential biological processes in the evolution of cancer diseases, e.g., tumor-cell invasion or the metastases-formation, are long-range interaction processes (cf. [113]) and can thus not be described by means of the local model. In this regard, one replaces $-\Delta$ in (1.6.1b) by the nonlocal operator $\mathcal{L}$, in order to take into account long-range interactions and make the model more accurate. This leads to the following nonlocal Cahn–Hilliard model for tumor growth:

$$\begin{aligned}
\partial_t c &= \Delta\mu + (\mathcal{P}\sigma - \mathcal{A})h(c) && \text{in } \Omega_T, && (1.7.1a)\\
\mu &= \mathcal{L}c + f'(c) && \text{in } \Omega_T, && (1.7.1b)\\
\partial_t \sigma &= \Delta\sigma + \mathcal{B}(\sigma_S - \sigma) - \mathcal{C}\sigma h(c) && \text{in } \Omega_T, && (1.7.1c)\\
\partial_{\mathbf{n}}\mu &= \partial_{\mathbf{n}}\sigma = 0 && \text{on } \partial\Omega_T, && (1.7.1d)\\
c(0) &= c_0, \quad \sigma(0) = \sigma_0 && \text{in } \Omega. && (1.7.1e)
\end{aligned}$$

Here, the interpretation of the functions c, μ, σ and f, h, σ_S as well as the constants $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ is analogous to the local model (1.6.1a)-(1.6.1e) above. Moreover, the nonlocal operator $\mathcal{L}$ is defined by the following nonlocal operator,

$$\mathcal{L}c(x) := \text{P.V.} \int_\Omega J(x-y)\,(c(x) - c(y))\; dy \quad \text{for all } x \in \Omega. \tag{1.7.2}$$

The nonlocal system (1.7.1a)-(1.7.1e) was introduced in Scarpa, Signori [113], where the authors considered a more general model with chemotaxis and active transport as well as relaxation parameters. Their paper yields an existence and uniqueness result for weak solutions of (1.7.1a)-(1.7.1e), cf. Theorem 6.2.2 below. For references in the direction of nonlocal Cahn–Hilliard-type models, we refer to the references in [60, 113] and Davoli et al. [42]. Moreover, let us also note the results in [52], where the author studied the optimal control problem for a viscous non-local tumor growth model.

The system (1.7.1a)–(1.7.1e) consists of a coupling of the nonlocal Cahn–Hilliard equation together with a reaction-diffusion equation for the nutrient σ. Note that the nonlocal Cahn–Hilliard equation is a second order differential equation. Thus, we do not need to impose any boundary condition for the order parameter c. The total energy of system (1.7.1a)–(1.7.1e) is given by

$$\begin{aligned}
E(c,\sigma) := \frac{1}{4}\int_\Omega\int_\Omega J(x-y)|c(x)-c(y)|^2\;\mathrm{d}y\;\mathrm{d}x + \int_\Omega f(c(x))\;\mathrm{d}x \\
+\frac{1}{2}\int_\Omega |\sigma(x)|^2\;\mathrm{d}x,
\end{aligned} \tag{1.7.3}$$

where we substituted the classical Ginzburg-Landau free energy with the nonlocal Helmholtz free energy (1.3.1).

1.8. The sharp interface limit

Diffuse interface models typically involve a small parameter η, which is proportional to the thickness of the interface. If the thickness η is sent to zero, one obtains a sharp interface model in the limit. Therefore, the limit $\eta \to 0$ is also referred to as *the sharp interface limit*.

Both diffuse and sharp interface models can be derived and motivated by physical principles and experimental observations. Since the motion of the free boundary is more visual, sharp interface models appear to be more qualitative and simpler. However, free boundaries might develop singularities at a finite time and thus cause both analytical and numerical difficulties. On the other hand, with diffuse interface models, singularities of the interface can be overcome more easily. Therefore, solutions typically have better analytical properties and so, the phase-field approach provides an appropriate alternative to sharp interface models. In this regard, it is an important task to relate these models. This can be done via their sharp interface limit (see [22,28,29,102] for more information).

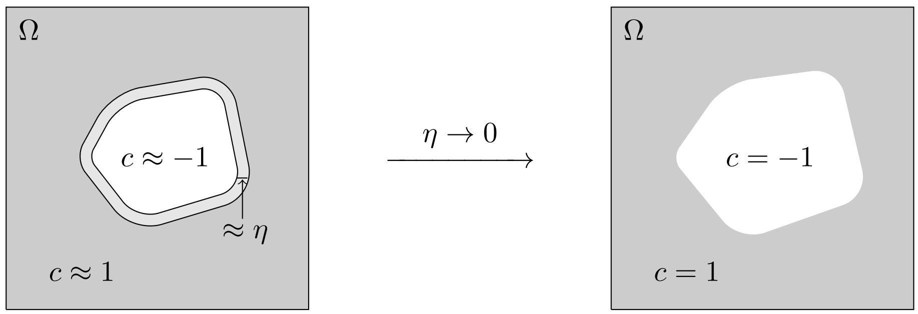

Figure 1.5: Sharp interface limit.

There are four principle methods to perform the sharp interface limit:

1. *Linearization techniques*: One derives suitable error estimates between the solution (depending on the parameter η) and the solution to the limit problem (or some quantity depending on the solution to the limit problem). Typically, this will be done with the aid of a suitable spectral estimate for a linear operator associated to the diffuse interface equation.

2. *Weak convergence methods*: One derives a priori estimates for the solutions in suitable function spaces. Then, based on these estimates, one extracts weakly convergent subsequences and passes to the limit in the weak formulation of the system in a suitable way.

3. *Relative entropy methods*: Based on a sufficiently regular solution, one constructs a relative entropy functional, in order to estimate the distance between any arbitrary weak solution and the regular solution. The main goal is to show a stability estimate on the relative entropy functional. This will be done by deriving a differential inequality for the functional and applying Gronwall's inequality.

4. *Comparison principle*: In the case of the Allen–Cahn equation, one suitably constructs sub- and supersolutions. Afterwards, one uses a parabolic comparison theorem, in order to prove local-in-time convergence as long as the interface remains smooth.

The sharp interface limit has already been shown for many phase-field models. For an overview, we refer to Section 8.2 as well as [13,19,28,29,97,101,102,114] and the references therein.

1.9. Nonlocal-to-local convergence

In the theory of nonlocal phase-field models, one typically investigates nonlocal operators of the form

$$\begin{aligned}\mathcal{L}_\varepsilon c(x) &:= \text{P.V.} \int_\Omega J_\varepsilon(x-y)(c(x)-c(y))\,\mathrm{d}y, \\ &= \lim_{r\to 0} \int_{\Omega\cap\{|x-y|\geq r\}} J_\varepsilon(x-y)\big(c(x)-c(y)\big)\,\mathrm{d}y \quad x\in\Omega, \quad \varepsilon>0,\end{aligned} \tag{1.9.1}$$

where the function $J_\varepsilon : \mathbb{R}^n \to \mathbb{R}$,

$$J_\varepsilon(x-y) = \varepsilon^{-n}\,\frac{\rho_\varepsilon(x-y)}{|x-y|^2} \tag{1.9.2}$$

is a prescribed interaction kernel and $(\rho_\varepsilon)_{\varepsilon>0}$ a suitable sequence of mollifiers. With this choice for the interaction kernel, it is well-known that the corresponding nonlocal energy functionals of the form

$$\mathcal{E}_\varepsilon(c) := \frac{1}{4}\int_\Omega J_\varepsilon(x-y)|c(x)-c(y)|^2\,\mathrm{d}y\,\mathrm{d}x \tag{1.9.3}$$

converge to local energy functionals as ε tends to zero. First results concerning these nonlocal-to-local asymptotics were obtained by J. Bourgain, H. Brezis and P. Mironescu [24, 25]. These results were then extended by A.C. Ponce in [109, 110] where also results on Γ-convergence and Poincaré-inequalities were shown.

If we additionally assume that the sequence of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$ is radially symmetric, it is well-known that nonlocal energies as in (1.9.3) converge to a multiple of the local free energy

$$\mathcal{E}(c) := \frac{1}{2}\int_\Omega |\nabla c(x)|^2\,\mathrm{d}x$$

as ε tends towards zero. Moreover, it is known that the nonlocal operator as in (1.9.1) converges to the negative Laplacian $-\Delta$ as ε goes to zero, cf. [41, 43, 44, 98].

If the sequence of mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$ is *not* radially symmetric but only assumed to be even, there are few results in the literature, so far. It is well-known by the contributions of Ponce [109, 110] that nonlocal energy functionals as in (1.9.3) converge to a local energy functional of the form

$$\mathcal{E}(c) = \frac{1}{4}\int_{\mathbb{R}^n}\left(\int_\Omega \left|Dc(x)\cdot\frac{z}{|z|^2}\right|^2\mathrm{d}x\right)\rho(z)\,\mathrm{d}z.$$

Here, the right-hand side is in fact equal to

$$\frac{1}{2}\int_\Omega \nabla c(x)^T M \nabla c(x)\,\mathrm{d}x,$$

where the matrix $M\in\mathbb{R}^{n\times n}$ is given by

$$M_{ij} := \frac{1}{2}\int_{\mathbb{R}^n}\frac{\rho(z)}{|z|^2} z_i z_j\,\mathrm{d}z \qquad \text{for all } i,j\in\{1,\ldots,n\}.$$

Based on the works in [24, 25, 109, 110], a whole framework of nonlocal-to-local convergence was developed. Recent results on convergence of nonlocal quadratic forms to local quadratic forms of gradient type and further references can be found in Foghem Gounoue et al. in [51]. Nonlocal-to-local asymptotics have already been studied in [123], where the author proved the convergence of weak solutions of the fractional heat equation to the fundamental solution as $t \to \infty$. In [39], the authors studied the limits $s \to 0^+$ and $s \to 1^-$ for s-fractional heat flows in a cylindrical domain with homogeneous Dirichlet boundary conditions.

Concerning the phase-field models introduced before, we have the following: Convergence of solutions to the nonlocal Cahn–Hilliard equation to solutions to the local Cahn–Hilliard equation has already been proved by Melchionna et al. in [98] in the case of periodic boundary conditions and a regular free energy density and by Davoli et al. in [41] in the case of periodic boundary conditions and a singular free energy density. In the case of Neumann boundary conditions, convergence has been proved by Davoli et al. in [44] with an additional viscosity term in the nonlocal Cahn–Hilliard equation and in [43] for $W^{1,1}$-kernels. A corresponding result for a singular phase-field system was proved by Kurima [91]. The authors in [48] proved convergence of the nonlocal to the local degenerate Cahn–Hilliard equation. In [15] and [94], the authors proved the nonlocal-to-local limit for a coupled Navier–Stokes/Cahn–Hilliard system, and in [42], the authors investigated a Cahn–Hilliard model for tumor growth.

In this regard, the local Cahn–Hilliard equation can be viewed as an approximation of its nonlocal counterpart. However, it is still unclear, how good (in terms of the parameter ε) this approximation is. To this end, one needs to derive suitable error estimates between these models. This can be done by means of an *energy method*, i.e., based on the energy functional of the system, we want to derive estimates to measure the distance between the solution to the nonlocal model and the solution to the local model. This will be done by proving a differential inequality for the functional and applying Gronwall's inequality. However, it turns out to be necessary to investigate the convergence of the nonlocal operator in the first step. More precisely, one needs to derive concrete rates of convergence. Even though the convergence of the nonlocal operator $\mathcal{L}_\varepsilon$ given as in (1.9.1) towards the negative Laplacian $-\Delta$ as ε tends to zero is well-known, no concrete rates of convergence have been established yet. Therefore, it is an interesting and important task to prove strong convergence of the nonlocal operator along with certain rates of convergence. Indeed, this will be the main novelty discussed within this thesis (see Chapter 2.1 below). Based on these estimates, one can then use an energy method and a Gronwall-type argument, in order to derive error estimates between the solutions to the nonlocal Cahn–Hilliard equation and the solution to the local Cahn–Hilliard equation, respectively. Indeed, this gives the desired error estimates and thus indicates the quality of the approximation. In fact, this can also be done for the phase-field models introduced before.

Moreover, we are also interested in nonlocal operators with even kernels. In this case, we prove convergence towards a local differential operator of the form $\mathcal{L}c := -\mathrm{div}(M\nabla c)$ together with concrete rates of convergence. This will be discussed in Chapter 2.1 below. As a consequence, this enables us to prove the nonlocal-to-local asymptotics and corresponding error estimates even for anisotropic phase-field models.

1.10. Outline

The outline of this thesis is as follows.

In Chapter 2, we provide an overview of the main results obtained in this thesis. We state the underlying assumptions and formulate the main results, which we also compare to the existing literature. In addition, we briefly explain the key ideas and difficulties in the proofs.

In Chapter 3, we introduce the notation and recall some essential mathematical results we use throughout this work.

In the first part of this thesis, we prove the nonlocal-to-local convergence of the nonlocal operator towards a local differential operator along with rates of convergence. In Chapter 4.3, we show convergence on the torus with respect to the L^2-norm. The analysis is done for a class of more singular kernels. The proof is based on methods from Fourier analysis. In Chapter 4.4, we then prove convergence on bounded domains. The analysis is done in the case of $W^{1,1}$-kernels as well as more singular kernels. Moreover, we show convergence with respect to the L^p-norm, where $p \in [1, \infty)$. The proof is based on localization and perturbation arguments.

In the second part of this thesis, we use the results from the first part to prove the nonlocal-to-local convergence for the phase-field models introduced in Sections 1.2–1.7. In each chapter, we first discuss well-posedness results for the respective models and then show the main result. The proofs are based on an energy method together with a Gronwall type argument. In Chapter 5, we consider the Cahn–Hilliard equation, while Chapter 5 concerns the Allen–Cahn equation. In Chapter 6, we investigate a Cahn–Hilliard model for tumor growth, and in Chapter 7, we study a coupled Cahn–Hilliard/Navier–Stokes system.

Finally, in Chapter 8, we study the nonlocal Allen–Cahn equation on the torus. We first prove a well-posedness result and afterwards, we investigate the case, where both the parameter in the nonlocal operator and the thickness of the diffuse interface are sent to zero. The proof combines the result from the first part of the thesis together with existing results for the sharp interface limit of the local Allen–Cahn equation.

1.11. Publications

This thesis mainly consists of the following publications and submitted papers:

[11] H. Abels, C. Hurm, *Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator*, J. Differential Equations, 402: 593-624, 2024.

[12] H. Abels, C. Hurm, M. Moser, *Convergence of the Nonlocal Allen-Cahn Equation to Mean Curvature Flow*, (2024), arXiv: 2410.08596.

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Chapter 2

Contributions of the thesis

In this chapter, we give an overview of the main results obtained in this thesis. The presentation is divided into six parts. The first part concerns the convergence of the nonlocal operator, whereas the remaining parts concern applications to the phase-field models described in the introduction. In each part, we first state the main assumptions and then formulate the main results, which we also compare with the existing literature.

2.1. Convergence of the nonlocal operator

For the convergence results of the nonlocal operator, we consider two different kinds of interaction kernels, namely anisotropic and isotropic kernels. In this regard, we now state the main assumptions on these kernels. For both kinds of kernels, we have different regularity assumptions. We note that the regularity assumptions for singular kernels can be relaxed when working on the full space. Thus, we split the assumptions for these kernels into two parts.

Assumptions on the isotropic kernel

We denote the class of isotropic interaction kernels by (I). Here, we make the following assumptions:

(I1) The function $\rho : \mathbb{R}^n \to [0,\infty)$ $\rho \not\equiv 0$, is measurable and radially symmetric (i.e., $\rho(x) = \rho(|x|)$ for almost all $x \in \mathbb{R}^n$) with

$$\int_{\mathbb{R}^n} |\rho(x)| \, (1 + |x|) \, \mathrm{d}x < \infty. \tag{2.1.1}$$

The associated kernel is given by

$$J : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad J(x) = \frac{\rho(x)}{|x|^2}. \tag{2.1.2}$$

For any $\varepsilon > 0$, we further introduce the functions

$$\rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \qquad \rho_\varepsilon(x) := \varepsilon^{-n} \, \rho(\tfrac{x}{\varepsilon}), \tag{2.1.3}$$

$$J_\varepsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad J_\varepsilon(x) = \frac{\rho_\varepsilon(x)}{|x|^2}. \tag{2.1.4}$$

Moreover, we assume that ρ_ε is normalized by

$$\int_0^\infty \rho_\varepsilon(r)\, r^{n-1}\, \mathrm{d}r = \frac{2}{C_n} \quad \text{for all } \varepsilon > 0, \tag{2.1.5}$$

where $C_n := \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 \, \mathrm{d}\mathcal{H}^{n-1}(\sigma)$.

(I2) In addition to assumption (I1), we assume that $\rho \in C^1(\mathbb{R}^n \setminus \{0\})$ is compactly supported and that there exist $c_0, c_1 > 0$ and $\alpha \in (0,2)$ such that

$$|\rho(x)| \le c_0 |x|^{2-\alpha-n} \quad \text{and} \quad |\nabla\rho(x)| \le c_1 |x|^{1-\alpha-n} \tag{2.1.6}$$

for all $x \in B_1(0) \setminus \{0\}$.

(I3) Let $J_\varepsilon : \mathbb{R}^n \to [0,\infty)$ be a non-negative function given by $J_\varepsilon(x) = \frac{\rho_\varepsilon(|x|)}{|x|^2}$ for all $x \in \mathbb{R}^n$ and $J_\varepsilon \in W^{1,1}(\mathbb{R}^n)$, where $(\rho_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers satisfying

$$\begin{aligned}
&\rho : \mathbb{R} \to [0,\infty), \quad \rho \in L^1(\mathbb{R}), \quad \rho(r) = \rho(-r) \quad \text{for all } r \in \mathbb{R},\\
&\rho_\varepsilon(r) = \varepsilon^{-n} \rho\left(\frac{r}{\varepsilon}\right) \qquad \text{for all } r \in \mathbb{R},\, \varepsilon > 0,\\
&\int_0^\infty \rho_\varepsilon(r)\, r^{n-1}\, \mathrm{d}r = \frac{2}{C_n} \quad \text{for all } \varepsilon > 0,\\
&\lim_{\varepsilon \searrow 0} \int_\delta^\infty \rho_\varepsilon(r)\, r^{n-1}\, \mathrm{d}r = 0 \quad \text{for all } \delta > 0,
\end{aligned}$$

where $C_n := \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 \, \mathrm{d}\mathcal{H}^{n-1}(\sigma)$.

Assumptions on the anisotropic kernel

The class of anisotropic kernels will be denoted by (J). Here, we make the following assumptions:

(J1) The function $\rho : \mathbb{R}^n \to [0,\infty)$, $\rho \not\equiv 0$, is measurable and even (i.e., $\rho(x) = \rho(-x)$ for almost all $x \in \mathbb{R}^n$) with

$$\int_{\mathbb{R}^n} |\rho(x)| \, (1 + |x|) \, \mathrm{d}x < \infty. \tag{2.1.7}$$

The associated kernel is given by

$$J : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad J(x) = \frac{\rho(x)}{|x|^2}. \tag{2.1.8}$$

For any $\varepsilon > 0$, we further introduce the functions

$$\rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \qquad \rho_\varepsilon(x) := \varepsilon^{-n} \rho(\tfrac{x}{\varepsilon}), \tag{2.1.9}$$

$$J_\varepsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad J_\varepsilon(x) = \frac{\rho_\varepsilon(x)}{|x|^2}. \tag{2.1.10}$$

Since $\rho \ge 0$ and $\rho \not\equiv 0$, the momentum matrix

$$M := \frac{1}{2} \int_{\mathbb{R}^n} J(z)\, z \otimes z \, \mathrm{d}z \tag{2.1.11}$$

is positive definite.

(J2) In addition to assumption (J1), we assume $\rho \in C^1(\mathbb{R}^n \setminus \{0\})$ is compactly supported and that there exist $c_0, c_1 > 0$ and $\alpha \in (0,2)$ such that

$$|\rho(x)| \leq c_0 |x|^{2-\alpha-n} \quad \text{and} \quad |\nabla \rho(x)| \leq c_1 |x|^{1-\alpha-n} \tag{2.1.12}$$

for all $x \in B_1(0) \setminus \{0\}$. Moreover, we assume that the associated kernel J fulfills the condition

$$\int_{\mathbb{R}^{n-1}} J\left(AQ\begin{pmatrix} z' \\ z_n \end{pmatrix}\right) z' \, \mathrm{d}z' = 0 \quad \text{for all } z_n \in \mathbb{R} \text{ and all } Q \in \mathrm{SO}(n), \tag{2.1.13}$$

where

$$A := \sqrt{M}. \tag{2.1.14}$$

(J3) In addition to assumption (J1), we assume that ρ is chosen in such a way that the corresponding kernel J satisfies $J \in W^{1,1}(\mathbb{R}^n)$ and condition (2.1.13).

Remark 2.1.1 (Discussion of the assumptions)**.**

1. The assumptions (I1) and (J1) hold true for the full space. In C^2-domains, however, we need to impose additional assumptions. In this case, we assume (I2) and (J2). The assumptions in (I3) and (J3) hold true for both the full space and C^2-domains. Note that the condition (2.1.13) in (J2) is only needed if we consider C^2-domains.

2. The regularity assumption for ρ (and thus for J) on the full space is minimal, cf. (I1) and (J1). In C^2-domains, however, we need to impose more regularity as well as a growth condition and compact support for the kernel, in order to derive suitable error estimates close to the boundary. This motivates condition (2.1.6).

3. In addition to (2.1.6), we further need to impose a condition for the first moments of anisotropic kernels in C^2-domains. In fact, assumption (2.1.13) is crucial to derive the desired error estimates. Observe that this condition is automatically satisfied for isotropic kernels due to their radial symmetry.

4. The normalization of ρ_ε as in (2.1.5) is necessary to get convergence towards the negative Laplacian. Otherwise, one would obtain a different scaling factor in the limit. Observe that this assumption is based on the works by Ponce [109, 110].

5. For the convergence of the nonlocal operator, we only consider kernels satisfying the assumptions in (I1) and (I2) or (J1) and (J2), respectively. The assumptions in (I3) and (J3) are then needed to prove the nonlocal-to-local convergence of solutions to the phase-field models introduced before. These regularity assumptions are based on the existing literature concerning these nonlocal-to-local asymptotics.

Remark 2.1.2 (Periodic extension)**.** In the case, where Ω is the torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$, we assume that the interaction kernel J is compactly supported in $(-\pi, \pi)^n$. Moreover, we identify J with its 2π-periodic extension $\tilde{J} : \mathbb{T}^n \to [0, \infty)$.

Example 2.1.3 (Examples for kernels)**.**

1. One famous example for nonlocal operators, whose kernel fulfills the assumptions in (I1), is the so-called *fractional Laplacian.* For $1 < \alpha < 2$, $x \in \mathbb{R}^n$ and $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$, the operator is defined by

$$(-\Delta)^{\frac{\alpha}{2}} u(x) := C(n,\alpha)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}}\,\mathrm{d}y,$$

where the constant $C(n,\alpha)$ is given by

$$C(n,\alpha) := \frac{2^\alpha \Gamma(\frac{\alpha}{2}+\frac{n}{2})}{\pi^{\frac{n}{2}}|\Gamma(-\frac{\alpha}{2})|},$$

and Γ denotes the Gamma function, cf. [40].

2. In order to construct a nontrivial, *anisotropic* density function ρ with associated kernel J, such that (J1) and (J2) are fulfilled, we proceed as follows:
Let $\mathring{\rho} \in C^1(\mathbb{R}^n \setminus \{0\}; [0,\infty))$ be radially symmetric function, which satisfies

$$0 < \int_{\mathbb{R}^n} \mathring{\rho}(x)(1+|x|)\,\mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla \mathring{\rho}(x)|\,|x|\,\mathrm{d}x < \infty. \tag{2.1.15}$$

Without loss of generality, we assume that $\|\mathring{\rho}\|_{L^1(\mathbb{R}^n)} = 2n$. This implies that

$$\frac{1}{2}\int_{\mathbb{R}^n} \frac{\mathring{\rho}(z)}{|z|^2} z \otimes z\,\mathrm{d}z = I. \tag{2.1.16}$$

Moreover, let $B \in \mathbb{R}^{n\times n}$ be a symmetric, positive definite matrix. To simplify the computations, we further assume, without loss of generality, that $\det B = 1$. We now define $\rho \in C^1(\mathbb{R}^n \setminus \{0\}; [0,\infty))$ with

$$\rho(x) := \mathring{\rho}(Bx)\frac{|x|^2}{|Bx|^2} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

In this way, ρ is clearly even but, in general, not any more radially symmetric. Its associated kernel can be expressed as

$$J(x) = \frac{\mathring{\rho}(Bx)}{|Bx|^2} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

It is easy to check that ρ and J satisfy (J1). Moreover, the change of variables $x \mapsto Bx$ yields

$$\begin{aligned} M = \frac{1}{2}\int_{\mathbb{R}^n} J(x) x\otimes x\,\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\mathring{\rho}(Bx)}{|Bx|^2} x \otimes x\,\mathrm{d}x \\ &= \frac{1}{2} B^{-1}\left(\int_{\mathbb{R}^n} \frac{\mathring{\rho}(x)}{|x|^2} x\otimes x\,\mathrm{d}x\right) B^{-T} = B^{-2}, \end{aligned}$$

where we used (2.1.16) and $\det B = 1$. This shows $M = B^{-2}$ and thus, (2.1.14) implies $A = B^{-1}$.

To verify (2.1.13), let $Q \in \mathrm{SO}(n)$ be arbitrary. Hence, recalling that $\mathring{\rho}$ is radially symmetric, we use the change of variables $z' \mapsto D'z'$ with $D' \coloneqq \operatorname{diag}(1/d_1, ..., 1/d_{n-1})$ to deduce that

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} J\left(AQ\begin{pmatrix} z' \\ z_n \end{pmatrix}\right) z' \,\mathrm{d}z' &= \int_{\mathbb{R}^{n-1}} \frac{\mathring{\rho}(BAQ(z', z_n)^T)}{\left|BAQ(z', z_n)^T\right|^2} z' \,\mathrm{d}z' \\ &= \int_{\mathbb{R}^{n-1}} \frac{\mathring{\rho}(z', z_n)}{\left|(z', z_n)\right|^2} z' \,\mathrm{d}z' = 0.\end{aligned}$$

This shows that J satisfies Condition (2.1.13) and consequently, (J2) is also fulfilled.

3. Eminent examples for radially symmetric kernels that satisfy $J \in W^{1,1}(\mathbb{R}^n)$ are Newtonian, Bessel and Riesz like potentials. In the case $n = 2$ and $x \in \mathbb{R}^2 \setminus \{0\}$, the *Bessel potential* is given by

$$b_s(|x|) := \frac{e^{-|x|}}{(2\pi)2^{\frac{s}{2}}\Gamma(\frac{s}{2})\Gamma(\frac{3-s}{2})} \int_0^\infty e^{-|x|t} \left(t + \frac{t^2}{2}\right)^{\frac{1-s}{2}} \,\mathrm{d}t,$$

where $s > 0$. For more details and examples, we refer to [65] and the references therein.

2.1.1. Convergence on the torus

The following result will be discussed in Chapter 4.3.

The first convergence result we present in this contribution, concerns the nonlocal-to-local convergence for operators on the torus $\mathbb{T}^n$. For radial symmetric kernels, it is already well-known that nonlocal operators as in (1.9.1) converge towards the negative Laplacian as ε is sent to zero, cf. for instance [98]. However, no concrete rates of convergence have been provided, so far. To the best of the authors knowledge, also anisotropic nonlocal operators have not been studied yet.

In the main result of this chapter, we want to prove the nonlocal-to-local convergence of operators as in (1.9.1), i.e.,

$$\begin{aligned}\mathcal{L}_\varepsilon^{\mathbb{T}^n} u(x) : &= \mathrm{P.V.} \int_{\mathbb{T}^n} \tilde{J}_\varepsilon(x - y)(u(x) - u(y)) \,\mathrm{d}y \\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} J_\varepsilon(x - y)(u(x) - u(y)) \,\mathrm{d}y \quad \text{for all } x \in \mathbb{T}^n,\end{aligned}$$

along with rates of convergence. Here, $\tilde{J}$ denotes a 2π-periodic extension of J as mentioned in Remark 2.1.2. In contrast to many works in the literature, we also cover the case of anisotropic kernels. This chapter is based on the result in [11]. However, we consider a more general setting where the kernel is assumed to have the regularity as in (J1) or (I1) instead of $W^{1,1}(\mathbb{R}^n)$.

The proof mainly uses methods from Fourier analysis. In particular, we apply Plancherel's theorem. Therefore, it suffices to prove the convergence for the Fourier coefficients. In order to provide certain rates of convergence, we need to derive error estimates. This will be done with the aid of Taylor's theorem applied to suitable auxiliary functions, which we construct by means of the Fourier coefficients.

Indeed, we then have the following convergence result:

Theorem 2.1.4. *Suppose that the assumptions in* (J1) *and Remark 2.1.2 hold.*

(a) *For all $u \in H^2(\mathbb{T}^n)$ it holds that*

$$\left\| \mathcal{L}_\varepsilon^{\mathbb{T}^n} u + \operatorname{div}(M\nabla u) \right\|_{L^2(\mathbb{T}^n)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.1.17}$$

where the matrix $M \in \mathbb{R}^{n\times n}$ is given by

$$M := \frac{1}{2}\int_{\mathbb{R}^n} J(z)\, z \otimes z \,\mathrm{d}z.$$

(b) *There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $u \in H^3(\mathbb{T}^n)$, it holds that*

$$\left\| \mathcal{L}_\varepsilon^{\mathbb{T}^n} u + \operatorname{div}(M\nabla u) \right\|_{L^2(\mathbb{T}^n)} \leq \varepsilon C \|u\|_{H^3(\mathbb{T}^n)}. \tag{2.1.18}$$

In addition, if we assume the kernel to be radially symmetric, we immediately obtain the following result:

Theorem 2.1.5. *Suppose that the assumptions in* (I1) *and Remark 2.1.2 hold.*

(a) *For all $u \in H^2(\mathbb{T}^n)$, it holds that*

$$\left\| \mathcal{L}_\varepsilon^{\mathbb{T}^n} u + \Delta u \right\|_{L^2(\mathbb{T}^n)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.1.19}$$

(b) *There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $u \in H^3(\mathbb{T}^n)$, it holds that*

$$\left\| \mathcal{L}_\varepsilon^{\mathbb{T}^n} u + \Delta u \right\|_{L^2(\mathbb{T}^n)} \leq \varepsilon C \|u\|_{H^3(\mathbb{T}^n)}. \tag{2.1.20}$$

2.1.2. Convergence in bounded domains

The following result is contained in

[11] H. Abels, C. Hurm, *Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator*, J. Differential Equations, 402: 593-624, 2024

and will be discussed in Chapter 4.4.

In this chapter, we want to extend the convergence results established in Section 2.1.1 to sufficiently smooth bounded domains in $\mathbb{R}^n$. Moreover, we also consider more singular kernels and intend to prove convergence with respect to the L^p-topology for $p \in [1, \infty)$. Similarly to the periodic case, it is already well-known that for radially symmetric kernels, nonlocal operators as in (1.9.1) converge towards the negative Laplacian as ε is sent to zero. Here, we additionally need to assume that the functions fulfill the homogeneous Neumann boundary condition, cf. [43, 44]. However, no concrete rates of convergence have been established, so far. To the best of the authors knowledge, also anisotropic nonlocal operators have not been investigated yet.

In the main result of this chapter, we show the nonlocal-to-local convergence of nonlocal operators as in (1.9.1) along with certain rates of convergence. In contrast to many works in the literature, we also consider anisotropic kernels, which can be even more singular.

The proof is based on localization. In the first step, we prove the convergence on the full space $\mathbb{R}^n$. This follows by a denseness argument and Taylor's theorem. Next, we show convergence on a bent half space. In the case of anisotropic kernels, we need to perform a suitable coordinate transform onto a reference domain, in order to simplify the proof. With the aid of this transform, the matrix M reduces to the identity matrix I. Therefore, we can use similar methods as for the isotropic case. In particular, we derive suitable error estimates by applying denseness and perturbation arguments. Finally, we close the proof via localization.

To simplify the notation, we start with a definition.

Definition 2.1.6. *Let $k \geq 2$, $p \in [1, \infty)$ and $D \in \mathbb{R}^{n\times n}$. We introduce the vector space*

$$W^{k,p}_D(\Omega) := \{u \in W^{k,p}(\Omega) : \; D\nabla u \cdot \mathbf{n}_{\partial\Omega} = 0 \text{ on } \partial\Omega\}. \tag{2.1.21}$$

We then have the following convergence result:

Theorem 2.1.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain with compact boundary of class C^3, let $\varepsilon \in (0, 1]$, and suppose that either assumption* (J1) *and* (J2) *hold and let $p \in [1, \infty)$ be arbitrary. Then, the nonlocal operator introduced in* (1.9.1) *gives a bounded linear operator*

$$\mathcal{L}^\Omega_\varepsilon : W^{2,p}_M(\Omega) \to L^p(\Omega) \tag{2.1.22}$$

with

$$\|\mathcal{L}^\Omega_\varepsilon u\|_{L^p(\Omega)} \leq C\|u\|_{W^{2,p}(\Omega)} \tag{2.1.23}$$

for all $u \in W^{2,p}_M(\Omega)$, where C is a positive constant depending only on Ω, ρ and p. Moreover, the following convergence properties hold true:

(a) *For all $u \in W^{2,p}_M(\Omega)$, it holds*

$$\|\mathcal{L}_\varepsilon u + \operatorname{div}(M\nabla u)\|_{L^p(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.1.24}$$

where the matrix $M \in \mathbb{R}^{n\times n}$ is given by

$$M := \frac{1}{2}\int_{\mathbb{R}^n} J(z)\, z \otimes z \,\mathrm{d}z.$$

(b) *There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $u \in W^{3,p}_M(\Omega)$ it holds*

$$\|\mathcal{L}_\varepsilon u + \operatorname{div}(M\nabla u)\|_{L^p(\Omega)} \leq C\sqrt[p]{\varepsilon}\|u\|_{W^{3,p}(\Omega)}. \tag{2.1.25}$$

In addition, if we assume the kernel to be radially symmetric, we obtain the following convergence properties:

Theorem 2.1.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain with compact boundary of class C^3, let $\varepsilon \in (0, 1]$, and suppose that either assumption* (I1) *and* (I2) *hold and let $p \in [1, \infty)$ be arbitrary. Then, the nonlocal operator introduced in* (1.9.1) *gives a bounded linear operator*

$$\mathcal{L}^\Omega_\varepsilon : W^{2,p}_I(\Omega) \to L^p(\Omega) \tag{2.1.26}$$

with

$$\|\mathcal{L}_\varepsilon^\Omega u\|_{L^p(\Omega)} \leq C\|u\|_{W^{2,p}(\Omega)} \tag{2.1.27}$$

for all $u \in W_I^{2,p}(\Omega)$, where C is a positive constant depending only on Ω, ρ and p. Moreover, the following convergence properties hold true:

(a) *For all $u \in W_I^{2,p}(\Omega)$, it holds*

$$\|\mathcal{L}_\varepsilon u + \Delta u\|_{L^p(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.1.28}$$

(b) *There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $u \in W_I^{3,p}(\Omega)$ it holds*

$$\|\mathcal{L}_\varepsilon u + \Delta u\|_{L^p(\Omega)} \leq C\sqrt[p]{\varepsilon}\|u\|_{W^{3,p}(\Omega)}. \tag{2.1.29}$$

Remark 2.1.9. (a) The assertions in Theorem 2.1.7 and 2.1.8, respectively, also hold true in the case of $W^{1,1}$-kernels, i.e., if (J3) or (I3) are assumed.

(b) Observe that Theorem 2.1.8 also covers the result in [11]. In fact, this immediately follows by choosing kernels of type (I3) and $p = 2$. However, the result in 2.1.8 is more general, since it also allows for more singular kernels and convergence in a more general setting.

Remark 2.1.10. We note that the rate of convergence in bounded domains is worse compared to the torus. This follows from the behavior of the nonlocal operator close to the boundary: In the proof of Theorem 2.1.7 (and Theorem 2.1.8, respectively), we use Taylor's theorem to derive suitable error estimates depending on ε. The problem then arises when estimating the first and second order terms in the Taylor expansion. The first order term only behaves of order $\mathcal{O}(|x-y|)$, whereas the singularity in the kernel is of order $\mathcal{O}(|x-y|^{-2})$. Using the boundary condition of u, we get an additional term of order $\mathcal{O}(|x-y|)$ such that the singularity vanishes. This follows by the fundamental theorem of calculus. By definition of the interaction kernel J_ε, cf. (2.1.10), the remaining term then is of the form

$$\sup_{x' \in B_1(0)'} \int_{\mathbb{R}^{n-1}\times(-1,0)} \rho_\varepsilon(x-y)\,\mathrm{d}y.$$

By construction and the properties of ρ, this term only depends on x_n. Computing the L^p-norm with respect to x_n and using the parametrization of ρ as in (2.1.9), one can perform a suitable change of variables both in y and x_n. In fact, this yields an additional scaling by $\varepsilon^{1/p}$ due to the Jacobian determinants and the L^p-norm. Similarly, one argues for the second term in the Taylor expansion. This motivates the rate of convergence in (2.1.25) and (2.1.29), respectively.

Since we do not have any boundary on the torus, this problem does not occur in this case. The same holds true on the full space. Therefore, the rate of convergence is better in these cases.

Remark 2.1.11. The rate of convergence obtained in Theorem 2.1.7 and 2.1.8 is optimal. Even in the simplest case, where $n = 1$, $\Omega = \mathbb{R}_+$ and $u \in C_0^\infty(\overline{\mathbb{R}_+})$ with $\partial_{\mathbf{n}} u = 0$ on $\partial\mathbb{R}_+$, we do not gain a better rate of convergence in $L^p(\mathbb{R}_+)$. This shows the following calculation: Let $x > 0$ and let $\tilde{u} \in C_0^\infty(\mathbb{R})$ be a suitable extension of u to $\mathbb{R}$. Then, the error term reads as

$$\begin{aligned}\mathcal{R}_\varepsilon \tilde{u}(x) &= \int_{-\infty}^{0} J_\varepsilon(x-y)(u(x) - \tilde{u}(y))\,\mathrm{d}y \\ &= \int_{B_\delta(x)\cap\mathbb{R}_-} J_\varepsilon(x-y)(u(x)-\tilde{u}(y)))\,\mathrm{d}y + \int_{(B_\delta(x)\cap\mathbb{R}_-)^c} J_\varepsilon(x-y)(u(x)-\tilde{u}(y)))\,\mathrm{d}y\end{aligned}$$

for some $\delta > 0$. In the second term on the right-hand side, we observe that $|x - y| \geq \delta$. Hence, computing the $L^p(\mathbb{R}_+)$-norm, this term behaves of order $\mathcal{O}(\varepsilon)$. This follows by the same arguments as in the proof of Theorem 2.1.7.

In the first term on the right-hand side, we apply Taylor's theorem. This yields

$$\begin{aligned}&\int_{B_\delta(x)\cap\mathbb{R}_-} J_\varepsilon(x-y)(u(x)-\tilde{u}(y)))\,\mathrm{d}y \\ &= \int_{B_\delta(x)\cap\mathbb{R}_-} J_\varepsilon(x-y)(-u'(x)(x-y) - R_2(x,y))\,\mathrm{d}y.\end{aligned}$$

Exploiting the boundary condition of u, the fundamental theorem of calculus implies

$$\int_{B_\delta(x)\cap\mathbb{R}_-} J_\varepsilon(x-y)u'(x)(x-y)\,\mathrm{d}y = \left(\int_0^1 u''(tx)\,\mathrm{d}t\right)\left(\int_{B_\delta(x)\cap\mathbb{R}_-} \rho_\varepsilon(x-y)\,\mathrm{d}y\right),$$

where $\int_{B_\delta(x)\cap\mathbb{R}_-} \rho_\varepsilon(x-y)\mathrm{d}y =: a_\varepsilon(x)$ and $\|a_\varepsilon(x)\|_{L^p(\mathbb{R}_+)} \geq K\sqrt[p]{\varepsilon}$ for $\varepsilon > 0$ small enough similar as in the proof of Theorem 2.1.7. The error term in the Taylor expansion can be treated similarly.

Hence, on account of Remark 2.1.10, these terms only behave of order $\mathcal{O}(\varepsilon^{1/p})$ unless $u''(0) = 0$. Therefore, even in the simplest case, we obtain the same rate as in Theorem 2.1.7 and 2.1.8, respectively.

2.2. Convergence of the Cahn–Hilliard equation

The following result is contained in

[11] H. Abels, C. Hurm, *Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator*, J. Differential Equations, 402: 593-624, 2024

and will be discussed in Chapter 5.

2.2.1. Main assumptions

We make the following general assumptions:

(A1) The set $\Omega \subset \mathbb{R}^n$ with $n \in \{2, 3\}$ is a bounded domain with C^3-boundary. Moreover, $T > 0$ is a fixed final time and we set $\Omega_T := \Omega \times (0, T)$ as well as $\partial\Omega_T := \partial\Omega \times (0, T)$.

(A2) Let the kernel satisfy either the assumption (I3) or (J3).

(A3) The potential $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Moreover, there exist constants $C_1, C_2, C_3, \alpha > 0$ such that

$$-C_1 \leq f(s) \leq C_2(1+|s|^4), \quad |f'(s)| \leq C_3(1+|s|^3) \quad \text{and} \quad f''(s) \geq \alpha$$

for all $s \in \mathbb{R}$.

(A4) The function $f : \mathbb{R} \to \mathbb{R}$ fulfills the growth assumptions in (A3) and allows for a splitting

$$f(s) = f_1(s) + f_2(s),$$

where f_1 is convex and f_2 is concave. In addition, there exists a constant $C_4 > 0$ such that for all $s_1, s_2 \in \mathbb{R}$,

$$(f_2'(s_1) - f_2'(s_2))(s_1 - s_2) \geq -C_4|s_1 - s_2|^2.$$

Remark 2.2.1. The splitting assumption in (A4) is crucial to prove uniqueness of weak solutions to the Cahn–Hilliard equation. In fact, this is needed for both the anisotropic and isotropic Cahn–Hilliard equation. A typical choice for potentials satisfying (A3)–(A4) is the double-well potential $f_{\text{reg}}(s) = \frac{1}{4}(1-s^2)^2$ for all $s \in \mathbb{R}$ as shown in Figure 1.4.

2.2.2. Main result

We are interested in the nonlocal-to-local asymptotics for the Cahn–Hilliard equation. It is well-known that solutions to the nonlocal Cahn–Hilliard equation converge to a solution to the local Cahn–Hilliard equation as the parameter in the nonlocal operator tends towards zero. In the case of periodic boundary conditions, this has been analyzed by [98] for regular potentials and [41] for singular potentials. In the case of Neumann boundary condition, the convergence has been shown in [43] with an additional viscosity term in the Cahn–Hilliard equation and in [44] for $W^{1,1}$-kernels. However, no concrete rates of convergence have been provided in these results. Moreover, the convergence has only been shown for the isotropic Cahn–Hilliard equation, whereas the anisotropic Cahn–Hilliard equation has not been investigated so far.

It is the main goal of this chapter to prove the nonlocal-to-local convergence for the Cahn–Hilliard equation along with certain rates of convergence. In the analysis, we consider both the anisotropic and the isotropic Cahn–Hilliard equation. The proof is based on an energy method combined with the convergence results established in Section 2.1.

We then have the following convergence result:

Theorem 2.2.2. *Let the assumptions* (A1), (A3)–(A4) *and* (J3) *hold. Let* $c_0 \in H^1(\Omega)$ *and* $c_{0,\varepsilon} \in L^2(\Omega)$ *for any* $\varepsilon > 0$ *be prescribed initial data. We assume that*

$$\overline{c_{0,\varepsilon}} = \overline{c_0} \quad \text{for any } \varepsilon > 0. \tag{2.2.1}$$

Moreover, there exists a constant $C_0 > 0$ *such that*

$$\|c_{0,\varepsilon} - c_0\|_{H^{-1}_{(0)}(\Omega)} \leq C_0\sqrt{\varepsilon}. \tag{2.2.2}$$

Then, there exists a constant $C_1 > 0$ such that the weak solution $(c_\varepsilon, \mu_\varepsilon)$ to the nonlocal Cahn–Hilliard equation from Theorem 5.2.5 and the unique weak solution (c, μ) to the local Cahn–Hilliard equation 5.2.1 satisfy:

$$\sup_{t\in[0,T]} \|c_\varepsilon(t) - c(t)\|_{H^{-1}_{(0)}(\Omega)} + \|c_\varepsilon - c\|_{L^2(0,T;L^2(\Omega))} \leq C_1\sqrt{\varepsilon}. \tag{2.2.3}$$

In addition, if we assume the interaction kernel to be radially symmetric, we obtain the following convergence result:

Theorem 2.2.3. *Let the assumptions* (A1)*,* (A3)*–*(A4) *and* (I3) *hold. Let $c_0 \in H^1(\Omega)$ and $c_{0,\varepsilon} \in L^2(\Omega)$ for any $\varepsilon > 0$ be prescribed initial data. We assume that*

$$\overline{c_{0,\varepsilon}} = \overline{c_0} \quad \text{for any } \varepsilon > 0. \tag{2.2.4}$$

Moreover, there exists a constant $C_0 > 0$ such that

$$\|c_{0,\varepsilon} - c_0\|_{H^{-1}_{(0)}(\Omega)} \leq C_0\sqrt{\varepsilon}.$$

Then, there exists a constant $C_1 > 0$ such that the weak solution $(c_\varepsilon, \mu_\varepsilon)$ to the nonlocal Cahn–Hilliard equation from Theorem 5.2.5 and the unique weak solution (c, μ) to the local Cahn–Hilliard equation 5.2.3 satisfy:

$$\sup_{t\in[0,T]} \|c_\varepsilon(t) - c(t)\|_{H^{-1}_{(0)}(\Omega)} + \|c_\varepsilon - c\|_{L^2(0,T;L^2(\Omega))} \leq C_1\sqrt{\varepsilon}. \tag{2.2.5}$$

Remark 2.2.4. Observe that the assumption (2.2.1) in Theorem 2.2.2 (and (2.2.4) in Theorem 2.2.3, respectively) is just for simplicity, since it allows us to test by $\mathcal{N}(c_\varepsilon - c)$ in the proof of convergence. Indeed, one can also omit this assumption and test by $\mathcal{N}(c_\varepsilon - c - (\overline{c_\varepsilon} - \overline{c}))$. In this case, one additionally needs to assume that there exists a constant $C_2 > 0$ such that

$$|\overline{c_{0,\varepsilon}} - \overline{c_0}| \leq C_2\sqrt{\varepsilon}$$

holds, in order to provide a rate of convergence.

Remark 2.2.5. Considering the torus $\mathbb{T}^n$ instead of bounded domains, we can even show convergence of order $\mathcal{O}(\varepsilon)$. This follows by the fact that the nonlocal operator converges faster on the torus, see Theorem 2.1.5 and Remark 2.1.10.

Remark 2.2.6. In [43, Theorem 2.3] is has been shown that the sequence $(c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0, T; L^2(\Omega))$. Thus, by interpolation and Theorem 2.1.8, it holds

$$\|c_\varepsilon - c\|_{L^\infty(0,T;H^s(\Omega))} \leq C\|c_\varepsilon - c\|^\theta_{L^\infty(0,T;L^2(\Omega))}\|c_\varepsilon - c\|^{1-\theta}_{L^\infty(0,T;H^{-1}_{(0)}(\Omega))}$$

for $s = \theta - 1$ and $\theta \in (0, 1)$.

2.3. Convergence for the Allen–Cahn equation

The following result is contained in

[11] H. Abels, C. Hurm, *Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator*, J. Differential Equations, 402: 593-624, 2024

and will be discussed in Chapter 5.

2.3.1. Main assumptions

We make the following general assumptions:

(A1) Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded domain with C^3-boundary. Moreover, $T > 0$ is a fixed final time and we set $\Omega_T := \Omega \times (0,T)$ as well as $\partial\Omega_T := \partial\Omega \times (0,T)$.

(A2) Let the interaction kernel satisfy condition (I3).

(A3) $f : \mathbb{R} \to \mathbb{R}$ is a smooth double-well potential with wells of equal depth, more precisely,

$$f \in C^\infty(\mathbb{R}), \quad f(\pm 1) = f'(\pm 1) = 0, \quad f''(\pm 1) > 0, \quad f > 0 \text{ in } (-1,1)$$

and $f' < 0$ in $(-\infty, -R_0)$ as well as $f' > 0$ in (R_0, ∞) for some $R_0 \geq 1$. Finally, we assume

$$|f'(r)| \leq C(1 + |r|^3) \quad \text{for all } r \in \mathbb{R}$$

and $f'' \geq -\alpha$ for some $\alpha \geq 0$.

Remark 2.3.1. A typical example for a potential satisfying the assumptions (A3) is the classical double-well potential $f_{\mathrm{reg}}(s) := \frac{1}{4}(1 - s^2)^2$ for all $s \in \mathbb{R}$ as shown in Figure 1.4.

2.3.2. Main result

We are interested in the nonlocal-to-local convergence of the Allen–Cahn equation. Even though it is not explicitly mentioned in the literature, it is well-known that solutions to the nonlocal Allen–Cahn equation converge to a solution to the local Allen–Cahn equation as the parameter in the nonlocal operator tends to zero. Indeed, since the analysis of the Allen–Cahn equation is less challenging compared to the Cahn–Hilliard equation, this follows by similar arguments as in [41, 43, 44, 98]. However, it is not known if there is any rate of convergence for these asymptotics.

In this chapter, we prove the nonlocal-to-local convergence of the Allen–Cahn equation along with concrete rates of convergence. The proof is based on an energy method together with the convergence result in Section 2.1.

We have the following convergence result:

Theorem 2.3.2. *Let the assumptions* (A1)–(A3) *hold. Let* $c_0 \in H^1(\Omega)$ *and* $c_{0,\varepsilon} \in L^2(\Omega)$ *for* $\varepsilon > 0$ *be prescribed initial data. We assume that there exists a constant* $C_0 > 0$ *such that*

$$\|c_{0,\varepsilon} - c_0\|_{L^2(\Omega)} \leq C_0\sqrt{\varepsilon}. \tag{2.3.1}$$

Let the weak solution $c \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ *to thelocal Allen–Cahn equation satisfy* $c \in L^2(0,T;H^3(\Omega))$. *Then, there exists a constant* $C_1 > 0$ *independent of* ε *such that the weak solution* c_ε *to the nonlocal Allen–Cahn equation* (1.3.7a)-(1.3.7b) *and the strong solution to the local Allen–Cahn equation* (1.2.6a)-(1.2.6c) *satisfy:*

$$\sup_{t\in[0,T]} \|c_\varepsilon(t) - c(t)\|_{L^2(\Omega)} \leq C_1\sqrt{\varepsilon}. \tag{2.3.2}$$

2.4. Convergence for a Cahn–Hilliard tumor growth model

The following result is contained in

[88] C. Hurm, M. Moser, *Nonlocal-to-Local Convergence for a Cahn-Hilliard Tumor Growth Model*, GAMM-Mitteilungen, 48 (2025), p. e70003

and will be discussed in Chapter 6.

2.4.1. Main assumptions

We make the following general assumptions.

(A1) Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded domain with C^3-boundary. Moreover, let $T > 0$ be fixed and $\Omega_T := \Omega \times (0,T)$ as well as $\partial\Omega_T := \partial\Omega \times (0,T)$.

(A2) Let the kernel satisfy the assumptions in (I3).

(A3) $\mathcal{P}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ are non-negative constants, $\sigma_S \in L^\infty(\Omega_T)$ and $0 \le \sigma_S \le 1$ a.e. in Ω_T.

(A4) The function $h : \mathbb{R} \to [0,1]$ is of class C^2, bounded and Lipschitz continuous.

(A5) The potential $f : \mathbb{R} \to [0,\infty)$ is of class C^3 and satisfies

$$|f'(s)| \le k_0 f(s) + k_1, \tag{2.4.1}$$

$$-k_4 \le f''(s) \le k_4(1 + |s|^2), \tag{2.4.2}$$

for all $s, t \in \mathbb{R}$ and some positive constants k_i, $i = 0, \dots, 4$.

(A6) There are constants $C_1, C_2 > 0$ such that for all $s \in \mathbb{R}$

$$f(s) \ge C_1|s|^4 - C_2.$$

Remark 2.4.1. A typical example for a potential satisfying the assumptions (A5)-(A6) is the classical double-well potential $f_{\text{reg}}(s) := \frac{1}{4}(1 - s^2)^2$ for all $s \in \mathbb{R}$ as shown in Figure 1.4.

Remark 2.4.2. Observe that functions $f : \mathbb{R} \to [0,\infty)$ as in (A5) also satisfy the inequality

$$|f'(s) - f'(t)| \le k_5(1 + |s|^2 + |t|^2)|s - t| \tag{2.4.3}$$

for all $s, t \in \mathbb{R}$. This follows from the mean value theorem and (2.4.2).

2.4.2. Main result

We are interested in the nonlocal-to-local convergence of the Cahn–Hilliard tumor growth model introduced in (1.7.1). It has already been shown by the authors in [42] that solutions to a nonlocal Cahn–Hilliard tumor growth model converge to a solution to its local counterpart. We point out that this has been done in a more general setting, since they also considered chemotaxis and viscosity coefficients as well as non-constant mobilities. However, they have not provided any rates of convergence.

In the main contribution of this chapter, we intend to prove the nonlocal-to-local asymptotics for the Cahn–Hilliard tumor growth model (1.7.1) along with certain rates of convergence. The main advantage of model (1.7.1) is that its local counterpart, i.e, system (1.6.1), admits a unique solution with the desired regularity, cf. Theorem 6.2.1. This allows us to use the convergence result established in Section 2.1. Thus, we can apply an energy method and prove convergence together with rates of convergence using a Gronwall-type argument.

Indeed, we have the following result:

Theorem 2.4.3. *Let the assumptions* (A1)–(A6) *hold, let* $n = 3$ *and* $\varepsilon_0 > 0$ *be as in Theorem 6.2.2. Moreover, for the initial data* (c_0, σ_0) *to the local system* (1.6.1a)-(1.6.1e) *we assume* $c_0 \in H^3(\Omega)$ *with* $\partial_{\mathbf{n}} c_0 = 0$ *on* $\partial\Omega$ *and* $\sigma_0 \in H^1(\Omega)$ *with* $0 \le \sigma_0 \le 1$ *a.e. in* Ω. *Additionally, for* $\varepsilon \in (0, \varepsilon_0]$ *let the initial data for the nonlocal system* (1.7.1a)-(1.7.1e) *satisfy* $c_{0,\varepsilon}, \sigma_{0,\varepsilon} \in L^2(\Omega)$, $\mathcal{E}_\varepsilon(c_{0,\varepsilon}) \le C$, $\int_\Omega f(c_{0,\varepsilon})\,\mathrm{d}x \le C$ *and*

$$\|c_{0,\varepsilon} - c_0\|_{H^1(\Omega)'} + \|\sigma_{0,\varepsilon} - \sigma_0\|_{L^2(\Omega)} + |\overline{c_{0,\varepsilon}} - \overline{c_0}| \le C\sqrt{\varepsilon} \tag{2.4.4}$$

for some constant $C > 0$ *independent of* $\varepsilon \in (0, \varepsilon_0]$.

Then there exists a constant $K > 0$ *independent of* ε *such that the weak solution* $(c_\varepsilon, \sigma_\varepsilon, \mu_\varepsilon)$ *to the nonlocal model* (6.1.1a)-(6.1.1e) *for* $\varepsilon \in (0, \varepsilon_0]$ *from Theorem 6.2.2 and the strong solution* (c, σ, μ) *to the local model* (6.1.3a)-(6.1.3e) *from Theorem 6.2.1 satisfy:*

$$\sup_{t\in[0,T]} \|c_\varepsilon(t) - c(t)\|_{H^1(\Omega)'} + \|c_\varepsilon - c\|_{L^2(0,T;L^2(\Omega))} \le K\sqrt{\varepsilon}, \tag{2.4.5}$$

$$\sup_{t\in[0,T]} \|\sigma_\varepsilon(t) - \sigma(t)\|_{L^2(\Omega)} + \|\nabla\sigma_\varepsilon - \nabla\sigma\|_{L^2(0,T;L^2(\Omega)^n)} \le K\sqrt{\varepsilon}. \tag{2.4.6}$$

Remark 2.4.4. We assumed $n = 3$ in the theorem because this is also the case in [73,113]. However, note that Theorem 6.2.1, Theorem 6.2.2 and Theorem 2.4.3 should also work in the case $n = 2$ because the required embeddings are improved.

2.5. Convergence for a Cahn–Hilliard/Navier–Stokes model

The following result is contained in

[87] C. Hurm, P. Knopf, A. Poiatti, *Nonlocal-to-local convergence rates for strong solutions to a Navier-Stokes-Cahn-Hilliard system with singular potential*, Commun. in Partial Differential Equations, 49(9), 832–871, 2024

and will be discussed in Chapter 7.

2.5.1. Main assumptions

The following general assumptions are supposed to hold throughout this section.

(A1) For $n \in \{2,3\}$, we either choose Ω to be a bounded domain in $\mathbb{R}^n$ of class C^3 or we take Ω to be the torus $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$.

(A2) The density ρ, the viscosity ν and the mobility m are positive constants. For convenience, we set $\rho = \nu = m = 1$. This does not mean any loss of generality as the explicit choice of these positive constants does not have any impact on the mathematical analysis.

(A3) Let Ω be given as in (A1). Let the kernel satisfy the assumptions in (I3). If $\Omega = \mathbb{T}^n$, we further demand that for all $\varepsilon > 0$, ρ_ε is compactly supported in $(-\pi, \pi)$.

For the singular potential in the free energy functional, we make the following assumptions, which not necessarily need to hold at the same time. We will specify further which of these assumptions are actually are needed in each stated result.

(S1) The potential $f : [-1,1] \to \mathbb{R}$ exhibits the decomposition

$$f(s) = F(s) - \frac{\theta_0}{2}s^2 \quad \text{for all } s \in [-1,1]$$

with a given constant $\theta_0 > 0$. Here, $F \in C([-1,1]) \cap C^2(-1,1)$ has the properties

$$\lim_{r\to-1} F'(r) = -\infty, \quad \lim_{r\to 1} F'(r) = +\infty, \quad F''(s) \geq \theta, \quad F'(0) = 0$$

for all $s \in (-1,1)$ and a prescribed constant $\theta \in (0, \theta_0)$. Without loss of generality, we further assume $F(0) = 0$. In particular, this means that $F(s) \geq 0$ for all $s \in [-1,1]$.

For convenience, we extend f and F onto $\mathbb{R} \setminus [-1,1]$ by defining $f(s) := +\infty$ and $F(s) := +\infty$ for all $s \in \mathbb{R} \setminus [-1,1]$.

(S2) In addition to (S1), there exists $\beta > \frac{1}{2}$ such that

$$\frac{1}{F'(1-2\delta)} = O\left(\frac{1}{|\ln(\delta)|^\beta}\right), \qquad \frac{1}{|F'(-1+2\delta)|} = O\left(\frac{1}{|\ln(\delta)|^\beta}\right). \tag{2.5.1}$$

as $\delta \to 0^+$.

(S3) In addition to (S1), it holds

$$\frac{1}{F'(1-2\delta)} = O\left(\frac{1}{|\ln(\delta)|}\right), \qquad \frac{1}{F''(1-2\delta)} = O(\delta), \tag{2.5.2}$$

$$\frac{1}{|F'(-1+2\delta)|} = O\left(\frac{1}{|\ln(\delta)|}\right), \quad \frac{1}{F''(-1+2\delta)} = O(\delta). \tag{2.5.3}$$

as $\delta \to 0^+$. Moreover, there exists $\gamma_0 > 0$ such that F'' is monotonously increasing on $(-1, -1+\gamma_0]$ and on $[1-\gamma_0, 1)$.

Remark 2.5.1. We point out that the logarithmic potential (also known as the *Flory–Huggins potential*), which is given by

$$f_{\log}(s) = F_{\log}(s) - \frac{\theta_0}{2}s^2 \quad \text{for all } s \in [-1,1] \tag{2.5.4}$$

with $F_{\log}(\pm 1) = \theta \ln(2)$ and

$$F_{\log}(s) = \frac{\theta}{2}((1+s)\ln(1+s) + (1-s)\ln(1-s)) \quad \text{for all } s \in (-1,1), \tag{2.5.5}$$

satisfies all assumptions (S1)–(S3). However, the assumptions (S1)–(S3) allow for a much more general class of potentials (see, e.g., [67] for a discussion).

2.5.2. Convergence result

We are interested in the nonlocal-to-local convergence of the Model H. It has already been shown by the author in [94] that weak solutions to the nonlocal Model H convergence to the weak solution to the local Model H. Even in the case of *unmatched* densities, the authors in [15] proved the nonlocal-to-local asymptotics for weak solutions. However, in these results no concrete rates of convergence were obtained.

In the main result of this chapter, we want to prove the nonlocal-to-local convergence of Model H along with certain rates of convergence. This will be done using an energy method together with the convergence results established in Section 2.1. However, this method turns out to be very challenging from a mathematical point of view. Since the nonlocal Cahn–Hilliard equation is a second order differential equation, the regularity of the phase field variable c is lower compared to the local Cahn–Hilliard equation. Thus, the coupling with the Navier–Stokes equation leads to additional difficulties compared to the results for nonlocal-to-local convergence of the Cahn–Hilliard equation. For example, in three dimensions, we can merely expect local-in-time existence of strong solutions to the Model H as the global existence of strong solutions of the Navier–Stokes equation is a well-known open problem. Moreover, the Korteweg force $\mu\nabla c$ acting on the fluid can have low regularity. In this regard, we need higher regularity of the solutions to both the nonlocal and local system, in order to provide suitable estimates for the differences of the solutions. Even though one can prove that the solution c_ε to the nonlocal Model H stays away from the pure phases ± 1 from a certain time on, it is not clear whether this holds true uniformly in ε. This property is also referred to as *strict separation property*, see Section 7.2.1 for details. Therefore, it is not straightforward to extend some of the techniques, which are used in the local case. In particular, we need to modify the proof, in order to estimate the difference $f'(c_\varepsilon) - f'(c)$.

The main result then is the following theorem.

Theorem 2.5.2. *Suppose that the assumptions* (A1)–(A3) *and* (S1) *hold, and if* $n = 2$, *we further assume that* (S2) *holds.*

We prescribe initial data $\mathbf{v}_0 \in H^1_\sigma(\Omega)$ *and* $c_0 \in H^2(\Omega)$ *with* $\|c_0\|_{L^\infty(\Omega)} \le 1$, $|\overline{c_0}| < 1$ *and* $-\Delta c_0 + f'(c_0) \in H^1(\Omega)$. *If* Ω *is a bounded domain, we additionally assume* $\partial_{\mathbf{n}} c_0 = 0$ *a.e. on* $\partial\Omega$, *and if* $n = 3$, *we further assume that* $\|c_0\|_{L^\infty(\Omega)} \le 1 - \delta_0$ *for some* $\delta_0 \in (0,1)$. *This ensures the existence of the corresponding unique right-maximal strong solution*

$$(\mathbf{v}, p, c, \mu) : \Omega \times [0, T_\star) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

to the local Model H, which satisfies the properties (iii)–(v) *of Proposition 7.2.1.*

For any $\varepsilon > 0$*, we prescribe initial data* $\mathbf{v}_{\varepsilon,0} \in H^1_\sigma(\Omega)$ *and* $c_{\varepsilon,0} \in L^\infty(\Omega)$ *with* $|\overline{c_{\varepsilon,0}}| < 1$*,* $F'(c_{\varepsilon,0}) \in L^2(\Omega)$ *and* $F''(c_{\varepsilon,0})\nabla c_{\varepsilon,0} \in L^2(\Omega)^n$*. We further demand that there exists constants* $C_0, C_1, C_2 > 0$ *independent of* ε *such that*

$$E_\varepsilon(\mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}) \le C_0, \tag{2.5.6}$$
$$\|D\mathbf{v}_{\varepsilon,0}\|_{L^2(\Omega)^{n\times n}} + \|\nabla\mu_{\varepsilon,0}\|_{L^2(\Omega)^n} \le C_1, \tag{2.5.7}$$
$$\|\mathbf{v}_{\varepsilon,0} - \mathbf{v}_0\|_\sigma + \|c_{\varepsilon,0} - c_0 - (\overline{c_{\varepsilon,0}} - \overline{c_0})\|_* + |\overline{c_{\varepsilon,0}} - \overline{c_0}| \le C_2\varepsilon^\alpha, \tag{2.5.8}$$

where $\mu_{\varepsilon,0} := \mathcal{L}_\varepsilon c_{\varepsilon,0} + f'(c_{\varepsilon,0})$*. Here,* $\alpha = \frac{1}{2}$ *if* Ω *is a bounded domain, and* $\alpha = 1$ *if* $\Omega = \mathbb{T}^n$*. This ensures the existence of the corresponding unique right-maximal strong solution*

$$(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon) : \Omega \times [0, T_{\varepsilon,*}) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

to the nonlocal Model H associated with ε*, which satisfies the properties* (iv) *and* (v) *of Theorem 7.2.4.*

We now choose $T_0 > 0$ *as in Proposition 7.2.1 and* $T_* > 0$ *as in Theorem 7.2.4, and we set* $T_\diamond := \infty$ *if* $n = 2$ *and* $T_\diamond := \min\{T_0, T_*\}$ *if* $n = 3$*. Then, for any* $T \in (0, T_\diamond)$*, there exists a constant* $C(T) > 0$ *independent of* ε *such that*

$$\begin{aligned} &\|\mathbf{v}_\varepsilon - \mathbf{v}\|_{L^\infty(0,T;(H^1_\sigma(\Omega))')} + \|c_\varepsilon - c\|_{L^\infty(0,T;H^1(\Omega)')} \\ &\quad + \|\mathbf{v}_\varepsilon - \mathbf{v}\|_{L^2(0,T;L^2(\Omega)^n)} + \|c_\varepsilon - c\|_{L^2(0,T;L^2(\Omega))} + \int_0^T \mathcal{E}_\varepsilon(c_\varepsilon - c)\,\mathrm{d}t \le C(T)\varepsilon^\alpha \end{aligned} \tag{2.5.9}$$

for all $\varepsilon \in (0, \varepsilon_s]$*, where* ε_s *is the number introduced in Theorem 7.2.4 and* α *is the exponent from* (2.5.8)*.*

Remark 2.5.3. (a) We note that the time $T_* > 0$ as in Theorem 7.2.4 satisfies

$$\inf_{\varepsilon\in(0,\varepsilon_s]} T_{\varepsilon,*} \ge T_* > 0.$$

This follows by Theorem 7.2.4(vi) and Remark 7.2.5(b).

(b) As the convergence rates are mainly inherited from Theorem 2.1.8 and Theorem 2.1.5, we obtain a higher convergence rate if $\Omega = \mathbb{T}^n$ than in the case of Ω being a bounded domain in $\mathbb{R}^n$.

(c) We point out that assuming a strictly separated initial datum c_0 in the case $n = 3$ is necessary to prove the assertion, as the strict separation property (7.2.6) is essential. In the case $n = 2$, however, the strict separation of the initial datum c_0 does not have to be imposed as an additional assumption (see also Remark 7.2.3(a)). Moreover, it is worth mentioning that assuming strict separation of the initial data $c_{\varepsilon,0}$, $\varepsilon > 0$, of the nonlocal Model H is not necessary to prove the nonlocal-to-local convergence, not even in three dimensions. This follows by the well-posedness result and the uniform estimates established in Theorem 7.2.4.

2.6. Convergence to mean curvature flow

The following result is contained in

[12] H. Abels, C. Hurm, M. Moser, *Convergence of the Nonlocal Allen-Cahn Equation to Mean Curvature Flow*, (2024), arXiv: 2410.08596

and will be discussed in Chapter 8.

2.6.1. Main assumptions

We make the following general assumptions:

(A1) Let $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, $n \in \{2,3\}$, be the n-dimensional torus.

(A2) Let the kernel satisfy the assumptions in (I3). In addition, let ρ be compactly supported in $(-\pi,\pi)$. Finally, let $\tilde{J}_\varepsilon : \mathbb{T}^n \to [0,\infty)$ be the 2π-periodic extension of J_ε.

(A3) $f : \mathbb{R} \to \mathbb{R}$ is a smooth double-well potential with wells of equal depth, more precisely,

$$f \in C^\infty(\mathbb{R}), \quad f(\pm 1) = f'(\pm 1) = 0, \quad f''(\pm 1) > 0, \quad f > 0 \text{ in } (-1,1)$$

and $f' < 0$ in $(-\infty, -R_0)$ as well as $f' > 0$ in (R_0, ∞) for some $R_0 \geq 1$. Finally, we assume

$$|f'(r)| \leq C(1+|r|^3) \quad \text{for all } r \in \mathbb{R}$$

and $f'' \geq -\alpha$ for some $\alpha \geq 0$.

Remark 2.6.1. A typical example for a potential satisfying the assumptions (A3) is the classical double-well potential $f_{\text{reg}}(s) := \frac{1}{4}(1-s^2)^2$ for all $s \in \mathbb{R}$ as shown in Figure 1.4.

2.6.2. Convergence result

We are interested in the sharp interface limit of the nonlocal Allen–Cahn equation, i.e, we intend to analyze the limit $\eta \to 0$ for initial data close to the interface configuration. To the best of the author's knowledge, there is no result for the rigorous sharp interface limit of the nonlocal Allen–Cahn equation so far. However, let us mention the work by Alberti and Bellettini [18], where the authors prove the existence of an optimal profile in both one and N dimensions as well as the the work by Davoli and Tasso [45], where the authors provide a sharp-interface analysis via Gamma-convergence for a nonlocal and non-homogeneous diffuse-interface model.

The sharp interface limit is a very challenging task from a mathematical point of view. Referring to the methods introduced in Section 1.8, it is not clear how these apply in the nonlocal case. For instance, it is not obvious how to employ linearization techniques, since the asymptotic expansions to construct an approximate solution are very difficult. So far, there is not even a spectral estimate for the corresponding linear operator, which is crucial to derive error estimates. Furthermore, there are no results for a relative entropy in the nonlocal setting. Therefore, also this method cannot be used in that case.

In order to prove the sharp interface limit for the nonlocal Allen–Cahn equation, we want to make use of the local situation where the sharp interface limit is already well-known. In this regard, we restrict our attention to the case where the parameter ε in the nonlocal operator is very small compared to the thickness η of the interface, in order to enforce closeness to the local case. Indeed, we then use the approximate solution and the spectral estimate from the local case and combine the latter with estimate (2.1.20) from Section 2.1, in order to derive suitable error estimates.

We have the following convergence result:

Theorem 2.6.2. *Let $n \in \{2,3\}$, $T_0 > 0$ and $(\Gamma_t)_{t\in[0,T_0]}$ with $\Gamma_t \subset (-\pi,\pi)^n$ for all $t \in [0,T_0]$ be a smoothly evolving compact closed hypersurface in $\mathbb{T}^n$ satisfying mean curvature flow, i.e. $V_{\Gamma_t} = H_{\Gamma_t}$ for all $t \in [0,T_0]$, where V_{Γ_t} is the normal velocity and H_{Γ_t} the mean curvature of Γ_t. Here, Γ_t separates $\mathbb{T}^n$ into two disjoint connected domains $\Omega_t^\pm$ for all $t \in [0,T_0]$. We set $\Gamma := \bigcup_{t\in[0,T_0]} \Gamma_t \times \{t\}$ and $\Omega^\pm := \bigcup_{t\in[0,T_0]} \Omega_t^\pm \times \{t\}$. Moreover, for sufficiently small $\eta > 0$, we denote by $\Gamma(\eta)$ the tubular neighbourhood around Γ, and the tangential and normal derivatives by ∇_τ and ∂_n, respectively. Finally, let $L \in \mathbb{N}$.*

Then, there exists an $\eta_0 > 0$ and $c_\eta^A : \mathbb{T}^n \times [0,T_0] \to \mathbb{R}$ smooth for $\eta \in (0,\eta_0]$ with $\lim_{\eta\to 0} c_\eta^A = \pm 1$ uniformly on compact subsets of $\Omega^\pm$ and the following holds:

1. *Let $L \geq 3$ and consider initial values $c_{0,\eta} \in H^3(\mathbb{T}^n)$, $\eta \in (0,\eta_0]$ for (1.3.7a)-(1.3.7b) with*

$$\sup_{\eta\in(0,\eta_0]} \|c_{0,\eta}\|_\infty < \infty \quad \text{and} \quad \|c_{0,\eta} - c_\eta^A(0))\|_{L^2(\mathbb{T}^n)} \leq R\eta^{L+\frac{1}{2}}, \quad R > 0 \text{ fixed},$$

$$\|c_{0,\eta}\|_{H^1(\mathbb{T}^n)} \leq \frac{C}{\eta^2}, \quad \|c_{0,\eta}\|_{H^2(\mathbb{T}^n)} \leq \frac{C}{\eta^4} \quad \text{and} \quad \|c_{0,\eta}\|_{H^3(\mathbb{T}^n)} \leq \frac{C}{\eta^{11}}.$$

   Then there are constants $c, \eta_1 > 0$ such that if $\varepsilon = \varepsilon(\eta) \leq cR\eta^{16+L+\frac{1}{2}}$, $\eta \in (0,\eta_1]$, then the solution c_η of the nonlocal Allen-Cahn equation (1.3.7a)-(1.3.7b) from Theorem 8.3.2 satisfies for all $\eta \in (0,\eta_1]$ and $T \in (0,T_0]$, where $\bar{c}_\eta := c_\eta - c_\eta^A$,

$$\sup_{t\in[0,T_0]} \|\bar{c}_\eta(t)\|_{L^2(\mathbb{T}^n)} + \|\nabla\bar{c}_\eta\|_{L^2(\mathbb{T}^n\times(0,T)\setminus\Gamma(\eta))} \leq C\eta^{L+\frac{1}{2}},$$

$$\|\nabla_\tau\bar{c}_\eta\|_{L^2(\mathbb{T}^n\times(0,T)\cap\Gamma(\eta))} + \eta\|\partial_n\bar{c}_\eta\|_{L^2(\mathbb{T}^n\times(0,T)\cap\Gamma(\eta))} \leq C\eta^{L+\frac{1}{2}}.$$

2. *Let $L = 2$. Then the same statement holds when time T is small.*

Remark 2.6.3.

1. The c_η^A in Theorem 2.6.2 is the approximate solution from the local case as given in Theorem 8.2.2 below. Here L determines the number of terms in the expansion, cf. Theorem 8.2.2 below for the details.

2. The condition $\varepsilon = \varepsilon(\eta) \leq cR\eta^{16+L+\frac{1}{2}}$ is needed technically in our proof. We expect that our result can be extended to a broader range of choices $\varepsilon(\eta)$ as long as at least $\varepsilon = O(\eta)$. Without this condition convergence and a possible limit system seem to be unclear.

Chapter 3

Preliminaries

3.1. Basic notation and function spaces

We start by introducing some notation.

(N1) **Notation for general Banach spaces.** For any normed space X of scalar-valued functions, we denote its norm by $\|\cdot\|_X$, its dual space by X' and the duality pairing between X' and X by $\langle\cdot,\cdot\rangle_X$. Besides, if X is a Hilbert space, we write $(\cdot,\cdot)_X$ to denote the corresponding inner product.

(N2) **Lebesgue and Sobolev spaces.** For any $n \in \mathbb{N}$, let now Ω be either a bounded domain in $\mathbb{R}^n$ of class C^l, $l \in \{2,3\}$, or the torus $\mathbb{T}^n$, which accounts for periodic boundary conditions. For $1 \le p \le \infty$ and $k \in \mathbb{N}$, the standard Lebesgue spaces and Sobolev spaces defined on Ω are denoted by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, and their standard norms are denoted by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. In the case $p=2$, we use the notation $H^k(\Omega) = W^{k,2}(\Omega)$. We point out that $H^0(\Omega)$ coincides with $L^2(\Omega)$. For simplicity, we just write $(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{H^1(\Omega)}$. In the case, where Ω is the torus $\mathbb{T}^n$, we recall that $H_2^s(\mathbb{T}^n)$ for $s \in \mathbb{R}$ are the usual Bessel-Potential spaces endowed with the norm

$$\|f\|^2_{H_2^s(\mathbb{T}^n)} := \sum_{k\in\mathbb{Z}^n} (1+|k|^2)^s |\hat{f}(k)|^2.$$

It is well-known that $H_2^k(\mathbb{T}^n)$ is isomorphic to $H^k(\mathbb{T}^n)$ for all $k \in \mathbb{Z}$.

Moreover, for any interval $I \subset \mathbb{R}$, any Banach space X, $1 \le p \le \infty$ and $k \in \mathbb{N}$, we write $L^p(I;X)$, $W^{k,p}(I;X)$ and $H^k(I;X) = W^{k,2}(I;X)$ to denote the Lebesgue and Sobolev spaces of functions with values in X. The standard norms are denoted by $\|\cdot\|_{L^p(I;X)}$, $\|\cdot\|_{W^{k,p}(I;X)}$ and $\|\cdot\|_{H^k(I;X)}$, respectively. We further define

$$L^p_{\mathrm{loc}}(I;X) := \{u : I \to X \mid u \in L^p(J;X) \text{ for every compact interval } J \subset I\}$$

$$L^p_{\mathrm{uloc}}(I;X) := \left\{ u : I \to X \,\middle|\, \begin{array}{l} u \in L^p_{\mathrm{loc}}(I;X) \text{ and } \exists C>0\ \forall t \in \mathbb{R} : \\ \|u\|_{L^p(I\cap[t,t+1);X)} \le C \end{array} \right\}.$$

The spaces $W^{k,p}_{\mathrm{loc}}(I;X)$, $H^k_{\mathrm{loc}}(I;X)$, $W^{k,p}_{\mathrm{uloc}}(I;X)$, $H^k_{\mathrm{uloc}}(I;X)$ are defined analogously.

(N3) **Spaces of continuous functions.** For any interval $I \subset \mathbb{R}$ and any Banach space X, $C(I;X)$ denotes the space of continuous functions mapping from I to X and $BC(I;X)$ denotes the space of functions in $C(I;X)$, which are additionally bounded. Moreover, $C_{\mathrm{w}}(I;X)$ denotes the space of functions mapping from I to X, which are continuous on I with respect to the weak topology on X, and $BC_{\mathrm{w}}(I;X)$ denotes the space of functions in $C_{\mathrm{w}}(I;X)$, which are additionally bounded.

(N4) **Spaces of functions with zero mean.** For any $f \in H^1(\Omega)'$, its generalized spatial mean is defined as

$$\overline{f} := |\Omega|^{-1}\langle f, 1\rangle,$$

where $|\Omega|$ stands for the n-dimensional Lebesgue measure of Ω. Using this definition, we introduce the following function spaces:

$$\begin{aligned} H^{-1}_{(0)}(\Omega) &:= \{u \in H^1(\Omega)' \,:\, \overline{u} = 0\} \subset H^1(\Omega)', \\ L^2_{(0)}(\Omega) &:= \{u \in L^2(\Omega) \,:\, \overline{u} = 0\} \subset L^2(\Omega), \\ H^1_{(0)}(\Omega) &:= \{u \in H^1(\Omega) \,:\, \overline{u} = 0\} \subset H^1(\Omega). \end{aligned}$$

As closed linear subspaces of the respective Hilbert space, these spaces are also Hilbert spaces.

(N5) **Spaces of divergence-free functions.** In the following, we denote vector-valued functions by $\mathbf{u}$. If Ω is a bounded domain, we define the closed linear subspaces

$$\begin{aligned} L^2_\sigma(\Omega) &:= \overline{\{\mathbf{u} \in C_0^\infty(\Omega)^n \mid \operatorname{div}\ \mathbf{u} = 0\}}^{L^2(\Omega)^n} \subset L^2(\Omega)^n, \\ H^1_\sigma(\Omega) &:= \overline{\{\mathbf{u} \in C_0^\infty(\Omega)^n \mid \operatorname{div}\ \mathbf{u} = 0\}}^{H^1(\Omega)^n} \subset H^1(\Omega)^n. \end{aligned}$$

In the case $\Omega = \mathbb{T}^n$, the corresponding closed linear subspaces are defined as

$$\begin{aligned} L^2_\sigma(\Omega) &:= \overline{\{\mathbf{u} \in C^\infty(\Omega)^n \mid \operatorname{div}\mathbf{u} = 0 \text{ and } \overline{\mathbf{u}} = 0\}}^{L^2(\Omega)^n} \subset L^2(\Omega)^n, \\ H^1_\sigma(\Omega) &:= \overline{\{\mathbf{u} \in C^\infty(\Omega)^n \mid \operatorname{div}\mathbf{u} = 0 \text{ and } \overline{\mathbf{u}} = 0\}}^{H^1(\Omega)^n} \subset H^1(\Omega)^n. \end{aligned}$$

In both cases, Korn's and Poinacré's inequality yield

$$\|\mathbf{u}\|_{L^2(\Omega)^n} \le C\|D\mathbf{u}\|_{L^2(\Omega)^{n\times n}} \le C\|\nabla\mathbf{u}\|_{L^2(\Omega)^{n\times n}} \quad \text{for all } \mathbf{u} \in H^1_\sigma(\Omega), \tag{3.1.1}$$

where the constant $C > 0$ depends on $|\Omega|$. Hence, $\|\nabla\cdot\|_{L^2(\Omega)^{n\times n}}$ is a norm on $H^1_\sigma(\Omega)$ that is equivalent to the standard norm $\|\cdot\|_{H^1(\Omega)^n}$.

3.2. Embedding and interpolation theorems

In this section, we report some results from the theory of Sobolev spaces, namely, embedding and interpolation inequalities. First of all, we recall the following embedding results from [21, Theorem 8.9 and Theorem 8.13].

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz-domain. Let $m_1, m_2 \ge 0$ be integers, let $p_1, p_2 \in [0,\infty)$ and $\alpha \in [0,1]$. Then, the following hold true:*

1. *If* $m_1 - \frac{n}{p_1} \geq m_2 - \frac{n}{p_2}$ *and* $m_1 \geq m_2$*, then the embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow W^{m_2,p_2}(\Omega)$$

exists and is continuous. Moreover, for all $u \in W^{m_1,p_1}(\Omega)$ *the following estimate holds true:*

$$\|u\|_{W^{m_2,p_2}(\Omega)} \leq C\|u\|_{W^{m_1,p_1}(\Omega)}.$$

2. *If* $m_1 - \frac{n}{p_1} > m_2 - \frac{n}{p_2}$ *and* $m_1 > m_2$*, then the embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow\hookrightarrow W^{m_2,p_2}(\Omega)$$

exists and is continuous and compact.

3. *If* $m_1 - \frac{n}{p_1} = m_2 + \alpha$ *and* $0 < \alpha < 1$ *(i.e.,* $\alpha \neq 0, 1$*), then the embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow C^{m_2,\alpha}(\overline{\Omega})$$

exists and is continuous. More precisely, for every $u \in W^{m_1,p_1}(\Omega)$*, there exists a unique continuous function (again denoted by* u*) that coincides almost everywhere with* u *such that*

$$\|u\|_{C^{m_2,\alpha}(\overline{\Omega})} \leq C\|u\|_{W^{m_1,p_1}(\Omega)}.$$

4. *If* $m_1 - \frac{n}{p_1} > m_2 + \alpha$*, then the embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow\hookrightarrow C^{m_2,\alpha}(\overline{\Omega})$$

exists and is continuous and compact. Here, $C^{m_2,0}(\overline{\Omega}) = C^{m_2}(\overline{\Omega})$ *for* $m_2 \geq 0$*.*

Furthermore, the following interpolation inequality can be found in [105].

Theorem 3.2.2 (Gagliardo–Nirenberg)**.** *Let* $u \in L^q(\mathbb{R}^n)$ *and let* $D^m u \in L^r(\mathbb{R}^n)$ *for all* $1 \leq q$*,* $r \leq \infty$*. For the derivatives* $D^j u$*,* $0 \leq j < m$*, the following inequalities hold*

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \leq C\|D^m u\|^a_{L^r(\mathbb{R}^n)}\|u\|^{1-a}_{L^q(\mathbb{R}^n)}, \tag{3.2.1}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$$

for all a *in the interval*

$$\frac{j}{m} \leq a \leq 1$$

(the constant only depending on n, m, j, p, q, r, a*) with the following exceptional cases:*

1. *If* $j = 0$*,* $rm < n$*,* $q = +\infty$*, then we make the additional assumption that either* u *tends to zero at infinity or* $u \in L^{\tilde{q}}(\Omega)$ *for some finite* $\tilde{q} > 0$*.*

2. *If $1 < r < \infty$, and $m - j - \frac{n}{r}$ is a non-negative integer, then* (3.2.1) *holds only for a satisfying $\frac{j}{m} \leq a < 1$.*

The following inequality concerns a special case of the Gagliardo–Nirenberg inequality, namely,

$$\|u\|_{L^4(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{4}} \|Du\|_{L^2(\mathbb{R}^n)^n}^{\frac{n}{4}} \quad \text{for all } u \in H^1(\mathbb{R}^n), \tag{3.2.2}$$

where $n \in \{2, 3\}$. This inequality is also referred to as *Ladyzhenskaya's inequality.* Next, we recall Agmon's inequality which is taken from [38, Lemma 4.10].

Theorem 3.2.3 (Agmon's inequality)**.** *Let Ω be an open, bounded set of class C^l. Assume $l > \frac{n}{2}$. Let l' be an integer with $l' < \frac{n}{2}$. Then, there exists a constant depending on Ω (scale invariant), l, l' such that for any $f \in H^l(\Omega)$*

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{H^{l'}(\Omega)}^{1-t} \|f\|_{H^l(\Omega)}^t, \tag{3.2.3}$$

where

$$t = \frac{\frac{n}{2} - l'}{l - l'}.$$

In particular, in the case $n = 2$, this implies

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|f\|_{H^2(\Omega)}^{\frac{1}{2}} \quad \text{for all } f \in H^2(\Omega), \tag{3.2.4}$$

and in the case $n = 3$,

$$\|f\|_{L^\infty(\Omega)} \leq C\|f\|_{H^1(\Omega)}^{\frac{1}{2}} \|f\|_{H^2(\Omega)}^{\frac{1}{2}} \quad \text{for all } f \in H^2(\Omega). \tag{3.2.5}$$

We recall the following interpolation result, which can be found in [27, Theorem II.5.5].

Theorem 3.2.4. *Let I be an interval on $\mathbb{R}$, let Ω be an open set of $\mathbb{R}^d$, and let p_1, q_1, p_2, q_2 be four real numbers in $[1, +\infty]$. If $f \in L^{p_1}(I; L^{q_1}(\Omega)) \cap L^{p_2}(I; L^{q_2}(\Omega))$, then for all $\theta \in (0, 1)$ the function f belongs to $L^p(I; L^q(\Omega))$ for p and q defined by*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \textit{and } \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

and we have

$$\|f\|_{L^p(I;L^q(\Omega))} \leq \|f\|_{L^{p_1}(I;L^{q_1}(\Omega))}^{\theta} \|f\|_{L^{p_2}(I;L^{q_2}(\Omega))}^{1-\theta}.$$

Finally, we recall the following compactness result taken from [27, Theorem II.5.16].

Theorem 3.2.5 (Aubin-Lions-Simon)**.** *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \leq p, r \leq +\infty$. For $T > 0$, we define*

$$E_{p,r} := \Big\{ v \in L^p(0, T; B_0), \frac{\mathrm{d}v}{\mathrm{d}t} \in L^r(0, T; B_2) \Big\}.$$

1. *If $p < +\infty$, the embedding of $E_{p,r}$ in $L^p(0, T; B_1)$ is compact.*

2. *If $p = +\infty$ and if $r > 1$, the embedding of $E_{p,r}$ in $C^0([0, T]; B_1)$ is compact.*

3.3. Tools from functional analysis

Here, we collect some useful convergence results from functional analysis. In order to prove convergence of the nonlocal operators, we first show the assertion for smooth functions, since it allows us to use Taylor's theorem and derive suitable estimates. Afterwards, we conclude the proof via a density argument. Therefore, we need the following theorem:

Theorem 3.3.1 (Banach–Steinhaus). *Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be Banach spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of linear bounded operators $T_n : X \to Y$, $n \in \mathbb{N}$. Then, the sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise to a linear bounded operator $T : X \to Y$ if and only if the following conditions hold true:*

1. *The operator norm of the sequence $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, i.e., there exists a constant $C > 0$ such that*
$$\|T_n\|_{\mathcal{L}(X,Y)} \leq C \quad \text{for all } n \in \mathbb{N}.$$
2. *There exists a dense subset $D \subseteq X$ such that for all $x \in D$ the sequence $(T_n(x))_{n\in\mathbb{N}}$ converges in Y.*

A proof can be found in [81, Theorem 4.4]. Next, we recall the generalized convergence theorem of Lebesgue, which is taken from [21, Theorem 1.25].

Theorem 3.3.2 (Generalized Convergence Theorem of Lebesgue). *Let $(S, \mathcal{B}, \mu)$ be a measure space and let Y be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $f_k, f : S \to Y$ be strongly μ-measurable and let $g_k \to g$ as $k \to \infty$ in $L^1(\mu, \mathbb{R})$. Further, let $1 \leq p < \infty$. Suppose that*

$$\begin{aligned} f_k \to f \quad & \mu\text{-almost everywhere as } k \to \infty, \\ |f_k|^p \leq g_k \quad & \mu\text{-almost everywhere for all } k \in \mathbb{N}, \end{aligned}$$

Then, it follows that $f_k, f \in L^p(\mu, Y)$ and $f_k \to f$ in $L^p(\mu, Y)$ as $k \to \infty$.

Furthermore, we recall the definition and properties of Nemytskii operators, since they are useful for the treatment of nonlinear functions. The following theory can be found in [112, Chapter 10.3.4]: Let $\Omega \subset \mathbb{R}^n$ be a domain. A function

$$f : \Omega \times \mathbb{R}^m \to \mathbb{R}, \quad (x, u) \mapsto f(x, u)$$

satisfies the *Carathéodory conditions* if

$$u \mapsto f(x, u) \quad \text{is continuous for almost every } x \in \Omega$$

and

$$x \mapsto f(x, u) \quad \text{is measurable for almost every } u \in \Omega.$$

Given any function f that satisfies the Carathéodory conditions and $u : \Omega \to \mathbb{R}^m$, we define the operator

$$T(u)(x) := f(x, u(x)). \tag{3.3.1}$$

This operator is then called *Nemytskii operator*. The main result about these operators concerns the boundedness and continuity from $L^p(\Omega)$ to $L^q(\Omega)$.

Theorem 3.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let*

$$f : \Omega \times \mathbb{R}^m \to \mathbb{R}, \quad (x, u) \mapsto f(x, u)$$

satisfies the Carathéodory conditions. In addition, let $p \in (1, \infty)$ and $g \in L^q(\Omega)$ be given, where $\frac{1}{p} + \frac{1}{q} = 1$, and let f satisfy

$$|f(x, u)| \leq C|u|^{p-1} + g(x) \quad \text{for a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}^m.$$

Then, the Nemytskii operator T defined by (3.3.1) is a bounded and continuous map from $L^p(\Omega)$ to $L^q(\Omega)$.

3.4. Fourier analysis

3.4.1. Fourier series

Here, we discuss some basic facts about the Fourier analysis on the torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. In the following, we identify $\mathbb{T}^n$ with $[-\pi, \pi]^n$ and every function $f : [-\pi, \pi]^n \cong \mathbb{T}^n \to \mathbb{C}$ with its periodic extension $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$, i.e.,

$$\tilde{f}(x_1, \ldots, x_j + 2\pi, \ldots, x_n) = \tilde{f}(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and all $j \in \{1, \ldots, n\}$. In this regard, a function $f : \mathbb{T}^n \to \mathbb{C}$ is continuous (differentiable or measurable, respectively) if its extension $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ is continuous (differentiable or measurable, respectively).

For any $f \in L^1(\mathbb{T}^n)$, we define the *k-th Fourier coefficient* of f by

$$\hat{f}_k := \hat{f}(k) := \int_{\mathbb{T}^n} f(x) e^{-ik \cdot x} \, \mathrm{d}x \qquad \text{for all } k \in \mathbb{Z}^n. \tag{3.4.1}$$

Then, the *Fourier series* of f at $x \in \mathbb{T}^n$ is given by

$$\frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{ik \cdot x}. \tag{3.4.2}$$

In the following, we want to present the most important properties of Fourier series, which we use throughout this thesis. A general introduction can be found, e.g., in the book by Grafakos [82, Chapters 3 and 4]. First of all, let us mention some elementary properties:

Lemma 3.4.1. *Let $f, g \in L^1(\mathbb{T}^n)$ and let $\lambda \in \mathbb{C}$. Then, for all $k \in \mathbb{Z}^n$ we have*

1. $\widehat{f + g}(k) = \hat{f}(k) + \hat{g}(k)$,
2. $\widehat{\lambda f}(k) = \lambda \hat{f}(k)$,
3. $\hat{f}(0) = \int_{\mathbb{T}^n} f(x) \, \mathrm{d}x$,
4. $\sup_{k \in \mathbb{Z}^n} |\hat{f}(k)| \leq \|f\|_{L^1(\mathbb{T}^n)}$,
5. $\hat{\bar{f}}(k) = \overline{\hat{f}(-k)}$.

For functions $f, g \in L^1(\mathbb{T}^n)$, we define the convolution of f and g by

$$f * g(x) := \int_{\mathbb{T}^n} f(x-y)g(y)\,\mathrm{d}y \quad \text{for } x \in \mathbb{T}^n.$$

In this case, Fourier coefficients have the following property:

Lemma 3.4.2. *For all $f \in L^1(\mathbb{T}^n)$ and $g \in L^p(\mathbb{T}^n)$ with $1 \le p \le \infty$ it holds $f * g \in L^p(\mathbb{T}^n)$ as well as*

$$\|f * g\|_{L^p(\mathbb{T}^n)} \le \|f\|_{L^1(\mathbb{T}^n)} \|g\|_{L^p(\mathbb{T}^n)}.$$

Moreover, it holds

$$\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k) \quad \textit{for } k \in \mathbb{Z}^n.$$

It turns out that the convolution operator is a very useful tool in the Fourier analysis, since it allows us to represent the partial sums of the Fourier series. For $x \in \mathbb{T}^n$, $N \in \mathbb{N}$, we introduce

$$S_N f(x) := \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n, |k_1|, \ldots, |k_n| \le N} \hat{f}(k) e^{ik \cdot x}.$$

Then, it holds

$$S_N f(x) = D_N^{(n)} * f(x),$$

where $D_N^{(n)}$ denotes the *Dirichlet kernel* given by

$$D_N^{(n)}(x) = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n, |k_1|, \ldots, |k_n| \le N} e^{ik \cdot x} \quad \text{for } x \in \mathbb{T}^n.$$

Under suitable assumptions on f, the Dirichlet kernel can be used to prove the convergence of the Fourier series in suitable norms. Indeed, we have the following result:

Theorem 3.4.3. *Let $1 < p < \infty$ and $f \in L^p(\mathbb{T}^n)$. Then, $D_N^{(n)} * f$ converges to f in $L^p(\mathbb{T}^n)$ as $N \to \infty$.*

For a proof, we refer to [82, Theorem 4.1.8]. As a consequence, the sequence of projections $(S_N)_{N \in \mathbb{N}}$ is bounded, i.e., there exists a constant $C > 0$ such that

$$\sup_{N \in \mathbb{N}} \|S_N\|_{\mathcal{L}(L^p(\mathbb{T}^n), L^p(\mathbb{T}^n))} \le C \quad \text{for all } 1 < p < \infty.$$

This will be crucial for the proof of Theorem 8.3.2, since we intend to use these projections in the Galerkin ansatz. In fact, also the converse implication holds true. This follows from [82, Theorem 4.1.1]. Another important result concerns the decay of Fourier coefficients. There, we have the following:

Proposition 3.4.4 (Riemann–Lebesgue Lemma)**.** *Given a function $f \in L^1(\mathbb{T}^n)$, it holds*

$$|\hat{f}(k)| \to 0 \quad \textit{as } |k| \to \infty.$$

In the case $p = 2$, the Hilbert space structure of $L^2(\mathbb{T}^n)$ has essential consequences for the Fourier series. The following result can be found in [82, Prop. 3.2.7]:

Proposition 3.4.5. *The following are valid for $f, g \in L^2(\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$:*

1. *(Plancherel's identity)*
$$\|f\|^2_{L^2(\mathbb{T}^n)} = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)|^2. \tag{3.4.3}$$

2. *(Parseval's relation)*
$$\int_{\mathbb{T}^n} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\overline{\hat{g}(k)}. \tag{3.4.4}$$

3. *The map $f \mapsto (\hat{f}(k))_{k \in \mathbb{Z}^n}$ is an isomorphism from $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$.*

3.4.2. Fourier transform

In this section, we want to extend the theory in Section 3.4.1 to functions defined on $\mathbb{R}^n$. For any function $f \in L^1(\mathbb{R}^n)$, we define its *Fourier transform* by

$$\hat{f}(\xi) := \mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}^n.$$

We observe that the basic properties of the Fourier series also hold true in case of the Fourier transform. Indeed, we have the following:

Proposition 3.4.6. *Let $f, g \in L^1(\mathbb{R}^n)$ and let $\lambda \in \mathbb{C}$. Then, we have*

1. $\widehat{f+g} = \hat{f} + \hat{g}$,
2. $\widehat{\lambda f} = \lambda \hat{f}$,
3. $\|\hat{f}\|_\infty = \sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi)| \le \|f\|_{L^1(\mathbb{R}^n)}$,
4. $\widehat{f * g} = \hat{f}\hat{g}$,
5. *$\mathcal{F} : L^1(\mathbb{R}^n) \to C_b^0(\mathbb{R}^n)$ is a bounded linear operator.*

Further properties can be found in [82, Prop. 2.2.11]. The last property follows, e.g., from [124, Satz V.2.2]. In analogue to the Riemann–Lebesgue lemma for Fourier coefficients, there is a similar result concerning the behavior of the Fourier transform at infinity. There, we have:

Proposition 3.4.7 (Riemann–Lebesgue Lemma)**.** *For a function f in $L^1(\mathbb{R}^n)$, it holds*

$$|\hat{f}(\xi)| \to 0 \quad \text{as } |\xi| \to \infty.$$

Finally, we note that $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isomorphism and satisfies *Plancherel's identity*, i.e.,

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,\mathrm{d}x = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi \quad \text{for all } f, g \in L^2(\mathbb{R}^n).$$

More details about the Fourier transform can be found, e.g., in [82, 124].

3.5. Basic inequalities

In this section, we collect some inequalities we need throughout this thesis. In order to prove the nonlocal-to-local convergence, we intend to use an energy method where we derive differential inequalities. Thus, to close the argument, we want to apply Gronwall's inequality. The following version can be found in [121, Lemma 2.7].

Lemma 3.5.1 (Generalized Gronwall's inequality)**.** *Suppose $\psi : [0,T] \to \mathbb{R}$ satisfies*

$$\psi(t) \leq \alpha(t) + \int_0^t \beta(s)\psi(s)\mathrm{d}s \quad \textit{for all } t \in [0,T],$$

with $\alpha(t) \in \mathbb{R}$ and $\beta(t) \geq 0$. Then,

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)\mathrm{d}r\right)\mathrm{d}s \quad \textit{for all } t \in [0,T].$$

Moreover, if in addition $\alpha(s) \leq \alpha(t)$ for $s \leq t$, then

$$\psi(t) \leq \alpha(t)\exp\left(\int_0^t \beta(s)\mathrm{d}s\right) \quad \textit{for all } t \in [0,T].$$

The next inequality is very useful in the context of measure theory. Indeed, we use a measure theoretical ansatz, in order to control the difference of the singular potentials in Chapter 7. The idea is based on the following inequality:

Lemma 3.5.2 (Chebyshev–Markov inequality)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary measure space. For every measurable numerical function f on Ω and every pair of real numbers $p > 0$ and $\alpha > 0$ the* Chebyshev–Markov inequality

$$\mu(\{|f| \geq \alpha\}) \leq \frac{1}{\alpha^p}\int_\Omega |f|^p \, d\mu \tag{3.5.1}$$

holds.

A proof can be found in the book of Bauer, cf. [23, Lemma 20.1]. Next, we recall Hardy's inequality taken from [85, §327].

Lemma 3.5.3 (Hardy's inequality)**.** *If $p > 1$, $f : (0,\infty) \to [0,\infty)$ locally integrable and $F(x) := \int_0^x f(t)\,\mathrm{d}t$, then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p \,\mathrm{d}x.$$

The next inequalities concern a nonlocal version of Ehrling's inequality. The following lemma is taken from [48, Lemma C.3].

Lemma 3.5.4. *Let $\varepsilon > 0$ and let J_ε satisfy assumption* (I3)*. For every $\gamma > 0$ there exist constants $C_\gamma > 0$ and $\varepsilon_\gamma > 0$ with the following properties:*

1. *For every sequence* $(f_\varepsilon)_{\varepsilon>0} \subset H^1(\Omega)$*, there holds*

$$\begin{aligned}\|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{H^1(\Omega)} \le \gamma \int_\Omega \int_\Omega J_{\varepsilon_1}(x,y)\big|\nabla f_{\varepsilon_1}(x) - \nabla f_{\varepsilon_2}(y)\big|^2 \,\mathrm{d}y\mathrm{d}x \\ + \gamma \int_\Omega \int_\Omega J_{\varepsilon_2}(x,y)\big|\nabla f_{\varepsilon_1}(x) - \nabla f_{\varepsilon_2}(y)\big|^2 \,\mathrm{d}y\mathrm{d}x \\ + C_\gamma \|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.5.2}$$

2. *For every sequence* $(f_\varepsilon)_{\varepsilon>0} \subset L^2(\Omega)$*, there holds*

$$\|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{L^2(\Omega)} \le \gamma \mathcal{E}_{\varepsilon_1}(f_{\varepsilon_1}) + \gamma \mathcal{E}_{\varepsilon_2}(f_{\varepsilon_2}) + C_\gamma \|f_{\varepsilon_1} - f_{\varepsilon_2}\|^2_{H^1(\Omega)'}. \tag{3.5.3}$$

In Lemma 3.5.4 it is not specified how the constant C_γ depends on γ. However, we need this dependence, in order to derive uniform estimates in Section 8.4. To this end, we prove the following theorem.

Theorem 3.5.5 (Nonlocal Ehrling Inequality)**.** *Let* $n \in \mathbb{N}$*,* $R \ge 1$*,* $0 < \varepsilon \le \frac{1}{R}$ *and* $\mathcal{E}_\varepsilon$ *be defined as in* (1.3.1) *with* J_ε *satisfying* (I3)*. Then for all* $u \in L^2(\mathbb{T}^n)$ *it holds with* $C > 0$ *independent of* R, ε, u*,*

$$\|u\|^2_{L^2(\mathbb{T}^n)} \le \frac{C}{R^2}\mathcal{E}_\varepsilon(u) + CR^2\|u\|^2_{H^{-1}(\mathbb{T}^n)}.$$

Proof. Let $R \ge 1$ and $0 < \varepsilon \le \frac{1}{R}$ be arbitrary. As already pointed out in Remark 2.1.2, we need to consider a 2π-periodic extension $\tilde{J}_\varepsilon : \mathbb{T}^n \to \mathbb{R}$ of J_ε. In this regard, the nonlocal operator is then given by

$$\begin{aligned}\mathcal{L}_\varepsilon u(x) &= \int_{\mathbb{T}^n} \tilde{J}_\varepsilon(x-y)\,(u(x) - u(y))\; dy \\ &= \int_{\mathbb{R}^n} J_\varepsilon(x-y)\,(u(x) - u(y))\; dy \quad \text{for all } x \in \mathbb{T}^n,\, u \in L^2(\mathbb{T}^n).\end{aligned}$$

Exploiting the symmetry of the interaction kernel J_ε, it follows

$$2\mathcal{E}_\varepsilon(u) = \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon u(x)u(x)\,\mathrm{d}x.$$

Then, using Parseval's relation (3.4.4), the Convolution Theorem 3.4.2 and the definition of J_ε, we obtain

$$\begin{aligned}\mathcal{E}_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{T}^n} \mathcal{L}_\varepsilon u(x)u(x)\,\mathrm{d}x \\ &= \frac{1}{2(2\pi)^n}\sum_{k\in\mathbb{Z}^n} \widehat{\mathcal{L}_\varepsilon u}(k)\overline{\widehat{u}(k)} = \frac{1}{2(2\pi)^n}\sum_{k\in\mathbb{Z}^n}\left(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k)\right)|\hat{u}(k)|^2\end{aligned}$$

for all $u \in L^2(\mathbb{T}^n)$. Let us note that $\widehat{J_\varepsilon}$ is indeed well-defined, since the interaction kernel is of class $W^{1,1}$. Observe that the interaction kernel J_ε is compactly supported in $(-\pi,\pi)^n$, cf. Remark 2.1.2. Therefore,

$$\widehat{J_\varepsilon}(k) = \int_{\mathbb{T}^n} \tilde{J}_\varepsilon(z)e^{-ik\cdot z}\,\mathrm{d}z = \int_{\mathbb{R}^n} J_\varepsilon(z)e^{-ik\cdot z}\,\mathrm{d}z = \mathcal{F}[J_\varepsilon](k) \tag{3.5.4}$$

for all $k \in \mathbb{Z}^n$. For the following, we define the auxiliary function

$$\Psi : \mathbb{R}^n \to \mathbb{C} : \xi \mapsto \frac{1}{2(2\pi)^n}\left(\widehat{J}(0) - \mathcal{F}[J](\xi)\right).$$

Due to (3.5.4) and since $\mathcal{F}[J]$ is a function on $\mathbb{R}^n$, it follows by standard Fourier analysis that $\Psi \in C^\infty(\mathbb{R}^n)$ and Ψ is $\mathbb{R}$-valued because J is even. Moreover, a scaling argument yields

$$\frac{1}{2(2\pi)^n}\left(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k)\right) = \frac{1}{\varepsilon^2}\Psi(k\varepsilon) \quad \text{for all } k \in \mathbb{Z}^n,\ 0 < \varepsilon \leq \frac{1}{R}.$$

We need to derive more properties of Ψ. It holds $\Psi \geq 0$ because of

$$|\mathcal{F}[J](\xi)| \leq \int_{\mathbb{R}^n} J(x)\,\mathrm{d}x = \widehat{J}(0) \quad \text{for all } \xi \in \mathbb{R}^n.$$

Moreover, the following contradiction argument shows $\Psi(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. We conversely assume that there exists some $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\Psi(\xi) = 0$. In particular, this also means that $\mathrm{Re}\Psi(\xi) = 0$. Thus, it holds

$$\begin{aligned} 0 = \mathrm{Re}\left(\widehat{J}(0) - \mathcal{F}[J](\xi)\right) &= \mathrm{Re}\int_{\mathbb{R}^n} J(x)(1 - e^{-i\xi\cdot x})\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} J(x)(1 - \cos(\xi \cdot x))\,\mathrm{d}x, \end{aligned}$$

which implies that $J(x)(1 - \cos(\xi \cdot x)) = 0$ for almost all $x \in \mathbb{R}^n$, since the integrand is non-negative. However, the properties of J then yield $1 - \cos(\xi \cdot x) = 0$ for almost all $x \in \mathbb{R}^n$ and thus $\xi = 0$, which contradicts our choice of $\xi \in \mathbb{R}^n \setminus \{0\}$. This shows $\Psi(\xi) > 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. For the gradient we obtain

$$\nabla\Psi(\xi) = -i\frac{1}{2}\int_{\mathbb{R}^n} J(x)e^{-ix\cdot\xi}x\,\mathrm{d}x.$$

In particular, this implies $\nabla\Psi(0) = 0$ since the interaction kernel J is even. Moreover, we have

$$D^2\Psi(\xi)_{k,j} = \frac{1}{2}\int_{\mathbb{R}^n} J(x)e^{-ix\cdot\xi}x_k x_j\,\mathrm{d}x,$$

and therefore $D^2\Psi(0) = I$ due to the properties of J. More precisely, the radial symmetry and normalization of ρ imply

$$D^2\Psi(0)_{k,j} = \frac{1}{2}\int_{\mathbb{R}^n} J(x)x_k x_j\,\mathrm{d}x = 0 \quad \text{if } k \neq j$$

and if $k = j$,

$$D^2\Psi(0)_{k,k} = \frac{1}{2}\int_{\mathbb{R}^n} J(x)x_k^2\,\mathrm{d}x = \frac{1}{2n}\int_{\mathbb{R}^n} J(x)|x|^2\,\mathrm{d}x = \frac{1}{2n}\int_{\mathbb{R}^n} \rho(x)\,\mathrm{d}x = 1.$$

Hence, a Taylor expansion around 0 shows that Ψ has quadratic growth in a small neighbourhood around 0. In particular, a compactness argument yields

$$\Psi(\xi) \geq c_0|\xi|^2 \quad \text{for all } |\xi| \leq 1,$$

where $c_0 > 0$ is a constant independent of R, ε, u. Finally, due to the Riemann-Lebesgue Lemma 3.4.4, we have $|\mathcal{F}[J](\xi)| \to 0$ for $|\xi| \to \infty$, which implies that $\Psi(\xi) \to \frac{1}{2(2\pi)^n}\widehat{J}(0)$ for $|\xi| \to \infty$. In particular, by continuity it holds

$$\Psi(\xi) \geq c_1 \quad \text{for all } |\xi| \geq 1,$$

where $c_1 > 0$ is a constant independent of R, ε, u.

Finally, we can prove the main assertion. Due to the arguments in the steps before, we have

$$\begin{aligned}
\mathcal{E}_\varepsilon(u) &= \sum_{k\in\mathbb{Z}^n} \frac{\Psi(k\varepsilon)}{\varepsilon^2}|\hat{u}(k)|^2 \\
&\geq \sum_{k\in\mathbb{Z}^n, |k|\leq R} c_0|k|^2|\hat{u}(k)|^2 + \sum_{k\in\mathbb{Z}^n, |k|\geq\frac{1}{\varepsilon}} \frac{c_1}{\varepsilon^2}|\hat{u}(k)|^2 + \sum_{k\in\mathbb{Z}^n, \frac{1}{\varepsilon}>|k|>R} c_0|k|^2|\hat{u}(k)|^2 \\
&\geq \sum_{k\in\mathbb{Z}^n, |k|\leq R} c_0|k|^2|\hat{u}(k)|^2 + c\sum_{k\in\mathbb{Z}^n, |k|>R} R^2|\hat{u}(k)|^2 \\
&\geq c\sum_{k\in\mathbb{Z}^n, |k|>R} R^2|\hat{u}(k)|^2 = c\sum_{k\in\mathbb{Z}^n} R^2|\hat{u}(k)|^2 - c\sum_{k\in\mathbb{Z}^n, |k|\leq R} R^2|\hat{u}(k)|^2,
\end{aligned}$$

where $c := \min\{c_0, c_1\} > 0$. Noting that $R^2 \leq \frac{2R^4}{1+|k|^2}$ for all $k \in \mathbb{Z}^n$ with $|k| \leq R$, we obtain

$$\begin{aligned}
\mathcal{E}_\varepsilon(u) &\geq c\sum_{k\in\mathbb{Z}^n} R^2|\hat{u}(k)|^2 - c\sum_{k\in\mathbb{Z}^n} \frac{2R^4}{1+|k|^2}|\hat{u}(k)|^2 \\
&= (2\pi)^n cR^2\|u\|^2_{L^2(\mathbb{T}^n)} - 2cR^4\|u\|^2_{H^{-1}(\mathbb{T}^n)},
\end{aligned}$$

and thus the assertion follows. □

3.6. Operator theory

3.6.1. The Laplace operator

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with Lipschitz-boundary. We consider the *Laplace equation* with homogeneous Neumann boundary condition, i.e., given $f \in L^2_{(0)}(\Omega)$,

$$\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
\partial_{\mathbf{n}} u &= 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{3.6.1}$$

It is well-known that there exists a unique solution $u \in H^1_{(0)}(\Omega) \cap L^2(\Omega)$, which solves (3.6.1) in a weak sense, cf. [27, Theorem III.4.3]. This can be shown using the Lax–Milgram theorem applied to the bilinear form

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \quad \text{for all } u, v \in H^1_{(0)}(\Omega).$$

Since the bilinear form a is continuous on $H^1_{(0)}(\Omega) \times H^1_{(0)}(\Omega)$, we can now define an operator $\mathcal{A} : H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ via

$$\langle \mathcal{A}u, v\rangle_{H^1_{(0)}(\Omega)} := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \quad \text{for all } u, v \in H^1_{(0)}(\Omega).$$

The operator $\mathcal{A}$ is then called the negative *Neumann Laplace operator*. By means of the Lax–Milgram theorem, we can prove that $\mathcal{A}$ is a continuous linear isomorphism. We denote the inverse of $\mathcal{A}$, which is a bounded linear operator, by

$$\mathcal{N} = \mathcal{A}^{-1} : H^{-1}_{(0)}(\Omega) \to H^1_{(0)}(\Omega). \tag{3.6.2}$$

For any $g, h \in H^{-1}_{(0)}(\Omega)$, we set

$$(g, h)_* := (\nabla \mathcal{N} g, \nabla \mathcal{N} h), \quad \|g\|_* := \|\nabla \mathcal{N} g\|_{L^2(\Omega)^n}. \tag{3.6.3}$$

This defines a bilinear form $(\cdot,\cdot)_*$ which is an inner product on the Hilbert space $H^{-1}_{(0)}(\Omega)$. Its induced norm $\|\cdot\|_*$ is equivalent to the standard norm on this space. Moreover, due to elliptic regularity theory, there exists a constant $C > 0$ such that for all $g \in L^2_{(0)}(\Omega)$,

$$\|\mathcal{N} g\|_{H^2(\Omega)} \le C \|g\|_{L^2(\Omega)}. \tag{3.6.4}$$

We further point out that the mapping $g \mapsto (\|g - \overline{g}\|_*^2 + |\overline{g}|^2)^{\frac{1}{2}}$ defines a norm on $H^1(\Omega)'$ that is equivalent to the standard operator norm on this space.

If Ω is the torus $\mathbb{T}^n$, we impose periodic boundary conditions in the Laplace equation (3.6.1). We note that in this setting the aforementioned properties still hold true.

3.6.2. The Stokes operator

We use the same notation as in Section 3.1. Moreover, let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with C^2-boundary. We start this section by introducing the homogeneous *Stokes problem* with Dirichlet boundary conditions, i.e., given $\mathbf{f} \in (H^1_\sigma(\Omega))'$,

$$\begin{aligned} -\Delta \mathbf{v} + \nabla p &= \mathbf{f}, && \text{in } \Omega, \\ \operatorname{div}(\mathbf{v}) &= 0, && \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0}, && \text{on } \partial\Omega. \end{aligned} \tag{3.6.5}$$

It is well-known that there exists a unique solution $(\mathbf{v}, p) \in H^1_\sigma(\Omega) \times L^2_{(0)}(\Omega)$ to (3.6.5), cf. [27, Theorem IV.5.1]. The existence of $\mathbf{v}$ can be shown by means of the Lax–Milgram theorem applied to the bilinear form

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, \mathrm{d}x \quad \text{for all } \mathbf{u}, \mathbf{v} \in H^1_\sigma(\Omega).$$

The proof of the existence of the pressure p, however, is more technical. Therefore, we only refer to [27, Theorem IV.2.3] for details. Since the bilinear form a is continuous on $H^1_\sigma(\Omega) \times H^1_\sigma(\Omega)$, this allows us to define an operator $A_{S,w} : H^1_\sigma(\Omega) \to (H^1_\sigma(\Omega))'$ via

$$\langle A_{S,w} \mathbf{u}, \mathbf{v} \rangle_{H^1_\sigma(\Omega)} := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, \mathrm{d}x \quad \text{for all } \mathbf{u}, \mathbf{v} \in H^1_\sigma(\Omega). \tag{3.6.6}$$

The operator $A_{S,w}$ is then called the *weak Stokes operator*.

For the study of the Navier–Stokes equation, it is important to develop an L^2-theory for the Stokes operator $A_{S,w}$. To this end, let $\mathbf{P}_\sigma : L^2(\Omega)^n \to L^2(\Omega)^n$ be the orthogonal projection onto $L^2_\sigma(\Omega)$. Then, $\mathbf{P}_\sigma$ is called the *Leray–Helmholtz projection*. We can now consider the Stokes operator as operator $A_S : \mathcal{D}(A_S) \subseteq L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ with domain $\mathcal{D}(A_S) := H^1_\sigma(\Omega) \cap H^2(\Omega)^n$. In this regard, we have the following definition

$$A_S\mathbf{v} = -\mathbf{P}_\sigma(\Delta\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathcal{D}(A_S).$$

It follows by the Lax–Milgram theorem that A_S is a continuous linear isomorphism. Moreover, there exists a constant $C > 0$ such that

$$\frac{1}{C}\|\mathbf{v}\|_{H^2(\Omega)^n} \le \|A_S\mathbf{v}\|_{L^2_\sigma(\Omega)} \le C\|\mathbf{v}\|_{H^2(\Omega)^n} \quad \text{for all } \mathbf{v} \in \mathcal{D}(A_S). \tag{3.6.7}$$

It is well-known (cf. [118]) that A_S is a positive self-adjoint operator. Moreover, A_S is invertible and its inverse $A_S^{-1} : \mathcal{D}(A_S^{-1}) \to L^2_\sigma(\Omega)$ with domain $\mathcal{D}(A_S^{-1}) = L^2_\sigma(\Omega)$ is again positive self-adjoint. For any $\mathbf{v}, \mathbf{w} \in (H^1_\sigma(\Omega))'$, we set

$$(\mathbf{v}, \mathbf{w})_\sigma := (\nabla A_S^{-1}\mathbf{v}, \nabla A_S^{-1}\mathbf{w}), \quad \|\mathbf{v}\|_\sigma := \|\nabla A_S^{-1}\mathbf{v}\|_{L^2(\Omega)^{n\times n}}.$$

This defines a bilinear form $(\cdot,\cdot)_\sigma$, which is an inner product on the Hilbert space $(H^1_\sigma(\Omega))'$. Its induced norm $\|\cdot\|_\sigma$ is equivalent to the standard operator norm on this space. In particular, due to Poincaré's inequality, there exists a constant $C_{S,1} > 0$ such that for all $\mathbf{u} \in (H^1_\sigma(\Omega))'$, it holds

$$\|A_S^{-1}\mathbf{u}\|_{H^1(\Omega)^n} \le C_{S,1}\|\mathbf{u}\|_\sigma. \tag{3.6.8}$$

Moreover, due to regularity theory for the Stokes operator, there exists a constant $C_{S,2} > 0$ such that for all $\mathbf{u} \in L^2_\sigma(\Omega)$, it holds

$$\|A_S^{-1}\mathbf{u}\|_{H^2(\Omega)^n} \le C_{S,2}\|\mathbf{u}\|_{L^2_\sigma(\Omega)}. \tag{3.6.9}$$

For a more detailed introduction into the theory of the Stokes operator, we refer to [27,118].

Instead of Dirichlet boundary conditions, one can also impose periodic boundary conditions for the Stokes problem (3.6.5). In this setting, the associated Stokes problem reads as

$$\begin{aligned} -\Delta\mathbf{v} + \nabla p &= \mathbf{f} && \text{in } \mathbb{T}^n, \\ \operatorname{div}(\mathbf{v}) &= 0 && \text{in } \mathbb{T}^n, \end{aligned} \tag{3.6.10}$$

given $\mathbf{f} \in (H^1(\mathbb{T}^n)^n)'$ or $\mathbf{f} \in L^2_{(0)}(\mathbb{T}^n)^n$. This problem can be solved explicitly by means of Fourier series, cf. [120, Chapter 2.2]. Similarly as before, we can now introduce the *Stokes operator associated to periodic boundary conditions*. The operator $A_S : \mathcal{D}(A_S) \to L^2_\sigma(\mathbb{T}^n)$ with domain $\mathcal{D}(A_S) := H^2(\mathbb{T}^n)^n \cap L^2_\sigma(\mathbb{T}^n)$ is then defined by

$$A_S\mathbf{v} = -\Delta\mathbf{v} \quad \text{for all } \mathbf{v} \in \mathcal{D}(A_S).$$

Recall from (N5) that functions in $L^2_\sigma(\mathbb{T}^n)$ have vanishing mean value. Thus, the operator A_S becomes an isomorphism if $\mathcal{D}(A_S)$ is endowed with the norm on $L^2_\sigma(\mathbb{T}^n)$. Furthermore, there exists a constant $C > 0$ such that

$$\frac{1}{C}\|\mathbf{v}\|_{H^2(\mathbb{T}^n)^n} \le \|A_S\mathbf{v}\|_{L^2_\sigma(\mathbb{T}^n)} \le C\|\mathbf{v}\|_{H^2(\mathbb{T}^n)^n} \quad \text{for all } \mathbf{v} \in \mathcal{D}(A_S). \tag{3.6.11}$$

For details, we refer to [120, Chapter 2.2]. Due to the vanishing mean value, cf. (N5), Poincaré's inequality implies that there exists a constant $C_{S,1} > 0$ such that for all $\mathbf{u} \in (H^1(\mathbb{T}^n)^n)'$, it holds

$$\|A_S^{-1}\mathbf{u}\|_{H^1(\mathbb{T}^n)^n} \leq C_{S,1}\|\mathbf{u}\|_\sigma. \tag{3.6.12}$$

Finally, due to regularity theory for the Stokes operator, there exists a constant $C_{S,2} > 0$ such that for all $\mathbf{u} \in L^2_\sigma(\mathbb{T}^n)$, it holds

$$\|A_S^{-1}\mathbf{u}\|_{H^2(\mathbb{T}^n)^n} \leq C_{S,2}\|\mathbf{u}\|_{L^2_\sigma(\mathbb{T}^n)}. \tag{3.6.13}$$

Part I

Strong convergence of the nonlocal operator

Chapter 4

Strong convergence of the nonlocal operator

4.1. Introduction

In this chapter, we want to investigate nonlocal operators of the form

$$\begin{aligned}\mathcal{L}u(x) :&= \text{P.V.} \int_{\Omega} J(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \\ &= \lim_{r\to 0} \int_{\Omega\cap\{|x-y|\geq r\}} J(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y,\end{aligned} \tag{4.1.1}$$

for all $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is either a sufficiently smooth bounded domain or the torus $\Omega = \mathbb{T}^n$ and

$$J : \mathbb{R}^n \setminus \{0\} \to [0,\infty), \quad J(x) := \frac{\rho(x)}{|x|^2} \tag{4.1.2}$$

is a prescribed interaction kernel. The function $\rho : \mathbb{R}^n \to [0,\infty)$ is assumed to be even and has to satisfy certain integrability conditions. If ρ is additionally assumed to be radially symmetric, the conditions in (I) have to be satisfied. In this case, the corresponding nonlocal operator $\mathcal{L}$ exhibits an *isotropic* behavior. Otherwise, the conditions in (J) have to be satisfied. Then, the nonlocal operator exhibits an *anisotropic* behavior.

The main goal of this chapter is to study the nonlocal-to-local convergence of operators as introduced in (4.1.1). In particular, we want to prove that the nonlocal operator converges to a local differential operator as ρ converges to a multiple of the δ-distribution. To this end, we introduce the functions

$$\rho_\varepsilon : \mathbb{R} \to [0,\infty), \qquad \rho_\varepsilon(x) := \varepsilon^{-n}\rho\left(\frac{x}{\varepsilon}\right), \tag{4.1.3}$$

$$J_\varepsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}, \quad J_\varepsilon(x) := \frac{\rho_\varepsilon(x)}{|x|^2} \tag{4.1.4}$$

for all $\varepsilon > 0$ and all functions ρ satisfying either (I) or (J). Hence, by construction, the sequence $(\rho_\varepsilon)_{\varepsilon>0}$ is a *Dirac sequence*, i.e., it converges to a multiple of the δ-distribution as $\varepsilon \to 0$, see Lemma 4.2.1. For any $\varepsilon > 0$, we now define the corresponding nonlocal operator by

$$\mathcal{L}_\varepsilon u(x) := \text{P.V.} \int_\Omega J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \tag{4.1.5}$$

for all $x \in \Omega$. We want to prove that in the limit $\varepsilon \to 0$, nonlocal operators as in (4.1.5) converge to a local differential operator which we motivate in the following lines.

For isotropic kernels, it is well-known that the corresponding nonlocal operator approaches the negative Neumann Laplacian $-\Delta$ as $\varepsilon \to 0$. For details, we refer to, e.g., [41,43,44,98].

For anisotropic kernels, this has not been investigated, so far. Based on the works by Ponce, cf. [109,110], it has been shown that the corresponding quadratic form satisfies

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon^{-n} &\int_\Omega \int_\Omega \frac{\rho(\frac{x-y}{\varepsilon})}{|x-y|^2} |u(x)-u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \left(\int_\Omega \left| Du(x) \cdot \frac{z}{|z|^2} \right|^2 \mathrm{d}x \right) \rho(z)\,\mathrm{d}z\end{aligned} \tag{4.1.6}$$

for all $u \in H^1(\Omega)$. We observe that the term on the right-hand side can be written as

$$\left| Du(x) \cdot \frac{z}{|z|^2} \right|^2 = \frac{1}{|z|^2} \sum_{i,j=1}^n \partial_{x_i} u(x) \partial_{x_j} u(x) z_i z_j.$$

Hence, it holds

$$\begin{aligned}\int_{\mathbb{R}^n} \left(\int_\Omega \left| Du(x) \cdot \frac{z}{|z|^2} \right|^2 \mathrm{d}x \right) \rho(z)\,\mathrm{d}z &= \sum_{i,j=1}^n \int_\Omega \left(\int_{\mathbb{R}^n} \frac{\rho(z)}{|z|^2} z_i z_j \,\mathrm{d}z \right) \partial_{x_i} u(x) \partial_{x_j} u(x)\,\mathrm{d}x \\ &= \int_\Omega \nabla u(x)^T M \nabla u(x)\,\mathrm{d}x,\end{aligned} \tag{4.1.7}$$

where we defined the matrix $M \in \mathbb{R}^{n\times n}$ by

$$M_{ij} := \int_{\mathbb{R}^n} \frac{\rho(z)}{|z|^2} z_i z_j \,\mathrm{d}z \qquad \text{for all } i,j \in \{1,\dots,n\}.$$

Noticing that by symmetry of the interaction kernel we have

$$\varepsilon^{-n} \int_\Omega \int_\Omega \frac{\rho(\frac{x-y}{\varepsilon})}{|x-y|^2} |u(x)-u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2} \int_\Omega \mathcal{L}_\varepsilon u(x) u(x)\,\mathrm{d}x, \tag{4.1.8}$$

this motivates to study local differential operators of the type

$$\mathcal{L}u(x) := -\operatorname{div}(M\nabla u)(x) = -\sum_{i,j=1}^n M_{ij}\,\partial_{x_j} \partial_{x_k} u(x) \quad \text{for all } x \in \Omega, \tag{4.1.9}$$

where now $M \in \mathbb{R}^{n\times n}$ is a symmetric matrix with entries

$$M_{ij} := \frac{1}{2} \int_{\mathbb{R}^n} J(z)\, z_i\, z_j \,\mathrm{d}z \quad \text{for all } i,j \in \{1,\dots,n\}. \tag{4.1.10}$$

Note that the additional scaling in M follows by the convergence (4.1.6), formula (4.1.8) and integration by parts in (4.1.7) – provided that u satisfies the boundary condition $M\nabla u \cdot \mathbf{n}_{\partial\Omega} = 0$ on $\partial\Omega$.

We intend to prove the convergence of the nonlocal operator towards a local differential operator as in (4.1.9) as $\varepsilon \to 0$. The main novelty in these results is that we provide certain rates of convergence. These will be measured with respect to the L^p-norm, where $p \in [1, \infty)$. In contrast to many works in the literature, we do not only consider $W^{1,1}$-kernels but also more singular kernels as specified in Section 2.1.

The structure of this section is the following: The first part concerns the well-definedness of the nonlocal operator. In the analysis, we consider both the full space and C^2-domains. We prove that the nonlocal operator is well-defined and linear for smooth functions and conclude that the operator can be extended to a bounded operator for functions in $W^{2,p}(\mathbb{R}^n)$ and $W^{2,p}(\Omega)$, respectively. In the remaining subsections, we then show the convergence of the nonlocal operator. First, we investigate the case, where Ω is the torus $\mathbb{T}^n$. There, we allow for both even and radially symmetric kernels which are more singular. We intend to prove strong convergence of the nonlocal operator with respect to the L^2-topology. The proof is based on Fourier analysis. In the last part, we show convergence of the nonlocal operator on smooth bounded domains in $\mathbb{R}^n$. There, we again consider both even and radially symmetric kernels. However, in contrast to the torus, we also prove convergence with respect to the L^p-topology, where $p \in [1, \infty)$. We first show convergence on the full space $\mathbb{R}^n$. Then, we consider a bent half space and complete the argument via localization.

This chapter is based on the work in:

[11] H. Abels, C. Hurm, *Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator*, J. Differential Equations, 402: 593-624, 2024.

We point out that this section is more general than the contribution in [11]. Therein, the authors proved convergence of nonlocal operators with radially symmetric kernels of $W^{1,1}$-regularity in the L^2-topology. In this section, however, we also allow for even kernels which can be more singular. Moreover, the convergence is shown in the L^p-topology, where $p \in [1, \infty)$.

4.2. Well-Definedness of the nonlocal operator

We start this section with a preliminary result concerning the properties of the sequence $(\rho_\varepsilon)_{\varepsilon>0}$ as introduced in (4.1.3). More precisely, we prove integrability and show that ρ_ε concentrates around the origin as ε tends towards zero. Throughout this section, we always assume the regularity of the kernel to be minimal.

Lemma 4.2.1. *Suppose that Assumption* (J1) *or* (I1) *holds.*

(a) *For any $\varepsilon > 0$, it holds $\rho_\varepsilon \in L^1(\mathbb{R}^n)$ with*

$$\|\rho_\varepsilon\|_{L^1(\mathbb{R}^n)} = \|\rho\|_{L^1(\mathbb{R}^n)} < \infty. \tag{4.2.1}$$

(b) *For any $\delta > 0$, it holds that*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\delta(0)} \rho_\varepsilon(x) \, \mathrm{d}x = 0. \tag{4.2.2}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\rho \in L^1(\mathbb{R}^n)$ and the identity in (4.2.1) follows from the change of variables $x \mapsto \frac{x}{\varepsilon}$. Thus, assertion (a) is verified.

Let now $\delta > 0$ be arbitrary. The change of variables $x \mapsto \frac{x}{\varepsilon}$ yields

$$\int_{\mathbb{R}^n \setminus B_\delta(0)} \rho_\varepsilon(x) = \int_{\mathbb{R}^n \setminus B_{\delta/\varepsilon}(0)} \rho(x) \, \mathrm{d}x.$$

Since $\rho \in L^1(\mathbb{R}^n)$, the left-hand side converges to zero as $\varepsilon \to 0$ due to Lebesgue's dominated convergence theorem. This verifies (b). □

The first result in this subsection states that the nonlocal operator $\mathcal{L}_\varepsilon$ as in (4.1.9) is well-defined for functions $u \in C_c^\infty(\mathbb{R}^n)$. Afterwards, the operator can be viewed as a linear bounded mapping from $W^{2,p}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In fact, we have the following theorem:

Theorem 4.2.2. *Suppose that the assumption* (I1) *or* (J1) *holds and let $p \in [1, \infty)$ be arbitrary. Then, the following statements hold.*

(a) *The operator*

$$\begin{aligned} &\mathcal{L}_\varepsilon : C_c^\infty(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \\ &\mathcal{L}_\varepsilon u(x) = \int_{\mathbb{R}^n} J_\varepsilon(x-y)\big(u(x) - u(y) - \nabla u(y) \cdot (x-y)\big) \, \mathrm{d}y \end{aligned} \tag{4.2.3}$$

is well-defined and linear. Moreover, for any $u \in C_c^\infty(\mathbb{R}^n)$, $\mathcal{L}_\varepsilon u$ can be expressed as

$$\mathcal{L}_\varepsilon u(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} J_\varepsilon(x-y)\big(u(x) - u(y)\big) \, \mathrm{d}y \tag{4.2.4}$$

for almost all $x \in \mathbb{R}^n$.

(b) *The operator from* (a) *can be extended to a bounded linear operator*

$$\mathcal{L}_\varepsilon : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad \textit{with} \quad \|\mathcal{L}_\varepsilon u\|_{L^p(\mathbb{R}^n)} \le C_* \|u\|_{W^{2,p}(\mathbb{R}^n)},$$

where C_* is a positive constant depending only on ρ and p.

By means of Theorem 4.2.2, we can also establish that the nonlocal operator $\mathcal{L}_\varepsilon$ is well-defined in the case Ω is a C^2-domain. Note that Ω does not need to be bounded. In fact, we have the following result:

Theorem 4.2.3. *Let Ω be a (not necessarily bounded) domain with C^2-boundary, suppose that* (I1) *or* (J1) *holds, and let $p \in [1, \infty)$ and $\varepsilon > 0$ be arbitrary. Then, the operator*

$$\begin{aligned} &\mathcal{L}_\varepsilon^\Omega : C_c^2(\overline{\Omega}) \to L^p_{\mathrm{loc}}(\Omega), \\ &\mathcal{L}_\varepsilon^\Omega u(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} J_\varepsilon(x-y)\big(u(x) - u(y)\big) \, \mathrm{d}y \end{aligned} \tag{4.2.5}$$

is well-defined and linear.

Remark 4.2.4. We note that Theorem 4.2.3 only states that the nonlocal operator is well-defined for functions in $C_c^2(\overline{\Omega})$. In order to extend it to the Sobolev space $W^{2,p}(\Omega)$, we still need to prove boundedness of the nonlocal operator for functions in $W^{2,p}(\Omega)$. However, the involved calculations are quite demanding. As we will see in Section 4.4.2, a transform onto a reference domain simplifies the computations a lot. Therefore, we prove the boundedness in Section 4.4.3.

In the remainder of this section, we present the proofs of Theorem 4.2.2 and Theorem 4.2.3.

Proof of Theorem 4.2.2. Let $p \in [1,\infty)$ be arbitrary. In the following, the letter C will denote generic positive constants depending only on ρ and p, which may change their value from line to line.

Proof of (a). Let $u \in C_c^\infty(\mathbb{R}^n)$ be arbitrary. By means of Taylor's theorem, we have

$$u(x) - u(y) - \nabla u(x) \cdot (x-y) = -R_2(x,y), \tag{4.2.6}$$

where the error term is given by

$$R_2(x,y) := \sum_{|\beta|=2} \frac{2}{\beta!} \int_0^1 (1-t) D^\beta u(y + t(x-y))\,\mathrm{d}t\,(x-y)^\beta. \tag{4.2.7}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| J_\varepsilon(x-y)\big(u(x) - u(y) - \nabla u(x)\cdot(x-y)\big)\right| \mathrm{d}y \\
&\leq \sum_{|\beta|=2} \frac{2}{\beta!} \int_{\mathbb{R}^n} \int_0^1 \left| J_\varepsilon(x-y)(1-t) D^\beta u(y+t(x-y))\,(x-y)^\beta \right| \mathrm{d}t\,\mathrm{d}y \\
&\leq C\,\|u\|_{W^{2,\infty}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\rho_\varepsilon(x-y)|\,\mathrm{d}y = C\,\|u\|_{W^{2,\infty}(\mathbb{R}^n)}\,\|\rho\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

This proves that for almost all $x \in \mathbb{R}^n$,

$$J_\varepsilon(x-\cdot)\big(u(x) - u(\cdot) - \nabla u(x)\cdot(x-\cdot)\big) \in L^1(\mathbb{R}^n) \tag{4.2.8}$$

and therefore, the integral in the definition of $\mathcal{L}_\varepsilon$ actually exists. Recalling (4.2.6) and (4.2.7), we use Hölder's inequality and Fubini's theorem to deduce

$$\begin{aligned}
&\|\mathcal{L}_\varepsilon u\|_{L^p(\mathbb{R}^n)}^p \\
&= \int_{\mathbb{R}^n} \left| \sum_{|\beta|=2} \frac{2}{\beta!} \int_{\mathbb{R}^n} \int_0^1 J_\varepsilon(x-y)(1-t) D^\beta u(y+t(x-y))\,(x-y)^\beta\,\mathrm{d}t\,\mathrm{d}y \right|^p \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^n} \sum_{|\beta|=2} \left(\int_{\mathbb{R}^n} \int_0^1 |\rho_\varepsilon(x-y)||D^\beta u(y+t(x-y))|\,\mathrm{d}t\,\mathrm{d}y \right)^p \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^n} \|\rho\|_{L^1(\mathbb{R}^n)}^{\frac{p-1}{p}} \sum_{|\beta|=2} \int_{\mathbb{R}^n} \int_0^1 |\rho_\varepsilon(x-y)||D^\beta u(y+t(x-y))|^p\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x \\
&\leq C \sum_{|\beta|=2} \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\rho_\varepsilon(x-y)||D^\beta u(y+t(x-y))|^p\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Applying first the change of variables $z = z(y) := x - y$ and afterwards the change of variables $w = w(x) := x - z + tz$, we infer

$$\begin{aligned}\|\mathcal{L}_\varepsilon u\|_{L^p(\mathbb{R}^n)}^p &\le C \sum_{|\beta|=2} \int_0^1 \left(\int_{\mathbb{R}^n} |\rho_\varepsilon(z)| \,\mathrm{d}z \right) \left(\int_{\mathbb{R}^n} |D^\beta u(w)|^p \,\mathrm{d}w \right) \,\mathrm{d}t \\ &\le C \|\rho\|_{L^1(\mathbb{R}^n)} \sum_{|\beta|=2} \left(\int_{\mathbb{R}^n} |D^\beta u(w)|^p \,\mathrm{d}w \right) \le C_*^p \, \|u\|_{W^{2,p}(\mathbb{R}^n)}^p \end{aligned} \tag{4.2.9}$$

for some constant $C_* > 0$ depending only on ρ and p. This proves that the operator $\mathcal{L}_\varepsilon$ is well-defined and bounded in the sense of (4.2.9). Moreover, it is easy to check that the operator $\mathcal{L}_\varepsilon$ is linear.

It remains to verify the representation (4.2.4) provided that ρ is an even function. Therefore, let $r > 0$ be arbitrary. Recalling (4.2.8), we deduce

$$\begin{aligned}&\int_{|x-y|\ge r} J_\varepsilon(x-y)\big(u(x) - u(y)\big) \,\mathrm{d}y \\ &= \int_{|x-y|\ge r} J_\varepsilon(x-y)\big(u(x) - u(y) - \nabla u(x)\cdot(x-y)\big) \,\mathrm{d}y \\ &\quad + \int_{|x-y|\ge r} J_\varepsilon(x-y) \nabla u(x)\cdot(x-y) \,\mathrm{d}y \end{aligned} \tag{4.2.10}$$

for almost all $x \in \mathbb{R}^n$. We point out that

$$\begin{aligned}&\int_{|x-y|\ge r} \big|J_\varepsilon(x-y)\nabla u(x)\cdot(x-y)\big| \,\mathrm{d}y \\ &\le \int_{|x-y|\ge r} |\rho_\varepsilon(x-y)| \, |\nabla u(x)| \, \frac{1}{|x-y|} \,\mathrm{d}y \le \frac{1}{r} \|u\|_{W^{1,\infty}(\mathbb{R}^n)} \|\rho\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

This means that the second integral on the right-hand side of (4.2.10) actually exists and therefore, the identity (4.2.10) is justified. As ρ is an even function so is the kernel J_ε.

This implies that

$$\int_{|x-y|\ge r} J_\varepsilon(x-y)\nabla u(x)\cdot(x-y) \,\mathrm{d}y = 0. \tag{4.2.11}$$

Moreover, invoking (4.2.8), we obtain

$$\int_{|x-y|\ge r} J_\varepsilon(x-y)\big(u(x) - u(y) - \nabla u(x)\cdot(x-y)\big) \,\mathrm{d}y \to \mathcal{L}_\varepsilon u(x) \tag{4.2.12}$$

as $r \to 0$ for almost all $x \in \mathbb{R}^n$ by means of Lebesgue's dominated convergence theorem. Eventually, combining (4.2.10), (4.2.11) and (4.2.12), we conclude that

$$\mathcal{L}_\varepsilon u(x) = \lim_{r \searrow 0} \int_{|x-y|\ge r} J_\varepsilon(x-y)\big(u(x)-u(y)\big) \,\mathrm{d}y.$$

By the definition of the principal value, this is exactly (4.2.4).

Proof of (b). We already know from (a) that the operator $\mathcal{L}_\varepsilon : C_c^\infty(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is well-defined, linear and bounded in the sense of (4.2.9). As $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{2,p}(\mathbb{R}^n)$, we conclude via estimate (4.2.9) that $\mathcal{L}_\varepsilon$ can be extended to a bounded linear operator

$$\mathcal{L}_\varepsilon : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad \text{with} \quad \|\mathcal{L}_\varepsilon u\|_{L^p(\mathbb{R}^n)} \le C_* \|u\|_{W^{2,p}(\mathbb{R}^n)}.$$

Hence, the proof is complete. □

Proof of Theorem 4.2.3. Without loss of generality, we assume that $\Omega \neq \mathbb{R}^n$ as this case was already handled in Theorem 4.2.2. Let $u \in C_c^2(\overline{\Omega})$ be arbitrary. Since Ω is of class C^2, we can find an extension $\tilde{u} \in C_c^2(\mathbb{R}^n)$ with $\tilde{u}|_\Omega = u$. We now fix an arbitrary $x \in \Omega$. Since $\Omega \subset \mathbb{R}^n$ is open, we can find a radius $r(x) > 0$ such that $B_{r(x)}(x) \subset \Omega$. Then, for any $r \in (0, r(x)]$, we have $B_r(x) \subset \Omega$ and thus,

$$\begin{aligned}&\int_{\Omega\setminus B_r(x)} J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n\setminus B_r(x)} J_\varepsilon(x-y)\big(\tilde{u}(x)-\tilde{u}(y)\big)\,\mathrm{d}y - \int_{\mathbb{R}^n\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(x)-\tilde{u}(y)\big)\,\mathrm{d}y\end{aligned} \tag{4.2.13}$$

We already know from Theorem 4.2.2(a) that

$$\int_{\mathbb{R}^n\setminus B_r(x)} J_\varepsilon(x-y)\big(\tilde{u}(x)-\tilde{u}(y)\big)\,\mathrm{d}y \to \mathcal{L}_\varepsilon^{\mathbb{R}^n}\tilde{u}(x) \quad \text{as } r\to 0.$$

Therefore, it remains to show that the second integral on the right-hand side of (4.2.13) actually exists. For $x \in B_{r(x)}(x) \subset \Omega$ and $y \in \mathbb{R}^n \setminus \Omega$, we obviously have

$$|x-y| \geq r(x).$$

Recalling the definition of J_ε, this implies that

$$\begin{aligned}\left|\int_{\mathbb{R}^n\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(x)-\tilde{u}(y)\big)\,\mathrm{d}y\right| &\leq \int_{\mathbb{R}^n\setminus\Omega} J_\varepsilon(x-y)\big|\tilde{u}(x)-\tilde{u}(y)\big|\,\mathrm{d}y \\ &\leq \frac{2}{r(x)^2}\,\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)}\|\rho_\varepsilon\|_{L^1(\mathbb{R}^n)} = \frac{2}{r(x)^2}\,\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)}\|\rho\|_{L^1(\mathbb{R}^n)} < \infty.\end{aligned}$$

Hence, the expression

$$\begin{aligned}\mathcal{L}_\varepsilon^\Omega u(x) &= \mathrm{P.V.}\int_\Omega J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \\ &= \lim_{r\searrow 0}\int_{\Omega\cap\{|x-y|\geq r\}} J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \\ &= \mathcal{L}_\varepsilon^{\mathbb{R}^n}\tilde{u}(x) - \int_{\mathbb{R}^n\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(x)-\tilde{u}(y)\big)\,\mathrm{d}y\end{aligned} \tag{4.2.14}$$

is well-defined for every $x \in \Omega$ provided that $u \in C_c^2(\overline{\Omega})$.

Let now K be an arbitrary compact subset of Ω. Since Ω is open, we know that $r(K) \coloneqq \operatorname{dist}(\mathbb{R}^n \setminus \Omega, K) > 0$. As the pointwise limit of a sequence of measurable functions is measurable, we further know that the mapping

$$\Omega \ni x \mapsto \mathcal{L}_\varepsilon^\Omega u(x) \in \mathbb{R}$$

is measurable. Using (4.2.14) along with Theorem 4.2.2, and proceeding similarly as above, we infer that

$$\begin{aligned}\|\mathcal{L}_\varepsilon^\Omega u\|_{L^p(K)}^p &\leq C\|\mathcal{L}_\varepsilon^{\mathbb{R}^n}\tilde{u}\|_{L^p(\mathbb{R}^n)}^p + \int_K \left|\int_{\mathbb{R}^n\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(x)-\tilde{u}(y)\big)\,\mathrm{d}y\right|^p \mathrm{d}x \\ &\leq C\|\mathcal{L}_\varepsilon^{\mathbb{R}^n}\tilde{u}\|_{L^p(\mathbb{R}^n)}^p + \frac{2^p|K|}{r(K)^{2p}}\,\|\tilde{u}\|_{L^\infty(\mathbb{R}^n)}^p\,\|\rho\|_{L^1(\mathbb{R}^n)}^p < \infty.\end{aligned}$$

As the compact subset K was arbitrary, this proves that $\mathcal{L}_\varepsilon^\Omega u \in L^p_{\mathrm{loc}}(\Omega)$. This means that the operator

$$\mathcal{L}_\varepsilon^\Omega : C_c^2(\overline{\Omega}) \to L^p_{\mathrm{loc}}(\Omega), \quad u \mapsto \mathcal{L}_\varepsilon^\Omega u$$

is well-defined. Moreover, the operator is obviously linear and thus, the proof is complete. □

4.3. Convergence on the torus

In this section, we investigate the case, where Ω is the torus $\mathbb{T}^n$. We assume that the interaction kernel J_ε satisfies the assumptions in (J1). As already mentioned in Remark 2.1.2, we assume that J is compactly supported in $(-\pi,\pi)^n$ and we consider a 2π-periodic extension $\tilde{J}_\varepsilon$ of the interaction kernel J_ε. Thus, the nonlocal operator is given by

$$\begin{aligned} \mathcal{L}_\varepsilon^{\mathbb{T}^n} u(x) :&= \mathrm{P.V.} \int_{\mathbb{T}^n} \tilde{J}_\varepsilon(x-y)(u(x)-u(y)) \,\mathrm{d}y \\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} J_\varepsilon(x-y)(u(x)-u(y)) \,\mathrm{d}y \quad \text{ for all } x \in \mathbb{T}^n, \end{aligned} \tag{4.3.1}$$

for suitable functions $u : \mathbb{T}^n \to \mathbb{R}$. On account of the identity in (4.3.1), we can use the same arguments as in the proof of Theorem 4.2.2 before to conclude that the nonlocal operator as in (4.3.1) is well-defined and linear.

In the main result of this section, we intend to prove convergence of the nonlocal operator as introduced in (4.3.1) towards a local differential operator as ε tends towards zero. The proof is based on methods from Fourier analysis. More precisely, we want to use Plancherel's theorem. Thus, it suffices to show the convergence for the corresponding Fourier coefficients. This will be done by means of Taylor's theorem applied to suitable auxiliary functions that can be constructed in terms of the Fourier coefficients.

4.3.1. Proof of Theorem 2.1.4

Proof of (a). Let $u \in H^2(\mathbb{T}^n)$ be arbitrary. We intend to prove this assertion using Plancherel's theorem. Thus, it suffices to show that

$$\left\| \left(\left(\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n} u} + \mathrm{div} \widehat{(M \nabla u)} \right)(k) \right)_{k \in \mathbb{Z}^n} \right\|^2_{\ell^2(\mathbb{Z}^n)} \to 0 \quad \text{as } \varepsilon \to 0.$$

In order to show this convergence, we will use Lebesgue's dominated convergence theorem. To this end, we will first show the pointwise convergence

$$\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n}} u(k) + \mathrm{div} \widehat{(M \nabla u)}(k) \to 0$$

for almost all $k \in \mathbb{Z}^n$. Recalling Theorem 3.4.2 and the definition of $\mathcal{L}_\varepsilon$, the Fourier transform of the nonlocal operator is given by

$$\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n}} u(k) = \left(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) \right) \hat{u}(k)$$

for all $k \in \mathbb{Z}^n$. Here, we have used the fact that $J_\varepsilon * 1$ is constant. Indeed, this follows from the calculation

$$(J_\varepsilon * 1)(x) = \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x-y)}{|x-y|^2}\,\mathrm{d}y = \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(z)}{|z|^2}\,\mathrm{d}z = (J_\varepsilon * 1)(0) =: a_\varepsilon$$

for almost all $x \in \mathbb{T}^n$. Then, by definition of the Fourier coefficients, it follows that $a_\varepsilon = \widehat{J_\varepsilon}(0)$. By definition of the Fourier coefficients, we obtain after integrating by parts

$$\begin{aligned}
\widehat{\operatorname{div}(M\nabla u)}(k) &= \int_{\mathbb{T}^n} \operatorname{div}(M\nabla u)(x) e^{-ik\cdot x}\,\mathrm{d}x \\
&= \sum_{m=1}^n \int_{\mathbb{T}^n} \partial_{x_m}(M\nabla u(x))_m\, e^{-ik\cdot x}\,\mathrm{d}x \\
&= i\sum_{m=1}^n \int_{\mathbb{T}^n} M\nabla u(x)\, e^{-ik\cdot x} k_m\,\mathrm{d}x \\
&= i\sum_{m,l=1}^n M_{ml} \int_{\mathbb{T}^n} \partial_{x_l} u(x)\, e^{-ik\cdot x} k_m\,\mathrm{d}x \\
&= -\sum_{m,l=1}^n M_{ml} \int_{\mathbb{T}^n} u(x)\, e^{-ik\cdot x} k_m k_l\,\mathrm{d}x \\
&= -\sum_{m,l=1}^n M_{ml} k_m k_l\ \hat{u}(k).
\end{aligned}$$

Combining the identities above, we then obtain

$$\begin{aligned}
\left\|\mathcal{L}_\varepsilon^{\mathbb{T}^n} u + \operatorname{div}(M\nabla u)\right\|_{L^2(\mathbb{T}^n)}^2 &= \left\|\left((\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n} u} + \widehat{\operatorname{div}(M\nabla u)})(k)\right)_{k\in\mathbb{Z}^n}\right\|_{\ell^2(\mathbb{Z}^n)}^2 \\
&= \sum_{k\in\mathbb{Z}^n} \left|\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n} u}(k) + \widehat{\operatorname{div}(M\nabla u)}(k)\right|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \left|\Big(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{m,l=1}^n M_{ml}k_m k_l\Big)\widehat{u}(k)\right|^2.
\end{aligned}$$

Thus, we need to show the pointwise convergence

$$\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) \to \sum_{m,l=1}^n M_{ml}k_m k_l \quad \text{as } \varepsilon \to 0$$

for almost all $k \in \mathbb{Z}^n$. Note that by construction, we have

$$\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) = \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2}(1 - e^{-ik\cdot x})\,\mathrm{d}x.$$

For any $k \in \mathbb{Z}^n$, we now introduce the auxiliary functions

$$f_k : \mathbb{T}^n \to \mathbb{C}, \quad f_k(x) := -e^{-ik\cdot x}.$$

Then, we can reformulate this term by means of the auxiliary functions f_k

$$\begin{aligned}\widehat{J_\varepsilon}(0)-\widehat{J_\varepsilon}(k) &= \int_{\mathbb{T}^n}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}(f_k(x)-f_k(0))\,\mathrm{d}x\\ &= \int_{\mathbb{T}^n}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}(f_k(x)-f_k(0)-\frac{1}{2}x^TD^2f_k(0)x)\,\mathrm{d}x\\ &\quad+\int_{\mathbb{T}^n}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}\frac{1}{2}x^TD^2f_k(0)x\,\mathrm{d}x\\ &:= I_\varepsilon^1+I_\varepsilon^2.\end{aligned}\tag{4.3.2}$$

In the following, we analyze the integrals I_ε^1 and I_ε^2 separately.

Ad I_ε^1: Since the interaction kernel J_ε is even, it follows that the function

$$\mathbb{T}^n\to\mathbb{C}^n,\quad x\mapsto J_\varepsilon(x)x$$

is odd. Thus, by definition of J_ε, it holds

$$\int_{\mathbb{T}^n}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}x\,\mathrm{d}x=0.\tag{4.3.3}$$

Therefore, taking the scalar product of (4.3.3) by $\nabla f_k(0)$, this yields

$$I_\varepsilon^1=\int_{\mathbb{T}^n}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}(f_k(x)-f_k(0)-\nabla f_k(0)\cdot x-\frac{1}{2}x^TD^2f_k(0)x)\,\mathrm{d}x.$$

Since the function f_k is smooth, we can apply Taylor's theorem. Then, for any $\tilde\varepsilon>0$, there exists some $\delta>0$ such that

$$|f_k(x)-f_k(0)-\nabla f_k(0)\cdot x-\frac{1}{2}x^TD^2f_k(0)x|<\tilde\varepsilon C|x|^2$$

for all $|x|<\delta$. Consequently,

$$\begin{aligned}|I_\varepsilon^1| &\le \int_{B_\delta(0)}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}\Big|f_k(x)-f_k(0)-\nabla f_k(0)\cdot x-\frac{1}{2}x^TD^2f_k(0)x\Big|\,\mathrm{d}x\\ &\quad+\int_{B_\delta(0)^c}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}\Big|f_k(x)-f_k(0)-\nabla f_k(0)\cdot x-\frac{1}{2}x^TD^2f_k(0)x\Big|\,\mathrm{d}x\\ &\le \int_{B_\delta(0)}\tilde\varepsilon C\tilde\rho_\varepsilon(x)\,\mathrm{d}x+\int_{B_\delta(0)^c}C\tilde\rho_\varepsilon(x)\,\mathrm{d}x.\end{aligned}$$

Since $\tilde\varepsilon$ can be chosen arbitrarily small, the first term on the right-hand side can be made arbitrarily small. For the second integral on the right-hand side, we again applied Taylor's theorem. Then, the properties of ρ_ε imply that also this integral vanishes as $\varepsilon\to0$. Altogether, this shows $I_\varepsilon^1\to0$ as $\varepsilon\to0$.

Ad I_ε^2: Recalling the definition of f_k, we can reformulate the integral I_ε^2 as

$$\begin{aligned}I_\varepsilon^2 &= \frac{1}{2}\sum_{l,m=1}^n\int_{\mathbb{T}^n}\frac{\tilde\rho_\varepsilon(x)}{|x|^2}x_m\,k_mk_l\,x_l\,\mathrm{d}x\\ &= \frac{1}{2}\sum_{l,m=1}^n\int_{\mathbb{R}^n}\frac{\rho_\varepsilon(x)}{|x|^2}x_m\,k_mk_l\,x_l\,\mathrm{d}x\\ &= \sum_{l,m=1}^n M_{ml}k_mk_l,\end{aligned}\tag{4.3.4}$$

where we used the definition of the matrix M in the last step. Altogether, we have thus shown the pointwise convergence

$$\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) \to \sum_{l,m=1}^{n} M_{ml} k_m k_l \quad \text{as } \varepsilon \to 0$$

for almost all $k \in \mathbb{Z}^n$. Using Taylor's theorem once more, we observe that

$$\begin{aligned}
&\Big|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml} k_m k_l\Big| \\
&= \Big| \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} (f_k(0) - f_k(x) - \nabla f_k(0)\cdot x - \frac{1}{2} x^T D^2 f_k(0) x) \,\mathrm{d}x \Big| \\
&\le C(1+|k|^2) \int_{\mathbb{T}^n} \tilde{\rho}_\varepsilon(x)\,\mathrm{d}x \le C(1+|k|^2)
\end{aligned}$$

for all $k \in \mathbb{Z}^n$ and $\varepsilon > 0$. This implies

$$\Big|\Big(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml} k_m k_l\Big)\widehat{u}(k)\Big|^2 \le C\Big(\big|\widehat{u}(k)\big|^2 + \big||k|^2\, \widehat{u}(k)\big|^2\Big)$$

for all $k \in \mathbb{Z}^n$ and $\varepsilon > 0$. Since $u \in H^2(\mathbb{T}^n)$, it follows that

$$\Big(\big|\widehat{u}(k)\big|^2 + \Big|\sum_{l,m=1}^{n} M_{ml} k_m k_l\, \widehat{u}(k)\Big|^2\Big)_{k\in\mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n).$$

Therefore, we can apply Lebesgue's dominated convergence theorem and obtain

$$\Big\| \big((\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n} u} + \mathrm{div}\widehat{(M\nabla u)})(k)\big)_{k\in\mathbb{Z}^n} \Big\|^2_{\ell^2(\mathbb{Z}^n)} \to 0$$

as $\varepsilon \to 0$. This shows the first assertion.
Proof of (b). Let $u \in H^3(\mathbb{T}^n)$ be arbitrary. Similarly as above, we intend to prove this assertion using Plancherel's theorem. Thus, it suffices to control the term

$$\begin{aligned}
\Big\| \big((\widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n} u} + \mathrm{div}\widehat{(M\nabla u)})(k)\big)_{k\in\mathbb{Z}^n} \Big\|^2_{\ell^2(\mathbb{Z}^n)} &= \sum_{k\in\mathbb{Z}^n} \Big| \widehat{\mathcal{L}_\varepsilon^{\mathbb{T}^n} u}(k) + \mathrm{div}\widehat{(M\nabla u)}(k)\Big|^2 \\
&= \sum_{k\in\mathbb{Z}^n} \Big|\Big(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml} k_m k_l\Big)\widehat{u}(k)\Big|^2.
\end{aligned}$$

For any $k \in \mathbb{Z}^n$, we again introduce the auxiliary functions

$$f_k : \mathbb{T}^n \to \mathbb{C}, \quad f_k(x) := -e^{-ik\cdot x}.$$

Then, using the symmetry of the interaction kernel J_ε, cf. (4.3.3), and the same arguments as in (4.3.2) and (4.3.4), we obtain

$$\begin{aligned}
&\Big|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml} k_m k_l\Big| \\
&= \Big| \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} \Big(f_k(0) - f_k(x) - \nabla f_k(0)\cdot x - \frac{1}{2} x^T D^2 f_k(0) x\Big) \,\mathrm{d}x \Big|.
\end{aligned}$$

Since the function f_k is smooth, we can use a third order Taylor expansion around $x = 0$ to get

$$\Big|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml}k_m k_l\Big| \leq C \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} \sup_{y\in[-\pi,\pi]^n} |D^3 f_k(y)||x|^3 \,\mathrm{d}x.$$

Recalling the definition of the auxiliary function f_k, we observe that

$$\sup_{y\in[-\pi,\pi]^n} |D^3 f_k(y)| \leq C|k|^3$$

for all $k \in \mathbb{Z}^n$. Therefore, we arrive at

$$\Big|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml}k_m k_l\Big| \leq C|k|^3 \int_{\mathbb{T}^n} \tilde{\rho}_\varepsilon(x)|x| \,\mathrm{d}x = C|k|^3 \int_{\mathbb{R}^n} \rho_\varepsilon(x)|x| \,\mathrm{d}x$$

for all $k \in \mathbb{Z}^n$ and $\varepsilon > 0$. Eventually, the change of variables

$$\mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \frac{x}{\varepsilon}$$

as well as the integrability properties of ρ imply

$$\Big|\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml}k_m k_l\Big| \leq \varepsilon\, C|k|^3 \int_{\mathbb{R}^n} \rho(z)|z| \,\mathrm{d}z \leq \varepsilon\, C|k|^3$$

for all $k \in \mathbb{Z}^n$ and $\varepsilon > 0$. In particular, this yields

$$\begin{aligned}\|\mathcal{L}_\varepsilon^{\mathbb{T}^n} u + \operatorname{div}(M\nabla u)\|_{L^2(\mathbb{T}^n)}^2 &= \sum_{k\in\mathbb{Z}^n} \Big|\Big(\widehat{J_\varepsilon}(0) - \widehat{J_\varepsilon}(k) - \sum_{l,m=1}^{n} M_{ml}k_m k_l\Big)\widehat{u}(k)\Big|^2 \\ &\leq C\varepsilon^2 \sum_{k\in\mathbb{Z}^n} \big||k|^3\widehat{u}(k)\big|^2 \\ &\leq C\varepsilon^2 \|u\|_{H^3(\mathbb{T}^n)}^2\end{aligned}$$

for all $u \in H^3(\mathbb{T}^n)$ and $\varepsilon > 0$. This concludes the proof. □

Remark 4.3.1 (The isotropic case)**.** If the kernel is assumed to be radially symmetric, then the integral I_ε^2 in (4.3.2) behaves as follows:

Recalling the definition of f_k, we can reformulate the integral I_ε^2 as

$$I_\varepsilon^2 = \frac{1}{2} \sum_{l,m=1}^{n} \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_m\, k_m k_l\, x_l \,\mathrm{d}x.$$

Now, we consider the case $m \neq l$. Then, the change of variables,

$$T_l : \mathbb{T}^n \to \mathbb{T}^n, \quad x \mapsto \tilde{x} := \begin{cases} x_j & \text{if } j \neq l, \\ -x_l & \text{if } j = l, \end{cases}$$

implies

$$\int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_m\, k_m k_l\, x_l \,\mathrm{d}x = -\int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_m\, k_m k_l\, x_l \,\mathrm{d}x.$$

In particular, this shows

$$\int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_m\, k_m k_l\, x_l \,\mathrm{d}x = 0$$

for all $m \neq l$. Therefore, the integral I_ε^2 simplifies to

$$I_\varepsilon^2 = \frac{1}{2} \sum_{m=1}^{n} k_m^2 \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_m^2 \,\mathrm{d}x.$$

In the next step, we introduce another change of variables,

$$\Phi_i : \mathbb{T}^n \to \mathbb{T}^n, \quad \Phi_i(x_1, \dots, x_i, \dots, x_n) := (x_i, \dots, x_1, \dots, x_n),$$

which interchanges the first and i-th component of x. This yields

$$\int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_i^2 \,\mathrm{d}x = \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(\Phi_i(x))}{|\Phi_i(x)|^2} (\Phi_i(x))_i^2 \,\mathrm{d}x = \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_1^2 \,\mathrm{d}x$$

for all $i = 1, \dots, n$. Hence, we obtain

$$\begin{aligned} I_\varepsilon^2 &= \frac{1}{2} \sum_{m=1}^{n} k_m^2 \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_m^2 \,\mathrm{d}x \\ &= \frac{1}{2} \sum_{m=1}^{n} k_m^2 \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_1^2 \,\mathrm{d}x \\ &= \frac{1}{2} \sum_{m=1}^{n} k_m^2 \, \frac{1}{n} \sum_{j=1}^{n} \int_{\mathbb{T}^n} \frac{\tilde{\rho}_\varepsilon(x)}{|x|^2} x_j^2 \,\mathrm{d}x \\ &= \frac{1}{2n} \sum_{m=1}^{n} k_m^2 \left(\int_{\mathbb{T}^n} \tilde{\rho}_\varepsilon(x) \,\mathrm{d}x \right) = |k|^2. \end{aligned}$$

Note that in the last equality, the normalization of ρ_ε, cf. (I1), implies

$$\frac{1}{2n} \int_{\mathbb{T}^n} \tilde{\rho}_\varepsilon(x) \,\mathrm{d}x = \frac{1}{2n} \int_{\mathbb{R}^n} \rho_\varepsilon(x) \,\mathrm{d}x = \frac{1}{2n}\, \omega_n \int_0^\infty \rho(r) r^{n-1} \,\mathrm{d}r = \frac{\omega_n}{n} \frac{1}{C_n} = 1,$$

where we also used $\omega_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ and that for any $j = 1, \dots, n$ it holds

$$C_n = \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 \,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = \int_{\mathbb{S}^{n-1}} |e_j \cdot \sigma|^2 \,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} |\sigma|^2 \,\mathrm{d}\mathcal{H}^{n-1}(\sigma) = \frac{\omega_n}{n}.$$

Let us recall that the Fourier transform of the Laplacian Δu is given by

$$\widehat{\Delta u}(k) = -|k|^2\, \widehat{u}(k)$$

for all $k \in \mathbb{Z}^n$. Thus, repeating the arguments in the proof before, this shows Theorem 2.1.5.

4.4. Convergence in bounded domains

In this section, we investigate the case, where Ω is a sufficiently smooth bounded domain in $\mathbb{R}^n$. Again, we allow for both anisotropic and isotropic kernels. However, in contrast to the previous section, we prove convergence in more general topologies, i.e., with respect to the L^p-norm, where $p \in [1, \infty)$.

The proof is based on a localization argument. Therefore, we first prove the statement in the full space $\mathbb{R}^n$. There, we derive suitable error estimates by means of Taylor's theorem. In the next step, we perform a suitable coordinate transform onto a reference domain, where the matrix M simplifies to the identity matrix I. This allows us to study the case $M = I$ in the remaining proofs. Afterwards, we study the nonlocal operator on a bent half space. Using a suitable coordinate transform, we can apply perturbation arguments, in order to derive the desired error estimates. Finally, we close the argument via localization.

4.4.1. Convergence on the full space

In this subsection, we consider the case, where Ω is the full space, i.e., $\Omega = \mathbb{R}^n$. Our main goal is to study the convergence of the nonlocal operator $\mathcal{L}_\varepsilon$, which is given by

$$\begin{aligned} \mathcal{L}_\varepsilon u(x) &= \text{P.V.} \int_{\mathbb{R}^n} J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \\ &:= \lim_{r\to 0} \int_{\mathbb{R}^n\setminus B_r(x)} J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y \end{aligned} \tag{4.4.1}$$

for all $x \in \Omega$ and any sufficiently regular function $u : \Omega \to \mathbb{R}$, to the local differential operator $\mathcal{L}$ as defined in (4.1.9), i.e.,

$$\mathcal{L}u(x) = -\operatorname{div}(M\nabla u) \tag{4.4.2}$$

for all $x \in \Omega$ and any sufficiently regular function $u : \Omega \to \mathbb{R}$, where the matrix M is defined as in (4.1.10).

Based on the results in Theorem 4.2.2, we obtain the following nonlocal-to-local convergence properties:

Theorem 4.4.1. *Suppose that the assumption* (J1) *holds and let $p \in [1, \infty)$ be arbitrary. Then, the following statements hold.*

(a) *For all $u \in W^{2,p}(\mathbb{R}^n)$,*

$$\mathcal{L}_\varepsilon u \to \mathcal{L}u \quad \text{in } L^p(\mathbb{R}^n) \text{ as } \varepsilon \to 0.$$

(b) *There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and all $u \in W^{3,p}(\mathbb{R}^n)$,*

$$\|\mathcal{L}_\varepsilon u - \mathcal{L}u\|_{L^p(\mathbb{R}^n)} \le C\varepsilon \|u\|_{W^{3,p}(\mathbb{R}^n)}.$$

Corollary 4.4.2. *Suppose that the assumption* (I1) *holds and let $p \in [1, \infty)$ be arbitrary. Then, the following statements hold.*

(a) *For all* $u \in W^{2,p}(\mathbb{R}^n)$,

$$\mathcal{L}_\varepsilon u \to -\Delta u \quad \text{in } L^p(\mathbb{R}^n) \text{ as } \varepsilon \to 0.$$

(b) *There exists a constant* $C > 0$ *such that for all* $\varepsilon > 0$ *and all* $u \in W^{3,p}(\mathbb{R}^n)$,

$$\|\mathcal{L}_\varepsilon u + \Delta u\|_{L^p(\mathbb{R}^n)} \leq C\varepsilon \|u\|_{W^{3,p}(\mathbb{R}^n)}.$$

Remark 4.4.3. In the following, we only prove Theorem 4.4.1, since the proof of Corollary 4.4.2 uses the same arguments with $M = I$. The latter identity follows from the properties of the kernel as in (I1). Indeed, it holds

$$M_{ij} = \frac{1}{2} \int_{\mathbb{R}^n} J(z) z_i z_j \, \mathrm{d}z = 0 \quad \text{if } i \neq j$$

due to the radial symmetry of J. In the case $i = j$, we first use the change of variables

$$\Phi_i : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi_i(x_1, \dots, x_i, \dots, x_n) = (x_i, \dots, x_1, \dots, x_n)$$

to conclude

$$\int_{\mathbb{R}^n} J(z) z_i^2 \, \mathrm{d}z = \int_{\mathbb{R}^n} J(z) z_1^2 \, \mathrm{d}z \quad \text{for all } i \in \{1, \dots, n\}.$$

This identity implies

$$\begin{aligned} M_{ii} = \frac{1}{2} \int_{\mathbb{R}^n} J(z) z_i^2 \, \mathrm{d}z &= \frac{1}{2} \int_{\mathbb{R}^n} J(z) z_1^2 \, \mathrm{d}z = \frac{1}{2n} \int_{\mathbb{R}^n} J(z) \sum_{j=1}^n z_j^2 \, \mathrm{d}z \\ &= \frac{1}{2n} \int_{\mathbb{R}^n} J(z) |z|^2 \, \mathrm{d}z = \frac{1}{2n} \int_{\mathbb{R}^n} \rho(z) \, \mathrm{d}z. \end{aligned}$$

Then, using the normalization of ρ, cf. (I1), we obtain

$$\frac{1}{2n} \int_{\mathbb{R}^n} \rho(z) \, \mathrm{d}z = \frac{1}{2n} \omega_n \int_0^\infty \rho(r) r^{n-1} \mathrm{d}r = \frac{\omega_n}{n} \frac{1}{C_n} = 1,$$

where we also used $\omega_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ and that for any $j = 1, \dots, n$ it holds

$$C_n = \int_{\mathbb{S}^{n-1}} |e_1 \cdot \sigma|^2 \, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = \int_{\mathbb{S}^{n-1}} |e_j \cdot \sigma|^2 \, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} |\sigma|^2 \, \mathrm{d}\mathcal{H}^{n-1}(\sigma) = \frac{\omega_n}{n}.$$

Eventually, this shows $M_{ii} = 1$ for all $i \in \{1, \dots, n\}$ and hence $M = I$.

In the remainder of this section, we present the proof of Theorem 4.4.1.

Proof of Theorem 4.4.1. Let now $p \in [1, \infty)$ and $\varepsilon > 0$ be arbitrary. In the following, the letter C will denote generic positive constants depending only on ρ and p, which may change their value from line to line.

Proof of (b). Since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{3,p}(\mathbb{R}^n)$, we first verify the assertion for functions $u \in C_c^\infty(\mathbb{R}^n)$. Therefore, let $u \in C_c^\infty(\mathbb{R}^n)$ be arbitrary. By the definitions of $\mathcal{L}_\varepsilon$ and $\mathcal{L}$ (see (4.2.3) and (4.4.2)), we obtain

$$\begin{aligned}
&\|\mathcal{L}_\varepsilon u - \mathcal{L}u\|_{L^p(\mathbb{R}^n)}^p \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} J_\varepsilon(x-y)\big(u(x) - u(y) + \nabla u(x)\cdot(x-y)\big)\,\mathrm{d}y + \frac{1}{2} M : D^2u(x) \right|^p \mathrm{d}x \\
&\le C \int_{\mathbb{R}^n} \left| \int_{B_1(x)} J_\varepsilon(x-y)\big(u(x) - u(y) + \nabla u(x)\cdot(x-y)\big)\,\mathrm{d}y + \frac{1}{2} M : D^2u(x) \right|^p \mathrm{d}x \\
&+ C \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n\setminus B_1(x)} J_\varepsilon(x-y)\big(u(x) - u(y) + \nabla u(x)\cdot(x-y)\big)\,\mathrm{d}y \right|^p \mathrm{d}x \\
&=: I_{1,\varepsilon} + I_{2,\varepsilon}.
\end{aligned} \tag{4.4.3}$$

We now estimate the terms $I_{1,\varepsilon}$ and $I_{2,\varepsilon}$ separately.

Ad $I_{1,\varepsilon}$: Applying Taylor's theorem, we obtain the expansion

$$u(x) - u(y) + \nabla u(x)\cdot(x-y) = -\frac{1}{2}(x-y)^T D^2u(x)(x-y) - R_3(x,y), \tag{4.4.4}$$

where the error term is given by

$$R_3(x,y) = \sum_{|\beta|=3} \frac{3}{\beta!} \int_0^1 (1-t) D^\beta u(y + t(x-y))\,\mathrm{d}t\,(x-y)^\beta. \tag{4.4.5}$$

Therefore, $I_{1,\varepsilon}$ can be estimated as

$$\begin{aligned}
I_{1,\varepsilon} &\le C \int_{\mathbb{R}^n} \left| -\frac{1}{2} \int_{B_1(x)} J_\varepsilon(x-y)(x-y)^T D^2u(x)(x-y)\,\mathrm{d}y + \frac{1}{2} M : D^2u(x) \right|^p \mathrm{d}x \\
&\quad + C \int_{\mathbb{R}^n} \left| \int_{B_1(x)} J_\varepsilon(x-y) R_3(x,y)\,\mathrm{d}y \right|^p \mathrm{d}x.
\end{aligned} \tag{4.4.6}$$

Employing the change of variables $z = x-y$ and recalling the definition of M (see (4.1.10)), we observe that

$$\begin{aligned}
&-\frac{1}{2} \int_{B_1(x)} J_\varepsilon(x-y)(x-y)^T D^2u(x)(x-y)\,\mathrm{d}y + \frac{1}{2} M : D^2u(x) \\
&= -\frac{1}{2} \int_{B_1(0)} J_\varepsilon(z) z^T D^2u(x) z\,\mathrm{d}z + \frac{1}{2} \int_{\mathbb{R}^n} J_\varepsilon(z) z^T D^2u(x) z\,\mathrm{d}z \\
&= \frac{1}{2} \int_{\mathbb{R}^n\setminus B_1(0)} J_\varepsilon(z) z^T D^2u(x) z\,\mathrm{d}z.
\end{aligned} \tag{4.4.7}$$

for all $x \in \mathbb{R}^n$. Plugging this identity into (4.4.6), we obtain

$$\begin{aligned}
I_{1,\varepsilon} &\le C \int_{\mathbb{R}^n} \left| \frac{1}{2} \int_{\mathbb{R}^n\setminus B_1(0)} J_\varepsilon(z) z^T D^2u(x) z\,\mathrm{d}z \right|^p \mathrm{d}x \\
&\quad + C \int_{\mathbb{R}^n} \left| \int_{B_1(x)} J_\varepsilon(x-y) R_3(x,y)\,\mathrm{d}y \right|^p \mathrm{d}x.
\end{aligned} \tag{4.4.8}$$

Since $|z| \geq 1$ for all $z \in \mathbb{R}^n \setminus B_1(0)$, the first summand on the right-hand side of (4.4.8) can be estimated as

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \frac{1}{2} \int_{\mathbb{R}^n \setminus B_1(0)} J_\varepsilon(z) z^T D^2 u(x) z \, \mathrm{d}z \right|^p \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus B_1(0)} |\rho_\varepsilon(z)| \, |z| \, \left| D^2 u(x) \right| \mathrm{d}z \right)^p \mathrm{d}x \\
&\leq C \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\rho_\varepsilon(z)| \, |z| \, \mathrm{d}z \right)^p \left| D^2 u(x) \right|^p \mathrm{d}x \leq C\varepsilon^p \|u\|^p_{W^{2,p}(\mathbb{R}^n)}.
\end{aligned}$$

Recalling the definition of the error term, we obtain the following estimate for the the second summand on the right-hand side of (4.4.8):

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \int_{B_1(x)} J_\varepsilon(x-y) R_3(x,y) \, \mathrm{d}y \right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left| \sum_{|\beta|=3} \frac{3}{\beta!} \int_{B_1(x)} \int_0^1 J_\varepsilon(x-y)(1-t) D^\beta u(y + t(x-y)) \, (x-y)^\beta \, \mathrm{d}t \, \mathrm{d}y \right|^p \mathrm{d}x \\
&\leq C \sum_{|\beta|=3} \int_{\mathbb{R}^n} \left(\int_{B_1(x)} \int_0^1 |\rho_\varepsilon(x-y)| \, |x-y| \, |D^\beta u(y + t(x-y))| \, \mathrm{d}t \, \mathrm{d}y \right)^p \mathrm{d}x
\end{aligned}$$

Next, we use the splitting

$$|\rho_\varepsilon(x-y)| \, |x-y| = \big(|\rho_\varepsilon(x-y)| \, |x-y| \big)^{\frac{1}{q}} \big(|\rho_\varepsilon(x-y)| \, |x-y| \big)^{\frac{1}{p}}, \tag{4.4.9}$$

where q denotes the conjugate exponent to p, i.e., $\frac{1}{q} + \frac{1}{p} = 1$. Then, using Hölder's inequality as well as the changes of variables $z = z(y) := x - y$ and $w = w(x) := x - z + tz$, we infer

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \int_{B_1(x)} J_\varepsilon(x-y) R_3(x,y) \, \mathrm{d}y \right|^p \mathrm{d}x \\
&\leq C \sum_{|\beta|=3} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\rho_\varepsilon(z)| \, |z| \, \mathrm{d}z \right)^{p-1} \\
&\qquad \cdot \left(\int_{\mathbb{R}^n} \int_0^1 |\rho_\varepsilon(z)| \, |z| \, \left| D^\beta u(x - z + tz) \right|^p \mathrm{d}t \, \mathrm{d}z \right) \mathrm{d}x \\
&\leq C\varepsilon^{p-1} \sum_{|\beta|=3} \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\rho_\varepsilon(z)| \, |z| \, \left| D^\beta u(w) \right|^p \mathrm{d}z \, \mathrm{d}w \, \mathrm{d}t \\
&\leq C\varepsilon^p \|u\|^p_{W^{3,p}(\mathbb{R}^n)}.
\end{aligned}$$

Altogether, this shows

$$I_{1,\varepsilon} \leq C\varepsilon^p \|u\|^p_{W^{3,p}(\mathbb{R}^n)}. \tag{4.4.10}$$

Ad $I_{2,\varepsilon}$: Applying to Taylor's theorem, we obtain the expansion

$$u(x) - u(y) - \nabla u(x) \cdot (x-y) = -R_2(x,y), \tag{4.4.11}$$

where the error term is given by

$$R_2(x,y) := \sum_{|\beta|=2} \frac{2}{\beta!} \int_0^1 (1-t) D^\beta u(y+t(x-y)) \,\mathrm{d}t\, (x-y)^\beta. \tag{4.4.12}$$

Hence, $I_{2,\varepsilon}$ can be estimated as

$$\begin{aligned} I_{2,\varepsilon} &= C \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n \setminus B_1(x)} J_\varepsilon(x-y) R_2(x,y) \,\mathrm{d}y \right|^p \mathrm{d}x \\ &\le C \sum_{|\beta|=2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus B_1(x)} \int_0^1 |\rho_\varepsilon(x-y)| \left| D^\beta u(y+t(x-y)) \right| \,\mathrm{d}y \,\mathrm{d}t \right)^p \mathrm{d}x \end{aligned}$$

Using once more the splitting (4.4.9), Hölder's inequality, and the changes of variables $z = z(y) := x - y$ and $w = w(x) := x - z + tz$, we infer

$$\begin{aligned} I_{2,\varepsilon} &\le C \sum_{|\beta|=2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\rho_\varepsilon(z)| \,\mathrm{d}z \right)^{p-1} \\ &\qquad \cdot \left(\int_{\mathbb{R}^n \setminus B_1(0)} \int_0^1 |\rho_\varepsilon(z)| \left| D^\beta u(x-z+tz) \right|^p \mathrm{d}t \,\mathrm{d}z \right) \mathrm{d}x \\ &\le C\varepsilon^{p-1} \sum_{|\beta|=2} \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B_1(0)} |\rho_\varepsilon(z)| \, |z| \left| D^\beta u(w) \right|^p \mathrm{d}z \,\mathrm{d}w \,\mathrm{d}t \\ &\le C\varepsilon^p \|u\|^p_{W^{2,p}(\mathbb{R}^n)}, \end{aligned} \tag{4.4.13}$$

Combining (4.4.3) with (4.4.10) and (4.4.13), we eventually conclude that

$$\|\mathcal{L}_\varepsilon u - \mathcal{L} u\|_{L^p(\mathbb{R}^n)} \le C\varepsilon \|u\|_{W^{3,p}(\mathbb{R}^n)} \tag{4.4.14}$$

holds for all $u \in C_c^\infty(\mathbb{R}^n)$.

Let now $u \in W^{3,p}(\mathbb{R}^n)$ be arbitrary. Then, because of density, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ such that $u_k \to u$ in $W^{3,p}(\mathbb{R}^n)$ as $k \to \infty$. In particular, this entails that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{3,p}(\mathbb{R}^n)$. It thus follows from (4.4.14) that

$$\Big(\mathcal{L}_\varepsilon u_k - \mathcal{L} u_k \Big)_{k\in\mathbb{N}} \subseteq L^p(\mathbb{R}^n)$$

is bounded. Hence, according to the Banach–Alaoglu theorem, there exists a function $w \in L^p(\mathbb{R}^n)$ such that

$$\mathcal{L}_\varepsilon u_k - \mathcal{L} u_k \rightharpoonup w \quad \text{in } L^p(\mathbb{R}^n) \text{ as } k \to \infty.$$

Because of Theorem 4.2.2, we know that

$$\mathcal{L}_\varepsilon u_k \to \mathcal{L}_\varepsilon u \quad \text{in } L^p(\mathbb{R}^n) \text{ as } k \to \infty.$$

Moreover, recalling the definition of $\mathcal{L}$, it is further clear that

$$\mathcal{L} u_k \to \mathcal{L} u \quad \text{in } L^p(\mathbb{R}^n) \text{ as } k \to \infty.$$

Consequently, due to uniqueness of the weak limit, we have $w = \mathcal{L}_\varepsilon u - \mathcal{L}u$. Thus, by means of weak lower semi-continuity, we conclude

$$\begin{aligned}\|\mathcal{L}_\varepsilon u - \mathcal{L}u\|_{L^p(\mathbb{R}^n)} &\le \liminf_{k\to\infty} \|\mathcal{L}_\varepsilon u_k - \mathcal{L}u_k\|_{L^p(\mathbb{R}^n)} \\ &\le \limsup_{k\to\infty} C\varepsilon\|u_k\|_{W^{3,p}(\mathbb{R}^n)} = C\varepsilon\|u\|_{W^{3,p}(\mathbb{R}^n)}.\end{aligned}$$

Since $u \in W^{3,p}(\mathbb{R}^n)$ was arbitrary, this proves (b).

Proof of (a). We already know from Theorem 4.2.2(b) that the operator norm of the nonlocal operator $\mathcal{L}_\varepsilon : W^{2,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded by the constant C_*, which is independent of ε. Together with the definition of $\mathcal{L}u$ (see (4.4.2)), this implies that

$$\|\mathcal{L}_\varepsilon u - \mathcal{L}u\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{L}_\varepsilon u\|_{L^p(\mathbb{R}^n)} + \|\mathcal{L}u\|_{L^p(\mathbb{R}^n)} \le C\|u\|_{W^{2,p}(\mathbb{R}^n)}$$

for all $u \in W^{2,p}(\mathbb{R}^n)$. We further know from part (b) that for any $u \in W^{3,p}(\mathbb{R}^n)$, it holds

$$\mathcal{L}_\varepsilon u \to \mathcal{L}u \quad \text{in } L^p(\mathbb{R}^n) \text{ as } \varepsilon \to 0.$$

As the inclusion $W^{3,p}(\mathbb{R}^n) \subset W^{2,p}(\mathbb{R}^n)$ is dense, the assertion of (a) directly follows from the Banach–Steinhaus theorem.

Hence, the proof of Theorem 4.4.1 is complete. □

4.4.2. Transformation onto a reference domain

The following section mainly concerns anisotropic kernels. Due to the matrix M involved in both the local differential operator and the boundary condition, it is very challenging to control the first order term in the Taylor expansion with the aid of the boundary condition for the function u. For isotropic kernels, however, the matrix M is given by the identity matrix I and hence, the computations are simpler in this case. In order to overcome this issue for anisotropic kernels, it turns out to be very useful to study the convergence on a reference domain. This will be motivated in the following lines.

Let $M \in \mathbb{R}^{n\times n}$ be as in (4.1.10) and let $u \in W^{2,p}_M(\Omega)$ (see Definition 2.1.21) be arbitrary. Since the matrix M is symmetric and positive definite, there exists a unique principal square root $A \in \mathbb{R}^{n\times n}$ such that $M = A^2$. Note that A is again symmetric and positive definite. By the definition of M and the change of variables $z \mapsto Az$, we observe that

$$AA^T = M = \frac{1}{2}\int_{\mathbb{R}^n} J(z) z\otimes z \,\mathrm{d}z = \frac{1}{2}A\Big(\int_{\mathbb{R}^n} J(Az) z\otimes z \,\mathrm{d}z\Big)A^T|\det A|,$$

which implies that

$$\frac{1}{2}\int_{\mathbb{R}^n} |\det A|\, J(Az) z\otimes z \,\mathrm{d}z = I, \tag{4.4.15}$$

where $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix. In the following, we introduce $\widehat{\Omega} := A^{-1}\Omega$ as the *reference domain*.

As Ω is a domain of class C^3, so is $\widehat{\Omega}$. Therefore, at least locally, it can be expressed as the zero level set of a function $g \in C^3(\mathbb{R}^n;\mathbb{R})$. Therefore, without loss of generality, we assume that there exists $g \in C^3(\mathbb{R}^n;\mathbb{R})$ such that

$$\partial\Omega = \{x \in \mathbb{R}^n : g(x) = 0\};$$

otherwise, we need to localize. Then, the boundary of $\widehat{\Omega}$ can be expressed as

$$\partial\widehat{\Omega} = \{A^{-1}x \in \mathbb{R}^n : g(x) = 0\} = \{\hat{x} \in \mathbb{R}^n : g(A\hat{x}) = 0\}.$$

Thus, thanks to the chain rule, the normal spaces satisfy the relation

$$\mathrm{N}_{\hat{x}}(\partial\widehat{\Omega}) = \mathrm{span}\{\nabla_{\hat{x}} g(A\hat{x})\} = \mathrm{span}\{A\,\nabla g(A\hat{x})\} = A\,\mathrm{span}\{(\nabla g)(A\hat{x})\} = A\mathrm{N}_{A\hat{x}}(\partial\Omega).$$

In particular, since $A^T = A$, this implies that

$$\mathbf{n}_{\partial\widehat{\Omega}}(\hat{x}) = A\,\mathbf{n}_{\partial\Omega}(A\hat{x}) \quad \text{ for all } x \in \partial\widehat{\Omega}.$$

We now define the function

$$w : \widehat{\Omega} \to \mathbb{R}, \quad w(\hat{x}) = u(A\hat{x}).$$

By means of the chain rule, it is straightforward to check that

$$w \in C_c^2(\,\overline{\widehat{\Omega}}\,) \cap W^{2,p}(\widehat{\Omega}) \quad \text{with} \quad \|w\|_{W^{2,p}(\widehat{\Omega})} \le C\|u\|_{W^{2,p}(\Omega)}\,. \tag{4.4.16}$$

Recalling the definition of $W^{2,p}_M(\Omega)$ (see (2.1.21)) and using the above identities along with the chain rule, we observe that

$$\begin{aligned} \nabla w(\hat{x}) \cdot \mathbf{n}_{\partial\widehat{\Omega}}(\hat{x}) &= A\nabla u(A\hat{x}) \cdot A\mathbf{n}_{\partial\Omega}(A\hat{x}) \\ &= A^2\nabla u(A\hat{x}) \cdot \mathbf{n}_{\partial\Omega}(A\hat{x}) = M\nabla u(A\hat{x}) \cdot \mathbf{n}_{\partial\Omega}(A\hat{x}) = 0 \end{aligned}$$

for all $x \in \partial\widehat{\Omega}$. This means that $w \in W^{2,p}_I(\widehat{\Omega})$. Moreover, by means of the chain rule, we further deduce that

$$\Delta w(\hat{x}) = \mathrm{div}(\nabla w(\hat{x})) = \mathrm{div}(A\nabla u(A\hat{x}))$$

for all $\hat{x} \in \widehat{\Omega}$. We now define $\widehat{\rho} : \mathbb{R}^n \to \mathbb{R}$ with

$$\widehat{\rho}(x) = \rho(Ax)\frac{|x|^2}{|Ax|^2}\,\det A \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{4.4.17}$$

and we write $\widehat{J}$ to denote its associated kernel. Since ρ and J satisfy (J1) and (J2), it is straightforward to check that $\widehat{\rho}$ and $\widehat{J}$ fulfill (J1) and (J2) with

$$\widehat{M} := \frac{1}{2}\int_{\mathbb{R}^n} \widehat{J}(z)\, z \otimes z \,\mathrm{d}z = I, \tag{4.4.18}$$

$$\widehat{A} := I \tag{4.4.19}$$

in place of M and A, respectively. In particular, for all $\varepsilon > 0$, we have

$$\widehat{J}(x) = J(Ax)\det A \quad \text{and} \quad \widehat{J_\varepsilon}(x) = J_\varepsilon(Ax)\det A \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

and in view of assumption (2.1.13), we have

$$\int_{\mathbb{R}^{n-1}} \widehat{J}\left(Q\begin{pmatrix} z' \\ z_n \end{pmatrix}\right) z'\,\mathrm{d}z' = 0 \quad \text{for almost all } z_n \in \mathbb{R} \text{ and all } Q \in \mathrm{O}(n). \tag{4.4.20}$$

This condition will be crucial in Step 1 of the proof of Theorem 2.1.7 and it is the reason, why (2.1.13) was demanded in Assumption (J2) in the first place.

Now, we further introduce the nonlocal operator

$$\begin{aligned} &\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} : C_c^2(\overline{\widehat{\Omega}}) \cap W^{2,p}(\widehat{\Omega}) \to L^p_{\text{loc}}(\widehat{\Omega}), \\ &\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} v(x) = \text{P.V.} \int_{\widehat{\Omega}} \widehat{J}_\varepsilon(x-y)\big(v(x)-v(y)\big)\,\mathrm{d}y. \end{aligned} \tag{4.4.21}$$

Since $\widehat{\rho}$ and $\widehat{J}$ satisfy (J1) and (J2), we already know from Corollary 4.2.3 that this operator is well-defined and linear. Thus, in order to verify the claims in Theorem 2.1.7, we need to establish the following key estimates:

$$\|\mathcal{L}_\varepsilon^{\widehat{\Omega}} v\|_{L^p(\widehat{\Omega})} \le C\|v\|_{W^{2,p}(\widehat{\Omega})} \qquad \text{for all } v \in C_c^2(\overline{\widehat{\Omega}}) \cap W_I^{2,p}(\widehat{\Omega}), \tag{4.4.22}$$

$$\|\mathcal{L}_\varepsilon^{\widehat{\Omega}} v + \Delta v\|_{L^p(\widehat{\Omega})} \le C\sqrt[p]{\varepsilon}\|v\|_{W^{3,p}(\widehat{\Omega})} \qquad \text{for all } v \in C_c^3(\overline{\widehat{\Omega}}) \cap W_I^{3,p}(\widehat{\Omega}). \tag{4.4.23}$$

Once this is achieved, we use the changes of variables $x \mapsto \hat{x} = A^{-1}x$ and $y \mapsto \hat{y} = A^{-1}y$ to derive the identity

$$\begin{aligned} &\|\mathcal{L}_\varepsilon^{\Omega} u - \mathcal{L}^{\Omega} u\|^p_{L^p(\Omega)} \\ &= \int_{A\widehat{\Omega}} \left| \text{P.V.} \int_{A\widehat{\Omega}} J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y + \operatorname{div}\big(A\nabla u(x)\big)\right|^p \,\mathrm{d}x \\ &= \det A \int_{\widehat{\Omega}} \left| \text{P.V.} \int_{\widehat{\Omega}} J_\varepsilon\big(A(\hat{x}-\hat{y})\big)\big(w(\hat{x})-w(\hat{y})\big) \det A\,\mathrm{d}\hat{y} + \Delta w(\hat{x})\right|^p \,\mathrm{d}\hat{x} \\ &= \det A\, \|\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} w + \Delta w\|^p_{L^p(\widehat{\Omega})}. \end{aligned} \tag{4.4.24}$$

In particular, using (4.4.22) and (4.4.16), we infer that

$$\begin{aligned} \|\mathcal{L}_\varepsilon^{\Omega} u\|_{L^p(\Omega)} &\le \|\mathcal{L}_\varepsilon^{\Omega} u - \mathcal{L}^{\Omega} u\|_{L^p(\Omega)} + \|\mathcal{L}^{\Omega} u\|_{L^p(\Omega)} \\ &\le \|\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} w + \Delta w\|_{L^p(\widehat{\Omega})} + \|\Delta w\|_{L^p(\widehat{\Omega})} \\ &\le \|\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} w\|_{L^p(\widehat{\Omega})} + 2\|\Delta w\|_{L^p(\widehat{\Omega})} \\ &\le C\|w\|_{W^{2,p}(\widehat{\Omega})} \le C\|u\|_{W^{2,p}(\Omega)}. \end{aligned} \tag{4.4.25}$$

Consequently, $\mathcal{L}_\varepsilon^{\Omega}$ can be extended to a bounded linear operator

$$\mathcal{L}_\varepsilon^{\Omega} : W_M^{2,p}(\Omega) \to L^p(\Omega)\,,$$

which fulfills (2.1.23). Now, we take an arbitrary $u \in C_c^3(\overline{\Omega}) \cap W^{3,p}(\Omega)$. This means that

$$w \in C_c^3(\overline{\widehat{\Omega}}) \cap W^{3,p}(\widehat{\Omega}) \quad \text{with} \quad \|w\|_{W^{3,p}(\widehat{\Omega})} \le C\|u\|_{W^{3,p}(\Omega)}\,. \tag{4.4.26}$$

Combining (4.4.23), (4.4.24) and (4.4.16), we further have

$$\begin{aligned} \|\mathcal{L}_\varepsilon^{\Omega} u - \mathcal{L}^{\Omega} u\|_{L^p(\Omega)} &= \|\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} w + \Delta w\|^p_{L^p(\widehat{\Omega})} \\ &\le C\sqrt[p]{\varepsilon}\|w\|_{W^{3,p}(\widehat{\Omega})} \le C\sqrt[p]{\varepsilon}\|u\|_{W^{3,p}(\Omega)}. \end{aligned} \tag{4.4.27}$$

Because of density, it follows that (2.1.25) holds. Proceeding as in Step 3 of the proof of Theorem 4.4.4, we apply the Banach–Steinhaus theorem to conclude that (2.1.24) is fulfilled. This means that all the claims are established.

Altogether, we thus make the following observation: On account of the global coordinate transform by A, we can reformulate the problem in terms of the reference domain $\widehat{\Omega}$, where our analysis reduces to the case $M = I$. Recalling the inequalities (4.4.25) and (4.4.27) as well as the key estimates (4.4.22) and (4.4.23), it suffices to study the nonlocal operator on the reference domain $\widehat{\Omega}$, in order to show the boundedness as well as the convergence along with rates of convergence. Hence, we may assume without loss of generality $M = I$ in the subsections below.

4.4.3. Convergence on a bent half space

We consider the bent half space

$$\mathbb{R}^n_\gamma := \{x \in \mathbb{R}^n : x_n > \gamma(x_1, \dots, x_{n-1})\}$$

for some prescribed function $\gamma \in C^3_b(\mathbb{R}^{n-1})$. In this chapter, we study the nonlocal operator on a bent half space as introduced before. To be more precise, we consider the operator given by

$$\begin{aligned}\mathcal{L}_\varepsilon u(x) &= \text{P.V.} \int_{\mathbb{R}^n_\gamma} J_\varepsilon(x-y)(u(x)-u(y))\,\mathrm{d}y \\ &= \lim_{r\to 0} \int_{\mathbb{R}^n_\gamma \cap B_r(x)} J_\varepsilon(x-y)(u(x)-u(y))\,\mathrm{d}y,\end{aligned}$$

where $x \in \mathbb{R}^n_\gamma$ and $u \in C^2_c(\overline{\mathbb{R}^n_\gamma})$. Thanks to Theorem 4.2.3, we already know that this operator is well-defined and linear. Hence, we still need to verify that this operator is bounded for functions $u \in W^{2,p}(\mathbb{R}^n_\gamma)$, in order to extend it to a bounded linear operator on $W^{2,p}(\mathbb{R}^n_\gamma)$. This will be one of the main results discussed in this section. Once having this, we can then prove the strong convergence of the nonlocal operator along with rates of convergence.

Owing to the arguments in Section 4.4.2, we may assume without loss of generality that $M = I$. Indeed, this assumption is crucial, since it simplifies the calculations below. Moreover, for ease in notation, we use the same notation as on Ω.

The main result of this chapter then is the following:

Theorem 4.4.4. *Let $\gamma \in C^3_b(\mathbb{R}^{n-1})$, $\varepsilon \in (0,1]$, and suppose that* (J1) *and* (J2) *hold with $M = I$. Then, if $\|\gamma\|_{C^1_b(\mathbb{R}^{n-1})}$ is sufficiently small, the operator introduced in* (4.2.5) *(restricted to $C^2_c(\overline{\mathbb{R}^n_\gamma}) \cap W^{2,p}_I(\mathbb{R}^n_\gamma)$) can be extended to a bounded linear operator*

$$\mathcal{L}^{\mathbb{R}^n_\gamma}_\varepsilon : W^{2,p}_I(\mathbb{R}^n_\gamma) \to L^p(\mathbb{R}^n_\gamma) \tag{4.4.28}$$

with

$$\|\mathcal{L}^{\mathbb{R}^n_\gamma}_\varepsilon u\|_{L^p(\mathbb{R}^n_\gamma)} \le C\|u\|_{W^{2,p}(\mathbb{R}^n_\gamma)}, \tag{4.4.29}$$

for all $u \in W^{2,p}_I(\mathbb{R}^n_\gamma)$, where C is a positive constant depending only on $\mathbb{R}^n_\gamma$, ρ and p. Moreover, there exists a positive constant C depending only on $\mathbb{R}^n_\gamma$, ρ and p such that for *all* $\varepsilon > 0$ *and all* $u \in W_I^{3,p}(\mathbb{R}^n_\gamma)$*, it holds*

$$\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u + \Delta u\|_{L^p(\mathbb{R}^n_\gamma)} \leq C \sqrt[p]{\varepsilon}\, \|u\|_{W^{3,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.30}$$

Furthermore, for all $u \in W_I^{2,p}(\mathbb{R}^n_\gamma)$*, it holds*

$$\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u \to -\Delta u \quad \text{in } L^p(\mathbb{R}^n_\gamma) \text{ as } \varepsilon \to 0. \tag{4.4.31}$$

Remark 4.4.5. Observe that this result also holds true for kernels satisfying (I1) and (I2), since $M = I$ due to the radial symmetry, cf. Remark 4.4.3. The proof is based on the same arguments and estimates. However, we do not need to impose condition (2.1.13), since it is automatically satisfied by isotropic kernels. Thus, the corresponding estimate is even simpler in the isotropic case. Let us also note that this result includes kernels with $W^{1,1}$-regularity, since we can perform the same estimates due to their regularity.

Corollary 4.4.6. *Let* $\gamma \in C_b^3(\mathbb{R}^{n-1})$*,* $Q \in \mathrm{SO}(n)$*,* $\varepsilon \in (0,1]$*, and suppose that* (J1) *and* (J2) *hold with* $M = I$*. Then, the results of Theorem 4.4.4 hold true for* $Q\mathbb{R}^n_\gamma$ *instead of* $\mathbb{R}^n_\gamma$ *and* $W_Q^{k,p}(Q\mathbb{R}^n_\gamma)$ *instead of* $W_I^{k,p}(\mathbb{R}^n_\gamma)$ *for* $k = 2, 3$*.*

In the proof of Theorem 4.4.4, we want use a suitable coordinate transform, in order to reduce the problem to the half space $\mathbb{R}^n_+$. However, it is crucial that this diffeomorphism maps the exterior normal vector $\mathbf{n}_{\partial\mathbb{R}^n_+}$ of $\partial\mathbb{R}^n_+$ to the exterior normal vector $\mathbf{n}_{\partial\mathbb{R}^n_\gamma}$ of $\partial\mathbb{R}^n_\gamma$. This is important to control the first order term in the Taylor expansion with the aid of the boundary condition of u. Since the construction of the diffeomorphism is very technical and non-trivial, we briefly explain the main idea of its construction:

Construction of the diffeomorphism F_γ

The setting. For $k \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ be a $C^{k,1}$-domain. This means, for every $x_0 \in \partial\Omega$, we can rotate and shift the coordinate system in such a way that its origin is x_0 and there exists a neighborhood $U = U(x_0) \subset \mathbb{R}^n$ of x_0 with

$$\Omega \cap U = \{x \in U : \; x_n > \gamma(x')\},$$

where $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ is a $C^{k,1}$-function. It is the goal to construct a suitable diffeomorphism $F_\gamma : V \to U$, where $V \subset \mathbb{R}^n$ is a neighborhood of 0, that maps the exterior normal vector $\mathbf{n}_{\partial\mathbb{R}^n_+}$ of $\partial\mathbb{R}^n_+$ to the exterior normal vector $\mathbf{n}_{\partial\Omega}$ of $\partial\Omega$ in the sense that

$$\partial_{x_n} F_\gamma(x', 0) = -\mathbf{n}_{\partial\Omega}(x') \quad \text{ for } (x', 0) \in V$$

and that conserves the $C^{k,1}$-regularity.

The result. The following result is due to Schumacher [115]. Given the setting as above, there exists a chart $F_\gamma : V \to U$ with the properties

$$F_\gamma(0) = x_0, \quad F_\gamma(V \cap (\mathbb{R}^{n-1} \times \{0\})) = U \cap \partial\Omega, \quad F_\gamma(V \cap \mathbb{R}^n_+) = U \cap \Omega$$

as well as

$$\partial_{x_n} F_\gamma(x',0) = -\mathbf{n}_{\partial\Omega}(x') \quad \text{ for } (x',0) \in V,$$

where $\mathbf{n}_{\partial\Omega}(x')$ denotes the exterior normal vector at $(x', \gamma(x')) \in \partial\Omega$, and F_γ conserves the $C^{k,1}$-regularity, cf. [115, Lemma 2.1].

Construction of the chart. After rotating and shifting the coordinate system, we can assume without loss of generality that $x_0 = 0$, $(0, \gamma(0)) = 0$ and $\nabla\gamma(0) = 0$. Moreover, let $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n-1})$ be radially symmetric such that

$$\mathrm{supp}\varphi \subset B_1(0) \quad \text{ and } \int_{\mathbb{R}^{n-1}} \varphi \,\mathrm{d}x = 1.$$

For $t \neq 0$, we introduce the notation $\varphi_t(x') := |t|^{-n-1}\varphi(x'/t)$. Then, we can define the chart F_γ by

$$F_\gamma(x', x_n) := \begin{cases} \begin{pmatrix} x' \\ \gamma(x') \end{pmatrix} - (x_n \varphi_{x_n} * \mathbf{n}_{\partial\Omega})(x') & \text{if } x_n \neq 0, \\ \begin{pmatrix} x' \\ \gamma(x') \end{pmatrix} & \text{if } x_n = 0, \end{cases} \tag{4.4.32}$$

where the convolution is taken in $\mathbb{R}^{n-1}$. The chart F_γ as introduced in (4.4.32) then fulfills the desired properties. For details, we refer to the proof of [115, Lemma 2.1].

Generalizations. The result in [115] has then been generalized by Abels and Terasawa in [14], who considered functions $\gamma \in W^{2-\frac{r}{2},r}(\mathbb{R}^{n-1})$ for $r > n$. In particular, they proved that there exists some $F_\gamma \in W^{2,r}(\mathbb{R}^n)^n$ such that $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism, $F_\gamma(\mathbb{R}^n_+) = \mathbb{R}^n_\gamma$, $F_\gamma(x',0) = (x', \gamma(x'))$ and $-\partial_{x_n} F_\gamma(x',0) = \mathbf{n}_{\partial\mathbb{R}^n_\gamma}(x', \gamma(x'))$, where $\mathbf{n}_{\partial\mathbb{R}^n_\gamma}$ denotes the exterior unit normal of $\partial\mathbb{R}^n_\gamma$, cf. [14, Prop. 1]. The proof is based on similar ideas as in [115, Lemma 2.1].

Consequences. In the following, we use the same notation as in [14], i.e, we define

$$\begin{aligned} (F_\gamma^* u)(x) &:= u(F_\gamma(x)) && \text{for } u : \mathbb{R}^n_\gamma \to \mathbb{R}, \\ (F_\gamma^{*,-1} v)(x) &:= v(F_\gamma^{-1}(x)) && \text{for } u : \mathbb{R}^n_+ \to \mathbb{R}. \end{aligned}$$

Let $U = U(x')$ be the orthonormal matrix that maps the exterior unit normal $\mathbf{n}_{\partial\mathbb{R}^n_\gamma}$ on $\partial\mathbb{R}^n_\gamma$ at $(x', \gamma(x'))$ to $-e_n$, which is the exterior unit normal on $\mathbb{R}^n_+$. Then, it holds

$$\nabla F_\gamma^{*,-1} v = F_\gamma^{*,-1} U^T(x') H(x) \nabla v(x),$$

where $H(x)\xi = U(x')(DF_\gamma(x))^{-1}\xi$ for $v \in C^1(\overline{\mathbb{R}^n_\gamma})$. From [14, Prop. 1], we then infer that $(H_{|x_n=0})^{-T}$ has the structure

$$H(x',0)^{-T} = \left(\begin{array}{ccc|c} & & & 0 \\ & H'(x',0)^{-T} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right) \quad \text{for all } x' \in \mathbb{R}^{n-1}, \tag{4.4.33}$$

where the matrix $H'(x',0)^{-T}$ depends smoothly on $\nabla'\gamma(x')$. Observe that the matrix $H_{|x_n=0}$ has the same structure with $H'(x',0)$ instead of $H'(x',0)^{-T}$.

Remark 4.4.7. The structure of the matrix H as in (4.4.33) is a crucial ingredient for the proof of Theorem 4.4.4 below. Using similar arguments as before, DF_γ can be decomposed as

$$DF_\gamma(x) = U(x')H(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Thanks to this decomposition, we can perform a suitable change of variables, in order to simplify the integral involving the first order term in the Taylor expansion. Then, we can exploit the boundary condition to derive error estimates depending on ε, which yield rates of convergence.

In the remainder of this section, we present the proof of Theorem 4.4.4 and Corollary 4.4.6.

Proof of Theorem 4.4.4. In this proof, the letter C will denote generic positive constants depending only on $\mathbb{R}^n_\gamma$, ρ and p, which may change their value from line to line. To provide a cleaner presentation, we will usually refrain from indicating the principal value by the symbol P.V. whenever the meaning is clear. The proof is split into three steps.

Step 1: Proof of Estimate (4.4.29) for $u \in C_c^2(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{2,p}(\mathbb{R}^n_\gamma)$.
Let $u \in C_c^2(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{2,p}(\mathbb{R}^n_\gamma)$ be arbitrary. Hence, according to Corollary 4.2.3, the expression

$$\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u(x) = \text{P.V.} \int_{\mathbb{R}^n_\gamma} J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y$$

is well-defined. Our first goal is to show that $\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u \in L^p(\mathbb{R}^n_\gamma)$ and that there exists a constant $C > 0$ depending only on $\mathbb{R}^n_\gamma$, ρ and p such that

$$\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u\|_{L^p(\mathbb{R}^n_\gamma)} \le C\|u\|_{W^{2,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.34}$$

By construction, it is clear that $\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u$ is measurable. Since $\mathbb{R}^n_\gamma$ is of class C^3, we can find an extension $\tilde{u} \in C_c^2(\mathbb{R}^n)$ with $u|_{\mathbb{R}^n_\gamma} = u$. In particular, $\tilde{u}$ can be chosen in such a way that

$$\|\tilde{u}\|_{W^{2,p}(\mathbb{R}^n)} \le C\|u\|_{W^{2,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.35}$$

Using Theorem 4.2.2, we obtain the estimate

$$\begin{aligned} \|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u\|^p_{L^p(\mathbb{R}^n_\gamma)} &\le C\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} \tilde{u}\|^p_{L^p(\mathbb{R}^n)} + C\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \\ &\le C\|u\|^p_{W^{2,p}(\mathbb{R}^n_\gamma)} + C\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)}, \end{aligned} \tag{4.4.36}$$

where the error term is given by

$$\mathcal{R}_\varepsilon \tilde{u}(x) := \int_{(\mathbb{R}^n_\gamma)^c} J_\varepsilon(x-y)\big(u(x)-\tilde{u}(y)\big)\,\mathrm{d}y \quad \text{for a.e. } x \in \mathbb{R}^n_\gamma.$$

Therefore, it remains to show that

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \le C\|u\|^p_{W^{2,p}(\mathbb{R}^n_\gamma)} \tag{4.4.37}$$

since (4.4.34) then follows by combining (4.4.36) and (4.4.37).

Since $\gamma \in C^3_b(\mathbb{R}^{n-1})$, we infer from [115, Lemma 2.1] that there exists a $C^{2,1}$-diffeomorphism $F_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ with $F_\gamma(\mathbb{R}^n_+) = \mathbb{R}^n_\gamma$, which satisfies

$$F_\gamma(x',0) = \begin{pmatrix} x' \\ \gamma(x') \end{pmatrix} \quad \text{and} \quad \partial_{x_n} F_\gamma(x)|_{x_n=0} = -\mathbf{n}_{\partial\mathbb{R}^n_\gamma}(x',\gamma(x')) \tag{4.4.38}$$

for all $x' \in \mathbb{R}^{n-1}$. Since $F_\gamma \in C^{2,1}(\mathbb{R}^n;\mathbb{R}^n)$, we further have $DF_\gamma \in W^{2,\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ with

$$\|DF_\gamma\|_{W^{2,\infty}}(\mathbb{R}^n;\mathbb{R}^{n\times n}) \le C\|\gamma\|_{C^3_b(\mathbb{R}^{n-1})} \le C. \tag{4.4.39}$$

In the following, we write

$$d_{F_\gamma} := |\det DF_\gamma|$$

as an abbreviation. In particular, due to (4.4.39), we have

$$\|d_{F_\gamma}\|_{L^\infty(\mathbb{R}^n)} \le C. \tag{4.4.40}$$

Moreover, assuming that $\|\gamma\|_{C^1_b(\mathbb{R}^{n-1})}$ to be sufficiently small, we can ensure that

$$\sup_{x\in\mathbb{R}^n} |DF_\gamma(x) - I| \le \frac{1}{6}. \tag{4.4.41}$$

Now, by the change of variables by $x \mapsto F_\gamma(x)$, we infer that

$$\begin{aligned} \|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} &= \int_{\mathbb{R}^n_\gamma} \left| \int_{(\mathbb{R}^n_\gamma)^c} J_\varepsilon(x-y)\big(u(x) - \tilde{u}(y)\big)\,\mathrm{d}y \right|^p \mathrm{d}x \qquad (4.4.42) \\ &= \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_-} J_\varepsilon\big(F_\gamma(x) - F_\gamma(y)\big)\Big(u(F_\gamma(x)) - \tilde{u}(F_\gamma(y))\Big)\, d_{F_\gamma}(y)\,\mathrm{d}y \right|^p d_{F_\gamma}(x)\,\mathrm{d}x, \end{aligned}$$

For ease in notation, we introduce the functions

$$w : \mathbb{R}^n \to \mathbb{R}^n, \qquad w(x) = \tilde{u}(F_\gamma(x)), \tag{4.4.43}$$

$$G_\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}, \quad G_\gamma(x,y) := \int_0^1 DF_\gamma(y + t(x-y))\,\mathrm{d}t. \tag{4.4.44}$$

In particular, this means that $G_\gamma(x,x) = DF_\gamma(x)$ for all $x \in \mathbb{R}^n$. Using the chain rule along with (4.4.35), (4.4.39) and (4.4.40), we deduce

$$\|w\|_{W^{2,p}(\mathbb{R}^n)} \le C\|\tilde{u}\|_{W^{2,p}(\mathbb{R}^n)} \le C\|u\|_{W^{2,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.45}$$

Moreover, by means of the fundamental theorem of calculus, we obtain

$$F_\gamma(x) - F_\gamma(y) = G_\gamma(x,y)\,(x-y).$$

Let now $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^n$ be arbitrary. Recalling that $G_\gamma(\tilde{x},\tilde{x}) = DF_\gamma(\tilde{x})$, we infer from (4.4.41) that

$$\big|[G_\gamma(\tilde{x},\tilde{x}) - I](x-y)\big| \le \frac{1}{6}|x-y|. \tag{4.4.46}$$

This implies that

$$\big|G_\gamma(\tilde{x},\tilde{x})(x-y)\big| \geq \Big||x-y| - \big|\big[G_\gamma(\tilde{x},\tilde{x}) - I\big](x-y)\big|\Big| \geq \frac{5}{6}|x-y|. \tag{4.4.47}$$

Moreover, recalling the definition of G_γ in (4.4.44) and invoking once more (4.4.41), we deduce that

$$\begin{aligned}&\big|\big[G_\gamma(\tilde{x},\tilde{y}) - G_\gamma(\tilde{x},\tilde{x})\big](x-y)\big| \\ &\qquad \leq \Big[\big|G_\gamma(\tilde{x},\tilde{y}) - I\big| + \big|G_\gamma(\tilde{x},\tilde{x}) - I\big|\Big]\,|x-y| \leq \frac{1}{3}|x-y|.\end{aligned} \tag{4.4.48}$$

Combining (4.4.47) and (4.4.48), we conclude that

$$\begin{aligned}&\big|G_\gamma(\tilde{x},\tilde{y})(x-y)\big| \\ &\quad \geq \Big|\big|G_\gamma(\tilde{x},\tilde{x})(x-y)\big| - \big|\big[G_\gamma(\tilde{x},\tilde{y}) - G_\gamma(\tilde{x},\tilde{x})\big](x-y)\big|\Big| \geq \frac{1}{2}|x-y|\end{aligned} \tag{4.4.49}$$

for all $x,\tilde{x},y,\tilde{y} \in \mathbb{R}^n$. Next, using (4.4.44), we can reformulate (4.4.42) as

$$\begin{aligned}&\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \\ &\quad = \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_-} J_\varepsilon\big(G_\gamma(x,y)(x-y)\big)\big(w(x)-w(y)\big)\, d_{F_\gamma}(y)\,\mathrm{d}y \right|^p d_{F_\gamma}(x)\,\mathrm{d}x.\end{aligned} \tag{4.4.50}$$

Since ρ is assumed to have compact support, we find $R>0$ such that $\operatorname{supp}\rho \subset B_R(0)$. Using this radius R, we introduce the sets

$$\mathcal{A}_{2R} \coloneqq \mathbb{R}^{n-1}\times(0,2R) \subset \mathbb{R}^n_+ \quad \text{and} \quad \mathcal{B}_{2R} \coloneqq \mathbb{R}^{n-1}\times(-2R,0) \subset \mathbb{R}^n_-\,.$$

Hence, we infer from (4.4.50) that

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C\big(I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon\big), \tag{4.4.51}$$

where

$$\begin{aligned}I^1_\varepsilon &\coloneqq \int_{\mathbb{R}^n_+\setminus\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon\big(G_\gamma(x,y)(x-y)\big)\big(w(x)-w(y)\big)\, d_{F_\gamma}(y)\,\mathrm{d}y \right|^p d_{F_\gamma}(x)\,\mathrm{d}x\,, \\ I^2_\varepsilon &\coloneqq \int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_-\setminus\mathcal{B}_{2R}} J_\varepsilon\big(G_\gamma(x,y)(x-y)\big)\big(w(x)-w(y)\big)\, d_{F_\gamma}(y)\,\mathrm{d}y \right|^p d_{F_\gamma}(x)\,\mathrm{d}x\,, \\ I^3_\varepsilon &\coloneqq \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon\big(G_\gamma(x,y)(x-y)\big)\big(w(x)-w(y)\big)\, d_{F_\gamma}(y)\,\mathrm{d}y \right|^p d_{F_\gamma}(x)\,\mathrm{d}x\,.\end{aligned}$$

These integral terms will now be handled separately.

Ad I^1_ε*:* We first observe that for all $x \in \mathbb{R}^n_+ \setminus \mathcal{A}_{2R}$ and all $y \in \mathcal{B}_{2R}$, we have

$$|x-y| \geq |x_n - y_n| = x_n - y_n \geq 2R.$$

Hence, due to (4.4.49), we have

$$\big|G_\gamma(x,y)(x-y)\big| \geq \frac{1}{2}|x-y| \geq R.$$

Since $\operatorname{supp}\rho \subset B_R(0)$, we have $\operatorname{supp}\rho_\varepsilon \subset B_{R\varepsilon}(0)$. For all $\varepsilon \in (0,1]$, we thus have

$$\rho_\varepsilon(G_\gamma(x,y)(x-y)) = 0$$

for all $x \in \mathbb{R}^n_+ \setminus \mathcal{A}_{2R}$ and all $y \in \mathcal{B}_{2R}$. This implies that $I^1_\varepsilon = 0$.

Ad I^2_ε*:* Here, we notice that for all $x \in \mathbb{R}^n_+$ and all $y \in \mathbb{R}^n_- \setminus \mathcal{B}_{2R}$, it holds

$$|x-y| \geq |x_n - y_n| = x_n - y_n \geq 2R.$$

Hence, arguing analogously as for the term I^1_ε, we conclude that $I^2_\varepsilon = 0$.

Ad I^3_ε*:* A straightforward computation yields

$$I^3_\varepsilon \leq C\big(I^{3,2}_\varepsilon + I^{3,1}_\varepsilon + I^{3,3}_\varepsilon\big) \tag{4.4.52}$$

where

$$\begin{aligned}
I^{3,1}_\varepsilon &:= \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(G_\gamma(x,x)(x-y))(w(x)-w(y))\, d_{F_\gamma}(x)\, \mathrm{d}y \right|^p \mathrm{d}x \\
I^{3,2}_\varepsilon &:= \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(G_\gamma(x,y)(x-y))(w(x)-w(y))(d_{F_\gamma}(y) - d_{F_\gamma}(x))\, \mathrm{d}y \right|^p \mathrm{d}x \\
I^{3,3}_\varepsilon &:= \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} \Big(J_\varepsilon(G_\gamma(x,y)(x-y)) - J_\varepsilon(G_\gamma(x,x)(x-y))\Big) \right. \\
&\qquad\qquad \left. \cdot (w(x)-w(y))\, d_{F_\gamma}(x)\, \mathrm{d}y \right|^p \mathrm{d}x
\end{aligned}$$

These terms will be handled separately.

Ad $I^{3,1}_\varepsilon$*:* Due to the construction of F_γ, there exist continuously differentiable functions $U : \mathbb{R}^n \to \mathrm{SO}(n)$, $H : \mathbb{R}^n \to \mathrm{GL}_n(\mathbb{R})$ and $H' : \mathbb{R}^n \to \mathrm{GL}_{n-1}(\mathbb{R})$ with

$$H(x) = \left(\begin{array}{ccc|c} & & & 0 \\ & H'(x) & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right) \quad \text{for all } x \in \mathbb{R}^n \tag{4.4.53}$$

such that

$$G_\gamma(x,x) = \mathrm{D}F_\gamma(x) = U(x)H(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{4.4.54}$$

For more details on this decomposition, we refer to [14, Proof of Corollary 2]. In particular, we have

$$H(x)^{-1} = \left(\begin{array}{ccc|c} & & & 0 \\ & H'(x)^{-1} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right) \quad \text{for all } x \in \mathbb{R}^n \tag{4.4.55}$$

and since $\det U(x) = 1$ for all $x \in \mathbb{R}^n$, it further holds that

$$\det H(x) = d_{F_\gamma(x)} \quad \text{and} \quad \det\big(H(x)^{-1}\big) = \big(d_{F_\gamma(x)}\big)^{-1} \quad \text{for all } x \in \mathbb{R}^n.$$

Recalling $G_\gamma(x,x) = DF_\gamma(x)$, we now apply Taylor's theorem to derive the estimate

$$I_\varepsilon^{3,1} \le C(I_\varepsilon^{3,1,1} + I_\varepsilon^{3,1,2}), \tag{4.4.56}$$

where

$$I_\varepsilon^{3,1,1} := \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(DF_\gamma(x)(x-y)) \nabla w(x) \cdot (x-y) d_{F_\gamma}(x) \, \mathrm{d}y \right|^p \mathrm{d}x,$$
$$I_\varepsilon^{3,1,2} := \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(DF_\gamma(x)(x-y)) R_2(x,y) d_{F_\gamma}(x) \, \mathrm{d}y \right|^p \mathrm{d}x,$$

and the error term is given by

$$R_2(x,y) = \sum_{|\beta|=2} \frac{2}{\beta!} \left(\int_0^1 (1-t) D^\beta w(y + t(x-y)) \, \mathrm{d}t \right) (x-y)^\beta. \tag{4.4.57}$$

Ad $I_\varepsilon^{3,1,1}$*:* Using the change of variables $y \mapsto z = x - y$, then $z \mapsto \tilde{z} = H(x)z$, and finally $\tilde{z} \mapsto y = x - \tilde{z}$, the term $I_\varepsilon^{3,1,1}$ can be reformulated as

$$\begin{aligned} I_\varepsilon^{3,1,1} &= \int_{\mathcal{A}_{2R}} \left| \int_{\mathbb{R}^{n-1} \times (x_n, x_n + 2R)} J_\varepsilon(U(x)H(x)z) \nabla w(x) \cdot z \, \mathrm{d}z \, d_{F_\gamma}(x) \right|^p \mathrm{d}x \\ &= \int_{\mathcal{A}_{2R}} \left| \int_{\mathbb{R}^{n-1} \times (x_n, x_n + 2R)} J_\varepsilon(U(x)\tilde{z}) \, H(x)^{-T} \nabla w(x) \cdot \tilde{z} \, \mathrm{d}\tilde{z} \right|^p \mathrm{d}x \\ &= \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(U(x)(x-y)) \, H(x)^{-T} \nabla w(x) \cdot (x-y) \, \mathrm{d}y \right|^p \mathrm{d}x \\ &= \int_{\mathcal{A}_{2R}} \left| H(x)^{-T} \nabla w(x) \cdot \int_{\mathcal{B}_{2R}} J_\varepsilon(U(x)(x-y)) \, (x-y) \, \mathrm{d}y \right|^p \mathrm{d}x. \end{aligned}$$

Thus, exploiting the structure of $H(x)^{-T}$, we use Condition (2.1.13) to infer that

$$I_\varepsilon^{3,1,1} = \int_{\mathcal{A}_{2R}} \left| \partial_{x_n} w(x) \int_{\mathcal{B}_{2R}} J_\varepsilon(U(x)(x-y))(x_n - y_n) \, \mathrm{d}y \right|^p \mathrm{d}x.$$

Since $u \in C_c^2(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{2,p}(\mathbb{R}^n_\gamma)$, we deduce by means of the chain rule and (4.4.38) that

$$\partial_{x_n} w(x',0) = \partial_{x_n}(u \circ F_\gamma)(x',0) = -\nabla u(x', \gamma(x')) \cdot \mathbf{n}(x', \gamma(x')) = 0 \tag{4.4.58}$$

for all $x' \in \mathbb{R}^{n-1}$. Moreover, we have

$$|x_n| \le |x_n - y_n| \quad \text{for all } x \in \mathcal{A}_{2R} \text{ and } y \in \mathcal{B}_{2R}.$$

Thus, invoking the fundamental theorem of calculus, we have

$$\partial_{x_n} w(x) = \partial_{x_n} w(x) - \partial_{x_n} w(x',0) = \int_0^1 \partial_{x_n}^2 w(x', t x_n) \, x_n \, \mathrm{d}t$$

for all $x = (x', x_n) \in \mathbb{R}^n$. Hence, we obtain

$$\begin{aligned}
I_\varepsilon^{3,1,1} &= \int_{\mathcal{A}_{2R}} \left| (\partial_{x_n} w(x) - \partial_{x_n} w(x',0)) \int_{\mathcal{B}_{2R}} J_\varepsilon(U(x)(x-y))\,(x_n - y_n)\,\mathrm{d}y \right|^p \mathrm{d}x \\
&= \int_{\mathcal{A}_{2R}} \left| \int_0^1 \int_{\mathcal{B}_{2R}} J_\varepsilon(U(x)(x-y))\,\partial_{x_n}^2 w(x', tx_n)\, x_n\,(x_n - y_n)\,\mathrm{d}y\,\mathrm{d}t \right|^p \mathrm{d}x \\
&\le \int_{\mathcal{A}_{2R}} \left(\int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon(U(x)(x-y))\,|\partial_{x_n}^2 w(x', tx_n)|\,\mathrm{d}y\,\mathrm{d}t \right)^p \mathrm{d}x \qquad (4.4.59) \\
&= \int_{\mathcal{A}_{2R}} \left[\left(\int_0^1 |\partial_{x_n}^2 w(x', tx_n)|\,\mathrm{d}t \right) \left(\int_{\mathcal{B}_{2R}} \rho_\varepsilon(U(x)(x-y))\,\mathrm{d}y \right) \right]^p \mathrm{d}x
\end{aligned}$$

Applying the change of variables $y \mapsto z = U(x)(x-y)$ along with Lemma 4.2.1(a), we deduce that

$$\int_{\mathcal{B}_{2R}} \rho_\varepsilon(U(x)(x-y))\,\mathrm{d}y \le \int_{\mathbb{R}^n} \rho_\varepsilon(z)\,\mathrm{d}z = \|\rho\|_{L^1(\mathbb{R}^n)}. \qquad (4.4.60)$$

Hence, recalling the definition of $\mathcal{A}_{2R}$ and using the change of variables $t \mapsto s = tx_n$, we get

$$I_\varepsilon^{3,1,1} \le C \int_{\mathbb{R}^{n-1}} \int_0^\infty \left(\frac{1}{x_n} \int_0^{x_n} |\partial_{x_n}^2 w(x', s)|\,\mathrm{d}s \right)^p \mathrm{d}x_n\,\mathrm{d}x' \qquad (4.4.61)$$

Finally, applying Hardy's inequality, we conclude that

$$I_\varepsilon^{3,1,1} \le C \left(\frac{p}{p-1} \right)^p \int_{\mathbb{R}^{n-1}} \int_0^\infty |\partial_{x_n}^2 w(x)|^p\,\mathrm{d}x_n\,\mathrm{d}x' \le C\|w\|_{W^{2,p}(\mathbb{R}^n_+)}^p. \qquad (4.4.62)$$

Ad $I_\varepsilon^{3,1,2}$*:* Recalling the definition of R_2 (see (4.4.57)) and (4.4.49), we use Hölder's inequality and Lemma 4.2.1(a) to obtain

$$\begin{aligned}
I_\varepsilon^{3,1,2} &= \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(DF_\gamma(x)(x-y))\,R_2(x,y)\,d_{F_\gamma}(x)\,\mathrm{d}y \right|^p \mathrm{d}x \\
&\le C \sum_{|\beta|=2} \int_{\mathcal{A}_{2R}} \left(\int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon(DF_\gamma(x)(x-y))\,|D^\beta w(y + t(x-y))|\,\mathrm{d}y\,\mathrm{d}t \right)^p \mathrm{d}x \\
&\le C \sum_{|\beta|=2} \int_{\mathcal{A}_{2R}} \left(\int_{\mathcal{B}_{2R}} \rho_\varepsilon(DF_\gamma(x)(x-y))\,\mathrm{d}y \right)^{p-1} \\
&\qquad \cdot \left(\int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon(DF_\gamma(x)(x-y))\,|D^\beta w(y + t(x-y))|^p\,\mathrm{d}y\,\mathrm{d}t \right) \mathrm{d}x. \qquad (4.4.63)
\end{aligned}$$

Using the change of variables $y \mapsto DF_\gamma(x)(x-y)$ as well as Lemma 4.2.1(a), we deduce that

$$\int_{\mathcal{B}_{2R}} \rho_\varepsilon(DF_\gamma(x)(x-y))\,\mathrm{d}y \le \int_{\mathbb{R}^n} \rho_\varepsilon(z)\,\mathrm{d}z = \|\rho\|_{L^1(\mathbb{R}^n)}.$$

Consequently, we have

$$I_\varepsilon^{3,1,2} \le C \sum_{|\beta|=2} \int_{\mathcal{A}_{2R}} \int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon(DF_\gamma(x)(x-y))\, |D^\beta w(y+t(x-y))|^p \,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}x.$$

We now apply the change of variables

$$\mathbb{R}^{2n} \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} x-y \\ y+t(x-y) \end{pmatrix} \in \mathbb{R}^{2n}. \tag{4.4.64}$$

Note that

$$\det\left(D_{(x,y)} \left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix} \right) \right) = \det \begin{pmatrix} I & -I \\ tI & (1-t)I \end{pmatrix} = 1.$$

In this way, we obtain

$$I_\varepsilon^{3,1,2} \le C \int_0^1 \sum_{|\beta|=2} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \rho_\varepsilon\Big(DF_\gamma(\eta+(1-t)\xi)\xi\Big)\,\mathrm{d}\xi \right) |D^\beta w(\eta)|^p \,\mathrm{d}\eta\,\mathrm{d}t. \tag{4.4.65}$$

Applying (4.4.49), we deduce that

$$\Big| DF_\gamma(\eta+(1-t)\xi)\xi \Big| \ge \frac{1}{2}|\xi| \tag{4.4.66}$$

for all $\xi \in \mathbb{R}^n$ and $t \in (0,1)$. Consequently, since $\operatorname{supp}\rho_\varepsilon \subset B_{R\varepsilon}(0)$, we have

$$\rho_\varepsilon\Big(DF_\gamma(\eta+(1-t)\xi)\xi\Big) = 0 \quad \text{if } |\xi| \ge 2R\varepsilon. \tag{4.4.67}$$

Invoking Condition (2.1.6) from Assumption (J2) and applying the change of variables $\xi \mapsto \zeta = \xi/\varepsilon$, we infer that

$$\begin{aligned} \int_{\mathbb{R}^n} \rho_\varepsilon\Big(DF_\gamma(\eta+(1-t)\xi)\xi\Big)\,\mathrm{d}\xi &= \int_{B_{2R\varepsilon}} \rho_\varepsilon\Big(DF_\gamma(\eta+(1-t)\xi)\xi\Big)\,\mathrm{d}\xi \\ \le C\varepsilon^{-n} \int_{B_{2R\varepsilon}(0)} &\left(1+\left|\frac{\xi}{\varepsilon}\right|^{2-\alpha-n}\right)\mathrm{d}\xi = C\int_{B_{2R}(0)} \Big(1+|\zeta|^{2-\alpha-n}\Big)\,\mathrm{d}\zeta \end{aligned} \tag{4.4.68}$$

Since $0<\alpha<2$, we have $1-\alpha>-1$ and thus, a transformation to spherical coordinates yields

$$\begin{aligned} \int_{\mathbb{R}^n} \rho_\varepsilon\Big(DF_\gamma(\eta+(1-t)\xi)\xi\Big)\,\mathrm{d}\xi &\le \int_{B_{2R}(0)} \Big(1+r^{2-\alpha-n}\Big)\,\mathrm{d}\xi \\ &= C\int_0^{2R} \Big(1+r^{2-\alpha-n}\Big) r^{n-1}\,\mathrm{d}r \le C. \end{aligned} \tag{4.4.69}$$

Hence, in view of (4.4.72), we obtain

$$I_\varepsilon^{3,1,2} \le C \sum_{|\beta|=2} \int_{\mathbb{R}^n} |D^\beta w(z)|^p \,\mathrm{d}z \le C\|w\|^p_{W^{2,p}(\mathbb{R}^n)}.$$

In summary, recalling (4.4.56), we finally conclude that

$$I_\varepsilon^{3,1} \le C\|w\|^p_{W^{2,p}(\mathbb{R}^n)}. \tag{4.4.70}$$

Ad $I_\varepsilon^{3,2}$*:* It follows from (4.4.39) that d_{F_γ} is Lipschitz continuous. Hence, using the fundamental theorem of calculus and Hölder's inequality, we deduce that

$$\begin{aligned} I_\varepsilon^{3,2} &= \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} J_\varepsilon(G_\gamma(x,y)(x-y))(w(x)-w(y))(d_{F_\gamma}(y)-d_{F_\gamma}(x)) \,\mathrm{d}y \right|^p \mathrm{d}x \\ &\le C \int_{\mathcal{A}_{2R}} \left(\int_{\mathcal{B}_{2R}} \int_0^1 \rho_\varepsilon(G_\gamma(x,y)(x-y)) |\nabla w(y+t(x-y))| \,\mathrm{d}t\,\mathrm{d}y \right)^p \,\mathrm{d}x \\ &\le C \int_{\mathcal{A}_{2R}} \left(\int_{\mathcal{B}_{2R}} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \,\mathrm{d}y \right)^{p-1} \\ &\qquad \cdot \left(\int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \left|\nabla w(y+t(x-y))\right|^p \,\mathrm{d}y\,\mathrm{d}t \right) \,\mathrm{d}x \end{aligned} \tag{4.4.71}$$

Applying the change of variables $y \mapsto z = x - y$, we obtain

$$\int_{\mathcal{B}_{2R}} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \,\mathrm{d}y \le \int_{\mathbb{R}^n} \rho_\varepsilon(G_\gamma(x,x-z)z) \,\mathrm{d}z. \tag{4.4.72}$$

Invoking (4.4.49) and recalling that $\operatorname{supp} \rho_\varepsilon \subset B_{R\varepsilon}(0)$, we know that

$$\rho_\varepsilon(G_\gamma(x,x-z)\,z) = 0 \quad \text{if } |z| \ge 2R\varepsilon. \tag{4.4.73}$$

Hence, proceeding similarly as in (4.4.68) and (4.4.69), we infer that

$$\int_{\mathcal{B}_{2R}} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \,\mathrm{d}y \le C. \tag{4.4.74}$$

This implies that

$$I_\varepsilon^{3,2} \le C \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \left|\nabla w(y+t(x-y))\right|^p \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$

Applying the change of variables (4.4.64) and proceeding similarly as in (4.4.65)–(4.4.69), we finally conclude that

$$I_\varepsilon^{3,2} \le C\|w\|_{W^{2,p}(\mathbb{R}^n)}^p. \tag{4.4.75}$$

Ad $I_\varepsilon^{3,3}$*:* In order to estimate $I_\varepsilon^{3,3}$, we first notice that the fundamental theorem of calculus yields

$$\begin{aligned} &J_\varepsilon(G_\gamma(x,y)(x-y)) - J_\varepsilon(G_\gamma(x,x)(x-y)) \\ &\quad = \int_0^1 \nabla J_\varepsilon(z_s) \cdot (G_\gamma(x,y) - G_\gamma(x,x))(x-y) \,\mathrm{d}s, \end{aligned} \tag{4.4.76}$$

for all $x, y \in \mathbb{R}^n$, where

$$z_s = z_s(x,y) \coloneqq G_\gamma(x,x)(x-y) + s\big[G_\gamma(x,y) - G_\gamma(x,x)\big](x-y)$$

for all $s \in [0,1]$. Recalling the definition of G_γ in (4.4.44) and using the Lipschitz continuity of DF_γ, which follows from (4.4.39), we find that

$$|G_\gamma(x,y) - G_\gamma(x,x)| \le C|x-y| \tag{4.4.77}$$

for all $x, y \in \mathbb{R}^n$. Plugging this estimate into (4.4.76), we infer that

$$\big|J_\varepsilon(G_\gamma(x,y)(x-y)) - J_\varepsilon(G_\gamma(x,x)(x-y))\big| \leq C \int_0^1 |\nabla J_\varepsilon(z_s)|\, |x-y|^2 \,\mathrm{d}s \tag{4.4.78}$$

for all $x, y \in \mathbb{R}^n$. Combining (4.4.47) and (4.4.48), we further deduce that

$$\begin{aligned} |z_s| &\geq \Big| |G_\gamma(x,x)(x-y)| - s\big|[G_\gamma(x,y) - G_\gamma(x,x)](x-y)\big| \Big| \\ &\geq \frac{5}{6}|x-y| - \frac{1}{3} s\, |x-y| \geq \frac{1}{2}|x-y| \end{aligned} \tag{4.4.79}$$

for all $x, y \in \mathbb{R}^n$. Using the chain rule, we derive the estimate

$$|\nabla J_\varepsilon(x)| \leq \frac{|\nabla \rho_\varepsilon(x)|}{|x|^2} + 2\frac{|\rho_\varepsilon(x)|}{|x|^3}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. In view of (4.4.79), we thus obtain

$$|\nabla J_\varepsilon(z_s)| \leq C\frac{|\nabla \rho_\varepsilon(z_s)|}{|x-y|^2} + C\frac{|\rho_\varepsilon(z_s)|}{|x-y|^3} \tag{4.4.80}$$

for all $x, y \in \mathbb{R}^n \setminus \{0\}$. Invoking the fundamental theorem of calculus, we further have

$$w(x) - w(y) = \int_0^1 \nabla w(y + t(x-y)) \,\mathrm{d}t \cdot (x-y) \tag{4.4.81}$$

for all $x, y \in \mathbb{R}^n$. Combining (4.4.78), (4.4.80) and (4.4.81), we now conclude that

$$\begin{aligned} I_\varepsilon^{3,3} &\leq C \int_{\mathcal{A}_{2R}} \left| \int_{\mathcal{B}_{2R}} \int_0^1 \int_0^1 |\nabla J_\varepsilon(z_s)| |\nabla w(y + t(x-y))| |x-y|^3 \,\mathrm{d}s \,\mathrm{d}t \,\mathrm{d}y \right|^p \mathrm{d}x. \\ &\leq C \int_{\mathcal{A}_{2R}} \Bigg(\int_{\mathcal{B}_{2R}} \int_0^1 \int_0^1 \big[|\rho_\varepsilon(z_s)| + |\nabla \rho_\varepsilon(z_s)|\, |x-y|\big] \\ &\qquad\qquad \cdot \big|\nabla w(y + t(x-y))\big| \,\mathrm{d}s \,\mathrm{d}t \,\mathrm{d}y \Bigg)^p \mathrm{d}x \end{aligned}$$

Hence, by means of Hölder's inequality, we obtain

$$\begin{aligned} I_\varepsilon^{3,3} \leq C \int_{\mathcal{A}_{2R}} &\left(\int_{\mathcal{B}_{2R}} \int_0^1 \big[|\rho_\varepsilon(z_s)| + |\nabla \rho_\varepsilon(z_s)|\, |x-y|\big] \,\mathrm{d}s \,\mathrm{d}y \right)^{p-1} \\ &\cdot \Bigg(\int_{\mathcal{B}_{2R}} \int_0^1 \int_0^1 \big[|\rho_\varepsilon(z_s)| + |\nabla \rho_\varepsilon(z_s)|\, |x-y|\big] \\ &\qquad \cdot \big|\nabla w(y + t(x-y))\big|^p \,\mathrm{d}s \,\mathrm{d}t \,\mathrm{d}y \Bigg) \mathrm{d}x \end{aligned} \tag{4.4.82}$$

Due to (4.4.79) and Condition (2.1.6) from Assumption (J2), we proceed as in the estimates for $I_\varepsilon^{3,1,2}$ and $I_\varepsilon^{3,2}$ to conclude that

$$I_\varepsilon^{3,3} \leq C\|w\|_{W^{2,p}(\mathbb{R}^n)}^p. \tag{4.4.83}$$

Now, combining (4.4.70), (4.4.75) and (4.4.83), we infer from (4.4.52) that

$$I_\varepsilon^3 \leq C\|w\|^p_{W^{2,p}(\mathbb{R}^n)}. \tag{4.4.84}$$

Since $I_\varepsilon^1 = I_\varepsilon^2 = 0$, we conclude from (4.4.51) and (4.4.45) that

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C\|w\|^p_{W^{2,p}(\mathbb{R}^n)} \leq C\|u\|^p_{W^{2,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.85}$$

In view of (4.4.36), this means that (4.4.34) is finally verified.

Step 2: Proof of Estimate (4.4.30) **for** $u \in C_c^3(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{3,p}(\mathbb{R}^n_\gamma)$**.**
Let now $u \in C_c^3(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{3,p}(\mathbb{R}^n_\gamma)$ be arbitrary. Our goal is to show that there exists a constant $C > 0$ depending only on $\mathbb{R}^n_\gamma$, ρ and p such that

$$\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u + \Delta u\|_{L^p(\mathbb{R}^n_\gamma)} \leq C \sqrt[p]{\varepsilon}\, \|u\|_{W^{3,p}(\mathbb{R}^n_\gamma)}.$$

Since $\mathbb{R}^n_\gamma$ is of class C^3, we can find an extension $\tilde{u} \in C_c^3(\mathbb{R}^n)$ with $u|_{\mathbb{R}^n_\gamma} = u$. In particular, $\tilde{u}$ can be chosen in such a way that

$$\|\tilde{u}\|_{W^{3,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{3,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.86}$$

By means of Theorem 4.4.1, we derive the estimate

$$\begin{aligned}
\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u + \Delta u\|^p_{L^p(\mathbb{R}^n_\gamma)} &\leq C\|\mathcal{L}_\varepsilon^{\mathbb{R}^n} \tilde{u} + \Delta \tilde{u}\|^p_{L^p(\mathbb{R}^n)} + \|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \\
&\leq C\varepsilon^p \|\tilde{u}\|^p_{W^{3,p}(\mathbb{R}^n)} + \|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \\
&\leq C\varepsilon^p \|u\|^p_{W^{3,p}(\mathbb{R}^n_\gamma)} + \|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)},
\end{aligned} \tag{4.4.87}$$

where the error term is given by

$$\mathcal{R}_\varepsilon \tilde{u}(x) := \int_{{\mathbb{R}^n_\gamma}^c} J_\varepsilon(x-y)\big(u(x) - \tilde{u}(y)\big)\,\mathrm{d}y \quad \text{ for all } x \in \mathbb{R}^n_\gamma.$$

Therefore, it remains to show that

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C\varepsilon \|u\|^p_{W^{3,p}(\mathbb{R}^n_\gamma)} \tag{4.4.88}$$

since (4.4.34) then follows by combining (4.4.36) and (4.4.37). To this end, we define the radius $R > 0$, the functions F_γ, G_γ and w and the sets $\mathcal{A}_r$ and $\mathcal{B}_r(x)$ with $r > 0$ and $x \in \mathbb{R}^n$ as in Step 1. In particular, using the chain rule along with (4.4.86), (4.4.39)and (4.4.40), we deduce

$$\|w\|_{W^{3,p}(\mathbb{R}^n)} \leq C\|\tilde{u}\|_{W^{3,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{3,p}(\mathbb{R}^n_\gamma)}. \tag{4.4.89}$$

Moreover, we obtain the estimate

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C I_\varepsilon^3, \tag{4.4.90}$$

where I_3 is the integral term introduced in Step 1. Recall that we have already shown that $I_\varepsilon^1 = I_\varepsilon^2 = 0$ and thus, only the contribution I_ε^3 remains. Consequently, we have

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C\big(I_\varepsilon^{3,1,1} + I_\varepsilon^{3,1,2} + I_\varepsilon^{3,2} + I_\varepsilon^{3,3}\big). \tag{4.4.91}$$

Now, having $W^{3,p}$-regularity at hand, we intend to improve the estimates from Step 1 for $I_\varepsilon^{3,1,1}$, $I_\varepsilon^{3,1,2}$, $I_\varepsilon^{3,2}$ and $I_\varepsilon^{3,3}$ such that the desired rate with respect to ε is obtained.

Ad $I_\varepsilon^{3,1,1}$*:* According to (4.4.59), we have

$$I_\varepsilon^{3,1,1} = C \int_{\mathcal{A}_{2R}} \left[\left(\int_0^1 |\partial_{x_n}^2 w(x', tx_n)| \,\mathrm{d}t \right) \cdot \left(\int_{\mathcal{B}_{2R}} \rho_\varepsilon(U(x)(x-y)) \,\mathrm{d}y \right) \right]^p \mathrm{d}x.$$

Recalling the definition of $\mathcal{A}_{2R}$ and $\mathcal{B}_{2R}$, we infer that

$$\begin{aligned} I_\varepsilon^{3,1,1} \leq C &\int_{\mathbb{R}^{n-1}} \|\partial_{x_n}^2 w(x', \cdot)\|_{L^\infty((0,2R))}^p \\ &\cdot \int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon(U(x)(x-y)) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^p \mathrm{d}x_n \,\mathrm{d}x'. \end{aligned} \tag{4.4.92}$$

Recalling the definition of ρ_ε, we use the changes of variables $y \mapsto z = x - y$, $z \mapsto \varepsilon z$, $x_n \mapsto \varepsilon x_n$ and $z \mapsto y = x - z$ to compute

$$\begin{aligned} &\int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon(U(x)(x-y)) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^p \mathrm{d}x_n \\ &\quad = \int_0^{2R} \left(\varepsilon^{-n} \int_{\mathbb{R}^{n-1}} \int_{x_n}^{x_n+2R} \rho\big(U(x)\tfrac{z}{\varepsilon}\big) \,\mathrm{d}z_n \,\mathrm{d}z' \right)^p \mathrm{d}x_n \\ &\quad = \int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{x_n/\varepsilon}^{(x_n+2R)/\varepsilon} \rho\big(U(x)z\big) \,\mathrm{d}z_n \,\mathrm{d}z' \right)^p \mathrm{d}x_n \\ &\quad = \varepsilon \int_0^{2R/\varepsilon} \left(\int_{\mathbb{R}^{n-1}} \int_{x_n}^{x_n+(2R/\varepsilon)} \rho\big(U(x)z\big) \,\mathrm{d}z_n \,\mathrm{d}z' \right)^p \mathrm{d}x_n \\ &\quad \leq \varepsilon \int_0^{\infty} \left(\int_{\mathbb{R}^{n-1}} \int_{x_n}^{x_n+(2R/\varepsilon)} \rho\big(U(x)z\big) \,\mathrm{d}z_n \,\mathrm{d}z' \right)^p \mathrm{d}x_n \\ &\quad = \varepsilon \int_0^{\infty} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R/\varepsilon}^0 \rho\big(U(x)(x-y))\big) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^p \mathrm{d}x_n \,. \end{aligned}$$

for all $x' \in B_{2R}(0)$. Since $U(x) \in \mathrm{SO}(n)$ for all $x \in \mathbb{R}^n$, we know that

$$\big|U(x)(x-y)\big| = |x-y| \quad \text{for all } x, y \in \mathbb{R}^n.$$

As $\operatorname{supp} \rho \subset B_R(0)$, this implies that

$$\rho\big(U(x)(x-y)\big) = 0 \quad \text{if } \; x_n \geq 2R \; \text{ or } \; y_n \leq -2R.$$

Consequently, thanks to the above computation, we have

$$\begin{aligned} &\int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon(U(x)(x-y)) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^p \mathrm{d}x_n \\ &\quad \leq \varepsilon \int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho\big(U(x)(x-y))\big) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^p \mathrm{d}x_n \\ &\quad \leq \varepsilon \int_0^{2R} \left(\int_{\mathbb{R}^n} \rho\big(U(x)(x-y))\big) \,\mathrm{d}y \right)^p \mathrm{d}x_n \,. \end{aligned}$$

Recalling (4.4.60), this proves that

$$\int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon\big(U(x)(x-y)\big) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^p \mathrm{d}x_n \le C\varepsilon \tag{4.4.93}$$

for all $x' \in B_{2R}(0)$. Plugging this estimate into (4.4.92), we infer that

$$I_\varepsilon^{3,1,1} \le C\varepsilon \int_{\mathbb{R}^{n-1}} \|\partial_{x_n}^2 w(x',\cdot)\|_{L^\infty((0,2R))}^p \,\mathrm{d}x'.$$

As the embedding $W^{1,p}((0,2R)) \hookrightarrow L^\infty((0,2R))$ is continuous, this implies that

$$\begin{aligned} I_\varepsilon^{3,1,1} &\le C\varepsilon \int_{\mathbb{R}^{n-1}} \|\partial_{x_n}^2 w(x',\cdot)\|_{W^{1,p}((0,2R))}^p \,\mathrm{d}x' \\ &\le C\varepsilon \, \|w\|_{W^{3,p}(\mathbb{R}^n)}^p \,. \end{aligned} \tag{4.4.94}$$

Ad $I_\varepsilon^{3,1,2}$*:* In view of (4.4.63), we have

$$\begin{aligned} I_\varepsilon^{3,1,2} \le C \sum_{|\beta|=2} \int_{\mathcal{A}_{2R}} & \left(\int_{\mathcal{B}_{2R}} \rho_\varepsilon\big(DF_\gamma(x)(x-y)\big) \,\mathrm{d}y \right)^{p-1} \\ & \cdot \left(\int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon\big(DF_\gamma(x)(x-y)\big) \, |D^\beta w(y+t(x-y))|^p \,\mathrm{d}y \,\mathrm{d}t \right) \mathrm{d}x. \end{aligned}$$

Recalling the definitions of $\mathcal{A}_{2R}$ and $\mathcal{B}_{2R}$, we use the change of variables $y \mapsto z = x - y$ to deduce that

$$\begin{aligned} I_\varepsilon^{3,1,2} &\le C \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon\big(DF_\gamma(x)(x-y)\big) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^{p-1} \\ &\qquad \cdot \Bigg(\int_0^1 \int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon\big(DF_\gamma(x)(x-y)\big) \\ &\qquad\qquad \cdot \|D^\beta w(y'+t(x'-y'),\cdot)\|_{L^\infty((0,2R))}^p \,\mathrm{d}y_n \,\mathrm{d}y' \,\mathrm{d}t \Bigg) \mathrm{d}x_n \,\mathrm{d}x' \\ &\le C \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon\big(DF_\gamma(x)(x-y)\big) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^{p-1} \\ &\qquad \cdot \Bigg(\int_0^1 \int_{\mathbb{R}^{n-1}} \int_0^{4R} \rho_\varepsilon\big(DF_\gamma(x)z\big) \\ &\qquad\qquad \cdot \|D^\beta w(x'-z'+tz',\cdot)\|_{L^\infty((-2R,2R))}^p \,\mathrm{d}z_n \,\mathrm{d}z' \,\mathrm{d}t \Bigg) \mathrm{d}x_n \,\mathrm{d}x' \end{aligned}$$

We already know from (4.4.49) that

$$|DF_\gamma(x)z| \ge \frac{1}{2}|z| \quad \text{for all } x, z \in \mathbb{R}^n.$$

This implies that

$$\rho_\varepsilon\big(DF_\gamma(x)z\big) = 0 \quad \text{if } \ |z'| \ge 2R\varepsilon \ \text{ or } \ z_n \ge 2R\varepsilon.$$

Hence, invoking Condition (2.1.6) from Assumption (J2), we infer that

$$I_\varepsilon^{3,1,2} \leq C \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon(DF_\gamma(x)(x-y)) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^{p-1} \mathrm{d}x_n \cdot \left(\varepsilon^{-n} \int_0^1 \int_{B_{2R\varepsilon}(0)} \int_0^{2R\varepsilon} \left(1 + \left|\frac{z}{\varepsilon}\right|^{2-\alpha-n}\right) \cdot \|D^\beta w(x'-z'+tz',\cdot)\|^p_{L^\infty((-2R,2R))} \,\mathrm{d}z_n \,\mathrm{d}z' \,\mathrm{d}t \right) \mathrm{d}x'$$

Proceeding similarly to the derivation of (4.4.93), we show that

$$\int_0^{2R} \left(\int_{\mathbb{R}^{n-1}} \int_{-2R}^0 \rho_\varepsilon(DF_\gamma(x)(x-y)) \,\mathrm{d}y_n \,\mathrm{d}y' \right)^{p-1} \mathrm{d}x_n \leq C\varepsilon$$

for all $x' \in B_{2R}(0)$. Consequently, we have

$$I_\varepsilon^{3,1,2} \leq C\varepsilon \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \left(\varepsilon^{-n} \int_0^1 \int_{B_{2R\varepsilon}(0)} \int_0^{2R\varepsilon} \left(1 + \left|\frac{z}{\varepsilon}\right|^{2-\alpha-n}\right) \cdot \|D^\beta w(x'-z'+tz',\cdot)\|^p_{L^\infty((-2R,2R))} \,\mathrm{d}z_n \,\mathrm{d}z' \,\mathrm{d}t \right) \mathrm{d}x'$$

Now, employing first the change of variables $x' \mapsto x'-z'+tz'$ and afterwards $z \mapsto \varepsilon z$, we infer that

$$\begin{aligned} I_\varepsilon^{3,1,2} &\leq C\varepsilon \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \left(\varepsilon^{-n} \int_{B_{2R\varepsilon}(0)} \int_0^{2R\varepsilon} \left(1 + \left|\frac{z}{\varepsilon}\right|^{2-\alpha-n}\right) \mathrm{d}z_n \,\mathrm{d}z' \cdot \|D^\beta w(x',\cdot)\|^p_{L^\infty((-2R,2R))} \right) \mathrm{d}x' \\ &\leq C\varepsilon \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \int_{B_{4R}(0)} \left(1 + |z|^{2-\alpha-n}\right) \mathrm{d}z \, \|D^\beta w(x',\cdot)\|^p_{L^\infty((-2R,2R))} \,\mathrm{d}x' \\ &\leq C\varepsilon \sum_{|\beta|=2} \int_{\mathbb{R}^{n-1}} \|D^\beta w(x',\cdot)\|^p_{L^\infty((-2R,2R))} \,\mathrm{d}x'. \end{aligned}$$

Finally, using the continuous embedding $W^{1,p}((0,2R)) \hookrightarrow L^\infty((0,2R))$ and proceeding similarly to (4.4.94), we conclude that

$$I_\varepsilon^{3,1,2} \leq C\varepsilon \, \|w\|^p_{W^{3,p}(\mathbb{R}^n)}. \tag{4.4.95}$$

Ad $I_\varepsilon^{3,2}$*:* We have already shown in (4.4.71) that

$$I_\varepsilon^{3,2} \leq C \int_{\mathcal{A}_{2R}} \left(\int_{\mathcal{B}_{2R}} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \,\mathrm{d}y \right)^{p-1} \cdot \left(\int_0^1 \int_{\mathcal{B}_{2R}} \rho_\varepsilon(G_\gamma(x,y)(x-y)) \, |\nabla w(y+t(x-y))|^p \,\mathrm{d}y \,\mathrm{d}t \right) \mathrm{d}x.$$

As we know from (4.4.49) that

$$|DF_\gamma(x)z| \geq \frac{1}{2}|z| \quad \text{for all } x, z \in \mathbb{R}^n,$$

the integral term $I_\varepsilon^{3,2}$ can be estimated analogously to $I_\varepsilon^{3,1,2}$. In this way, we obtain

$$I_\varepsilon^{3,2} \leq C\varepsilon \, \|w\|^p_{W^{3,p}(\mathbb{R}^n)}. \tag{4.4.96}$$

Ad $I_\varepsilon^{3,3}$*:* According to (4.4.82), we have

$$\begin{aligned} I_\varepsilon^{3,3} \leq C \int_{\mathcal{A}_{2R}} \bigg(\int_{\mathcal{B}_{2R}} \int_0^1 \left[|\rho_\varepsilon(z_s)| + |\nabla \rho_\varepsilon(z_s)| \, |x-y| \right] \mathrm{d}s \, \mathrm{d}y \bigg)^{p-1} \\ \cdot \bigg(\int_{\mathcal{B}_{2R}} \int_0^1 \int_0^1 \left[|\rho_\varepsilon(z_s)| + |\nabla \rho_\varepsilon(z_s)| \, |x-y| \right] \\ \cdot \big| \nabla w(y + t(x-y)) \big|^p \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}y \bigg) \, \mathrm{d}x \, . \end{aligned}$$

Moreover, we have shown in (4.4.79) that $|z_s| \geq \frac{1}{2}|x-y|$ for all $x, y \in \mathbb{R}^n$. Therefore, the integral term $I_\varepsilon^{3,3}$ can be estimated analogously to $I_\varepsilon^{3,1,2}$ and we obtain

$$I_\varepsilon^{3,3} \leq C\varepsilon \, \|w\|^p_{W^{3,p}(\mathbb{R}^n)}. \tag{4.4.97}$$

Finally, plugging the estimates (4.4.94), (4.4.95), (4.4.96) and (4.4.97) into (4.4.91), we have shown that

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C\varepsilon \, \|w\|^p_{W^{3,p}(\mathbb{R}^n)},$$

which verifies (4.4.88). Combining this estimate with (4.4.87) and (4.4.89), we finally conclude that

$$\|\mathcal{R}_\varepsilon \tilde{u}\|^p_{L^p(\mathbb{R}^n_\gamma)} \leq C\varepsilon \, \|u\|^p_{W^{3,p}(\mathbb{R}^n_\gamma)}.$$

This means that (4.4.30) is established for $u \in C_c^3(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{3,p}(\mathbb{R}^n_\gamma)$.

Step 3: Construction of the operator and completion of the proof.
As the inclusion $C_c^2(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{2,p}(\mathbb{R}^n_\gamma) \subset W_I^{2,p}(\mathbb{R}^n_\gamma)$ is dense, we can use the result established in Step 1 to extend

$$\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} : C_c^2(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{2,p}(\mathbb{R}^n_\gamma) \to L^p_{\mathrm{loc}}(\mathbb{R}^n_\gamma)$$

to a bounded linear operator

$$\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} : W_I^{2,p}(\mathbb{R}^n_\gamma) \to L^p(\mathbb{R}^n_\gamma) \tag{4.4.98}$$

with

$$\|\mathcal{L}_\varepsilon^{\mathbb{R}^n_\gamma} u\|_{L^p(\mathbb{R}^n_\gamma)} \leq C \|u\|_{W^{2,p}(\mathbb{R}^n_\gamma)} \tag{4.4.99}$$

for a suitable constant $C > 0$ that depends only on $\mathbb{R}^n_\gamma$, ρ and p. In particular, the extension (4.4.98) is unique. We further know from Step 2 that (4.4.30) holds for all $u \in C_c^3(\overline{\mathbb{R}^n_\gamma}) \cap W_I^{3,p}(\mathbb{R}^n_\gamma)$. Because of density, it is clear that (4.4.30) holds true for all $u \in W_I^{3,p}(\mathbb{R}^n_\gamma)$. Finally, proceeding similarly to the proof of Theorem 4.4.1(a), we use the Banach–Steinhaus theorem to conclude the convergence (4.4.31).

This means that all claims are established and thus, the proof is complete. □

Proof of Corollary 4.4.6. First, let $u \in C_c^2(\overline{Q\mathbb{R}^n_\gamma}) \cap W_Q^{2,p}(Q\mathbb{R}^n_\gamma)$ be arbitrary. We now introduce the function

$$w : \mathbb{R}^n_\gamma \to \mathbb{R}, \quad w(x) = u(Qx).$$

As the transformation $x \mapsto Qx$ is linear, it is clear that $w \in W^{3,p}(\mathbb{R}^n_\gamma)$. In particular, since $Q \in \mathrm{SO}(n)$, we obtain

$$\|w\|_{W^{2,p}(\mathbb{R}^n_\gamma)} = \|u\|_{W^{2,p}(Q\mathbb{R}^n_\gamma)}. \tag{4.4.100}$$

Moreover, using the chain rule, we obtain that

$$\nabla w(x) \cdot e_n = Q\,\nabla u(Qx) \cdot e_n = 0 \quad \text{for all } x \in \partial\mathbb{R}^n_\gamma$$

since $Qx \in \partial Q\mathbb{R}^n_\gamma$. Thus, we have $w \in W_I^{3,p}(\mathbb{R}^n_\gamma)$. Moreover, as the Laplace operator is invariant under rotations, we have

$$\Delta w(x) = \Delta u(Qx) \quad \text{for all } x \in \mathbb{R}^n_\gamma.$$

This can also be easily verified by means of the chain rule. We now define $\widehat{\rho} : \mathbb{R}^n \to \mathbb{R}$ with

$$\widehat{\rho}(x) = \rho(Qx) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Hence, its associated kernel is given by $\widehat{J} = J(Q\,\cdot\,)$. Accordingly, we write $\widehat{J}_\varepsilon = J_\varepsilon(Q\,\cdot\,)$ for all $\varepsilon > 0$. Since ρ and J satisfy (J1) and (J2) and $QQ^T = I$, it is straightforward to check that $\widehat{\rho}$ and $\widehat{J}$ fulfill (J1) and (J2) with

$$\begin{aligned}
\widehat{M} &:= \frac{1}{2}\int_{\mathbb{R}^n} \widehat{J}(z)\, z \otimes z \,\mathrm{d}z = \frac{1}{2}\, Q \int_{\mathbb{R}^n} J(Qz)\,(Qz) \otimes (Qz)\,\mathrm{d}z\, Q^T \\
&= \frac{1}{2}\, Q \int_{\mathbb{R}^n} J(Qz)\,(Qz) \otimes (Qz)\,\mathrm{d}z\, Q^T = \frac{1}{2}\, Q \int_{\mathbb{R}^n} J(z)\, z \otimes z \,\mathrm{d}z\, Q^T = I, \\
\widehat{A} &:= I
\end{aligned}$$

in place of M and A. In particular, according to Theorem 4.4.4 and Theorem 4.2.3, the nonlocal operators

$$\begin{aligned}
&\widehat{\mathcal{L}}_\varepsilon^{\mathbb{R}^n_\gamma} : W^{2,p}(\mathbb{R}^n_\gamma) \to L^p(\mathbb{R}^n_\gamma), \\
&\widehat{\mathcal{L}}_\varepsilon^{\mathbb{R}^n_\gamma} v(x) = \mathrm{P.V.} \int_{\mathbb{R}^n_\gamma} \widehat{J}_\varepsilon(x-y)\big(v(x) - v(y)\big)\,\mathrm{d}y
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} : C_c^2(\overline{Q\mathbb{R}^n_\gamma}) \cap W_Q^{2,p}(Q\mathbb{R}^n_\gamma) \to L^p_{\mathrm{loc}}(Q\mathbb{R}^n_\gamma), \\
&\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} v(x) = \mathrm{P.V.} \int_{Q\mathbb{R}^n_\gamma} J_\varepsilon(x-y)\big(v(x) - v(y)\big)\,\mathrm{d}y
\end{aligned}$$

are well-defined for every $\varepsilon > 0$. Employing the changes of variables $x \mapsto Qx$ and $y \mapsto Qy$, we deduce that

$$
\begin{aligned}
&\|\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} u + \Delta u\|^p_{L^p(Q\mathbb{R}^n_\gamma)} \\
&= \int_{Q\mathbb{R}^n_\gamma} \left| \text{P.V.} \int_{Q\mathbb{R}^n_\gamma} J_\varepsilon(x-y)(u(x)-u(y)) \, \mathrm{d}y + \Delta u(x) \right|^p \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n_\gamma} \left| \text{P.V.} \int_{\mathbb{R}^n_\gamma} J_\varepsilon(Q(x-y))(w(x)-w(y)) \, \mathrm{d}y + \Delta w(x) \right|^p \, \mathrm{d}x \\
&= \|\widehat{\mathcal{L}}_\varepsilon^{\mathbb{R}^n_\gamma} w + \Delta w\|^p_{L^p(\mathbb{R}^n_\gamma)}.
\end{aligned}
\tag{4.4.101}
$$

In particular, invoking the estimate (4.4.29) from Theorem 4.4.4 and the identity (4.4.100), this entails

$$
\begin{aligned}
\|\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} u\|_{L^p(Q\mathbb{R}^n_\gamma)} &\le \|\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} u + \Delta u\|_{L^p(Q\mathbb{R}^n_\gamma)} + \|\Delta u\|_{L^p(Q\mathbb{R}^n_\gamma)} \\
&= \|\widehat{\mathcal{L}}_\varepsilon^{\mathbb{R}^n_\gamma} w + \Delta w\|_{L^p(\mathbb{R}^n_\gamma)} + \|\Delta w\|_{L^p(\mathbb{R}^n_\gamma)} \\
&\le \|\widehat{\mathcal{L}}_\varepsilon^{\mathbb{R}^n_\gamma} w\|_{L^p(\mathbb{R}^n_\gamma)} + 2\|\Delta w\|_{L^p(\mathbb{R}^n_\gamma)} \\
&\le C\|w\|_{W^{2,p}(\mathbb{R}^n_\gamma)} = C\|u\|_{W^{2,p}(Q\mathbb{R}^n_\gamma)}.
\end{aligned}
\tag{4.4.102}
$$

Consequently, $\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma}$ can be extended to a bounded linear operator

$$
\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} : W^{2,p}_Q(Q\mathbb{R}^n_\gamma) \to L^p(Q\mathbb{R}^n_\gamma) \, .
$$

Now, we additionally assume that $u \in W^{3,p}_Q(Q\mathbb{R}^n_\gamma)$. Hence, $u \in W^{3,p}_I(\mathbb{R}^n_\gamma)$ with

$$
\|w\|_{W^{3,p}(\mathbb{R}^n_\gamma)} = \|u\|_{W^{3,p}(Q\mathbb{R}^n_\gamma)}. \tag{4.4.103}
$$

Using the inequality (4.4.30) from Theorem 4.4.4 to estimate the right-hand side of (4.4.101), we obtain

$$
\begin{aligned}
\|\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} u + \Delta u\|_{L^p(Q\mathbb{R}^n_\gamma)} &= \|\widehat{\mathcal{L}}_\varepsilon^{\mathbb{R}^n_\gamma} w + \Delta w\|_{L^p(\mathbb{R}^n_\gamma)} \\
\le C_\gamma \sqrt[p]{\varepsilon} \, \|w\|_{W^{3,p}(\mathbb{R}^n_\gamma)} &= C_\gamma \sqrt[p]{\varepsilon} \, \|u\|_{W^{3,p}(Q\mathbb{R}^n_\gamma)}
\end{aligned}
$$

as desired. Proceeding as in Step 3 of the proof of Theorem 4.4.4, we apply the Banach–Steinhaus theorem to conclude that

$$
\mathcal{L}_\varepsilon^{Q\mathbb{R}^n_\gamma} u \to -\Delta u \quad \text{for all } u \in W^{2,p}(Q\mathbb{R}^n_\gamma).
$$

Therefore, the proof of Corollary 4.4.6 is complete. □

4.4.4. Proof of Theorem 2.1.7

To provide a cleaner presentation, we will usually refrain from indicating the principal value by the symbol P.V. whenever the meaning is clear. The proof is split into two steps.

Recalling the arguments in Subsection 4.4.2, we can study the problem on the reference domain $\widehat{\Omega}$, where the matrix M reduces to the identity matrix I. Therefore, in order to complete the proof, it remains to establish the key estimates (4.4.22) and (4.4.23).

Step 1. Localization of the reference domain.
In order to verify the key estimates (4.4.22) and (4.4.23), we need to localize the reference domain $\widehat{\Omega}$. For this purpose, we need to construct a suitable partition of unity.

We start by fixing an arbitrary $z \in \partial\widehat{\Omega}$. Then, since $\partial\widehat{\Omega}$ is of class C^3, there exists an open set $U_z \subseteq \mathbb{R}^n$, a function $\gamma_z \in C_b^3(\mathbb{R}^{n-1})$ and a matrix $Q_z \in \mathrm{SO}(n)$ such that

$$\widehat{\Omega} \cap U_z = Q_z \mathbb{R}^n_{\gamma_z} \cap U_z.$$

Since $\gamma_z \in C_b^3(\mathbb{R}^{n-1})$, we infer from [115, Lemma 2.1] that there exists a $C^{2,1}$-diffeomorphism $F_z : \mathbb{R}^n \to \mathbb{R}^n$ with $F_{\gamma_z}(\mathbb{R}^n_+) = \mathbb{R}^n_{\gamma_z}$, $F_z(x',0) = (x', \gamma(x'))$ and

$$\big(DF_z(x',0)\big)^{-1}\,\mathbf{n}_{\partial\mathbb{R}^n_{\gamma_z}}\big(F_z(x',0)\big) \in \mathrm{span}\{e_n\} \tag{4.4.104}$$

for all $x' \in \mathbb{R}^{n-1}$. For any $\delta > 0$, we now introduce the functions $\phi_z \in C_c^\infty(\mathbb{R}^{n-1};[0,1])$ and $\psi_z \in C_c^\infty(\mathbb{R};[0,1])$ with

$$\begin{aligned} \phi_z &= 1 \ \text{ on } B_{\delta/2}(z'), \qquad && \operatorname{supp}\phi_z \subset B_\delta(z') \subset \mathbb{R}^{n-1}, \\ \psi_z &= 1 \ \text{ on } \big(-\tfrac{\delta}{2}, \tfrac{\delta}{2}\big), \qquad && \operatorname{supp}\psi_z \subset (-\delta,\delta)\,. \end{aligned}$$

Next, we define

$$\widehat{\varphi}_z : \mathbb{R}^n \to \mathbb{R}, \quad \widehat{\varphi}_z(y) = \phi_z(y')\psi_z(y_n) \quad \text{and} \quad \tilde{\varphi}_z := \widehat{\varphi}_z \circ \big(Q_z F_z\big)^{-1}.$$

Since $(Q_zF_z)^{-1}(U) \subset \mathbb{R}^n$ is open, we can ensure that

$$\operatorname{supp}\widehat{\varphi}_z \subset (Q_zF_z)^{-1}(U_z) \quad \Leftrightarrow \quad \operatorname{supp}\tilde{\varphi}_z \subset U_z$$

by choosing δ_z sufficiently small. Moreover, as $F_z^{-1} \in C^{2,1}(\mathbb{R}^n)$, we infer that

$$\tilde{\varphi}_z \in C_c^2(\mathbb{R}^n;[0,1]) \cap W^{3,\infty}(\mathbb{R}^n).$$

Let now $y \in \partial\widehat{\Omega} \cap U_z = \partial\big(Q_z\mathbb{R}^n_{\gamma_z}\big) \cap U_z$ be arbitrary. Hence, there exists $x' \in \mathbb{R}^{n-1}$ such that $Q_zF_z(x',0) = y$. By the construction of F_z, we have

$$D\big((Q_zF_z)^{-1}\big)(y) = \big(D(Q_zF_z)(x',0)\big)^{-1} = \big(DF_z(x',0)\big)^{-1} Q_z^T \tag{4.4.105}$$

Moreover, we have

$$Q_z^T\,\mathbf{n}_{\partial(Q_z\mathbb{R}^n_{\gamma_z})}(y) = \mathbf{n}_{\partial(\mathbb{R}^n_{\gamma_z})}(Q_z^T y) = \mathbf{n}_{\partial\mathbb{R}^n_{\gamma_z}}\big(F_z(x',0)\big) \tag{4.4.106}$$

Now, using the chain rule as well as (4.4.105) and (4.4.106), we derive the identity

$$\begin{aligned} \nabla\tilde{\varphi}_z(y) \cdot \mathbf{n}_{\partial\widehat{\Omega}}(y) &= D\tilde{\varphi}_z(y)\,\mathbf{n}_{\partial\widehat{\Omega}}(y) = D\tilde{\varphi}_z(y)\,\mathbf{n}_{\partial(Q_z\mathbb{R}^n_{\gamma_z})}(y) \\ &= D\widehat{\varphi}_z\big((Q_zF_z)^{-1}(y)\big)\,\big(DF_z(x',0)\big)^{-1} Q_z^T\,\mathbf{n}_{\partial(Q_z\mathbb{R}^n_{\gamma_z})}(y) \\ &= D\widehat{\varphi}_z\big((Q_zF_z)^{-1}(y)\big)\,\big(DF_z(x',0)\big)^{-1}\,\mathbf{n}_{\partial\mathbb{R}^n_{\gamma_z}}\big(F_z(x',0)\big) \end{aligned} \tag{4.4.107}$$

By the construction of $\widehat{\varphi}_z$, we have

$$D\widehat{\varphi}_z(y) = \left(\begin{array}{ccc|c} & & & 0 \\ & D\phi_z(y') & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & \psi_z'(y_n) \end{array}\right) \quad \text{for all } y \in \mathbb{R}^n \tag{4.4.108}$$

Consequently, since $\psi_z'(0) = 0$, it holds that

$$D\widehat{\varphi}_z\big((Q_z F_z)^{-1}(y)\big)e_n = D\widehat{\varphi}_z(x', 0)e_n = 0$$

Thus, in view of (4.4.104), we conclude from (4.4.107) that

$$\nabla\tilde{\varphi}_z(y) \cdot \mathbf{n}_{\partial\widehat{\Omega}}(y) = 0$$

for all $y \in \partial\widehat{\Omega} \cap U_z$.

By the above construction, it is clear that the sets $\operatorname{supp}\tilde{\varphi}_z$, $z \in \partial\widehat{\Omega}$, are an open cover of $\partial\widehat{\Omega}$. Hence, since $\partial\widehat{\Omega}$ is compact, we can select $z_1, \dots, z_N$ such that

$$\partial\widehat{\Omega} \subset \bigcup_{j=1}^{N} \{\tilde{\varphi}_{z_j} > 0\} \subset \bigcup_{j=1}^{N} U_{z_j}.$$

Therefore, in the following, we simply write

$$U_j \coloneqq U_{z_j}, \quad Q_j \coloneqq Q_{z_j}, \quad \gamma_j \coloneqq \gamma_{z_j}, \quad \tilde{\varphi}_j \coloneqq \tilde{\varphi}_{z_j}$$

for all $j \in \{1, \dots, N\}$. We further find an open set $U_0 \subset \mathbb{R}^n$ with $\overline{U_0} \subset \widehat{\Omega}$ and a function $\tilde{\varphi}_0 \in C_c^\infty(\mathbb{R}^n; [0,1])$ with $\operatorname{supp}\tilde{\varphi}_0 \subset U_0$ such that

$$\overline{\widehat{\Omega}} \subset \bigcup_{j=0}^{N} \{\tilde{\varphi}_j > 0\} \subset \bigcup_{j=0}^{N} U_j.$$

To simplify the notation, we further set $Q_0 \coloneqq I$ and $\mathbb{R}^n_{\gamma_0} = \mathbb{R}^n$. Finally, we define

$$\varphi_j : \mathbb{R}^n \to \mathbb{R}^n, \quad \varphi_j(x) = \begin{cases} \dfrac{\tilde{\varphi}_j(x)}{S(x)} & \text{if } S(x) > 0\,, \\ 0 & \text{if } S(x) = 0 \end{cases}$$

for all $j \in \{0, \dots, N\}$, where

$$S(x) = \sum_{j=0}^{N} \tilde{\varphi}_j(x) \quad \text{for all } x \in \mathbb{R}^n.$$

This ensures that

$$\sum_{j=0}^{N} \varphi_j = 1 \quad \text{on } \overline{\widehat{\Omega}}.$$

By the above construction, the family

$$\{\varphi_j\}_{j=0,\dots,N} \subset C_c^2(\mathbb{R}^n; [0,1]) \cap W^{3,\infty}(\mathbb{R}^n)$$

is a partition of unity of $\widehat{\Omega}$, which satisfies

$$\widehat{\Omega} \cap U_j = Q_j \mathbb{R}^n_{\gamma_j} \cap U_j, \quad \operatorname{supp}\varphi_j \subset U_j, \quad \nabla\varphi_j \cdot \mathbf{n}_{\partial\widehat{\Omega}} = 0 \text{ on } \partial\widehat{\Omega} \tag{4.4.109}$$

for all $j \in \{0, \dots, N\}$.

Step 2. Proof of the key estimates (4.4.22) **and** (4.4.23)**.**
Let $v \in C_c^2(\overline{\widehat{\Omega}}) \cap W_I^{2,p}(\widehat{\Omega})$ be arbitrary. For every $j \in \{0, \dots, N\}$, we further choose a function

$$\psi_j \in C_0^\infty(\mathbb{R}^n; [0,1]) \quad \text{with} \quad \operatorname{supp} \psi_j \subseteq U_j \quad \text{and} \quad \psi_j = 1 \text{ on } \operatorname{supp} \varphi_j.$$

Now, we set $v_j := v\varphi_j$ for $j = 0, \dots, N$. Using the product rule along with (4.4.109), we find that

$$v_j \in C_c^2(\overline{Q_j \mathbb{R}^n_{\gamma_j}}) \cap W_I^{2,p}(Q_j \mathbb{R}^n_{\gamma_j}) \quad \text{for all } j \in \{0, \dots, N\}.$$

Recalling that $\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}}$ is linear, we use the decomposition $v = \sum_{j=0}^N v_j$, to obtain

$$\widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} v = \sum_{j=0}^N \widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} v_j = \sum_{j=0}^N \Big(J_{j,1}^\varepsilon + J_{j,2}^\varepsilon \Big). \tag{4.4.110}$$

with

$$J_{j,1}^\varepsilon := \psi_j \, \widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} v_j, \qquad J_{j,2}^\varepsilon := (1 - \psi_j) \, \widehat{\mathcal{L}}_\varepsilon^{\widehat{\Omega}} v_j \tag{4.4.111}$$

for $j = 0, \dots, N$. We will now estimate these summands separately.

Ad $J_{j,1}^\varepsilon$*:* Let $j \in \{0, \dots, N\}$ be arbitrary. We obtain

$$\begin{aligned} J_{j,1}^\varepsilon(x) &= \psi_j(x) \int_{\widehat{\Omega}} \widehat{J}_\varepsilon(x-y)\big(v_j(x) - v_j(y)\big) \, \mathrm{d}y \\ &= J_{j,1,1}^\varepsilon + J_{j,1,2}^\varepsilon - J_{j,1,3}^\varepsilon \end{aligned}$$

with

$$\begin{aligned} J_{j,1,1}^\varepsilon(x) &:= \psi_j(x) \int_{Q_j \mathbb{R}^n_{\gamma_j}} \widehat{J}_\varepsilon(x-y) \, \big(v_j(x) - v_j(y)\big) \, \mathrm{d}y, \\ J_{j,1,2}^\varepsilon(x) &:= \psi_j(x) \int_{\widehat{\Omega} \setminus U_j} \widehat{J}_\varepsilon(x-y) \, v_j(x) \, \mathrm{d}y, \\ J_{j,1,3}^\varepsilon(x) &:= \psi_j(x) \int_{Q_j \mathbb{R}^n_{\gamma_j} \setminus U_j} \widehat{J}_\varepsilon(x-y) \, v_j(x) \, \mathrm{d}y \end{aligned}$$

for all $x \in \widehat{\Omega}$. Since $\widehat{\rho}$ and $\widehat{J}$ satisfy (J1) and (J2), Theorem 4.4.1 (for $j = 0$) and Corollary 4.4.6 (for $j > 0$) imply that

$$\|J_{j,1,1}^\varepsilon\|_{L^p(\widehat{\Omega})} \le C \|v_j\|_{W^{2,p}(\widehat{\Omega})} \le C \|v\|_{W^{2,p}(\widehat{\Omega})}. \tag{4.4.112}$$

We further recall that $\operatorname{supp} v_j \subseteq \operatorname{supp} \varphi_j \subset U_j$. Thus, for $x \in \operatorname{supp} \varphi_j$ and $y \in \mathbb{R}^n \setminus U_j$, it holds that

$$|x - y| \ge \delta_j \quad \text{with} \quad \delta_j := \operatorname{dist}(\operatorname{supp} \varphi_j, \mathbb{R}^n \setminus U_j).$$

Moreover, invoking the definition of $\widehat{\rho}$ (see (4.4.17)) and Lemma 4.2.1(b), a straightforward computation yields

$$\begin{aligned} |J_{j,1,2}^\varepsilon(x)| &\le \delta_j^{-3} \, |v_j(x)| \int_{\mathbb{R}^n} \widehat{\rho}_\varepsilon(x-y) \, |x-y| \, \mathrm{d}y \\ &= \varepsilon \, \delta_j^{-3} \, |v_j(x)| \int_{\mathbb{R}^n} \widehat{\rho}(x-y) \, |x-y| \, \mathrm{d}y \\ &\le C\varepsilon \, \delta_j^{-3} \, |v_j(x)| \, . \end{aligned}$$

Consequently, we obtain

$$\|J^{\varepsilon}_{j,1,2}\|_{L^p(\widehat{\Omega})} \le C\varepsilon\|v_j\|_{L^p(\widehat{\Omega})} \le C\varepsilon\|v\|_{L^p(\widehat{\Omega})}. \tag{4.4.113}$$

The term $J^{\varepsilon}_{j,1,3}$ can be estimated completely analogously and we end up with

$$\|J^{\varepsilon}_{j,1,3}\|_{L^p(\widehat{\Omega})} \le C\varepsilon\|v\|_{L^p(\widehat{\Omega})}. \tag{4.4.114}$$

Combining (4.4.115) and (4.4.116) and recalling that $\varepsilon \in (0,1]$, we conclude that

$$\|J^{\varepsilon}_{j,1}\|_{L^p(\widehat{\Omega})} \le C\|v\|_{W^{2,p}(\widehat{\Omega})}. \tag{4.4.115}$$

for all $j \in \{0,\dots,N\}$.

Ad $J^{\varepsilon}_{j,2}$*:* Let $j \in \{0,\dots,N\}$ be arbitrary. We first notice that $1-\psi_j(x) = 0$ if $x \in \operatorname{supp}\varphi_j$. If $x \in \widehat{\Omega} \setminus \operatorname{supp}\psi_j$ and $y \in \operatorname{supp}\varphi_j$, we have

$$|x-y| \ge \tilde{\delta}_j \quad \text{with} \quad \tilde{\delta}_j := \operatorname{dist}(\mathbb{R}^n \setminus \operatorname{supp}\psi_j, \operatorname{supp}\varphi_j).$$

Hence, for all $x \in \widehat{\Omega}$, it holds that

$$|J^{\varepsilon}_{j,2}(x)| \le (1-\psi_j(x))\tilde{\delta}_j^{-3} \int_{\widehat{\Omega}} \widehat{\rho}_{\varepsilon}(x-y)\,|x-y||v_j(y)|\,\mathrm{d}y\,.$$

This directly implies

$$\begin{aligned}\|J^{\varepsilon}_{j,2}\|^p_{L^p(\widehat{\Omega})} &\le C\tilde{\delta}_j^{-3p}\int_{\mathbb{R}^n}\left(\int_{\widehat{\Omega}}\widehat{\rho}_{\varepsilon}(x-y)\,|x-y|\,\mathrm{d}y\right)^{p-1} \\ &\qquad\cdot\left(\int_{\widehat{\Omega}}\widehat{\rho}_{\varepsilon}(x-y)\,|x-y||v_j(y)|^p\,\mathrm{d}y\right)\mathrm{d}x \\ &\le C\tilde{\delta}_j^{-3p}\varepsilon^{p-1}\int_{\widehat{\Omega}}\left(\int_{\mathbb{R}^n}\widehat{\rho}_{\varepsilon}(x-y)\,|x-y|\,\mathrm{d}x\right)|v_j(y)|^p\,\mathrm{d}y \\ &\le C\varepsilon^p\|v_j\|^p_{L^p(\widehat{\Omega})} \le C\varepsilon^p\|v\|^p_{L^p(\widehat{\Omega})}.\end{aligned} \tag{4.4.116}$$

In summary, combining (4.4.115) and (4.4.116) to bound the right-hand side of (4.4.110), we infer the estimate

$$\|\widehat{\mathcal{L}}^{\widehat{\Omega}}_{\varepsilon} v_j\|_{L^p(\widehat{\Omega})} \le C\|v\|_{W^{2,p}(\widehat{\Omega})}$$

for all $v \in C^2_c(\overline{\widehat{\Omega}}) \cap W^{2,p}_I(\widehat{\Omega})$. This means that key estimate (4.4.22) is established.

In order to verify key estimate (4.4.23), let now $v \in C^3_c(\overline{\widehat{\Omega}}) \cap W^{3,p}_I(\widehat{\Omega})$ be arbitrary. Using the decomposition (4.4.110) along with the linearity of the Laplacian, we derive the identity

$$\widehat{\mathcal{L}}^{\widehat{\Omega}}_{\varepsilon} v + \Delta v = \sum_{j=0}^{N}\left((J^{\varepsilon}_{j,1,1} + \Delta v_j) + J^{\varepsilon}_{j,1,2} + J^{\varepsilon}_{j,1,3} + J^{\varepsilon}_{j,2}\right), \tag{4.4.117}$$

where $J^{\varepsilon}_{j,1,1}$, $J^{\varepsilon}_{j,1,2}$, $J^{\varepsilon}_{j,1,3}$ and $J^{\varepsilon}_{j,2}$, $j = 0, \ldots, N$, are defined as above. Since $\psi_j = 1$ on $\operatorname{supp} v_j$, we have

$$J^{\varepsilon}_{j,1,1} + \Delta v_j = \psi_j(x) \left(\int_{Q_j \mathbb{R}^n_{\gamma_j}} \widehat{J}_{\varepsilon}(x-y) \left(v_j(x) - v_j(y)\right) \mathrm{d}y + \Delta v_j \right).$$

Hence, applying Theorem 4.4.1 (for $j = 0$) and Corollary 4.4.6 (for $j > 0$), we deduce

$$\|J^{\varepsilon}_{j,1,1} + \Delta v_j\|_{L^p(\widehat{\Omega})} \leq C \sqrt[p]{\varepsilon}\, \|v_j\|_{W^{3,p}(\widehat{\Omega})} \leq C \sqrt[p]{\varepsilon}\, \|v\|_{W^{3,p}(\widehat{\Omega})}. \tag{4.4.118}$$

In combination with (4.4.113), (4.4.113) and (4.4.116), we conclude from (4.4.117) that

$$\|\widehat{\mathcal{L}}^{\widehat{\Omega}}_{\varepsilon} v + \Delta v\|_{L^p(\widehat{\Omega})} \leq C \sqrt[p]{\varepsilon}\, \|v\|_{W^{3,p}(\widehat{\Omega})}.$$

This means that key estimate (4.4.23) is established and thus, the proof of Theorem 2.1.7 is complete.

Part II

Application to phase-field models

Chapter 5

Nonlocal-to-local convergence for the Cahn–Hilliard and Allen–Cahn equations

5.1. Introduction

In this chapter, we study the Cahn–Hilliard and Allen–Cahn equations as introduced in Section 1.2 and 1.3, respectively. For $\varepsilon > 0$, we recall that the nonlocal Cahn–Hilliard equation (see Section 1.3) reads as

$$\begin{aligned}
\partial_t c_\varepsilon &= \Delta \mu_\varepsilon && \text{in } \Omega_T, && (5.1.1a)\\
\mu_\varepsilon &= \mathcal{L}_\varepsilon c_\varepsilon + f'(c_\varepsilon) && \text{in } \Omega_T, && (5.1.1b)\\
\partial_{\mathbf{n}} \mu_\varepsilon &= 0 && \text{on } \partial\Omega_T, && (5.1.1c)\\
c_\varepsilon(0) &= c_{\varepsilon,0} && \text{in } \Omega, && (5.1.1d)
\end{aligned}$$

where the nonlocal operator $\mathcal{L}_\varepsilon$ is given as in (4.1.5), i.e,

$$\mathcal{L}_\varepsilon u(x) := \text{P.V.} \int_\Omega J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y, \quad x \in \Omega, \qquad (5.1.2)$$

for a prescribed interaction kernel J_ε as in (4.1.4). We are interested in the asymptotics $\varepsilon \to 0$ in system (5.1.1). If the interaction kernel of the nonlocal operator $\mathcal{L}_\varepsilon$ is radially symmetric, it is well-known that solutions to the nonlocal Cahn–Hilliard equation (5.1.1) converge to a solution to the local isotropic Cahn–Hilliard equation

$$\begin{aligned}
\partial_t c &= \Delta \mu && \text{in } \Omega_T, && (5.1.3a)\\
\mu &= -\Delta c + f'(c) && \text{in } \Omega_T, && (5.1.3b)\\
\partial_{\mathbf{n}} c = \partial_{\mathbf{n}} \mu &= 0 && \text{on } \partial\Omega_T, && (5.1.3c)\\
c(0) &= c_0 && \text{in } \Omega. && (5.1.3d)
\end{aligned}$$

This has been shown, e.g., in [44], where the authors even considered additional viscosity terms. In case of periodic boundary conditions, the convergence has been shown in [98]. Let us mention that also singular potentials have been studied in the nonlocal-to-local asymptotics, cf. the contributions in [41, 43, 44].

On account of the convergence result in Theorem 2.1.7, we are also interested in the case, where the interaction kernel of $\mathcal{L}_\varepsilon$ is *not* radially symmetric but only even. In this setting, we intend to prove that solutions to the nonlocal Cahn–Hilliard equation (5.1.1) converge to a solution to the local *anisotropic* Cahn–Hilliard equation

$$\partial_t c = \Delta\mu \quad \text{in } \Omega_T, \tag{5.1.4a}$$

$$\mu = -\operatorname{div}(M\nabla c) + f'(c) \quad \text{in } \Omega_T, \tag{5.1.4b}$$

$$\mathbf{n}\cdot M\nabla c = \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \partial\Omega_T, \tag{5.1.4c}$$

$$c(0) = c_0 \quad \text{in } \Omega, \tag{5.1.4d}$$

where the matrix $M \in \mathbb{R}^{n\times n}$ is given by

$$M = \frac{1}{2}\int_{\mathbb{R}^n} J(z) z\otimes z \,\mathrm{d}z.$$

To the best of the author's knowledge, this convergence has not been studied yet. We also want to mention that the convergence results in [41, 43, 44, 98] have not provided any rate of convergence.

Furthermore, we intend to study the Allen–Cahn equation. For $\varepsilon > 0$, we recall that the nonlocal Allen–Cahn equation (see Section 1.3) reads as

$$\partial_t c_\varepsilon + \mathcal{L}_\varepsilon c_\varepsilon + f'(c_\varepsilon) = 0 \quad \text{in } \Omega\times(0,T), \tag{5.1.5a}$$

$$c_\varepsilon(0) = c_{\varepsilon,0} \quad \text{in } \Omega. \tag{5.1.5b}$$

We are interested in the asymptotic behavior of the solutions to (5.1.5a)–(5.1.5b) as ε tends to 0. More precisely, we want to show that the sequence of solutions converges to the solution to the local Allen–Cahn equation, i.e.,

$$\partial_t c - \Delta c + f'(c) = 0 \quad \text{in } \Omega\times(0,T), \tag{5.1.6}$$

$$\partial_{\mathbf{n}} c = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{5.1.7}$$

$$c(0) = c_0 \quad \text{in } \Omega. \tag{5.1.8}$$

It is the main goal of this section to prove that solutions to the nonlocal Cahn–Hilliard equation (5.1.1) and the nonlocal Allen–Cahn equation (5.1.5), respectively, converge to a solution to its local counterpart. The essential novelty in these results is that concrete rates of convergence can be shown. Moreover, in the case of the Cahn–Hilliard equation, convergence can be shown even for anisotropic kernels.

The structure of this section is as follows: In the first part, we present well-posedness results for both the Cahn–Hilliard and Allen–Cahn equation. There, we discuss both the local and nonlocal models. In the second part, we then prove the main results concerning the nonlocal-to-local asymptotics. First, we prove convergence for the Cahn–Hilliard equation (see Theorem 2.2.2). The proof is based on an energy method together with the convergence result obtained in Theorem 2.1.7. Then, we show convergence for the Allen–Cahn equation (see Theorem 2.3.2). The proof is based on an energy method combined with the convergence result from Theorem 2.1.8.

This chapter is based on the works in:

[11] H. Abels, C. Hurm, *Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator*, J. Differential Equations, 402: 593-624, 2024.

Let us note that the contribution in [11] only proves convergence for the isotropic Cahn–Hilliard equation since they only allow for isotropic kernels. In this chapter, however, the analysis is more general, since we also allow for anisotropic kernels. Therefore, we can prove convergence even for the anisotropic Cahn–Hilliard equation.

5.2. Well-Posedness Results

5.2.1. Well-Posedness of the Cahn–Hilliard equation

In this subsection, we recall some well-posedness results for the Cahn–Hilliard equation as introduced in Section 1.2 and 1.3, respectively. In the first part, we discuss the local Cahn–Hilliard equation. There, we consider both anisotropic and isotropic variants of the system. In the second part, we then focus on its nonlocal counterpart.

Well-Posedness of the local Cahn–Hilliard equation

Here, we briefly recall the well-posedness results for both the anisotropic and isotropic Cahn–Hilliard equation. Both systems have already been studied very well. For the anisotropic Cahn–Hilliard equation, the existence of weak solutions has been shown, e.g., in [68]. Uniqueness and regularity theory can be found, e.g., in [69]. We note that in these contributions also more general situations have been studied, since they also allow for singular potentials like the logarithmic potential as in (1.2.3) or even for the double-obstacle potential. They also considered non-constant mobilities and more general anisotropies. More precisely, they studied functions $\Phi : \mathbb{R}^n \to \mathbb{R}$ satisfying suitable growth and monotonicity conditions, cf. [69, Assumption (A2)], and their corresponding anisotropies given by $\Phi(p)$ for $p \in \mathbb{R}^n$. In our case, the anisotropy is given by

$$\Phi(p) := \frac{1}{2} p^T M p \quad \text{for all } p \in \mathbb{R}^n. \tag{5.2.1}$$

It remains to prove that anisotropies as in (5.2.1) satisfy the assumptions in [69, Assumption (A2)]. Indeed, $\Phi : \mathbb{R}^n \to \mathbb{R}$ is smooth by our construction. Moreover, Φ is positive in $\mathbb{R}^n \setminus \{0\}$ which follows by positive definiteness of the matrix M. For $\lambda > 0$, we observe that

$$\Phi(\lambda p) = \frac{1}{2}(\lambda p)^T M(\lambda p) = \lambda^2 \frac{1}{2} p^T M p = \lambda^2 \Phi(p)$$

for all $p \in \mathbb{R}^n$. Again, since M is positive definite, there exist constants $\Phi_0, \Phi_1 > 0$ such that

$$\Phi_0 |p|^2 \leq \Phi(p) \leq \Phi_1 |p|^2$$

holds for all $p \in \mathbb{R}^n$. Note that the gradient $\Phi' : \mathbb{R}^n \to \mathbb{R}^n$ of Φ is given by $\Phi'(p) = Mp$. Thus, it immediately follows that there exists a constant $a_0 > 0$ such that

$$|\Phi'(p)| \leq a_0 |p|$$

for all $p \in \mathbb{R}^n$. Moreover, we observe that

$$(\Phi'(p) - \Phi'(q)) \cdot (p - q) = (p - q)^T M (p - q) \geq a_1 |p - q|^2$$

for some constant $a_1 > 0$ and all $p, q \in \mathbb{R}^n$. This again follows by the properties of M. Hence, all the assumptions in [69, Assumption (A2)] are fulfilled. In particular, we then have the following well-posedness result:

Theorem 5.2.1. *Let the assumptions* (A1) *and* (A3)–(A4) *hold, let* $M \in \mathbb{R}^{n\times n}$ *be as in* (4.1.10) *and let* $c_0 \in H^1(\Omega)$ *by any initial datum. We further assume that there exists* $\mu_0 \in H^1(\Omega)$ *such that for all* $\xi \in H^1(\Omega)$*, it holds*

$$\int_\Omega \mu_0\, \xi \,\mathrm{d}x = \int_\Omega M\nabla c_0 \cdot \nabla\xi \,\mathrm{d}x + \int_\Omega f'(c_0)\xi \,\mathrm{d}x.$$

Then, there exists a unique weak solution (c, μ) *to* (5.1.4) *with the regularity*

$$\begin{aligned} c &\in C(\overline{\Omega_T}) \cap L^\infty(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)),\\ \mu &\in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)),\\ f'(c) &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^\infty(\Omega)) \end{aligned}$$

such that $c(0) = c_0$ *a.e. in* Ω *and it holds*

$$\begin{aligned} \langle \partial_t c, \zeta\rangle &= -\int_\Omega \nabla\mu\cdot\nabla\zeta \,\mathrm{d}x,\\ \int_\Omega \mu\,\xi \,\mathrm{d}x &= \int_\Omega M\nabla c\cdot\nabla\xi \,\mathrm{d}x + \int_\Omega f'(c)\xi \,\mathrm{d}x \end{aligned}$$

a.e. in $[0,T]$ *and for all* $\zeta, \xi \in H^1(\Omega)$.

As already mentioned above, the results in the literature are more general than the setting in Theorem 5.2.1. For analytic results, we refer to the contributions in [68, 69].

In order to apply Theorem 2.1.7 and prove convergence along with certain rates, we need higher regularity for the phase-field c. In fact, the regularity in Theorem 5.2.1 can be improved. We then obtain the following:

Remark 5.2.2. Owing to the chain rule, the growth assumption of f, cf. (A3), and the regularity of c in Theorem 5.2.1, it follows that

$$\|\nabla f'(c)\|^2_{L^2(\Omega)^n} = \int_\Omega |f''(c)\nabla c|^2 \,\mathrm{d}x \le C\int_\Omega (1+|c|^2)^2|\nabla c|^2 \,\mathrm{d}x \le C\int_\Omega |\nabla c|^2 \,\mathrm{d}x,$$

which yields $f'(c) \in L^\infty(0,T;H^1(\Omega))$. Recalling that we have

$$-\nabla\cdot(M\nabla c(t)) = \mu(t) - f'(c(t)) \in H^1(\Omega)$$

for almost all $t \in [0,T]$, regularity theory for quasilinear elliptic partial differential equations, e.g., [116, Korollar 6.15], implies $c(t) \in H^3(\Omega)$ for almost all $t \in [0,T]$. Finally, since

$$\|c(t)\|_{H^3(\Omega)} \le C\big(\|\mu(t) - f'(c(t))\|_{H^1(\Omega)} + \|c(t)\|_{L^2(\Omega)}\big)$$

for almost all $t \in [0,T]$, it follows that $c \in L^\infty(0,T;H^3(\Omega))$.

Considering the case $M = I$ in (5.1.4), we obtain the isotropic Cahn–Hilliard equation (5.1.3). This system is already well analyzed and there exists an intense literature about its analysis. It covers more general cases such as non-constant and degenerate mobilities, singular potentials as well as regularity of solutions, separation from the pure phases and long-time behavior. Moreover, different types of boundary conditions have been studied in the literature. Instead of the homogeneous Neumann boundary condition for c, there are also results about homogeneous Dirichlet boundary conditions or dynamic boundary conditions. We refer to [17, 19, 49, 50, 99, 100, 104, 125] to name a few of them.

In the case of constant mobility (e.g., $m = 1$) and regular potentials, we then have the following well-posedness result:

Theorem 5.2.3. *Let the assumptions* (A1), (A3)–(A4) *hold and let* $c_0 \in H^1(\Omega)$ *be any initial datum. Then, there exists a unique weak solution* (c, μ) *to* (5.1.3) *with the regularity*

$$\begin{aligned} c &\in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap L^2(0,T;H^3(\Omega)), \\ \mu &\in L^2(0,T;H^1(\Omega)), \\ f'(c) &\in L^2(0,T;L^2(\Omega)) \end{aligned}$$

such that $c(0) = c_0$ *a.e. in* Ω *and it holds*

$$\begin{aligned} \langle \partial_t c, \zeta \rangle &= -\int_\Omega \nabla\mu \cdot \nabla\zeta \,\mathrm{d}x, \\ \int_\Omega \mu\,\xi \,\mathrm{d}x &= \int_\Omega \nabla c \cdot \nabla\xi \,\mathrm{d}x + \int_\Omega f'(c)\xi \,\mathrm{d}x \end{aligned}$$

a.e. in $[0,T]$ *and for all* $\zeta, \xi \in H^1(\Omega)$.

Remark 5.2.4. The existence of weak solutions can be easily shown using a Faedo–Galerkin scheme. Note that this can be done even for regular potentials satisfying more general growth assumptions and also less regular bounded domains Ω (Lipschitz-regularity is sufficient for the existence proof). The uniqueness then follows by standard arguments. Here, it is crucial to assume a splitting of f as in (A4). Higher regularity of c can be shown by means of standard elliptic regularity theory. Here, we need that the second derivative of f grows at most quadratically, cf. (A3).

Well-Posedness of the nonlocal Cahn–Hilliard equation

This subsection concerns the well-posedness of the nonlocal Cahn–Hilliard equation (5.1.1). In recent years, this system has gained more interest and has already been studied very well. The existence of weak solutions in the case of periodic boundary conditions has been shown in [77, 98] for regular potentials and in [41] for singular potentials. The latter also allows for an additional viscosity term. In case of Neumann boundary conditions, existence has been shown in [43, 44, 61]. Well-posedness and large-time behavior for more singular kernels and singular free energies can be found in [3] and well-posedness together with regularity and strict separation property in [63]. Further results on the separation property have been provided in [107]. The nonlocal Cahn–Hilliard equation with nonlocal dynamic boundary condition has been studied in [90].

In case of constant mobility and regular potentials, we have the following well-posedness result, see, e.g., [43, 63]:

Theorem 5.2.5. *Let $\varepsilon > 0$ and let the assumptions* (A1)–(A4) *hold. Furthermore, let $c_{0,\varepsilon} \in L^2(\Omega)$ be any initial datum. Then, there exists a weak solution c_ε to* (5.1.1) *with the regularity*

$$\begin{aligned} &c_\varepsilon \in H^1(0,T;H^1(\Omega)') \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \\ &\mu_\varepsilon = \mathcal{L}_\varepsilon c_\varepsilon + f'(c_\varepsilon) \in L^2(0,T;H^1(\Omega)) \end{aligned}$$

such that $c_\varepsilon(0) = c_{0,\varepsilon}$ in Ω and it holds

$$\langle \partial_t c_\varepsilon, v\rangle + \int_\Omega \nabla\mu_\varepsilon \cdot \nabla v \,\mathrm{d}x = 0 \quad \textit{for all } v \in H^1(\Omega), \quad \textit{a.e. in } (0,T).$$

Remark 5.2.6. Let us mention that the well-posedness result Theorem 5.2.5 holds true for both anisotropic and isotropic kernels. In fact, the proof only uses that the kernel is even. Thus, the radial symmetry is not needed for well-posedness but only for asymptotics towards a local differential operator.

5.2.2. Well-Posedness of the Allen–Cahn equation

This subsection is devoted to the well-posedness of the Allen–Cahn equation. In the first part, we consider the local Allen–Cahn equation. Since this follows from standard theory of partial differential equations, we will only explain the key ideas. In the second part, we then discuss the nonlocal Allen–Cahn equation.

Well-Posedness of the local Allen–Cahn equation

The well-posedness of the local Allen–Cahn equation (5.1.6)–(5.1.8) is already well-known. In particular, this follows by standard theory for partial differential equations. For instance, we refer to the book by Smoller [117, Chapter 14], where systems of reaction-diffusion equations of the form

$$\partial_t \mathbf{u} = D\Delta\mathbf{u} + f(\mathbf{u}) \quad \text{in } \Omega \subseteq \mathbb{R}^k,\ t > 0 \tag{5.2.2}$$

are considered. Here, $\mathbf{u} \in \mathbb{R}^n$ is a vector-valued function, $D \in \mathbb{R}^{n\times n}$ a matrix and $f(\mathbf{u})$ a smooth function. Moreover, the function $\mathbf{u}$ needs to satisfy an initial condition and specific boundary conditions, e.g., Dirichlet or Neumann boundary conditions. We observe that system (5.2.2) reduces to the Allen–Cahn equation (5.1.6)–(5.1.8) if we set $n = D = 1$ and impose homogeneous Neumann boundary conditions.

It then follows by the theory in [117, Chapter 14 §A] that system (5.2.2) has a local solution u on a small time interval $[0,\tau)$ depending only on the initial data. If the solution is a-priori bounded with respect to the L^∞-norm, it even exists globally in time. The latter is fulfilled if system (5.2.2) admits bounded *invariant regions*, i.e., closed subsets $\Sigma \subset \mathbb{R}^n$ with the property that if any solution $u(x,t)$ having all of its boundary and initial values in Σ, satisfies $u(x,t) \in \Sigma$ for all $x \in \Omega$ and for all $t \in [0,\tau)$. In particular, this provides a-priori bounds for the solution with respect to the L^∞-norm and thus, if the initial datum lies in Σ, the solution exists for all times, cf. [117, Chapter 14 §B].

In the context of the Allen–Cahn equation, invariant regions can also be used to show that the solution stays in the interval $[-1,1]$ provided that the initial datum lies in $[-1,1]$. To this end, one needs to prove that

$$\Sigma := \{u : |u| \leq 1\}$$

is an invariant region for the Allen–Cahn equation (5.1.6)–(5.1.8). However, since f' identically vanishes on $\partial\Sigma$, cf. Assumption (A3), we can not directly apply the theory developed in [117]. Instead, we first need to consider disks, which contain Σ in its interior. For instance (see [117, 14§B, Example 3]), we can take

$$\Sigma_\delta := \{u : |u| \leq 1+\delta\}, \quad \delta > 0.$$

Then, [117, Corollary 14.8(b)] implies that Σ_δ is invariant for every $\delta > 0$. If we assume that $u_0(x) = u(x,0) \in \Sigma$ for all $x \in \Omega$, it holds that $u_0(x) \in \Sigma_\delta$ for all $\delta > 0$ and all $x \in \Omega$. Since Σ_δ is invariant for all $\delta > 0$, it then follows that $u(x,t) \in \Sigma_\delta$ for all $\delta > 0$ and all $(x,t) \in \Omega_T$. This entails $u(x,t) \in \Sigma$ for all $(x,t) \in \Omega_T$.

Therefore, we do not need to consider singular potentials to show that the solution stays in $[-1,1]$. We also observe that the potential does not need to satisfy any growth condition, in order to prove the existence of solutions. However, we restrict our attention to polynomial double-well potentials as discussed in Section 1.2.

Well-Posedness of the nonlocal Allen–Cahn equation

Regarding the well-posedness of the nonlocal Allen–Cahn equation, there are not many references in the literature. In the case of periodic boundary conditions, the well-posedness is addressed in Lemma 8.3.1 and Theorem 8.3.2, respectively. In the case Ω being a bounded domain, most of the arguments can be adapted. However, one needs to adjust the arguments in Step 3 in a suitable way, in order to get uniform estimates in $L^\infty(0,T;H^1(\Omega))$. This can be done similarly as in estimate (7.3.4) in the proof of Theorem 7.2.4, namely testing the weak formulation of (5.1.5a) by $-\Delta c$. This yields [1]

$$\begin{aligned}
(\mathcal{L}_\varepsilon c + f'(c), -\Delta c) &= (\nabla \mathcal{L}_\varepsilon c + f''(c)\nabla c, \nabla c) \\
&= (-\nabla J_\varepsilon * c + (J_\varepsilon * 1)\nabla c + \nabla(J_\varepsilon * 1)c + f''(c)\nabla c, \nabla c) \\
&\geq \alpha \|\nabla c\|^2_{L^2(\Omega)^n} - 2\|\nabla J_\varepsilon\|_{L^1(\mathbb{R}^n)}\|c\|_{L^2(\Omega)}\|\nabla c\|_{L^2(\Omega)^n} \\
&\geq \frac{\alpha}{2}\|\nabla c\|^2_{L^2(\Omega)^n} - k\|c\|^2_{L^2(\Omega)}.
\end{aligned}$$

Then, we can follow the remaining arguments in the proof of Theorem 8.3.2 line by line, in order to prove the existence of a weak solution. The uniqueness follows by the same ideas as in the proof of Theorem 8.3.1.

[1] at least formally; more precisely, one needs to work on the Galerkin level with aid of eigenfunctions of the Neumann Laplace operator similar to the proof of Theorem 8.3.2

5.3. Proof of Theorem 2.2.2

For any $\varepsilon > 0$, let $(c_\varepsilon, \mu_\varepsilon)$ be the weak solution to the nonlocal Cahn–Hilliard equation (5.1.1) given by Theorem 5.2.5. Moreover, let (c, μ) be the unique weak solution to the local anisotropic Cahn–Hilliard equation (5.1.4) given by Theorem 5.2.1. We define the differences $u := c_\varepsilon - c$, $w := \mu_\varepsilon - \mu$ and $u_0 := c_{0,\varepsilon} - c_0$. Then, the pair (u, w) solves

$$\partial_t u = \Delta w \qquad \text{in } \Omega_T, \tag{5.3.1}$$

$$w = \mathcal{L}_\varepsilon c_\varepsilon + \nabla \cdot (M \nabla c) + f'(c_\varepsilon) - f'(c) \qquad \text{in } \Omega_T, \tag{5.3.2}$$

$$u(0) = u_0 \qquad \text{in } \Omega \tag{5.3.3}$$

in a weak sense, i.e.,

$$\langle \partial_t u, \zeta \rangle = -\int_\Omega \nabla w \cdot \nabla \zeta \, \mathrm{d}x,$$

$$\int_\Omega w \, \xi \, \mathrm{d}x = \int_\Omega \Big(\mathcal{L}_\varepsilon c_\varepsilon + \nabla \cdot (M \nabla c) \Big) \xi \, \mathrm{d}x + \int_\Omega \Big(f'(c_\varepsilon) - f'(c) \Big) \xi \, \mathrm{d}x$$

a.e. in $[0, T]$ for all $\zeta, \xi \in H^1(\Omega)$. Observe that the assumptions on the mean value imply $\overline{u} = 0$. Thus, we can test (5.3.1) by $\mathcal{N} u$. This yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{H^{-1}_{(0)}(\Omega)} = -\int_\Omega uw \, \mathrm{d}x.$$

Moreover, testing (5.3.2) by u, we obtain

$$\begin{aligned} \int_\Omega uw \, \mathrm{d}x &= \int_\Omega (\mathcal{L}_\varepsilon c_\varepsilon + \nabla \cdot (M \nabla c)) u \, \mathrm{d}x + \int_\Omega (f'(c_\varepsilon) - f'(c)) u \, \mathrm{d}x \\ &= 2\mathcal{E}_\varepsilon(u) + \int_\Omega (\mathcal{L}_\varepsilon c + \nabla \cdot (M \nabla c)) u \, \mathrm{d}x + \int_\Omega (f'(c_\varepsilon) - f'(c)) u \, \mathrm{d}x. \end{aligned}$$

Here, we used the identity

$$\int_\Omega \mathcal{L}_\varepsilon u u \, \mathrm{d}x = \frac{1}{2} \int_\Omega \int_\Omega J_\varepsilon(x - y) |u(x) - u(y)|^2 \, \mathrm{d}y \, \mathrm{d}x = 2\mathcal{E}_\varepsilon(u),$$

which follows by symmetry of the interaction kernel J_ε. Next, owing to assumption (A3), it holds

$$\int_\Omega (f'(c_\varepsilon) - f'(c)) u \, \mathrm{d}x \geq -\alpha \|u\|^2_{L^2(\Omega)}.$$

Invoking Young's inequality, we obtain

$$\int_\Omega (\mathcal{L}_\varepsilon c + \nabla \cdot (M \nabla c)) u \, \mathrm{d}x \leq \frac{1}{2} \|\mathcal{L}_\varepsilon c + \nabla \cdot (M \nabla c)\|^2_{L^2(\Omega)} + \frac{1}{2} \|u\|^2_{L^2(\Omega)}.$$

Therefore, combining all these estimates, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{H^{-1}_{(0)}(\Omega)} + 2\mathcal{E}_\varepsilon(u) + \frac{1}{2} \|u\|^2_{L^2(\Omega)} \leq (\alpha + 1) \|u\|^2_{L^2(\Omega)} + \frac{1}{2} \|\mathcal{L}_\varepsilon c + \nabla \cdot (M \nabla c)\|^2_{L^2(\Omega)}.$$

In the next step, we employ the nonlocal Ehrling lemma 3.5.4 with $\gamma = \frac{1}{\alpha+1}$ to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{H^{-1}_{(0)}(\Omega)} + \mathcal{E}_\varepsilon(u) + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|\mathcal{L}_\varepsilon c + \nabla\cdot(M\nabla c)\|^2_{L^2(\Omega)} + C\|u\|^2_{H^{-1}_{(0)}(\Omega)}.$$

Thus, Gronwall's lemma yields

$$\begin{aligned}&\frac{1}{2}\sup_{t\in[0,T]}\|u(t)\|^2_{H^{-1}_{(0)}(\Omega)} + \int_0^T \mathcal{E}_\varepsilon(u(t))\,\mathrm{d}t + \frac{1}{2}\int_0^T \|u(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\qquad\leq \Big(\frac{1}{2}\|u_0\|^2_{H^{-1}_{(0)}(\Omega)} + \frac{1}{2}\int_0^T \|\mathcal{L}_\varepsilon c(t) + \nabla\cdot(M\nabla c(t))\|^2_{L^2(\Omega)}\,\mathrm{d}t\Big)e^{CT}.\end{aligned}$$

Since $c \in L^2(0,T;H^3(\Omega))$ (see Remark 5.2.2), Theorem 2.1.7 implies

$$\int_0^T \|\mathcal{L}_\varepsilon c(t) + \nabla\cdot(M\nabla c(t))\|^2_{L^2(\Omega)}\,\mathrm{d}t \leq C\varepsilon\|c\|^2_{L^2(0,T;H^3(\Omega))}$$

Recalling assumption (2.2.2), we then arrive at

$$\frac{1}{2}\sup_{t\in[0,T]}\|u(t)\|^2_{H^{-1}_{(0)}(\Omega)} + \int_0^T \mathcal{E}_\varepsilon(u(t))\,\mathrm{d}t + \frac{1}{2}\int_0^T \|u(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \leq C\varepsilon.$$

Hence, the proof is complete. □

Remark 5.3.1. Using the same arguments as in the proof of Theorem 2.2.2, one can also prove Theorem 2.2.3. In fact, since $M = I$, one only needs to apply the convergence result in Theorem 2.1.8 instead of Theorem 2.1.7.

5.4. Proof of Theorem 2.3.2

For $\varepsilon > 0$, let c_ε be the weak solution to the nonlocal Allen–Cahn equation and c be the unique strong solution to the local Allen–Cahn equation. We denote the difference by $u := c_\varepsilon - c$. Then, the function u solves

$$\partial_t u = -\mathcal{L}_\varepsilon c_\varepsilon - f'(c_\varepsilon) - \Delta c + f'(c) \qquad \text{in } \Omega_T, \tag{5.4.1}$$

$$u(0) = c_{\varepsilon,0} - c_0 \qquad \text{in } \Omega \tag{5.4.2}$$

in a weak sense. Testing (5.4.1) by u, this yields

$$\begin{aligned}\int_\Omega \partial_t u u\,\mathrm{d}x &= -\int_\Omega (\mathcal{L}_\varepsilon c_\varepsilon + \Delta c)u\,\mathrm{d}x - \int_\Omega (f'(c_\varepsilon) - f'(c))u\,\mathrm{d}x \\ &= -\int_\Omega (\mathcal{L}_\varepsilon c + \Delta c)u\,\mathrm{d}x - \int_\Omega \mathcal{L}_\varepsilon u u\,\mathrm{d}x - \int_\Omega (f'(c_\varepsilon) - f'(c))u\,\mathrm{d}x.\end{aligned}$$

In the left-hand side, we use the identity

$$\int_\Omega \partial_t u u\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|^2_{L^2(\Omega)}.$$

Employing the properties of f, cf. (A3), it follows that

$$\int_\Omega (f'(c_\varepsilon) - f'(c))u \,\mathrm{d}x \geq -\alpha \|u\|^2_{L^2(\Omega)}.$$

Finally, we use that J_ε is even to get

$$\int_\Omega \mathcal{L}_\varepsilon u u \,\mathrm{d}x = \frac{1}{2}\int_\Omega\int_\Omega J_\varepsilon(x-y)|u(x)-u(y)|^2 \,\mathrm{d}y \,\mathrm{d}x = 2\mathcal{E}_\varepsilon(u).$$

Therefore, combining the arguments above, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u\|^2_{L^2(\Omega)} + \mathcal{E}_\varepsilon(u) \leq \alpha\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\Big\|\mathcal{L}_\varepsilon c + \Delta c\Big\|^2_{L^2(\Omega)}.$$

Thus, Gronwall's lemma implies

$$\frac{1}{2}\sup_{t\in[0,T]}\|u(t)\|^2_{L^2(\Omega)} + \int_0^T \mathcal{E}_\varepsilon(u) \,\mathrm{d}t \leq \Big(\frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \int_0^T \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} \,\mathrm{d}t\Big)e^{CT}.$$

Since $c \in L^2(0,T;H^3(\Omega))$, Theorem 2.1.8 yields

$$\int_0^T \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} \,\mathrm{d}t \leq C\varepsilon\|c\|^2_{L^2(0,T;H^3(\Omega))}.$$

Together with assumption (2.3.1), we thus have

$$\sup_{t\in[0,T]}\|u(t)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}.$$

In particular, this concludes the proof. □

Chapter 6

Nonlocal-to-local convergence for a Cahn–Hilliard tumor growth model

6.1. Introduction

For $\varepsilon > 0$, we consider the nonlocal Cahn–Hilliard tumor growth model from Section 1.7,

$$\begin{aligned}
\partial_t c_\varepsilon &= \Delta\mu_\varepsilon + (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(c_\varepsilon) && \text{in } \Omega_T, && (6.1.1a)\\
\mu_\varepsilon &= \mathcal{L}_\varepsilon c_\varepsilon + f'(c_\varepsilon) && \text{in } \Omega_T, && (6.1.1b)\\
\partial_t \sigma_\varepsilon &= \Delta\sigma_\varepsilon + \mathcal{B}(\sigma_S - \sigma_\varepsilon) - \mathcal{C}\sigma h(c_\varepsilon) && \text{in } \Omega_T, && (6.1.1c)\\
\partial_{\mathbf{n}}\mu_\varepsilon &= \partial_{\mathbf{n}}\sigma_\varepsilon = 0 && \text{on } \partial\Omega_T, && (6.1.1d)\\
c_\varepsilon(0) &= c_{\varepsilon,0}, \quad \sigma_\varepsilon(0) = \sigma_{\varepsilon,0} && \text{in } \Omega, && (6.1.1e)
\end{aligned}$$

where the nonlocal operator $\mathcal{L}_\varepsilon$ is given as in (4.1.5), i.e,

$$\mathcal{L}_\varepsilon u(x) := \text{P.V.} \int_\Omega J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y, \quad x \in \Omega, \tag{6.1.2}$$

for a prescribed interaction kernel J_ε as in (4.1.4). In this section, we want to investigate the asymptotics of the solutions to system (6.1.1) as $\varepsilon \to 0$. More precisely, we intend to show that weak solutions converge to a solution to the following system:

$$\begin{aligned}
\partial_t c &= \Delta\mu + (\mathcal{P}\sigma - \mathcal{A})h(c) && \text{in } \Omega_T, && (6.1.3a)\\
\mu &= -\Delta c + f'(c) && \text{in } \Omega_T, && (6.1.3b)\\
\partial_t \sigma &= \Delta\sigma + \mathcal{B}(\sigma_S - \sigma) - \mathcal{C}\sigma h(c) && \text{in } \Omega_T, && (6.1.3c)\\
\partial_{\mathbf{n}} c &= \partial_{\mathbf{n}}\mu = \partial_{\mathbf{n}}\sigma = 0 && \text{on } \partial\Omega_T, && (6.1.3d)\\
c(0) &= c_0, \quad \sigma(0) = \sigma_0 && \text{in } \Omega. && (6.1.3e)
\end{aligned}$$

The structure of this chapter is the following: In the first part, we recall well-posedness results for both the nonlocal and local Cahn–Hilliard tumor growth model. In the second part, we then prove the convergence result in Theorem 2.4.3. This will be done using an energy method together with the convergence result established in Section 2.1.

This chapter is based on the contribution in:

[88] C. Hurm, M. Moser, *Nonlocal-to-Local Convergence for a Cahn–Hilliard Tumor Growth Model*, GAMM-Mitteilungen, 48 (2025), p. e70003.

6.2. Well-Posedness results

In this section, we collect well-posedness results for the Cahn–Hilliard tumor growth models introduced in Sections 1.6.1 and 1.7.1, respectively. First, we discuss the well-posedness of the local model. Afterwards, we focus on its nonlocal counterpart.

6.2.1. Well-Posedness of the local model

The local Cahn-Hilliard model (6.1.3a)-(6.1.3e) has already been studied intensively. Well-posedness of (6.1.3a)-(6.1.3e) is available in a slightly more general setting due to [73], where the authors considered the problem with an additional control function. In [72], a similar system with chemotaxis and active transport has been analyzed. For positive viscosity coefficients, the authors in [35] proved well-posedness and studied long-time behavior and in [36, 37], the authors investigated the asymptotic analysis as the positive viscosity coefficients tend towards zero. For well-posedness of similar Cahn–Hilliard tumor growth models, we mention the results in [59, 70, 71].

In particular, we have the following well-posedness result for the system (6.1.3a)–(6.1.3e).

Theorem 6.2.1. *Let the assumptions* (A1)*,* (A3)-(A5) *hold and let* $n = 3$*. Let the initial data* (c_0, σ_0) *satisfy* $c_0 \in H^3(\Omega)$ *with* $\partial_{\mathbf{n}} c_0 = 0$ *on* $\partial\Omega$ *and* $\sigma_0 \in H^1(\Omega)$ *with* $0 \leq \sigma_0 \leq 1$ *a.e. in* Ω*.*

Then there is a unique solution (c, μ, σ) *of* (6.1.3a)-(6.1.3e) *with the regularity*

$$\begin{aligned}
&c \in L^\infty(0,T,H^2(\Omega)) \cap L^2(0,T,H^3(\Omega)) \cap H^1(0,T,L^2(\Omega)) \cap C^0(\overline{\Omega_T}),\\
&\mu \in L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,L^2(\Omega)),\\
&\sigma \in L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)) \cap H^1(0,T,L^2(\Omega)), \quad 0 \leq \sigma \leq 1 \text{ a.e. in } \Omega_T,
\end{aligned}$$

such that $c(0) = c_0$*,* $\sigma(0) = \sigma_0$ *and for a.e.* $t \in (0,T)$ *and all* $\xi \in H^1(\Omega)$ *it holds*

$$0 = \int_\Omega \partial_t c \xi + \nabla\mu \cdot \nabla\xi - (\mathcal{P}\sigma - \mathcal{A})h(c)\xi \,\mathrm{d}x, \tag{6.2.1}$$

$$0 = \int_\Omega \mu\xi - f'(c)\xi - \nabla c \cdot \nabla\xi \,\mathrm{d}x, \tag{6.2.2}$$

$$0 = \int_\Omega \partial_t \sigma \xi + \nabla\sigma \cdot \nabla\xi + \mathcal{C}h(c)\xi + \mathcal{B}(\sigma - \sigma_S)\xi \,\mathrm{d}x. \tag{6.2.3}$$

Proof. We refer to [73, Theorem 2.1]. □

6.2.2. Well-Posedness of the nonlocal model

Existence of weak solutions of the nonlocal system (6.1.1a)–(6.1.1e) has already been shown in [113], where the authors considered a more general Cahn-Hilliard system including chemotaxis and active transport, as well as possible relaxation terms for the Cahn–Hilliard equation. In [60], the authors considered similar Cahn–Hilliard-type tumor growth model. In the case of non-degenerate mobilities and smooth potentials, they proved well-posedness for the system, and in the case of degenerate mobilities and singular potentials, they proved the existence of weak solutions. Let us also mention the contribution in [52], where the author studied the optimal control problem for a viscous non-local tumor growth model.

The following result is obtained:

Theorem 6.2.2. *Let the assumptions* (A1)-(A6) *hold and let* $n = 3$. *Moreover, we assume that* $c_{0,\varepsilon}, \sigma_{0,\varepsilon} \in L^2(\Omega)$. *Then for* $\varepsilon_0 > 0$ *small and all* $\varepsilon \in (0, \varepsilon_0]$ *there exists a unique weak solution* $(c_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ *of* (6.1.1a)-(6.1.1e) *with the regularity*

$$\begin{aligned}
&c_\varepsilon \in H^1(0,T,H^1(\Omega)') \cap L^2(0,T,H^1(\Omega)),\\
&\mu_\varepsilon \in L^2(0,T,H^1(\Omega)),\\
&\sigma_\varepsilon \in H^1(0,T,H^1(\Omega)') \cap L^2(0,T,H^1(\Omega)), \quad 0 \le \sigma_\varepsilon(t) \le 1 \textit{ a.e. in } \Omega, \textit{ for all } t \in [0,T],
\end{aligned}$$

such that $c_\varepsilon(0) = c_{0,\varepsilon}$, $\sigma_\varepsilon(0) = \sigma_{0,\varepsilon}$ *and for all* $\xi \in H^1(\Omega)$ *and a.e.* $t \in (0,T)$ *it holds*

$$0 = \langle \partial_t c_\varepsilon, \xi \rangle_{H^1(\Omega)} + \int_\Omega \nabla \mu_\varepsilon \cdot \nabla \xi \, \mathrm{d}x - \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A}) h(c_\varepsilon) \xi \, \mathrm{d}x, \tag{6.2.4}$$

$$\mu_\varepsilon = \mathcal{L}_\varepsilon c_\varepsilon + f'(c_\varepsilon), \tag{6.2.5}$$

$$0 = \langle \partial_t \sigma_\varepsilon, \xi \rangle_{H^1(\Omega)} + \int_\Omega \nabla \sigma_\varepsilon \cdot \nabla \xi + \mathcal{C} h(c_\varepsilon) \xi + \mathcal{B}(\sigma_\varepsilon - \sigma_S) \xi \, \mathrm{d}x. \tag{6.2.6}$$

Proof. This follows from [113], Theorem 2.14 and Theorem 2.15. Here, $\varepsilon_0 > 0$ is such that for some $c_0 > 0$ it holds

$$f''(s) + \inf_{x \in \Omega} \int_\Omega J_\varepsilon(x-y) \, \mathrm{d}y \ge c_0 \quad \text{for all } s \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0].$$

The existence of such an $\varepsilon_0 > 0$ follows from (2.4.2) and

$$\inf_{x \in \Omega} \int_\Omega J_\varepsilon(x-y) \, \mathrm{d}y \ge \frac{c}{\varepsilon^2} \quad \text{for all } \varepsilon \in (0,1]$$

for some $c > 0$. Let us briefly remark the ideas for the latter estimate here. First, note that by applying the transformation rule and a rescaling argument together with the properties of J_ε from (A2), we obtain

$$\int_{B_\delta(x)} J_\varepsilon(x-y) \, \mathrm{d}y \ge \frac{c}{\varepsilon^2} \quad \text{for all } \varepsilon \in (0,1],$$

where $B_\delta(x) \subseteq \mathbb{R}^n$ is the ball with radius $\delta > 0$ around x. Here $c > 0$ is a positive constant that can be chosen uniformly for all $x \in \mathbb{R}^n$ and $\delta \ge \delta_0 > 0$, where δ_0 is any fixed positive constant. This holds analogously if the balls are replaced by sectors of such balls based on angles uniformly bounded away from zero. Hence the task reduces to assign to each point $x \in \Omega$ such an object contained in Ω. For points $x \in \Omega$ outside a tubular neighbourhood of $\partial\Omega$ this is directly clear. For points close to the boundary one can employ the uniform interior ball condition for $\partial\Omega$. $\square$

6.3. Proof of Theorem 2.4.3

We denote by $(c_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)$ the weak solution to the nonlocal model (6.1.1a)-(6.1.1e) given by Theorem 6.2.2 and with (c, μ, σ) the unique solution to the local model (6.1.3a)-(6.1.3e) provided by Theorem 6.2.1. Then, the functions $\tilde{c} := c_\varepsilon - c, \tilde{\mu} := \mu_\varepsilon - \mu, \tilde{\sigma} := \sigma_\varepsilon - \sigma$ solve the system

$$\partial_t \tilde{c} = \Delta\tilde{\mu} + (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(c_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(c) \quad \text{in } \Omega_T, \tag{6.3.1}$$

$$\tilde{\mu} = \mathcal{L}_\varepsilon c_\varepsilon + \Delta c + f'(c_\varepsilon) - f'(c) \quad \text{in } \Omega_T, \tag{6.3.2}$$

$$\partial_t \tilde{\sigma} = \Delta\tilde{\sigma} - \mathcal{B}\tilde{\sigma} - \mathcal{C}\sigma_\varepsilon h(c_\varepsilon) + \mathcal{C}\sigma h(c) \quad \text{in } \Omega_T, \tag{6.3.3}$$

$$\tilde{c}(0) = c_{\varepsilon,0} - c_0, \quad \tilde{\sigma}(0) = \sigma_{\varepsilon,0} - \sigma_0 \quad \text{in } \Omega. \tag{6.3.4}$$

in a weak sense. More precisely, the weak formulation is obtained by testing with functions in $H^1(\Omega)$, cf. the weak formulations (6.2.1)-(6.2.3) and (6.2.4)-(6.2.6).

Testing (6.3.1) with $\mathcal{N}(\tilde{c} - \overline{\tilde{c}}) \in H^1_{(0)}(\Omega)$, cf. the property (3.6.2) for the inverse Neumann Laplacian, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c} - \overline{\tilde{c}}\|^2_{H^1(\Omega)'} &= -\int_\Omega \tilde{\mu}(\tilde{c} - \overline{\tilde{c}})\,\mathrm{d}x \\ &\quad + \int_\Omega \Big[(\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(c_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(c)\Big]\mathcal{N}(\tilde{c} - \overline{\tilde{c}})\,\mathrm{d}x. \end{aligned} \tag{6.3.5}$$

For the second term on the right-hand side of (6.3.5), we have

$$\begin{aligned} &\int_\Omega \Big[(\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(c_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(c)\Big]\mathcal{N}(\tilde{c} - \overline{\tilde{c}})\,\mathrm{d}x \\ &= \int_\Omega \mathcal{P}\sigma_\varepsilon(h(c_\varepsilon) - h(c))\mathcal{N}(\tilde{c} - \overline{\tilde{c}})\,\mathrm{d}x \\ &\quad - \int_\Omega \mathcal{A}(h(c_\varepsilon) - h(c))\mathcal{N}(\tilde{c} - \overline{\tilde{c}})\,\mathrm{d}x \\ &\quad + \int_\Omega h(c)\mathcal{P}\tilde{\sigma}\mathcal{N}(\tilde{c} - \overline{\tilde{c}})\,\mathrm{d}x =: I_1 + I_2 + I_3. \end{aligned}$$

Using the Lipschitz continuity of h and the boundedness of h, cf. assumption (A4), and $|\sigma_\varepsilon| \le 1$ a.e. in Ω_T due to Theorem 6.2.2, we obtain the following estimates, where L_h denotes the Lipschitz constant of h:

$$\begin{aligned} |I_1| &\le \mathcal{P}L_h\|\tilde{c}\|_{L^2(\Omega)}\|\mathcal{N}(\tilde{c} - \overline{\tilde{c}})\|_{L^2(\Omega)} \\ &\le \tfrac{1}{36}\|\tilde{c}\|^2_{L^2(\Omega)} + \mathcal{P}K\|\tilde{c} - \overline{\tilde{c}}\|^2_{H^1(\Omega)'}, \end{aligned} \tag{6.3.6}$$

$$|I_2| \le \tfrac{1}{36}\|\tilde{c}\|^2_{L^2(\Omega)} + \mathcal{A}K\|\tilde{c} - \overline{\tilde{c}}\|^2_{H^1(\Omega)'}, \tag{6.3.7}$$

$$|I_3| \le \tfrac{1}{2}\|\tilde{\sigma}\|^2_{L^2(\Omega)} + \mathcal{P}K\|\tilde{c} - \overline{\tilde{c}}\|^2_{H^1(\Omega)'}. \tag{6.3.8}$$

In the next step, we test (6.3.2) with $\tilde{c} - \tilde{c}_\Omega$. This yields

$$\int_\Omega \tilde{\mu}(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x = \int_\Omega (\mathcal{L}_\varepsilon c_\varepsilon + \Delta c)(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x + \int_\Omega (f'(c_\varepsilon) - f'(c))(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x. \tag{6.3.9}$$

For the first term on the right-hand side of (6.3.9), it holds

$$\int_\Omega (\mathcal{L}_\varepsilon c_\varepsilon + \Delta c)(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x = \int_\Omega (\mathcal{L}_\varepsilon c + \Delta c)(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x + \int_\Omega \mathcal{L}_\varepsilon \tilde{c}(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x. \tag{6.3.10}$$

Recalling that $\mathcal{L}_\varepsilon \overline{\tilde{c}} = 0$, we can add the term $-\int_\Omega \mathcal{L}_\varepsilon \overline{\tilde{c}}(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x$ to the right-hand side in (6.3.10). Then, by the symmetry of the interaction kernel J_ε, we observe that

$$\int_\Omega \mathcal{L}_\varepsilon(\tilde{c} - \overline{\tilde{c}})(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x = 2\mathcal{E}_\varepsilon(\tilde{c} - \overline{\tilde{c}}). \tag{6.3.11}$$

For the second term in (6.3.9), we first observe that

$$\int_\Omega (f'(c_\varepsilon) - f'(c))(\tilde{c} - \overline{\tilde{c}}) \,\mathrm{d}x = \int_\Omega (f'(c_\varepsilon) - f'(c))\tilde{c} \,\mathrm{d}x - \int_\Omega (f'(c_\varepsilon) - f'(c))\overline{\tilde{c}} \,\mathrm{d}x.$$

For the first term on the right-hand side, we use the fundamental theorem of calculus. Then, the assumption $f'' \geq -k_4$ yields

$$\int_\Omega (f'(c_\varepsilon) - f'(c))\tilde{c} \,\mathrm{d}x \geq -k_4 \|\tilde{c}\|^2_{L^2(\Omega)}.$$

For the second term, we use assumption (2.4.3) and get

$$\int_\Omega (f'(c_\varepsilon) - f'(c))\overline{\tilde{c}} \,\mathrm{d}x \leq k_5 \int_\Omega (1 + |c_\varepsilon|^2 + |c|^2)|\tilde{c}||\overline{\tilde{c}}| \,\mathrm{d}x.$$

Invoking the inequalities of Hölder and Young, we infer

$$\begin{aligned} k_5 \int_\Omega (1 + |c_\varepsilon|^2 + |c|^2)|\tilde{c}||\overline{\tilde{c}}| \,\mathrm{d}x &\leq k_5 \|\tilde{c}\|_{L^2(\Omega)} \big\|(1 + |c_\varepsilon|^2 + |c|^2)|\overline{\tilde{c}}|\big\|_{L^2(\Omega)} \\ &\leq k_5 \|\tilde{c}\|^2_{L^2(\Omega)} + K|\overline{\tilde{c}}|^2 \|(1 + |c_\varepsilon|^2 + |c|^2)\|^2_{L^2(\Omega)} \\ &\leq k_5 \|\tilde{c}\|^2_{L^2(\Omega)} + K|\overline{\tilde{c}}|^2 \big(1 + \|c_\varepsilon\|^4_{L^4(\Omega)} + \|c\|^4_{L^4(\Omega)}\big). \end{aligned}$$

We need to control the term $\|c_\varepsilon\|_{L^4(\Omega)}$ uniformly in ε. To this end, we test equation (6.2.4) by μ_ε, (6.2.5) by $-\partial_t c_\varepsilon$ and sum the resulting equations. This yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{E}_\varepsilon(c_\varepsilon) + \int_\Omega f(c_\varepsilon) \,\mathrm{d}x\Big) + \|\nabla \mu_\varepsilon\|^2_{L^2(\Omega)^n} \\ &= \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon)\mu_\varepsilon \,\mathrm{d}x \\ &= \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon)(\mu_\varepsilon - \overline{\mu_\varepsilon}) \,\mathrm{d}x + \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(\varphi_\varepsilon)\overline{\mu_\varepsilon} \,\mathrm{d}x. \end{aligned} \tag{6.3.12}$$

Since the function h is bounded, we can control the terms on the right-hand side using the assumptions in (A3), Young's inequality and the fact that $|\sigma_\varepsilon| \leq 1$ a.e. in Ω_T. Indeed, we have the following estimates:

$$\Big|\int_\Omega \mathcal{C}h(c_\varepsilon)\sigma_\varepsilon \,\mathrm{d}x\Big| + \Big|\int_\Omega \mathcal{B}(\sigma_\varepsilon - \sigma_S)\sigma_\varepsilon \,\mathrm{d}x\Big| \leq C \int_\Omega |\sigma_\varepsilon| \,\mathrm{d}x \leq C.$$

For the last term, we again use $|\sigma_\varepsilon| \leq 1$ a.e. in Ω_T. This gives

$$\left| \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A}) h(c_\varepsilon)\mu_\varepsilon \,\mathrm{d}x \right| \leq C\|\mu_\varepsilon\|_{L^2(\Omega)}$$

We intend to use the Poincaré-Wirtinger inequality for $\|\mu_\varepsilon\|_{L^2(\Omega)}$, in order to absorb the term $\|\nabla\mu_\varepsilon\|_{L^2(\Omega)}$ into the left-hand side in (6.3.12). To this end, we need to show that μ_ε has bounded mean value. Testing (6.2.5) by 1, we get

$$\int_\Omega \mu_\varepsilon \,\mathrm{d}x = \int_\Omega \mathcal{L}_\varepsilon c_\varepsilon \,\mathrm{d}x + \int_\Omega f'(c_\varepsilon) \,\mathrm{d}x.$$

However, due to the symmetry of the interaction kernel J_ε, it follows that

$$\int_\Omega \mathcal{L}_\varepsilon c_\varepsilon \,\mathrm{d}x = 0.$$

Recalling the growth assumption (2.4.1) in (A5), we have

$$\left| \int_\Omega f'(c_\varepsilon) \,\mathrm{d}x \right| \leq k_0 \int_\Omega f(c_\varepsilon) \,\mathrm{d}x + C$$

and therefore,

$$|\overline{\mu_\varepsilon}| \leq C \int_\Omega f(c_\varepsilon) \,\mathrm{d}x + C.$$

Hence, we obtain

$$\begin{aligned} \left| \int_\Omega (\mathcal{P}\sigma_\varepsilon - \mathcal{A}) h(c_\varepsilon)\mu_\varepsilon \,\mathrm{d}x \right| &\leq C\|\mu_\varepsilon - \overline{\mu_\varepsilon}\|_{L^2(\Omega)} + C|\overline{\mu_\varepsilon}| \\ &\leq \frac{1}{2}\|\nabla\mu_\varepsilon\|^2_{L^2(\Omega)^n} + C\int_\Omega f(c_\varepsilon) \,\mathrm{d}x + C, \end{aligned}$$

where we also used Young's inequality. Eventually, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{E}_\varepsilon(c_\varepsilon) + \int_\Omega f(c_\varepsilon) \,\mathrm{d}x\Big) + \frac{1}{2}\|\nabla\mu_\varepsilon\|^2_{L^2(\Omega)^n} \leq C + C\int_\Omega f(c_\varepsilon) \,\mathrm{d}x.$$

Finally, the Gronwall lemma yields uniform estimates independent of ε. In particular, we have that

$$\int_\Omega f(c_\varepsilon) \,\mathrm{d}x \leq C.$$

Then, assumption (A6) yields that the sequence $(c_\varepsilon)_{\varepsilon>0} \subset L^\infty(0,T;L^4(\Omega))$ is bounded.

Next, we test equation (6.3.4) with $\tilde{\sigma}$. Then, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\sigma}\|^2_{L^2(\Omega)} + \|\nabla\tilde{\sigma}\|^2_{L^2(\Omega)^n} = -\mathcal{B}\|\tilde{\sigma}\|^2_{L^2(\Omega)} - \int_\Omega (\mathcal{C}\sigma_\varepsilon h(c_\varepsilon) - \mathcal{C}\sigma h(c))\tilde{\sigma} \,\mathrm{d}x. \tag{6.3.13}$$

For the second term in (6.3.13), we observe

$$\begin{aligned} \int_\Omega (\mathcal{C}\sigma_\varepsilon h(c_\varepsilon) - \mathcal{C}\sigma h(c))\tilde{\sigma} \,\mathrm{d}x &= \int_\Omega \mathcal{C}\sigma_\varepsilon\big(h(c_\varepsilon) - h(c)\big)\tilde{\sigma} \,\mathrm{d}x + \int_\Omega \mathcal{C}h(c)|\tilde{\sigma}|^2 \,\mathrm{d}x \\ &\leq K\mathcal{C}\|\tilde{\sigma}\|^2_{L^2(\Omega)} + \frac{1}{36}\|\tilde{c}\|^2_{L^2(\Omega)}, \end{aligned} \tag{6.3.14}$$

where we again used $|\sigma_\varepsilon| \le 1$ a.e. in Ω_T and the Lipschitz continuity of h and the boundedness of h. Combining the previous estimates, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\tilde c - \bar{\tilde c}\|^2_{H^1(\Omega)'} + \|\tilde\sigma\|^2_{L^2(\Omega)}\Big) + 2\mathcal{E}_\varepsilon(\tilde c - \bar{\tilde c}) + \|\nabla\tilde\sigma\|^2_{L^2(\Omega)^n} + \frac{1}{2}\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)}$$
$$\le \frac{1}{12}\|\tilde c\|^2_{L^2(\Omega)} + K\|\tilde\sigma\|^2_{L^2(\Omega)} - \int_\Omega (\mathcal{L}_\varepsilon c + \Delta c)(\tilde c - \bar{\tilde c})\,\mathrm{d}x + (k_4 + k_5)\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)}$$
$$+ K\|\tilde c - \bar{\tilde c}\|^2_{H^1(\Omega)'} + \frac{1}{2}\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)} + K|\bar{\tilde c}|^2. \tag{6.3.15}$$

Finally, we test (6.3.1) with the mean value $\bar{\tilde c}$ and obtain

$$\frac{|\Omega|}{2}\partial_t|\bar{\tilde c}|^2 = \int_\Omega \partial_t\tilde c\bar{\tilde c}\,\mathrm{d}x = \int_\Omega [(\mathcal{P}\sigma_\varepsilon - \mathcal{A})h(c_\varepsilon) - (\mathcal{P}\sigma - \mathcal{A})h(c)]\,\bar{\tilde c}\,\mathrm{d}x.$$

With analogous estimates as before, it follows that

$$\frac{1}{2}\partial_t|\bar{\tilde c}|^2 \le \frac{1}{12}\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)} + C(\|\tilde\sigma\|^2_{L^2(\Omega)} + |\bar{\tilde c}|^2).$$

Moreover, Hölder's and Young's inequalities imply

$$\int_\Omega (\mathcal{L}_\varepsilon c + \Delta c)(\tilde c - \bar{\tilde c})\,\mathrm{d}x \le \frac{1}{6}\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)} + K\|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)}.$$

Owing to the inequality,

$$\|\tilde c\|^2_{L^2(\Omega)} \le 2\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)} + C|\bar{\tilde c}|^2,$$

we can now collect the terms on the right-hand side in (6.3.15) and use inequality (3.5.4) with $\gamma = \frac{12}{12(k_4+k_5)+11}$ to obtain

$$(k_4 + k_5 + \tfrac{11}{12})\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)} \le \mathcal{E}_\varepsilon(\tilde c - \bar{\tilde c}) + C\|\tilde c - \bar{\tilde c}\|^2_{H^1(\Omega)'}.$$

Eventually, we then arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\tilde c - \bar{\tilde c}\|^2_{H^1(\Omega)'} + \|\tilde\sigma\|^2_{L^2(\Omega)} + |\bar{\tilde c}|^2\Big) + \mathcal{E}_\varepsilon(\tilde c - \bar{\tilde c}) + \|\nabla\tilde\sigma\|^2_{L^2(\Omega)^n} + \frac{1}{2}\|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)}$$
$$\le K\Big(\|\tilde c - \bar{\tilde c}\|^2_{H^1(\Omega)'} + \|\tilde\sigma\|^2_{L^2(\Omega)} + |\bar{\tilde c}|^2\Big) + K\|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)}. \tag{6.3.16}$$

Thus, Gronwall's lemma implies

$$\frac{1}{2}\sup_{t\in[0,T]}\|\tilde c(t) - \bar{\tilde c}(t)\|^2_{H^1(\Omega)'} + \frac{1}{2}\sup_{t\in[0,T]}\|\tilde\sigma(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\sup_{t\in[0,T]}|\bar{\tilde c}(t)|^2$$
$$+ \int_0^T \mathcal{E}_\varepsilon(\tilde c - \bar{\tilde c})\,\mathrm{d}t + \int_0^T \|\nabla\tilde\sigma\|^2_{L^2(\Omega)^n}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)}\,\mathrm{d}t$$
$$\le \Big(\frac{1}{2}\|\tilde c_0 - \bar{\tilde c}_0\|^2_{H^1(\Omega)'} + \frac{1}{2}\|\tilde\sigma_0\|^2_{L^2(\Omega)} + \frac{1}{2}|\bar{\tilde c}_0|^2 + \int_0^T \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)}\,\mathrm{d}t\Big)e^{CT}.$$

Hence, recalling the assumptions on the initial data, cf. (2.4.4), and Theorem 2.1.8, we obtain

$$\frac{1}{2}\sup_{t\in[0,T]}\|\tilde c(t) - \bar{\tilde c}(t)\|^2_{H^1(\Omega)'} + \frac{1}{2}\sup_{t\in[0,T]}\|\tilde\sigma(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\sup_{t\in[0,T]}|\bar{\tilde c}(t)|^2$$
$$+ \int_0^T \mathcal{E}_\varepsilon(\tilde c - \bar{\tilde c})\,\mathrm{d}t + \int_0^T \|\nabla\tilde\sigma\|^2_{L^2(\Omega)^n}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\tilde c - \bar{\tilde c}\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C\varepsilon.$$

Finally, observe that it holds

$$\begin{aligned}\|\tilde{c}\|_{L^\infty(0,T;H^1(\Omega)')} &\leq \|\tilde{c}-\bar{\tilde{c}}\|_{L^\infty(0,T;H^1(\Omega)')} + \|\bar{\tilde{c}}\|_{L^\infty(0,T)} \\ &\leq \sup_{t\in[0,T]} \|\tilde{c}(t)-\bar{\tilde{c}}(t)\|_{H^1(\Omega)'} + \sup_{t\in[0,T]} |\bar{\tilde{c}}(t)| \leq C\sqrt{\varepsilon},\end{aligned}$$

which shows the convergence in (2.4.5). Hence, the proof is complete. □

Chapter 7

Nonlocal-to-local convergence for a Cahn–Hilliard/Navier–Stokes model

7.1. Introduction

For $\varepsilon > 0$, we consider the nonlocal Cahn–Hilliard/Navier–Stokes system (1.5.2) as introduced in Section 1.5,

$$\rho(\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon) - \nu\Delta\mathbf{v}_\varepsilon + \nabla p_\varepsilon = \mu_\varepsilon \nabla c_\varepsilon \qquad \text{in } \Omega_T, \tag{7.1.1a}$$
$$\operatorname{div}(\mathbf{v}_\varepsilon) = 0 \qquad \text{in } \Omega_T, \tag{7.1.1b}$$
$$\partial_t c_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon = m\Delta\mu_\varepsilon \qquad \text{in } \Omega_T, \tag{7.1.1c}$$
$$\mu_\varepsilon = \mathcal{L}_\varepsilon c_\varepsilon + f'(c_\varepsilon) \qquad \text{in } \Omega_T, \tag{7.1.1d}$$
$$\mathbf{v}_\varepsilon(0) = \mathbf{v}_{\varepsilon,0}, \quad c_\varepsilon(0) = c_{\varepsilon,0} \qquad \text{in } \Omega, \tag{7.1.1e}$$

where the nonlocal operator $\mathcal{L}_\varepsilon$ is given as in (4.1.5), i.e,

$$\mathcal{L}_\varepsilon u(x) := \text{P.V.} \int_\Omega J_\varepsilon(x-y)\big(u(x) - u(y)\big)\, \mathrm{d}y, \quad x \in \Omega, \tag{7.1.1f}$$

for a prescribed interaction kernel J_ε as in (4.1.4). In case Ω is a bounded domain, we further impose the standard boundary conditions

$$\mathbf{v} = \mathbf{0}, \quad \partial_{\mathbf{n}}\mu = 0 \quad \text{on } \partial\Omega_T, \tag{7.1.1g}$$

and if $\Omega = \mathbb{T}^n$, we assume periodic boundary conditions. In this setting, it is already well-known that weak solutions to the nonlocal Model H (7.1.1) converge to the weak solution to the local Model H as $\varepsilon \to 0$, cf. [7, 94]. Thus, the sequence of solutions converges to a solution of the following system:

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \nu\Delta\mathbf{v} + \nabla p = \mu \nabla c \qquad \text{in } \Omega_T, \tag{7.1.2a}$$
$$\operatorname{div}(\mathbf{v}) = 0 \qquad \text{in } \Omega_T, \tag{7.1.2b}$$
$$\partial_t c + \mathbf{v} \cdot \nabla c = m\Delta\mu \qquad \text{in } \Omega_T, \tag{7.1.2c}$$
$$\mu = -\Delta c + f'(c) \qquad \text{in } \Omega_T, \tag{7.1.2d}$$
$$\mathbf{v}(0) = \mathbf{v}_0, \quad c(0) = c_0 \qquad \text{in } \Omega, \tag{7.1.2e}$$

where, in case Ω is a bounded domain, we further impose the standard boundary conditions

$$\mathbf{v} = \mathbf{0}, \quad \partial_{\mathbf{n}}\mu = 0, \quad \partial_{\mathbf{n}} c = 0 \quad \text{on } \partial\Omega_T, \tag{7.1.2f}$$

and if $\Omega = \mathbb{T}^n$, we assume periodic boundary conditions.

It is the main goal of this chapter to prove the convergence result in Theorem 2.5.2. To this end, we first address the well-posedness results for both the nonlocal and local Model H. For the nonlocal system, we also give a proof about the existence of a unique local-in-time strong solution (for any $\varepsilon > 0$), also in the case $n = 3$. Furthermore, for $n = 2, 3$, we show that weak and strong solutions are bounded in suitable norms by a constant, which is independent of ε. These estimates will be crucial for the proof of the nonlocal-to-local convergence for the Model H . In the last part of this chapter, we then prove our main result. This will be done by an energy method.

This chapter is based on the following contribution:

[87] C. Hurm, P. Knopf, A. Poiatti, *Nonlocal-to-local convergence rates for strong solutions to a Navier-Stokes-Cahn-Hilliard system with singular potential*, Commun. in Partial Differential Equations, 49(9), 832–871, 2024.

7.2. Well-Posedness results

For the sake of completeness, we state here the already-known results about the Model H. In the first part of this section, we recall the separation property, which is crucial for both the well-posedness and the nonlocal-to-local asymptotics. Then, we discuss the well-posedness of the local and nonlocal Model H, respectively.

7.2.1. The separation property

One essential ingredient for the analysis of Cahn–Hilliard type equations involving singular potentials is the so-called *(strict) separation property.* This means that the order parameter c stays away from the pure phases ± 1 from a certain time t_* on. More precisely, this means that the order parameter stays within a compact interval of $(-1, 1)$, i.e.,

$$\|c(t)\|_{L^\infty(\Omega)} \leq 1 - \delta \quad \text{ for all } t > t_* \tag{7.2.1}$$

for some $\delta > 0$ and $t_* > 0$. This property is in fact of great physical and mathematical interest. From a physical point of view, this means that no pure phases are present anymore and a complete mixing between the fluids takes place. From a mathematical point of view, this property is crucial to prove higher regularity of the order parameter and to study its long-time behavior. It is also important to prove uniqueness, since it allows us to estimate differences involving the singular potential.

Depending on the time t_*, one can distinguish between the following types of separation properties:

1. *Local separation property:* If the initial datum c_0 is strictly separated from the pure phases and if c is continuous in space and time, the separation property (7.2.1) holds

for small times, i.e., there exists some finite time T_0 and $\delta > 0$ such that

$$\sup_{t\in[0,T_0]} \|c(t)\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

2. *Asymptotic separation property:* The separation property (7.2.1) holds from a positive time on, i.e., there exists some time T_1 (depending on the initial datum and the system parameters) and $\delta > 0$ such that

$$\sup_{t\in[T_1,\infty)} \|c(t)\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

3. *Instantaneous separation property:* The separation property (7.2.1) holds for *any* positive time on, i.e., for all $\tau > 0$ there exists $\delta = \delta(\tau) > 0$ such that

$$\sup_{t\in[\tau,\infty)} \|c(t)\|_{L^\infty(\Omega)} \leq 1 - \delta.$$

   Observe that if the initial datum is sufficiently regular and separated from the pure phases, one can even show that this property even holds for $\tau = 0$. Thus, the solution is strictly separated for almost any time $\tau \geq 0$.

For a more detailed overview, we refer the reader to [63, 67, 106, 107] and the references therein. For results on the separation property, we also refer to the sections below.

7.2.2. Well-Posedness of the local model

For the local Model H, i.e., system (7.1.2), there already exists an extensive literature. If Ω is a bounded domain, the existence of weak solution has been shown in [2, Theorem 1]. The author also addresses the uniqueness and regularity of weak and strong solutions. More precisely, in $n = 2$ he proves the uniqueness of weak solutions as well as the existence and uniqueness of global strong solutions. In $n = 3$, he proves the existence and uniqueness of local-in-time strong solutions. Similar results can be found in, e.g., [80]. The asymptotic behavior of solutions in $n = 2$ has been investigated by the authors in [66]. Even the more general case, where the fluids have different densities, has been studied very well. Among the well-posedness, also regularity properties of solutions and long-time behavior have been addressed. Let us mention the contributions in [1,4,5,7–9,16,62,78,79] to name a few of them. A variant of the same model allowing to treat multi-phase fluids is also studied in [10].

If Ω is the torus $\mathbb{T}^n$, the same results can be obtained by adapting the arguments in the aforementioned literature to the periodic setting. For instance, in the case $n = 2$, the existence of a unique global strong solution was established in [78].

Let us also mention, that in the case $n = 2$, the strict separation property (7.2.4) can be shown in a similar way as in [67, Theorem 3.3]. In the case $n = 3$, the local strict separation property (7.2.6) can be shown similarly as in [100, Corollary 4.4]. For details, we refer to [87].

In the following theorem, we collect the most important results concerning weak and strong well-posedness as well as separation properties. For references, we refer to Remark 7.2.2 below.

Theorem 7.2.1. *Suppose that the assumptions* (A1)–(A3) *and* (S1) *hold. We prescribe initial data* $v_0 \in L^2_\sigma(\Omega)$ *and* $c_0 \in L^\infty(\Omega) \cap H^1(\Omega)$ *with* $\|c_0\|_{L^\infty(\Omega)} \leq 1$ *and* $|\overline{c_0}| < 1$. *Then there exists a global weak solution*

$$(\mathbf{v}, c, \mu) : \Omega \times [0, \infty) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$$

to (7.1.2) *with the following properties:*

(i) *For any* $T > 0$, *it holds*

$$\begin{cases} \mathbf{v} \in C_{\mathrm{w}}([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega)), \\ c \in L^\infty(\Omega \times (0,T)) \cap L^4(0,T; H^2(\Omega)) \text{ with } |c| < 1 \text{ a.e. in } \Omega_T, \\ \partial_t \mathbf{v} \in L^{\frac{4}{n}}(0,T; (H^1_\sigma(\Omega))'), \\ \partial_t c \in L^2(0,T; H^1(\Omega)'), \\ \mu \in L^2(0,T; H^1(\Omega)). \end{cases} \tag{7.2.2}$$

(ii) *For any* $T > 0$, *the triplet* $(\mathbf{v}, c, \mu)$ *fulfills the equations* (7.1.2a)–(7.1.2d) *in the weak sense, whereas the initial conditions* (7.1.2e) *are fulfilled a.e. in* Ω. *If* Ω *is a bounded domain, it further holds* $\mathbf{v} = \mathbf{0}$ *and* $\partial_{\mathbf{n}} c = 0$ *a.e. on* $\partial\Omega_T$.

If $n = 2$, *the weak solution is unique.*

Now, we additionally assume $\mathbf{v}_0 \in H^1_\sigma(\Omega)$, $c_0 \in H^2(\Omega)$ *and* $\mu_0 := -\Delta c_0 + f'(c_0) \in H^1(\Omega)$. *If* Ω *is a bounded domain, we further assume* $\partial_{\mathbf{n}} c_0 = 0$ *a.e. on* $\partial\Omega$. *Then, there exists a unique right-maximal strong solution*

$$(\mathbf{v}, p, c, \mu) : \Omega \times [0, T_\star) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

of system (7.1.2). *If* $n = 2$, *it holds* $T_\star = \infty$. *This strong solution has the following properties:*

(iii) *It holds*

$$\begin{cases} \mathbf{v} \in BC([0,T_\star); H^1_\sigma(\Omega)) \cap L^2_{\mathrm{uloc}}([0,T_\star); H^2(\Omega)^n \cap H^1_\sigma(\Omega)) \\ \qquad \cap H^1_{\mathrm{uloc}}([0,T_\star); L^2_\sigma(\Omega)), \\ p \in L^2_{\mathrm{uloc}}([0,T_\star); H^1_{(0)}(\Omega)), \\ c \in L^\infty(0,T_\star; L^\infty(\Omega)) \cap BC_{\mathrm{w}}([0,T_\star); W^{2,p}(\Omega)), \\ |c(x,t)| < 1 \text{ for almost all } x \in \Omega \text{ and all } t \in [0,T_\star), \\ \partial_t c \in L^\infty(0,T_\star; H^1(\Omega)') \cap L^2_{\mathrm{uloc}}([0,T_\star); H^1(\Omega)), \\ F'(c) \in L^\infty(0,T_\star; L^p(\Omega)), \\ \mu \in L^\infty(0,T_\star; H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([0,T_\star); H^3(\Omega)). \end{cases} \tag{7.2.3}$$

for all $p \in [2, \infty)$ *if* $n = 2$ *and all* $p \in [2, 6]$ *if* $n = 3$.

(iv) *The quadruplet* $(\mathbf{v}, p, c, \mu)$ *fulfills the equations* (7.1.2a)–(7.1.2d) *a.e. in* $\Omega \times [0, T_\star)$ *and the initial condition* (7.1.2e) *a.e. in* Ω. *If* Ω *is a bounded domain, it further holds* $\mathbf{v} = \mathbf{0}$ *and* $\partial_{\mathbf{n}} c = \partial_{\mathbf{n}} \mu = 0$ *a.e. on* $\partial\Omega \times (0, T_\star)$.

(v) *If* $n = 2$ *and assumption* (S2) *additionally holds, there exists* $\delta_\star > 0$ *such that the strict separation property*

$$\sup_{t \in [0,\infty)} \|c(t)\|_{L^\infty(\Omega)} \leq 1 - \delta_\star \tag{7.2.4}$$

is fulfilled. In particular, this entails

$$c \in L^\infty(0,T;H^3(\Omega)) \quad \text{for all } T > 0. \tag{7.2.5}$$

If $n = 3$ *and* $\|c_0\|_{L^\infty(\Omega)} \leq 1 - \delta_0$ *holds for some* $\delta_0 \in (0,1)$*, there exist* $0 < T_0 < T_\star$ *such that the strict separation property*

$$\sup_{t\in[0,T_0]} \|c(t)\|_{L^\infty(\Omega)} \leq 1 - \frac{\delta_0}{2} \tag{7.2.6}$$

is fulfilled. In particular, this entails

$$c \in L^\infty(0,T_0;H^3(\Omega)). \tag{7.2.7}$$

Remark 7.2.2. If Ω is a bounded domain, the existence of a weak solution was established in [2, Theorem 1]. In the case $n = 2$, the uniqueness of the weak solution was shown, e.g., in [80]. Concerning the assertions on strong well-posedness we refer to [80, Sections 4-5]. Note that the compatibility condition $\mu_0 = -\Delta c_0 + f'(c_0) \in H^1(\Omega)$ is crucial for obtaining strong solutions.

If Ω is the torus $\mathbb{T}^n$, the same results can be obtained by adapting the arguments in the aforementioned literature to the periodic setting. For instance, in the case $n = 2$, the existence of a unique global strong solution was established in [78].

In the case $n = 2$, the strict separation property (7.2.4) can be established by following the line of argument in [67, Theorem 3.3], which is based on De Giorgi iterations. A crucial ingredient in this proof is the estimate

$$\sup_{t\geq 0} \|F'(c(t))\|_{L^p(\Omega)} \leq C\sqrt{p}, \quad \text{for all } p \in [2,\infty). \tag{7.2.8}$$

It can be derived by means of a Gagliardo–Nirenberg type estimate, which can be found, e.g., in [122, p. 479]. For more details, we refer to the derivation of (7.3.61) below, which is obtained by similar computations. Once (7.2.8) is established, one can proceed as in [67, Theorem 3.3] to deduce the strict separation property (7.2.4) in the case $n = 2$. In this context, we recall that, as pointed out in Remark 7.2.3(a), the assumption $\mu_0 = -\Delta c_0 + f'(c_0) \in H^1(\Omega)$ already entails that the initial datum c_0 is strictly separated. As (7.2.4) directly implies $f'(c) \in L^\infty(0,\infty;H^1(\Omega))$, we apply elliptic regularity theory to the equation $-\Delta c = \mu - f'(c)$ in $\Omega \times (0,\infty)$ to conclude (7.2.5).

In the case $n = 3$, the local strict separation property (7.2.6) can be shown similarly as in [100, Corollary 4.4] by means of a continuity argument. In view of the regularities in (7.2.3), we deduce

$$\|c(t) - c(s)\|_* \leq \left| \int_s^t \|\partial_t c\|_* \,\mathrm{d}\tau \right| \leq C|t-s|$$

for all $s,t \in [0,T_\star)$. This entails $c \in C^{0,1}([0,T_\star);H^1(\Omega)')$. Using once more (7.2.3), we infer via interpolation that

$$\|c(t) - c(s)\|_{L^\infty(\Omega)} \leq C\|c(t) - c(s)\|_*^\omega \, \|c(t) - c(s)\|_{H^2(\Omega)}^{1-\omega} \leq C|t-s|^\omega$$

holds for all $s, t \in [0, T_\star)$ and some suitably chosen $\omega \in (0,1)$. Therefore, we thus have $c \in C^{0,\omega}([0,T_\star); L^\infty(\Omega))$, and hence, if there exists $\delta_0 > 0$ such that $\|c_0\|_{L^\infty(\Omega)} \leq 1 - \delta_0$, then (7.2.6) holds for $T_0 = \min\{(\delta_0/2)^{1/\omega}, T_\star\}$. As a direct consequence of (7.2.7), we have $F'(c) \in L^\infty(0, T_0; H^1(\Omega))$. Hence, by applying elliptic regularity theory to the equation $-\Delta c = \mu - f'(c)$ in $\Omega \times (0, T_0)$, we conclude (7.2.7).

Remark 7.2.3. (a) To obtain the strict separation properties (7.2.4) and (7.2.6) on an interval including the initial time, it is crucial that the initial datum c_0 is already strictly separated (i.e., $\|c_0\|_{L^\infty(\Omega)} \leq 1 - \delta_0$ for some $\delta_0 \in (0,1)$). In the case $n = 2$, this already follows from the assumption $\mu_0 = -\Delta c_0 + f'(c_0) \in H^1(\Omega)$ by means of De Giorgi iterations as employed in [67, Theorem 4.3].

In the case $n = 3$, at least up to now, the separation property (7.2.6) can merely be obtained on a local neighborhood of the initial time. For this result, it is sufficient to assume that the potential f satisfies (S1). If $n = 2$, assuming both (S1) and (S2), even a strict separation property on the entire interval $[0, \infty)$ can be established. The question, whether this property can also be proven for $n = 3$, is a challenging open problem. As shown in [67, Section 6.1.1], a strict separation property on the entire right-maximal interval $[0, T_\star)$ can also be obtained in the case $n = 3$ if slightly more singular potentials f than the Flory–Huggins potential (see Remark 2.5.1) are used.

The strict separation properties (7.2.4) and (7.2.6) will be an essential ingredient in the proof of Theorem 2.5.2.

(b) As pointed out in Theorem 7.2.1, the unique strong solution exists globally in time (i.e, $T_\star = \infty$) if $n = 2$. In the case $n = 3$, due to the involved Navier–Stokes equation, only local existence of the strong solution (i.e, $T_\star < \infty$) for general initial data is known so far. However, if the initial data are sufficiently close to a stationary point (i.e., a minimizer of the total energy), the global existence of the strong solution can still be ensured. If we additionally assume c_0 to be strictly separated, up to reducing the size of some norms of the initial data (i.e., assuming a sufficiently small initial energy), the strict separation property (7.2.6) for system (7.1.2) can also be established globally in time by using similar techniques as in [67, Theorem 6.4]. Indeed, although [67] refers to the single Cahn-Hilliard equation, it is based on the application of a De Giorgi iteration scheme for the equation of the chemical potential μ, which has exactly the same definition also in the case of a Navier–Stokes–Cahn–Hilliard system.

(c) We point out that most of the results in the literature concerning the local version of the Model H consider the case of bounded domains. However, it is clear that these results can usually be transferred to the case $\Omega = \mathbb{T}^n$ by slightly adapting the arguments.

Note that, if $\Omega = \mathbb{T}^n$, the assumption $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ already includes the condition $\overline{\mathbf{v}_0} = \mathbf{0}$. This then implies $\int_\Omega \mathbf{v}(t)\, \mathrm{d}x = \int_\Omega \mathbf{v}_0\, \mathrm{d}x = 0$ (cf. (1.4.4)) and therefore, we may apply the inverse Stokes operator A_S^{-1} (see Section 3.6.2) directly on $\mathbf{v}(t)$ for every $t \geq 0$ for which the solution exists.

However, the assumption $\mathbf{v}_0 \in L^2_\sigma(\Omega)$ really does not mean any loss of generality as we could simply consider the difference $\mathbf{v} - \overline{\mathbf{v}}$ instead of $\mathbf{v}$, which would not have a major impact on our mathematical analysis.

7.2.3. Well-Posedness of the nonlocal model

This subsection is concerned with the existence and uniqueness of solutions to the nonlocal Model H, i.e., system (7.1.1).

For $n = 2, 3$ and any fixed $\varepsilon > 0$, the existence of weak solutions of the nonlocal Model H has already been shown in [56, Theorem 1]. Furthermore, in the case $n = 2$, the strong well-posedness theory of this Model H has been developed in [63] and, more in details, in [65, Theorem 1.5, Theorem 1.9], which even deals with the more general case of unmatched densities (i.e., ρ is not constant and depends on the phase-field). Again, all these results are obtained for bounded domains, but as in the local case, they can be easily adapted to the case $\Omega = \mathbb{T}^n$.

Regarding the strict separation property for the nonlocal Cahn–Hilliard equation, we refer to [63,107], in which the first results in $n = 2$ and $n = 3$, respectively, are shown (see also, for instance, [64,67] and references therein).

In the following theorem, we show the existence of a unique local-in-time strong solution to the nonlocal Model H (for any $\varepsilon > 0$) also in the case $n = 3$. Moreover, for $n = 2, 3$, we want to bound weak and strong solutions in suitable norms by a constant independent of ε, at least provided that the considered ε is sufficiently small. These uniform estimates will be an essential ingredient in the nonlocal-to-local convergence of the Model H. The following theorem can be found in [87].

Theorem 7.2.4. *Let $\varepsilon > 0$ and suppose that the assumptions* (A1)–(A3) *and* (S1) *hold. We prescribe initial data $\mathbf{v}_{\varepsilon,0} \in L^2_\sigma(\Omega)$ and $c_{\varepsilon,0} \in L^\infty(\Omega)$ with $|\overline{c_{\varepsilon,0}}| < 1$. We further assume that there exists a constant $C_0 > 0$ independent of ε such that*

$$E_\varepsilon(\mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}) \le C_0, \tag{7.2.9}$$

where E_ε is the energy functional defined in (1.5.1)*. Then there exists a global weak solution*

$$(\mathbf{v}_\varepsilon, c_\varepsilon, \mu_\varepsilon) : \Omega \times [0, \infty) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$$

to the nonlocal Model H (7.1.1) *associated with ε, and for any $T > 0$, the following properties hold.*

(i) *It holds*

$$\begin{cases} \mathbf{v}_\varepsilon \in C_{\mathrm{w}}([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_\sigma(\Omega)), \\ \partial_t \mathbf{v}_\varepsilon \in L^{\frac{4}{n}}(0,T; (H^1_\sigma(\Omega))'), \\ c_\varepsilon \in L^\infty(\Omega_T) \text{ with } |c_\varepsilon| < 1 \text{ a.e. in } \Omega \times (0,T), \\ c_\varepsilon \in L^2(0,T; H^1(\Omega)) \quad \text{if } \Omega = \mathbb{T}^n, \\ \partial_t c_\varepsilon \in L^2(0,T; H^1(\Omega)'), \\ \mu_\varepsilon \in L^2(0,T; H^1(\Omega)). \end{cases} \tag{7.2.10}$$

(ii) *The triplet $(\mathbf{v}_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ fulfills the equations* (7.1.1a)–(7.1.1d) *in weak sense and the initial conditions $\mathbf{v}_\varepsilon(\cdot, 0) = \mathbf{v}_{\varepsilon,0}$ and $c_\varepsilon(\cdot, 0) = c_{\varepsilon,0}$ hold in Ω.*

(iii) *There exists a constant $C_1(T) > 0$ such that*

$$\begin{aligned} &\|\mathbf{v}_\varepsilon\|_{L^2(0,T;H^1_\sigma(\Omega))} + \|\partial_t \mathbf{v}_\varepsilon\|_{L^{4/n}(0,T;(H^1_\sigma(\Omega))')} \\ &\quad + \|c_\varepsilon\|_{L^\infty(\Omega\times(0,T))} + \|\partial_t c_\varepsilon\|_{L^2(0,T;H^1(\Omega)')} + \|\mu_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C_1(T). \end{aligned} \tag{7.2.11}$$

There further exist $\varepsilon_w = \varepsilon_w(\theta_0) > 0$ and a constant $C_2(T) > 0$ such that

$$\|c_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C_2(T) \quad \text{if } \Omega = \mathbb{T}^n \text{ and } \varepsilon \in (0, \varepsilon_w]. \tag{7.2.12}$$

We additionally assume $\mathbf{v}_{\varepsilon,0} \in H^1_\sigma(\Omega)$, $c_{\varepsilon,0} \in H^1(\Omega)$, with $|\overline{c_{\varepsilon,0}}| < 1$, $F'(c_{\varepsilon,0}) \in L^2(\Omega)$ and $F''(c_{\varepsilon,0})|\nabla c_{\varepsilon,0}|^2 \in L^1(\Omega)$. We further demand that there exists a constant $c_0 > 0$ independent of ε such that

$$\|D\mathbf{v}_{\varepsilon,0}\|_{L^2(\Omega)^{n\times n}} + \|\nabla \mu_{\varepsilon,0}\|_{L^2(\Omega)^n} \le c_0, \tag{7.2.13}$$

where $\mu_{\varepsilon,0} := \mathcal{L}_\varepsilon c_{\varepsilon,0} + f'(c_{\varepsilon,0})$. Then, there exists a unique right-maximal strong solution

$$(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon) : \Omega \times [0, T_{\varepsilon,*}) \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

of system (7.1.1) associated with ε. If $n = 2$, it holds $T_{\varepsilon,*} = \infty$, but if $n = 3$, the maximal existence time $T_{\varepsilon,*}$ may depend on ε. This strong solution has the following properties:

(iv) *For any $T \in (0, T_{\varepsilon,*})$, it holds*

$$\begin{cases} \mathbf{v}_\varepsilon \in BC([0,T]; H^1_\sigma(\Omega)) \cap L^2(0,T; H^2(\Omega)^n \cap H^1_\sigma(\Omega)) \cap H^1(0,T; L^2_\sigma(\Omega)), \\ p_\varepsilon \in L^2(0,T; H^1_{(0)}(\Omega)), \\ c_\varepsilon \in L^\infty(0,T; H^1(\Omega) \cap L^\infty(\Omega)) \text{ with } |c_\varepsilon| < 1 \text{ a.e. in } \Omega \times (0,T), \\ \partial_t c_\varepsilon \in L^\infty(0,T; H^1(\Omega)') \cap L^2(0,T; L^2(\Omega)), \\ F'(c_\varepsilon) \in L^\infty(0,T; L^p(\Omega)), \\ \mu_\varepsilon \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \end{cases} \tag{7.2.14}$$

for all $p \in [2, \infty)$ if $n = 2$ and all $p \in [2, 6]$ if $n = 3$.

(v) *$(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ fulfills the equations (7.1.1a)–(7.1.1d) a.e. in $\Omega \times [0, T_{\varepsilon,*})$ and the initial condition (7.1.1e) a.e. in Ω. If Ω is a bounded domain, it further holds $\mathbf{v}_\varepsilon = \mathbf{0}$ and $\partial_{\mathbf{n}} \mu_\varepsilon = 0$ a.e. on $\partial\Omega \times (0, T_{\varepsilon,*})$.*

(vi) *If $n = 3$, there exist $\varepsilon_s \in (0, \varepsilon_w]$ and $T_* \in (0, T_{\varepsilon,*})$ independent of ε such that for any $T \in (0, T_*]$, there exist constants $C_3(T), C_4(T) > 0$ such that*

$$\begin{aligned} &\|\mathbf{v}_\varepsilon\|_{L^\infty(0,T;H^1_\sigma(\Omega))} + \|\mathbf{v}_\varepsilon\|_{L^2(0,T;H^2(\Omega)^n)} + \|\mathbf{v}_\varepsilon\|_{H^1(0,T;L^2_\sigma(\Omega)^n)} \\ &\quad + \|p_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t c_\varepsilon\|_{L^\infty(0,T;H^1(\Omega)')} + \|\partial_t c_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\ &\quad + \|F'(c_\varepsilon)\|_{L^\infty(0,T;L^p(\Omega))} + \|\mu_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le C_3(T) \qquad \text{if } \varepsilon \in (0, \varepsilon_s], \end{aligned} \tag{7.2.15}$$

for all $p \in [2, 6]$, and

$$\|c_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|\mu_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le C_4(T) \quad \text{if } \Omega = \mathbb{T}^n \text{ and } \varepsilon \in (0, \varepsilon_s]. \tag{7.2.16}$$

If $n = 2$, there exists $\varepsilon_s \in (0, \varepsilon_w]$ such that (7.2.15) and (7.2.16) even hold for every $T > 0$ and every $p \in [2, \infty)$.

(vii) *If $n = 2$, we now additionally assume that (S2) holds, and if $n = 3$, we additionally assume that (S3) is fulfilled. Then, for any $\tau > 0$, there exists $\delta_{\varepsilon,\tau} \in (0, 1)$ such that the strict separation property*

$$\sup_{t \in [\tau, T_{\varepsilon,*})} \|c_\varepsilon(t)\|_{L^\infty(\Omega)} \le 1 - \delta_{\varepsilon,\tau} \tag{7.2.17}$$

holds. Moreover, if we further assume that $\|c_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq 1 - \delta_{\varepsilon,0}$ *for some* $\delta_{\varepsilon,0} \in (0,1)$*, then there exists* $\delta^*_{\varepsilon,0} > 0$ *such that the strict separation property*

$$\sup_{t\in[0,T_{\varepsilon,*})} \|c_\varepsilon(t)\|_{L^\infty(\Omega)} \leq 1 - \delta^*_\varepsilon \tag{7.2.18}$$

holds. In this case, we further have $\partial_t \mu_\varepsilon \in L^2(0,T;L^2(\Omega))$ *for every* $T \in (0, T_{\varepsilon,*})$*.*

We point out that the constants $C_1(T), ..., C_4(T)$ *may depend on the choice of* Ω*, the initial data and the system parameters, but are independent of* ε*.*

Remark 7.2.5. (a) We remark that, in the case $\Omega = \mathbb{T}^n$, the assumptions on the initial data for strong solutions already entail that

$$\int_\Omega F''(c_{\varepsilon,0})|\nabla c_{\varepsilon,0}|^2 \,\mathrm{d}x + \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla c_{\varepsilon,0}(x) - \nabla c_{\varepsilon,0}(y)|^2 \,\mathrm{d}x\,\mathrm{d}y$$
$$\leq C(1 + \|\nabla \mu_{\varepsilon,0}\|^2_{L^2(\Omega)^n}) \leq C$$

with a constant $C > 0$ that does not depend on ε, as long as ε is sufficiently small. In fact, this estimate can be shown similarly as estimate (7.3.27), which will be derived in the proof of Theorem 7.2.4.

(b) We point out that even though the maximal existence times $T_{\varepsilon,*}$ may depend on ε, Theorem 7.2.4(vi) entails that

$$\inf_{\varepsilon\in(0,\varepsilon_s]} T_{\varepsilon,*} \geq T_* > 0.$$

This means that all strong solutions $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ with $\varepsilon \in (0, \varepsilon_s]$ exist at least on the time interval $[0, T_*]$, whose length is independent of ε.

7.3. Proof of Theorem 7.2.4

In the whole proof, we will set the constants ρ, ν and m of system (7.1.1) to one as their concrete values do not have any impact on the mathematical analysis.

7.3.1. Uniqueness of the right-maximal strong solution

We start the proof of Theorem 7.2.4 by discussing the uniqueness of the right-maximal strong solution. In the case $n = 2$, the proof of uniqueness of weak solutions to (7.1.1) is quite standard, and we refer, for instance, to [63, Theorem 6.2]. In $n = 2$, the uniqueness of strong solutions to (7.1.1) (even in the more general case of unmatched viscosities) then follows by the result for weak solutions. For a proof, we refer to [65, Theorem 1.9]. In the case $n = 3$, the uniqueness of weak solutions is of course an open problem due to the involved Navier–Stokes equation. However, we are able to prove the uniqueness of the right-maximal strong solution.

Therefore, in the remainder of this subsection, we choose $n = 3$ and fix an arbitrary $\varepsilon > 0$. As the choice of ε does not matter in this subsection, the index ε will simply be omitted. Furthermore, in this subsection, the letter C denotes generic positive constants that may depend on the choice of Ω, the initial data and the system parameters including ε. The exact value of C may vary in the subsequent line of argument.

We consider two sets of initial data $(\mathbf{v}_{0,1}, c_{0,1})$ and $(\mathbf{v}_{0,2}, c_{0,2})$ which satisfy the assumptions for the existence of strong solutions imposed in Theorem 7.2.4. In addition, we assume that $\overline{c_{0,1}} = \overline{c_{0,2}}$. For each $i = 1, 2$, we assume that $(\mathbf{v}_i, p_i, c_i, \mu_i)$ is a strong solution of (7.1.1) associated with ε corresponding to the initial data $(\mathbf{v}_{0,i}, c_{0,i})$, which exists on the time-interval $[0, T_i)$. We now set $T := \min\{T_1, T_2\}$ and we further write

$$\begin{aligned}(\mathbf{v}_0, c_0) &:= (\mathbf{v}_{0,1}, c_{0,1}) - (\mathbf{v}_{0,2}, c_{0,2}),\\ (\mathbf{v}, p, c, \mu) &:= (\mathbf{v}_1, p_1, c_1, \mu_1) - (\mathbf{v}_2, p_2, c_2, \mu_2).\end{aligned}$$

This means that the quadruplet $(\mathbf{v}, p, c, \mu)$ fulfills the following system of equations in the strong sense:

$$\partial_t \mathbf{v} + (\mathbf{v}_1 \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}_2 - \Delta \mathbf{v} + \nabla p = \mu_1 \nabla c_1 - \mu_2 \nabla c_2 \quad \text{in } \Omega_T, \tag{7.3.1a}$$

$$\operatorname{div}(\mathbf{v}) = 0 \quad \text{in } \Omega_T, \tag{7.3.1b}$$

$$\partial_t c + \mathbf{v}_1 \cdot \nabla c + \mathbf{v} \cdot \nabla c_2 = \Delta \mu \quad \text{in } \Omega_T, \tag{7.3.1c}$$

$$\mu = \mathcal{L}_\varepsilon c + F'(c_1) - F'(c_2) + \theta_0 c \quad \text{in } \Omega_T, \tag{7.3.1d}$$

$$\mathbf{v}(0) = \mathbf{v}_0, \quad c(0) = c_0 \quad \text{in } \Omega. \tag{7.3.1e}$$

If Ω is a bounded domain, $(\mathbf{v}, p, c, \mu)$ also satisfies the boundary conditions

$$\mathbf{v} = 0, \quad \partial_{\mathbf{n}} \mu = 0 \quad \text{on } \partial\Omega_T. \tag{7.3.1f}$$

Integrating (7.3.1c) over Ω and recalling that $\mathbf{v}$ and $\mathbf{v}_1$ are divergence-free, we first observe

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega c \,\mathrm{d}x = \int_\Omega \partial_t c \,\mathrm{d}x = \int_\Omega \operatorname{div}\left(\nabla \mu - \mathbf{v}_1 c - \mathbf{v} c_2\right) \mathrm{d}x = 0$$

by means of Gauß's divergence theorem. This means that

$$\overline{c}(t) = \overline{c_0} = 0 \quad \text{for all } t \in [0, T]. \tag{7.3.2}$$

We now test (7.3.1a) by $A_S^{-1}\mathbf{v}$ (cf. Section 3.6.2) and (7.3.1c) by $\mathcal{N}c$ (cf. 3.6.1), and we add the resulting equations. Integrating by parts and invoking the identities

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_\sigma^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla A_S^{-1}\mathbf{v}\|_{L^2(\Omega)^{n\times n}}^2 = \langle \partial_t \mathbf{v}, A_S^{-1}\mathbf{v}\rangle_{H_\sigma^1(\Omega)},\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c\|_*^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \mathcal{N}c\|_{L^2(\Omega)^n}^2 = \langle \partial_t c, \mathcal{N}c\rangle_{H_{(0)}^1(\Omega)},\end{aligned}$$

we infer

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{v}\|_\sigma^2 + \|c\|_*^2\Big) + \|\mathbf{v}\|_{L^2(\Omega)^n} + (\mu, c)\\ &\quad = (\mathbf{v}_1 \otimes \mathbf{v}, \nabla A_S^{-1}\mathbf{v}) + (\mathbf{v} \otimes \mathbf{v}_2, \nabla A_S^{-1}\mathbf{v}) - (\mathbf{v}_1 \cdot \nabla c, \mathcal{N}c)\\ &\qquad - (\mathbf{v} \cdot \nabla c_2, \mathcal{N}c) + (\mu_1 \nabla c_1 - \mu_2 \nabla c_2, A_S^{-1}\mathbf{v}).\end{aligned} \tag{7.3.3}$$

Replacing μ by means of (7.3.1d) and recalling the monotonicity of F', the definition of $\mathcal{L}_\varepsilon$ and the properties of J_ε (see (A3)), we use Young's inequality for products as well as

Young's inequality for convolutions (see, e.g., [95, Section 4.2]) to derive the estimate

$$\begin{aligned}(\mu, c) &\geq \theta_0 \|c\|^2_{L^2(\Omega)} + \big((J_\varepsilon * 1)c, c\big) - (J_\varepsilon * c, c) \\ &\geq \theta_0 \|c\|^2_{L^2(\Omega)} - \big((\nabla J_\varepsilon) * c, \nabla \mathcal{N}(c)\big) \\ &\geq \theta_0 \|c\|^2_{L^2(\Omega)} - \|J_\varepsilon\|_{W^{1,1}(\mathbb{R}^n)} \|c\|_{L^2(\Omega)} \|c\|_* \\ &\geq \frac{10}{16}\theta_0 \|c\|^2_{L^2(\Omega)} - C\|c\|^2_*.\end{aligned} \tag{7.3.4}$$

Furthermore, recalling that the velocity fields $\mathbf{v}_i$, $i = 1, 2$, are divergence-free, and using integration by parts, the Gagliardo–Nirenberg inequality, estimate (3.6.4) and Young's inequality, we deduce

$$\begin{aligned}|(\mathbf{v}_1 \cdot \nabla c, \mathcal{N}c)| = |(\mathbf{v}_1 c, \nabla \mathcal{N}c)| &\leq \|\mathbf{v}_1\|_{L^6(\Omega)^n} \|c\|_{L^2(\Omega)} \|\nabla \mathcal{N}c\|_{L^3(\Omega)^n} \\ &\leq C\|\mathbf{v}_1\|_{L^6(\Omega)^n} \|c\|_{L^2(\Omega)} \|\nabla \mathcal{N}c\|^{1/2}_{H^1(\Omega)^n} \|\nabla \mathcal{N}c\|^{1/2}_{L^2(\Omega)^n} \\ &\leq C\|\mathbf{v}_1\|_{L^6(\Omega)^n} \|c\|^{3/2}_{L^2(\Omega)} \|c\|^{1/2}_* \\ &\leq \frac{\theta_0}{16}\|c\|^2_{L^2(\Omega)} + C\|\mathbf{v}_1\|^4_{L^6(\Omega)^n} \|c\|^2_*.\end{aligned} \tag{7.3.5}$$

Proceeding similarly, we obtain

$$\begin{aligned}\left|(\mathbf{v}_1 \otimes \mathbf{v}, \nabla A_S^{-1}\mathbf{v})\right| &\leq \|\mathbf{v}_1\|_{L^6(\Omega)^n} \|\mathbf{v}\|_{L^2(\Omega)^n} \|\nabla A_S^{-1}\mathbf{v}\|_{L^3(\Omega)^{n\times n}} \\ &\leq \frac{1}{8}\|\mathbf{v}\|^2_{L^2(\Omega)^n} + C\|\mathbf{v}_1\|^4_{L^6(\Omega)^n} \|\mathbf{v}\|^2_\sigma,\end{aligned} \tag{7.3.6}$$

and analogously, we get

$$\left|(\mathbf{v} \otimes \mathbf{v}_2, \nabla A_S^{-1}\mathbf{v})\right| \leq \frac{1}{8}\|\mathbf{v}\|^2_{L^2(\Omega)^n} + C\|\mathbf{v}_2\|^4_{L^6(\Omega)^n} \|\mathbf{v}\|^2_\sigma. \tag{7.3.7}$$

Recalling that $|c| \leq |c_1| + |c_2| < 2$ a.e. in Ω_T and that $\mathbf{v}$ is divergence free, we further deduce

$$\begin{aligned}|(\mathbf{v} \cdot \nabla c_2, \mathcal{N}c)| = |(\mathbf{v}c_2, \nabla \mathcal{N}c)| &\leq 2\|\mathbf{v}\|_{L^2(\Omega)^n} \|\nabla \mathcal{N}c\|_{L^2(\Omega)^n} \\ &\leq \frac{1}{8}\|\mathbf{v}\|^2_{L^2(\Omega)^n} + C\|c\|^2_*.\end{aligned} \tag{7.3.8}$$

Furthermore, expressing μ_1 and μ_2 by means of (7.1.2d), we deduce

$$\begin{aligned}(\mu_1 \nabla c_1 - \mu_2 \nabla c_2, A_S^{-1}\mathbf{v}) &= \big((J_\varepsilon * 1)c_1 \nabla c_1 - (J_\varepsilon * 1)c_2 \nabla c_2, A_S^{-1}\mathbf{v}\big) \\ &\quad + \big((J_\varepsilon * c_1) \nabla c_1 - (J_\varepsilon * c_2) \nabla c_2, A_S^{-1}\mathbf{v}\big) \\ &\quad + \big(\nabla F(c_1) - \nabla F(c_2), A_S^{-1}\mathbf{v}\big) \\ &\quad + \tfrac{1}{2}\theta_0 \big(\nabla(c_1^2) - \nabla(c_2^2), A_S^{-1}\mathbf{v}\big).\end{aligned}$$

As $A_S^{-1}\mathbf{v}$ is divergence-free, the last two lines of the right-hand side vanish after integrating by parts. Moreover, reformulating the first two lines and using integration by parts, we obtain

$$\begin{aligned}(\mu_1 \nabla c_1 - \mu_2 \nabla c_2, A_S^{-1}\mathbf{v}) &= \big((J_\varepsilon * 1)c_1 \nabla c + (J_\varepsilon * 1)c \nabla c_2, A_S^{-1}\mathbf{v}\big) \\ &\quad + \big((J_\varepsilon * c_1) \nabla c + (J_\varepsilon * c) \nabla c_2, A_S^{-1}\mathbf{v}\big) \\ &= -\big((\nabla J_\varepsilon * 1)c_1 c, A_S^{-1}\mathbf{v}\big) - \big((J_\varepsilon * 1)c \nabla c_2, A_S^{-1}\mathbf{v}\big) \\ &\quad + \big((\nabla J_\varepsilon * c_1) c, A_S^{-1}\mathbf{v}\big) - \big((J_\varepsilon * c) \nabla c_2, A_S^{-1}\mathbf{v}\big) \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

We now recall that $J_\varepsilon \in W^{1,1}(\mathbb{R}^n)$ (cf. (A3)) and that $|c_1| < 1$ a.e. in Ω_T. Invoking Hölder's inequality, Young's inequality (both for products and for convolutions) and Agmon's inequality along with the properties of the operator A_S^{-1} (cf. Section 3.6.2), the terms $I_1, ..., I_4$ can be estimated as follows:

$$|I_1| \le \|J_\varepsilon\|_{W^{1,1}(\mathbb{R}^n)} \|c_1\|_{L^\infty(\Omega)} \|c\|_{L^2(\Omega)} \|A_S^{-1}\mathbf{v}\|_{L^2(\Omega)^n} \le \frac{\theta_0}{16}\|c\|^2_{L^2(\Omega)} + C\|\mathbf{v}\|^2_\sigma ,$$

$$\begin{aligned} |I_2| &\le \|J_\varepsilon\|_{W^{1,1}(\mathbb{R}^n)} \|c\|_{L^2(\Omega)} \|\nabla c_2\|_{L^2(\Omega)^n} \|A_S^{-1}\mathbf{v}\|_{L^\infty(\Omega)^n} \\ &\le C\|c\|_{L^2(\Omega)} \|A_S^{-1}\mathbf{v}\|^{1/2}_{H^2(\Omega)^n} \|A_S^{-1}\mathbf{v}\|^{1/2}_{H^1(\Omega)^n} \le C\|c\|_{L^2(\Omega)} \|\mathbf{v}\|^{1/2}_{L^2(\Omega)^n} \|\mathbf{v}\|^{1/2}_\sigma \\ &\le \frac{\theta_0}{16}\|c\|^2_{L^2(\Omega)} + \frac{1}{4}\|\mathbf{v}\|^2_{L^2(\Omega)^n} + C\|\mathbf{v}\|^2_\sigma , \end{aligned}$$

$$|I_3| \le \|J_\varepsilon\|_{W^{1,1}(\mathbb{R}^n)} \|c_1\|_{L^\infty(\Omega)} \|c\|_{L^2(\Omega)} \|A_S^{-1}\mathbf{v}\|_{L^2(\Omega)^n} \le \frac{\theta_0}{16}\|c\|^2_{L^2(\Omega)} + C\|\mathbf{v}\|^2_\sigma ,$$

$$|I_4| \le \|J_\varepsilon\|_{W^{1,1}(\mathbb{R}^n)} \|c\|_{L^2(\Omega)} \|\nabla c_2\|_{L^2(\Omega)^n} \|A_S^{-1}\mathbf{v}\|_{L^2(\Omega)^n} \le \frac{\theta_0}{16}\|c\|^2_{L^2(\Omega)} + C\|\mathbf{v}\|^2_\sigma .$$

In summary, we thus have

$$\big|(\mu_1\nabla c_1 - \mu_2\nabla c_2, A_S^{-1}\mathbf{v})\big| \le \frac{\theta_0}{4}\|c\|^2_{L^2(\Omega)} + \frac{1}{8}\|\mathbf{v}\|^2_{L^2(\Omega)^n} + C\|\mathbf{v}\|^2_\sigma . \tag{7.3.9}$$

Combining (7.3.3)–(7.3.9), we conclude

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\mathbf{v}\|^2_\sigma + \|c\|^2_*\Big) + \frac{1}{2}\|\mathbf{v}\|^2_{L^2(\Omega)^n} + \frac{\theta_0}{2}\|c\|^2_{L^2(\Omega)} \\ &\quad \le C\Big(1 + \|\mathbf{v}_1\|^4_{L^6(\Omega)^n} + \|\mathbf{v}_2\|^4_{L^6(\Omega)^n}\Big)\Big(\|\mathbf{v}\|^2_\sigma + \|c\|^2_*\Big). \end{aligned}$$

Applying Gronwall's lemma, and recalling that $\mathbf{v}_i \in L^4(0,T;L^6(\Omega)^n)$, $i = 1, 2$, we eventually obtain

$$\|\mathbf{v}(t)\|^2_\sigma + \|c(t)\|^2_* \le (\|\mathbf{v}_0\|^2_\sigma + \|c_0\|^2_*) \exp\left(\int_0^t \Big(1 + \|\mathbf{v}_1(s)\|^4_{L^6(\Omega)^n} + \|\mathbf{v}_2(s)\|^4_{L^6(\Omega)^n}\Big)\,\mathrm{d}s\right)$$

for all $t \in [0, T]$. As the right-hand side vanishes if $\mathbf{v}_{0,1} = \mathbf{v}_{0,2}$ and $c_{0,1} = c_{0,2}$ a.e. in Ω, this proves the uniqueness of the corresponding strong solution.

7.3.2. Existence of weak and strong solutions with uniform bounds

In the case $n = 2$, under the respective assumptions made in Theorem 7.2.4, the existence of a weak solution satisfying (i) and (ii) was established in [53], and the existence of a strong solution satisfying (iv) and (v) was shown in [63, 65]. In fact, in [53] and [65] even the more general case of unmatched densities is considered.

In the case $n = 3$, the existence of a weak solution satisfying (i) and (ii) was also proven in [53]. The existence of a strong solution satisfying (iv) and (v) can be shown by proceeding similarly as in [65]. However, in contrast to the two-dimensional case, it cannot be shown that the constructed right-maximal strong solution exists for all times. This is, of course, due to the involved Navier–Stokes equation for which global existence of regular solutions in three dimensions is still an open problem. For the maximal existence time of strong solutions, a concrete lower bound T_* that is uniform in ε will be derived in Subsection 7.3.2.

Discretization by a semi-Galerkin scheme

To prove the existence of weak and strong solutions which fulfill the properties claimed in Theorem 7.2.4, we proceed as in [65, Section 5] and employ a semi-Galerkin scheme, where only the velocity field is discretized via a Galerkin ansatz.

It is well known that the Stokes operator A_S (see Section 3.6.2) possesses a sequence of eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ along with a sequence of corresponding eigenfunctions $(\mathbf{w}_j)_{j\in\mathbb{N}} \subset H^1_\sigma(\Omega)$, which can be chosen in such a way that they form an orthonormal basis of $L^2_\sigma(\Omega)$ as well as an orthogonal Schauder basis of $H^1_\sigma(\Omega)$. By means of regularity theory for the Stokes operator, we further infer that $\mathbf{w}_j \in H^3(\Omega)^n$. For any $m \in \mathbb{N}$, we now introduce the finite-dimensional subspace

$$(V_m)^n := \operatorname{span}\{\mathbf{w}_1, ..., \mathbf{w}_m\} \subset H^1_\sigma(\Omega),$$

and we write $\mathbb{P}_m : H^1_\sigma(\Omega) \to (V_m)^n$ to denote the $L^2_\sigma(\Omega)$-orthogonal projection of $H^1_\sigma(\Omega)$ onto $(V_m)^n$. Proceeding as in [65], one can show that for any $m \in \mathbb{N}$ and any $T > 0$, there exists an approximate solution $(\mathbf{v}^m_\varepsilon, c^m_\varepsilon, \mu^m_\varepsilon)$, which has the regularities

$$\begin{cases} \mathbf{v}^m_\varepsilon \in C([0,T];(V_m)^n) \cap H^1(0,T;(V_m)^n), \\ c^m_\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega_T) \text{ with } |c^m_\varepsilon| < 1 \text{ a.e. in } \Omega \times (0,T), \\ \partial_t c^m_\varepsilon \in L^\infty(0,T;H^1(\Omega)') \cap L^2(0,T;L^2(\Omega)), \\ F'(c^m_\varepsilon) \in L^\infty(0,T;H^1(\Omega)), \\ \mu^m_\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \end{cases} \tag{7.3.10}$$

satisfies the equations

$$(\partial_t \mathbf{v}^m_\varepsilon, \mathbf{w}) + ((\mathbf{v}^m_\varepsilon \cdot \nabla)\mathbf{v}^m_\varepsilon, \mathbf{w}) + (\nabla \mathbf{v}^m_\varepsilon, \nabla \mathbf{w}) = (\mu^m_\varepsilon \nabla c^m_\varepsilon, \mathbf{w}) \tag{7.3.11a}$$

in $[0,T]$ for all $\mathbf{w} \in (V_m)^n$,

$$\partial_t c^m_\varepsilon + \mathbf{v}^m_\varepsilon \cdot \nabla c^m_\varepsilon = \Delta \mu^m_\varepsilon \qquad \text{a.e. in } \Omega_T, \tag{7.3.11b}$$

$$\mu^m_\varepsilon = \mathcal{L}_\varepsilon c^m_\varepsilon + f'(c^m_\varepsilon) \qquad \text{a.e. in } \Omega_T, \tag{7.3.11c}$$

$$\mathbf{v}^m_\varepsilon|_{t=0} = \mathbb{P}_m \mathbf{v}_{\varepsilon,0}, \quad c^m_\varepsilon|_{t=0} = c_{\varepsilon,0} \qquad \text{a.e. in } \Omega, \tag{7.3.11d}$$

and if Ω is a bounded domain, it further satisfies the boundary conditions

$$\mathbf{v}^m_\varepsilon = \mathbf{0}, \quad \partial_\mathbf{n} \mu^m_\varepsilon = 0 \quad \text{a.e. on } \partial\Omega_T. \tag{7.3.11e}$$

In case Ω is a bounded domain, the existence of such approximate solutions follows directly from [65, Section 5], when $n = 2$, and from the same argument, but substituting [65, Theorem 4.1] (valid in $n = 2$) with its three-dimensional counterpart [108, Theorem 2.2], when $n = 3$. If $\Omega = \mathbb{T}^n$, their existence can by shown analogously.

From now on, the letter C denotes generic positive constants that may depend only on the choice of Ω, the initial data and the system parameters, but neither on ε nor on m. The exact value of C may vary throughout this proof.

Existence of a global weak solution with uniform bounds

We first prove the existence of a global-in-time weak solution, which satisfies the properties (i)–(iii) of Theorem 7.2.4. Therefore, let $T > 0$ be arbitrary. Note that we cannot just rely on the existence result from [53] since, on the one hand, we do not know whether the weak solution constructed there is unique, and on the other hand, we need the semi-Galerkin scheme anyway to derive uniform bounds with respect to ε.

Let now $m \in \mathbb{N}$ be arbitrary. Invoking condition (7.2.9), we first observe that

$$E_\varepsilon(\mathbb{P}_m \mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}) \leq C E_\varepsilon(\mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}) \leq C_0.$$

Using this estimate and testing the equations (7.3.11a) by $\mathbf{v}_\varepsilon^m \in (V_m)^n$, (7.1.1c) by μ_ε^m and (7.1.1d) by $\partial_t c_\varepsilon^m$, and using integration by parts, we derive the energy inequality

$$E_\varepsilon\big(\mathbf{v}_\varepsilon^m(t), c_\varepsilon^m(t)\big) + \int_0^t \|D\mathbf{v}_\varepsilon^m(s)\|_{L^2(\Omega)^{n\times n}}^2 \,\mathrm{d}s + \int_0^t \|\nabla \mu_\varepsilon^m(t)\|_{L^2(\Omega)^n}^2 \,\mathrm{d}s \leq C_0 \qquad (7.3.12)$$

for all $t \geq 0$. In the following, let $C(T)$ denote generic positive constants that may depend only on T, Ω, the initial data and the system parameters, but neither on ε nor on m. The exact value of $C(T)$ may vary in the subsequent line of argument.

In view of (S1), the boundedness of the energy resulting from (7.3.12) already entails

$$|c_\varepsilon^m| \leq 1 \quad \text{a.e. in } \Omega_T. \qquad (7.3.13)$$

Using this bound as well as Korn's inequality, we further conclude from (7.3.12) that

$$\begin{aligned} &\|c_\varepsilon^m\|_{L^\infty(\Omega_T)} + \|\nabla \mu_\varepsilon^m\|_{L^2(0,T;L^2(\Omega)^n)} \\ &\quad + \|\mathbf{v}_\varepsilon^m\|_{L^\infty(0,T;L^2(\Omega)^n)} + \|\mathbf{v}_\varepsilon^m\|_{L^2(0,T;H^1(\Omega)^n)} \leq C(T). \end{aligned} \qquad (7.3.14)$$

We now recall the inequality

$$\int_\Omega |F'(c_\varepsilon^m)| \,\mathrm{d}x \leq C \int_\Omega F'(c_\varepsilon^m)(c_\varepsilon^m - \overline{c_\varepsilon^m}) \,\mathrm{d}x + C \quad \text{a.e. in } [0,T], \qquad (7.3.15)$$

which can be found, e.g., in [99, Proposition 4.3.]. Testing (7.1.1d) by $c_\varepsilon^m - \overline{c_\varepsilon^m}$, we obtain

$$\int_\Omega \mu_\varepsilon (c_\varepsilon^m - \overline{c_\varepsilon^m}) \,\mathrm{d}x = \int_\Omega \mathcal{L}_\varepsilon c_\varepsilon^m (c_\varepsilon^m - \overline{c_\varepsilon^m}) \,\mathrm{d}x + \int_\Omega \big(F'(c_\varepsilon^m) - \theta_0\big)(c_\varepsilon^m - \overline{c_\varepsilon^m}) \,\mathrm{d}x. \qquad (7.3.16)$$

By the definition of the mean, the left-hand side can be reformulated as

$$\int_\Omega \mu_\varepsilon^m (c_\varepsilon^m - \overline{c_\varepsilon^m}) \,\mathrm{d}x = \int_\Omega (\mu_\varepsilon - \overline{\mu_\varepsilon^m}) c_\varepsilon^m \,\mathrm{d}x.$$

In view of the properties of $\mathcal{L}_\varepsilon$, the first term on the right-hand side of (7.3.16) is non-negative. Due to (7.3.14), the Poincaré–Wirtinger inequality yields

$$\left| \int_\Omega F'(c_\varepsilon^m)(c_\varepsilon^m - \overline{c_\varepsilon^m}) \,\mathrm{d}x \right| \leq C\big(1 + \|\nabla \mu_\varepsilon^m\|_{L^2(\Omega)^n}\big). \qquad (7.3.17)$$

Hence, by means of (7.3.15), we conclude

$$\int_\Omega |F'(c_\varepsilon^m)| \,\mathrm{d}x \leq C\big(1 + \|\nabla \mu_\varepsilon^m\|_{L^2(\Omega)^n}\big). \qquad (7.3.18)$$

Consequently, it holds

$$|\overline{\mu_\varepsilon^m}| = |\overline{F'(c_\varepsilon^m)} - \theta_0 \overline{c_\varepsilon^m}| \leq C(1 + \|\nabla \mu_\varepsilon^m\|_{L^2(\Omega)^n}). \tag{7.3.19}$$

Recalling (7.3.14) and applying Poincaré's inequality, we thus conclude

$$\|\mu_\varepsilon^m\|_{L^2(0,T;H^1(\Omega))} \leq C(T). \tag{7.3.20}$$

By comparison in (7.1.1c), we further have

$$\|\partial_t c_\varepsilon^m\|_{L^2(0,T;H^1(\Omega)')} \leq C(T) \tag{7.3.21}$$

with the help of (7.3.14) and (7.3.20). Furthermore, using again (7.3.14) and (7.3.20), and recalling the definition of $\mathbf{P}_\sigma$ (see Section 3.6.2), we deduce

$$\left\|\mathbf{P}_\sigma\left[\mu_\varepsilon^m \nabla c_\varepsilon^m\right]\right\|_\sigma \leq \|\nabla \mu_\varepsilon^m\|_{L^2(\Omega)^n}.$$

Performing the usual estimates for the remaining terms in the Navier–Stokes equation (7.1.1a), we conclude

$$\left\|\mathbf{P}_\sigma\left[(\mathbf{v}_\varepsilon^m \cdot \nabla)\mathbf{v}_\varepsilon^m + 2\operatorname{div}(D\mathbf{v}_\varepsilon^m)\right]\right\|_{L^{\frac{4}{n}}(0,T;(H_\sigma^1(\Omega))')} \leq C(T),$$

which directly yields

$$\|\partial_t \mathbf{v}_\varepsilon^m\|_{L^{\frac{4}{n}}(0,T;(H_\sigma^1(\Omega))')} \leq C(T) \tag{7.3.22}$$

by a further comparison argument. Combining (7.3.14), (7.3.20), (7.3.21) and (7.3.22), we have thus verified (7.2.11).

If $\Omega = \mathbb{T}^n$, we further take the gradient of (7.3.11c) and test it by ∇c_ε^m. We then use the identity

$$\begin{aligned} \int_\Omega \nabla(\mathcal{L}_\varepsilon c_\varepsilon^m) \cdot \nabla c_\varepsilon^m \,\mathrm{d}x &= \int_\Omega \mathcal{L}_\varepsilon \nabla c_\varepsilon^m \cdot \nabla c_\varepsilon^m \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla c_\varepsilon^m(x) - \nabla c_\varepsilon^m(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{7.3.23}$$

to deduce

$$\begin{aligned} \int_\Omega F''(c_\varepsilon^m)|\nabla c_\varepsilon^m|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla c_\varepsilon^m(x) - \nabla c_\varepsilon^m(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ = \int_\Omega \nabla \mu_\varepsilon^m \cdot \nabla c_\varepsilon^m \,\mathrm{d}x + \theta_0 \|\nabla c_\varepsilon^m\|_{L^2(\Omega)^n}^2. \end{aligned} \tag{7.3.24}$$

We point out that (7.3.23) follows from the relation $\nabla(J_\varepsilon * c_\varepsilon^m) = J_\varepsilon * \nabla c_\varepsilon^m$, which holds if $\Omega = \mathbb{T}^n$, but is (in general) not valid if Ω is a bounded domain. Exploiting Lemma 3.5.43.5.2 with $\gamma = \frac{1}{4\theta_0}$, we find $\varepsilon_w = \varepsilon_w(\theta_0) > 0$ such that

$$\theta_0 \|\nabla c_\varepsilon^m\|_{L^2(\Omega)^n}^2 \leq \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla c_\varepsilon^m(x) - \nabla c_\varepsilon^m(y)|^2 \,\mathrm{d}x\,\mathrm{d}y + C\|c_\varepsilon^m\|_{L^2(\Omega)}^2 \tag{7.3.25}$$

if $\varepsilon \in (0, \varepsilon_w]$. Combining (7.3.24) and (7.3.25), we infer

$$\begin{aligned} &\int_\Omega F''(c_\varepsilon^m)|\nabla c_\varepsilon^m|^2 \,\mathrm{d}x + \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla c_\varepsilon^m(x) - \nabla c_\varepsilon^m(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \int_\Omega \nabla\mu_\varepsilon^m \cdot \nabla c_\varepsilon^m \,\mathrm{d}x + C\|c_\varepsilon^m\|^2_{L^2(\Omega)}. \end{aligned} \tag{7.3.26}$$

provided that $\varepsilon \in (0, \varepsilon_w]$. Recalling (7.3.14) and that $F'' \geq \theta$, we use Young's inequality to infer from (7.3.26) that

$$\begin{aligned} \theta\|\nabla c_\varepsilon^m\|^2_{L^2(\Omega)^n} &\leq \int_\Omega F''(c_\varepsilon^m)|\nabla c_\varepsilon^m|^2 \,\mathrm{d}x + \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)|\nabla c_\varepsilon^m(x) - \nabla c_\varepsilon^m(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leq C(1 + \|\nabla\mu_\varepsilon^m\|^2_{L^2(\Omega)^n}) + \frac{\theta}{2}\|\nabla c_\varepsilon^m\|^2_{L^2(\Omega)^n}, \end{aligned} \tag{7.3.27}$$

if $\varepsilon \in (0, \varepsilon_w]$. In combination with (7.3.14), we thus conclude

$$\|c_\varepsilon^m\|_{L^2(0,T;H^1(\Omega))} \leq C \quad \text{if } \Omega = \mathbb{T}^n \text{ and } \varepsilon \in (0, \varepsilon_w]. \tag{7.3.28}$$

Based upon the uniform estimates (7.3.13), (7.3.14), (7.3.20), (7.3.21), (7.3.22) and (7.3.28), we use standard compactness results to eventually pass to the limit $m \to \infty$ in (7.3.11) along a suitable subsequence. In this way, we show that the limit functions $(\mathbf{v}_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ are a weak solution of system (7.1.1) in the sense of Theorem 7.2.4, which satisfy (i)–(iii). In particular, since $T > 0$ was arbitrary, this weak solution exists globally in time.

Existence of a right-maximal strong solution with uniform bounds

We next prove the existence of a right-maximal strong solution, which satisfies the properties (iv)–(vi) of Theorem 7.2.4. Again, we make use of the semi-Galerkin scheme introduced above. Therefore, let $\varepsilon \in (0, \varepsilon_w]$ and $m \in \mathbb{N}$ be arbitrary. Moreover, we take $T_{\varepsilon,*} > 0$, which will be specified later. Invoking regularity theory for the Stokes operator, we further infer the existence of a pressure $p_\varepsilon^m \in C([0, T_{\varepsilon,*}); H^1(\Omega))$ such that

$$-\Delta\mathbf{v}_\varepsilon^m + \nabla p_\varepsilon^m = A_S\mathbf{v}_\varepsilon^m \quad \text{a.e. in } \Omega \times [0, T_{\varepsilon,*}). \tag{7.3.29}$$

Step 1: Uniform estimates for $\mathbf{v}_\varepsilon^m$ and p_ε^m. Our first step is to derive higher order bounds on the velocity field $\mathbf{v}_\varepsilon^m$, which are uniform with respect to ε. In the case $\Omega = \mathbb{T}^n$, it is well known that testing the momentum equation (7.1.1a) by $-\Delta\mathbf{v}_\varepsilon^m$ (instead of $A_S\mathbf{v}_\varepsilon^m$) is sufficient to bound $\mathbf{v}_\varepsilon^m$ in the $H^2(\Omega)^n$-norm (see, e.g., [120]). However, as we also want to cover the case of Ω being a bounded domain, we use a more general approach and test the momentum equation by $A_S\mathbf{v}_\varepsilon^m = -\mathbf{P}_\sigma\Delta\mathbf{v}_\varepsilon^m \in (V_m)^n$. Performing this testing procedure and employing the well-known identity

$$\frac{1}{2}\|D\mathbf{v}_\varepsilon^m(t)\|^2_{L^2(\Omega)^{n\times n}} = \frac{1}{2}\|D\mathbb{P}_m\mathbf{v}_{\varepsilon,0}\|^2_{L^2(\Omega)^{n\times n}} + \int_0^t\int_\Omega \partial_t\mathbf{v}_\varepsilon^m \cdot A_S\mathbf{v}_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}s$$

for almost all $t \in [0, T_{\varepsilon,*})$, we obtain

$$\begin{aligned} \frac{1}{2}\|D\mathbf{v}_\varepsilon^m(t)\|^2_{L^2(\Omega)^{n\times n}} &\leq C + \int_0^t\int_\Omega \mu_\varepsilon^m\nabla c_\varepsilon^m \cdot A_S\mathbf{v}_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t\int_\Omega (\mathbf{v}_\varepsilon^m \cdot \nabla)\mathbf{v}_\varepsilon^m \cdot A_S\mathbf{v}_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega |D\mathbf{v}_\varepsilon^m|^2 \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{7.3.30}$$

for almost all $t \in [0, T_{\varepsilon,*})$. Here, we further used that

$$\begin{aligned}\frac{1}{2}\|D\mathbb{P}_m \mathbf{v}_{\varepsilon,0}\|^2_{L^2(\Omega)^{n\times n}} &\le \frac{1}{2}\|\mathbb{P}_m \mathbf{v}_{\varepsilon,0}\|^2_{H^1_\sigma(\Omega)} \le C\|\mathbf{v}_{\varepsilon,0}\|^2_{H^1_\sigma(\Omega)} \\ &\le C\big(\|\mathbf{v}_{\varepsilon,0}\|^2_{L^2(\Omega)^n} + \|D\mathbf{v}_{\varepsilon,0}\|^2_{L^2(\Omega)^{n\times n}}\big) \le C\end{aligned} \tag{7.3.31}$$

due to Korn's inequality and the conditions (7.2.9) and (7.2.13).

Performing an integration by parts, and applying Hölder's inequality, Young's inequality and (7.3.14), we get

$$\begin{aligned}\Big|\int_\Omega \mu_\varepsilon^m \nabla c_\varepsilon^m \cdot A_S \mathbf{v}_\varepsilon^m \,\mathrm{d}x\Big| &= \Big|\int_\Omega \nabla\mu_\varepsilon^m c_\varepsilon^m \cdot A_S \mathbf{v}_\varepsilon^m \,\mathrm{d}x\Big| \\ &\le C\|\nabla\mu_\varepsilon^m\|^2_{L^2(\Omega)^n} + \frac{1}{4}\|A_S\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n}.\end{aligned} \tag{7.3.32}$$

If $n = 2$, we use Hölder's inequality, Young's inequality, the Gagliardo–Nirenberg inequality as well as (7.3.14) to deduce

$$\begin{aligned}\Big|\int_\Omega (\mathbf{v}_\varepsilon^m \cdot \nabla)\mathbf{v}_\varepsilon^m \cdot A_S \mathbf{v}_\varepsilon^m \,\mathrm{d}x\Big| &\le C\|\mathbf{v}_\varepsilon^m\|_{L^4(\Omega)^n}\|D\mathbf{v}_\varepsilon^m\|_{L^4(\Omega)^{n\times n}}\|A_S\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^n} \\ &\le C\|\mathbf{v}_\varepsilon^m\|^{\frac{1}{2}}_{L^2(\Omega)^n}\|D\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^{n\times n}}\|A_S\mathbf{v}_\varepsilon^m\|^{\frac{3}{2}}_{L^2(\Omega)^n} \\ &\le \frac{1}{4}\|A_S\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n} + C\|D\mathbf{v}_\varepsilon^m\|^4_{L^2(\Omega)^{n\times n}}.\end{aligned} \tag{7.3.33}$$

In fact, if $\Omega = \mathbb{T}^n$, the above integral even vanishes (see, e.g., [120, Lemma 3.1]). In the case $n = 3$, we proceed similarly to derive the estimate

$$\begin{aligned}&\Big|\int_\Omega (\mathbf{v}_\varepsilon^m \cdot \nabla)\mathbf{v}_\varepsilon^m \cdot A_S \mathbf{v}_\varepsilon^m \,\mathrm{d}x\Big| \le C\|\mathbf{v}_\varepsilon^m\|_{L^6(\Omega)^n}\|D\mathbf{v}_\varepsilon^m\|_{L^3(\Omega)^{n\times n}}\|A_S\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^n} \\ &\quad\le C\|D\mathbf{v}_\varepsilon^m\|^{\frac{3}{2}}_{L^2(\Omega)^{n\times n}}\|A_S\mathbf{v}_\varepsilon^m\|^{\frac{3}{2}}_{L^2(\Omega)^n} \le \frac{1}{4}\|A_S\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n} + C\|D\mathbf{v}_\varepsilon^m\|^6_{L^2(\Omega)^{n\times n}}.\end{aligned} \tag{7.3.34}$$

Furthermore, testing the momentum equation by $\partial_t \mathbf{v}_\varepsilon^m$, we derive the identity

$$\begin{aligned}\int_0^t \|\partial_t\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n}\,\mathrm{d}s &= -\int_0^t\int_\Omega (\mathbf{v}_\varepsilon^m\cdot\nabla)\mathbf{v}_\varepsilon^m \cdot \partial_t\mathbf{v}_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}s + 2\int_0^t\int_\Omega \Delta\mathbf{v}_\varepsilon^m \partial_t\mathbf{v}_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega \mu_\varepsilon^m\nabla c_\varepsilon^m \cdot \partial_t\mathbf{v}_\varepsilon^m\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{7.3.35}$$

for almost all $t \in [0, T_{\varepsilon,*})$. Using Young's inequality as well as integration by parts, we obtain

$$\begin{aligned}\Big|\int_\Omega \nabla\mathbf{v}_\varepsilon^m : \nabla\partial_t\mathbf{v}_\varepsilon^m\,\mathrm{d}x\Big| &= \Big|\int_\Omega \Delta\mathbf{v}_\varepsilon^m \cdot \partial_t\mathbf{v}_\varepsilon^m\,\mathrm{d}x\Big| \\ &\le C_0\|A_S\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n} + \frac{1}{4}\|\partial_t\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n},\end{aligned} \tag{7.3.36}$$

$$\begin{aligned}\int_\Omega \mu_\varepsilon^m\nabla c_\varepsilon^m \cdot \partial_t\mathbf{v}_\varepsilon^m\,\mathrm{d}x &= -\int_\Omega c_\varepsilon^m\nabla\mu_\varepsilon^m \cdot \partial_t\mathbf{v}_\varepsilon^m\,\mathrm{d}x \\ &\le C\|\nabla\mu_\varepsilon^m\|^2_{L^2(\Omega)^n} + \frac{1}{4}\|\partial_t\mathbf{v}_\varepsilon^m\|^2_{L^2(\Omega)^n}.\end{aligned} \tag{7.3.37}$$

for some positive constant C_0 depending on the same quantities as the constants denoted by C. Now, proceeding similarly as in the derivation of (7.3.33) and (7.3.34), we deduce

$$\begin{aligned}&\left|\int_\Omega (\mathbf{v}_\varepsilon^m \cdot \nabla)\mathbf{v}_\varepsilon^m \cdot \partial_t \mathbf{v}_\varepsilon^m \,\mathrm{d}x\right| \\ &\qquad \le \frac{1}{4}\|\partial_t \mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^n}^2 + C_0\|A_S\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^n}^2 + C\|D\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^{n\times n}}^\gamma,\end{aligned} \tag{7.3.38}$$

where $\gamma = 4$ if $n = 2$ and $\gamma = 6$ if $n = 3$. We now add inequality (7.3.30) and inequality (7.3.35) multiplied by $\frac{1}{8C_0}$. By means of (7.3.20), (7.3.33)–(7.3.32) and (7.3.36)–(7.3.37), we infer

$$\begin{aligned}&\frac{1}{2}\|D\mathbf{v}_\varepsilon^m(t)\|_{L^2(\Omega)^{n\times n}}^2 + \frac{1}{16C_0}\int_0^t \|\partial_t \mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^n}^2 \,\mathrm{d}s + \frac{1}{16}\int_0^t \|A_S\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^n}^2 \,\mathrm{d}s \\ &\quad \le C + C\int_0^t \|D\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^{n\times n}}^2 \|D\mathbf{v}_\varepsilon^m\|_{L^2(\Omega)^{n\times n}}^{\gamma-2} \,\mathrm{d}s,\end{aligned}$$

for almost all $t \in [0, T_{\varepsilon,*})$ with γ as introduced above. We now recall (7.3.14), the uniform estimates stated in (iii) as well as assumption (2.5.7). Applying Bihari's inequality (see, e.g., [26, Lemma II.4.12] with $f(y) = y^{\gamma-2}$) and recalling (7.3.12), we conclude that $T_{\varepsilon,*} > 0$ can be fixed in a unique (maximal) way such that the estimate

$$\|\mathbf{v}_\varepsilon^m\|_{L^\infty(0,T;H^1_\sigma(\Omega))} + \|\mathbf{v}_\varepsilon^m\|_{L^2(0,T;H^2(\Omega)^n)} + \|\partial_t \mathbf{v}_\varepsilon^m\|_{L^2(0,T;L^2_\sigma(\Omega)^n)} \le C(T) \tag{7.3.39}$$

holds for all $T \in (0, T_{\varepsilon,*})$. If $n = 2$, we have $T_{\varepsilon,*} = \infty$. In the case $n = 3$, due to the uniform bound assumed in (2.5.7), Bihari's inequality (as stated in [26, Lemma II.4.12]) further implies the existence of a time $T_* \in (0, T_{\varepsilon,*})$, which is independent of ε, such that (7.3.39) holds true for all $T \in (0, T_*]$. By a comparison argument, we eventually obtain

$$\|p_\varepsilon^m\|_{L^2(0,T;H^1(\Omega))} \le C(T) \tag{7.3.40}$$

for any $T \in (0, T_{\varepsilon,*})$.

Step 2: Uniform estimates for c_ε^m and μ_ε^m. In the following, let $T \in (0, T_{\varepsilon,*})$ be arbitrary. The next goal is to derive further uniform bounds on c_ε^m and μ_ε^m. Therefore, in order to obtain the desired estimates, we need to truncate the initial datum $c_{\varepsilon,0}$ as it was done in [108]. For any $k \in \mathbb{N}$, we define the Lipschitz continuous truncation

$$\sigma_k : \mathbb{R} \to \mathbb{R}, \quad s \mapsto \begin{cases} -1 + \frac{1}{k} & \text{if } s < -1 + \frac{1}{k}, \\ s & \text{if } -1 + \frac{1}{k} < s < 1 - \frac{1}{k}, \\ 1 - \frac{1}{k} & \text{if } s > 1 - \frac{1}{k}, \end{cases}$$

and we set $c_{\varepsilon,0}^k := \sigma_k \circ c_{\varepsilon,0}$ and $\mu_{\varepsilon,0}^k := \mathcal{L}_\varepsilon c_{\varepsilon,0}^k + f'(c_{\varepsilon,0}^k)$. The strong solution corresponding to the initial datum $(\mathbb{P}_m \mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}^k)$ will be denoted as $(\mathbf{v}_\varepsilon^{m,k}, p_\varepsilon^{m,k}, c_\varepsilon^{m,k}, \mu_\varepsilon^{m,k})$. We point out that the estimates (7.3.12), (7.3.14), (7.3.20), (7.3.21), (7.3.22), (7.3.39) and (7.3.40) remain valid for the solution $(\mathbf{v}_\varepsilon^{m,k}, p_\varepsilon^{m,k}, c_\varepsilon^{m,k}, \mu_\varepsilon^{m,k})$ and are uniform with respect to k as long as k is sufficiently large. Indeed, for the $\varepsilon \in (0, \varepsilon_w]$ chosen above, there exists $k_0 = k_0(\varepsilon)$ such that we have $E_\varepsilon(\mathbb{P}_m \mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}^k) \le C E_\varepsilon(\mathbf{v}_{\varepsilon,0}, c_{\varepsilon,0}) + C$ for all $k \ge k_0$, and clearly the initial datum $\mathbf{v}_{\varepsilon,0}$ does not depend on k. In the following, we thus consider $k \ge k_0(\varepsilon)$.

Due to the assumptions in Theorem 7.2.4, we clearly have $\mu_{\varepsilon,0}^{k} \in H^1(\Omega)$. Moreover, as shown in [108, Formula (3.9)], the function $\mu_\varepsilon^{m,k}$ has the additional regularity

$$\mu_\varepsilon^{m,k} \in C([0,T];H^1(\Omega)). \tag{7.3.41}$$

In view of the regularities stated in (7.2.14), we further have

$$c_\varepsilon^{m,k} \in C_{\mathrm{w}}([0,T];H^1(\Omega)) \tag{7.3.42}$$

thanks to an embedding result, which can be found, e.g., in [119, Corollary 2.1].

Arguing as in [65, Proof of Theorem 4.1] ($n=2$) or [108, Proof of Theorem 2.2] ($n=3$), we derive the identity

$$\begin{aligned}
&\frac{1}{2}\|\nabla\mu_\varepsilon^{m,k}(t)\|_{L^2(\Omega)^n}^2 + \int_0^t\int_\Omega \mathbf{v}_\varepsilon^{m,k}\cdot\nabla c_\varepsilon^{m,k}\,\partial_t\mu_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad + \int_0^t\int_\Omega (\mathcal{L}_\varepsilon\partial_t c_\varepsilon^{m,k})\,\partial_t c_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega F''(c_\varepsilon^{m,k})|\partial_t c_\varepsilon^{m,k}|^2\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad = \frac{1}{2}\|\nabla\mu_{\varepsilon,0}^{k}\|_{L^2(\Omega)^n}^2 + \int_0^t\int_\Omega \theta_0|\partial_t c_\varepsilon^{m,k}|^2\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{7.3.43}$$

for almost all $t\in[0,T]$. Formally, (7.3.43) can be obtained as follows: We differentiate (7.1.1d) with respect to time, which yields

$$\partial_t\mu_\varepsilon^{m,k} = \mathcal{L}_\varepsilon\partial_t c_\varepsilon^{m,k} + f''(c_\varepsilon^{m,k})\partial_t c_\varepsilon^{m,k}. \tag{7.3.44}$$

Now, we test (7.1.1c) by $-\partial_t\mu_\varepsilon^{m,k}$ and (7.3.44) by $\partial_t c_\varepsilon^{m,k}$. Adding and integrating the resulting equations with respect to time from 0 to t, and using the identity

$$\begin{aligned}
\frac{1}{2}\|\nabla\mu_\varepsilon^{m,k}(t)\|_{L^2(\Omega)^n}^2 &= \frac{1}{2}\|\nabla\mu_{\varepsilon,0}^{k}\|_{L^2(\Omega)^n}^2 - \int_0^t\int_\Omega \partial_t\mu_\varepsilon^{m,k}\,\Delta\mu_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s\\
&= \frac{1}{2}\|\nabla\mu_{\varepsilon,0}^{k}\|_{L^2(\Omega)^n}^2 - \int_0^t\int_\Omega \partial_t c_\varepsilon^{m,k}\,\partial_t\mu_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}$$

we arrive at (7.3.43).

By a straightforward computation, the second summand on the left-hand side of (7.3.43) can be reformulated as

$$\begin{aligned}
&\int_0^t\int_\Omega \mathbf{v}_\varepsilon^{m,k}\cdot\nabla c_\varepsilon^{m,k}\partial_t\mu_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad = \int_\Omega \mathbf{v}_\varepsilon^{m,k}(t)\cdot\nabla c_\varepsilon^{m,k}(t)\mu_\varepsilon^{m,k}(t)\,\mathrm{d}x - \int_\Omega \mathbb{P}_m\mathbf{v}_{\varepsilon,0}\cdot\nabla c_{\varepsilon,0}^{k}\mu_{\varepsilon,0}^{k}\,\mathrm{d}x\\
&\qquad - \int_0^t\int_\Omega \partial_t\mathbf{v}_\varepsilon^{m,k}\cdot\nabla c_\varepsilon^{m,k}\mu_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \mathbf{v}_\varepsilon^{m,k}\partial_t c_\varepsilon^{m,k}\cdot\nabla\mu_\varepsilon^{m,k}\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{7.3.45}$$

for almost all $t\in[0,T]$. Due to the properties of $\mathcal{L}_\varepsilon$, we further have

$$\int_\Omega \mathcal{L}_\varepsilon\partial_t c_\varepsilon^{m,k}\,\partial_t c_\varepsilon^{m,k}\,\mathrm{d}x = 2\mathcal{E}_\varepsilon(\partial_t c_\varepsilon^{m,k}) \tag{7.3.46}$$

a.e. in $[0,T]$. We now introduce the function

$$H_\varepsilon^k : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \frac{1}{2}\|\nabla \mu_\varepsilon^{m,k}(t)\|_{L^2(\Omega)^n}^2 - \int_\Omega \mathbf{v}_\varepsilon^{m,k}(t) \cdot \nabla \mu_\varepsilon^{m,k}(t)\, c_\varepsilon^{m,k}(t)\; \mathrm{d}x.$$

Due to the regularities (7.2.14), (7.3.41) and (7.3.42), we know that H_ε^k is continuous and it thus holds

$$H_\varepsilon^k(0) = \frac{1}{2}\|\nabla \mu_{\varepsilon,0}^k\|_{L^2(\Omega)^n}^2 - \int_\Omega \mathbb{P}_m \mathbf{v}_{\varepsilon,0} \cdot \nabla \mu_{\varepsilon,0}^k\, c_{\varepsilon,0}^k\; \mathrm{d}x. \tag{7.3.47}$$

Recalling $F'' \geq \theta$, combining (7.3.43), (7.3.45) and (7.3.46), and using integration by parts, we conclude

$$\begin{aligned} &H_\varepsilon^k(t) + \int_0^t \int_\Omega \theta |\partial_t c_\varepsilon^{m,k}|^2 \; \mathrm{d}x\; \mathrm{d}s + 2\int_0^t \mathcal{E}_\varepsilon(\partial_t c_\varepsilon^{m,k})\; \mathrm{d}s \\ &\quad \leq H_\varepsilon^k(0) + \int_0^t \int_\Omega \partial_t \mathbf{v}_\varepsilon^{m,k} \cdot \nabla c_\varepsilon^{m,k} \mu_\varepsilon^{m,k}\; \mathrm{d}x\; \mathrm{d}s \\ &\qquad - \int_0^t \int_\Omega \mathbf{v}_\varepsilon^{m,k} \partial_t c_\varepsilon^{m,k} \cdot \nabla \mu_\varepsilon^{m,k}\; \mathrm{d}x\; \mathrm{d}s + \int_0^t \int_\Omega \theta_0 |\partial_t c_\varepsilon^{m,k}|^2\; \mathrm{d}x\; \mathrm{d}s. \end{aligned} \tag{7.3.48}$$

for almost all $t \in [0,T]$. Exploiting (7.3.14) and using integration by parts as well as Young's inequality, we obtain

$$\begin{aligned} \left| \int_\Omega \partial_t \mathbf{v}_\varepsilon^{m,k} \cdot \nabla c_\varepsilon^{m,k} \mu_\varepsilon^{m,k}\; \mathrm{d}x \right| &= \left| \int_\Omega \partial_t \mathbf{v}_\varepsilon^{m,k} \cdot c_\varepsilon^{m,k} \nabla \mu_\varepsilon^{m,k}\; \mathrm{d}x \right| \\ &\leq C\|\nabla \mu_\varepsilon^{m,k}\|_{L^2(\Omega)^n}^2 + C\|\partial_t \mathbf{v}_\varepsilon^{m,k}\|_{L^2(\Omega)^n}^2. \end{aligned} \tag{7.3.49}$$

Moreover, invoking the continuous embedding $H^2(\Omega)^n \hookrightarrow L^\infty(\Omega)^n$ and Young's inequality, we deduce

$$\begin{aligned} \left| \int_\Omega \mathbf{v}_\varepsilon^{m,k} \cdot \partial_t c_\varepsilon^{m,k} \nabla \mu_\varepsilon^{m,k}\; \mathrm{d}x \right| &\leq C\|\mathbf{v}_\varepsilon^{m,k}\|_{L^\infty(\Omega)^n} \|\partial_t c_\varepsilon^{m,k}\|_{L^2(\Omega)} \|\nabla \mu_\varepsilon^{m,k}\|_{L^2(\Omega)^n} \\ &\leq \theta_0 \|\partial_t c_\varepsilon^{m,k}\|_{L^2(\Omega)}^2 + C\|\mathbf{v}_\varepsilon^{m,k}\|_{H^2(\Omega)^n}^2 \|\nabla \mu_\varepsilon^{m,k}\|_{L^2(\Omega)^n}^2. \end{aligned} \tag{7.3.50}$$

Applying Lemma 3.5.43.5.3 with $\gamma = \frac{1}{2\theta_0}$, we find $\varepsilon_s = \varepsilon_s(\theta_0) > 0$ such that

$$2\theta_0 \|\partial_t c_\varepsilon^{m,k}\|_{L^2(\Omega)}^2 \leq \mathcal{E}_\varepsilon(\partial_t c_\varepsilon^{m,k}) + C\|\partial_t c_\varepsilon^{m,k}\|_*^2 \tag{7.3.51}$$

if $\varepsilon \in (0, \varepsilon_s]$. Without loss of generality, we assume $\varepsilon_s \leq \varepsilon_w$ and from now on, we further demand that $\varepsilon \in (0, \varepsilon_s]$.

Combining the inequalities (7.3.48), (7.3.49), (7.3.50) and (7.3.51), we conclude that the estimate

$$\begin{aligned} &H_\varepsilon^k(t) + \theta \int_0^t \|\partial_t c_\varepsilon^{m,k}\|_{L^2(\Omega)}^2\; \mathrm{d}s + \int_0^t \mathcal{E}_\varepsilon(\partial_t c_\varepsilon^{m,k})\; \mathrm{d}s \\ &\quad \leq H_\varepsilon^k(0) + C\int_0^t \|\partial_t \mathbf{v}_\varepsilon^{m,k}\|_{L^2(\Omega)^n}^2\; \mathrm{d}s + C\int_0^t \|\partial_t c_\varepsilon^{m,k}\|_*^2\; \mathrm{d}s \\ &\qquad + C\int_0^t \Big(1 + \|\mathbf{v}_\varepsilon^{m,k}\|_{H^2(\Omega)^n}^2\Big) \|\nabla \mu_\varepsilon^{m,k}\|_{L^2(\Omega)^n}^2\; \mathrm{d}s. \end{aligned}$$

holds for almost all $t \in [0,T]$. Note that, due to (7.3.14), we have

$$\left|\int_\Omega \mathbf{v}_\varepsilon^{m,k} \cdot c_\varepsilon^{m,k} \nabla \mu_\varepsilon^{m,k} \, \mathrm{d}x\right| \le \|\mathbf{v}_\varepsilon^{m,k}\|_{L^2(\Omega)^n} \|\nabla \mu_\varepsilon^{m,k}\|_{L^2(\Omega)^n} \le C + \frac{1}{4}\|\nabla \mu_\varepsilon^{m,k}\|_{L^2(\Omega)^n}^2. \tag{7.3.52}$$

Hence, there exist positive constants K and $\tilde{K}$ that may depend on the same quantities as C such that

$$\frac{1}{4}\|\nabla \mu_\varepsilon^{m,k}(t)\|_{L^2(\Omega)^n}^2 - \tilde{K} \le H_\varepsilon^k(t) \le \|\nabla \mu_\varepsilon^{m,k}(t)\|_{L^2(\Omega)^n}^2 + K \quad \text{for almost all } t \in [0,T].$$

This allows us to apply Gronwall's lemma, which yields

$$\begin{aligned} &H_\varepsilon^k(t) + \theta \int_0^t \|\partial_t c_\varepsilon^{m,k}(s)\|_{L^2(\Omega)}^2 \, \mathrm{d}s \\ &\le \left(H_\varepsilon^k(0) + C\int_0^t (\|\partial_t \mathbf{v}_\varepsilon^{m,k}(s)\|_{L^2(\Omega)^n} + \|\partial_t c_\varepsilon^{m,k}\|_*^2) \, \mathrm{d}s\right) \\ &\quad \cdot \exp\left(\int_0^t C(1 + \|\mathbf{v}_\varepsilon^{m,k}(s)\|_{H^2(\Omega)^n}^2) \, \mathrm{d}s\right) \\ &\le C(T)(1 + H_\varepsilon^k(0)) \end{aligned} \tag{7.3.53}$$

for almost all $t \in [0,T]$, thanks to (7.3.14), (7.3.21) and (7.3.39). It thus remains to control $H_\varepsilon^k(0)$ uniformly with respect to ε. Recalling $|c_{\varepsilon,0}^k| < 1$, the representation (7.3.47) as well as (7.3.31), we deduce

$$\begin{aligned} |H_\varepsilon^k(0)| &\le \|\nabla \mu_{\varepsilon,0}^k\|_{L^2(\Omega)^n}^2 + \|\mathbb{P}_m \mathbf{v}_{\varepsilon,0}\|_{L^2(\Omega)^n} \|\nabla \mu_{\varepsilon,0}^k\|_{L^2(\Omega)^n} \|c_{\varepsilon,0}^k\|_{L^\infty(\Omega)} \\ &\le C(1 + \|\nabla \mu_{\varepsilon,0}^k\|_{L^2(\Omega)^n}^2). \end{aligned} \tag{7.3.54}$$

Now, following [65, 108], we can prove that $\|\nabla \mu_{\varepsilon,0}^k\|_{L^2(\Omega)^n} \to \|\nabla \mu_{\varepsilon,0}\|_{L^2(\Omega)^n}$ as $k \to \infty$. Therefore, for the $\varepsilon \in (0, \varepsilon_s]$ that was chosen above, there exists $\overline{k} = \overline{k}(\varepsilon) \ge k_0(\varepsilon)$ such that

$$\|\nabla \mu_{\varepsilon,0}^k - \nabla \mu_{\varepsilon,0}\|_{L^2(\Omega)^n} \le \frac{1}{2} \quad \text{for all } k \ge \overline{k}(\varepsilon).$$

Hence, from (7.3.54) and the assumptions on $\mu_{\varepsilon,0}$ and $\mathbf{v}_{\varepsilon,0}$, we conclude

$$|H_\varepsilon^k(0)| \le C\big(1 + \|\nabla \mu_{\varepsilon,0}^k - \nabla \mu_{\varepsilon,0}\|_{L^2(\Omega)^n}^2 + \|\nabla \mu_{\varepsilon,0}\|_{L^2(\Omega)^n}^2\big) \le C$$

for any $k \ge \overline{k}(\varepsilon)$. Consequently, for every $k \ge \overline{k}(\varepsilon)$, (7.3.53) provides the bound

$$\|\nabla \mu_\varepsilon^{m,k}\|_{L^\infty(0,T;L^2(\Omega)^n)} + \|\partial_t c_\varepsilon^{m,k}\|_{L^2(0,T;L^2(\Omega))} \le C(T). \tag{7.3.55}$$

As the estimates (7.3.14), (7.3.20), (7.3.21), (7.3.22), (7.3.39) and (7.3.40) remain valid for the solution $(\mathbf{v}_\varepsilon^{m,k}, p_\varepsilon^{m,k}, c_\varepsilon^{m,k}, \mu_\varepsilon^{m,k})$ and are uniform with respect to k, it follows by standard compactness arguments that $(\mathbf{v}_\varepsilon^{m,k}, p_\varepsilon^{m,k}, c_\varepsilon^{m,k}, \mu_\varepsilon^{m,k})$ converges to $(\mathbf{v}_\varepsilon^m, p_\varepsilon^m, c_\varepsilon^m, \mu_\varepsilon^m)$, as $k \to \infty$, in the corresponding function spaces. For more details, we also refer to [108, Proof of Theorem 2.2]. In particular, we conclude that the strong solution $(\mathbf{v}_\varepsilon^m, p_\varepsilon^m, c_\varepsilon^m, \mu_\varepsilon^m)$ satisfies the uniform bound

$$\|\nabla \mu_\varepsilon^m\|_{L^\infty(0,T;L^2(\Omega)^n)} + \|\partial_t c_\varepsilon^m\|_{L^2(0,T;L^2(\Omega))} \le C(T). \tag{7.3.56}$$

Using Poincaré's inequality along with (7.3.19), we further obtain

$$\|\mu_\varepsilon^m\|_{L^\infty(0,T;H^1(\Omega))} \leq C(T). \tag{7.3.57}$$

In the case $\Omega = \mathbb{T}^n$, it further follows from (7.3.27) that

$$\|c_\varepsilon^m\|_{L^\infty(0,T;H^1(\Omega))} \leq C(T). \tag{7.3.58}$$

By comparison in (7.1.1c), we now use the uniform estimates (7.3.39) and (7.3.58) to deduce

$$\begin{aligned}&\|\Delta\mu_\varepsilon^m\|_{L^2(0,T;L^2(\Omega))}\\ &\quad\leq \|\partial_t c_\varepsilon^m\|_{L^2(0,T;L^2(\Omega))} + \|\mathbf{v}_\varepsilon^m\|_{L^2(0,T;L^\infty(\Omega)^n)}\,\|\nabla c_\varepsilon^m\|_{L^\infty(0,T;L^2(\Omega)^n)} \leq C(T).\end{aligned}$$

As $\Omega = \mathbb{T}^n$, we have $\|D^2\mu_\varepsilon^m\|^2_{L^2(\Omega)^{n\times n}} = \|\Delta\mu_\varepsilon^m\|^2_{L^2(\Omega)}$ a.e. in $[0,T]$. Hence, in combination with (7.3.57), we conclude the uniform bound

$$\|\mu_\varepsilon^m\|_{L^2(0,T;H^2(\Omega))} \leq C(T). \tag{7.3.59}$$

Step 3: A uniform estimate for $F'(c_\varepsilon^m)$. In the following, let $p \in [2,\infty)$ if $n = 2$ and let $p \in [2,6]$ if $n = 3$. As a consequence of (7.3.57), we obtain the estimate

$$\|\mu_\varepsilon^m\|_{L^\infty(0,T;L^p(\Omega))} \leq C(T)\sqrt{p}. \tag{7.3.60}$$

In the case $n = 3$, this inequality simply follows from the continuous embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and the fact that $\sqrt{p} \geq 1$. In the case $n = 2$, (7.3.59) follows from the following Sobolev type inequality, which can be found, e.g., in [122, p. 479]: there exists a constant $C_\Omega > 0$ depending only on Ω such that for all $u \in H^1(\Omega)$ and all $p \in [2,\infty)$, it holds

$$\|u\|_{L^p(\Omega)} \leq C_\Omega\sqrt{p}\,\|u\|_{H^1(\Omega)}.$$

We now intend to derive a uniform bound on $F'(c_\varepsilon^m)$ in $L^\infty(0,T;L^p(\Omega))$. Therefore, we test equation (7.1.1d) by $|F'(c_\varepsilon^m)|^{p-2}F'(c_\varepsilon^m)$. If $p = 2$, this test function is simply to be interpreted as $F'(c_\varepsilon^m)$. We obtain

$$\begin{aligned}\int_\Omega \mu_\varepsilon^m |F'(c_\varepsilon^m)|^{p-2}F'(c_\varepsilon^m)\,\mathrm{d}x &= \int_\Omega \mathcal{L}_\varepsilon c_\varepsilon^m |F'(c_\varepsilon^m)|^{p-2}F'(c_\varepsilon^m)\,\mathrm{d}x + \|F'(c_\varepsilon^m)\|^p_{L^p(\Omega)}\\ &\quad - \theta_0\int_\Omega c_\varepsilon^m|F'(c_\varepsilon^m)|^{p-2}F'(c_\varepsilon^m)\,\mathrm{d}x.\end{aligned}$$

Using Hölder's and Young's inequalities, and recalling that $|c_\varepsilon^m| < 1$ a.e. in Ω_T, we observe

$$\begin{aligned}\int_\Omega \mu_\varepsilon^m |F'(c_\varepsilon^m)|^{p-2}F'(c_\varepsilon^m)\,\mathrm{d}x &\leq C\|\mu_\varepsilon^m\|^p_{L^p(\Omega)} + \frac{1}{4}\|F'(c_\varepsilon^m)\|^p_{L^p(\Omega)},\\ \theta_0\int_\Omega c_\varepsilon^m|F'(c_\varepsilon^m)|^{p-2}F'(c_\varepsilon^m)\,\mathrm{d}x &\leq C|\Omega|^{\frac{1}{p}} + \frac{1}{4}\|F'(c_\varepsilon^m)\|^p_{L^p(\Omega)} \leq C + \frac{1}{4}\|F'(c_\varepsilon^m)\|^p_{L^p(\Omega)}.\end{aligned}$$

Since F' is strictly increasing, so is $g(r) := |F'(r)|^{p-2}F'(r)$ for $r \in (-1,1)$. This implies

$$\big(c_\varepsilon^m(x) - c_\varepsilon^m(y)\big)\big[g(c_\varepsilon^m(x)) - g(c_\varepsilon^m(y))\big] \geq 0 \quad \text{for almost all } x,y \in \Omega.$$

Consequently, since $J_\varepsilon \geq 0$, we have

$$\begin{aligned} &\int_\Omega \mathcal{L}_\varepsilon c_\varepsilon^m |F'(c_\varepsilon^m)|^{p-2} F'(c_\varepsilon^m) \, \mathrm{d}x \\ &= \frac{1}{2} \int_\Omega \int_\Omega J_\varepsilon(x-y) \left(c_\varepsilon^m(x) - c_\varepsilon^m(y)\right) \left[g(c_\varepsilon^m(x)) - g(c_\varepsilon^m(y))\right] \, \mathrm{d}y \, \mathrm{d}x \geq 0. \end{aligned}$$

Altogether, this implies

$$\|F'(c_\varepsilon^m)\|_{L^p(\Omega)} \leq C\big(1 + \|\mu_\varepsilon^m\|_{L^p(\Omega)}\big).$$

Hence, in combination with (7.3.60), we eventually conclude

$$\|F'(c_\varepsilon^m)\|_{L^\infty(0,T;L^p(\Omega))} \leq C\sqrt{p}. \tag{7.3.61}$$

Having all these uniform estimates at hand, we argue as in [65] and pass to the limit $m \to \infty$ in (7.3.11) along a suitable subsequence. This shows that the corresponding limit functions $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ are a strong solution of system (7.1.1) in the sense of Theorem 7.2.4, which satisfies (iv)–(vi). In particular, from what was observed at the end of Step 1, we have also shown that

$$\inf_{\varepsilon \in (0,\varepsilon_s]} T_{\varepsilon,*} \geq T_* > 0,$$

justifying Remark 7.2.5(b).

7.3.3. Strict separation properties

The last step is to verify the strict separation properties stated in item (vii).

In the case $n = 2$, it has already been proven in (7.3.61) (see also [65, Theorem 1.4]) that

$$\|F'(c_\varepsilon)\|_{L^\infty(0,\infty;L^p(\Omega))} \leq C_\varepsilon\sqrt{p}, \quad \text{for all } p \in [2,\infty), \tag{7.3.62}$$

where C_ε is a constant that may depend on the usual quantities as well as on ε. Therefore, one can proceed as in the the proof of [67, Theorem 4.3] to conclude that (7.2.17) holds. Assuming that the initial datum is strictly separated, we can repeat the same argument as in [107, Corollary 4.5] (i.e., the De Giorgi iteration scheme without the use of a cutoff function in time) to show that there exists $T_S > 0$ such that the solution is strictly separated on $[0, T_S]$. Combined with (7.2.17), the result (7.2.18) is verified. We point out that the dependence of δ_ε^* on ε is not only due to the constant C_ε in estimate (7.3.62) (which could be avoided if we restrict ourselves to finite time intervals, see (7.3.61)), but also results from the fact that the $W^{1,1}(\mathbb{R}^n)$-norm of J_ε is not bounded uniformly with respect to ε.

In the case $n = 3$, thanks to estimate (7.3.61), one can argue exactly as in [107, Theorem 4.3] (see also [107, Remarks 4.7, 4.9]) to prove the validity of (7.2.17) and (7.2.18). As in the two-dimensional setting, the dependence of δ_ε^* on ε cannot be avoided using this method.

We remark that in the aforementioned proofs, the presence of the additional convective term $\mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon$ in (7.1.1c) does not disturb the line of argument, since in the De Giorgi iteration scheme this term simply vanishes as the velocity field is divergence-free and vanishes at the boundary if Ω is a bounded domain. For more details, we refer to [107, Remark 4.7].

In summary, all statements of Theorem 7.2.4 are now established, and thus the proof is complete. □

7.4. Proof of Theorem 2.5.2

As in the proof of Theorem 7.2.4, we will set the constants ρ, ν and m of systems (7.1.1) and (7.1.2) to one as their concrete values do not have any impact on the mathematical analysis.

Let $\varepsilon_s > 0$ be the real number introduced in Theorem 7.2.4. For any $\varepsilon \in (0, \varepsilon_s]$, let $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ be the unique right-maximal strong solution to the nonlocal Model H (i.e., (7.1.1) with $\varepsilon \in (0, \varepsilon_s]$) given by Theorem 7.2.4. Moreover, let $(\mathbf{v}, p, c, \mu)$ denote the unique strong solution to the local Model H, i.e., (7.1.2), given by Theorem 7.2.1.

Note that the definition of $T_\diamond > 0$ ensures that the strong solutions $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ with $\varepsilon \in (0, \varepsilon_s]$ and the strong solution $(\mathbf{v}, p, c, \mu)$ exist on the time interval $[0, T_\diamond)$. In particular, the strong solution $(\mathbf{v}, p, c, \mu)$ fulfills the strict separation property stated in Theorem 7.2.1(v), and for any $\varepsilon \in (0, \varepsilon_s]$, the strong solution $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ fulfills the uniform estimates stated in Theorem 2.5.2(iii) and (vi).

From now on, in order to verify the convergence property (2.5.9), let $T \in (0, T_\diamond)$ and $\varepsilon \in (0, \varepsilon_s]$ be arbitrary. Moreover, we use the notation

$$\begin{aligned} (\tilde{\mathbf{v}}_0, \tilde{c}_0) &:= (\mathbf{v}_{0,\varepsilon}, c_{0,\varepsilon}) - (\mathbf{v}_0, c_0), \\ (\tilde{\mathbf{v}}, \tilde{p}, \tilde{c}, \tilde{\mu}) &:= (\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon) - (\mathbf{v}, p, c, \mu). \end{aligned}$$

This means that the quadruplet $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{c}, \tilde{\mu})$ fulfills the following system of equations in the strong sense:

$$\partial_t \tilde{\mathbf{v}} + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon + (\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta\tilde{\mathbf{v}} + \nabla\tilde{p} = \mu_\varepsilon \nabla c_\varepsilon - \mu\nabla c \quad \text{in } \Omega_T, \tag{7.4.1a}$$

$$\operatorname{div}(\tilde{\mathbf{v}}) = 0 \quad \text{in } \Omega_T, \tag{7.4.1b}$$

$$\partial_t \tilde{c} + \mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon + \mathbf{v} \cdot \nabla c = \Delta\tilde{\mu} \quad \text{in } \Omega_T, \tag{7.4.1c}$$

$$\tilde{\mu} = \mathcal{L}_\varepsilon c_\varepsilon + \Delta c + F'(c_1) - F'(c_2) + \theta_0 \tilde{c} \quad \text{in } \Omega_T, \tag{7.4.1d}$$

$$\tilde{\mathbf{v}}(0) = \tilde{\mathbf{v}}_0, \quad \tilde{c}(0) = \tilde{c}_0 \quad \text{in } \Omega. \tag{7.4.1e}$$

If Ω is a bounded domain, $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{c}, \tilde{\mu})$ also satisfies the boundary conditions

$$\tilde{\mathbf{v}} = 0, \quad \partial_{\mathbf{n}}\tilde{\mu} = 0 \quad \text{on } \partial\Omega_T. \tag{7.4.1f}$$

Step 1: An estimate for the difference $f'(c_\varepsilon) - f'(c)$. We first intend to derive an estimate for the difference $f'(c_\varepsilon) - f'(c)$ in the $L^1(\Omega)$-norm. As the separation properties for the solutions $(\mathbf{v}_\varepsilon, p_\varepsilon, c_\varepsilon, \mu_\varepsilon)$ are not uniform with respect to ε, the derivation of such an estimate is not straightforward. However, we can still exploit the strict separation property of the solution $(\mathbf{v}, p, c, \mu)$ to the local Model H. Therefore, let $\delta_\star$ be the constant from (7.2.4) and let δ_0 be the constant from (7.2.6). In the following, we set $\delta := \delta_\star/2$ if $n = 2$ and $\delta := \delta_0/4$ if $n = 3$. Hence, in view of Theorem 7.2.1(v), we have,

$$\|c(t)\|_{L^\infty(\Omega)} \le 1 - 2\delta \quad \text{for all } t \in [0, T] \tag{7.4.2}$$

due to the choices of $T_\diamond$ and T. For any $t \in [0,T]$, we now define

$$A_\delta(t) := \{x \in \Omega : \ |c_\varepsilon(x,t)| \geq 1-\delta\},$$
$$B_\delta(t) := \{x \in \Omega : \ |c(x,t) - c_\varepsilon(x,t)| \geq \delta\}.$$

Exploiting (7.4.2), we observe

$$1-\delta \leq |c_\varepsilon(x,t)| \leq |c(x,t)| + |c(x,t)-c_\varepsilon(x,t)| \leq 1-2\delta + |c(x,t)-c_\varepsilon(x,t)| \tag{7.4.3}$$

for all $t \in [0,T]$ and all $x \in A_\delta(t)$. This entails that

$$|c(x,t)-c_\varepsilon(x,t)| \geq \delta \tag{7.4.4}$$

for all $t \in [0,T]$ and all $x \in A_\delta(t)$. Consequently, for every $t \in [0,T]$, we have the inclusion

$$A_\delta(t) \subset B_\delta(t).$$

Therefore, invoking Chebyshev's inequality, we conclude

$$|A_\delta(t)| \leq \int_{B_\delta(t)} 1 \,\mathrm{d}x \leq \int_{B_\delta(t)} \frac{|c_\varepsilon(t)-c(t)|^2}{\delta^2} \,\mathrm{d}x \leq \int_\Omega \frac{|c_\varepsilon(t)-c(t)|^2}{\delta^2} \,\mathrm{d}x \tag{7.4.5}$$

for all $t \in [0,T]$, where $|A_\delta(t)|$ denotes the n-dimensional Lebesgue measure of the set $A_\delta(t)$. Using the Cauchy–Schwarz inequality as well as the fundamental theorem of calculus, we deduce

$$\begin{aligned}
&\|f'(c_\varepsilon) - f'(c)\|_{L^1(\Omega)} \\
&\leq \|f'(c_\varepsilon) - f'(c)\|_{L^1(A_\delta)} + \|f'(c_\varepsilon) - f'(c)\|_{L^1(\Omega\setminus A_\delta)} \\
&\leq \|f'(c_\varepsilon) - f'(c)\|_{L^2(A_\delta)} |A_\delta|^{\frac{1}{2}} + \int_{\Omega\setminus A_\delta} \left| \int_0^1 f''(sc_\varepsilon + (1-s)c)(c_\varepsilon - c) \,\mathrm{d}s \right| \mathrm{d}x
\end{aligned} \tag{7.4.6}$$

for all $t \in [0,T]$. By (7.4.2) and the definition of A_δ, we have

$$|sc_\varepsilon(t) + (1-s)c(t)| \leq s|c_\varepsilon(t)| + (1-s)|c(t)| \leq 1-\delta \quad \text{a.e. in } \Omega \setminus A_\delta(t)$$

for all $t \in [0,T]$ and all $s \in [0,1]$. Recalling $F'' \in C(-1,1)$, we thus have

$$\begin{aligned}
&\int_0^1 f''(sc_\varepsilon(t) + (1-s)c(t))(c_\varepsilon(t) - c(t)) \,\mathrm{d}s \\
&\quad \leq \left(\max_{|s|\leq 1-\delta} F''(s) + \theta_0 \right) |c_\varepsilon(t) - c(t)| =: C_\delta |c_\varepsilon(t) - c(t)| \quad \text{a.e. in } \Omega \setminus A_\delta(t).
\end{aligned} \tag{7.4.7}$$

Plugging this estimate into (7.4.6) and using again (7.4.5), we deduce

$$\begin{aligned}
&\|f'(c_\varepsilon) - f'(c)\|_{L^1(\Omega)} \\
&\leq \|f'(c_\varepsilon) - f'(c)\|_{L^2(A_\delta)} |A_\delta|^{\frac{1}{2}} + \int_{\Omega\setminus A_\delta} \left| \int_0^1 f''(sc_\varepsilon + (1-s)c)(c_\varepsilon - c) \; ds \right| \; \mathrm{d}x \\
&\leq \frac{1}{\delta} \|f'(c_\varepsilon) - f'(c)\|_{L^2(\Omega)} \|c_\varepsilon - c\|_{L^2(\Omega)} + C_\delta \|c_\varepsilon - c\|_{L^1(\Omega)} \\
&\leq \frac{1}{\delta} \big(\|f'(c_\varepsilon)\|_{L^2(\Omega)} + \|f'(c)\|_{L^2(\Omega)} \big) \|c_\varepsilon - c\|_{L^2(\Omega)} + C_\delta \|c_\varepsilon - c\|_{L^1(\Omega)} \\
&\leq \left(\frac{C}{\delta} + |\Omega|^{\frac{1}{2}} \, C_\delta \right) \|c_\varepsilon - c\|_{L^2(\Omega)} =: K_\delta \|c_\varepsilon - c\|_{L^2(\Omega)}.
\end{aligned} \tag{7.4.8}$$

Here, we used that $f'(c) \in L^\infty(0,T;L^2(\Omega))$ (see (7.2.3)) and that $f'(c_\varepsilon)$ is bounded in $L^\infty(0,T;L^2(\Omega))$ uniformly with respect to ε (see (7.2.14)).

Step 2: Estimates for the Navier–Stokes equation. From now on, the letter C denotes generic positive constants that may depend only on the choice of Ω, the number δ from (7.4.2), the initial data and the system parameters, but not on ε. The exact value of C may vary throughout this proof.

Testing (7.4.1a) by $A_S^{-1}\tilde{\mathbf{v}}$ and invoking the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\mathbf{v}}\|_\sigma^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^2(\Omega)^{n\times n}}^2 = (\partial_t\tilde{\mathbf{v}}, A_S^{-1}\tilde{\mathbf{v}}),$$

we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\mathbf{v}}\|_\sigma^2 + \int_\Omega [(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon - (\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x + \int_\Omega \nabla\tilde{\mathbf{v}} : \nabla A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x \\ &= \int_\Omega (\mu_\varepsilon\nabla c_\varepsilon - \mu\nabla c)\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x. \end{aligned} \tag{7.4.9}$$

Recalling that $\mathbf{v}$, $\mathbf{v}_\varepsilon$ and $\tilde{\mathbf{v}}$ are divergence-free, the second term on the left-hand side of (7.4.9) can be reformulated as

$$\begin{aligned} &\int_\Omega [(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon - (\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x \\ &= \int_\Omega (\mathbf{v}_\varepsilon\cdot\nabla)\tilde{\mathbf{v}}\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x + \int_\Omega (\tilde{\mathbf{v}}\cdot\nabla)\mathbf{v}\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x \\ &= -\int_\Omega (\mathbf{v}_\varepsilon\otimes\tilde{\mathbf{v}}) : \nabla A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x - \int_\Omega (\tilde{\mathbf{v}}\otimes\mathbf{v}) : \nabla A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x. \end{aligned} \tag{7.4.10}$$

Using the Gagliardo–Nirenberg inequality, Young's inequality and the uniform bound (7.2.15), the first term can be estimated as

$$\begin{aligned} &\left|\int_\Omega (\mathbf{v}_\varepsilon\otimes\tilde{\mathbf{v}}) : \nabla A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x\right| \le \|\mathbf{v}_\varepsilon\|_{L^6(\Omega)^n}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^3(\Omega)^{n\times n}}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n} \\ &\quad \le C\|\mathbf{v}_\varepsilon\|_{H^1_\sigma(\Omega)}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^2(\Omega)^{n\times n}}^{\frac{1}{2}}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^{\frac{3}{2}} \le \frac{1}{16}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + C\|\tilde{\mathbf{v}}\|_\sigma^2. \end{aligned} \tag{7.4.11}$$

Proceeding similarly, we deduce

$$\begin{aligned} &\left|\int_\Omega (\tilde{\mathbf{v}}\cdot\nabla)\mathbf{v}\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x\right| \le \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}\|\mathbf{v}\|_{L^6(\Omega)^n}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^3(\Omega)^{n\times n}} \\ &\quad \le C\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}\|\mathbf{v}\|_{H^1_\sigma(\Omega)}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^2(\Omega)^{n\times n}}^{\frac{1}{2}}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{H^1(\Omega)^{n\times n}}^{\frac{1}{2}} \\ &\quad \le C\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^{\frac{3}{2}}\|\tilde{\mathbf{v}}\|_\sigma^{\frac{1}{2}} \le \frac{1}{16}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + C\|\tilde{\mathbf{v}}\|_\sigma^2. \end{aligned} \tag{7.4.12}$$

Combining (7.4.10)–(7.4.12), we conclude

$$\left|\int_\Omega [(\mathbf{v}_\varepsilon\cdot\nabla)\mathbf{v}_\varepsilon - (\mathbf{v}\cdot\nabla)\mathbf{v}]\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x\right| \le \frac{1}{8}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + C\|\tilde{\mathbf{v}}\|_\sigma^2. \tag{7.4.13}$$

Using integration by parts and recalling once more that $\tilde{\mathbf{v}}$ is divergence-free, we further obtain

$$\int_\Omega \nabla\tilde{\mathbf{v}} : \nabla A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x = \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2. \tag{7.4.14}$$

Via integration by parts, the right-hand side of (7.4.9) can be reformulated as

$$\int_\Omega (\mu_\varepsilon \nabla c_\varepsilon - \mu \nabla c) \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x = -\int_\Omega \nabla \mu_\varepsilon \tilde{c} \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x + \int_\Omega \tilde{\mu} \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x. \tag{7.4.15}$$

Employing the Gagliardo–Nirenberg inequality, Young's inequality and the uniform bound (7.2.15), the first term can be estimated as

$$\begin{aligned}
\left| \int_\Omega \nabla \mu_\varepsilon \tilde{c} \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x \right| &\leq \|\nabla \mu_\varepsilon\|_{L^2(\Omega)^n} \|\tilde{c}\|_{L^2(\Omega)} \|A_S^{-1}\tilde{\mathbf{v}}\|_{L^\infty(\Omega)^n} \\
&\leq C \|\nabla \mu_\varepsilon\|_{L^2(\Omega)^n} \|\tilde{c}\|_{L^2(\Omega)} \|A_S^{-1}\tilde{\mathbf{v}}\|_{H^1(\Omega)^n}^{\frac{1}{2}} \|A_S^{-1}\tilde{\mathbf{v}}\|_{H^2(\Omega)^n}^{\frac{1}{2}} \leq C \|\tilde{c}\|_{L^2(\Omega)} \|\tilde{\mathbf{v}}\|_\sigma^{\frac{1}{2}} \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^{\frac{1}{2}} \\
&\leq \frac{\theta_0}{16} \|\tilde{c}\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + C \|\tilde{\mathbf{v}}\|_\sigma^2 \\
&\leq \frac{\theta_0}{8} \|\tilde{c} - \overline{\tilde{c}}\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + C|\overline{\tilde{c}}|^2 + C \|\tilde{\mathbf{v}}\|_\sigma^2.
\end{aligned} \tag{7.4.16}$$

We point out that this estimate is one of the main reasons for which the solution $(\mathbf{v}_\varepsilon, c_\varepsilon)$ has to be strong since otherwise we would only have $\mu_\varepsilon \in L^2(0,T;H^1(\Omega))$. Let us also note that the dependence on θ_0 is important to apply the nonlocal Ehrling lemma in Step 4 since it allows us to explicitly choose the constant γ as in (3.5.3). By means of (7.4.1d), the second term in (7.4.15) can be expanded as

$$\begin{aligned}
\int_\Omega \tilde{\mu} \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x = &\int_\Omega (\mathcal{L}_\varepsilon c + \Delta c) \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x + \int_\Omega \mathcal{L}_\varepsilon \tilde{c}\, \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x \\
&+ \int_\Omega (f'(c_\varepsilon) - f'(c)) \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x.
\end{aligned} \tag{7.4.17}$$

Using Young's inequality, Agmon's inequality and the uniform estimate (7.2.15), the first summand on the right-hand side can be estimated as

$$\begin{aligned}
&\int_\Omega (\mathcal{L}_\varepsilon c + \Delta c) \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}} \,\mathrm{d}x \\
&\quad\leq \frac{1}{2} \|\mathcal{L}_\varepsilon c + \Delta c\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla c\|_{L^2(\Omega)^n}^2 \|A_S^{-1}\tilde{\mathbf{v}}\|_{L^\infty(\Omega)^n}^2 \\
&\quad\leq \frac{1}{2} \|\mathcal{L}_\varepsilon c + \Delta c\|_{L^2(\Omega)}^2 + C \|\nabla c\|_{L^2(\Omega)^n}^2 \|A_S^{-1}\tilde{\mathbf{v}}\|_{H^1(\Omega)^n} \|A_S^{-1}\tilde{\mathbf{v}}\|_{H^2(\Omega)^n} \\
&\quad\leq \frac{1}{2} \|\mathcal{L}_\varepsilon c + \Delta c\|_{L^2(\Omega)}^2 + C \|\tilde{\mathbf{v}}\|_\sigma \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n} \\
&\quad\leq \frac{1}{2} \|\mathcal{L}_\varepsilon c + \Delta c\|_{L^2(\Omega)}^2 + \frac{1}{8} \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + C \|\tilde{\mathbf{v}}\|_\sigma^2,
\end{aligned} \tag{7.4.18}$$

where we exploited (7.2.3) for the $L^\infty(0,T;H^1(\Omega))$-regularity of μ. To estimate the second summand on the right-hand side of (7.4.17), we use the following Poincaré type inequality, which can be found in [110, Theorem 1]: there exists $C > 0$, such that for all $f \in H^1(\Omega)$,

$$\mathcal{E}_\varepsilon(f) \leq C \|f\|_{H^1(\Omega)}. \tag{7.4.19}$$

We further recall (7.2.5), (7.2.7) and (7.2.15), the definition of $\mathcal{L}_\varepsilon$, and the continuous embeddings $H^3(\Omega) \hookrightarrow W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and $H^1(\Omega)^n \hookrightarrow L^4(\Omega)^n$. With the help of these results, we derive the estimate

$$
\begin{aligned}
&\int_\Omega \mathcal{L}_\varepsilon \tilde{c}\, \nabla c \cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x \\
&\quad \le 2\sqrt{\mathcal{E}_\varepsilon(\tilde{c})}\,\sqrt{\mathcal{E}_\varepsilon(\nabla c\cdot A_S^{-1}\tilde{\mathbf{v}})} \le C\sqrt{\mathcal{E}_\varepsilon(\tilde{c})}\,\|\nabla c\cdot A_S^{-1}\tilde{\mathbf{v}}\|_{H^1(\Omega)} \\
&\quad \le C\sqrt{\mathcal{E}_\varepsilon(\tilde{c})}\Big(\|\nabla c\|_{L^4(\Omega)^n}\|A_S^{-1}\tilde{\mathbf{v}}\|_{L^4(\Omega)^n} + \|D^2c\|_{L^4(\Omega)^{n\times n}}\|A_S^{-1}\tilde{\mathbf{v}}\|_{L^4(\Omega)^n} \\
&\qquad\qquad\qquad + \|\nabla c\|_{L^\infty(\Omega)^n}\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^2(\Omega)^{n\times n}}\Big) \\
&\quad \le C\sqrt{\mathcal{E}_\varepsilon(\tilde{c})}\,\|c\|_{H^3(\Omega)}\|A_S^{-1}\tilde{\mathbf{v}}\|_{H^1(\Omega)^n} \\
&\quad \le \frac{1}{2}\mathcal{E}_\varepsilon(\tilde{c}) + C\|\tilde{\mathbf{v}}\|_\sigma^2 = \frac{1}{2}\mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}}) + C\|\tilde{\mathbf{v}}\|_\sigma^2,
\end{aligned}
\tag{7.4.20}
$$

recalling $\mathcal{E}_\varepsilon(\tilde{c}) = \mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}})$. Here, the first inequality follows by exploiting the properties of the interaction kernel J_ε (cf. [44, p. 128]). Invoking (7.4.8), Agmon's inequality and Young's inequality, the third summand on the right-hand side of (7.4.17) can be bounded via the estimate

$$
\begin{aligned}
&\int_\Omega (f'(c_\varepsilon)-f'(c))\nabla c\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x \\
&\quad \le C\|f'(c_\varepsilon)-f'(c)\|_{L^1(\Omega)}\|\nabla c\|_{L^\infty(\Omega)^n}\|\tilde{\mathbf{v}}\|_\sigma^{\frac{1}{2}}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^{\frac{1}{2}} \\
&\quad \le CK_\delta\|c_\varepsilon - c\|_{L^2(\Omega)}\,\|\nabla A_S^{-1}\tilde{\mathbf{v}}\|_{L^2(\Omega)^{n\times n}}^{\frac{1}{2}}\|A_S^{-1}\tilde{\mathbf{v}}\|_{H^2(\Omega)^n}^{\frac{1}{2}} \\
&\quad \le \frac{1}{8}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + \frac{\theta_0}{16}\|\tilde{c}\|_{L^2(\Omega)}^2 + CK_\delta^4\|\tilde{\mathbf{v}}\|_\sigma^2 \\
&\quad = \frac{1}{8}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + \frac{\theta_0}{2}\|\tilde{c}-\overline{\tilde{c}}\|_{L^2(\Omega)}^2 + C|\overline{\tilde{c}}|^2 + C\|\tilde{\mathbf{v}}\|_\sigma^2.
\end{aligned}
\tag{7.4.21}
$$

In view of (7.4.17),(7.4.18), (7.4.20) and (7.4.21), we thus have

$$
\begin{aligned}
&\left|\int_\Omega \tilde{\mu}\nabla c\cdot A_S^{-1}\tilde{\mathbf{v}}\,\mathrm{d}x\right| \\
&\qquad \le \frac{1}{2}\mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}}) + \frac{1}{4}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 + \frac{\theta_0}{2}\|\tilde{c}-\overline{\tilde{c}}\|_{L^2(\Omega)}^2 + C\|\tilde{\mathbf{v}}\|_\sigma^2 + C|\overline{\tilde{c}}|^2.
\end{aligned}
\tag{7.4.22}
$$

Eventually, combining (7.4.9) with (7.4.13), (7.4.14) and (7.4.22), we conclude

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\mathbf{v}}\|_\sigma^2 + \frac{3}{4}\|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n}^2 &\le C\big(\|\tilde{\mathbf{v}}\|_\sigma^2 + \|\tilde{c}-\overline{\tilde{c}}\|_*^2\big) + \frac{1}{2}\mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}}) \\
&\quad + \frac{\theta_0}{2}\|\tilde{c}-\overline{\tilde{c}}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\mathcal{L}_\varepsilon c + \Delta c\|_{L^2(\Omega)}^2 + C|\overline{\tilde{c}}|^2.
\end{aligned}
\tag{7.4.23}
$$

Step 3: Estimates for the convective Cahn–Hilliard system. Testing (7.4.1c) with $\mathcal{N}(\tilde{c}-\overline{\tilde{c}})$ and using the identity

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}-\overline{\tilde{c}}\|_*^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\|_{L^2(\Omega)^n}^2 = \big(\partial_t\tilde{c}, \mathcal{N}(\tilde{c}-\overline{\tilde{c}})\big),
$$

we derive the equation

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}-\overline{\tilde{c}}\|_*^2 = -\int_\Omega (\mathbf{v}_\varepsilon\cdot\nabla c_\varepsilon - \mathbf{v}\cdot\nabla c)\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x - \int_\Omega \tilde{\mu}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x,
\tag{7.4.24}
$$

Expressing $\tilde{\mu}$ via (7.4.1d), the second term on the right-hand side can be reformulated as

$$-\int_\Omega \tilde{\mu}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x = -\int_\Omega (\mathcal{L}_\varepsilon c_\varepsilon + \Delta c)(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x - \int_\Omega (f'(c_\varepsilon)-f'(c))(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x. \tag{7.4.25}$$

Recalling the definition of f and the condition $F'' \geq \theta$ (see (S1)), we use Young's inequality along with (7.4.8) to obtain

$$\begin{aligned}
&-\int_\Omega (f'(c_\varepsilon)-f'(c))(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x \\
&= -\int_\Omega (F'(c_\varepsilon)-F'(c))\tilde{c}\,\mathrm{d}x + \theta_0\|\tilde{c}\|^2_{L^2(\Omega)} - \int_\Omega (f'(c_\varepsilon)-f'(c))\overline{\tilde{c}}\,\mathrm{d}x \\
&\leq -\theta\|\tilde{c}\|^2_{L^2(\Omega)} + \theta_0\|\tilde{c}\|^2_{L^2(\Omega)} + |\overline{\tilde{c}}|\,\|f'(c_\varepsilon)-f'(c)\|_{L^1(\Omega)} \\
&\leq -\theta\|\tilde{c}\|^2_{L^2(\Omega)} + \frac{9}{8}\theta_0\|\tilde{c}-\overline{\tilde{c}}\|^2_{L^2(\Omega)} + C|\overline{\tilde{c}}|^2.
\end{aligned} \tag{7.4.26}$$

We next use that the kernel J_ε is even (see, e.g., (6.3.11)) to derive the identity

$$\int_\Omega \mathcal{L}_\varepsilon\tilde{c}\,(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x = 2\mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}}). \tag{7.4.27}$$

Using this result, we deduce

$$\begin{aligned}
-\int_\Omega (\mathcal{L}_\varepsilon c_\varepsilon + \Delta c)(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x &= -\int_\Omega (\mathcal{L}_\varepsilon c + \Delta c)(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x - \int_\Omega \mathcal{L}_\varepsilon\tilde{c}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x \\
&\leq \frac{1}{2}\|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} + \frac{\theta_0}{8}\|\tilde{c}-\overline{\tilde{c}}\|^2_{L^2(\Omega)} - 2\mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}}).
\end{aligned} \tag{7.4.28}$$

Combining (7.4.25), (7.4.25) and (7.4.28), we have

$$\begin{aligned}
&-\int_\Omega \tilde{\mu}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x \\
&\leq \frac{5}{4}\theta_0\|\tilde{c}-\overline{\tilde{c}}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} + C|\overline{\tilde{c}}|^2 - 2\mathcal{E}_\varepsilon(\tilde{c}-\overline{\tilde{c}}) - \theta\|\tilde{c}\|^2_{L^2(\Omega)}.
\end{aligned} \tag{7.4.29}$$

Recalling that $\mathbf{v}_\varepsilon$ and $\mathbf{v}$ are divergence-free and using integration by parts, the second summand on the right-hand side of (7.4.24) can be expressed as

$$\begin{aligned}
&\int_\Omega (\mathbf{v}_\varepsilon\cdot\nabla c_\varepsilon - \mathbf{v}\cdot\nabla c)\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x \\
&\quad= -\int_\Omega \mathbf{v}_\varepsilon\tilde{c}\cdot\nabla\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x - \int_\Omega c\,\tilde{\mathbf{v}}\cdot\nabla\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x.
\end{aligned} \tag{7.4.30}$$

Invoking Hölder's inequality, the Gagliardo–Nirengberg inequality, the embedding $H^1_\sigma(\Omega)\hookrightarrow L^4(\Omega)^n$ and the uniform bound (7.2.15), the first term on the right-hand side can be estimated as

$$\begin{aligned}
&\left|\int_\Omega \mathbf{v}_\varepsilon\tilde{c}\cdot\nabla\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\,\mathrm{d}x\right| \leq \|\mathbf{v}_\varepsilon\|_{L^6(\Omega)^n}\|\tilde{c}\|_{L^2(\Omega)}\|\nabla\mathcal{N}(\tilde{c}-\overline{\tilde{c}})\|_{L^3(\Omega)^n} \\
&\leq C\|\mathbf{v}_\varepsilon\|_{H^1_\sigma(\Omega)}\|\tilde{c}-\overline{\tilde{c}}\|^{\frac{3}{2}}_{L^2(\Omega)}\|\tilde{c}-\overline{\tilde{c}}\|^{\frac{1}{2}}_* + C|\overline{\tilde{c}}|\,\|\mathbf{v}_\varepsilon\|_{H^1_\sigma(\Omega)}\|\tilde{c}-\overline{\tilde{c}}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\tilde{c}-\overline{\tilde{c}}\|^{\frac{1}{2}}_* \\
&\leq \frac{\theta_0}{4}\|\tilde{c}-\overline{\tilde{c}}\|^2_{L^2(\Omega)} + C\|\tilde{c}-\overline{\tilde{c}}\|^2_* + C|\overline{\tilde{c}}|^2.
\end{aligned} \tag{7.4.31}$$

Moreover, employing (7.2.15) and Hölder's inequality, we show that the second summand on the right-hand side of (7.4.30) fulfills the estimate

$$\begin{aligned} \left| \int_\Omega c\, \tilde{\mathbf{v}} \cdot \nabla \mathcal{N}(\tilde{c} - \overline{\tilde{c}})\, \mathrm{d}x \right| &\le \|\tilde{\mathbf{v}}\|_{L^2(\Omega)^n} \|c\|_{L^\infty(\Omega)} \|\nabla \mathcal{N}(\tilde{c} - \overline{\tilde{c}})\|_{L^2(\Omega)^n} \\ &\le \frac{1}{4}\|\tilde{\mathbf{v}}\|^2_{L^2(\Omega)^n} + C\|\tilde{c} - \overline{\tilde{c}}\|^2_*. \end{aligned} \tag{7.4.32}$$

Using (7.4.31) and (7.4.32) to estimate the right-hand side of (7.4.30), we infer

$$\begin{aligned} &-\int_\Omega (\mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon - \mathbf{v} \cdot \nabla c) \mathcal{N}(\tilde{c} - \overline{\tilde{c}})\, \mathrm{d}x \\ &\le \frac{\theta_0}{4}\|\tilde{c} - \overline{\tilde{c}}\|^2_{L^2(\Omega)} + \frac{1}{4}\|\tilde{\mathbf{v}}\|^2_{L^2(\Omega)^n} + C|\overline{\tilde{c}}|^2 + C\|\tilde{c} - \overline{\tilde{c}}\|^2_*. \end{aligned} \tag{7.4.33}$$

Eventually, using (7.4.29) and (7.4.33) to bound the right-hand side of (7.4.24), we conclude

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\tilde{c} - \overline{\tilde{c}}\|^2_* + \theta\|\tilde{c}\|^2_{L^2(\Omega)} + 2\mathcal{E}_\varepsilon(\tilde{c} - \overline{\tilde{c}}) \\ &\le \frac{1}{2}\|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} + \frac{1}{4}\|\tilde{\mathbf{v}}\|^2_{L^2(\Omega)^n} + \frac{3}{2}\theta_0\|\tilde{c} - \overline{\tilde{c}}\|^2_{L^2(\Omega)} + C|\overline{\tilde{c}}|^2 + C\|\tilde{c} - \overline{\tilde{c}}\|^2_*. \end{aligned} \tag{7.4.34}$$

Step 4: Completion of the proof. Adding (7.4.22) and (7.4.34) we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\tilde{\mathbf{v}}\|^2_\sigma + \frac{1}{2}\|\tilde{c} - \overline{\tilde{c}}\|^2_*\Big) + \frac{1}{2}\|\tilde{\mathbf{v}}\|^2_{L^2(\Omega)^n} + \theta\|\tilde{c}\|^2_{L^2(\Omega)} + \frac{3}{2}\mathcal{E}_\varepsilon(\tilde{c} - \overline{\tilde{c}}) \\ &\quad \le C\Big(\|\tilde{\mathbf{v}}\|^2_\sigma + \|\tilde{c} - \overline{\tilde{c}}\|^2_*\Big) + \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} + 2\theta_0\|\tilde{c} - \overline{\tilde{c}}\|^2_{L^2(\Omega)} + C|\overline{\tilde{c}}|^2 \end{aligned} \tag{7.4.35}$$

in $[0, T]$. Recalling the definition of ε_s in (7.3.51), applying Lemma 3.5.43.5.3 with $\gamma = \frac{1}{2\theta_0}$ yields

$$2\theta_0\|\tilde{c} - \overline{\tilde{c}}\|^2_{L^2(\Omega)} \le \mathcal{E}_\varepsilon(\tilde{c} - \overline{\tilde{c}}) + C\|\tilde{c} - \overline{\tilde{c}}\|^2_*$$

as we consider $\varepsilon \in (0, \varepsilon_s]$. Plugging this estimate into (7.4.35) and recalling

$$\overline{\tilde{c}}(t) = \overline{\tilde{c}_0} \text{ for all } t \in [0, T] \quad \text{and} \quad \mathcal{E}_\varepsilon(\tilde{c}) = \mathcal{E}_\varepsilon(\tilde{c} - \overline{\tilde{c}}),$$

we infer

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\|\tilde{\mathbf{v}}\|^2_\sigma + \frac{1}{2}\|\tilde{c} - \overline{\tilde{c}}\|^2_*\Big) + \frac{1}{2}\|\tilde{\mathbf{v}}\|^2_{L^2(\Omega)^n} + \theta\|\tilde{c}\|^2_{L^2(\Omega)} + \frac{1}{2}\mathcal{E}_\varepsilon(\tilde{c}) \\ &\quad \le C\Big(\|\tilde{\mathbf{v}}\|^2_\sigma + \|\tilde{c} - \overline{\tilde{c}}\|^2_*\Big) + \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)} + C|\overline{\tilde{c}_0}|^2 \end{aligned} \tag{7.4.36}$$

in $[0, T]$. Thus, Gronwall's lemma implies

$$\begin{aligned} &\frac{1}{2}\sup_{t\in[0,T]}\|\tilde{\mathbf{v}}(t)\|^2_\sigma + \frac{1}{2}\sup_{t\in[0,T]}\|\tilde{c}(t) - \overline{\tilde{c}}(t)\|^2_* \\ &+ \theta\int_0^T \|\tilde{c}\|^2_{L^2(\Omega)}\, \mathrm{d}t + \frac{1}{2}\int_0^T \|\tilde{\mathbf{v}}\|^2_{L^2(\Omega)^n}\, \mathrm{d}t + \frac{1}{2}\int_0^T \mathcal{E}_\varepsilon(\tilde{c})\, \mathrm{d}t \\ &\le \left(\frac{1}{2}\|\tilde{\mathbf{v}}_0\|^2_\sigma + \frac{1}{2}\|\tilde{c}_0\|^2_* + C\int_0^T |\overline{\tilde{c}_0}|^2\, \mathrm{d}t + \int_0^T \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)}\, \mathrm{d}t\right) e^{CT}. \end{aligned} \tag{7.4.37}$$

Now, since $c \in L^\infty(0,T;H^3(\Omega))$ (see (7.2.5) and (7.2.7)), the Theorems 2.1.8 and 2.1.5 yield

$$\int_0^T \|\mathcal{L}_\varepsilon c + \Delta c\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C\varepsilon^{2\alpha}\|c\|^2_{L^2(0,T;H^3(\Omega))} \le C\varepsilon^{2\alpha},$$

where $\alpha = \frac{1}{2}$ in case Ω is a bounded domain and $\alpha = 1$ if $\Omega = \mathbb{T}^n$. Together with assumption (2.5.8), we thus have

$$\begin{aligned}
&\frac{1}{2}\sup_{t\in[0,T]}\|\tilde{\mathbf{v}}(t)\|^2_\sigma + \frac{1}{2}\sup_{t\in[0,T]}\|\tilde{c}(t) - \overline{\tilde{c}}(t)\|^2_*\\
&+ \theta\|\tilde{c}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\|\tilde{\mathbf{v}}\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\int_0^T \mathcal{E}_\varepsilon(\tilde{c})\,\mathrm{d}t \le C\varepsilon^{2\alpha}.
\end{aligned} \tag{7.4.38}$$

As the norms $\|\cdot\|_\sigma$ and $\|\cdot\|_{(H^1_\sigma(\Omega))'}$ on $(H^1_\sigma(\Omega))'$ are equivalent, this also yields

$$\|\tilde{\mathbf{v}}\|_{L^\infty(0,T;(H^1_\sigma(\Omega))')} \le C\sup_{t\in[0,T]}\|\tilde{\mathbf{v}}(t)\|^2_\sigma \le C\varepsilon^\alpha. \tag{7.4.39}$$

Moreover, since the norms $\|\cdot\|_*$ and $\|\cdot\|_{H^1(\Omega)'}$ on $H^{-1}_{(0)}(\Omega)$ are equivalent, we further have

$$\begin{aligned}
\|\tilde{c}\|_{L^\infty(0,T;H^1(\Omega)')} &\le \|\tilde{c} - \overline{\tilde{c}}\|_{L^\infty(0,T;H^1(\Omega)')} + \|\overline{\tilde{c}}\|_{L^\infty(0,T)}\\
&\le \sup_{t\in[0,T]}\|\tilde{c}(t) - \overline{\tilde{c}}(t)\|_* + |\overline{\tilde{c}_0}| \le C\varepsilon^\alpha.
\end{aligned} \tag{7.4.40}$$

Combining (7.4.38)–(7.4.40), we have thus verified estimate (2.5.9). Hence, the proof is complete. $\square$

Chapter 8

Convergence of the nonlocal Allen–Cahn equation to mean curvature flow

8.1. Introduction

We consider the nonlocal Allen–Cahn equation as introduced in Section 1.3 on the torus $\mathbb{T}^n$. The main topic we will address the sharp interface limit, i.e., we will study the situation when the thickness η of the interface tends to zero. We recall that the equation we study reads as:

$$\partial_t c_\eta + \mathcal{L}_\varepsilon c_\eta + \frac{1}{\eta^2} f'(c_\eta) = 0 \qquad \text{in } \mathbb{T}^n \times (0,T), \tag{8.1.1}$$

$$c_\eta(0) = c_{\eta,0} \qquad \text{in } \mathbb{T}^n. \tag{8.1.2}$$

Here, the nonlocal operator $\mathcal{L}\varepsilon$ is given as in (4.1.5), i.e,

$$\mathcal{L}_\varepsilon u(x) := \text{P.V.} \int_\Omega J_\varepsilon(x-y)\big(u(x)-u(y)\big)\,\mathrm{d}y, \quad x \in \Omega, \tag{8.1.3}$$

for a prescribed interaction kernel J_ε as in (4.1.4). The structure of this chapter is the following: In the first section, we give an overview of the sharp interface limit result of the local Allen–Cahn equation. In particular, we will present the method of de Mottoni and Schatzman. The second section is devoted to the well-posedness of the nonlocal Allen–Cahn equation. We will prove the existence of a unique weak solution together with regularity properties and a maximum principle. Then, we derive uniform estimates for the solution. These will be used to show error estimates between the solution to the nonlocal Allen–Cahn equation and the approximate solution to the local Allen–Cahn equation. The last section contains the proof of Theorem 2.6.2. Therein, we show stability estimates, which then imply convergence.

This chapter is based on the following contribution:

[12] H. Abels, C. Hurm, M. Moser, *Convergence of the Nonlocal Allen-Cahn Equation to Mean Curvature Flow*, (2024), arXiv: 2410.08596.

8.2. The local Allen–Cahn equation

In this chapter, we want to explain the basic idea for the sharp interface limit of the local Allen–Cahn equation based on *the Method of de Mottoni and Schatzman* [46]. This will be a substantial ingredient for the proof of Theorem 2.6.2. In particular, de Mottoni and Schatzman considered the Allen–Cahn equation on $\mathbb{R}^n$ and showed convergence of the Allen–Cahn equation towards mean curvature flow as long as there is a smooth solution of the mean curvature flow equation. For a more detailed overview, we refer to [102].

The main idea of de Mottoni and Schatzman is based on matched asymptotic expansions. To this end, we first give a basic explanation for this ansatz, see also [47, 102] and the references therein. Typically, one considers an equation, which is singularly perturbated in terms of a small parameter η. This means that the equation changes its type fundamentally when the parameter is set to zero. For instance, this could lead to a change in the order of differentiation. In this situation, there usually exist different regions in the domain of definition (which are often denoted by inner and outer regions), where the solutions have different qualitative properties, which overlap in some transition regions. Then, the solution is separately expanded into η-series in the inner and outer regions, respectively. Let us note that one has to introduce a change of variables depending on η in one of the regions (typically the inner region). This demonstrates the difference between these regions. Moreover, also the equations need to be expanded into η-series. Afterwards, the expansions have to be put together suitably. To this end, one needs to impose a "matching condition". Eventually, for each order of η, one can derive conditions for the coefficients in the η-series, which can be used for the construction of the coefficients in the series and then give an approximation for the exact solution. If the construction of the coefficients is rigorous and the approximation error can be estimated, the asymptotic expansion is called "*rigorous*".

Let us note that in our situation, the diffuse interface model can be seen as singularly perturbed in terms of the small parameter η, which typically appears in the equation. In this context, the term "inner region" refers to the part of the bulk domain that is close to the diffuse interface. Accordingly, the parts of the bulk domain that are far away from the interface are referred to as the "outer region".

The method of de Mottoni and Schatzman follows the following strategy. First, let us assume that there exists a local smooth solution to the limit sharp interface problem. Then, the method consists of the following steps:

1. An approximate solution to the diffuse interface model is rigorously constructed by means of matched asymptotic expansions. These expansions are based on the evolving surface, which is part of the solution to the limit problem. In this step, one also needs to solve model problems, in order to construct the coefficients of the series.

2. Suitable estimates between the exact and approximate solution are derived. This will be done using a spectral estimate of the linearized Allen–Cahn operator.

Besides the Allen–Cahn equation, the method by de Mottoni and Schatzman has already been applied to many other phase-field models. These are based on more general spectral estimates. For instance, there are results on the Cahn–Hilliard equation by Alikakos, Bates, Chen [19], the mass-conserving Allen–Cahn equation by Chen, Hilhorst, Logak [33], a Navier–Stokes/Allen–Cahn system by Abels, Fischer and Moser [6] and a Stokes/Cahn–Hilliard system by Marquardt [97]. For more results that rely on this method, we refer the reader to [102].

In the subsequent sections, we explain the two steps from above in more detail. In the first part, we explain the construction of the approximate solution for the Allen–Cahn equation. In the second part, we then focus on the spectral estimate of the linearized Allen–Cahn operator.

First of all, we introduce some notation, which will be used throughout this chapter.

8.2.1. Preliminaries: Smooth evolving hypersurfaces

We identify $\mathbb{T}^n \cong [-\pi,\pi]^n$. Let $T_0 > 0$,let $\Sigma \subset \mathbb{T}^n$ be a smooth, connected, compact and orientable hypersurface, and let $X_0 : \Sigma \times [0,T_0] \to \mathbb{T}^n$ be a smooth parametrization such that $X_0(.,t)$ is an injective immersion for all $t \in [0,T_0]$. Since $X_0(.,t)$ is an injective continuous map from a compact space to a Hausdorff space, it follows that $X_0(.,t)$ is a topological embedding for all $t \in [0,T_0]$. In addition, since the map $X_0(.,t)$ is an injective immersion, $X_0(.,t)$ is a smooth embedding for all $t \in [0,T_0]$, cf. [93, Appendix A]. Hence, $\Gamma_t := X_0(\Sigma,t) \subset \mathbb{T}^n$ is a smooth, orientable, compact and connected hypersurface for all $t \in [0,T_0]$. This means that

$$\Gamma := \bigcup_{t\in[0,T_0]} \Gamma_t \times \{t\}$$

is a smooth evolving hypersurface and the map

$$\overline{X}_0 := (X_0, \mathrm{pr}_t) : \Sigma \times [0,T_0] \to \Gamma, \quad (s,t) \mapsto (X_0(s,t),t)$$

is a smooth diffeomorphism. In this regard, Γ_t separates $\mathbb{T}^n$ into two disjoint connected domains $\Omega_t^\pm$ for all $t \in [0,T_0]$. Next, we choose a smooth normal field $\vec{n} : \Sigma \times [0,T_0] \to \mathbb{T}^n$, i.e., the map $\vec{n}$ is smooth and $\vec{n}_{\Gamma_t} := \vec{n}(.,t)$ describes the exterior normal of Γ_t at $X_0(s,t)$ for all $t \in [0,T_0]$ in the sense that $\vec{n}(.,t)$ points into Ω_t^+, and we set

$$\Omega^\pm := \bigcup_{t\in[0,T_0]} \Omega_t^\pm \times \{t\}.$$

Roughly speaking, we consider a smooth evolving hypersurface in the cell $[-\pi,\pi]^n$. Then, we extend this cell periodically as shown in Figure 8.1 below.

Moreover,

$$V(s,t) := V_{\Gamma_t}(s) := \partial_t X_0(s,t) \cdot \vec{n}(s,t) \quad \text{for } (s,t) \in \Sigma \times [0,T_0]$$

is the *normal velocity* and $H(s,t) := H_{\Gamma_t}(s)$ for $(s,t) \in \Sigma \times [0,T_0]$ is the *mean curvature* (w.r.t. $\vec{n}_{\Gamma_t}$), i.e., the sum of the principal curvatures.

In the following, let

$$X : [-a_0,a_0] \times \Sigma \times [0,T_0] \to \mathbb{T}^n : (r,s,t) \mapsto X_0(s,t) + r\,\vec{n}(s,t)$$

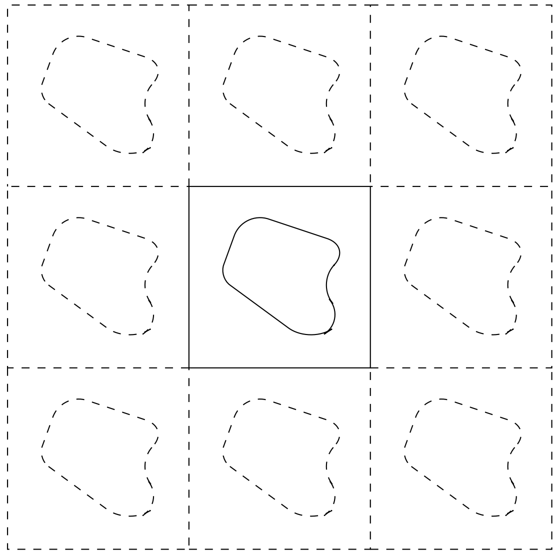

Figure 8.1: Evolving hypersurface in $\mathbb{T}^n$.

for $a_0 > 0$ small be the standard tubular neighbourhood coordinate system, where for all

$$x \in \Gamma_t(a_0) := \{x \in \mathbb{T}^n : \operatorname{dist}(x, \Gamma_t) < a_0\}$$

we define

$$r := \operatorname{sdist}(\Gamma_t, x), \quad s := X_0^{-1}(P_{\Gamma_t}(x), t) = S(x,t).$$

Here, $\Gamma_t(a_0)$ is the *tubular neighbourhood* of Γ_t of width a_0, P_{Γ_t} denotes the *orthogonal projection* onto Γ_t and

$$d_\Gamma(x,t) := \operatorname{sdist}(\Gamma_t, x) := \begin{cases} \operatorname{dist}(\Omega_t^-, x) & \text{if } x \notin \Omega_t^-, \\ -\operatorname{dist}(\Omega_t^+, x) & \text{if } x \in \Omega_t^- \end{cases}$$

denotes the *signed distance*. For the following, we define for $a \in (0, a_0]$

$$\Gamma(a) := \bigcup_{t \in [0,T_0]} \Gamma_t(a) \times \{t\}.$$

Since Γ is of class C^2, it is well-known (see [111, Chapter 2.3]) that there exists a number $a_0 > 0$ such that for all $a \in (0, a_0]$, the mapping $\overline{X} := (X, \operatorname{pr}_t)$ is a smooth diffeomorphism onto the neighbourhood $\overline{\Gamma(a)}$ of Γ. We point out that the functions above satisfy the following identities:

$$\nabla d_\Gamma(x,t) = \vec{n}_{\Gamma_t}(P_{\Gamma_t}(x)), \quad \partial_t d_\Gamma(x,t) = -V_{\Gamma_t}(P_{\Gamma_t}(x)), \quad \Delta d_\Gamma(q,t) = -H_{\Gamma_t}(q)$$

for all $(x,t) \in \Gamma(a_0)$ and $(q,t) \in \Gamma$, respectively. This follows by differentiating the identity

$$d_\Gamma(X_0(s,t) + r\,\vec{n}(s,t), t) = r,$$

see [13] for details. Moreover, the components of the inverse of $\overline{X}$ can be expressed as

$$(r, s, \operatorname{pr}_t) := \overline{X}^{-1} : \overline{\Gamma(a)} \to [-a, a] \times \Sigma \times [0, T_0].$$

Let $a \in (0, a_0]$. Then, for sufficiently smooth $f : \Gamma(a) \to \mathbb{R}$, we define the *tangential* and *normal derivative* by

$$\nabla_\tau f := (D_x s)^\top \left[\nabla_\Sigma (f \circ \overline{X}) \circ \overline{X}^{-1}\right] \quad \text{and} \quad \partial_n f := \partial_r (f \circ \overline{X}) \circ \overline{X}^{-1},$$

respectively. For further details about evolving hypersurfaces, we refer to [13,93,102,111].

8.2.2. Approximate solution for the Allen–Cahn equation

In the following, we use the same notation as in Section 8.2.1. Moreover, let $a > 0$ be such that $\overline{X} := (X, \mathrm{pr}_t)$ is a smooth diffeomorphism onto the neighbourhood $\overline{\Gamma(2a)}$ of Γ and let $(r, s) : \overline{\Gamma(2a)} \to [-2a, 2a] \times \Sigma$ be the coordinates that describe the tubular neighbourhood $\overline{\Gamma(2a)}$ of Γ. Furthermore, we assume that Γ evolves according to mean curvature flow, i.e.,

$$V_{\Gamma_t} = H_{\Gamma_t} \quad \text{on } \Gamma_t.$$

We follow and explain the ideas in [102, Sections 5.1 and 5.2]. Based on the hypersurface Γ, we intend to construct an approximate solution c_η^A to (1.2.6a)–(1.2.6c) with the properties that $c_\eta^A = \pm 1$ in $\mathbb{T}^n \times (0, T_0) \setminus \Gamma(2a)$, c_η^A rapidly (but smoothly) changes its value from -1 to 1 where the slope of c_η^A increases as $\eta \to 0$ and the zero-level set $\{c_\eta^A = 0\}$ approximates Γ as $\eta \to 0$ (in the sense that the maximal distance goes to zero).

The construction of the approximate solution c_η^A is based on an inner and outer expansion of the Allen–Cahn equation. It is well-known that in the outer expansion one obtains the values ± 1 for the 0-th order coefficient in the η-series and 0 for all the remaining coefficients. Therefore, we only focus on the inner expansion. Let $L \in \mathbb{N}$ with $L \geq 2$ correspond to the number of terms in the expansion. For every $j = 1, \ldots, L$, we introduce height functions

$$h_j : \Sigma \times [0, T_0] \to \mathbb{R} \text{ and } h_\eta := \sum_{j=1}^{L} \eta^{j-1} h_j$$

and the stretched variable

$$\hat{\rho}_\eta := \frac{r(x,t) - \eta h_\eta(s(x,t), t)}{\eta}.$$

Then, we use the following ansatz for the inner expansion: Let $\eta > 0$ be small and let

$$c_\eta^I := \sum_{j=0}^{L+1} \eta^j c_j^I, \quad c_j^I(x,t) := \hat{c}_j^I(\hat{\rho}_\eta(x,t), s(x,t), t) \tag{8.2.1}$$

for all $(x,t) \in \overline{\Gamma(2a)}$. Here,

$$\hat{c}_j^I : \mathbb{R} \times \Sigma \times [0, T_0] \to \mathbb{R}, \quad (\hat{\rho}, s, t) \mapsto \hat{c}_j^I(\hat{\rho}, s, t)$$

for all $j = 0, \ldots, L+1$ and we define $\hat{c}_\eta^I := \sum_{j=0}^{L+1} \eta^j \hat{c}_j^I$. Next, we insert the ansatz (8.2.1) into the Allen–Cahn equation (1.2.6a) and expands it into η-series (with coefficients in $(\hat{\rho}_\eta, s, t)$). Note that one only needs to insert terms up to order $\mathcal{O}(\eta^{L-1})$ in order to get an error of order $\mathcal{O}(\eta^L)$. Then, collecting terms of the same order, we obtain ordinary differential equations in $\hat{\rho}$ for the coefficient functions $\hat{c}_j^I$, $j = 0, \ldots, L+1$. At the lowest order, i.e., equating terms of order $\mathcal{O}(\eta^{-2})$, this yields

$$-\partial_{\hat{\rho}}^2 \hat{c}_0^I(\hat{\rho}, s, t) + f'(\hat{c}_0^I(\hat{\rho}, s, t)) = 0. \tag{8.2.2}$$

As already mentioned in the beginning of this chapter, we need to impose a matching condition to relate the inner expansion and the outer expansion. In this situation, the condition reads as

$$\lim_{\hat{\rho} \to \pm\infty} \hat{c}_0^I(\hat{\rho}, s, t) = \pm 1, \tag{8.2.3}$$

$$\lim_{\hat{\rho} \to \pm\infty} \hat{c}_j^I(\hat{\rho}, s, t) = 0 \quad \text{for all } j \geq 1. \tag{8.2.4}$$

Moreover, we want the zero-level set of $c_\eta^A = \hat{c}_0^I(\hat{\rho}, s, t) + \mathcal{O}(\eta)$ to approximate Γ sufficiently well. To this end, we additionally need to impose the condition

$$\hat{c}_0^I(0, s, t) = 0. \tag{8.2.5}$$

Hence, at the lowest order, we have to solve the ordinary differential equation (8.2.2)–(8.2.5), in order to construct the coefficient function $\hat{c}_0^I$. This has already been discussed in [102, 114].

Remark 8.2.1. For functions f as in (A3), there exists a unique solution $\vartheta_0 \in C^2(\mathbb{R})$ to the ordinary differential equation

$$-\vartheta_0'' + f'(\vartheta_0) = 0 \quad \text{in } \mathbb{R}, \quad \lim_{\hat{\rho} \to \pm\infty} \vartheta_0(\hat{\rho}) = \pm 1, \quad \vartheta_0(0) = 0. \tag{8.2.6}$$

The solution ϑ_0 is also referred to as the *optimal profile*.

Owing to Remark 8.2.1, we will take the optimal profile ϑ_0 for the coefficient of lowest order $\hat{c}_0^I$. Finally, let us also mention the equation of second lowest order, i.e., terms of order $\mathcal{O}(\eta^{-1})$. Here, we obtain

$$-\partial_{\hat{\rho}}^2 \hat{c}_1^I(\hat{\rho}, s, t) + f''(\vartheta_0(\hat{\rho}))\, \hat{c}_1^I(\hat{\rho}, s, t) = \vartheta_0'(\hat{\rho})(V_{\Gamma_t} - H_{\Gamma_t}). \tag{8.2.7}$$

It follows by [114, Lemma 2.6.2] that there exists a unique bounded solution to (8.2.7) if and only if

$$V_{\Gamma_t} = H_{\Gamma_t} \quad \text{on } \Gamma_t.$$

The latter is fulfilled, since we assume that Γ is evolved by mean curvature flow. Indeed, this also shows that evolving Γ by the mean curvature flow is necessary for the construction of the coefficient functions.

Now, we are in a position to define the approximate solution c_η^A. Let

$$c_\eta^I(x, t) := \vartheta_0(\hat{\rho}_\eta(x, t)) + \sum_{k=2}^{L+1} \eta^j c_{j+1}(\hat{\rho}_\eta(x, t), s(x, t), t) \quad \text{in } \Gamma(a)$$

be the function constructed in the inner expansion. Moreover, let $\varphi : \mathbb{R} \to [0,1]$ be a smooth cutoff-function with $\varphi(r) = 1$ for $|r| \leq 1$, $\varphi(r) = 0$ for $|r| \geq 2$. Then, we set

$$c_\eta^A := \begin{cases} \varphi(\frac{r}{a})c_\eta^I + (1 - \varphi(\frac{r}{a}))\mathrm{sign}(r) & \text{in } \Gamma(2a), \\ \pm 1 & \text{in } \mathbb{T}^n \times (0, T_0) \setminus \Gamma(2a). \end{cases}$$

This construction yields an approximate solution in the sense that the following error estimates hold:

Theorem 8.2.2. *The function c_η^A is smooth, uniformly bounded for small η and for $r_\eta^A := \partial_t c_\eta^A - \Delta c_\eta^A + \frac{1}{\eta^2} f'(c_\eta^A)$, the remainder of c_η^A in the local Allen–Cahn equation* (1.2.6a)*, it holds*

$$\begin{aligned} \left| r_\eta^A \right| &\leq C(\eta^L e^{-c|\hat{\rho}_\eta|} + \eta^{L+1}) && \text{in } \Gamma(a_0), \\ r_\eta^A &= 0 && \text{in } \mathbb{T}^n \times (0, T_0) \setminus \Gamma(a_0), \end{aligned}$$

where $c, C > 0$ are some constants and $\eta > 0$ is small.

Proof. This essentially follows from the computations in Chen, Hilhorst, Logak [33] and Abels, Liu [13], see also the expansions in Moser [101]. □

8.2.3. Spectral estimate for the local Allen–Cahn operator

In the second step in the method of de Mottoni and Schatzman, we want to derive error estimates between the exact c^A and approximate solution c_η^A. This will be done with the help of suitable estimates for the linearized Allen–Cahn operator. Basically, the idea is to linearize the Allen–Cahn equation around the approximate solution c_η^A, since its structure is well-known from the chapter before. This linearization is then represented by the *linearized Allen–Cahn operator* given by

$$\mathcal{A}_\eta u := -\Delta u + \frac{1}{\eta^2} f''(c^A)u \quad \text{for all } u \in H^2(\mathbb{T}^n). \tag{8.2.8}$$

It is well-known that this operator is self-adjoint. Moreover, the spectrum of $\mathcal{L}_\eta$ lies in $\mathbb{R}$ and is bounded from below. The latter is a consequence from the following theorem:

Theorem 8.2.3 (**Spectral Estimate for Allen-Cahn Operator**)**.** *Let $a \in (0, \frac{a_0}{2}]$ and $h_\eta : \Sigma \times [0, T_0] \to \mathbb{R}$ be such that $h_\eta(., t)$ is continuous for all $t \in [0, T_0]$ and $\|h_\eta\|_\infty \leq C_0$, where $\eta > 0$ is small and $C_0 > 0$ is a fixed constant. Moreover, we define the scaled variable*

$$\hat{\rho}_\eta := \frac{r - \eta h_\eta(s,t)}{\eta} \qquad \text{in } \overline{\Gamma(2a)}.$$

For $\eta > 0$ small, we consider $c_\eta^A : \mathbb{T}^n \times [0, T_0] \to \mathbb{R}$ measurable with the property

$$c_\eta^A = \begin{cases} \vartheta_0(\hat{\rho}_\eta) + \mathcal{O}(\eta^2) & \text{in } \Gamma(a), \\ \pm 1 + \mathcal{O}(\eta) & \text{in } [\mathbb{T}^n \times (0, T_0)] \setminus \Gamma(a), \end{cases}$$

where ϑ_0 is the optimal profile from (8.2.6)*.*

Then there exist $\eta_0, \overline{C} > 0$ *independent of* h_η *for fixed* C_0 *such that for all* $\eta \in (0, \eta_0]$, $t \in [0, T_0]$ *and all* $f \in H^1(\mathbb{T}^n)$ *it holds*

$$\int_{\mathbb{T}^n} |\nabla f|^2 + \frac{1}{\eta^2} f''(c_\eta^A(.,t)) f^2 \, \mathrm{d}x \geq -\overline{C} \|f\|^2_{L^2(\mathbb{T}^n)} + \|\nabla f\|^2_{L^2(\mathbb{T}^n \setminus \Gamma_t(a))} + \|\nabla_\tau f\|^2_{L^2(\Gamma_t(a))}.$$

Proof. This goes back to Chen [32] and Abels, Liu [13, Theorem 2.13]. □

With the help of this spectral estimate, we can now control the error between the exact and approximate solution. Observe that the difference $u := c_\eta^A - c^A$ fulfills the equation

$$\partial_t u + \mathcal{A}_\eta u = \frac{1}{\eta^2} \mathcal{N}(c_\eta^A, c^A) + r_\eta.$$

Testing this equation by u, we can estimate all the terms in a suitable way. Finally, a Gronwall-type argument concludes the proof.

8.3. The nonlocal Allen–Cahn equation

This chapter is devoted to the well-posedness of the nonlocal Allen–Cahn equation. For this purpose, we first address the uniqueness of weak solutions. Afterwards, we then prove the existence of a weak solution together with regularity properties as well as a maximum principle.

Lemma 8.3.1 (Uniqueness). *Let the assumptions* (A1)-(A3) *hold, let* $T > 0$ *and* $\varepsilon > 0$. *For initial values in* $H^1(\mathbb{T}^n)$, *solutions of* (8.1.1)-(8.1.2) *with the regularity*

$$c \in H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^1(\mathbb{T}^n))$$

are unique.

Proof. Let $c_0 \in H^1(\mathbb{T}^n)$ and $c_1, c_2 \in H^1(0, T; L^2(\mathbb{T}^n)) \cap L^\infty(0, T; H^1(\mathbb{T}^n))$ be weak solutions of

$$\begin{aligned} \partial_t c + \mathcal{L}_\varepsilon c + f'(c) &= 0 && \text{in } \mathbb{T}^n \times (0, T), \\ c(0) &= c_0 && \text{in } \mathbb{T}^n. \end{aligned}$$

Then, the difference $\tilde{c} := c_1 - c_2$ solves

$$\begin{aligned} \partial_t \tilde{c} + \mathcal{L}_\varepsilon \tilde{c} + f'(c_1) - f'(c_2) &= 0 && \text{in } \mathbb{T}^n \times (0, T), \\ \tilde{c}(0) &= 0 && \text{in } \mathbb{T}^n \end{aligned}$$

in a weak sense. Testing the latter by $\tilde{c}$ gives

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{c}\|^2_{L^2(\mathbb{T}^n)} + 2\mathcal{E}_\varepsilon(\tilde{c}) + \int_{\mathbb{T}^n} (f'(c_1) - f'(c_2)) \tilde{c} \, \mathrm{d}x = 0,$$

where we note that $f'(c_i) \in L^2(0,T;L^2(\mathbb{T}^n))$ for $i = 1, 2$, because of (A3) and Sobolev embeddings. Moreover, the growth assumption of f from (A3) entails

$$\int_{\mathbb{T}^n} (f'(c_1) - f'(c_2))\tilde{c}\,\mathrm{d}x \geq -\alpha\|\tilde{c}\|^2_{L^2(\mathbb{T}^n)}.$$

Therefore, it holds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{c}\|^2_{L^2(\mathbb{T}^n)} + 2\mathcal{E}_\varepsilon(\tilde{c}) \leq \alpha\|\tilde{c}\|^2_{L^2(\mathbb{T}^n)}$$

and hence the Gronwall Lemma yields $c_1 = c_2$. □

In the following, we will prove the existence of a weak solution to the nonlocal Allen–Cahn equation. This will be done using a Galerkin ansatz. Moreover, we will study regularity properties of the nonlocal Allen–Cahn equation. More precisely, we will show that the solution belongs to $H^3(\mathbb{T}^n)$ for almost all times. This will be crucial for the proof of the main result, since we want to employ the convergence result in Section 2.1 for the solution to the nonlocal Allen–Cahn equation. However, we require that the nonlocal operator commutes with derivatives, in order to prove higher regularity. Therefore, we only consider the case $\Omega = \mathbb{T}^n$. Indeed, we have

Theorem 8.3.2 (**Existence, Higher Regularity and Boundedness**). *Let the assumptions* (A1)-(A3) *hold, $T > 0$ and $\eta > 0$. Moreover, let $c_{\eta,0} \in H^1(\mathbb{T}^n)$. Then, there exists a unique solution c_η to* (8.1.1)–(8.1.2) *with the regularity*

$$c_\eta \in H^1(0,T;L^2(\mathbb{T}^n)) \cap L^\infty(0,T;H^1(\mathbb{T}^n)) \cap C^0([0,T],L^p(\mathbb{T}^n))$$

for all $p \in [1,\infty)$ if $n = 2$ and $p \in [1, 6)$ if $n = 3$. Additionally, if $c_{\eta,0} \in H^3(\mathbb{T}^n)$ and $|c_{\eta,0}| \leq R_0$ with $R_0 \geq 1$ as in (A3)*, then the solution c_η has the regularity*

$$c_\eta \in H^1(0,T;L^2(\mathbb{T}^n)) \cap L^\infty(0,T;H^3(\mathbb{T}^n)) \cap C^1([0,T],C^0(\mathbb{T}^n)) \cap C^0([0,T],H^2(\mathbb{T}^n))$$

and $|c_\eta|$ is uniformly bounded by R_0.

Proof. For simplicity we set $\eta = 1$ in the proof and omit the index η in the notation. All the arguments can be done analogously for the case of arbitrary $\eta > 0$. Moreover, $\varepsilon > 0$ is fixed in the following.

We prove this theorem using a Galerkin ansatz, in particular we project the equations and solution spaces onto finite dimensional subspaces of $H^1(\mathbb{T}^n)$. For the approximation scheme, we consider the finite-dimensional subspaces

$$V_N := \operatorname{span}\{1, \cos(k\cdot .), \sin(k\cdot .) : k = (k_1,\ldots,k_n) \in \mathbb{Z}^n, |k_1|,\ldots,|k_n| \leq N\}, \quad N \in \mathbb{N},$$

generated by trigonometric functions. One can directly show that the Laplacian is diagonizable on V_N and we denote the eigenfunctions by $w_1, ..., w_{\dim V_N}$. Since $V_{N_1} \subseteq V_{N_2}$ for $N_1 \leq N_2$ this is consistent for all $N \in \mathbb{N}$ and it is well-known that $(w_i)_{i\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{T}^n)$. We denote with $(\lambda_i)_{i\in\mathbb{N}}$ the corresponding eigenvalues. Moreover, note that we have the following representation of the L^2-projection for (real-valued) $f \in L^2(\mathbb{T}^n)$ and all $N \in \mathbb{N}$:

$$P_N f := P^{L^2}_{V_N} f = S_N f = D_N * f, \tag{8.3.1}$$

where

$$S_N f := \frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n, |k_1|,\dots,|k_N|\le N} \hat{f}_k e^{ik\cdot\cdot} \quad \text{and} \quad D_N := \frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n, |k_1|,\dots,|k_N|\le N} e^{ik\cdot\cdot}$$

for all $N \in \mathbb{N}$ and $f \in L^1(\mathbb{T}^n)$. Boundedness and convergence of S_N are well-known from Fourier-Analysis, cf. [82]. Recalling that convolution operators commute, we observe that $\mathcal{L}_\varepsilon P_N u = P_N \mathcal{L}_\varepsilon u$ for all $u \in L^2(\mathbb{T}^n)$. Moreover, we note that $\sup_{N\in\mathbb{N}} \|P_N\|_{\mathcal{L}(L^p(\mathbb{T}^n))} < \infty$, cf. [82, Theorem 4.1.8]. In particular, $\operatorname{span}\{w_i : i \in \mathbb{N}\}$ is dense in $H^k(\mathbb{T}^n)$ for all $k \in \mathbb{N}$. Finally, for vector-valued $f \in L^2(\mathbb{T}^n)^m, m \in \mathbb{N}$ equation (8.3.1) is understood componentwise.

Step 1: Finite dimensional approximation. In this step, we consider approximate solutions c^N of (8.1.1)–(8.1.2) of the form

$$c^N = \sum_{i=1}^N g_i^N w_i,$$

where $\mathbf{g}^N := (g_i^N)_{i=1}^N : [0,T] \to \mathbb{R}$ is assumed to be continuously differentiable. In particular, it holds $c^N(t) \in V_N$ for all $t \in [0,T]$. We will construct $\mathbf{g}^N$ such that c^N is a solution of system

$$\int_{\mathbb{T}^n} \partial_t c^N(t) w_i \,\mathrm{d}x + \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon c^N(t) w_i \,\mathrm{d}x + \int_{\mathbb{T}^n} f'(c^N(t)) w_i \,\mathrm{d}x = 0 \tag{8.3.2}$$

for all $t \in [0,T]$ and $i = 1, \dots, N$ together with the initial condition

$$c^N(0) = P_N c_0 = \sum_{i=1}^N (c_0, w_i)_{L^2(\mathbb{T}^n)} w_i. \tag{8.3.3}$$

Since the functions $(w_i)_{i\in\mathbb{N}}$ are an orthonormal basis of $L^2(\mathbb{T}^n)$, the approximate system (8.3.2)-(8.3.3) is equivalent to the following ordinary differential equation for $\mathbf{g}^N$:

$$\frac{\mathrm{d}}{\mathrm{d}t} g_i^N(t) = -\sum_{k=1}^N g_k^N(t) \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon w_k w_i \,\mathrm{d}x - \int_{\mathbb{T}^n} f'\left(\sum_{k=1}^N g_k^N(t) w_k\right) w_i \,\mathrm{d}x \tag{8.3.4}$$

together with the initial condition

$$g_i^N(0) = (c_0, w_i)_{L^2} \tag{8.3.5}$$

for all $i = 1, \dots, N$. Since the right-hand side in (8.3.4) depends continuously on $\mathbf{g}^N$, Peano's Theorem guarantees the existence of a local C^1-solution of this initial value problem on a right-maximal interval $[0, T_N^*) \cap [0,T]$ for some $T_N^* > 0$. In the next two steps, we will derive an energy estimate and show global existence of the Galerkin approximation. Therefore we show uniform bounds on $[0, T_N]$ for $T_N \in [0, T_N^*) \cap [0,T]$ arbitrary.

Step 2: Energy estimate. First of all, we define

$$E^N(t) := \frac{1}{4} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\varepsilon(x-y) \left|c^N(x,t) - c^N(y,t)\right|^2 \mathrm{d}y \,\mathrm{d}x + \int_{\mathbb{T}^n} f(c^N(t)) \,\mathrm{d}x$$

for all $t \in [0, T_N]$. Then, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t} E^N(t) = \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\varepsilon(x-y)(c^N(x,t) - c^N(y,t))(\partial_t c^N(x,t) - \partial_t c^N(y,t)) \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathbb{T}^n} f'(c^N(t)) \partial_t c^N(t) \,\mathrm{d}x = - \int_{\mathbb{T}^n} |\partial_t c^N(t)|^2 \,\mathrm{d}x \leq 0,$$

where we used that $g_1^N, \ldots, g_N^N$ are C^1-functions and for the second equality we multiplied (8.3.2) by $(g_i^N)'(t)$ and summed over all $i = 1, \ldots, N$. Integrating the inequality above with respect to time yields for all $s \in [0, T_N]$

$$E^N(s) + \int_0^s \int_{\mathbb{T}^n} \left| \partial_t c^N(x,t) \right|^2 \,\mathrm{d}x \,\mathrm{d}t \leq E^N(0) \tag{8.3.6}$$

for all $N \in \mathbb{N}$. In order to establish uniform bounds on our solutions, we need to verify that $E^N(0)$ can be bounded by some constant $C > 0$, which does not depend on N.

First of all, because of $c_0 \in H^1(\mathbb{T}^n)$, the initial condition (8.3.3) and the convolution representation in (8.3.1), it follows that the sequence $(c^N(0))_{N \in \mathbb{N}} \subseteq H^1(\mathbb{T}^n)$ is bounded. By [24, Theorem 1], there exists a constant $C > 0$ such that

$$\frac{1}{4} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\varepsilon(x-y) |c^N(x,0) - c^N(y,0)|^2 \,\mathrm{d}y \,\mathrm{d}x = \mathcal{E}_\varepsilon(c^N(0)) \leq C \|\nabla c^N(0)\|^2_{L^2(\mathbb{T}^n)^n}.$$

Moreover, the assumptions on f from (A3) and the Sobolev embedding theorem imply that

$$\int_{\mathbb{T}^n} f(c^N(0)) \,\mathrm{d}x \leq C \left(1 + \|c^N(0)\|^4_{L^4(\mathbb{T}^n)} \right)$$

is bounded uniformly in $N \in \mathbb{N}$. Finally, we obtain the following energy estimate

$$E^N(s) + \int_0^s \int_{\mathbb{T}^n} |\partial_t c^N(x,t)|^2 \,\mathrm{d}x\mathrm{d}t \leq E^N(0) \leq C \tag{8.3.7}$$

for all $s \in [0, T_N]$ and all $N \in \mathbb{N}$, where C is independent of $T_N \in [0, T_N^*) \cap [0, T]$ and $N \in \mathbb{N}$.

Step 3: Uniform estimates. From the energy inequality (8.3.7) we obtain the uniform estimate $\|\partial_t c^N\|_{L^2(0,T_N;L^2(\mathbb{T}^n))} \leq C$. In the next step we prove that the sequence $(c^N)_{N \in \mathbb{N}}$ is bounded in $L^\infty(0, T_N; H^1(\mathbb{T}^n))$. To this end, we first test (8.3.2) by Δc^N. More precisely, we multiply (8.3.2) by $\lambda_i g_i^N(t)$ and sum over all $i = 1, \ldots, N$. This yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla c^N\|^2_{L^2(\mathbb{T}^n)^n} + \frac{1}{2} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} J_\varepsilon(x-y) |\nabla c^N(x,t) - \nabla c^N(y,t)|^2 \,\mathrm{d}y \,\mathrm{d}x + \int_{\mathbb{T}^n} \nabla f'(c^N) \cdot \nabla c^N \,\mathrm{d}x = 0.$$

Due to the assumption on f'' from (A3), it holds

$$\int_{\mathbb{T}^n} \nabla f'(c^N) \cdot \nabla c^N \,\mathrm{d}x \geq -\alpha \|\nabla c^N\|^2_{L^2(\mathbb{T}^n)^n}.$$

Thus, we can apply Gronwall's inequality to conclude that $\left(\nabla c^N\right)_{N\in\mathbb{N}} \subseteq L^\infty(0,T_N;L^2(\mathbb{T}^n)^n)$ is bounded. Next, we test (8.3.2) by c^N. On the Galerkin level, we multiply (8.3.2) by $g_i^N(t)$ and sum over all $i=1,\dots,N$. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c^N\|^2_{L^2(\mathbb{T}^n)} + \frac{1}{2}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} J_\varepsilon(x-y)\left|c^N(x,t)-c^N(y,t)\right|^2 \mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{T}^n} f'(c^N)c^N\,\mathrm{d}x = 0.$$

In order to control the third term on the right-hand side, we use the growth assumption of f from (A3), the Hölder inequality as well as the Gagliardo-Nirenberg inequality. Hence we obtain

$$\begin{aligned}\int_{\mathbb{T}^n} f'(c^N)c^N\,\mathrm{d}x &\le \|f'(c^N)\|_{L^2(\mathbb{T}^n)}\|c^N\|_{L^2(\mathbb{T}^n)}\\ &\le C\|c^N\|_{L^2(\mathbb{T}^n)} + C\|c^N\|^3_{L^6(\mathbb{T}^n)}\|c^N\|_{L^2(\mathbb{T}^n)}\\ &\le C\|c^N\|_{L^2(\mathbb{T}^n)} + C\|\nabla c^N\|^n_{L^2(\mathbb{T}^n)^n}\|c^N\|^{4-n}_{L^2(\mathbb{T}^n)}.\end{aligned}$$

Since $\left(\nabla c^N\right)_{N\in\mathbb{N}}$ is bounded in $L^\infty(0,T_N;L^2(\mathbb{T}^n)^n)$, we apply Young's inequality and obtain from Gronwall's inequality that $(c^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,T_N;L^2(\mathbb{T}^n))$ is bounded.

Altogether, the sequence $(c^N)_{N\in\mathbb{N}}$ is bounded in $C^0([0,T_N],H^1(\mathbb{T}^n))\cap H^1(0,T_N;L^2(\mathbb{T}^n))$ uniformly for all $T_N\in[0,T_N^*)\cap[0,T]$ and $N\in\mathbb{N}$. In particular, this yields that the solution $\mathbf{g}^N$ is bounded on $[0,T_N]$ uniformly for all $T_N\in[0,T_N^*)\cap[0,T]$ and all $N\in\mathbb{N}$. An extension argument yields that $\mathbf{g}^N$ and hence c^N exist globally on $[0,T]$ and $T_N=T$ can be chosen above.

Step 4: Passage to the limit. Due to the uniform estimates derived in the step before, there exists a subsequence, again denoted by $(c^N)_{N\in\mathbb{N}}$, such that for $N\to\infty$ it holds

$$\begin{aligned} c^N &\rightharpoonup c && \text{in } H^1(0,T;L^2(\mathbb{T}^n)),\\ c^N &\to c && \text{in } C^0([0,T];L^p(\mathbb{T}^n)),\end{aligned}$$

for some $c\in H^1(0,T;L^2(\mathbb{T}^n))\cap C^0([0,T];L^p(\mathbb{T}^n))$ and for all $p\in[1,\infty)$ if $n=2$ and all $p\in[1,6)$ if $n=3$. Here, the last convergence follows by the Aubin-Lions-Simon Lemma. We need to show equation (8.1.1) for c.

Let $\xi\in C_0^\infty(0,T)$. We multiply (8.3.2) by $\xi(t)$ and integrate with respect to time. Hence

$$\int_0^T\int_{\mathbb{T}^n}\partial_t c^N w_i\xi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^n}\mathcal{L}_\varepsilon c^N w_i\xi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^n} f'(c^N)w_i\xi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

For the first term on the left-hand side, we use the weak convergence $c^N \rightharpoonup c$ in $H^1(0,T;L^2(\mathbb{T}^n))$ for $N\to\infty$ to conclude that

$$\int_0^T\int_{\mathbb{T}^n}\partial_t c^N w_i\xi\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_{\mathbb{T}^n}\partial_t c\, w_i\xi\,\mathrm{d}x\,\mathrm{d}t.$$

For the second term we prove in the following that $\mathcal{L}_\varepsilon c^N \to \mathcal{L}_\varepsilon c$ in $L^2(0,T;L^2(\mathbb{T}^n))$ for

$N \to \infty$. By definition and Young's convolution inequality, we have

$$\begin{aligned}
&\|\mathcal{L}_\varepsilon c^N - \mathcal{L}_\varepsilon c\|^2_{L^2(0,T;L^2(\mathbb{T}^n))} \\
&\leq \int_0^T \|J_\varepsilon * (c^N(t) - c(t))\|^2_{L^2(\mathbb{T}^n)} \,\mathrm{d}t + \int_0^T \|(J_\varepsilon * 1)(c^N(t) - c(t))\|^2_{L^2(\mathbb{T}^n)} \,\mathrm{d}t \\
&\leq \int_0^T \|J_\varepsilon\|^2_{L^1(\mathbb{T}^n)} \|(c^N(t) - c(t))\|^2_{L^2(\mathbb{T}^n)} \,\mathrm{d}t + \int_0^T \|(J_\varepsilon * 1)(c^N(t) - c(t))\|^2_{L^2(\mathbb{T}^n)} \,\mathrm{d}t.
\end{aligned}$$

Since the term $J_\varepsilon * 1$ is constant on the torus, the assertion follows from the strong convergence $c^N \to c$ in $C^0([0,T]; L^2(\mathbb{T}^n))$ for $N \to \infty$. Altogether, this implies for $N \to \infty$ that

$$\int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon c^N w_i \xi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon c \, w_i \xi \,\mathrm{d}x \,\mathrm{d}t.$$

Finally, for the last term we apply the General Lebesgue Convergence Theorem. Due to the growth assumption of f' from (A3), it holds

$$|f'(c)| \leq C|c|^3 + C.$$

Thus, the operator T defined by

$$T(x, c(x)) := f'(c(x))$$

is a Nemytskii operator and by Theorem 3.3.3 it follows that T is a bounded and continuous operator from $L^4(\mathbb{T}^n)$ to $L^{4/3}(\mathbb{T}^n)$. Therefore, thanks to the strong convergence $c^N \to c$ in $C^0([0,T]; L^4(\mathbb{T}^n))$, we can pass to the limit and get

$$\int_0^T \int_{\mathbb{T}^n} f'(c^N) w_i \xi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^n} f'(c) w_i \xi \,\mathrm{d}x \,\mathrm{d}t$$

for $N \to \infty$. Finally, we obtain

$$\int_0^T \int_{\mathbb{T}^n} \partial_t c w_i \xi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon c w_i \xi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^n} f'(c) w_i \xi \,\mathrm{d}x \,\mathrm{d}t = 0$$

for all $i \in \mathbb{N}$ and all $\xi \in C_0^\infty(0,T)$. Since $\mathrm{span}\{w_i : i \in \mathbb{N}\}$ is dense in $H^1(\mathbb{T}^n)$ we obtain

$$\int_0^T \int_{\mathbb{T}^n} \partial_t c w \xi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon c w \xi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^n} f'(c) w \xi \,\mathrm{d}x \,\mathrm{d}t = 0$$

for all $w \in H^1(\mathbb{T}^n)$ and all $\xi \in C_0^\infty(0,T)$. Therefore, the Fundamental Lemma of Calculus of Variations implies that (8.1.1) holds for c.

Finally, we show that the initial condition holds. We have already seen that $c^N(0) \to c_0$ in $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ for $N \to \infty$. Since we also have the convergence $c^N \to c$ in $C^0([0,T]; L^2(\mathbb{T}^n))$, we conclude that $c(0) = c_0$ in $L^2(\mathbb{T}^n)$ for $N \to \infty$.

Step 5: Higher order estimates. In this part we prove higher regularity for the solution obtained in the steps before provided that the initial value satisfies $c_0 \in H^3(\mathbb{T}^n)$ and $|c_0| \leq R_0$ with $R_0 \geq 1$ as in (A3). In the end we prove that the solution is confined to $[-R_0, R_0]$ in this situation and hence we can change the potential outside of this interval suitably. Therefore from now on we assume without loss of generality that the potential f and its derivatives are uniformly bounded.

We first prove that $c \in L^\infty(0,T;H^2(\mathbb{T}^n))$. For the proof, we again use the Galerkin scheme introduced in Step 1. We test equation (8.3.2) by $\Delta^2 c^N$, which means, on the Galerkin level, that we multiply (8.3.2) by $\lambda_i^2 g_i^N(t)$ and sum over all $i = 1, \dots, N$. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta c^N\|^2_{L^2(\mathbb{T}^n)} + \frac{1}{2}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} J_\varepsilon(x-y)\left|\Delta c^N(x,t) - \Delta c^N(y,t)\right|^2 \mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{T}^n} \Delta f'(c^N)\Delta c^N\,\mathrm{d}x = 0.$$

For the third term on the left-hand side, we observe

$$\begin{aligned}\int_{\mathbb{T}^n} \Delta f'(c^N)\Delta c^N\,\mathrm{d}x &= \int_{\mathbb{T}^n} f'''(c^N)|\nabla c^N|^2\Delta c^N\,\mathrm{d}x + \int_{\mathbb{T}^n} f''(c^N)|\Delta c^N|^2\,\mathrm{d}x\\ &\geq \int_{\mathbb{T}^n} f'''(c^N)|\nabla c^N|^2\Delta c^N\,\mathrm{d}x - \alpha\|\Delta c^N\|^2_{L^2(\mathbb{T}^n)}.\end{aligned}$$

By our assumption on the potential f, it holds $\|f^{(k)}\|_{L^\infty(\mathbb{R})} \leq C$ for all $k \geq 3$. Thus, the remaining term can be controlled as follows:

$$\left|\int_{\mathbb{T}^n} f'''(c^N)|\nabla c^N|^2\Delta c^N\,\mathrm{d}x\right| \leq \|f'''\|_\infty \|\nabla c^N\|^2_{L^4(\mathbb{T}^n)^n}\|\Delta c^N\|_{L^2(\mathbb{T}^n)}.$$

In order to control $\|\nabla c^N\|_{L^4(\mathbb{T}^n)^n}$, we infer from (8.3.2) that

$$0 = P_N(\partial_t c^N + \mathcal{L}_\varepsilon c^N + f'(c^N)) = \partial_t c^N + \mathcal{L}_\varepsilon c^N + P_N(f'(c^N)),$$

where we used that P_N and $\mathcal{L}_\varepsilon$ commutes due to the representation (8.3.1) and properties of convolutions. We differentiate the latter equation with respect to the space variable and test by $P_N(|\nabla c^N|^{p-2}\nabla c^N)$, $p > 2$. This yields

$$\int_{\mathbb{T}^n} \partial_t \nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N)\,\mathrm{d}x + \int_{\mathbb{T}^n} \mathcal{L}_\varepsilon \nabla c^N \cdot |\nabla c^N|^{p-2}\nabla c^N\,\mathrm{d}x + \int_{\mathbb{T}^n} f''(c^N)\nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N)\,\mathrm{d}x = 0, \tag{8.3.8}$$

where we used $\mathcal{L}_\varepsilon \nabla c^N = \mathcal{L}_\varepsilon P_N \nabla c^N = P_N \mathcal{L}_\varepsilon \nabla c^N$ for the second term. This follows from the fact that convolution operators commute with the gradient. For the first term, we use $\partial_t \nabla c^N \in (V_N)^n$ to obtain

$$\int_{\mathbb{T}^n} \partial_t \nabla c^N \cdot P_N(|\nabla c^N|^{p-2}\nabla c^N)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^n} \frac{1}{p}|\nabla c^N|^p\,\mathrm{d}x.$$

For the term involving the nonlocal operator, we use a symmetry argument to conclude

$$\begin{aligned}&\int_{\mathbb{T}^n} \mathcal{L}_\varepsilon \nabla c^N \cdot |\nabla c^N|^{p-2}\nabla c^N\,\mathrm{d}x\\ &= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} J_\varepsilon(x-y)(\nabla c^N(x) - \nabla c^N(y)) \cdot \nabla b(\nabla c^N(x))\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} J_\varepsilon(x-y)(\nabla c^N(x) - \nabla c^N(y)) \cdot \Big(\nabla b(\nabla c^N(x)) - \nabla b(\nabla c^N(y))\Big)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

where the function b is defined as $b(z) := \frac{|z|^p}{p}$ for all $z \in \mathbb{R}^n$. Since ∇b is monotone and since J_ε is nonnegative, this implies that

$$\int_{\mathbb{T}^n} \mathcal{L}_\varepsilon \nabla c^N \cdot |\nabla c^N|^{p-2} \nabla c^N \,\mathrm{d}x \geq 0.$$

Finally, for the last term in (8.3.8), we use Hölder's inequality to get

$$\left| \int_{\mathbb{T}^n} f''(c^N) \nabla c^N \cdot P_N(|\nabla c^N|^{p-2} \nabla c^N) \,\mathrm{d}x \right| \leq C \|\nabla c^N\|_{L^p(\mathbb{T}^n)^n} \|P_N(|\nabla c^N|^{p-2}\nabla c^N)\|_{L^{p'}(\mathbb{T}^n)^n}.$$

From Fourier Analysis and (8.3.1) it is well-known that $\sup_{N\in\mathbb{N}} \|P_N\|_{\mathcal{L}(L^{p'}(\mathbb{T}^n)^n)} \leq K$ for every $1 < p < \infty$, where $K > 0$ is independent of $N \in \mathbb{N}$. Moreover, we observe that $\||\nabla c^N|^{p-2}\nabla c^N\|_{L^{p'}(\mathbb{T}^n)^n} = \|\nabla c^N\|^{p-1}_{L^p(\mathbb{T}^n)^n}$. Hence, (8.3.8) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^n} \frac{1}{p} |\nabla c^N|^p \,\mathrm{d}x \leq C \|\nabla c^N\|^p_{L^p(\mathbb{T}^n)^n}.$$

Using the relation $c^N(0) = P_N c_0$ and (8.3.1), we conclude that the sequence $(c^N(0))_{N\in\mathbb{N}}$ is bounded in $H^3(\mathbb{T}^n)$ provided that $c_0 \in H^3(\mathbb{T}^n)$.

Thus by Gronwall's inequality the sequence $(\nabla c^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,T;L^p(\mathbb{T}^n)^n)$ is bounded for all $p > 2$. Therefore it holds

$$\int_{\mathbb{T}^n} f'''(c^N)|\nabla c^N|^2 \Delta c^N \,\mathrm{d}x \leq C(1 + \|\Delta c^N\|^2_{L^2(\mathbb{T}^n)})$$

and thus, again by Gronwall's inequality, $(c^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,T;H^2(\mathbb{T}^n))$ is bounded since the second order derivatives can be controlled by the Laplacian via integration by parts.

In the next step, we prove $c \in L^\infty(0,T;H^3(\mathbb{T}^n))$. To this end, we test equation (8.3.2) by $\Delta^3 c^N$. More precisely, we multiply (8.3.2) by $\lambda_i^3 g_i^N(t)$ and sum over all $i = 1,\dots,N$. This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta c^N\|^2_{L^2(\mathbb{T}^n)^n} + \frac{1}{2}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} J_\varepsilon(x-y)\left|\nabla\Delta c^N(t,x) - \nabla\Delta c^N(t,y)\right|^2 \mathrm{d}y\mathrm{d}x + \int_{\mathbb{T}^n} \nabla\Delta f'(c^N)\cdot\nabla\Delta c^N \,\mathrm{d}x = 0.$$

We need to control the last term on the left-hand side. First of all, the chain rule gives

$$\begin{aligned} \nabla\Delta f'(c^N) = &f^{(4)}(c^N)|\nabla c^N|^2\nabla c^N + 2f'''(c^N)D^2c^N\nabla c^N \\ &+ f'''(c^N)\Delta c^N\nabla c^N + f''(c^N)\nabla\Delta c^N. \end{aligned} \tag{8.3.9}$$

In order to estimate the terms we use that we assumed without loss of generality that the derivatives of f are uniformly bounded. Moreover,

$$\begin{aligned} \left\||\nabla c^N|^2\nabla c^N\right\|_{L^2(\mathbb{T}^n)^n} &= \|\nabla c^N\|^3_{L^6(\mathbb{T}^n)^n} \leq C, \\ \left\|D^2c^N\nabla c^N\right\|_{L^2(\mathbb{T}^n)^n} &\leq \|D^2c^N\|_{L^2(\mathbb{T}^n)^{n\times n}}\|\nabla c^N\|_{L^\infty(\mathbb{T}^n)^n} \leq C\|c^N\|_{H^2(\mathbb{T}^n)}\|c^N\|_{H^3(\mathbb{T}^n)} \\ &\leq C\|c^N\|_{H^2(\mathbb{T}^n)}\left(1 + \|\nabla\Delta c^N\|_{L^2(\mathbb{T}^n)^n}\right), \\ \left\|\Delta c^N\nabla c^N\right\|_{L^2(\mathbb{T}^n)^n} &\leq C\|c^N\|_{H^2(\mathbb{T}^n)}\left(1 + \|\nabla\Delta c^N\|_{L^2(\mathbb{T}^n)^n}\right). \end{aligned}$$

Therefore we can control the corresponding parts in the integral above using Young's inequality. For the last term, we use the growth condition of f from (A3), which implies

$$\int_{\mathbb{T}^n} f''(c^N)|\nabla\Delta c^N|^2 \,\mathrm{d}x \geq -\alpha\|\nabla\Delta c^N\|^2_{L^2(\mathbb{T}^n)^n}.$$

Altogether, Gronwall' lemma implies that $(c^N)_{N\in\mathbb{N}} \subseteq L^\infty(0,T;H^3(\mathbb{T}^n))$ is bounded.

From the previous estimates we obtain $c \in H^1(0,T;L^2(\mathbb{T}^n)) \cap L^\infty(0,T;H^3(\mathbb{T}^n))$. Now, due to the Aubin–Lions–Simon Lemma, we get

$$H^1(0,T;L^2(\mathbb{T}^n)) \cap L^\infty(0,T;H^3(\mathbb{T}^n)) \hookrightarrow C^0([0,T];H^2(\mathbb{T}^n)),$$

which in particular yields $c \in C^0([0,T],C^0(\mathbb{T}^n))$. Finally, from the equation (8.1.1) for c we obtain $c \in C^1([0,T];C^0(\mathbb{T}^n))$.

Step 6: Boundedness. Finally, we prove that the solution $c \in C^1([0,T];C^0(\mathbb{T}^n))$ of (8.1.1)–(8.1.2) is bounded by R_0 provided that $|c_0| \leq R_0$, where $R_0 \geq 1$ is from (A3). Assume that there exists $(x_0,t_0) \in \mathbb{T}^n \times [0,T]$ such that c attains its maximum in (x_0,t_0) with $c(x_0,t_0) > R_0$. Then $\partial_t c(x_0,t_0) \geq 0$ and it holds

$$\partial_t c(x_0,t_0) + \mathcal{L}_\varepsilon c(x_0,t_0) + f'(c(x_0,t_0)) > 0,$$

since $f'(c(x_0,t_0)) > 0$ if $c(x_0,t_0) > R_0$ and

$$\mathcal{L}_\varepsilon c(x_0,t_0) = \int_{\mathbb{T}^n} J_\varepsilon(x_0-y)(c(x_0,t_0)-c(y,t_0))\,\mathrm{d}y \geq 0,$$

since $J_\varepsilon \geq 0$. This is a contradiction since c solves (8.1.1). Analogously, one shows $c \geq -R_0$.

This concludes the proof of Theorem 8.3.2. □

8.4. Uniform estimates

Let $\varepsilon > 0$ be small. In order to close the proof of the main result with a Gronwall argument, it remains to prove that the solutions from Theorem 8.3.2 are bounded uniformly in ε. In fact, we have the following estimates:

Theorem 8.4.1 (**Uniform H^3-Bounds**). *Let $T > 0$ be fixed. For any $\eta \in (0,1)$, let c_η be the solution to* (8.1.1)-(8.1.2) *from Theorem 8.3.2 for initial values $c_{\eta,0} \in H^3(\mathbb{T}^n)$ such that $|c_{\eta,0}| \leq R_0$ with $R_0 \geq 1$ as in* (A3) *and such that with some constant $K > 0$ independent of $\eta \in (0,1)$*

$$\|c_{\eta,0}\|_{H^1(\mathbb{T}^n)} \leq \frac{K}{\eta^2}, \quad \|c_{\eta,0}\|_{H^2(\mathbb{T}^n)} \leq \frac{K}{\eta^4} \quad \text{and} \quad \|c_{\eta,0}\|_{H^3(\mathbb{T}^n)} \leq \frac{K}{\eta^{11}}.$$

Then there exist constants $\eta_0 \in (0,1)$, $c, C > 0$ independent of $c_\eta, \eta, \varepsilon$ such that for all $0 < \varepsilon \leq c\eta^3$, $\eta \in (0,\eta_0)$ it holds

$$\|c_\eta\|_{L^\infty(0,T;H^3(\mathbb{T}^n))} \leq \frac{C}{\eta^{11}}.$$

Proof. Let c_η be the solution of (8.1.1)-(8.1.2) according to Theorem 8.3.2. We have to control the H^3-norm of c_η in dependence of η. To this end, we proceed in several steps.

Step 1: H^1-estimate. We test (8.1.1) with $-\Delta c_\eta$. After integration by parts, we obtain for all $t \in [0,T]$ that

$$\frac{1}{2}\|\nabla c_\eta(t)\|^2_{L^2(\mathbb{T}^n)^n} + \int_0^t \mathcal{E}_\varepsilon(\nabla c_\eta)\,\mathrm{d}s \leq \frac{1}{2}\|\nabla c_{\eta,0}\|^2_{L^2(\mathbb{T}^n)^n} + \frac{\alpha}{\eta^2}\int_0^t \|\nabla c_\eta\|^2_{L^2(\mathbb{T}^n)^n}\,\mathrm{d}s$$

where we also used the condition for f'' from (A3). Next, we apply Theorem 3.5.5 (Non-local Ehrling Inequality) which yields for all $R \geq 1$, $0 < \varepsilon \leq \frac{1}{R}$ that

$$\begin{aligned}&\frac{1}{2}\|\nabla c_\eta(t)\|^2_{L^2(\mathbb{T}^n)^n} + \int_0^t \mathcal{E}_\varepsilon(\nabla c_\eta)\,\mathrm{d}s\\ &\leq \frac{1}{2}\|\nabla c_{\eta,0}\|^2_{L^2(\mathbb{T}^n)^n} + \int_0^t \left[\frac{C\alpha}{\eta^2R^2}\mathcal{E}_\varepsilon(\nabla c_\eta) + \frac{C\alpha}{\eta^2}R^2\|c_\eta\|^2_{L^2(\mathbb{T}^n)}\right]\,\mathrm{d}s.\end{aligned}$$

We choose $R = 2\frac{\sqrt{C\alpha}}{\eta}$ and η small such that $R \geq 1$. Thus, using that $|c_\eta|$ is bounded thanks to the maximum principle, we obtain

$$\|\nabla c_\eta\|_{L^\infty(0,T;L^2(\mathbb{T}^n)^n)} \leq \frac{C}{\eta^2}$$

for all $0 < \varepsilon \leq \frac{1}{2\sqrt{C\alpha}}\eta$ and η small.

Step 2: H^2-estimate. We test (8.1.1) with $\Delta^2 c_\eta$. This yields for all $t \in [0,T]$

$$\begin{aligned}&\frac{1}{2}\|\Delta c_\eta(t)\|^2_{L^2(\mathbb{T}^n)} + \int_0^t \mathcal{E}_\varepsilon(\Delta c_\eta)\,\mathrm{d}s\\ &\leq \frac{1}{2}\|\Delta c_{\eta,0}\|^2_{L^2(\mathbb{T}^n)} + \frac{\alpha}{\eta^2}\int_0^t \|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s - \frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^n} f'''(c_\eta)|\nabla c_\eta|^2\Delta c_\eta\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Because of the maximum principle in Theorem 8.3.2, we have that $|c_\eta|$ is uniformly bounded. Moreover, the Gagliardo-Nirenberg interpolation inequality gives for all $t \in [0,T]$

$$\|\nabla c_\eta\|_{L^4(\mathbb{T}^n)^n} \leq C\|c_\eta\|^{1/2}_{L^\infty(\mathbb{T}^n)}\|c_\eta\|^{1/2}_{H^2(\mathbb{T}^n)} \leq C\|c_\eta\|^{1/2}_{H^2(\mathbb{T}^n)}$$

for both $n \in \{2,3\}$, where we omit the time-dependence for simplicity. Then by Young's inequality

$$\begin{aligned}\frac{1}{\eta^2}\int_{\mathbb{T}^n} f'''(c_\eta)|\nabla c_\eta|^2\Delta c_\eta\,\mathrm{d}x &\leq \frac{C}{\eta^2}\|\nabla c_\eta\|^4_{L^4(\mathbb{T}^n)^n} + \frac{C}{\eta^2}\|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}\\ &\leq \frac{C}{\eta^2}\|c_\eta\|^2_{H^2(\mathbb{T}^n)} + \frac{C}{\eta^2}\|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}\end{aligned}$$

By definition of the H^2-norm, the boundedness of c_η and the estimate derived in Step 1, we have

$$\|c_\eta\|^2_{H^2(\mathbb{T}^n)} \leq C\left(1 + \frac{1}{\eta^4} + \|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}\right).$$

This yields

$$\frac{1}{\eta^2}\int_{\mathbb{T}^n} f'''(c_\eta)|\nabla c_\eta|^2\Delta c_\eta\,\mathrm{d}x \leq \frac{C}{\eta^6}+\frac{C}{\eta^2}\|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}.$$

Together with the Nonlocal Ehrling Inequality, cf. Theorem 3.5.5, we get for all $R \geq 1$, $0<\varepsilon\leq \frac{1}{R}$ and all $t\in[0,T]$ the estimate

$$\begin{aligned}
&\frac{1}{2}\|\Delta c_\eta(t)\|^2_{L^2(\mathbb{T}^n)}+\int_0^t \mathcal{E}_\varepsilon(\Delta c_\eta)\,\mathrm{d}s\\
&\leq \frac{1}{2}\|\Delta c_{\eta,0}\|^2_{L^2(\mathbb{T}^n)}+\frac{C}{\eta^6}+\frac{C}{\eta^2}\int_0^t \|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s\\
&\leq \frac{1}{2}\|\Delta c_{\eta,0}\|^2_{L^2(\mathbb{T}^n)}+\frac{C}{\eta^6}+\int_0^t\left[\frac{C}{\eta^2R^2}\mathcal{E}_\varepsilon(\Delta c_\eta)+\frac{C}{\eta^2}R^2\|\nabla c_\eta\|^2_{L^2(\mathbb{T}^n)^n}\right]\mathrm{d}s.
\end{aligned}$$

Choosing R proportional to $\frac{1}{\eta}$ to absorb the $\mathcal{E}_\varepsilon$-term, we get

$$\|\Delta c_\eta\|_{L^\infty(0,T;L^2(\mathbb{T}^n))}\leq \frac{C}{\eta^4}$$

for all $0<\varepsilon\leq c\eta$ and $\eta>0$ small, where $c>0$ is some fixed constant. This also controls the $L^2(\mathbb{T}^n)$-norm of the second derivatives due to integration by parts.

Step 3: H^3-estimate. In the final step, we test (8.1.1) with $-\Delta^3 c_\eta$. Then, for almost all $t\in[0,T]$ we have

$$\begin{aligned}
&\frac{1}{2}\|\nabla\Delta c_\eta(t)\|^2_{L^2(\mathbb{T}^n)^n}-\frac{1}{2}\|\nabla\Delta c_{\eta,0}\|^2_{L^2(\mathbb{T}^n)^n}\\
&+\int_0^t \mathcal{E}_\varepsilon(\nabla\Delta c_\eta)\,\mathrm{d}s+\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^n}\nabla\Delta f'(c_\eta)\cdot\nabla\Delta c_\eta\,\mathrm{d}x\,\mathrm{d}s=0.
\end{aligned}$$

For simplicity, we often omit the time-dependence in the following. The estimates are uniform with respect to time. Using the chain rule as in (8.3.9) and that c_η is uniformly bounded, we obtain

$$\|f^{(4)}(c_\eta)|\nabla c_\eta|^2\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n}\leq C\||\nabla c_\eta|^2\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n}\leq C\|\nabla c_\eta\|^3_{L^6(\mathbb{T}^n)^n}.$$

In particular, this gives

$$\begin{aligned}
\left|\frac{1}{\eta^2}\int_{\mathbb{T}^n} f^{(4)}(c_\eta)|\nabla c_\eta|^2\nabla c_\eta\cdot\nabla\Delta c_\eta\,\mathrm{d}x\right| &\leq \frac{C}{\eta^2}\|c_\eta\|^3_{H^2(\mathbb{T}^n)}\|\nabla\Delta c_\eta\|_{L^2(\mathbb{T}^n)^n}\\
&\leq \frac{C}{\eta^{22}}+\frac{C}{\eta^6}\|\nabla\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)^n}.
\end{aligned}$$

Here, we used the estimate derived in Step 2 together with Young's inequality for the terms η^{-11} and $\eta^{-3}\|\nabla\Delta c_\eta\|_{L^2(\mathbb{T}^n)}$. For the remaining terms, we have

$$\begin{aligned}
\|f'''(c_\eta)D^2c_\eta\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n} &\leq C\|\Delta c_\eta\|_{L^2(\mathbb{T}^n)}\|\nabla c_\eta\|_{H^2(\mathbb{T}^n)^n}\\
&\leq \frac{C}{\eta^4}\left(\|\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n}+\|\nabla\Delta c_\eta\|_{L^2(\mathbb{T}^n)^n}\right),\\
\|f'''(c_\eta)\Delta c_\eta\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n} &\leq \frac{C}{\eta^4}\left(\|\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n}+\|\nabla\Delta c_\eta\|_{L^2(\mathbb{T}^n)^n}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\frac{1}{\eta^2}\int_{\mathbb{T}^n}\Big(f'''(c_\eta)D^2c_\eta\nabla c_\eta+f'''(c_\eta)\Delta c_\eta\nabla c_\eta\Big)\cdot\nabla\Delta c_\eta\,\mathrm{d}x\right|\\&\le\frac{C}{\eta^6}\left(\|\nabla c_\eta\|_{L^2(\mathbb{T}^n)^n}+\|\nabla\Delta c_\eta\|_{L^2(\mathbb{T}^n)^n}\right)\|\nabla\Delta c_\eta\|_{L^2(\mathbb{T}^n)^n}\\&\le\frac{C}{\eta^6}\|\nabla\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)^n}+\frac{C}{\eta^{10}},\end{aligned}$$

where we applied Young's inequality in the last step. For the last term, (A3) implies

$$\frac{1}{\eta^2}\int_{\mathbb{T}^n}f''(c_\eta)|\nabla\Delta c_\eta|^2\,\mathrm{d}x\ge-\frac{\alpha}{\eta^2}\|\nabla\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)^n}.$$

Combining these estimates with Theorem 3.5.5 (Nonlocal Ehrling Inequality), we obtain for all $R\ge1$, $0<\varepsilon\le\min\{c\eta^2,\frac{1}{R}\}$ and for a.e. $t\in[0,T]$ the estimate

$$\begin{aligned}&\frac12\|\nabla\Delta c_\eta(t)\|^2_{L^2(\mathbb{T}^n)^n}+\int_0^t\mathcal{E}_\varepsilon(\nabla\Delta c_\eta)\,\mathrm{d}s\\&\le\frac12\|\nabla\Delta c_{\eta,0}\|_{L^2(\mathbb{T}^n)^n}+\frac{C}{\eta^{22}}+\frac{C}{\eta^6}\int_0^t\|\nabla\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)^n}\,\mathrm{d}s\\&\le\frac12\|\nabla\Delta c_{\eta,0}\|_{L^2(\mathbb{T}^n)^n}+\frac{C}{\eta^{22}}+\int_0^t\left[\frac{C}{\eta^6R^2}\mathcal{E}_\varepsilon(\nabla\Delta c_\eta)+\frac{C}{\eta^6}R^2\|\Delta c_\eta\|^2_{L^2(\mathbb{T}^n)}\right]\,\mathrm{d}s.\end{aligned}$$

Finally, we choose R proportional to $\frac{1}{\eta^3}$ to absorb the $\mathcal{E}_\varepsilon$-term. Then we obtain for all $0<\varepsilon\le c\eta^3$ and $\eta>0$ small, where the constant c is possibly smaller than before, that

$$\|\nabla\Delta c_\eta\|_{L^\infty(0,T;L^2(\mathbb{T}^n)^n)}\le\frac{C}{\eta^{11}}.$$

The L^2-norm of the third derivatives is then also controlled via integration by parts. □

8.5. Proof of Theorem 2.6.2

We use the notation from Theorem 2.6.2. Moreover, let c_η^A for $\eta>0$ small be the approximate solution from the local case, constructed for the evolving hypersurface $(\Gamma_t)_{t\in[0,T_0]}$ and the parameter $L\in\mathbb{N}$, cf. Theorem 8.2.2. Note that we require Γ to evolve according to mean curvature flow in order to have the remainder estimates for c_η^A in Theorem 8.2.2 for the local Allen-Cahn equation available. For the latter we set $r_\eta^A:=\partial_tc_\eta^A-\Delta c_\eta^A+\frac{1}{\eta^2}f'(c_\eta^A)$. Moreover, let $\beta\ge0$ be fixed (to be chosen later) and set $g_\beta(t):=e^{-\beta t}$ for all $t\ge0$. Finally, we recall that $\bar{c}_\eta:=c_\eta-c_\eta^A$. Our goal is to verify the estimates

$$\begin{aligned}\sup_{t\in[0,T]}\|g_\beta\bar{c}_\eta(t)\|^2_{L^2(\mathbb{T}^n)}+\|g_\beta\nabla\bar{c}_\eta\|^2_{L^2(\mathbb{T}^n\times(0,T)\setminus\Gamma(a))}\le2R^2\eta^{2L+1},\\\|g_\beta\nabla_\tau\bar{c}_\eta\|^2_{L^2(\mathbb{T}^n\times(0,T)\cap\Gamma(a))}+\eta^2\|g_\beta\partial_n\bar{c}_\eta\|^2_{L^2(\mathbb{T}^n\times(0,T)\cap\Gamma(a))}\le2R^2\eta^{2L+1},\end{aligned}\tag{8.5.1}$$

where $\eta,R,L>0$ and $T\in(0,T_0]$. To this end, we define

$$T_{\eta,\beta,R}:=\sup\{T\in(0,T_0]:(8.5.1)\text{ holds for }\eta,\beta,R\}.$$

In the different cases in Theorem 2.6.2, we know, due to continuity, that $T_{\eta,\beta,R}$ is well-defined and positive. For the two cases, we need to verify the following claims:

1. Let $L \geq 3$. There are $\beta, \eta_1, c > 0$ such that if $\varepsilon = \varepsilon(\eta) \leq cR\eta^{16+L+\frac{1}{2}}$, then it holds $T_{\eta,\beta,R} = T_0$ for all $\eta \in (0, \eta_1]$.

2. Let $L = 2$. The analogous statement as in the first case holds for $\beta = 0$ if T is sufficiently small.

In the following, we first carry out a general computation that works for every case. Taking the difference of the nonlocal Allen-Cahn equation (8.1.1) for c_η and the local Allen-Cahn equation (1.2.6a) for c_η^A with remainder r_η^A, we obtain

$$\partial_t \overline{c}_\eta - \Delta \overline{c}_\eta + f''(c_\eta^A)\overline{c}_\eta + \mathcal{L}_\varepsilon c_\eta + \Delta c_\eta = -r_\eta^A - r_\eta(c_\eta, c_\eta^A), \tag{8.5.2}$$

where we have set $r_\eta(c_\eta, c_\eta^A) := \frac{1}{\eta^2}\left[f'(c_\eta) - f'(c_\eta^A) - f''(c_\eta^A)\overline{c}_\eta\right]$. Testing (8.5.2) with $g_\beta^2 \overline{c}_\eta$ and integrating over $\mathbb{T}^n \times (0,T)$ for $T \in (0, T_{\eta,\beta,R,L}]$ yields

$$\begin{aligned}
&\frac{1}{2} g_\beta^2(T)^2 \|\overline{c}_\eta(T)\|_{L^2(\mathbb{T}^n)}^2 - \frac{1}{2}\|\overline{c}_\eta(0)\|_{L^2(\mathbb{T}^n)}^2 + \beta \int_0^T g_\beta^2 \|\overline{c}_\eta\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}t \\
&+ \int_0^T g_\beta^2 \int_{\mathbb{T}^n} |\nabla \overline{c}_\eta|^2 + \frac{1}{\eta^2} f''(c_\eta^A)|\overline{c}_\eta|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T g_\beta^2 \int_{\mathbb{T}^n} (\mathcal{L}_\varepsilon c_\eta + \Delta c_\eta)\overline{c}_\eta \,\mathrm{d}x\,\mathrm{d}t \\
&= - \int_0^T g_\beta^2 \int_{\mathbb{T}^n} r_\eta^A \overline{c}_\eta + r_\eta(c_\eta, c_\eta^A)\overline{c}_\eta \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where we used $\frac{\mathrm{d}}{\mathrm{d}t} g_\beta = -\beta g_\beta$. We have $\frac{1}{2}\|\overline{c}_\eta(0)\|_{L^2(\mathbb{T}^n)}^2 \leq \frac{1}{2}R^2\eta^{2L+1}$ due to the assumption in the theorem. Moreover, the spectral estimate for the local case from Theorem 8.2.3 yields

$$\begin{aligned}
&\int_0^T g_\beta^2 \int_{\mathbb{T}^n} |\nabla \overline{c}_\eta|^2 + \frac{1}{\eta^2} f''(c_\eta^A)|\overline{c}_\eta|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\geq -\overline{C} \int_0^T g_\beta^2 \|\overline{c}_\eta\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}t + \int_0^T g_\beta^2 \left(\|\nabla \overline{c}_\eta\|_{L^2(\mathbb{T}^n \setminus \Gamma_t(a))}^2 + \|\nabla_\tau \overline{c}_\eta\|_{L^2(\Gamma_t(a))}^2 \right) \mathrm{d}t.
\end{aligned}$$

Furthermore, we use the estimate $\|\mathcal{L}_\varepsilon c_\eta + \Delta c_\eta\|_{L^2(\mathbb{T}^n)} \leq C\varepsilon \|c_\eta\|_{H^3(\mathbb{T}^n)}$ from Theorem 2.1.5 and the uniform H^3-estimate for c_η from Theorem 8.4.1 for $0 < \varepsilon \leq c\eta^4$ to deduce

$$\begin{aligned}
&\left| \int_0^T g_\beta^2 \int_{\mathbb{T}^n} (\mathcal{L}_\varepsilon c_\eta + \Delta c_\eta)\overline{c}_\eta \,\mathrm{d}x\,\mathrm{d}t \right| \leq \int_0^T g_\beta^2 C\varepsilon\eta^{-16} \|\overline{c}_\eta\|_{L^2(\mathbb{T}^n)} \,\mathrm{d}t \\
&\leq \frac{\beta}{2} \int_0^T g_\beta^2 \|\overline{c}_\eta\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}t + C\|g_\beta\|_{L^2(0,T)}^2 (\varepsilon\eta^{-16})^2.
\end{aligned}$$

Moreover, using the remainder estimate for r_η^A from Theorem 8.2.2 and an integral transformation in tubular neighbourhood coordinates along with (8.5.1), we obtain

$$\left| \int_0^T g_\beta^2 \int_{\mathbb{T}^n} r_\eta^A \overline{c}_\eta \,\mathrm{d}x\,\mathrm{d}t \right| \leq C \int_0^T g_\beta^2 \eta^{L+\frac{1}{2}} \|\overline{c}_\eta\|_{L^2(\mathbb{T}^n)} \leq \overline{C}_1 \|g_\beta\|_{L^1(0,T)} R\eta^{2L+1}$$

for all $t \in (0, T_{\eta,\beta,R}]$. Additionally, we use the uniform boundedness of c_η and c_η^A from Theorem 8.3.2 and Theorem 8.2.2 as well as Taylor's theorem to derive the estimate

$$\left| \int_0^T g_\beta^2 r_\eta(c_\eta, c_\eta^A)\overline{c}_\eta \,\mathrm{d}t \right| \leq \frac{C}{\eta^2} \int_0^T g_\beta^2 \|\overline{c}_\eta\|_{L^3(\mathbb{T}^n)}^3 \,\mathrm{d}t.$$

The right-hand side can be estimated by splitting $\mathbb{T}^n$ into $\mathbb{T}^n \backslash \Gamma(a)$ and $\Gamma(a)$, using tubular neighbourhood coordinates for the latter set as well as Gagliardo-Nirenberg estimates, cf. Moser [101], Lemma 6.6. This implies

$$\left| \int_0^T g_\beta^2 r_\eta(c_\eta, c_\eta^A) \overline{c}_\eta \, \mathrm{d}t \right| \le C R^3 \eta^{2L+1} \eta^{L-2} \| g_\beta^{-1} \|_{L^{\frac{4}{4-n}}(0,T)}.$$

Finally, we control the $\partial_n \overline{c}_\eta$-term in (8.5.1) by the spectral term via

$$\begin{aligned}\eta^2 \| g_\beta \partial_n \overline{c}_\eta \|^2_{L^2(\mathbb{T}^n \times (0,T) \cap \Gamma(a))} \le C \eta^2 \int_0^T g_\beta^2 \int_{\mathbb{T}^n} |\nabla \overline{c}_\eta|^2 + \frac{1}{\eta^2} f''(c_\eta^A) \overline{c}_\eta^2 \, \mathrm{d}x \, \mathrm{d}t \\ + C \int_0^T g_\beta^2 \| \overline{c}_\eta \|^2_{L^2(\mathbb{T}^n)} \, \mathrm{d}t,\end{aligned}$$

where we used that c_η^A is uniformly bounded with respect to small η. The first term on the right hand side is absorbed by half of the spectral term for η small. Altogether, this yields

$$\begin{aligned}&\frac{1}{2} g_\beta(T)^2 \| \overline{c}_\eta(T) \|^2_{L^2(\mathbb{T}^n)} + \frac{1}{2} \| g_\beta \nabla \overline{c}_\eta \|^2_{L^2(\mathbb{T}^n \times (0,T) \backslash \Gamma(a))} \\ &+ \frac{1}{2} \| g_\beta \nabla_\tau \overline{c}_\eta \|^2_{L^2(\mathbb{T}^n \times (0,T) \cap \Gamma(a))} + \frac{\eta^2}{2} \| g_\beta \partial_n \overline{c}_\eta \|^2_{L^2(\mathbb{T}^n \times (0,T) \cap \Gamma(a))} \\ &\le \frac{R^2}{2} \eta^{2L+1} + \int_0^T g_\beta^2 (-\frac{\beta}{2} + \overline{C}_0) \| \overline{c}_\eta \|^2_{L^2(\mathbb{T}^n)} \, \mathrm{d}t + C \| g_\beta \|^2_{L^2(0,T)} (\varepsilon \eta^{-16})^2 \\ &+ \overline{C}_1 \| g_\beta \|_{L^1(0,T)} R \eta^{2L+1} + C R^3 \eta^{2L+1} \eta^{L-2} \| g_\beta^{-1} \|_{L^{\frac{4}{4-n}}(0,T)}\end{aligned} \tag{8.5.3}$$

for all $T \in (0, T_{\eta,\beta,R}]$ and $\eta \in (0, \tilde{\eta}_0]$, where $\tilde{\eta}_0 > 0$ is small (independent of β, R, T).

Now we consider the distinct cases in the theorem.

Ad 1. Let $L \ge 3$. We choose $\beta \ge 2\overline{C}_0$ sufficiently large such that $\frac{\overline{C}_1}{\beta} \le \frac{R}{8}$. We now roughly estimate $\| g_\beta \|^2_{L^2(0,T)} \le T_0^2$ for this case. Then, if $\varepsilon \le c R \eta^{16+L+\frac{1}{2}}$ for some sufficiently small $c > 0$, the right hand side in (8.5.3) is bounded by $\frac{3}{4} R^2 \eta^{2L+1}$ for all $T \in (0, T_{\eta,\beta,R}]$ and $\eta \in (0, \eta_1]$, where $\eta_1 > 0$ is small (depending on R, β). Finally, a contradiction and continuity argument shows $T_{\eta,\beta,R} = T_0$ for all $\eta \in (0, \eta_1]$. More precisely, if we assume that $T_{\eta,\beta,R} \neq T_0$,. Then, by continuity of the left-hand side in (8.5.1), there exists some $\tilde{T} \in (T_{\eta,\beta,R}, T_0)$ such that (8.5.1) still holds true for $\tilde{T}$. However, this contradicts the definition of $T_{\eta,\beta,R}$.

Ad 2. Let $L = 2$. Then we set $\beta = 0$ and obtain for $\varepsilon \le c R \eta^{16+L+\frac{1}{2}}$ with (8.5.1) that the right hand side in (8.5.3) is estimated by

$$\left[\frac{R^2}{2} + \overline{C}_0 T + C R^2 T^2 + \overline{C}_1 T R + C R^3 T^{\frac{4-n}{4}} \right] \eta^{2L+1}.$$

We choose $R > 0$ small enough such that the right-hand side in (8.5.3) is bounded by $\frac{3}{4} R^2 \eta^{2L+1}$. Then, by continuity of the left-hand side in (8.5.1), there exist $\eta_1 > 0$ and $T_1 > 0$ such that estimate (8.5.1) holds true for all $T \in (0, \min\{T_{\eta,0,R}, T_1\}]$ and $\eta \in (0, \eta_1]$. This implies $T_{\eta,0,R} \ge T_1$ by a similar contradiction argument as before. Namely, if

$T_{\eta,0,R} < T_1$ for all $\eta \in (0, \eta_1]$, there exists some $\tilde{T} \in (T_{\eta,0,R}, T_1)$ such that (8.5.1) still holds true for T_1 and all $\eta \in (0, \eta_1]$, which follows by continuity of the left-hand side in (8.5.1). However, this contradicts the definition of $T_{\eta,0,R}$.

This shows Theorem 2.6.2. $\square$

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