Semiclassical Mechanism for the Quantum Decay in Open Chaotic Systems

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We address the decay in open chaotic quantum systems and calculate semiclassical corrections to the classical exponential decay. We confirm random matrix predictions and, going beyond, calculate Ehrenfest time effects. To support our results we perform extensive numerical simulations. Within our approach we show that certain (previously unnoticed) pairs of interfering, correlated classical trajectories are of vital importance. They also provide the dynamical mechanism for related phenomena such as photoionization and photodissociation, for which we compute cross-section correlations. Moreover, these orbits allow us to establish a semiclassical version of the continuity equation.

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Besides their relevance to many areas of physics, open quantum systems play an outstanding role in gaining an improved understanding of the relation between classical and quantum physics [1]. For a closed quantum system the spatially integrated probability density

$$\rho(t) = \int_V d\mathbf{r} \psi(\mathbf{r}, t) \psi^*(\mathbf{r}, t)$$  (1)

of a wave function $\psi(\mathbf{r}, t)$ in the volume $V$ is constant, i.e., $\rho(t) = 1$. This fact is naturally retained when taking the classical limit in a semiclassical evaluation of Eq. (1), reflecting particle conservation in the quantum and classical limit. However, when opening up the system, $\rho(t)$, then representing the quantum survival probability, exhibits deviations from its classical counterpart $\rho_{cl}(t)$; in other words, certain quantum properties of the closed system can be unveiled upon opening it.

For an open quantum system with a completely chaotic classical counterpart, i.e., fully hyperbolic dynamics, the classical survival probability is asymptotically $\rho_{cl}(t) = \exp(-t/\tau_d)$, with classical dwell time $\tau_d$. This has been observed in various disciplines, either directly, as in atom billiards [2,3], or indirectly in the spectral regime of Ericson fluctuations in electron [4] or microwave [5] cavities, and in atomic photoionization [6].

However, it was found numerically [7] and confirmed with supersymmetry techniques [8] that the difference between $\rho(t)$ and $\rho_{cl}(t)$ becomes significant at times close to the quantum relaxation time $t^* = \sqrt{\tau_d}\hbar$. In the semiclassical limit $t^*$ is shorter than the Heisenberg time $t_H = 2\pi\hbar/\Delta$ (with $\Delta$ the mean level spacing). It was shown in Ref. [8] that $\rho(t)$ is a universal function depending only on $\tau_d$ and $t_H$ in the random matrix theory (RMT) limit [9]. Though the leading quantum deviations from $\rho_{cl}(t)$ were reproduced semiclassically for graphs [10], a general understanding of its dynamical origin is still lacking.

In this Letter we present a semiclassical calculation of $\rho(t)$ for $t < t^*$ for general, classically chaotic systems. It reveals the mechanism underlying the appearance of quantum corrections upon opening the system. In our calculation we go beyond the so-called diagonal approximation and evaluate contributions from correlated trajectory pairs [11]. This technique has been extended and applied to calculate various spectral [12,13] and scattering [14–19] properties of quantum chaotic systems. We find, however, that for calculating $\rho(t)$ a new class of correlated trajectory pairs, “one-leg-loops” (1lI), has to be considered along with the previously known loop diagrams. They prove particularly crucial for ensuring unitarity in problems involving semiclassical propagation along open trajectories inside a system and, moreover, allow one to semiclassically recover the continuity equation.

We present the dominant quantum corrections to $\rho_{sc}(t)$ for systems with and without time reversal symmetry. Going beyond RMT, we calculate Ehrenfest time effects on $\rho(t)$ which are particularly pronounced in the time domain, compare with quantum simulations of billiard dynamics, and extend our approach to photoionization and photodissociation cross sections.

Semiclassical approach.—We consider $\rho(t)$, Eq. (1), for a two-dimensional system of area $A$ and express $\psi(\mathbf{r}, t) = \int d\mathbf{r}' K(\mathbf{r}, \mathbf{r}'; t) \psi_0(\mathbf{r}')$ through the initial wave function $\psi_0(\mathbf{r}')$ and the time-dependent propagator $K(\mathbf{r}, \mathbf{r}'; t)$ that we approximate semiclassically by [20]

$$K_{se}(\mathbf{r}, \mathbf{r}'; t) = \frac{1}{2\pi \imath \hbar} \sum_{\gamma(r'\rightarrow r)} D_\gamma e^{\imath S_\gamma/\hbar}. \quad (2)$$

Here $S_\gamma = S_\gamma(\mathbf{r}', \mathbf{r}; t)$ is the classical action along the path $\gamma$ connecting $\mathbf{r}'$ and $\mathbf{r}$ in time $t$, and $D_\gamma = |\det(\partial^2 S_\gamma/\partial \mathbf{r} \partial \mathbf{r}')|^{1/2} e^{-\imath \pi \mu_\gamma/2}$ with Morse index $\mu_\gamma$.

The semiclassical survival probability, $\rho_{sc}(t)$, obtained by expressing the time evolution of $\psi(\mathbf{r}, t)$ and $\psi^*(\mathbf{r}, t)$ in Eq. (1) through $K_{se}$, Eq. (2), is given by three spatial integrals over a double sum over trajectories $\gamma$, $\gamma'$ starting at initial points $\mathbf{r}$ and $\mathbf{r}'$, weighted by $\psi_0(\mathbf{r}')$ and $\psi_0^*(\mathbf{r}'')$, and ending at the same point $\mathbf{r}$ inside $A$. For simplicity of presentation we here assume $\psi_0$ to be spatially localized,
so that contributions originate from points \( r' \) close to \( r'' \) [21]; generalizations are given below. Introducing \( r_0 = (r' + r'')/2 \) and \( q = (r' - r'') \), we replace the original paths \( \gamma, \gamma' \), by nearby trajectories \( \gamma \) and \( \gamma' \) connecting \( r_0 \) and \( r \) in time \( t \). Then, upon expanding the action \( S_\gamma(r, r'; t) \approx S_\gamma(r_0, r_0; t) - q^2/2 \) (with \( p_0 \) the initial momentum of path \( \gamma \)) and \( S_{\gamma'} \) analogously, we obtain

\[
\rho_{\text{sc}}(t) = \frac{1}{(2\pi\hbar)^2} \int d\mathbf{r} d\mathbf{r}' dq \psi_0^* (\mathbf{r}_0 + q/2) \psi_0 (\mathbf{r}_0 - q/2) \times \sum_{\gamma, \gamma'} D_{\gamma} D_{\gamma'} e^{i/h[H(\mathbf{r}_0 - \mathbf{r}_0') + p_0 \cdot q]/2}. \tag{3}
\]

The double sums in Eq. (3) contain rapidly oscillating phases \((S_\gamma - S_{\gamma'})/\hbar\) which are assumed to vanish unless \( \gamma \) and \( \gamma' \) are correlated. The main, diagonal contribution to \( \rho_{\text{sc}} \) arises from pairs \( \gamma = \gamma' \) that, upon employing a sum rule [14], yield the classical decay \( \rho_{\text{cl}}(t) = (e^{-t/\tau_c}) \). Here \( (F) = (2\pi\hbar)^{-2} \int d\mathbf{r} d\mathbf{p} F(\mathbf{r}, \mathbf{p}) \rho_{\text{P}}(\mathbf{r}, \mathbf{p}) \), where \( \rho_{\text{P}}(\mathbf{r}, \mathbf{p}) = \int dq \psi_0^* (\mathbf{r}_0 + q/2) \psi_0 (\mathbf{r}_0 - q/2) e^{i(h/\hbar)p_0} \) is the Wigner transform of the initial state and \( \tau_d = \Omega(E)/2\pi p \), with \( \Omega(E) = \int d\mathbf{r} d\mathbf{p} \delta(E - H(\mathbf{r}, \mathbf{p})) \) and \( w \) the size of the opening. For two-dimensional chaotic billiards \( \tau_d(p) = m\pi A/\hbar p \). For initial states with small energy dispersion \( \rho_{\text{cl}}(t) = e^{-t/\tau_d(p)} \). Our subsequent analysis is valid for times \( \lambda t \gg 1 \) where \( \lambda \) denotes the Lyapunov exponent. Furthermore we assume a small opening such that \( \lambda \tau_d \gg 1 \), while the number of channels \( N = t_H/\tau_d \) is still large.

For systems with time reversal symmetry, leading-order quantum corrections to \( \rho_{\text{cl}}(t) \) arise from off-diagonal contributions to the double sum in Eq. (3), given by pairs of correlated orbits depicted as full and dashed line in Fig. 1(a), as in related semiclassical treatments [11,14–19]. The two orbits are exponentially close to each other along the two open “legs” and along the loop [14], but deviate in the intermediate encounter region [box in Fig. 1(a)]. Its length is \( t_{\text{enc}} = \lambda^{-1}\ln(c^2/|su|) \) [12], where \( c \) is a classical constant, and \( s \) and \( u \) are the stable and unstable coordinates in a Poincaré surface of section (PSS) in the encounter region. Such “two-leg-loops” (2ll) are based on orbit pairs with \( S_\gamma - S_{\gamma'} = su \) and a density \( w_{2ll}(s, u, t) = [t - 2t_{\text{enc}}(s, u)]^2/[2\Omega(E)t_{\text{enc}}(s, u)] \) [19]. Invoking the sum rule, the double sum in Eq. (3) is replaced by \( \int du \int ds e^{i(h/\hbar)s} w_{2ll}(s, u, t) e^{i(h/\hbar)su} \). Here \( e^{i\omega/\tau_d} \) accounts for the fact that if the first encounter stretch is inside \( A \) the second must also be inside \( A \). This gives the 2ll contribution [Fig. 1(a)] to \( \rho(t) \) (for \( t < t' \)):

\[
\rho_{2ll}(t) = e^{-t/\tau_d} \left(-2 \frac{t}{t_H} + \frac{t^2}{2\tau_d t_H}\right). \tag{4}
\]

The linear term in Eq. (4) violates unitarity, since it does not vanish upon closing the system, i.e., as \( \tau_d \to \infty \). This is cured by considering a new type of diagrams. These orbit pairs, to which we refer as one-leg-loops, are characterized by an initial or final point inside the encounter region (Figs. 1(b) and 1(c)). They are relevant for open orbits starting or ending inside \( A \) and hence have not arisen in conductance treatments based on lead-connecting paths, since at an opening the exit of one encounter stretch implies the exit of the other one. For their evaluation consider the time \( t' \) between the initial or final point of the trajectory and the PSS, defined in the zoom into Fig. 1(b). Then \( t_{\text{enc}}(t', u) = t' + \lambda^{-1}\ln(c/|u|) \) and \( S_\gamma - S_{\gamma'} = su \) for any position of the PSS. The density of encounters is \( w_{1ll}(s, u, t) = 2 \lambda^{-1}\ln(c/|u|) dt'[t - 2t_{\text{enc}}(t', u)]/[\Omega(E)t_{\text{enc}}(t', u)] \), where the prefactor 2 accounts for the two cases of beginning or ending in an encounter region. We evaluate this contribution by modifying \( \rho_{\text{cl}}(t) \) by \( e^{i\omega/\tau_d} \) as before and integrating over \( u \) and \( t' \). To this end we substitute [17] \( t' = t + \lambda^{-1}\ln(c/|u|) \), \( \sigma = c/u \) and \( s = su/c^2 \), with integration domains \(-1 < x < 1, 1 < \sigma < c/\sqrt{\sigma} \) and \( 0 < t' < \lambda^{-1}\ln(1/|x|) \). Note that the limits for \( t' \) include the case when the paths do not have a self-crossing in configuration space [Fig. 1(c)]. The integration yields

\[
\rho_{1ll}(t) = 2 \frac{t}{t_H} e^{-t/\tau_d}. \tag{5}
\]

It precisely cancels the linear term in \( \rho_{2ll} \), Eq. (4), i.e.,

\[
\rho_{2ll}(t) + \rho_{1ll}(t) = e^{-t/\tau_d} t^2/(2\tau_d t_H), \quad \text{recovering unitarity.}
\]

The next-order quantum corrections are obtained by calculating [22] 1ll and 2ll contributions of diagrams as discussed in [12]. Together with Eqs. (4) and (5), this yields for systems with time reversal symmetry

\[
\rho_{w}(t) \approx e^{-t/\tau_d} \left(1 + \frac{t^2}{2\tau_d t_H} - \frac{t^2}{3\tau_d^2 t_H} + \frac{5t^4}{24\tau_d^3 t_H^3}\right). \tag{6}
\]

The term quadratic in \( t \) represents the weak-localization-type enhancement of the quantum survival probability. The expansion in Eq. (6) agrees with RMT [8].

For systems without time reversal symmetry the calculation of the relevant one- and two-leg-loops gives, again in...
accordance with RMT [8],
\[
\rho_{w}(t) \simeq e^{-t/\tau_{d}} \left(1 + \frac{t^{4}}{24\tau_{d}^{4}}t^{2}\right). \tag{7}
\]

We finally note that our restriction to localized initial states can be lifted and the results generalized to arbitrary initial states by considering an additional local time average of \( \rho(t) \). This amounts to selecting in Eq. (3) trajectory pairs \( \gamma, \gamma' \) starting at adjacent points [22].

**Continuity equation.**—It is instructive to reformulate the decay problem in terms of paths crossing the opening. To this end we consider the integral version of the continuity equation, \( \frac{\partial}{\partial t} \rho(r, t) + \nabla \cdot \mathbf{j}(r, t) = 0 \), namely

\[
\frac{\partial}{\partial t} \rho(t) = -\int_{S} \mathbf{j}(r, t) \cdot \hat{n}_{s} dx, \tag{8}
\]

where \( S \) is the cross section of the opening with a normal vector \( \hat{n}_{s} \). In Eq. (8), the current density \( \mathbf{j}(r, t) = \frac{1}{(2\pi)^{3}m} \text{Re} \left\{ \overline{\psi}(r) \psi'(r) \nabla \psi(r) \right\} \) can be semiclassically expressed through Eq. (2) in terms of orbit pairs connecting points inside \( A \) with the opening. In the diagonal approximation we obtain \( \int_{S} \mathbf{j}_{\text{diag}} \cdot \hat{n}_{s} dx = e^{-t/\tau_{d}}/\tau_{d} \), consistent with \( \rho_{cl}(t) \). Loop contributions are calculated analogously to those of \( \rho_{w} \) from Eq. (3), giving

\[
\int_{S} (\mathbf{j}_{1\text{ll}} + \mathbf{j}_{1\text{II}}) \cdot \hat{n}_{s} dx = e^{-t/\tau_{d}} \frac{t^{2} - 2t\tau_{d}}{2\tau_{d}^{3}t}. \tag{9}
\]

Time integration of Eq. (8) leads to \( \rho_{2\text{II}}(t) + \rho_{1\text{II}}(t) = e^{-t/\tau_{d}}t^{2}/(2\tau_{d}^{3}t) \), consistent with Eq. (6). The 1II contributions enter into Eq. (9) with half the weight, since 1I's with a short leg (encounter box) at the opening must be excluded. These “missing” paths assure the correct form of quantum deviations from \( \rho_{cl}(t) \).

Higher 2II and 1II corrections to \( \mathbf{j} \) lead to Eqs. (6) and (7). We conclude that both, 2II and 1II contributions to \( \mathbf{j} \) are essential to achieve a unitary flow and thereby to establish a semiclassical version of the continuity equation.

**Ehrenfest time effects.**—The Ehrenfest time \( \tau_{E} \) [23] separates the evolution of wave packets following essentially the classical dynamics from longer time scales dominated by wave interference. While \( \tau_{E} \) effects have been mainly considered for stationary processes involving time integration [15–18,24,25], signatures of \( \tau_{E} \) should appear most directly in the time domain [13,26], i.e., for \( \rho(t) \). Here we semiclassically compute the \( \tau_{E} \) dependence of the weak-localization correction to \( \rho(t) \) in Eq. (6). To this end we distinguish between \( \tau_{E} = \lambda^{-1} \text{ln}(\mathcal{L}/\Lambda_{B}) \), where \( \mathcal{L} \) is the typical system size and \( \Lambda_{B} \) the de Broglie wavelength, and \( \tau_{E}^{\text{cl}} = \lambda^{-1} \text{ln}(w^{2}/(\mathcal{L}\lambda_{B})) \), related to the width \( w \) of the opening [18]. As before we consider that the densities \( w_{2\text{II}}(s, u, t) \) contain the Heaviside function \( \theta(t - 2t_{\text{enc}}) \) (negligible for \( \tau_{E}^{\text{cl}} \ll \tau_{E} \) assuring that the time required to form a 1II or 2II is larger than \( 2t_{\text{enc}} \). Our calculation gives (for \( \tau_{E}^{\text{cl}} \lambda \gg 1 \) with \( \tau_{E} = \tau_{E}^{\text{cl}} + \tau_{E}^{\text{E}} \))

\[
\rho_{2\text{II}}(t) + \rho_{1\text{II}}(t) = e^{-t/\tau_{d}}\left(t - 2\tau_{E}^{2}\right)/(2\tau_{d}^{3}t). \tag{10}
\]

**Numerical simulation.**—The leading-order quantum corrections in Eqs. (6) and (7) were confirmed by numerical simulations for graphs [10]. Here we compare our semiclassical predictions with quantum calculations of \( \rho(t) \) based on the numerical propagation of Gaussian wave packets inside a billiard, a setup much closer to experiment. We chose the desymmetrized diamond billiard (inset Fig. 2) [27] that is classically chaotic (\( \lambda^{-1} = 3\tau_{f} \), with \( \tau_{f} \) the mean free flight time). Its opening \( \omega \) corresponds to \( N = 10 \) channels and \( \tau_{d} = 15\tau_{f} \) for \( \lambda_{B} = 3 \). For the simulations we reach \( t_{H} = 10.6\tau_{d} \) implying \( t' = 3.3\tau_{d}, \tau_{E}' = 0.17\tau_{d} \), and \( \tau_{E}^{\text{cl}} = 0.55\tau_{d} \) (with \( \mathcal{L} = \sqrt{A} \)).

In the upper inset of Fig. 2 we compare the decay \( \rho_{\text{qm}}^{\text{sim}}(t) \) (red full line) for a representative wave packet simulation with the corresponding classical, \( \rho_{\text{cl}}^{\text{sim}}(t) \) (dashed line), obtained from an ensemble of trajectories with the same phase space distribution as the Wigner function of the initial quantum state. \( \rho_{\text{cl}}^{\text{sim}}(t) \) merges into the exponential decay \( e^{-t/\tau_{d}} \), and \( \rho_{\text{qm}}^{\text{sim}}(t) \) coincides with \( \rho_{\text{cl}}^{\text{sim}}(t) \) up to scales of \( t' \). For a detailed analysis of the quantum deviations we consider the ratio \( R(t) = \rho_{\text{qm}}^{\text{sim}}(t) - \rho_{\text{cl}}^{\text{sim}}(t)/\rho_{\text{cl}}^{\text{sim}}(t) \). The red dots in Fig. 2 represent an average of \( R(t) \) over 27 different opening positions and initial momentum directions. The dashed and full curve depict the semiclassical results based on the quadratic term in Eq. (6).
(dominant for the $t/\tau_d$ range displayed) and on Eq. (10). The overall agreement of the numerical data with the full curve indicates $\tau_E$ signatures. We note, however, that we cannot rule out other nonuniversal effects (e.g., due to scars [9], short orbits, diffraction, or fluctuations of the effective $\tau_d$ [7]) that may also yield time shifts. Furthermore the individual numerical traces $R(t)$ exhibit strong fluctuations (reflected in a large standard deviation in Fig. 2). A numerical confirmation of the $\log(1/\hbar)$ dependence of $\tau_E$ seems to date impossible for billiards.

**Photodissociation and photodissociation cross sections.**—Related to the decay problem are photodissolution processes where a molecule [28] (or correspondingly an atom) is excited into a classically chaotic, subsequently decaying resonant state. In dipole approximation, the photodissociation cross section of the molecule excited from the ground state $|g\rangle$ is $\sigma(e) = \text{ImTr}[AG(e)]$, where $G(e)$ is the retarded molecule Green function, $\hat{A} = [\varepsilon/(\varepsilon + \hat{e}_0)](\hat{\phi}\rangle|\hat{\phi}\rangle$ and $|\hat{\phi}\rangle = D|\hat{g}\rangle$, with $D = \mathbf{d} \cdot \hat{e}$ the projection of the dipole moment on the light polarization $\hat{e}$. The two-point correlation function of $\sigma(e)$ is defined as

$$C(\omega) = \langle \sigma(e + \omega \Delta / 2)\sigma(e - \omega \Delta / 2) \rangle_{\omega} / \langle \sigma(e) \rangle_{\omega}^2 - 1. \quad (11)$$

Here $\langle \sigma(e) \rangle_{\omega} = \pi(2\pi\hbar)^{-1} \int d\mathbf{r} d\mathbf{p} A_{\omega}(\mathbf{r}, \mathbf{p}) \delta(e - H(\mathbf{r}, \mathbf{p}))$ semiclassically, with Wigner transform $A_{\omega}$ of $\hat{A}$. Previous semiclassical treatments of $C(\omega)$ [29,30] were limited to the diagonal approximation. To compute off-diagonal (loop) terms we consider $Z(\tau) = \int_{-\infty}^{\infty} d\omega e^{2\pi i\omega \tau} C(\omega)$ with $\tau = t/t_f$. Semiclassically, $Z_{\text{sc}}(\tau)$ is again given by a double sum over orbits with different initial and final points in an open system with $N$ decay channels. Because of rapidly oscillating phases from the action differences, only two possible configurations of those points contribute [29]: (i) orbits in a sum similar to Eq. (3) leading to a contribution as for $\rho_{\text{sc}}(t)$; (ii) trajectories in the vicinity of a periodic orbit. Expanding around it, as in [30], leads to the spectral form factor $K_{\text{sc}}(\tau)$ of an open system. From (i) and (ii) we have $Z_{\text{sc}}(\tau) = K_{\text{sc}}(\tau) + 2\rho_{\text{sc}}(\tau)$ for the time reversal case. Up to second order in $\tau > 0$ we find $K_{\text{sc}}(\tau) = e^{-N\tau}N(2\tau - 2\tau^2)$ [31] and $\rho_{\text{sc}}(\tau) = e^{-N\tau}(1 + N\tau^2/2)$ [Eq. (6)]. Thereby $Z_{\text{sc}}(\tau) = e^{-N\tau}[2 + 2\tau + (N - 2)\tau^2]$, confirming a conjecture of [32]. Its inverse Fourier transform yields the two-point correlation (with $\Gamma = 2\pi\omega/N$)

$$C_{\text{sc}}(\Gamma) = \frac{4}{N} \frac{1}{1 + \Gamma^2} \left[ 1 + \frac{1 - \Gamma^2}{N} \frac{1}{1 + \Gamma^2} + \frac{N - 2}{N^2} \frac{1 - 3\Gamma^2}{(1 + \Gamma^2)^2} \right]. \quad (12)$$

The first two diagonal terms agree with [29]; the third term represents the leading quantum correction.

To conclude, we presented a general semiclassical approach to the problems of quantum decay and photo cross section statistics in open chaotic quantum systems.

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