A six-functor formalism for syntomic cohomology of p-adic formal schemes

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Contents

Introduction v
- Leitfaden viii
- Conventions viii
- Acknowledgments ix
- Addendum ix

Chapter 1. Remarks on Six-Functor Formalisms 1
- 1.1. Recollections on Six-Functor Formalisms 2
- 1.1.1. The Definition 2
- 1.1.2. Constructing six-functor formalisms 2
- 1.1.3. Extending to stacks 5
- 1.2. Additively oriented six-functor formalisms 7
- 1.2.1. Cohomologically smoothness 7
- 1.2.2. Additive orientations and the Projective Bundle formula 9
- 1.2.3. The Motivic realization 11
- 1.2.4. Cycle classes 16
- 1.2.5. Computing the dualizing sheaf 19

Chapter 2. Adic stacks 25
- 2.1. Recollections on Analytic stacks 26
- 2.1.1. Analytic rings 26
- 2.1.2. Analytic stacks 29
- 2.1.3. Proper maps of analytic stacks 32
- 2.1.4. Examples of surjections 33
- 2.2. Derived Huber pairs 35
- 2.2.1. Affine derived adic spaces 35
- 2.2.2. Valuation spectra of affine derived adic spaces 38
- 2.2.3. Derived adic spaces and their compactifications 46
- 2.2.4. Discrete adic spaces 50
- 2.3. Classical Tate p-adic spaces 52
- 2.3.1. Bounded solid affinoids 52
- 2.3.2. Étale morphisms of Tate adic spaces 53
- 2.4. Derived formal schemes 55
- 2.4.1. Completely descendable morphisms 55
- 2.4.2. Open immersions and étale morphisms 58
- 2.4.3. The deformation to the normal bundle 61
- 2.5. Formal schemes as analytic stacks 63
- 2.5.1. Locally proper analytification 63
- 2.5.2. Geometric analytification 64
- 2.5.3. The generic fibre via admissible blowups 66

2.5.4. Proper maps of formal schemes 73

Chapter 3. The Prismatisation and its friends as analytic stacks 77
3.1. Preliminaries 78
3.1.1. Recollections on integral perfectoids 78
3.1.2. The naive syntomic topology 83
3.1.3. Prisms and the Prismatisation of $\mathbf{Z}_p$ 85
3.1.4. Prismatic cohomology of semiperfectoids 88
3.1.5. Perfectoidisation of Tate algebras 89
3.1.6. The naive syntomic topology on Tate algebras 92
3.2. Analytifying the Prismatisation 94
3.2.1. Recollections on the prismatisation 94
3.2.2. Recollections on the Hodge-Tate locus 99
3.2.3. Locally proper prismatisation 103
3.2.4. The Hodge-Tate locus 106
3.2.5. The étale locus 108
3.2.6. The Prismatisation 113
3.3. Analytifying the syntomification 119
3.3.1. Recollections on filtered objects 119
3.3.2. Recollections on the Nygaard filtration 120
3.3.3. The Nygaard filtered Prismatisation 126
3.3.4. The Syntomification 131

Chapter 4. Six functors for syntomic cohomology 133
4.1. The six-functor formalism 133
4.1.1. The Construction 133
4.1.2. The additive orientation 134

Bibliography 137

Introduction

To understand the dualities one obtains from a fixed cohomology theory it turns out that, instead of considering just the cohomology theory, one should directly build a whole coefficient theory and build/understand duality on the level of this coefficient theory. This was first embraced by Pierre Deligne, using étale cohomology in his proof of the Weil conjectures, and runs under the name *six-functor formalism.*

Fix a prime p. In this text, we construct such a coefficient theory for syntomic cohomology of p-adic formal schemes as invented by Bhatt-Morrow-Scholze [**BMS16**] [**BMS18**] and Bhatt-Scholze [**BS19**]. The main theorem can be summarized as follows (see 4.1.1.2):

Theorem (A). *There exists a six-functor formalism*

$$X \mapsto \mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\mathbb{\Delta}}(X)$$

on the category of derived p-adic formal schemes satisfying the following:

(A) For any derived p-adic formal scheme X,

$$\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\mathbb{\Delta}}(X)$$

is a stable presentable ∞-category[1]*. Furthermore, the full subcategory of dualizable objects identifies with the category of perfect F-gauges as defined in [**Bha22**][6.1.]. In particular, there is an identification*

$$R\Gamma_{syn}(X, \mathbf{Z}_p(n)) \simeq \mathrm{Hom}_{\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\mathbb{\Delta}}(X)}(\mathbf{1}_X, \mathbf{1}_X(n))$$

*functorial in X, where the left-hand side is the syntomic cohomology as defined in [**BL22a**][7.4.] and the twist is given by the Breuil-Kisin twist.*

(B) The functor $\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\mathbb{\Delta}}(\_)^*$ is an étale sheaf and the functor $\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\mathbb{\Delta}}(\_)_!$ and étale cosheaf.

(C) Any map which locally in the étale topology factors as an integral map followed by a map of finite type is !-able. Proper maps are cohomologically proper, and étale maps are cohomologically étale.

(D) For any derived p-adic formal scheme X, the object

$$\mathbf{1}_X(-1) := cof(\mathbf{1}_X \to f_*\mathbf{1}_{\mathbf{P}^1_X})$$

is $\otimes$-invertible and identifies with the $\otimes$-inverse of the Breuil-Kisin twist. Furthermore, any smooth morphism $f\colon X \to S$ is cohomologically smooth, and there is a canonical identification

$$\omega_f := f^!(\mathbf{1}_S) \simeq \mathbf{1}_X(d)$$

of the dualizing sheaf, where d is given by the relative dimension of f.

[1] For the rest of this text we will refer to ∞-categories as categories.

The starting idea to prove, or rather construct, the above theorem is the combination of two developments.

On one hand Bhatt-Lurie [**BL22a**] [**BL22b**] and Drinfeld [**Dri20**] introduced a formal stack X^{syn}, one can associate to a p-adic formal scheme X, together with a line bundle $\mathcal{O}_{X^{\mathrm{syn}}}\{1\}$ on this stack. From this data, one can recover the syntomic cohomology of X in weight n as the mapping spectrum

$$\mathrm{Hom}(\mathcal{O}_{X^{\mathrm{syn}}}, \mathcal{O}_{X^{\mathrm{syn}}}\{n\})$$

in the category of quasi-coherent sheaves on X^{syn}. This gives a good theory of coefficients, which has one downside, though. Namely, how to define compactly supported cohomology for quasi-coherent cohomology of algebraic geometric stacks or schemes is a subtle question, and for example, it can not be obtained as a functor acting on quasi-coherent sheaves.

Here, the second development comes in. To understand compactly supported quasi-coherent cohomology as a functor acting on a category of quasi-coherent sheaves, Clausen-Scholze enlarge the latter category and also consider certain completed topological modules [**CS19b**]. Their theory of compactly supported quasi-coherent cohomology not only works for algebraic geometric stacks, but also for stacks of a more analytic nature. The objects they consider are called analytic stacks [**CS24**], and the content of the above theorem then becomes answering the question of how one should interpret the syntomification as an analytic stack.

The syntomification can be constructed as a pushout

$$\begin{tikzcd} X^{\Delta} \coprod X^{\Delta} \arrow[r] \arrow[d, "can"'] & X^{\mathcal{N}} \arrow[d] \\ X^{\Delta} \arrow[r] & X^{\mathrm{syn}} \end{tikzcd}$$

where the upper vertical map is given by two disjoint open immersions. In particular to interpret this stack as an analytic stack, it is enough to interpret each term in the defining cospan. Here X^{Δ} denotes the Prismatisation of X, which is a stack whose quasi-coherent sheaves recover the prismatic cohomology of X and the stack $X^{\mathcal{N}}$ represents the prismatic cohomology of X together with its Nygaard filtration in the same way. One way to understand those stacks is to descend them from so-called quasi-regular semiperfectoid rings. Those are p-complete ring S who's prismatic cohomology Δ_S becomes a static ring, such that one can define

$$S^{\Delta} \simeq \mathrm{Spf}(\Delta_S).$$

Similar the Nygaard filtration on Δ_S becomes static and the stack $S^{\mathcal{N}}$ is then given by the Rees construction of this Nygaard filtration. In order to capture all derived p-adic formal schemes, we weaken the notion of a quasi-regular semiperfectoid.

Definition. An animated ring is called *semiperfectoid*, if it is derived p-complete and admits a π_0-surjection from an integral perfectoid ring.

On those semiperfectoid rings, the prismatic cohomology and its Nygaard filtration are connective, such that the above constructions still make sense if one interprets them in the world of derived formal schemes. We can then associate an analytic stack with these derived affine formal schemes.

One then glues these stacks using the so-called quasi-syntomic topology. In order to make the so obtained surjections stay surjections in the world of analytic stacks, we also weaken this notion slightly:

Definition. A morphism of derived p-complete animated rings is called a *naive syntomic cover*, if it can be refined by a map, which lives in the smallest class stable under pushouts along arbitrary maps and composition generated by maps of the form

$$\mathbf{Z}_p\langle x_i | i \in I\rangle \to \mathbf{Z}_p\langle x_i^{\frac{1}{p^\infty}} | i \in I\rangle$$

for some set I.

Using the naive syntomic topology, one can still cover any derived p-complete animated ring by a semiperfectoid, and the main point of making the construction of the above stacks work is the following (see 3.2.1.8 and 3.3.2.10):

Proposition. *Given a naive syntomic cover $S \to S'$ of semiperfectoids, we have the following:*

- *The map $\Delta_S \to \Delta_{S'}$ is descendable in the category $\mathcal{D}_{(p,I)\text{-}comp}(\Delta_S)$ of derived (p, I)-complete Δ_S-modules.*
- *The map $Fil^\bullet_{\mathcal{N}}\Delta_S \to Fil^\bullet_{\mathcal{N}}\Delta_{S'}$ is descendable in the category $\mathcal{DF}_{(I,p)\text{-}comp}(Fil^\bullet_{\mathcal{N}}\Delta_S)$ of (I, p)-complete objects in filtered modules over the Nygaard filtered prismatic cohomology of S.*

Here we write $Fil^\bullet_{\mathcal{N}}\Delta_S$ for the Nygaard filtration and the notion of descendability is a strong descent assertion [**Mat16**] which is closely related to the notion of coverings used in the formalism of analytic stacks.

After having the construction, the remaining part is to prove Poincaré duality. Here we give a general strategy slightly extending results from [**Zav23**]. Given a six-functor formalism $\mathcal{D}$ on a category $\mathcal{C}$, where we write $\mathcal{C}_E$ for the subcategory generated by morphisms for which one has defined the compactly supported cohomology. Then one can assume that there exists a finite limit preserving functor

$$\mathcal{S}m_B^{\mathrm{sep}} \to \mathcal{C}_E$$

from the category of separated smooth schemes over a fixed scheme B, which preserved étale and proper maps. Then one can make sense of the Tate twist

$$\mathbf{1}_X(1) \in \mathcal{D}(X)$$

for any $X \in \mathcal{C}$ and further ask for the existence of a theory of first Chern classes. A way to formulate this is to ask for a natural transformation

$$R\Gamma_{\text{dét}}(\_, \mathbf{G}_m)[1] \to \mathrm{Hom}_{(\_)}(\mathbf{1}_{(\_)}, \mathbf{1}_{(\_)}(1))$$

where there dét-topology is generated by those cohomologically étale morphism for which $\mathcal{D}^*$ satisfies descent. Using these first Chern classes, one can construct a morphism

$$\bigoplus_{i=0}^{d} \mathfrak{C}^*(B)(d-i) \to \mathfrak{C}^*(\mathbf{P}^d_B)(d)$$

which we assume to be an isomorphism. This is referred to as the *Projective bundle formula* and essentially the only assertion we use to prove Poincaré duality. Let us call the above data an *additive orientation* of the six-functor formalism $\mathcal{D}$, then we prove the following (see 1.2.4.8 and 1.2.5.13):

Theorem (B). *Given an additively oriented six-functor formalism. Then any map, which locally in the dét-topology factors as a cohomologically étale morphism followed by a projection $\mathbf{A}^n_S \to S$ is cohomologically smooth. Furthermore, if we assume the existence of the deformation to the normal bundle for sections of smooth morphisms*[2]*, for any smooth morphism*[3] *$f\colon X \to S$ in $\mathcal{C}$, we obtain a canonical identification*

$$\omega_f := f^!(\mathbf{1}_S) \simeq \mathbf{1}_X(d)$$

where d denotes the relative dimension of f.

Leitfaden. In section 1.1 of Chapter 1 we recall the notion of a six-functor formalism and in section 1.2 we explain the proof of the second theorem stated above.

In section 2.1, 2.2, and 2.3 of Chapter 2, we recall the notion of an analytic stack and discuss how to understand Huber pairs as such. Most of the material is a recollection from [**CS24**], and the reader familiar with this theory can easily skip these sections. In the sections 2.4 of Chapter 2, we recall some notions of derived formal schemes, and in section 2.5 we explain two ways to understand derived formal schemes as analytic stacks. In the end, we discuss the compatibility of the notions of properness in the two worlds 2.5.4.5. Here, the generality in which this is done is not strictly necessary to deduce the main theorem, but it might be of independent interest.

In section 3.1 of Chapter 3, we recall a bit about the theory around prismatic cohomology. Most importantly for this text, we define the naive syntomic topology on p-adic formal schemes 3.1.2 and Tate p-adic spaces 3.1.6. In section 3.2, we prove one half of the above descendability result 3.2.1.8 and define the solid Prismatisation 3.2.6. Finally, in section 3.3 we prove the other half of the above descendability assertion 3.3.2.10 and construct the solid Nygaard filtered Prismatisation 3.3.3 and the solid syntomification 3.3.4.

In the last Chapter 4 we deduce the main theorem.

Conventions. Any ring is commutative, and p always denotes a prime number.

Essentially all categories in this text are honest ∞-categories. In order to avoid so many "$\infty's$" we decided to refer to those just as categories. Sometimes we will use the term ∞-topos to highlight that we use an assertion, which needs the ∞. Still, by a topos we mean an ∞-topos.

We often consider "big topoi" like sheaves on a non-small category. Here the convention is that we consider the respective κ small versions for κ a strong limit cardinal, and then take the colimit over all strong limit cardinals. Note that this way, all exactness properties of a topos involving finite limits and small colimits survive.

We often refer to an effective epimorphism in a topos as a surjection.

[2]Note that this existence is a property.

[3]We give assertions that the class of smooth morphisms has to satisfy, which essentially follow from a Jacobi criterion.

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Addendum. The main objective of this text is to understand syntomification as a solid analytic stack. The former is a formal stack (in the sense of formal schemes), and there is a natural way to understand formal stacks as analytic stacks analogous to how one can understand formal schemes as adic spaces. Originally, the author thought that if one uses this to understand the syntomification, there will be closed immersions, which are not proper through the eyes of the so-obtained six-functor formalism. It was pointed out to the author by Johannes Anschütz, that the type of example of this phenomenon is not an example, and the current state is that the author believes that there is a more natural and more interesting solid syntomification. The interested reader should either contact the author or go to the hopefully already existing arXiv version of this work.

CHAPTER 1

Remarks on Six-Functor Formalisms

In the following chapter, we will first recall some elementary notions concerning six-functor formalisms.

Afterwards, we will explain a general strategy to prove cohomologically smoothness for smooth morphisms, reducing the problem to an often-satisfied computation of the cohomology of projective space (normally referred to as the *Projective bundle formula*). This is strongly motivated and a slight generalization of [**Zav23**] using also ideas from [**AI22**] [**AHI24**] [**AHI25**] and a construction due to Longke Tang.

1.1. Recollections on Six-Functor Formalisms

1.1.1. The Definition. First, let us clarify what we mean by a six-functor formalism. There are many discussions on what a six-functor formalism is supposed to capture [**HM24**] [**GR17**] [**CD09**] [**Sch23**]. In particular, we will focus on the formal definition we will use. The definition is taken from [**HM24**].

1.1.1.1. Consider a pair $(\mathcal{C}, \mathcal{C}_E)$ where $\mathcal{C}$ is a category and $\mathcal{C}_E \subset \mathcal{C}$ a wide subcategory satisfying the following:

(a) Morphisms in $\mathcal{C}_E$ are closed under pullbacks along morphisms in $\mathcal{C}$.
(b) $\mathcal{C}_E$ admits pullbacks and the inclusion $\mathcal{C}_E \subset \mathcal{C}$ preserves those.

Such data is called a *geometric setup* and given such data we can construct the category

$$\mathcal{S}pan(\mathcal{C}, \mathcal{C}_E)$$

which informally can be described as follows (see [**HM24**][2.2] for an honest construction):

- Objects are given by the objects in $\mathcal{C}$.
- A morphism from X to Y is given by a span

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

  where the right leg lives in $\mathcal{C}_E$. To compose such spans, one takes fibre products of the inner cospan.

Assuming that $\mathcal{C}$ admits finite products, the category $\mathcal{S}pan(\mathcal{C}, \mathcal{C}_E)$ can be equipped with a symmetric monoidal structure induced by the cartesian product in $\mathcal{C}$.

Definition 1.1.1.2. Given a geometric setup $(\mathcal{C}, \mathcal{C}_E)$, such that $\mathcal{C}$ admits finite products. The a *six-functor formalism* on $(\mathcal{C}, \mathcal{C}_E)$ is given by a lax symmetric monoidal functor

$$\mathcal{D} \colon \mathcal{S}pan(\mathcal{C}, \mathcal{C}_E) \to \mathcal{P}r^L.$$

Where $\mathcal{P}r^L$ denotes the category of presentable categories with colimit-preserving functors as morphisms and equipped with the Lurie-tensor product.

1.1.2. Constructing six-functor formalisms. We now recall the most common strategy to construct six-functor formalisms.

1.1.2.1. Consider a geometric setup $(\mathcal{C}, \mathcal{C}_E)$. Then, often the subcategory $\mathcal{C}_E$ often can be controlled by two further wide subcategories

$$\mathcal{C}_I, \mathcal{C}_P \subset \mathcal{C}_E$$

satisfying the following:

(a) Morphisms in $\mathcal{C}_I$ as well as morphisms in $\mathcal{C}_P$ are closed under base change along morphisms in $\mathcal{C}$. Furthermore, both of those classes are left cancellable[1].

[1] This means that if a composition $g \circ f$ and g are in this class, then f is in this class as well.

(b) For any morphism f in $\mathcal{C}_E$, there exists a factorization $j \circ p \simeq f$, such that j lives in $\mathcal{C}_I$ and p lives in $\mathcal{C}_P$.
(c) Any morphism in $\mathcal{C}_I \cap \mathcal{C}_P$ is n-truncated for some $n \geq -2$ (possibly depending on the morphism).

For the rest of the subsection, we will assume that such wide subcategories are given.

Remark 1.1.2.2. If $\mathcal{C}$ is some category of geometric objects, morphisms in $\mathcal{C}_I$ will often be open immersions and morphisms in $\mathcal{C}_P$ proper morphisms. Morphisms in $\mathcal{C}_E$ are then given by those maps for which one can find a compactification. This will be the situation for all examples in this text.

1.1.2.3. Let us assume we have given a functor

$$\mathcal{D}^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$$

such that for any morphism $f\colon X \to S$ the functor $f^*\colon \mathcal{D}(S) \to \mathcal{D}(X)$ admits a right adjoint f_*. Then to obtain the !-functors one normally asks for the following.

- For any morphism $j\colon U \to X$ in $\mathcal{C}_I$ the functor

$$j^*\colon \mathcal{D}(X) \to \mathcal{D}(U)$$

also admits a left adjoint $j_!$.
- For any morphism $p\colon X \to S$ in $\mathcal{C}_P$ the functor

$$p_*\colon \mathcal{D}(X) \to \mathcal{D}(S)$$

admits a right adjoint $p^!$.

Having this, for any morphism $f \simeq j \circ p$ in $\mathcal{C}_E$, on then sets

$$f_! \simeq j_! \circ p_*$$

which is supposed to be independent of the choice of compactification and admits a right adjoint $f^!$. Let us make this more accurate.

1.1.2.4. Consider a functor

$$\mathcal{D}^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}r^L.$$

Note that this in particular means that for any morphism $f\colon X \to S$ the functor f^* admits a right adjoint f_*. Then we will require the following:

(1) For any morphism $j\colon U \to X$, the functor j^* admits a left adjoint $j_!$. Furthermore, for any Cartesian square

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "j'"'] & U \arrow[d, "j"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

with j in $\mathcal{C}_I$ the Beck–Chevalley transformation

$$BC_!\colon j'_!(f')^* \to j'_!(f')^*j^*j_! \simeq j'_!(j')^*f^*j_! \to f^*j_!$$

is an isomorphism.

(2) For any morphism $p\colon X \to S$ in $\mathcal{C}_P$, the right adjoint p_* of the functor p^* preserves colimits. Furthermore for any Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

with p in $\mathcal{C}_P$ the Beck-Chevalley transformation

$$BC_*\colon f^*p_* \to p'_*(p')^*f^*p_* \simeq p'_*(f')^*p^*p_* \to p'_*(f')^*$$

is an isomorphism.

(3) For any cartesian square

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ S' \arrow[r, "j"'] & S \end{tikzcd}$$

with j in $\mathcal{C}_I$ and p in $\mathcal{C}_P$, by (1), we obtain a commutative square

$$\begin{tikzcd} \mathcal{D}(S') \arrow[r, "j_!"] \arrow[d, "(p')^*"'] & \mathcal{D}(S) \arrow[d, "p^*"] \\ \mathcal{D}(X') \arrow[r, "j'_!"'] & \mathcal{D}(X). \end{tikzcd}$$

From this, we obtain a double Beck-Chevalley map

$$BC_{!,*}\colon j_!p'_* \to p_*p^*j_!p'_* \simeq p_*j'_!(p')^*p'_* \to p_*j'_!$$

and we will ask this map to be an isomorphism.

Recall that, to obtain a full six-functor formalism, the categories $\mathcal{D}(X)$ should be equipped with a symmetric monoidal structure compatible with the other functors. To capture this, let us first, recall that $\mathcal{P}r^L$ admits a symmetric monoidal structure by restricting the tensor product on cocomplete categories to presentable categories [**Lur17**].

1.1.2.5. Let us now assume our functor can be enhanced to a lax symmetric monoidal functor

$$\mathcal{D}\colon (\mathcal{C}^{\mathrm{op}})^{\otimes} \to (\mathcal{P}r^L)^{\otimes}.$$

Then we will say $\mathcal{D}$ is lax symmetric monoidal (I, P)-biadjointable, if the underlying functor is (I, P)-biadjointable and the following assertions hold:

(A) For any morphism $j\colon U \to X$ in $\mathcal{C}_I$ the natural transformation[2]

$$j_!(j^* \otimes id) \to id \otimes j_!$$

is an isomorphism.

(B) For any morphism $p\colon X \to S$ in $\mathcal{C}_P$ the natural transformation[3]

$$id \otimes p_* \to p_*(p^* \otimes id)$$

is an isomorphism.

[2]This is adjoint to the natural transformation $j^* \otimes id \to j^* \otimes j^*j_!$ induced by the unit.

[3]This is adjoint to the natural transformation $p^* \otimes p^*p_* \to p^* \otimes id$ induced by the counit.

1.1.2.6. Given a lax monoidal functor

$$\mathcal{D}^*\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{P}r^L$$

satisfying the assertions from 1.1.2.4 and 1.1.2.5 using [**Man22**][A.5] or [**CLL25**] we can extend this functor to a six functor formalism on the geometric setup $(\mathcal{C}, \mathcal{C}_E)$. All six functor formalisms in this text will arise in this way, and we want to point out that we thus have pre-specified cohomologically proper as well as cohomologically étale morphisms.

1.1.3. Extending to stacks. Let us fix a six-functor formalism, which has been constructed as recalled in the last section. Then, following [**HM24**][3.4], we will now recall a way to extend the six-functor formalism to certain sheaves on $\mathcal{C}$.

Definition 1.1.3.1. We make the following definitions:

(a) A sieve on $\mathcal{C}$ is called a *universal $*$-cover*, if the functor $\mathcal{D}^*$ universally descends along it.
(b) A sieve on $\mathcal{C}$ is called a *universal !-cover*, if it is generated by a small family of !-able maps and $\mathcal{D}^!$ universally descends along it.

We will call the topology generated by universal $*$-covers and universal !-covers the $\mathcal{D}$-topology.

Remark 1.1.3.2. Assuming the six functor formalism

$$\mathcal{D}\colon \mathcal{S}pan(\mathcal{C}, \mathcal{C}_E) \to \mathcal{P}r^L$$

is monoidal, any !-cover is automatically a universal. That is if $\mathcal{D}^!$ descends along a covering it automatically descends along any pullback. Furthermore any !-cover is automatically a $*$-cover, such that in that case the $\mathcal{D}$-topology is generated by !-covers. If this is the case, we will refer to the $\mathcal{D}$-topology also as !-topology [**CS24**][Lecture 17].

1.1.3.3. Let us assume that the $\mathcal{D}$-topology is sub-canonical, then by [**HM24**][3.4.2.+3.4.11.] there exists a wide subcategory

$$\mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{E'} \subset \mathcal{S}h_{\mathcal{D}}(\mathcal{C})$$

spanned by a minimal class of morphisms such that:

(a) The right Kan-extension of $\mathcal{D}$ gives a unique extension to a six-functor formalism on $(\mathcal{S}h_{\mathcal{D}}(\mathcal{C}), \mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{E'})$.
(b) A map who's pullback to any object in $\mathcal{C}$ lives in $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{\mathbf{E}'}$, lives in $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{E'}$.
(c) A map who !-locally on source and target lives in $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{\mathbf{E}}$, lives in $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{E'}$.
(d) Any map $f\colon X \to S$ in $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})_{E'}$ with $S \in \mathcal{C}$ is !-locally on X in $\mathcal{C}_E$.

We will denote the six-functor formalism on $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})$ obtained in this way also by $\mathcal{D}$.

Construction 1.1.3.4. Given a morphism $f\colon X \to S$ in $\mathcal{S}h_{\mathcal{D}}(\mathcal{C})$, then we can consider the diagram

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
& X \times_S X \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & S.
\end{tikzcd}$$

- Assuming we have an identification $\Delta_* \xrightarrow{\sim} \Delta_!$, then we obtain a natural transformation
$$f^* f_! \simeq q_! p^* p_* \Delta_* \to q_! \Delta_* \simeq id$$
coming from the co-unit. This map is adjoint to a map
$$f_! \to f_*.$$
- Assuming we have an identification $\Delta^! \xrightarrow{\sim} \Delta^*$. Then the co-unit gives us a natural transformation
$$p_! q^* f^! \simeq f^* f_! f^! \to f^*.$$
Then, using the above identification and the map adjoint to the latter map, we get a natural transformation
$$f^! \simeq \Delta^* q^* f^! \to \Delta^* p^! f^* \simeq \Delta^! p^! f^* \simeq f^*.$$

Definition 1.1.3.5. A morphism $f\colon X \to S$ in $\mathcal{S}h_D(\mathcal{C})$ is called:

(a) *cohomologically* 0*-proper*, if it $\mathcal{D}$-locally on the target lives in $\mathcal{C}_P$.
(b) *cohomologically* n*-proper* for some $n \geq 1$, if Δ_f is cohomologically $n-1$-proper and the natural transformation
$$f_! \to f_*$$
constructed in 1.1.3.4 is an isomorphism.
(c) *cohomologically proper*, if it is cohomologically n-proper for some $n \geq 0$.
(d) *cohomologically* 0*-étale*, if it $\mathcal{D}$-locally in the target lives in $\mathcal{C}_I$.
(e) *cohomologically* n*-étale* for some $n \geq 1$, if Δ_f is cohomologically $n-1$-étale and the natural transformation
$$f^! \to f^*$$
constructed in 1.1.3.4 is an isomorphism.
(f) *cohomologically étale*, if it is cohomologically n-étale for some $n \geq 0$.

Remark 1.1.3.6. It is easy to check that cohomologically proper morphisms, as well as cohomologically étale morphisms are stable under base change and composition.

Remark 1.1.3.7. Being cohomologically proper as well as being cohomologically étale is local on the target in the $\mathcal{D}$-topology. This follows as in the proof of [**HM24**][4.6.3].

Remark 1.1.3.8. To check that the natural transformations appearing in the definition of cohomologically proper and étale are isomorphisms, it is enough to check that they are isomorphisms after applying the symmetric monoidal unit [**HM24**][4.6.4].

1.2. Additively oriented six-functor formalisms

In the following, we will understand a geometric setup as honestly coming from geometry and explain how to deduce that a six-functor formalism acts on smooth morphisms in this geometry, as expected from the so-called *Projective Bundle formula*. The discussion is highly inspired and will use results from [**Zav23**] and the paper series [**AI22**] [**AHI25**] [**AHI24**].

1.2.1. Cohomologically smoothness. Let us fix a geometric setup $(\mathcal{C}, \mathcal{C}_E)$ and a six-functor formalism $\mathcal{D}$ on it. Then recall the following definition.

Definition 1.2.1.1. A morphism $f\colon X \to S$ in $\mathcal{C}_E$ is called *weakly cohomologically smooth*, if the following holds:

(a) The natural transformation[4]

$$f^!(\mathbf{1}_S) \otimes f^* \to f^!$$

is an isomorphism.

(b) The *dualizing complex* $\omega_f := f^!(\mathbf{1}_S)$ is $\otimes$-invertible and commutes with arbitrary base change. That is for any Cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S \end{tikzcd}$$

the canonical morphism $(g')^* f^!(\mathbf{1}_S) \to (f')^!(\mathbf{1}_{S'})$ is an isomorphism.

A morphism in $\mathcal{C}_E$ is called *cohomologically smooth* if any base change of this morphism is weakly cohomologically smooth.

Remark 1.2.1.2. For a cohomologically smooth morphism $f\colon X \to S$ the induced isomorphism

$$\mathrm{Hom}_X(\mathbf{1}_X, \omega_f \otimes \mathbf{1}_X) \simeq \mathrm{Hom}_S(f_!(\mathbf{1}_X), \mathbf{1}_S)$$

in examples recovers *Poincaré Duality* isomorphisms. Note that if the morphism is also cohomologically proper, the right-hand side recovers homology.

Being cohomologically smooth is local on the source and target with respect to the following topology.

Definition 1.2.1.3. We will say a morphism $U \to X$ in $\mathcal{C}$ is a *dét-cover*, if it is a cohomologically étale !-cover (with respect to $\mathcal{D}$).

We now want to implement some classical geometric objects into our geometric setup.

Definition 1.2.1.4. We will say a geometric setup $(\mathcal{C}, \mathcal{C}_E)$ is *geometrized*, if it comes with finite limit preserving functor

$$\mathcal{S}m_B^{\mathrm{sep}} \to \mathcal{C}_E$$

from the category of separated smooth schemes over some (possibly derived) scheme B.

[4] This is adjoint to the natural transformation $f_!(f^!(\mathbf{1}_S) \otimes f^*) \simeq f_! f^!(\mathbf{1}_S) \otimes id \to id$ induced by the counit.

Furthermore, a six-functor formalism $\mathcal{D}$ on a geometrized geometric setup is called *geometric*, if:

- For any object $S \in \mathcal{C}$ the category $\mathcal{D}(S)$ is stable.
- Any étale morphism in $\mathcal{S}m_B$ is cohomologically étale with respect to $\mathcal{D}$.
- Any proper morphism in $\mathcal{S}m_B$ is cohomologically proper with respect to $\mathcal{D}$.
- Any Nisnevich covering gets sent to a dét-cover.

Remark 1.2.1.5. We will normally not write the functor $\mathcal{S}m_B \to \mathcal{C}$. That is, for example, we will write $\mathbf{A}^n$ for the n-dimensional affine space seen as an object in $\mathcal{C}$ via this functor. We can also define the affine space over an arbitrary object $X \in \mathcal{C}$ via base change.

Remark 1.2.1.6. The Nisnevich topology is famously used in motivic homotopy theory and sits in between the Zariski and the étale topology. The étale topology would be enough for all applications in this text. We just chose this definition for the sake of generality.

Definition 1.2.1.7. We will say a morphism $X \to S$ in $\mathcal{C}$ is *smooth*, if locally on X and S in the dét-topology, it factors as

$$X \to \mathbf{A}^n_S \to S$$

where the first map is cohomologically étale.

How to check if any smooth map becomes cohomologically smooth was answered elegantly in [**Zav23**]. To recall this, let us fix some notations.

Consider a morphism $f\colon X \to S$ in $\mathcal{C}_E$. Then we consider the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_S X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & S.
\end{tikzcd}$$

Definition 1.2.1.8. A *trace-cycle theory* on a morphism $f\colon X \to S$ in $\mathcal{C}_E$ consists of a triple $(\omega_f, \mathrm{tr}_f, \mathrm{cl}_\Delta)$ of:

(a) A $\otimes$-invertible object ω_f in $\mathcal{D}(X)$.
(b) A *trace morphism* $\mathrm{tr}_f\colon f_!\omega_f \to \mathbf{1}_S$ in $\mathcal{D}(S)$.
(c) A *cycle* morphism $\mathrm{cl}_\Delta\colon \Delta_!\mathbf{1}_X \to (p_2)^*\omega_f$ in $\mathcal{D}(X \times_S X)$.

Such that the following hold:

(1) The composition

$$\mathbf{1}_X \xrightarrow{\simeq} (p_1)_!\Delta_!\mathbf{1}_X \xrightarrow{(p_1)_!(\mathrm{cl}_\Delta)} (p_1)_!(p_2)^*\omega_f \xrightarrow{\mathrm{tr}_{p_1}} \mathbf{1}_X$$

is the identity. Where we write $\mathrm{tr}_{p_1} \simeq f^*(\mathrm{tr}_f)$.

(2) The composition

$$\omega_f \xrightarrow{\simeq} (p_2)_!(p_1^*\omega_f \otimes \Delta_!\mathbf{1}_X) \xrightarrow{(p_2)_!(id\otimes \mathrm{cl}_\Delta)} (p_2)_!(p_1^*\omega_f \otimes p_2^*\omega_f) \simeq (p_2)_!p_2^*\omega_f \otimes \omega_f \xrightarrow{\mathrm{tr}_f\otimes id} id \otimes \omega_f \simeq \omega_f$$

is the identity.

Then section 3 in [**Zav23**] implies the following.

Theorem 1.2.1.9. *Consider a geometrized geometric setup $(\mathcal{C}, \mathcal{C}_E)$ and a geometric six-functor formalism $\mathcal{D}$ on it. Then a morphism $f\colon X \to S$ in $\mathcal{C}_E$ is cohomologically smooth, if and only if it admits a trace-cycle theory, and in that case, we can compute the dualizing sheaf as*

$$f^! \mathbf{1}_S \simeq \omega_f$$

. Furthermore the following are equivalent:

(a) Any smooth morphism is cohomologically smooth.
(b) The morphism $\mathbf{A}^1 \to B$ is cohomologically smooth.
(c) The morphism $\mathbf{P}^1 \to B$ is cohomologically smooth.

1.2.2. Additive orientations and the Projective Bundle formula. Let us fix a geometrized geometric setup $(\mathcal{C}, \mathcal{C}_0)$ and a geometric six-functor formalism $\mathcal{D}$ on it.

Definition 1.2.2.1. Let us write $f\colon \mathbf{P}^1 \to B$ for the projection. We say $\mathcal{D}$ admits a *Tate twists*, if the object

$$\mathbf{1}_B(-1) := cof(\mathbf{1}_B \to f_* \mathbf{1}_{\mathbf{P}^1_B})$$

is $\otimes$-invertible. In that case, we will write

$$\mathbf{1}_B(1)$$

for it's $\otimes$-inverse and call it the *Tate twist*.

Remark 1.2.2.2. Using proper base change, the Tate twist defines a cartesian section for $\mathcal{D}^*$. That is, we can define

$$\mathbf{1}_S(1)$$

for an arbitrary $S \in \mathcal{C}$, either the same way or via base change.

The following definition is taken from [**Zav23**][5.2.4].

Definition 1.2.2.3. We will say that $\mathcal{D}$ admits a *theory of first chern classes*, if it admits Tate twists, and it comes with a natural transformation

$$c_1\colon R\Gamma_{\text{d\'et}}(\_, \mathbf{G}_m)[1] \to \operatorname{Hom}(\mathbf{1}_{(\_)}, \mathbf{1}_{(\_)}(1))$$

of sheaves of spectra on $\mathcal{C}$.

Remark 1.2.2.4. Note that there is a natural transformation

$$R\Gamma_{\text{Nis}}(\_, \mathbf{G}_m)[1] \to R\Gamma_{\text{d\'et}}(\_, \mathbf{G}_m)[1]$$

of sheaves of spectra on $\mathcal{S}m_B^{\text{sep}}$. Furthermore, π_0 of the left-hand side computes the Picard group. In particular, any line bundle $\mathcal{L}$ on a smooth separated scheme S over B gives rise to a map

$$c_1(\mathcal{L})\colon \mathbf{1}_S \to \mathbf{1}_S(1).$$

In practice, π_0 of the right-hand side will essentially compute the group of line bundles on an object $S \in \mathcal{C}$.

Remark 1.2.2.5. The assignment of first Chern classes is compatible with base change and we have the formula

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) \simeq c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2).$$

Construction 1.2.2.6. Consider a morphism $f\colon X \to S$ of separated smooth schemes over B and a line bundle $\mathcal{L}$ on X. Then, by adjunction, the morphism $c_1(\mathcal{L})$ induces a morphism

$$c_1(\mathcal{L})\colon \mathbf{1}_S \to f_*\mathbf{1}_X(1)$$

which we will denote the same way. Furthermore, taking iteratively $\otimes$-powers of this morphism with itself, we obtain a map

$$c_1(\mathcal{L})^d\colon \mathbf{1}_S \to f_*\mathbf{1}_X(d).$$

Now, if we specialize to $f\colon \mathbf{P}_S(\mathcal{E}) \to S$ being the projection from the projective space associated to a vector bundle $\mathcal{E}$ of rank $d+1$ on S, we can use these constructions to obtain a morphism

$$\sum_{i=0}^{d} c_1(\mathcal{O}(1))^i(d-i)\colon \bigoplus_{i=0}^{d} \mathbf{1}_S(d-i) \to f_*\mathbf{1}_{\mathbf{P}_S(\mathcal{E})}(d)$$

where $\mathcal{O}(1)$ denotes the universal line bundle and $c_1(\mathcal{O}(1))^0$ by adjunction corresponds to the identity.

The following definition is taken from [**Zav23**][5.2.8].

Definition 1.2.2.7. We will say that a theory of first Chern classes for $\mathcal{D}$ is an *additive orientation*, if for each $d \geq 1$ the map

$$\sum_{i=0}^{d} c_1(\mathcal{O}(1))^i(d-i)\colon \bigoplus_{i=0}^{d} \mathbf{1}_B(d-i) \to f_*\mathbf{1}_{\mathbf{P}^d_B}(d)$$

is an isomorphism.

Remark 1.2.2.8. We will often refer to this isomorphism or versions thereof as the *Projective Bundle formula*.

Remark 1.2.2.9. Note that the Projective bundle formula isomorphism base changes to its counterpart over an arbitrary object $S \in \mathcal{C}$. In particular, the Projective Bundle formula holds over any object in $\mathcal{C}$ if we have an additive orientation. As we can check isomorphisms locally, we also obtain a Projective Bundle formula for an arbitrary projective bundle over a smooth separated scheme over S.

Remark 1.2.2.10. For any object $S \in \mathcal{C}$ and any sheaf $\mathbf{E} \in \mathcal{D}(S)$ we obtain an isomorphism

$$\sum_{i=0}^{d} c_1(\mathcal{O}(1))^i(d-i)\colon \bigoplus_{i=0}^{d} \mathbf{E}(d-i) \to f_*f^*\mathbf{E}(d)$$

by tensoring the sheaf on the Projective Bundle formula isomorphism.

Example 1.2.2.11. Consider the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[drr, equal] & \mathbf{P}^1_S \arrow[r] \arrow[dr, "p_1"] & \mathbf{P}^2_S \arrow[r] \arrow[d, dashed, "p_2"] & \mathbf{P}^3_S \arrow[r] \arrow[dl, "p_3"] & \dots \\
 & & S & &
\end{tikzcd}$$

over some separated smooth scheme S over B. We can compute the homology of $\mathbf{P}^\infty_S$ by the formula

$$\mathrm{colim}_n (p_n)_!(p_n)^!\mathbf{1}_S \simeq \bigoplus_{i=0}^{\infty} \mathbf{1}_S(i) \in \mathcal{D}(S).$$

And dually, we can compute the cohomology as

$$\lim_n (p_n)_*(p_n)^*\mathbf{1}_S \simeq \prod_{i=0}^{\infty} \mathbf{1}_S(-i) \in \mathcal{D}(S).$$

We now have enough structure to construct the trace morphism for the morphism $f\colon \mathbf{P}^1_B \to B$. For this, we follow [**Zav23**][5.6.1].

Construction 1.2.2.12. As a dualizing sheaf, we want to consider the Tate twist. That is, we have to construct a morphism

$$\mathrm{tr}_f\colon f_*\mathbf{1}_{\mathbf{P}^1_B}(1) \to \mathbf{1}_B.$$

Such a morphism is given by the composition

$$f_*\mathbf{1}_{\mathbf{P}^1_B}(1) \to \mathbf{1}_B \oplus \mathbf{1}_B(1) \to \mathbf{1}_B.$$

The first map is the inverse of the Projective Bundle formula isomorphism, and the second is the projection to the first factor.

1.2.3. The Motivic realization.

In the following, we also want to construct the cycle class map for the morphism $\mathbf{P}^1_B \to B$. To explain the idea, let us make a little detour.

1.2.3.1. In the paper series [**AI22**] [**AHI25**] [**AHI24**] the authors present a category called motivic spectra together with a symmetric monoidal functor

$$\mathcal{S}m^{\mathrm{sep}}_B \to \mathcal{MS}(B).$$

In this category, one is meant to present all reasonable cohomology theories on $\mathcal{S}m^{\mathrm{sep}}_B$. Formally, this functor is the initial symmetric monoidal functor to a presentable stable category satisfying the following:

- (Nisnevich descent) The functor is a cosheaf for the Nisnevich topology[5].
- (Elementary Blow-up excision) For any separated smooth scheme S over B and any $d \geq 1$ the Blow-up square

$$\begin{tikzcd} \mathbf{P}^{d-1}_S \arrow[r] \arrow[d] & \mathbf{V}_{\mathbf{P}^{d-1}_S}(\mathcal{O}(1)) \arrow[d] \\ \{0\} \arrow[r] & \mathbf{A}^d_S \end{tikzcd}$$

  gets sent to a pushout square by this functor.
- (Tate-twist) The cofibre of the zero section $B \to \mathbf{P}^1_B$ becomes $\otimes$-invertible.

Remark 1.2.3.2. The relation to what we have done earlier comes from the fact that for those motivic spectra, which admit a theory of first Chern classes (so called *oriented motivic spectra*), satisfying Elementary Blow-up Excision is equivalent to the Projective Bundle formula morphism to be an isomorphism.

[5] Again, for this text the reader can replace this topology by the étale topology.

Now, one can construct the needed duality data for the morphism $\mathbf{P}^1_B \to B$ already in $\mathcal{MS}(B)$ [**AHI24**] [**Tan**]. Furthermore, there should be a functor induced by taking homology

$$\mathcal{MS}(B) \to \mathcal{D}(B), (f\colon S \to B) \mapsto f_! f^! (\mathbf{1}_B)$$

which transports the duality data we need to our setting. Unfortunately, it is a priori not clear that homology produces a symmetric monoidal functor[6], and thus that we have such a functor. A posteriori, we will have such a functor as all the maps in question will be cohomologically smooth, and thus, homology will be symmetric monoidal. In any case, this idea will still be our guiding principle.

To start, let us now assume that our geometric six-functor formalism admits an additive orientation.

1.2.3.3. Given a separated smooth scheme S over B, we will write

$$\mathcal{S}ch_S^{B\text{-sm}}$$

for the category of those S-schemes, which are smooth and separated over B. Then for any object $\mathbf{E}_S \in \mathcal{D}(S)$, homology induces a functor

$$\mathfrak{C}_*(\_, \mathbf{E}_S)\colon \mathcal{S}ch_S^{B\text{-sm}} \to \mathcal{D}(S)$$

assigning to a morphism $f\colon X \to S$ the object $f_! f^!(\mathbf{E}_S)$. Furthermore, we will write

$$\mathfrak{C}^*(\_, \mathbf{E}_S)\colon (\mathcal{S}ch_S^{B\text{-sm}})^{\mathrm{op}} \to \mathcal{D}(S)$$

for the given by cohomology. That is the functor which assigns to a scheme $f\colon X \to S$ over S the object $f_* f^*(\mathbf{E}_S)$.

1.2.3.4. We will call a blow-up square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbf{Bl}_X(Z) \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

associated to a closed immersion $Z \to X$ in $\mathcal{S}m_B^{\mathrm{sep}}$ a *smooth blowup square*. We obtain the following crucial corollary.

Proposition 1.2.3.5. *For any separated smooth scheme S over B and any object $\mathbf{E}_S \in \mathcal{D}(S)$, the functor*

$$\mathfrak{C}_*(\_, \mathbf{E}_S)\colon \mathcal{S}ch_S^{B\text{-}sm} \to \mathcal{D}(S)$$

as well as the functor

$$\mathfrak{C}^*(\_, \mathbf{E}_S)\colon (\mathcal{S}ch_S^{B\text{-}sm})^{op} \to \mathcal{D}(S)$$

sent smooth blow-up squares to pushout and pullback squares.

---

[6]This is at least not formal: As an example, one can consider the six-functor formalism which assigns to a locally compact Hausdorff space the category of postikov complete sheaves of (derived) abelian groups. Then, assuming that the assignment $(f\colon S \to *) \mapsto f_! f^! \mathbf{Z}$ from profinite sets to abelian groups is symmetric monoidal, which would imply that the global sections of the solid tensor product in the abstract tensor product.

Proof. We first prove the claim for cohomology. As we assume that cohomology is a Nisnevich sheaf, by [**AHI25**][2.2] it suffices to check that for any S-scheme Z smooth over B Blow-up squares of the form

$$\begin{tikzcd}
\mathbf{P}^{d-1}_Z \arrow[r] \arrow[d] & \mathbf{V}_{\mathbf{P}^{d-1}_Z}(\mathcal{O}(1)) \arrow[d] \\
Z \arrow[r, "0"'] & \mathbf{A}^d_Z
\end{tikzcd}$$

get send to (co)cartesian squares by $\mathbf{E}_S$-cohomology. Now consider the commutative diagram

$$\begin{tikzcd}
\mathbf{P}^{d-1}_Z \arrow[rr] \arrow[dd] \arrow[dr, equal] & & \mathbf{V}_{\mathbf{P}^{d-1}_Z}(\mathcal{O}(1)) \arrow[dd] \arrow[dr] & & W \arrow[ll] \arrow[dd, equal] \arrow[dr] & \\
& \mathbf{P}^{d-1}_Z \arrow[rr, crossing over] & & \mathbf{P}_{\mathbf{P}^{d-1}_Z}(\mathcal{O}(1) \oplus \mathcal{O}) & & U \arrow[ll, crossing over] \arrow[dd, equal] \\
Z \arrow[rr] \arrow[dr, equal] & & \mathbf{A}^d_Z \arrow[dr] & & W \arrow[ll] \arrow[dr] & \\
& Z \arrow[rr] \arrow[from=uu, crossing over] & & \mathbf{P}^d_Z \arrow[from=uu, crossing over] & & U \arrow[ll]
\end{tikzcd}$$

where we write U and W for the respective complements of the zero sections. Now, by Zariski descent, the upper and lower squares in the right cube get send to (co)cartesian squares by $\mathbf{E}_S$-cohomology. From this one sees that on $\mathbf{E}_S$-cohomology the square in the back of the left cube becomes (co)cartesian if and only if the square in the front of the left cube becomes (co)cartesian. Thus, it suffices to check the claim for squares of the form

$$\begin{tikzcd}
\mathbf{P}^{d-1}_Z \arrow[r] \arrow[d] \arrow[dr, "h"] & \mathbf{P}_{\mathbf{P}^{d-1}_Z}(\mathcal{O}(1) \oplus \mathcal{O}) \arrow[d, "f"] \\
Z \arrow[r, "0"'] & \mathbf{P}^d_Z.
\end{tikzcd}$$

For the proof, let us refer to such squares as a projective Blow-up squares. Then we first claim the following:

($*$) The proposition holds for projective Blow-up squares with $Z = S$.

Let us write $p\colon \mathbf{P}^d_S \to S$ for the projection. Then we have to check that the square

$$\begin{tikzcd}
p_*p^*\mathbf{E}_S \arrow[r] \arrow[d] & p_*0_*0^*p^*\mathbf{E}_S \arrow[d] \\
p_*f_*f^*p^*\mathbf{E}_S \arrow[r] & p_*h_*h^*p^*\mathbf{E}_S
\end{tikzcd}$$

is (co)cartesian. Using the Projective Bundle formula, we see that this square identifies with the square

$$\begin{tikzcd}
\oplus_{i=0}^{d}\mathbf{E}_S(-i) \arrow[r] \arrow[d] & \mathbf{E}_S \arrow[d] \\
\oplus_{i=0}^{d-1}\mathbf{E}_S(-i) \oplus \oplus_{i=1}^{d}\mathbf{E}_S(-i) \arrow[r] & \oplus_{i=0}^{d-1}\mathbf{E}_S(-i)
\end{tikzcd}$$

which is easily seen to be (co)cartesian. To see that the map induced on the fibres is the identity one uses that
$$f^*\mathcal{O}(1) \simeq \mathcal{O}(1).$$
Now, for a general projective Blow-up square, let us write $g\colon Z \to S$ for the structure map. Then applying $*$ for $Z = S$ with coefficients in $g^*\mathbf{E}_S$ and using that g_* preserves (co)cartesian squares, we win. This finishes the argument for cohomology. To see the claim for homology, we reduce the claim to projective Blow-up squares over the base in the same way, and for those, one uses that for a proper map $f\colon X \to S$, we have an identification
$$f_! f^! \mathbf{E}_S \simeq \underline{\mathrm{Hom}}_S(f_*\mathbf{1}_X, \mathbf{E}_S)$$
such that the claim follows from the case of cohomology applied to the unit. $\square$

1.2.3.6. For any separated smooth scheme S over B let us write
$$\mathcal{P}_{\mathrm{ebu}}(S)_* := \mathcal{P}_{\mathrm{ebu}}(\mathcal{S}ch_S^{B\text{-sm}})_*$$
for the localization of pointed Nisnevich sheaves obtained by forcing Elementary Blow-up squares to become pushouts. Then by 1.2.3.5, we obtain colimit resp. limit preserving functors
$$\mathfrak{C}_*(\_/S)\colon \mathcal{P}_{\mathrm{ebu}}(S)_* \to \mathcal{D}(S)$$
and
$$\mathfrak{C}^*(\_/S)\colon \mathcal{P}_{\mathrm{ebu}}(S)_*^{\mathrm{op}} \to \mathcal{D}(S)$$
induced by taking homology resp. cohomology. Furthermore for any vector bundle $\mathcal{E}$ on S we can consider the *Thom space*
$$\mathbf{Th}_S(\mathcal{E}) := cof(\mathbf{P}_S(\mathcal{E})_+ \to \mathbf{P}_S(\mathcal{E} \oplus \mathcal{O})_+) \in \mathcal{P}_{\mathrm{ebu}}(S)_*.$$
This construction contravariant functorial along surjections of vector bundles and by [**AHI25**][3] can be promoted to a symmetric monoidal functor. In particular, we have the formula
$$\mathbf{Th}_S(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq \mathbf{Th}_S(\mathcal{E}_1) \otimes \mathbf{Th}_S(\mathcal{E}_2).$$

Proposition 1.2.3.7. *For any separated smooth scheme S over B and any vector bundle $\mathcal{E}$ on S the Thom space and its dual*
$$\mathfrak{C}_*(\mathbf{Th}_S(\mathcal{E})/S), \mathfrak{C}^*(\mathbf{Th}_S(\mathcal{E})/S) \in \mathcal{D}(S)$$
are $\otimes$-invertible.

Proof. By adjunction, we have an identification
$$\mathfrak{C}_*(\mathbf{Th}_S(\mathcal{E})/S) \simeq (\mathfrak{C}^*(\mathbf{Th}_S(\mathcal{E})/S))^\vee$$
and as taking duals preserves $\otimes$-invertible objects, it is enough to show the claim for the cohomology. But restricted to proper schemes over S cohomology is symmetric monoidal, so by Zariski descent and symmetric monoidality of the Thom space, we can assume $\mathcal{E} \simeq \mathcal{O}$. But then
$$\mathfrak{C}^*(\mathbf{Th}_S(\mathcal{O})/S) \simeq \mathbf{1}_S(-1)$$
which is $\otimes$-invertible by assumption. $\square$

Remark 1.2.3.8. Using our orientation, one can construct Thom isomorphism for any vector bundle in the classical way (see, for example [**AHI25**][6]). That is, we have isomorphisms
$$\mathfrak{C}_*(\mathbf{Th}_S(\mathcal{E})/S) \simeq \mathbf{1}_S(\mathrm{rk}(\mathcal{E})).$$

Remark 1.2.3.9. We will now start just writing things out in the case of homology. The dual concepts exist, and the dual statements hold.

Corollary 1.2.3.10. *Consider a separated smooth scheme S over B together with a vector bundle $\mathcal{E}$ on S and a linear map $\sigma\colon \mathcal{E} \to \mathcal{O}$. Then there exists a canonical homotopy $h(\sigma)$ in $\mathcal{D}(S)$ between the induced map*

$$\sigma\colon \mathfrak{C}_*(S/S) \to \mathfrak{C}_*(\mathbf{V}_S(\mathcal{E})/S) \to \mathfrak{C}_*(\mathbf{P}_S(\mathcal{E} \oplus \mathcal{O})/S)$$

and the analogous map induced by the zero section. Furthermore, this homotopy is functorial in $(\mathcal{E}, \sigma)$ and is the identity if σ is the zero section.

Proof. The author does not see how to formally apply [**AHI25**][4.1], but one can now do the same proof using 1.2.3.5 and 1.2.3.7. □

Definition 1.2.3.11. We will say a $\mathbf{P}^1_S$-*homotopy* between two morphisms $f, g\colon \mathbf{E} \to \mathbf{E}'$ in $\mathcal{D}(S)$ is a morphism $h\colon \otimes\,\mathfrak{C}_*(\mathbf{P}^1_S/S) \otimes \mathbf{E} \to \mathbf{E}'$ making the following diagram commute

$$\begin{tikzcd}
\mathbf{E} \arrow[dr, "0"'] \arrow[drr, bend left, "f"] & & \\
& \mathbf{E} \otimes \mathfrak{C}_*(\mathbf{P}^1_S/S) \arrow[r, "h"] & \mathbf{E}' \\
\mathbf{E}. \arrow[ur, "1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

We will say two morphisms are $\mathbf{P}^1_S$-*homotopic*, if they are related by a zigzag of $\mathbf{P}^1_S$-homotopies. Furthermore, we will say a morphism in $\mathcal{D}(S)$ is a homotopy equivalence if it admits an inverse up to $\mathbf{P}^1_S$-homotopies.

Corollary 1.2.3.12. *Consider a separated smooth scheme S over B. Then $\mathbf{P}^1_S$-homotopy equivalences are isomorphisms in $\mathcal{D}(S)$.*

Proof. This easily follows from 1.2.3.10. □

This is quite useful to justify that certain maps are isomorphisms in $\mathcal{D}(S)$. Let us record some examples.

Example 1.2.3.13. Note that there is a map of bi-pointed objects $\mathbf{P}^1_S \to \mathbf{A}^1/\mathbf{G}_m$, where on the right we consider the action coming from the multiplication on $\mathbf{A}^1$. From this one deduces that also $\mathbf{G}_m$-equivariant $\mathbf{A}^1$-homotopies (with non-trivial action!) are isomorphisms in $\mathcal{D}(S)$. One example of such a map is

$$\mathfrak{C}_*((\mathbf{A}^n_S/\mathbf{G}_m)/S) \to \mathfrak{C}_*(\mathbf{B}\mathbf{G}_m/S)$$

coming from the projection.

Example 1.2.3.14. Consider a vector bundle $\mathcal{E}$ on S for which we can find a surjective map $\mathcal{E} \to \mathcal{O}$. The using 1.2.3.12 and 1.2.3.5 one proves the same way as in [**AHI25**][5.3] that the projection

$$\mathfrak{C}_*(\mathbf{Gr}_d(\mathcal{E}^\infty)/S) \to \mathfrak{C}_*(\mathbf{BGL}_d/S)$$

from the infinite Grassmannian is an isomorphism in $\mathcal{D}(S)$.

Example 1.2.3.15. A special case of the last example is the map

$$\mathfrak{C}_*(\mathbf{P}^\infty_S/S) \to \mathfrak{C}_*(\mathbf{BG}_m/S).$$

Thus, using 1.2.2.11, we see that we have an isomorphism

$$\bigoplus_{i=0}^{\infty} \mathbf{1}_S(i) \simeq \mathfrak{C}_*(\mathbf{BG}_m/S).$$

To end the section, let us give an alternative construction of the first Chern classes following the classical strategy.

Construction 1.2.3.16. Consider a separated smooth scheme S over B and a line bundle $\mathcal{L}$ on S. Then $\mathcal{L}$ corresponds to a section

$$i_{\mathcal{L}} \colon S \to \mathbf{BG}_m$$

and we can consider the composition

$$\mathrm{c}_1'(\mathcal{L}) \colon \mathbf{1}_S \to \prod_{i=0}^{\infty} \mathbf{1}_S(1-i) \simeq \mathfrak{C}^*(\mathbf{BG}_m/S)(1) \to \mathbf{1}_S(1)$$

where the first map is the inclusion into the first factor, and the second map is induced by $i_{\mathcal{L}}$ on cohomology.

Proposition 1.2.3.17. *For any separated smooth scheme S over B and any line bundle $\mathcal{L}$ on S the map $\mathrm{c}_1'(\mathcal{L})$ constructed in 1.2.3.16 identifies with the first Chern class associated to $\mathcal{L}$.*

Proof. We will implicitly use the identification

$$\mathfrak{C}^*(\mathbf{BG}_m/S) \xrightarrow{\sim} \mathfrak{C}^*(\mathbf{P}^\infty_S/S).$$

Using proper base change, one sees that the construction of the map $\mathrm{c}_1'(\mathcal{L})$ is stable under base change. In particular it suffices to identify the construction for the universal line bundle on $\mathbf{BG}_m$. This easily follows the Projective Bundle formula. □

1.2.4. Cycle classes. Let us now come to the construction of the cycle class map. For this, we will use a construction of Gysin maps by Longke Tang [**Tan**]. All geometric objects in the following will be considered in $\mathcal{P}_{\mathrm{ebu}}(S)_*$.

Let us start with a well-known construction. The so-called *deformation to the normal bundle*.

Proposition 1.2.4.1. *Consider a closed immersion $i \colon Z \to S$ of smooth separated schemes over B. Then there exists a closed immersion of stacks*

$$(\mathbf{A}^1/\mathbf{G}_m)_Z \to \mathbf{D}_{Z/S}$$

over $(\mathbf{A}^1/\mathbf{G}_m)_S$. Furthermore, this construction satisfies the following:

(a) *It is contravariant functorial along cartesian maps of closed immersions.*

(b) *If i is given by the zero section into a vector bundle V, then there is a cartesian square*

$$\begin{tikzcd} \mathbf{D}_{Z/S} \arrow[r] \arrow[d] & (V/^{-1}\mathbf{G}_m)_S \arrow[d] \\ (\mathbf{A}^1/\mathbf{G}_m)_S \arrow[r] & (\mathbf{BG}_m)_S \end{tikzcd}$$

where we use the action of weight -1 on V.

(c) *The fibre over $1\colon S \simeq (\mathbf{G}_m/\mathbf{G}_m)_S \to (\mathbf{A}^1/\mathbf{G}_m)_S$ recovers the closed immersion i.*

(d) *The fibre over $0\colon (\mathbf{BG}_m)_S \to (\mathbf{A}^1/\mathbf{G}_m)_S$ is given by the zero section*

$$0\colon (\mathbf{BG}_m)_Z \to \mathcal{N}_{Z/X}/^{-1}(\mathbf{G}_m)_Z$$

into the normal bundle. Where in the normal bundle, we use the action of weight -1.

(e) *If $i\colon D \to S$ is given by an effective cartier divisor, the inclusion of the fibre at 0 factors as*

$$\begin{tikzcd} \mathcal{N}_{D/S}/^{-1}(\mathbf{G}_m)_D \arrow[rr] \arrow[dr] & & \mathbf{D}_{D/S} \\ & \mathbf{V}_S(\mathcal{O}(-D))/^{-1}(\mathbf{G}_m)_S. \arrow[ur] & \end{tikzcd}$$

Where the first map is the canonical inclusion, and, if we restrict the second map to the 1-section, it identifies the composition of the 1-section in $\mathcal{D}_{D/S}$ with the structure map to $(\mathbf{BG}_m)_S$ with the map

$$\mathcal{O}(D)\colon S \to (\mathbf{BG}_m)_S$$

corresponding to the line bundle $\mathcal{O}(D)$.

Proof. The construction is quite standard. See, for example, [**KR25**]. Note that (a) and (b) determine the construction locally. For the last claim, note that there is an identification

$$\mathbf{V}_S(\mathcal{O}(-D))^*/^{-1}(\mathbf{G}_m)_S \simeq \mathbf{V}_S(\mathcal{O}(D))^*/(\mathbf{G}_m)_S$$

over $(\mathbf{BG}_m)_S$. □

1.2.4.2. Given a closed immersion $i\colon Z \to S$ of separated smooth schemes over B, let us write $\mathcal{D} = \mathbf{D}_{Z/S}, \mathcal{A} = (\mathbf{A}^1/\mathbf{G}_m)_Z$ and $\mathcal{D}_0, \mathcal{A}_0$ for the respective fibres over 0. Then we have a map

$$\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0) \to \mathcal{D}/(\mathcal{D} - \mathcal{A})$$

in $\mathcal{P}_{\mathrm{ebu}}(S)$.

Proposition 1.2.4.3. *Given a closed immersion $i\colon Z \to S$ of separated smooth schemes over B, the map*

$$\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0) \to \mathcal{D}/(\mathcal{D} - \mathcal{A})$$

constructed in 1.2.4.2 induces an isomorphism on homology and cohomology.

Proof. The argument is exactly the same as given in [**Tan**][5] using 1.2.3.5 and 1.2.3.12. □

Construction 1.2.4.4. Now let us consider a closed immersion $i\colon Z \to S$ of separated smooth schemes over B. Note that the map

$$\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0) \to (\mathbf{BG}_m)_Z/\mathbf{P}_Z(\mathcal{N}_{Z/S})$$

is Zariski locally a $\mathbf{P}^1$-homotopy equivalence. In particular, we obtain an isomorphism

$$\mathfrak{C}^*((\mathbf{BG}_m)_Z/\mathbf{P}_Z(\mathcal{N}_{Z/S})/S) \xrightarrow{\simeq} \mathfrak{C}^*((\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0))/S)$$

in $\mathcal{D}(S)$. Furthermore, using the isomorphism observed in 1.2.3.15 we see that

$$\mathfrak{C}^*((\mathbf{BG}_m)_Z/\mathbf{P}_Z(\mathcal{N}_{Z/S})/S) \simeq \prod_{i=d}^{\infty} i_*i^*\mathbf{1}_S(-i)$$

where d is given by the rank of the normal bundle of i. Including in the first factor, we obtain a map

$$i_*\mathbf{1}_Z \to \mathfrak{C}^*((\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0))/S)(d).$$

Using this map, we consider the composition

$$i_*\mathbf{1}_Z \longrightarrow \mathfrak{C}^*((\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0))/S)(d) \xleftarrow{\simeq} \mathfrak{C}^*((\mathcal{D}/(\mathcal{D} - \mathcal{A}))/S)(d) \xrightarrow{1} \mathbf{1}_S(d)$$

where the last map is induced on cohomology by the 1-section and the second map uses the isomorphism from 1.2.4.3.

Definition 1.2.4.5. We will call the map

$$\mathrm{cl}_i\colon i_*\mathbf{1}_Z \to \mathbf{1}_S(d)$$

constructed in 1.2.4.4 the *cycle class map* associated to $i\colon Z \to S$.

Remark 1.2.4.6. Using proper base change, one sees that the construction of cycle classes is stable under base change.

The following says that our theory of first Chern classes underlies our theory of cycle class maps in the sense of [**Zav23**][5.3.3].

Proposition 1.2.4.7. *Consider an effective cartier divisor $i\colon D \to S$ of separated smooth schemes over B. Then the triangle*

$$\begin{tikzcd}
\mathbf{1}_S \arrow[rr, "c_1(\mathcal{O}(D))"] \arrow[dr, "ad_*"'] & & \mathbf{1}_S(1) \\
& i_!\mathbf{1}_D \arrow[ur, "cl_i"'] &
\end{tikzcd}$$

commutes.

Proof. Unwinding what we have to do, we arrive at the diagram, which is induced on cohomology by

$$\begin{tikzcd}
& \mathcal{D}_0 \arrow[dl] \arrow[dr] \arrow[rr, equal] \arrow[dd, shift left] & & (\mathbf{BG}_m)_D \arrow[dr] \arrow[dd, shift left] & \\
\mathbf{V}_S(\mathcal{O}(-D))/^{-1}\mathbf{G}_m \arrow[rr] & & \mathcal{D} \arrow[rr] \arrow[dd, shift left] & & (\mathbf{BG}_m)_S \arrow[dd, shift left] \\
& \mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0) \arrow[uu, dotted, shift left] \arrow[rr, equal] \arrow[dr, equal] & & (\mathbf{BG}_m)_D/\mathbf{P}(\mathcal{N}) \arrow[uu, dotted, shift left] \arrow[dr] & \\
S \arrow[uu, "1"] \arrow[rr] & & \mathcal{D}/(\mathcal{D} - \mathcal{A}) \arrow[uu, dotted, shift left] \arrow[rr, dotted] & & (\mathbf{BG}_m)_S/\mathbf{P}(-D) \arrow[uu, dotted, shift left]
\end{tikzcd}$$

where we indicate the cohomology isomorphisms as equal. Now note that there are sections on cohomology as indicated by dotted arrows, and as we just have to check commutativity after precomposing with the inclusion into the first factor, we are free to insert the idempotents induced by these sections as often as we want. Furthermore, there exists a dotted arrow, making the right square commute up to this idempotency, as indicated. Now it suffices to check that the large square in the front commutes (up to idempotency), which easily follows by diagram chase from 1.2.4.1. □

We now arrive at the following:

Theorem 1.2.4.8. *Consider a geometrized geometric setup $(\mathcal{C}, \mathcal{C}_0)$ together with a geometric six-functor formalism on it, which admits an additive orientation. Then any smooth morphism is cohomologically smooth. Furthermore, for any smooth morphism, which globally admits a factorization*

$$f\colon U \to \mathbf{A}^n_S \to S$$

where the first map is cohomologically étale and the second map is the projection, we have an identification

$$\omega_f \simeq \mathbf{1}_U(1).$$

Proof. By 1.2.1.9, to deduce the first claim, it suffices to check the following:

(∗) The triple $(\mathbf{1}_{\mathbf{P}^1_B}(1), \mathrm{tr}_f, \mathrm{cl}_\Delta)$ constructed in 1.2.2.12 and 1.2.4.5 gives a trace-cycle theory for the projection $f\colon \mathbf{P}^1_B \to B$.

But by 1.2.4.7 we see that our theory of Chern classes underlie our theory of cycle classes so that we can apply [**Zav23**][5.6.6]. The computation of the dualizing sheaf now easily follows by base change and the fact that for a cohomologically étale morphism $f\colon U \to S$, we have

$$f^* \simeq f^!.$$

□

Remark 1.2.4.9. In practice, the computation of the dualizing sheaf will hold for general smooth morphisms. The obstruction to arguing for this lies in finding a global comparison map. We will find this by using the deformation to the normal bundle in the respective geometric setting.

1.2.5. Computing the dualizing sheaf. To finish the section, we want to explain how to compute the dualizing complex of a smooth morphism in an additively oriented six-functor formalism.

We will continue to fix a geometric six functor formalism $\mathcal{D}$, which admits an additive orientation. Now we will need to assume the existence of certain geometric constructions. Let us start with the following definition:

Definition 1.2.5.1. A subclass of smooth morphisms is called *geometrically smooth* if it satisfies the following:

(a) The class is stable under base change along arbitrary morphisms.

(b) For a morphism $e\colon U \to X$, which is geometrically smooth and cohomologically étale, the diagonal morphism factors

$$U \xrightarrow{i} V \xrightarrow{j} U \times_X U.$$

where i is a Zariski closed immersion and j a cohomological étale monomorphism. We will call such morphisms *geometrically étale.*

(c) For any geometrically smooth morphism $f\colon X \to S$, there exists a $\mathcal{D}$-cover of X consisting of geometrically étale morphism, such that restricted to this covering the morphism factors as

$$X \xrightarrow{e} \mathbf{A}^n_S \xrightarrow{p} S.$$

where e is a geometrically étale morphism and p the projection.

(d) Given a geometrically smooth morphism $f\colon X \to S$ then any section $s\colon S \to X$ factors as

$$S \xrightarrow{i} U \xrightarrow{j} X.$$

where j is a cohomologically étale monomorphism and i locally in the dét-topology arises as a cartesian square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & U \arrow[d] \\ \{0\} \arrow[r] & \mathbf{A}^n_B \end{tikzcd}$$

.

(e) Assume we have a commutative triangle

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

where g and f are geometrically smooth and i locally in the dét-topology arises as the pullback of some zero section. Then we can find an open covering $\{U_i\}_I$ of X, such that the restriction of i sits in a cartesian square

$$\begin{tikzcd} Z \cap U_i \arrow[r, "i"] \arrow[d] & U_i \arrow[d] \\ \mathbf{A}^{d(Z)}_B \arrow[r] & \mathbf{A}^{d(X)}_B \end{tikzcd}$$

where the vertical maps are cohomologically étale and the lower horizontal map is given by

$$(a_1, \ldots, a_{d(Z)}) \mapsto (a_1, \ldots, a_{d(Z)}, 0, \ldots, 0).$$

(d) For any commutative triangle

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

where g and f are geometrically smooth and i locally in the dét-topology arises as the pullback of a zero section. Locally on X in the dét topology, we can find cartesian squares

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "i"'] & Z \arrow[r, equal] \arrow[d, "t"'] & Z \arrow[d, "0"] \\ X & U \arrow[l] \arrow[r] & \mathbf{A}^n_Z \end{tikzcd}$$

with the horizontal morphisms étale.

Remark 1.2.5.2. In practice, assertion (c) in the definition of geometrically smooth morphisms follows from the Jacobian criterion.

Remark 1.2.5.3. Assertion (d) in the definition of geometrically smooth morphisms, in practice follows from assertion (d) together with the fact that the diagonal of an étale morphism is an open embedding and that the image of the complement of an open along a Zariski closed immersion (a map locally arising as the pullback of a zero section) has an open complement [**MV99**][Section 3. Lemma 2.28].

Remark 1.2.5.4. Given a Zariski closed immersion $Z \to X$ then locally on X Z admits a complement by pulling back the complement of the zero section. As these are monomorphisms we can glue those local complements to a global complement $U \subset X$. Furthermore for any etale morphism $V \to X$, such that the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d, equal] & V \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

is Cartesian, we see that the pair

$$V \amalg U \to X$$

is a dét covering, as this is true locally. Note also that as $U \subset X$ is a monomorphism, satisfying descent for this dét cover, is equivalent to sending the square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

to a pullback.

Remark 1.2.5.5. Assume that we have a square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbf{Bl}_X(Z) \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

of geometrically smooth objects over $S \in \mathcal{C}$, which locally on X comes as the pullback of an Elementary Blow up square. Then as in the proof of 1.2.3.5 we can use assertion (d) in 1.2.5.1 and 1.2.5.4 to see that cohomology as well as homology of this square becomes a (co)cartesian square in $\mathcal{D}(S)$.

1.2.5.6. We will now write

$$(\mathbf{A}^1/\mathbf{G}_m)_X := (\mathbf{A}^1/\mathbf{G}_m)_B \times X$$

for the quotient stack in the dét-topology on X and similar for $(\mathbf{B}\mathbf{G}_m)_X$. Note that by 1.2.4.8 we already know that the maps

$$\mathbf{A}^1_X \to (\mathbf{A}^1/\mathbf{G}_m)_X \text{ and } X \to (\mathbf{B}\mathbf{G}_m)_X$$

are cohomologically smooth. As they have local sections, it is easy to see that the from !-covers 1.1.3.1.

We will now assume that in the geometry $\mathcal{C}$, we can construct the deformation to the normal bundle. Let us formulate this in the following way.

1.2.5.7 (Existence of the deformation to the normal bundle). Given a geometrically smooth morphism $f\colon X \to S$ in $\mathcal{C}$ which admits a section $s\colon S \to X$, there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, shift left, "f"] & \mathbf{D}_s \arrow[d, shift left, "\tilde{f}"] & \mathcal{N}_s/^{-1}\mathbf{G}_m \arrow[l] \arrow[d, shift left, "f_0"] & \mathcal{N}_s \arrow[l] \arrow[d, shift left, "pr"] \\
S \arrow[r, "1"'] \arrow[u, bend left, "s"] & (\mathbf{A}^1/\mathbf{G}_m)_S \arrow[u, bend left, "\tilde{s}"] & (\mathbf{B}\mathbf{G}_m)_S \arrow[l, "0"] \arrow[u, bend left, "s_0"] & S \arrow[l] \arrow[u, bend left, "0"]
\end{tikzcd}$$

which on the locus on X, where $s\colon S \to X$ comes as the pullback of a zero section $\{0\} \to \mathbf{A}^n_B$ is comes from pulling back the deformation to the normal bundle 1.2.4.1 of this zero section along the map $X \to \mathbf{A}^n_B$.

Construction 1.2.5.8. Consider a Zariski closed immersion $i\colon Z \to X$ and let us assume we can associate the deformation to the normal bundle to it. Then let us write again $\mathcal{D} = \mathbf{D}_i, \mathcal{A} = (\mathbf{A}^1/\mathbf{G}_m)_Z$ and $\mathcal{D}_0, \mathcal{A}_0$ for the fibres over 0. Then the induced map

$$\mathcal{D}_0/(\mathcal{D}_0 - \mathcal{A}_0) \to \mathcal{D}/(\mathcal{D} - \mathcal{A})$$

will induce an isomorphism on homology as well as cohomology.

To see this, one does the same argument as in 1.2.4.3 using 1.2.5.5 and 1.2.5.4 together with the fact that one can pullback the identifications of $\mathbf{P}^1$-homotopies from B 1.2.3.10.

In particular, as in 1.2.4.4, we can associate to i a cycle class map

$$\mathrm{cl}_i\colon i_*\mathbf{1}_Z \to \mathbf{1}_X(d)$$

where d denotes the rank of the normal bundle.

Remark 1.2.5.9. Consider the composition of two Zariski closed immersions

$$h \simeq g \circ i\colon Z \to X \to Y$$

then the triangle

$$\begin{tikzcd}
h_!\mathbf{1}_Z \arrow[rr, "\mathrm{cl}_h"] \arrow[dr, "g_!(\mathrm{cl}_i)"'] & & \mathbf{1}_Y(d+d') \\
& g_!\mathbf{1}_X(d) \arrow[ur, "\mathrm{cl}_g \otimes id(d)"'] &
\end{tikzcd}$$

commutes, where d and d' denote the respective codimensions [**Tan22**][5.29].

Remark 1.2.5.10. Using proper base and that the deformation to the normal bundle construction is stable under base change, we see that given a Cartesian square

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

where i is a closed immersion, we have an identification

$$f^*(\mathrm{cl}_i) \simeq \mathrm{cl}_{i'}.$$

Construction 1.2.5.11. Let us consider a geometrically smooth morphism $f\colon X \to S$, then we can produce the following diagram:

$$\begin{tikzcd} X \arrow[dr, "\Delta"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\ & X \times_S X \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & S. \end{tikzcd}$$

Furthermore by assertion (a) in 1.2.5.1, there is a factorization

$$\Delta \simeq j \circ \tilde{\Delta}\colon X \to U \to X \times_S X$$

into a Zariski closed immersion followed by an open immersion. Thus, using 1.2.5.8, we obtain a cycle class map

$$\mathrm{cl}_{\tilde{\Delta}}\colon \tilde{\Delta}_! \mathbf{1}_X \to \mathbf{1}_U(d).$$

Furthermore, applying $j_!$ and using the counit, we obtain a map

$$\mathrm{cl}_{\Delta}\colon \Delta_! \mathbf{1}_X \to \mathbf{1}_{X\times_S X}(d)$$

which we will also refer to as *cycle class map.*

Theorem 1.2.5.12. *Given a geometrically smooth morphism* $f\colon X \to S$ *in* $\mathcal{C}_E$*, then we have an adjunction*

$$f_! : \mathcal{D}(X) \rightleftarrows \mathcal{D}(S) : f^*(\_) \otimes \mathbf{1}_X(d)$$

where d *denotes the relative dimension.*

Proof. Applying $q_!$ to the cycle class map cl_Δ 1.2.5.11, we obtain a map $\mathbf{1}_X \to f^* f_!(\mathbf{1}_X(d))$ and tensoring with this map gives a natural transformation

$$id \to f^*(f_!(\_)) \otimes \mathbf{1}_X(d).$$

We claim that this map is the unit of the claimed adjunction. That, for any two objects $\mathbf{E}_S \in \mathcal{D}(S)$ and $\mathbf{E}_X \in \mathcal{D}(X)$, the induced map

$$\mathrm{Hom}_S(f_!\mathbf{E}_X, \mathbf{E}_S) \to \mathrm{Hom}_X(f^*f_!\mathbf{E}_X(d), f^*\mathbf{E}_S(d)) \to \mathrm{Hom}_X(\mathbf{E}_X, f^*\mathbf{E}_S(d))$$

is an isomorphism. This can be checked geometrically étale locally on X, thus we can assume that f factors as a geometrically étale map $e\colon X \to \mathbf{A}^n_S$ followed by the projection. Using 1.2.5.10 and (the proof of) [**Zav23**][3.2.8], it suffices to check that in this case, the map

$$\mathrm{cl}_{\Delta}\colon \Delta_! \mathbf{1}_X \to \mathbf{1}_{X\times_S X}(d)$$

is part of a trace-cycle theory. We can pull pack such a trace-cycle theory along e so using 1.2.5.10 and 1.2.5.9 we reduce to check this to the case where f is given by a projection $p\colon \mathbf{A}^n_S \to S$. Using 1.2.5.10 and 1.2.5.9 again we reduce to the case where $n = 1$ and thus to the case where f is the projection $\mathbf{P}^1_S \to S$. This was explained in 1.2.4.8. □

Corollary 1.2.5.13. *For a geometrically smooth morphism $f\colon X \to S$, there is a canonical identification*

$$\omega_f := f^!(\mathbf{1}_S) \simeq \mathbf{1}_X(d)$$

where d denotes the relative dimension.

Proof. As both functors are right adjoint to $f_!$ by 1.2.5.12, we obtain a canonical identification

$$f^! \simeq f^*(\_)(d).$$

Applying this to the unit gives what we want. □

CHAPTER 2

Adic stacks

The main part of this chapter is to recall some aspects of the theory of analytic stacks (mainly in the adic setting), developed by Clausen-Scholze. In particular, the only originality we claim is for suboptimal exposition and mistakes. The other part is to recall some notions of derived formal schemes, most crucially the construction of the deformation to the normal bundle. In the end, we explain that understanding a formal scheme as an analytic stack is compatible with proper maps.

2.1. Recollections on Analytic stacks

2.1.1. Analytic rings.

2.1.1.1. Recall that the *weight* of a profinite set $S \simeq \lim_I S_i$ is given by the cardinality of the set of continuous functions

$$\mathrm{Cont}(S, \mathbf{F}_2) \simeq \mathrm{colim}_I\, \mathbf{F}_2^{S_i}$$

from S to $\mathbf{F}_2$. We will say a profinite set is *light* if its weight is countable and write

$$\mathcal{P}ro_{\mathbf{N}}(\mathcal{F}in)$$

for the category of light profinite sets. This category can also be described as the full subcategory of those profinite sets whose indexing category is given by the natural numbers.

2.1.1.2. We equip $\mathcal{P}ro_{\mathbf{N}}(\mathcal{F}in)$ with a pretopology, where coverings are generated by disjoint unions and surjective maps. Now a *light condensed object* in a category $\mathcal{D}$ is a presheaf

$$X : \mathcal{P}ro_{\mathbf{N}}(\mathcal{F}in)^{\mathrm{op}} \to \mathcal{D}$$

such that for any hypercovering $U_\bullet \to S$ the induced map

$$X(S) \xrightarrow{\sim} \lim_\Delta X(U_\bullet)$$

is an isomorphism. We will write $\mathcal{C}ond(\mathcal{D})$ for the category of condensed objects in $\mathcal{D}$.

Example 2.1.1.3. Any topological space X gives rise to a light condensed anima via the assignment

$$S \mapsto \mathrm{Cont}(S, X).$$

This construction is fully faithful on metrizable compactly generated topological spaces [**CS24**][Lecture 2]. Note that this construction preserves products, so any topological algebraic object (like topological abelian group, topological ring, etc.) gives us a condensed algebraic object.

Example 2.1.1.4. The light condensed objects appearing in this text will be either

$$\mathcal{C}ond^{\mathrm{lgt}}(\mathbf{Z})$$

which is our notation for the category of light condensed objects in the derived category of $\mathbf{Z}$. Or

$$\mathcal{C}ond^{\mathrm{lgt}}(\mathcal{R}ing)$$

which is our notation for light condensed animated rings. Note that any topological abelian group gives rise to a light condensed derived abelian group, as any derived abelian group. Similar for light condensed animated rings.

Note also that to any light condensed animated ring A, we can associate the category of modules over the underlying light condensed $\mathcal{E}_\infty$-ring

$$\mathcal{C}ond^{\mathrm{lgt}}(A) := \mathcal{M}od_A(\mathcal{C}ond^{\mathrm{lgt}}(\mathbf{Z})).$$

2.1.1.5. The category

$$\mathcal{C}ond^{\mathrm{lgt}}(\mathbf{Z})$$

admits a t-structure whose heart is given by light condensed abelian groups [**Lur18**][1.3.2.7.]. Furthermore, as we consider hypersheaves, one can describe the connective part $\mathcal{C}ond^{\mathrm{lgt}}(\mathbf{Z})_{\geq 0}$ (resp. the coconnevtive part $\mathcal{C}ond^{\mathrm{lgt}}(\mathbf{Z})_{\leq 0}$)

as those objects N whose homotopy sheaves $\pi_n N \simeq 0$ vanish for $n < 0$ (resp. $n > 0$) [**Lur18**][1.3.3.3.]. This induces a t-structure on

$$\mathcal{C}ond^{\mathrm{lgt}}(A)$$

for any light condensed animated ring A.

Remark 2.1.1.6. The category of light condensed abelian groups is not just a Grothendieck abelian category. It behaves even better than that, similar to the classical category of abelian groups itself. For example, arbitrary products and sums are exact and filtered colimits distribute over products [**CS19b**].

Definition 2.1.1.7 ((Clausen-Scholze))**.** An *analytic ring* A is a pair $(A^{\triangleright}, \mathcal{D}(A))$, where $A^{\triangleright}$ is a light condensed animated ring and

$$\mathcal{D}(A) \subset \mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright})$$

is a full subcategory, such that:

(1) $\mathcal{D}(A)$ is closed under limits and colimits.
(2) For $N \in \mathcal{D}(A)$ and $M \in \mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright})$ we have

$$\underline{\mathrm{Hom}}_{A^{\triangleright}}(M, N) \in \mathcal{D}(A).$$

(3) The left adjoint to the inclusion

$$\widehat{(\_)} \colon \mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright}) \to \mathcal{D}(A)$$

seen as an endofunctor on the domain preserves connective objects.
(4) $A^{\triangleright} \in \mathcal{D}(A)$.

A morphism of analytic rings $A \to B$ is a morphism of light condensed animated rings

$$A^{\triangleright} \to B^{\triangleright}$$

such that the induced restriction of scalars functor

$$\begin{tikzcd}
\mathcal{C}ond^{\mathrm{lgt}}(B^{\triangleright}) \arrow[r] & \mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright}) \\
\mathcal{D}(B) \arrow[r] \arrow[u, hook] & \mathcal{D}(A) \arrow[u, hook]
\end{tikzcd}$$

restricts to a functor on the derived categories.

Remark 2.1.1.8. Note that the existence of the localization functor in (3) is automatic by [**RS22**].

2.1.1.9. Let us reformulate these conditions to describe the category of analytic rings more honestly. Note first of all that as $\mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright})$ is the stabilization of $\mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright})_{\geq 0}$ (which implies the analogous statement for $\mathcal{D}(A)$), we could have dropped condition (3) and formulate the other conditions using the respective connective parts. Now for any light condensed animated ring $A^{\triangleright}$ the category $\mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright})_{\geq 0}$ is a commutative algebra in the category of compactly generated categories[1]. In this language condition (1) exactly means that $\mathcal{D}(A)_{\geq 0}$ is compactly

[1]More concretely, it is compactly generated and admits a symmetric monoidal structure, such that the tensor product preserves filtered colimits in each variable, and the tensor product of two compact objects is again compact

generated and the completion functor

$$\widehat{(\_)}\colon \mathcal{C}ond^{\mathrm{lgt}}(A^{\triangleright})_{\geq 0} \to \mathcal{D}(A)_{\geq 0}$$

is a localization which preserves compact objects (i.e., a morphism of compactly generated categories which is a localization) and condition (3) exactly boils down to the assertion that this completion functor induces a (unique) symmetric monoidal structure on $\mathcal{D}(A)_{\geq 0}$ making the functor symmetric monoidal. The category of such objects is presentable and comes with a colimit-preserving functor

$$s\colon \mathcal{L}oc(\mathcal{C}at^{\mathrm{cptgen},\otimes}) \to \mathcal{C}at^{\mathrm{cptgen},\otimes}$$

to commutative algebras in compactly generated categories, which picks out the source of the localization.

Now let us call the category of objects like in 2.1.1.7, but without assuming condition (4), the category of *pre-analytic rings*. We have just explained that this category sits in a Cartesian square

$$\begin{tikzcd}
\mathcal{A}n\mathcal{R}ing^{\mathrm{pre}} \arrow[r] \arrow[d] & \mathcal{L}oc(\mathcal{C}at^{\mathrm{cptgen}}_{\otimes}) \arrow[d, "s"] \\
\mathcal{C}ond^{\mathrm{lgt}}(\mathcal{R}ing) \arrow[r] & \mathcal{C}at^{\mathrm{cptgen}}_{\otimes}
\end{tikzcd}$$

where the lower horizontal functor is given by $A \mapsto \mathcal{C}ond^{\mathrm{lgt}}(A)$. In particular, the category is presentable, and to deduce the same for analytic rings, one observes that there is a localization

$$\widehat{(\_)}\colon \mathcal{A}n\mathcal{R}ing^{\mathrm{pre}} \to \mathcal{A}n\mathcal{R}ing$$

given by the formula $(A^{\triangleright}, \mathcal{D}(A)) \mapsto (\widehat{A^{\triangleright}}.\mathcal{D}(A))$, which admits a fully faithful right adjoint preserving filtered colimits.

2.1.1.10. Many interesting analytic rings are constructed using the following construction. Consider the profinite set

$$\mathbf{N} \cup \{\infty\} := \lim_n \{0, \dots, n, \infty\}$$

where the transition maps sent the highest number to ∞. To this we can associate its free (light) condensed derived abelian group $\mathbf{Z}[\mathbf{N} \cup \{\infty\}]$ and we will write $\mathbf{P}$ for the cofibre

$$\mathbf{Z}[\{\infty\}] \to \mathbf{Z}[\mathbf{N} \cup \{\infty\}] \to \mathbf{P}$$

of the canonical inclusion. Note that this is a split exact sequence.

Remark 2.1.1.11. Understanding $\mathbf{N} \cup \{\infty\}$ as a topological space via the limit topology, a continuous map

$$\mathbf{N} \cup \{\infty\} \to X$$

to a topological space exactly corresponds to a sequence $(x_i)_{i \in \mathbf{N}}$ of points in X together with a limit point $x_\infty \in X$, such that the sequence converges to x_∞. In particular $\mathbf{P}$ parameterizes null sequences in condensed derived abelian groups.

Proposition 2.1.1.12. *The object $\mathbf{P}$ is internally compact in $\mathcal{C}ond^{lgt}(\mathbf{Z})$. That is the endofunctor*

$$\underline{\mathrm{Hom}}_{\mathbf{Z}}(\mathbf{P}, \_)\colon \mathcal{C}ond^{lgt}(\mathbf{Z}) \to \mathcal{C}ond^{lgt}(\mathbf{Z})$$

preserves colimits.

PROOF. This is explained in [CS24][Lecture 3]. □

2.1.1.13. One can use $\mathbf{P}$ to construct new analytic rings out of old ones. Namely, if we consider an analytic ring A 2.1.1.12 formally implies that

$$\mathbf{P}_A := \mathbf{P} \otimes_{\mathbf{Z}} A \in \mathcal{D}(A)$$

is internally compact as well. Furthermore if we have given any collection W of endomorphisms of $\mathbf{P}_A$ and write

$$\widehat{(\_)} \colon \mathcal{D}(A) \to \mathcal{D}(A)[W^{-1}]$$

for the Bousfield localization obtained by inverting maps in W. Then it is easy to check that

$$(\widehat{A^{\triangleright}}, \mathcal{D}(A)[W^{-1}])$$

defines a new analytic ring. All analytic rings in this text will arise in this way.

2.1.2. Analytic stacks.

Definition 2.1.2.1. We define the category of *affine analytic stacks* to be

$$\mathcal{A}n\mathcal{S}tack^{\mathrm{aff}} := \mathcal{A}n\mathcal{R}ing^{\mathrm{op}}$$

the opposite category of analytic rings. For an object corresponding to an analytic ring A, we will write $\mathrm{AnSpec}(A)$.

Proposition 2.1.2.2. *Consider a map $f \colon AnSpec(B) \to AnSpec(A)$ of affine analytic rings. Then the following are equivalent:*

- *The induced map $f_* \colon \mathcal{D}(B) \to \mathcal{D}(A)$ satisfies the projection formula. That is for $M \in \mathcal{D}(A)$ and $N \in \mathcal{D}(B)$ we have*

$$f_*(f^*(M) \otimes_B N) \simeq M \otimes_A f_*(N) \in \mathcal{D}(A).$$

- *The analytic ring structure on B is induced from A via f. That is, we have the canonical functor*

$$\mathcal{M}od_{B^{\triangleright}}(\mathcal{D}(A)) \xrightarrow{\sim} \mathcal{D}(B)$$

  is an equivalence.

PROOF. This is explained in [CS24][Lecture 16]. □

Definition 2.1.2.3. A map of affine analytic stacks is called *proper* if it satisfies the equivalent conditions from 2.1.2.2.

Using the second definition, one easily obtains the following:

Corollary 2.1.2.4. *Proper morphisms of affine analytic stacks are stable under composition and base change. Furthermore, for any pair*

$$g \circ f \colon X \to Y \to Z$$

of composable morphisms, if g and $g \circ f$ are proper f is proper as well.

Definition 2.1.2.5. A map $j\colon \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ of affine analytic stacks is called an *open immersion*, if the functors

$$j^*\colon \mathcal{D}(A) \to \mathcal{D}(B)$$

admits a fully faithful left adjoint $j_!$ which satisfies the projection formula. That is for $M \in \mathcal{D}(A)$ and $N \in \mathcal{D}(B)$ we have

$$j_!(j^*(M) \otimes_B N) \simeq M \otimes_A j_!(N) \in \mathcal{D}(A).$$

2.1.2.6. Given an open immersion $j\colon U \to X$ of affine analytic stacks, one obtains a recollement

$$\begin{tikzcd} \mathcal{D}_{\mathrm{qc}}(U) \arrow[r, hook, shift left=2, "j_!"] \arrow[r, hook, shift right=2, "j_*"'] & \mathcal{D}_{\mathrm{qc}}(X) \arrow[l, "j^*" description] \arrow[r, shift left=2, "i^*"] \arrow[r, shift right=2, "i^!"'] & \mathcal{D}_{\mathrm{qc}}(Z). \arrow[l, hook, "i_*" description] \end{tikzcd}$$

That is a diagram as indicated, where $\mathcal{D}_{\mathrm{qc}}(Z)$ is constructed as the cofibre of the functor $j_!$, every functor above is left adjoint to the functor below, and the functors indicated as injective are fully faithful. Having the projection formula in this situation exactly boils down to $\mathcal{D}_{\mathrm{qc}}(Z)$ being the category of modules over an idempotent algebra

$$\mathcal{O}_Z := i_*i^*\mathcal{O}_X \in \mathcal{D}_{qc}(X).$$

On the other hand, if we have such an idempotent given, we can produce the recollement by taking the fibre. Asking $(\mathcal{O}_U, \mathcal{D}_{qc}(U))$ to be an analytic ring can be phrased by asking

$$\underline{\mathrm{Hom}}_X(\mathcal{O}_Z, \_)[-1]$$

to preserve colimits and connective objects. This explains the following.

Proposition 2.1.2.7. *Associating to an open immersion $j\colon AnSpec(B) \to AnSpec(A)$ the object*

$$cof(j_!B^{\triangleright} \to A^{\triangleright}) \in \mathcal{D}(A)$$

induces a bijection between the poset of affine open immersions into $AnSpec(A)$ and idempotent algebras $C \in \mathcal{D}(A)$ such that $\underline{\mathrm{Hom}}_A(C, \_)[-1]$ preserves all colimits and connective objects.

Corollary 2.1.2.8. *Open immersions of affine analytic stacks are stable under composition and base change. Furthermore, for any pair*

$$g \circ f\colon X \to Y \to Z$$

of composable morphisms, if g and $g \circ f$ are open immersions, f is an open immersion as well.

Example 2.1.2.9. In most cases, we will produce open immersions by starting with the idempotent algebra

$$\mathcal{O}_Z \in \mathcal{D}_{\mathrm{qc}}(X).$$

Let us point out that in this situation, one can compute the functors appearing in the recollement 2.1.2.6 quite explicitly. Via the functor j_* the category $\mathcal{D}_{\mathrm{qc}}(U)$ identifies as with the full subcategory of those objects M such that

$$\underline{\mathrm{Hom}}(\mathcal{O}_Z, M) \simeq 0.$$

Now, if we write $F \simeq fib(\mathcal{O}_X \to \mathcal{O}_Z)$ for the fibre of (the only) ring map , then it is not hard to check that the formula

$$j^* \simeq \mathrm{Hom}(F, \_)\colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(U)$$

localizes to this full subcategory. But now j^* has an obvious left adjoint given by $j_! \simeq F \otimes \_$.

Definition 2.1.2.10. A morphism of affine analytic stacks is called *!-able* if one can factor it as an open immersion followed by a proper map.

Remark 2.1.2.11. Note that for any map of affine analytic stacks $f\colon \mathrm{AnSpec}(C) \to \mathrm{AnSpec}(A)$ there is a canonical factorization

$$\begin{tikzcd} \mathrm{AnSpec}(C) \arrow[r, "j"] \arrow[dr, "f"'] & \mathrm{AnSpec}(A \otimes_{A^\triangleright} C^\triangleright) \arrow[d, "p"] \\ & \mathrm{AnSpec}(A) \end{tikzcd}$$

where the complete modules for $A \otimes_{A^\triangleright} C^\triangleright$ are given by $\mathcal{M}od_{C^\triangleright}(\mathcal{D}(A))$. In particular, the map p is proper. We claim that j is an open immersion if and only if f is !-able. To see this let us consider a factorization $\mathrm{AnSpec}(C) \to \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ of f into an open immersion followed by a proper map. Then we can produce the following Cartesian square

$$\begin{tikzcd} \mathrm{AnSpec}(C \otimes_B A \otimes_{A^\triangleright} C^\triangleright) \arrow[r, hook] \arrow[d, hook, shift left] & \mathrm{AnSpec}(A \otimes_{A^\triangleright} C^\triangleright) \arrow[d, hook] \\ \mathrm{AnSpec}(C) \arrow[r, hook] \arrow[u, dashed, shift left] \arrow[ur, hook, "j"] & \mathrm{AnSpec}(B) \end{tikzcd}$$

where all maps are -1-truncated and the horizontal maps are open immersions. So the left vertical map is a -1-truncated map which admits a section and thus an isomorphism.

Proposition 2.1.2.12. *!-able morphisms of affine analytic stacks are stable under base change and composition. Furthermore, for any pair*

$$g \circ f\colon X \to Y \to Z$$

of composable morphisms, if g and $g \circ f$ are !-able, f is !-able as well.

PROOF. Stability by base change easily follows from 2.1.2.8 and 2.1.2.4. For stability by composition, note that for any composable pair of maps of affine analytic stacks $\mathrm{AnSpec}(C) \to \mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ the square

$$\begin{tikzcd} \mathrm{AnSpec}(B \otimes_{B^\triangleright} C^\triangleright) \arrow[r] \arrow[d] & \mathrm{AnSpec}(A \otimes_{A^\triangleright} C^\triangleright) \arrow[d] \\ \mathrm{AnSpec}(B) \arrow[r] & \mathrm{AnSpec}(A \otimes_{A^\triangleright} B^\triangleright) \end{tikzcd}$$

is Cartesian. So we can use 2.1.2.11 2.1.2.8 and 2.1.2.4. Also, the last claim easily follows using the functorial compactifications from 2.1.2.11. □

Construction 2.1.2.13. As explained in 1.1.2, we can extend the functor which assigns to an affine analytic stack $\mathrm{AnSpec}(A)$ its category of quasi-coherent sheaves

$$\mathcal{D}_{\mathrm{qc}}(\mathrm{AnSpec}(A)) := \mathcal{D}(A)$$

to a six-functor formalism. Here we choose open immersions as the class $\mathcal{C}_I$, proper morphisms as the class $\mathcal{C}_P$, and !-able maps as the class $\mathcal{C}_E$. The assertions needed to do this were just explained.

Remark 2.1.2.14. Note that the six-functor formalism 2.1.2.13 is monoidal, such that the $\mathcal{D}$-topology identifies with the !-topology 1.1.3.2.

Definition 2.1.2.15. An *analytic stack* X is a sheaf

$$X\colon (\mathcal{A}n\mathcal{S}tack^{\mathrm{aff}})^{\mathrm{op}} \to \mathcal{A}ni$$

for the !-topology[2].

Remark 2.1.2.16. The official definition of the category of analytic stacks is a further localization of what we defined here as analytic stacks. At no point in this text will we need these further identifications, so we chose this slightly simplified definition.

Construction 2.1.2.17. The !-topology on affine analytic stacks is sub-canonical [**CS24**][Lecture 17]. So, as explained in 1.1.3, we can extend the six-functor formalism from analytic rings to analytic stacks. We will write

$$\mathcal{D}_{\mathrm{qc}}$$

for this six-functor formalism.

2.1.3. Proper maps of analytic stacks.

Definition 2.1.3.1. Consider a map of analytic stacks $f\colon X \to S$. We will say f is:

- *affine proper*, if it is locally on the target, affine and proper.
- *proper*, if it is cohomologically proper 1.1.3.5 in the six-functor formalism 2.1.2.17.
- *locally proper*, if there exists a proper surjection $g\colon Y \to X$, such that the composition $f \circ g$ is affine proper.

Remark 2.1.3.2. It is easy to check that proper and locally proper morphisms are stable under pullbacks and composition.

Remark 2.1.3.3. Being proper is local on the target in analytic stacks, which follows as in the proof of [**HM24**][4.6.3].

We will use the following criterion for properness.

Proposition 2.1.3.4. *Consider a locally proper map $f\colon X \to AnSpec(A)$ to an affine analytic stack, such that the diagonal $\Delta_f\colon X \to X \times_A X$ is proper. Then f is proper if and only if $\mathcal{O}_X \in \mathcal{D}_{qc}(X)$ is compact.*

Proof. If f is proper, then f_* is a left adjoint and thus preserves colimits. So the claim follows as $A^{\triangleright} \in \mathcal{D}(A)$ is compact.

To see the other implication, recall that we have a natural transformation

$$f_! \to f_*$$

and it suffices to check that this natural transformation becomes an isomorphism, if we apply it to the unit 1.1.3.8. Now choose a proper surjection $g\colon \mathrm{AnSpec}(B) \to X$, such that $f \circ g$ is proper. Then the full subcategory of those objects in $\mathcal{D}_{\mathrm{qc}}(X)$ for which the above natural transformation is an isomorphism is stable under retracts and contains all objects in the image of the functor g_*. As g is a surjection, we can write

$$\mathcal{O}_X \simeq \mathrm{colim}_{\bullet \in \Delta}\, g(\bullet)_* g(\bullet)^! \mathcal{O}_X \simeq \mathrm{colim}_{n \in \mathbf{N}}\, \mathrm{colim}_{\bullet \in \Delta_{\leq n}}\, g(\bullet)_* g(\bullet)^! \mathcal{O}_X$$

[2]This is just a topos up to size issues. See the Conventions

where $g(\bullet)$ denote the respective maps in the Čech nerve. So, as $\mathcal{O}_X$ is compact, it also lives in this subcategory. □

Definition 2.1.3.5. If it is proper and a monomorphism, we will say a morphism $Z \to X$ is a *closed immersion*.

2.1.3.6. Consider an analytic stack X that admits a surjection

$$U_1 \amalg U_2 \to X$$

where both $U_i \to X$ are monomorphisms[3]. Then one easily checks that the square

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] & U_1 \arrow[d] \\ U_2 \arrow[r] & X \end{tikzcd}$$

is a pushout square of analytic stacks[4]. In particular the The induced square on quasi-coherent sheaves becomes a pullback.

The following argument will be used several times in this text, so let us record it in a corollary.

Corollary 2.1.3.7. *Consider a locally proper map $X \to AnSpec(A)$ to an affine analytic stack, such that the diagonal of f is proper. Assume there exists a surjection*

$$\amalg_{1 \leq i \leq n} Z_i \to X$$

such that each $Z_i \to X$ is a closed immersion and $\mathcal{O}_{Z_i} \in \mathcal{D}_{qc}(Z_i)$ is compact for all $1 \leq i \leq n$. Then f is proper.

Proof. By 2.1.3.4 we just have to check that $\mathcal{O}_X \in \mathcal{D}_{\mathrm{qc}}(X)$ is compact. Not as we consider closed immersions the restriction of $\mathcal{O}_X$ to any finite intersection of the Z_i becomes compact as well. Now the claim follows by induction on n using the Cartesian squares

$$\begin{tikzcd} \mathcal{D}_{\mathrm{qc}}(X) \arrow[r] \arrow[d] & \mathcal{D}_{\mathrm{qc}}(V) \arrow[d] \\ \mathcal{D}_{\mathrm{qc}}(Z_n) \arrow[r] & \mathcal{D}_{\mathrm{qc}}(V \cap Z_n) \end{tikzcd}$$

2.1.3.6 where $V = \cap_{1 \leq i \leq n-1} Z_i$. □

2.1.4. Examples of surjections.

Let us collect some examples of surjections of analytic stacks for later use. We start with the case of proper maps.

2.1.4.1. Given a stable presentable symmetric monoidal category $\mathcal{C}$, recall from [**BS17**][11.2] that an $\mathcal{E}_\infty$-algebra A in $\mathcal{C}$ is called *descendable*, if the multiplication map

$$fib(\mathbf{1}_{\mathcal{C}} \to A)^{\otimes n} \to \mathbf{1}_{\mathcal{C}}$$

for some n identifies with the 0-map. This is equivalent to the assertion that the functor which assigns to an algebra B the category

$$\mathcal{M}od_{\mathcal{M}od_B(\mathcal{C})}(\mathcal{P}r^L_{\mathrm{st}})$$

[3]In most cases both U_i will be either closed immersions or open immersions. Finding a surjection from a mixture of such might be hard.

[4]For example by pulling back to the Z_i.

descends along the morphism $\mathbf{1}_{\mathcal{C}} \to A$ [**Mat16**][3.3].

Example 2.1.4.2. A affine proper map $\mathrm{AnSpec}(B) \to \mathrm{AnSpec}(A)$ of analytic stacks, is a surjection, iff $B^{\triangleright}$ is a descendable algebra in $\mathcal{D}(A)$ [**CS24**][Lecture 18].

2.1.4.3. We will call a map $j\colon U \to X$ of analytic stacks an *open immersion*, if it is cohomologically étale and a monomorphism. This implies that there exists an idempotent algebra

$$\mathcal{O}_Z \in \mathcal{D}_{\mathrm{qc}}(X)$$

such that the category $\mathcal{D}_{\mathrm{qc}}(U)$ via the functor $j_!$ identifies with the kernel of the functor $(\_) \otimes_{\mathcal{O}_X} \mathcal{O}_Z$.

Given a finite collection of open substacks $\{U_i \to X\}$, then the map

$$\amalg_I U_i \to X$$

is a surjection of analytic stacks, if and only if

$$\mathcal{O}_{Z_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{O}_{Z_n} \simeq 0.$$

This is explained in [**CS24**][Lecture 18].

Example 2.1.4.4. Given an analytic stack X let us write $\mathcal{S}(X)$ for the locale of idempotent algebras in $\mathcal{D}_{\mathrm{qc}}(X)$ [**CS22**][5]. For any map of locales $\mathcal{S}(X) \to T$ and any open $U \subset T$, we obtain an open substack of X given by the complement of the idempotent algebra obtained by pulling back U. Furthermore, if a finite collection of opens covers T, then the obtained open substack will be jointly surjective in analytic stacks.

Example 2.1.4.5. Given a cohomologically etale morphism $j\colon U \to X$ of analytic stacks, it is a surjection if and only if the functor

$$j^*\colon \mathcal{D}_{\mathrm{qc}}(X) \to \mathcal{D}_{\mathrm{qc}}(U)$$

is conservative [**HM24**][4.7.1].

2.2. Derived Huber pairs

2.2.1. Affine derived adic spaces.

2.2.1.1. Recall that there is an endomorphism

$$\mathrm{shift}\colon \mathbf{N}\cup\{\infty\}\to\mathbf{N}\cup\{\infty\}$$

which shifts the points in $\mathbf{N}$ one up and is the identity on the limit point. In particular, we can consider the endomorphism

$$1_{\square} := \mathrm{id}-\mathrm{shift}\colon \mathbf{P}\to\mathbf{P}$$

and using 2.1.1.13 we can invert $1_{\square}$ in light condensed derived abelian groups to obtain an analytic ring

$$\mathbf{Z}_{\square}$$

which we will call *the solid integers*. Note that for an analytic stack it is a property to live over the corresponding affine analytic stack

$$\mathrm{Spa}(\mathbf{Z})$$

which we will call *solid*. We will restrict our attention from now on to solid analytic stacks.

Example 2.2.1.2. Concretely, a light condensed derived abelian group M is solid if the map

$$1_{\square}^{*}\colon\ \underline{\mathrm{Hom}}(\mathbf{P},M)\xrightarrow{\sim}\underline{\mathrm{Hom}}(\mathbf{P},M)$$

is an isomorphism. The inverse of this map should be thought of as the assignment taking a null sequence $(m_0,m_1\ldots)$ to the sequence

$$(\sum_{i\geq 0}^{\infty}m_i,\sum_{i\geq 1}^{\infty}m_i,\ldots).$$

This condition states that we can sum null sequences in solid modules. One example where these sums become finite is discrete derived abelian groups, in particular, those are solid.

Example 2.2.1.3. As solid modules are stable under limits and cofibers, for any solid condensed ring $A^{\triangleright}$ and any function (or finite collection of functions) $f\in A^{\triangleright}(*)$, the derived f-adic completion

$$M_{\widehat{f}}\simeq \lim_n M/(f^n)$$

of a solid $A^{\triangleright}$-module is again solid.

2.2.1.4. Recall that the one-point compactification, as an assignment from locally compact Hausdorff spaces to pointed compact Hausdorff spaces is a functor for proper maps and symmetric monoidal (i.e., sends finite products to smash products) [**Jam84**][Chapter 3. Prop 3.7]. As the addition map on the natural numbers is proper (it has finite fibres), we can use this to understand

$$\mathbf{N}\cup\{\infty\}$$

as a commutative monoid in pointed topological spaces and thus as a condensed object in monoids in pointed sets. In formulas, we mean

$$*\wedge\infty=\infty=\infty\wedge * \text{ and } n\wedge m=n+m.$$

As a pointed object, the free condensed abelian group on $\mathbf{N} \cup \{\infty\}$ is given by $\mathbf{P}$, which, using the above, can be seen as a condensed ring. As such, it comes with a ring map

$$\mathbf{Z}[x] \to \mathbf{P}$$

by choosing 1.

2.2.1.5. The solidification (that is the localization to $\mathcal{D}(\mathbf{Z}_\square)$) of a free condensed derived abelian group on a light profinite set $S = \lim_I S_i$ can be identified with

$$\mathbf{Z}[S]^\square \simeq \lim_I \mathbf{Z}[S_i] \simeq \prod_{\mathbf{N}} \mathbf{Z}$$

[**CS24**][Lecture 5]. Using this, one checks that the ring map from 2.2.1.4 after solidification becomes the canonical map

$$\mathbf{Z}[x] \to \mathbf{Z}[[x]].$$

Via these computations, one also obtains the most important formula in the solid setting

$$\mathbf{Z}[[x]] \otimes_{\mathbf{Z}_\square} \mathbf{Z}[[y]] := (\mathbf{Z}[[x]] \otimes \mathbf{Z}[[y]])^\square \simeq \mathbf{Z}[[x, y]].$$

2.2.1.6. Let us write $\mathbf{A}^1_\square$ for the affine analytic stack corresponding to the solid analytic ring

$$(\mathbf{Z}[y], \mathcal{M}od_{\mathbf{Z}[Y]}(\mathcal{D}(\mathbf{Z}_\square))).$$

This admits an affine open substack $\mathbf{D}^1_\square \subset \mathbf{A}^1_\square$ which we will now construct. The underlying condensed ring is given by $\mathbf{Z}[y]$ itself, and to obtain its category of complete modules, we invert the morphism

$$y_\square := \mathrm{id} - y \cdot \mathrm{shift} \colon \mathbf{Z}[[x]][y] \to \mathbf{Z}[[x]][y]$$

in $\mathcal{M}od_{\mathbf{Z}[y]}(\mathcal{D}(\mathbf{Z}_\square))$. Note that the cofibre of this map is given by

$$\mathbf{Z}((y^{-1})) := \mathbf{Z}[[y^{-1}]] \otimes_{\mathbf{Z}[y^{-1}]} \mathbf{Z}[y^{-1}, y],$$

which is an idempotent algebra by 2.2.1.5 and this localization is obtained by killing this idempotent algebra as in 2.1.2.9. In particular, we can describe the localization to the complete modules by the functor

$$j^* \simeq \underline{\mathrm{Hom}}_{\mathbf{A}^1_\square}(\mathbf{Z}((y^{-1}))/\mathbf{Z}[y][-1], \_).$$

Using the already mentioned fibre sequence one checks that $\underline{\mathrm{Hom}}(\mathbf{Z}((y^{-1})), \_)[-1]$ preserves colimits and connective objects, so we have produced an affine open immersion by 2.1.2.7.

Remark 2.2.1.7. Note that as an analytic stack, the affine line $\mathbf{A}^1_\square$ is proper and not smooth over $\mathrm{Spa}(\mathbf{Z})$. The unit disc $\mathbf{D}^1_\square$ in contrast becomes smooth [**CS19b**][Lecture XI]. In particular, the pullback functor

$$f^* \colon \mathcal{D}_{\mathrm{qc}}(\mathrm{Spa}(\mathbf{Z})) \to \mathcal{D}_{\mathrm{qc}}(\mathbf{D}^1_\square)$$

admits a left adjoint and thus commutes with all limits (and also colimits). This fact can be seen more directly on its own. Namely, one can compute this pullback functor as

$$f^* \simeq \underline{\mathrm{Hom}}_{\mathbf{Z}_\square}(x\mathbf{Z}[[x]], \_)$$

where the $\mathbf{Z}[x]$ action is induced by $x^n \mapsto x^{n-1}$ [**CS24**][Lecture 7].

Definition 2.2.1.8. Any analytic ring A has an underlying animated ring $A^{\triangleright}(*)$ and if A is solid an element $f \in \mathbf{A}^{\triangleright}(*)$ precisely corresponds to a map of analytic stacks

$$\begin{tikzcd} & \mathrm{AnSpec}(A) \arrow[dl, dotted] \arrow[d, "f"] \\ \mathbf{D}^1_{\square} \arrow[r, hook] & \mathbf{A}^1_{\square} \end{tikzcd}$$

and we say a function is bounded by 1, if this map factors through $\mathbf{D}^1_{\square}$. We will write

$$A^+ \subset \pi_0 A^{\triangleright}(*)$$

for the subset of functions bounded by 1.

Remark 2.2.1.9. The subset A^+ forms an integrally closed subring [**CS24**][Lecture 8]. Furthermore, by pulling it back to $A^{\triangleright}$, we will also understand it as a sub-animated ring of $A^{\triangleright}$ [**Cam24**][2.6].

2.2.1.10. Consider a solid analytic ring A then we can construct a new solid analytic ring $(A^{\triangleright}, A^+)_{\square}$ together with a map

$$(A^{\triangleright}, A^+)_{\square} \to A.$$

As an underlying condensed ring, we take $A^{\triangleright}$, and to obtain the complete modules, we invert the maps

$$f_{\square} := \mathrm{id} - f \cdot \mathrm{shift}\colon \mathbf{P}_{A^{\triangleright}} \to \mathbf{P}_{A^{\triangleright}}$$

in $\mathcal{M}od_{A^{\triangleright}}(\mathcal{D}(\mathbf{Z}_{\square}))$ for all $f \in A^+$.

Now we can recall the following definition from [**Cam24**][2.6.6].

Definition 2.2.1.11. A solid analytic ring A is called *solid affinoid*, if the map

$$(A^{\triangleright}, A^+)_{\square} \xrightarrow{\sim} A$$

is an isomorphism. We will write $\mathcal{A}ff\mathcal{R}ing_{\square}$ for the full subcategory of solid affinoid analytic rings. Furthermore, we will write $\mathrm{Spa}(A^{\triangleright}, A^+)$ for the corresponding analytic stack of a solid affinoid analytic.

Remark 2.2.1.12. The category $\mathcal{A}ff\mathcal{R}ing_{\square}$ is compactly generated and stable under colimits and finite products in all analytic rings [**Cam24**][2.6.8].

Example 2.2.1.13. Any Huber pair (A, A^+) in the sense of [**Hub96**] gives rise to a solid affinoid analytic ring by considering the topological ring A as a condensed ring and A^+ as the subring of functions bounded by 1. This defines a fully faithful functor from Huber pairs to solid affinoid analytic rings [**And21**][3.34].

Before we start gluing affine derived adic spaces, let us record the following simple but useful characterization of proper maps between affine derived adic spaces.

Proposition 2.2.1.14. *A map $f\colon Spa(B^{\triangleright}, B^+) \to Spa(A^{\triangleright}, A^+)$ of affine derived adic spaces is proper if the map*

$$f^+\colon A^+ \to B^+$$

is integral.

PROOF. The map is proper if $\mathcal{D}((B^{\triangleright}, B^+)_{\square})$ can be obtained from $\mathcal{M}od_{B^{\triangleright}}(\mathcal{D}(A^{\triangleright}, A^+))$ by inverting the maps

$$f^+(a)_{\square} \colon \mathbf{P}_{B^{\triangleright}} \to \mathbf{P}_{B^{\triangleright}}$$

for $a \in A^+$. The set of elements $b \in \pi_0 B^{\triangleright}(*)$ such that $b_{\square}$ is an isomorphism in $\mathcal{D}(B^{\triangleright}, B^+)$ is integrally closed in $\pi_0 B^{\triangleright}(*)$ and is given by B^+ 2.2.1.9. $\square$

2.2.2. Valuation spectra of affine derived adic spaces.

2.2.2.1. Let us write $\mathcal{S}(A)$ for the locale of idempotent algebras in $\mathcal{D}(A)$ the category of quasi-coherent sheaves on a solid affinoid ring A. In this section, we will produce maps

$$\mathcal{S}(A) \to X$$

to topological spaces. By 2.1.4.4, this will produce coverings of analytic stacks by open substacks by pulling back open coverings from X.

2.2.2.2. Given a solid affinoid algebra A, we will write

$$|\mathrm{Spv}(A^{\triangleright}, A^+)| := \{x \colon \pi_0 A^{\triangleright}(*) \to \Gamma_x \text{ s.t. } |f(x)| \leq 1 \text{ for all } f \in A^+\}/\simeq$$

for the set of those valuations which understand elements in A^+ as ≤ 1 and equip it with the coarsest topology where the subsets

$$|U(\frac{f}{g})| := \{x \in |\mathrm{Spv}(A^{\triangleright}, A^+)| \text{ s.t. } |f(x)| \leq |g(x)| \neq 0|\}$$

are open for all $f, g \in \pi_0 A^{\triangleright}(*)$. This defines a quasi-compact spectral topological space [**Wed19**][4.7] and it is easy to check that it admits a basis by the *rational opens*

$$|U(\frac{f_1, \ldots, f_n}{g})| := \{|f_1(x)|, \ldots, |f_n(x)| \leq |g(x)| \neq 0\}$$

for $f_1, \ldots, f_n, g \in \pi_0 A^{\triangleright}(*)$, which are quasi-compact and stable under intersections.

Example 2.2.2.3. We will call the rational opens of the form

$$|U(\frac{f_1, \ldots, f_n}{g})|$$

such that $f_1, \ldots, f_n, g$ generated the unit ideal *standard rational opens.* Note that a collection of standard rational opens that cover is given by

$$\{|U(\frac{f_1, \ldots, f_n}{f_i})|\}_{1 \leq i \leq n}$$

where $f_1, \ldots, f_n$ are functions generating the unit ideal. We will call such a covering *standard rational cover.*

Let us summarize how to understand these topological spaces in terms of sheaves in the following proposition.

Proposition 2.2.2.4. *Consider the topological space $X = |Spv(A, A^+)|$ associated to a solid affinoid analytic ring. Then the standard rational opens form a basis of X which is stable under intersections. Furthermore, a presheaf $\mathcal{E}$ on X is a sheaf if and only if one of the following equivalent assertions holds.*

(1) We have
   (a) $\mathcal{E}(\emptyset) \simeq *$

(b) For any Mayer-Vietoris square consisting of standard rational opens the square

$$\begin{tikzcd}
\mathcal{E}(V \cup U) \arrow[r] \arrow[d] & \mathcal{E}(U) \arrow[d] \\
\mathcal{E}(V) \arrow[r] & \mathcal{E}(V \cap U)
\end{tikzcd}$$

is Cartesian.

(2) For any standard rational open U and any standard rational cover $\{V_i \to U\}_I$ tha canonical map

$$\mathcal{E}(U) \xrightarrow{\sim} \lim_{n\in\Delta} \prod_{I^n} \mathcal{E}(V_{1_i} \cap \cdots \cap V_{n_i})$$

is an isomorphism.

(3) We have

(a) $\mathcal{E}(\emptyset) \simeq *$
(b) For any standard rational open U and any function $f \in \pi_0 A^{\flat}(*)$ the squares

$$\begin{tikzcd}
\mathcal{E}(U) \arrow[r] \arrow[d] & \mathcal{E}(U(\frac{1}{f})) \arrow[d] \\
\mathcal{E}(U(\frac{1}{1-f})) \arrow[r] & \mathcal{E}(U(\frac{1}{1-f}) \cap U(\frac{1}{f}))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}(U) \arrow[r] \arrow[d] & \mathcal{E}(U(\frac{1}{f})) \arrow[d] \\
\mathcal{E}(U(\frac{f}{1})) \arrow[r] & \mathcal{E}(U(\frac{f}{1}) \cap U(\frac{1}{f}))
\end{tikzcd}$$

are Cartesian.

Proof. The fact that standard rational opens form a basis which is stable under intersections is [**Hub93a**][2.6]. Using this "unfolding" tells that a sheaf is determined on its values on these standard rational opens[5].

The characterization (2) follows as any covering of rational opens admits a refinement by a standard rational cover [**Hub94**][2.6].

To see (1) and (2) note first that both types of Mayer-Vietoris squares appearing there form cd-structures. So by [**AHW15**][3.2.5], to finish the proof, we have to check that any standard rational cover admits a refinement by a composition of covers appearing in (3).

For this, let us first prove the following claim:

($*_1$) For any rational open U and any collection of rational functions $f_1, \dots, f_n \in \pi_0 A^{\flat}(*)$ such that $f_1 + \cdots + f_n = 1$ on U the covering

$$\{U(\frac{1}{f_i})\}_{1\le i\le n}$$

admits a refinement of a composition of coverings appearing in (3).

We prove this by induction on n. The case $n = 2$ holds by assumption. To see the induction step, note that by assumption we have a covering of U given by

$$\{U(\frac{1}{f_n}), U(\frac{1}{f_1 + \cdots + f_{n-1}}) \cap U(\frac{f_n}{1}) := V\}.$$

[5]Note that this also tells us what to do with infinite unions of open subsets

The pullback of the original covering to $U(\frac{1}{f_n})$ becomes split so we just have to argue for V. On V the function $f_1+\cdots+f_{n-1}$ becomes invertible, let us write x for the inverse. Then

$$\{U(\frac{1}{xf_i})\}_{1\le i\le n-1}$$

as open on V refines the original cover and can be refined by the covers appearing in (3) by the inductive hypothesis.

For the rest of the argument, we closely follow [**CS19b**][10.3]. We have already seen that a standard rational covering can refine any covering

$$\{U(\frac{f_1,\dots,f_n}{f_i})\}_{1\le i\le n}$$

so we have to refine such a covering. Thus we can find functions x_i such that $x_1f_1+\cdots+x_nf_n=1$ and by $(*_1)$ it suffices to check the assertion after pulling back to the $U(\frac{1}{x_kf_k})$. Over such a rational open, we get

$$U(\frac{x_kf_kf_1,\dots,x_kf_kf_n}{x_kf_kf_i})=U(\frac{f_k^{-1}f_1,\dots,1,\dots,f_k^{-1}f_n}{f_k^{-1}f_i})$$

for all i. So it suffices to check the following:

$(*_2)$ For any rational open U and any collection of rational functions $f_1,\dots,f_n\in\pi_0A^{\triangleright}(*)$, such that $1=f_1$ on U, the standard rational cover

$$\{U(\frac{f_1,\dots,f_n}{f_i})\}_{1\le i\le n}$$

admits a refinement by a composition of coverings appearing in (3).

We again prove the statement by induction on n, the case $n=2$ holding by assumption. For the induction step note that the assumption tells us that we can prove the assertion after pulling back to $U(\frac{1}{f_n})$ and $U(\frac{f_n}{1})$, that is we can assume that either $f_n^{-1}\le 1$ or $f_n\le 1$. In the first case, the collection

$$U(\frac{1,\dots,f_n}{f_i})=U(\frac{1,\dots,f_{n-1}}{f_i})$$

for $1\le i\le n-1$ gives a refinement. So we can apply the inductive hypothesis. In the second case, the collection

$$U(\frac{1,\dots,f_n}{f_i})=U(\frac{f_n^{-1}f_2,\dots,f_n^{-1}f_{n-1},1}{f_n^{-1}f_i})$$

for $2\le i\le n$ gives a refinement, we can again apply the inductive hypothesis. $\square$

Example 2.2.2.5. The universal case of a rational open of the form $U(\frac{1}{f})$ is given by

$$U(\frac{1}{x})\subset\mathbf{A}^1_{\square}.$$

We claim that the corresponding solid affinoid ring is obtained by killing the idempotent algebra $\mathbf{Z}[[x]]$ in $\mathcal{D}_{\mathrm{qc}}(\mathbf{A}^1_{\square})$ as in 2.1.2.9.

For this it suffices to see that for any M such that $\underline{\mathrm{Hom}}_{\mathbf{Z}[x]}(\mathbf{Z}[[x]],M)\simeq 0$ the map

$$M\to M[x^{-1}]$$

is an isomorphism. But note that the fibre is x-torsion and any x-torsion module is a module over $\mathbf{Z}[[x]]$. As the latter is an idempotent algebra, this implies that for any such x-torsion module N_{tors} we have

$$\underline{\mathrm{Hom}}_{\mathbf{Z}[x]}(\mathbf{Z}[[x]], N_{\mathrm{tors}}) \xrightarrow{\sim} N_{\mathrm{tors}}.$$

This easily implies what we want.

2.2.2.6. To transport the rational open topology on adic spaces to the world of analytic stacks, we have to determine open substacks of the affine line. These open substacks can be summarized in the following Cartesian square

$$\begin{tikzcd} (\mathbf{G}_m^{\mathrm{an}})_\square \arrow[r] \arrow[d] & U(\frac{1}{x}) \arrow[d] \\ \mathbf{D}^1_\square \arrow[r] & \mathbf{A}^1_\square. \end{tikzcd}$$

Here, the right vertical immersion was just discussed in 2.2.2.5 and the lower horizontal immersion in 2.2.1.6.

2.2.2.7. Consider a solid affinoid analytic ring $A = (A^\triangleright, A^+)$. Then, to any collection of functions defining a standard rational open

$$U := |U(\frac{f_1, \ldots, f_n}{g})| \subset |\mathrm{Spv}(A)|$$

we can associate and affine open substack of $U \subset \mathrm{Spa}(A)$ which is represented by a solid affinoid analytic ring $\mathcal{O}_U(U)$. An inducing pre-analytic ring can be described as follows:

The underlying condensed ring is given by

$$\mathcal{O}_U(U)^\triangleright \simeq A^\triangleright[\frac{1}{g}]$$

and we obtain the category of complete modules $\mathcal{D}_{\mathrm{qc}}(U)$ by inverting the endomorphisms

$$\frac{f_i}{g}_\square : \mathbf{P}_A[\frac{1}{g}] \to \mathbf{P}_A[\frac{1}{g}]$$

for all f_i in $\mathcal{M}od_{A^\triangleright[\frac{1}{g}]}(\mathcal{D}(A))$.

Let us recall the argument why this gives an open immersion of analytic stacks from [**MW24**][5.3]. Note first that the defining functions of the rational open generate the unit ideal by assumption. So the function g becomes invertible in the algebra

$$A^\triangleright[x_1, \ldots, x_n]/(gx_1 - f_1, \ldots, gx_n - f_n).$$

From this, one sees that we can compute $\mathcal{O}_U(U)$ also using this ring and then solidify the functions x_i. Using what we know, how to compute the pullback functor to the unit disc 2.2.1.7, we see that the pullback to $\mathcal{O}_U(U)$ then is given by $\underline{\mathrm{Hom}}_A(Q, \_)$ where

$$Q \simeq \otimes_{1=i}^n((x_i\mathbf{Z}[[x_i]] \otimes_{\mathbf{Z}} A)/(gx_i - f_i)[-1])$$

which has a left adjoint given by $Q \otimes_A \_$. To check the projection formula, one first observes that Q is a module over $A^\triangleright[\frac{1}{g}]$ so one can check the projection formula in

$\mathcal{M}od_{A^\triangleright}(\mathcal{D}(A))$ but from this category $\mathcal{D}_{\mathrm{qc}}(U)$ is obtained by successfully killing the idempotent algebras

$$\mathbf{Z}((x_i^{-1})) \otimes_{\mathbf{Z}[x_i]} A[\frac{1}{g}]$$

where on the right we use the ring map induced by $x_i \mapsto \frac{f_i}{g}$ 2.2.1.5. So the projection formula holds for formal reasons 2.1.2.6 and 2.1.2.9.

The following is a version of [**Hub94**][1.3]. It gives a universal property of the just-constructed solid affinoid, which in particular says that this construction relative to A just depends on the rational open and not on the defining functions.

Proposition 2.2.2.8. *Consider the situation in 2.2.2.7. Then the solid affinoid A-algebra $\mathcal{O}_U(U)$ is initial among those solid affinoid A-algebras B for which the induced map*

$$\begin{tikzcd} {|Spv(B)|} \arrow[r] \arrow[dr, dotted] & {|Spv(A)|} \\ & U \arrow[u] \end{tikzcd}$$

factors through U.

Proof. Note first that there is a map $|\mathrm{Spv}(\mathcal{O}_U(U))| \to U$. Now the A-algebra $\mathcal{O}_U(U)$ has an obvious universal property. Namely if we consider an A-algebra B we have to check that g becomes invertible on $B^\triangleright$ and then that $\frac{f_i}{g}$ lands in B^+ for all i. Both of these conditions can be checked on $\pi_0 B^\triangleright(*)$. Now note that $(\pi_0 B^\triangleright(*), B^+)$ is a discrete Huber pair, so the rest is standard. The first condition can be checked using trivial valuations for maximal ideals, and the second condition follows as B^+ consists exactly of those elements which have value ≤ 1 on all valuations in $|\mathrm{Spv}(B)|$ [**Hub93a**][3.3]. $\square$

We can now prove the central proposition of this section.

Proposition 2.2.2.9. *For a solid affinoid A assigning a standard rational open $U \subset |Spv(A)|$ to the solid affinoid $\mathcal{O}_U(U)$ induces a continuos map of locals*

$$\mathcal{S}(A) \to |Spv(A)|$$

where $\mathcal{S}(A)$ denotes the locale of idempotent algebras un $\mathcal{D}(A)$. This construction is functorial in A (i.e., it produces a functor from solid affinoid analytic rings to categorified locales).

In particular the induced functor

$$U \mapsto \mathcal{D}_{qc}(U)$$

defines a sheaf on the topological space $|Spv(A)|$.

Proof. The in particular part follows from [**CS22**][5.5]. Recall that taking sheaves defines a fully faithful functor from locales to topoi, so to produce the continuous map in the statement, we can produce a morphism of topoi

$$f^*\colon \mathcal{S}h(|\mathrm{Spv}(A)|) \to \mathcal{S}h(\mathcal{S}(A)).$$

That is, we have to produce a cosheaf $Op(|\mathrm{Spv}(A)|) \to \mathcal{S}h(\mathcal{S}(A))$. Using 2.2.2.4 we have to define the functor on standard rational opens, where we use 2.2.2.8. Now, by 2.2.2.4(3), to check that we have produced a cosheaf boils down to the following computations, where we write $\mathcal{O}_{Z(\frac{1}{f})}$ respectively for the corresponding idempotent:

(1) $\mathcal{O}(|\mathrm{Spv}(A)|) \simeq A$.
(2) $\mathcal{O}_{Z(\frac{1}{f})} \otimes \mathcal{O}_{Z(\frac{1}{1-f})} \simeq 0$ on U for a local function f.
(3) $\mathcal{O}_{Z(\frac{1}{f})} \otimes \mathcal{O}_{Z(\frac{f}{1})} \simeq 0$ on U for a local function f.

The first isomorphism is obvious. For (2) and (3), it suffices to check these isomorphisms in the universal cases. For (2), by 2.2.2.5, this means we have to check that

$$\mathbf{Z}[[x]] \otimes_{\mathbf{Z}[x]} \mathbf{Z}[[1-x]] \simeq \mathbf{Z}[[x,y]]/(1-(x+y)) \simeq 0$$

where the first isomorphism follows using 2.2.1.5 and the second as the geometric series

$$\sum_{n}^{\infty} (x+y)^n$$

gives an inverse for $1-(x+y)$. Similarly, one checks that

$$\mathbf{Z}[[x]] \otimes_{\mathbf{Z}[x]} \mathbf{Z}((x^{-1})) \simeq \mathbf{Z}[[x,y]]/(1-yx) \simeq 0$$

which shows (3) by 2.2.1.6.

The functoriality becomes clear as soon as we see that a map of solid affinoid analytic rings $A \to B$ gives rise to a map of categorified locales. That is, we have to check that the diagram

$$\begin{tikzcd} \mathcal{S}(B) \arrow[r] \arrow[d] & \mathcal{S}(A) \arrow[d] \\ |\mathrm{Spv}(B)| \arrow[r] & |\mathrm{Spv}(A)| \end{tikzcd}$$

commutes. Similar to the above argument, we have to check this on standard rational opens, where it easily follows using the respective universal properties. □

We want to consider certain subspaces of these valuation spectra following Huber to get topological spaces that know something about the "topology" of the solid affinoid. For this, we make the following definitions, which are taken from [**CS24**][Lecture 8].

Definition 2.2.2.10. Consider a solid condensed animated ring $A^{\triangleright}$. Then a function $f^{\triangleright}\colon \mathbf{Z}[x] \to A^{\triangleright}$ is called

(1) *topologically nilpotent* if $f^{\triangleright}$ factors as

$$\begin{tikzcd} \mathbf{Z}[x] \arrow[r, "f^{\triangleright}"] \arrow[d] & A^{\triangleright} \\ \mathbf{Z}[[x]]. \arrow[ur, dotted] & \end{tikzcd}$$

(2) *power-bounded* if $A^{\triangleright} \in \mathcal{D}_{\mathrm{qc}}(\mathbf{D}^1_{\square})$ via the induced map $f^{\triangleright}\colon \mathbf{Z}[x] \to A^{\triangleright}$.

We will write $A^{\circ\circ} \subset A^{\triangleright}(*)$ for the subset of topological nilpotent elements and $A^{\circ} \subset A^{\triangleright}(*)$ for the subset of power-bounded elements.

Remark 2.2.2.11. A° forms an integrally closed subring of $A^{\triangleright}(*)$ and $A^{\circ\circ}$ a radical ideal in A° [**CS24**][Lecture 8].

Example 2.2.2.12. In a discrete animated ring, the topologically nilpotent elements are exactly the nilpotent elements.

2.2.2.13. For a solid affinoid A, let us consider the closed subspace

$$|\mathrm{Spv}(A, A^{\circ\circ})| \subset |\mathrm{Spv}(A)|$$

of those valuations x such that $|f(x)| < 1$ for all $f \in A^{\circ\circ}$. Then the first observation is that the map from 2.2.2.9 factors as

$$\begin{tikzcd} \mathcal{S}(A) \arrow[r] \arrow[dr, dotted] & {|\mathrm{Spv}(A)|} \\ & {|\mathrm{Spv}(A, A^{\circ\circ})|.} \arrow[u, hook] \end{tikzcd}$$

To see this, note that we can cover the complement by standard rational opens $U(\frac{1}{f})$ for $f \in A^{\circ\circ}$. Now, by 2.2.2.5, we obtain the open substack corresponding to this rational open by looking at those $M \in \mathcal{D}(A)$ such that $\underline{\mathrm{Hom}}_A(\mathbf{Z}[[x]] \otimes_{\mathbf{Z}[x]} A, M) \simeq 0$. But we have

$$0 \simeq \underline{\mathrm{Hom}}_A(\mathbf{Z}[[x]] \otimes_{\mathbf{Z}[x]} A, M) \simeq \underline{\mathrm{Hom}}_A(\mathbf{Z}[[x]] \otimes_{\mathbf{Z}[[x]]} A, M) \simeq M$$

as f is topologically nilpotent. This open Substack is the empty stack.

Definition 2.2.2.14. Given a solid affinoid A, we define

$$|\mathrm{Spa}(A)| \subset |\mathrm{Spv}(A, A^{\circ\circ})|$$

as the subspace of continuous valuations. That is those valuations $x \in |\mathrm{Spv}(A, A^{\circ\circ})|$, such that for all $f \in A^{\circ\circ}$ and any $\gamma \in \Gamma_x$ there exists some natural number N such that

$$|f(x)^N| < \gamma$$

.

Remark 2.2.2.15. The space $|\mathrm{Spa}(A)|$ is a spectral space but the inclusion $|\mathrm{Spa}(A)| \subset |\mathrm{Spv}(A, A^{\circ\circ})|$ is not spectral. To still make use of this space, we will use the fact that, in certain situations, Huber has constructed a retraction in the other direction. For this, we need to move slightly closer to his theory.

Definition 2.2.2.16. Consider a solid condensed animated ring $A^{\triangleright}$. Then we say a subring

$$A^{\circ\circ} \subset A_0 \subset A^{\circ}$$

is a *ring of definition*, if it admits a finitely generated ideal $I \subset A_0$ such that

$$\sqrt{I} = A^{\circ\circ}.$$

If it admits a ring of definition, we will say a solid affinoid A is *adic*. Furthermore, a map of solid affinoids $A \to B$ is called *adic*, if there exist rings of definition A_0, B_0 for A and B together with an ideal of definition $I \subset A_0$, such that

$$f^{\triangleright} \colon \pi_0 A^{\triangleright}(*) \to \pi_0 B^{\triangleright}(*)$$

maps A_0 into B_0 and $IB_0 \subset B_0$ defines an ideal of definition.

Remark 2.2.2.17. The main reason we introduce this definition is to obtain the retraction

$$r\colon |\mathrm{Spv}(A, A^{\circ\circ})| \to |\mathrm{Spa}(A)|$$

for a solid affinoid. For this, Huber's proof uses the fact that $A^{\circ\circ}$ is the radical of a finitely generated ideal. Also, this retraction will just be functorial for adic morphisms.

2.2.2.18. To obtain opens in the space $|\mathrm{Spa}(A)|$ which behave well with respect to the already mentioned retraction, we have to add a condition. We will say an open $U \subset |Spa(A)|$ is a *basic rational open*, if it is of the form

$$U = U(\frac{f_1, \dots, f_n}{g})$$

where $f_1, \dots, f_n, g \in A^{\triangleright}(*)$ are functions such that

$$A^{\circ\circ} \cdot \pi_0 A^{\triangleright}(*) \subset \sqrt{(f_1, \dots, f_n)}$$

.

Let us now summarize Huber's construction in the following proposition.

Proposition 2.2.2.19. *Consider an adic solid affinoid analytic ring A. Then we have the following:*

(1) *The basic rational opens form a basis for the topology on $|Spa(A)|$, which is stable under intersections.*
(2) *There exists a spectral quasi-compact retraction*

$$r\colon |Spv(A, A^{\circ\circ})| \to |Spa(A)|$$

of the inclusion.
(3) *For a basic rational open $U = U(\frac{f_1,\dots,f_n}{g}) \subset |Spa(A)|$ we have that*

$$r^{-1}(U) = U(\frac{f_1, \dots, f_n}{g}) \subset |Spv(A, A^{\circ\circ})|.$$

Proof. This is [**Hub93a**][2.6 + 3.1]. □

Now we can deduce that for an adic solid affinoid A, the category also localizes over $|\mathrm{Spa}(A)|$.

Proposition 2.2.2.20. *Assigning to an adic solid affinoid A the composition*

$$\mathcal{S}(A) \to |Spv(A, A^{\circ\circ})| \to |Spa(A)|$$

where the second map is the retraction from 2.2.2.19, induces a functor from adic solid affinoid analytic rings with adic maps as morphisms to categorified locales. In particular any open $U \subset |Spa(A)|$ (resp. open covering) induces an open substack

$$U \hookrightarrow Spa(A)$$

(resp. open covering of substacks).

PROOF. By 2.2.2.9, it suffices to check that for any adic morphism $A \to B$ of adic solid affinoids, the diagram of locales

$$\begin{tikzcd} {|\mathrm{Spv}(B, B^{\circ\circ})|} \arrow[r] \arrow[d, "r"'] & {|\mathrm{Spv}(A, A^{\circ\circ})|} \arrow[d, "r"] \\ {|\mathrm{Spa}(B)|} \arrow[r] & {|\mathrm{Spa}(A)|} \end{tikzcd}$$

commutes. By 2.2.2.19(1), it suffices to check this on basic rational opens in $|\mathrm{Spa}(A)|$. Now, as the map is adic, the pre-image of a basic rational open along the map

$$|\mathrm{Spa}(B)| \to |\mathrm{Spa}(A)|$$

is a basic rational. So the claim easily follows from 2.2.2.19(3). □

2.2.3. Derived adic spaces and their compactifications. We can now define derived adic spaces.

Definition 2.2.3.1. An open immersion of analytic stacks $U \to \mathrm{Spa}(A, A^+)$ into an adic solid affinoid is called *basic rational*, if U admits a open covering of adic solid affinoids $Spa(B, B^+)$ such that each map

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$$

arises from construction 2.2.2.20. An open immersion of analytic stacks is called *basic rational* if it is representable by basic rational opens. A *derived adic space* is an analytic stack X which admits a surjection

$$\amalg_I Spa(A, A^+)_i \to X$$

such that each map $\mathrm{Spa}(A, A^+) \to X$ is a basic rational open immersion.

A morphism of derived adic spaces is a morphism of analytic stacks, which locally on the source and target comes from an adic map of adic solid affinoids.

Example 2.2.3.2. A derived discrete adic space is a derived adic space that locally is of the form

$$\mathrm{Spa}(A, A^+)$$

where A is a (discrete) animated ring. Note that for a standard rational open

$$U \subset |\mathrm{Spa}(A, A^+)|$$

the ring $\mathcal{O}_U(U)^{\triangleright} := A[\frac{1}{g}]$ considered in 2.2.2.7 is discrete as well. For this one easily sees that

$$\mathcal{O}_U(U)^{\triangleright} \in \mathcal{D}_{\mathrm{qc}}(U)$$

2.2.1.2. In particular, we see that

$$U \simeq |\mathrm{Spa}(\mathcal{O}_U(U))|.$$

This tells us that one can associate a well-defined underlying topological space with a derived discrete adic space.

Example 2.2.3.3. Let us consider those derived adic spaces, which are glued along adic maps from affine derived adic spaces of the form

$$\mathrm{Spa}(A, A^+)$$

where A is the derived I-adic completion along some finitely generated ideal in a discrete animated ring. Then, using the formula for the functor

$$f^*\colon \mathcal{D}_{\mathrm{qc}}(\mathrm{Spa}(A, A^+)) \to \mathcal{D}_{\mathrm{qc}}(\mathbf{D}^1_{\mathrm{Spa}(A,A^+)})$$

given in 2.2.1.7, one sees that this functor sends $A^{\triangleright}$ to

$$(A[x])_{\widehat{I}}$$

the derived I-adic completion of the polynomial ring. In particular, for a basic rational open $U(\frac{f_1,\ldots,f_n}{g}) \subset |\mathrm{Spa}(A, A^+)|$, we see that

$$\mathcal{O}_U(U)^{\triangleright} \simeq (A[x_1, \ldots, x_n]/(gx_1 - f_1, \ldots, gx_n - f_n))_{\widehat{I}}$$

(note that the solid tensor product of the I-adically complete objects here stays I-adically complete [**Bos23**][A.3]). This ring, though, lives in $\mathcal{D}_{\mathrm{qc}}(U)$, such that we obtain an isomorphism

$$U \simeq |\mathrm{Spa}(\mathcal{O}_U(U))|$$

as the right-hand side just depends on the Hausdorff quotient of the completion [**Hub93a**][3.9]. This tells us we get a well-defined underlying topological space of such derived adic spaces. This discussion, for example, applies to the derived adic spaces coming from derived formal schemes via 2.5.1.1 or 2.5.2.

Definition 2.2.3.4. We make the following definitions:

- A derived adic space is called *quasi-compact* if any covering of basic rational opens admits a finite subcovering.
- A derived adic space is called *quasi-separated* if any intersection of two quasi-compact opens is quasi-compact.
- A morphism of derived adic spaces is called *quasi-compact* if the preimage of any quasi-compact open is quasi-compact.
- A morphism of derived adic spaces is called *quasi-separated* if the diagonal is quasi-compact.

We will now construct compactifications of derived adic spaces.

Definition 2.2.3.5. A map of solid affinoids $A \to B$ is called an *elementary closed immersion*, if it sits in a cartesian square

$$\begin{tikzcd} \mathrm{Spa}(B) \arrow[r] \arrow[d] & \mathrm{Spa}(A) \arrow[d, "f"] \\ \mathrm{Spa}(\mathbf{Z}[x^{\pm}], \mathbf{Z}) \arrow[r] & \mathrm{Spa}(\mathbf{Z}[x], \mathbf{Z}) \end{tikzcd}$$

for some function $f \in A^{\triangleright}(*)$. We will say a collection of such maps is an *elementary closed covering*, if the corresponding functions generate the unit ideal.

Remark 2.2.3.6. Note that an elementary closed covering gives a closed covering of analytic stacks.

Remark 2.2.3.7. Note that an elementary closed covering

$$\{\mathrm{Spa}(A[\frac{1}{f_i}], \widetilde{A^+}) \to \mathrm{Spa}(A, A^+)\}_I$$

is refined by the rational open covering

$$\{\mathrm{Spa}(A[\frac{1}{f_i}], \widetilde{A^+[\frac{1}{f_i}]}) \to \mathrm{Spa}(A, A^+)\}_I.$$

2.2.3.8. Consider the functor

$$\overline{(\_)}\colon \mathcal{A}dic\mathcal{S}p^{\mathrm{aff}} \to \mathcal{A}n\mathcal{S}tack$$

which assigns to a adic solid affinoid $(A^{\triangleright}, A^{+})_{\square}$ the analytic stack

$$\mathrm{Spa}(A, \widetilde{\mathbf{Z}})$$

.

Proposition 2.2.3.9. *The functor*

$$\overline{(\_)}\colon \mathcal{A}dic\mathcal{S}p^{\mathrm{aff}} \to \mathcal{A}n\mathcal{S}tack$$

sends rational open coverings to elementary closed coverings.

Proof. By 2.2.2.4, we just have to check that squares of the form

$$\begin{tikzcd} U(\frac{1}{1-f}) \cap U(\frac{1}{f}) \arrow[r] \arrow[d] & U(\frac{1}{f}) \arrow[d] \\ U(\frac{1}{1-f}) \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} U(\frac{f}{1}) \cap U(\frac{1}{f}) \arrow[r] \arrow[d] & U(\frac{1}{f}) \arrow[d] \\ U(\frac{f}{1}) \arrow[r] & U \end{tikzcd}$$

get sent to pushouts coming from elementary closed coverings. Write $U \simeq \mathrm{AnSpec}(A)$ then the fist square gets send to

$$\begin{tikzcd} \mathrm{Spa}(A[\frac{1}{(1-f)f}], \widetilde{\mathbf{Z}}) \arrow[r] \arrow[d] & \mathrm{Spa}(A[\frac{1}{f}], \widetilde{\mathbf{Z}}) \arrow[d] \\ \mathrm{Spa}(A[\frac{1}{1-f}], \widetilde{\mathbf{Z}}) \arrow[r] & \mathrm{Spa}(A, \widetilde{\mathbf{Z}}) \end{tikzcd}$$

and the second square to

$$\begin{tikzcd} \mathrm{Spa}(A[\frac{1}{f}], \widetilde{\mathbf{Z}}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Spa}(A[\frac{1}{f}], \widetilde{\mathbf{Z}}) \arrow[d] \\ \mathrm{Spa}(A, \widetilde{\mathbf{Z}}) \arrow[r, "\simeq"'] & \mathrm{Spa}(A, \widetilde{\mathbf{Z}}). \end{tikzcd}$$

Both of these squares are easily seen to be such pushouts of analytic stacks. $\square$

2.2.3.10. Using 2.2.3.9, we obtain a functor

$$\overline{(\_)}\colon \mathcal{A}dic\mathcal{S}p \to \mathcal{A}n\mathcal{S}tack$$

and by 2.2.3.7 we see that this functor lands in derived adic spaces.

Now, for a morphism of derived adic spaces $f\colon X \to S$, we define a derived adic space $\overline{X}^{/S}$ by the cartesian square

$$\begin{tikzcd} \overline{X}^{/S} \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ S \arrow[r] & \overline{S}. \end{tikzcd}$$

Note that we have a factorization of the original morphism

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X}^{/S} \arrow[d, "p"] \\ & S. \end{tikzcd}$$

Example 2.2.3.11. Given a morphism $\mathrm{Spa}(B, B^+) \to \mathrm{AnSpec}(A, A^+)$ of affine derived adic spaces, the factorization constructed in 2.2.3.10 is given by

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(B, \widetilde{A^+}) \to \mathrm{Spa}(A, A^+)$$

.

Definition 2.2.3.12. A morphism $X \to S$ of derived adic spaces is called *locally proper*, if the map

$$X \to \overline{X}^{/S}$$

is an isomorphism. It is called *proper* if it is locally proper and quasi-compact.

Remark 2.2.3.13. A morphism of classical derived Tate adic spaces is proper if and only if it is proper as a map of analytic stacks.

Definition 2.2.3.14. A map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affine derived adic spaces is called of $^+$*finite type*, if there exist finitely many functions

$$f_i \colon \mathrm{Spa}(B, B^+) \to \mathbf{D}_\square$$

bounded by 1, such that the induced map

$$A^+[f_1, \dots, f_n] \to B^+$$

is integral.

A map $X \to S$ of classical derived adic spaces is called locally of $^+$*finite type*, if it is locally on X and S in the rational open topology of $^+$finite type.

Proposition 2.2.3.15. *Consider a map $f \colon X \to S$ of derived adic spaces, which is locally of $^+$finite type. Then the map*

$$X \to \overline{X^{/S}}$$

is a rational open immersion.

Proof. The question is local on X and $\overline{X^{/S}}$ in the rational open topology. So we can assume that f is a map of $^+$finite type between affines $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$. Then the map in question is given by

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(B, \widetilde{A^+})$$

which is the intersection of the finitely many opens

$$\begin{tikzcd} U(\frac{f_i}{1}) \arrow[r] \arrow[d] & \mathrm{Spa}(B, \widetilde{A^+}) \arrow[d] \\ \mathbf{D}^1_\square \arrow[r] & \mathbf{A}^1_\square \end{tikzcd}$$

and thus open. $\square$

2.2.4. Discrete adic spaces. We now collect some results from the literature in the case that the functions on our adic spaces do not carry a topology. These results will be used later.

2.2.4.1. To an affine derived scheme $\mathrm{Spec}(A)$, we can associate the derived discrete adic space

$$\mathrm{Spa}(A).$$

One observes that an affine Zariski covering $\{U(f_i) \to \mathrm{Spec}(A)\}$ via this assignment gets mapped to the rational cover

$$\{U(\frac{1}{f_i}) \to \mathrm{Spa}(A)\}.$$

So by 2.2.2.20 we obtain a functor

$$(\_)^{\mathrm{an}} \colon \mathcal{S}ch \to \mathcal{A}n\mathcal{S}tack$$

from schemes to analytic stacks, which preserves open immersions and lands in derived discrete adic spaces.

Proposition 2.2.4.2. *The functor*

$$(\_)^{an} \colon \mathcal{S}ch \to \mathcal{A}n\mathcal{S}tack$$

preserves étale morphisms (resp. étale covers).

Proof. The preservation of étale morphisms is explained in [**CS19b**][XI]. Now, to show that it preserves étale coverings, by 2.1.4.5, we have to check that the assignment

$$\mathrm{Spec}(A) \mapsto \mathcal{D}_{\mathrm{qc}}(\mathrm{Spa}(A))$$

is an étale sheaf (where we use the upper-$*$-functoriality). This follows from [**Man22**][2.10.6+2.6.3]. □

2.2.4.3. For an affine derived discrete adic space $X \simeq \mathrm{Spa}(A, A^+)$, any rational open becomes a basic rational open. Furthermore, it is not hard to see that for a standard rational open

$$U(\frac{f_1, \dots, f_n}{g}) = U \subset X$$

the underlying condensed animated ring of the analytic ring $\mathcal{O}_U(U)$ is given by the ring $A[\frac{1}{g}]$. That is, the localization to $\mathcal{D}_{\mathrm{qc}}(U)$ does not affect the unit. In particular, one sees that the map

$$|\mathrm{Spv}(\mathcal{O}_U(U))| \to U$$

from 2.2.2.8 is an isomorphism. As any map of affine derived discrete adic spaces is adic, we also get the analogous statement for $|\mathrm{Spa}(\mathcal{O}_U(U))|$.

In total, we have argued that the assignment

$$\mathrm{Spa}(A, A^+) \mapsto |\mathrm{Spa}(A, A^+)|$$

gives a well-defined functor from affine derived discrete adic spaces to topological spaces which preserves open immersions, such that for any derived discrete adic spaces X, we obtain an underlying topological space $|X|$. A more detailed discussion of this fact can also be found in [**Man22**][2.9.20].

2.2.4.4. Recall from [**Man22**][2.9.25] or 2.2.3.10, that for any map $f\colon X \to S$ of discrete adic spaces, we have a canonical "compactification"

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X}^{/S} \arrow[d, "p"] \\ & S \end{tikzcd}$$

where $\overline{X}^{/S}$ is a discrete adic spaces. If $X = \mathrm{Spa}(B, B^+)$ and $S = \mathrm{Spa}(A, A^+)$ then $\overline{X}^{/S} = \mathrm{Spa}(B, \widetilde{A^+})$.

Let us recall the following from [**Man22**][2.9.29].

Proposition 2.2.4.5. *Consider a map $f\colon X \to S$ of discrete adic spaces together with the factorization*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X}^{/S} \arrow[d, "p"] \\ & S \end{tikzcd}$$

from 2.2.4.4. Then we have the following:

(a) If f is quasi-compact p is proper.
(b) If f is of ${}^+$finite type then j is an open immersion.

In the case (b) holds the following are equivalent:

- *j is an isomorphism.*
- *For any valuation ring V with fraction field K and any solid diagram*

$$\begin{tikzcd} Spa(K) \arrow[r] \arrow[d] & X \arrow[d] \\ Spa(K, V) \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

there exists a unique dashed arrow making the diagram commute.

In particular, if f is also quasi-compact, it is proper.

Proof. To see (a) we can assume $S \simeq \mathrm{Spa}(A, A^+)$, then recall [**Man22**][2.9.25] that for an open covering

$$\{\mathrm{Spa}(B, B^+)_i \to X\}_I$$

we get a covering $\{\mathrm{Spa}(B, \widetilde{A^+})_i \to \overline{X}^{/S}\}$ which in the sense of analytic Stacks give a closed covering. Besides the last "in particular" statement, which follows using 2.1.3.7, the rest follows from [**Man22**][2.9.29]. $\square$

2.3. Classical Tate p-adic spaces

We will now filter out the type of (derived) Tate adic spaces, we want to use later.

2.3.1. Bounded solid affinoids.

2.3.1.1. Note that any topological nilpotent unit $\pi \in A$ in a solid affinoid algebra induces a map of analytic rings

$$\mathbf{Z}((\pi))_\square = (\mathbf{Z}((\pi)), \mathbf{Z}[[\pi]])_\square \to A.$$

2.3.1.2. Recall from [**Cam24**][2.6.10] that a solid affinoid $\mathbf{Z}((q))_\square$-algebra $A = (A^\triangleright, A^+)$ is called *bounded*, if the canonical map

$$A^b = (A^\circ[\frac{1}{\pi}], A^+) \to A$$

is an isomorphism.

2.3.1.3. Recall that an object x in a compactly generated symmetric monoidal category $\mathcal{C}$ is called *nuclear*, if for any compact object $c \in \mathcal{C}$ the canonical map

$$\mathrm{Hom}(\mathbf{1}, \underline{\mathrm{Hom}}(c, \mathbf{1}) \otimes x) \xrightarrow{\sim} \mathrm{Hom}(c, x)$$

is an isomorphism [**CS22**][8] [**CS19a**][13].

Example 2.3.1.4. Given a discrete animated ring R then for any integrally closed subring R^+ the nuclear objects in $\mathcal{D}_{\mathrm{qc}}(\mathrm{Spa}(R, R^+))$ identify with the classical derived category $\mathcal{D}(R)$ of R.

Definition 2.3.1.5. We will say a bounded solid affinoid $\mathbf{Z}((\pi))_\square$-algebra A is *classical*, if it is nuclear as a $\mathbf{Z}((\pi))_\square$-module.

Furthermore we will say a classical bounded solid affinoid A is called *p-adic* if it admits a topologically nilpotent unit $\pi' \in A^\circ$, which divides p in A° and A admits a ring of definition, which is derived π'-adically complete.

A derived adic space is called *classical p-adic Tate*, if it locally is represented by classical p-adic bounded solid affinoids.

Remark 2.3.1.6. The category of nuclear $\mathbf{Z}((\pi))_\square$-modules is generated by the objects

$$\mathrm{Cont}(S, \mathbf{Z}((\pi)))$$

for S a (light) profinite set. In particular, for a classical p-adic bounded solid affinoid, we can find a ring of definition, which is the derived π'-adic completion of a discrete animated ring.

Remark 2.3.1.7. Given a classical bounded solid affinoid algebra A and a topological nilpotent unit $\pi' \in A$. Then A is bounded with respect to the induced map

$$\mathbf{Z}((\pi'))_\square \to A.$$

To see this, take a pseudouniformizer $\pi \in A$ justifying A to be bounded. Then π' is nilpotent in A°/π and invertible in $A^\circ[\frac{1}{\pi}]$. From this, one sees that

$$A^\triangleright \simeq A^\circ[\frac{1}{\pi}] \simeq A^\circ[\frac{1}{\pi'}].$$

Remark 2.3.1.8. Classical bounded solid affinoid rings are stable under colimits in solid affinoid rings [**Cam24**][2.6.14].

Example 2.3.1.9. For a Tate Huber pair (A, A^+) the topology on A° is given by the π-adic topology for some pseudouniformizer π. In particular, the associated bounded solid affinoid ring is classical.

2.3.1.10. Combining the last example with 2.2.1.13, we see that the category of Tate Huber pairs sits fully faithful in the category of classical bounded solid affinoid rings. The latter category will be denoted by

$$\mathcal{B}nd_{\square}^{\mathrm{cl}}.$$

Furthermore we will write $\mathcal{B}nd^{\mathrm{cl}}_{(\mathbf{Z}_p)_\square/}$ for the category of p-adic classical bounded solid affinoid algebras.

Remark 2.3.1.11. Given a ring of definition A_0 for a classical bounded solid affinoid A with pseudouniformizer $\pi \in A_0$, we can compute

$$A^{\triangleright} \simeq A^\circ[\frac{1}{\pi}] \simeq A_0[\frac{1}{\pi}]$$

.

2.3.2. Étale morphisms of Tate adic spaces.

In the following, we want to compare two natural notions of étale morphisms of classical derived Tate adic spaces. In this section any classical derived Tate adic space will be p-adic.

2.3.2.1. Recall from [**Cam24**][3.4], that to any map $f\colon X \to S$ of classical derived Tate adic spaces, we can associate the *cotangent complex*

$$\mathbf{L}_f \in \mathcal{D}_{qc}(X).$$

Following [**Cam24**][3.5], we now make the following definition.

Definition 2.3.2.2. We will say a morphism $f\colon X \to S$ of classical derived Tate adic spaces is *étale*, if it is locally of finite presentation and

$$\mathbf{L}_f \simeq 0.$$

Remark 2.3.2.3. Often in Tate adic geometry, one says a morphism is étale if it locally on source and target factors as a rational open, followed by a finite étale map. At least in favorable situations, this will also be true for us.

Definition 2.3.2.4. An étale map $f\colon \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affine classical derived Tate adic spaces is called *finite étale*, is B is a finite projective A-module and B^+ the integral closure of A^+ is B.

Furthermore, a morphism of classical derived Tate adic spaces is called *finite étale*, if it is affine and locally on the target given by a finite étale morphism of affine classical derived Tate adic spaces.

2.3.2.5. To formulate what we want, it will be convenient to use *sous-perfectoid* adic spaces following [**HK20**][7.] [**SW20**][6.3]. Recall that a Tate algebra A is called sous-perfectoid if it admits a split injection

$$A \to \tilde{A}$$

of topological A-modules into a perfectoid Tate algebra. Huber pairs coming from sous-perfectoid Tate algebras are sheafy, and we will call an adic space sous-perfectoid if it admits a rational covering by affines coming from sous-perfectoid Tate algebras.

Example 2.3.2.6. For a sous-perfectoid Tate algebra A, the Tate algebra

$$A\langle x_i | i \in I\rangle$$

in arbitrary many variables is again a sous-perfectoid Tate algebra.

Example 2.3.2.7. A rational open in a sous-perfectoid Tate adic space is again a sous-perfectoid Tate adic space.

Example 2.3.2.8. A classical derived Tate adic space, finite étale over a sous-perfectoid Tate adic space, is a sous-perfectoid Tate adic space [**HK20**][7.5.].

Proposition 2.3.2.9. *Consider a map $f\colon X \to S$ of classical derived Tate adic spaces, such that locally in the rational open topology on S, one can find a map $S \to S'$ to a sous-perfectoid Tate adic space. Then if f is étale, it locally on X and S factors as an open immersion followed by a finite étale map.*

Proof. As the question is local, we can assume that we have a map $S \to S'$ to a sous-perfectoid. Now by [**Cam24**][3.5.6] locally on X and S we can find functions $g_1, \dots, g_n$ on $\mathbf{D}^n_\square$, such that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & (\mathbf{D}^n_\square)_{S'} \arrow[d, "{(g_1,\dots,g_n)}"] \\ S \arrow[r, "0"'] & (\mathbf{D}^n_\square)_{S'} \end{tikzcd}$$

is cartesian and $\Delta := \det((\frac{\partial g_i}{\partial x_j})_{1\leq i,j\leq n})$ becomes invertible on X. Now locally on X, we can assume that either $|\Delta| \leq 1$ or $|\Delta^{-1}| \leq 1$, so after possibly shrinking X we find a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U(\frac{1}{\Delta}) \arrow[d, "{(g_1,\dots,g_n)}"] \\ S \arrow[r, "0"'] & (\mathbf{D}^n_\square)_{S'} \end{tikzcd}$$

where the right upper corner denotes the rational open in $(\mathbf{D}^n_\square)_{S'}$. But now the map on the right is a map between sous-perfectoid adic spaces and thus admits a local factorization as wanted by [**FS21**][IV.4.15] and the claim follows by base change. □

Corollary 2.3.2.10. *Consider an étale map $f\colon X \to S$ of classical derived Tate adic spaces, such that S is sous-perfectoid (resp. perfectoid). Then X is sous-perfectoid (resp. perfectoid) as well.*

Proof. For sous-perfectoids, this now follows from 2.3.2.9, 2.3.2.6 and 2.3.2.7 and for the case of perfectoids, this is explained in [**Sch12**][6.3] and [**KL16**][3.3.18]. □

2.4. Derived formal schemes

The following two sections will discuss some aspects of the relationship between (derived) formal schemes and analytic stacks. Our main issue is that the fibre products in the respective worlds are not generally compatible. To resolve this issue, we will work with affine formal schemes, which are adic over some fixed formal stack, and then define appropriate topologies on those. The concrete goal is to construct "analytification" functors for formal stacks.

2.4.1. Completely descendable morphisms. For the definition of (derived) formal schemes, we will follow [**Lur18**][II.8] with the difference that we work with animated rings instead of $\mathcal{E}_\infty$-rings. Concretely, that means our ambient category at first will be the category of functors[6] from animated rings to anima. We often refer to such an object as *presheaf on affine derived schemes.*

Definition 2.4.1.1. Any ideal I in a ring A determines a topology where a basis of open neighborhoods of 0 is given by the subsets I^n for $n \geq 0$. We will call this topology the *I-adic topology.*

For a topological ring A, we say an ideal $I \subset A$ is an *ideal of definition*, if the topology coincides with the I-adic topology.

An *adic ring* is a ring A equipped with a topology that admits a finitely generated ideal of definition. Furthermore, a morphism of adic rings is a continuous ring morphism.

An *adic animated ring* is an animated ring A, such that $\pi_0 A$ is an adic ring. More concretely, we define the category of *adic animated rings* as the limit of the cospan

$$\mathcal{R}ing^{adic} \longrightarrow \mathcal{R}ing \xleftarrow{\pi_0} \mathcal{A}ni(\mathcal{R}ing)$$

where the functor on the left forgets the topology.

2.4.1.2. Any adic animated ring A gives rise to a presheaf on affine derived schemes by the formula

$$\mathrm{Spf}(A)(B) := \mathrm{Hom}_{\mathcal{A}ni(\mathcal{R}ing)^{adic}}(B, A)$$

where we equip B with the discrete topology. These presheaves build the building blocks for (derived) formal schemes and can be described more concretely as follows. Choose an ideal of definition I of A and a finite family of generators $f_1, \dots, f_n$ of I. Then

$$\mathrm{Spf}(A) \simeq \mathrm{colim}_m \, \mathrm{Spec}(A/(f_1^m, \dots, f_n^m))$$

where the quotients are defined as the pushouts

$$\begin{tikzcd}
\mathbf{Z}[x_1, \dots, x_n] \arrow[r, "x_i \mapsto 0"] \arrow[d, "x_i \mapsto f_i^m"'] & \mathbf{Z} \arrow[d] \\
A \arrow[r] & A/(f_1^m, \dots, f_n^m)
\end{tikzcd}$$

Note that from this description one also observes that $\mathrm{Spf}(A)$ just depends on the (derived) I-adic completion

$$A_{\widehat{I}} \simeq \lim_m A/(f_1^m, \dots, f_n^m)$$

[6] By convention, we consider those which are small colimits of affine schemes.

.

2.4.1.3. Right Kan extending the assignment $A \mapsto \mathcal{D}(A)$ from affine derived schemes to presheaves on those, one obtains

$$\mathcal{D}(\mathrm{Spf}(A)) \simeq \lim_m \mathcal{D}(A/(f_1^m \dots, f_n^m)) \simeq \mathcal{D}_{I\text{-}comp}(A) \subset \mathcal{D}(A)$$

the full subcategory of derived I-complete A-modules. That is the full subcategory of those objects for which the canonical map

$$M \to M_{\widehat{I}} \simeq \lim_m M/(f_1^m, \dots, f_n^m)$$

is an isomorphism. Alternatively, one can describe I-complete objects in the following way. Let us write $\mathcal{D}(A)^{Loc(I)} \subset \mathcal{D}(A)$ for the full subcategory of those modules for which the canonical map $M \to M[x^{-1}]$ is an isomorphism for some $x \in I$. These objects are called I-local objects, and one obtains I-complete objects by killing those. That is, there is an exact sequence of stable categories

$$\mathcal{D}(A)^{Loc(I)} \to \mathcal{D}(A) \to \mathcal{D}_{I\text{-}comp}(A)$$

.

We need to extend some standard notations for adic rings to more general (pre)sheaves. There will be several topologies on the category of animated rings considered later, and the following definitions make sense for all of them.

Definition 2.4.1.4. A morphism between adic animated rings $f\colon A \to B$ is called *adic* if there exists some finitely generated ideal of definition $I \subset \pi_0 A$ such that $f(I)\pi_0 B \subset \pi_0 B$ defines an ideal of definition.

A morphism of $f\colon X \to S$ (pre)sheaves on affine derived schemes is called representable by affine schemes if for any map $\mathrm{Spec}(A) \to S$ the fibre product $\mathrm{Spec}(A) \times_S X$ is representable by an affine scheme.

A morphism $f\colon X \to S$ of (pre)sheaf on affine derived schemes is called *adic* if f as well as the diagonal of f $\Delta_f\colon X \to X \times_S X$ are representable by affine schemes.

Lemma 2.4.1.5. A morphism of adic animated rings $f\colon A \to B$ with B being complete is adic, if and only if the induced morphism $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is representable by affine schemes. Furthermore, for any adic morphism of adic animated rings, the morphism $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ as well as the diagonal $\mathrm{Spf}(B) \to \mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B)$ are representable by affine schemes.

Proof. We start with the first claim. Let us assume we have given finitely generated ideals of definition I and J for A and B such that $I\pi_0 B \subset J$ (note that $\pi_0 A \to \pi_0 B$ is continuous). We show that the ideal $J/I\pi_0 B \subset \pi_0 B/I\pi_0 B$ is nilpotent and thus $I\pi_0 B$ sandwiches in between two ideals of definition. As J is finitely generated, we have to show that it is nilpotent for any element $x \in J/I\pi_0 B$. Let us write $A/^L I$ for the derived quotient computed by choosing a family of generators and then take the derived quotient with respect to those (any such choice of generators will make the following work. Then, we consider the Cartesian square

$$\begin{tikzcd} \mathrm{Spf}(B/^L IB) \arrow[r] \arrow[d] & \mathrm{Spf}(B) \arrow[d] \\ \mathrm{Spec}(A/^L I) \arrow[r] & \mathrm{Spf}(A). \end{tikzcd}$$

As $\mathrm{Spf}(B/^L IB)$ is an affine scheme the global sections functor

$$\Gamma\colon \mathcal{D}(\mathrm{Spf}(B/^L IB)) \to \mathcal{D}(A/^L I)$$

commutes with colimits. In particular, we have

$$(B/^L IB)[x^{-1}] \simeq \Gamma(B/^L IB)[x^{-1}] \simeq \Gamma(B/^L IB[x^{-1}]) \simeq 0$$

which shows what we want. For the second and third claim, observe that for an adic map $A \to B$, we have cartesian squares

$$\begin{tikzcd} \mathrm{Spf}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spf}(A) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spf}(B) \arrow[r] \arrow[d] & \mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spf}(B) \arrow[d] \\ \mathrm{Spec}(B) \arrow[r] & \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B). \end{tikzcd}$$

From this, it is easy to obtain the claims. ☐

Lemma 2.4.1.6. Consider a commutative triangle of (pre)sheaves on affine derived schemes

$$\begin{tikzcd} \mathrm{Spf}(B) \arrow[r, "h"] \arrow[dr, "f"'] & \mathrm{Spf}(A) \arrow[d, "g"] \\ & S. \end{tikzcd}$$

If f and g are adic, then h is also adic.

Proof. Using 2.4.1.5, we have to show that h is representable by affine schemes. This is a standard trick: We factor h via the graph, followed by the projection. In this factorization, both maps are pulled back from maps representable by affine schemes. ☐

The following is the primary definition of this section.

Definition 2.4.1.7. A adic map $f\colon A \to B$ of adic animated rings is called *completely descendable* if

$$B_{\widehat{I}} \in \mathcal{D}_{I\text{-}comp}(A)$$

defines a descendable algebra.

Example 2.4.1.8. Any faithfully flat étale map of animated rings is descendable as it is finitely presented and faithfully flat [**Mat16**][3.33]. We will see later that any map $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$, which is representable by affine étale covers, comes as a pullback of such a descendable map 2.4.2.11 and thus is completely descendable.

Example 2.4.1.9. Recall that the presheaf $\mathbf{G}_a$ seen as a spectrum valued functor on animated rings can be written as[7]

$$A \mapsto \mathrm{Hom}_{\mathcal{D}(A)}(A, A).$$

As associating the category of modules already admits descent for descendable maps [**Mat16**][3.22], we see that this also defines a sheaf.

[7] We can, of course, even remember the $\mathcal{E}_\infty$-structure

Example 2.4.1.10. An example we will use in this text is freely adjoining p-th roots of elements[8]. By this, we mean pushouts of maps of the form

$$\mathbf{Z}[x_i | i \in I] \to \mathbf{Z}[x_i^{\frac{1}{p^\infty}} | i \in I]$$

for some set I. To see that such a map is descendable, one observes that the fiber of this map F is a free module over the domain sitting in homological degree -1. In particular any map $F \to \mathbf{Z}[x_i]$ vanishes and one can apply [**BS17**][11.20].

We want to generate a (Grothendieck) topology on the category of adic animated rings. So, for convenience, let us recall the following stability properties.

Lemma 2.4.1.11. Completely descendable morphisms of adic animated rings are stable under composition and base change. Furthermore, for any composable pair of (adic) morphisms of adic animated rings

$$f \circ g \colon A \to B \to C.$$

If $f \circ g$ is completely descendable, then g is also completely descendable.

Proof. See for example [**Mat16**][3.24]. □

Construction 2.4.1.12. We will often consider the following situation. Fix a (pre)sheaf S on affine schemes, and let us write

$$f\mathcal{S}ch^{\mathrm{aff}}_{/S}$$

for the full subcategory of presheaves over S of the form

$$\mathrm{Spf}(A) \to S$$

where the structure morphism is adic. We equip this category with the (Grothendieck) topology generated by adic descendable morphisms 2.4.1.7 and write

$$S_{\mathrm{fdesc}}$$

for the induced topos[9].

2.4.2. Open immersions and étale morphisms.

For this section, we fix a presheaf S on animated rings and write

$$f\mathcal{S}ch^{\mathrm{aff}}_{S}$$

for the category of affine formal schemes adic over S.

Definition 2.4.2.1. A morphism $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ in $f\mathcal{S}ch^{\mathrm{aff}}_{S}$ is called a *principal open immersion* if it sits in a Cartesian square

$$\begin{tikzcd} \mathrm{Spf}(B) \arrow[r] \arrow[d] & \mathrm{Spf}(A) \arrow[d, "f"] \\ \mathbf{G}_m \arrow[r] & \mathbf{A}^1 \end{tikzcd}$$

of presheaves on affine derived schemes. We will denote such an open by $U(f)$.

[8] One can adjoin arbitrary functions, which are annihilated by monic polynomials.

[9] By convention, this is just a topos up to size issues.

Construction 2.4.2.2. It is easy to see that principal open immersions are stable under composition and pullbacks. Thus, by declaring a finite collection

$$\{U(f_i) \to \mathrm{Spf}(A)\}_I$$

to be a covering if the ideal (f_i) generates the ring $\pi_0 A$ we produce a pretopology. The induced topos will be denoted by

$$S_{Zar}$$

and called the (big) *Zariski topos* of S. We can now use this to define a derived formal scheme.

Definition 2.4.2.3. Given an affine formal scheme X over S, a morphism $U \to X$ from a Zariski sheaf is called an *open immersion* if it is a monomorphism and there exists an effective epimorphism

$$\amalg_I V_i \to U$$

in S_{Zar} such that each of the induced maps $U_i \to X$ is a principal open immersion.

If X is a Zariski sheaf, a map $U \to X$ is called an *open immersion* if for any affine formal scheme $\mathrm{Spf}(A)$ over S and any map $\mathrm{Spf}(A) \to X$ the pullback map $U \times_X \mathrm{Spf}(A) \to \mathrm{Spf}(A)$ is an open immersion.

A Zariski sheaf X over S is called a *derived formal scheme* over S, if there exists an effective epimorphism

$$\amalg_I U_i \to X$$

such that each of the maps $U_i \to X$ is an open immersion and U_i affine formal schemes over S for all $i \in I$.

We will write $f\mathcal{S}ch_S$ for the category of derived formal schemes over S.

Definition 2.4.2.4. A formal scheme X over S is called *quasi-compact* if any Zariski covering admits a finite sub-covering. Furthermore, it is called *quasi-separated* if the intersection of two quasi-compact opens is again quasi-compact. A morphism of formal schemes is called *quasi-compact* if the preimage of a quasi-compact open is quasi-compact.

Let us continue with étale morphisms.

Definition 2.4.2.5. Recall that a map of animated rings $A \to B$ is *étale*, if it is flat and the underlying map $\pi_0 A \to \pi_0 B$ is an étale map of static rings.

A map of affine formal schemes $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is called *étale* if for any morphism $\mathrm{Spec}(C) \to \mathrm{Spf}(A)$ from an affine scheme the pullback

$$\mathrm{Spf}(B) \times_{\mathrm{Spf}(A)} \mathrm{Spec}(C) \to \mathrm{Spec}(C)$$

is an étale map of affine schemes.

A morphism of derived formal schemes is *étale* if it is Zariski locally on the domain and the target is an étale morphism of affine formal schemes.

Remark 2.4.2.6. Note that for an étale map $A \to B$ of animated rings, there is an idempotent $e \in B \otimes_A B$ such that

$$B \otimes_A B[\frac{1}{e}] \simeq B.$$

From this, we see that the diagonal of an étale map of formal schemes is an open immersion, and a standard argument shows that the cotangent complex of an étale morphism vanishes [**Sta**, Tag 08R2]. This is the main property of an étale morphism we will use.

Example 2.4.2.7. As in non-derived algebraic geometry, there is a generic example of an étale R-algebra map of animated rings $g\colon A \to B$ (see [**Lur18**][B.1.1.3] for the case of $\mathcal{E}_\infty$-rings). That is, any such map can be written as a pushout of a map

$$g\colon R[x_1, \dots, x_n] \to R[y_1, \dots, y_n][\frac{1}{\triangle}]$$

where $\triangle$ denotes the determinant of the Jacobian matrix of the functions $g(x_i)$.

2.4.2.8. Using the vanishing of the cotangent complex 2.4.2.6 and the structure theory of étale maps 2.4.2.7, one can prove the following classical fact from deformation theory (see [**Lur18**][17.1.35] for the case of $\mathcal{E}_\infty$-rings). For any square zero extension $\bar{A} \to A$, taking pushouts induces an equivalence

$$\bar{A}_{\text{ét}} \xrightarrow{\sim} A_{\text{ét}}$$

of categories of étale algebras over the respective rings. There are two fundamental examples of this phenomenon regarding derived formal algebraic geometry.

Example 2.4.2.9. In derived algebraic geometry, fundamental examples of square zero extensions are the maps in the Postnikov tower of an animated ring

$$A \to \dots \to \tau_{\leq n} A \to \dots \tau_{\leq 1} A \to \pi_0 A$$

(see [**Lur17**][7.4.1.28] for the case of $\mathcal{E}_\infty$-rings).

Note that, as in the category of anima, Postnikov towers converge, and the truncations preserve finite products, we have an equivalence

$$\mathcal{A}ni(\mathcal{R}ing)_A \simeq \lim_n \tau_{\leq n} \mathcal{A}ni(\mathcal{R}ing)_{\tau_{\leq n} A}$$

of categories where the transition functors are given by truncating. Further observing that for a flat map, these truncations can be identified with scalar extending along $\tau_{\leq n+1} A \to \tau_{\leq n} A$, we can use 2.4.2.8 to see that we have a chain of equivalences

$$A_{\text{ét}} \xrightarrow{\sim} \dots \xrightarrow{\sim} \tau_{\leq n} A_{\text{ét}} \xrightarrow{\sim} \dots \xrightarrow{\sim} \tau_{\leq 1} A_{\text{ét}} \xrightarrow{\sim} \pi_0 A_{\text{ét}}$$

of étale algebras over the respective rings.

Example 2.4.2.10. Any surjective map $\bar{A} \to A$ of static, animated rings with nilpotent kernel gives a square zero extension and thus induces an equivalence of étale algebras over the respective rings.

2.4.2.11. Let us summarize how one can understand an affine formal scheme étale over an affine formal scheme. So let $\mathrm{Spf}(A)$ be an affine formal scheme and $\mathrm{Spf}(A)_{\text{ét}}$ the category of affine formal schemes étale over $\mathrm{Spf}(A)$. As in 2.4.1.2, we can choose a presentation

$$\mathrm{Spf}(A) \simeq \mathrm{colim}_n \, \mathrm{Spec}(A/(f_i^n))$$

where (f_i) generates some ideal of definition for $\pi_0 A$. Thus, using, for example, descent for ∞-topoi, we see that there is an equivalence

$$f\mathcal{S}ch^{\text{aff}}_{\mathrm{Spf}(A)} \simeq \lim_n \mathcal{A}ni(\mathcal{R}ing)_{A/(f_i^n)}$$

and restricting this equivalence to étale algebras, we obtain the following diagram:

$$\begin{tikzcd} A_{\text{ét}} \arrow[r] \arrow[d, "\sim"'] & \mathrm{Spf}(A)_{\text{ét}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \cdots \arrow[r, "\sim"] & A/(f_i^n)_{\text{ét}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \cdots \arrow[r, "\sim"] & A/(f_i)_{\text{ét}} \arrow[d, "\sim"] \\ \pi_0 A_{\text{ét}} \arrow[r] & \mathrm{Spf}(A)_{\text{ét}}^{\text{cl}} \arrow[r, "\sim"'] & \cdots \arrow[r, "\sim"'] & \pi_0 A/(f_i^n)_{\text{ét}} \arrow[r, "\sim"'] & \cdots \arrow[r, "\sim"'] & \pi_0 A/(f_i)_{\text{ét}} \end{tikzcd}$$

where the vertical equivalences come from 2.4.2.9 and the lower horizontal equivalences from 2.4.2.10. This shows that the upper horizontal functors are equivalences as indicated. Furthermore we claim that the horizontal functors on the left are essentially surjective. This for example follows from [**Sta**, Tag 0AN8] for the lower one and thus also for the upper one. Alternatively one directly uses the structure theory of étale morphisms 2.4.2.7.

2.4.3. The deformation to the normal bundle. In the following, we explain how to construct the deformation to the normal bundle for derived formal schemes.

Let us fix an affine derived formal scheme $\mathrm{Spf}(A)$. Any formal scheme in this section will implicitly assumed to be adic over $\mathrm{Spf}(A)$.

Definition 2.4.3.1. We will call a morphism $Z \to X$ of derived formal schemes a *regular closed immersion*, if it Zariski locally on X sits in a cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ \{0\} \arrow[r] & \mathbf{A}^n_{\mathrm{Spf}(A)} \end{tikzcd}$$

where the lower horizontal map is the zero-section.

Construction 2.4.3.2. Consider a regular closed immersion $i\colon Z \to X$ of derived schemes, which arises as the pullback of a zero-section and let us write $\mathcal{I}$ for the fibre of the map

$$\mathcal{O}_X \to i_*\mathcal{O}_Z.$$

Then as explain in [**Tan22**][5.1], we can associate to $\mathcal{I}$ its Rees algebra

$$R(\mathcal{I}^\bullet) := \bigoplus_{n\in\mathbf{Z}} \mathcal{I}^{-n}t^n$$

which is a generated animated ring. In particular, its associated derived scheme comes with a $\mathbf{G}_m$-action and we will write

$$\mathcal{R}(\mathcal{I}^\bullet) := R(\mathcal{I}^\bullet)/\mathbf{G}_m$$

for the quotient stack of this action. Furthermore, the base change along the map $\mathrm{Spf}(A) \to \mathrm{Spec}(A)$ defined a formal stack, which we will denote by

$$\mathbf{D}_{Z/X}.$$

The above construction is functorial in $\mathcal{I}$. In particular given a general regular closed immersion of formal schemes $i\colon Z \to X$, the quasi-coherent sheaf

$$fib(\mathcal{O}_X \to i_*\mathcal{O}_Z)$$

provides the gluing data to globalize the above construction to obtain the *deformation to the normal bundle*

$$\mathbf{D}_{Z/X}$$

for i.

The structure of the above construction can be summarized as follows:

Proposition 2.4.3.3. *Consider a regular closed immersion $i\colon Z \to X$ of formal schemes adic over Spf(A). Then there exists a regular closed immersion of stacks*

$$(\mathbf{A}^1/\mathbf{G}_m)_Z \to \mathbf{D}_{Z/X}$$

over $(\mathbf{A}^1/\mathbf{G}_m)_X$. Furthermore, this construction satisfies the following:

(a) It is contravariant functorial along cartesian maps of regular closed immersions.

(b) If i is given by the zero section into a vector bundle V, then there is a cartesian square

$$\begin{tikzcd} \mathbf{D}_{Z/X} \arrow[r] \arrow[d] & (V/^{-1}\mathbf{G}_m)_X \arrow[d] \\ (\mathbf{A}^1/\mathbf{G}_m)_X \arrow[r] & (\mathbf{BG}_m)_X \end{tikzcd}$$

where we use the action of weight -1 on V.

(c) The fibre over $1\colon S \simeq (\mathbf{G}_m/\mathbf{G}_m)_X \to (\mathbf{A}^1/\mathbf{G}_m)_X$ recovers the closed immersion i.

(d) The fibre over $0\colon (\mathbf{BG}_m)_X \to (\mathbf{A}^1/\mathbf{G}_m)_X$ is given by the zero section

$$0\colon (\mathbf{BG}_m)_Z \to \mathcal{N}_{Z/X}/^{-1}(\mathbf{G}_m)_Z$$

into the normal bundle. Where in the normal bundle, we use the action of weight -1.

(e) If $i\colon D \to X$ is given by an effective cartier divisor, the inclusion of the fibre at 0 factors as

$$\begin{tikzcd} \mathcal{N}_{D/X}/^{-1}(\mathbf{G}_m)_D \arrow[rr] \arrow[dr] & & \mathbf{D}_{D/X} \\ & \mathbf{V}_X(\mathcal{O}(-D))/^{-1}(\mathbf{G}_m)_X. \arrow[ur] & \end{tikzcd}$$

Where the first map is the canonical inclusion, and, if we restrict the second map to the 1-section, it identifies the composition of the 1-section in $\mathbf{D}_{D/X}$ with the structure map to $(\mathbf{BG}_m)_X$ with the map

$$\mathcal{O}(D)\colon S \to (\mathbf{BG}_m)_X$$

corresponding to the line bundle $\mathcal{O}(D)$.

Proof. This follows the same way as in [**Tan22**][5.1] and [**KR25**][4.1.13]. $\square$

2.5. Formal schemes as analytic stacks

2.5.1. Locally proper analytification.

Construction 2.5.1.1. To any affine formal scheme $\mathrm{Spf}(A)$, we can associate the adic solid affinoid analytic stack $\mathrm{Spa}(A, \widetilde{\mathbf{Z}})$. That is: The underlying condensed ring is given by the (derived) I-adic completion of A for some (and thus all) ideal of definition [**AM24**][2.2].

$$A_{\widehat{I}} \simeq \lim_n A/(f_1^n, \dots, f_m^n)$$

where $(f_1, \dots, f_n) = I$ is some generating set.

The ring of integral elements is given by the integral closure of $\mathbf{Z}$ in $A_{\widehat{I}}$. Or in other words, the category of complete modules of this analytic ring is given by

$$\mathcal{M}od_{A_{\widehat{I}}}(\mathcal{D}(\mathbf{Z}_\square))$$

This construction is functorial and thus, for any presheaf on affine schemes S, produces a functor

$$(\_)^{lp} \colon f\mathcal{S}ch^{\mathrm{aff}}_{/S} \to \mathcal{A}n\mathcal{S}tack$$

.

The following proposition tells us that this functor induces a functor

$$S_{\mathrm{fdesc}} \to \mathcal{A}n\mathcal{S}tack$$

which preserves fiber products. In particular, if there exists an adic descendable cover

$$\mathrm{Spf}(A) \to S$$

we obtain a finite limit preserving functor to $\mathcal{A}n\mathcal{S}tack_{S^{lp}}$.

Proposition 2.5.1.2. *The functor $(\_)^{lp}$ preserves fiber products and sends completely descendable morphisms to proper surjections (i.e., effective epimorphisms of the associated analytic stacks).*

Proof. Note first that any map of adic animated rings gets mapped to a proper morphism of analytic rings 2.1.2.2. The first claim boils down to the following: involving 2.4.1.6, to see that all maps considered are adic. For any cospan $C \leftarrow A \to B$ of adic rings and $I \subset \pi_0 A$ an ideal of definition, the canonical map

$$C_{\widehat{I}} \otimes^{\square}_{A_{\widehat{I}}} B_{\widehat{I}} \xrightarrow{\sim} (C \otimes^{\square}_{A} B)_{\widehat{I}}$$

is an isomorphism. This is explained in [**Bos23**][A.3].

For the second claim, consider a descendable map $A \to B$ of adic animated rings and I an ideal of definition for A. By 2.1.4.2 we have to see that $B_{\widehat{I}}$ is a descendable algebra in $\mathcal{D}((A, \mathbf{Z})_\square) = \mathcal{M}od_{\widehat{A}}(\mathcal{D}(\mathbf{Z}_\square))$. But the map $A_{\widehat{I}} \to B_{\widehat{I}}$ lies in the essential image of the lax monoidal functor

$$\mathcal{D}_{I\text{-}comp}(A) \to \mathcal{D}((A, \mathbf{Z})_\square)_{\widehat{I}} \subset \mathcal{D}((A, \mathbf{Z}_\square))$$

where the middle term denotes the full subcategory of I-complete objects. The claim now follows from [**BS17**][11.20]. $\square$

2.5.1.3. As already mentioned 2.5.1.2 gives us a functor $(\_)^{lp}\colon S_{\text{fdesc}} \to \mathcal{A}n\mathcal{S}tack$ which preserves fibre products. Not that by 2.4.1.8 we also have a functor

$$(\_)^{lp}\colon f\mathcal{S}ch_S \to S_{\text{fdesc}}$$

through which we will understand formal schemes in the following.

Proposition 2.5.1.4. *The functor $(\_)^{lp}\colon S_{fdesc} \to \mathcal{A}n\mathcal{S}tack$ respects the following types of morphisms:*

(a) Affine formal schemes get sent to adic solid affinoids. Furthermore, any map between affine formal schemes becomes adic and proper.
(b) Zariski open immersions (resp. coverings) get mapped to closed immersions (resp. coverings) of analytic stacks.
(c) Quasi-compact and quasi-separated morphisms get mapped to proper morphisms of analytic stacks.

Proof. For the first claim, note that for an affine formal scheme $\mathrm{Spf}(A)$ with ideal of definition I, the topologically nilpotent elements in $A_{\widehat{I}}$ are exactly the radical of I. An element in I is topologically nilpotent, and a topologically nilpotent element becomes nilpotent modulo I 2.2.2.12.

The second claim follows from 2.5.1.2 as any map of affine formal schemes becomes proper 2.1.2.2 and Zariski open immersions are monomorphisms.

To see (c), let us consider a quasi-compact and quasi-separated map of formal schemes $f\colon X \to Y$ and argue that $X^{lp} \to Y^{lp}$ is proper. We first prove the following

($*$) For any separated morphism $f\colon X \to Y$ the map $X^{lp} \to Y^{lp}$ is proper.

By (b) and 2.1.3.3 The problem is local on the target, so we can assume $Y \simeq \mathrm{Spf}(A)$. As the map is separated, its diagonal is a closed immersion and thus, in particular, affine. So the diagonal becomes proper by (a). Now we can find a finite Zariski cover of X by affines, and on each affine analytic stack, the structure sheaf is compact. So the claim follows from (b) and 2.1.3.7.

For a general quasi-compact quasi-separated map, we can again assume Y to be affine, such that X becomes quasi-compact quasi-separated. Now the diagonal is proper by $(*)$ and the claim again follows from (b) and 2.1.3.7 as we can choose a finite cover by affines. $\square$

2.5.2. Geometric analytification.

Construction 2.5.2.1. To any affine formal scheme $X = \mathrm{Spf}(B)$, we can associate an affine adic solid affinoid analytic stack

$$\mathrm{Spa}(B).$$

The condensed ring is given by $B_{\widehat{I}}$ where I is an ideal of definition and the completion is interpreted derived and in the condensed world 2.4.1.2 (in particular the "topology" is nontrivial).

As a ring of integral elements, we take $B_{\widehat{I}}$. That is the category of complete modules $\mathcal{D}((B,B)_\square)$ is given by the localization of $\mathcal{M}od_{B_{\widehat{I}}}(\mathcal{D}(\mathbf{Z}_\square))$ at morphisms of the form

$$f_\square := id\text{-}f \cdot \mathrm{shift}\colon \mathbf{P}_{B_{\widehat{I}}} \to \mathbf{P}_{B_{\widehat{I}}}$$

for all $f \in \pi_0 B_{\widehat{I}}$ 2.1.1.13.

Using this construction, we obtain a functor

$$(\_)^{\mathrm{an}}\colon f\mathcal{S}ch_S^{\mathrm{aff}} \to \mathcal{A}n\mathcal{S}tack$$

and the goal of this section is to understand what geometric structures and properties it preserves.

Proposition 2.5.2.2. *The functor*

$$(\_)^{an}\colon f\mathcal{S}ch_S^{aff} \to \mathcal{A}n\mathcal{S}tack$$

preserves fiber products and open immersions. Furthermore, a Zariski covering gets sent to an open covering of analytic stacks.

Proof. We start with the claim on pullbacks. Let us consider a Cartesian square

$$\begin{tikzcd} \mathrm{Spf}(D) \arrow[r] \arrow[d] & \mathrm{Spf}(C) \arrow[d] \\ \mathrm{Spf}(B) \arrow[r] & \mathrm{Spf}(A) \end{tikzcd}$$

in the domain. Then, in 2.5.1.2, we have already seen that the statement is true on underlying solid condensed rings. That is, we have to check that

$$\mathcal{D}((D,D)_\square) \simeq \mathcal{D}((B,B)_\square \otimes_{(A,A)_\square} (C,C)_\square).$$

Both of these categories are full subcategories of $\mathcal{M}od_{(B\otimes_A^\square C)_{\widehat{I}}}(\mathcal{D}(\mathbf{Z}_\square))$. The one on the left is the localization along morphisms of the form

$$f_\square := id\text{-}f\cdot \mathrm{shift}\colon \mathbf{P}_{D_{\widehat{I}}} \to \mathbf{P}_{D_{\widehat{I}}}$$

with $f\in \pi_0 D_{\widehat{I}}$ and the one on the right by morphisms of the same form, but with $f = 1\otimes g$ for $g\in C_{\widehat{I}}$ or $f = g\otimes 1$ for $g \in B_{\widehat{I}}$. Thus, the claim follows as the set of functions bounded by 1 in a solid analytic ring is integrally closed 2.2.1.9/. Note also that the completions here do not affect the complete modules for the same reason, and Mittag-Leffler methods.

For the claim on open immersions, let us write $\mathbf{D}^1_\square \simeq \mathrm{Spa}(\mathbf{Z}[x])$ and $(\mathbf{G}_m^{\mathrm{an}})_\square \simeq \mathrm{Spa}(\mathbf{Z}[x^\pm])$ then the map $j\colon (\mathbf{G}_m^{\mathrm{an}})_\square \to \mathbf{D}^1_\square$ is an open immersion by 2.2.2.6. We claim that for any affine open immersion $U(f) \to \mathrm{Spf}(A)$ the square

$$\begin{tikzcd} U(f)^{\mathrm{an}} \arrow[r] \arrow[d] & \mathrm{Spf}(A)^{\mathrm{an}} \arrow[d, "f"] \\ (\mathbf{G}_m^{an})_\square \arrow[r] & \mathbf{D}^1_\square \end{tikzcd}$$

is Cartesian, which implies the claim 2.1.2.8. Note that as j is an open immersion j^* preserves limits such that

$$j^*(A_{\widehat{I}}) \simeq \lim_n (A/I^n[\frac{1}{f}]) \in \mathcal{D}_{\mathrm{qc}}((\mathbf{G}_m^{\mathrm{an}})_\square).$$

This shows the identification of underlying condensed rings as soon as we identify the categories of complete modules as subcategories of $\mathcal{M}od_{j^*(A_{\widehat{I}})}(\mathcal{D}_{\mathrm{qc}}((\mathbf{G}_m^{\mathrm{an}})_\square))$. This now works as in the first part of the proposition.

For the last statement note that a Zariski cover $\{U(f_i) \to \mathrm{Spf}(A)\}_I$ becomes the rational covering

$$\{U(\frac{1}{f_i}) \to \mathrm{Spa}(A)\}$$

so the claim follows from 2.2.2.20. $\square$

2.5.2.3. By 2.5.2.2, we obtain a functor

$$(\_)^{\mathrm{an}}\colon f\mathcal{S}ch_S \to S_{Zar} \to \mathcal{A}n\mathcal{S}tack$$

which preserves fibre products and open immersions (resp. coverings). It also preserves etale morphisms, as we are checking now.

Proposition 2.5.2.4. *The functor*

$$(\_)^{an}\colon f\mathcal{S}ch_S \to \mathcal{A}n\mathcal{S}tack$$

from 2.5.2.3 preserves étale morphisms as well as étale coverings (i.e., morphisms which are represented by surjective étale morphisms of schemes, get sent to étale effective epimorphisms).

Proof. Using 2.5.2.3, we have to check the claim for an affine étale map $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$. Now using 2.4.2.11 we can find an étale A-algebra $\tilde{B}$ which pulls back to $\mathrm{Spf}(B)$. Note also that we can assume the induced map on spectra to be surjective if the original map was, by adding the complement of $\mathrm{Spec}(A/I) \to \mathrm{Spf}(A)$ to the cover, where I is some ideal of definition. We claim that there is a Cartesian square

$$\begin{tikzcd} \mathrm{Spf}(B)^{\mathrm{an}} \arrow[r] \arrow[d] & \mathrm{Spf}(A)^{\mathrm{an}} \arrow[d] \\ \mathrm{Spa}(\tilde{B}) \arrow[r, "j"'] & \mathrm{Spa}(A) \end{tikzcd}$$

where the lower line is equipped with the discrete topology. The claim then follows from 2.2.4.2. To see this note that the vertical maps are proper, such that the pullback is given by

$$\mathcal{M}od_{j^*(A_{\widehat{I}})}(\mathcal{D}((\tilde{B},\tilde{B})_{\square}))$$

with underlying condensed ring given by the unit. But j is étale 2.2.4.2 such that j^* commutes with limits and we have

$$j^*(A_{\widehat{I}}) \simeq \tilde{B}_{\widehat{I}} \simeq B_{\widehat{I}}.$$

This shows what we want as completions are integral on π_0. $\square$

2.5.3. The generic fibre via admissible blowups. When defining formal schemes as specific stacks on rings, a proper map is typically defined as a map represented by proper maps of schemes. In particular, if we look at the induced map of adic spaces, it is not directly clear (at least to the author) why this map stays proper (say in the sense of analytic stacks). The main issue is that on the adic spaces we can localize towards Tate-points (also called analytic points). In the following, we argue why such a map remains proper. The intuition should be that a "generic fiber functor" in the sense of Reynauld should preserve proper maps, and the concrete input is a construction of Bhargav Bhatt [**Bha17**][8.1].

Construction 2.5.3.1. For an affine derived formal scheme $\mathrm{Spf}(A)$ with ideal of definition $I \subset \pi_0 A$, we will call a proper morphism

$$X \to \mathrm{Spf}(A)$$

a *local admissible blowup*, if it sits in a cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ \mathrm{Spf}(A) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

where $p\colon \tilde{X} \to \mathrm{Spec}(A)$ is a proper morphism of schemes, which becomes an isomorphism over $\mathrm{Spec}(A) - V(I)$.

Definition 2.5.3.2. A morphism of derived formal schemes

$$X \to S$$

is called an *admissible blowup*, if it locally on S is a local admissible blow up 2.5.3.1.

Remark 2.5.3.3. This notion of admissible blowup might sound more general than needed. We choose this notion because we will need possibly non-finite type closed immersions in our construction. A restriction one could make for our purpose is that we could work with projective ones instead of considering arbitrary proper morphisms.

Remark 2.5.3.4. Note that for a formal scheme X, the category $\mathcal{A}dm(X)$ of admissible blowups over X admits fibre products and a terminal object. In particular, it is cofiltered.

2.5.3.5. From now on, let us consider the following situation. We fix a microbial valuation ring[10] V with fraction field K and pseudouniformizer π. Furthermore, let us fix an adic quasi-compact and quasi-separated morphism of derived formal schemes

$$X \to \mathrm{Spf}(V)$$

where the target is equipped with the π-adic topology. Then we can associate two invariances with this situation.

On one hand, using 2.5.2, the map induces a map of derived adic spaces. So we can consider the generic fiber as an derived adic space

$$\begin{tikzcd} X_\eta \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spa}(K,V) \arrow[r] & \mathrm{Spa}(V). \end{tikzcd}$$

This is a classical derived Tate adic spaces and thus admits an underlying topological space $|X_\eta|$.

On the other hand, let us consider the category $\mathcal{A}dm(X)$ of admissible blowups over X. Now for each such admissible blowup $Y_i \to X$ we write $\bar{Y}_i$ for the reduction

[10] By this we mean a valuation ring which contains a non-zero element π, which is contained in a height one prime. Such an element is called a pseudouniformizer.

modulo π

$$\begin{tikzcd} \bar{Y}_i \arrow[r] \arrow[d] & Y_i \arrow[d] \\ \mathrm{Spec}(V/\pi) \arrow[r] & \mathrm{Spf}(V) \end{tikzcd}$$

which we understand as a derived scheme.

The goal for the rest of the section is to prove the following, which is a global version of [**Bha17**][8.1].

Theorem 2.5.3.6. *Consider the situation described in 2.5.3.5. Then there is a canonical isomorphism of topological spaces*

$$|X_\eta| \simeq \lim_{\mathcal{A}dm(X)} |\bar{Y}_i|$$

functorial in X.

Corollary 2.5.3.7. *Consider a universally closed morphism $X \to S$ of quasi-compact, quasi-separated derived formal schemes adic over Spf(V). Then the induced morphism*

$$|X_\eta| \to |S_\eta|$$

is closed.

Proof. Using 2.5.3.6, we can write this map as an inverse limit of closed morphisms of spectral spaces. Such a map is closed [**FK13**][2.2.13]. □

To describe the maps involved, let us first recall how to understand points in the topological space $|X_\eta|$.

2.5.3.8. Recall that we can understand points in the underlying topological spaces $|X|$ of a adic space as maps

$$\mathrm{Spa}(K, V) \to X$$

of adic spaces where $\mathrm{Spa}(K, V)$ is a valued field. That is, V is a valuation ring together with its fraction field $V \subset K$, and the adic space is understood in one of the following ways

- The topology on V is the π-adic topology for a pseudouniformizer π and $K \simeq V[\frac{1}{\pi}]$. That is V is microbial.
- V and K are discrete.

The first type of point is called *analytic* or *Tate*, and the second type just *non-analytic* or *non-Tate*. Here, two maps are identified if they build a commutative triangle and the square

$$\begin{tikzcd} \tilde{V} \arrow[r] \arrow[d] & \tilde{K} \arrow[d] \\ V \arrow[r] & K \end{tikzcd}$$

is Cartesian. In the case that the adic space is Tate, just Tate points will appear.

Construction 2.5.3.9. Let us describe the map $\phi\colon |X_\eta| \to \lim_{\mathcal{A}dm(X)} |\bar{Y}_i|$. For this, let us be given a point

$$\mathrm{Spa}(E, W) \to X_\eta$$

then we can find an affine open $\mathrm{Spf}(A) \to X$, such that this point lives in $|\mathrm{Spa}(A[\frac{1}{\pi}], A)| \subset |X_\eta|$. Now, any admissible blowup $Y_i \to X$ can be pulled back to $\mathrm{Spf}(A)$ and, by refining $\mathrm{Spec}(A)$ around the closed point of $\mathrm{Spec}(W)$ if necessary, we can assume that this pullback is a local admissible blow-up. No we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & Y_i \arrow[d] \\ \mathrm{Spec}(W) \arrow[r] \arrow[ur, dashed, "!"] & \mathrm{Spec}(A) \end{tikzcd}$$

where we can find a unique dashed lift using the valuative criterion for properness. Now the closed point in $\mathrm{Spec}(V)$ lands in $|\bar{Y}_i|$ and and if we vary Y_i, we obtain a compatible collection of points. That is a point in $\lim_{\mathcal{A}dm(X)} |\bar{Y}_i|$.

To describe the inverse map, let us consider the spaces

$$Y := \lim_{\mathcal{A}dm(X)} Y_i \text{ and } \bar{Y} := \lim_{\mathcal{A}dm(X)} \bar{Y}_i$$

as locally ringed spaces. Then the crucial input is the following, a version of [**Bha17**][8.1.3].

Proposition 2.5.3.10. *For any point $y \in \bar{Y}$ the Hausdorff quotient*

$$S_y := \pi_0 \mathcal{O}_{Y,y} / \cap_n \pi^n \pi_0 \mathcal{O}_{Y,y}$$

is a microbial valuation ring with pseudouniformizer π.

Before doing the proof, let us recall some examples of admissible blowups, we want to use.

2.5.3.11. Recall from [**KR25**] that a closed immersion of derived schemes $Z \to X$ is called a *quasi-regular of finite type*[11], if it locally on X sits in a cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, "(f_1,\dots,f_n)"] \\ \{0\} \arrow[r] & \mathbf{A}^n. \end{tikzcd}$$

Any such closed immersion can be blown up. That is, there is a projective morphism of derived schemes

$$\mathbf{Bl}_Z(X) \to X$$

which becomes an isomorphism over the complement of Z. The derived scheme $\mathbf{Bl}_Z(X)$ is called the *blowup of Z in X* and locally on X can be constructed as the fibre product

$$\begin{tikzcd} \mathbf{Bl}_Z(X) \arrow[r] \arrow[d] & \mathbf{Bl}_{\{0\}}(\mathbf{A}^n) \arrow[d] \\ X \arrow[r, "(f_1,\dots,f_n)"'] & \mathbf{A}^n \end{tikzcd}$$

where we use the classical blowup in the right upper corner. One then globalizes this construction for general quasi-regular closed immersions.

[11] In the given reference, these maps are called *quasi-smooth closed immersions*. We decided to change The name is used in order to avoid confusion with the notion of quasi-regular closed immersions used in the theory of prismatic cohomology. The relation is that for now we just stick to those quasi-regular closed immersions which are also of finite type.

Construction 2.5.3.12. For an affine derived formal scheme $\mathrm{Spf}(A)$, we will call a finitely generated ideal $I \subset \pi_0 A$ *admissible*, if it contains an ideal of definition. Let us consider functions $(f_1, \ldots, f_n) = I$ generating such an admissible ideal and write

$$A/^L I$$

for the derived quotient with respect to the functions $(f_1, \ldots, f_n)$. In this situation, the pullback

$$\begin{CD} \mathrm{Spf}(A)_I @>>> \mathbf{Bl}_{\mathrm{Spec}(A/^L I)}(\mathrm{Spec}(A)) \\ @V{\pi_I}VV @VVV \\ \mathrm{Spf}(A) @>>> \mathrm{Spec}(A). \end{CD}$$

gives an example of an admissible blowup. Note the formal scheme $\mathrm{Spf}(A)_I$ just depends on the ideal and not on the choice of generating functions. In particular, any admissible quasi-coherent ideal sheaf on a formal scheme gives rise to an admissible blow-up.

2.5.3.13. We will use these blowups in the following way. Note that the blow up

$$\mathbf{Bl}_{\{0\}}(\mathbf{A}^2)$$

admits affine charts given by the schemes

$$\mathrm{Spec}(\mathbf{Z}[x, \frac{y}{x}]) \text{ and } \mathrm{Spec}(\mathbf{Z}[\frac{x}{y}, y]).$$

In particular, if we cut out two functions $Z = \mathrm{Spec}(A/(f,g)) \to \mathrm{Spec}(A) = X$ then on the blow up $\mathbf{Bl}_Z(X)$, we have either $f|g$ or $g|f$.

Proof of 2.5.3.10. Note that we can write

$$T_y := \pi_0 \mathcal{O}_{Y,y} \simeq \mathrm{colim}_{Y_i \in \mathcal{A}dm(X)} \mathrm{colim}_{U \subset Y_i} \pi_0 \mathcal{O}_{Y_i}(U)$$

where the second colimit runs over opens. Then the first observation we do is that we can assume $\mathcal{O}_Y$ to be π-torsion free. This follows as the system of those admissible blow-ups with π-torsion-free functions is limit cofinal in all admissible blowups as in an admissible blow-up, we can kill the π-torsion to obtain a new admissible blowup. We now claim:

$(*_1)$ $S_y \neq 0$.

Assuming the statement would imply that $1 \in \pi T_y$, but this is impossible as T_y is a filtered colimit of local rings (with local transition maps), such that in each term π is contained in the maximal ideal (as $y \in \bar{Y}$).

Now we claim the following, which in particular tells us that π is topological nilpotent and thus will be a pseudouniformizer.

$*_2$ For $\bar{f} \in S_y$ we have either $\bar{f} = 0$ or $\bar{f}|\pi^n$ for large n.

Let us lift $\bar{f}$ to $f \in T_y$ and fix n. Then we claim that either $f|\pi^n$ or $\pi^n|f$. We can represent $f \in \mathcal{O}_{Y_i}(U)$ and consider the ideal

$$(f, \pi^n) \subset \mathcal{O}_{Y_i}(U).$$

This ideal can be lifted to a quais-coherent ideal sheaf $\mathcal{I}$ on Y_i which contains π^n (we can add it if necessary). In particular, using 2.5.3.12 we obtain a blow up $Y_{\mathcal{I}} \to Y_i$ which is admissible as π^n is contained in $\mathcal{I}$. Now using 2.5.3.13 we see that in $Y_{\mathcal{I}}$

around the point corresponding to y, we either have $f|\pi^n$ or $\pi^n|f$. To get $*_2$, we are done in the first case, and in the second, we increase n by one and redo the argument.

Now we claim the following, which is one part of being a valuation ring.

($*_3$) For each $\bar{f}, \bar{g} \in S_y$, we have either $\bar{f}|\bar{g}$ or $\bar{g}|\bar{f}$.

We can assume both are non-zero. Thus by $*_2$, if we choose lifts $f, g \in T_y$, we can find n such that $f|\pi^n$ and similar for g. In particular, we can find some open in some admissible blowup $U \subset Y_i$, such that the ideal

$$(f, g) \subset \mathcal{O}_{Y_i}(U)$$

contains π^n for some n. We now use the same argument as in the proof of $*_2$ to get the claim.

To finish the proposition, we are left to show the following, which is the other half of being a valuation ring.

($*_4$) The ring S_y is a domain.

Let us take two non-zero functions $\bar{f}, \bar{g} \in S_y$, such that $\bar{f}\bar{g} = 0$. We will derive a contradiction. Using $*_2$, we can find n and m such that $\bar{f}|\pi^n$ and $\bar{g}|\pi^m$. Now, if we choose lifts $f, g \in T_y$, the same holds for those as the kernel of the map $T_y \to S_y$ is π-divisible. This means we obtain a chain of inclusions

$$(\pi^{n+m}) \subset (f, g) \subset \cap_k \pi^k T_y.$$

So as π is a non zero-divisor, we see that $1 \in \cap_k \pi^k T_y$, which implies $S_y \simeq 0$ and thus contradicts $*_1$. $\square$

Construction 2.5.3.14. As a result of 2.5.3.10, we obtain a map

$$\psi\colon \lim_{\mathcal{A}dm(X)} |\bar{Y}_i| \to |X_\eta|,$$

which we construct now. Given a point y in the left-hand side, we can find an affine cover on $\mathrm{Spf}(A) \to X$ such that

$$y \in \lim_{\mathcal{A}dm(X)} |\bar{Y}_{i\mathrm{Spec}(A/\pi)}|.$$

On functions, this gives us a map $A \to S_y$, where S_y denotes the ring from 2.5.3.10. This map is continuos for the π-adic topology, and inverting π gives a map

$$\mathrm{Spa}(S_y[\frac{1}{\pi}], \widetilde{S_y}) \to \mathrm{Spa}(A[\frac{1}{\pi}], \widetilde{A}) \to X_\eta$$

which produces a point in $|X_\eta|$ as S_y is a microbial valuation ring 2.5.3.10.

Let us now come to the proof.

Proof of 2.5.3.6. Using the valuative criterion for properness, it is easy to see that $\phi \circ \psi = id$. To check that these two maps are mutually inverse to one another, we make the following claim (see [**Bha17**][8.1.4]). This claim implies that we have $\psi \circ \phi = id$.

($*_1$) For any point $x \in |X_\eta|$ we have a canonical isomorphism $\kappa(x)^+ \simeq S_{\psi(x)}$, where $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$ denotes the terminal representative of the point x.

Consider a point $x \in |X_\eta|$. Then we can find an affine open $\mathrm{Spf}(A) \to X$, such that $x \in |\mathrm{Spf}(A)_\eta|$. We did define $\phi(x)$ by taking compatible lifts of the map $\mathrm{Spf}(\kappa(x)^+) \to \mathrm{Spf}(A)$ using the valuative criterion for properness. These lifts give us a compatible family of maps of local rings $\mathcal{O}_{Y_i,\phi(x)} \to \kappa(x)^+$ and thus a map of local rings $\mathcal{O}_{Y,\phi(x)} \to \kappa(x)^+$. As the target is static and Hausdorff, this map induces a map

$$S_{\phi(x)} \to \kappa(x)^+.$$

This map is injective, as the kernel is a prime ideal and thus would contain π if it were not zero. Thus, using 2.5.3.10, we see that this map is an injective local map between valuation rings. Such maps are faithfully flat. Now consider the composition

$$A[\frac{1}{\pi}] \to S_{\phi(x)}[\frac{1}{\pi}] \to \kappa(x).$$

Using the faithful flatness, we see that the kernel of the first map is the same as the kernel of the composition, which is given by the support of x. Thus, we obtain a factorization

$$\kappa(x) \to S_{\phi(x)}[\frac{1}{\pi}] \to \kappa(x)$$

where the composition is identity. As all terms are fields, this shows that the second map is an isomorphism. But then $S_{\phi(x)} \to \kappa(x)$ is a local inclusion of valuation rings with the same field of fractions and thus an isomorphism.

We now prove that ψ is continuous and spectral by identifying basic rational opens in $|X_\eta|$ with quasi-compact opens in Y via this map. Note that we can do this locally on X, so let us take some affine open $\mathrm{Spf}(A) \to X$, and some basic rational open

$$U(\frac{f_1, \dots, f_n}{g}) \subset |\mathrm{Spa}(A[\frac{1}{\pi}], \widetilde{A})|.$$

That is we have we can assume $g, f_1, \dots, f_n \in A$ and $\pi^N \in (f_1, \dots, f_n)$ for some large N [**Bha17**][7.4.3]. Now consider the ideal

$$I = (g, f_1, \dots, f_n) \subset A.$$

Extending this ideal to an ideal sheaf $\mathcal{I}$ on X containing π^N, we can use 2.5.3.12 to obtain an admissible blowup $X_{\mathcal{I}} \to X$. Furthermore, note that we have an open immersion

$$\mathbf{Bl}_I(\mathrm{Spf}(A)) \to X_{\mathcal{I}}.$$

Now, by construction of the blowup, we have a closed immersion

$$\mathbf{Bl}_I(\mathrm{Spf}(A)) \to \mathbf{Bl}_{\{0\}}(\mathbf{A}^{n+1}) \simeq \mathbf{P}^n_{\mathrm{Spf}(A)}$$

mapping g to some homogenous coordinate x_k. Let $U \subset \mathbf{Bl}_I(\mathrm{Spf}(A))$ be the pullback along this closed immersion of the complement of the hyperplane $x_k \neq 0$ and consider the preimage

$$\begin{tikzcd} W \arrow[rr] \arrow[d] & & \bar{Y} \arrow[d] \\ {|U/\pi|} \arrow[r] & {|\mathbf{Bl}_I(\mathrm{Spec}(A/\pi))|} \arrow[r] & {|\bar{X_{\mathcal{I}}}|} \end{tikzcd}$$

Then we claim (see [**Bha17**][8.1.5]):

($*_2$) $\psi^{-1}(U(\frac{f_1,\dots,f_n}{g})) = W$.

We first check the inclusion $\supset$. Note as in 2.5.3.13 we see that by the choice of U, we have the relation $g|f_i$ on $\mathcal{O}_{X_{\mathcal{I}}}(U)$ for all i. In particular, for any $y \in Y$ lying over $|U|$, we have the same relation in $\mathcal{O}_{Y,y}$ and thus also in S_y. But then the point

$$\mathrm{Spa}(S_y[\frac{1}{\pi}], \widetilde{S_y}) \to X_\eta$$

lives in $U(\frac{f_1,\dots,f_n}{g})$. Now, to see the inclusion $\subset$, we note that for a point $y \in \bar{Y}$, the condition that the point corresponding to the map

$$\mathrm{Spa}(S_y[\frac{1}{\pi}], \widetilde{S_y}) \to \mathrm{Spa}(A[\frac{1}{\pi}], \widetilde{A})$$

lands in $U(\frac{f_1,\dots,f_n}{g})$, tells us that we have the relations $g|f_i$ for all i in S_y. Using the valuative criterion for properness and the closed immersion from above, we obtain a map

$$\begin{tikzcd} & \mathbf{Bl}_I(\mathrm{Spf}(A)) \arrow[r] \arrow[d] & \mathbf{P}^n_{\mathrm{Spf}(A)} \\ \mathrm{Spf}(S_y) \arrow[r] \arrow[ur, dashed] & \mathrm{Spf}(A) & \end{tikzcd}$$

corresponding to the line bundle $(x_0 = f_1, \dots, x_k = g, \dots, x_n = f_n)$ with chosen sections as indicated. But using the relations $g|f_i$, we see that as a point in projective space, this is equivalent to the line bundle

$$(x_0 = \frac{f_1}{g}, \dots, x_k = 1, \dots, x_n = \frac{f_n}{g})$$

with chosen sections as indicated. But this point clearly lands in U, which shows what we want.

We have now seen that ψ is a bijective spectral map between spectral spaces. So by [**Sta**, Tag 09XU], to check that it is a homeomorphism, it suffices to see that generalizations lift along ψ. This can be done locally on X and then the same way as in [**Bha17**][8.1.6]. □

2.5.4. Proper maps of formal schemes. To conclude the section, we will analyze proper morphisms of formal schemes. Let us start with the definition. The following notion of finite type is strongly inspired by [**Man22**][2.9.28].

Definition 2.5.4.1. An (adic) morphism of affine formal schemes $\mathrm{Spf}(B) \to \mathrm{Spf}(A)$ is called:

- of $^+$*finite type* if there exists finitely many functions $f_i \in B$ such that the induced map
$$A[x_1, \dots, x_n] \to B$$
is integral on π_0.
- of *finite type* if it is representable by finite type morphisms of animated rings.

An (adic) morphism of formal schemes is called:

- locally of $^+$*finite type* if it is so locally on source and target.
- locally of *finite type* if it is so locally on the source and target.
- $^+$*proper* if it is locally of $^+$finite type, quasi-compact, and represented by separated and universally closed morphisms of schemes.

- *proper* if it is $^+$proper and of locally of finite type.

Example 2.5.4.2. Any integral map of animated rings is a $^+$proper map. A handy example will be that of naive syntomic covers 3.1.2.1.

Construction 2.5.4.3. Note that for any affine formal scheme $\mathrm{Spf}(A)$ over S there is a map

$$\mathrm{Spa}(A) \to \mathrm{Spa}(A, \widetilde{\mathbf{Z}})$$

natural in $\mathrm{Spf}(A)$. Thus, using the functors 2.5.2.3 and 2.5.1.1 for any formal scheme X we obtain a map

$$X^{\mathrm{an}} \to X^{lp}$$

natural in X.

Furthermore, if we consider an adic morphism $X \to Y$ of formal schemes over S, we can define $\overline{X^{\mathrm{an}}}^{/Y^{\mathrm{an}}} := Y^{\mathrm{an}} \times_{Y^{lp}} X^{lp}$ and obtain a factorization

$$X^{\mathrm{an}} \to \overline{X^{\mathrm{an}}}^{/Y^{\mathrm{an}}} \to Y^{\mathrm{an}}.$$

As before, this construction should be thought of as a "compactification". The following proposition tells us in which cases it actually is a compactification.

2.5.4.4. In the following argument, we will deal with specializations in derived Tate adic spaces. For such a Tate adic space X and a point $x \in X$, the set of specializations is given by the set of valuation rings $V \subset \kappa(x)^+\kappa(x)$ with fractions field $\kappa(x)$, which are contained in $\kappa(x)^+$ [**Wed19**][4.12].

From this one sees that a map of Tate adic spaces is universally specializing, if and only if for any commutative square

$$\begin{tikzcd}
\mathrm{Spa}(K,V) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spa}(K,W) \arrow[r] \arrow[ur, dashed] & S,
\end{tikzcd}$$

where on the left we have valued fields, there exists a dashed arrow as indicated.

Proposition 2.5.4.5. *Consider an adic morphism $f\colon X \to S$ of formal schemes and write*

$$p \circ j\colon X^{an} \to \overline{X^{an}}^{/S^{an}} \to S^{an}$$

for the factorization constructed in 2.5.4.3. Then we have:

(a) If f is quasi-compact, p is proper.
(b) If f is locally of $^+$finite type. Then j is an open immersion.
(c) If f is $^+$proper. Then, j is an isomorphism.

PROOF. Note that all assertions are local on S^{an}, so by 2.5.2.4 we can assume $S \simeq \mathrm{Spa}(A)$ to be affine with ideal of definition I. Now by 2.5.1.4 any Zariski covering of X induces a closed covering of $\overline{X^{\mathrm{an}}}^{/S^{\mathrm{an}}}$. For (a), we first show the claim in the case X is separated. Then the diagonal is affine and thus becomes proper, so we can use 2.1.3.7. In the general case, we do the same argument now using the fact that the diagonal is separated.

Note that by 2.5.1.4 and 2.5.2.4, the assertion in (b) is local on X as well. That is, we can assume a factorization

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(B, \widetilde{A^+}) \to \mathrm{Spa}(A, A^+)$$

such that we have finitely many $f_i \in B^+$ inducing an integral map $A^+[x_i, i \in I] \to B^+$. But then $\mathrm{Spa}(B, B^+)$ is given by the intersection of the pullbacks

$$\begin{tikzcd} U(|f_i| \leq 1) \arrow[r] \arrow[d] & \mathrm{Spa}(B, \widetilde{A^+}) \arrow[d, "f_i"] \\ \mathbf{D}^1_\square \arrow[r] & \mathbf{A}^1_\square \end{tikzcd}$$

which is a finite intersection of opens and thus open.

We now check (c). We first claim:

($*_1$) The map $|X^{\mathrm{an}}| \to |\overline{X^{\mathrm{an}}}^{/S^{\mathrm{an}}}|$ is surjective.

As the adic fiber product surjects onto the topological fibre product (see for example [**Hub93b**][3.9.1] [**Hub94**][3.10.4]), we can check this fibered over a valued field

$$x \colon \mathrm{Spa}(K, V) \to S^{\mathrm{an}}.$$

Now, there are two cases. Let us first assume x corresponds to a non-Tate point. Then K and V are discrete, such that any ideal of definition in A vanishes on V and the fibered triangle lives in discrete adic spaces. Then the claim follows using 2.2.4.5 and the valuative criterion for properness in algebraic geometry.

In the case x corresponds to a Tate point, the fibered triangle

$$\begin{tikzcd} X^{\mathrm{an}}_\eta \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X^{\mathrm{an}}}^{/S^{\mathrm{an}}}_\eta \arrow[d, "p"] \\ & \mathrm{Spa}(K, V) \end{tikzcd}$$

lives in Tate adic spaces. At any point in the target of j is a specialization of a point in the domain [**Wed19**][7.41+7.42], it suffices to check that the map is specializing. For this, we first claim:

($*_2$) The map $X^{\mathrm{an}}_\eta \to \mathrm{Spa}(K, V)$ is universally specializing.

Note that to see this, it suffices to check that for any map of valued fields $\mathrm{Spa}(E, W) \to \mathrm{Spa}(K, V)$ the induced map

$$|X^{\mathrm{an}}_\eta \times_{\mathrm{Spa}(K,V)} \mathrm{Spa}(E, W)| \to |\mathrm{Spa}(E, W)|$$

is closed. This fibre product can be computed by first taking the pullback

$$\begin{tikzcd} X_W \arrow[r] \arrow[d] & X_V \arrow[d] \\ \mathrm{Spf}(W) \arrow[r] & \mathrm{Spf}(V) \end{tikzcd}$$

in formal schemes, taking the analytification and then the generic fibre. So the claimed closedness of the above map and thus also the claim $*_2$ follows from 2.5.3.7.

Let us consider a square

$$\begin{tikzcd} \mathrm{Spa}(K,W) \arrow[r] \arrow[d] & X^{\mathrm{an}}_{\eta} \arrow[d] \\ \mathrm{Spa}(K,V) \arrow[r] \arrow[ur, dashed] & \overline{X^{\mathrm{an}}}^{/S^{\mathrm{an}}}_{\eta} \end{tikzcd}.$$

We can find a dashed arrow as indicated, making the upper left triangle commute, by further mapping down to $\mathrm{Spa}(K,V)$ and using $*_2$. Now the lower right triangle commutes, as the diagonal of the map $\overline{X^{\mathrm{an}}}^{/S^{\mathrm{an}}}_{\eta} \to \mathrm{Spa}(K,V)$ is universally specializing.

Let us now finish the argument. By (b), we know that j is an open immersion and thus corresponds to an idempotent algebra in

$$\mathcal{D}_{\mathrm{qc}}(\overline{X^{\mathrm{an}}}^{/S^{\mathrm{an}}}).$$

We claim that this idempotent algebra is 0. Using $*_1$ and 2.2.2.20 we can check this after pulling back to $\mathcal{D}_{\mathrm{qc}}(X^{\mathrm{an}})$ where it is true by construction. □

Corollary 2.5.4.6. *Consider a map of formal schemes $f\colon X \to S$. Then we have:*

(a) If f is of ${}^{+}$finite type then

$$X^{an} \to S^{an}$$

is !-able.

(b) If f is ${}^{+}$proper then

$$X^{an} \to S^{an}$$

is proper. Furthermore, in that case, the square

$$\begin{tikzcd} X^{an} \arrow[r] \arrow[d] & X^{lp} \arrow[d] \\ S^{an} \arrow[r] & S^{lp} \end{tikzcd}$$

is Cartesian.

CHAPTER 3

The Prismatisation and its friends as analytic stacks

The goal of this chapter is to construct a six-functor formalism for prismatic cohomology and its variants. The strategy we want to follow is to produce analytic stacks out of a p-adic formal scheme (or more honestly it's Prismatisation, etc.), such that the categories involved in these formalisms come as categories of quasi-coherent sheaves on these stacks.

$$f\mathcal{S}ch^{\mathrm{op}}_{\mathrm{Spf}(\mathbf{Z}_p)} \xrightarrow{(\_)^{\Delta,\square}} \mathcal{A}n\mathcal{S}tack^{\mathrm{op}} \xrightarrow{\mathcal{D}_{\mathrm{qc}}} \mathcal{C}at$$

Such stacks have been defined by Bhatt-Lurie and Drinfeld (see, for example, [BL22a] [BL22b] [Dri20]) using the language of formal geometry. So, the first thing to consider is understanding formal stacks as analytic stacks. At least two options are coming to mind:

Option one comes from associating with an affine formal scheme $\mathrm{Spf}(A)$ the analytic stack

$$\mathrm{Spa}(A, \widetilde{\mathbf{Z}})$$

as discussed in 2.5.1. This will produce a six-functor formalism where nearly any map is cohomologically proper, similar to the situation in "classical" algebraic geometry. In particular, the amount of, for example, cohomologically étale maps will be very restrictive. So this will not produce what we want.

Option two might be a less naive guess. Here, we associate with $\mathrm{Spf}(A)$ the analytic stack

$$\mathrm{Spa}(A)$$

as discussed in 2.5.2. This will produce a six-functor formalism on formal stacks, which behaves way more "geometric" than option one. We will see that one can obtain a formalism for quasi-coherent cohomology of formal schemes with the correct amount of cohomologically proper and smooth maps.

As pointed out in the addendum, at the point in time where this text is written, it is not clear to the author why, using the second variant, one obtains the correct amount of cohomologically proper morphisms.

This means that in this text, we want to find an option between one and two. Here, the general philosophy will be that on integral perfectoids, we want to use option two, and for quasiregular semiperfectoids, we want to use a relative (over a perfectoid) version of option one. As the reader familiar with these objects knows, this should determine everything.

3.1. Preliminaries

3.1.1. Recollections on integral perfectoids. In the following, we will recall the notion of integral perfectoid rings from [**BMS16**][3]. The main point of this section is that Fontaine's ring still makes sense if we work in the animated setting.

3.1.1.1. In the following, we will make extensive use of the ring of (p-typical) Witt Vectors of an animated ring [**BL22b**][A] and [**Hub24**]. That is, we will consider the functor, which assigns to an animated ring S over $\mathbf{Z}_{(p)}$ its ring of Witt Vectors $\mathrm{W}(S)$. This functor is represented by the scheme

$$\mathrm{Spec}(\mathbf{Z}_{(p)}\{x\}) = \mathrm{Spec}(\mathbf{Z}_{(p)}[x, \delta(x), \delta^2(x), \dots])$$

corresponding to the free δ-ring on one generator, thus defining a (derived) ring scheme.

There also exists n-truncated versions W_n such that

$$\mathrm{W} \simeq \lim_n \mathrm{W}_n$$

and all the classical operations on the Witt vectors give rise to W-linear natural transformations, see [**BL22b**][A], [**Bha22**][2.6] and [**Dri20**][3]:

- There are projections $R\colon \mathrm{W}_n \to \mathrm{W}_{n-1}$. Furthermore $\mathrm{W}_1 \simeq \mathbf{G}_a$.
- There is the *Witt vector Frobenius* $F\colon \mathrm{W}_n \to F_*\mathrm{W}_{n-1}$ which gives the Frobenius after base change to $\mathbf{F}_p$.
- There are the *Verschiebung maps* $V\colon \mathrm{W}_{n-1} \to \mathrm{W}_n$. Furthermore, there are fibre sequences $\mathrm{W}_{n-1} \to \mathrm{W}_n \to \mathbf{G}_a$ where the first map is Verschiebung and the second projection.
- We have the equation $FV \simeq p$ and in characteristic p the equation $VF \simeq p$.

Remark 3.1.1.2. Note that there is an isomorphism of schemes

$$\mathrm{W}_n \simeq \mathbf{A}^n$$

for each n. In particular, we see that

$$\pi_0\mathrm{W}_n(S) \simeq \mathrm{W}_n(\pi_0 S).$$

Remark 3.1.1.3. For any derived p-complete animated ring S, the ring of Witt vectors $\mathrm{W}(S)$ is derived p-complete. To see this, note that p-complete objects are stable under limits, so as the Witt vectors commute with limits, we can check the claim where p is nilpotent in S. But then $\mathrm{W}(S) \simeq \lim_R \mathrm{W}_n(S)$ and p is nilpotent in each of the rings $\mathrm{W}_n(S)$ [**BMS16**][3.2] which implies the claim.

Remark 3.1.1.4. The Frobenius morphisms give rise to a Frobenius endomorphism

$$F\colon \mathrm{W} \to \mathrm{W}$$

on the whole, Witt vectors. This endomorphism can also be understood in the following way. As a functor on (animated) rings, taking the Witt vectors gives a right adjoint to the forget functor from δ-rings to rings [**Joy85**] [**Hub24**]. In particular $\mathrm{W}(S)$ carries a δ-structure given by

$$(s_0, s_1, s_2, \dots) \mapsto (s_1, s_2, \dots)$$

and the induced Frobenius endomorphism gives the above Frobenius [**Haz78**][17]. This also means we recover the classical Frobenius on $\mathrm{W}(S)/p$.

3.1.1.5. Recall from (the proof of) [**BS17**][11.6], that on an animated rings S of characteristic p the Frobenius acts as the 0-map on $\pi_n S$ for all $n \geq 1$.

We now come to the main observation of this section, which says that Fontaine's ring still makes sense for certain animated rings. We follow [**BMS16**][3].

3.1.1.6. We fix an animated ring S, which is derived π-adically complete for some function, such that π^p divides p. We will write $\varphi\colon S/p \to S/p$ for the Frobenius and

$$S^\flat \simeq \lim_\varphi S/p$$

for the inverse limit perfection of S/p. Then we can consider the following.

Definition 3.1.1.7. We will write

$$\mathbb{A}_{\mathrm{inf}}(S) := \mathrm{W}(S^\flat)$$

for Fontaine's ring, which comes with a Frobenius automorphism φ.

We now have the following:

Proposition 3.1.1.8. *Consider an animated ring S as in 3.1.1.6. Then we have the following isomorphisms:*

(a) *The morphism*
$$S^\flat \xrightarrow{\sim} (\pi_0 S)^\flat$$
is an isomorphism.

(b) *For $T \simeq S$ or $T \simeq S/\pi^n$ the morphism*
$$\lim_F W_n(T) \xrightarrow{\sim} \lim_F W_n(\pi_0 T)$$
is an isomorphism.

(c) *The morphism*
$$\varphi_\infty\colon \lim_F W_n(S^\flat) \xrightarrow{\sim} \lim_R W_n(S^\flat)$$
induced by the morphisms $\varphi_n\colon W_n(S^\flat) \to W_n(S^\flat)$ coming from the Frobenius on $S^\flat$ for $n \geq 1$, is an isomorphism.

(d) *The morphism*
$$\lim_F W_n(S^\flat) \xrightarrow{\sim} \lim_F W_n(S/\pi)$$
induced by the canonical morphism $S^\flat \to S/\pi$, is an isomorphism.

(e) *The morphism*
$$\lim_F W_n(S) \xrightarrow{\sim} \lim_F W_n(S/\pi)$$
induced by the canonical morphism $S \to S/p$, is an isomorphism.

Proof. To see (a) note that by 3.1.1.5 both rings are static, so the only obstruction is a $\lim^1$-term. But using 3.1.1.5 again, one sees that the map

$$\prod_n \pi_i S/p \to \prod_n \pi_i S/p$$

used to produced the $\lim^1$ is the identity (for $i \geq 1$). So this obstruction vanishes.

To see (b) note that both sides are (derived) p-complete 3.1.1.3 so we can check the claim modulo p. Now, using the isomorphism

$$\lim_F \mathrm{W}_n(T) \simeq \lim_F \mathrm{W}(T)$$

respectively, both rings are perfect of characteristic p 3.1.1.4 and thus static 3.1.1.5. That means again, the obstruction is a $\lim^1$-term. Using the same isomorphism again, we see that we can compute this $\lim^1$-term using the map

$$\prod_n \pi_i \mathrm{W}(T)/p \to \prod_n \pi_i \mathrm{W}(T)/p$$

(for $i = 1$) which again is the identity by 3.1.1.5.

Using (a) and (b), we can now replace S by $\pi_0 S$ respectively in all claims, so they follow the same way as in [**BMS16**][3.2]. ☐

3.1.1.9. Consider an animated ring S as in 3.1.1.6. Then by 3.1.1.8 we obtain isomorphisms

$$\mathbb{A}_{\inf}(S) \xleftarrow{\simeq} \lim_F \mathrm{W}_n(S^\flat) \xrightarrow{\simeq} \lim_F \mathrm{W}_n(S/\pi) \xleftarrow{\simeq} \lim_F \mathrm{W}_n(S).$$

Precomposing this composition with the Frobenius automorphism $\varphi\colon \mathbb{A}_{\inf}(S) \to \mathbb{A}_{\inf}(S)$ and then using the projection, we obtain Fontaine's map

$$\theta\colon \mathbb{A}_{\inf}(S) \to S.$$

We can now recall the definition of integral perfectoid rings from [**BMS16**][3.5] and record some needed examples.

Definition 3.1.1.10. We will say an animated ring S is *integral perfectoid*, if it is derived π-adically complete for some function, such that π^p divides p and the fibre of Fontaine's map

$$\theta\colon \mathbb{A}_{\inf}(S) \to S$$

is static and generated by one element.

Remark 3.1.1.11. Note that asking the Frobenius $\varphi\colon S/p \to S/p$ to be a π_0-surjection is equivalent to asking Fontaine's map $\mathbb{A}_{\inf}(S) \to S$ to be a π_0-surjection. In particular, the above definition recovers the definition from [**BMS16**][3.5].

Remark 3.1.1.12. By 3.1.1.8, any integral perfectoid is static, so the definition does not give more examples than [**BMS16**][3.5]. This is how it should be, as perfect prisms are also static [**BL22b**][2.16].

3.1.1.13. We will recall the notion of a prism [**BS19**] [**BL22b**] later on. Let us use it for now. Recall that the construction

$$R \mapsto (\mathbb{A}_{\inf}(R), \ker(\theta))$$

gives an equivalence of categories

$$\{\text{Integral perfectoids}\} \simeq \{\text{perfect prisms}\}$$

[**BS19**][3.10] [**BL22b**][2.16].

Example 3.1.1.14. Integral perfectoids of characteristic p are exactly perfect rings.

Example 3.1.1.15. The (derived) p-completion of the cyclotomic numbers

$$R = \mathbf{Z}_p^{\mathrm{cyc}} := \mathbf{Z}_p\langle \mu_{p^\infty}\rangle$$

give an integral perfectoid. If we choose a multiplicative lift ϵ of a compatible system $(1, \xi_p, \xi_{p^2}, \dots) \in R^\flat \simeq \mathbf{F}_p[[x^{\frac{1}{p^\infty}}]]$ of roots of unity with ξ_p primitive, then one can compute

$$\mathbb{A}_{\inf}(R) \simeq \mathbf{Z}_p\langle \epsilon^{\frac{1}{p^\infty}}\rangle, \ker(\theta) \simeq (1 + \epsilon^{\frac{1}{p}} + \epsilon^{\frac{2}{p}} + \cdots + \epsilon^{\frac{p-1}{p}}).$$

Example 3.1.1.16. The ring

$$R = \mathbf{Z}_p\langle p^{\frac{1}{p^\infty}}\rangle$$

is integral perfectoid. Choosing a compatible system of p-th roots of p $\pi^\flat = (p^{\frac{1}{p}}, p^{\frac{1}{p^2}}, \dots) \in R^\flat$[1], one can compute

$$\mathbb{A}_{\mathrm{inf}}(R) \simeq \mathbf{Z}_p\langle [\pi^\flat]^{\frac{1}{p^\infty}}\rangle, \ker(\theta) \simeq (p + [\pi^\flat]^p).$$

Example 3.1.1.17. For any span of integral perfectoids, the pushout

$$\begin{tikzcd} R_0 \arrow[r] \arrow[d] & R_1 \arrow[d] \\ R_2 \arrow[r] & R_2 \widehat{\otimes}_{R_0} R_1 \end{tikzcd}$$

is integral perfectoid. That follows from [**BS19**][8.13].

Example 3.1.1.18. For any étale map of affine p-adic formal schemes

$$\mathrm{Spf}(S) \to \mathrm{Spf}(R)$$

we have that if R is integral perfectoid, then S is integral perfectoid as well. To see this, note that by 2.4.2.8 there exists a unique etale map $\mathrm{Spf}(\tilde{S}) \to \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(R))$ fitting into a cartesian square

$$\begin{tikzcd} \mathrm{Spf}(S) \arrow[r] \arrow[d] & \mathrm{Spf}(R) \arrow[d] \\ \mathrm{Spf}(\tilde{S}) \arrow[r] & \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(R)). \end{tikzcd}$$

Now by [**BS19**][2.18] the δ-structure on $\mathbb{A}_{\mathrm{inf}}(R)$ extends uniquely to a δ-structure on $\tilde{S}$, thus by [**BL22b**][2.10] $\tilde{S}$ carries a unique structure of a Prism compatible with the étale map. This prism is perfect as the Frobenius becomes an isomorphism modulo p and thus is an isomorphism. This shows what we want by 3.1.1.13.

Example 3.1.1.19. Given an integral perfectoid R, the ring

$$R\langle x^{\frac{1}{p^\infty}}\rangle$$

is an integral perfectoid with

$$\mathbb{A}_{\mathrm{inf}}(R\langle x^{\frac{1}{p^\infty}}\rangle) \simeq \mathbb{A}_{\mathrm{inf}}(R)\langle x^{\frac{1}{p^\infty}}\rangle.$$

To see this equip $S = \mathbb{A}_{\mathrm{inf}}(R)[\langle x^{\frac{1}{p^\infty}}\rangle$ with the trivial δ-structure at x, then there exists a unique Prism structure on S making the map $\mathbb{A}_{\mathrm{inf}}(R) \to S$ a map of prisms [**BL22b**][2.10]. But this prism is perfect, which shows the claim by 3.1.1.13. Note that this also works in multiple variables.

Later on, we will be interested in the generic fibres of integral perfectoids. For this it will be useful to systematical neglect $R^{\circ\circ}$-torsion, where $R^{\circ\circ}$ denotes the ideal of topological nilpotent elements in an integral perfectoid R.

---

[1] Here we write $R^\flat \simeq \lim_{r \mapsto r^p} R$ as monoid.

3.1.1.20. Recall from [**GR02**] that an ideal $I \subset R$ in a ring is called an *almost ideal*, if it is flat and the canonical map $I \otimes_R I \xrightarrow{\sim} I$ is an isomorphism. In this situation, the algebra R/I is idempotent over R, and if we kill this idempotent algebra 2.1.2.9, we obtain a localization

$$\mathcal{M}od_R \to \mathcal{M}od_{R^{\mathrm{a}}}$$

of the category of R-modules, which we will call *almost* R^{a}-modules. The right adjoint of this localization is given by the formula

$$M_* \simeq \mathrm{Hom}_R(I, M)$$

and this adjunction induces an adjunction on algebra objects. We will say a morphism is an *almost isomorphism* if it becomes an isomorphism in $\mathcal{M}od_{R^{\mathrm{a}}}$.

The example of the above situation we will care about is the following.

Lemma 3.1.1.21. Consider an integral perfectoid R, then the ideal $R^{\circ\circ}$ of topological nilpotent elements in R defines a (derived p-completed) almost ideal.

Proof. By [**BMS16**][3.9] we can assume that π admits a compatible system of p-power roots. Then $(\pi^{\frac{1}{p^\infty}}) = R^{\circ\circ}$ and one easily sees that $R^{\circ\circ} \otimes_R R^{\circ\circ} \simeq R^{\circ\circ}$. To see the ($p$-complete) flatness, we write M for the p-completed colimits of the diagram

$$R \xrightarrow{\pi^{1-\frac{1}{p}}} R \xrightarrow{\pi^{\frac{1}{p}-\frac{1}{p^2}}} R \xrightarrow{\pi^{\frac{1}{p^2}-\frac{1}{p^3}}} \dots .$$

There is a map $M \to R^{\circ\circ}$ induced by the elements $\pi^{\frac{1}{p^n}}$ and we claim that this map is an isomorphism. To see this, recall that we have a commutative triangle

$$\begin{tikzcd} \mathrm{W}(R^\flat) \arrow[r] \arrow[d] & R \\ R^\flat \arrow[ur, dashed, "\sharp"'] & \end{tikzcd}$$

where the upper horizontal map is given by Fontaine's map, the left vertical map is given by quotienting modulo p, and the dashed arrow is a map of (multiplicative) monoids. Now note that $R/(\pi^{\frac{1}{p^\infty}})$ is an integral perfectoid with a perfect prism given by

$$\mathrm{W}(R^\flat/(\omega^{\frac{1}{p^\infty}}))$$

with $\omega^\sharp = \pi$ and the Hodge-Tate ideal induced from the Hodge-Tate ideal $(d) \subset \mathrm{W}(R^\flat)$. In particular (d) acts as a non-zero divisor on the latter quotient. Thus to check the claimed isomorphism, we can check that the sequence

$$\tilde{M} \to \mathrm{W}(R^\flat) \to \mathrm{W}(R^\flat/(\omega^{\frac{1}{p^\infty}}))$$

is a fibre sequence, where $\tilde{M}$ denotes the p-completed colimit of the sequence

$$\mathrm{W}(R^\flat) \xrightarrow{\omega^{1-\frac{1}{p}}} \mathrm{W}(R^\flat) \xrightarrow{\omega^{\frac{1}{p}-\frac{1}{p^2}}} \mathrm{W}(R^\flat) \xrightarrow{\omega^{\frac{1}{p^2}-\frac{1}{p^3}}} \dots .$$

This we can check modulo p, where it follows as in [**Bha17**][4.1.3]. $\square$

Proposition 3.1.1.22. *Consider an integral perfectoid R with tilt $R^\flat$, then there is an equivalence of categories*

$$\mathcal{P}erfd^{int}_{R^\flat/} \simeq \mathcal{P}erfd^{int}_{R/}$$

between the category of integral perfectoid $R^\flat$-algebras and the category of integral perfectoid R-algebras. The functor in one direction assigns to an R-algebra, its tilt, and the functor in the other direction assigns to a $R^\flat$-algebra T the R-algebra

$$W(T) \otimes_{W(R^\flat)} R$$

. Furthermore, this equivalence preserves almost isomorphisms, where we do almost mathematics with respect to the ideals of topological nilpotent elements 3.1.1.21.

PROOF. The category of integral perfectoids over R is equivalent to the category of perfect prisms over $\mathrm{W}(R^\flat)$ [**BS19**][3.10]. This category is equivalent to the category of perfect δ-rings over $\mathrm{W}(R^\flat)$ [**BL22b**][2.10+2.16], which is equivalent to the category of integral perfectoid rings over $R^\flat$ [**BS19**][2.31].

To see that the equivalence respects almost isomorphisms, recall first that we can assume that $\pi \in R$ admits a compatible system of p-th roots and thus can be lifted to an element $\omega = \pi^\flat \in R^\flat$ [**BMS16**][3.9]. Now $R^{\circ\circ} = (\pi^{\frac{1}{p^\infty}})$ and $(R^\flat)^{\circ\circ} = (\omega^{\frac{1}{p^\infty}})$. Let us take a map $S \to T$ of integral perfectoid R algebras, then we have to show that the map $(\pi^{\frac{1}{p^\infty}})S \to (\pi^{\frac{1}{p^\infty}})T$ is an isomorphism if and only if the map $(\omega^{\frac{1}{p^\infty}})S^\flat \to (\omega^{\frac{1}{p^\infty}})T^\flat$ is an isomorphism. But the proof of 3.1.1.21 shows that, by derived Nakayama, both are equivalent to the assertion that the map

$$([\omega]^{\frac{1}{p^\infty}})\mathrm{W}(S^\flat) \to ([\omega]^{\frac{1}{p^\infty}})\mathrm{W}(T^\flat)$$

is an isomorphism. □

3.1.2. The naive syntomic topology. A crucial input in the theory of prismatic cohomology is the so-called syntomic topology. On the other hand, we would like to use the descendable topology, as this topology is compatible with the theory of analytic stacks. The author does not know if the syntomic topology is coarser than the descendable topology so we will work with a coarser topology. This topology will be called the *naive syntomic topology*.

In the following, we work with the category

$$f\mathcal{S}ch^{\mathrm{aff}}_{\mathbf{Z}_p}$$

of affine p-adic formal schemes. In the end, all outputs will satisfy Zariski or even étale descent, so there is no harm in restricting to affine objects.

Definition 3.1.2.1. An (adic) map $X \to S$ of affine derived p-adic formal schemes is called a *naive syntomic cover*, if it can be refined by a map living in the smallest class, which is stable under base change and composition, and contains maps of the form

$$\mathrm{Spf}(\mathbf{Z}\langle x_i^{\frac{1}{p^\infty}} | i \in I\rangle) \to \mathrm{Spf}(\mathbf{Z}\langle x_i | i \in I\rangle)$$

with I some set.

Remark 3.1.2.2. It is not clear how to characterize all naive syntomic covers. This makes it hard to verify that a functor is a sheaf for the naive syntomic topology in general. We will only need to verify this in particular easy situations, where this problem reduces to the generic examples. One is defining a functor of topoi, that is, a left exact left adjoint. The other will be associating a cohomology ring, which satisfies the Künneth formula, sends naive syntomic covers to a type of map that is stable under refining, base change, and composition.

Example 3.1.2.3. Any naive syntomic cover is a (derived version) of a syntomic cover [**BMS18**][4.2]. In particular, any naive syntomic cover is a (p-completely) faithfully flat cover. Note that the situation is even slightly better, namely, any naive syntomic cover is pulled back from a faithfully flat map of (derived) schemes.

Example 3.1.2.4. Any naive syntomic cover is a p-completely descendable cover 2.4.1.10.

3.1.2.5. By construction, naive syntomic covers build a pretopology[2] on $f\mathcal{S}ch^{\mathrm{aff}}_{\mathbf{Z}_p}$ and we will write

$$(\mathbf{Z}_p)_{\mathrm{nsyn}}$$

for the induces topos[3].

The main goal of this section is now to explain why the "unfolding" idea in [**BMS18**][4.31] still goes through using the naive symmetric topology. For this, we will make the following definition, which uses the concept of *integral perfectoid* rings from [**BMS16**][3]. We will also recall this notion in more detail in the next section.

Definition 3.1.2.6. A (derived) p-complete animated ring S is called *semiperfectoid*, if it admits a map

$$R \to S$$

from an integral perfectoid ring and the Frobenius $\varphi\colon S/p \to S/p$ is a π_0-surjection.

Remark 3.1.2.7. A derived p-complete animated ring S is semiperfectoid, if and only if it admits a π_0-surjection from an integral perfectoid. To see this note that, if we have some map $R \to S$ from an integral perfectoid, the map

$$\mathbb{A}_{\mathrm{inf}}(R) \to \mathbb{A}_{\mathrm{inf}}(S)$$

induces a perfect Prism structure on $\mathbb{A}_{\mathrm{inf}}(S)$. If the Frobenius on S/p is surjective, the corresponding integral perfectoid surjects onto S.

The following should be thought of as a version of [**BMS18**][4.31] in the animated setting.

Proposition 3.1.2.8. *Any p-complete animated ring $\tilde{S}$ admits a naive syntomic covering by a semiperfectoid S, such that each term in the Čech nerve*

$$\tilde{S} \to S \rightrightarrows S \widehat{\otimes}_{\tilde{S}} S \cdots$$

is semiperfectoid as well. In particular, the inclusion of semiperfectoids into affine p-adic formal schemes induces an isomorphism of categories of naive syntomic sheaves. The inverse of this equivalence is given by right Kan extension.

Proof. The argument for the first claim is just the same as in [**BMS18**][4.28+4.30] and the second claim is standard (see, for example [**BMS18**][4.31]). □

[2] Do not forget to also add the map from the empty presheaf to the empty scheme as a cover.
[3] By convention, this is just a topos up to size issues.

3.1.3. Prisms and the Prismatisation of $\mathbf{Z}_p$. Let us recall the notion of an (animated) prism introduced in [**BL22b**]. See also [**BS19**] for the original definition of a Prism.

Construction 3.1.3.1. Recall that the moduli of *generalized Cartier divisors* is given by the quotient stack $\mathbf{A}^1/\mathbf{G}_m$. That is a map $\mathrm{Spec}(A) \to \mathbf{A}^1/\mathbf{G}_m$ can for example be understood in the following ways:

- An A-linear map $\alpha\colon I \to A$ where I is given by an invertible A-module.
- A map of animated rings $A \to \overline{A}$ with fibre given by an invertible A-module. Here one obtains $\mathrm{Spec}(\bar{A})$ by pulling back along the map $\mathbf{BG}_m \to \mathbf{A}^1/\mathbf{G}_m$.

Note that for a generalized Cartier divisor $\mathrm{Spec}(A) \to \mathbf{A}^1/\mathbf{G}_m$ we can equip A with the I-adic topology by pulling back along the completion at the origin

$$\widehat{\mathbf{A}^1}/\mathbf{G}_m \subset \mathbf{A}^1/\mathbf{G}_m.$$

In what follows, if we refer to a generalized Cartier divisor, we refer to the produced map

$$\mathrm{Spf}(A) \to \widehat{\mathbf{A}^1}/\mathbf{G}_m.$$

Definition 3.1.3.2. An *animated prism* is an animated δ-ring in the sense of [**BL22b**][A.11] [**Hub24**][2.4] together with a generalized Cartier divisor $A \to \overline{A}$ such that:

(i) A is (p, I)-complete.
(ii) For any perfect field k of characteristic p and any map $A \to k$ of animated rings which annihilates I, we have

$$\mathrm{W}(k) \otimes_A \overline{A} \simeq k.$$

Here $\mathrm{W}(k)$ is understood as an A-algebra via the adjoint map of δ-rings.

3.1.3.3. Recall that an animated δ-algebra is given by an animated ring A together with an endomorphism

$$\varphi\colon A \to A$$

which identifies with the Frobenius modulo p. The category of those objects admits all colimits, and the forgetful functor to animated rings preserves those [**Hub24**][2.8].

Example 3.1.3.4. An animated prism $A \to \overline{A}$ is called *perfect* if the Frobenius lift

$$\varphi\colon A \xrightarrow{\sim} A$$

is an isomorphism. As already mentioned, these prisms are exactly the ones that are of the form

$$\theta\colon \mathbb{A}_{\mathrm{inf}}(R) \to R$$

for R an integral perfectoid ring.

3.1.3.5. One can prove [**BL22b**][2.10] that a map of animated prisms is just a map of δ-algebras. That is, for an animated prism $A \to \overline{A}$, the category of animated prisms over $A \to \overline{A}$ is equivalent to the category of derived (p, I)-complete δ-algebras over A. This observation has the following Corollary.

Corollary 3.1.3.6. *The category of animated prisms has nonempty coproducts and pushouts.*

3.1.3.7. Instead of giving a formal proof, let us explain how to compute them. Using 3.1.3.5 we can compute the pushout as the pushout of δ-algebras, induce the Cartier divisor, and then complete. Note that the underlying ring is just the completed pushout of the underlying Rings.

Given two animated prisms $I \to A$ and $I' \to A'$, we can consider the pushout

$$A \otimes^{\delta}_{\mathbf{Z}} A'$$

in δ-rings, whose underlying ring is just given by the coproduct of animated rings. Then we take the Cartier divisor $I \otimes A' \to A \otimes_{\mathbf{Z}} A$ and complete to this divisor and p. This computes the coproduct.

The moduli of animated prisms gives the Prismatisation of $\mathbf{Z}_p$.

Definition 3.1.3.8. Let us fix a p-nilpotent animated ring R and $\mathrm{W}(R)$ the associated (animated) ring of Witt Vectors. A *Cartier-Witt divisor* on R is an animated prism $\mathrm{W}(R) \to \overline{\mathrm{W}(R)}$ (where we use the canonical δ-structure on $\mathrm{W}(R)$), such that the composition

$$\pi_0(I) \to \pi_0 \mathrm{W}(R) \to \pi_0 R$$

has a nilpotent image. The second map is given by taking the 1-th Witt component. We write

$$\mathbf{Z}_p^{\Delta}(R)$$

for the groupoid of Cartier Witt divisors on R. Note that this construction is functorial in R and thus defines an object in $\mathrm{Spf}(\mathbf{Z}_p)_{fdesc}$ which we refer to as the *prismatisation* of $\mathrm{Spf}(\mathbf{Z}_p)$.

Construction 3.1.3.9. For any animated prism $I \to A$, there is a map

$$\rho_A \colon \mathrm{Spf}(A) \to \mathbf{Z}_p^{\Delta}.$$

This map comes as follows. As A admits a δ-structure any map $A \to R$ to a p-nilpotent ring factors by adjunction as

$$A \to \mathrm{W}(R) \to R$$

where the first map is a map of δ-rings. Now

$$I \otimes_A \mathrm{W}(R) \to \mathrm{W}(R)$$

gives a Cartier-Witt divisor on R.

Example 3.1.3.10. An important example of a prism is the universal oriented prism. For an animated p-nilpotent ring R we write $\mathrm{W}_0(R) \subset \mathrm{W}(R)$ for the subspace of those Verschiebung expansions $\sum_{n \geq 0} V^n[a_n]$ for which a_0 is nilpotent and a_1 is a unit. Note that, as a functor, this assignment is represented by an affine formal scheme

$$\mathrm{Spf}(\mathbf{Z}_p\langle a_0, a_1^{\pm 1}, a_2, \dots \rangle)$$

where we also complete to (a_0), and that the Frobenius on the Witt Vectors restricts to this subspace, such that we obtain a δ-structure on the representing ring. Thus choosing (a_0) as an ideal we have produced a prism and using 3.1.3.9 we get a map

$$\mathrm{W}_0 \to \mathbf{Z}_p^{\Delta}.$$

By (the proof of) [**BL22a**][3.2.3] (see also [**BL22b**][8.5]), this map identifies the target as the quotient, in the Zariski topology, of the domain by the canonical action of the affine group scheme representing the functor $R \mapsto \mathrm{W}(R)^\times$. Where we write $\mathrm{W}(R)^\times$ for the units in the Witt Vectors. This group scheme is represented by the free delta ring on a unit $\mathbf{Z}_p\{u^\pm\}$.

In particular, using 2.4.1.8, we see that we have found a presentation of

$$\mathbf{Z}_p^{\Delta} \in (\mathbf{Z}_p^{\Delta})_{fdesc}.$$

The following proposition is a slight generalization of [**BL22a**][3.2.8]. It was essentially pointed out there that it holds in this generality.

Proposition 3.1.3.11. *Consider two animated prisms $I \to A$ and $J \to B$. Furthermore let $K \to C$ be their coproduct in the category of prisms. Then the square*

$$\begin{tikzcd} \mathit{Spf}(C) \arrow[r] \arrow[d] & \mathit{Spf}(B) \arrow[d, "\rho_B"] \\ \mathit{Spf}(A) \arrow[r, "\rho_A"'] & \mathbf{Z}_p^{\Delta} \end{tikzcd}$$

is Cartesian.

Proof. We can prove the assertion Zariski locally, so that we can assume I and J correspond to functions d_A and d_B. Then we can factor the square as

$$\begin{tikzcd} \mathrm{Spf}(C) \arrow[r] \arrow[d] & \mathrm{Spf}(B\{u^\pm\}) \arrow[r] \arrow[d] & \mathrm{Spf}(B) \arrow[d] \\ \mathrm{Spf}(A\{u^\pm\}) \arrow[r] \arrow[d] & \mathrm{W}^\times \times \mathrm{W}_0 \arrow[r] \arrow[d] & \mathrm{W}_0 \arrow[d] \\ \mathrm{Spf}(A) \arrow[r] & \mathrm{W}_0 \arrow[r] & \mathbf{Z}_p^{\Delta}. \end{tikzcd}$$

By [**BL22a**][3.2.8], the lower right square is Cartesian and corresponds to a coproduct of prisms. Now the upper right square is Cartesian. Thus, it corresponds to a pushout of prisms (as all maps are maps of prisms). From this we see that $B\{u^\pm\}$ gives the coproduct of B and the prism representing W_0. So the whole square on the left corresponds to a pushout of prisms and thus is cartesian 3.1.3.6. □

Corollary 3.1.3.12. *Consider two integral perfectoids R_1 and R_2, then the pullback*

$$\begin{tikzcd} \mathit{Spf}(\mathbb{A}_{inf}(\tilde{R})) \arrow[r] \arrow[d] & \mathit{Spf}(\mathbb{A}_{inf}(R_1)) \arrow[d] \\ \mathit{Spf}(\mathbb{A}_{inf}(R_2)) \arrow[r] & \mathbf{Z}_p^{\Delta} \end{tikzcd}$$

is represented by $\mathbb{A}_{inf}(\tilde{R})$ of an integral perfectoid $\tilde{R}$ (i.e. corresponds to a perfect prism).

Proof. Using the equivalence between perfect prisms and integral perfectoids 3.1.3.4, this follows from 3.1.3.11 as a coproduct of perfect prisms is perfect. □

3.1.4. Prismatic cohomology of semiperfectoids. The upcoming proposition will be crucial when defining the prismatisation as an analytic stack.

3.1.4.1. Recall that we called a derived p-complete animated ring S semiperfectoid if there exists a map $R \to S$ from an integral perfectoid and the Frobenius $\varphi\colon S/p \to S/p$ is a π_0-surjection.

Example 3.1.4.2. A useful fact is that any étale algebra

$$S \to \tilde{S}$$

over a semiperfectoid is semiperfectoid as well. Let us point out for later use that for such a map, the map

$$\mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(\tilde{S})) \to \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(S))$$

is an étale map of formal schemes. This follows, for example, using 3.1.1.8 and the fact that the maps $\mathrm{W}_n(S) \to \mathrm{W}_n(\tilde{S})$ are étale [**Bor10**][15.2]. If the original map is a surjection, the map on $\mathbb{A}_{\mathrm{inf}}(\_)$ is as well [**Bor10**][16.11].

3.1.4.3. Recall that to a semiperfectoid S we can associate the (derived) absolute prismatic cohomology

$$\mathbb{\Delta}_S$$

[**BS19**] [**BL22a**]. We stick to semiperfectoids for now because, in this case, this object can be understood as an (completed) animated ring. We will recall the argument for this fact later on 3.2.1.15.

Even better, $\mathbb{\Delta}_S$ carries the structure of an animated prism. One way to see this is that, as it is prismatic cohomology, it carries a δ-structure [**Hol24**] and to obtain the Cartier divisor, we can choose an integral perfectoid R mapping to S and induce the divisor via the map

$$\mathbb{A}_{\mathrm{inf}}(R) \simeq \mathbb{\Delta}_R \to \mathbb{\Delta}_S.$$

It is easy to check that this construction does not depend on the chosen integral perfectoid. In particular, as in 3.1.3.9, we obtain a map

$$\rho_{\mathbb{\Delta}_S}\colon \mathrm{Spf}(\mathbb{\Delta}_S) \to \mathbf{Z}_p^{\mathbb{\Delta}}.$$

Remark 3.1.4.4. In the following argument, we will also use the prismatisation of a general p-adic formal scheme. This is a stack whose construction we will recall in the next section. The one thing we will need here is that, as a functor from p-adic formal schemes to stacks, it preserves fibre products.

Proposition 3.1.4.5. *For a semiperfectoid ring S, there exists a map of δ-rings*

$$\mathbb{A}_{inf}(S) \to \mathbb{\Delta}_S$$

uniquely determined by the following requirements:

(a) It is functorial in maps between semiperfectoids.
(b) For any perfectoid R mapping to S, the square

$$\begin{tikzcd} \mathbb{\Delta}_R \arrow[r] & \mathbb{\Delta}_S \\ \mathbb{A}_{inf}(R) \arrow[u, "\simeq"] \arrow[r] & \mathbb{A}_{inf}(S) \arrow[u] \end{tikzcd}$$

commutes.

Proof. If we show the existence, the uniqueness is clear from 3.1.3.5. Consider a map $R \to S$ from an integral perfectoid. Then, by 3.1.3.5 and 3.1.3.4, the induced map

$$\mathbb{A}_{\mathrm{inf}}(R) \to \mathbb{A}_{\mathrm{inf}}(S)$$

induces a perfect prism structure on $\mathbb{A}_{\mathrm{inf}}(S)$. Let us write $\tilde{S}$ for the corresponding integral perfectoid. Using Fontaine's map $\theta\colon \mathbb{A}_{\mathrm{inf}}(S) \to S$ 3.1.1.9, we obtain a map

$$\tilde{S} \to S.$$

Now, the map in the proposition is given by the composition

$$\mathbb{A}_{\mathrm{inf}}(S) \simeq \mathbb{A}_{\mathrm{inf}}(\mathbb{A}_{\mathrm{inf}}(S)) \to \mathbb{A}_{\mathrm{inf}}(\tilde{S}) \simeq \Delta_{\tilde{S}} \to \Delta_S$$

and we are done, if we check that the prism structure on $\mathbb{A}_{\mathrm{inf}}(S)$ does not depend on R.

For this let us consider two integral perfectoids R_1 and R_2 mapping to S. Then we have a Cartesian square

$$\begin{tikzcd}
\mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(S)) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (R_1 \widehat{\otimes}_{\mathbf{Z}_p} R_2)^{\Delta} \arrow[r] \arrow[d] & \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(R_2)) \arrow[d, "\rho_{\mathbb{A}_{\mathrm{inf}}(R_2)}"] \\
& \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(R_1)) \arrow[r, "\rho_{\mathbb{A}_{\mathrm{inf}}(R_1)}"'] & \mathbf{Z}_p^{\Delta}.
\end{tikzcd}$$

Now, by 3.1.3.11, the pullback can be computed via the pushout of δ-rings, so that there exists a dashed arrow making the diagram commute as indicated. By 3.1.3.12 we have

$$(R_1 \widehat{\otimes}_{\mathbf{Z}_p} R_2)^{\Delta} \simeq \mathrm{Spf}(\mathbb{A}_{\mathrm{inf}}(\tilde{R}))$$

for some integral perfectoid $\tilde{R}$. In particular, the prism structure on $\mathbb{A}_{\mathrm{inf}}(S)$ induced by $\mathbb{A}_{\mathrm{inf}}(R_1)$ is the same as the prism structure induced by $\mathbb{A}_{\mathrm{inf}}(\tilde{R})$ which is the same as the one induced by $\mathbb{A}_{\mathrm{inf}}(R_2)$. □

Remark 3.1.4.6. As the reader probably has observed, the argument for 3.1.4.5 could have been written in more elementary terms. The reason we choose this formulation is that, for the author, the reason this argument works is that $\mathbf{Z}_p^{\Delta}$ should be thought of as the "initial prism".

3.1.5. Perfectoidisation of Tate algebras.

3.1.5.1. Recall from that a uniform Tate Huber pair (A, A^+) is called perfectoid if there exists a topological nilpotent unit $\pi \in A^\circ$ such that π^p divides p and the absolute Frobenius

$$\varphi\colon A^\circ/\pi \to A^\circ/\pi^p$$

is surjective.

3.1.5.2. From 2.3.1.10, we obtain a fully faithful functor

$$\mathcal{P}erfd \to \mathcal{B}nd^{\mathrm{cl}}_{(\mathbf{Z}_p)_\square/}$$

from the category of perfectoid Tate Huber pairs to classical bounded solid affinoids.

Definition 3.1.5.3. We will say a bounded solid affinoid algebra A is *perfectoid* if it lies in the essential image of the above functor.

Example 3.1.5.4. Given a perfectoid Tate algebra (A, A^+), then A^+ is integral perfectoid. This is explained in [**BMS16**][3.20].

Perfectoid Tate algebras are controlled by Perfectoid Tate algebras of characteristic p in the following way (see [**SW20**] [**KL13**]).

3.1.5.5. Let us write $\mathcal{FF}$ for the category of triples

$$(R, R^+, I \subset \mathrm{W}(R^+))$$

where (R, R^+) defines a perfectoid Tate Huber pair of characteristic p and $I \subset \mathrm{W}(R^+)$ a perfect prism. Note that this makes sense by 3.1.5.4.

Theorem 3.1.5.6 (Scholze, Kedlaya-Liu)**.** *There is an equivalence of categories*

$$\mathcal{P}erfd \simeq \mathcal{FF}$$

where one assigns to a perfectoid Tate algebra (R, R^+) the triple

$$(R^\flat, (R^+)^\flat, I \subset \mathbb{A}_{inf}(R^+))$$

where the prism is the one corresponding to R^+. The inverse assigns to a triple (S, S^+, I) the Tate algebra

$$(W(S^+)/I[\frac{1}{\omega}], W(S^+)/I)$$

.

Proof. Given an integral perfectoid R, which is complete for some non-zero divisor π such that π^p divides p, the ring $R[\frac{1}{\pi}]$ is a perfectoid Tate algebra and the map $R \to (R[\frac{1}{\pi}])^\circ$ is an almost isomorphism of integral perfectoids [**BMS16**][3.21]. In particular, we can write the category $\mathcal{P}erfd$ as the category of those integral perfectoid rings R, such that R is π-adically complete for some non-zero divisor as above and R is integrally closed in $(R[\frac{1}{\pi}])^\circ$. Both of these conditions are preserved under the Tilting equivalence 3.1.1.22. For the non-zero divisor condition see [**Mor17**][1.7] and and the integral closedness follows from [**Mor17**][2.5] as

$$(R^\flat)^\circ \simeq (R^\circ)^\flat$$

. The claim now follows from the equivalence between integral perfectoids and perfect prisms [**BS19**][3.10]. ☐

Corollary 3.1.5.7. *Consider an integral perfectoid R with topologically nilpotent element π, such that π^p divides p. Then the Huber pair*

$$(R[\frac{1}{\pi}], R^+)$$

is a perfectoid Tate algebra for any ring of integral elements R^+.

Proof. Recall first that the statement just depends on $R[\frac{1}{\pi}]$ and not on the ring of integral elements. Let us assume that $\pi \in R$ admits a compatible system of p-th roots and choose a lift $\omega = \pi^\flat \in R^\flat$. Then we claim

(∗) The ring $(R^\flat)_*$ is integral perfectoid and ω-torsion free. Where $(\_)_*$ denotes the right adjoint of the almostification.

We argue first for the ω-torsion freeness. For this we claim that the power ω-torsion ideal $R^\flat[\omega^\infty] \subset R^\flat$ is almost zero. Consider some element $r \in R^\flat$ such that $\omega^c r = 0$ for large enough c. Then $\omega^c r^{p^n} = 0$ for all n as well and by perfectness we see that $\omega^{\frac{c}{p^n}} t = 0$ for all n, which shows the almost vanishing. To prove that the ring is integral perfectoid by [**BMS16**][3.10], we just have to check that the frobenius

$$(R^\flat)_*/\omega \to (R^\flat)_*/\omega^p$$

is an isomorphism. For this, we argue as in [**Sch12**][5.6].

Now by the claim and [**BMS16**][3.21] the pair

$$((R^\flat)_*[\frac{1}{\omega}], \widetilde{(R^\flat)_*})$$

defines a perfectoid Tate algebra such that $R^\flat \to \widetilde{(R^\flat)_*} := T$ is an almost isomorphism. Using 3.1.1.22, we obtain an almost isomorphism $R \to S$ of integral perfectoid algebras, such that

$$S \simeq \mathrm{W}(T) \otimes_{\mathrm{W}(R^\flat)} R$$

. In particular, inducing a prism structure on $\mathrm{W}(T)$ from the prism structure on $\mathrm{W}(R^\flat)$, we obtain a triple as in 3.1.5.6 corresponding to the perfectoid Tate algebra

$$(S[\frac{1}{\pi}], \widetilde{S}) \simeq (R[\frac{1}{\pi}], \widetilde{S})$$

. This shows what we want. $\square$

Remark 3.1.5.8. Perfectoid bounded solid affinoids are stable under pushouts in solid affinoids. affinoid rings. This follows, for example, by combining the fact that A° of a perfectoid algebra is integral perfectoid [**BMS16**][3.20] with the fact that integral perfectoids are stable under pushouts [**BS19**][8.13] and the compatibility of the solid tensor-product with the completed tensor product [**Bos23**][A.3].

Definition 3.1.5.9. A classical bounded solid affinoid algebra (A, A^+) is called *semiperfectoid* if there exists ring of definition $A_0 \subset A$ and a topologically nilpotent element $\pi \in A_0$, such that the derived π-adic completion of A_0 is an integral semiperfectoid ring.

Example 3.1.5.10. Consider an integral semiperfectoid S. Then, for any topological nilpotent element $\pi \in S$, the ring

$$S[\frac{1}{\pi}]$$

defined a semiperfectoid Tate algebra. This follows as killing the π-power torsion $S/S[\pi^\infty]$ produces an integral semiperfectoid.

Proposition 3.1.5.11. *The inclusion $\mathcal{P}erfd \subset \mathcal{S}emi\mathcal{P}erfd$ admits a left adjoint*

$$(\_)_{perfd} \colon \mathcal{S}emi\mathcal{P}erfd \to \mathcal{P}erfd$$

called perfectoidisation. Furthermore, as a functor

$$(\_)_{perfd} \colon \mathcal{S}emi\mathcal{P}erfd \to \mathcal{B}nd^{cl}_\square$$

it preserves pushouts.

Proof. Given a semiperfectoid Tate algebra B, we have to construct a map

$$B \to B_{\mathrm{perfd}}$$

to a perfectoid Tate algebra, such that for any other perfectoid algebra C the map

$$\mathrm{Hom}_{\mathcal{B}nd_{\square}}(B_{\mathrm{perfd}}, C) \to \mathrm{Hom}_{\mathcal{B}nd_{\square}}(B, C)$$

is an isomorphism.

Consider a semiperfectoid (B, B^+) with semiperfectoid ring of definition B_0 and topologically nilpotent element π as in the definition. Then the derived π-adic completion of B_0 admits an integral perfectoidisation $(B_0)_{\mathrm{perfd}}$ by [**BS19**][8.14] and by 3.1.5.7 the Tate algebra

$$B_{\mathrm{perfd}} := ((B_0)_{\mathrm{perfd}}[\frac{1}{\pi}], \widetilde{B^+})$$

is perfectoid.

To see that the map $B \to B_{\mathrm{perfd}}$ induces an isomorphism on hom anima, it is enough to observe that for a perfectoid algebra C the map $C \xrightarrow{\sim} C_{\mathrm{perfd}}$ is an isomorphism 3.1.5.4 and that for a semiperfectoid B, we have the formula

$$(B_{\mathrm{perfd}} \otimes_B B_{\mathrm{perfd}})_{\mathrm{perfd}} \simeq B_{\mathrm{perfd}}$$

which follows from [**BS19**][8.13] and [**Bos23**][A.3].

The second claim follows as perfectoid algebras are stable under pushouts in $\mathcal{B}nd^{\mathrm{cl}}_{\square}$ 3.1.5.8. $\square$

Remark 3.1.5.12. Note that for a semiperfectoid B, the map $B \to B_{\mathrm{perfd}}$ is proper by construction. With the notation of the above proof, the ring $\widetilde{B^+}$ can also be presented as the integral closure of $(B_0)_{\mathrm{perfd}}$ in B_{perfd} as the map $B_0 \to (B_0)_{\mathrm{perfd}}$ is surjective [**BS19**][7.4].

Remark 3.1.5.13. Note that the Proposition in particular says that the perfectoidisation does not depend on the chosen ring of definition in the construction.

3.1.6. The naive syntomic topology on Tate algebras. We now want to define a version of the naive syntomic topology on p-adic classical bounded solid affinoids. This topology will be defined on the category

$$(\mathcal{B}nd^{\mathrm{cl}})_{(\mathbf{Q}_p)_{\square}/}$$

of those p-adic classical bounded solid affinoids which admit a $(\mathbf{Q}_p)_{\square}$-algebra structure. Note that the latter algebra is idempotent over $\mathbf{Z}_{\square}$, such that this category forms a full subcategory of all classical bounded solid affinoids.

Definition 3.1.6.1. We will say a map $A \to B$ of p-adic classical bounded solid affinoid rings is a *naive syntomic cover*, if it can be refined by a map which lives in the smallest class stable under composition and pullbacks generated by maps of the form

$$\mathbf{Q}_p\langle x_i | i \in I\rangle \to \mathbf{Q}_p\langle x_i^{\frac{1}{p^{\infty}}} | i \in I\rangle$$

for some set I.

We now equip $(\mathcal{B}nd^{\mathrm{cl}})_{(\mathbf{Q}_p)_{\square}/}$ with the naive syntomic topology.

Proposition 3.1.6.2. *The subcategory*

$$\mathcal{S}emi\mathcal{P}erfd_{(\mathbf{Q}_p)_\square/} \subset (\mathcal{B}nd^{cl})_{(\mathbf{Q}_p)_\square/}$$

forms a basis for the naive syntomic topology.

Proof. Consider a p-adic classical bounded solid affinoid A. Then we need to produce a naive syntomic cover

$$A \to B$$

such that all terms $\otimes_A^n B$ in the Čech nerve are semiperfectoid. This can be done the same way as in the integral case: Consider a π_0-surjection

$$\mathbf{Q}_p\langle x_i | i \in I\rangle \to A$$

and produce the pushout

$$\begin{tikzcd}
\mathbf{Q}_p\langle x_i | i \in I\rangle \arrow[r] \arrow[d] & A \arrow[d] \\
\mathbf{Q}_p\langle p^{\frac{1}{p^\infty}}, x_i^{\frac{1}{p^\infty}} | i \in I\rangle \arrow[r] & B.
\end{tikzcd}$$

Then the right vertical map in the square defines such a cover. $\square$

3.2. Analytifying the Prismatisation

In [**Dri20**] [**BL22a**] [**BL22b**], Drinfeld and Bhatt-Lurie define stacks whose quasi-coherent cohomology computes prismatic cohomology in the sense of [**BS19**]. The goal of this section is to produce an analytic stack out of their construction in such a way that one obtains the expected geometric behavior.

The idea will first be to consider a "compact" version of this analytic stack. Here, essentially any map obtained will be cohomologically proper, which, for example, simplifies the question of $\mathcal{D}_{qc}$-covers, and we define a "decompactification" afterward whose geometry will behave more as we would like.

3.2.1. Recollections on the prismatisation.

We start by recalling the prismatisation as constructed in [**BL22b**].

Construction 3.2.1.1. For a (derived) p-adic formal scheme, the prismatisation is defined as an object of

$$(\mathbf{Z}_p^{\mathbb{\Delta}})_{fdesc}$$

via so-called transmutation. That is, for the affine line $\mathbf{G}_a$, we define the ringed stack $\mathbf{G}_a^{\mathbb{\Delta}}$, which takes a point $\mathrm{Spec}(R) \to \mathbf{Z}_p^{\mathbb{\Delta}}$ corresponding to a Cartier-Witt divisor $I \to \mathrm{W}(R)$ to the animated ring $\mathrm{W}(R)/I$.

Now for general X, we define $X^{\mathbb{\Delta}}$ via the assignment

$$R \mapsto \mathrm{Hom}_{\mathrm{Spf}(\mathbf{Z}_p)}(\mathrm{Spec}(\mathbf{G}_a^{\mathbb{\Delta}}(R)), X).$$

Remark 3.2.1.2. Note that $\mathbf{Z}_p^{\mathbb{\Delta}}$ is not just p-adically complete but also "I-adically". As can, for example, be seen by observing that the map

$$\mathbf{Z}_p^{\mathbb{\Delta}} \to \mathbf{A}^1/\mathbf{G}_m$$

coming from point wise scalar extending along the projection $\mathrm{W}(R) \to R$, factors through $\widehat{\mathbf{A}^1}/\mathbf{G}_m$. Actually, the reader should think of the prismatisation of $\mathbf{Z}_p$ as the initial prism, which gives an intuitional reason to work in the ambient category $(\mathbf{Z}_p^{\mathbb{\Delta}})$. The necessity of doing this manifests in the fact that the fiber products in this category will be compatible with the fiber products we take in the category of analytic stacks.

Definition 3.2.1.3. Let us write $\mathcal{O}(-1) \to \mathcal{O}$ for the universal generalized Cartier divisor on $\mathbf{A}^1/\mathbf{G}_m$. Then pulling back the line bundle $\mathcal{O}(-1)$ via the map

$$\mathbf{Z}_p^{\mathbb{\Delta}} \to \mathbf{A}^1/\mathbf{G}_m$$

defines a line bundle on $\mathbf{Z}_p^{\mathbb{\Delta}}$ which we will denote by $\mathcal{O}\{1\}$ an call the *Breuil-Kisin twist*. We will denote several pullbacks of this line bundle in the same way.

Let us fix a derived affine p-adic formal scheme $\mathrm{Spf}(S)$ and understand the following objects in the category

$$\mathcal{D}_{fdesc}(\mathrm{Spf}(S), \mathrm{W}(S))$$

of sheaves in the formally descendable topology on this affine formal scheme with values in $\mathrm{W}(S)$-modules. The following lemma will also be used later on.

Lemma 3.2.1.4. Given an affine p-adic formal scheme $\mathrm{Spf}(S)$ and a finite projective $\mathrm{W}(S)$-module P we have

$$R\Gamma_{fdesc}(S, P \otimes_{\mathrm{W}(S)} \mathrm{W}) \in \mathcal{D}(\mathrm{W}(S))_{\geq 0}$$

and the same holds for the truncated Witt vectors.

Proof. Writing P as a summand of a finite free $\mathrm{W}(S)$-module this reduces to showing $R\Gamma(S, \mathrm{W}) \in \mathcal{D}(\mathrm{W}(S))_{\geq 0}$. Furthermore writing W as a limit of the truncated Witt Vectors and observing that the transition maps are surjective, the Milnor sequence tells us that this reduces to the claim

$$R\Gamma(S, \mathrm{W}_n) \in \mathcal{D}(\mathrm{W}(S))_{\geq 0}.$$

This follows by induction on n using the fiber sequences $\mathrm{W}_{n-1} \to \mathrm{W}_n \to \mathbf{G}_a$ coming from the Verschiebung and the corresponding claim for $\mathbf{G}_a$ 2.4.1.9. $\square$

Corollary 3.2.1.5. *For a Cartier-Witt divisor $I \to W(S)$, we have*

$$R\Gamma_{fdesc}(S, \mathbf{G}_a^{\mathbb{A}}) \in \mathcal{D}(\,W(S))_{\geq 0}$$

and the latter admits the structure of an animated ring.

Proof. The first claim follows from 3.2.1.4 and the cofiber sequence

$$I \otimes_{\mathrm{W}(S)} \mathrm{W} \to \mathrm{W} \to \overline{\mathrm{W}}$$

of sheaves on $(S)_{fdesc}$ with values in $\mathcal{D}(\mathrm{W}(S))$. For the second claim, note that the Cartier-Witt divisor can be seen as a map $\mathrm{Spec}(\mathrm{W}(S)) \to \mathbf{A}^1/\mathbf{G}_m$ and we can pull back this map to $\mathbf{BG}_m$. $\square$

Construction 3.2.1.6. Given a derived p-adic formal scheme X together with an animated prism $I \to A$ and a map $\mathrm{Spf}(\overline{A}) \to X$ (i.e. an object in the derived prismatic site of X), there is a map

$$\mathrm{Spf}(A) \to X^{\mathbb{A}}$$

where we equip A with the (p, I)-adic topology. Let us recall how to obtain this map following [**BL22a**][3.2.4] and [**BL22b**][3.10]. Given a p-nilpotent animated ring R then any map $f\colon A \to R$ factors as $pr \circ \tilde{f}\colon A \to \mathrm{W}(R) \to R$, where $\tilde{f}$ is a map of δ-rings. Thus the base change along $\tilde{f}$

$$I \otimes_A \mathrm{W}(R) \to \mathrm{W}(R)$$

defines a generalized Cartier divisor, which gives a Cartier-Witt divisor on R as the image of $I + (p)$ becomes nilpotent in R by assumption. Furthermore we also have a map $\mathrm{Spf}(\overline{\mathrm{W}(R)}) \to \mathrm{Spf}(\overline{A}) \to X$.

Example 3.2.1.7. Given a integral perfectoid R with corresponding perfect prism (A, I), then 3.2.1.6 gives a map

$$\mathrm{Spf}(A) \to R^{\mathbb{A}}.$$

As explained in [**BL22b**][3.12], this map is an isomorphism of functors.

Proposition 3.2.1.8. *The functor*

$$(\_)^{\Delta}\colon f\mathcal{S}ch_{Spf(\mathbf{Z}_p)} \to (\mathbf{Z}_p^{\Delta})_{fdesc}$$

preserves finite limits and sends the following types of covers to completely descendable covers.

- *Étale covers.*
- *Naive syntomic covers.*

Proof. For the case of étale covers, we consult [**BL22b**][3.9] to see that the prismatisation preserves étale covers. Then the claim follows from 2.4.1.8.

Note that the assertion is stable under pullbacks and compositions, so we have to check the claim for a universal naive syntomic cover. Now we closely follow [**BL22b**][6.3]. Let us write $g\colon \mathrm{Spf}(B) \to \mathrm{Spf}(A)$ for such a cover. We will use two facts about this cover:

- The cofibre of the map $A \to B$ is a free A module.
- The p-completed cotangent complex $\mathbf{L}_{B/A}[1]$ is a free B-module.

Note that both of these conditions are stable under base change, so we have to check them for a cover of the form

$$\mathrm{Spf}(\mathbf{Z}[x_i^{\frac{1}{p^\infty}}|i \in I]_{\widehat{(p)}}) \to \mathrm{Spf}(\mathbf{Z}[x_i|i \in I]_{\widehat{(p)}}).$$

Then the first claim is clear, and the second follows, as mod p, we have a (co)fiber sequence

$$\mathbf{L}_{\mathbf{F}_p[x_i]/\mathbf{F}_p} \otimes_{\mathbf{F}_p[i]} \mathbf{F}_p[x_i]_{perf} \to \mathbf{L}_{\mathbf{F}_p[x_i]_{perf}/\mathbf{F}_p} \to \mathbf{L}_{\mathbf{F}_p[x_i]_{perf}/\mathbf{F}_p[x_i]}$$

where the left term is free in the set I, and the middle term vanishes by perfectness, from the derived Nakayama.

Let us now use these facts to prove what we want. For this we take a point $f\colon \mathrm{Spec}(R) \to A^{\Delta}$ and want to lift it to a point $\mathrm{Spec}(T) \to B^{\Delta}$ along a completely descendable cover $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$. The point f corresponds to a Cartier-Witt divisor $\alpha\colon I \to \mathrm{W}(R)$ and a map $\tilde{f}\colon R \to \overline{\mathrm{W}(R)}$. Now pushing out g along f gives a map $g'\colon \overline{\mathrm{W}(R)} \to C$ and the proof of [**BL22b**][2.17] shows that the map

$$\mathrm{W}(R) \to \Delta_{C/\mathrm{W}(R)}$$

is a map of δ-rings which mod I factors as $\overline{\mathrm{W}(R)} \to C \to \overline{\Delta}_{C/\mathrm{W}(R)}$. To obtain T, we now take the pushout

$$\begin{tikzcd}
\mathrm{W}(R) \arrow[r] \arrow[d] & \Delta_{C/\mathrm{W}(R)} \arrow[d] \\
R \arrow[r] & T
\end{tikzcd}$$

and note that, by adjunction, the right vertical map factors through a map of δ-rings $\Delta_{C/\mathrm{W}(R)} \to \mathrm{W}(T) \to T$ such that the composition $\mathrm{W}(R) \to \mathrm{W}(T)$ is the induced map on Witt Vectors. Putting all the data together, we have produced a Cartier-Witt divisor $\alpha\colon I \otimes_{\mathrm{W}(R)} \mathrm{W}(T) \to \mathrm{W}(T)$ and a map $B \to \overline{\mathrm{W}(T)}$. This gives the desired lift $\mathrm{Spec}(T) \to B^{\Delta}$ and we are left to show that the map

$$\mathrm{W}(R) \to \Delta_{C/\mathrm{W}(R)}$$

is a completely descendable cover.

This we can prove Zariski locally on $\mathrm{W}(R)$ by the first part, and can thus assume that I is a free module. Let us write F for the fiber of the map; then we will check that

$$\mathrm{Hom}(F, \mathrm{W}(R)) \in \mathcal{D}_{I\text{-}comp}(\mathrm{W}(R))_{\geq 1}.$$

By completeness, we can check this mod I, where the map in question factors as

$$\overline{\mathrm{W}(R)} \to C \to \overline{\Delta}_{C/\mathrm{W}(R)}.$$

Now the target can be written as the colimit of its conjugate filtration [**BL22a**][4.1.7]

$$C \to Fil_1^{conj} \Delta_{C/\mathrm{W}(R)} \to Fil_2^{conj} \Delta_{C/\mathrm{W}(R)} \to Fil_3^{conj} \Delta_{C/\mathrm{W}(R)} \to \dots$$

and the second map can be seen as the structure map into this colimit. This filtration has graded pieces given by

$$gr_i^{conj} \Delta_{C/\mathrm{W}(R)} \simeq \wedge^i \mathbf{L}_{C/\mathrm{W}(R)}[i]$$

Based on the second of the abovementioned observations, these are free C-modules. Using this, one computes that the cofibre of the second map is given by

$$\bigoplus_{i\geq 1} \wedge^i \mathbf{L}_{C/\mathrm{W}(R)}[i]$$

which is a free C-module. In particular, $\overline{\Delta}_{C/\mathrm{W}(R)}$ is a free C-module, and the algebra map picks out a basis element. By the first of the two observations made in the beginning, the same also holds for the map $\overline{\mathrm{W}(R)} \to C$, such that in total, we get that the fiber of the composition has the form

$$\bigoplus_{J} \overline{\mathrm{W}(R)}[1]$$

for some set J. The claim now follows as $\overline{\mathrm{W}(R)}$ is connective. □

Construction 3.2.1.9. Consider an animated prism $I \to A$, then as in 3.2.1.6 we can construct a map $\mathrm{Spf}(A) \to \overline{A}^{\Delta}$. Using this, for any p-adic formal scheme X over $\mathrm{Spf}(\overline{A})$, we define the *relative Prismatisation* as the fibre product

$$\begin{tikzcd}
(X/A)^{\Delta} \arrow[r] \arrow[d] & X^{\Delta} \arrow[d] \\
\mathrm{Spf}(A) \arrow[r] & \overline{A}^{\Delta}
\end{tikzcd}$$

where the right vertical map comes from the structure map $X \to \mathrm{Spf}(\overline{A})$.

Example 3.2.1.10. Given an integral perfectoid R with corresponding perfect prism (A, I). Then for any derived p-adic formal scheme X over $\mathrm{Spf}(R)$, using 3.2.1.7 we see that the map

$$(X/A)^{\Delta} \to X^{\Delta}$$

is an isomorphism of functors.

3.2.1.11. In the upcoming statement, we will make use of the notion of derived algebras in the sense of Bhatt-Mathew. These objects form a category $\mathcal{DA}lg$, which should be understood as a generalization of animated rings to the non-connective setting [**Rak20**][4]. That is, any derived algebra has an underlying object in $\mathcal{D}(\mathbf{Z})$ and animated rings identify with those derived algebras whose underlying derived abelian group is connective. Furthermore, any derived algebra has an underlying $\mathcal{E}_\infty$-ring in $\mathcal{D}(Z)$, and this forgetful functor commutes with all limits and colimits.

The example of such an object we have in mind is the relative prismatic cohomology

$$\mathbb{\Delta}_{S/A}$$

of an animated $\overline{A}$-algebra where $I \to A$ is an animated prism. This forms a derived A-algebra.

Definition 3.2.1.12. Given an animated prism $I \to A$ and an animated $\overline{A}$-algebra S, we write

$$\mathrm{Spf}(\mathbb{\Delta}_{S/A}) := \mathrm{Hom}_{\mathcal{DA}lg_A}(\mathbb{\Delta}_{S/A}, \_)$$

seen as an object in $\mathrm{Spf}(A)_{fdesc}$.

Let us recall the following theorem from [**Hol24**][3.3.14+3.3.7].

Theorem 3.2.1.13. *(Holeman) Given a prism (A, I) and a derived affine p-adic formal scheme $Spf(S) \to Spf(\overline{A})$. Then, there is a canonical isomorphism*

$$Spf(\mathbb{\Delta}_{S/A}) \simeq (S/A)^{\mathbb{\Delta}}$$

of functors on affine derived formal schemes adic over $Spf(A)$.

Remark 3.2.1.14. In [**BL22b**][7.17], the authors obtain a similar statement. Their argument would work in our setting as well using a descendability result from [**BS19**][8.6] but would give a minimal weaker statement. We would get the stated isomorphism in $\mathrm{Spf}(A)_{fdesc}$.

Corollary 3.2.1.15. *Consider a prism (A, I) and an affine p-adic formal scheme $Spf(S)$ such that $\Omega^1_{(\pi_0 S/p)/(\bar{A}/p)} \simeq 0$. Then $\mathbb{\Delta}_{S/A}$ is an animated A-algebra and corepresents $(S/A)^{\mathbb{\Delta}}$ as an object of $Spf(A)_{fdesc}$.*

Proof. As in [**BL22b**][7.18] this follows from 3.2.1.13 and the Hodge-Tate comparison. $\square$

Example 3.2.1.16. Let S be a semiperfectoid animated ring. Then the frobenius on $\pi_0(S/p)$ is surjective and thus induces a surjective map on $\Omega^1_{\pi_0(S/p)/\mathbf{F}_p}$. As it also induces the 0-map we see that $\Omega^1_{\pi_0(S/p)/\mathbf{F}_p} \simeq 0$ and as this module surjects onto

$$\Omega^1_{\pi_0(S/p)/R/p} \simeq 0$$

we see that also the latter vanishes.

Now applying 3.2.1.15 we obtain an animated algebra $\mathbb{\Delta}_S := \mathbb{\Delta}_{S/A}$ (where A denotes any perfect prism with corresponding integral perfectoid R mapping to

S [**BL22a**][4.4.12]). By comparing fibers and applying the other part of 3.2.1.15 we see that there is a cartesian square

$$\begin{tikzcd} \mathrm{Spf}(\Delta_S) \arrow[r] \arrow[d] & S^{\Delta} \arrow[d] \\ \mathrm{Spf}(A) \arrow[r] & R^{\Delta} \end{tikzcd}$$

in $(\mathbf{Z}_p^{\Delta})_{fdesc}$. As the lower horizontal map is an iso 3.2.1.10, the upper horizontal map is an iso as well. This shows that the Prismatisation of a semiperfectoid is affine and represented by its prismatic cohomology.

3.2.2. Recollections on the Hodge-Tate locus.

Construction 3.2.2.1. Recall from 3.2.1.2 that there is a map

$$\mathbf{Z}_p^{\Delta} \to \widehat{\mathbf{A}^1}/\mathbf{G}_m$$

and we will denote by $\mathbf{Z}_p^{HT}$ the fibre of this map over $\mathbf{BG}_m$. For a general derived p-adic formal scheme X, we define the so-called *Hodge-Tate* stack of X via the cartesian square

$$\begin{tikzcd} X^{HT} \arrow[r] \arrow[d] & X^{\Delta} \arrow[d] \\ \mathbf{Z}_p^{HT} \arrow[r] & \mathbf{Z}_p^{\Delta}. \end{tikzcd}$$

Note that these stacks no longer carry the "I-adic" topology and thus should be seen as "just" p-adic formal stacks. Concretely, we mean that we will most of the time understand them as objects in $\mathrm{Spf}(\mathbf{Z}_p)_{fdesc}$.

One can understand the geometry of the Hodge-Tate stack quite well. In order to make use of this later, we need to recall some aspects of these stacks. For this, we will essentially copy [**BL22b**][5], but taking care that we work in a slightly different topology. More concretely, note that the Hodge-Tate stacks are defined via transmutation using the ring stack

$$\mathbf{G}_a^{\Delta} \to \mathbf{Z}_p^{\Delta}$$

restricted to the Hodge-Tate locus, and understanding the Hodge-Tate stacks means understanding this ring stack better.

3.2.2.2. Let $\mathbf{G}_a^{\sharp}$ be the PD-hull of the origin in $\mathbf{G}_a$ over $\mathbf{Z}$. Concretely we have

$$\mathbf{G}_a^{\sharp} \simeq \mathrm{Spec}(\mathbf{Z}[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots]).$$

Note that there is a map $\mathbf{G}_a^{\sharp} \to \mathbf{G}_a$ and that the equalities

- $\frac{(x+y)^n}{n!} = \sum_{i+j=n} \frac{x^i}{i!}\frac{y^j}{j!}$
- $\frac{(xy)^n}{n!} = \frac{x^n}{n!} \cdot y^n$

show that the additive group action on $\mathbf{G}_a$ induces an action on $\mathbf{G}_a^{\sharp}$ and the the multiplicative group action on $\mathbf{G}_a$ induces via the above map an action on $\mathbf{G}_a^{\sharp}$

which makes it into a $\mathbf{G}_a$-module scheme. In particular we can understand $\mathbf{G}_a^\sharp$ as a W-module scheme via the projection $\mathrm{W} \to \mathbf{G}_a$ and thus as an object in

$$\mathcal{D}_{fdesc}(\mathrm{Spf}(S), \mathrm{W}(S)).$$

Lemma 3.2.2.3. [**Dri20**][3.4] [**Bha22**][2.6.1] [**BL22a**][3.4.11] The Frobenius $F\colon \mathrm{W} \to F^*\mathrm{W}$ is a π_0 surjection in the completely descendable topology. Furthermore, if we write $\mathrm{W}[F]$ for the fibre of the Frobenius, the composition $\mathrm{W}[F] \subset \mathrm{W} \to \mathbf{G}_a$ of the inclusion with the projection lifts uniquely to an isomorphism

$$\mathrm{W}[F] \simeq \mathbf{G}_a^\sharp.$$

In particular, there is a fiber sequence

$$\mathbf{G}_a^\sharp \to \mathrm{W} \to F_*\mathrm{W}$$

in $\mathcal{D}_{fdesc}(\mathrm{Spf}(S), \mathrm{W}(S))$ where the second map is given by the Frobenius.

Proof. The proof of [**Bha22**][2.6.1] shows that the Frobenius is faithfully flat; thus it is descendable by [**Mat16**][3.31] as the domain is countable. This shows π_0-surjectivity.

For the rest, note that the ordinary scheme represents $\mathrm{W}[F]$ also on derived schemes

$$\mathrm{Spec}(\mathbf{Z}_{(p)}[x_0, x_1, x_2, \dots]/(x_0^p + px_1, x_1^p + px_2, \dots))$$

and the proof of [**Bha22**][2.6.1] gives an isomorphism $\mathrm{W}[F] \simeq \mathbf{G}_a^\sharp$ of representing objects. □

Lemma 3.2.2.4. [**BL22b**][5.7] For any affine p-adic formal scheme $\mathrm{Spf}(S)$ and any finite projective $\mathrm{W}(S)$-module P there are fibre sequences

- $R\Gamma_{fdesc}(\mathrm{Spf}(S), P \otimes_{\mathrm{W}} \mathrm{W}_n[F]) \to P \otimes_{\mathrm{W}} \mathrm{W}_n(S) \to P \otimes_{\mathrm{W}} F_*\mathrm{W}_{n-1}(S)$ $(n \geq 2)$.
- $R\Gamma_{fdesc}(\mathrm{Spf}(S), P \otimes_{\mathrm{W}} \mathrm{W}[F]) \to P \otimes_{\mathrm{W}} \mathrm{W}(S) \to P \otimes_{\mathrm{W}} F_*\mathrm{W}(S)$

in $\mathcal{D}(\mathrm{W}(S))$.

Proof. We claim that these fiber sequences arise as the global sections of the fiber sequence

$$\mathrm{W}[F] \to \mathrm{W} \to F_*\mathrm{W}$$

tensored with P (and the analog fiber sequence with the truncated Witt Vectors). That is the essential claim is that $R\Gamma(\mathrm{Spf}(S), P \otimes_{\mathrm{W}(S)} \mathrm{W}) \in \mathcal{D}(\mathrm{W}(S))_{\geq 0}$. This was explained in 3.2.1.4. □

3.2.2.5. Let us write

$$\mathbf{G}_a^\sharp\{1\} \in (\mathbf{Z}_p^{HT})_{fdesc}$$

for the sheaf which takes a Hodge-Tate divisor $I \to \mathrm{W}(S)$ on $\mathrm{Spec}(S)$ to $\mathbf{G}_a^\sharp(R)\{1\}$ where the Breuil-Kisin twist was defined in 3.2.1.3. This defines a group object and we will write $\mathbf{BG}_a^\sharp\{1\}$ for the sheaf of torsors on it. Not that using 3.2.2.4 and 3.2.2.3 we obtain an identification

$$\mathbf{BG}_a^\sharp\{1\} \simeq R\Gamma(\_, \mathrm{W}[F]\{1\}[1])$$

where the twist $\{1\}$ on the right is defined analogously.

3.2.2.6. In the following argument, we will use the fact that there exists a basis in the descendable topology on static rings (we can also assume flatness), for with the Frobenius maps

$$F\colon \mathrm{W}_n(S) \to F_*\mathrm{W}_{n-1}(S)$$

are surjective for all $n \geq 2$. Let us explain here why this is true. In [**DK14**][3.2] the authors explain that for this surjectivity to hold it is enough that the Frobenius $F\colon \mathrm{W}(S) \to F_*\mathrm{W}(S)$ on the whole, Witt Vectors hit all multiplicative lifts $[r]$ of elements $s \in S$. As we have the formula $F([r]) = [r^p]$ this is true for any ring admitting all p-th roots of its elements, such that our claim follows from 2.4.1.10.

Let us write $\mathbf{G}_a^{HT}$ for the ring stack over $\mathbf{Z}_p^{HT}$ obtained by pulling back $\mathbf{G}_a^{\mathbb{\Delta}}$. The following, then, is an absolute version of [**BL22b**][5.10] with the same proof.

Proposition 3.2.2.7. *The ring stacks $\mathbf{G}_a^{HT}$ is a square-zero extension of $\mathbf{G}_a$ by $\mathbf{BG}_a^{\sharp}\{1\}[-1]$. That is there exists a natural W-linear derivation $\partial\colon \mathbf{G}_a \to \mathbf{G}_a \oplus \mathbf{BG}_a^{\sharp}\{1\}$ fitting into a cartesian square*

$$\begin{tikzcd}
\mathbf{G}_a^{HT} \arrow[r] \arrow[d, "\pi^{HT}"'] & \mathbf{G}_a \arrow[d, "\partial"] \\
\mathbf{G}_a \arrow[r, "\partial_{triv}"'] & \mathbf{G}_a \oplus \mathbf{BG}_a^{\sharp}\{1\}
\end{tikzcd}$$

of ring stacks over $\mathbf{Z}_p^{HT}$.

Proof. First, all ring stacks in the square define sheaves for the completely descendable topology. For the lower right corner, this follows from the identification 3.2.2.5, for the left upper corner from the same argument as 3.2.1.5, and for $\mathbf{G}_a$ from 2.4.1.9. That means to show the claim for the values on an animated Hodge-Tate divisor $\alpha\colon I \to \mathrm{W}(S)$, we can resolve $\mathbf{Z}_p^{HT}$ by (ordinary) prisms and show the claim on the induced cover of Spec(S). Concretely, we can, for example, take the covering from 3.1.3.10 then all objects appearing in the Čech nerve are represented by (ordinary) prisms [**BL22a**][3.2.8+3.2.10]. This reduces the claim to the situation relative to a prism, in which case the argument is given in [**BL22b**][5.10]. For the convenience of the reader, we recall this argument now.

We fix a prism (A, I). Using 3.2.2.5 we can replace $\mathbf{BG}_a^{\sharp}\{1\}$ by $R\Gamma(\_, \mathrm{W}[F]\{1\}[1])$ and by taking limits it suffices to prove the corresponding claim for $R\Gamma(\_, \mathrm{W}_n[F]\{1\}[1])$ functorial in n.

We first prove this claim evaluated at an discrete $\overline{A}$-algebra S for which the Frobenius maps

$$F\colon \mathrm{W}_n(S) \to F_*\mathrm{W}_{n-1}(S)$$

are surjective for all $n \geq 2$. To see this, we first claim:

(∗) The map $\alpha\colon I \otimes_{\mathrm{W}(S)} \mathrm{W}_n(S) \to \mathrm{W}_n(S)$ maps surjectivity onto $V\mathrm{W}_n(S)$.

Using this claim, we can compute

$$\pi_0\overline{\mathrm{W}_n}(S) \simeq \mathrm{W}_n(S)/V\mathrm{W}_n(S) \simeq S$$

and

$$\pi_1\overline{\mathrm{W}_n}(S) \simeq ker(I \otimes_{\mathrm{W}(S)} \mathrm{W}_n(S) \to \mathrm{W}_n(S)) \simeq I \otimes_{\mathrm{W}(S)} \mathrm{W}_n[F](S).$$

So the claim of the proposition, in this case, follows as any 1-truncated animated ring is naturally a square zero extension of its π_0 by its $\pi_1[1]$ 2.4.2.9. To see $(*)$ we can work zariski locally on $\mathrm{W}(S)$ and thus assume α corresponds to a distinguished element $d = (x_0, x_1, \dots) \in \mathrm{W}(S)$. The assumption that the Cartier-Witt divisor lives in the Hodge-Tate stack tells us that $x_0 = 0$, such that $d = V(u)$ where u is a unit as d is distinguished. Thus the formula $V(u) \cdot \underline{x} = V(u \cdot F(\underline{x}))$ shows that α maps into the image of the Verschiebung. On the other hand, as the Frobenius is surjective, we can write any $\underline{y} \in \mathrm{W}_{n+1}(S)$ as $\underline{y} = u \cdot F(\underline{x})$ for some $\underline{x}$ and the surjectivity follows from the equality

$$V(\underline{y}) = V(u \cdot F(\underline{x})) = V(u) \cdot \underline{x}.$$

To deduce the claim of the proposition, we first deduce the claim for polynomial algebras over $\overline{A}$ by descent 3.2.2.6 using what we have done above, and that all functors in question are sheaves for the descendable topology. For a general animated $\overline{A}$-algebra, we observe that all functors in question are left Kan extended from polynomial algebras (for $R\Gamma(\_, \mathrm{W}_n[F]\{1\})[1]$ use 3.2.2.4). □

3.2.2.8. For any morphism $f\colon X \to S$ of p-adic formal schemes precomposing with the map $\pi_{HT}\colon \mathbf{G}_a^{HT} \to \mathbf{G}_a$ induces a map

$$\pi_f^{HT}\colon X^{HT} \to X \times_S S^{HT}$$

which we will call the *Hodge-Tate structure map* of f. Using 3.2.2.7, we can understand the geometry of this map in terms of the cotangent complex of f.

3.2.2.9. Given a square zero extension $\tilde{B} \to B$ along a connective B-module N. Then, for any morphism $X \to S$ of (derived formal) schemes and any point $\eta \in X(B) \times_{S(B)} S(\tilde{B})$ the fibre at η of the map

$$X(\tilde{B}) \to X(B) \times_{S(B)} S(\tilde{B})$$

gives a torsor over $\mathrm{Hom}_X(\mathbf{L}_f, N)$. Let us informally[4] describe the action. The point η corresponds to a commutative square like the outer square in the following diagram

$$\begin{tikzcd}
\mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_X \times_B \tilde{B} \arrow[r] \arrow[d] & \tilde{B} \arrow[d] \\
\mathcal{O}_X \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & \mathcal{O}_X \arrow[r] & B.
\end{tikzcd}$$

Thus, the fiber is given by the anima of dashed lifts in the square. This anima is equivalent to the dashed lifts in the left square, but now choosing such a lift gives an identification of $\mathcal{O}_X \times_B \tilde{B}$ with the trivial square zero extension. Via this identification, the anima of lifts, by definition, becomes the anima of $\mathcal{O}_S$-linear derivations in N, which is isomorphic to $\mathrm{Hom}_X(\mathbf{L}_f, N)$.

Notation 3.2.2.10. Consider a morphism $f\colon X \to S$ of p-adic formal schemes. Then, we will write

$$\mathbf{V}(\mathbf{L}_f\{1\})^\sharp \to X \times_S S^{HT}$$

for the bundle which takes a point $\mathrm{Spec}(R) \to X \times_S S^{HT}$ to

$$\mathrm{Hom}_R(\eta_X^* \mathbf{L}_f, \mathbf{G}_a^\sharp\{1\}(R))$$

[4]To make this coherent, we have to specify a point in $\mathbf{B}\,\mathrm{Hom}_X(\mathbf{L}_f, N) \simeq \mathrm{Hom}_X(\mathbf{L}_f, N[-1])$. But the trivial map does the job.

where η corresponds to the point in X and the Breuil-Kisin twist is defined via the point in S^{HT}. Note that this defines a group object. The following proposition is just a reformulation of [**BL22b**][5.12] in its natural generality.

Proposition 3.2.2.11. *Given a map $f\colon X \to S$ of derived p-adic formal schemes, the associated Hodge-Tate structure map*

$$\pi_f^{HT}\colon X^{HT} \to X \times_S S^{HT}$$

defined a gerbe baned by $\mathbf{V}(\mathbf{L}_f\{1\})^\sharp$.

Proof. We claim that for any point $\eta\colon \mathrm{Spec}(R) \to X \times_S S^{HT}$ the fibre of the map

$$X^{HT}(R) \to X(R) \times_{S(R)} S^{HT}(R)$$

at η defines a torsor over

$$\mathbf{BV}(\mathbf{L}_f)^\sharp(R) \simeq \mathrm{Hom}_R(\eta_X^*\mathbf{L}_f, \mathbf{BG}_a^\sharp(R)).$$

This follows by combining 3.2.2.9 and 3.2.2.7. □

Corollary 3.2.2.12. *For any map $f\colon X \to S$ of p-adic formal schemes, which has vanishing cotangent complex the square*

$$\begin{tikzcd} X^{HT} \arrow[r] \arrow[d, "\pi^{HT}"'] & S^{HT} \arrow[d, "\pi^{HT}"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

is Cartesian.

3.2.3. Locally proper prismatisation. We now come to the definition of the prismatisation as a locally proper analytic stack.

Construction 3.2.3.1. Using 2.5.1.1 and 3.1.3.10, we obtain a colimit and finite limit preserving functor

$$(\_)^{lp}\colon (\mathbf{Z}_p^{\Delta})_{fdesc} \to \mathcal{A}n\mathcal{S}tack_{\mathbf{Z}_p^{\Delta,lp}}.$$

Furthermore, we can precompose with the construction given in 3.2.1.1 and then base changing along the map

$$\mathrm{colim}_{n\in\mathbf{N}}\,\mathrm{Spa}(\mathbf{Z}/p^n) \to \mathrm{Spa}(\mathbf{Z}_p)$$

to obtain a finite limit preserving functor

$$(\_)^{\Delta,lp}\colon f\mathcal{S}ch_{\mathrm{Spf}(\mathbf{Z}_p)} \to \mathcal{A}n\mathcal{S}tack_{\mathbf{Z}_p^{\Delta,lp}}.$$

Definition 3.2.3.2. For a derived p-adic formal scheme X, we call the analytic stack

$$X^{\Delta,lp}$$

the *locally proper prismatisation* of X.

We have already proven the following.

Proposition 3.2.3.3. *The functor*

$$(\_)^{\Delta,lp}\colon f\mathcal{S}ch_{Spf(\mathbf{Z}_p)} \to \mathcal{A}n\mathcal{S}tack_{\mathbf{Z}_p^{\Delta,lp}}$$

from 3.2.3.1 preserves finite limits and sends étale covers, as well as naive syntomic covers, to locally proper surjections.

Proof. Combine 3.2.1.8 and 2.5.1.2. □

Let us unwind how to access these stacks.

3.2.3.4. For a derived p-adic formal scheme X, we want to find a presentation of the analytic stack

$$X^{\Delta,lp}.$$

By 3.2.3.3, we can find a presentation of X by affines using a Zariski cover. So let us assume $X \simeq \mathrm{Spf}(T)$ is affine. Now we choose a π_0 surjection to T from a polynomial ring and consider the pushout

$$\begin{tikzcd} \mathbf{Z}_p\langle x_I\rangle \arrow[r] \arrow[d] & T \arrow[d, "f"] \\ \mathbf{Z}_p\langle x_I^{\frac{1}{p^\infty}}, p^{\frac{1}{p^\infty}}\rangle \arrow[r] & S \end{tikzcd}$$

to obtain a naive syntomic cover of T. Thus, by the other part of 3.2.3.3, we see that the map

$$S^{\Delta,lp} \to T^{\Delta,lp}$$

gives us a (locally proper) surjection of analytic stacks. That is, to compute the target, we want to compute the Čech nerve of this map. To do this note that all the tensor products $\otimes_T^n S$ are semiperfectoid and thus fit into 3.2.1.16 which shows that

$$(\otimes_T^n S)^{\Delta} \simeq \mathrm{Spf}(\Delta_{\otimes_T^n S}).$$

So, unwinding the definitions, we have found a presentation

$$\mathrm{colim}_{\bullet\in\Delta}\,\mathrm{Spf}(\Delta_{\otimes_T^\bullet S}, \widetilde{\mathbf{Z}_p}) \simeq T^{\Delta,lp}$$

where we write

$$\mathrm{Spf}(A, A^+) := \mathrm{colim}_n\,\mathrm{Spa}(A/p^n, \widetilde{A^+/p^n}).$$

Let us record the following easy consequences.

Corollary 3.2.3.5. *The functor*

$$(\_)^{\Delta,lp}\colon f\mathcal{S}ch_{Spf(\mathbf{Z}_p)} \to \mathcal{A}n\mathcal{S}tack_{\mathbf{Z}_p^{\Delta,lp}}$$

respects the following geometries:

(a) Zariski open immersions (resp. covers) get sent to closed immersions (resp. covers).
(b) Zariski closed immersions get sent to affine maps of analytic stacks.
(c) Naive syntomic covers get sent to affine proper surjections.

PROOF. (a) directly follows from 3.2.3.3. To see (b) we take a closed immersion $Z \to X$. Now by 3.2.3.3 we can assume $X \simeq Spf(T)$ and thus also $Z \simeq \mathrm{Spf}(T')$ to be affine. But now, if we construct a naive syntomic cover as in 3.2.3.4 and consider the pushout

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f"'] & T' \arrow[d, "f'"] \\ S \arrow[r] & S', \end{tikzcd}$$

f' gives a naive syntomic cover as well and S' is semiperfect (i.e fits into 3.2.1.16). As we can check the claim after pulling back to S 3.2.3.3 we win by 3.2.1.16.

It suffices to check (c) in the universal case. Now by 3.2.3.3 we can check the claim after pulling back the cover along it selfs. If we do this in the pushout

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "f"'] & S \arrow[d] \\ S \arrow[r] & S \otimes_T S \end{tikzcd}$$

for f a universal naive syntomic cover, it is easy to see that $S \otimes_T S$ is semiperfectoid. So the claim follows from 3.2.1.16. $\square$

3.2.3.6. Consider a morphism of $f\colon X \to S$ of p-adic formal schemes. Then

$$f_*\mathcal{O}_{X^{\Delta,lp}} \in \mathcal{D}_{\mathrm{qc}}(S^{\Delta,lp})$$

defines an $\mathcal{E}_\infty$-algebra. Furthermore, we have a canonical symmetric monoidal functor

$$\mathcal{M}od_{f_*\mathcal{O}_{X^{\Delta,lp}}}(\mathcal{D}_{\mathrm{qc}}(S^{\Delta,lp})) \to \mathcal{D}_{\mathrm{qc}}(X^{\Delta,lp}).$$

Example 3.2.3.7. Consider a prism (A, I) and a p-adic formal scheme $f\colon X \to \mathrm{Spf}(A/I)$, then we claim that

$$f_*\mathcal{O}_{(X/A)^{\Delta,lp}} \in \mathcal{D}_{\mathrm{qc}}(\mathrm{Spf}(A)^{lp})$$

can be identified with the relative prismatic cohomology of X over A, understood as an $\mathcal{E}_\infty$-algebra. Note that both of these algebras arise as limits from Zariski descent and descent via naive syntomic covers, and the only subtlety is that classically this limit is taken in the (p, I)-complete category. But (p, I)-complete objects are stable under limits, and locally both algebras are (p, I)-complete as f_* commutes with limits.

3.2.3.8. Consider a semiperfectoid S and let us write $T = S[x_1, \ldots, x_n]$ for a polynomial ring over S and $T_\infty = S[x_1^{\frac{1}{p^\infty}}, \ldots, x_n^{\frac{1}{p^\infty}}]$ for the naive perfection. Then the map

$$f_*\mathcal{O}_{T^{\Delta,lp}} \to f_*^\infty\mathcal{O}_{T_\infty^{\Delta,lp}} \in \mathcal{D}_{\mathrm{qc}}(S^{\Delta,lp})$$

is descendable. In the (p, I)-completed category this easily follows from [**BS19**][8.6] but as the inclusion of (p, I)-complete objects is lax monoidal this is enough [**BS17**][11.20].

Proposition 3.2.3.9. *Consider a finite type map of affine p-adic formal schemes Spf$(T) \to$ Spf(S). Then the canonical functor*

$$\mathcal{M}od_{f_*\mathcal{O}_{T^{\Delta,lp}}}(\mathcal{D}_{qc})(S^{\Delta,lp}) \xrightarrow{\sim} \mathcal{D}_{qc}(T^{\Delta,lp})$$

is an isomorphism.

Proof. Choosing a naive syntomic cover of $\mathrm{Spf}(S)$ and using 3.2.3.5 with proper base change, we can assume S to be semiperfectoid. Furthermore, using 3.2.3.5[(b)], we can assume that $T = S[x_1, \ldots, x_n]$ is a finite polynomial ring. But in this case we can resolve $T \to T_\infty$ by the naive perfection and get

$$\mathcal{D}_{\mathrm{qc}}(T^{\Delta,lp}) \simeq \lim_{\Delta} \mathcal{D}_{\mathrm{qc}}((T_\infty^{\otimes_T^\bullet})^{\Delta,lp}).$$

But using 3.2.3.8 we see that the same limit computes $\mathcal{M}od_{f_*\mathcal{O}_{T^{\Delta,lp}}}(\mathcal{D}_{\mathrm{qc}})(S^{\Delta,lp})$. □

This has the following Corollary.

Corollary 3.2.3.10. *The functor*

$$(\_)^{\Delta,lp} \colon f\mathcal{S}ch_{Spf(\mathbf{Z}_p)} \to \mathcal{A}n\mathcal{S}tack_{\mathbf{Z}_p^{\Delta,lp}}$$

sends maps of p-adic formal schemes, which are quasi-compact and locally of finite type, to proper morphisms of analytic stacks.

Proof. Let us write $f \colon X \to S$ for such a morphism. We first assume that f is separated. Note that the problem is local on the target, so by 3.2.3.5 we can assume S to be affine and then semiperfectoid. Now the diagonal is a closed immersion and becomes proper by 3.2.3.5, so by 2.1.3.7 we have to find a finite cover of closed substacks of $X^{\Delta,lp}$ such that the restriction of the structure sheaf becomes compact. As f is quasi-compact and locally of finite type, we can find a finite Zariski cover of X such that the composition to S becomes finite type and affine. This cover does the job by 3.2.3.5 and 3.2.3.9. For a general morphism, we do the same argument again, now using that the diagonal is separated. □

3.2.4. The Hodge-Tate locus.

Construction 3.2.4.1. Given an derived p-adic formal scheme X, choosing a presentation of X^{Δ} as in 3.2.3.4 gives by base change a presentation of X^{HT}. In the case where $X \simeq \mathrm{Spf}(T)$ is affine one sees that the objects in the Čech nerve take the form

$$(\otimes_T^n S)^{HT} \simeq \mathrm{Spf}(\overline{\Delta}_{\otimes_T^n S})$$

and using the locally proper analytification

$$(\_)^{lp} \colon (\mathrm{Spf}(\mathbf{Z}_p))_{fdesc} \to \mathcal{A}n\mathcal{S}tack_{(\mathbf{Z}_p)_\square}$$

we get a presentation

$$\mathrm{colim}_{\bullet \in \Delta} \mathrm{Spa}(\overline{\Delta}_{\otimes_T^n S}, \widetilde{\mathbf{Z}_p}) \simeq T^{HT,lp}$$

in analytic stacks.

Note also that from this, it is easy to see that for any derived p-adic formal scheme X, there is a cartesian square

$$\begin{tikzcd} X^{HT,lp} \arrow[r] \arrow[d] & X^{\Delta,lp} \arrow[d] \\ \mathbf{Z}_p^{HT,lp} \arrow[r] & \mathbf{Z}_p^{\Delta,lp} \end{tikzcd}$$

of analytic stacks.

Definition 3.2.4.2. For a derived p-adic formal scheme X, we will call the analytic stack

$$X^{HT,lp}$$

the *locally proper Hodge-Tate stack*.

Remark 3.2.4.3. The results for the Hodge-Tate locus work without completing into the p-direction. That is why we work with this version of the Hodge-Tate stack. This is not necessary for the proof of the main theorem.

Construction 3.2.4.4. Recall from 2.5.4.3 that for any p-adic formal scheme X, we have a map

$$X^{\mathrm{an}} \to X^{lp}$$

natural in X. Using the Hodge-Tate structure map, we can build the Cartesian square

$$\begin{tikzcd} X^{HT,\square} \arrow[r] \arrow[d] & X^{HT,lp} \arrow[d, "\pi^{HT}"] \\ X^{\mathrm{an}} \arrow[r] & X^{lp}. \end{tikzcd}$$

This produces a functor

$$(\_)^{HT,\square} \colon f\mathcal{S}ch_{\mathrm{Spf}(\mathbf{Z}_p)} \to \mathcal{A}n\mathcal{S}tack_{\mathbf{Z}_p^{HT,\square}}.$$

Definition 3.2.4.5. For a p-adic formal scheme X, we will call the analytic stack

$$X^{HT,\square}$$

the *solid Hodge-Tate* stack associated to X.

Proposition 3.2.4.6. *The functor from 3.2.4.4 preserves finite limits. Furthermore for a map $f\colon X \to S$ of p-adic formal schemes we have the following cases:*

If f is étale, then we have a cartesian square

$$\begin{tikzcd} X^{HT,\square} \arrow[r] \arrow[d] & S^{HT,\square} \arrow[d] \\ X^{an} \arrow[r] & S^{an} \end{tikzcd}$$

of analytic stacks. In particular, assigning the Hodge-Tate stack preserves open immersion (resp. open coverings) as well as étale morphisms (resp. étale coverings).

If f is either proper or affine integral, then we have a Cartesian square

$$\begin{tikzcd} X^{HT,\square} \arrow[r] \arrow[d] & X^{HT,lp} \arrow[d] \\ S^{HT,\square} \arrow[r] & S^{HT,lp} \end{tikzcd}$$

of analytic stacks. In particular, assigning the Hodge-Tate stack preserves proper morphisms, and any naive syntomic cover becomes a proper covering.

Proof. For the preservation of finite limits, note that all three corners in the defining cospan preserve fibre products 2.5.2.3 2.5.1.2 3.2.4.1.

For the two Cartesian squares, consider the cube

$$\begin{tikzcd}
X^{HT,\square} \arrow[rr] \arrow[dd] \arrow[dr] & & X^{HT,lp} \arrow[dd] \arrow[dr] & \\
& X^{\mathrm{an}} \arrow[rr] \arrow[dd] & & X^{lp} \arrow[dd] \\
S^{HT,\square} \arrow[rr] \arrow[dr] & & S^{HT,lp} \arrow[dr] & \\
& S^{\mathrm{an}} \arrow[rr] & & S^{lp}
\end{tikzcd}$$

where the upper and the lower squares are Cartesian by definition.

Now, in the case of f being étale, the square on the right is cartesian by 3.2.2.12 and 2.5.1.2. Thus, the square on the left is Cartesian as well.

In the case of f being proper, the square in the front is Cartesian by 2.5.4.6, which shows that the square in the back is Cartesian as well.

The "in particular" part for the first square now follows from 2.5.2.3 and 2.5.2.4. For the second square, we use 3.2.3.10 and 3.2.3.5. □

Example 3.2.4.7. Let us be given an integral perfectoid R with corresponding prism (A, I). Then, the Hodge-Tate structure map

$$R \to A/I \simeq R$$

is the identity. In particular, using 3.2.4.1 we see that

$$R^{HT,\square} \simeq \mathrm{Spa}(R).$$

Furthermore, if S is a semiperfectoid, then using 3.2.1.16 and 3.2.4.1 we see that $S^{HT,\square}$ is represented by an analytic ring whose underlying condensed rings is given by $\overline{\Delta}_S$ (with it's p-adic topology). Now we can choose a map $R \to S$ from an integral perfectoid, which is surjective on π_0. Thus the integral elements just depend on the later 2.2.1.9, we further deduce that

$$\mathcal{D}_{\mathrm{qc}}(S^{HT,\square}) \simeq \mathcal{M}od_{\overline{\Delta}_S}(\mathcal{D}_{\mathrm{qc}}(\mathrm{Spa}(R))).$$

3.2.5. The étale locus. Recall from [**BS19**][9.1] the prismatic cohomology of a p-adic formal scheme recovers the étale cohomology of the generic fibre. In the following, we will give an interpretation of this étale part on the level of stacks.

Construction 3.2.5.1. Consider a semiperfectoid bounded solid affinoid $(\mathbf{Q}_p)_\square$-algebra A. Then by 3.1.5.11 this algebra admits a perfectoidisation A_{perfd}, which by 3.1.5.6 corresponds to a triple

$$(R, R^+, I \subset \mathrm{W}(R^+))$$

consisting of an affinoid perfectoid in characteristic p and a (perfect) prism structure. To this data, we can associate the analytic stack

$$\mathcal{X}_A := \mathrm{colim}_n \, \mathrm{Spa}((\mathrm{W}(R^+)[\frac{1}{I}])/p^n, \widetilde{\mathrm{W}(R^+)}/p^n).$$

Note that the frobenius $\varphi\colon \mathrm{W}(R^+) \to \mathrm{W}(R^+)$ modulo p sends I to I^p. From this, one sees that it induces an endomorphism of the just constructed analytic stack, and we set

$$A^{\text{ét}} := \mathcal{X}_A/\varphi^{\mathbf{Z}}$$

to be the quotient stack of this action.

In total, we have produced a functor

$$(\_)^{\text{ét}} \colon (\mathcal{S}emi\mathcal{P}erfd_{(\mathbf{Q}_p)_\square/})^{\text{op}} \to \mathcal{A}n\mathcal{S}tack.$$

Remark 3.2.5.2. The category of quasi-coherent sheaves of $A^{\text{ét}}$ is given by the full subcategory of derived p-complete objects in the equalizer of the diagram

$$\mathcal{D}_{\text{qc}}(\text{Spa}(\text{W}(R^+)[\tfrac{1}{I}], \widetilde{\widehat{\text{W}(R^+)}})) \underset{\varphi^*}{\overset{id}{\rightrightarrows}} \mathcal{D}_{\text{qc}}(\text{Spa}(\text{W}(R^+)[\tfrac{1}{I}], \widetilde{\widehat{\text{W}(R^+)}})).$$

In particular as explained in [**BS21**][3.7] the endomorphisms of the unit compute the p-adic étale cohomology of A.

Remark 3.2.5.3. Consider a semiperfectoid bounded solid affinoid $(\mathbf{Q}_p)_\square$-algebra A with integral semiperfectoid A_0 ring of definition. Then we can understand

$$\mathcal{X}_A \simeq \text{colim}_n \, \text{Spa}((\mathbb{\Delta}_{A_0})_{\text{perf}}[\frac{1}{I}]/p^n, \widetilde{(\mathbb{\Delta}_{A_0})_{\text{perf}}}/p^n)$$

where we write $(\mathbb{\Delta}_{A_0})_{\text{perf}}$ for the (completed) colimit over taking the Frobenius iteratively.

Remark 3.2.5.4. Consider an affinoid perfectoid $(\mathbf{Q}_p)_\square$-algebra (A, A^+). Then the kernel of Fontaine's map

$$\text{W}((A^+)^\flat) := \text{W}(R^+) \to A^+$$

is generated by an element $\xi = p + [\pi^\flat]$ with $\pi \in A$ a topological nilpotent unit, such that $\pi^p = p$. From this, we learn that

$$\text{Spa}((\text{W}(R^+)[\frac{1}{I}])/p, \widetilde{\text{W}(R^+)}/p) \simeq \text{Spa}(A, A^+)^\flat.$$

3.2.5.5. We also need a locally proper version of the étale locus. In the same way as above, we can associate to a semiperfectoid bounded solid affinoid $(\mathbf{Q}_p)_\square$-algebra A a triple

$$(R, R^+, I \subset \text{W}(R^+))$$

corresponding to its perfectoidisation A_{perfd}. Now we will write

$$\mathcal{X}_A^{lp} := \text{colim}_n \, \text{Spa}((\text{W}(R^+)[\frac{1}{I}])/p^n, \widetilde{\mathbf{Z}_p})$$

and

$$A^{\text{ét},lp} := \mathcal{X}_A^{lp}/\varphi^{\mathbf{Z}}$$

for the locally proper versions.

3.2.5.6. We will often use the following observation. Consider a map

$$B^{\text{ét}} \to A^{\text{ét}}$$

induced by a map of perfectoid bounded solid affinoids, and assume we want to verify a property, stable under base change and local on the target in the topos of analytic stacks, for any base change along a map

$$\text{AnSpec}(C) \to A^{\text{ét}}.$$

First of all, it is enough to show this property for any base change along a map

$$\text{AnSpec}(C) \to \mathcal{X}_A$$

and by definition such a map factors through some $\mathrm{Spa}((\mathrm{W}(R^+)[\frac{1}{I}])/p^n, \mathrm{W}(\widetilde{R^+)}/p^n)$. Now by descent for ∞-topoi, we have cartesian squares

$$\begin{tikzcd} \mathrm{Spa}(\mathrm{W}((B^+)^\flat)[\frac{1}{I_B}]/p^n, \mathrm{W}(\widetilde{(B^+)^\flat}/p^n) \arrow[r] \arrow[d] & \mathcal{X}_B \arrow[d] \\ \mathrm{Spa}(\mathrm{W}((A^+)^\flat)[\frac{1}{I_A}]/p^n, \mathrm{W}(\widetilde{(A^+)^\flat}/p^n) \arrow[r] & \mathcal{X}_A \end{tikzcd}$$

for all n. In particular, we have to check our property for the left vertical map. For this note, for any n, the map

$$\mathrm{Spa}(A^\flat, (A^+)^\flat) \to \mathrm{Spa}((\mathrm{W}(R^+)[\frac{1}{I}])/p^n, \mathrm{W}(\widetilde{R^+)}/p^n)$$

is a surjection of analytic stacks, as it is proper and on algebras given by a nilpotent extension (and thus descendable [**Mat16**][3.35]). In total, we have reduced the problem to checking the property in question for the map

$$\mathrm{Spa}(B^\flat, (B^+)^\flat) \to \mathrm{Spa}(A^\flat, (A^+)^\flat).$$

The analog trick also works for the locally proper version.

Remark 3.2.5.7. Using 3.1.5.11 and 3.2.5.6 one deduces that the functor $(\_)^{\text{ét}}$, as well as its locally proper version, preserve fibre products.

Proposition 3.2.5.8. *The functors*

$$\mathcal{X}^{lp}_{(\_)}, \mathcal{X}_{(\_)}, (\_)^{ét,lp}, (\_)^{ét} \colon (\mathcal{S}emi\mathcal{P}erfd_{(\mathbf{Q}_p)_\square/})^{op} \to \mathcal{A}n\mathcal{S}tack$$

sends naive syntomic covers to proper surjections of analytic stacks.

Proof. Note first that for a naive syntomic cover $A \to B$, the ring of integral elements B^+ for B is given by the integral closure of A^+ in B. So by 3.1.5.12, the same also holds for the map

$$A_{\text{perfd}} \to B_{\text{perfd}}.$$

But then the same also holds for the map

$$(A_{\text{perfd}})^\flat \to (B_{\text{perfd}})^\flat$$

by [**Mor17**][2.5]. From here, we see that to prove the proposition, it suffices to prove descendability for the map on underlying rings. As the functors preserve fibre products 3.2.5.7, we just have to check the descendability for a generic example. That is, we can assume that our naive syntomic cover $A \to B$ comes from a naive syntomic cover

$$A^\circ \to B_0$$

of integral semiperfectoids, where B_0 is some ring of definition for B. Now, using 3.2.5.3, we see that it is enough to see that the map

$$(\Delta_{A^\circ})_{\text{perf}}[\frac{1}{I}] \to (\Delta_{B_0})_{\text{perf}}[\frac{1}{I}]$$

is descendable. So by [**BS17**][11.22] it suffices to check that the map on prismatic cohomology is descendable, which follows from 3.2.1.8. $\square$

3.2.5.9. Using 3.2.5.8 and 3.1.6.2 we obtain functors

$$(\_)^{\text{ét},lp}, (\_)^{\text{ét}} \colon (\mathcal{B}nd^{\text{cl}}_{(\mathbf{Q}_p)_\square/})^{\text{op}} \to \mathcal{A}n\mathcal{S}tack$$

by Kan extension, which preserves fibre products and sends naive syntomic covers to proper surjections of analytic stacks.

Proposition 3.2.5.10. *The functors*

$$\mathcal{X}_{(\_)}, (\_)^{ét} \colon (\mathcal{B}nd^{cl}_{(\mathbf{Q}_p)_\square/})^{op} \to \mathcal{A}n\mathcal{S}tack$$

respect the following maps:

- *It sends rational opens (resp. covers) to open immersions (resp. covers).*
- *It sends elementary closed immersions (resp. covers) to closed immersions (resp. covers).*
- *It sends étale morphisms (resp. covers) to étale morphisms (resp. covers).*

Furthermore the functors

$$\mathcal{X}^{lp}_{(\_)}, (\_)^{ét,lp} \colon (\mathcal{B}nd^{cl}_{(\mathbf{Q}_p)_\square/})^{op} \to \mathcal{A}n\mathcal{S}tack$$

send rational opens (resp. covers), as well as elementary closed immersions (resp. covers), to closed immersions (resp. covers).

Proof. Consider a map in question $A \to B$. Then, as the statements are local on the target 1.1.3.7 by using naive syntomic descent, we can assume that A is semiperfectoid, and thus admits perfectoidisation A_{perfd}. Now the base change

$$B_{\text{perfd}} := B \otimes_A A_{\text{perfd}}$$

in all three cases is already perfectoid. For open immersions, this is [Sch12][6.3], for elementary closed immersion, this follows from the case of open immersions and the fact that being perfectoid only depends on the underlying ring, and for étale morphism this is 3.1.1.18. Now using 3.2.5.6, we see that we have to check that tilting preserves the maps in question, which is [Sch12][7.12] [KL13][3.6.21]. $\square$

3.2.5.11. Using 3.2.5.10 we obtain functors

$$\mathcal{X}^{lp}_{(\_)}, \mathcal{X}_{(\_)}, (\_)^{\text{ét},lp}, (\_)^{\text{ét}} \colon \mathcal{T}a\mathcal{A}dic\mathcal{S}p^{\text{cl}}_{/\text{Spa}(\mathbf{Q}_p)} \to \mathcal{A}n\mathcal{S}tack$$

such that the locally proper versions send rational covers to closed covers of analytic stacks and the "decompactified" versions preserve étale coverings.

Definition 3.2.5.12. For a p-adic classical derived Tate adic space X, we call the analytic stacks

$$X^{\text{ét}}, X^{\text{ét},lp}$$

the *étale locus* and the *locally proper étale locus.*

Let us now come to the discussion of proper morphisms.

Proposition 3.2.5.13. *Consider a map $f \colon X \to S$ of classical derived Tate adic spaces over Spa($\mathbf{Z}_p$). Then we have the following:*

(a) If f is locally of finite type and quasi compact, quasi separated, then the induced maps

$$\mathcal{X}^{lp}_X \to \mathcal{X}^{lp}_S, X^{ét,lp} \to S^{ét,lp}$$

are proper.

(b) *If f is locally proper, we have cartesian squares*

$$\begin{tikzcd} \mathcal{X}_X \arrow[r] \arrow[d] & \mathcal{X}_X^{lp} \arrow[d] & X^{\acute{e}t} \arrow[r] \arrow[d] & X^{\acute{e}t,lp} \arrow[d] \\ \mathcal{X}_S \arrow[r] & \mathcal{X}_S^{lp} & S^{\acute{e}t} \arrow[r] & S^{\acute{e}t,lp}. \end{tikzcd}$$

(c) *If f is proper, the induced maps*

$$\mathcal{X}_X \to \mathcal{X}_S, X^{\acute{e}t} \to S^{\acute{e}t}$$

are proper.

(d) *If f is locally of $^+$finite type, then the maps*

$$\mathcal{X}_X \to \mathcal{X}_S, X^{\acute{e}t} \to S^{\acute{e}t}$$

are $!$-able.

PROOF. Note first that (c) is implied by (a) and (b) and as a map locally of $^+$finite type locally can be factored as an open immersion followed by a locally proper map (d) is implied by (b) and 3.2.5.10. So we have to prove (a) and (b).

Note also that both assertions are local on the target, so we will always implicitly make the following reduction steps without further mentioning them. First, we have to check the claims of the functor $\mathcal{X}_{(\_)}$ and its locally proper version. Furthermore we can assume $S = \mathrm{Spa}(A)$ to be affine by 3.2.5.10 and perfectoid by 3.2.5.8. The last reduction we are going to make is that, as explained in 3.2.5.6, we can check the claims modulo p.

To prove (a) we first claim:

($*_1$) Assume f is a Zariski-closed immersion. Then (a) holds.

In this case, $X \simeq \mathrm{Spa}(B)$ is affine and thus semiperfectoid. In particular, we see that

$$\mathcal{X}_X^{lp}/p \simeq \mathrm{Spa}((B_{\mathrm{perfd}})^\flat, \widetilde{\mathbf{Z}_p})$$

from which one easily deduces the claim.

Now we claim:

($*_2$) Assume f is separated. Then (a) holds.

To see this note first, that by $*_1$ the diagonal induces a proper morphism. Furthermore, using the quasi-compactness and 3.2.5.10, we can use a finite cover by affine rational opens $\mathrm{Spa}(B_i) \simeq U_i \subset X$ to obtain a closed cover

$$\{\mathcal{X}_{U_i}^{lp}/p \subset \mathcal{X}_X^{lp}/p\}_I$$

such that the compositions $f_i \colon U_i \subset X \to S$ factor as

$$A \to A\langle x_1, \dots, x_n\rangle \to B_i$$

where the second map is a Zariski closed immersion. So by 2.1.3.4 we have to check that

$$\mathcal{O}_{\mathcal{X}_{U_i}^{lp}/p} \in \mathcal{D}_{\mathrm{qc}}(\mathcal{X}_{U_i}^{lp}/p)$$

is compact. As $\mathcal{X}_S^{lp}/p$ is affine, this follows from the next lemma. To prove (a) for a general map, we do the same argument as for $(*_2)$, now using that the diagonal is separated.

We now prove (b). As f is locally proper, we can find a covering by elementary closed subspaces of X which are affine. Thus, using 3.2.5.10 we can assume $\mathrm{Spa}(B) \simeq X$ to be affine. Furthermore, using 3.2.5.8 we can assume B to be perfectoid by choosing a naive syntomic cover. But then B^+ is given by the integral closure of A^+ in B so $(B^\flat)^+$ is given by the integral closure of $(A^\flat)^+$ in $B^\flat$ by [**Mor17**][2.5]. From this one easily sees the claim. $\square$

Lemma 3.2.5.14. We use notation as in the proof of 3.2.5.13. The canonical functor induces an equivalence

$$\mathcal{D}_{\mathrm{qc}}(\mathcal{X}_{U_i}^{lp}/p) \simeq \mathcal{M}od_{(f_i)_*\mathcal{O}_{\mathcal{X}_{U_i}^{lp}/p}}(\mathcal{D}_{\mathrm{qc}}(\mathcal{X}_S^{lp}/p)).$$

Proof. We note that the statement is stable under compositions, and we have already seen that it holds for Zariski-closed immersions while proving $(*_1)$ in 3.2.5.13. So we can assume that $f_i := g\colon A \to A\langle x_1, \dots, x_n\rangle$ is given by the projection from the unit ball. In this case, we have a naive syntomic cover

$$\tilde{U} := \mathrm{Spa}(A\langle x_1^{\frac{1}{p^\infty}}, \dots, x_n^{\frac{1}{p^\infty}}\rangle) \to U := \mathrm{Spa}(A\langle x_1, \dots, x_n\rangle)$$

by a perfectoid. Using the Čech nerve of this cover to compute the left side of the equivalence, we see that it suffices to observe that the map

$$g_*\mathcal{O}_{\mathcal{X}_U^{lp}/p} \to \tilde{g}_*\mathcal{O}_{\mathcal{X}_{\tilde{U}}^{lp}/p}$$

is descendable. This follows as in 3.2.3.10 using 3.2.5.3. $\square$

3.2.6. The Prismatisation. We now want to "decompactify" the locally proper prismatisation. We will do this using naive syntomic descent.

Construction 3.2.6.1. Let us write $f\mathcal{S}ch_{\mathbf{Z}_p}^{\mathrm{aff,sperf}}$ for the category of affine p-adic formal schemes, whose ring of functions gives a semiperfectoid ring.

Now we consider the functor, which we will refer to as the *Prismatisation*,

$$(\_)^{\mathbb{\Delta},\square}\colon f\mathcal{S}ch_{\mathbf{Z}_p}^{\mathrm{aff,sperf}} \to \mathcal{A}n\mathcal{S}tack$$

which assigns to a semiperfectoid S the analytic stack

$$\mathrm{colim}_n\,\mathrm{Spa}(\mathbb{\Delta}_S/p^n, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)/p^n})$$

where the map $\mathbb{A}_{\mathrm{inf}}(S) \to \mathbb{\Delta}_S$ comes from 3.1.4.5. Similarly, we will write

$$S^{\mathbb{\Delta},lp} := \mathrm{colim}_n\,\mathrm{Spa}(\mathbb{\Delta}_S/p^n, \widetilde{\mathbf{Z}/p^n})$$

for its locally proper variant. Then we have a map

$$S^{\mathbb{\Delta},\square} \to S^{\mathbb{\Delta},lp}$$

natural in S.

Example 3.2.6.2. For an integral perfectoid R the map $\mathbb{A}_{\mathrm{inf}}(R) \xrightarrow{\sim} \mathbb{\Delta}_S$ is an isomorphism. In particular, we see that

$$R^{\mathbb{\Delta},\square} \simeq \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(R)).$$

To start, we want to relate the Prismatisation of a semiperfectoid to the étale locus of its generic fiber. For this, we will consider the following construction.

Construction 3.2.6.3. Given an integral semiperfectoid S, we will write

$$\overline{U(S^{HT})} \subset S^{\mathbb{\Delta},\square}$$

for the closed substack given by the closure of the open complement of the Hodge-Tate locus. In formulas, this means

$$\overline{U(S^{HT})} \simeq \mathrm{colim}_n \operatorname{Spa}(\mathbb{\Delta}_S[\frac{1}{I}]/p^n, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)/p^n}).$$

where we write I for the Hodge-Tate ideal. Similarly, we will denote by

$$\overline{U(S^{HT})}^{lp}$$

its locally proper counterpart.

Remark 3.2.6.4. Note that $\varphi(I) = I^p \bmod p$. In particular the frobenius on $S^{\mathbb{\Delta},\square}$ restricts to an endomorphism on $\overline{U(S^{HT})}$.

The following corollary can be explained most transparently using the *Nygaard filtered Prismatisation*, which we will discuss later in the text 3.3.3.4. That is why we will state it for now and explain the argument later.

Corollary 3.2.6.5. *Given an integral semiperfectoid S, the Frobenius acts as an automorphism on*

$$\overline{U(S^{HT})} \text{ and } \overline{U(S^{HT})}^{lp}.$$

3.2.6.6. Using 3.2.6.5, it makes sense to consider the quotient stack

$$\overline{U(S^{HT})}/\varphi^{\mathbf{Z}}$$

and similar to its locally proper variant.

Corollary 3.2.6.7. *Given an integral semiperfectoid S, we have identifications*

- $\overline{U(S^{HT})} \simeq \mathcal{X}_{(S[\frac{1}{p}],S)_\square}$.
- $\overline{U(S^{HT})}^{lp} \simeq \mathcal{X}^{lp}_{(S[\frac{1}{p}],S)_\square}$.
- $\overline{U(S^{HT})}/\varphi^{\mathbf{Z}} \simeq Spa(S[\frac{1}{p}], \widetilde{S})^{\acute{e}t}$
- $\overline{U(S^{HT})}^{lp}/\varphi^{\mathbf{Z}} \simeq Spa(S[\frac{1}{p}], \widetilde{S})^{\acute{e}t,lp}$

functorial in S.

PROOF. Using 3.2.5.3 we see that the difference lies in the perfection. So the claims follow from 3.2.6.5. ☐

Corollary 3.2.6.8. *Consider an integral map $S \to T$ of integral semiperfectoids. Then the square*

$$\begin{tikzcd} T^{\mathbb{\Delta},\square} \arrow[r] \arrow[d] & T^{\mathbb{\Delta},lp} \arrow[d] \\ S^{\mathbb{\Delta},\square} \arrow[r] & S^{\mathbb{\Delta},lp} \end{tikzcd}$$

is Cartesian.

PROOF. We can check the claim on a stratification. That is, it suffices to prove the analogous claim for the Hodge-Tate locus and for the closure of its complement.

More precisely, note that the category of quasi-coherent sheaves on the open complement of the substack

$$\overline{U(S^{HT})} \subset S^{\Delta,\square}$$

is given by the full subcategory of $\mathcal{D}_{\mathrm{qc}}(S^{\Delta,\square})$ consisting of the I-adically complete objects. To reduce to the Hodge-Tate locus, one can thus use the derived Nakayama lemma and the fact that associating quasi-coherent sheaves is a conservative operation on affines.

Now, for the Hodge-Tate locus, the claim follows from 3.2.4.6 together with 3.2.4.7. The case of the closure of its complement was explained in 3.2.5.13 using the identification 3.2.6.7. □

Proposition 3.2.6.9. *The functor*

$$(\_)^{\Delta,\square} \colon f\mathcal{S}ch_{\mathbf{Z}_p}^{\mathit{aff},\mathit{sperf}} \to \mathcal{A}n\mathcal{S}tack$$

from 3.2.6.1 preserve fibre products and sends naive syntomic covers to proper surjections. In particular it is a cosheaf for the naive syntomic topology.

PROOF. The claim on fibre products can be checked on a stratification. Thus, as in the proof of 3.2.6.8 the claim reduced to the analogous claim for the Hodge-Tate locus and the analogous claim for the closure of its complement. For the former, this was explained in 3.2.4.6, and for the latter, this follows from 3.2.6.7 and 3.2.5.7.

The claim of being a cosheaf follows from 3.2.6.8 and the analogous claim for the locally proper version 3.2.3.10. □

3.2.6.10. Using 3.2.6.9 and 3.1.2.8 we see that there exists a unique cosheaf

$$(\_)^{\Delta,\square} \colon f\mathcal{S}ch_{\mathbf{Z}_p}^{\mathrm{aff}} \to \mathcal{A}n\mathcal{S}tack$$

on the naive syntomic site, which on a semiperfectoid is given by 3.2.6.1. We now extend this construction to all (derived) p-adic formal schemes.

Note that the proof also shows that there is a map

$$(\_)^{\Delta,\square} \to (\_)^{\Delta,lp}$$

of cosheaves.

Proposition 3.2.6.11. *The functor*

$$(\_)^{\Delta,\square} \colon f\mathcal{S}ch_{\mathbf{Z}_p}^{\mathit{aff}} \to \mathcal{A}n\mathcal{S}tack$$

preserves open immersion (resp. covers) and étale morphisms (resp. covers).

PROOF. We first claim:

($*_1$) For any (affine) etale map $U \to \mathbf{A}^n_{\mathbf{Z}_p}$ the induced map $U^{\Delta,\square} \to (\mathbf{A}^n_{\mathbf{Z}_p})^{\Delta,\square}$ is an étale map of analytic stacks.

The assertion is local on the target, so we are allowed to pull back along a naive syntomic cover $\mathrm{Spf}(R) \to \mathbf{A}^n_{\mathbf{Z}_p}$ 3.2.6.9. Using 3.1.1.16 and 3.1.1.19 we see that we can choose R to be an integral perfectoid. But then

$$U \times_{\mathbf{A}^n_{\mathbf{Z}_p}} \mathrm{Spf}(R) \simeq \mathrm{Spf}(\tilde{R})$$

for an integral perfectoid $\tilde{R}$ 3.1.1.18 and we have to check that the map

$$\mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(\tilde{R})) \to \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(R))$$

is an étale map of analytic stacks 3.2.6.2. This follows from 3.1.4.2 and 2.5.2.4.

Now we claim:

($*_2$) For any etale cover $\mathrm{Spf}(\tilde{S}) \to \mathrm{Spf}(S)$, the induced map $\tilde{S}^{\mathbb{\Delta},\square} \to S^{\mathbb{\Delta},\square}$ is a surjection of analytic stacks.

Again, the assertion is local on the target, so we can assume S to be semiperfectoid 3.2.6.9. Thus by 3.1.4.2 we can assume $\tilde{S}$ to be semiperfectoid as well. Then we have to check that the map

$$\mathrm{Spa}(\mathbb{\Delta}_{\tilde{S}}, \widetilde{\mathbb{A}_{\mathrm{inf}}(\tilde{S})}) \to \mathrm{Spa}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})$$

is surjective. This map is refined by

$$\mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(\tilde{S})\widehat{\otimes}_{\mathbb{A}_{\mathrm{inf}}(S)}\mathbb{\Delta}_{\tilde{S}}, \widetilde{\mathbb{A}_{\mathrm{inf}}(\tilde{S})}) \to \mathrm{Spa}(\mathbb{\Delta}_{\tilde{S}}, \widetilde{\mathbb{A}_{\mathrm{inf}}(\tilde{S})}) \to \mathrm{Spa}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)}).$$

So it suffices to check that on one hand, the map

$$\mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(\tilde{S})) \to \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(S))$$

is surjective, which follows from 3.1.4.2 and 2.5.2.4. And on the other hand, that

$$\mathrm{Spa}(\mathbb{\Delta}_{\tilde{S}}, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)}) \to \mathrm{Spa}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})$$

is surjective, which follows as $\mathbb{\Delta}_S \to \mathbb{\Delta}_{\tilde{S}}$ is descendable as a (p, I)-completely faithfully flat etale map of animated rings [**BL22b**][3.9].

Now ($*_1$) and ($*_2$) together with 2.4.2.7 and the fact that the prismatisation preserves fibre products 3.2.6.9 imply all claims. ☐

3.2.6.12. Using 3.2.6.11, we see that there exists a unique cosheaf

$$(\_)^{\mathbb{\Delta},\square} \colon f\mathcal{S}ch_{\mathbf{Z}_p} \to \mathcal{A}n\mathcal{S}tack$$

which on affines is given by 3.2.6.10. Furthermore, this extension preserves open immersions (resp. covers) and étale morphisms of p-adic formal schemes (resp. covers).

Definition 3.2.6.13. For a p-adic formal scheme X, we call the analytic stack

$$X^{\mathbb{\Delta},\square}$$

the *Prismatiastion*.

3.2.6.14. Let us explain that this construction matches the construction of the (completed) Hodge-Tate stack. That is, we claim that for any p-adic formal scheme X, the square

$$\begin{tikzcd} X^{HT,\square} \arrow[r] \arrow[d] & X^{\mathbb{\Delta},\square} \arrow[d] \\ \mathbf{Z}_p^{HT,\square} \arrow[r] & \mathbf{Z}_p^{\mathbb{\Delta},\square} \end{tikzcd}$$

is cartesian. Using 3.2.6.11, 3.2.4.6 and naive syntomic descent, this assertion can be checked locally on X, and we can assume $X \simeq \mathrm{Spf}(S)$ for a semiperfectoid. But then the claim follows from 3.2.4.7.

3.2.6.15. Using naive syntomic descent 3.2.5.8 3.2.6.9, we can use the identification 3.2.6.7 to obtain an isomorphism

$$\overline{U(\mathbf{Z}_p^{HT})} \simeq \mathcal{X}_{(\mathbf{Q}_p)_\square}.$$

Furthermore also involving Zariski descent 3.2.6.11 and 3.2.5.10, we see that for any p-adic formal scheme X, there is a cartesian square

$$\begin{tikzcd} \mathcal{X}_{X_\eta} \arrow[r] \arrow[d] & X^{\Delta,\square} \arrow[d] \\ \mathcal{X}_{(\mathbf{Q}_p)_\square} \arrow[r] & \mathbf{Z}_p^{\Delta,\square} \end{tikzcd}$$

functorial in X. Where X_η denotes the generic fibre understood as a classical derived Tate adic space.

Let us now come to the case of proper maps.

Proposition 3.2.6.16. *Consider a map of p-adic formal schemes $f\colon X \to S$ and let us write $f^{\Delta,\square}\colon X^{\Delta,\square} \to S^{\Delta,\square}$ for the induced map. Then we have the following:*

(a) If f is $^+$proper, the square

$$\begin{tikzcd} X^{\Delta,\square} \arrow[r] \arrow[d] & X^{\Delta,lp} \arrow[d] \\ S^{\Delta,\square} \arrow[r] & S^{\Delta,lp} \end{tikzcd}$$

is cartesian.

(b) If f is locally of $^+$finite type, $f^{\Delta,\square}$ is !-able.

(c) If f is $^+$proper, $f^{\Delta,\square}$ is locally proper.

(d) If f is proper, $f^{\Delta,\square}$ is proper.

Proof. Note that (c) is implied by (a) and (d) is implied by (a) together with 3.2.3.10.

Now we prove (a). As explained in the proof of 3.2.6.8 we can check the assertion on a stratification given by the Hodge-Tate locus and the closure of its complement. For the Hodge-Tate locus, the claim follows from 3.2.4.6. On the other hand, we can use 3.2.6.15 to reduce the claim now to the analogous claim for the stack $\mathcal{X}_{(\_)}$ applied to the generic fibre of f. As f is $^+$proper the generic fibre is locally proper 2.5.4.6, so the claim follows from 3.2.5.13.

The (b) assertion is local on the source and target. So using 3.2.6.11 and naive syntomic descent we can assume $X \simeq \mathrm{Spf}(\tilde{S})$ and $S \simeq \mathrm{Spf}(S)$ with S semiperfectoid and that the map factors as

$$S \to S\langle x_1, \dots, x_n\rangle \to \tilde{S}$$

where the second map is integral. So by (a), we just have to check the assertion for the completed polynomial algebra, and using base change stability, we can assume $S \simeq R$ is an integral perfectoid. But now using 3.1.1.19 we have a naive syntomic cover

$$R\langle x_1, \dots, x_n\rangle \to R\langle x_1^{\frac{1}{p^\infty}}, \dots, x_n^{\frac{1}{p^\infty}}\rangle$$

by an integral perfectoid with

$$\mathbb{A}_{\mathrm{inf}}(R\langle x_1^{\frac{1}{p^\infty}}, \dots, x_n^{\frac{1}{p^\infty}}\rangle) \simeq \mathbb{A}_{\mathrm{inf}}(R)\langle x_1^{\frac{1}{p^\infty}}, \dots, x_n^{\frac{1}{p^\infty}}\rangle := \tilde{A}.$$

So we have reduced the question to

$$\mathrm{Spa}(\tilde{A}) \to \mathrm{Spa}(\mathbb{A}_{\mathrm{inf}}(R))$$

being !-able. But the domain is the intersection of the pullbacks

$$\begin{tikzcd} U(\frac{1}{x_i}) \arrow[r] \arrow[d] & \mathrm{Spa}(\tilde{A}, \widetilde{\mathbb{A}_{\mathrm{inf}}(R)}) \arrow[d, "x_i"] \\ \mathbf{D}^1_\square \arrow[r] & \mathbf{A}^1_\square \end{tikzcd}$$

which is a finite intersection of opens and thus open in $\mathrm{Spa}(A, \mathbb{A}_{\mathrm{inf}}(R))$. This implies the claim. $\square$

3.3. Analytifying the syntomification

3.3.1. Recollections on filtered objects. Let us collect some facts on filtered objects.

Definition 3.3.1.1. Given an animated ring R, we will write

$$\mathcal{DF}(R) := \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathcal{D}(R))$$

for the category of *filtered objects* in $\mathcal{D}(R)$. Here we understand $\mathbf{Z}^{\mathrm{op}}$ as a poset.

Remark 3.3.1.2. We often work with the full subcategory of derived I-complete objects in $\mathcal{DF}(R)$ for some ideal in R. But it will be obvious how to adopt everything we say about this category.

3.3.1.3. Taking the colimit and taking the cofiber of each map in the filtration induces functors

$$\mathcal{D}(R) \xleftarrow{\mathrm{colim}} \mathcal{DF}(R) \xrightarrow{\mathrm{gr}^\bullet} \mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R))$$

where we understand $\mathbf{Z}$ as a discrete category. We will refer to the category $\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R))$ as *graded objects* in $\mathcal{D}(R)$.

3.3.1.4. Given two filtered objects $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in \mathcal{DF}(R)$, the mapping spectrum

$$\mathrm{Hom}_{\mathcal{DF}(R)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$$

can be computed as the equalizer of the two maps

$$\textstyle\prod_{i \in \mathbf{Z}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}(R)}(\mathcal{F}^i, \mathcal{E}^i) \rightrightarrows \prod_{j \in \mathbf{Z}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{D}(R)}(\mathcal{F}^j, \mathcal{E}^{j-1})$$

one coming from postcomposing along the target filtration and one from precomposing along the source filtration. This follows as one can write the category $\mathbf{Z}^{\mathrm{op}}$ as the Segal completion of the simplicial anima

$$\dots \Delta^1 \coprod_{\Delta^0} \Delta^1 \coprod_{\Delta^0} \Delta^1 \dots .$$

3.3.1.5. Day-convolution equips the category $\mathcal{DF}(R)$ with a symmetric monoidal structure. That is, we obtain the formula

$$(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet)^n \simeq \mathrm{colim}_{i+j \geq n} \mathcal{F}^i \otimes \mathcal{E}^j .$$

Furthermore, this monoidal structure is closed, so we have an internal hom. Note also that the colimit functor from 3.3.1.3 is symmetric monoidal.

Day-convolution also equips the category $\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R))$ of graded objects with a symmetric monoidal structure. For two graded objects $\mathcal{F}^\bullet, \mathcal{E}^\bullet$ this produces the formula

$$(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet)^n \simeq \mathrm{colim}_{i+j=n} \mathcal{F}^i \otimes \mathcal{E}^j \simeq \bigoplus_{j+i=n} \mathcal{F}^i \otimes \mathcal{E}^j .$$

Furthermore, the associated graded functor from 3.3.1.3 is symmetric monoidal for these structures. Also, the symmetric monoidal structure on graded objects is closed. Unwinding the formula for the internal hom, one gets

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R))}(\mathcal{F}^\bullet, \mathcal{E}^\bullet)^n \simeq \prod_{m \in \mathbf{Z}} \underline{\mathrm{Hom}}_{\mathcal{D}(R)}(\mathcal{F}^m, \mathcal{E}^{m-n}).$$

The associated graded functor is also compatible with the internal homs [**GP16**][2.28], that is, we have the formula

$$\mathrm{gr}^\bullet \underline{\mathrm{Hom}}_{\mathcal{DF}(R)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \simeq \underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R))}(\mathrm{gr}(\mathcal{F}^\bullet), \mathrm{gr}(\mathcal{E}^\bullet)).$$

3.3.1.6. A filtered object $\mathcal{F}^\bullet$ in $\mathcal{D}(R)$ is called complete, if

$$\lim_{n \in \mathbf{Z}^{\mathrm{op}}} F^n \simeq 0.$$

The full subcategory $\widehat{\mathcal{DF}}(R) \subset \mathcal{DF}(R)$ of complete filtered objects admits a symmetric monoidal left adjoint [**GP16**][2.25]. Furthermore, the associated grade functor is conservative when restricted to complete objects.

3.3.1.7. There is a t-structure on $\mathcal{DF}(R)$, called the *standard t-structure*. The connective objects are given by those filtered objects $\mathcal{F}^\bullet$, for which each $\mathcal{F}^i$ is connective in the standard t-structure on $\mathcal{D}(R)$.

3.3.1.8. Recall from [**Mou19**] that there is also the following geometric viewpoint on the category $\mathcal{DF}(R)$. There are symmetric monoidal equivalences

- $\mathrm{Fun}(\mathbf{Z}, \mathcal{D}(R)) \simeq \mathcal{D}_{\mathrm{qc}}((\mathbf{BG}_m)_R)$
- $\mathcal{DF}(R) \simeq \mathcal{D}_{\mathrm{qc}}((\mathbf{A}^1/\mathbf{G}_m)_R)$.

Under those equivalences, the associated graded functor corresponds to the pullback along the zero-section $\mathbf{BG}_m \to \mathbf{A}^1/\mathbf{G}_m$ and the colimit functor corresponds to the pullback along the one-section $\mathbf{G}_m/\mathbf{G}_m \to \mathbf{A}^1/\mathbf{G}_m$.

3.3.2. Recollections on the Nygaard filtration. The absolute prismatic complex Δ_S of an animated ring comes equipped with a filtration called the *Nygaard filtration*. In the following, we will recollect some facts about this filtration. Let us start with the construction following [**BS19**] [**BL22a**].

3.3.2.1. Given a bounded (static) prism (A, I) and a (static) quasi-regular semiperfectoid $\overline{A}$-algebra S, the relative prismatic complex $\Delta_{S/A}$ is static and comes with a relative Frobenius

$$\varphi\colon F^*\Delta_{S/A} := \Delta_{S/A} \otimes_{A,\varphi} A \to \Delta_{S/A}.$$

In particular we can equip $F^*\Delta_{S/A}$ with a filtration by considering the ideals

$$Fil^i_{\mathcal{N}} F^*\Delta_{S/A} := \{x | \varphi(x) \in I^i\} \subset F^*\Delta_{S/A}.$$

Now a polynomial ring over $\overline{A}$, can be resolved by a (static) quasi-regular semiperfectoid, such that each term in the Čech nerve stays quasi-regular semiperfectoid. Thus, to obtain the *Nygaard filtration* on

$$F^*\Delta_{\overline{A}[x_1,\ldots,x_n]/A}$$

we can descend the above filtration from the Čech nerve. For a general animated $\overline{A}$-algebra, one then left Kan-extends the above construction from polynomial algebras. This is well defined and comes with a map of filtered complexes

$$\varphi\colon Fil^\bullet_{\mathcal{N}} F^*\Delta_{S/A} \to I^\bullet$$

called the *filtered frobenius*. Using the filtered Frobenius, one can characterize the Nygaard filtration uniquely in the following way [**BL22a**][5.1.1]:

(a) The functor
$$Fil^{\bullet}_{\mathcal{N}} F^* \Delta_{(\_)/A} \colon \mathcal{A}ni(\mathcal{R}ing)_{\overline{A}/} \to \mathcal{DF}_{(I,p)\text{-comp}}(A)$$
preserves sifted colimits.

(b) For every integer i, the induced map on graded pieces
$$gr^i_{\mathcal{N}} F^* \Delta_{S/A} \to \overline{\Delta}_{S/A}\{i\}$$
identifies with the inclusion of the i-piece of the conjugate filtration.
$$Fil^{\mathrm{conj}}_i \overline{\Delta}_{S/A}\{i\} \to \overline{\Delta}_{S/A}\{i\}$$
functorially in S.

3.3.2.2. Using the fact that the construction of the Nygaard filtration is stable under base change along prisms and thus defines a filtered quasi-coherent sheaf on $\mathbf{Z}_p^{\Delta}$, on obtains a Nygaard filtration
$$\varphi \colon Fil^{\bullet}_{\mathcal{N}} \Delta_S \to I^{\bullet}$$
together with a filtered Frobenius on the absolute prismatic cohomology of an animated ring (see [**BL22a**][5.5.] for details). In the case the prism (A, I) is perfect, the identification $\Delta_S \xrightarrow{\sim} \Delta_{S/A}$ can be extended to an isomorphism
$$Fil^{\bullet}_{\mathcal{N}} \Delta_S \xrightarrow{\sim} Fil^{\bullet}_{\mathcal{N}} F^* \Delta_{S/A}$$
in $\mathcal{CA}lg(\mathcal{DF}_{(I,p)\text{-comp}}(A))$ [**BL22a**][5.6.2].

Example 3.3.2.3. Given an integral perfectoid R, the frobenius on $\mathbb{A}_{\mathrm{inf}}(R)$ induces an isomorphism between the Nygaard filtration and the filtration, which in positive degrees is given by
$$Fil^i_{\mathcal{N}} \mathbb{A}_{\mathrm{inf}}(R) \simeq (d^i) \subset \mathbb{A}_{\mathrm{inf}}(R)$$
and in negative degrees by $\mathbb{A}_{\mathrm{inf}}(R)$ sitting in the respective degree.

3.3.2.4. Given a semiperfectoid S, the filtered frobenius
$$Fil^{\bullet}_{\mathcal{N}} \Delta_S \to I^{\bullet}$$
is a (I, p)-complete Zariski localization. To see this, one uses the identifications made recalled in 3.3.2.1 (see [**Bha22**][5.5.1]). That is we can understand the situation relative to a perfectoid R. Then multiplication by a generator of
$$g \in Fil^1_{\mathcal{N}} \Delta_R \simeq \varphi^{-1}(d) \mathbb{A}_{\mathrm{inf}}(R)$$
on graded pieces, it induces the morphism
$$Fil^{\mathrm{conj}}_i \overline{\Delta}_S\{i\} \xrightarrow{e} Fil^{\mathrm{conj}}_i \overline{\Delta}_S\{i+1\} \longrightarrow Fil^{\mathrm{conj}}_{i+1} \overline{\Delta}_S\{i+1\}$$
coming from the conjugate filtration. So we can invert this generator to obtain the claim as the conjugate filtration is exhaustive.

We want to consider the Rees algebra associated with the Nygaard filtration of a semiperfectoid. We will need the following for this to still live in the connective setting.

Proposition 3.3.2.5. *For an animated integral semiperfectoid S, each piece of the Nygaard filtration*
$$Fil^i_{\mathcal{N}} \Delta_S \to \Delta_S$$
is connective, which lives in homological degrees ≥ 0.

Proof. Let us choose an integral perfectoid R with a surjective map $R \to S$ and identify the absolute Nygaard filtration with the relative Nygaard filtration over R. Then as the Kähler differentials of S over R vanish, we see that each graded piece in the conjugate filtration lives in homological degree ≥ 0. By induction, this implies that each filtered piece in the conjugate filtration and thus each graded piece in the Nygaard filtration lives in the same degrees. This implies that each filtered piece in the Nygaard filtration lives in homological degrees ≥ -1, and to prove the proposition, we have to check that for each i, the map

$$Fil^i_{\mathcal{N}} {\Delta}_S \to gr^i_{\mathcal{N}} {\Delta}_S$$

induces a surjection on π_0. This statement is stable under sifted colimits, so by 3.3.2.6 it suffices to observe this in the case

$$S \simeq A/(f_1, \dots, f_n)$$

being a derived quotient by finitely many functions. By Andre's flatness lemma [**BS19**][7.14] we can find a faithfully flat extension $R \to \tilde{R}$ of integral perfectoid such that $\tilde{R}$ is absolutely integrally closed. Thus, by base change, we can assume that the functions $f_1, \dots, f_n$ admit compatible systems of p-th roots and R contains all p-th roots of unity. The claim follows by base change from the universal case discussed in [**BS19**][12.3]. □

For an animated ring A, let us write

$$\mathcal{C}losed(A)$$

for the full subcategory of those animated A-algebras $A \to S$, for which the structure map induces a surjection on π_0. Then we used the following lemma.

Lemma 3.3.2.6. For an animated ring A, the category $\mathcal{C}losed(A)$ is projectively generated under sifted colimits by pushouts of the form

$$\begin{tikzcd} A[x_1, \dots, x_n] \arrow[r, "0"] \arrow[d, "{(f_1,\dots,f_n)}"'] & A \arrow[d] \\ A \arrow[r] & A/(f_1, \dots, f_n) \end{tikzcd}$$

for finitely many functions $f_1, \dots, f_n \in \pi_0 A$. That is, it is the animation of those A-algebras.

Proof. First, observe that the category $\mathcal{C}losed(A)$ has all colimits, and they are computed in A-algebras. So it suffices to check that the algebras $A/(f_1, \dots, f_n)$ are stable under coproducts, projective, and mapping out of them is conservative. The first assertion is obvious.

We check that the A-algebra $A/(f_1, \dots, f_n)$ is projective, that is mapping out of it preserves sifted colimits. Consider such a sifted colimit

$$\operatorname{colim}_I S_i \in \mathcal{C}losed(A)$$

then, as polynomial A-algebras are projective, the assertion boils down to observing that the square

$$\begin{tikzcd}
\operatorname{colim}_I \operatorname{Hom}_A(A/(f_1,\dots,f_n),S_i) \arrow[r] \arrow[d] & \operatorname{colim}_I \operatorname{Hom}_A(A,S_i) \arrow[d] \\
\operatorname{colim}_I \operatorname{Hom}_A(A,S_i) \arrow[r] & \operatorname{colim}_I \operatorname{Hom}_A(A[x_1,\dots,x_n,S_i])
\end{tikzcd}$$

is Cartesian. This follows from descent for ∞-topoi.

Let us check the conservativity. For this, we consider a map $S \to \tilde{S}$, which becomes an isomorphism when mapping all algebras of the form $A/(f)$ into it. First of all, it suffices to check that the map $S \to \tilde{S}$ is an isomorphism of A-modules. By stability, to observe this, it suffices that the fibre of the map $A \to S$ is isomorphic to the fibre of the map $A \to \tilde{S}$ via the canonical map. As both of these fibres live in homological degree ≥ 0, we can check this on the underlying anima, which can be checked fibered over A. But the fibre at a point $f \in A$ of the map $fib(A \to S) \to A$ is computed by

$$\operatorname{Hom}_A(A/(f),S)$$

and similar for the map $A \to \tilde{S}$, which shows what we want. $\square$

For the following, let us fix an integral perfectoid R with corresponding perfect prism (A,I). Furthermore let us write $T = R[x_1,\dots,x_d]$ for a finitely generated polynomial ring over R and $T^\infty = R[x_1^{\frac{1}{p^\infty}},\dots,x_d^{\frac{1}{p^\infty}}]$ for its naive perfection. Then we have the following:

Proposition 3.3.2.7. *The map*

$$Fil^\bullet_{\mathcal{N}} F^* \mathbb{\Delta}_{T/A} \to Fil^\bullet_{\mathcal{N}} F^* \mathbb{\Delta}_{T^\infty/A}$$

is descendable in $\mathcal{DF}_{(I,p)\text{-}comp}(A)$.

Proof. Let us write $F^\bullet$ for the fibre of the map in question, then by [**BS17**][11.20] it suffices to check that

$$\pi_0 \operatorname{Hom}_{\mathcal{DF}}((F^\bullet)^{\otimes d+3}, Fil^\bullet_{\mathcal{N}} F^* \mathbb{\Delta}_{T/A}) \simeq 0$$

where the tensor product is taken over the Nygaard filtration on $F^*\mathbb{\Delta}_{T/A}$. To check this, let us first make some reduction steps. First of all, we can take the tensor product over A. Also as the Hodge filtration on the De Rham cohomology of a finite polynomial ring is complete, by [**BL22a**][5.2.10], the Nygaard filtration on the latter is complete as well, so we can take the completed tensor product and thus assume that both sides are complete. Now to see the above, by 3.3.1.4, it suffices to check that

$$\underline{\operatorname{Hom}}_{\mathcal{DF}}((F^\bullet)^{\otimes d+3}, Fil^\bullet_{\mathcal{N}} F^* \mathbb{\Delta}_{T/A}) \in \mathcal{DF}_{>1}$$

where we use the standard t-structure. As both sides are (assumed to be) complete, we can use 3.3.1.6 and the compatibility of taking associated graded with the internal Hom 3.3.1.5, to reduce this to showing that

$$\underline{\operatorname{Hom}}_{\mathcal{D}_{\text{graded}}}(gr^\bullet(F^\bullet)^{\otimes d+3}, gr^\bullet_{\mathcal{N}} F^* \mathbb{\Delta}_{T/A}) \in (\mathcal{D}_{\text{graded}})_{>2}.$$

Unwinding the internal Hom 3.3.1.5 and the definition of the t-structure, this boils down to showing that for each pair of integers i,j, we have

$$\operatorname{Hom}_{\mathcal{D}}(gr^i(F^\bullet)^{\otimes d+3}, gr^j_{\mathcal{N}} F^* \mathbb{\Delta}_{T/A}) \in \mathcal{D}_{>2}.$$

For this, we now claim the following:

(∗) The fibre of the map
$$gr^n_{\mathcal{N}} F^* \Delta_{T/A} \to gr^n_{\mathcal{N}} F^* \Delta_{T^\infty/A}$$
takes the form
$$\bigoplus_{i=1}^{n} \bigoplus_{S(i)} T[-i]$$
for some sets $S(i)$ if $n \geq 0$ and vanishes otherwise.

First, note that using the identification of the graded pieces on the Nygaard filtration with the filtered pieces of the conjugate filtration and the fact that the cohomology of the later vanishes above the (relative) dimension of T, we see that $(*)$ implies what we want (note also that T is a free R-module).

To observe $(*)$, we identify the graded pieces with the filtered pieces of the conjugate filtration. Now note that in positive degrees
$$Fil_n^{\mathrm{conj}} \overline{\Delta}_{T^\infty/A} \simeq T^\infty$$
by relative perfectness. Using this we inductively observe that there are pushout squares
$$\begin{tikzcd}
Fil_{n-1}^{\mathrm{conj}} \overline{\Delta}_{T/A} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
Fil_{n}^{\mathrm{conj}} \overline{\Delta}_{T/A} \arrow[r] \arrow[d] & \bigoplus_{S'(n)} T[-n] \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\
Fil_{n}^{\mathrm{conj}} \overline{\Delta}_{T^\infty/A} \arrow[r] & M \arrow[r] & \bigoplus_{i=0}^{n-1} \bigoplus_{S(n)} T[-i]
\end{tikzcd}$$
and the map $Fil_{n-1}^{\mathrm{conj}} \overline{\Delta}_{T/A} \to Fil_{n}^{\mathrm{conj}} \overline{\Delta}_{T^\infty/A}$ identifies with the map
$$T \oplus \bigoplus_{i=1}^{n} \bigoplus_{S'(i)} T[-i] \to T^\infty$$
which includes the first factor and maps everything else to 0. Here, the upper pushout square comes from the identification of the graded pieces in the conjugate filtration with (shifts) of exterior powers of the cotangent complex. This shows what we want. □

Given a prism (A, I), let us write $\mathcal{DF}_{(I,p)\text{-comp}}(Fil^\bullet_{\mathcal{N}} A)$ for the category of modules over the Nygaard filtration on A in the filtered derived category of A.

3.3.2.8. Given a semiperfectoid S living over an integral perfectoid R, we can consider the Rees algebra
$$Rees(Fil^\bullet_{\mathcal{N}} \Delta_S) := \bigoplus_{i \in \mathbf{Z}} Fil^i_{\mathcal{N}} \Delta_S t^{-i}$$
which is an animated S-algebra coming with a $\mathbf{G}_m$-action. In particular, we can consider the quotient stack
$$\mathcal{R}(Fil^\bullet_{\mathcal{N}} \Delta_S) := \mathrm{Spf}(Rees(Fil^\bullet_{\mathcal{N}} \Delta_S))/\mathbf{G}_m.$$

For this stack, one obtains an equivalence

$$\mathcal{D}_{\mathrm{qc}}(\mathcal{R}(Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_S)) \simeq \mathcal{DF}_{(I,p)\text{-comp}}(Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_S)$$

and it is explained in [**Bha22**][5.5.10] that in the case S is quasi-regular semiperfectoid it can be defined via transmutation over $\mathcal{R}(Fil^{\bullet}_{\mathcal{N}}\mathbb{A}_{\mathrm{inf}}(R))$. In particular, given two quasi-regular semiperfectoids S and $\tilde{S}$ over R, we obtain a cartesian square

$$\begin{tikzcd} \mathcal{R}(Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_{S\otimes_R\tilde{S}}) \arrow[r] \arrow[d] & \mathcal{R}(Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_S) \arrow[d] \\ \mathcal{R}(Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_{\tilde{S}}) \arrow[r] & \mathcal{R}(Fil^{\bullet}_{\mathcal{N}}\mathbb{A}_{\mathrm{inf}}(R)). \end{tikzcd}$$

On the level of functions, this translates into the formula

$$Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_S \otimes Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_{\tilde{S}} \simeq Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_{S\otimes_R\tilde{S}}$$

in $\mathcal{DF}_{(I,p)\mathrm{comp}}(Fil^{\bullet}_{\mathcal{N}}\mathbb{A}_{\mathrm{inf}}(R))$. From this and the last proposition we obtain the following corollary.

Corollary 3.3.2.9. *Given an integral perfectoid R with corresponding perfect prism (A, I), the functor*

$$Fil^{\bullet}_{\mathcal{N}}F^*{\mathbb{\Delta}}_{(\_)/A} : \mathcal{A}ni(\mathcal{R}ing)_{R/} \to \mathcal{CA}lg(\mathcal{DF}_{(I,p)\text{-}comp}(Fil^{\bullet}_{\mathcal{N}}A))$$

commutes with colimits.

Proof. By construction, it preserves sifted colimits, so by the universal property of the animation, it suffices to check that the restriction to finite polynomial R-algebras preserves coproducts.

Let us write T_1, T_2 for two such polynomial R-algebras and

$$F^{\bullet}T_i := Fil^{\bullet}_{\mathcal{N}}F^*{\mathbb{\Delta}}_{T_i/A}$$

respectively. Then, using 3.3.2.7 twice, we obtain an equivalence

$$\mathcal{M}od_{F^{\bullet}T_1\otimes F^{\bullet}T_2}(\mathcal{DF}) \simeq \lim_{n\in\Delta}\mathcal{M}od_{F^{\bullet}(T_1^{\infty})^{\otimes_{T_1}n}\otimes F^{\bullet}(T_2^{\infty})^{\otimes_{T_2}n}}(\mathcal{DF})$$

where we write $\mathcal{DF} := \mathcal{DF}(Fil^{\bullet}_{\mathcal{N}}A)$. Furthermore, using 3.3.2.8, we see that the limit gives the right-hand side of this equivalence

$$\lim_{n\in\Delta}\mathcal{M}od_{F^{\bullet}((T_1\otimes T_2)^{\infty})^{\otimes_{T_1\otimes T_2}n}}(\mathcal{DF})$$

so running through this equivalence once gives the claim. □

We will also need the following.

Proposition 3.3.2.10. *Given a naive syntomic cover $S \to \tilde{S}$ between semiperfectoids. Then the algebra*

$$Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_{\tilde{S}} \in \mathcal{DF}_{(I,p)\text{-}comp}(Fil^{\bullet}_{\mathcal{N}}{\mathbb{\Delta}}_S)$$

is descendable.

Proof. We choose an integral perfectoid R mapping to S, then we can identify the absolute Nygaard filtrations with the relative Nygaard filtrations over R 3.3.2.2. Furthermore, using 3.3.2.9, we can assume that the map $S \to \tilde{S}$ comes as a base change from a universal naive syntomic cover. But then, using 3.3.2.9 again, we can assume $S = R$. From now on, we will write S for $\tilde{S}$.

Now let us write $F^\bullet$ for the fibre of the map

$$Fil^\bullet_{\mathcal{N}}\mathbb{\Delta}_R \to Fil^\bullet_{\mathcal{N}}\mathbb{\Delta}_S$$

and we claim that

$$\pi_0 \operatorname{Hom}_{\mathcal{DF}}((F^\bullet)^{\otimes 3}, Fil^\bullet_{\mathcal{N}}\mathbb{\Delta}_R) \simeq 0.$$

This will finish the proof by 3.3.2.7. As in the proof of 3.3.2.7, to observe the above, it suffices to observe that

$$\operatorname{Hom}_{\mathcal{D}(R)}(gr^i(F^\bullet)^{\otimes 3}, gr^j_{\mathcal{N}}\mathbb{\Delta}_R) \in \mathcal{D}_{>2}$$

for all i, j. For this, we recall the following two observations from the proof of 3.2.1.8:

(a) The cofibre of the map $R \to S$ is a free R-module.
(b) $\mathbf{L}_{S/R}[-1]$ is a free S-module.

Now, let us identify the graded pieces with the filtered pieces in the conjugate filtration. Then using (b), we observe that the conjugate filtration on $\overline{\mathbb{\Delta}}_S$ takes the form

$$S \to \bigoplus_{I_1} S \to \bigoplus_{I_2} S \to \bigoplus_{I_3} S \to \dots$$

where each map is an inclusion of a direct summand. In particular using (b), we learn that

$$gr^i(F^\bullet) \simeq \bigoplus_K R[-1]$$

for some set K for all i. From this, one can easily observe what we want. $\square$

3.3.3. The Nygaard filtered Prismatisation. Let us now explain how we want to understand the Nygaard filtered Prismatisation as an analytic stack.

Construction 3.3.3.1. Consider an integral semiperfectoid S then the Nygaard filtration on $\mathbb{\Delta}_S$ gives us a filtration on the solid analytic ring

$$Fil^\bullet_{\mathcal{N}}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})_\square \to (\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})_\square.$$

Furthermore, let us write

$$Rees(Fil^\bullet_{\mathcal{N}}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})_\square) := ((\bigoplus_{i\in\mathbf{Z}} Fil^i_{\mathcal{N}}\mathbb{\Delta}_S x^{-i})_{\widehat{I}}, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})^{x\square}$$

for the graded $(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})$-algebra obtained by I-adically completing the Rees algebras associated with the Nygaard filtration, and then solidify along the variable, remembering the grading. Out of this algebra, we can produce the analytic stack

$$\operatorname{Spf}(Rees(Fil^\bullet_{\mathcal{N}}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})_\square)) := \operatorname{colim}_n \operatorname{Spa}(Rees(Fil^\bullet_{\mathcal{N}}(\mathbb{\Delta}_S/p^n, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)}/p^n)_\square)).$$

Now the grading provides an $(\mathbf{G}_m^{\mathrm{an}})_\square$-action on this analytic stack, where we interpret $(\mathbf{G}_m^{\mathrm{an}})_\square$ living over $\operatorname{Spf}((\mathbf{Z}_p)_\square) := \operatorname{colim}_n \operatorname{Spa}(\mathbf{Z}/p^n)$. Now we set

$$S^{\mathcal{N},\square} := \operatorname{Spf}(Rees(Fil^\bullet_{\mathcal{N}}(\mathbb{\Delta}_S, \widetilde{\mathbb{A}_{\mathrm{inf}}(S)})_\square))/(\mathbf{G}_m^{\mathrm{an}})_\square$$

and refer to this stack as the *Nygaard filtered Prismatisation* of S.

3.3.3.2. Let us unwind a bit on how this stack looks.

(a) We have a structure morphism

$$\pi\colon S^{\mathcal{N},\square} \to S^{\Delta,\square}$$

coming from the inclusion into degree zero.

(b) There is a Cartesian square

$$\begin{tikzcd} S^{\Delta,\square} \arrow[r] \arrow[d] & S^{\mathcal{N},\square} \arrow[d] \\ \mathrm{Spf}((\mathbf{Z}_p)_\square) \arrow[r] & \mathbf{D}^1_\square/(\mathbf{G}_m^{\mathrm{an}})_\square \end{tikzcd}$$

where the right vertical map comes from the graded structure and the lower horizontal map from the identification

$$\mathrm{Spf}((\mathbf{Z}_p)_\square) \simeq (\mathbf{G}_m^{\mathrm{an}})_\square/(\mathbf{G}_m^{\mathrm{an}})_\square.$$

One way to see that this square is Cartesian is that the fibre product is a $(\mathbf{G}_m^{\mathrm{an}})_\square$-torsor over $S^{\Delta,\square}$, which admits a section. In particular, the upper horizontal map gives an open immersion

$$j_{dR}\colon S^{\Delta,\square} \to S^{\mathcal{N},\square}$$

which, composed with the structure map, recovers the identity of the Prismatisation.

(c) Observe that the Rees stack of the I-adic filtration identifies with the Prismatisation. In particular using 3.3.2.4, we see that the filtered frobenius gives an open immersion

$$j_{HT}\colon S^{\Delta,\square} \to S^{\mathcal{N},\square}$$

for any semiperfectoid S. Furthermore, the composition

$$S^{\Delta,\square} \xrightarrow{j_{HT}} S^{\mathcal{N},\square} \xrightarrow{\pi} S^{\Delta,\square}$$

identifies with the Frobenius on the prismatisation.

(d) Note that if we invert I in the Nygaard filtration, it becomes a filtration with structure maps isomorphisms. In particular, we see that there is a Cartesian square

$$\begin{tikzcd} \overline{U(S^{\mathrm{HT}})} \arrow[r] \arrow[d, dashed, "\simeq"] & S^{\mathcal{N},\square} \arrow[d, "\pi"] \\ \overline{U(S^{\mathrm{HT}})} \arrow[r] & S^{\Delta,\square} \end{tikzcd}$$

for any semiperfectoid S.

3.3.3.3. Doing the same construction as in 3.3.3.1, but starting with the analytic ring

$$(\Delta_S, \widetilde{\mathbf{Z}_p})_\square$$

instead, we obtain a locally proper version of the Nygaard filtered Prismatisation. We will write

$$S^{\mathcal{N},lp}$$

for this analytic stack and call it the *locally proper Nygaard filtered Prismatisation*. The analogous statements of 3.3.3.2 hold as well.

We now obtain the following, which was partially used in the last section already 3.2.6.5.

Proposition 3.3.3.4. *Consider a map $S \to \tilde{S}$ of semiperfectoids, then we have the following:*

(1) The squares

$$\begin{tikzcd} \tilde{S}^{\mathbb{\Delta},\square} \arrow[r, "j_{dR}"] \arrow[d] & \tilde{S}^{\mathcal{N},\square} \arrow[d] \\ S^{\mathbb{\Delta},\square} \arrow[r, "j_{dR}"'] & S^{\mathcal{N},\square} \end{tikzcd} \qquad \begin{tikzcd} \tilde{S}^{\mathbb{\Delta},lp} \arrow[r, "j_{dR}"] \arrow[d] & \tilde{S}^{\mathcal{N},lp} \arrow[d] \\ S^{\mathbb{\Delta},lp} \arrow[r, "j_{dR}"'] & S^{\mathcal{N},lp} \end{tikzcd}$$

are Cartesian.

(2) The squares

$$\begin{tikzcd} \tilde{S}^{\mathbb{\Delta},\square} \arrow[r, "j_{HT}"] \arrow[d] & \tilde{S}^{\mathcal{N},\square} \arrow[d] \\ S^{\mathbb{\Delta},\square} \arrow[r, "j_{HT}"'] & S^{\mathcal{N},\square} \end{tikzcd} \qquad \begin{tikzcd} \tilde{S}^{\mathbb{\Delta},lp} \arrow[r, "j_{HT}"] \arrow[d] & \tilde{S}^{\mathcal{N},lp} \arrow[d] \\ S^{\mathbb{\Delta},lp} \arrow[r, "j_{HT}"'] & S^{\mathcal{N},lp} \end{tikzcd}$$

are Cartesian.

(3) The frobenius acts as an isomorphism on $\overline{U(S^{HT})}$ and $\overline{U(S^{HT})}^{lp}$.

Proof. (1) is obvious as j_{dR} is defined via global pullback. To see (2) choose an integral perfectoid R mapping to S. Then as explained in 3.3.2.4 we can construct j_{HT} for S as well as for $\tilde{S}$ by inverting a generator in

$$Fil^1_{\mathcal{N}} \mathbb{\Delta}_R.$$

This shows the claim.

Let us proof (3) for $\overline{U(S^{HT})}$, the other case goes the same. Recall from 3.3.3.2, that we can factor the Frobenius on the Prismatisation as

$$S^{\mathbb{\Delta},\square} \xrightarrow{j_{HT}} S^{\mathcal{N},\square} \xrightarrow{\pi} S^{\mathbb{\Delta},\square}.$$

Thus pulling back to $\overline{U(S^{HT})}$ and using 3.3.3.2(d), we see that by (2), for any map of semiperfectoids $S \to \tilde{S}$, the square

$$\begin{tikzcd} \overline{U(\tilde{S}^{HT})} \arrow[r, "\varphi"] \arrow[d] & \overline{U(\tilde{S}^{HT})} \arrow[d] \\ \overline{U(S^{HT})} \arrow[r, "\varphi"'] & \overline{U(S^{HT})} \end{tikzcd}$$

is Cartesian. But now for a semiperfectoid S, we can find an integral perfectoid R together with a map $R \to S$. In this situation, the lower horizontal map in the square will be an isomorphism, and thus the upper horizontal map will also be. $\square$

We now extend the construction to affine formal schemes.

Proposition 3.3.3.5. *The functors*

$$(\_)^{\mathcal{N},\square}, (\_)^{\mathcal{N},lp} \colon f\mathcal{S}ch^{aff,sperf}_{\mathbf{Z}_p} \to \mathcal{A}n\mathcal{S}tack$$

preserve pullbacks and define cosheaves for the naive syntomic topology. Furthermore for an integral map of semiperfectoids $S \to \tilde{S}$ the square

$$\begin{tikzcd} \tilde{S}^{\mathcal{N},\square} \arrow[r] \arrow[d] & \tilde{S}^{\mathcal{N},lp} \arrow[d] \\ S^{\mathcal{N},\square} \arrow[r] & S^{\mathcal{N},lp} \end{tikzcd}$$

is Cartesian.

Proof. Recall the compatibility of the solid tensor product with the completed tensor product for connective objects [**Bos23**][A.3] and that the Rees algebra of the Nygaard filtration on the prismatic cohomology of a semiperfectoid is connective 3.3.2.5. So that we can make use of the first fact. We will use those assertions implicitly.

Now, for the claim on preservation of pullbacks, consider a pushout

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d] & S_1 \arrow[d] \\ S_2 \arrow[r] & S_3 \end{tikzcd}$$

of semiperfectoids. Then we can choose an integral perfectoid R mapping to S_0 and identify the absolute Nygaard filtrations on $\mathbb{\Delta}_{S_i}$ with the relative Nygaard filtrations over $\mathbb{A}_{\inf}(R)$ 3.3.2.2. Now the claim for $(\_)^{\mathcal{N},lp}$ follows from 3.3.2.9 and to deduce the claim for $(\_)^{\mathcal{N},\square}$ we additionally argue in the same way as for the Prismatisation 3.2.6.9.

We now prove the claimed Cartesian square. For this, let us be given an integral map $S \to \tilde{S}$ of semiperfectoids. Then we consider the following cube:

$$\begin{tikzcd} \tilde{S}^{\mathcal{N},\square} \arrow[rr] \arrow[dr, "\pi"] \arrow[dd] & & \tilde{S}^{\mathcal{N},lp} \arrow[dr, "\pi"] \arrow[dd] & \\ & \tilde{S}^{\mathbb{\Delta},\square} \arrow[rr] \arrow[dd] & & \tilde{S}^{\mathbb{\Delta},lp} \arrow[dd] \\ S^{\mathcal{N},\square} \arrow[rr] \arrow[dr, "\pi"] & & S^{\mathcal{N},lp} \arrow[dr, "\pi"] & \\ & S^{\mathbb{\Delta},\square} \arrow[rr] & & S^{\mathbb{\Delta},lp} \end{tikzcd}$$

In this cube, the horizontal squares above and below are Cartesian by construction, and the square in the front is Cartesian by 3.2.6.16, so the square in the back is Cartesian as well.

To prove the statement about cosheaves, it is enough to check that both functors send naive syntomic covers to cohomologically proper surjections of analytic stacks. By the already proven Cartesian square, it is enough to check this for the locally proper version. This boils down to the assertion that, for a naive syntomic cover $S \to \tilde{S}$, the map of filtrations

$$Fil^{\bullet}_{\mathcal{N}} \mathbb{\Delta}_S \to Fil^{\bullet}_{\mathcal{N}} \mathbb{\Delta}_{\tilde{S}}$$

induces a descendable map on Rees algebras. This follows from 3.3.2.10. $\square$

3.3.3.6. Using 3.3.3.5 and 3.1.2.8 we obtain preserving functors

$$(\_)^{\mathcal{N},\square}, (\_)^{\mathcal{N},lp} \colon f\mathcal{S}ch^{\mathrm{aff}}_{\mathbf{Z}_p} \to \mathcal{A}n\mathcal{S}tack.$$

To extend them to all p-adic formal schemes, we need to observe compatibility with etale morphisms. For this, we prove the following.

Proposition 3.3.3.7. *For any étale morphism $S \to \tilde{S}$ of p-adic formal schemes, the squares*

$$\begin{tikzcd} \tilde{S}^{\mathcal{N},\square} \arrow[r, "\pi"] \arrow[d] & \tilde{S}^{\Delta,\square} \arrow[d] \\ S^{\mathcal{N},\square} \arrow[r, "\pi"'] & S^{\Delta,\square} \end{tikzcd} \qquad \begin{tikzcd} \tilde{S}^{\mathcal{N},lp} \arrow[r, "\pi"] \arrow[d] & \tilde{S}^{\mathcal{N},lp} \arrow[d] \\ S^{\mathcal{N},lp} \arrow[r, "\pi"'] & S^{\mathcal{N},lp} \end{tikzcd}$$

are Cartesian. Furthermore, we have the following:

- *The functor $(\_)^{\mathcal{N},\square}$ sends open immersions (resp. covers) to open immersions (resp. covers) and étale morphisms (resp. covers) to étale morphisms (resp. covers).*
- *The functor $(\_)^{\mathcal{N},lp}$ sends open immersions (resp. covers) to closed immersions (resp. covers) and étale covers to proper surjections.*

PROOF. Using descent for ∞-topoi and 1.1.3.7, we see that all assertions in the statement are local on S. So by naive syntomic descent, we can assume S to be semiperfectoid.

We now prove the claimed pullback squares in the case $S \to \tilde{S}$ is an open immersion. In that case we can choose an integral perfectoid R and a surjection $R \to S$, but then the map in question comes as a pushout of an open immersion $R \to \tilde{R}$ so that, as all functors preserve pullbacks 3.3.3.5, it is enough to check the assertion for the later map. Now $\tilde{R}$ is also perfectoid 3.1.1.18 and the Nygaard filtration identifies with the I-adic filtration 3.3.2.3, which makes this case an easy computation.

Using that, we have the claimed cartesian squares for open immersions now, the assertions about preservation of those follow from the case of the Prismatisation 3.2.6.11.

As we have proven the proposition for open covers now, the assertions about étale morphisms can be checked Zariski locally. But Zariski locally any étale morphism $S \to \tilde{S}$ from a semiperfectoid comes as a pushout of an étale morphism $R \to \tilde{R}$ from an integral perfectoid 2.4.2.7. Again, $\tilde{R}$ is integral perfectoid as well 3.1.1.18 and we can run the argument above again. The assertions about preservation of étale morphisms then follow from the case of the Prismatisation 3.2.6.11 using the proven cartesian squares. □

Using 3.3.3.7, we can extend the Nygaard filtered prismatisation to all p-adic formal schemes.

Definition 3.3.3.8. For a p-adic formal scheme X, we will call the analytic stack

$$X^{\mathcal{N},\square}$$

the *Nygaard filtered Prismatisation*. The analytic stack

$$X^{\mathcal{N},lp}$$

will be called the *locally proper Nygaard filtered Prismatisation*.

To finish the section, let us discuss proper morphisms.

Proposition 3.3.3.9. *Consider a map of p-adic formal schemes $f\colon X \to S$ and let*

$$f^{\mathcal{N},\square}\colon X^{\mathcal{N},\square} \to S^{\mathcal{N},\square}$$

be the induced map on the solid Nygaard filtered prismatisation. Then we have the following:

(a) If f is $^+$proper, the square

$$\begin{tikzcd} X^{\mathcal{N},\square} \arrow[r] \arrow[d] & X^{\mathcal{N},lp} \arrow[d] \\ S^{\mathcal{N},\square} \arrow[r] & S^{\mathcal{N},lp} \end{tikzcd}$$

is Cartesian.
(b) If f is of $^+$finite type, $f^{\mathcal{N},\square}$ is !-able.
(c) If f is $^+$proper, $f^{\mathcal{N},\square}$ is locally proper.
(d) If f is proper, $f^{\mathcal{N},\square}$ is proper.

Proof. (a) follows the same way as in 3.3.3.5 using 3.2.6.16 and (c) follows from (a).

For (b) and (d) one argues the same way as for the solid Prismatisation 3.2.6.16 using 3.3.2.7. □

3.3.4. The Syntomification.

We now come to the solid syntomification.

Definition 3.3.4.1. Given a derived p-adic formal scheme X, we write $X^{\mathrm{syn},\square}$ for the pushout

$$\begin{tikzcd} X^{\mathbb{\Delta},\square} \coprod X^{\mathbb{\Delta},\square} \arrow[r] \arrow[d, "can"'] & X^{\mathcal{N},\square} \arrow[d] \\ X^{\mathbb{\Delta},\square} \arrow[r] & X^{\mathrm{syn},\square} \end{tikzcd}$$

where the upper horizontal map is induced by the two open immersions j_{dR} and j_{HT}. We will refer to this analytic stack as the *solid Syntomification.*

3.3.4.2. Using that the open immersions j_{dR} and j_{HT} are stable under base change 3.3.3.4 and descent for ∞-topoi, we see that for any morphism $f\colon X \to S$ of derived p-adic formal schemes, the induced square

$$\begin{tikzcd} X^{\mathcal{N},\square} \arrow[r] \arrow[d] & S^{\mathcal{N},\square} \arrow[d] \\ X^{\mathrm{syn},\square} \arrow[r] & S^{\mathrm{syn},\square} \end{tikzcd}$$

is Cartesian. Furthermore for any derived p-adic formal scheme X the map

$$X^{\mathcal{N},\square} \to X^{\mathrm{syn},\square}$$

is an étale surjection. Combining these two observations, one can deduce most properties for the solid syntomification from the solid Nygaard filtered Prismatisation.

Proposition 3.3.4.3. *Consider a morphism $f\colon X \to S$ of derived p-adic formal schemes and write*

$$f^{syn,\square}\colon X^{syn,\square} \to S^{syn,\square}$$

for the induced morphism. Then we have the following:

(a) *If f is a naive syntomic cover, then $f^{syn,\square}$ is a locally proper surjection of analytic stacks.*
(b) *If f is étale (resp. a étale cover), then $f^{syn,\square}$ is étale (resp. a étale surjection).*
(c) *If f is of $^+$finite type, then $f^{syn,\square}$ is !-able.*
(d) *If f is $^+$proper, then $f^{syn,\square}$ is locally proper.*
(e) *If f is proper, then $f^{syn,\square}$ is proper.*

PROOF. All the assertions are local on the target, so using 3.3.4.2, all statements follow from the analogous statements for the solid Nygaard filtered Prismatisation. For (a) this is 3.3.3.5, for (b) this is 3.3.3.7 and for (c),(d) and (e) this is 3.3.3.9. $\square$

CHAPTER 4

Six functors for syntomic cohomology

4.1. The six-functor formalism

In this section, we will apply what we have done so far to obtain well-behaved categorifications of syntomic cohomology.

4.1.1. The Construction.

Definition 4.1.1.1. Given a p-adic formal scheme X, will write

$$\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\Delta}(X) := \mathcal{D}_{\mathrm{qc}}(X^{\mathrm{syn},\square})$$

for the category of quasi-coherent sheaves on the syntomification of X and refer to it as *solid prismatic F-gauges* on X.

The aim of this subsection is now to prove the following:

Theorem 4.1.1.2. *The functor*

$$\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\Delta}(\_)\colon f\mathcal{S}ch^{op}_{Spf(\mathbf{Z}_p)} \to \mathcal{P}r^L_{st}$$

can be extended to a six-functor formalism on the category of (derived) p-adic formal schemes satisfying the following:

(A) Morphisms locally of ${}^{+}$finite type are !-able.
(B) Étale morphisms are cohomologically étale.
(C) Proper morphisms are cohomologically proper.
(D) The functor $\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\Delta}(\_)^*$ is an étale sheaf and the functor $\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\Delta}(\_)_!$ an étale cosheaf.
(E) It admits Tate twists. That is for any p-adic formal scheme X the object

$$\mathbf{1}_X(-1) := cof(\mathbf{1}_X \to f_*\mathbf{1}_{\mathbf{P}^1})$$

is $\otimes$-invertible. Furthermore the inverse $\mathbf{1}_X(1)$ understood as a line bundle on $X^{syn,\square}$ identifies with the Breuil-Kisin twist.
(F) Any smooth morphism is cohomologically smooth. Furthermore, for such a smooth morphism $f\colon X \to S$, we have an identification

$$f^!\mathbf{1}_S := \omega_f \simeq \mathbf{1}_X(d)$$

of the dualizing sheaf, where d is the relative dimension of f.
(G) There is a functorial identification

$$R\Gamma^{BMS}_{syn}(\_, \mathbf{Z}_p(n)) \simeq \mathrm{Hom}_{(\_)}(\mathbf{1}_{(\_)}, \mathbf{1}_{(\_)}(n))$$

*of the mapping spectrum with the syntomic cohomology of p-adic formal schemes as defined in [**BL22a**].*

Construction 4.1.1.3. Precomposing the six-functor formalism $\mathcal{D}_{\mathrm{qc}}$ on analytic stacks 2.1.2.17 with the solid syntomification $(\_)^{\mathrm{syn},\square}$, we obtain the six-functor formalism
$$\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\Delta}.$$

Proof of 4.1.1.2(A)(B)(C)(D). This follows from 3.3.4.3. $\square$

4.1.2. The additive orientation. In order to prove the rest, we will construct an additive orientation.

4.1.2.1. For a p-adic formal scheme X, we can compute the category of solid prismatic F-gauges as the equalizer
$$\mathcal{F}\text{-}\mathcal{G}auge^{\square}_{\Delta}(X) \longrightarrow \mathcal{D}_{\mathrm{qc}}(X^{\mathcal{N},\square}) \underset{j^*_{HT}}{\overset{j^*_{dR}}{\rightrightarrows}} \mathcal{D}_{\mathrm{qc}}(X^{\Delta,\square}).$$
As dualizable considering dualizable objects preserves limits [**Lur17**][4.6.1.11] and the dualizable object in quasi-coherent sheaves on an affine derived adic space are given by perfect complexes on the underlying ring, we see that the dualizable object in solid F-gauges on X recover perfect F-gauges as defined in [**Bha22**][6.1].

4.1.2.2. Recall from [**BL22a**][2.2.11+3.3.8] that there is a line bundle
$$\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \in \mathcal{P}erf(\mathbf{Z}_p^{\Delta})$$
called the *Breuil-Kisin twist*, which comes together with a frobenius automorphism [**BL22a**][2.2.14]
$$\varphi\colon F^*\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \simeq \mathcal{I}^{-1} \otimes \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}$$
where $\mathcal{I}$ denotes the Hode-Tate ideal. Heuristically, this line bundle should be thought of as
$$\otimes_{k\geq 0}(F^k)^*\mathcal{I}$$
where the Frobenius isomorphism is the evident one. Using 4.1.2.1, we can understand this object as a line bundle on $\mathbf{Z}_p^{\Delta,\square}$.

Recall that we had two maps
$$\mathbf{Z}_p^{\Delta,\square} \xleftarrow{\ \pi\ } \mathbf{Z}_p^{\mathcal{N},\square} \xrightarrow{\ t\ } \mathbf{D}^1_{\square}/(\mathbf{G}_m)^{\mathrm{an}}_{\square}$$
where π is the structure map and t comes from the construction as a Rees stack. Using these maps, the *Nygaard filtered Breuil-Kisin twist* can be defined via
$$\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} := \pi^*\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \otimes t^*\mathcal{O}(-1).$$
We will understand this line bundle on $\mathbf{Z}_p^{\mathcal{N},\square}$.

Note that on the one hand we have
$$j^*_{dR}\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} \simeq \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}$$
as composing j_{dR} with the structure map gives the identity, and the composition $t \circ j_{dR}$ factors over $(\mathbf{G}_m)^{\mathrm{an}}_{\square}/(\mathbf{G}_m)^{\mathrm{an}}_{\square}$. On the other hand, we have
$$j^*_{HT}\mathcal{O}_{\mathbf{Z}_p^{\mathcal{N}}}\{1\} \simeq F^*\mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\} \otimes \mathcal{I} \simeq \mathcal{O}_{\mathbf{Z}_p^{\Delta}}\{1\}.$$

Where, for the first isomorphism, we use that composing j_{HT} with the structure map recovers the frobenius and that the composition $t \circ j_{HT}$ classifies the Hodge-Tate locus $\mathbf{Z}_p^{HT} \subset \mathbf{Z}_p^{\Delta}$. And the second Isomorphism comes from the Frobenius automorphism on the Breuil-Kisin twist recalled above. In particular, using this identification, we obtain a line bundle

$$\mathcal{O}_{\mathbf{Z}_p^{\mathrm{syn}}}\{1\}$$

on $\mathbf{Z}_p^{\mathrm{syn},\square}$ which we will refer to as the *syntomic Breuil-Kisin twist*.

By base change, we also obtain line bundles

$$\mathcal{O}_{X^{\Delta}}\{1\}, \mathcal{O}_{X^{\mathcal{N}}}\{1\}, \mathcal{O}_{X^{\mathrm{syn}}}\{1\}$$

for an arbitrary derived p-adic formal scheme X.

4.1.2.3. Given a p-adic formal scheme X, then the mapping spectrum

$$\mathrm{Hom}_{X^{\mathrm{syn},\square}}(\mathcal{O}_{X^{\mathrm{syn},\square}}, \mathcal{O}_{X^{\mathrm{syn},\square}}\{n\})$$

via the presentation given in 4.1.2.1 become the fibre of the map

$$\varphi\{n\} - \mathrm{can}\colon Fil^n_{\mathcal{N}}\Delta_X\{n\} \to \Delta_X\{n\}$$

where $\varphi\{n\}$ is the twisted filtered frobenius and "can" comes from the inclusion of the filtration. In particular, this mapping spectrum identifies with the syntomic cohomology

$$R\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(n))$$

for p-adic formal schemes as defined in [**BL22a**][7.4]. This is statement (G) in 4.1.1.2.

4.1.2.4. Using 4.1.2.3, we recall that there is a map

$$\mathrm{c}_1^{\mathrm{syn}}\colon R\Gamma_{\text{ét}}(X, \mathbf{G}_m)[1] \to \mathrm{Hom}_{X^{\mathrm{syn},\square}}(\mathcal{O}_{X^{\mathrm{syn},\square}}, \mathcal{O}_{X^{\mathrm{syn},\square}}\{1\})$$

coming from the prismatic logarithm [**BL22a**][7.5.2]. This map is natural in X and identifies the target with the derived p-completion of the source [**BL22a**][7.5.6]. In particular, this gives us a theory of first Chern classes.

Construction 4.1.2.5. Using the first Chern classes from 4.1.2.4, as in 1.2.2.6 we can construct a morphism

$$\sum_{i=0}^{d} c_1^{\mathrm{syn}}(\mathcal{O}(1))^i\{d-i\}\colon \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbf{Z}_p^{\mathrm{syn},\square}}\{d-i\} \to f_*\mathcal{O}_{(\mathbf{P}^d)^{\mathrm{syn},\square}}\{d\}$$

where we write $f\colon \mathbf{P}^d \to \mathrm{Spf}(\mathbf{Z}_p)$ for the projection.

Proposition 4.1.2.6. *The morphism*

$$\sum_{i=0}^{d} c_1^{syn}(\mathcal{O}(1))^i\{d-i\}\colon \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbf{Z}_p^{syn,\square}}\{d-i\} \to f_*\mathcal{O}_{(\mathbf{P}^d)^{syn,\square}}\{d\}$$

is an isomorphism.

Proof. First, note that by naive syntomic descent and proper base change, we can assume that $\mathbf{Z}_p = R$ for R a semiperfectoid (even perfectoid).

As a proper push forward preserves limits, all objects appearing in the statement are (p, I)-complete. So we can check the assertion modulo (I, p). Now all objects are discrete and the proper push forward restricts to the push forward from classical quasi-coherent sheaves. This shows that we can check the assertion in $\mathcal{D}_{\mathrm{qc}}(R^{\mathrm{syn}})$.

Using the formula of mapping spectra recalled in 4.1.2.1, it suffices to check that the maps

(a) $\bigoplus_{i=0}^{d} \mathcal{O}_{R^{\Delta}}\{d-i\} \to f_*\mathcal{O}_{(\mathbf{P}^d)^{\Delta}}\{d\}$ in $\mathcal{D}_{\mathrm{qc}}(R^{\Delta})$
(b) $\bigoplus_{i=0}^{d} \mathcal{O}_{R^{\mathcal{N}}}\{d-i\} \to f_*\mathcal{O}_{(\mathbf{P}^d)^{\mathcal{N}}}\{d\}$ in $\mathcal{D}_{\mathrm{qc}}(R^{\mathcal{N}})$

are isomorphism. Now (a) follows from [**BL22a**][9.1.4.(4)] as mapping out of the unit is conservative on the category $\mathcal{D}_{\mathrm{qc}}(R^{\Delta})$ and (b) follows from [**BL22a**][9.1.4.(5)] as considering the filtered pieces is jointly conservative on $\mathcal{D}_{\mathrm{qc}}(R^{\mathcal{N}})$. □

Proof of 4.1.1.2(E)(F). (E) easily follows from the Projective bundle formula 4.1.2.6.

By p-completing, we obtain a finite limit preserving functor

$$\mathcal{S}m_{\mathbf{Z}}^{\mathrm{sep}} \to f\mathcal{S}ch_{\mathbf{Z}_p}.$$

Thus using $(A),(B),(C),(D)$ of 4.1.1.2, we see that the six-functor formalism $\mathcal{F}\text{-}\mathcal{G}auge_{\Delta}^{\square}$ is a geometrized geometric six-functor formalism. Furthermore, by 4.1.2.6 the map given in 4.1.2.4 gives us an additive orientation. Thus to prove (F) we can apply 1.2.4.8 and 1.2.5.13 using the deformation to the normal bundle for formal schemes 2.4.3.3. □

This finishes the proof of 4.1.1.2.

Bibliography

[AHI24] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Atiyah duality for motivic spectra. *preprint arXiv:2403.01561*, 2024.

[AHI25] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Algebraic cobordism and a Conner-Floyd isomorphism for algebraic K-theory. *Journal of the American Mathematical Society*, 38(1):243–289, arXiv:2303.02051 2025.

[AHW15] Aravind Asok, Marc Hoyois, and Matthias Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles. *Duke Mathematical Journal*, 166:1923–1953, arXiv:1506.07093 2015.

[AI22] Toni Annala and Ryomei Iwasa. Motivic spectra and universality of K-theory. *preprint arXiv:2204.03434*, 2022.

[AM24] Johannes Anschutz and Lucas Mann. Descent for solid quasi-coherent sheaves on perfectoid spaces. *preprint arXiv:2403.01951v2*, 2024.

[And21] Grigory Andreychev. Pseudocoherent and Perfect Complexes and Vector Bundles on Analytic Adic Spaces. *preprint arXiv:2105.12591*, 2021.

[Bha17] Bhargav Bhatt. Lecture notes for a class on perfectoid spaces. websites.umich.edu/ bhattb/teaching/mat679w17/lectures.pdf, 2017.

[Bha22] Bhargav Bhatt. Prismatic F-gauges (Lecture notes). math.ias.edu/ bhatt/teaching/mat549f22/lectures.pdf, 2022.

[BL22a] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *preprint arXiv:2201.06120*, 2022.

[BL22b] Bhargav Bhatt and Jacob Lurie. The prismatization of p-adic formal schemes. *preprint arXiv:2201.06124*, 2022.

[BMS16] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publications mathématiques de l'IHÉS*, 128:219–397, arXiv:1602.03148 2016.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publications mathématiques de l'IHÉS*, 129:199–310, arXiv:1802.03261 2018.

[Bor10] James M. Borger. The basic geometry of Witt vectors. II: Spaces. *Mathematische Annalen*, 351:877–933, arXiv:1006.0092 2010.

[Bos23] Guido Bosco. Rational p-adic Hodge theory for rigid-analytic varieties. *preprint arXiv:2306.06100*, 2023.

[BS17] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine Grassmannian. *Inventiones mathematicae*, 209(2):329–423, arXiv:1507.06490 2017.

[BS19] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Annals of Mathematics*, arXiv:1905.08229 2019.

[BS21] Bhargav Bhatt and Peter Scholze. Prismatic F-crystals and crystalline Galois representations. *Cambridge Journal of Mathematics*, arXiv:2106.14735 2021.

[Cam24] Juan Esteban Rodríguez Camargo. The analytic de Rham stack in rigid geometry. *preprint arXiv:2401.07738*, 2024.

[CD09] Denis-Charles Cisinski and Frédéric Déglise. Triangulated Categories of Mixed Motives. *Springer Monographs in Mathematics*, arXiv:0912.2110 2009.

[CLL25] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens. Universality of span 2-categories and the construction of 6-functor formalisms. *preprint arXiv:2505.19192*, 2025.

[CS19a] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry (Lecture notes). people.mpim-bonn.mpg.de/scholze/Analytic.pdf, 2019.

[CS19b] Dustin Clausen and Peter Scholze. Lectures on Condensed Mathematics (Lecture notes). people.mpim-bonn.mpg.de/scholze/Condensed.pdf, May 2019.

[CS22] Dustin Clausen and Peter Scholze. Condensed Mathematics and Complex Geometry. people.mpim-bonn.mpg.de/scholze/Complex.pdf, 2022.

[CS24] Dustin Clausen and Peter Scholze. Analytic Stacks Lecture (Lecture). youtube, 2024.

[DK14] Christopher Davis and Kiran S. Kedlaya. On the Witt vector Frobenius. *Proceedings of the American Mathematical Society*, 142(7):2211–2226, arXiv:1409.7530 2014.

[Dri20] Vladimir Drinfeld. Prismatization. *preprint arXiv:2005.04746*, 2020.

[FK13] Kayuhiro Fujiwara and Fumiharu Kato. Foundations of rigid geometry I. *preprint arXiv:1308.4734*, 2013.

[FS21] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence. *preprint arXiv:2102.13459*, 2021.

[GP16] Owen Gwilliam and Dmitri Pavlov. Enhancing the filtered derived category. *Journal of Pure and Applied Algebra*, 222(11):3621–3674, arXiv:1602.01515 2016.

[GR02] Ofer Gabber and Lorenzo Ramero. Almost ring theory - sixth release. *preprint arXiv:0201175*, 2002.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A Study in Derived Algebraic Geometry, Volume I: Correspondences and Duality*, volume 1 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2017.

[Haz78] Michiel Hazewinkel. *Formal Groups and Applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press, 1978.

[HK20] David Hansen and Kiran S. Kedlaya. Sheafiness criteria for Huber rings. *preprint kskedlaya.org/papers/criteria.pdf*, 2020.

[HM24] Claudius Heyer and Lucas Mann. 6-Functor Formalisms and Smooth Representations. *preprint arXiv:2410.13038*, 2024.

[Hol24] Adam Holeman. *Derived δ-Rings and Relative Prismatic Cohomology*. PhD thesis, Northwestern University, arXiv:2303.17447 2024.

[Hub93a] R. Huber. Continuous valuations. *Mathematische Zeitschrift*, 212(1):455–477, 1993.

[Hub93b] Roland Huber. *Bewertungsspektrum und rigide Geometrie*. PhD thesis, Regensburg University, 1993.

[Hub94] Roland Huber. A generalization of formal schemes and rigid analytic varieties. *Mathematische Zeitschrift*, 217(4):513–552, 1994.

[Hub96] Roland Huber. *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, volume 30 of *Aspects of Mathematics*. Vieweg+Teubner Verlag, 1996.

[Hub24] Edith Hubner. Animated λ-rings and Frobenius lifts. *preprint arXiv:2404.15040*, 2024.

[Jam84] Ioan Mackenzie James. *General topology and homotopy theory*. Springer New York, 1984.

[Joy85] André Joyal. δ-anneaux et vecteurs de Witt. *CR Math. Rep. Acad. Sci. Canada*, 7(3):177–182, 1985.

[KL13] Kiran S. Kedlaya and Ruochuan Liu. Relative p-adic Hodge Theory: Foundations. *preprint arXiv:1301.0792*, 2013.

[KL16] Kiran S. Kedlaya and Ruochuan Liu. Relative p-adic Hodge theory, II: Imperfect period rings. *preprint arXiv:1602.06899*, 2016.

[KR25] Adeel A. Khan and David Rydh. Virtual Cartier divisors and blow-ups. *Selecta Mathematica*, 31(4):1–28, arXiv:1802.05702 2025.

[Lur17] Jacob Lurie. Higher Algebra. math.ias.edu/ lurie/papers/HA.pdf 2017.

[Lur18] Jacob Lurie. Spectral Algebraic Geometry. math.ias.edu/ lurie/papers/SAG-rootfile.pdf 2018.

[Man22] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. PhD thesis, Rheinische Friedrich-Wilhelms-Universitaet Bonn (Germany), arXiv:2206.02022 2022.

[Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Advances in Mathematics*, 291:403–541, arXiv:1404.2156 2016.

[Mor17] Matthew Morrow. Foundations of perfectoid spaces (Lecture notes). math.ias.edu/ lurie/ffcurve/Lecture6-8-Perfectoid.pdf, 2017.

[Mou19] Tasos Moulinos. The geometry of filtrations. *Bulletin of the London Mathematical Society*, 53(5):1486–1499, arXiv:1907.13562 2019.

[MV99] Fabien Morel and Vladimir Voevodsky. A1-homotopy theory of schemes. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 90(1):45–143, 1999.

[MW24] Samuel Meyer and Ferdinand Wagner. Derived q-Hodge complexes and refined TC^-. *preprint arXiv:2410.23115*, 2024.

[Rak20] Arpon Raksit. Hochschild homology and the derived de Rham complex revisited. *preprint arXiv:2007.02576*, 2020.

[RS22] S. S. Ragimov and Tomer M. Schlank. The ∞-Categorical Reflection Theorem and Applications. *preprint arXiv:2207.09244*, 2022.

[Sch12] Peter Scholze. Perfectoid Spaces. *Publications mathématiques de l'IHÉS*, 116(1):245–313, arXiv:1111.4914 2012.

[Sch23] Peter Scholze. Six-Functor Formalisms (Lecture notes). people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf, 2023.

[Sta] The Stacks Project Authors Stacks. Stacks Project. `https://stacks.math.columbia.edu`.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley Lectures on p-adic Geometry*, volume 207 of *Annals of Mathematics Studies*. Princeton University Press, 2020.

[Tan] Longke Tang. Gysin maps for oriented $P1-$spectra.

[Tan22] Longke Tang. Syntomic cycle classes and prismatic Poincaré duality. *Compositio Mathematica*, 160(10):2322–2365, arXiv:2210.14279 2022.

[Wed19] Torsten Wedhorn. Adic spaces. *preprint arXiv:1910.05934*, 2019.

[Zav23] Bogdan Zavyalov. Poincaré Duality in abstract 6-functor formalisms. *preprint arXiv:2301.03821*, 2023.