The quest for a theory of quantum gravity remains one of the central challenges in theoretical physics. A promising approach to this problem in our four-dimensional universe lies in first understanding lower-dimensional theories. This thesis therefore focuses on studying quantum gravity in two dimensions, primarily considering Jackiw-Teitelboim (JT) gravity, for which a connection with random matrix ensembles was recently uncovered. The central idea is to investigate the implications of "random matrix universality" — a property of such ensembles — in JT gravity and related theories. These implications manifest as specific mathematical properties of the geometric objects underlying the studied quantum gravity theories and can be understood as imprints of quantum chaoticity of those theories. To pursue this program across all varieties of the considered gravitational theories, this thesis extends and implements computational methods for these geometric objects, particularly to settings involving unorientable geometries.

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LIST OF PUBLICATIONS

This thesis aims at putting together the results obtained by the author and collaborators during the last years in a coherent and, to a reasonable amount, self-contained way. Said results that are either already published or will be soon after the colloquium of this thesis are given in the following list. There, the works which will be the primary focus of this thesis are indicated with an asterisk.

PUBLISHED ARTICLES

∗ **T. Weber**, F. Haneder, K. Richter and J. D. Urbina, 'Constraining Weil-Petersson volumes by universal random matrix correlations in low-dimensional quantum gravity', Journal of Physics A: Mathematical and Theoretical 56 (20), 205206 (2023) [1]

∗ **T. Weber**, J. Tall, F. Haneder, J. D. Urbina and K. Richter, 'Unorientable topological gravity and orthogonal random matrix universality', Journal of High Energy Physics 2024 (7), 267 (2024) [2]

∗ J. Tall, **T. Weber**, J. D. Urbina and K. Richter, 'Chaos and moduli space volumes in unorientable JT gravity', Journal of High Energy Physics 2025 (7), 46 (2025) [3]

- F. Haneder, J.D. Urbina, C. Moreno, **T. Weber** and K. Richter, 'Beyond the ensemble paradigm in low-dimensional quantum gravity: Schwarzian density, quantum chaos, and wormhole contributions', Physical Review D 111, 126015 (2025) [4]

∗ **T. Weber**, M. Lents, J. Dieplinger, J. D. Urbina and K. Richter, ´Topological gravity for arbitrary Dyson index´, arXiv:2507.03172, (July 2025), Journal of High Energy Physics 2025 (11), 88 (2025) [5]

PUBLICATIONS IN PREPARATION

- **T. Weber**, J. Dieplinger, C. Forster, L. Resch, F. Evers and K. Richter, Universality of spectral long-range correlations of localizing Hamiltonians (in preparation)

∗ **T. Weber**, J. D. Urbina and K.Richter, Topological gravity for arbitrary Dyson index: The microcanonical story (in preparation) [6]

CONTENTS

1

INTRODUCTION: QUANTUM CHAOS AND QUANTUM GRAVITY

Quantum chaos has emerged as one of the central themes of modern theoretical physics, bridging such diverse areas as atomic, nuclear and condensed matter physics, quantum information and quantum gravity. The origin of this field is the observation that a system which is classically chaotic behaves radically different from a classically integrable system upon quantisation. There are several examples showcasing this difference of behaviour, e.g. in the structure of wave functions, which in the chaotic case exhibit random wave behaviour [7] or scarring [8], or in dynamical probes like the observation of a decay of the Loschmidt echo after a small perturbation of the quantum Hamiltonian [9]. Most relevant for our application, however, is the manifestation of chaos in a classical system in the spectrum of the associated quantum system. This hallmark of chaos is given by the fidelity of certain "universal" features of the spectra to the predictions of *random matrix theory* (RMT) as implied by a well-known conjecture due to Bohigas, Giannoni and Schmit (BGS) [10]. Notably, this also allows the characterisation of quantum systems without a classical counterpart as quantum chaotic by checking for agreement with RMT.

By RMT we mean an ensemble of matrices of large dimension drawn "randomly" from a certain matrix space i.e. according to a probability distribution on the respective set. This theory was proposed by Wigner [11, 12] to study the spectra of large nuclei and later developed further by Dyson [13, 14] and others (a rather complete review of the historic development can be found in [15, 16]). Notably, for the spectral observables used to characterise quantum chaos, the choice of probability distribution is not relevant, a property referred to as *random matrix universality*[1], and hence the easiest (Gaussian) distribution is used to evaluate the predictions from the matrix ensemble. Hence, in the following we shall reserve the name "RMT" for the ensemble drawn from a Gaussian distribution, while we will denote an ensemble of matrices with an arbitrary probability distribution as a *matrix model*. Furthermore, let us already remark that there are three standard sets of matrices (the Wigner-

1 This is one of the most important starting points of this thesis and hence we give a more complete treatment of this in section 2.4.

Dyson classes) from which to choose the matrices for a matrix model. They are distinguished by the so-called *Dyson index* $\beta \in \{1, 2, 4\}$ and in the application for characterising chaos via the BGS conjecture the choice of the appropriate class is determined by the presence or absence of time-reversal symmetry and further symmetry properties of the considered Hamiltonian (cf. section 3.1). RMT and matrix models, apart from their relation to quantum chaos, have several other important applications in physics: in the theory of disordered metals (cf. [17–19] and references therein), for studying the Dirac operator in QCD (cf. [20] and references therein) etc. ; a more complete list of applications can be found in [16, 21]. Even outside physics it has some useful applications, for example in biology (e.g. for RNA folding [22]) or in quantitative finance (cf. [23] and refs. therein). Most notable, being in the same spirit as the work presented in this thesis, there is an application of RMT in number theory, more precisely, to the Riemann ζ function. Here, the starting point is the conjectured RMT-like statistical distribution of the imaginary parts of the non-trivial zeroes of the Riemann zeta function [24], supported by a substantial amount of numerical evidence [25], which one can use to obtain number-theoretical results [26, 27].

1.1 QUANTUM GRAVITY AND HOLOGRAPHY

In this thesis we are concerned with the application of RMT in quantum gravity, also deriving new results for mathematical objects from this reasoning, specifically by applying universality, i.e. the quantum chaotic nature of the theory. Finding a theory of quantum gravity is, of course, one of the most sought for goals of theoretical physics. Finding such a theory would potentially enable one to give an answer to such fundamental questions as how the universe began, what happens inside a black hole when you fall in (beyond the sure fact that you will never be able to tell anyone outside) and how spacetime looks like at the Planck scale. Many proposals exist to tackle this problem, a (biased) selection of which is given by string theory (e.g. [28, 29]) and loop quantum gravity (LQG) [30]. As it is well-known, none of these theories (or the other ones not mentioned here) has so far provided a satisfactory answer to the question how a complete theory of quantum gravity in at least four spacetime dimensions should look like. While here this observation is made purely from a theoretical point of view, it has to be mentioned that in the end a theory needs to produce falsifiable predictions that then have to be tested experimentally. To this end, a variety of tests has already been proposed to check the predictions of some proposed theories of quantum gravity. These proposals include the use of astrophysical measurements to check e.g. small violations of Lorentz invariance of spacetime, proposed by some theories of quantum gravity (see [31] for a recent review). Further tests are not concerned with checking specific features of proposed theories but rather whether gravity is, in fact, a "quantum" phenomenon, i.e. can for example induce entanglement. This is especially interesting, since the assumption of a quantum nature of gravity, unsurprisingly, underlies all theories of quantum gravity. Hence, an experimental verification of this paradigm, recently challenged theoretically by proposed non-quantum theories of gravity (e.g. [32]), would be most useful (see [33] for a recent review).

Returning to a purely theoretical point of view, we note that considerable progress has been made in the last two decades in understanding quantum gravity through the access to it via the *holographic principle*. The holographic principle proposes that the information contained within a gravitational system is encoded on its (lower-dimensional) boundary [34, 35]. The idea for this arises from the famous observation that the entropy of a black hole is proportional not to its volume, as one would expect for an extensive quantity like entropy, but to the area of its event horizon [36, 37]. This idea was given a precise and far-reaching realisation in the Anti–de Sitter/Conformal Field Theory (AdS_d/CFT_{d-1}) correspondence, which relates, in the sense of a duality, a gravitational theory in a d-dimensional AdS (i.e. constant negative curvature) spacetime, the "bulk" theory, with a non-gravitational conformal field theory living on its ($d-1$ dimensional) boundary [38–40]. We will not attempt to give an overview of the enormous amount of applications this has had in quantum gravity; such an exposition can for example be found in [41]. Furthermore, mainly due to the weak coupling regime of the bulk theory corresponding to the large coupling regime of the boundary theory, the correspondence could be used to gain information on strongly-correlated complex quantum systems, like for example the upper bound on shear viscosity [42, 43] or the Lyapunov exponent (as computed via out-of-time-correlators (OTOCs)) [44], or to study genuine strong-coupling phenomena that are hard (or impossible) to access by other methods. Examples for the latter in condensed matter are the study of non-Fermi liquids (e.g. [45]), holographic superconductors [46, 47], in particular physics that of relativistic heavy-ion collisions (e.g. [48–51]).

1.2 JT GRAVITY: A BRIDGE BETWEEN QUANTUM CHAOS AND QUANTUM GRAVITY

We shall here be concerned with the application of the idea of holography for the setting of $d = 2$. Regarding the goal of treating gravity in the actual universe, the motivation for studying two dimensions comes from the thought that it might be a good idea to first study a "simpler" lower-dimensional theory rather than full-fledged four-dimensional quantum gravity to gain some understanding of how gravity works there and to then increase the dimensionality subsequently to go to the real goal of treating actual quantum black holes. The theory which we will (mostly) study here is referred to as *Jackiw-Teitelboim* (JT) gravity. It was first discussed by Jackiw [52] and Teitelboim [53], hence the name, and is a theory of gravity in two (euclidean) dimensions containing a scalar field denoted as the *dilaton* as an additional field to the metric. As we will discuss in detail in the main text (section 2.1), the effect of the dilaton is essentially that of a Lagrange multiplier enforcing solutions of the equations of motion (when studying the theory classically) or contributions in the gravitational path-integral (when studying the quantum theory) to be of constant negative curvature[2]. There are several reasons to be interested in this theory. First, it arises as the low energy description of the geometry near the boundary of extremal black holes in $d = 4$ spacetime dimensions [54–56] and is hence directly related to higher-dimensional gravity. Second, the theory has been of large interest in recent years also due to the well-known Sachdev-Ye-Kitaev (SYK) model [57–59] in the small energy regime being described by the same action one finds in JT gravity, the so-called Schwarzian action [60]. This relation was then realised to be understandable in the sense of the AdS_2/CFT_1 duality [61–64]. This motivated intense study of the theory itself in the following years, an extensive list of references can be found in [65]. Among other notable results, such as the resolution of the black hole information paradox (in JT gravity) [66, 67], these studies revealed the surprising fact that there is another theory to which JT gravity is dual: A matrix model [68].

The setting where this duality can be seen most direct is that of computing correlation functions of n boundary operators via bulk JT gravity. This translates the computation to the evaluation of the gravitational path-integral over hyperbolic surfaces with n boundaries[3], which naturally implies the correlators to have a *topological expansion*, i.e. a (perturbative) expansion where every contribution corresponds to a geometry with a certain number of "handles", the surface's *genus*. The individual contributions to this expansions can then be shown to be determined by the "volume of inequivalent metrics that can be put on a hyperbolic surface of genus g and n boundaries", i.e. the volume of the so-called *moduli space* of hyperbolic surfaces with genus g and n (geodesic) boundaries [68, 69]. The computation of these volumes, the *Weil-Petersson volumes*, is a well-studied subject in mathematics and a way to compute all the volumes, taking the form of a recursion relation, has famously been found by Mirzakhani [70], leading to her being awarded the

2 For details on the gravitational path-integral in two euclidean dimensions the reader is referred to section 2.1. Let us only note here that it means a path-integral over the metric (which, surprisingly, turns out to be a mathematically well-defined operation here) together with a sum over all topologies one chooses to include.

3 The choice of the boundary operators enters here via defining the boundary conditions for the surface.

2014 Fields medal. Remarkably, this recursion is equivalent to a recursion relation computing correlation functions in a certain matrix model [71], which hence can be used to compute the topological expansion of the considered JT gravity observables by a matrix model computation, as first done in [68]. This is the (more) precise statement of the aforementioned duality of JT gravity with a matrix model.

Let us briefly remark that this connection of quantum gravity with matrix models has a long tradition, originally starting with the idea of studying the gravitational path-integral by considering "random polygonisations" of surfaces in the 1980's. Here, the connection to matrix models arises by the observation that they provide a generating function for such polygonisations (cf. section 2.2). An introductory review to this body of work with an extensive list of references is given in [72], a more recent one in [73]. The maybe most well-known result in this context was the proof of the Witten conjecture [74], a result concerning certain correlation functions, so-called *intersection numbers*, in a theory tightly related with JT gravity, known as *topological* gravity. It was found by Kontsevich [75] using matrix model techniques and was one of the main reasons for awarding him the 1998 Fields medal.

1.3 MAIN IDEAS, STRUCTURE AND RESULTS OF THIS THESIS

Building on the duality of JT gravity with a matrix model, we present in this thesis our contributions to the investigation of moduli space volumes of hyperbolic bordered surfaces and hence, two dimensional quantum gravity. We can already state the essential observation for (most of) this work: JT gravity correlation functions are determined by the Weil-Petersson volumes or, equivalently, by a dual matrix model. Random matrix universality implies universal results for certain observables of any matrix model, in particular that dual to JT gravity. Consequently, we can study the implications of random matrix universality on moduli space volumes via the intermediate step of the JT gravity matrix model dual. Notably, using the BGS conjecture, this can be formulated as investigating the implications of quantum chaos on moduli space volumes. Throughout our work we have focussed on a specific universal observable: the two-point function of densities of states, evaluated at energies with a difference that is, in a well-defined sense, "small". For more details on this, the reader is referred to section 2.4.

It is important to note that this idea can be pursued for all three Wigner-Dyson classes, mentioned above. In contrast to this, the duality between JT gravity and a matrix model was worked out to a degree such that this study can be performed only for one of these classes, the unitary one ($\beta = 2$), which corresponds to including only orientable manifolds into the gravitational path-integral. Consequently, a major part of our work was concerned with building on the already established ideas of extension of the duality to the other Wigner-Dyson classes, given in [69, 76], to pursue our investigation of the implications of universality also for the gravitational theories occurring in this setting. On the gravitational side this extension is performed by including also unorientable manifolds into the gravitational path integral. We succeeded not only in extending the knowledge about the moduli space volumes for geometries dual to the matrix models for $\beta \in \{1, 4\}$ to the degree required to implement our program of investigating the implications of universality, but also proposed and investigated the extension of this to arbitrary values of the Dyson index, i.e. allowing for a smooth interpolation between all symmetry classes. Also in the latter setting we were able to find certain imprints of random matrix universality, in a generalised sense. Let us, however, already note that due to several reasons, given in the main text, we restrict our reasoning for the unorientable setting for large parts of the discussion to the aforementioned theory of topological gravity, arising from JT gravity in a specific limit. This essentially enables us to capture most of the relevant behaviour of full unorientable JT gravity, while avoiding some of the from a certain perspective "unnecessary" complications this theory entails.

To present these our results, the main part of this thesis is divided into three parts:

FUNDAMENTAL CONCEPTS. In chapter 2 the necessary background to understand the rest of the main part is recalled concisely but in sufficient detail. In particular, we will first recall in section 2.1 the definition of JT gravity and the derived theory of topological gravity, being the two gravitational theories of most interest in this work. Here we shall, for the sake of clarity, restrict ourselves only to the case of orientable manifolds in the gravitation path-integral, defining the correlation functions of these theories. We will recall how to relate their computation for JT gravity to that of the Weil-Petersson volumes, already mentioned above, and explain how to find those using Mirzakhani's celebrated recursion. For topological gravity we give a way to compute the relevant moduli space volumes using the technique of Kontsevich diagrammatics, highlighting the connection of this theory with intersection numbers.

Second, we precisely define in section 2.2 what shall be known as a matrix model in this thesis and define the observables of such a model which we are interested in. Furthermore, we gather from the literature how to find the perturbative expansion of all of these observables using the so-called *loop equations* approach and give basic examples. Additionally, we discuss a simplification of this approach in a particular class of matrix models, those for $\beta = 2$, leading to the *topological recursion*, a simplified version of the loop equations.

Third, we study in section 2.3 how a particular matrix model is dual to JT gravity in the aforementioned sense of agreeing with its perturbative topological expansion to all orders. The reason for this, as we shall discuss, is rooted in the topological recursion for this particular matrix model being actually equivalent to a certain integral transform of Mirzakhani's recursion. Using these results, we explore how this extends to a duality for topological gravity with the Airy (matrix) model, which arises from the matrix model dual to JT gravity in a specific limit. We conclude this discussion with applying the duality to compute the WP volumes of topological gravity for surfaces with one geodesic boundary for arbitrary topology.

Fourth, in section 2.4 we give a more detailed recollection of the BGS conjecture as a deep relation of chaos and random matrices and the precise sense in which it is understood here. Furthermore, we discuss in detail the notion of *universality* in matrix models, i.e. the emergence of independence on specific properties of a matrix model for a class of observables in their respective universal regime.

BEYOND ORIENTABILITY IN 2D QUANTUM GRAVITY. Building on the background material discussed in the first part, we shall consider in chapter 3 the first larger set of new results obtained in the work leading to this thesis, i.e. those concerned with studying the generalisation of the duality beyond the orientable/unitary case. To give these results, in section 3.1 we first recall from the literature how the generalisation of the duality should be performed, i.e. what types of surfaces have to be included in the gravitational path-integral to find gravitational theories dual to the remaining two Wigner-Dyson classes (and, beyond this, for arbitrary values of the Dyson index).

Second, we recall in section 3.2 the study of a specific case of including unorientable manifolds into JT gravity. Specifically it is relevant, what problems arise for the moduli space volumes, how to circumvent them and that one can directly prove the duality to a certain matrix model as in the orientable case. Here, we can start to discuss our own results. The guiding principle of our computations there, is to work on the matrix model side, which in the orientable setting has proven itself as being the side of the duality where computations are easiest. We will state our work's two-thonged approach of *a):* Circumventing the difficulties inherent to the JT moduli space volumes by studying topological gravity instead, allowing for the development of computational methods to compute moduli space volumes and study their basic structure to relatively large genus and numbers of boundaries. And *b):* Using the matrix model dual to the so-called *minimal string*, which serves as an interpolation between topological and JT gravity, to extend said methods to the JT setting, facilitating strongly the relevant computations and enabling the computation to higher orders than possible with previous approaches. One particularly convenient thing about the method we use is that it contains a parameter (the Dyson index β, introduced above) which allows one to tune between the different varieties of the gravitation theories.

Third, in section 3.3 we first discuss this method of computing correlation functions in a matrix model which is based on the loop equations and give some examples (section 3.3.1). Furthermore, in section 3.3.2 we discuss and prove the generic structure of said correlation functions in terms of the Dyson index for all matrix models, in particular for those dual to topological/JT gravity. For the Airy model, we give in section 3.3.3 the general structure of an, arguably, the most important observable for all numbers of boundaries and topologies and work out many explicit results relevant for the discussions of the next chapter. For the case of highest importance for this discussion we provide a diagrammatic proof of the structure. For JT gravity we give in section 3.3.4 an alternative proof of the general structure in terms of the Dyson index using a modified Mirzakhani-like recursion, providing the structure with a geometric interpretation. Furthermore, we provide the results relevant for the subsequent discussion of chaos in JT gravity and explain how to obtain them. We then conclude this chapter in section 3.4 by giving some outlook at potentially interesting further developments of these our geometric considerations.

THE INTERPLAY OF CHAOS AND GRAVITY: CONSISTENCY AND CONSTRAINTS. Using now the geometric results found in chapter 3, we go on to investigate in chapter 4 what the imprints of chaos, in the guise of universal random matrix behaviour of spectral two-point functions, on the discussed geometric objects, the Weil-Petersson volumes, are. We start this discussion off by showing in section 4.1 the explicit implications of the universal behaviour of said two-point functions on a related quantity, which is known as the canonical *spectral form factor* (SFF).

In section 4.2 we first present in section 4.2.1 the literature result that in the orientable case there is a perfect direct match between the canonical SFF obtained from JT/topological gravity and the mentioned prediction from the universal random matrix behaviour. We go on to clarify the surprising nature of this matching by discussing a related microcanonical observable. Most important, however, we show in the subsequent section 4.2.2 how this matching, which can be seen as the first sign of RMT universality in JT/topological gravity, leads to non-trivial constraints on the WP volumes, being the second imprint of RMT universality.

Going over to the unorientable setting first for topological gravity in section 4.3 we encounter in section 4.3.1 behaviour of the canonical SFF, as computed from the moduli space volumes, that is, at first sight, not consistent with the expected universal random matrix behaviour. We find a way to resolve this and give strong evidence for the presence of universal behaviour also in the setting of unorientable topological gravity. We further show that for unorientable JT gravity, except for the topological gravity behaviour inherent to the results there, no additional difficulties appear and the expected agreement can be observed. In section 4.3.2 we again investigate constraints, now imposed on the unorientable moduli space volumes for topological gravity, arising from the matching to universal random matrix behaviour.

Finally, in section 4.4 we discuss the behaviour of the gravitational theories and their dual matrix models upon tuning the Dyson index to values in between those corresponding to purely orientable or specific unorientable incarnations of the theory. In section 4.4.1 we give support to our suspicion that universal random matrix behaviour persists to be present in these incarnations of the model by showing that all the constraints found for the purely unorientable/orientable settings are, except certain understood exceptions, fulfilled by the topological gravity moduli space volumes generalised to arbitrary Dyson index. Furthermore, we give numerical evidence of the model reproducing the universal behaviour of the microcanonical SFF found from the Gaussian matrix model at arbitrary β, which is not fully known analytically. Building on this, we show in section 4.4.2 how the general structure in terms of the Dyson index and the known universal random matrix results can be used to determine an important part of this universal result for arbitrary Dyson index analytically. We underpin this computation again by numerical evidence and additionally by a comparison with analytical literature results for the microcanonical SFF in the Gaussian matrix model at arbitrary β. We then conclude this chapter in section 4.5, again by pointing out interesting future directions of research, extending the reasoning presented here.

We then conclude the whole thesis in chapter 5 by discussing more broadly applications and possibilities of extension of the results described in this thesis.

This thesis also includes 5 appendices, which contain: an example for the computation of a WP volume by using Mirzakhani's recursion (appendix A), additional proofs and examples regarding the relation of the matrix model observables studied in section 2.2 (appendix B), a short introduction to the concept of *hyperfunctions* (appendix C), details of the numerical implementation of the Gaussian matrix model at arbitrary values of the Dyson index (appendix D) and the collection of the (quite lengthy) constraints for the unorientable Airy WP volumes (appendix E). Apart from this, a collection of all the moduli space volumes we have computed for this work is included as electronic supplementary material.

2

FUNDAMENTAL CONCEPTS

This chapter gives an overview of the topics relevant for understanding the main part of this thesis. These are not the author's own results so this chapter is mostly based on literature sources. The most important of these are the original papers [10, 68–71] and the reviews [21, 65]. Additionally, the author's master thesis [77] contains an enlarged background section discussing the presented concepts in detail with all relevant proofs. We refer the interested reader there for more details.

2.1 JT AND TOPOLOGICAL GRAVITY: THE ESSENTIALS

ACTION AND BOUNDARY CONDITIONS. JT gravity is a theory of two-dimensional gravity which, in this thesis, is studied purely in euclidean signature. The theory is constructed by adding an additional complex scalar field ϕ known as the *dilaton* to Einstein gravity in two dimensions and minimally coupling it to the Ricci scalar in such a way that its classical equations of motion give a constant negative Ricci scalar [52, 53]. Consequently, the bulk part of the action is given by

$$I_{\text{JT, Bulk}}[\phi, \mathbf{g}] = -\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}\mu\,\sqrt{g}R\right] - \left[\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}\mu\,\sqrt{g}\phi(R+2)\right], \quad (2.1.1)$$

where $\mathrm{d}\mu\,\sqrt{g}$ is the volume form induced by the metric $\mathbf{g}$ and $\mathcal{M}$ is a 2-manifold, for which it is of crucial importance that it can have boundaries. This action can be motivated by studying the near-horizon geometry of an extremal four-dimensional Reissner-Nordström (i.e. charged but not rotating) black hole, which is review e.g. in [54, 55, 78].

As in the standard treatment of the Einstein-Hilbert action, having boundaries implies that one has to add a Gibbons-Hawking-York boundary term to make the action principle well defined. Adding the standard term for the

Einstein-Hilbert part of the action and a boundary term for the dilaton part of the action, derived in a similar way (cf. e.g. [77]), one finds

$$I_{\text{JT, t(otal)}}[\phi, \mathbf{g}] := I_{\text{top}}[\phi, \mathbf{g}] + I_{\text{JT}}[\phi, \mathbf{g}], \tag{2.1.2}$$

$$I_{\text{top(ological)}}[\phi, \mathbf{g}] = -\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}\mu\, \sqrt{g}R + \int_{\partial\mathcal{M}} \mathrm{d}\omega\, \sqrt{h}K\right], \tag{2.1.3}$$

$$I_{\text{JT}}[\phi, \mathbf{g}] = -\left[\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}\mu\, \sqrt{g}\phi(R+2) + \int_{\partial\mathcal{M}} \mathrm{d}\omega\, \sqrt{h}\phi(K-1)\right], \tag{2.1.4}$$

where h is the (1-dimensional) metric induced by $\mathbf{g}$ on the boundary of $\mathcal{M}$, denoted as $\partial\mathcal{M}$, $\mathrm{d}\omega\,\sqrt{h}$ the associated volume form and K the extrinsic curvature. Note that in I_{JT}, a counterterm was introduced to cancel a divergence which will be explained more closely later on. Furthermore, by the Gauss-Bonnet theorem, one finds

$$I_{\text{top}}[\phi, \mathbf{g}] = I_{\text{top}} = -S_0\chi(\mathcal{M}), \tag{2.1.5}$$

with the Euler characteristic of $\mathcal{M}$ denoted as $\chi(\mathcal{M})$. Thus, one can see clearly that this part of the action solely depends on the topology of the manifold $\mathcal{M}$ which, as this component of the action corresponds to the usual Einstein-Hilbert functional, is one way to explain why gravity in $2d$ is a topological theory.
Putting everything together, one can write the action of JT gravity as

$$I_{\text{JT,t}}[\phi, \mathbf{g}] = -S_0\chi(\mathcal{M}) - \left[\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}\mu\, \sqrt{g}\phi(R+2) + \int_{\partial\mathcal{M}} \mathrm{d}\omega\, \sqrt{h}\phi(K-1)\right], \tag{2.1.6}$$

which is the final form of the action that will be considered here.

As necessary for any theory, a set of boundary conditions for its fields, here the metric and the dilaton, are needed. For the simplest and arguably most useful incarnation of the theory, one chooses Dirichlet boundary conditions on both fields [63]

$$h = \frac{1}{\epsilon^2}, \quad \phi\Big|_{\partial M} = \frac{\gamma}{\epsilon}, \tag{2.1.7}$$

where ϵ is introduced as a regulator and γ is a constant that can be interpreted as the (regularised) boundary value of the dilaton. This can be motivated by studying the classical theory on a manifold with, for simplicity, one boundary of a finite length. As one can see directly from the action, the classical equation of motion for the dilaton constrains the Ricci scalar to be constant negative. Hence, the simplest classical solution is AdS_2, which has a natural conformal boundary. Since the length of this boundary diverges, this necessitates a regularisation. To perform this regularisation and discuss a boundary of finite (renormalised) length, one can cut off AdS_2 near the conformal boundary at some curve and take this curve, $\zeta : [0, \beta] \to AdS_2$ as the boundary. For this to make sense, it has to hold that $\zeta(0) = \zeta(\beta)$ or, equivalently, one can define $\zeta : \mathbb{R} \to AdS_2$ and require $\zeta(x + \beta) = \zeta$. Computing the length of said curve, one finds

$$l(\zeta) = \int_0^\beta du \sqrt{g_{\mu\nu} \frac{d\zeta^\mu}{du} \frac{d\zeta^\nu}{du}} = \int_0^\beta du \sqrt{h} = \frac{\beta}{\epsilon}, \tag{2.1.8}$$

which motivates one to consider β as the regularised length of the boundary. Alternatively to the boundary condition on the metric, one can require this statement about the length of the cut-off boundary, being a proper-time curve, which also fixes the value of the boundary's induced metric. It is interesting to note that also other choices of boundary conditions are possible, all of which are discussed in [79]. We will briefly revisit them in section 2.3.

THE GRAVITATIONAL PATH-INTEGRAL. Having given now the action and boundary conditions for the theory we would like to study, we can now introduce the observable we are mainly interested in. For the chosen boundary conditions, it makes sense to consider the theory on manifolds with n boundaries that, as we will see in a second, also for the quantum theory will be hyperbolic and can thus be cut off in the same way as discussed above to get boundaries of regularised lengths β_i. Using the $(N)AdS_2/(N)CFT_1$ dictionary [63] or by extending the reasoning that computing the euclidean path integral in a field theory with a finite-length circular boundary gives rise to the expectation value of the thermal partition function at an inverse temperature determined by the boundary's length, one defines [68]

$$\langle Z(\beta_1) \dots Z(\beta_n) \rangle_c =: \int_{\text{BC}} \mathcal{D}[\phi, \mathbf{g}] e^{-I_{\text{JT},t}[\phi, \mathbf{g}]}, \tag{2.1.9}$$

where by BC we denote the boundary conditions discussed above. For the path integral on the right hand side, it is important to note that the "integration" of the metric also includes a sum over topologies. In fact, since we would like to study connected correlation functions, as indicated by the index c on the left hand side, the types of manifold appearing in the path integral are connected manifolds with n boundaries[1]. To keep the introductory discussion as simple as possible, we will also only consider the appearance of orientable manifolds, deferring the inclusion of unorientable ones to section 3.1. For the computation of the right hand side of eq. (2.1.9), it is most convenient to first perform the path integral over the dilaton first.

1 At this point, this statement is a definition. It will turn out in the following that this choice precisely corresponds to the standard notion of "connectedness" of correlation functions.

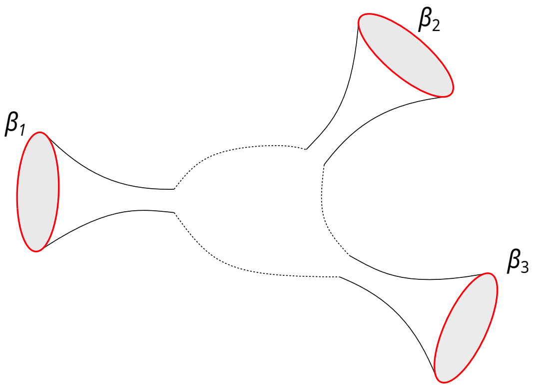


Figure 2.1.1: Graphical interpretation of the correlation function in eq. (2.1.9) for $n = 3$. Adapted from [77].

As seen from eq. (2.1.6), the dilaton appears linearly in the action and one can thus integrate it out (along the imaginary axis) directly, yielding a Dirac δ-function of $(R+2)$, i.e. restricting the manifolds appearing in the path integral to be of constant negative curvature [68, 80]. Knowing this, one can make sense of the arguments β_i of the partition functions on the left hand side of eq. (2.1.9) as the regularized lengths of the cut-off boundaries in the sense discussed above.

Having integrated out the dilaton field, the remaining path integral is given by

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle_c = \int_{\text{BC},\ R=-2} \mathcal{D}[\mathbf{g}] \quad e^{S_0\chi(\mathcal{M})+\int_{\partial\mathcal{M}}\mathrm{d}\omega\sqrt{h}\phi(K-1)}. \tag{2.1.10}$$

At this point, one can make the sum over geometries explicit by using the classification of the topology of hyperbolic 2-manifolds in terms of their Euler characteristic [81]. For the present setting, where the number of boundaries of the surfaces to be summed over in the path integral is known (n), one can further use the well known relation [82]

$$\chi(\mathcal{M}) = 2-2g-n, \tag{2.1.11}$$

relating the Euler characteristic directly to the genus. Hence, one can write the sum of geometries explicitly as

$$\begin{aligned}\langle Z(\beta_1)\dots Z(\beta_n)\rangle_c &= \sum_{g=0}^{\infty} e^{S_0(2-2g-n)} \int_{\substack{\text{BC},\ R=-2\\ g(\mathcal{M})=g}} \mathcal{D}[\mathbf{g}] \quad e^{\int_{\partial\mathcal{M}}\mathrm{d}\omega\sqrt{h}\phi(K-1)}\\ &:= \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1\dots\beta_n)}{(e^{S_0})^{2g+n-2}},\end{aligned} \tag{2.1.12}$$

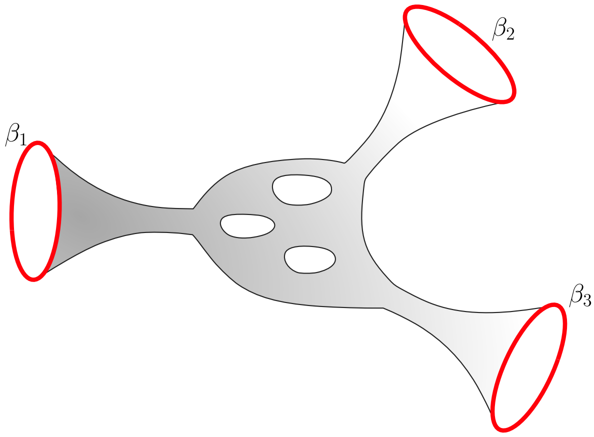


Figure 2.1.2: Depiction of the manifold contributing to the correlation function of three partition functions of inverse temperatures β_1, β_2 and β_3 at $g = 3$. Adapted from [77].

which is commonly referred to as the *topological* or genus expansion, where we defined the contribution to the correlation function at genus g in the last line. Graphically, one can depict one of these contributions as filling in the blank space in between the boundaries (cf.fig. 2.1.1) by a manifold of some genus. We show the contribution to the 3-point-function at $g = 3$ as an example in fig. 2.1.2.

The remaining computation of the path-integral is now based on a splitting of the contributing manifolds into two types of parts. Those that contain the boundaries of regularized lengths, denoted as the external boundaries in the following, and those that contain the potentially non-zero genus part. One can see the motivation for this split from eq. (2.1.12), since the remaining part of the action purely depends on the external boundaries, while the path-integral over the non-zero genus part leads, as we will see momentarily, to moduli space volumes of hyperbolic surfaces, being mathematically rather well studied objects. This separation of the surface can literally be thought of as splitting off the external boundaries along new internal geodesic boundaries, as depicted for our example of $(g, n) = (3, 3)$ in fig. 2.1.3. The split off part that contains a geodesic boundary and the external boundary is referred to as the *trumpet*, due to its appearance, while we will denote the remaining part of the surface, having n geodesic boundaries and being potentially of non-zero genus, as the *convex core*. The technical argumentation why this splitting is possible relies on a rewriting of the theory as a $PSL(2, \mathbb{R})$ BF theory and is omitted here since it is not necessary for the discussion of the main part. It can be found in [68]. Consequently, we will recall now how the path-integral for the trumpet geometry (depicted in fig. 2.1.4a) can be evaluated and then concern ourselves with discussing how to glue the convex core back into the surface. Before going into the computation, one has to note that for reasons apparent later on, there is one exception from the splitting prescription. That

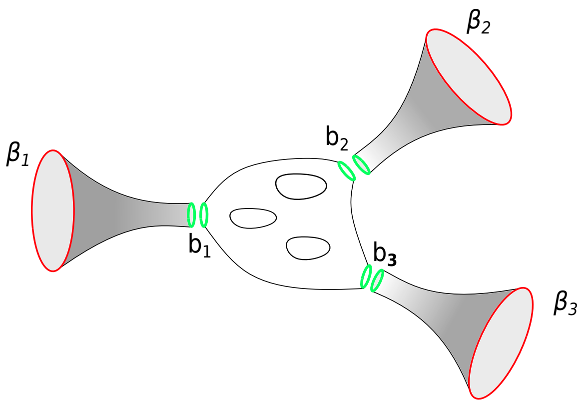


Figure 2.1.3: Splitting of the contribution at $g = 3$ and $n = 3$ into trumpets and convex core. Adapted from [5].

is the $n = 1$, $g = 0$ contribution, known as the disk and shown in fig. 2.1.4b, where one has to perform the path-integral directly.

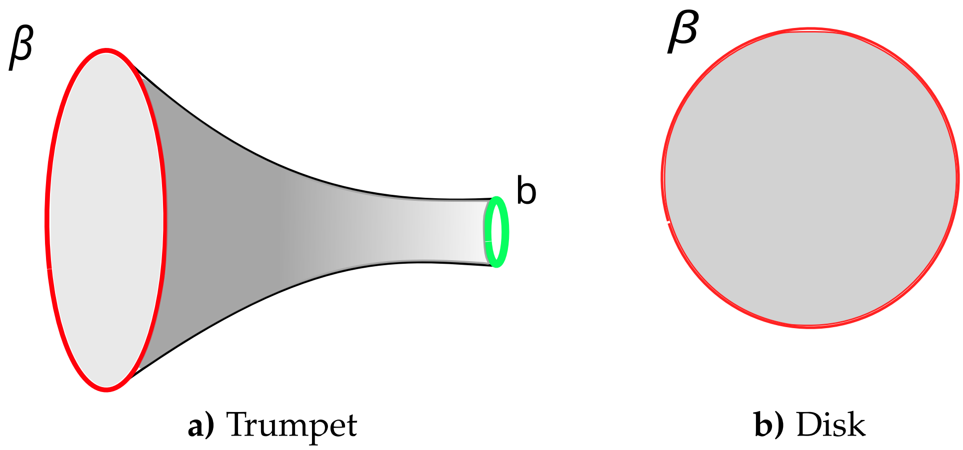


Figure 2.1.4: Manifolds containing an external boundary in JT gravity. Adapted from [77].

THE DISK AND TRUMPET PARTITION FUNCTIONS. For the discussion of the path integral for the parts of the manifolds that contain the external boundaries, it is necessary to go into more detail for the cut-off procedure by which the external boundaries are obtained. For this discussion, it is most convenient to choose Poincaré coordinates for AdS_2, i.e.

$$ds^2 = \frac{dt^2 + dz^2}{z^2}, \quad \text{with } z \in (0, \infty), \ t \in (-\infty, \infty). \tag{2.1.13}$$

Note that these coordinates are essentially the upper half-plane with the actual boundary of AdS_2 at $z = 0$. Now one can make an explicit choice for the boundary curve $\zeta : [0, \beta] \to AdS_2, u \mapsto (t(u), z(u))$, with periodicity conditions as above. Using this parametrisation, one can compute the induced metric on this curve as

$$h = g_{\mu\nu} \frac{d\zeta^\mu}{du} \frac{d\zeta^\nu}{du} = \frac{t'^2 + z'^2}{z^2}, \tag{2.1.14}$$

denoting the derivative by u by a prime. The boundary condition on the induced metric consequently implies

$$z^2 = \epsilon^2 \left(t'^2 + z'^2 \right) \implies z = \epsilon t' + \mathcal{O}\left(\epsilon^3\right), \tag{2.1.15}$$

which relates the two, a priori independent, choices of $t(u)$ and $z(u)$. This means that by just giving $t(u)$, the cut-off curve is determined. To compute the remaining part of the action, using this parametrisation of the boundary, one has to evaluate the extrinsic curvature. This yields [63]

$$\begin{aligned} K &= \frac{t'\left(t'^2 + z'^2 + zz''\right) - zz't''}{(t'^2 + z'^2)^{\frac{3}{2}}} = \frac{t'\left(t'^2 + \epsilon^2 t''^2 + \epsilon^2 t' t'''\right) - \epsilon^2 t' t''^2}{t'^3 (1 + \epsilon^2 (\frac{t''}{t'})^2)^{\frac{3}{2}}} = \\ &\approx \left[1 + \epsilon^2 \frac{t'''}{t'}\right] \left[1 - \frac{3}{2}\epsilon^2 \left(\frac{t''}{t'}\right)^2\right] = 1 + \epsilon^2 \left[-\frac{1}{2}\left(\frac{t''}{t'}\right)^2 + \left(\frac{t''}{t'}\right)'\right] \\ &=: 1 + \epsilon^2 \mathrm{Sch}(t, u), \end{aligned} \tag{2.1.16}$$

where we inserted the relation eq. (2.1.15), expanded up to second order in ϵ and defined in the last line the *Schwarzian* derivative of the function t. Plugging this result now in the remaining boundary part of the action one finds, using also the boundary condition for the dilaton,

$$\int_{\partial\mathcal{M}} d\omega \sqrt{h}\phi(K - 1) = \int_0^\beta \frac{du}{\epsilon} \frac{\gamma}{\epsilon} \left(1 + \epsilon^2 \mathrm{Sch}(t, u) - 1\right) = \gamma \int_0^\beta du \ \mathrm{Sch}(t, u). \tag{2.1.17}$$

This is the so-called Schwarzian action which also appears as a low energy effective action of the SYK model [60, 83]. This appearance is the basis of the deep connection between JT gravity and the SYK model, which is reviewed in detail in [65]. Furthermore, in the second step of this computation, one could see that the counter-term, which was included in the JT gravity action eq. (2.1.6), cancels the would-be divergent term, rendering the final form of the action independent of the regulator ϵ.

Having found this result, one can nearly compute the path-integrals for the disk and trumpet geometries. However, one should first clarify what will actually be integrated here. For the external boundary, the only remaining dependence of the action is that on the choice of t for the cut-off boundary. Hence, one should integrate over these choices in the path-integral. It is best to illustrate this in going over to the actual coordinates for the disk geometry, given by the Poincaré disk model, whose time coordinate θ can be related to the time coordinate t in the half-plane model via [65]

$$t(u) = \tan\left(\frac{\theta(u)}{2}\right). \tag{2.1.18}$$

By construction, the geometry is periodic in the coordinate θ with a periodicity of 2π. Hence, it has to hold that

$$\theta(u+\beta) = \theta(u) + 2\pi, \tag{2.1.19}$$

which, combined with the reasonable assumption that $\theta' \geq 0$, makes it apparent that θ is essentially a reparametrisation of S_1. Those will in the following be denoted as $\mathrm{diff}(S_1)$. Furthermore, it is important to note that due to the invariance of AdS_2 under $SL(2,\mathbb{R})$, acting by Möbius transformations, one has to mod out these transformations, i.e. integrate over the quotient space $\mathrm{diff}(S_1)/SL(2,\mathbb{R})$ [84]. The path-integral over this space with the Schwarzian action has been shown in [84], for the disk as well as the trumpet geometry, to be one-loop exact. This statement, proven by applying the Duistermaat-Heckman theorem [85], essentially means that the full path-integral can be computed by integrating only the linear fluctuations around the classical solution. As one might imagine, this tremendously simplifies the computation of the path-integral. For its computation we are missing now only one ingredient, the measure on the linear fluctuations. This measure, denoted as $\mu[t]$ in the following, is induced by a symplectic form on $\mathrm{diff}(S_1)/SL(2,\mathbb{R})$ that is discussed in depth in [86] and is given[2] by

$$\Omega = \frac{\alpha}{2}\int_0^\beta \mathrm{d}u \left[\mathrm{d}\epsilon'(u) \wedge \mathrm{d}\epsilon''(u) - 2\,\mathrm{Sch}(t(u),u)\;\mathrm{d}\epsilon(u)\wedge \mathrm{d}\epsilon'(u)\right]. \tag{2.1.20}$$

Remarkably, this could also be derived using the BF theory formulation of JT gravity in [68].

Having gathered all necessary ingredients, we will now go through the main steps of the computation of the path-integral for the disk, based on [68], with the details worked out e.g. in [77]. First, using eq. (2.1.18), one can rewrite the Schwarzian action in terms of θ to find

$$\gamma\int_0^\beta \mathrm{d}u\mathrm{Sch}(t,u) = -\frac{\gamma}{2}\int_0^\beta \mathrm{d}u\left[\frac{\theta''^2}{\theta'^2} - \theta'^2\right]. \tag{2.1.21}$$

As suggested by the one-loop exactness of the path-integral we study the classical solution of this action. Varying the action one finds it to be given by $\theta_c(z) = Cu$. Now, one can use the periodicity conditions on θ to find

$$\theta_c(u) = \frac{2\pi}{\beta}u. \tag{2.1.22}$$

2 With α introduced as an additional scaling.

This condition is easy to put into the action, to find its saddle-point value as

$$\frac{\gamma}{2}\int_0^\beta \mathrm{d}u\,\frac{4\pi^2}{\beta^2} = \frac{2\gamma\pi^2}{\beta}. \tag{2.1.23}$$

Due to the one-loop exactness, one now has to perform the integral over the linear fluctuations around this solution with the measure as induced by eq. (2.1.20). This is conveniently done by writing

$$\theta(u) = \frac{2\pi}{\beta}\left(u+\epsilon(u)\right), \tag{2.1.24}$$

and expanding the fluctuations in Fourier modes, i.e.

$$\epsilon(u) = \sum_{n=-\infty}^{\infty} e^{-\frac{2\pi}{\beta}inu}\left[\epsilon_n^R + i\epsilon_n^I\right], \tag{2.1.25}$$

choosing the coefficients such that ϵ is real. Evaluating the symplectic form eq. (2.1.20) and simplifying the action using this expansion, one finds

$$\begin{aligned} Z_{0,1}(\beta) &= e^{\frac{2\gamma\pi^2}{\beta}}\prod_{n=2}^{\infty} 2\alpha\frac{(2\pi)^3}{\beta^2}\left(n^3-n\right)\int \mathrm{d}\epsilon^R \mathrm{d}\epsilon^I e^{\gamma\frac{(2\pi)^4}{\beta^3}\left[n^4-n^2\right]\left[\left(\epsilon^R\right)_n^2+\left(\epsilon^I\right)_n^2\right]} \\ &= e^{\frac{2\gamma\pi^2}{\beta}}\prod_{n=2}^{\infty}\frac{\alpha\beta}{\gamma n} =: e^{\frac{2\gamma\pi^2}{\beta}}\prod_{n=2}^{\infty}\frac{\tilde{a}}{n}, \end{aligned} \tag{2.1.26}$$

where in the last step the Gaussian integrals over the expansion coefficients were computed. This can now be simplified by first noting

$$\prod_{n=2}^{\infty}\frac{a}{n} = \exp\left[\sum_{n=2}^{\infty}\log\left(\frac{a}{n}\right)\right] = \exp\left[\sum_{n=1}^{\infty}\log\left(\frac{a}{n}\right) - \log(a)\right], \tag{2.1.27}$$

and then rewriting the sum of logarithms as

$$\begin{aligned} \sum_{n=1}^{\infty}\log\left(\frac{a}{n}\right) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\sum_{n=1}^{\infty}\left(\frac{a}{n}\right)^s = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} a^s\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^s = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} a^s\zeta(s) \\ &= \zeta(0)\left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} a^s\right) + \zeta'(0) = -\frac{1}{2}\log(a) - \frac{1}{2}\log(2\pi) \\ &= \log\left(\frac{1}{\sqrt{2\pi a}}\right), \end{aligned} \tag{2.1.28}$$

where $\zeta(s)$ is the Riemann Zeta-function, whose values and values of its derivative are taken from [87]. Combining again, one finds

$$\begin{aligned} \prod_{n=2}^{\infty}\frac{\tilde{a}}{n} &= \exp\left[\sum_{n=1}^{\infty}\log\left(\frac{\tilde{a}}{n}\right) - \log(\tilde{a})\right] \\ &= \exp\left[-\frac{3}{2}\log(\tilde{a}) - \frac{1}{2}\log(2\pi)\right] = \frac{1}{\sqrt{2\pi}}\left(\frac{\gamma}{\beta\alpha}\right)^{\frac{3}{2}}, \end{aligned} \tag{2.1.29}$$

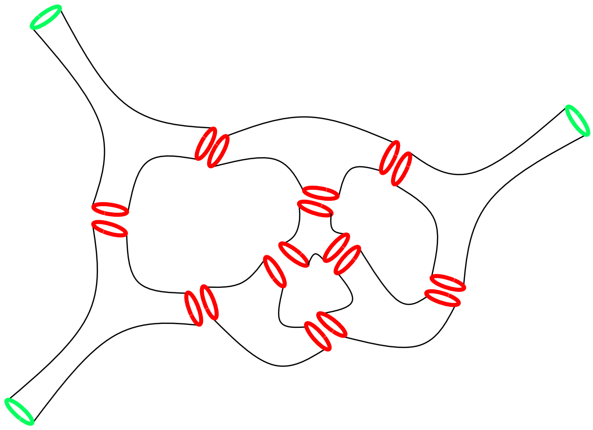


Figure 2.1.5: Example for a pants-decomposition: Decomposition of a surface with $g = 3$ and $n = 3$ into $2g + n - 2 = 7$ 3-holed spheres. Adapted from [77].

which yields the final result

$$Z_\alpha^d(\beta) := Z_{0,1} = \frac{1}{\sqrt{2\pi}} \left(\frac{\gamma}{\beta\alpha}\right)^{\frac{3}{2}} e^{\frac{2\gamma\pi^2}{\beta}}. \tag{2.1.30}$$

Along the same lines of reasoning one finds for the trumpet geometry [68]

$$Z_\alpha^t(\beta, b) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\gamma}}{\sqrt{\alpha}\sqrt{\beta}} e^{-\frac{\gamma b^2}{2\beta}}. \tag{2.1.31}$$

Having recalled how to perform the computation of the necessary path-integrals for the disk and trumpet geometries, we now turn to reviewing how to incorporate the convex core.

DEALING WITH THE CONVEX CORE. The part of the full path-integral for a given genus g and number of boundaries n is, after cutting off n trumpets along geodesic boundaries, essentially given by the integral over all possible metrics one can put on a hyperbolic surface with n geodesic boundaries of given lengths $L \in \mathbb{R}_+^n$. This is precisely what is known as the volume of the moduli space of Riemann surfaces[3], a well-studied subject in mathematics. In order to abbreviate the discussion of this subject as much as possible, we do not go into rigorously defining moduli space. Such details can be found in the author's words in [77] and in an exhaustive form in [89, 90].

One can understand the moduli space intuitively by giving a set of coordinates for it, known as the Fenchel-Nielsen or twist-length coordinates. The definition of these coordinates builds on the possibility to perform a so-called pants-decomposition of any hyperbolic Riemann surface. An example for such a decomposition of a surface into "pants", more academically referred

3 Note that every orientable surface with a metric, i.e. a Riemannian 2-manifold is a Riemann surface since a complex structure is induced by the metric [88].

to as 3-holed spheres, is given in fig. 2.1.5. It is easy to show that for such a decomposition, one has to cut a surface of given genus g and n boundaries $k(g,n) := 3g - 3 + n$ times[4]. The idea, why this is a useful thing to do derives from the fact in hyperbolic geometry that three lengths determine a unique hyperbolic surface of genus 0 with three geodesic boundaries of these lengths [89], which is precisely the 3-holed sphere. Hence, they can be regarded as the building blocks for a general hyperbolic surface. The "glueing" of the building blocks now has two degrees of freedom that will give the Fenchel-Nielsen coordinates. First, for each pair of boundaries one can choose a positive real number as the length. Second, one can perform a twist of the boundaries with respect to one another, illustrated in fig. 2.1.6, prior to the glueing. Thus, the Fenchel-Nielsen coordinates can be defined on $\mathbb{R}_+^{k(g,n)} \times \mathbb{R}^{k(g,n)}$ [5]. In fact, they can be introduced by essentially mapping each pants-decomposition of a surface to these coordinates and showing that the mapping is a homeomorphism [91], actually even a diffeomorphism between $\mathbb{R}_+^{k(g,n)} \times \mathbb{R}^{k(g,n)}$ and, speaking not rigorously, the space of inequivalent realisations of a surface of genus g and n geodesic boundaries of the given lengths (the Teichmüller-space $\mathcal{T}_{g,n}(L)$)[92]. This also shows that the real dimension of $\mathcal{T}_{g,n}(L)$ is given by $2k(g,n) = 6g - 6 + 2n$. One of the many virtues of these coordinates is that they allow one to define a symplectic form on $\mathcal{T}_{g,n}$ by pulling back the canonical symplectic form on $\mathbb{R}_+^{k(g,n)} \times \mathbb{R}^{k(g,n)}$, i.e. in the twist-length coordinates. This so-called *Weil-Petersson* symplectic form, including a scaling factor α, is hence given by

$$\omega_{g,n} = \alpha \sum_{i=1}^{k(g,n)} \mathrm{d}l_i \wedge \mathrm{d}\tau_i. \tag{2.1.32}$$

It is important to note that in addition to the notion of equivalence of surfaces already taken into account to define $\mathcal{T}_{g,n}(L)$, surfaces related by a diffeomorphism are still distinct points in this space. From the perspective of the path-integral (and, a posteriori, to get finite moduli space volumes) one would like to divide out this equivalence under the so-called *mapping-class group* $\mathrm{Mod}_{g,n}$ as well, which can be done mathematically rigorously by defining the moduli space of bordered hyperbolic surfaces of genus g and n geodesic boundaries ($\mathcal{M}_{g,n}(L)$) as

$$\mathcal{M}_{g,n}(L) := \mathcal{T}_{g,n}(L) / \mathrm{Mod}_{g,n}.$$

Remarkably, ω_{WP} can be shown to be invariant under the action of $\mathrm{Mod}_{g,n}$ and hence it also induces a symplectic form on $\mathcal{M}_{g,n}(L)$ [92]. Thus, a volume

4 One way to perform this proof is by observing that the surface's Euler characteristic $(2 - 2g - n)$ has to be given by the sum of that of the decomposing 3-holed spheres, each of which has Euler characteristic of -1. This determines the number of 3-holed spheres to be $2g + n - 2$, which in turns implies the result for the number of cuts.

5 We define $\mathbb{R}_+ := [0, \infty) \subset \mathbb{R}$.

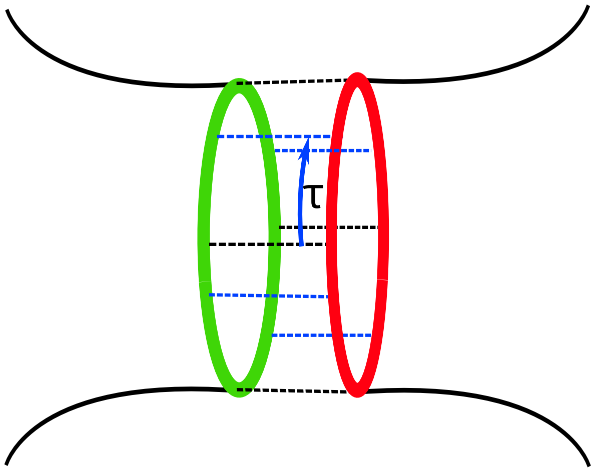


Figure 2.1.6: Illustration of the glueing of two geodesic boundaries. The black and blue lines illustrate the degree of freedom to twist one of the boundaries with respect to the glueing along the black lines by a length τ to glue along the blue lines. Adapted from [77].

form on $\mathcal{M}_{g,n}(L)$ can be constructed as $\frac{\omega_{g,n}^{k(g,n)}}{k(g,n)!}$ and it can be used to define the Weil-Petersson volumes $V_{g,n}^{\alpha}(L)$ as the volume of $\mathcal{M}_{g,n}(L)$ as computed with this volume form. In the mathematical literature, it is not customary to include the scaling factor α. However, one can easily restore it from the results where it has been put as unity via

$$V_{g,n}^{\alpha}(L) = \alpha^{k(g,n)} V_{g,n}^{1}(L) =: \alpha^{k(g,n)} V_{g,n}(L), \tag{2.1.33}$$

which is obvious from the definition of the volume form.

Working out the volume integral in twist-length coordinates is, however, quite complicated. Mainly, this is due to one essentially having to restrict the integration to a fundamental domain of the action of $\text{Mod}_{g,n}$. This is quite hard to do, though examples where it has been done exist (e.g. [93]). Fortunately, these problems were overcome in Mirzakhani's seminal work [70] where she proved a recursion relation for the Weil-Petersson volumes enabling their actual computation, of central importance for the developments in JT gravity. We state the main theorem of her work without proof, which can be found in the original paper [70] and e.g. in a form directed at a physics audience in [69].

Theorem 2.1 (Mirzakhani's recursion)**.**
The $V_{g,n}(L)$ with $L = (L_1, \ldots, L_n)$ (and $\widehat{L} = (L_2, \ldots, L_n)$) with $2g - 2 + n > 0$ (thus there is a pants decomposition with at least one cutting geodesic) are determined by a recursion relation.

The input to this recursion are the volumes $V_{0,3}(L)$ and $V_{1,1}(L)$ and the relation is given by

$$\frac{\partial}{\partial L_1}\left(L_1 V_{g,n}(L)\right) = \mathcal{A}^{con}_{g,n}(L_1,\widehat{L}) + \mathcal{A}^{dcon}_{g,n}(L_1,\widehat{L}) + \mathcal{B}_{g,n}(L_1,\widehat{L}),$$

with

$$\begin{aligned}\mathcal{A}^{(d)con}_{g,n}(L_1,\widehat{L}) &:= \frac{1}{2}\int_0^\infty\int_0^\infty x\mathrm{d}xy\mathrm{d}y\widehat{\mathcal{A}}^{(d)con}_{g,n}(x,y,L_1,\widehat{L}),\\ \mathcal{B}_{g,n}(L_1,\widehat{L}) &:= \int_0^\infty x\mathrm{d}x\widehat{\mathcal{B}}_{g,n}(x,L_1,\widehat{L}),\end{aligned}$$

and

$$\begin{aligned} m(a,b) &:= \delta_{x-1,0}\delta_{y-1,0},\\ \widehat{\mathcal{A}}^{con}_{g,n}(x,y,L_1,\widehat{L}) &:= \frac{1}{2^{m(g-1,n+1)}}V_{g-1,n+1}(x,y,\widehat{L})H(x+y,L_1),\\ \widehat{\mathcal{B}}_{g,n}(x,L_1,\widehat{L}) &:= \frac{1}{2^{m(g,n-1)}}\sum_{j=2}^{n}\frac{1}{2}\left[H(x,L_1+L_j)+H(x,L_1-L_j)\right]\times\\ &\qquad V_{g,n-1}(x,L_2,\ldots,\hat{L}_j,\ldots,L_n),\\ \widehat{\mathcal{A}}^{dcon}_{g,n}(x,y,L_1,\widehat{L}) &:= \sum_{a\in\mathcal{I}_{g,n}} V(a,x,y,\widehat{L})H(x+y,L_1),\end{aligned}$$

where $\mathcal{I}_{g,n}$ is the set of ordered pairs $a = ((g_1,I_1),(g_2,I_2))$, $I_1,I_2 \subset \{2,\ldots,n\}$ with requiring $0 \le g_1,\ g_2 \le g$ and

1. $I_1 \cap I_2 = \emptyset$ *and* $I_1 \cup I_2 = \{2,\ldots,n\}$,

2. $g_1 + g_2 = g,\ \underset{i\in\{1,2\}}{\forall}\ 2 \le 2g_i + |I_i|.$

For this, one defines

$$V(a,x,y,\widehat{L}) = \frac{V_{g_1,|I_1|+1}(x,L_{I_1})}{2^{m(g_1,|I_1|+1)}}\cdot\frac{V_{g_2,|I_2|+1}(y,L_{I_2})}{2^{m(g_2,|I_2|+1)}}.$$

Furthermore, the notation $\hat{L}_j$ in the definition of $\hat{\mathcal{B}}_{g,n}$ means that the j-th element of L is left out. Finally, the function $H(x,y)$ is defined as

$$H(x,y) := \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}. \tag{2.1.34}$$

Since the moduli space for $(g,n) = (0,3)$ consists of only one point its volume is seen to be $V_{0,3} = 1$, giving the first input to the recursion. It is more work to find $V_{1,1}$ and it was first found in [93] as

$$V_{1,1}(L) = \frac{\pi^2}{6} + \frac{L^2}{24}, \tag{2.1.35}$$

with a simplified argument for its computation given in [70]. It is important to note that the surface of genus one with one boundary possesses a $\mathbb{Z}_2$ symmetry and hence the "real" moduli space volume given here is twice what has to be used in the glueing construction [65, 70]. In Mirzakhani's convention, this is incorporated into the recursion by adding a factor of $\frac{1}{2}$ whenever $V_{1,1}(L)$ occurs. In the physics literature, this factor is often taken into the definition of the volume. In this introduction we will use Mirzakhani's convention but already define the rescaled volumes for later use as

$$\tilde{V}_{1,1}(L) = \frac{1}{2} V_{1,1}(L). \tag{2.1.36}$$

For the application of the recursion and its geometrical origin it is useful to understand the individual terms in the recursion as specific ways to glue some surface(s) to a 3-holed sphere. We depict in fig. 2.1.7 the three terms occurring in theorem 2.1, where the splitting in fig. 2.1.7a corresponds to $\mathcal{A}^{\text{con}}_{g,n}$, that in fig. 2.1.7b to $\mathcal{A}^{dcon}_{g,n}$ and that in fig. 2.1.7c to $\mathcal{B}_{g,n}$. The correspondence of the terms in the recursion with this ways of decomposing the surface makes writing down the terms occurring in the recursion rather intuitive.

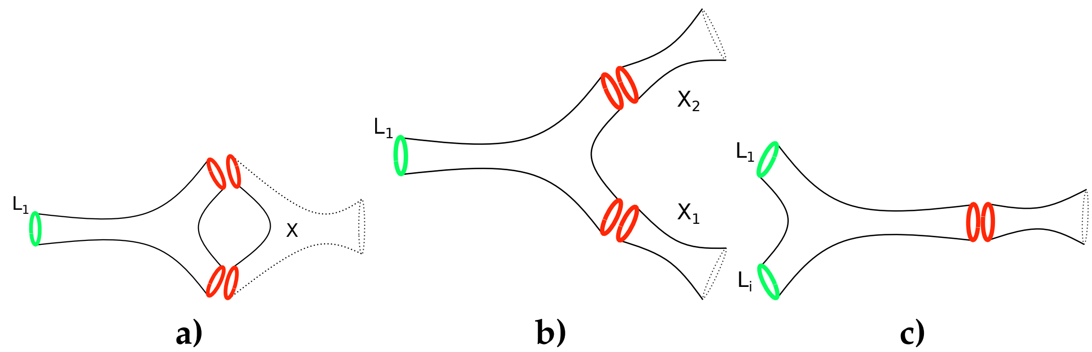


Figure 2.1.7: Splittings of a surface of genus g and n geodesic boundaries corresponding to the individual terms occurring in theorem 2.1. In a): Decomposition into a pair of pants containing the geodesic boundary of length L_1 and a surface X of genus $g-1$ and $n+1$ geodesic boundaries. In b): Decomposition into a pair of pants containing the geodesic boundary of length L_1 and surfaces X_1 and X_2 which have combined genus g and $n+1$ geodesic boundaries. In c): Decomposition into a pair of pants containing the geodesic boundaries of lengths L_1 and L_i and a surface of genus g and n-1 geodesic boundaries. Adapted from [77].

Equipped with this intuition, the recursion can now be iterated to get (with some effort) any moduli space volume one would like to have. We give an example of such a computation in appendix A.

Besides giving a way to find explicit results, [70] also showed the general structure of the Weil-Petersson volumes to be determined as

Theorem 2.2 (Polynomial structure of the Weil-Petersson volumes).
It holds that

$$V_{g,n}(\vec{L}) = \sum_{\vec{a}}^{\|\vec{a}\|_1 \leq k(g,n)} C_{g,n}^{\vec{a}} \prod_{i=1}^{n} L_i^{2a_i}. \tag{2.1.37}$$

with $\vec{L} \in \mathbb{R}_+^n$, $\vec{a} \in \mathbb{N}_0^n$, $\|\cdot\|_1$ *denoting the sum-norm and* $k(g,n) = 3g - 3 + n$, *as above.*
Furthermore, $C_{g,n}^{\vec{a}} \in \mathbb{Q}_{>0} \cdot \pi^{2k(g,n)-2\|\vec{a}\|_1}$ *and the coefficients are required to be symmetric under interchanging the positions of elements of* $\vec{a}$, *i.e. the polynomial is totally symmetric.*

This is a very strong theorem, which will be vital for the considerations of section 4.2.2. Having now discussed a way to obtain the moduli space volumes, we will next review how to put them together with the results of the previous paragraph to get the final result for the topological expansion of the JT gravity correlation functions of partition functions.

Before doing this, however, it is important to note the exceptions from the formalism discussed in this paragraph. Indeed, as the attentive reader might have noticed already, there is an obvious problem with the pants-decomposition of a hyperbolic Riemann surface if the number of pants needed to decompose it, $2g + n - 2$, vanishes or becomes negative. Surfaces for which this is the case are denoted as *unstable*, those for which a decomposition is possible are, not surprisingly, denoted as *stable*. Fortunately, there are not many unstable surfaces. In fact, the only ones are those with $(g, n) = (0, 0); (0, 1); (0, 2); (1, 0)$. Those contributions with $n = 0$ will not be relevant for our considerations, hence we shall not discuss them. The two surfaces that are relevant are that with $g = 0$ and $n = 1$ (the disk) which we already discussed, and that with $g = 0$ and $n = 2$, known as the double-trumpet, which we will discuss momentarily.

GLUEING ALL TOGETHER. Indeed, we will start discussing glueing of the trumpets to the convex core in the simplest case of the double-trumpet, where there is no convex core. Glueing together the two trumpets along their geodesic boundaries is, however, intuitively the same thing as the internal glueings of the 3-holed spheres performed to build the convex core. In fact, one would suspect the measure to be the same measure, the Weil-Petersson measure, which one can prove by using the aforementioned formulation of JT gravity as a BF theory [68]. This formulation also shows that the measure that arose for the integration of the boundary modes has the same origin and hence, the scaling factor α introduced there and that introduced in the Weil-Petersson measure coincide. Consequently, it holds that[6]

$$\begin{aligned} Z^{\alpha}_{0,2}(\beta_1,\beta_2) &= \alpha \int_0^\infty \mathrm{d}b \int_0^b \mathrm{d}\tau \, Z^t_\alpha(\beta_1,b) Z^t_\alpha(\beta_2,b) \\ &= \alpha \int_0^\infty b \, \mathrm{d}b \, Z^t_\alpha(\beta_1,b) Z^t_\alpha(\beta_2,b) \\ &= \alpha^2 \int_0^\infty b_1 \, \mathrm{d}b_1 \int_0^\infty b_2 \, \mathrm{d}b_2 \underbrace{\frac{\delta(b_1-b_2)}{\alpha b_1}}_{:=V^\alpha_{0,2}(b_1,b_2)} Z^t_\alpha(\beta_1,b) Z^t_\alpha(\beta_2,b), \end{aligned} \tag{2.1.38}$$

where to the second line we computed the integral over the twist parameter, which was possible due to the integrand not depending on it. This is always the case and hence the correct measure for glueing a trumpet to a surface along a geodesic boundary is given by $\alpha b \, \mathrm{d}b$. To the last line, we rewrote the expression in such a way that it can be viewed as integrating two trumpets with the glueing measure against the (formal) Weil-Petersson volume for $(g,n) = (0,2)$. Working out the integrals, one finds

$$Z^{\alpha}_{0,2}(\beta_1,\beta_2) = \frac{1}{2\pi} \frac{\sqrt{\beta_1\beta_2}}{\beta_1+\beta_2}, \tag{2.1.39}$$

which is, notably, independent of α and γ. Generalising the reasoning leading to this result, one finds

$$\begin{aligned} Z^{\alpha}_{g,n}(\beta_1,\dots,\beta_n) &= \alpha^n \prod_{i=1}^n \left[\int_0^\infty b_i \mathrm{d}b_i Z^t_\alpha(\beta_i,b_i) \right] V^\alpha_{g,n}(b_1,\dots,b_n) \\ &= \alpha^{k(g,n)+n-\frac{n}{2}} \prod_{i=1}^n \left[\int_0^\infty b_i \mathrm{d}b_i Z^t_1(\beta_i,b_i) \right] V_{g,n}(b_1,\dots,b_n) \\ &= \alpha^{-\frac{3}{2}\chi(\mathcal{M})} \prod_{i=1}^n \left[\int_0^\infty b_i \mathrm{d}b_i Z^t_1(\beta_i,b_i) \right] V_{g,n}(b_1,\dots,b_n), \end{aligned} \tag{2.1.40}$$

where it is important to note that, as we had remarked above, for the case of $(g,n) = (1,1)$ the volume to be put into this construction is $\tilde{V}_{1,1}$ (cf. eq. (2.1.36)). For the disk, which has Euler characteristic $2-1=1$ it also holds that

$$Z^d_\alpha = \alpha^{-\frac{3}{2}\chi(\mathcal{M})} Z^d_1. \tag{2.1.41}$$

Hence, the whole effect of α can be captured by shift of S_0 by $-\frac{3}{2}\log(\alpha)$ [68]. Consequently, one can without loss of generality use $\alpha = 1$, which we will do

6 Note that we introduce here a superscript α to indicate that the whole result depends on the choice of α.

in the following. Hence, we will drop the various superscripts and indices of α for the remainder of this thesis. This concludes the discussion of what is needed to be known about orientable JT gravity to understand the main part of this thesis.

To recapitulate briefly, we recall that the topological expansion of the connected correlation function of partition functions in JT gravity is given in eq. (2.1.12) as

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle_c := \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1\dots\beta_n)}{(e^{S_0})^{2g+n-2}}, \tag{2.1.42}$$

with the contribution to the expansion given by

$$Z_{g,n}(\beta_1,\dots,\beta_n) = \prod_{i=1}^{n}\left[\int_0^\infty b_i \mathrm{d}b_i Z^t(\beta_i,b_i)\right] V_{g,n}(b_1,\dots,b_n), \tag{2.1.43}$$

for all $n>0$, with the special case of

$$Z^d(\beta) = Z_{0,1}(\beta) = \frac{1}{\sqrt{2\pi}}\left(\frac{\gamma}{\beta}\right)^{\frac{3}{2}} e^{\frac{2\gamma\pi^2}{\beta}}. \tag{2.1.44}$$

Note that we retained up to now the dependence on γ, the renormalized boundary value of the dilaton field. This adds no further difficulty to the computations, so one can decide to track this dependence, which, notably, cannot be removed like the one on α. However, in most of the JT literature the choice $\gamma = \frac{1}{2}$ is taken which we will also take in the following, whenever not explicitly including γ. In expressions dependent on β it can easily be restored by replacing $\beta \to \frac{\beta}{2\gamma}$.

TOPOLOGICAL GRAVITY. To conclude our brief introduction to the gravity theories considered here, we will now discuss what is known as *topological* or Witten-Kontsevich gravity. This theory can most easily be thought of as arising from JT gravity in the limit of large boundary lengths, i.e. small temperatures in the arguments of the partition function correlators. Studying this limit for the JT correlation functions written as in eq. (2.1.43), one can see from the functional form of the trumpet partition function that the main contribution to the result will come from the region of large boundary length of the WP volumes. Consequently, one can compute correlation functions in topological gravity by replacing the WP volumes by their leading order behaviour, i.e. by defining the *Airy* WP volumes as

$$V_{g,n}^{\text{Airy}}(\vec{L}) := \sum_{\vec{a}}^{\|a\|_1=k(g,n)} C_{g,n}^{\vec{a}} \prod_{i=1}^{n} L_i^{2a_i}, \tag{2.1.45}$$

with the notation from theorem 2.2 and the coefficients those of the WP volumes. Note that it will be clear in a moment, why this name has been chosen for the volumes.

Geometrically it is interesting to think about what happens with a hyperbolic surface of given genus g and number of geodesic boundaries n if one takes the limit of very large boundary lengths. Due to the surface being of constant negative curvature, the Gauss-Bonnet theorem implies[7]

$$-A = 2\pi(2-2g-n) \implies A = 2\pi(n+2g-2). \tag{2.1.46}$$

Hence, for stable surfaces, the surface's area is constrained to be some positive number. Blowing up now the lengths of the boundaries necessarily implies that the surface connecting the boundaries becomes very "thin", i.e. the surface will look like an assembly of thin strips. As we will see in section 2.2, that is precisely the way the diagrammatic expressions for the perturbative expansion of a matrix model look like, the so-called ribbon-graphs. This already gives a hint at the connection of matrix models with moduli space volumes. In fact, the matrix model whose duality with topological gravity we will discuss in section 2.3, is known as the Airy model, which motivates our choice of name for the topological gravity WP volumes.

Indeed, such a relation was first seen in topological gravity, starting with the pioneering work of Witten [74] and set in its final form by Kontsevich [75]. To write down this relation, we need to introduce the concept of *intersection numbers*, being the fundamental correlation functions of interest in topological gravity. As the moduli space volumes, they are defined as certain integrals over the moduli space of Riemann surfaces, however, here only of those with punctures $\mathcal{M}_{g,n}(0)$. In fact, they are defined as integrals over its compactification, the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$, details of which are discussed in an accessible form in [90]. What is now integrated on this space are the first Chern-classes of specific complex line-bundles over it. These bundles are constructed by considering a punctured Riemann surface with n punctures at points z_i and considering the cotangent spaces at the punctures. Doing this for all elements of $\overline{\mathcal{M}}_{g,n}$, this yields a complex line bundle over $\overline{\mathcal{M}}_{g,n}$ for each puncture, whose first Chern-classes will be denoted as ψ_i [73, 94]. Using them, one can define the intersection numbers as

$$\langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}, \tag{2.1.47}$$

where the right hand side is understood to be non-vanishing iff the integrated form is of top degree, i.e. $6g-6+2n$, the (real) dimension of $\mathcal{M}_{g,n}(L)$.

7 Note that the boundary term vanishes due to the vanishing of the extrinsic curvature of geodesic boundaries.

Since the Chern classes, or rather their representatives, are two-forms, this implies that the intersection numbers vanish unless $\sum_{i=1}^{n} \alpha_i = 3g - 3 + n$. The notation on the left-hand side is due to Witten [74] and is more compact since it encodes n in the number of τs. Furthermore, we note that intersection numbers are symmetric with respect to interchanging the inserted τ, as it is obvious from the definition.

Examples for the computation of intersection numbers from their definition are discussed in [74], while their computation is involved in general. Kontsevich's contribution to simplifying this, was to show that one can compute them using diagrammatics of 3-valent ribbon graphs.

Before stating the theorem that establishes this connection, we should briefly discuss the occurring graphs. A ribbon graph essentially is a diagram whose propagators are double-lines and the vertices are hence such that double-lines come in and double-lines go out (cf. examples 2.1 and 2.2). In such a diagram, one can identify its boundaries by colouring one side of a double-line propagator and continuing with the same colour through the vertices until one arrives where the colouring started[8]. Then, one continues in the same manner with another starting point until all sides of all propagators are coloured. A line of a given colour is then denoted as a boundary. Denoting the number of boundaries as n, and associating each colour with a number in $\{1, \ldots, n\} \subset \mathbb{N}$ one can label each side of a propagator by the number of the boundary component it is part of. To relate this graph to the Riemann surfaces we are interested in, it is necessary to also associate with it a genus. This is done by using the Euler characteristic and determining it on the one hand by thinking of the ribbon graph as a surface ($\chi = 2 - 2g - n$) and on the other by using that its Euler characteristic is that of the single-line graph one can construct from it by collapsing the double-line propagators and vertices. The second reasoning yields $\chi = \#\text{Vertices} - \#\text{Edges}$ by the formula for the Euler characteristic for graphs[9]. For the specific case of 3-valent graphs it is quickly seen that $\#\text{Edges} = \frac{3}{2}\#\text{Vertices}$, and hence it holds that $\chi = -\frac{1}{2}\#\text{Vertices}$. Thus, the genus g for a given ribbon graph is given by $g = 1 - \frac{1}{2}n + \frac{1}{4}\#\text{Vertices}$.

In practice, if one is looking for a ribbon graph for a given genus and number of boundaries, it is more useful to invert this relation to find

$$\#\text{Vertices}(g, n) = 4g + 2n - 4. \tag{2.1.48}$$

8 Under the assumption that the diagram has only finitely many propagators, which is the case we are interested in, this has to happen.

9 Here, we treat the graph as having no faces since the n faces are taken to be the boundaries of the surface formed by the ribbon graph. Alternatively, one can take into account the faces on the side of the ribbon graph (i.e. $\chi = \#\text{Vertices} - \#\text{Edges} + \#\text{Faces}$) and consider the surface without taking the faces out, i.e. $\chi = 2 - 2g$. Evidently, this yields the same results.

Using this, one can construct the ribbon graphs with the number of vertices for the desired values of g and n and then do the colouring to determine by the number of boundaries those that are needed. Having given this introduction we can state Kontsevich's theorem as

Theorem 2.3 (Kontsevich [75]). *Denote by $\Gamma_{g,n}$ the set of ribbon graphs built from 3-valent vertices having n boundaries and genus g. Then it holds that*

$$\sum_{|\vec{\alpha}|=3g-3+n} \langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \rangle \prod_{k=1}^{n} \frac{(2\alpha_k - 1)!!}{z_k^{2\alpha_k+1}} = \sum_{\gamma \in \Gamma_{g,n}} \frac{2^{2g-2+n}}{|\mathrm{Aut}(\gamma)|} \prod_{k=1}^{6g-6+3n} \frac{1}{z_{l(k)} + z_{r(k)}},$$

where the product on the RHS is over the edges/propagators of the individual ribbon graph, thus $l(k)$ denotes the boundary component to which the left edge of the respective propagator belongs, $r(k)$ the right edge, respectively.

This is the relation we advertised above. To use it in practice, it is useful to know [90]

$$|\mathrm{Aut}(\gamma)| = \frac{3^{\#\text{ Vertices}} \#\text{ Vertices}!}{\#\text{ Glueings}}, \tag{2.1.49}$$

which connects the order of the graphs' automorphism groups to the numbers of ways of glueing them starting from just the vertices, the counting procedure one encounters for regular QFT Feynman-graphs (e.g. [95]).

Before giving some examples for the computation of intersection numbers by ribbon-graphs, we would like to connect the intersection numbers with the large-boundary limit of the WP volumes which, as we claimed above, was related to topological gravity. This is most easily done by using an expression of the full WP volumes in terms of intersection numbers due to Mirzakhani [96]:

$$V_{g,n}(L_1, \dots, L_n) = \sum_{|\vec{\alpha}|+m=3g-3+n} \frac{(2\pi^2)^m}{2^{|\vec{\alpha}|}\vec{\alpha}!m!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} \omega^m \vec{L}^{2\vec{\alpha}}, \tag{2.1.50}$$

with $\vec{\alpha}! := \prod_i \alpha_i!$, $\vec{L}^{\vec{\alpha}} = \prod_i L_i^{\alpha_i}$ and ω the Weil-Petersson form on $\mathcal{M}_{g,n}(0)$. Note that we have chosen here to include the ambiguous factor of $\frac{1}{2}$ for $(g,n) = (1,1)$ into the volume. The leading order behaviour in the lengths is now given by the terms with $m = 0$, which precisely are the intersection numbers. Hence, the relation of the leading order behaviour of the WP volumes with topological gravity can also be seen from the intersection theory picture. Using this relation, one can now directly relate the Airy WP volumes with the ribbon graphs using theorem 2.3. For this, it is easiest to perform a Laplace transformation on the volumes to find

$$\begin{aligned}\mathcal{L}\left[V_{g,n}^{\text{Airy}}(L_1,\dots,L_n);(z_1,\dots,z_n)\right] &= \sum_{|\vec{\alpha}|=3g-3+n} \frac{1}{2^{|\vec{\alpha}|}\vec{\alpha}!}\langle \tau_{\alpha_1}\dots\tau_{\alpha_n}\rangle \prod_{i=1}^{n}\frac{(2a_i+1)!}{\left(z_i^2\right)^{a_i+1}} \\ &= \sum_{|\vec{\alpha}|=3g-3+n} \langle \tau_{\alpha_1}\dots\tau_{\alpha_n}\rangle \prod_{k=1}^{n}\frac{(2\alpha_k-1)!!}{z_k^{2\alpha_k+1}} \\ &= \sum_{\gamma\in\Gamma_{g,n}} \frac{2^{2g-2+n}}{|\text{Aut}(\gamma)|}\prod_{k=1}^{6g-6+3n}\frac{1}{z_{l(k)}+z_{r(k)}},\end{aligned} \tag{2.1.51}$$

where in the last line we used theorem 2.3. Having discussed now also this direction, we are ready to give some examples to illustrate the usefulness of this relation to compute intersection numbers/Airy WP volumes.

Example 2.1 ($g=0, n=3$, orientable). *As a first example, we consider the case $(g,n)=(0,3)$, which should lead to $V_{0,3}=1$. The relevant ribbon graphs have 2 vertices and 3 edges. One finds that there are two types of such ribbon graphs having 3 boundaries which are depicted in fig. 2.1.8. For the first of those graphs, one finds*

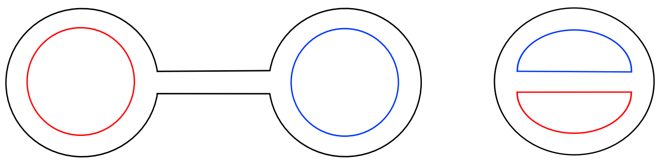


Figure 2.1.8: Diagrams for $g=0, n=3$

by eq. (2.1.49) the order of the automorphism group to equal 2, while for the second one, it is of order 6. In the sum over $\Gamma_{0,3}$, all possible labellings of the graph are also included. In general, there are $3!=6$ ways to label 3 distinct boundaries with 3 labels. For the second graph, all possible labellings yield the same graph, while for the first graph the contribution of the diagram depends only on the labelling of the middle propagator, while the 2 possibilities for each choice for this yield the same contribution. Thus, one finds using eq. (2.1.51)

$$\begin{aligned}\mathcal{L}\left[V_{0,3}^{\text{Airy}}(L_1,L_2,L_3);(z_1,z_2,z_3)\right] &= 2^1\left\{\frac{6}{6}\frac{1}{z_1+z_2}\frac{1}{z_1+z_3}\frac{1}{z_2+z_3}+\right. \\ &\left.+\frac{2}{2}\left(\frac{1}{z_1+z_1}\frac{1}{z_1+z_2}\frac{1}{z_1+z_3}+(\textit{all permutations})\right)\right\} \\ &=\dots=\frac{1}{z_1z_2z_3}.\end{aligned} \tag{2.1.52}$$

By an inverse Laplace transform, this yields the expected result for the volume.

Example 2.2 ($g = 1$, $n = 1$, orientable)**.** *As the next example, we consider* $(g,n) = (1,1)$. *Again, one finds #Edges=3, #Vertices =2, as one would expect due to* $\chi_{0,3} = \chi_{1,1} = -1$. *Now one finds only one graph, depicted in fig. 2.1.9. Computing*

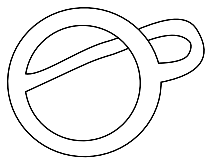


Figure 2.1.9: Diagram for $g = 1, n = 1$

again the order of the automorphism group, one finds it to be 6. As there is only one boundary to label, there is no further multiplicity and one finds

$$\mathcal{L}\left[V_{1,1}^{\text{Airy}}(L_1);(z_1)\right] = 2\frac{1}{6}\frac{1}{(2z)^3} = \frac{1}{24z^3}. \tag{2.1.53}$$

Inverting the Laplace transform, one finds

$$V_{1,1}^{\text{Airy}}(L_1) = \frac{L^2}{48}, \tag{2.1.54}$$

agreeing with eq. (2.1.35)*, noting that we have included here the ambiguous factor of* $\frac{1}{2}$ *for* $(g,n) = (1,1)$ *into the volume.*

This concludes our discussion of topological gravity. The relation of this theory with ribbon graphs and through them with a matrix model can be thought of as being the precursor of the relation of full JT gravity with a matrix model. To better understand this relation and to state that for JT gravity, it is useful to recall the basics of matrix models, as done in the next section.

2.2 BASICS OF MATRIX MODELS

DEFINITION OF MATRIX MODELS. A matrix model is essentially defined as an ensemble of random matrices. This means that generically, one considers an ensemble of matrices $\mathbb{E}$ endowed with a measure $\mathrm{d}\mu$ such that one can define the matrix model partition function as

$$\mathcal{Z} := \int_{\mathbb{E}} \mathrm{d}\mu\,(H). \tag{2.2.1}$$

Given a function $h : \mathbb{E} \to \mathbb{C}$, one can define its expectation value by insertion into this integral, i.e.

$$\langle g \rangle := \frac{1}{\mathcal{Z}} \int_{\mathbb{E}} \mathrm{d}\mu\,(H) g(H). \tag{2.2.2}$$

In this way, we shall define all correlation functions relevant for this thesis. For most of the cases studied here, the ensembles can be specified as certain types of square matrices. These are the *unitary* ensemble of hermitian, the *orthogonal* ensemble of real symmetric and the *symplectic* ensemble of real quaternionic hermitian matrices[10]. These ensembles are known as the Wigner-Dyson ensembles. One often distinguishes them, for reasons apparent momentarily, by their *Dyson index* β where $\beta = 1$ corresponds to the orthogonal, $\beta = 2$ to the unitary and $\beta = 4$ to the symplectic ensemble[11]. In the following, we shall denote the ensembles of the $N \times N$ matrices of the ensemble of Dyson index β by $\mathbb{E}_N^\beta$. These ensembles share the property that all its elements are diagonalisable, i.e. [15]

$$\begin{aligned} &\underset{H \in \mathbb{E}_N^\beta}{\forall} \quad \underset{\substack{U \in U_N^\beta \\ \lambda_1,\dots,\lambda_N \in \mathbb{R}}}{\exists} : H = U \Lambda U^{-1}, \\ &\qquad\text{with} \begin{cases} \Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_N) & \beta \in \{1,2\}, \\ \Lambda := \mathrm{diag}(\lambda_1, \lambda_1, \dots, \lambda_N, \lambda_N) & \beta = 4. \end{cases} \end{aligned} \tag{2.2.3}$$

where $U_N^1 = O(N)$, $U_N^2 = U(N)$ and $U_N^4 = \mathrm{Sp}(2N)$, which are the groups that for historic reasons give the ensembles their names and are denoted as the *circular ensembles* U_N^β . Hence, all elements of each ensemble have N independent real eigenvalues.

In the following, we shall only be interested in observables that depend on these eigenvalues, i.e. they are invariant under conjugation by the circular ensembles. For such observables, it is intuitive that one might "diagonalise" the ensembles like it can be done for their elements (eq. (2.2.3)) and integrate out the circular ensembles which, conveniently, are all compact Lie groups. More rigorously, one can do this by starting with the standard Lebesgue measure on the ensembles $\mathrm{d}H_N^\beta$, given essentially by one real integration variable per real degree of freedom of the matrices, i.e. [21, 97]

$$\mathrm{d}H_N^\beta = \prod_{i=1}^N \mathrm{d}H_{i,i}^{(0)} \prod_{i<j} \prod_{\alpha=0}^{\beta-1} \mathrm{d}H_{i,j}{}^{(\alpha)} \,. \tag{2.2.4}$$

Here, we denote by $\mathrm{d}H_{i,j}{}^{(k)}$ the k-th entry of the matrix entry $H_{i,j}$ i.e. for $\beta = 1$ $H_{i,j} = H_{i,j}^{(0)}$, for $\beta = 2$ $H_{i,j} = H_{i,j}^{(0)} + iH_{i,j}^{(1)}$, and for $\beta = 4$ $H_{i,j} =$

10 Note that each entry of such a matrix is a quaternion, written here as a 2×2 matrix with complex entries. Hence, an $N \times N$ quaternionic matrix is a $2N \times 2N$ complex matrix. "Real quaternionic" now means that the coefficients of the quaternionic entries, when expanded in e.g. a basis of Pauli matrices, are real [97].

11 Note that β should not be confused for β that throughout this thesis denotes an inverse temperature.

$H^{(0)}_{i,j}\mathbb{1}_{2\times 2} + H^{(k)}_{i,j}\tau_k$ with $\tau_k = -i\sigma_k$ and the Pauli-matrices σ_k. Now, one has to compute the Jacobian of the change from these coordinates to those given by the N (non-degenerate) eigenvalues and an element of the respective circular ensemble, as suggested by the diagonalisation. This can be done by standard methods (cf.[21, 97]), which yields

$$\mathrm{d}H^{\beta}_{N} = \underbrace{\left[\prod_{i=1}^{N}\mathrm{d}\lambda_i\right]}_{:=\mathrm{d}\Lambda} |\Delta(\Lambda)|^{\beta}\,\mathrm{d}U^{\beta}_{N}, \quad \text{with} \quad \Delta(\Lambda) := \prod_{1\leq i<j\leq N}\left(\lambda_j - \lambda_i\right), \tag{2.2.5}$$

where the *Vandermonde*-determinant $\Delta(\Lambda)$ has been introduced and we denote the Haar measure on the circular ensembles by $\mathrm{d}U^{\beta}_{N}$. To find the version of the measure $\mathrm{d}\mu$ we are interested in, one now has to slightly generalise this by including the exponential of a (polynomial) potential V. This leads to[12]

$$\mathrm{d}\mu\,(H) = \mathrm{d}H^{\beta}_{N} e^{-N\frac{\beta}{2}\,\mathrm{tr}(V(H))}. \tag{2.2.6}$$

Using these steps, one can rewrite the partition function of eq. (2.2.1) as

$$\mathcal{Z} = \mathcal{N}\int_{\mathbb{R}^N}\mathrm{d}\Lambda\,|\Delta(\Lambda)|^{\beta} e^{-N\frac{\beta}{2}\sum_{i=1}^{N}V(\lambda_i)}, \tag{2.2.7}$$

where we already integrated out the circular ensembles which, due to being compact Lie groups, have a finite volume. This volume, though computable, is not relevant in the following and hence put into the constant $\mathcal{N}$.

It is now apparent that the only remnant of the choice of ensemble is the Dyson index β. Hence, it is a natural generalisation of matrix models defined in the present sense to consider $\beta \in \mathbb{R}_+$. This generalisation has several important advantages. From a purely practical side, we will see that in the discussion of the perturbative expansion of matrix model correlation functions, it will turn out that one can perform all computations while keeping the Dyson index variable. This will enable the computation of the correlation functions for all Wigner-Dyson ensembles and beyond in one step, which is discussed in section 3.3.1. Also, there is an explicit realisation of arbitrary β matrix ensembles as tridiagonal matrices with the matrix entries drawn from specific distributions [98] (see appendix D). This will be considered in more detail in section 4.4.

OBSERVABLES. Having defined now the notion of a matrix model that will be relevant for this thesis, we go on by defining the observables we are interested in. There are mainly three objects, in fact their expectation values

12 The included factors of N and β are conventional. We follow the convention of [69].

and correlation functions in the matrix ensemble, which we are interested in. Those are the density of states

$$\rho(x) := \sum_{i=1}^{N} \delta(x - \lambda_i), \tag{2.2.8}$$

the resolvent

$$R(x) := \sum_{i=1}^{N} \frac{1}{x - \lambda_i}, \tag{2.2.9}$$

and the canonical partition function

$$Z(\beta) := \sum_{i=1}^{N} e^{-\beta\lambda_i}. \tag{2.2.10}$$

As mentioned above, these observables depend only on the N matrix eigenvalues and can hence can be inserted in the matrix model partition function written as an eigenvalue integral (eq. (2.2.7)). Between the observables there are various well-known relations via integral transforms. Obviously, one can compute all observables from the density of states via

$$R(x) = \int_{\mathrm{supp}(\rho)} \mathrm{d}\lambda \frac{\rho(\lambda)}{x - \lambda}, \tag{2.2.11}$$

$$Z(\beta) = \int_{\mathrm{supp}(\rho)} \mathrm{d}x \rho(x) e^{-\beta x}, \tag{2.2.12}$$

where the first transformation is known as a Stieltjes transform, while the second one is a Laplace transform in the case of $\mathrm{supp}(\rho) = \mathbb{R}_+$, which will be the case in most of the cases considered here. In the following, we will denote the Laplace transform of the function $f(x)$ to the variable y by the standard notation $\mathcal{L}[f(x), x, y]$. Less obviously (see appendix B for a proof), it holds that

$$\rho(x) = \frac{1}{-2\pi i} \lim_{\epsilon \to 0} \left[R(x + i\epsilon) - R(x - i\epsilon)\right], \tag{2.2.13}$$

$$R(x) = -\mathcal{L}[Z(\beta), \beta, -x], \tag{2.2.14}$$

which are all the relations between the observables that we will need.

In some cases, the expectation values of certain correlation functions, in fact the contributions to their perturbative expansions, will not be regular functions but rather distributions. In these cases, it will be necessary to treat the contributions as such and also be cautious when transforming them. Here, we shall use the author's favourite way of considering distributions, the formalism of hyperfunctions (e.g. [99], cf. appendix C for an introduction), and use the definition of the integral transforms for these generalised functions. We will discuss examples of these and the relations of observables in section 2.3.

Having defined the relevant observables, we can define correlation functions of them by insertion into the partition function. For later use, it will be convenient to define observables of n variables x_i that are collected in $I := (x_1, \dots, x_n)$. We define, for a class of observables generically written as $\mathcal{O}(x)$,

$$\mathcal{O}(I) := \prod_{i=1}^{n} \mathcal{O}(x_i). \tag{2.2.15}$$

Another statistical concept we need is that of a "connected" correlation function. For a single occurrence, the connected correlation function is just the expectation value. For more than one argument we define connected correlation functions via

$$\langle \mathcal{O}(I) \rangle := \sum_{k=1}^{n} \sum_{\substack{I_1,\dots,I_k \subset I \\ \bigcup_{i=1}^{n} I_i = I}} \prod_{j=1}^{k} \langle \mathcal{O}(I_j) \rangle_c \,. \tag{2.2.16}$$

For example, it holds that

$$\langle \mathcal{O}(x_1, x_2) \rangle = \langle \mathcal{O}(x_1, x_2) \rangle_c + \langle \mathcal{O}(x_1) \rangle \langle \mathcal{O}(x_2) \rangle \,, \tag{2.2.17}$$

which implies

$$\langle \mathcal{O}(x_1, x_2) \rangle_c = \langle \mathcal{O}(x_1, x_2) \rangle - \langle \mathcal{O}(x_1) \rangle \langle \mathcal{O}(x_2) \rangle \,. \tag{2.2.18}$$

As a further example, one can show in the same way

$$\begin{aligned} \langle \mathcal{O}(x_1, x_2, x_3) \rangle_c = {} & \langle \mathcal{O}(x_1, x_2, x_3) \rangle - \langle \mathcal{O}(x_1, x_2) \rangle \langle \mathcal{O}(x_3) \rangle - \langle \mathcal{O}(x_2, x_3) \rangle \langle \mathcal{O}(x_1) \rangle \\ & - \langle \mathcal{O}(x_1, x_3) \rangle \langle \mathcal{O}(x_2) \rangle + 2 \langle \mathcal{O}(x_1) \rangle \langle \mathcal{O}(x_2) \rangle \langle \mathcal{O}(x_3) \rangle \,. \end{aligned} \tag{2.2.19}$$

We will see later on that the notion of connected correlation functions defined here coincides with that in the previous section.

ANOTHER TOPOLOGICAL EXPANSION. We have defined the observables we are interested in and can now concern ourselves with their computation. For most of this thesis, "computation" of an object will mean computing contributions to a, in a specific sense, perturbative expansion of said object. As a first step towards this, we show the existence of such an expansion, discussing the actual method of computing it next.

Since all the objects we are interested in are linked to each other by a linear transformation, a perturbative expansion of one of these objects induces one for all the others. Here, we choose the resolvents as our starting point. To study their perturbative expansion, it is useful to rewrite the resolvent in the following way

$$\langle R(x)\rangle = \left\langle \operatorname{tr}\left(\frac{1}{x-H}\right)\right\rangle = \sum_{k=0}^{\infty} x^{-1-k}\left\langle \operatorname{tr}\left(H^k\right)\right\rangle . \tag{2.2.20}$$

Hence, it is sufficient to give a perturbative expansion for the expectation value of traces of powers of the matrix to get such an expansion for the resolvent. Also, for the computation of $\langle R(I)\rangle$, it suffices to know $\left\langle \operatorname{tr}\left(H^{k_1}\right)\cdots\operatorname{tr}\left(H^{k_n}\right)\right\rangle$ for $k_i \in \mathbb{N}_0$.

For simplicity, we restrict ourselves to the unitary case here and point out how to go to the case of the other symmetry classes after having established the existence of the expansion for this case. In order to do this, it is worthwhile to write down the matrix model partition function with an explicit choice of the potential as

$$\mathcal{Z} = \mathcal{N}' \int_{\mathbb{E}_N^2} \mathrm{d}H_N^2 e^{-N\left[\frac{1}{2}\operatorname{tr}\left(H^2\right)-\sum_{k=3}^{\infty}\frac{t_k}{k}\operatorname{tr}\left(H^k\right)\right]}, \tag{2.2.21}$$

where we have scaled the measure such that the quadratic term in the action has this precise form and have shifted the integration such that a potential linear term vanishes. Furthermore, since in the (Gaussian) case, where all powers in the potential beyond the quadratic vanish, the integral is a Gaussian integral and can be directly evaluated, one can choose $\mathcal{N}'$ such that in that case $\mathcal{Z} = 1$. This is reminiscent of the starting point of the derivation of Feynman-diagram perturbation theory in a Quantum Field Theory in the path-integral formalism (e.g [80, 95]), where the quadratic term would determine the propagator, while the higher-order terms determine the possible vertices. One can indeed view the present theory as a (0-dimensional) field theory with matrix-valued fields and apply the same diagrammatic formalism to compute perturbative expansions.

Here, the quadratic term is given by

$$N\operatorname{tr}\left(H^2\right) = NH^{ij}H_{ji} = H^{ij}\left[N\delta_{ik}\delta_{jl}\right]H^{lk}. \tag{2.2.22}$$

Hence, one finds the propagator

$$\left\langle H^{ij}H^{lk}\right\rangle = \frac{1}{N}\delta_{ik}\delta_{jl}, \tag{2.2.23}$$

which, in the purely Gaussian case that can be viewed as the free theory from a field theory perspective, determines all correlation functions of traces of powers of the matrix by Wick's theorem, i.e. the statement that the expectation value of a composite object is given by the sum of all possible ways to contract the components pairwise[13]. As an example, we consider the following expectation value in the Gaussian theory, which we denote by a subscript G on the expectation value:

$$\begin{aligned}\left\langle N \operatorname{tr}\left(H^4\right)\right\rangle_G &= N \left\langle H_{ij} H^j{}_m H^m_k H^{ki} \right\rangle_G \\ &= N\Big[\left\langle H_{ij} H^j{}_m \right\rangle_G \left\langle H^m_k H^{ki} \right\rangle_G + \left\langle H_{ij} H^{ki} \right\rangle_G \left\langle H^j{}_m H^m_k \right\rangle_G \\ &+ \langle H_{ij} H^m_k \rangle_G \left\langle H^j{}_m H^{ki} \right\rangle_G \Big] \\ &= \frac{1}{N}\left[\delta_{im}\delta^j_j \delta^{mi}\delta^k_k + \delta^i_i \delta_j{}^k \delta^j{}_k \delta^m_m + \delta_{ik}\delta_j{}^m \delta^{ji}\delta_m{}^k \right] \\ &= N^2 + N^2 + 1.\end{aligned} \tag{2.2.24}$$

By this reasoning one can, in principle, work out the expectation value in the Gaussian theory of all $\operatorname{tr}\left(H^k\right)$ and hence it is tempting to perform a Taylor expansion of the exponential in eq. (2.2.21) and to define a so-called *formal* matrix integral via

$$\begin{aligned}\mathcal{Z}_{(\text{formal})} &:= \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle \left[\sum_{k=3}^{\infty} N \frac{t_k}{k} \operatorname{tr}\left(H^k\right) \right]^n \right\rangle_G \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k_1,\ldots,k_m \geq 3} \left\langle \prod_{n=1}^{m} N \frac{t_{k_n}}{k_m} \operatorname{tr}\left(H^{k_m}\right) \right\rangle_G .\end{aligned} \tag{2.2.25}$$

This way of interpreting the partition function will be the way we understand the perturbative expansion of a matrix model here[14].

Working out the contractions determining the expectation values needed for the computation of the partition function is in general rather tedious, so it is useful to come up with a diagrammatic expansion simplifying this. Since one is dealing with matrix valued objects (i.e. that have two indices), the propagator is visualized as a double-line, shown in fig. 2.2.1a. To find the diagrammatic representation of the traces of powers of the matrix, it is useful to rewrite the trace as the contraction of k copies of the matrix with k Kronecker deltas. Taking the expectation value of this amounts to taking Wick contractions of the matrices, leaving the Kronecker deltas unchanged. To represent the trace, the deltas are such that upon contraction with the matrices, the first index of the first matrix is contracted with the second index

13 Essentially, this is a statement about Gaussian integrals. This is why it appears in this context, where the partition function (in the Gaussian case) is just a Gaussian integral over a random (matrix-valued) variable and in the QFT context, where the partition function for the free theory is a functional Gaussian integral.

14 Further comments on this way of defining the expansion and its relation to the others ways to do so are discussed in [90].

of the last matrix and in between, each second index of a matrix is contracted with the first index of the next one. Hence, the diagrammatic representation of the trace of the matrix to the power k has k double-line legs, where the legs implement the structure of the Kronecker deltas by forming the vertex, as seen exemplarily for $k = 4$ in fig. 2.2.1b. Performing the Wick-contractions can now

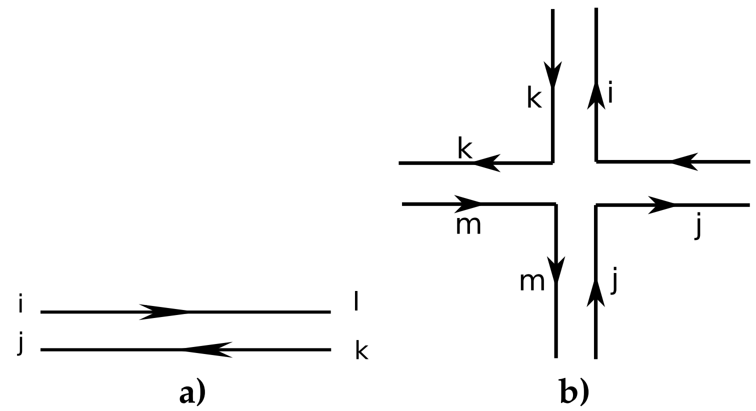


Figure 2.2.1: a): Double-line propagator for a matrix model (in the unitary symmetry class), b): Vertex with four legs. Adapted from [77].

be visualized by connecting the legs pairwise by propagators, yielding so-called *ribbon*-graphs or fatgraphs. For the example of one four-valent vertex, this yields the contributions given in fig. 2.2.2 that directly correspond to the terms obtained in the direct computation of the expectation value performed above. The power of N associated with any diagram that can possibly arise in

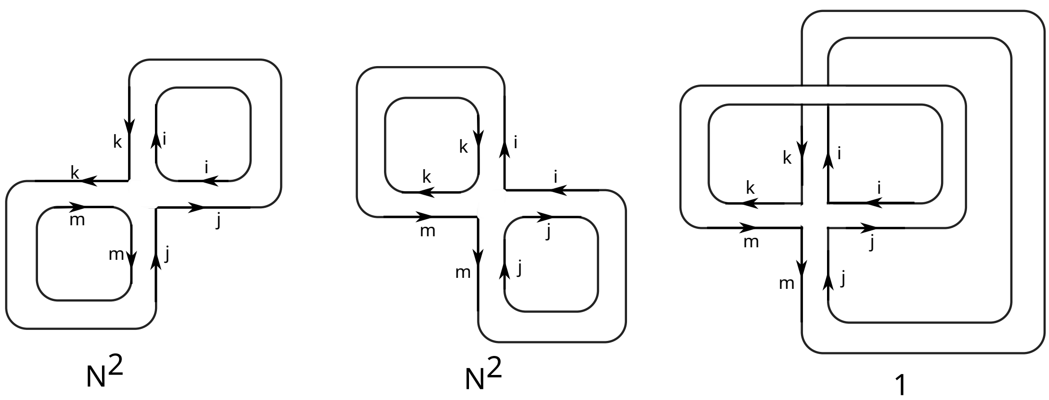


Figure 2.2.2: Ribbon graphs for the expectation value of $N\,\mathrm{tr}(M^4)$. Adapted from [77].

the computation of the expectation values relevant for eq. (2.2.25) is composed of one factor of N^{-1} for each propagator, i.e. an edge of the graph, one factor of N for each vertex and a factor of N for each closed line, which is denoted as a face of the graph. Hence, the power of N of a graph G is given by

$$\#\text{Vertices} - \#\text{Edges} + \#\text{Faces} = \chi(G), \tag{2.2.26}$$

where we notice that this is just the graph's Euler characteristic. This fact was famously observed by t'Hooft [100] in the context of studying the large N limit of $SU(N)$ Yang-Mills theory. This observation is interesting insofar as it allows to interpret the $\frac{1}{N}$ expansion of the correlation functions in the free theory as a topological expansion, which consequently induces a topological expansion of the formal partition function. Explicitly, one can write the partition function as [101]

$$\mathcal{Z} = \sum_{G \in \mathcal{R}} \mathcal{C}(G) N^{\chi(G)} \prod_k t_k^{\#\text{k-vertices}}, \tag{2.2.27}$$

where $\mathcal{R}$ denotes the set of all ribbon-graphs (also the disconnected ones) that can be built with the vertices one has at disposal, i.e. those of valencies k such that $t_k \neq 0$. The prefactor $\mathcal{C}(G)$ can be determined explicitly but since we are only interested in finding the form of the $\frac{1}{N}$ expansion of resolvents, they are not of relevance for our presentation. Notably, this expansion is a topological expansion since the power of N with which an individual graph is weighted is given by its Euler characteristic.

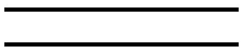
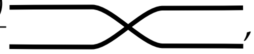
Up to now, we have only discussed the perturbative expansion of the partition function. However, the expectation values of the $\operatorname{tr}\left(H^k\right)$ which were needed for this are also the expectation values appearing in the expansion of the expectation value of the resolvent in eq. (2.2.20). Hence, one can work out the contributions to this by the ribbon-graph expansion we discussed for the partition function and by this find a topological $\frac{1}{N}$ expansion also for them. Analogously, one can argue that in the case of inserting more than one trace, which is relevant for the correlation function of products of resolvents, it holds that

$$\left\langle \operatorname{tr}\left(H^{k_1}\right) \cdots \operatorname{tr}\left(H^{k_n}\right) \right\rangle_c = \sum_{g=0}^{\infty} \frac{K(g, k_1, \ldots, k_n)}{N^{2g+n-2}}, \tag{2.2.28}$$

whose derivation in full detail is given e.g. in [21, 77, 90] but is omitted here due to the details not being relevant for the following discussion. This suffices to show the sought for statement that there is an expansion of multi-resolvents as

$$\langle R(I) \rangle_c =: \sum_{g=0}^{\infty} \frac{R_g(I)}{N^{2g+n-2}}. \tag{2.2.29}$$

Having found this expansion, we briefly comment on going beyond the case of $\beta = 2$. For this case, the propagator in the diagrammatic expansion can be worked out to be given by [21]

 $+ \frac{(2-\beta)}{\beta}$  , (2.2.30)

i.e. the sum of a propagator as for the unitary case and a "twisted" propagator weighted with a factor such that its contribution vanishes in the case of $\beta = 2$ while being given by 1 in the orthogonal and $-\frac{1}{2}$ in the symplectic case. The presence of the twisted propagator in the non-unitary cases implies that the occurring graphs can be *unorientable*, which results, among other implications, in extending the set of possible genus beyond integers by the inclusion of half-integer (i.e. $g = \frac{m}{2}$ with m an odd integer) values. In the following, we will denote the set of half-integers by $\frac{\mathbb{N}}{2}$. Since the treatment of unorientable surfaces is one of the main points of this thesis, we will discuss them in more detail in chapter 3 and hence do not expand on them here. As we discussed above, a definition of a matrix model makes sense for $\beta \in \mathbb{R}_+$ and we would like to remark here that also the propagator in eq. (2.2.30) is easily generalised beyond the Wigner-Dyson classes, which will be important in section 3.3.3.

We can thus summarise the discussion of the perturbative expansion of matrix model observables by saying that any of the observables we are interested in, schematically written as $X(I)$, has a topological expansion of its expectation value of the form

$$\langle X(I) \rangle_c^\beta = \sum_{g=0,\frac{1}{2},\cdots} \frac{X_g^\beta(I)}{N^{2g+n-2}}, \tag{2.2.31}$$

where the dependence on the symmetry class is made explicit by including the Dyson index. Notably, for unitary matrix models all contributions at half-integer genus vanish. This expansion is, of course, not of very great value up to now, since we are lacking a method of computing the coefficient functions. This gap will be filled by discussing our method of choice to compute them, the *loop equations*.

THE PERTURBATIVE LOOP EQUATIONS. The starting point for the derivation of these very useful equations is the representation of the partition function as an eigenvalue integral eq. (2.2.7). In anticipation of the following, we admit that the range of eigenvalues, which up to now was considered to be the whole of $\mathbb{R}$, can be different and one might, for reasons of making the integral absolutely convergent, want the integration over eigenvalues to be along a general contour in the complex plane. Hence, we slightly generalise eq. (2.2.7) to [21]

$$\mathcal{Z} = \mathcal{N} \int_\Gamma \mathrm{d}\Lambda \, |\Delta(\Lambda)|^\beta e^{-N\frac{\beta}{2}\sum_{i=1}^N V(\lambda_i)}, \tag{2.2.32}$$

where $\Gamma = (\gamma_1, \cdots, \gamma_N)$ with paths γ_i such that the integral for the given potential is absolutely convergent and the integrand vanishes on $\partial\Gamma$. The discussion of the appropriate choice of integration contour is rather involved in general and explored in detail in [21]. For the derivation of the loop equations, it is however sufficient to note that [69]

$$0 = \sum_{a=1}^{N} \int_{\Gamma} \mathrm{d}\Lambda \frac{\partial}{\partial \lambda_a} \left[\frac{1}{x - \lambda_a} R(I) |\Delta|^{\beta} e^{-N\frac{\beta}{2}\sum_{k=1}^{N} V(\lambda_k)} \lambda_i^k \right], \tag{2.2.33}$$

due to the integral being over a total derivative with the integrand vanishing on the boundary of Γ by assumption. Distributing the derivatives inside the integral and writing the expressions deriving from this as expectation values in the matrix integral with given potential V and Dyson index β as $\langle\rangle$, it is shown in [69] that

$$\begin{aligned} -N \langle P(x, I) \rangle_c = &\left(1 - \frac{2}{\beta}\right) \partial_x \langle R(x, I) \rangle_c + \langle R(x, x, I) \rangle_c + \sum_{J \supseteq I} \langle R(x, J) \rangle_c \langle R(x, I \backslash J) \rangle_c \\ &- NV'(x) \langle R(x, I) \rangle_c + \frac{2}{\beta} \sum_{k=2}^{n} \partial_{x_k} \left[\frac{\langle R(x, I \backslash \{x_k\}) \rangle_c - \langle R(I) \rangle_c}{x - x_k} \right], \end{aligned} \tag{2.2.34}$$

where $P(x, I)$ collects terms that are analytic in x, which is, as we shall see momentarily, the only relevant information about this object.

From these equations[15], one can not directly compute the expectation values of multi-resolvents recursively since e.g. for the computation of the resolvent with one argument one needs that with two arguments etc. . However, upon plugging in the perturbative expansion of resolvents, this yields the sought for recursive procedure determining the coefficient functions of the resolvents' expansion. To explain how this is done we give some examples that are actually special cases and proceed then by giving the general result.

Example 2.3 ($I = \emptyset$).
Putting in $I = \emptyset$ into eq. (2.2.34) and expanding the correlation functions, taking into account only the leading order contributions, one finds:

$$(analytic) = \left(1 - \frac{2}{\beta}\right) \partial_x \frac{R_0^{\beta}(x)}{N^{-1}} + \frac{R_0^{\beta}(x, x)}{1} + \left(\frac{R_0^{\beta}(x)}{N^{-1}}\right)^2 - NV'(x) \frac{R_0^{\beta}(x)}{N^{-1}}, \tag{2.2.35}$$

where by analytic we mean the contribution arising from the expansion of $\langle P(x) \rangle_c$ whose exact form is not of interest beyond the fact that it is analytic in x. To leading order, $\mathcal{O}(N^2)$, this yields

$$(analytic) = R_0^{\beta}(x)^2 - V'(x) R_0^{\beta}(x). \tag{2.2.36}$$

15 There is one equation for every $n \in \mathbb{N}$.

The other occurring terms and those from higher genus contributions will be included into other equations of this type.

Using eq. (2.2.36), we can define the central object for solving matrix models via loop equations, the *spectral curve*. This is done by completing the square on the left hand side by adding and subtracting the analytic term $\frac{V'^2}{4}$ and moving the subtracted term to the right hand side. This then yields

$$y(x)^2 = (\text{analytic}), \tag{2.2.37}$$

$$\text{with} \quad y(x) := R_0^\beta(x) - \frac{V'(x)}{2}, \tag{2.2.38}$$

where $y(x)$ is referred to as the spectral curve of the matrix model. It is a hyperelliptic curve over $\mathbb{C}$, in fact defined on a double-cover of the complex plane. This can essentially be seen from the fact that if $y(x)$ is a solution of the equation, also $-y(x)$ is and hence eq. (2.2.37) defines a multi-valued function with a branch cut that for all cases considered here will be a compact interval $[a_-, a_+] \subset \mathbb{R}$ i.e. we consider only so-called *one-cut* matrix models.

One may wonder now whether this definition is useful, since one has essentially expressed the spectral curve by something that hasn't been computed yet. However, it will turn out that all the higher genus contributions to (multi-)resolvents are determined solely by the spectral curve and hence, it is convenient to drop the potential as defining the matrix model and define it directly by the spectral curve. Furthermore, one can give a more physical interpretation of this object by relating it to the leading order density of states. The density of states can be expressed using the resolvent via eq. (2.2.13) and hence, plugging the topological expansion on both sides of this equation, one finds

$$\begin{aligned}
\rho_0(x) &= \frac{-1}{2\pi i} \lim_{\epsilon \to 0} \left[R_0^\beta(x + i\epsilon) - R_0^\beta(x - i\epsilon) \right] \\
&= \frac{-1}{2\pi i} \lim_{\epsilon \to 0} \left[y(x + i\epsilon) - y(x - i\epsilon) \right] \\
&= -\frac{1}{2\pi i} \begin{cases} 2 \lim\limits_{\epsilon \to 0} \pm y(x \pm i\epsilon) & , \quad \text{for} \quad x \in [a_-, a_+], \\ 0 & , \quad \text{else}, \end{cases}
\end{aligned} \tag{2.2.39}$$

where we have used the analyticity of $V'(x)$ in the first and the explicit branch-cut structure in the second line. Here and in the following, we choose this structure such that the spectral curve has the positive sign when going from the branch-cut into the upper half-plane and the negative sign when going to the lower half-plane. This means that, when given the leading order density of states of a matrix model, this implies the position of the cut, since by eq. (2.2.39) it is just given by the support of the density of states. The leading order density of states also determines the spectral curve via the relation

$$\underset{x\in\text{supp}(\rho)}{\forall} \mp i\pi\rho_0(x) = \lim_{\epsilon\to 0} y(x\pm i\epsilon). \tag{2.2.40}$$

Thus, the spectral curve and with it, in fact, the whole topological expansion of the multi-resolvents, is determined by giving the leading-order density of states for the matrix model. This will be a crucial insight for the connection of a matrix model with JT/topological gravity.

Example 2.4 ($I = \{x_1\}$)**.**
Considering now $I = \{x_1\}$ in eq. (2.2.34) and expanding again to the first orders, one finds as the leading order ($\mathcal{O}(N)$) equation

$$R_0^\beta(x,x_1)\underbrace{\left[2R_0^\beta(x) - V'(x)\right]}_{=2y(x)} + \frac{2}{\beta}\partial_{x_1}\left[\frac{R_0^\beta(x)+R_0^\beta(x_1)}{x-x_1}\right] = (\textit{analytic}). \tag{2.2.41}$$

For the method we would like to use to solve this equation, we can assume that x_1 is away from the cut and hence, the only non-analyticity is in the cut of $R_0^\beta(x)$. Noticing this, one can perform the derivative, move analytic terms to the right hand side and add some analytic terms to the left hand side to find

$$2y(x)R_0^\beta(x,x_1) + \frac{2}{\beta}\frac{y(x)}{(x-x_1)^2} = (\textit{analytic in } x \textit{ near the cut}). \tag{2.2.42}$$

This can be solved for $R_0^\beta(x,x_1)$ using a dispersion-relation argument originally due to Migdal [102]. The principal idea of this is to write $R_0^\beta(x,x_1)$ as a contour integral and then use eq. (2.2.42) and a deformation of the contour to write this integral in terms of the (known) second term of eq. (2.2.42). In order to do this, one would like to divide eq. (2.2.42) by $y(x)$. This is, however, not possible directly since $\frac{1}{y(x)}$ diverges at the end-points of the cut and hence the right hand side of eq. (2.2.42) would no longer be analytic. To work around this, one defines

$$\sigma(x) := (x-a_+)(x-a_-), \tag{2.2.43}$$

which is chosen such that $\sqrt{\sigma(x)}$ has the same branch-cut structure as $y(x)$. Hence, $\frac{\sqrt{\sigma(x)}}{y(x)}$ is analytic near the cut. Multiplying with this factor, one finds

$$\sqrt{\sigma(x)}R_0^\beta(x,x_1) + \frac{1}{\beta}\frac{\sqrt{\sigma(x)}}{(x-x_1)^2} = (\text{analytic in } x \text{ near the cut}). \tag{2.2.44}$$

Having established this, one computes

$$\begin{aligned}
R_0^\beta(x,x_1)\sqrt{\sigma(x)} \overset{\text{Cauchy}}{=} & \frac{1}{2\pi i}\oint_{\mathcal{C}'}\frac{\mathrm{d}x'}{x'-x}R_0^\beta(x',x_1)\sqrt{\sigma(x')} \\
= & \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}x'}{x'-x}R_0^\beta(x',x_1)\sqrt{\sigma(x')} \\
= & -\frac{1}{2\pi i}\frac{1}{\beta}\oint_{\mathcal{C}}\frac{\mathrm{d}x'}{x'-x}\frac{\sqrt{\sigma(x')}}{(x'-x_1)^2} \\
=: & -\frac{1}{\beta}\frac{1}{2\pi i}\oint_{\mathcal{C}}\mathrm{d}x' f(x',x_1),
\end{aligned} \tag{2.2.45}$$

which uses first Cauchy's theorem to rewrite the left hand side of the equation as a contour integral over the contour $\mathcal{C}'$ depicted in fig. 2.2.3a. Note that this contour circumvents x in an anti-clockwise fashion. To the second line, the contour was deformed to the contour $\mathcal{C}$ shown in fig. 2.2.3b, which can be done since $R_0^\beta(x',x_1)$ is holomorphic in x' away from the cut. This modification is divided in two steps. First, one deforms the contour such that it is given by going first the green circle, followed by the black contour around the cut. Then, one brings the end points of the green contour together at the real line, outside the cut. After this, the radius of the green circle is sent to infinity where the integrand vanishes due to $R_0^\beta(x',x_1) \propto \frac{1}{x'}$ for large modulus of x[16]. Thus, one is left with just the contribution of the contour encircling the cut which, as a last step, is brought to the final form depicted in fig. 2.2.3b. In the third line, we use eq. (2.2.44) to exchange the integrand by -1 times the second term of this equation by noting that the contour integral of a function analytic near the cut vanishes. To simplify the notation a bit in the computation of this integral, we rename $x \to x_1$ and $x_1 \to x_2$. Considering the case where both of these arguments are away from the cut one can deform the contour $\mathcal{C}$ depicted again in fig. 2.2.4a to the contour $\mathcal{C}''$ depicted in fig. 2.2.4b. To reduce this to residues, one brings the lines connecting the small circles around the x_i to the large circle very close to one another which will make them cancel each other as they are in opposite direction and enclose a region where the integrand is analytic. Having done

16 This can be seen, for example, by writing the contribution to the topological expansion of the multi-resolvent as an integral of the corresponding contribution to the correlation function of densities of states via eq. (2.2.11) and noting that the coefficient function of this expansion have to be finite at large x' due to the normalisation of the density of states [21, 69].

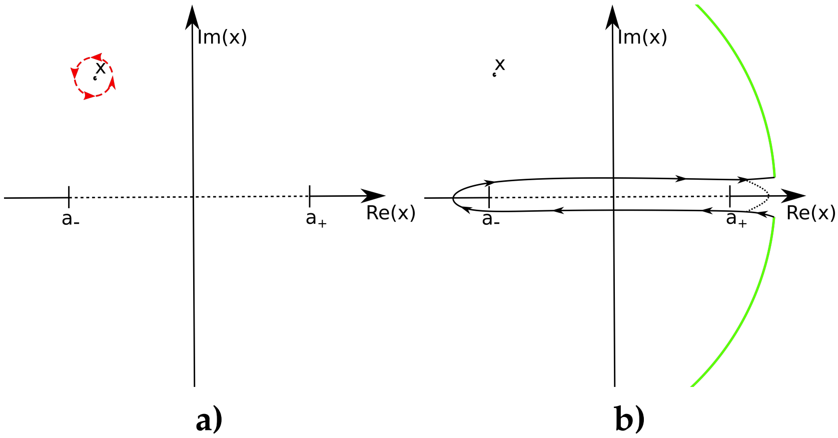


Figure 2.2.3: Illustration of the contours appearing in eq. (2.2.45). The branch cut a one-cut matrix integral is depicted as the broken line with the edges a_- and a_+. In a) the contour $\mathcal{C}'$ is shown in red, in b) first deformation is shown in black with the final deformation $\mathcal{C}$ found as the completion of the contour circumventing the cut by the dotted line (black). Adapted from [2].

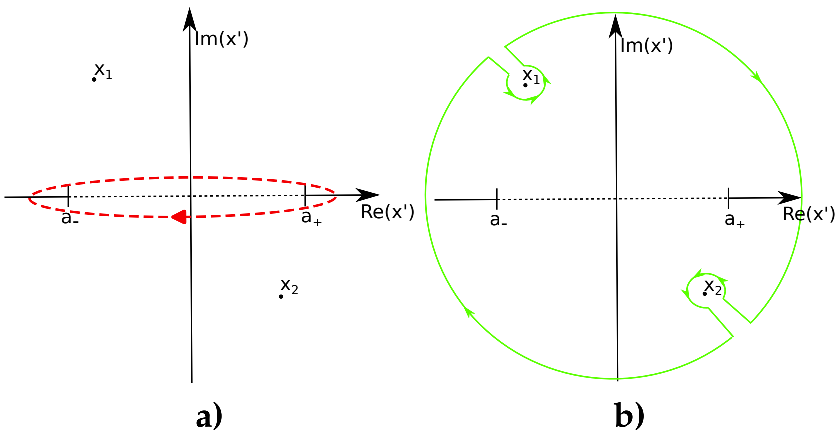


Figure 2.2.4: Modification of the integration contour needed to evaluate the integral. In a) the original contour $\mathcal{C}$ (red), in b) its deformation $\mathcal{C}''$ (green). Adapted from [2].

this, the radius of the large circle is sent to infinity which, as above, lets its contribution vanish. Hence, it holds that

$$\begin{aligned} R_0^\beta(x_1,x_2)\sqrt{\sigma(x_1)} &= -\frac{1}{\beta}\left(\operatorname*{Res}_{x'=x_1} + \operatorname*{Res}_{x'=x_2}\right) f(x',x_2) \\ &= -\frac{1}{\beta}\left[\frac{\sqrt{\sigma(x_1)}}{(x_1-x_2)^2} - \frac{x_1x_2 + a_+a_- - \frac{a_++a_-}{2}(x_1+x_2)}{(x_1-x_2)^2\sqrt{\sigma(x_2)}}\right] \\ &= \frac{1}{\beta(x_1-x_2)^2}\left[\frac{x_1x_2 + a_+a_- - \frac{a_++a_-}{2}(x_1+x_2)}{\sqrt{\sigma(x_2)}} - \sqrt{\sigma(x_1)}\right], \end{aligned} \tag{2.2.46}$$

which yields as a final result

$$R_0^\beta(x_1,x_2) = \frac{1}{\beta(x_1-x_2)^2}\left[\frac{x_1x_2 + a_+a_- - \frac{a_++a_-}{2}(x_1+x_2)}{\sqrt{\sigma(x_1)}\sqrt{\sigma(x_2)}} - 1\right]. \quad (2.2.47)$$

Interestingly, this result has no dependence on the spectral curve but is rather determined solely through the cut's end points.

Example 2.5 ($I = \{x_1, x_2\}$).
This, the final special case, is done analogously to the two previous examples and yields an expression for $R_0^\beta(x,x_1,x_2)$ *via a contour integral. It is given by [69]*

$$R_0^\beta(x,x_1,x_2) = -\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dx'}{x'-x}\frac{\sqrt{\sigma(x')}}{\sqrt{\sigma(x)}}\frac{1}{y(x')}\times \\ \left[R_0^\beta(x',x_1)R_0^\beta(x',x_2) + \frac{1}{\beta}\left(\frac{R_0^\beta(x',x_2)}{(x'-x_1)^2} + \frac{R_0^\beta(x',x_1)}{(x'-x_2)^2}\right)\right]. \quad (2.2.48)$$

Having discussed now the special cases and illustrated how to get to the representation of the $R_g^\beta(I)$ as contour integrals we can state the relation valid for all other cases, which we shall refer to as the *perturbative* loop equations. The relation is given by [69]

$$2y(x)R_0^\beta(x,I) + F_g^\beta(x,I) = (\text{analytic in } x), \quad (2.2.49)$$

with

$$F_g^\beta(x,I) = \left(1-\frac{2}{\beta}\right)\partial_x R_{g-\frac{1}{2}}^\beta(x,I) + R_{g-1}^\beta(x,x,I) \\ + \sum_{J\supseteq I,h}' R_h^\beta(x,J)R_{g-h}^\beta(x,I\backslash J) \\ + 2\sum_{k=1}^n\left[R_0^\beta(x,x_k) + \frac{1}{\beta}\frac{1}{(x-x_k)^2}\right]R_g^\beta(x,I\backslash\{x_k\}), \quad (2.2.50)$$

where the resolvents for negative genus are defined to vanish and the primed sum means that contributions containing $R_0^\beta(x)$ or $R_0^\beta(x,x_k)$ are excluded. Using the dispersion relation argument discussed above for the case of $(g,n) = (0,2)$, the relation can be rewritten as

$$R_g^\beta(x,I) = -\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{dx'}{x'-x}F_g^\beta(x',I)\frac{1}{2y(x')}\frac{\sqrt{\sigma(x')}}{\sqrt{\sigma(x)}}. \quad (2.2.51)$$

This is now in essence the recursive procedure that we will use in section 3.3.1 to compute the correlation functions for specific matrix models. A prime advantage of this recursion is that it already incorporates, with no additional effort, the dependence on the Dyson index β. Hence, we will be able to compute perturbative expansions for matrix models determined by a spectral curve/leading order density of states directly in the setting of arbitrary Dyson index.

THE DOUBLE-COVER COORDINATE. Specifically, the matrix models we will be interested in all have in common that one can set $a_- = 0$ and send $a_+ \to \infty$. In this setting, the genus 0 two-point function of resolvents drastically simplifies as one can see by putting a_- directly into eq. (2.2.47) and then taking the limit on a_+:

$$\begin{aligned} R_0^\beta(x_1, x_2) &= \lim_{a_+ \to \infty} \frac{1}{\beta(x_1 - x_2)^2} \left(\frac{x_1 x_2 - \frac{a_+}{2}(x_1 + x_2)}{\sqrt{x_1^2 - x_1 a_+}\sqrt{x_2^2 - x_2 a_+}} - 1 \right) \\ &= \frac{1}{2\beta[(\sqrt{-x_1} - \sqrt{x_2})(\sqrt{-x_1} + \sqrt{-x_2})]^2} \left(\frac{-x_1 - x_2 - 2\sqrt{-x_1}\sqrt{-x_2}}{\sqrt{-x_1}\sqrt{-x_2}} \right) \\ &= \frac{1}{2\beta} \frac{1}{(\sqrt{-x_1} + \sqrt{-x_2})^2 \sqrt{-x_1}\sqrt{-x_2}}, \end{aligned} \tag{2.2.52}$$

One might ask whether for such models, it still makes sense to speak of a contour integral "around the cut" if one of its end points is at real infinity. This can be resolved by noting that, due to the regularity of the resolvent at infinity on both of the sheets before taking the limit on a_+, they can both be compactified to the Riemann sphere $\mathbb{P}^1$ [21]. Taking the limit on a_+ in this setting resolves the question, since on the Riemann sphere one can go around the infinite point.

In this setting, it is useful to introduce a so-called *double-cover* coordinate $z \in \mathbb{P}^1$. This coordinate is chosen such that the spectral curve is a single-valued function. This is achieved by defining

$$x := -z^2. \tag{2.2.53}$$

Here, we choose the sign in this way to simplify the form of eq. (2.2.52). In the new coordinate it becomes

$$R_0^\beta(z_1, z_2) = \frac{1}{2\beta} \frac{1}{z_1 z_2 (z_1 + z_2)^2}. \tag{2.2.54}$$

Apart from this, also the perturbative loop equations simplify considerably in the double-cover coordinate, as we worked out in [2] and it was also reported in [76]. To get them, one first has to take the necessary limit on the quotient of the two σ as

$$\frac{\sqrt{\sigma(x')}}{\sqrt{\sigma(x)}} = \lim_{a_+\to\infty} \frac{\sqrt{x'(x'-a_+)}}{\sqrt{x(x-a_+)}} = \frac{\sqrt{-x'}}{\sqrt{-x}} = \frac{z'}{z}, \tag{2.2.55}$$

which enables one to write down the perturbative loop equations in double-cover coordinates as

$$\begin{aligned} R_g^\beta(-z^2, I) &= -\frac{1}{2\pi i}\oint_{\mathcal{C}_z} \frac{2z'\mathrm{d}z'}{z'^2 - z^2}\frac{z'}{z}\frac{1}{2y(-z'^2)} F_g^\beta(-z'^2, I) \\ &= -\frac{1}{2\pi i z}\oint_{\mathcal{C}_z} \frac{z'^2\mathrm{d}z'}{z'^2 - z^2}\frac{1}{y(-z'^2)} F_g^\beta(-z'^2, I) \\ &= \frac{1}{2\pi i z}\oint_{[-i\infty+\epsilon, i\infty+\epsilon]} \frac{z'^2\mathrm{d}z'}{z'^2 - z^2}\frac{1}{y(-z'^2)} F_g^\beta(-z'^2, I), \end{aligned} \tag{2.2.56}$$

where we have denoted the preimage of the contour $\mathcal{C}$ under $z \to x(z)$ as $\mathcal{C}_z$ and used $\mathrm{d}x' = -2z'\,\mathrm{d}z'$. In the last line, this was made explicit as being the interval $[-i\infty + \epsilon, i\infty + \epsilon]$ which, since the contour is defined on the Riemann sphere, is indeed a closed contour, as $-i\infty$ and $i\infty$ are one point on the sphere. The contour is that which, with a small offset into the hemisphere of positive real value from the origin, winds once around the sphere. That this is indeed the preimage of the contour in the coordinate x can be argued as follows. The contour $\mathcal{C}$ goes around the cut once in a clock-wise fashion (cf.fig. 2.2.4a). Hence, it intersects the negative real axis at the point $-\delta$ and before (after) this intersection point has negative (positive) imaginary part. Using the branch-cut structure of the square root, which can be written as (e.g. [99])

$$\lim_{\epsilon\to 0} \sqrt{-x \pm i\epsilon} = (\pm i)^{\theta(x)}\sqrt{|x|}, \tag{2.2.57}$$

with θ denoting the Heaviside function, one can quickly convince oneself that the contour in the double cover coordinate has to go from positive imaginary infinity to negative imaginary infinity with a small offset ϵ which gives the claimed contour [2]. Reversing now the direction of traversing the contour gives the last line of eq. (2.2.56). In the following, we will write this contour as $i\mathbb{R} + \epsilon$.

Having transformed the contour integral, one also has to transform the expression for the function $F_g^\beta(-z'^2, I)$. To do this, one has to note

$$\partial_x f(x) = \frac{1}{-2z}\partial_z f(z), \tag{2.2.58}$$

which one can put in the definition eq. (2.2.50) to find

$$F_g^\beta(-z^2, I) = \left(1 - \frac{2}{\beta}\right)\frac{1}{-2z}\partial_z R^\beta_{g-\frac{1}{2}}\left(-z^2, I\right) + R^\beta_{g-1}\left(-z^2, -z^2, I\right) + \sum_{I \supseteq J, h}' R_h^\beta\left(-z^2, J\right) R^\beta_{g-h}\left(-z^2, I \backslash J\right) + 2\sum_{k=1}^{n}\left[R_0^\beta(-z^2, -z_k^2) + \frac{1}{\beta}\frac{1}{\left(z_k^2 - z^2\right)^2}\right] R_g^\beta\left(-z^2, I \backslash \left\{-z_k^2\right\}\right). \tag{2.2.59}$$

As the notation becomes quite cluttered by carrying the explicit dependence of the resolvents on $-z_i^2$, we will in the following write this dependence as just a dependence on z_i, leaving the true dependence implicit.

For completeness, we also give the transformed version of the remaining special case as

$$R_0^\beta(z_1, z_2, z_3) = \frac{1}{2\pi i z_1}\oint_{i\mathbb{R}+\epsilon}\frac{z'^2 \mathrm{d}z'}{z'^2 - z_1^2}\frac{2}{y(z')} \times \left[R_0^\beta(z', z_2)R_0^\beta(z', z_3) + \frac{1}{\beta}\left(\frac{R_0^\beta(z', z_3)}{\left(z_2^2 - z'^2\right)^2} + \frac{R_0^\beta(z', z_2)}{\left(z_3^2 - z'^2\right)^2}\right)\right]. \tag{2.2.60}$$

This and the general expression above are now the version of the perturbative loop equations that we will use to explicitly perform computations in the main text for all symmetry classes at once.

A BRIEF EXCURSION: THE TOPOLOGICAL RECURSION. However, for the unitary case, there is a vast simplification of the recursion due to Eynard [103], known as the *topological recursion*. Before going over to the discussion of the duality of matrix models with JT/topological gravity we will briefly recall it, largely based on the recollection of it in [1], without going into its derivation which can be found in the original paper or, in a more pedagogical presentation, in [90].

Topological recursion is most naturally formulated using differential forms and thus one defines[17]

$$\omega_{g,n}(x_1, \ldots, x_n) := R_g(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n + \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1 - x_2)^2}, \tag{2.2.61}$$

17 $\omega_{g,n}$ is to be understood as an element of the tensor product of n one-forms. Due to this, not notational sloppiness, there are no wedge-product signs.

with the convention that a missing superscript on a contribution to the topological expansion of a resolvent means $\beta = 2$. Going over to the double-cover coordinates, it holds for $(g, n) \neq (0, 2)$ that

$$\begin{aligned}\omega_{g,n}(z_1, \dots, z_n) =& (-2)^n z_1 \dots z_n R_{g,n}(z_1, \dots, z_n) \mathrm{d}z_1 \dots \mathrm{d}z_n \\ :=& W_{g,n}(z_1, \dots, z_n) \mathrm{d}z_1 \dots \mathrm{d}z_n.\end{aligned} \tag{2.2.62}$$

For the excluded case, $(g, n) = (0, 2)$, one defines $W_{0,2}$ analogously including the additional summand in eq. (2.2.61)[18]. We proceed to stating the topological recursion determining the $\omega_{g,n}$ for a spectral curve y:
First, by definition[19]

$$\omega_{0,1}(z) = 2zy(z)dz, \tag{2.2.63}$$

$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}, \tag{2.2.64}$$

which serve as the input to the recursion. Now, for each branch point a of the spectral curve one defines [21]

$$K_a(z_1, z) := \frac{1}{2} \frac{\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma_a(z))}, \tag{2.2.65}$$

where $\sigma_a(z)$ is a so-called local Galois involution, which denotes a function locally exchanging the two sheets meeting at a branch point[20]. Furthermore, the notation $\int \omega(z_1, \cdot)$ with $\omega \propto \mathrm{d}z_1 \mathrm{d}z_2$ denotes the integration of the one-form whose position is replaced by the $\cdot$. Having defined this, one can write the topological recursion for the unitary symmetry class determining the other $\omega_{g,n}$ as [21]

$$\begin{aligned}\omega_{g,n}(z_1, I) = \sum_a \Big[&\operatorname*{Res}_{z=a} K_a(z_1, z) [\omega_{g-1,n+1}(z, \sigma_a(z), I) \\ &+ \sideset{}{'}\sum_{\substack{h+h'=g \\ J \cup J' = I}} \omega_{h,1+|J|}(z, J) \omega_{h',1+|J'|}(\sigma_a(z), J') \Big],\end{aligned} \tag{2.2.66}$$

18 Adding the additional term to $\omega_{0,2}$ seems unreasonable at first sight, but as one can see from the relation of the correlation function of products of ρ with that of multi-resolvents via the generalisation of eq. (2.2.13), only the discontinuity of R through the cut matters. The additional term does not contribute to this. Thus, it is valid to use $W_{0,2}$ to compute e.g. the corresponding contribution to $\langle Z(\beta_1) Z(\beta_2) \rangle$. However, one should keep in mind that the added term actually introduces a singularity as $x_1 \to x_2$ which was not present in the original $R_{0,2}$.

19 The definition of $\omega_{0,1}$ is modified by a sign compared to [21] to follow our sign convention. Using the general result for $R_{0,2}$ for one-cut matrix models eq. (2.2.54) one can verify directly that $\omega_{0,2}$ is indeed of this form.

20 Put plainly, it is a function mapping the coordinate z to the other coordinate corresponding to the same x. For the coordinate defined in eq. (2.2.53), $\sigma_0(z) = -z$, which is the only branch point in the cases we are interested in.

with $\sum'$ denoting the sum excluding the cases $(h, I) = (0, \emptyset)$ and $(h, I) = (g, J)$.
To apply this result, it remains to compute K_a for the branch points of the spectral curves of interest. For those there is only one branch point, $a = 0$, and thus only K_0 has to be computed. As noted in footnote 20, $\sigma_0(z) = -z$, implying

$$\begin{aligned} K_0(z_1, z) &= \frac{1}{2}\frac{\int_{-z}^{z} \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(-z)} = \frac{\frac{1}{2}\mathrm{d}z_1 \int_{-z}^{z} \frac{\mathrm{d}z'}{(z_1 - z')^2}}{2z(y(z) - y(-z))\mathrm{d}z} \\ &= \frac{\frac{1}{2}\mathrm{d}z_1 \left[\frac{1}{z_1 - z} - \frac{1}{z_1 + z}\right]}{2z(y(z) - y(-z))\mathrm{d}z} \\ &= \frac{\mathrm{d}z_1}{z_1^2 - z^2}\frac{1}{2(y(z) - y(-z))\mathrm{d}z}. \end{aligned} \tag{2.2.67}$$

All the spectral curves that will concern us, at least in the orientable part of this thesis, are antisymmetric, as it will be obvious momentarily in the next section. Hence,

$$K_0(z_1, z) = \frac{1}{z_1^2 - z^2}\frac{1}{4y(z)}\frac{\mathrm{d}z_1}{\mathrm{d}z}. \tag{2.2.68}$$

Putting this into eq. (2.2.66) one can, after noting that, on the RHS, one of the dz is cancelled by the denominator of K_0, drop the differentials and write the equation in terms of the $W_{g,n}$, yielding

$$\begin{aligned} W_{g,n}(z_1, J) =& \operatorname*{Res}_{z=0} \left\{ \frac{\mathrm{d}z}{z_1^2 - z^2}\frac{1}{4y(z)} \Big[W_{g-1,n+1}(z, -z, J) \right. \\ &\left. + \sum_{\substack{h+h'=g \\ I \cup I' = J}}' W_{h,1+|I|}(z, I) W_{h',1+|I'|}(-z, I') \Big] \right\}, \end{aligned} \tag{2.2.69}$$

which is the form of the topological recursion relevant for the application to JT/topological gravity.

2.3 THE JT/MM DUALITY IN A NUTSHELL

Having now introduced the basics of JT gravity and of matrix models, we can finally concern ourselves with their relation. To establish this relation, to which we have alluded now several times, one can start with asking what a matrix model should look like that has a relation to JT gravity. As we discussed in section 2.2, a matrix model is specified by giving a spectral curve and choosing a Dyson index. Here, we will consider only the case of $\beta = 2$, since it will turn out that the correct symmetry class for studying orientable theories on the gravitational side is the unitary one. We will consider the reasoning leading to this result in section 3.1.

Giving a spectral curve is, by eq. (2.2.40), equivalent to requiring a specific function for the leading-order density of states. Hence, to find a matrix model related to JT gravity, one has to find the leading-order contribution to the expectation value of the density of states there. Inverting eq. (2.2.12), one finds [68]

$$\begin{aligned} \rho_0^{\mathrm{JT}}(x) = \mathcal{L}^{-1}[Z_{0,1}(\beta), \beta, x] &= \frac{\gamma}{2\pi^2} \sinh\left(2\pi\sqrt{2\gamma x}\right) \\ &\overset{\gamma=\frac{1}{2}}{=} \frac{1}{4\pi^2} \sinh\left(2\pi\sqrt{x}\right). \end{aligned} \tag{2.3.1}$$

This function has real support on the positive real axis, which is why we specialised parts of our discussion of the perturbative expansion of matrix models to this case. To find the corresponding spectral curve for it, one rewrites (assuming $x > 0$) the left hand side of eq. (2.2.40) as

$$\mp i\pi\rho_0^{\mathrm{JT}}(x) = \frac{1}{4\pi} \sin\left(\mp i 2\pi\sqrt{x}\right) = \lim_{\epsilon\to 0} \frac{1}{4\pi} \sin\left(2\pi\sqrt{-(x \pm i\epsilon)}\right), \tag{2.3.2}$$

where we first used the definition and antisymmetry of the hyperbolic sine and then the continuity of sine and the branch-cut structure of the square-root, eq. (2.2.57). This implies via eq. (2.2.40)

$$y^{\mathrm{JT}}(x) = \frac{1}{4\pi} \sin\left(2\pi\sqrt{-x}\right), \tag{2.3.3}$$

which in double-cover coordinates can be written as

$$y^{\mathrm{JT}}(z) = \frac{1}{4\pi} \sin\left(2\pi z\right). \tag{2.3.4}$$

This could now be put into the topological recursion or the loop equations to find the perturbative expansions of the observables we are interested in.

DOUBLE-SCALING. However, as the attentive reader will have noticed, $\rho_0^{\mathrm{JT}}(x)$ is not normalised and hence in conflict to our definition of the density of states in matrix models eq. (2.2.8), for which it holds that

$$\int_{\mathrm{supp}(\rho)} \mathrm{d}x \left\langle \rho^{\mathrm{M}}(x) \right\rangle = \sum_{i=1}^{N} 1 = N. \tag{2.3.5}$$

Here, we have included the superscript M to indicate that this is the density of states for a matrix model with a finite value of N. From the perturbative expansion of matrix model correlation functions, eq. (2.2.31), it follows that

$$\left\langle \rho^{\mathrm{M}}(x) \right\rangle = \sum_{g=0}^{\infty} \frac{\rho_g^{\mathrm{M}}(x)}{N^{2g-1}} = \underbrace{N\rho_0^{\mathrm{M}}(x)}_{:=\rho_0^t(x)} + \mathcal{O}\left(N^{-1}\right), \tag{2.3.6}$$

and hence for large N it has to hold that

$$\int_{\text{supp}(\rho)} \mathrm{d}x\, \rho_0^{\mathrm{M}}(x) = 1, \tag{2.3.7}$$

which for the choice of $\rho_0^{\mathrm{M}} = \rho_0^{\mathrm{JT}}$ cannot be fulfilled. However, one can define the matrix model having this leading-order density of states by a so-called *double-scaling* procedure [68]. To illustrate this procedure, we give a simplified argument for it. For this, consider a matrix model which in the large N limit fulfils

$$\left\langle \rho^{\mathrm{M}}(x) \right\rangle = \frac{\kappa}{4\pi^2} \sinh\left(2\pi \sqrt{x - \frac{x^2}{2a}} \right), \tag{2.3.8}$$

with $a, \kappa \in \mathbb{R}$. Since this density of states has real support on $[0, a] \subset \mathbb{R}$, i.e. a compact interval, and is continuous, its integral over said interval is finite and one can define

$$C(a) := \int_0^a \frac{1}{4\pi^2} \sinh\left(2\pi \sqrt{x - \frac{x^2}{2a}} \right). \tag{2.3.9}$$

Hence, the normalisation for $\langle \rho^{\mathrm{M}} \rangle$ can be expressed as

$$N = \kappa C(a), \tag{2.3.10}$$

or equivalently

$$\kappa = \frac{N}{C(a)}. \tag{2.3.11}$$

The reason, why we have chosen this specific form for the large N behaviour of $\langle \rho^{\mathrm{M}} \rangle$ is that

$$\lim_{a \to \infty} \left\langle \rho^{\mathrm{M}}(x) \right\rangle = \kappa \rho_0^{\mathrm{JT}}(x), \tag{2.3.12}$$

i.e. upon identifying $\kappa = e^{S_0}$, this is the leading-order behaviour of the expectation value for the density of states in JT gravity, which is given by

$$\left\langle \rho^{\mathrm{JT}}(x) \right\rangle = \sum_{g=0}^{\infty} \frac{\rho_g^{\mathrm{JT}}(x)}{(e^{S_0})^{2g-1}} = e^{S_0} \rho_0^{\mathrm{JT}}(x) + \mathcal{O}\left(\left(e^{S_0} \right)^{-1} \right). \tag{2.3.13}$$

Hence, upon taking simultaneously $N \to \infty$ and $a \to \infty$ (thus the name "double-scaling") while keeping κ constant, the matrix model reproduces the behaviour of JT gravity to leading order in e^{S_0}.

This motivates that to make sense of a matrix model with support of the density of states on the whole real axis, the topological expansion has to be considered as an expansion in a new effective parameter rather than in $\frac{1}{N}$. The extension of the leading-order discussion given here to higher orders is rather involved [65] and we will hence refrain from extending on it. A selection of articles giving an extensive discussion of the double-scaling procedure for general matrix models is [72, 74, 90].

Another perspective on the discussion for higher orders [69] starts from observing that the double-scaling essentially turns the leading-order approximation to $\langle \rho^{\mathrm{M}} \rangle$ from

$$\rho_0^t(x) = N\rho_0^{\mathrm{M}}(x), \tag{2.3.14}$$

to

$$\rho_0^t(x) = e^{S_0}\rho_0^{\mathrm{JT}}(x). \tag{2.3.15}$$

As we have shown above, the perturbative expansion for the expectation value of correlation functions is determined by the loop equations that require as an input only the leading-order density of states. In fact, they determine the coefficient functions of an expansion of the form eq. (2.2.31). However, one could also put the dependence of N into the coefficient functions, e.g. write

$$\left\langle \rho^{\mathrm{M}}(x) \right\rangle = \sum_{g=0}^{\infty} \rho_g^t(x), \tag{2.3.16}$$

where then also the dependence on N would be determined by the recursion. Iterating the recursion with the input eq. (2.3.15) would then yield precisely what we claimed to happen, an expansion with $N \to e^{S_0}$. Actually, taking the factors of e^{S_0} again out of the coefficient functions, one sees that the new coefficient functions are those determined by putting in the original recursion $\rho_0^{\mathrm{M}} \to \rho_0^{\mathrm{JT}}$[21].

Consequently, the process of double-scaling allows one to make sense of putting ρ_0^{JT} as the input density of states for a matrix model and also turns the matrix model topological expansion in $\frac{1}{N}$ into one in $\frac{1}{e^{S_0}}$, i.e. enabling direct comparison of the matrix model with the JT gravity results.

THE JT GRAVITY/MATRIX MODEL DUALITY. For the orientable case, it is most economical to compare the results of the topological recursion eq. (2.2.69) with choosing y^{JT} (eq. (2.3.4)) as the spectral curve to JT gravity. In the following, we will refer to the matrix model with this spectral curve as the JT gravity matrix model. Since the topological recursion is formulated in terms of the $W_{g,n}(I)$, it is useful to determine the corresponding objects in JT gravity. By the direct generalisation of eq. (2.2.14), one can relate the perturbative

21 Note that this argument still holds when considering different values of the Dyson index than $\beta = 2$.

contributions to the expectation value of multi-resolvents to those for the product of canonical partition functions via

$$R_{g,n}(z_1,\dots,z_n) = (-1)^n \prod_{i=1}^{n} \left[\int_0^\infty \mathrm{d}\beta_i e^{-\beta_i z_i^2}\right] Z_{g,n}(\beta_1,\dots,\beta_n). \tag{2.3.17}$$

Using eq. (2.2.62), this implies

$$W_{g,n}(z_1,\dots,z_n) = \prod_{i=1}^{n} \left[2z \int_0^\infty \mathrm{d}\beta_i\, e^{-\beta_i z_i^2}\right] Z_{g,n}(\beta_1,\dots,\beta_n). \tag{2.3.18}$$

For the case of JT gravity, the partition functions are expressed through the WP volumes via eq. (2.1.43) and hence

$$W^{\mathrm{JT}}_{g,n}(z_1,\dots,z_n) = \prod_{i=1}^{n} \left[\int_0^\infty b_i \mathrm{d}b_i 2z_i \int_0^\infty \mathrm{d}\beta_i\, e^{-\beta_i z_i^2} Z^t(\beta_i, b_i)\right] V_{g,n}(b_1,\dots,b_n). \tag{2.3.19}$$

Evaluating the Laplace transform of the trumpet partition function this yields

$$W^{\mathrm{JT}}_{g,n}(z_1,\dots,z_n) = \prod_{i=1}^{n} \left[\int_0^\infty b_i \mathrm{d}b_i e^{-b_i z_i}\right] V_{g,n}(b_1,\dots,b_n). \tag{2.3.20}$$

Due to the polynomial structure of the WP volumes, this implies

$$W^{\mathrm{JT}}_{g,n}(z_1,\dots,z_n) = \sum_{\vec{a}}^{\|\vec{a}\|_1 \leq k(g,n)} C^{\vec{a}}_{g,n} \prod_{i=1}^{n} \frac{(2a_i+1)!}{\left(z_i^2\right)^{a_i+1}}, \tag{2.3.21}$$

with the $C^{\vec{a}}_{g,n}$ from theorem 2.2.

Now we can state the duality very easily. It is given by

$$W^{\mathrm{JT}}_{g,n}(I) = W_{g,n}(I), \tag{2.3.22}$$

where the $W_{g,n}(I)$ on the right hand side are those determined by the topological recursion with the spectral curve y^{JT}. It is important to note that this statement is a *perturbative* duality, i.e. the statement is equality of the coefficient functions of the topological expansion of both sides. The proof for this is due to Eynard and Orantin [71] and starts by showing that rewriting the topological recursion as a recursion for the WP volumes using eq. (2.3.20) yields Mirzakhani's recursion (cf. theorem 2.1)[22]. Then, to show eq. (2.3.22), it remains to derive that the inputs agree, i.e. $W^{\mathrm{JT}}_{0,3}(I) = W_{0,3}(I)$

22 With the modification that the factor of $\frac{1}{2}$ that is present in the recursion for appearances of $V_{1,1}$ is defined into the volume, i.e. $\tilde{V}$ (eq. (2.1.36) is used and there are no additional factors of $\frac{1}{2}$ in the recursion.

and $W_{1,1}^{\text{JT}}(z) = W_{1,1}(z)$. We will show this explicitly, also to give some examples for the application of the topological recursion.

First, for $(g,n) = (0,3)$ eq. (2.2.69) yields

$$\begin{aligned} W_{0,3}(z_1,z_2,z_3) =& \operatorname*{Res}_{z=0}\left\{\frac{\mathrm{d}z}{z_1^2-z^2}\frac{1}{4y(z)}[W_{0,2}(z,z_2)W_{0,2}(-z,z_3)+W_{0,2}(z,z_3)W_{0,2}(-z,z_2)]\right\} \\ =& \frac{2}{z_1^2z_2^2z_3^2}\operatorname*{Res}_{z=0}\left\{\frac{\mathrm{d}z}{4y(z)}\right\}, \end{aligned} \tag{2.3.23}$$

where we used that $\frac{1}{y(z)}$ has a simple pole at $z=0$. For $(g,n) = (1,1)$, it implies

$$W_{1,1}(z_1) = \operatorname*{Res}_{z=0}\left\{\frac{\mathrm{d}z}{z_1^2-z^2}\frac{1}{4y(z)}W_{0,2}(z,-z)\right\} = \operatorname*{Res}_{z=0}\left\{\frac{\mathrm{d}z}{16z^2(z_1^2-z^2)y(z)}\right\}. \tag{2.3.24}$$

Putting in the spectral curve for JT gravity one finds

$$\begin{aligned} W_{0,3}(z_1,z_2,z_3) &= \frac{2}{z_1^2z_2^2z_3^2}\operatorname*{Res}_{z=0}\left\{\frac{\pi\mathrm{d}z}{\sin(2\pi z)}\right\} = \frac{2}{z_1^2z_2^2z_3^2}\lim_{z\to 0}\left\{\frac{1}{2\cos(2\pi z)}\right\} \\ &= \frac{1}{z_1^2z_2^2z_3^2}, \end{aligned} \tag{2.3.25}$$

and

$$\begin{aligned} W_{1,1}(z_1) =& \operatorname*{Res}_{z=0}\left\{\frac{\mathrm{d}z}{16z^2(z_1^2-z^2)y(z)}\right\} = \frac{\pi}{4}\frac{1}{2!}\frac{\partial^2}{\partial z^2}\bigg|_{z=0}\frac{z^3}{z^2(z_1^2-z^2)\sin(2\pi z)} \\ =& \cdots = \frac{3+2\pi^2z_1^2}{24z_1^4}. \end{aligned} \tag{2.3.26}$$

Now we use eq. (2.3.21) to see that these results for the $W_{g,n}$ of the JT gravity matrix model correspond to

$$V_{0,3}^{\text{MM}}(b_1,b_2,b_3) = 1, \quad \text{and} \quad V_{1,1}^{\text{MM}}(b) = \frac{1}{48}b^2 + \frac{\pi^2}{12}, \tag{2.3.27}$$

which are precisely the volumes given as input to Mirzakhani's recursion (note footnote 22).

This duality, which for JT gravity was first stated in [68], now leads to the drastic simplification of the computation of WP volumes, as seen from the following example.

Example 2.6 ($V_{1,2}(b_1, b_2)$)**.** *Using the topological recursion for the JT gravity spectral curve one finds*

$$\begin{aligned} W_{1,2}(z_1,z_2) =& \operatorname*{Res}_{z=0} \left\{ \frac{\mathrm{d}z}{z_1^2 - z^2} \frac{1}{4y^{JT}(z)} [W_{0,3}(z,-z,z_2) + W_{0,2}(z,z_2)W_{1,1}(-z) + (z \leftrightarrow -z)] \right\} \\ =& \operatorname*{Res}_{z=0} \left\{ \frac{\mathrm{d}z}{z_1^2 - z^2} \frac{\pi}{\sin(2\pi z)} \left[\frac{1}{z^4 z_2^2} + \frac{3 + 2\pi^2 z^2}{24 z^4} \left(\frac{1}{(z-z_2)^2} + \frac{1}{(z+z_2)^2} \right) \right] \right\} \\ =& \ldots \\ =& \frac{1}{8 z_1^6 z_2^6} \left[5\left(z_1^4 + z_2^4\right) + 3 z_1^2 z_2^2 + 2\pi^4 z_1^4 z_2^4 + 4\pi^2 \left(z_1^2 z_2^4 + z_2^2 z_1^4 \right) \right]. \end{aligned} \tag{2.3.28}$$

By the duality and eq. (2.3.21) *this implies*

$$V_{1,2}(L_1, L_2) = \frac{1}{192}\left(L_1^4 + L_2^4\right) + \frac{1}{96} L_1^2 L_2^2 + \pi^2 \frac{1}{12}\left(L_1^2 + L_2^2\right) + \frac{\pi^4}{4}, \tag{2.3.29}$$

in agreement with the result found from Mirzakhani's recursion in appendix A.

In the main part of this thesis, it will be most convenient to work in terms of multi-resolvents on the matrix model side. Hence, it is useful to directly relate the resolvents with the WP volumes. Inverting eq. (2.3.20) and using the definition of the $W_{g,n}$, eq. (2.2.62), this is found to be

$$\begin{aligned} V_{g,n}(\vec{b}) &= \mathcal{L}^{-1}\left[\prod_{i=1}^{n} \left(\frac{-2z_i}{b_i} \right) R_g(I), \vec{b} \right] \\ &= (-1)^n \int_{\delta + i\mathbb{R}} R_{g,n}(I) \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \frac{2z_j}{b_j} e^{b_j z_j}, \end{aligned} \tag{2.3.30}$$

with the usual convention of $I = z_1, \ldots, z_n$, $\vec{b} \in \mathbb{R}_+^n$ and $\delta \in \mathbb{R}_+$. To the second line, we used the Bromwich integral to write out the inverse Laplace transform (e.g. [87])[23].

THE DUALITY FOR TOPOLOGICAL GRAVITY. As we discussed above, topological gravity can be viewed as a limit of JT gravity. Thus, one finds a duality in the sense of that for JT gravity also for this theory. Actually, we have already discussed the dual matrix model when discussing the Kontsevich ribbon graphs in section 2.1 but it is more convenient to compute the correlation functions in this model also by using the loop equations/topological recursion. For this, one needs as an input the leading-order density of

23 Note that this way of writing the integral assumes that all singularities of the integrand are left of the integration contour. This is not obvious at the moment and we will make sure that it holds true before using the formula.

states. Since topological gravity emerges from JT gravity in the limit of large boundary lengths/inverse temperatures, the limit in terms of quantities depending on energies has to be taken as the limit of small energies. Hence, the leading-order (in the topological expansion) density of states for topological gravity is found as the leading-order (in terms of energy) behaviour of that for JT gravity, i.e. $\rho_0^{\mathrm{JT}}(x) = \frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{x}\right)$. Thus:

$$\rho_0^{\mathrm{Airy}}(x) = \frac{1}{2\pi}\sqrt{x}. \tag{2.3.31}$$

As a sanity check, one can compute from this the Airy version of the disk-contribution to the expectation value of the partition function as

$$Z_{0,1}^{\mathrm{Airy}}(x) = \mathcal{L}[\rho_0^{\mathrm{Airy}}(x), x, \beta] = \frac{1}{2\pi}\mathcal{L}[\sqrt{x}, x, \beta] = \frac{1}{2\pi}\frac{\Gamma\left(\frac{3}{2}\right)}{\beta^{\frac{3}{2}}} = \frac{1}{4\sqrt{\pi}\beta^{\frac{3}{2}}}. \tag{2.3.32}$$

This precisely coincides with the leading-order behaviour at large β of the JT gravity result (eq. (2.1.44))

$$Z^d(\beta) = Z_{0,1}(\beta) = \frac{1}{4\sqrt{\pi}\beta^{\frac{3}{2}}}e^{\frac{\pi^2}{\beta}}. \tag{2.3.33}$$

The spectral curve for the Airy model can, via eq. (2.2.40), be derived from $\rho_0^{\mathrm{Airy}}(x)$ to be

$$y^{\mathrm{Airy}}(z) = \frac{z}{2}, \tag{2.3.34}$$

which could also be seen by taking the leading-order contribution to the spectral curve for JT gravity.

One can quickly convince oneself that the iteration of this spectral curve in the topological expansion or the loop equations gives indeed the part of the $W_{g,n}^{\mathrm{JT}}$ that corresponds to the Airy WP volumes. The quickest way to perform this reasoning is, to expand

$$\frac{1}{y^{\mathrm{JT}}(z)} = \frac{4\pi}{\sin\left(2\pi z\right)} = \frac{1}{y^{\mathrm{Airy}}(z)} + \sum_{k=1}^{n}\frac{2\left(2^{k-1}-1\right)|B_{2k}|}{(2k)!}(2\pi)^{2k}z^{2k-1}, \tag{2.3.35}$$

using the series expansion for the cosecant around $z = 0$ [87][1.411 eq.11] and denoting by B_m the m-th Bernoulli number. From this expansion, it is now obvious that the terms in the $W_{g,n}^{\mathrm{JT}}$ containing powers of π^2 are those arising from the sum, while those without dependence on π arise from the Airy spectral curve. Since the relation of the $W_{g,n}^{\mathrm{JT}}$ with the WP volumes, eq. (2.3.20), is linear, this translates directly to the statement that the terms arising from the Airy spectral curve are precisely the Airy WP volumes, since those are the only terms in the full WP volumes independent on π.

An alternative route to obtain this statement would be, to go the same way as for full JT gravity and derive first the analogue of Mirzakhani's recursion for the Airy WP volumes, as done e.g. in appendix C of [94], and then show that this corresponds to the topological recursion with the Airy spectral curve.

Having said this, we have now introduced the two main gravitational theories that we will be interested in in the main text and stated their duality to certain double-scaled matrix models. We will close this part of the background chapter by considering an example linking many of the concepts we have introduced here, the considerations of the expansion to all orders of the Airy model's one-point function.

THE AIRY MODEL FOR $n = 1$. For starters, let us consider the topological expansion of the one-point function of resolvents in this model. From eq. (2.3.21), we can infer that the coefficient function of this expansion can, in the double-cover coordinate, be written as

$$R_g^{\text{Airy}}(z) = -C_g^R \frac{1}{z^{2k(g,1)+3}} = -\frac{C_g^R}{z^{6g-1}}, \tag{2.3.36}$$

or as

$$R_g^{\text{Airy}}(z) = -\frac{C_g^R}{\sqrt{-x}^{6g-1}}, \tag{2.3.37}$$

in the energy coordinate.

The coefficients C_g^R could in principle be determined from e.g. the WP volumes. Here, we keep them as a variable, since, in general, one is concerned with inferring all other quantities from the resolvents. As we advertised above, we shall discuss this inference in detail for the case of the Airy model.

The relation to the matrix-model quantities was discussed in section 2.2, while the relation with the Airy WP volumes arises in the case of the Airy matrix model due to the duality. We briefly recall these relations, valid also for JT gravity, to have them assembled at one place for future reference.

For the relation $R \to Z$, it is easiest to invert eq. (2.2.14) in the energy variable x to find

$$Z_{g,1} = -\mathcal{L}^{-1}\left[R_g(-x), x, \beta\right]. \tag{2.3.38}$$

The relation $R \to V$ we have already put in eq. (2.3.30), which for $n = 1$ reads

$$V_{g,1}(b) = \frac{1}{b}\mathcal{L}^{-1}\left[-2zR_g(z), b\right]. \tag{2.3.39}$$

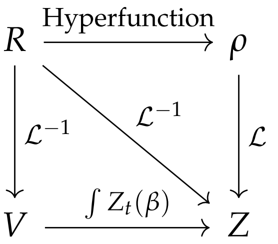


Figure 2.3.1: Illustration of the relation of observables in the matrix models dual to JT gravity/topological gravity.

The final relation $R \to \rho$ is given via eq. (2.2.13), i.e.

$$\rho_{g,1}(x) = \frac{1}{-2\pi i} \lim_{\epsilon \to 0} \left[R_g(x + i\epsilon) - R_g(x - i\epsilon) \right] . \tag{2.3.40}$$

It is interesting to note that via this definition, $\rho_{g,1}(x)$ can be regarded as the *hyperfunction* corresponding to $R_g(x)$, which is defined precisely in this manner (cf. appendix C), i.e. as the difference of a complex valued function across a cut in the complex plane. Using the notation for this introduced in appendix C, one can thus write

$$\rho_{g,1}(x) = \left[\frac{-1}{2\pi i} R_g(y) \right]. \tag{2.3.41}$$

We give a graphical representation of the relation between the different observables of the matrix models dual to JT/topological gravity in fig. 2.3.1. It is important to note that the generalisation to an arbitrary number of boundaries is straightforward for all relations except that for $R \to \rho$. Here, the relation can be derived by using eq. (2.2.13) for every occurrence of the density of states in the correlation function and then taking the limits outside. We shall only need the expression for $n = 2$, for which the resulting relation is given by [68]

$$\rho_{g,2}(x_1, x_2) = \frac{-1}{4\pi^2} \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} [R(++) + R(--) - R(-+) - R(+-)], \tag{2.3.42}$$

where we defined $R(\pm, \mp) := R(x_1 \pm \epsilon_1, x_2 \mp \epsilon_2)$. We will give explicit examples for this type of computation when it becomes relevant in chapter 4. As a final comment on the relation of the observables in the matrix model dual to JT gravity, we remark that all of the correlation functions can be directly realised as arising from the gravitational path-integral via different boundary conditions than the ones discussed in section 2.1. This discussion can be found in [79].

Returning to our example of the Airy model for $n = 1$, we can now work out the topological expansion of all observables starting with that for the resolvent. For the Airy WP volumes (for $g > 0$) and the partition function this yields

$$V_{g,1}^{\text{Airy}}(b) = \frac{2C_g^R}{(6g-3)!} b^{6g-4}, \tag{2.3.43}$$

$$Z_{g,1}^{\text{Airy}}(\beta) = C_g^R \frac{2^{6g-3}(3g-2)!}{\sqrt{\pi}(6g-3)!} \beta^{3\left(g-\frac{1}{2}\right)}. \tag{2.3.44}$$

For the density of states one can use eq. (C.9)[24]

$$\theta(x) x^{-\frac{2k+1}{2}} = \left[\frac{(-1)^k}{2i(-y)^{k+\frac{1}{2}}} \right], \tag{2.3.45}$$

to find

$$\rho_g(x) = C_g^R \frac{(-1)^{3g+1}}{\pi x^{3g-\frac{1}{2}}} \theta(x). \tag{2.3.46}$$

It is easy to check that also the computations of the partition function via the density of states or the Airy WP volume give consistent results, justifying the abuse of the commutative diagram notation in fig. 2.3.1.

Additionally, using the relation of the WP volumes with intersection numbers (eq. (2.1.50)), it holds for the Airy WP volumes that

$$V_{g,1}^{\text{Airy}}(b) = \sum_{\alpha = 3g-2} \frac{1}{2^{|\alpha|}\alpha!} \langle \tau_\alpha \rangle b^{2\alpha} = \frac{1}{2^{3g-2}(3g-2)!} \left\langle \tau_{3g-2} \right\rangle_g b^{6g-4}. \tag{2.3.47}$$

This means that one can express the intersection number via the coefficient of the resolvent as

$$\left\langle \tau_{3g-2} \right\rangle_g = \frac{2^{3g-1}(3g-2)!}{(6g-3)!} C_g^R. \tag{2.3.48}$$

This finalises our example of relating all the one-point observables for the orientable Airy model. We shall revisit this computation when talking about its unorientable variant in the main part of this thesis.

Before leaving the discussion of gravity and random matrices and going over to the relation of matrix models with quantum chaos, we would like to do a final computation in the orientable Airy model, actually finding the expression for the intersection numbers in eq. (2.3.48). In the orientable Airy model, it is actually possible to work out the resummed result for the expectation value of the thermal partition function using the so-called

24 The reader unwilling to follow the hyperfunction route, may find the same result using an intermediate result for the computation of $\rho_{1,1}^{\text{Airy}}$ in appendix B.

Korteweg-de-Vries (KdV) hierarchy underlying both topological gravity and the matrix model. Since it will not be important for the main part of this work, we will not comment on it and refer to the excellent review in [94]. The resummed result is found as [94, 104]

$$\begin{aligned}\langle Z(\beta)\rangle^{\text{Airy}} &= \frac{e^{S_0}}{4\sqrt{\pi}\beta^{\frac{3}{2}}} e^{\frac{1}{3}\beta^3 e^{-2S_0}} \\ &= \sum_{g=0}^{\infty} \underbrace{\frac{\beta^{3\left(g-\frac{1}{2}\right)}}{4\sqrt{\pi}3^g g!}}_{Z_{g,1}^{\text{Airy}}(\beta)} \frac{1}{\left(e^{S_0}\right)^{2g-1}}.\end{aligned} \tag{2.3.49}$$

Comparing with the expression of the coefficient functions in terms of the C_g^R one deduces

$$C_g^R = \frac{1}{4\sqrt{\pi}3^g g!} \frac{(6g-3)!}{\sqrt{\pi}2^{6g-3}(3g-2)!} = \frac{(6g-3)!}{2^{3g-1}24^g g!(3g-2)!}. \tag{2.3.50}$$

Via eq. (2.3.48) this conveniently implies

$$\left\langle \tau_{3g-2} \right\rangle_g = \frac{2^{3g-1}(3g-2)!}{(6g-3)!} \frac{(6g-3)!}{2^{3g-1}24^g g!(3g-2)!} = \frac{1}{24^g g!}, \tag{2.3.51}$$

i.e. a very simple result for the one-point intersection numbers found in different ways in e.g. [105].

This concludes our discussion of the orientable Airy model and already gives a hint at the general principle which is one of the main guidelines for the main part of this thesis: Working on the matrix model side is for almost all cases more economical than working on the gravity side.

2.4 CHAOS AND RANDOM MATRICES

In this background section we give details on the view on quantum chaos that will give rise to the considerations of the main part of this thesis regarding the application of random matrix universality.

THE BGS CONJECTURE. This view is given by the conjecture due to Bohigas, Giannoni and Schmit (BGS) [10] that certain statistical correlations in the spectrum of a quantum system arising from the quantisation of a classically chaotic system are described by *random matrix theory* (RMT). Let us repeat that by "RMT" we denote here the matrix model with a Gaussian potential, for which many quantities that we discussed perturbatively above can be evaluated exactly [15].

In order to state examples for (statistical) observables[25] obeying this conjecture, it is necessary to introduce *spectral unfolding*. By this, one means scaling the spectrum in such a way that its mean level spacing is unity. Unfolding is commonly done by redefining the energy as [16, 97]

$$e(x) := \int_{-\infty}^{x} \mathrm{d}x \, \langle \rho(x) \rangle = \langle N(x) \rangle \,, \tag{2.4.1}$$

where $N(x) := \#\{x_i \text{ with } x_i \leq x\}$, the "spectral staircase". For a matrix model, this definition is clear, but for a theory that is not a matrix model, which is what one would like to compare to, one has to understand this average as the "smooth part" of the level staircase, i.e.

$$N(x) = \langle N(x) \rangle + N_{\text{fluc}}(x), \tag{2.4.2}$$

where the smooth part is in some cases known analytically (e.g. Weyl's law for Billiard systems [106, 107]) or has to be approximated by an appropriate scheme, e.g. in numerical computations by a polynomial fit. It makes sense that one has to transform the spectrum in such a way in order to find information that is, in a sense, universal, i.e. arising purely from the quantum chaoticity of the system and not from system specific features. That the unfolding precisely does this, can be nicely seen by writing for an eigenvalue x_i

$$e(x_i) = N(x_i) - N_{\text{fluc}}(x_i) = i - N_{\text{fluc}}(x_i), \tag{2.4.3}$$

which means that the unfolded energy is the sum of an line increasing with unit slope and the fluctuations of the original spectrum. From this, it is obvious that the mean level-spacing in the unfolded energies is given by unity (likewise, $\langle \rho(e) \rangle = 1$ [16]).

WIGNER'S SURMISE. Having defined unfolding, we can turn to the maybe most commonly used observable, that is the distribution of unfolded level spacings. The distribution of spacings of unfolded eigenvalues is given by the famous *Wigner surmise* [97]

$$P_\beta(s) = \mathcal{N}(\beta) s^\beta \times \begin{cases} e^{-\frac{\pi}{4}s^2}, & \text{for} \quad \beta = 1, \\ e^{-\frac{4}{\pi}s^2}, & \text{for} \quad \beta = 2, \\ e^{-\frac{64}{9\pi}s^2}, & \text{for} \quad \beta = 4, \end{cases} \tag{2.4.4}$$

25 Let us clarify that, though we will omit the explicit mention of "statistical" in the following, when speaking about an "observable" in connection with the BGS conjecture we always mean a quantity arising from a correlation function in the ensemble of eigenvalues of a quantum mechanical system/a matrix model.

where the normalisation constant $\mathcal{N}(\beta)$ is determined by requiring $\int_0^\infty P(s) = 1$. This result is exact for the case of 2×2 matrices and holds to very good approximation also for arbitrarily large values of the matrix size [15]. The fidelity of quantum systems arising from a chaotic classical theory with this prediction has been checked for many different instances, like the Sinai-, Bunimovitch- [10] and hyperbola billiards [108] as well as the Hydrogen atom in a strong magnetic field [109], to name only a few examples.

THE SPECTRAL FORM FACTOR. Another important example for an observable considered in this context is the *spectral form factor* (SFF), which will be the main object of focus for our considerations. There are mainly two variations of this object, i.e. a canonical version defined as

$$\kappa_\beta(t,\beta) := \langle Z(\beta+it)Z(\beta-it)\rangle_c\,, \tag{2.4.5}$$

and a microcanonical incarnation defined as

$$\kappa_\beta(t,E) := \frac{1}{\langle\rho(E)\rangle}\int_{-\infty}^{\infty} \mathrm{d}\Delta\, e^{-it\Delta}\left\langle \rho\left(E+\frac{\Delta}{2}\right)\rho\left(E-\frac{\Delta}{2}\right)\right\rangle_c. \tag{2.4.6}$$

For the comparison with topological/JT gravity, the canonical variety is accessible more directly, while for the comparison of the RMT predictions with e.g. the semiclassical analysis of quantum chaotic systems [110–112] the microcanonical variant is better suited. We will discuss the relation of the two observables, in the setting we are interested in, in section 4.1[26].

Here, we shall focus on the microcanonical SFF. One notes that it is based on the expectation value of the spectral two-point function $\langle\rho(E_1)\rho(E_2)\rangle_c$. In the Gaussian matrix model, i.e. random matrix theory, this can be evaluated exactly in the large N limit as [15, 97][27]

$$\frac{\langle\rho(E_1)\rho(E_2)\rangle_c}{\langle\rho(E_1)\rangle\,\langle\rho(E_2)\rangle} = \Big[\delta(e_1-e_2) - \mathrm{Y}^\beta(e_1-e_2)\Big], \tag{2.4.7}$$

26 Let us already note that we will find the microcanonical SFF to have a universal form, while the canonical SFF is by definition system dependent and hence can't have a universal form. However, we will be able to compute it from the microcanonical SFF provided the input of the leading order density of states.

27 Note that in these references the density of states is assumed to be normalised to unity, while ours is normalised to N for finite N matrix integrals or e^{S_0} for the double-scaled case. This convention is also used in [16]. The quotient in the following equation is, however, invariant under the choice of convention.

for $E_1, E_2 \in \operatorname{supp}(\langle\rho\rangle)$ and e_i the corresponding unfolded energies. The functions $Y^\beta(x)$ for the three Wigner-Dyson values of β are for $x \geq 0$ given by

$$Y^1(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2} + \left(\frac{1}{2} - \frac{1}{\pi}\operatorname{Si}(\pi x)\right)\left(\frac{d}{dx}\frac{\sin(\pi x)}{\pi x}\right), \tag{2.4.8}$$

$$Y^2(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2}, \tag{2.4.9}$$

$$Y^4(x) = \frac{\sin^2(2\pi x)}{4\pi^2 x^2} - \left(\frac{d}{dx}\frac{\sin(2\pi x)}{2\pi x}\right) \times \frac{1}{\pi}\operatorname{Si}(2\pi x), \tag{2.4.10}$$

where Si denotes the Sine integral and they are defined for $x \leq 0$ by requiring them to be symmetric functions [15]. Since we would like to use these results to study the SFF, it is natural to define the *form factor*

$$b^\beta(x) := \int_{-\infty}^{\infty} dr\, e^{-2\pi i x r} Y^\beta(r), \tag{2.4.11}$$

which can be worked out to yield [15]

$$b_1(x) = \begin{cases} 1 - 2x + x\log(1+2x) & \text{if } x \leq 1, \\ -1 + x\log\left(\frac{2x+1}{2x-1}\right) & \text{if } x \geq 1, \end{cases} \tag{2.4.12}$$

$$b_2(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases} = \max(1-x, 0), \tag{2.4.13}$$

$$b_4(x) = \begin{cases} 1 - \frac{x}{2} + \frac{x}{4}\log(|1-x|) & \text{if } x \leq 2, \\ 0 & \text{if } x \geq 2. \end{cases} \tag{2.4.14}$$

The universal behaviour we are interested in appears in the late-time behaviour of the microcanonical spectral form factor, i.e. at the order of the so-called Heisenberg time [97]

$$T_H := 2\pi \langle \rho(E) \rangle. \tag{2.4.15}$$

This timescale arises due to being conjugate to the mean-level spacing $\frac{1}{\langle\rho(E)\rangle}$. Measuring the time in the microcanonical SFF in units of this quantity, one obtains

$$\kappa_\beta(t = xT_H, E) = \frac{1}{\langle\rho(E)\rangle}\int_{-\infty}^{\infty} d\Delta\, e^{-2\pi i x\langle\rho(E)\rangle\Delta}\left\langle \rho\left(E + \frac{\Delta}{2}\right)\rho\left(E - \frac{\Delta}{2}\right)\right\rangle_c. \tag{2.4.16}$$

It is now important to remember that we had normalised $\rho(E)$ in such a way that it grows as N when $N \to \infty$. Hence, the integral will in this limit be dominated by the contributions at small Δ, i.e. of the order of the mean level-spacing. The prediction of random matrix theory for this limit can be evaluated from eq. (2.4.7). For this computation, it is relevant to work out the difference of the unfolded energies corresponding to $E \pm \frac{\Delta}{2}$. This is conveniently done as

$$e\left(E+\frac{\Delta}{2}\right)-e\left(E-\frac{\Delta}{2}\right)=\int_{E-\frac{\Delta}{2}}^{E+\frac{\Delta}{2}}\mathrm{d}E'\,\langle\rho(E')\rangle=\Delta\,\langle\rho(E)\rangle+\mathcal{O}\left(\Delta^2\right). \tag{2.4.17}$$

In RMT, this could also have been obtained by noting that in the large N limit an equivalent unfolding could have been performed by defining $\tilde{e}(E)=E\,\langle\rho(E)\rangle$ [16, 97]. Hence, using eq. (2.4.7), one finds

$$\frac{\left\langle\rho\left(E+\frac{\Delta}{2}\right)\rho\left(E-\frac{\Delta}{2}\right)\right\rangle_c}{\langle\rho(E)\rangle^2}=\left[\delta(\Delta\,\langle\rho(E)\rangle)-\mathrm{Y}^{\beta}(\Delta\,\langle\rho(E)\rangle)\right], \tag{2.4.18}$$

where we used the approximation $\left\langle\rho\left(E\pm\frac{\Delta}{2}\right)\right\rangle=\langle\rho(E)\rangle$. Transforming to the scaled[28] difference $r:=\Delta\,\langle\rho(E)\rangle$ one finds as the RMT prediction for the microcanonical SFF

$$\begin{aligned}\kappa_\beta(t=xT_\mathrm{H},E)&=\int_{-\infty}^{\infty}\mathrm{d}r\,e^{-2\pi ixr}\left[\delta(r)-\mathrm{Y}^{\beta}(r)\right]\\&=1-b^{\beta}(x)\\&=:\kappa^{s}_{\beta}(x).\end{aligned} \tag{2.4.19}$$

Hence, for $t\leq T_\mathrm{H}$ one finds

$$\kappa_1(t,E)=\frac{t}{T_\mathrm{H}}-\frac{t}{T_\mathrm{H}}\log\left(1+2\frac{t}{T_\mathrm{H}}\right), \tag{2.4.20}$$

$$\kappa_2(t,E)=\frac{t}{T_\mathrm{H}}, \tag{2.4.21}$$

$$\kappa_4(t,E)=\frac{1}{2}\frac{t}{T_\mathrm{H}}-\frac{1}{4}\frac{t}{T_\mathrm{H}}\log\left(1-\frac{t}{T_\mathrm{H}}\right). \tag{2.4.22}$$

Remarkably, it was possible to show this behaviour for quantised classically chaotic systems using the *periodic orbit theory* approach, starting with Gutzwiller's celebrated trace formula for the fluctuating part of the density of states in such systems [113, 114]. This was achieved for the leading order, the *diagonal approximation*, by Berry [110], extended to the quadratic order by Sieber and Richter [111] by introducing the concept of "orbit encounters" and finalised in [112, 115] using this concept to compute the full series of

28 Or unfolded in the sense of the $\tilde{e}$-unfolding.

contributions for the three Wigner-Dyson classes. Hence, though a mathematically rigorous proof has not yet been provided, the agreement of the late-time microcanonical spectral form factor of quantised classically chaotic systems with the RMT prediction is very well motivated.

To better illustrate the universal behaviour of the SFF in RMT, we perform an explicit numerical simulation of matrices in the three Wigner-Dyson classes and compute the microcanonical SFF for these ensembles of matrices. They are displayed in fig. 2.4.1 in comparison to their respective analytic predictions deriving from eqs. (2.4.12) to (2.4.14) via eq. (2.4.19). In the numerical computation, we chose $N = 300$ as the size of the matrices and averaged over $N_{\text{avg}} = 5000$ realisations; details of the numerical computation can be found in appendix D. As it can be seen quite clearly, the numerical results are in almost perfect agreement with the analytic computation.

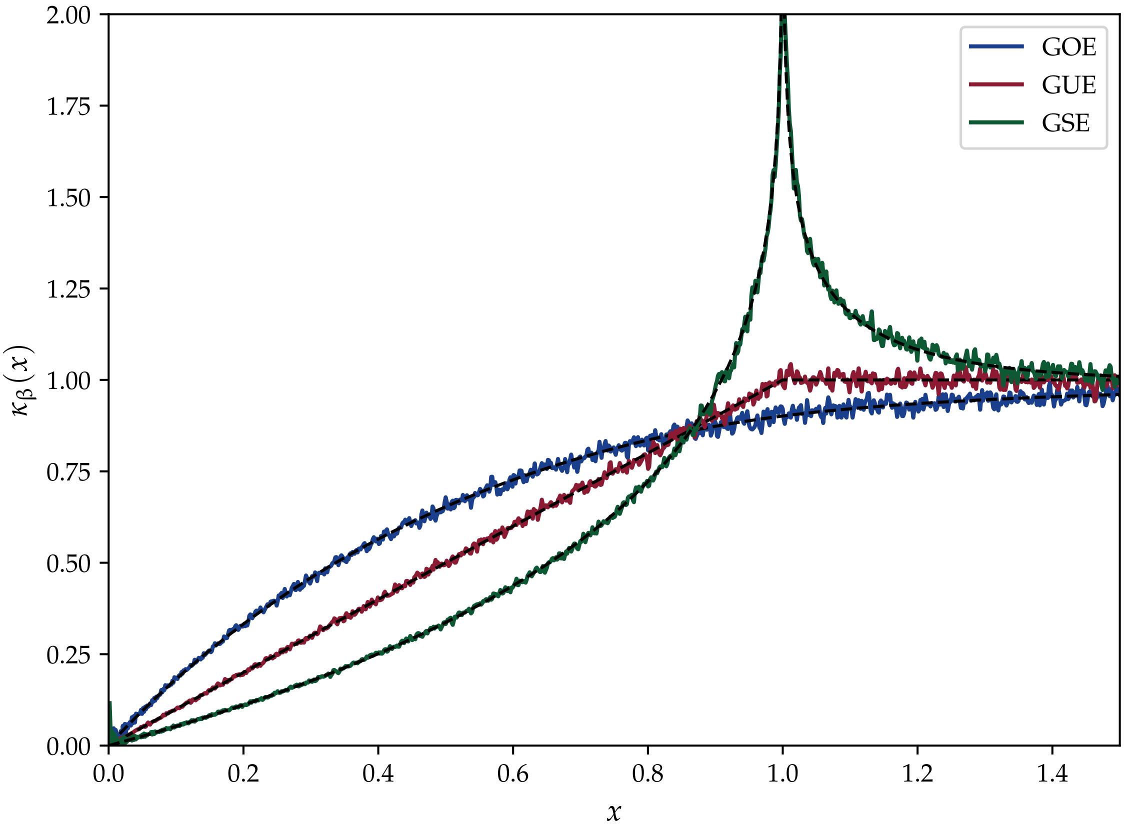


Figure 2.4.1: Comparison of numerical results for the microcanonical SFF in the Gaussian matrix models of the three Wigner-Dyson classes (indicated colours) with their respective analytic prediction (black lines). Note that we denote the microcanonical SFF as $\kappa_\beta(x)$ since it is independent on the energy with x in units of T_H.

INVERSE BGS AND RMT UNIVERSALITY. Building on the intricate relation between random matrix statistics and quantum chaos given by the BGS conjecture, the check for fidelity to the RMT prediction of the level spacing distribution, the SFF and other observables we have not mentioned here (see [16] for a more complete overview) for generic quantum system is used as a way to quantify quantum chaos there. This is especially interesting for such systems that do not have a classical limit where one could check for the presence of chaos in the classical sense. In this sense, one can view the fidelity to the predictions of random matrix theory as a *definition* of quantum chaos for this case. Prime examples for systems where this reasoning has been applied are the SYK model [57–59, 116, 117], Spin chains e.g. [118–120] and other many-body systems like the Bose-Hubbard model[29] [121, 122] and similar fermionic systems [123, 124].

The most important examples of theories not having an obvious classically chaotic limit for our purpose are matrix models with a generic potential. Specifically, we are interested in the presence of the universal behaviour of the microcanonical SFF in these models. The presence of this is assured by *random matrix universality* which can be put as the statement that eq. (2.4.18) holds true (in the large N limit) for all choices of the potential V (cf. eq. (2.2.7)) bar some exotic examples (cf.[16])[30]. For a proof of this statement in the non-double-scaled setting one can use Efetov's supersymmetric σ-model approach [17] (for an introduction see [97] or [18]). The (physics) proof for universality in the form we want to use it was given by Hackenbroich and Weidenmüller in [126], using this approach. Notably, there are other approaches not using the supersymmetric method and working directly with matrix models showing the universal behaviour, e.g. [127] for a mathematical physics proof for the unitary case and [128, 129] for a mathematically rigorous treatment of this case. For the orthogonal and symplectic symmetry classes, mathematically rigorous proofs are found in [130, 131] for the more general setting of so-called Wigner-matrices. A review of the development of these proofs in relation to other work was given in [125] and more recently also a rigorous treatment of the SFF in [132].

This closes the recollection of concepts that are needed for the main part of this thesis, especially chapter 4, whose objective it will be to study the implication of random matrix universality in the matrix models dual to topological/JT gravity for these theories. It is already apparent that these implications primarily affect the contributions to the two-point function of partition functions and hence the (Airy) WP volumes with $n = 2$ boundaries. The proven presence of universal behaviour in all Wigner-Dyson classes

29 Note that for the Bose-Hubbard model, a semiclassical approach exists, using which one can view the quantum chaoticity of this system as established by the fidelity to random matrix theory also from the traditional perspective of finding a chaotic theory in the system's classical limit [121].

30 The precise conditions on the potential that are required in order for the matrix model with this potential to obey RMT universality are given in the rigorous proofs for this statement that are cited below (see [125] for a broad review).

suggests the ability to follow this reasoning in all of these symmetry classes. Consequently, since so far we have introduced the duality only for the unitary symmetry class, we will first discuss its extension to the other two Wigner-Dyson classes and beyond in the next chapter.

3

BEYOND ORIENTABILITY IN 2D QUANTUM GRAVITY

Up to now, we have only discussed JT/topological gravity restricting the gravitational path-integral to orientable contributions and the duality of these theories with a hermitian matrix integral (i.e. $\beta = 2$). For matrix models we have already discussed the three Wigner-Dyson classes and mentioned that it is possible to consider matrix models with an arbitrary, potentially non-Wigner-Dyson, value of the Dyson index β. The aim of the first main part of this thesis is to consider the corresponding setting on the gravitational side, that of unorientable manifolds, to then discuss the implications of random matrix universality on these geometries in chapter 4. Our presentation of this is split into three main parts.

First, in section 3.1 we discuss how to go beyond the geometries dual to $\beta = 2$ matrix models in JT/topological gravity. For this, we first recall the origin of the three Wigner-Dyson classes as the ensembles of matrices associated with presence/absence of time-reversal symmetry, where the $\beta = 2$ case corresponds to the absence of time-reversal symmetry. Having understood this, we then go on to explain how this implies the geometries dual to the matrix models with $\beta \in \{1, 4\}$ to be unorientable surfaces, as expected already from the diagrammatic expansion of those matrix models discussed in section 2.2. We then go on to introduce the theory of unorientable JT gravity and discuss how the computation of correlation functions again reduces to the determination of moduli space volumes, now that of unorientable hyperbolic surfaces.

Second, in section 3.2 we discuss these moduli space volumes, which turn out to be divergent, though regularisable, objects in the JT setting. Specifically, we will explain the emergence of these divergences, present the equivalent of Mirzakhani's recursion for the unorientable case due to Stanford and discuss its geometric interpretation. Furthermore, we explain how the recursion enables one to prove the duality of unorientable JT/topological gravity with the corresponding matrix model. This duality is now the main point of attack of our work, whose approach we will then explain in detail. In brief, we compute all quantities on the gravitational side by doing computations on the matrix model side and then using the duality. As a first application of this, we generalise the Mirzakhani-like recursion to arbitrary Dyson index as induced by the generalisation of the matrix models to this setting, enabling the tuning between the Wigner-Dyson classes and beyond by changing a single parameter.

Third, section 3.3 contains our main results regarding moduli space volumes. We first explain the technique we use to perform these computations and give illustrative examples for it in section 3.3.1. Before going into explicit computations, we discuss the dependence of the matrix model correlation functions on the Dyson index β in section 3.3.2. We provide and prove a full classification of this dependence for general one-cut matrix models via several structure theorems. Notably, we will find these to imply specific contributions to the moduli space volumes to be genuinely non-Wigner-Dyson, i.e. associated with a part of moduli space that is, in a sense,"in between" orientability and unorientability. Explicit results are then computed, strongly facilitated by the knowledge of their dependence on the Dyson index, for topological gravity and JT gravity in the general β setting. Here, the focus is on topological gravity (section 3.3.3) since this theory is of highest relevance for the discussion of geometric implications of universality. Due to this, we discuss several explicit examples in detail and give the full general form of matrix model correlation functions in the matrix model dual to topological gravity. Furthermore, by a generalisation of Kontsevich diagrammatics to the arbitrary β setting, we prove the equivalent of theorem 2.2, i.e. the precise (polynomial) form of orientable WP volumes, for the case of general β and $n = 2$. This will be one of the most important ingredients for the study in chapter 4. Results for full JT gravity are then given in section 3.3.4, explaining first our method to compute the full unorientable WP volumes and then how to uplift the results obtained for $\beta = 1$ to the general β setting. Regarding this, we provide another (geometric) proof for the main structure theorem of section 3.3.2 for JT gravity (implying the same statement for topological gravity). By this we give it a more geometric interpretation, enabling a more explicit geometric discussion of the non-Wigner-Dyson parts of the moduli space volumes.

We then conclude the geometric part of this thesis and give an outlook to future perspectives of this line of research.

3.1 OF SURFACES AND SYMMETRIES

To get started, we discuss which gravitational theories, i.e. what types of surface in the path-integral, have to be considered when choosing a Wigner-Dyson class other than $\beta = 2$. To understand why, as expected, unorientable surfaces emerge as the correct generalisation of the orientable setting, it is worthwhile to recall the origin of the three-fold classification of random matrices in the symmetries of quantum mechanics.

SYMMETRIES IN RANDOM MATRICES. The origin of random matrix theory is the idea that one may model the complicated actual Hamiltonian of a complex quantum system, say of a large nucleus, by a large matrix M with random entries. Since one would like to model a quantum Hamiltonian, the matrix cannot be completely arbitrary but it has to be hermitian, i.e. $M^\dagger = M$. Furthermore, in the presence of unitary symmetries ((lattice) rotation, (lattice) translation, etc.), i.e. unitary operators U_i which commute with the Hamiltonian, there is a orthonormal basis in which the Hamiltonian is block-diagonal.

It is thus apparent that a random matrix ansatz for the whole Hamiltonian makes no sense, but one has to restrict to the individual blocks and study each individually. After performing this restriction, by Wigner's theorem [133] the only possible further symmetries are anti-unitary symmetries, the prime example for which is time-reversal. Anti-unitarity of an operator T means

$$\underset{|\psi\rangle,|\chi\rangle}{\forall} \langle T\psi|T\chi\rangle = \langle\chi|\psi\rangle\,. \tag{3.1.1}$$

One directly observes that this implies T^2 to be unitary. Furthermore, one easily infers that T is anti-linear and T^2 is linear. Let T now be an anti-unitary symmetry, i.e. it commutes with the considered block of the Hamiltonian, which is equivalent to requiring $THT^{-1} = H$ for the block Hamiltonian. Then, T^2 also commutes with the Hamiltonian in the considered block. We shall interpret T in the following as implementing time-reversal symmetry and for this it is reasonable to require [97]

$$\underset{\alpha\in\mathbb{C}}{\exists}\ \underset{|\psi\rangle\in\mathcal{H}}{\forall} T^2\,|\psi\rangle = \alpha\,|\psi\rangle\,, \tag{3.1.2}$$

with $\mathcal{H}$ denoting the (finite dimensional) Hilbert space associated with the considered block. Since T^2 is unitary, $|\alpha| = 1$. Furthermore, for $|\psi\rangle \in \mathcal{H}$

$$\begin{aligned} TT^2\,|\psi\rangle &= T\alpha\,|\psi\rangle = \alpha^* T\,|\psi\rangle \\ &= T^2T\,|\psi\rangle = \alpha T\,|\psi\rangle\,. \end{aligned} \tag{3.1.3}$$

Hence, on $\mathcal{H}$ it holds that

$$T^2 = \pm 1. \tag{3.1.4}$$

Classical examples for systems in both cases are a spin-less free particle, for which $T^2 = 1$, or a free spin-$\frac{1}{2}$ particle, for which it holds that $T^2 = -1$. In extension of this examples, one can consider a system of k spin-$\frac{1}{2}$ particles. Coupling the spins, the block diagonal form of the Hamiltonian is given by blocks of total spin j of dimension $2j+1$ furnished by states $|jm\rangle$. These states, as derived e.g. in [134] transform under the squared time-reversal operator as

$$T^2 |jm\rangle = (-1)^{2j} |jm\rangle \,, \tag{3.1.5}$$

i.e. with positive sign for integer and a negative sign for half-integer total spin j.

Equation (3.1.4) has strong implications for the form of the Hamiltonian. For the case of $T^2 = 1$, one can show that the Hamiltonian is real and symmetric, while for the case of $T^2 = -1$ it is a real quaternionic hermitian matrix [15, 97]. In the absence of time-reversal symmetry, the matrix is just hermitian with complex entries. Hence, the presence/absence of time-reversal symmetry in a quantum system in combination with the sign of the squared time-reversal operator classifies these systems into precisely one of the three Wigner-Dyson classes[1], Dyson's well-known "threefold way" [13]. This means that the appropriate random matrix ansatz for each block of the full system's Hamiltonian should be drawn from the ensemble as dictated by the block's behaviour under time-reversal.

To present a complete picture, it is interesting to note that beyond the possibility to have an anti-unitary operator commuting with the Hamiltonian (i.e. time-reversal) one could also have an operator C anti-commuting with the Hamiltonian. This has the implication that for an eigenstate $|\psi\rangle$ of the Hamiltonian H with the eigenvalue E it holds that

$$HC |\psi\rangle = -CH |\psi\rangle = -EC |\psi\rangle \,, \tag{3.1.6}$$

hence $C |\psi\rangle$ is an eigenstate of the Hamiltonian with eigenvalue $-E$, i.e. the spectrum is mirror-symmetric. The incorporation of (anti-)unitary operators with this property and that additionally commute with the time-reversal symmetry operator lead to more symmetry classes of random matrices, classified in the "tenfold way" of Altland and Zirnbauer [135]. Naturally, there is a formulation of matrix models also for these symmetry classes, conveniently reviewed in [69]. We will, however, only be concerned with the Wigner-Dyson classes and hence refer to the literature for a deeper discussion of the 7 "non-standard" symmetry classes.

Coming back to our aim to understand all Wigner-Dyson classes geometrically, we conclude that in order to understand what type of gravitational theory can be dual to the matrix models for $\beta \neq 2$ it is important to study in what sense time-reversal symmetry is absent in the orientable case and how to incorporate it in the gravitational theory.

UNORIENTABLE JT GRAVITY. In AdS/CFT and related dualities, such as the duality studied here, it is admissible to have a global symmetry like

1 Of course, this is the reason why those matrix ensembles were defined in the first place.

time-reversal invariance of the boundary theory, while it is not admissible to have a global symmetry of the bulk-theory [136]. Hence, one considers the case of time-reversal as a global symmetry of the boundary theory, i.e. invariance under reversing the orientation of the (cut-off) boundary discussed in section 2.1. This leads to the realisation of time-reversal symmetry as a gauge symmetry in the bulk theory [39, 136]. This means that upon glueing together surfaces one is free to glue circles with different orientations. This implies that the surfaces one needs to consider in a two-dimensional gravitational theory if one wishes the dual theory to possess time-reversal invariance are *unorientable surfaces*. Also it implies that theories which are not time-reversal invariant are related to orientable surfaces. Consequently, one can define correlation functions of partition functions in the theory we refer to as "unorientable JT gravity" by eq. (2.1.9), lifting the restriction to include only orientable manifolds. Note that we are careful here not to speak, as for the orientable case, of hyperbolic *Riemann* surfaces since Riemann surfaces are always orientable. Hence, to be precise, one includes such unorientable surfaces that can be endowed with a constant negative curvature metric, which is always the case for surfaces with negative Euler characteristic [69][2].

From the matrix model side, the relevance of unorientable manifolds is not unexpected, since the diagrammatic expansion of matrix models for $\beta \neq 2$ required the inclusion of a twisted propagator (cf. eq. (2.2.30)), leading to the graphs and the surfaces dual to them becoming unorientable. Before going over to the discussion of the duality with matrix models, we would like to recall briefly basic facts about unorientable surfaces relevant for the subsequent discussion.

First, we would like to know how to build such surfaces. In fact, this requires only the addition of the so-called *crosscap* to the building blocks of orientable surfaces, the 3-holed spheres [81]. This surface can be constructed in the way depicted in fig. 3.1.1, where it is attached to a 3-holed sphere along the broken black line. To construct it, one starts with a hyperbolic cylinder with two geodesic boundaries and identifies antipodal points on one of its boundaries. The resulting surface hence has only one remaining geodesic boundary and can be glued to the 3-holed sphere along it. We demonstrate now that this surface is indeed unorientable. A convenient classification of (un)orientability is that a surface is unorientable iff it contains an embedded Möbius strip. Such a Möbius strip is found as the neighbourhood of the green line depicted in fig. 3.1.1. To see this, one needs to observe that a point just to the right of the non-broken depiction of this line is mapped by the antipodal identification to a point just to the left of the broken depiction of it on the other side of the blue circle. Following the green line now back to the place

2 Note that the surfaces with vanishing or positive Euler characteristic are hence special cases. This was also true for the orientable case, cf. section 2.1.

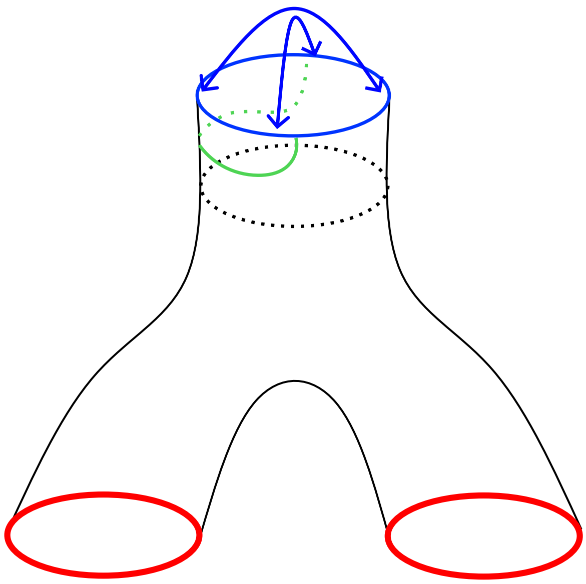


Figure 3.1.1: Illustration of glueing a crosscap to a 3-holed sphere. Adapted from [3].

where it intersects the blue circle as a non-broken line, one observes that one is now on the left side of the point one started from, which is precisely what happens on a Möbius strip. Hence, a neighbourhood of the green line is homeomorphic to a Möbius strip and the surface is consequently unorientable. Equivalently, this surface can be thought of as being the real-projective plane $\mathbb{R}P^2$ with a disk removed, or the Möbius strip itself [69]. Using the crosscap geometry, one can now build any unorientable hyperbolic surface by glueing together 3-holed spheres, potentially along orientation reversed boundaries, and crosscaps. Note that the decomposition of a given unorientable surface does however not necessarily contain a crosscap (like in the example of $g = 0$ surfaces, discussed momentarily) and there are in general more than one different decompositions of a surface, differing by an even number of contained crosscaps. Examples for this are discussed in section 3.3.4.

Having established how to build unorientable surfaces, the next question one would like to answer is, what genus such a surface has. To study this, it is useful to recall that the genus arises from the Euler characteristic (χ) which for a given surface can be computed from a polygon with its edges identified such that it is homeomorphic to the surface [82][3] via the formula

$$\chi = \#\text{Vertices} - \#\text{Edges} + \#\text{Faces}. \tag{3.1.7}$$

The simplest way to do this for the crosscap, is to view it as a Möbius strip which is easiest constructed as arising from the identification of opposite sides of a rectangle as depicted in fig. 3.1.2. Using this explicit polygonal

3 Alternatively, one can use a triangulation of the surface (a simplicial chain-complex) or some other realisation of it as a collection of polygons. The invariance of the Euler characteristic under the chosen way to compute it derives from it being homeomorphism invariant.

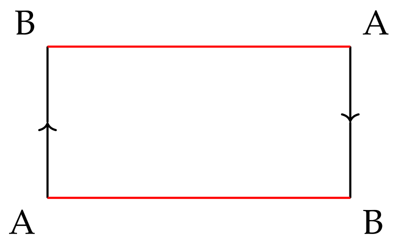


Figure 3.1.2: Depiction of the construction of a Möbius strip from identifying the left and right sides of a rectangle (black) with reversed orientations as indicated by the arrow.

representation of it, one can work out the Euler characteristic to be given by $\chi_{\text{Möbius}} = 2 - 3 + 1 = 0$. The relation with genus now comes in by our convention that the formula $\chi = 2 - 2g - n$ continues to hold for the unorientable case[4]. Using this, and noting that the crosscap (or, equivalently, the Möbius strip) has one boundary, one infers its genus to be given by $g = \frac{1}{2}$. Hence, for unorientable surfaces the genus is no longer necessarily an integer but can also be given by a half-integer (i.e. $g = \frac{k}{2}$ with k odd) as noted already above when discussing the diagrammatic expansion of matrix model correlation functions.

Coming back to unorientable JT gravity, the discussion leading to the topological expansion of correlation functions of this theory in the orientable case, recalled in section 2.1, can in large parts be repeated without changes for the unorientable case. In fact, one can generalise eq. (2.1.43), valid for $\beta = 2$, to all three Wigner-Dyson classes as

$$Z^{\beta}_{g,n}(\beta_1, \dots, \beta_n) = \prod_{i=1}^{n} \left[\int_0^{\infty} b_i \mathrm{d}b_i Z^t(\beta_i, b_i) \right] V^{\beta}_{g,n}(b_1, \dots, b_n), \qquad (3.1.8)$$

i.e. unorientable JT gravity boils down to studying the moduli space volumes of unorientable surfaces [69][5]. Note that there is no modification for the disk contribution, $Z^{(\beta)}_{0,1}(\beta)$, since reversing the orientation globally (which would be the only option here) leaves the geometry invariant. This has the implication that the leading-order contribution to the density of states, determining the spectral curve, also is not changed when considering unorientable JT gravity. In extension of the duality for the orientable case, it is thus well-motivated to suspect that the unorientable moduli-space volumes can be computed from a

4 In the mathematical literature, this formula is often modified to $\chi = 2 - k - n$ with the "unorientable genus" k, having the advantage of $k \in \mathbb{N}_0$, like the genus for orientable surfaces. In contrast to this, in our convention half-integer values of the genus can appear for unorientable surfaces, as seen momentarily.

5 Note that, in contrast to [69], we put all additional factors appearing when going beyond the orientable case into the volumes to have the same relations between the observables as in the orientable case.

matrix model with the JT gravity spectral curve (cf. eq. (2.3.4)) and $\beta \in \{1,4\}$ via

$$V^{\beta}_{g,n}(\vec{b}) = \mathcal{L}^{-1}\left[\prod_{i=1}^{n}\left(\frac{-2z_i}{b_i}\right)R^{\beta}_{g}(I), I, \vec{b}\right], \tag{3.1.9}$$

as it was proposed first in [69]. As we will see later from theorem 3.2, the matrix models with $\beta = 1$ and $\beta = 4$ are directly connected and hence we will restrict our discussion to the case of $\beta = 1$. Having discussed and proven the duality for this case, we can infer results for $\beta = 4$ by using the relation of the matrix models.

As a first sanity check of the extended duality, we consider the case of the $g = 0$ contributions with n boundaries in unorientable JT gravity. In this case, the surface cannot contain a crosscap since the total genus of a surface cannot be smaller than that of the constituent parts. Hence, one can start with the orientable surface with n boundaries and $g = 0$ and introduce unorientability by inserting a reversal of orientation at each boundary, i.e. the orientable volume is multiplied by a factor of 2 for each boundary. Since changing the overall orientation does not matter for the geometry, it holds that

$$V^{1}_{0,n}(\vec{b}) = 2^{n-1}V^{2}_{0,n}(\vec{b}). \tag{3.1.10}$$

This is precisely the relation between $R^1_0(I)$ and $R^2_0(I)$ for double-scaled matrix models as we will see in eq. (3.3.13) from the more general result that is theorem 3.1.

As for the orientable case, the duality can be proven by relating the loop equations for the matrix model directly with a Mirzakhani-like recursion for the unorientable WP volumes. We will discuss this in the next section. Before going into this deeper study of unorientable WP volumes, we would like to point out that the extension of the duality to all Wigner-Dyson classes via eq. (3.1.9) opens up the possibility to define a dual gravitational theory for *all* values of the Dyson index $\beta \in \mathbb{R}_+$ by defining the gravitational theory via the duality with a matrix model of a given, not necessarily Wigner-Dyson value of β, as we proposed in [5].

3.2 THE DIFFICULTIES INHERENT TO THE MODULI SPACE OF UNORIENTABLE SURFACES AND WAYS TO OVERCOME THEM

The discussion of the recursion and the extension of the duality to arbitrary Dyson index are largely based on [5].

The reduction of the computation of unorientable JT gravity correlation functions to the evaluation of the moduli space volumes of unorientable surfaces warrants a deeper discussion of these volumes. In this discussion, we will first recall from the literature the extension of the WP measure to the unorientable case and the difficulties arising from this as compared to the orientable case. In fact, the main problem of this discussion is that unorientable WP volumes are divergent and hence require regularisation. Considering a regularisation of these volumes, [76] established a Mirzakhani-like recursion for them and showed it to be equivalent to the loop equations for the orthogonal symmetry class. We give a brief discussion of the recursion, while not going into too much details, and explain why we actually do not use this recursion to compute moduli space volumes. Building on this discussion we then explain what we do instead to compute unorientable WP volumes. Finally, building on the recursion, we discuss its extension to arbitrary values of the Dyson index β and by this give a geometric interpretation for general β matrix models.

UNORIENTABLE WP VOLUMES. We recall from the discussion of orientable WP volumes in section 2.1 that one of the most important ingredients for their computation was the definition of the Weil-Petersson measure as arising from the symplectic structure of the moduli space of orientable surfaces. This measure arose via the pants-decomposition of the orientable surfaces. Hence, to find the corresponding measure for unorientable surfaces, one should start with the corresponding decomposition for these surfaces. As we discussed above, this decomposition includes as an additional building block to the 3-holed spheres (pants) the crosscap geometry. Hence, the generalisation of the WP measure to the unorientable moduli space needs to encompass the possibility to glue in a crosscap. The derivation of the appropriate measure, to our knowledge, works either by working out the possible measures on the unorientable moduli space that are mapping-class group invariant [137, 138] or via the formulation of JT gravity as a BF theory, where the measure can be evaluated by computing the so-called *torsion* of the crosscap [69][6]. The result of both computations is the same, namely that to glue in a crosscap with circumference of the remaining boundary a, the correct measure is

$$\mathrm{d}\mu_{CC} = \mathrm{d}a\,\frac{1}{2}\coth\left(\frac{a}{4}\right). \tag{3.2.1}$$

With this additional ingredient, one can write down a volume form on the moduli-space of unorientable surfaces by including the length-twist measures required to glue boundaries of three-holed-spheres that are the same for orientable and unorientable surfaces [137]. Notably, this volume form is invariant under the mapping-class group and hence one can apply the same reasoning as for the orientable case to define the unorientable WP volumes.

6 An accessible introduction to this quantity can be found in [139], discussing mainly the case of compact gauge groups. The case of non-compact gauge groups, specifically that of $PSL(2,\mathbb{R})$, is discussed in [69].

Furthermore, using this volume form, it is possible to derive a generalisation of Mirzakhani's recursion relation for moduli space volumes to the unorientable case, as done in [76], which we will recall momentarily. First, however, one should have a closer look at said volume form, specifically its crosscap part.

For the part of the measure associated with glueing a crosscap, i.e. eq. (3.2.1), it is already evident that it will give rise to a divergence of the moduli space volumes since the integral of this measure diverges as $a \to 0$, which is a point always contained in the necessary integration region, even when restricting to a fundamental domain of the action of the mapping-class group on the moduli space [76] as one has to do when computing the moduli space volumes. A possible regularisation for this is given by restricting the integration to values of the boundary length larger than a chosen ϵ, i.e. defining a regularised measure for the crosscap as [76, 137]

$$\mathrm{d}\mu_{CC}^{\epsilon} = \theta(a-\epsilon)\,\mathrm{d}\mu_{CC}\,, \tag{3.2.2}$$

where θ denotes the Heaviside function. Another way to write this, would be to write a regularised crosscap WP volume, i.e. $V^{1}_{\frac{1}{2},1}(b)$, as

$$V^{1,\epsilon}_{\frac{1}{2},1}(b) = \frac{\theta(b-\epsilon)}{2b}\coth\left(\frac{b}{4}\right), \tag{3.2.3}$$

with the convention that it is glued with the usual length-twist measure $b\,\mathrm{d}b$. In fact, this small crosscap divergence is the only divergence encountered in the computation of unorientable moduli space volumes but upon iterating it in the generalised recursion it will also appear in higher unorientable WP volumes. In fact, the structure of the WP volumes in terms of divergences can be found to be [76]

$$V^{1,\epsilon}_{g,n}(\vec{b}) = \sum_{k=0}^{2g}\log\left(\frac{1}{\epsilon}\right)^{k} v_{g,k}\left(\vec{b}\right) + \mathcal{O}(\epsilon). \tag{3.2.4}$$

STANFORD'S MIRZAKHANI-LIKE RECURSION. As announced, we shall cite now the actual recursion relation for the unorientable moduli space volumes from [76]. Its derivation can be found in said reference and goes beyond the scope of this thesis. This is due to the fact that we will actually not use the recursion to compute any regularised WP volumes. This is caused by the appearing integrals being quite tedious to deal with, having the consequence that the recursion could so far only be iterated to rather small values of g and n. From the intuition from the orientable theory, this was expected since the iteration of Mirzakhani's recursion is even there more tedious than working out the WP volumes via a matrix model computation. Precisely due to this, we worked around iterating the recursion by using the matrix model way to compute the moduli space volumes in [3], extending the known results from the iteration of the recursion. However, this uses the duality with a matrix model which can only be established by utilising the recursion.

The recursion relation is given by [76]

$$b_1 V^1_{g,n}(b_1, B) = 2\sum_{k=2}^{|B|}\int_0^\infty b'\,\mathrm{d}b'\,\left[b_1 - \mathsf{T}(b_1 \to b'; b_k)\right] V^1_{g,n-1}(b', B\backslash b_k) \tag{3.2.5a}$$

$$+\frac{1}{2}\int_0^\infty b'\,\mathrm{d}b' \int_0^\infty b''\,\mathrm{d}b''\,\mathsf{D}(b_1, b', b'')\times \left[V^1_{g-1,n+1}(b', b'', B) + \sum_{\substack{h+h'=g\\ B_1\cup B_2 = B}}' V^1_{h_1,|B_1|+1}(b', B_1) V^1_{h_2,|B_2|+1}(b'', B_2)\right] \tag{3.2.5b}$$

$$+\frac{1}{2}\int_0^\infty b'\,\mathrm{d}b'\,\mathsf{c}(b_1; b') V^1_{g-\frac{1}{2},n+1}(b', B), \tag{3.2.5c}$$

where we denote by $\sum'$ the sum excluding the appearance of $V^1_0(b', b'')$, $V^1_0(b')$, $B = (b_2, \ldots, b_n) \in \mathbb{R}^{n-1}_+$ and wrote down the recursion for the non-regularised volumes to declutter the formulae. Furthermore, the appearing functions T, D and c are given by

$$\mathsf{T}(b_1 \to b_2; b_3) = \log\left[\frac{\cosh\left(\frac{b_3}{2}\right) + \cosh\left(\frac{b_1+b_2}{2}\right)}{\cosh\left(\frac{b_3}{2}\right) + \cosh\left(\frac{b_1-b_2}{2}\right)}\right], \tag{3.2.6}$$

$$\mathsf{D}(b_1; b_2, b_3) = b_1 - \mathsf{T}(b_1 \to b_2, b_3) - \mathsf{T}(b_1 \to b_3; b_2), \tag{3.2.7}$$

$$\mathsf{c}(b_1, b_2) = b_1 b_2 - 2b_2 \log\left[\frac{\cosh\left(\frac{b_1+b_2}{4}\right)}{\cosh\left(\frac{b_1-b_2}{4}\right)}\right] + 4\left[\mathrm{Li}_2\left(-e^{-\frac{b_1+b_2}{2}}\right) - \mathrm{Li}_2\left(-e^{-\frac{b_1-b_2}{2}}\right)\right], \tag{3.2.8}$$

where Li_2 denotes the polylogarithm of second order, also known as dilogarithm or Spence's function. As input to the recursion it suffices to give first $V^1_{0,2}(b_1, b_2)$ and $V^1_{0,3}(b_1, b_2, b_3)$ which can be determined from the orientable result via eq. (3.1.10). Furthermore, the crosscap moduli space volume is needed, the regularised version of which we gave in eq. (3.2.3).

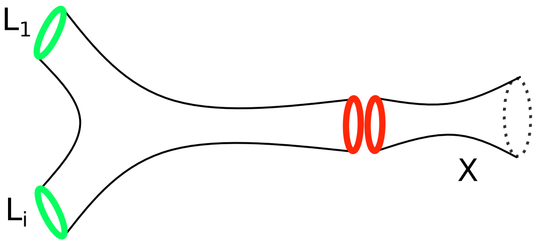
Notably, there is one exception to the applicability of the recursion, given by the case of $(g, n) = (\frac{1}{2}, 2)$. There, one can however pursue the explicit route of computing the WP volume by restricting to a fundamental domain of the mapping-class group directly [76, 140]. This boils down to restricting the length of the crosscap glueing boundary to be smaller than a^*, determined by $\sinh^2\left(\frac{a^*}{4}\right) = \cosh\left(\frac{b_1+b_2}{4}\right)\cosh\left(\frac{b_1-b_2}{4}\right) = \frac{1}{2}\left[\cosh\left(\frac{b_1}{2}\right) + \cosh\left(\frac{b_2}{2}\right)\right]$ [76]. Here, the connection between the length of the glueing boundary and that of the external geodesic lengths comes about by an explicit geometric consideration of the respective surface, pursued in [137]. The resulting moduli space volume is hence given as [76]

$$\begin{aligned} V^{1,\epsilon}_{\frac{1}{2},2}(b_1, b_2) &= 2\int_0^{a^*} b\,\mathrm{d}b\, V^{1,\epsilon}_{\frac{1}{2},1}(b) \\ &= 4\log\left[\frac{\sinh\left(\frac{a^*}{4}\right)}{\sinh\left(\frac{\epsilon}{4}\right)}\right] \\ &= 2\log\left[\frac{\cosh\left(\frac{b_1}{2}\right) + \cosh\left(\frac{b_2}{2}\right)}{2}\right] - 4\log\left(\frac{\epsilon}{4}\right) + \mathcal{O}\left(\epsilon^2\right), \end{aligned} \tag{3.2.9}$$

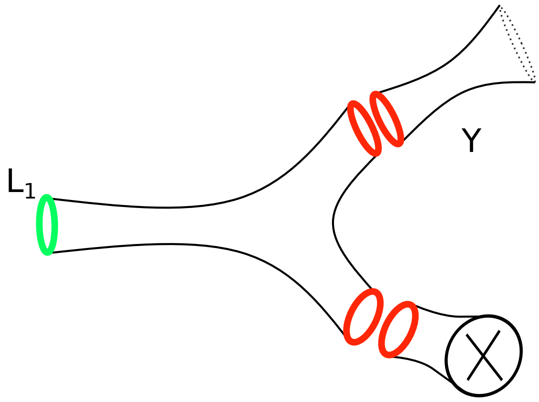
which is a good example to observe the general structure of the small-crosscap divergences of volumes (eq. (3.2.4)) arising from the crosscap upon iterating the recursion (or here a special case of it).

The main result of [76], however, is the proof that this recursion is precisely equivalent to the recursion for the resolvents in the suspected dual matrix model, i.e. eqs. (2.2.56) and (2.2.59) (the perturbative loop equations) for $\beta = 1$ and the spectral curve $y^{\mathrm{JT}}(z) = \frac{1}{4\pi}\sin(2\pi z)$ (eq. (2.3.4)). As for the orientable case, this proof works by performing a Laplace transform of the recursion for the volumes i.e. using the inverse of the suspected relation eq. (3.1.9). This transform precisely yields the perturbative loop equations, hence showing the validity of eq. (3.1.9). This result rigorously justifies the line of research we have been pursuing in [2, 3, 5], i.e. accessing unorientable WP volumes via matrix model techniques.

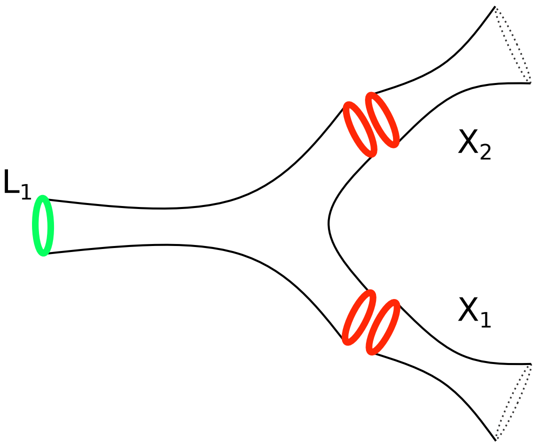
Before discussing how we managed to avoid iterating the generalised recursion in these works, we would like to point out an interpretation of it that will be quite useful for the geometric interpretation of arbitrary β matrix models. That is the interpretation of each term in the recursion as a splitting of the surface, similar to what we had already discussed for Mirzakhani's original recursion in fig. 2.1.7. Said interpretation for the unorientable case is depicted in fig. 3.2.1. To be precise, each term of the recursion represents a certain way to attach a 3-holed sphere (a pair of pants) to a surface of a specific genus and number of boundaries to build the desired surface. In fig. 3.2.1, we collect all 4 ways that are possible, where the option a) corresponds to eq. (3.2.5a), b) to eq. (3.2.5c), d) to the sole volume and c) to the sum over the product of two volumes in eq. (3.2.5b). Note that the special case of the



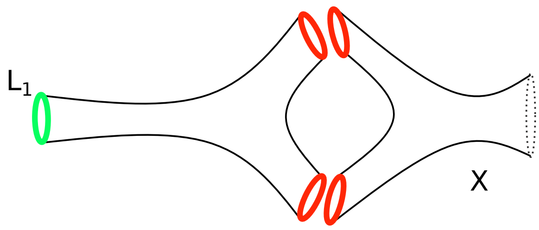
a) Glueing corresponding to eq. (3.2.5a). X has genus g with $n-1$ boundaries.



b) Glueing corresponding to eq. (3.2.5c). Y has genus $g - \frac{1}{2}$ and n boundaries. The surface attached to the other boundary is a crosscap.



c) Glueing corresponding to eq. (3.2.5b) (Disconnected part). X_1/X_2 have genera h_1/h_2 and n_1/n_2 boundaries with $h_1 + h_2 = g$ and $n_1 + n_2 = n - 1$



d) Glueing corresponding to eq. (3.2.5b) (Connected part). X has genus $g-1$ and $n+1$ boundaries.

Figure 3.2.1: Depiction of the different "glueings" i.e. the separation of a hyperbolic surface of genus g and n geodesic boundaries of lengths $L_1, \ldots, L_n$ into a 3-holed sphere and one or more hyperbolic surfaces. Adapted from [5].

recursion pointed out above can nevertheless be interpreted in the pictorial way (glueing a crosscap to a 3-holed sphere as in case a)).

HOW TO AVOID USING THE RECURSION. As we already mentioned, the idea of our work regarding unorientable WP volumes is to start on the matrix model side of the duality as expressed through eq. (3.1.9). In doing this, our approach has two main directions. These are determined by the different ways of dealing with the divergences of the moduli space volumes.

One way to do this is not to consider unorientable JT gravity but the large boundary limit of this theory, given by unorientable topological gravity. On the matrix model side this is, for the same reasons as in the orientable case discussed in section 2.1, easily implemented by using the topological gravity spectral curve $y^{\text{Airy}}(z) = \frac{z}{2}$ (eq. (2.3.34)). Using this to evaluate the crosscap Airy WP volume, we will find in section 3.3.3 that it is now integrable and hence also the volumes of higher genus and numbers of boundaries are non-divergent. In fact, the unorientable Airy WP volumes compute the leading-order part of the non-divergent contributions to the unorientable WP volumes. These, as we will see momentarily, already entail much of the modification of moduli space volumes due to unorientability and hence studying the unorientable Airy model, as we did in [2], is a good way to consider their properties without the interference of divergences. Another advantage of this strategy is the accessibility to volumes of much higher genus and number of boundaries than studied in [76] and, in fact, also in our approach to this [3], in finite time. Furthermore, by generalising the Kontsevich diagrammatics discussed in section 2.1 to the unorientable setting, to our knowledge first proposed in [140], one has another handle on the unorientable Airy WP volumes beyond solving the perturbative loop equations. This will enable us to derive statements about the structure of these volumes alike to theorem 2.2.

The already mentioned other way we used to deal with the divergences, was again not to use the JT gravity spectral curve directly but rather that for the so-called $(2, 2p + 1)$ *minimal string*. It is given by [141, 142]

$$y^{(p)}(z) = \frac{(-1)^p}{4\pi} T_{2p+1}\left(\frac{2\pi z}{2p+1}\right), \tag{3.2.10}$$

where T_n denotes the n-th Chebychev polynomial of the first kind. We will give more details for this model in section 3.3.4. The reasons why we choose it is its reduction to y^{Airy} for $p = 0$ and to y^{JT} in the large p limit. Like topological gravity, the crosscap WP volume for finite p will be integrable and hence p regularises the volumes' divergences. In fact, we will be able to connect p directly with the ϵ-regularisation of [76]. Pursuing this route in [3], we explicitly implemented the minimal string as a regularisation of unorientable JT gravity, which was, to our knowledge, first proposed in [69].

In this thesis, the focus will be put strongly on the study of unorientable topological gravity, in fact its generalisation to arbitrary β, introduced momentarily. There are several reasons to do so. First, the absence of divergences in topological gravity coincides with the absence of divergences in the universal result for the canonical SFF, as we will discuss in section 4.1. Hence, we already know that divergent contributions won't play an important role in comparing with and studying the implication of the predictions of universal RMT[7]. Conversely, studying the leading order of the finite part of the unorientable WP volumes will turn out to have the largest impact on this comparison, being the main object of this thesis. A second reason is, of course, the much higher order in g to which we could pursue the computations in topological gravity. Third, but not of least relevance, the method to find the explicit results for unorientable topological gravity and the actual results were worked out solely by the author[8] while the application of the method to JT gravity was done in collaboration with J.Tall, who did the heavy lifting building on the author's ideas. Hence, we shall go into great detail when considering explicitly topological gravity in section 3.3.3, while restricting for JT gravity to a presentation of the main results relevant for the later discussion and the main ideas of computation (section 3.3.4).

Before going into the discussion of these our approaches to unorientable WP volumes that avoid iterating the recursion, we briefly discuss its extension to the setting of arbitrary Dyson index.

A GEOMETRIC INTERPRETATION FOR GENERAL β. Having discussed the duality of a certain double-scaled matrix model in the three Wigner-Dyson classes with JT gravity upon including appropriate classes of surfaces in the gravitational path-integral, we can now present our proposal how to generalise this duality even further, beyond the Wigner-Dyson classes.

The idea for this, unsurprisingly, is to start on the matrix model side and to perform the generalisation of the theory to general Dyson index β there. This can be done by starting with the partition function of the matrix model as in eq. (2.2.7). The most general way to extend this expression, so far used only for $\beta \in \{1,2,4\} := \mathfrak{W}$ to $\beta \in \mathbb{R}_+$, is to introduce two functions $f, h : \mathbb{R}_+ \to \mathbb{R}_+$, with the property $f\big|_{\mathfrak{W}} = h\big|_{\mathfrak{W}} = \mathrm{id}_{\mathbb{R}_+}$ and define the generalised partition function as

$$\begin{aligned}\mathcal{Z} &= \mathcal{N}\int_{\mathbb{R}^N} \mathrm{d}\Lambda |\Delta(\Lambda)|^{f(\beta)} e^{-N\frac{h(\beta)}{2}\sum_{i=1}^N V(\lambda_i)} \\ &= \mathcal{N}\int_{\mathbb{R}^N} \mathrm{d}\Lambda |\Delta(\Lambda)|^{f(\beta)} e^{-N\frac{f(\beta)}{2}\sum_{i=1}^N \frac{h(\beta)}{f(\beta)} V(\lambda_i)}.\end{aligned} \tag{3.2.11}$$

Hence, this can equivalently be viewed as generalising the dependence on β only via the function f, while redefining the potential as $\tilde{V}(x) := \frac{h(\beta)}{f(\beta)} V(x)$. In the derivation of the perturbative loop equations, we saw that the dependence of a matrix model on the defining potential can be exchanged by the dependence on the spectral curve, which is the more economical way to define a matrix model. Applying this reasoning, we can infer that there

7 Though, of course, this vanishing of the divergent contributions in the universal limit is a non-trivial check on RMT universality (or quantum chaos) in unorientable JT gravity.

8 For $\beta = 1$, while for arbitrary β they were computed by an algorithm implemented by the author's master student M. Lents under the author's guidance.

is no dependence on the function h for the perturbative expansion of matrix model correlation functions for a generalised matrix model in the sense of eq. (3.2.11). Consequently, one can choose the direct generalisation of eq. (2.2.7), i.e. $f = \mathrm{id}_{\mathbb{R}_+}$ for all computations one wishes to perform and then introduce some other mapping $\hat{f}(\beta)$ afterwards. Specifically for the case of topological gravity, we will find an equivalent ambiguity when generalising the ribbon-graph propagator eq. (2.2.30) to arbitrary Dyson index as discussed around eq. (3.3.73).

Choosing this route, we can thus use the perturbative loop equations, allowing now for all values $\beta \in \mathbb{R}_+$, without any modification to compute the perturbative expansion of the resolvents, and hence all other interesting observables, in dependence on the Dyson index. Using these, we then *define* WP volumes for $\beta \in \mathbb{R}_+$, (β WP volumes) via the extension of eq. (3.1.9) to this setting. Using this, we can then define β JT/topological gravity. The moduli space volumes of these theories can be interpreted as constituting a crossover between orientable and unorientable manifolds in the gravitational path-integral. This is mathematically challenging insofar, as there is a dichotomy of surfaces, i.e. a surface is either orientable or unorientable. Consequently, the knowledge of the moduli space volumes for orientable and unorientable surfaces seems to be all there is to know and thus also the crossover should be determined by it. Hence, one could have the intuition of the β WP volumes being an interpolation between orientable and unorientable moduli space volumes by defining the β WP volume as the sum of that for $\beta = 2$ and $\beta = 1$ with prefactors such that the known results are obtained for $\beta \in \mathfrak{W}$. We will find this not to be true and that there are contributions to the moduli space volumes that appear only at non-Wigner-Dyson values of β and hence are contributions appearing only in the transitional regime between orientable and unorientable. We will see this from a pure matrix-model reasoning in section 3.3.2 and also from a geometric argument in JT gravity (the proof of theorem 3.5).

Hence, it is interesting to understand the generalised volumes geometrically. For this, we will use Stanford's unorientable Mirzakhani-like recursion eqs. (3.2.5a) to (3.2.5c). Specifically, we use the equivalence of this recursion for $\beta = 1$ with the perturbative loop equations for the JT gravity spectral curve y^{JT}. The perturbative loop equations formalism computes the unorientable WP volumes by solving the contour integral in eq. (2.2.56), for which one needs as an input $F_g^{\beta}(z, I)$ eq. (2.2.59). Notably, the difference between $\beta = 1$ and the case of general Dyson index is just additional factors for some of the terms. Due to the linearity of the Laplace transform, these additional factors can just be put in front of the transformation of the individual terms of $F_g^{\beta}(z, I)$ when transforming this recursion to the generalised Mirzakhani-like recursion for the unorientable WP volumes. Hence, by the same reasoning as applied for the $\beta = 1$ case in [76], we infer that the β WP volumes obey the recursion

$$b_1 V^{\beta}_{g,n}(b_1, B) = \sum_{k=2}^{|B|} \frac{2}{\beta} \int_0^{\infty} b'\,\mathrm{d}b' \left[b_1 - \mathsf{T}(b_1 \to b'; b_k)\right] V^{\beta}_{g,n-1}(b', B \backslash b_k) \quad (3.2.12a)$$

$$+ \frac{1}{2} \int_0^{\infty} b'\,\mathrm{d}b' \int_0^{\infty} b''\,\mathrm{d}b''\,\mathsf{D}(b_1, b', b'') \times \left[V^{\beta}_{g-1,n+1}(b', b'', B) + \sum_{\substack{h+h'=g \\ B_1 \cup B_2 = B}}' V^{\beta}_{h_1,|B_1|+1}(b', B_1) V^{\beta}_{h_2,|B_2|+1}(b'', B_2) \right] \quad (3.2.12b)$$

$$+ \frac{1}{2} \frac{2-\beta}{\beta} \int_0^{\infty} b'\,\mathrm{d}b'\,\mathsf{c}(b_1; b')\, V^{\beta}_{g-\frac{1}{2},n+1}(b', B), \quad (3.2.12c)$$

with the same notation as above and keeping in mind that there is a modification of the input volumes upon going to arbitrary Dyson index (cf. eq. (2.2.54)). We will derive all of these modifications in section 3.3.2. Essentially, the recursion is not much altered, i.e. the terms are the same up to prefactors and hence still have the interpretation depicted in fig. 3.2.1. The only modifications, except of course the input volumes, arise from the splittings of type a) and b), which are multiplied by $\frac{1}{\beta}$ (a) and $\frac{2-\beta}{\beta}$ (b). This derives from the definition of the volumes via the matrix model, but both factors can be motivated from geometrical considerations. Easiest, as we will see in section 3.3.2, $\frac{2-\beta}{\beta}$ is just the prefactor of the β crosscap, explaining why this factor occurs when splitting off a crosscap as in b). The additional factor for the splitting in a) can be motivated by noting that the factor of 2 present in the $\beta = 1$ case as compared to the orientable setting occurred due to the possibility, in the unorientable case, to glue the 3-holed sphere with or without a change of orientation for a geodesic going from the boundary of length L_1 to that of length L_i [69]. There is only one possibility in the case of $\beta = 2$, hence this factor has to be cancelled, which suggests that the additional factor is $\frac{1}{\beta}$.

Hence, we are able to understand the β WP volumes in two ways. First, as arising from the arbitrary β matrix model correlation functions via eq. (3.1.9), which is how we defined them. Second, as arising from the general β generalisation of the recursion relation for unorientable WP volumes eqs. (3.2.12a) to (3.2.12c), giving them a geometric interpretation.

As a finial comment, it is useful to remark that we introduced the generalisation to arbitrary Dyson index for the full WP volumes to establish a direct connection with the Mirzakhani-like recursion. However, one can perform the same reasoning also for the Airy WP volumes, i.e. use y^{Airy} in the perturbative loop equations. The advantages of doing this, as compared to considering JT gravity, remain the same as those discussed above.

We have now discussed why we can use matrix models to compute moduli space volumes for the two gravitational theories we are interested in. We finally come to describing how to do so and what the result actually are.

3.3 THE MATRIX MODEL WAY

This section deals with the computation of (Airy) WP volumes by using the dual matrix models. To illustrate the method, in section 3.3.1 we first give some examples how to solve the perturbative loop equations in double-cover coordinates. We will see in these examples that the answer for all symmetry classes is in some cases determined by computing the result in one of the Wigner-Dyson classes. Motivated by this, we study the structure in terms of the Dyson index of the perturbative expansion of matrix model correlation functions and provide results for this, valid for all one-cut matrix models, in section 3.3.2. Equipped with this, in section 3.3.3 we compute the actual results for matrix model correlation functions in topological gravity up to $g = 4, n = 1$ and prove several general statements regarding their structure. In section 3.3.4 we then extend the reasoning to full unorientable JT gravity. Overall, having already in mind the main goal of this thesis, i.e. to study the implications of the universal form of the canonical SFF onto moduli space volumes, we will discuss in most detail those correlation functions for $n = 2$ boundaries.

3.3.1 *How to solve a matrix model perturbatively for all symmetry classes at once*

This discussion is largely based on [2] and [5].

As we discussed above, the perturbative loop equations enable the computation of the perturbative contributions to multi-resolvents for all three Wigner-Dyson classes and actually also their extension to arbitrary Dyson index in one computation by keeping the Dyson index β variable. For the reader's convenience we present again the recursive procedure meant by this, given in eqs. (2.2.56) and (2.2.59). The recursion prescription is given by

$$R_g^{\beta}(z, I) = \frac{1}{2\pi i z} \oint_{[-i\infty+\epsilon, i\infty+\epsilon]} \frac{z'^2 dz'}{z'^2 - z^2} \frac{1}{y(z')} F_g^{\beta}(z', I), \tag{3.3.1}$$

with

$$
\begin{aligned}
F_g^\beta(z,I) =&\left(1-\frac{2}{\beta}\right)\frac{1}{-2z}\partial_z R^\beta_{g-\frac{1}{2}}(z,I) + R^\beta_{g-1}(z,z,I)\\
&+\sum_{I\supseteq J,h}{}' R_h^\beta(z,J)R^\beta_{g-h}(z,I\backslash J)\\
&+2\sum_{k=1}^{n}\left[R_0^\beta(z,z_k)+\frac{1}{\beta}\frac{1}{\left(z_k^2-z^2\right)^2}\right]R_g^\beta(z,I\backslash\{z_k\}),
\end{aligned}
\tag{3.3.2}
$$

where the primed sum means that summands containing $R_0^\beta(z)$ or $R_0^\beta(z_1,z_2)$ are excluded. Iterating the recursion for given values of the number of boundaries $n\in\mathbb{N}$ and the genus $g\in\mathbb{N}_0\cup\frac{\mathbb{N}}{2}$ now entails first computing all the terms needed to compute $F_g^\beta(-z^2,I)$, where we recall our convention of $I=\{z_1,\dots,z_n\}$. For this, only terms are needed that have either a smaller genus and the same or higher number of boundaries or the same genus but fewer boundaries. This illustrates that we are dealing indeed with a recursive prescription. The second step is then to perform the contour integral eq. (3.3.1), easiest done by using the residue theorem. To illustrate how to do this in the double-cover coordinates we are using we recall the special case of $(g,n)=(0,2)$ we had discussed already above.

Specifically, we start with the expression of $R_0^\beta(x,x_1)$ as a contour integral in eq. (2.2.46). Transforming this to double-cover coordinates, we find

$$
R_0^\beta(z_1,z_2)=\frac{1}{2\pi i\beta z_1}\oint_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}z'}{z'^2-z_1^2}\frac{2z'^2}{\left(z'^2-z_2^2\right)^2}. \tag{3.3.3}
$$

The integrand has poles at $\pm z_1$ and $\pm z_2$, considered as a function of z'. We consider the case of $\mathrm{Re}(z_1)>0$ and $\mathrm{Re}(z_2)>0$[9]. We will use this convention (applying it also for all other occurring z_i) also in the subsequent computations to fix the position of the poles. After the computation, we analytically continue the result to the whole complex plane in all arguments. With this choice, the poles at $-z_1$ and $-z_2$ are in the hemisphere of the negative real part (left) while the other two are on the other hemisphere (right). There are now two ways to deform the contour. First, surrounding the two poles on the left hemisphere in counter-clockwise direction and second, surrounding those on

9 To be precise, larger than ϵ. By sending $\epsilon\to 0$ after the computation this drops out and hence we do not include this.

the right hemisphere clockwise. Using the residue theorem, the first option implies

$$\begin{aligned} R_0(z_1,z_2) &= \frac{1}{\beta z_1}\left[\operatorname*{Res}_{z'=-z_1} + \operatorname*{Res}_{z'=-z_2}\right]\frac{dz'}{z'^2-z_1^2}\frac{2z'^2}{\left(z'^2-z_2^2\right)^2} \\ &= \frac{1}{2\beta}\frac{1}{z_1z_2(z_1+z_2)^2}, \end{aligned} \tag{3.3.4}$$

while the second yields

$$\begin{aligned} R_0(z_1,z_2) &= \frac{-1}{\beta z_1}\left[\operatorname*{Res}_{z'=z_1} + \operatorname*{Res}_{z'=z_2}\right]\frac{dz'}{z'^2-z_1^2}\frac{2z'^2}{\left(z'^2-z_2^2\right)^2} \\ &= \frac{1}{2\beta}\frac{1}{z_1z_2(z_1+z_2)^2}, \end{aligned} \tag{3.3.5}$$

Both results agree with each other and also with eq. (2.2.54), which had to be the case.

Compared with the modifications of the integration contour needed to perform this computation in the x-coordinates, as we discussed in section 2.2, we see the clear advantage of using the double-cover coordinate. In fact, the consideration relating the contour integral to a sum of residues presented here can be directly generalised to more than two boundaries as we will see in the explicit computations in sections 3.3.3 and 3.3.4.

Before going there, let us first consider the remaining special case of the perturbative loop equations, i.e. $(g,n) = (0,3)$. We already gave the expression of the corresponding contribution to the 3-point function of resolvents as a contour integral in double-cover coordinates in eq. (2.2.60), which we recall here for convenience

$$\begin{aligned} R_0^\beta(z_1,z_2,z_3) = \frac{1}{2\pi i z_1}\oint_{i\mathbb{R}+\epsilon}\frac{z'^2dz'}{z'^2-z_1^2}\frac{2}{y(z')}\times \\ \left[R_0^\beta(z',z_2)R_0^\beta(z',z_3) + \frac{1}{\beta}\left(\frac{R_0^\beta(z',z_3)}{\left(z_2^2-z'^2\right)^2} + \frac{R_0^\beta(z',z_2)}{\left(z_3^2-z'^2\right)^2}\right)\right]. \end{aligned} \tag{3.3.6}$$

When putting into this expression the result for $R_0^\beta(z_1,z_2)$ (eq. (3.3.5)) one directly finds

$$R_0^\beta(z_1,z_2,z_3) = \frac{1}{\beta^2}R_0^1(z_1,z_2,z_3), \tag{3.3.7}$$

which shows explicitly that one needs to know only the result for the case of $\beta = 1$ (in fact, for an arbitrary value of β) to determine that for all values of β.

Were this true for all other contributions to multi-resolvents, it would in particular imply the determination of the contributions for $\beta = 1$ by the orientable contributions ($\beta = 2$) and render our investigation of unorientable moduli space volumes rather trivial. Fortunately, the statement does not generalise to the general setting but it is nevertheless an interesting question whether one can determine the general structure of all contributions to the topological expansion of correlation functions. Consequently, we will study this before coming to the actual computations in the matrix models dual to JT/topological gravity.

3.3.2 *The general structure of matrix model correlation functions*

This section is largely based on [5].
In this section, we give and prove the general structure of the contributions to the topological expansion of matrix model correlation functions in terms of β. We will need this structure only for the case of double-scaled matrix models but the proof also extends to the more general setting of one-cut matrix models and so no harm is done by giving it for this more general case.

We will now state the main theorems of this structure. For these, let $R_g^\beta(I)$ be the contribution at genus g to the n-point correlation function of resolvents for a one-cut matrix model with spectral curve $y(x)$ and Dyson index β. Then, the following statements hold:

Theorem 3.1 (General structure ($g = 0$)). *The contribution for Dyson index β at genus 0 and n arguments is determined by the contribution for Dyson index $\beta = 1$ of the same genus and arguments via*

$$R_0^\beta(I) = \frac{1}{\beta^{|I|-1}} R_0^1(I). \tag{3.3.8}$$

Theorem 3.2 (Behaviour under $\beta \to \frac{4}{\beta}$). *The contribution at genus g and n arguments with Dyson index $\frac{4}{\beta}$ is determined by the corresponding contribution with Dyson index β via*

$$R_g^{\frac{4}{\beta}}(I) = (-1)^{2g} \left(\frac{\beta}{2}\right)^{2(g+|I|-1)} R_g^\beta(I). \tag{3.3.9}$$

Theorem 3.3 (General structure). *The contribution at genus g and n arguments with Dyson index β is given by*

$$R_g^\beta(I) = \frac{1}{\beta^{2g+n-1}} \left(\mathcal{R}_g^0(I)\beta^g + (2-\beta)^2 \sum_{i=1}^{g} \mathcal{R}_g^i(I)\beta^{i-1}((1-\beta)(4-\beta))^{g-i} \right), \tag{3.3.10}$$

for integer g and

$$R_g^\beta(I) = \frac{1}{\beta^{2g+n-1}} \left((2-\beta) \sum_{i=1}^{g+\frac{1}{2}} \mathcal{R}_g^i(I) \beta^{i-1} ((1-\beta)(4-\beta))^{g+\frac{1}{2}-i} \right), \tag{3.3.11}$$

for half-integer g. It is determined by $g+1$ *(integer genus)* $g+\frac{1}{2}$ *(half-integer genus) functions* $\mathcal{R}_g^i : \mathbb{C}^n \to \mathbb{C}^n$ *which do not depend on* β*.*

The rest of this section will now discuss and prove these statements.

THE STRUCTURE AT $g = 0$. The dependence of the $g = 0$ contributions on the Dyson index β is a direct generalisation of the behaviour we had encountered already for $n = 2$ in eq. (3.3.5) and for $n = 3$ in eq. (3.3.7).

It has an important corollary, namely that one can compute $R_0^\beta(I)$ not only from the $\beta = 1$ result but actually from any value $\hat{\beta}$ of the Dyson index. Explicitly

$$R_0^\beta(I) = \left(\frac{\hat{\beta}}{\beta}\right)^{|I|-1} R_0^{\hat{\beta}}(I), \tag{3.3.12}$$

which in particular implies

$$R_0^\beta(I) = \left(\frac{2}{\beta}\right)^{|I|-1} R_0^2(I). \tag{3.3.13}$$

The latter result is very useful since there is a way to compute the $g = 0$ orientable WP volumes very fast by using properties of the intersection numbers, given in [143]. We used an implementation of this algorithm to compute the orientable WP volumes for $n > 6$ and to cross-check our results for smaller values of n in [1, 2]. Then, via eq. (3.3.13), we could translate these results to the $\beta = 1$ case [2] and to the general β case in [5].

Furthermore, the structure will be the basis for our proof of theorem 3.3. Also, the idea we will use in this proof of the general case is already apparent in the proof of the $g = 0$ case which we will give now.

Proof of theorem 3.1. Let the assumptions be as in the statement of the theorem. The starting point for the perturbative loop equations for $n > 3$ is given by eq. (2.2.49), which for the case of $g = 0$ can be put as

$$2y(x)R_0^\beta(x, I) + F_0^\beta(x, I) = (\text{analytic in } x), \tag{3.3.14}$$

where

$$\begin{aligned} F_0^\beta(x,I) &= +\sum_{J\supseteq I}{}' R_0^\beta(x,J)R_0^\beta(x,I\backslash J) \\ &\quad +2\sum_{k=1}^{n}\left[R_0^\beta(x,x_k)+\frac{1}{\beta}\frac{1}{(x-x_k)^2}\right]R_0^\beta(x,I\backslash\{x_k\}) \\ &= \sum_{\substack{J\subseteq I\\ J\neq\emptyset\\ J\neq I}} R_0^\beta(x,J)R_0^\beta(x,I\backslash J)+\frac{2}{\beta}\sum_{k=1}^{n}\frac{R_0^\beta(x,I\backslash x_k)}{(x-x_k)^2}, \end{aligned} \tag{3.3.15}$$

which in the form of the last line is valid also in the special cases of $n=2$ and $n=3$ as one can see by a direct comparison of the relations[10]. Since the integral equation eq. (2.2.51) deriving from this is a linear relation between $R_0^\beta(x,I)$ and $F_0^\beta(x,I)$, it suffices to show

$$F_0^\beta(x,I) = \frac{1}{\beta^n}F_0^1(x,I), \tag{3.3.16}$$

which notably doesn't have any dependence on the cut-structure and hence is applicable in the general one-cut case.

We prove this statement by induction. The base case is given by $I=\{x_1\}$ for which one finds

$$F_0^\beta(x,I) = \frac{2}{\beta}\frac{R_0^\beta(x)}{(x-x_1)^2}, \tag{3.3.17}$$

cf. example 2.4, obviously fulfilling eq. (3.3.16). Now choose $n\in\mathbb{N}$ and assume eq. (3.3.16) for all lengths of I, $k\in\mathbb{N}$, with $k<n$. This implies theorem 3.1 for all $k\leq n$. Hence, for $|I|=n$ we find

$$\begin{aligned} F_0^\beta(x,I) &= \sum_{\substack{J\subseteq I\\ J\neq\emptyset\\ J\neq I}} R_0^\beta(x,J)R_0^\beta(x,I\backslash J)+\frac{2}{\beta}\sum_{k=1}^{n}\frac{R_g^\beta(x,I\backslash x_k)}{(x-x_k)^2} \\ &= \sum_{\substack{J\subseteq I\\ J\neq\emptyset\\ J\neq I}} \frac{R_0^1(x,J)R_0^1(x,I\backslash J)}{\beta^{|J|+|I\backslash J|}}+\frac{2}{\beta}\sum_{k=1}^{n}\frac{R_g^1(x,I\backslash x_k)}{\beta^{|I|-1}(x-x_k)^2} \\ &= \frac{1}{\beta^n}F_0^1(x,I), \end{aligned} \tag{3.3.18}$$

where we used the assumption to go to the second line. This is the statement of eq. (3.3.16) for $|I|=n$ and hence completes the induction step. □

10 In [5] we give a separate proof for the expression of $F_0^\beta(x,I)$ in the last line from the full loop equations.

The relation of β with $\frac{4}{\beta}$. Having now discussed the general structure at $g = 0$, we would like to find out a bit more about the dependence of generic contributions to the topological expansion of multi-resolvents on the Dyson index to find a good Ansatz for a general structure. Specifically, we would like to explore the extension of the relation of the $\beta = 1$ with the $\beta = 4$ Wigner-Dyson classes to the arbitrary β case. This relation was given in [68] as

$$R_g^1(I) = (-1)^{2g} 2^{2(g+n-1)} R_g^4(I). \tag{3.3.19}$$

In fact, this is an example of the general relation of matrix models with Dyson index β and those with Dyson index $\frac{4}{\beta}$. This relation is rooted in the invariance of the integral definition of the correlation functions under $(\beta, N) \leftrightarrow \left(\frac{4}{\beta}, -\frac{N\beta}{2}\right)$ (e.g [21] and references therein). We are interested in the behaviour of the perturbative contributions to the correlation functions of resolvents, which led us to study their transformation under the mapping $\beta \to \frac{4}{\beta}$, leading to theorem 3.2.

Proof of theorem 3.2. Let the assumptions be as in the statement of the theorem. Furthermore, let the $R_g^\beta(I)$ be the solutions to the loop equations for β. For the case of $g = 0$ the statement we would like to prove can be inferred directly from eq. (3.3.12) by choosing $\hat{\beta} = \frac{4}{\beta}$. For $g > 0$ it is easiest to use the expression of the respective contribution to the n-boundary resolvent as a contour integral, eq. (2.2.51). For convenience we recall this expression to be

$$R_g^\beta(x, I) = -\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}x'}{x' - x} F_g^\beta(x', I) \frac{1}{2y(x')} \frac{\sqrt{\sigma(x')}}{\sqrt{\sigma(x)}}, \tag{3.3.20}$$

with $\mathcal{C}$ being a contour encircling the cut clockwise and

$$\begin{aligned} F_g^\beta(x, I) := & \underbrace{\left(1 - \frac{2}{\beta}\right) \partial_x R^\beta_{g-\frac{1}{2}}(-x^2, I)}_{I} \\ & + \underbrace{R^\beta_{g-1}(-x^2, -x^2, I)}_{II} \\ & + \underbrace{\sum_{I \supseteq J, h}{}' R_h^\beta(-x^2, J) R^\beta_{g-h}(-x^2, I \backslash J)}_{III} \\ & + \underbrace{2 \sum_{k=1}^{n} \left[R_0^\beta(-x^2, -x_k^2) + \frac{1}{\beta} \frac{1}{\left(x_k^2 - x^2\right)^2} \right] R_g^\beta\left(-x^2, I \backslash \left\{-x_k^2\right\}\right)}_{IV}, \end{aligned} \tag{3.3.21}$$

where $\sum'$ is a notation for excluding $R_0(x)$ and $R_0(x, x_k)$ from the sum. Due to the recursive nature of this way of computing the topological expansion, we use the cases of $g = 0$, which we have shown above, as base clauses and perform the proof by induction. Assuming thus that for all resolvents necessary to compute $R_g^\beta(x, I)$, with $|I| =: n$, our claim holds, it remains to show that this implies our claim for $R_g^\beta(x, I)$. To do this, due to eq. (3.3.20), it suffices to show [11]

$$F_g^\beta(x, I) = (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} F_g^{\frac{4}{\beta}}(x, I), \tag{3.3.22}$$

which we shall prove by considering each line of eq. (3.3.21) separately and plugging our base assumption that the relation between β and $\frac{4}{\beta}$ holds for all lower contributions to the multi-resolvents.

$$\begin{aligned} I &= \left(1 - \frac{2}{\beta}\right)(-1)^{2g-1}\left(\frac{2}{\beta}\right)^{2g+2n-1} \partial_x R_{g-\frac{1}{2}}^{\frac{4}{\beta}}(x, I) \\ &= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)}\left(1 - \frac{\beta}{2}\right)\partial_x R_{g-\frac{1}{2}}^{\frac{4}{\beta}}(x, I) \\ &= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} I(\beta \leftrightarrow \frac{4}{\beta}). \end{aligned} \tag{3.3.23}$$

$$\begin{aligned} II &= (-1)^{2g-2}\left(\frac{2}{\beta}\right)^{2(g-1+n+2-1)} R_{g-1}^{\frac{4}{\beta}}(x, x, I) = (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} R_{g-1}^{\frac{4}{\beta}}(x, x, I) \\ &= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} II(\beta \leftrightarrow \frac{4}{\beta}). \end{aligned} \tag{3.3.24}$$

$$\begin{aligned} III &= \sum'_{I \supseteq J, h} (-1)^{2h+2g-2h}\left(\frac{2}{\beta}\right)^{2(h+|J|+g-h+n-|J|)} R_h^{\frac{4}{\beta}}(x, J) R_{g-h}^{\frac{4}{\beta}}(x, I \backslash J) \\ &= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} \sum'_{I \supseteq J, h} R_h^{\frac{4}{\beta}}(x, J) R_{g-h}^{\frac{4}{\beta}}(x, I \backslash J) \\ &= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} III(\beta \leftrightarrow \frac{4}{\beta}). \end{aligned} \tag{3.3.25}$$

11 Note that $R_g^\beta(x, I)$ has $n + 1$ arguments.

$$
\begin{aligned}
IV &= \sum_{k=1}^{n}\left[\left(\frac{2}{\beta}\right)^2 R_0^{\frac{4}{\beta}}(x,x_k) + \frac{1}{\beta}\frac{1}{\left(x_k^2-x^2\right)^2}\right](-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n-1)} R_g^{\frac{4}{\beta}}(x, I\backslash\{x_k\}) \\
&= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} \sum_{k=1}^{n}\left[R_0^{\frac{4}{\beta}}(x,x_k) + \frac{\beta}{4}\frac{1}{\left(x_k^2-x^2\right)^2}\right] R_g^{\frac{4}{\beta}}(x, I\backslash\{x_k\}) \\
&= (-1)^{2g}\left(\frac{2}{\beta}\right)^{2(g+n)} IV(\beta \leftrightarrow \frac{4}{\beta}).
\end{aligned}
\tag{3.3.26}
$$

□

HOW TO FIND THE GENERAL STRUCTURE. Equipped with theorem 3.1 and theorem 3.2, we can finally come to the general structure we advertised above. Before giving a proof for it that is again based on the recursion, we would like to briefly motivate it from a different perspective. This perspective arises from theorem 3.2 and the insight that the β dependence of the contributions to the topological expansion of resolvents arises purely from factors of $\frac{1}{\beta}$ and $\frac{2-\beta}{\beta}$ as one can quickly convince oneself looking at the explicit form of $F_g^{\beta}(x, I)$. From this insight, it is a well-motivated conjecture to write

$$
R_g^{\beta}(I) = \frac{1}{\beta^{2g+n-1}} \sum_{i=1}^{k} \mathcal{P}_i(\beta) g_i(I), \tag{3.3.27}
$$

with the $\mathcal{P}_i$ being a basis of the polynomials with coefficients in $\mathbb{Z}$ of maximal order k and the g_i denoting the dependence on the x_i or z_i. A proof of this would go precisely along the same lines as the proof for theorem 3.3, presented momentarily, so we will not give it here and prove the stronger statement instead. The combination of eq. (3.3.27) with theorem 3.2 now motivates one to choose a polynomial basis that is *invariant* under $\beta \to \frac{4}{\beta}$, i.e. such that the $g_i(I)$ are not mixed in this transformation. This is quickly found as the requirement on the polynomials that

$$
\mathcal{P}_i\left(\frac{4}{\beta}\right) = (-1)^{2g}\left(\frac{2}{\beta}\right)^{2g} \mathcal{P}_i(\beta). \tag{3.3.28}
$$

This is a quite strong requirement for the polynomials, as one can see from explicitly writing one of the relevant polynomials i.e. $\mathcal{P}_{\vec{b}}(\beta) := \sum_{i=0}^{k} b_i \beta^i$ with $\vec{b} \in \mathbb{Z}^k$. One can now directly write out $\mathcal{P}_{\vec{b}}(\frac{4}{\beta})$ as

$$
\mathcal{P}_{\vec{b}}\left(\frac{4}{\beta}\right) = (-1)^{2g}\left(\frac{2}{\beta}\right)^{2g} \sum_{i=2g-k}^{2g} b_{2g-i}(-1)^{2g} 2^{2(g-i)} \beta^i. \tag{3.3.29}
$$

Consequently, the polynomial is invariant iff

$$\underset{i\in[0,k]}{\forall} b_i = (-1)^{2g} 2^{2(g-i)} b_{2g-i}. \tag{3.3.30}$$

From this, one can infer two facts. First, for the polynomial to be invariant, it has to hold that $k \leq 2g$. This, i.e. $k = 2g$ in eq. (3.3.27), is indeed what one finds from theorem 3.3. Second, choosing the b_i for $i \in [0, g]$ in the integer genus and $i \in \left[0, g - \frac{1}{2}\right]$ in the half-integer genus case fixes the other b_i. This implies that the space of invariant polynomials is $g + 1$ dimensional in the integer genus and $g + \frac{1}{2}$ dimensional in the half-integer genus case. This is an interesting observation, since it shows the invariance under $\beta \to \frac{4}{\beta}$ to heavily constrain the polynomial dependence on β within the contributions to the multi-resolvents. Technically, finding the required number of invariant basis polynomials fixes a complete basis for the space of invariant polynomials occurring for genus g and hence a candidate for a general structure.

A first candidate for a basis of the polynomials would just be the monomials, i.e. $\mathcal{P}_i^m(x) := x^i$ for which it holds that

$$\mathcal{P}_i^m\left(\frac{4}{\beta}\right) = \left(\frac{2}{\beta}\right)^{2i} \beta^i. \tag{3.3.31}$$

Hence, for half-integer values of m this cannot be invariant and for integer genus $\mathcal{P}_i^m$ is invariant iff $i = g$.

A more sophisticated idea for basis entries would be given by choosing $\vec{a} \in \mathbb{Z}^m, \vec{n} \in \mathbb{N}^m$ with $m \in \mathbb{N}$ and then considering $\prod_{i=1}^m \left[(a_i - \beta)\left(\frac{4}{a_i} - \beta\right)\right]^{n_i}$. Here, the common exponent n_i has been chosen as this is the only way the expression can reproduce itself upon transforming β. This choice, however, is only consistent with our requirement of the polynomials' coefficients being purely from $\mathbb{Z}$ if $a_i \in \{1, 2, 4\}$. Since the basis choice for $a_i = 2$ is invariant by itself, one can allow it to appear without the $\frac{4}{a_i}$ term and hence candidates for elements of an invariant basis are given by

$$\mathcal{P}^i_{a,b,c}(\beta) := \beta^a (2 - \beta)^b [(1 - \beta)(4 - \beta)]^c, \tag{3.3.32}$$

with $(a, b, c) \in \mathbb{N}_0^3$. They transform as

$$\mathcal{P}^i_{a,b,c}\left(\frac{4}{\beta}\right) = (-1)^b \left(\frac{2}{\beta}\right)^{2(a+c)+b} \mathcal{P}^i_{a,b,c}(\beta), \tag{3.3.33}$$

which can solve the invariance condition for both the integer and half-integer case. We have to decrease the number of degrees of freedom for this choice of basis since $(1 - \beta)(4 - \beta) = (2 - \beta)^2 - \beta$ which implies that as it is chosen now, there is an overcounting. This can be avoided by choosing b to be the minimal value compatible with the invariance condition. Thus, we set $b = 1$ for the case of half-integer genus, which in turn implies $a + c = g - \frac{1}{2}$ to be equivalent to invariance. For integer genus we could set $b = 0$, but for reasons that will be clear momentarily we set $b = 2$, implying $a + c = g - 1$ for the basis entry to be invariant.

Hence, for half-integer genus a basis for the invariant polynomials is given by the $g + \frac{1}{2}$ polynomials $\mathcal{P}^i_{k,1,g-\frac{1}{2}-k}(\beta)$ for $k \in [0, g - \frac{1}{2}]$. In the integer genus case, the g polynomials $\mathcal{P}^i_{k,2,g-1-k}(\beta)$ for $k \in [0, g-1]$ only form a basis together with the monomial $\mathcal{P}^m_g$ that was also possible here. In all cases, the polynomials are linearly independent since they are of mutually different degree.

This "invariant" basis is now precisely the proposed form of the general structure in theorem 3.3, which we now prove.

Proof of theorem 3.3. Let the assumptions be as in the statement of the theorem. Furthermore, let the $R^\beta_g(I)$ be the solutions to the loop equations for β. For $g = 0$ the statement we would like to prove is proven already as theorem 3.1. As in the proof of this, we will proceed also here by induction, the base case of which can be given by the case of $(g, n) = (0, 2)$ or all of the $g = 0$ contributions. For $n \in \mathbb{N}$ we thus assume that the statement of theorem 3.3 holds for all resolvents determining $R^\beta_g(x, x_1, \ldots, x_n)$, i.e. the genus g contribution to the $n + 1$ boundary resolvent, via eq. (2.2.51) and eq. (2.2.50). Hence, we have to perform the induction step by showing that this implies the general structure for $R^\beta_g(x, x_1, \ldots, x_n)$.

We will thus consider each line of eq. (2.2.50) separately (using the notation we introduced in eq. (3.3.21)) and show that the claimed general form is present in $F^\beta_g(x, x_1, \ldots, x_n)$. Due to the general form being split into the case of integer and half-integer genus we treat these as two cases for each line.

I For the case of integer g, $g - \frac{1}{2}$ is half-integer and combining factors one finds that

$$I = \frac{(2-\beta)^2}{\beta^{2g+n}} \sum_{i=1}^{g} \beta^{i-1}[(1-\beta)(4-\beta)]^{g-i}(-1)\partial_x \mathcal{R}^i_{g-\frac{1}{2}}(x, I), \tag{3.3.34}$$

reproducing the claimed structure.

For half-integer g, $g - \frac{1}{2}$ is integer. For this case, as for many of the following, it is convenient to use the rewriting

$$(2-\beta)^2 = (1-\beta)(4-\beta) + \beta. \tag{3.3.35}$$

By use of this and shifting the summation index, one finds

$$I = \frac{(2-\beta)}{\beta^{2g+n}}(-1)\partial_x \Bigg[\sum_{i=1}^{g+\frac{1}{2}} \mathcal{R}^i_{g-\frac{1}{2}}(x,I)\beta^{i-1}[(1-\beta)(4-\beta)]^{g+\frac{1}{2}-i} + \sum_{i=2}^{g+\frac{1}{2}} \mathcal{R}^{i-1}_{g-\frac{1}{2}}(x,I)\beta^{i-1}[(1-\beta)(4-\beta)]^{g+\frac{1}{2}-i} \Bigg], \tag{3.3.36}$$

where in the first line we set $\mathcal{R}^{g+\frac{1}{2}}_{g-\frac{1}{2}} = \mathcal{R}^0_{g-\frac{1}{2}}$ which is possible as $\mathcal{R}^{g+\frac{1}{2}}_{g-\frac{1}{2}}$ was not defined before. Taking the derivative inside the bracket and combining the two sums it is apparent that the claimed structure is reproduced.

II For the case of integer g, $g-1$ is integer as well and by combining factors and shifting the index one finds

$$II = \frac{1}{\beta^{2g+n}} \Bigg[\mathcal{R}^0_{g-1}(x,x,I)\beta^g + (2-\beta)^2 \sum_{i=2}^{g} \beta^{i-1}[(1-\beta)(4-\beta)]^{g-i} R^{i-1}_g(x,x,I) \Bigg], \tag{3.3.37}$$

reproducing the expected structure. For half-integer g, $g-1$ is half-integer and in a similar fashion as for the integer case one finds

$$II = \frac{(2-\beta)}{\beta^{2g+n}} \sum_{i=2}^{g+\frac{1}{2}} \beta^{i-1}[(1-\beta)(4-\beta)]^{g+\frac{1}{2}-i}\mathcal{R}^{i-1}_g(x,x,I), \tag{3.3.38}$$

being in correspondence with the expected structure for half-integer g.

III For the case of half-integer genus, there is the possibility of h being an integer, then $g-h$ is half-integer and of h being half-integer which then implies $g-h$ to be integer. For the determination of the structure in terms of β it suffices, however, to consider the case of half-integer h as for every integer h the case of h' (half-integer) such that $g-h' = h$ has already been considered. Keeping in mind the sum over $J \subseteq I$, for a fixed half-integer h one has to evaluate $R^\beta_h(x,J)R^\beta_{g-h}(x,I\backslash J) := \star$. To abbreviate the following discussion we will drop the arguments of the resolvents, as they are uniquely reconstructible by the lower index. For the evaluation of $\star$ is will be useful to recall the form of the Cauchy product for finite sums, given by

$$\sum_{i=1}^{m}\sum_{j=1}^{k} a_i b_j y^{i+j} = \sum_{l=2}^{m+k} y^l \sum_{n=\max(1,l-k)}^{\min(m,l)} a_n b_{l-n}. \tag{3.3.39}$$

Using this and eq. (3.3.35), one can evaluate the product of the respective general expressions to find

$$\begin{aligned}\star =&\frac{(2-\beta)}{\beta^{2g+n}}\sum_{i=1+g-h}^{g+\frac{1}{2}}\mathcal{R}_h^0\mathcal{R}_{g-h}^{i-g+h}\beta^{i-1}[(1-\beta)(4-\beta)]^{g+\frac{1}{2}-l}+\\&-\frac{(2-\beta)}{\beta^{2g+n}}\sum_{l=2}^{g+\frac{1}{2}}\beta^{l-1}[(1-\beta)(4-\beta)]^{g+\frac{1}{2}-l}\sum_{a=\max(1,l-g+h)}^{\min\left(g+\frac{1}{2},l\right)}\mathcal{R}_h^a\mathcal{R}_{g-h}^{l-a}+\\&+\frac{(2-\beta)}{\beta^{2g+n}}\sum_{l=1}^{g-\frac{1}{2}}\beta^{l-1}[(1-\beta)(4-\beta)]^{g+\frac{1}{2}-l}\sum_{a=\max(1,l+1-g+h)}^{\min\left(g+\frac{1}{2},l+1\right)}\mathcal{R}_h^a\mathcal{R}_{g-h}^{l+1-a},\end{aligned}\tag{3.3.40}$$

which is of the form claimed for half-integer g, showing the claim for this case. Coming now to the case of integer genus g, we have to consider two cases, as now this can be split into an integer h and consequently an integer $g-h$ or a half-integer h which also implies $g-h$ to be half-integer. Considering again the case of half-integer h first, one can evaluate

$$\star = \frac{(2-\beta)^2}{\beta^{2g+n}}\sum_{l=1}^{g}\beta^{l-1}[(1-\beta)(4-\beta)]^{g-l}\sum_{a=\max\left(1,l+\frac{1}{2}-g+h\right)}^{\min\left(h+\frac{1}{2},l+1\right)}\mathcal{R}_h^a\mathcal{R}_{g-h}^{l+1-a},\tag{3.3.41}$$

which is of the claimed form. Coming now to the case of integer h, the expansion yields

$$\begin{aligned}\star =&\frac{1}{\beta^{2g+n}}\mathcal{R}_h^0\mathcal{R}_{g-h}^0+\\&+(2-\beta)^2\sum_{i=1+g-h}^{g}\mathcal{R}_h^{i+h-g}\mathcal{R}_h^0\beta^{i-1}[(1-\beta)(4-\beta)]^{g-i}+\\&+(2-\beta)^2\sum_{i=1+h}^{g}\mathcal{R}_h^{j-h}\mathcal{R}_h^0\beta^{i-1}[(1-\beta)(4-\beta)]^{g-i}+\\&-(2-\beta)^2\sum_{l=2}^{g}\beta^{l-1}[(1-\beta)(4-\beta)]^{g-l}\sum_{a=\max(1,l-g+h)}^{\min(g,l)}\mathcal{R}_h^a\mathcal{R}_{g-h}^{l-a}+\\&+(2-\beta)^2\sum_{l=1}^{g-1}\beta^{l-1}[(1-\beta)(4-\beta)]^{g-l}\sum_{a=\max(1,l+1-g+h)}^{\min(g,l+1)}\mathcal{R}_h^a\mathcal{R}_{g-h}^{l+1-a}.\end{aligned}\tag{3.3.42}$$

This result is of the claimed form, showing the statement also for the case of integer g, concluding the consideration of contribution III.

IV Finally, this contribution is dealt with rather quickly. First, we note that the appearing resolvent $R_g^\beta(x, I\backslash\{x\})$ is of genus g. Consequently, as it is one for n boundaries it only lacks a factor of $\frac{1}{\beta}$ to acquire the expected form, which is provided by the terms in bracket of contribution IV after noting (cf. eq. (2.2.52))

$$R_0^\beta(x_1, x_2) = \frac{1}{\beta} R_0^1(x_1, x_2). \tag{3.3.43}$$

With the last argument we have thus shown that for the two cases of integer and half-integer genus g $F_g^\beta(x, I)$ indeed is of the form claimed for the respective case. As stated above, this structure is not modified upon computing the actual resolvent from this and thus also $R_g^\beta(x, I)$ is of the claimed form. This concludes the induction step. □

IMPLICATIONS OF THE GENERAL STRUCTURE. Having now proven the general structure of the topological expansion of resolvents in terms of the Dyson index, we can study its implications.

Perhaps most striking of all is the fact that, in the Wigner–Dyson classes, all but one term (in the half-integer genus) or two terms (in the integer genus) vanish. From this, it is apparent that the generalisation to arbitrary Dyson index is not a mere interpolation between the Wigner-Dyson classes but it produces genuinely non-Wigner-Dyson contributions which are not computable from only the Wigner-Dyson results. For JT gravity we will give in section 3.3.4 a different proof for the general structure from which we can see the geometric origin of the non-Wigner-Dyson contributions.

Regarding the geometric interpretation of the terms that do contribute in the Wigner-Dyson classes, we can already say a lot here. For half-integer genus, this is just one term ($i = g + \frac{1}{2}$) and it vanishes in the case of $\beta = 2$. This was expected, since in the orientable case there are no manifolds with half-integer genus and those should thus not contribute. Hence, on the gravitational side there are only unorientable contributions to begin with. Explicitly, since for $\beta = 1$ the prefactors can be worked out to give unity, it holds that

$$\underset{g \in \frac{\mathbb{N}}{2}}{\forall} \mathcal{R}_g^{g+\frac{1}{2}}(I) = R_g^1(I), \tag{3.3.44}$$

For integer genus, there are two terms. The term outside the sum is the only contribution in the case of $\beta = 2$ and hence, geometrically, it arises purely from orientable manifolds. Hence, it holds that

$$\underset{g \in \mathbb{N}_0}{\forall} \mathcal{R}_g^0(I) = 2^{g+|I|-1} R_g^2(I), \tag{3.3.45}$$

In fact, this can be thought of as the generalisation to higher genus of eq. (3.1.10), i.e. the translation of the orientable moduli space volumes to the unorientable setting for $g = 0$. For $g = 0$ there were, however, only orientable contributions which could be incorporated with different relative orientations of boundaries. In the general case, there are genuinely unorientable manifolds and hence a genuinely unorientable part of moduli space which is captured by the term in the sum for $i = g$. Consequently, it holds that

$$\underset{g\in\mathbb{N}_0}{\forall}\mathcal{R}_g^g(I) = R_g^1(I) - \mathcal{R}_g^0(I) = R_g^1(I) - 2^{g+|I|-1}R_g^2(I). \tag{3.3.46}$$

Note that this distinction was possible by using solely the dependence on the Dyson index. This was the reason why we chose this basis for the general structure, i.e. chose $b = 2$ instead of $b = 0$ for the invariant basis eq. (3.3.32). Had we not done so, the sum would not have vanished as $\beta \to 2$ and the orientable term would not have been split off so conveniently.

This concludes our discussion of the behaviour of resolvents in dependence on the Dyson index. Apart from the implications we just discussed, theorem 3.3 will be most useful in section 4.4, where we will be able to use it to infer a statement about an important part of the arbitrary β (microcanonical) SFF from the Wigner-Dyson results. Furthermore, it provides structure to the explicit results we obtain for the two gravitational theories we are interested in. In particular this is useful for topological gravity, as we will discuss momentarily when considering the explicit results for topological gravity in the unorientable Wigner-Dyson classes and beyond in the next section.

3.3.3 *Unorientable topological gravity*

This section is based mainly on [5] for general β *and [2] for* $\beta = 1$.

We will now compute explicitly moduli space volumes for unorientable topological gravity. In fact, we will directly work out the results for the case of general Dyson index. As we already discussed above, we do this by working out the topological expansion of multi-resolvents in the matrix model dual to topological gravity, i.e. determined by the spectral curve $y^{\text{Airy}} = \frac{z}{2}$ and using then eq. (3.1.9) to find the Airy WP volumes from them. We do this by using the method introduced in section 3.3.1.

We will hence start off the computation by giving some examples for low values of g and n, where we will already see the simplifications due to the knowledge of the structure of the results in terms of β (theorem 3.3), many of which continue to apply for general one-cut matrix models, in particular the matrix model dual to JT gravity. Since the case of $n = 2$ will be most relevant in chapter 4, we will then use the extension of Kontsevich diagrammatics to the setting of arbitrary Dyson index to prove the generalisation of theorem 2.2 to this setting. We will then continue our discussion by studying the generic form of the resolvents for an arbitrary number of boundaries and then conclude our study of unorientable topological gravity for now by presenting, as for the orientable case, all observables we are interested in for the case of one boundary.

COMPUTATION OF AIRY WP VOLUMES FOR ARBITRARY β. The easiest volumes to compute are, of course, those for $g = 0$ where we can infer all of them from the orientable results by applying theorem 3.1, in fact its corollary eq. (3.3.13). Explicitly put[12]

$$V^{\beta}_{0,n}(\vec{b}) = \left(\frac{2}{\beta}\right)^{n-1} V^{2}_{0,n}(\vec{b}), \tag{3.3.47}$$

with $\vec{b} \in \mathbb{R}_+$. Hence, for example since we already know $V^2_{0,3}(b_1, b_2, b_3) = 1$ from section 2.1 we can directly infer

$$V^{\beta}_{0,3}(b_1, b_2, b_3) = \left(\frac{2}{\beta}\right)^{3-1} V^{2}_{0,3}(b_1, b_2, b_3) = \frac{4}{\beta^2}. \tag{3.3.48}$$

This can also be seen by using eq. (3.3.7) and then working out $R^1_0(z_1, z_2, z_3)$ by using eq. (2.2.60) for $\beta = 1$. Assuming, as in section 3.3.1, $\mathrm{Re}\,(z_i) > 0$ and deforming the contour such that the poles in the left hemisphere are surrounded, this yields

$$\begin{aligned} R^1_0(z_1, z_2, z_3) &= \frac{1}{z_1}\left[\underset{z'=0}{\mathrm{Res}} + \underset{z'=-z_1}{\mathrm{Res}} + \underset{z'=-z_2}{\mathrm{Res}} + \underset{z'=-z_3}{\mathrm{Res}}\right] \frac{z'^2}{z'^2 - z_1^2} \frac{2}{y(z')} \times \\ &\quad \left[R^1_0(z', z_2) R^1_0(z', z_3) + \left(\frac{R^1_0(z', z_3)}{\left(z_2^2 - z'^2\right)^2} + \frac{R^1_0(z', z_2)}{\left(z_3^2 - z'^2\right)^2} \right) \right] \\ &= \frac{1}{z_1}\left[\underset{z'=0}{\mathrm{Res}} + \sum_{i=1}^{3} \underset{z'=-z_i}{\mathrm{Res}}\right] \frac{z_3^2 z'^2 - 4 z_2 z_3 z'^2 + z_2^2\left(z_3^2 + z'^2\right) + z'^4}{z_2 z_3 z' \left(z'^2 - z_1^2\right)\left(z'^2 - z_2^2\right)^2 \left(z'^2 - z_3^2\right)^2} \\ &= -\frac{1}{2 z_1^3 z_2^3 z_3^3}. \end{aligned} \tag{3.3.49}$$

Of course, one finds the same result when using the deformation of the contour surrounding the poles at z_i in the right hemisphere. Note that the

12 Note that this is valid also for the full WP volumes. In the following, we shall consistently denote results valid only for topological gravity by the superscript "Airy" while leaving this away for results valid for JT gravity as well. In fact, whenever talking about contributions to resolvents, the absence of a superscript indicates the validity of the result for all one-cut matrix models.

function can be continued analytically to the whole complex plane in all variables and that is has exactly one pole of order 3 at $z = 0$ in all three variables. Hence, one can use the Bromwich contour to perform the inverse Laplace transform in eq. (3.1.9) to find the WP volume for $\beta = 1$

$$V^1_{0,3}(b_1, b_2, b_3) = 4, \tag{3.3.50}$$

which using eq. (3.3.7) yield precisely eq. (3.3.48). We will find that all the results we obtain are extendable to the whole complex plane after the computation and also the pole-structure is such that one can apply the Bromwich contour integral version of the inverse Laplace transform. We shall thus not repeat this statement for every example.

Coming to a more complicated example, let us consider the case of $n = 1, g = \frac{1}{2}$, i.e. the topological gravity equivalent of the crosscap.

Example 3.1 ($(g, n) = (\frac{1}{2}, 1)$)**.** *By applying theorem 3.3 we infer*

$$R^\beta_{\frac{1}{2}}(z) = \frac{2-\beta}{\beta}\mathcal{R}^1_{\frac{1}{2}}(z) \overset{\text{eq. (3.3.44)}}{=} \frac{2-\beta}{\beta}R^1_{\frac{1}{2}}(z), \tag{3.3.51}$$

which is actually valid for any one-cut matrix model. Hence, we only have to work out the case of $\beta = 1$*. For this, we compute (using eq.* (2.2.59)*)*

$$F^1_{\frac{1}{2}}(z) = \frac{1}{2z}\partial_z R^1_0(-z^2) \to \frac{1}{2z}\partial_z y(z) \overset{\text{for } y^{Airy}}{=} \frac{1}{4z}, \tag{3.3.52}$$

where $R_0(z)$ *was replaced by the spectral curve by adding analytic terms which vanish under the contour integration (cf. eq.* (2.2.38)*). Thus, by eq.* (2.2.56)

$$\begin{aligned} R^{1,\text{Airy}}_{\frac{1}{2}}(z) &= \frac{1}{2\pi i z}\int_{i\mathbb{R}+\epsilon} \frac{z'^2\,\mathrm{d}z'}{z'^2 - z^2}\frac{2}{z'}\frac{1}{4z'} = \frac{1}{2}\frac{1}{2\pi i z}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}z'}{(z'-z)(z+z')} \\ &= \frac{1}{2z}\operatorname*{Res}_{z'=-z}\frac{1}{(z'-z)(z'+z)} = -\frac{1}{2z}\operatorname*{Res}_{z'=z}\frac{1}{(z'-z)(z'+z)} \\ &= -\frac{1}{4z^2}, \end{aligned} \tag{3.3.53}$$

where in the evaluation of the contour integral, the assumption $\text{Re}\{z\} > 0$ *was used.*

To compare with the unorientable JT gravity result for the crosscap, eq. (3.2.3), we compute the corresponding unorientable Airy WP volume, i.e. [13]

$$V^{1,\text{Airy}}_{\frac{1}{2}}(b) = \frac{1}{b}\mathcal{L}^{-1}\left[-2zR^{\text{Airy}}_{\frac{1}{2}}(z), b\right] = \frac{1}{2b}\mathcal{L}^{-1}\left[\frac{1}{z}, b\right] = \frac{1}{2b}. \tag{3.3.54}$$

13 Note that in contrast to [2] we use the convention, used already in our discussion of unorientable JT gravity, of glueing the crosscap with the standard glueing measure $b\mathrm{d}b$. The difference is a factor of b that is here put into the volume while in [2] it was put into the glueing measure.

Using $\lim_{b\to\infty}\coth\left(\frac{b}{4}\right) = 1$, wee see that the topological gravity result agrees with the JT gravity one when going to the large b limit for the JT result. Furthermore, there is no divergence upon integrating the topological gravity crosscap against the glueing measure bdb and hence the Airy WP volumes are non-divergent, as we claimed above.

We are most interested in the WP volumes for $n = 2$ and hence we consider now this volume for $g = \frac{1}{2}$.

Example 3.2 ($(g, n) = (\frac{1}{2}, 2)$). *Again, we apply theorem 3.3 to find*

$$R^{\beta}_{\frac{1}{2}}(z_1, z_2) = \frac{2-\beta}{\beta^2}\mathcal{R}^1_{\frac{1}{2}}(z_1, z_2) \overset{eq.\ (3.3.44)}{=} \frac{2-\beta}{\beta^2}R^1_{\frac{1}{2}}(z_1, z_2). \tag{3.3.55}$$

Hence, we again have to work out only the result for $\beta = 1$. To do this, one finds from eq. (2.2.59)

$$F^1_{\frac{1}{2}}(z', z_2) = \frac{1}{2z'}\partial_{z'}R^1_0(z', z_2) + 2R^1_{\frac{1}{2}}(z')\left[R^1_0(z', z_2) + \frac{1}{\left(z'^2 - z_2^2\right)^2}\right], \tag{3.3.56}$$

and thus by eq. (2.2.56)

$$\begin{aligned} R^{1,\text{Airy}}_{\frac{1}{2}}(z_1, z_2) &= \frac{1}{z_1}\left[\underset{z'=0}{\text{Res}} + \underset{z'=-z_1}{\text{Res}} + \underset{z'=-z_2}{\text{Res}}\right](-1)\frac{z_2\left(z_2\left(z_2+z'\right) - 2z'^2\right) + 2z'^3}{z_2 z'^2\left(z'^2 - z_1^2\right)\left(z'-z_2\right)^2\left(z_2+z'\right)^3} \\ &= -\frac{1}{z_1}\left[\underset{z'=z_1}{\text{Res}} + \underset{z'=z_2}{\text{Res}}\right](-1)\frac{z_2\left(z_2\left(z_2+z'\right) - 2z'^2\right) + 2z'^3}{z_2 z'^2\left(z'^2 - z_1^2\right)\left(z'-z_2\right)^2\left(z_2+z'\right)^3} \\ &= \frac{z_1^4 + 3z_2z_1^3 + 3z_2^2z_1^2 + 3z_2^3z_1 + z_2^4}{2z_1^4z_2^4\left(z_1+z_2\right)^3}. \end{aligned} \tag{3.3.57}$$

Having found the contribution to the resolvent, we again use eq. (3.1.9) to find the Airy WP volume as

$$V^{1,\text{Airy}}_{\frac{1}{2}}(b_1, b_2) = \theta(b_1 - b_2)b_1 + \theta(b_2 - b_1)b_2 = \max\left(b_1, b_2\right), \tag{3.3.58}$$

which agrees with the result derived in [140] as the large length limit of eq. (3.2.9). Note that this first example of an unorientable Airy WP volume for $n = 2$ we present already shows a clear deviation from the orientable volumes since it is not a polynomial but now contains Heaviside functions. We will find this to be the generic behaviour of the unorientable Airy WP volumes. Furthermore, it is important to note the appearance of a power of $(z_1 + z_2)$ in the denominator of the resolvent. This gives rise to the Heaviside functions and hence can be thought of as the hallmark of the genuinely unorientable part of a contribution to the correlation function of resolvents.

Before studying this, however, we work out a final explicit example. This is due to all examples so far having reduced to the computation of the $\beta = 1$ case. This is not possible in the case we study now.

Example 3.3 ($(g, n) = (1, 1)$). *Applying theorem 3.3 in this case yields for the first time two terms, i.e.*

$$R_1^\beta(z) = \frac{1}{\beta}\mathcal{R}_1^0(z) + \frac{(2-\beta)^2}{\beta^2}\mathcal{R}_1^1(z). \tag{3.3.59}$$

Hence, we will directly work out both terms by using the general β *recursion directly. Doing this, one finds from eq. (2.2.59)*

$$\begin{aligned} F_1^\beta(z') &= \frac{2-\beta}{\beta}\frac{\partial_{z'} R^\beta_{\frac{1}{2}}(z')}{2z'} + R^\beta_{0,2}(z', z') + R^\beta_{\frac{1}{2}}(z')R^\beta_{\frac{1}{2}}(z') \\ &= \left[\left(\frac{2-\beta}{\beta}\right)^2\left(\frac{1}{4}+\frac{1}{16}\right) + \frac{1}{8\beta}\right]\frac{1}{z'^4} \\ &= \left[\frac{(2-\beta)^2}{\beta^2}\frac{5}{4} + \frac{1}{2\beta}\right]\frac{1}{4z'^4}. \end{aligned} \tag{3.3.60}$$

Using now eq. (2.2.56), one can compute the resolvent as

$$\begin{aligned} R_1^{\text{Airy}}(z) &= \left[\frac{(2-\beta)^2}{\beta^2}\frac{5}{4} + \frac{1}{2\beta}\right]\frac{1}{2\pi i z}\int_{i\mathbb{R}+\epsilon}\frac{z'^2\,\mathrm{d}z'}{z'^2 - z^2}\frac{2}{z'}\frac{1}{4z'^4} \\ &= -\left[\frac{(2-\beta)^2}{\beta^2}\frac{5}{2} + \frac{1}{\beta}\right]\frac{1}{4z}\operatorname*{Res}_{z'=z}\frac{\mathrm{d}z'}{(z'-z)(z'+z)z'^3} \\ &= -\left[\frac{(2-\beta)^2}{\beta^2}\frac{5}{2} + \frac{1}{\beta}\right]\frac{1}{8z^5}, \end{aligned} \tag{3.3.61}$$

from which one can read off the functions $\mathcal{R}_1^{0,\text{Airy}}(z)$ *and* $\mathcal{R}_1^{1,\text{Airy}}(z)$ *directly.*

As for the previous examples, one can compute the corresponding Airy WP volume via eq. (3.1.9). Using $\mathcal{L}^{-1}\left[\frac{1}{z^k}, z, b\right] = \frac{b^{k-1}}{(k-1)!}$ one finds

$$V_1^{\beta,\text{Airy}}(b) = \frac{(2-\beta)^2}{\beta^2}\frac{5b^2}{48} + \frac{1}{\beta}\frac{b^2}{24}, \tag{3.3.62}$$

which reproduces the result we had found in the orientable case in eq. (2.1.35) upon setting $\beta = 2$ and that found first in [76] for the unorientable case ($\beta = 1$).

Note that we could also have obtained this result by computing first the cases of $\beta = 2$ and $\beta = 1$ and then using eqs. (3.3.45) and (3.3.46). However, this ceases to be the case if there are more than two terms in the structure emerging from theorem 3.3, which is generically the case. Hence, we work out the contribution to higher genus and number of boundaries using the procedure from example 3.3. We will naturally not give all of the computations for this, but would rather present one of these contributions and refer to [2] for the results for $\beta = 1$ and to [5] for those for general Dyson index. Furthermore, unarguably more useful than extending this thesis by many pages containing mostly numbers, a Mathematica notebook containing all resolvents and volumes we have computed is included in the supplementary material to this thesis. These go up to $g = 4$, $n = 1$ and hence contain the contributions to the $n = 2$ correlation function of resolvents up to $g = \frac{7}{2}$. We would like to remark that for all volumes we did not present as explicitly worked out examples here, we were the first to compute them and there are, to the author's knowledge, so far no other approaches to determine them than the one followed in [2, 5] and here.

Before leaving briefly the explicit results and proving again structural statements, now concerning the generic form of the $\mathcal{R}^i_g(I)$, we would like to give a final example of a resolvent in β topological gravity. Performing the same steps as in example 3.3 for $(g,n) = (1,2)$ and computing first all resolvents needed for the computation of $F^{\beta}_1(z', z_2)$ one finds

$$\begin{aligned} R^{\beta,\text{Airy}}_1(z_1,z_2) =& \frac{1}{16\beta^3 z_1^7 z_2^7 (z_1+z_2)^4}\Big[5(\beta(5\beta-18)+20)(z_1^8+z_2^8) \\ &+20(\beta(5\beta-18)+20)\left(z_2 z_1^7+z_1 z_2^7\right)+ \\ &+33(\beta(5\beta-18)+20)\left(z_2^2 z_1^6+z_1^2 z_2^6\right) \\ &+16(\beta(11\beta-40)+44)\left(z_2^3 z_1^5+z_1^3 z_2^5\right)+ \\ &+8(\beta(23\beta-85)+92) z_2^4 z_1^4\Big], \end{aligned} \tag{3.3.63}$$

which is not at all in the form of theorem 3.3. This makes the dependence on β rather spurious and hence it is advisable to bring this to the ordered form we had proven to exist above. In order to do this automatically, we decompose the β dependent coefficient of each monomial in the z_i in some polynomial basis and then transform to the invariant basis of the polynomial space discussed above, before the proof of theorem 3.3. Details for this are given in [5][app. F]. For the example at hand, this results in

$$R_1(z_1, z_2) = (2-\beta)^2 \frac{f(z_1, z_2; 25, 100, 165, 176, 184)}{16\beta^3 z_1^7 z_2^7 (z_1+z_2)^4} + \frac{5z_1^4 + 3z_2^2 z_1^2 + 5z_2^4}{8\beta^2 z_1^7 z_2^7}, \tag{3.3.64}$$

where we used the notation

$$\begin{aligned} f(z_1, z_2; a_0, \dots, a_n) :=& a_0 z_1^{2n} + a_1 z_1^{2n-1} z_2 + \dots + a_{n-1} z_1^{n+1} z_2^{n-1} + \\ &+ a_n z_1^n z_2^n + a_{n-1} z_1^{n-1} z_2^{n+1} \dots, \end{aligned} \tag{3.3.65}$$

to simplify the presentation. From this form of the results, one can now easily read off the result for $\beta = 2$ which one then finds to agree with the Airy part of the resolvent deriving from the orientable WP volume computed in eq. (2.3.29).

AIRY WP VOLUMES FOR $n = 2$ AND ARBITRARY β. As motivated in section 2.4, we are interested in the canonical SFF of topological gravity. Hence, the unorientable Airy WP volumes for $n = 2$ are of highest interest. As we already mentioned in example 3.2, those volumes have a very specific structure generalising the structure for the orientable volumes, given in theorem 2.2.

Before studying this, to fix notation we briefly give the structure of the (Airy) WP volumes with respect to the Dyson index, as induced by theorem 3.3 as

$$V_{g,n}^{\beta}(\vec{b}) = \frac{1}{\beta^{2g+n-1}} \begin{cases} \mathcal{V}_{g,n}^0(\vec{b})\beta^g + (2-\beta)^2 \sum_{i=1}^{g} \mathcal{V}_{g,n}^i(\vec{b})\beta^{i-1}((1-\beta)(4-\beta))^{g-i} & g \in \mathbb{N}_0, \\ (2-\beta) \sum_{i=1}^{g+\frac{1}{2}} \mathcal{V}_{g,n}^i(\vec{b})\beta^{i-1}((1-\beta)(4-\beta))^{g+\frac{1}{2}-i} & g \in \frac{\mathbb{N}}{2}. \end{cases} \tag{3.3.66}$$

Deriving from the corresponding statement for resolvents, eq. (3.3.45), $\mathcal{V}_{g,n}^0(\vec{b})$ is fully determined by the orientable result. Hence, we can uplift the structure of the orientable WP volumes (theorem 2.2) to the setting of general Dyson index by inferring (for stable surfaces)

$$\mathcal{V}_{g,n}^0(\vec{b}) = 2^{g+n-1} \sum_{\vec{a} \in \mathbb{N}_0^n}^{\|\vec{a}\|_1 \leq 3g-3+n} C_{g,n}^{\vec{a}} \prod_{i=1}^{n} b_i^{2a_i}, \tag{3.3.67}$$

with $C_{g,n}^{\vec{a}} \in \mathbb{Q}_{>0} \cdot \pi^{2k(g,n)-2\|\vec{a}\|_1}$, totally symmetric in the elements of $\vec{a}$, as in theorem 2.2. The topological gravity part of this part of the β WP volume is, as discussed in section 2.1, given by the leading order behaviour, i.e. $\|\vec{a}\|_1 = 3g - 3 + n$.

For the other contributions to the $V_{g,n}^{\beta}(\vec{b})$ in eq. (3.3.66), even those for topological gravity, one can not infer anything from the orientable volumes.

Hence, the only handle we have on them so far is via explicit computation. For later use, we are nevertheless interested in obtaining a general structure for them in the case of $n = 2$. Using the explicit examples, we can formulate a conjecture for the structure of the $\mathcal{V}^{i,\text{Airy}}_{g,2}(b_1, b_2)$ as

Theorem 3.4 (Structure of unorientable Airy WP volumes). *For a given value of the genus g it holds that*

$$\mathcal{V}^{i,\text{Airy}}_{g,2}(b_1, b_2) = \mathcal{V}^{i,>}_g(b_1, b_2)\theta(b_1 - b_2) + \mathcal{V}^{i,>}_g(b_2, b_1)\theta(b_2 - b_1). \quad (3.3.68)$$

with

$$\mathcal{V}^{i,>}_g(b_1, b_2) = \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{N}_0 \\ \alpha_1+\alpha_2=6g-2}} C^{g,i}_{\alpha_1,\alpha_2} b_1^{\alpha_1} b_2^{\alpha_2}, \quad (3.3.69)$$

where the $C^{g,i}_{\alpha_1,\alpha_2} \in \mathbb{Q}_{\geq 0}$ are not necessarily symmetric under $\alpha_1 \leftrightarrow \alpha_2$.

This seems surprising at first, since quite obviously the resolvents and hence any other n-point functions in matrix models are symmetric upon interchanging variables and hence also the Airy WP volumes should have this property. In fact, they do have this property as one can quite clearly see from eq. (3.3.68). Only when considering the case of e.g. $b_1 > b_2$ the volumes are no longer symmetric polynomials, which is however not in conflict with their symmetry.

To give one of the examples motivating this conjecture, let us consider the Airy WP volume for $n = 2$ at $g = 1$. We already gave the corresponding resolvent in eq. (3.3.64). Computing the Airy WP volume from this, we find

$$V^{>}_1(b_1, b_2) = \frac{\left(5b_1^4 + 10b_2^2b_1^2 + 8b_2^3b_1 + b_2^4\right)(2-\beta)^2}{96\beta^3} + \frac{\left(b_1^2 + b_2^2\right)^2}{48\beta^2}, \quad (3.3.70)$$

where we directly wrote both $V^{i,>}_1(b_1, b_2)$ and collected them with the β dependent prefactors to $V^{>}_1(b_1, b_2)$. Apparently, this is of the claimed form, like all the examples we published in [5][App. B] and that can be found in the supplementary material.

However, to prove this form, we use a different method to study unorientable moduli space volumes, which is given by Kontsevich diagrammatics that we discussed for the orientable setting in section 2.1. The relevant relation there, relating the Laplace transform of the orientable Airy WP volumes to a sum of diagrams is given by eq. (2.1.51), which we recall here for convenience as

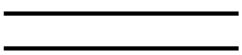
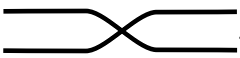
$$\mathcal{L}\left[V^{\text{Airy}}_{g,n}(\vec{b}); I\right] = \sum_{\gamma \in \Gamma_{g,n}} \frac{2^{2g-2+n}}{|\text{Aut}(\gamma)|} \prod_{k=1}^{6g-6+3n} \frac{1}{z_{l(k)} + z_{r(k)}}, \quad (3.3.71)$$

with $\vec{b} \in \mathbb{R}_+^n$ and the usual convention for I. Here, the sum is over trivalent ribbon graphs having $4g + 2n - 4$ vertices or, equivalently, $6g + 3n - 6$ propagators. We recall that the number of boundaries of said graph arose by giving labels to the two lines of the propagators and identifying those that are connected via a vertex. This determined then the number of boundaries as the number of occurring labels. This diagrammatic method can now be generalised to the unorientable setting by adding the possibility to have a twisted propagator, as we already discussed for general matrix model diagrammatics in section 2.2. This was first proposed in [140]. In fact, one can actually generalise it to the setting of arbitrary Dyson index by allowing a β dependent propagator of the form depicted in eq. (2.2.30) as we proposed in [5]. This amounts, effectively, to adding an additional factor of $\frac{2-\beta}{\beta}$ for every twisted propagator. This then leads to the generalisation of the Kontsevich diagrammatics to arbitrary Dyson index as

$$\mathcal{L}\left[V_{g,n}^{\beta,\text{Airy}}(\vec{b}); I\right] = \sum_{\gamma \in \tilde{\Gamma}_{g,n}} \frac{2^{2g-2+n}}{|\text{Aut}(\gamma)|} \prod_{k=1}^{6g-6+3n} \left(\frac{2-\beta}{\beta}\right)^{\#\text{twists}} \frac{1}{z_{l(k)} + z_{r(k)}}, \tag{3.3.72}$$

where we denote the set of Ribbon graphs with genus g and n boundaries with the propagators allowed to be twisted as $\tilde{\Gamma}_{g,n}$.

Before going into an example for this, let us briefly comment on the ambiguity in generalising the propagator eq. (2.2.30), valid for all three Wigner-Dyson classes, to arbitrary Dyson index. This ambiguity arises in a way similar to the ambiguity discussed in the definition of the arbitrary β matrix model using the eigenvalue integral eq. (3.2.11). It follows from the fact that the function of the Dyson index multiplying the twisted propagator could be any function $b(\beta) : \mathbb{R}_+ \to \mathbb{R}_+$ coinciding with $\frac{2-\beta}{\beta}$ at the Wigner-Dyson values of β, and additionally there could be a function $a(\beta) : \mathbb{R}_+ \to \mathbb{R}_+$ multiplying the untwisted propagator that assumes the value 1 at those values. Hence, the most general way to write an extension to arbitrary Dyson index of eq. (2.2.30) would be

$a(\beta)$  $+\, b(\beta)$  . (3.3.73)

To reduce this to an ambiguity dependent on only one function, we can factor out $a(\beta)$ and absorb it into a redefinition of N due to it appearing now in front of every diagram with a power $\#\text{Edges} = 6g - 6 + 3n = -3\chi_{g,n}$. The ambiguity remains in the prefactor of the twisted propagator $\frac{b(\beta)}{a(\beta)}$. It is convenient to rewrite this function as

$$\frac{2 - h(\beta)}{h(\beta)}, \tag{3.3.74}$$

which exemplifies that one can perform all computations with choosing h as the identity and then reintroduce an arbitrary function h afterwards. This is precisely our procedure of dealing with the ambiguity when using the loop equations to compute correlation functions. One might expect, intuitively, that the choice of $h = \mathrm{id}_{\mathbb{R}_+}$ coincides with the choice $f = \mathrm{id}_{\mathbb{R}_+}$ in the eigenvalue integral. This is supported by noting that the twisting of the propagator in a graph is equivalent to inserting a crosscap in the surface dual to the graph. For $f = \mathrm{id}_{\mathbb{R}_+}$ the dependence of this on the Dyson index is given by multiplying the result for $\beta = 1$ by $\frac{2-\beta}{\beta}$ (example 3.2), coinciding precisely with the factor of the twisted propagator for $h = \mathrm{id}_{\mathbb{R}_+}$. In this way, one can also justify not adding any β dependence to the prefactor of the non-twisted propagator, since it corresponds to inserting nothing additional in the dual surface. Hence, the two ambiguities are inter-related. To get coinciding results from the diagrammatics with the loop equations one has to choose $h = f$. As above, we continue by setting $h = \mathrm{id}_{\mathbb{R}_+}$, keeping the ambiguity in mind.

Having settled the discussion of the ambiguity, we consider an example illustrating the usefulness of the diagrammatics.

Example 3.4 ($V^{\mathrm{Airy},\beta}_{\frac{1}{2},2}$ by diagrammatics)**.** *For* $(g, n) = (\frac{1}{2}, 2)$ *one needs to consider diagrams with* $4g + 2n - 4 = 2$ *vertices and possibly twisted propagators. Drawing those diagrams, one finds the ones with* $n = 2$ *boundaries as those depicted in fig. 3.3.1. We now note that the graphs in the first and third column of fig. 3.3.1 have*

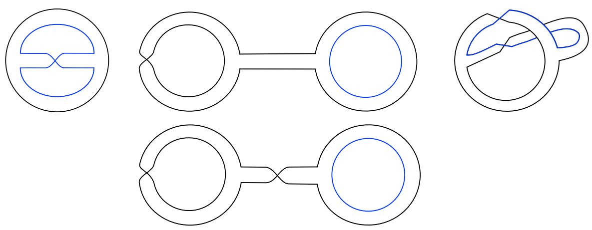


Figure 3.3.1: Diagrams contributing to $V_{\frac{1}{2},2}$, i.e. the elements of $\tilde{\Gamma}_{\frac{1}{2},2}$. Figure adapted from [5].

each two propagators with differently labelled sides (we shall refer to such as (12)-propagators) and one with only one label (a (11) or (22)-propagator). For those in the second column, we observe two (11) and one (12) propagator. Furthermore, in each of the pairs one diagram has one and the other diagram two twisted propagators. Hence, up to factors of $\frac{2-\beta}{\beta}$ *and possibly different orders of the respective automorphism groups, the diagrams in the two groups have the same contribution to the Laplace transform of the Airy WP volume. To work out this contribution, we have to find the orders of the automorphism groups. Conveniently, the order of the automorphism group does not depend on the presence or absence of twisting and we can hence use those orders we worked out in examples 2.1 and 2.2, where the same types of diagrams appeared without twisting. Thus, the diagram in the first column has an* *automorphism group of order 6 while those of the second column have one of order 2. The order of the automorphism group of the diagram in the third column is again 6. Furthermore, one has to include the number of possibilities to insert a twist into the "untwisted" diagram to find the present one. For the first column, there are 3 possibilities, for the second 2 each and again 3 for the third column. Hence, summing also over the two possible labellings of each diagram we find*[14]

$$\begin{aligned}\mathcal{L}\left[V^{\text{Airy},\beta}_{\frac{1}{2},2};(z_1,z_2)\right] =&2\left(\frac{2-\beta}{\beta}+\frac{(2-\beta)^2}{\beta^2}\right)\times \\ &\left[\frac{3}{6}\frac{1}{2z_1}\frac{1}{(z_1+z_2)^2}+(1\leftrightarrow 2)+\frac{2}{2}\frac{1}{4z_1^2(z_1+z_2)}+(1\leftrightarrow 2)\right] \\ =&\frac{2-\beta}{\beta^2}\frac{z_1^2+z_2z_1+z_2^2}{z_1^2z_2^2\,(z_1+z_2)}.\end{aligned} \tag{3.3.75}$$

Performing the inverse Laplace transform, we arrive at the result found from the loop equations by adding the β *dependence to eq.* (3.3.58).

From this example, it is already obvious that the computation of Airy WP volumes via the diagrammatic formulation is quite more tedious than the computation using the loop equations. This is the main reason, why we chose this way to compute explicit results. The diagrammatic formulation, however, has the distinct advantage of providing directly a result for the Laplace transform of the Airy WP volumes in dependence on the coefficients deriving from the order of the automorphism group, the number of twists in the diagram, etc. . Though it is in principle possible to study this for an arbitrary number of boundaries, we restrict here to the case of $n = 2$, as this is the most important case for us. For this case, one can classify the occurring diagrams by the number of (11) and (22) propagators which we shall denote as n and m in the following . Since this determines also the number of (12) propagators via the knowledge of the total number of propagators, we find

$$\mathcal{L}\left[V^{\text{Airy},\beta}_{g,2};(z_1,z_2)\right] = \sum_{n,m} C^{k(g)}_{n,m}(\beta)\frac{1}{z_1^n z_2^m (z_1+z_2)^{k(g)-n-m}}, \tag{3.3.76}$$

with $k(g) = \#\text{Edges} = 6g$ and $C^k_{n,m}(\beta)$ a function dependent on β. Due to theorem 3.3, we can transform the $C^k_{n,m}(\beta)$ to the invariant basis. In this basis, the diagrammatics induce the following form for the functions

$$\mathcal{L}\left[\mathcal{V}^{\text{Airy},i}_{g,2};(z_1,z_2)\right] = \sum_{n,m} C^{k(g),i}_{n,m}\frac{1}{z_1^n z_2^m (z_1+z_2)^{k(g)-n-m}}, \tag{3.3.77}$$

14 For reasons of compactness, we often leave out the argument of a function that is Laplace transformed since the number of arguments is clear from the number of arguments after the transform.

where the $\mathcal{C}^{k(g),i}_{n,m}$ arise from the decomposition of the $C^k_{n,m}(\beta)$ in the invariant basis. From eq. (3.3.72) we know that the prefactor of a diagram's dependence on β and the z_i is a non-negative rational number, and hence we can deduce $\mathcal{C}^{k(g),i}_{n,m} \in \mathbb{Q}_{\geq 0}$.

Using this, we can now nearly prove theorem 3.4. It is convenient, to compute first the contribution of a generic diagram to the Airy WP volume. For this, we define the contribution of the diagram (without the prefactors) with k propagators of which n are of type (11) and m are of type (22) as

$$\mathcal{D}^k_{n,m}(b_1, b_2) := \int_{\mathcal{C}_1} \frac{\mathrm{d}z_1}{2\pi i} \int_{\mathcal{C}_2} \frac{\mathrm{d}z_2}{2\pi i} \frac{e^{b_1 z_1} e^{b_2 z_2}}{z_1^n z_2^m (z_1 + z_2)^{k-n-m}}, \tag{3.3.78}$$

where the contours are chosen such that they go from $-i\infty$ to $i\infty$ on the right side of all poles of the integrand. From this definition, one can directly infer

$$\mathcal{D}^k_{m,n}(b_1, b_2) = \mathcal{D}^k_{n,m}(b_2, b_1). \tag{3.3.79}$$

Furthermore, it is useful to note that one can restrict in this computation to the cases where n and m do not vanish mutually. This is due to the fact that one cannot construct a ribbon diagram with 3-valent vertices and two boundaries that has only type (12) propagators. A quick way to see this, is to consider one 3-valent vertex which in such a diagram should only have incoming type (12) propagators. For two legs of the propagator this works fine, but the third one necessarily is of type (11) or (22), which is a contradiction.

Restricting to this setting, one can perform the contour integrals explicitly, worked out in [2][app. C.2], to find

$$\mathcal{D}^k_{n,m}(b_1, b_2) = \mathcal{A}^k_{n,m}(b_1, b_2) + \theta(b_1 - b_2)\mathcal{B}^k_{n,m}(b_1, b_2), \tag{3.3.80}$$

where $\mathcal{A}^k_{n,m}$ and $\mathcal{B}^k_{n,m}$ are both polynomials in b_1 and b_2 of combined order $k-2$ and rational coefficients[15]. Using this, we can write down the proof.

Proof of theorem 3.4. Fixing a value for the genus g, the generalised Kontsevich diagrammatics yield eq. (3.3.77). Hence, inverting the Laplace transform, one finds

$$\mathcal{V}^{\text{Airy},i}_{g,2}(b_1, b_2) = \sum_{n,m} \mathcal{C}^{k(g),i}_{n,m} \mathcal{D}^{k(g)}_{n,m}(b_1, b_2). \tag{3.3.81}$$

15 We have obtained explicit results for this, to be found in [2][app. C.2], but since they are not relevant in the following, we do not report them here.

Now we use the fact that for the diagram of type (n, m) the diagram of type (m, n) is just the other possibility to label the diagram and hence $\mathcal{C}^{k(g),i}_{n,m} = \mathcal{C}^{k(g),i}_{m,n}$. Combined with eq. (3.3.79), this yields

$$\begin{aligned}\mathcal{V}^{\text{Airy},i}_{g,2}(b_1, b_2) = &\sum_{n<m} \mathcal{C}^{k(g),i}_{n,m} \left[\mathcal{D}^{k(g)}_{n,m}(b_1, b_2) + \mathcal{D}^{k(g)}_{n,m}(b_2, b_1)\right] \\ &+ \sum_{n=1}^{k(g)} \mathcal{C}^{k(g),i}_{n,n} \mathcal{D}^{k(g)}_{n,n}(b_1, b_2).\end{aligned} \tag{3.3.82}$$

Here, the first line is apparently symmetric in $b_1 \leftrightarrow b_2$ while the second line is symmetric due to eq. (3.3.79) for $m = n$. Using now the explicit form of $\mathcal{D}^{k(g)}_{n,m}(b_1, b_2)$, one can insert the identity $\theta(b_1 - b_2) + \theta(b_2 - b_1) = 1$ after $\mathcal{A}^k_{n,m}(b_1, b_2)$ to write the summands in the first line in the claimed form. For the second line, it is easiest to rewrite $\mathcal{D}^{k(g)}_{n,n}(b_1, b_2) = \frac{1}{2}\left[\mathcal{D}^{k(g)}_{n,n}(b_1, b_2) + \mathcal{D}^{k(g)}_{n,n}(b_2, b_1)\right]$, using the symmetry in $b_1 \leftrightarrow b_2$. Having done so, one can proceed analogous to the first line. Using now that the combined order of $\mathcal{A}^k_{n,m}(b_1, b_2)$ as well as $\mathcal{B}^k_{n,m}(b_1, b_2)$ is given by $k(g) - 2 = 6g - 2$, this shows the claimed structure after noting that the $\mathcal{C}^{k(g),i}_{n,m}$ and all the polynomial coefficients are rational. □

As it will be apparent in section 4.3.2, theorem 3.4 is one of the, if not the, most important results about unorientable topological gravity if one wishes to study the implications of RMT universality on this theory.

The volumes for a higher number of boundaries than $n = 2$ are, of course, also computable. Their structure is, however, not as pleasant as that for the case of $n = 2$ and also not relevant for the discussion of chapter 4. The tediousness arises from the fact that e.g. for $n = 3$ Heaviside functions of more complicated arguments than in the case of $n = 2$ arise. We give an illustrative example for a contribution to a $n = 3$ Airy WP volume in [5], clarifying this point. While we will hence not study further the structure of the volumes for $n > 2$, we can study the resolvents, whose structure we will find in the next section to be tractable.

GENERIC FORM OF THE $\mathcal{R}^i_g$. After having discussed the general form of the β Airy WP volumes, we briefly present how this translates to the corresponding contributions to the topological expansion of multi-resolvents.

For this, we again begin with the orientable part of the contributions. For those, one can directly infer from eq. (3.3.67) via eq. (3.1.9) that it holds for stable values of (g, n), i.e. such that $\chi(n, g) > 0$, that

$$\begin{aligned} \mathcal{R}_g^0(I) &= 2^{g+n-1} \sum_{\vec{\alpha} \in \mathbb{N}_0^n}^{\|\vec{\alpha}\|_1 \leq 3g-3+n} C_{\vec{\alpha}}^g \prod_{i=1}^n \frac{(2\alpha_i + 1)!}{(-2) z^{2\alpha_i + 3}} \\ &= \sum_{k=0}^{3g-3+n} \prod_{j=1}^n \frac{1}{z_j^{2k+3}} \sum_{\vec{\alpha} \in \mathbb{N}_0^n}^{\|\vec{\alpha}\|_1 = k} C_{\vec{\alpha}}^g (-1)^n (2)^{g-1} \prod_{i=1}^n (2\alpha_i + 1)! z^{2k - 2\alpha_i} \\ &:= \sum_{k=0}^{3g-3+n} \frac{P_{g,n}^{0,k}(I)}{\prod_{i=1}^n (z_i)^{6g+2n-3}}, \end{aligned} \tag{3.3.83}$$

with the combined order of the polynomial $P_{g,n}^{0,k}$ given by $2(n-1)k$. The topological gravity part of this arises from the leading order contribution of the volumes, i.e. the $k = 3g - 3 + n = k(g)$ term. Hence, it holds that

$$\mathcal{R}_g^{0,\text{Airy}}(I) = \frac{P_{g,n}^{0,k(g)}(I)}{\prod_{i=1}^n (z_i)^{6g+2n-3}}, \tag{3.3.84}$$

with the order of the polynomial given by $2(n-1)(3g-3+n)$.

For the "unorientable" contributions, i.e. those vanishing for $\beta = 2$, we observed above that, in the case of $n = 2$, there are additional factors of $z_1 + z_2$ in the denominator of the resolvents. The appearance of these factors is apparent from the Kontsevich diagrammatics which can, of course, also be used to compute resolvents[16]. We pursue this way of studying the general form of resolvents in [2] but present another way here. For this, we translate the structure of the unorientable Airy WP volumes (theorem 3.4) to resolvents in an equivalent way as done above for $\mathcal{R}_g^{0,\text{Airy}}(I)$. This yields

$$\mathcal{R}_g^i(z_1, z_2) = \frac{P_{g,2}^i(z_1, z_2)}{(z_1 z_2)^{6g+1} (z_1 + z_2)^{2g+2}}, \tag{3.3.85}$$

with symmetric polynomials $P_g^i(z_1, z_2)$ of combined degree $8g$. This structure, unlike that for the WP volumes, can easily be generalised to more than n boundaries.

The main ingredient for this generalisation is to include all possible linearly independent sums of two arguments in the denominator. The presence of those is motivated by such contributions appearing generically in the

16 The relevant relation is given by $\prod_{i=1}^n (-2z_i) R_g^\beta(z_1, \ldots, z_n)$ $= (-1)^n \prod_{i=1}^n \frac{\partial}{\partial z_i} \mathcal{L}\Big[V_{g,n}^\beta(b_1, \ldots, b_n), (z_1, \ldots, z_n)\Big]$.

Kontsevich diagrams or by inspection of the explicit results. Furthermore, due to the symmetry requirement on the resolvents (translating directly to their constituent β independent parts) all of them have to be included with the same power. Furthermore, the structure of the $\mathcal{R}_g^i(I)$ should be the generic structure of the full $R_g^\beta(I)$ and hence the structure of the $\mathcal{R}_g^{0,\text{Airy}}(I)$ should arise in the case of the polynomial cancelling the factors of sums of arguments characteristic for the unorientable contributions. Hence, the generalised form should contain a dependence on the "single" arguments as the expression for $\mathcal{R}_g^{0,\text{Airy}}(I)$. This motivates us to conjecture the generalisation to be given by

$$\mathcal{R}_g^{i,\text{Airy}}(I) = \frac{P_{g,n}^i(I)}{\prod_{j=1}^n (z_j)^{6g+2n-3} \prod_{j<k}^n (z_j+z_k)^{(2g+2)}}, \tag{3.3.86}$$

where $P_{g,n}^i$ is again a polynomial. Here, the power of the sums of arguments is assumed to be the same for all values of n. This is an observation we make from the explicit results and we do not pursue a proof of this here. The order of the $P_{g,n}^i$ we can, however, infer by noting again that this should be the structure for the dependence on the arguments collected in I of the whole contribution to the resolvent, and hence parts of it should reduce to $\mathcal{R}_g^{0,\text{Airy}}(I)$. Namely, this implies the order of the $P_{g,n}^i$ to be given by the sum of orders of $P_{g,n}^{0,k(g)}(I)$, given by $2(n-1)(3g-3+n)$ and the product of linearly independent sums in the denominator. There are $\frac{n}{2}(n-1)$ distinct unordered pairs, and thus distinct linearly independent sums of two arguments, to be chosen from the n arguments. Hence

$$\begin{aligned}\text{comb. order}(P_{g,n}^i) &= 2(n-1)(3g-3+n) + \frac{n}{2}(n-1)(2g+2)\\ &= (n-1)[3(n-2)+g(n+6)],\end{aligned} \tag{3.3.87}$$

which is in agreement with all the resolvents we have computed.

ALL OBSERVABLES FOR $n=1$ IN β TOPOLOGICAL GRAVITY. So far, we have considered only resolvents and WP volumes as observables in unorientable topological gravity. Since we want to give a complete picture of this theory, we would like to finish the discussion of topological gravity in the arbitrary β setting in the same way as our discussion of the orientable case, by giving results for all correlation functions we are interested in for the case of $n=1$.

Starting again with the resolvents, we note that there is no change to the orientable case, except of course the inclusion of half-integer genus contributions. This can be understood easiest by employing Kontsevich diagrammatics, where any $n=1$ diagram just produces a dependence on z as $\frac{C(\beta)}{z^{\#\text{Edges}}}$ with $C(\beta)$ some polynomial in β and $\#\text{Edges} = 6g-3$ if one considers the contribution at genus g. In fact, for a structural discussion it suffices to study the case of just a constant $C(\beta)$ since we know the dependence on β from theorem 3.3. Computing the whole Laplace transformed β Airy WP volume then amounts to counting all the multiplicities, orders of automorphism groups etc. and summing all of this up to a prefactor of the known z dependence. From this, one can directly compute the corresponding resolvent (cf. footnote 16) to find (including, for convenience, a sign)

$$R_g^{\beta,\text{Airy}}(z) = -\frac{C_g^{\beta,R}}{z^{6g-1}}, \tag{3.3.88}$$

where we denoted the β dependent sum of prefactors of diagrams by $C_g^{\beta,R}$, which obeys a decomposition induced by theorem 3.3. Of course, we can read them off from the explicit results computed via the loop equations. The functional dependence on z now coincides with the result found above in the orientable setting in eq. (2.3.36), implying our initial statement[17].

Employing the relations of the resolvent with the other observables discussed in section 2.3, we can now directly compute the coefficient functions of their topological expansion from eq. (3.3.88). For the volumes, where the "coefficient functions" are indeed the interesting observables, this yields (for $g > 0$), via eq. (2.3.39)

$$V_{g,1}^{\beta,\text{Airy}}(b) = \frac{2C_g^{\beta,R}}{(6g-3)!} b^{6g-4}, \tag{3.3.89}$$

coinciding with the orientable result. For the partition function, we find via eq. (2.3.38)

$$Z_g^{\beta,\text{Airy}}(\beta) = C_g^{\beta,R} \beta^{3\left(g-\frac{1}{2}\right)} \begin{cases} \frac{1}{\left[3\left(g-\frac{1}{2}\right)\right]!} & \text{for} \quad g \in \frac{\mathbb{N}}{2}, \\ \frac{2^{6g-2}(3g-1)!}{\sqrt{\pi}(6g-2)!} & \text{for} \quad g \in \mathbb{N}, \end{cases} \tag{3.3.90}$$

which could, of course, also be found by integrating the Airy WP volumes. For the density of states, we use eq. (2.3.40). For integer genus we have already worked out the result above (eq. (2.3.46)), which does not change due to the unaltered form of the resolvent, except of course by replacing C_g^R by $C_g^{\beta,R}$. For the half-integer genus case, we need another identity of hyperfunctions than the one we applied for the integer genus case above. This is given by

$$\delta^{(n)}(x) = \left[\frac{n!}{2\pi i(-z)^{n+1}}\right], \tag{3.3.91}$$

17 One could have found this also directly as an application of eq. (3.3.86). However, since the direct derivation from the diagrammatics is rather short, we chose this (independent) way to derive this result.

discussed in example C.2. Using this, we can deduce

$$\underset{g\in\frac{\mathbb{N}}{2}}{\forall}\rho_g^{\beta,\text{Airy}}(E)=\frac{C_g^{\beta,R}}{\left[3\left(g-\frac{1}{2}\right)\right]!}\delta^{(3(g-\frac{1}{2}))}(E), \tag{3.3.92}$$

which means that this contribution surprisingly has "support" only at $E=0$. This, like the appearance of the θ functions for the $n=2$ case, exemplifies the higher level of complexity of correlation functions appearing in the unorientable setting as compared to the orientable one. As an easy sanity check, one can use this result to find e.g. the half-integer genus contribution to the partition function as

$$\begin{aligned}Z_g^{\beta,\text{Airy}}(\beta)&=\int_{-\infty}^{\infty}\mathrm{d}E\,e^{-\beta E}\rho_g^{\beta,\text{Airy}}(E)=\frac{C_g^{\beta,R}}{\left[3\left(g-\frac{1}{2}\right)\right]!}\int_{-\infty}^{\infty}\mathrm{d}E\,e^{-\beta E}\delta^{(3(g-\frac{1}{2}))}(E)\\&=\frac{C_g^{\beta,R}}{\left[3\left(g-\frac{1}{2}\right)\right]!}\beta^{3\left(g-\frac{1}{2}\right)},\end{aligned} \tag{3.3.93}$$

which coincides with the result reported above.

In the orientable case, we could now use these results to compute intersection numbers and derive then a result for these from a non-perturbative result for the partition function deriving from the KdV-hierarchy. While the extension of this reasoning to the unorientable setting is an interesting future route of research (cf. section 3.4), it goes beyond the scope of this thesis and we will hence leave it for future work.

This concludes our considerations of unorientable topological gravity, being the gravitational theory of highest interest for this thesis. We will now continue by studying full unorientable JT gravity by using the methods developed in studying topological gravity.

3.3.4 *Unorientable JT gravity*

This discussion is largely based on [3] and [5].

This section will now deal with the computation of the "full" unorientable WP volumes, i.e. with unorientable JT gravity. As mentioned and explained already, we will not go into as much detail as for topological gravity when discussing this computation. This means that our main goal is to present the results we found for the unorientable WP volumes to the extent that the method used to compute them can be understood and the results needed for the discussion of the next chapter are provided. Furthermore, as an extension of our published work, we find the generalisations of the found results to arbitrary Dyson index.

This section will hence be threefold: First, we give a short explanation of the idea and the technique used to find the unorientable WP volumes in [3] for $\beta = 1$ and give the found results. Second, we provide the geometric proof of the dependence of β WP volumes on the Dyson index advertised above and discuss the geometric interpretation of the arising structure. Third, using the firmly established β dependence, we uplift the results for $\beta = 1$ to those for arbitrary β and comment on how to do this in general.

HOW TO COMPUTE UNORIENTABLE WP VOLUMES. As we mentioned above, our approach to unorientable JT gravity is first working out matrix model results for the spectral curve $y^{(p)}(z)$, defined in eq. (3.2.10) as

$$y^{(p)}(z) = \frac{(-1)^p}{4\pi} T_{2p+1}\left(\frac{2\pi z}{2p+1}\right), \tag{3.3.94}$$

where $p \in \mathbb{N}_0$. This is useful to compute quantities in JT gravity, since one can use the property of Chebychev polynomials, $T_{2p+1}(x) = (-1)^p \sin\left[(2p+1) \arcsin x\right]$, and then take the limit of large p to find [141]

$$\lim_{p\to\infty} y^{(p)}(z) = y^{\mathrm{JT}}(z). \tag{3.3.95}$$

We choose this specific matrix model to approach JT gravity, since it has the property that for every value of p the matrix model is dual to the so-called $(2, 2p+1)$-minimal string. This is another well-defined theory of 2d gravity, arising from coupling a $(2, 2p+1)$ minimal CFT with Liouville gravity and is reviewed e.g. in [142]. We will not use any specific features of this theory and hence won't go into details. As it will turn out momentarily, for finite values of p the crosscap divergence of the moduli space volumes computed from the matrix model quantities is regularised. Hence, the matrix model defined by the spectral curve $y^{(p)}$ has a dual geometric interpretation and regularises the moduli space volumes, making it the ideal candidate to implement the regularisation of moduli space volumes in a matrix model setting[18]. To our knowledge, this was first noted in [69]. Another convenient property of $y^{(p)}$ is the reduction to topological gravity for $p = 0$, as it can be seen directly from

$$y^{(0)}(z) = \frac{1}{4\pi} T_1(2\pi z) = \frac{z}{2} = y^{\mathrm{Airy}}(z). \tag{3.3.96}$$

Hence, by tuning p we can interpolate between topological gravity, which we have studied in detail in the previous section, and JT gravity for all values of the Dyson index. Notationally, we shall indicate quantities deriving from the $(2, 2p+1)$ minimal string with $\beta = 1$ by a superscript (p). Resolvents that

18 Note that the ϵ regularisation we used above was genuinely geometrical and without an obvious way how to translate it to a matrix model computation.

arise when taking the limit $p \to \infty$ of $(2, 2p+1)$ minimal string quantities we shall denote by the superscript $(p), \mathrm{JT}$, while for the WP volumes we will have the superscript $\beta, (p)$, since we understand them as arising only in the JT gravity limit.

In principle, we could study arbitrary β JT gravity directly without any additional complications compared to treating the case of $\beta = 1$ which we considered in [3]. For the presentation here, it is however more convenient to treat first the case of $\beta = 1$, closely following [3], which we will then see to be directly extendable to the arbitrary β setting by using theorem 3.3.

Staying hence with $\beta = 1$, we make the first step to computing unorientable WP volumes by considering the $(g, n) = (\frac{1}{2}, 1)$ contribution for the $(2, 2p+1)$ minimal string and the moduli space volume of the regularised crosscap deriving from it. Due to the aim of this section of not giving a complete picture of the relevant computations, but rather one sufficient to understand them, we only give the main results without intermediate steps. All of them can be found in [3].

Example 3.5 (The p regularised crosscap). *We can essentially follow the reasoning of example 3.1, with the only difference of having to put in $y^{(p)}(z)$ as the spectral curve. For the computation of the contour integral determining $R^{(p)}_{\frac{1}{2}}(z)$, this requires the knowledge of the spectral curve's zeroes. For a given value of $p \in \mathbb{N}_0$ there are $2p+1$ zeroes, located at $x_k = \frac{2p+1}{2\pi} \sin\left[\frac{\pi k}{2p+1}\right]$ for $k \in \{-p, -p+1, \dots p\}$. Using this, analogous to the computation above, only with more residua to compute, one finds*

$$R^{(p)}_{\frac{1}{2}}(z) = \frac{-1}{4z^2} - \sum_{k=1}^{p} \frac{1}{2z(z+x_k)}, \tag{3.3.97}$$

valid actually for all values of p. We are most interested in JT gravity and hence consider the limit $p \to \infty$. This results in [19]

$$R^{(p),JT}_{\frac{1}{2}}(z) = \frac{-1}{4z^2} + \frac{1}{z} H_{2z} - \frac{1}{z}\log(p) + \mathcal{O}\left(p^{-1}\right), \tag{3.3.98}$$

with H_x denoting the (analytically continued) harmonic number. In this computation it was convenient to do a rescaling of p with the "new" p given by e^γ times the "old" p with γ the Euler-Mascheroni constant. We shall refer to the "new" p just by p in the following.

19 By taking this limit, we have, unsurprisingly, lost access to the $p = 0$ limit. Hence, the divergences as $p \to 0$ which is observed in the following result and those following from it are an artefact of the inaccessibility of this limit, not an indicator of divergences in unorientable topological gravity.

The appearance of $\log(p)$ *in this result is already quite reminiscent of the structure of the* ϵ*-regularised unorientable WP volumes eq.* (3.2.4)*. Computing the actual WP volume corresponding to eq.* (3.3.98) *via eq.* (3.1.9) *one finds* [20]

$$V^{1,(p)}_{\frac{1}{2},1}(b) = \frac{1}{2b} + 2\log(p)\frac{\delta(b)}{b} - \frac{1}{b}\sum_{k=1}^{\infty}\left(\frac{2}{k}\delta(b) - e^{-\frac{kb}{2}}\right) + \mathcal{O}\left(p^{-1}\right), \tag{3.3.99}$$

where one can perform the easy consistency check of the topological gravity result being produced as the leading-order polynomial contribution of the non-divergent part by comparing with eq. (3.3.54)*.*

This volume now has precisely the structure in terms of divergences as $p \to \infty$ *as the* ϵ *regularisation of the volumes (eq.* (3.2.4)*) when* $\epsilon \to 0$*. However, the explicit form of the* ϵ*-regularised crosscap eq.* (3.2.3) *is actually the one exception to this form. Nevertheless, the p-regularisation yields results consistent with the* ϵ*-regularisation as one can see for example by following the computation for the exception of the recursion for* $(g,n) = (\frac{1}{2},2)$*. One can do this, by replacing the* ϵ*-regularised crosscap in eq.* (3.2.9) *by the p-regularised one to find*

$$V^{1,(p)}_{\frac{1}{2},2}(b_1,b_2) = 2\log\left(\frac{\cosh\left(\frac{b_1}{2}\right) + \cosh\left(\frac{b_2}{2}\right)}{2}\right) + 4\log(2p) + \mathcal{O}\left(p^{-1}\right), \tag{3.3.100}$$

consistent with the result in ϵ *regularisation upon identifying* $p = \frac{2}{\epsilon}$*.*

Having now found the p-regularised crosscap, the matrix model recursion can now actually be iterated with the JT gravity spectral curve since, as it was noted above, the divergences of the moduli space volumes are purely due to the divergent crosscap and hence it is sufficient to regularise this. Furthermore it is important to note that for the gravitational computation we are most interested in the moduli space volumes and not the resolvents. While for the topological gravity computation this was not problematic and one could compute first the resolvents and then from them the moduli space volumes, for JT gravity it is more economical to work out the volumes directly.

Hence, the steps to compute volumes of genus g and number of boundaries n are as follows:

1. Compute $F^1_g(z,I)$ (eq. (2.2.59)) for $|I| = n-1$, $y = y^{\mathrm{JT}}$ and replace all occurrences of $R^1_{\frac{1}{2}}(z)$ by $R^{(p),\mathrm{JT}}_{\frac{1}{2}}(z)$.

20 In writing down the result, the harmonic series appears. We write it down as a formal expression, knowing that it is indeed a divergent series which however in the subsequent iterations of the recursion, as in the example we give, always cancels.

2. Use eq. (2.2.56) to evaluate $R_g^{(p)}(z, I)$ as a sum of residues.

3. Use eq. (3.1.9) to write $V_{g,n}^{1,(p)}(\vec{b})$ as an n-fold (Bromwich) contour integral of $R_g^{(p)}(z, I)$.

4. Close the integration contours in the left half-plane and work out the result (again) as a sum of residues.

It is important to note that in the last step one can disregard many terms upon realising that the regularised volumes cannot be divergent as any length goes to 0, since this would add a divergence to the volume that is not regularised by regularising the crosscaps.

Following these steps, in [3] it was possible to work out results for the WP volumes at $(g, n) \in \{(1, 1), (\frac{1}{2}, 2), (1, 2)\}$, of which only the case $(g, n) = (\frac{1}{2}, 2)$ was known fully analytically before (eq. (3.3.100))[21]. We refer to all further details of the computations, following the explained route, to [3] and now present the results in the notation

$$V_{g,n}^{1,(p)}(b_1, \ldots, b_n) = \sum_{k=0}^{2g} \log(p)^k v_{g,n,k}^1(b_1, \ldots, b_n) + \mathcal{O}\left(p^{-1}\right). \tag{3.3.101}$$

For $(g, n) = (1, 1)$ we find

$$v_{1,1,0}^1(b) = \frac{7b^2}{48} + \frac{\pi^2}{4} + b \log\left(1 + e^{-\frac{b}{2}}\right) + 2\log\left(1 + e^{-\frac{b}{2}}\right)^2, \tag{3.3.102}$$

$$v_{1,1,1}^1(b) = \left(b + 4\log\left(1 + e^{-\frac{b}{2}}\right)\right), \tag{3.3.103}$$

$$v_{1,1,2}^1(b) = 2, \tag{3.3.104}$$

which agrees with the results found in [76] upon identifying $p = \frac{2}{\epsilon}$. Note that in this reference $v_{1,1,0}^1(b)$ was determined not analytically but numerically.

For $(g, n) = (\frac{1}{2}, 2)$ we find the result already found above in eq. (3.3.100) also using the loop equations.

21 Note that we already derived the results for the unorientable volumes ($\beta = 1$) at $g = 0$ as arising from the orientable results for these volumes in eq. (3.1.10).

For $(g,n) = (1,2)$, we find

$$v_{1,2,0}(b_1,b_2) = \frac{\tilde{v}^1_{1,2,0}(b_1,b_2)}{2} + \frac{\tilde{v}^1_{1,2,0}(b_2,b_1)}{2}, \tag{3.3.105}$$

$$\begin{aligned} v_{1,2,1}(b_1,b_2) = &\frac{1}{3}\left(2b_1^3 + 3b_2^2 b_1 + b_2^3 + 4\pi^2\left(3b_1+b_2\right) + 168\zeta(3)\right) + \\ &- 32\mathrm{Li}_3\left(-e^{-\frac{b_2}{2}}\right) - 32\mathrm{Li}_3\left(-e^{-\frac{b_1}{2}}\right) + 32\mathrm{Li}_3\left(e^{-\frac{b_1}{2}-\frac{b_2}{2}}\right) + 32\mathrm{Li}_3\left(e^{\frac{1}{2}(b_2-b_1)}\right) \\ &+ 8\left(b_1-b_2\right)\mathrm{Li}_2\left(e^{\frac{1}{2}(b_2-b_1)}\right) + 8\left(b_1+b_2\right)\mathrm{Li}_2\left(e^{-\frac{b_1}{2}-\frac{b_2}{2}}\right), \end{aligned} \tag{3.3.106}$$

$$v_{1,2,2}(b_1,b_2) = 2b_1^2 + 2b_2^2 + 8\pi^2, \tag{3.3.107}$$

with the n-th polylogarithm $\mathrm{Li}_n(z)$, the Riemann ζ function and

$$\begin{aligned} \tilde{v}^1_{1,2,0}(b_1,b_2) = &\frac{1}{12}b_1\left(b_2^3 + 4\pi^2 b_2 + 96\zeta(3)\right) + \frac{7b_1^4}{96} \\ &+ \left(\frac{7b_2^2}{48} + \frac{5\pi^2}{6}\right)b_1^2 + \frac{b_2^4}{32} + \frac{b_2^2\pi^2}{2} + \frac{3\pi^4}{2} \\ &- 8b_1\mathrm{Li}_3\left(-e^{-\frac{1}{2}b_2}\right) - 8b_2\mathrm{Li}_3\left(e^{-\frac{1}{2}b_1}\right) \\ &+ \left(2b_1^2 + 2b_2^2 + 2b_1b_2\right)\mathrm{Li}_2\left(e^{-\frac{b_1}{2}-\frac{b_2}{2}}\right) + 8b_1\mathrm{Li}_3\left(e^{\frac{1}{2}(b_2-b_1)}\right) \\ &+ 8\left(b_1+b_2\right)\left(\mathrm{Li}_3\left(e^{-\frac{b_1}{2}-\frac{b_2}{2}}\right) - \mathrm{Li}_{2,1}\left(e^{-\frac{b_1}{2}-\frac{b_2}{2}},1\right)\right) \\ &+ \left(2b_1^2 - 2b_2b_1\right)\mathrm{Li}_2\left(e^{\frac{1}{2}(b_2-b_1)}\right) \\ &+ 8\left(b_2-b_1\right)\mathrm{Li}_{2,1}\left(e^{\frac{1}{2}(b_2-b_1)},1\right) + 16\mathrm{Li}_4\left(e^{\frac{1}{2}(b_2-b_1)}\right) \\ &+ 64\mathrm{Li}_4\left(e^{-\frac{1}{2}(b_1+b_2)}\right) - 32\mathrm{Li}_{3,1}\left(e^{-\frac{1}{2}(b_1+b_2)},1\right) \\ &- 32\mathrm{Li}_{3,1}\left(e^{-\frac{1}{2}(b_1-b_2)},1\right) + 16\mathrm{Li}_{2,2}\left(e^{-\frac{1}{2}(b_1+b_2)},1\right) \\ &+ 32\mathrm{Li}_{2,2}\left(1,e^{-\frac{1}{2}(b_1+b_2)}\right) + 16\mathrm{Li}_{2,2}\left(1,e^{-\frac{1}{2}(b_1-b_2)}\right) \\ &- 16\mathrm{Li}_{2,2}\left(-e^{-\frac{1}{2}b_2},e^{-\frac{1}{2}(b_1-b_2)}\right) + 16\mathrm{Li}_{2,2}\left(e^{-\frac{1}{2}(b_1+b_2)},-e^{\frac{1}{2}b_2}\right) \\ &+ 16\mathrm{Li}_{2,2}\left(e^{-\frac{1}{2}(b_1+b_2)},-e^{-\frac{1}{2}b_2}\right) + 16\mathrm{Li}_{2,2}\left(-e^{-\frac{1}{2}b_2},e^{-\frac{1}{2}(b_1+b_2)}\right) \\ &- 16\mathrm{Li}_{3,1}\left(e^{-\frac{1}{2}(b_1+b_2)},-e^{-\frac{1}{2}b_2}\right) - 16\mathrm{Li}_{1,3}\left(-e^{-\frac{1}{2}b_2},e^{-\frac{1}{2}(b_1+b_2)}\right) \\ &+ 16\mathrm{Li}_{3,1}\left(e^{-\frac{1}{2}(b_1+b_2)},-e^{\frac{1}{2}b_2}\right) - 16\mathrm{Li}_{1,3}\left(-e^{-\frac{1}{2}b_2},e^{-\frac{1}{2}(b_1-b_2)}\right) \\ &+ 32\mathrm{Li}_{3,1}\left(-e^{-\frac{b_1}{2}},-e^{-\frac{b_2}{2}}\right) + 32\mathrm{Li}_{1,3}\left(-e^{-\frac{b_2}{2}},-e^{-\frac{b_1}{2}}\right), \end{aligned} \tag{3.3.108}$$

where $\mathrm{Li}_{s_1,\ldots,s_d}(z_1,\ldots,z_d)$ denotes the multiple polylogarithm, defined as

$$\mathrm{Li}_{s_1,\ldots,s_d}(z_1,\ldots,z_d) = \sum_{k_1>k_2>\cdots>k_d>0} \frac{z_1^{k_1}\ldots z_d^{k_d}}{k_1^{s_1}\ldots k_d^{s_d}}. \tag{3.3.109}$$

For all of these results we checked that the leading-order behaviour of the non-divergent part of the volumes matches with the results from topological gravity. This is apparent by direct inspection of the results for $n=1$, obtained by a brief computation for $(g,n) = (\frac{1}{2},2)$ and quite tedious to find for the case of $(g,n) = (\frac{1}{2},2)$, where we used the implementation of multiple polylogarithms in Mathematica (PolyLogTools) [144] to verify the agreement.

It is manifest that the unorientable WP volumes have a more complicated dependence on the lengths than their orientable incarnation, where the structure was given by theorem 2.2. Also, the dependence is more complicated than that for topological gravity, studied in the preceding section. From the considered examples, one could, however, already deduce that each contribution to the unorientable WP volumes has a polynomial part and a part being a sum of (multiple) polylogarithms of various arguments. We give a conjecture for both the polynomial and the "polylogarithmic" part of the volumes, but we will not use it in the following and hence refer the interested reader to [3] to find it. We would, however, for completeness like to point out that the polynomial part of all the results computed explicitly is a direct generalisation of the structure of the orientable volumes (theorem 2.2) by allowing also odd powers of the lengths (as we observed already for topological gravity) and the coefficients being no longer positive rational numbers multiplied by a power of π^2 but rather such numbers multiplying a value of the ζ function at some integer m. That the values of the ζ function are a direct generalisation of the powers of π^2 can be seen from $\zeta(2a) \propto \pi^{2a}$ for all integers a (cf. eq. (A.4)). Furthermore, as for the orientable case, the sum of the total order in the lengths and m is fixed to $6g-6+2n$ for the finite part, while for the part diverging as $\log(p)^k$ this sum is given by $6g-6+2n-k$.

Hence, for $\beta=1$ we understand the structure of unorientable WP volumes rather well, though up to now we have not proven it. What we can prove, however, is the dependence on the Dyson index of the WP volumes. We already gave a proof of the corresponding statement for resolvents in section 3.3.2, using matrix model techniques. For the specific case of JT gravity, we are however able to give a fully geometric proof of this, which will be the subject of the next paragraph, before we go on to study the extension of the results obtained for $\beta=1$ to the arbitrary β setting using only the knowledge of the general structure.

THE STRUCTURE OF WP VOLUMES IN TERMS OF β. This geometric derivation of the structure of the WP volumes, which was given in eq. (3.3.66), builds on the generalisation of the Mirzakhani-like recursion given in eqs. (3.2.12a) to (3.2.12c). We already discussed the geometric interpretation of the individual terms which we gave in fig. 3.2.1. For the reader's convenience, we repeat this figure here, in fig. 3.3.2, adding also the additional dependence on β for the different cuttings in the recursion for general β, as discussed above.

To iterate the recursion, the input of $V^{\beta}_{0,3}, V^{\beta}_{0,2}, V^{\beta}_{\frac{1}{2},1}$ is needed[22]. All of them, by employing the explicit computation of the respective $F^{\beta}_{g}(x, I)$ needed for the computation of the corresponding resolvent via eq. (2.2.56) (or eq. (3.3.66)), can be related to the result for $\beta = 1$. Explicitly put

$$V^{\beta}_{0,3}(b_1, b_2, b_3) = \frac{1}{\beta^2} V^{1}_{0,3}(b_1, b_2, b_3), \tag{3.3.110}$$

deriving from eq. (3.3.7),

$$V^{\beta}_{0,2}(b_1, b_2) = \frac{1}{\beta} V^{1}_{0,2}(b_1, b_2), \tag{3.3.111}$$

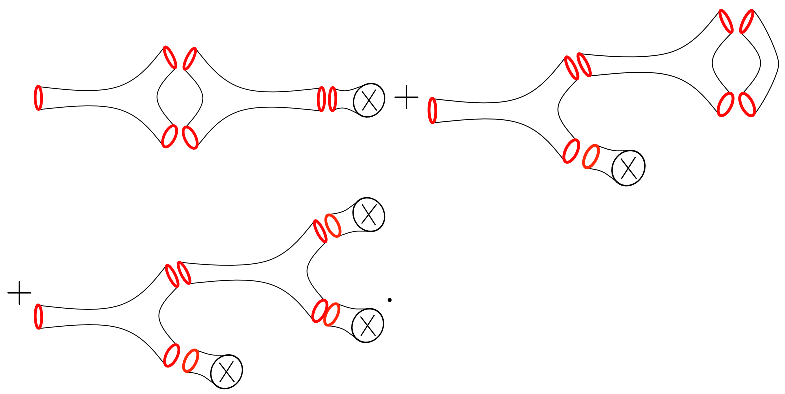
following actually also directly from the explicit form of $R^{\beta}_{0}(z_1, z_2)$ (e.g. eq. (3.3.5)). Finally, by eq. (3.3.51),

$$V^{\beta}_{\frac{1}{2},1}(b) = \frac{2-\beta}{\beta} V^{1}_{\frac{1}{2},1}(b). \tag{3.3.112}$$

Having now gathered the input to the recursion, we give an example illustrating the way we will use the recursion to prove the general structure of volumes geometrically.

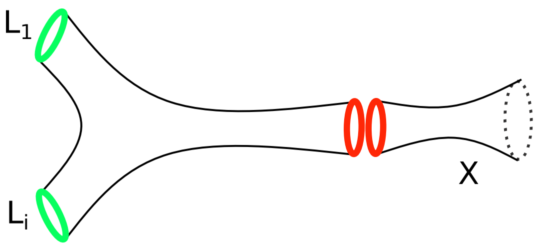
Example 3.6 (Decomposition of the surface with $(g, n) = (\frac{3}{2}, 1)$)**.** *We use the geometric interpretation of the recursion, given in fig. 3.3.2. This enables finding all occurring terms in the recursion by determining all decompositions of a surface into glueings of 3-holed spheres and crosscaps allowed by the splittings depicted in fig. 3.3.2. We find the following decompositions, decomposing also the manifolds glued to the 3-holed sphere, to build a surface of genus $\frac{3}{2}$ and one boundary in the Mirzakhani-like recursion:*

$$V^{\beta}_{\frac{3}{2},1} \cong$$

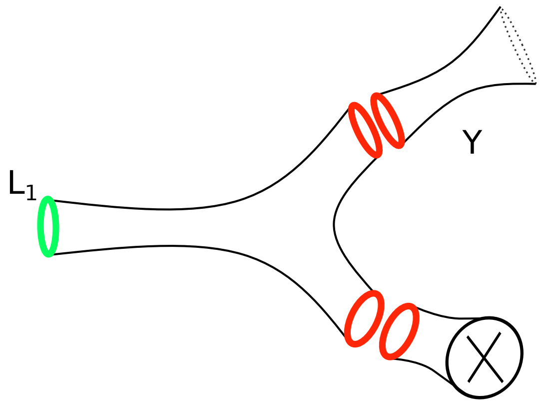


(3.3.113)

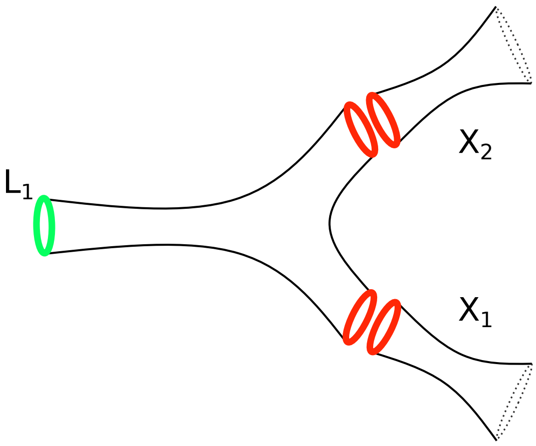
22 Note that when talking about the β dependence, the choice of regularisation does not matter since essentially one is dealing only with the prefactors of the occurring objects, which are independent of the regularisation. Hence, we do not add any indication of regularisation here and for the rest of this paragraph.



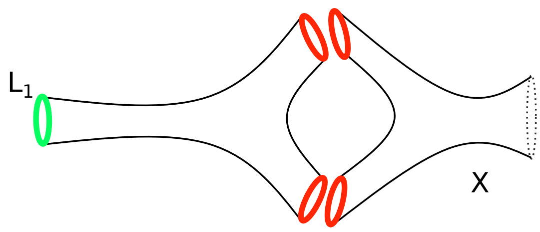
a) Glueing corresponding to eq. (3.2.12a). X has genus g with $n-1$ boundaries. There is an additional factor of $\frac{1}{\beta}$ arising from the recursion prescription.



b) Glueing corresponding to eq. (3.2.12c). Y has genus $g-\frac{1}{2}$ and n boundaries. The surface attached to the other boundary is a crosscap. There is an additional factor of $\frac{2-\beta}{\beta}$ arising from the crosscap.



c) Glueing corresponding to eq. (3.2.12b) (Disconnected part). X_1/X_2 have genera h_1/h_2 and n_1/n_2 boundaries with $h_1+h_2=g$ and $n_1+n_2=n-1$. There is no additional factor arising from the recursion prescription.



d) Glueing corresponding to eq. (3.2.12b) (Connected part). X has genus $g-1$ and $n+1$ boundaries. There is no additional factor arising from the recursion prescription.

Figure 3.3.2: Depiction of the different "glueings" i.e. the separation of a hyperbolic surface of genus g and n geodesic boundaries of lengths $L_1, \ldots, L_n$ into a 3-holed sphere and one or more hyperbolic surfaces. Eventually appearing additional dependence on β is indicated in the subcases. Note that further dependence on β arises from the volumes forming the input to the recursion. Adapted from [5].

For the individual terms, we are now interested only in the dependence on β. This can be found by putting in the factors arising from the rules for decomposition and *those from the input to the recursion. For the first term of the first line, there is one factor of* $\frac{1}{\beta}$ *from the second glueing (it appears whenever a 3-holed sphere is put with two legs to the left) and one factor of* $\frac{2-\beta}{\beta}$ *from the crosscap that is glued. Hence, the total dependence is given by* $\frac{2-\beta}{\beta^2}$. *For the second term of the first line, there is no additional factor from the glueings but the same total factor as for the first arises from the crosscap and the volume of genus* 0 *and 2 boundaries that is glued in. In the second line, we can read off the dependence to be given by the three glued crosscaps, i.e.* $\left(\frac{2-\beta}{\beta}\right)^3$.

The terms arising from the first line are thus of the general form already, while the second is not. However, we can use $(2-\beta)^2 = (1-\beta)(4-\beta)+\beta$ *to rewrite it as a sum of a term of the form of the first line and one of the form as in the contribution to the general structure apart from it.*

From this example it is now apparent that the general structure we have proven, in a sense chosen, above does not correspond to the decomposition of the volumes equivalent to that of the glueings of the Mirzakhani-like recursion. It is thus interesting to note that its distinct advantage, being the decomposition into Wigner-Dyson and non-Wigner-Dyson contributions, is consequently not a property immediately seen from the glueing construction inherent to the Mirzakhani-like recursion.

We note that this is not surprising, since the recursion can only produce factors of $\frac{(2-\beta)}{\beta}$ and $\frac{1}{\beta}$ and hence it is impossible to find the "invariant" structure given in eq. (3.3.66), deriving from theorem 3.3, without recombining terms. It is thus convenient to prove eq. (3.3.66) not directly, but to rather first derive another structure adapted for the recursion from it and prove this. To find this structure, we use $(1-\beta)(4-\beta) = (2-\beta)^2 - \beta$ to rewrite the invariant structure. For $g \in \mathbb{N}_0$ this yields, leaving the arguments implicit,

$$V^{\beta}_{g,n} = \sum_{m=0}^{g} \frac{1}{\beta^{g+n-m-1}} \left(\frac{2-\beta}{\beta}\right)^{2m} \bar{\mathcal{V}}^{m}_{g,n}, \tag{3.3.114}$$

with

$$\bar{\mathcal{V}}^{m}_{g,n} := (-1)^{m+g+1} \sum_{i=1}^{g-m+1} (-1)^i \binom{g-i}{m-1} \mathcal{V}^{i}_{g,n}, \tag{3.3.115}$$

for $m > 0$ and $\bar{\mathcal{V}}^{0}_{g,n} := \mathcal{V}^{0}_{g,n}$. For $g \in \frac{\mathbb{N}}{2}$ these considerations yield

$$V^{\beta}_{g,n} = \sum_{m=0}^{g-\frac{1}{2}} \frac{1}{\beta^{g-\frac{1}{2}+n-m-1}} \left(\frac{2-\beta}{\beta}\right)^{2m+1} \bar{\mathcal{V}}^{m}_{g,n}, \tag{3.3.116}$$

with

$$\bar{\mathcal{V}}^{m}_{g,n} := (-1)^{m+g+\frac{1}{2}} \sum_{i=1}^{g+\frac{1}{2}-m} (-1)^i \binom{g+\frac{1}{2}-i}{m} \mathcal{V}^{i}_{g,n}. \tag{3.3.117}$$

Hence, we can state the form of the dependence of the WP volumes on the Dyson index that we will then proceed to prove geometrically as

Theorem 3.5 (Structure of the β WP volumes). *For* $(g, n) \in \left(\mathbb{N}_0 \cup \frac{\mathbb{N}}{2}, \mathbb{N}\right)$ *the* β *WP volume is given by*

$$V^{\beta}_{g,n} = \begin{cases} \sum_{m=0}^{g} \frac{1}{\beta^{g+n-m-1}} \left(\frac{2-\beta}{\beta}\right)^{2m} \bar{\mathcal{V}}^{m}_{g,n} & \textit{for} \quad g \in \mathbb{N}_0, \\ \sum_{m=0}^{g-\frac{1}{2}} \frac{1}{\beta^{g-\frac{1}{2}+n-m-1}} \left(\frac{2-\beta}{\beta}\right)^{2m+1} \bar{\mathcal{V}}^{m}_{g,n} & \textit{for} \quad g \in \frac{\mathbb{N}}{2}, \end{cases} \tag{3.3.118}$$

with the $\bar{\mathcal{V}}^{m}_{g,n}$ *independent on* β.

It is important to remark that theorem 3.5 implies eq. (3.3.66) by inserting $(1-\beta)(4-\beta)+\beta = (2-\beta)^2$ and working out explicitly the inverse of the relation of the $\bar{\mathcal{V}}^{m}_{g,n}$ and the $\mathcal{V}^{m}_{g,n}$. Furthermore, it is illustrative to note that the β dependence in the summands in theorem 3.5 is such that one can associate $\bar{\mathcal{V}}^{m}_{g,n}$ for a given value of m with a contribution with $2m$ crosscaps for integer g and with one with $2m+1$ crosscaps for half-integer g.

Before we go into the geometric proof of theorem 3.5, we would like to remark that after the work leading to this proof was completed we became aware of [145], which actually provides the equivalent statement to theorem 3.5 for resolvents. However, they have a quite different proof and do not provide our main structural theorem 3.3, whose precise form is of highest relevance for most of this thesis. Consequently, our discussion of the preceding sections is not "just" a different proof for a known statement but rather an interesting statement on its own which happens to imply, after the rewriting we performed to find the statement of theorem 3.5, said known statement.

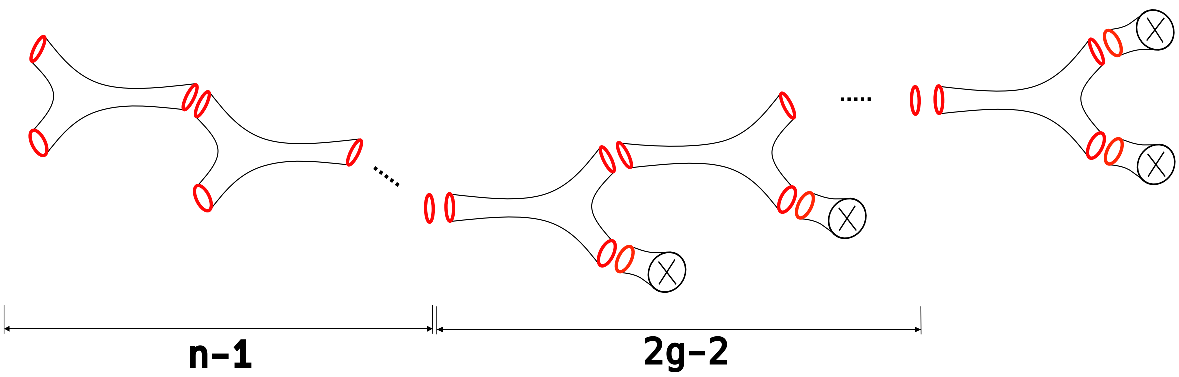
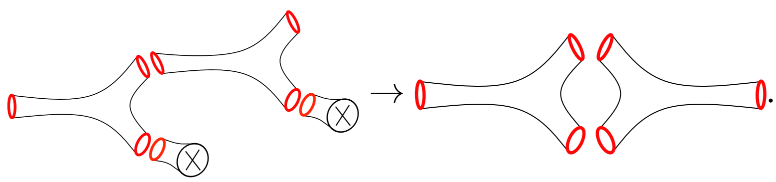


Figure 3.3.3: Possible decomposition of a surface of genus $g > \frac{1}{2}$ and $n > 0$ geodesic boundaries into 3-holed spheres and crosscaps with the maximal number of crosscaps possible. The numbers below the first and second part of this construction indicate how many 3-holed spheres/3-holed spheres with an attached crosscap are needed. Adapted from [5].

We proceed to work towards a geometric proof of theorem 3.5. To prove this structure, it is necessary to obtain all different decompositions of a surface of a given genus g and a number of boundaries n. Here, all decompositions are equivalent that yield the same dependence on β. Since the genus of the surface is fixed, we already know that the maximal number of crosscaps the surface can contain is given by $k_{\max} = 2g$. For this number of crosscaps, a decomposition of the surface is given in fig. 3.3.3. As it is evident from the figure, for this decomposition the surface is essentially cut into two parts. First, a part of genus 0 containing n external and one additional boundary to glue to the other part. Second, a part of genus g with one geodesic boundary that is glued to the first part along its additional boundary. As one can easily see, this implies the first part to be built from $n-1$ 3-holed spheres glued in the depicted way. The second part is built from $2g-2$ 3-holed spheres with one attached crosscap and one 3-holed sphere with two attached crosscaps, glued as depicted. The constructed surface is evidently of genus g with n geodesic boundaries. Note that for the cases of $g=0$ and $g=\frac{1}{2}$, that this decomposition does not capture, it is clear that the genus carrying part is either empty ($g=0$) with the boundary carrying part containing $n-2$ 3-holed spheres, or consists just of one crosscap ($g=\frac{1}{2}$).

To construct from this the decompositions with a smaller number of crosscaps, one can essentially trade two crosscaps for a hole, keeping the genus and number of boundaries constant. In the case of the "final" 3-holed sphere, being attached to two crosscaps, one can do this by taking out the two crosscaps and glueing the remaining two boundaries directly to one another. In the case of the other parts, one takes out two of the constituting blocks and replaces them with two 3-holed spheres as



(3.3.119)

To obtain a surface with k crosscaps, one obviously has to do a replacement of two crosscaps $\frac{1}{2}(k_{\max}-k)$ times, where we already found $k_{\max}$ to be given by $2g$.

We shall denote the illustrated decompositions of surfaces in the following as "ordered". Of course, these decompositions are not unique, but regarding the possible dependence on β they represent all possibilities, which we formulate as the following theorem.

Theorem 3.6 (Completeness of the surface decomposition)**.** *Any decomposition of a surface of genus g and $n>0$ geodesic boundaries into the parts prescribed by the arbitrary β Mirzakhani-like recursion, pictorially represented in fig. 3.3.2, can be "reshuffled" to yield either directly the decomposition in fig. 3.3.3, one deriving from* *it by adding holes as discussed above, or one of the special cases for* $g \in \left\{0, \frac{1}{2}\right\}$*. By "reshuffling" we mean moving the constituent parts of the surface around in a way that does not change the* β *dependence of the whole decomposition.*

Proof. Let a surface of genus g with n geodesic boundaries be decomposed into the allowed building blocks. Let us exclude unstable surfaces and the case $g = 0, n = 3$ at this point. We will now transform this to one of the ordered cases in two steps. First, we will move all parts containing external boundaries to the left, recreating the first part of fig. 3.3.3. Second, we will reshuffle the parts building up the remaining, genus carrying, part as to reproduce one of the ordered decompositions.

For the first part, if $n = 1$ we choose the external boundary as the starting point of the genus carrying part and have achieved our goal for the boundary part already since it's empty in this case.

If $n \geq 2$ there can be at most one 3-holed sphere containing two external boundaries to which something is glued as in case a) (Here and in the following, cases x) refer to the ones discussed in fig. 3.3.2). If it is there (case i), choose the boundary to which it is attached as the starting point for the genus carrying part and take out the 3-holed sphere. Then there are still $n_i = n - 2$ external boundaries left, that are not assigned as the starting point for the genus carrying part. If it is not there (case ii), there is one 3-holed sphere containing an external boundary to which something is glued in the way prescribe by a case that is not a). Choose this external boundary as the starting point for the genus carrying part. Now, there are still $n_{ii} = n - 1$ external boundaries that have not been assigned as the starting point for the genus carrying part.

The 3-holed spheres containing these boundaries can now either be glued as in

a) As X, then take out the 3-holed sphere. This multiplies the total dependence on β by β^2. If the 3-holed sphere it is glued to, is glued along two internal boundaries, remove it as well and glue the boundaries to which it was glued. This does not change the total dependence on β, hence there are two external boundaries less and the total dependence has been multiplied by β^2. If it is only glued along one internal boundary, "turn it around" i.e. make it such that is glued as X in case a). This multiplies the whole dependence by β but also cancels the removal of the initial 3-holed sphere in terms of the β dependence. Consequently, there is one external boundary less and the total dependence is multiplied by β.

b)/c) As Y or X_1/X_2. If it contains two external boundaries, take it out and also the 3-holed sphere it was attached to. Then glue the two other boundaries the latter one was attached to together. This multiplies the total dependence on β of the decomposition by β^2 (due to the β dependence of $V^{\beta}_{0,3}$) and there are two external boundaries less. If it contains one external boundary, take it out and glue the two boundaries it was attached to together. As it had to be oriented as the left side of case a), this multiplies the whole dependence by β and removes one external boundary.

d) As X, take it out and glue the boundaries together to which it was attached. This removes a total factor of $\frac{1}{\beta^2}$ and adds a factor of $\frac{1}{\beta}$, the total dependence is hence multiplied by β and there is one external boundary less.

If in one of those steps, the starting boundary for the genus carrying part is glued to something, this means identifying the glued boundary as the new starting point. If it is the only thing left after iterating the above steps, one necessarily is in the case of $g = 0$. In all cases where this is not the case, we have taken out a total of $n_i + 1 = n_{ii} = n - 1$ 3-holed spheres, with changing the total dependence on β by multiplying by a factor of β each time and have designated a starting boundary for the genus carrying part. This process has to terminate as n is finite. We can now assemble the 3-holed spheres in the way done in the first part of fig. 3.3.3 and attach them to the designated starting boundary, by which we have achieved the separation aimed at. Note that the β dependence of this is $\frac{1}{\beta^{n-1}}$, cancelling the change in total dependence caused by removing the 3-holed spheres. For the case of $g = 0$, the process removes $n - 2$ 3-holed spheres, while the removal of the last one modifies the total dependence on β by a multiplication by β^2. Hence, the total change of the dependence is a multiplication by β^{n-1}, which is precisely cancelled by the boundary containing part, assembled as in fig. 3.3.3, cf. the discussion above.

Having done this, we are left with a decomposition of a surface with one geodesic boundary to the left and genus g. We will bring it into one of the discussed forms by starting with the 3-holed sphere that contains this boundary and, potentially moving structures attached to it, render the objects attached to it into a form appearing in the "ordered" decomposition. We then proceed with the next attached 3-holed sphere in the same way. In order to do this, additionally to the cases a), b), c) and d) discussed in fig. 3.3.2 we introduce the additional cases i): like case b) with Y being a crosscap and ii): Like d) with X being the surface of genus 0 with two boundaries, i.e. the boundaries are glued to one another. Having introduced these cases, we can state our procedure. First, we note that case a) cannot appear with an unglued boundary since we already removed all external boundaries. It can only appear after a glueing of case d) or after a glueing that "skips" several intermediate glueings, which we denote as case iii) Consequently, we can define for every appearing 3-holed sphere, apart from these cases, a "left" boundary in the obvious way and choose an upper and lower boundary, where we take the convention that if a crosscap is glued to a 3-holed sphere, it is always glued to the lower boundary. The starting point for every step will be a 3-holed sphere S identified by its left boundary, while the starting point for the whole procedure is given by the unique 3-holed sphere containing the remaining external geodesic boundary.

Given S, the glueing is as in case

b) Then, the part is in the ordered form. Continue by setting S as the first 3-holed sphere of Y.

c) Proceed along the lower boundaries of the glued 3-holed spheres forming X_1, until either case i), ii) or iii) are assumed or a crosscap is attached.

 i) Attach a crosscap to the lower boundary of S, attach a 3-holed sphere (that from the end of X_1) to the upper boundary. Attach to the lower boundary of this another crosscap and to its upper boundary the rest of X_1 (i.e. with the final 3-holed sphere removed). Then attach X_2 to the remaining, now unglued boundary of X_1. Set S as the first 3-holed sphere of X_1.

 ii) Attach a 3-holed sphere in the a) way to S, attach X_1 (with the final 3-holed sphere removed) to it. Finally, glue X_2 to the unglued boundary of X_1 and proceed by setting S as the first 3-holed sphere of X_1.

 iii) This is only possible if there is an interconnection between X_2 and X_1. Denote the part of the decomposition traversed before reaching this point as A, having two glueing boundaries, the part of the decomposition that is adjoint to it like X in a) as B, having one glueing boundary. The other boundary of this 3-holed sphere is glued to a boundary of X_2, constituting said interconnection. Attach the 3-holed sphere in the d) way to S, glue A to the remaining boundary, glue X_2 to the other boundary of A and glue B to the former "interconnection" boundary. Set S as the first 3-holed sphere of A.

 crosscap Remove the 3-holed sphere to which the crosscap is glued, glue the two boundaries to which it was glued together. Insert the 3-holed sphere with the crosscap attached at its lower boundary before S. Restart with S unchanged.

d) Then, the part is in the ordered form. Continue by setting S as the 3-holed sphere attached to the type a) 3-holed sphere being glued to the present 3-holed sphere

i)/ii) Terminate, the decomposition is in one of the "ordered" forms. Also terminate if S is a not a 3-holed sphere but a crosscap.

All of these transformations do not change the β dependence of the decomposition since either the constituent parts are only moved around or in the case c),ii) the factor of $\frac{1}{\beta}$ from the self glueing of of the final 3-holed sphere is replaced by that from the a)-type glueing after the prescribed modification. Furthermore, it is important to note that in the reactions to the cases that do not terminate the procedure the genus of the (ordered) decomposition left of the new S is increased. Hence, the procedure is guaranteed to terminate since the genus of the decomposed surface is finite. Thus, since we have given all the possible cases how something can be glued to a given S, the procedure is complete and yields an ordered decomposition which shows our claim. □

This enables now the geometric proof of theorem 3.5.

Proof of theorem 3.5. Let values g and $n > 0$ be chosen as the genus and the number of boundaries. All the possible contributions to the β WP volume $V^{\beta}_{g,n}$ can then, by the pictorial interpretation of the recursion, be represented as a decomposition of the surface into the building blocks given in fig. 3.3.2. Theorem 3.6 shows that all possible dependences on β are exhausted by the ordered decompositions. These are characterised by the number of crosscaps contained in a decomposition. Hence, it suffices to study the dependence on β of the ordered decomposition with k crosscaps for all possible values of k to find the dependence of the whole β WP volume.

For the case of $k = k_{\text{max}}$, depicted in fig. 3.3.3, one finds the dependence to be given by[23]

$$\left(\frac{1}{\beta}\right)^{n-1}\left(\frac{2-\beta}{\beta}\right)^{2g-2+2} = \left(\frac{1}{\beta}\right)^{n-1}\left(\frac{2-\beta}{\beta}\right)^{k_{\text{max}}}, \tag{3.3.120}$$

As we noted above, to get the surface with k crosscaps, one has to replace two crosscaps $\frac{1}{2}(k_{\text{max}} - k) = g - \frac{k}{2}$ times. Hence, the dependence on β changes to

$$\left(\frac{1}{\beta}\right)^{n-1}\left(\frac{2-\beta}{\beta}\right)^{2g-2g+k}\left(\frac{1}{\beta}\right)^{g-\frac{1}{2}k} = \frac{1}{\beta^{g-\frac{k}{2}+n-1}}\left(\frac{2-\beta}{\beta}\right)^{k}. \tag{3.3.121}$$

In the case of half-integer genus, the maximal number of crosscaps is odd and hence also all possible numbers of crosscaps are odd since one can only replace two by a hole, leaving the total number odd. By the same argument, in the integer genus case all possible numbers of crosscaps are even. Hence, we parametrise the number of crosscaps as $2m + 1$ with $m \in \left[0, g - \frac{1}{2}\right]$ for $g \in \frac{\mathbb{N}}{2}$ and $2m$ with $m \in [0, g]$ for $g \in \mathbb{N}_0$. In terms of this parameter, the

23 Note that for $g = 0$, the "final" 3-holed sphere contributes as β^{-2}. Also, it is evident that this result holds for $g = \frac{1}{2}$, though the decomposition is not as in fig. 3.3.3.

possible dependences of the contributions to the WP volumes on the Dyson index are given by

$$\begin{cases} \frac{1}{\beta^{g-m+n-1}}\left(\frac{2-\beta}{\beta}\right)^{2m} & \text{for} \quad g \in \mathbb{N}_0, \\ \frac{1}{\beta^{g-m-\frac{1}{2}+n-1}}\left(\frac{2-\beta}{\beta}\right)^{2m+1} & \text{for} \quad g \in \frac{\mathbb{N}}{2}. \end{cases} \tag{3.3.122}$$

Defining the collection of the whole dependence on the n lengths $\vec{b} \in \mathbb{R}_+$ of the contributions with $k = 2m$ or $k = 2m+1$ crosscaps as $\bar{\mathcal{V}}^m_{g,n}(\vec{b})$, which are by construction independent on the Dyson index, this implies the claimed form for the β WP volumes. □

Notably, the proof for JT gravity also implies the same structure for topological gravity. Of course, we had already proven the structure there by the general matrix model argument. Using the geometric construction, however, gives us an insight to the origin of the dependence from the decomposition of surfaces which we could otherwise not have found.

Notably, the decomposition of surfaces is still built from surfaces that are either orientable or unorientable. Of course this was already expected, since there are only orientable or unorientable surfaces. Theorem 3.5 now gives an explicit decomposition of the β (Airy) WP volumes in this sense, i.e. as a sum of contributions of a well-defined number of crosscaps. In contrast to this, the "invariant" structure of the volumes, deriving from theorem 3.3 decomposes the correlation functions into Wigner-Dyson and non-Wigner-Dyson parts. Notably, the "orientable" part of both decompositions, i.e. the part contributing at $\beta = 2$ in theorem 3.3 or the contribution with no crosscaps in theorem 3.5 coincide. The other Wigner-Dyson contribution in theorem 3.3 ($i = g$ for $g \in \mathbb{N}_0$, $i = g + \frac{1}{2}$ for $g \in \frac{\mathbb{N}}{2}$) now arises from a sum of the contributions of well-defined number of crosscaps in theorem 3.5 (the precise form can be found upon inverting eqs. (3.3.114) and (3.3.116)). Hence, it gives a way to collect only the information relevant for the cases of $\beta \in \{1, 4\}$ from the decomposition of the volumes in theorem 3.5, where it is hidden in the various contributions. Likewise, the non-Wigner-Dyson contribution arise as a different sum of the contributions to the volumes in theorem 3.5.

Hence, we can conclude from theorem 3.5 that the β WP volumes are indeed determined by knowing the moduli space volumes for orientable and unorientable surfaces only, as it was our original reasoning. However, finding the correct result requires essentially decomposing the moduli space of unorientable surfaces into parts of well-defined number of crosscaps and then adding up the volumes of the constituent parts with different dependence on the Dyson index. A recombination of these contributions with the guiding principle of each contributing term being invariant under $\beta \to \frac{4}{\beta}$ then gives rise to our original structure for the moduli space volumes (eq. (3.3.66)) and hence induces a geometric interpretation for the volumes occurring in this decomposition. In chapter 4 the latter form will be of more use, due to its clear distinction between Wigner-Dyson and non-Wigner-Dyson contributions.

Having discussed now the geometric interpretation of the β dependence of the β WP volumes, we will now briefly discuss its interplay with their divergent structure an find the generalisation to general β for the explicit results we discussed above.

THE UPLIFT OF UNORIENTABLE WP VOLUMES TO ARBITRARY DYSON INDEX. In general, as we discussed above, it is not possible to infer the β WP volume from knowing only the results for $\beta = 1$ and $\beta = 2$. Only parts of the general contribution are determined by those results. Specifically, we showed in eq. (3.3.44)

$$\underset{g\in\frac{\mathbb{N}}{2}}{\forall} \mathcal{R}_g^{g+\frac{1}{2}}(I) = R_g^1(I), \tag{3.3.123}$$

in eq. (3.3.45)

$$\underset{g\in\mathbb{N}_0}{\forall} \mathcal{R}_g^{0}(I) = 2^{g+|I|-1} R_g^2(I), \tag{3.3.124}$$

and in eq. (3.3.46)

$$\underset{g\in\mathbb{N}_0}{\forall} \mathcal{R}_g^{g}(I) = R_g^1(I) - 2^{g+|I|-1} R_g^2(I). \tag{3.3.125}$$

By this, we already uplifted the orientable part of WP volumes for any values of the genus g and the number of boundaries (n) to arbitrary Dyson index in eq. (3.3.67). In particular, this implies the uplift for the whole of the WP volumes with $g = 0$ (cf.eq. (3.3.47)). Furthermore, we already uplifted the result for $(g,n) = (\frac{1}{2},1)$ in eq. (3.3.112). Here, we shall do the same for the explicit results found above for $(g,n) \in \{(1,1),(\frac{1}{2},2),(1,2)\}$.

For JT gravity for arbitrary Dyson index, apart from the structure in terms of β, there is also the structure of regularised divergences, arising for the same reasons as in the $\beta = 1$ case. This divergence structure is hence in general the same as for $\beta = 1$ and we introduce the following notation for it

$$V_{g,n}^{\beta,(p)}(b_1,\dots,b_n) = \sum_{k=0}^{2g} \log(p)^k v_{g,n,k}^{\beta}(b_1,\dots,b_n) + \mathcal{O}\left(p^{-1}\right). \tag{3.3.126}$$

Since the whole of $V_{g,n}^{\beta,(p)}(b_1,\dots,b_n)$ obeys eq. (3.3.66)/theorem 3.5, the same structure is induced for the $v_{g,n,k}^{\beta}$. We notate the contributions to this with the same convention as for the $V_{g,n}^{\beta}$, with the coefficient functions notated as $\mathfrak{v}_{g,n,k}^{i}$ for the invariant structure and $\bar{\mathfrak{v}}_{g,n,k}^{i}$ for the geometric one. Some of these coefficient functions vanish in general. In fact, due to eq. (3.3.45), for integer genus the "orientable" contribution to the volumes is determined by the $\beta = 2$ result solely and is hence finite. Consequently, there is no orientable contribution to the divergent parts of the volume. This implies for the coefficient functions

$$\underset{g\in\mathbb{N}_0}{\forall}\,\underset{n\in\mathbb{N}}{\forall}\,\underset{k>0}{\forall}\,\mathfrak{v}^0_{g,n,k} = \bar{\mathfrak{v}}^0_{g,n,k} = 0. \tag{3.3.127}$$

An alternative point of view, would of course be to consider the decomposition of the volumes according to theorem 3.5/eq. (3.3.66) and consider the divergence structure of the $\mathcal{V}^i_{g,n}$. Doing this, one could put eq. (3.3.127) as the $\mathcal{V}^0_{g,n}$ having only finite contributions. In the following, we choose the convention of eq. (3.3.126), however.

We now study explicitly the results for unorientable WP volumes computed above.

For $(g,n) = (\frac{1}{2},2)$ we directly see from theorem 3.3 or theorem 3.5 that there is only one contribution, having the prefactor $\frac{2-\beta}{\beta^2}$. This contribution, by eq. (3.3.44) is just the contribution for $\beta = 1$. Hence

$$V^\beta_{\frac{1}{2},2}(b_1,b_2) = \frac{2-\beta}{\beta^2} V^1_{\frac{1}{2},2}(b_1,b_2). \tag{3.3.128}$$

Note that the divergent structure of $V^1_{\frac{1}{2},2}$ thus translates directly to that of $V^\beta_{\frac{1}{2},2}$. Since this is true for any regularisation, we did not include any superscripts indicating one.

For $(g,n) = (1,1)$ there are now two contributions as seen from the invariant or the geometric structure. However, one of them is the "orientable" contribution and hence determined from $V^2_{1,1}(b) = \frac{\pi^2}{12} + \frac{L^2}{48}$ (eq. (2.1.35)[24]) via eq. (3.3.45). The other contribution is hence known by eq. (3.3.46) as [25]

$$\mathcal{V}^{1,(p)}_{1,1}(b) = \bar{\mathcal{V}}^{1,(p)}_{1,1}(b) = V^{1,(p)}_{1,1}(b) - 2V^2_{1,1}(b). \tag{3.3.129}$$

Hence, using eq. (3.3.127) we find

$$v^\beta_{1,1,0}(b) = \frac{2}{\beta} V^2_{1,1}(b) + \frac{(2-\beta)^2}{\beta^2}\left[\frac{5b^2}{48} + \frac{\pi^2}{12} + b\log\left(1+e^{-\frac{b}{2}}\right)\right], \tag{3.3.130}$$

$$v^\beta_{1,1,1}(b) = \frac{(2-\beta)^2}{\beta^2}\left(b + 4\log\left(1+e^{-\frac{b}{2}}\right)\right), \tag{3.3.131}$$

$$v^\beta_{1,1,2}(b) = 2\frac{(2-\beta)^2}{\beta^2}. \tag{3.3.132}$$

24 Note that here one has to use $\tilde{V}$ eq. (2.1.36).

25 Note that the superscript on the V indicates the Dyson index, while the superscript on the $\mathcal{V}/\bar{\mathcal{V}}$ indicates the summation index of the sum over β dependences in eq. (3.3.66)/theorem 3.5. Here, we made explicit the use of the p regularisation.

For $(g,n) = (1,2)$ one can follow the same reasoning, since there are still precisely two contributions in the dependence on β and the structure in terms of divergences also is the same as for the previous case. Due to the length of the expressions, we refrain here from writing down the whole of the results but rather give the equations determining them. For the non-divergent contribution one finds, using eq. (3.3.45) and eq. (3.3.46)

$$v^{\beta}_{1,2,0}(b_1,b_2) = \frac{4}{\beta^2} V^2_{1,2}(b_1,b_2) + \frac{(2-\beta)^2}{\beta^3} \mathfrak{v}^1_{1,2,0}(b_1,b_2), \tag{3.3.133}$$

with

$$\mathfrak{v}^1_{1,2,0}(b_1,b_2) = v^1_{1,2,0}(b_1,b_2) - \frac{4}{\beta^2} V^2_{1,2}(b_1,b_2). \tag{3.3.134}$$

$v^1_{1,2,0}$ is given in eq. (3.3.105), while $V^2_{1,2}$ is given in eq. (2.3.29). For the two divergent contributions, using eq. (3.3.127), we find only one contribution, which in both cases ($k \in \{1,2\}$) is given by

$$v^{\beta}_{1,2,k}(b_1,b_2) = \frac{(2-\beta)^2}{\beta^3} v^1_{1,2,k}(b_1,b_2). \tag{3.3.135}$$

This finishes the uplifting of the results we had found to the setting of general Dyson index. That this was possible without any further computations, is due to the low genus of the computed WP volumes. In fact, eqs. (3.3.44) to (3.3.46) enable the computation of the β WP volume from the $\beta = 1$ result for $g = \frac{1}{2}$ and $g = 1$ for all numbers of boundaries. For higher genus, there are more than one contributions to the genuinely unorientable part of the volumes, i.e. contributions with different numbers of crosscaps, and those cannot be distinguished from one another by just knowing the $\beta = 1$ result. Hence, starting with $g = \frac{3}{2}$, it is necessary to include factors of β into the computation of unorientable WP volumes.

Concerning those results, there is one final point we would like to clarify. This pertains the interpretation of the full volume as a sum over contributions with a well-defined number of crosscaps. For the structure in terms of β, we can use theorem 3.5 to infer from which part of moduli space, partitioned into parts of well-defined crosscap number, a particular part of the volume originates. However, since the divergences, like the factors of $\frac{2-\beta}{\beta}$, originate solely from the crosscaps, one could have the impression that the divergence structure (eq. (3.3.101) for the p regularisation) is also a split into such contributions. We would like to clarify that this is not true. This can be seen from the fact that the crosscap resolvent in the p regularisations eq. (3.3.98) is clearly split into a convergent and a divergent part (for $p \to \infty$). Note that this is one of the virtues of the p regularisation, though, of course, for higher genus and numbers of boundaries the same behaviour is found from the ϵ

regularisation. Hence, upon iteration in the recursion, contributions with more than one crosscap will contain products of multiple crosscaps (in their volume form for the Mirzakhani-like recursion, in their resolvent form for the loop equations) and hence it is apparent that the structure of eq. (3.3.101) arises, simply by collecting the orders of divergence of the product of crosscaps. Note that one still has to sum over all possible decompositions of a surface, i.e. different numbers of contained crosscaps. Consequently, each order of the divergence contains contributions from all possible sectors of well-defined number of crosscaps. This also explains, why there are also odd powers of $\log(p)$ in the divergence structure for integer genus/even powers for half-integer genus, while there are no decompositions of the respective surfaces with an odd/even number of crosscaps.

Consequently, the decomposition of moduli space in the sectors of well-defined number of crosscaps is, using the methods we discuss, possible only by using the β dependence. Having clarified this, we have given all our results concerning unorientable JT gravity which are relevant for this thesis.

3.4 CONCLUSION OF THE GEOMETRIC PART

In this chapter we have discussed the key geometric results of this thesis, the (Airy) WP volumes for an arbitrary value of the Dyson index β, in particular for the two non-unitary Wigner Dyson classes. These will be of pivotal relevance for the rest of this thesis, foremost the general structure of matrix model correlation functions in terms of the Dyson index (theorem 3.3) and the structure of Airy WP volumes for $n = 2$ (theorem 3.4).

Before continuing with these considerations, we will give some ideas how to extend and better understand these geometric results, which go beyond the scope of this thesis. Foremost, we would like to point out that the mathematical foundation of unorientable moduli space volumes is much less explored than that of the orientable moduli space volumes. This has very good reasons, for example the divergent structure of those volumes and the moduli space not being symplectic as in the orientable case[26]. However, since one can evidently go very far in the computation of the moduli space volumes, it is a valid question whether one can make the mathematical structure beneath these computations as rigorous as in the orientable case. It would be an important step in this direction, to understand the generalisation of intersection numbers to the unorientable setting. In the orientable setting, they were directly related to the WP volumes, specifically with their topological gravity part (cf. eq. (2.1.50)). Since we know a lot about the topological

26 This is easiest seen by noticing that when glueing in a crosscap the corresponding measure depends on only one coordinate (eq. (3.2.1)) and hence the dimension of the moduli space can be odd. Consequently, it cannot be symplectic.

gravity moduli space volumes, it would be interesting to study whether assuming their relation with putative "unorientable" intersection numbers being the same as in the orientable case yields useful objects. Notably, also their generalisation to arbitrary β along the same lines is possible. Using this, one could study whether an analogue of the KdV structure underlying topological gravity in the orientable setting can be found for the unorientable setting. Such a structure could then be used to derive the unorientable (or even general β) analogue of eq. (2.3.49) which would enable the generalisation of our reasoning at the end of section 2.3, leading to a closed result for the intersection numbers for one puncture to the unorientable/arbitrary β setting.

Apart from understanding better the results we have already computed, it would be interesting to find new results for geometries/symmetry classes we have not yet studied. We have already studied all the symmetry classes/generalisations thereof that can be covered by varying the Wigner-Dyson index β. However, as we mentioned in section 3.1, there are more symmetry classes of random matrices than the three Wigner-Dyson ones, the seven Altland-Zirnbauer classes. Studying a matrix model in one of these classes requires modifying the partition function defining it (eq. (2.2.7)) by an additional factor depending on the eigenvalues with a power α [69]. The symmetry class is then determined by the tuple (α, β). The variants of JT/topological gravity dual to these matrix models are those that include Riemann supersurfaces in the gravitational path-integral (for details cf. [69]). Alike for the matrix models depending only on β there are (perturbative) loop equations by which one can study correlation functions, also for the more general matrix models. Hence, one is led to suspect that one can apply the reasoning by which we defined JT gravity for arbitrary Dyson index to the full Altland-Zirnbauer case and study the generalisation to arbitrary values of the tuple of indices there, i.e. $(\alpha, \beta) \in \mathbb{R}_+^2$.

We will leave these interesting points for future work to investigate and continue by applying our results for JT/topological gravity for all Wigner-Dyson classes and beyond in the consideration, which caused us to derive them: The explicit verification of the presence of quantum chaos in this theories and the implications of its presence on moduli space volumes.

4

THE INTERPLAY OF CHAOS AND GRAVITY: CONSISTENCY AND CONSTRAINTS

In this second main part of the work presented in this thesis we will concern ourselves with its main objective, studying the implications of random matrix universality (recalled in section 2.4) on the studied theories of two-dimensional gravity. This discussion has two main parts.

First, we will study in section 4.1 what the predictions of RMT universality for the matrix models dual to JT/topological gravity actually are. For this purpose we will study the canonical SFF, introduced in section 2.4, and work out the implications of RMT universality, as expressed through the late-time behaviour of the microcanonical SFF, in both theories. Naturally, this prediction will depend on the choice of the Dyson index and hence the following second part of our discussion is split into sections according to this choice.

For the choices corresponding to purely orientable/unorientable manifolds on the gravitational side, this discussion will begin by first establishing that the gravitational theories are indeed consistent with the predictions of universal RMT, a statement quite non-trivial to establish in the unorientable setting. This sanity check is necessary since we are dealing with double-scaled theories and hence it is a priori a non-trivial step to apply the rigorously proven results for the non-double-scaled matrix models in this setting. We will comment on why we see evidence for the extension of these results from formal arguments. This agreement can, in the sense of the BGS conjecture, be interpreted as an explicit demonstration of quantum chaos in the gravitational theories. Having convinced ourselves of the applicability of the universal RMT results, we investigate their implications on the Weil-Petersson volumes for the respective choice of Dyson index which will take the form of constraints on sums of their coefficients.

For the non-Wigner-Dyson choices for the Dyson index, this discussion has to be approached, in a sense, the other way round due to the absence of a full analytic expression for the universal RMT behaviour in this case. Hence, we will first study the extension of applicability of the constraints found in the purely (un)orientable setting to the general β WP volumes. The fact that the constraints are, with some understood exceptions, indeed applicable also to the genuinely non-Wigner-Dyson parts of the generalised WP volumes we interpret as a strong sign for the presence of quantum chaos in the sense of the BGS conjecture. Building on this reasoning, we use the found results for the β WP volumes, specifically their general structure in terms of β (eq. (3.3.66)), to infer an analytic statement for (an important part of) the universal microcanonical SFF in the setting of arbitrary Dyson index.

4.1 THE LATE TIME CANONICAL SPECTRAL FORM FACTOR

This section is largely based on [1, 2]. Note that in this thesis we consistently use the normalisation of the spectral density induced by its definition in eq. (2.2.8). This is denoted as ρ^T in the papers, while the density normalised to unity was denoted as ρ there.

The observable of choice to study the implications of universal RMT behaviour in the gravitational theories we are interested in is the canonical SFF[1] defined above as

$$\kappa_\beta(t,\beta) = \langle Z(\beta+it)Z(\beta-it)\rangle_c =: \sum_{g=0,\frac{1}{2},1,\ldots} \frac{\kappa_\beta^g(t,\beta)}{(e^{S_0})^{2g}}, \tag{4.1.1}$$

where on the RHS we made explicit the topological expansion of this quantity and gave a name to its coefficient functions. As advertised above, we will now derive the prediction for $\kappa_\beta(t,\beta)$ as arising from the universal behaviour of the spectral two-point function for small energy differences. To this end, we first express the SFF as an energy integral via eq. (2.2.12):

$$\begin{aligned}\kappa_\beta(t,\beta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}E_1\,\mathrm{d}E_2\,\langle\rho(E_1)\rho(E_2)\rangle\, e^{-\beta(E_1+E_2)-it(E_1-E_2)} \\ &= \int_{-\infty}^{\infty}\mathrm{d}E\, e^{-2\beta E}\int_{-2E}^{2E}\mathrm{d}\Delta\, e^{-i\Delta t}\left\langle\rho\left(E+\frac{\Delta}{2}\right)\rho\left(E-\frac{\Delta}{2}\right)\right\rangle_c,\end{aligned} \tag{4.1.2}$$

where we changed variables to $E_{1/2} = E \pm \frac{\Delta}{2}$ in the second line. Universal behaviour is expected in the limit of large times, i.e. times of the order of the Heisenberg time T_H defined in eq. (2.4.15), which for double-scaled theories is proportional to e^{S_0} (cf. eq. (2.3.15)). For double-scaled matrix models, the most convenient way to access this limit is via so-called *τ-scaling*, to our knowledge first used in [140][2]. This is done by defining

$$\tau := e^{-S_0}t, \tag{4.1.3}$$

1 In this section, we will mean the canonical variety whenever writing "SFF".
2 Note that before the publication of the preprint leading to this publication the authors communicated this idea and the result for the unitary ensemble with us and the authors of [94]. Hence, this and our paper [1] are published earlier.

and considering the limit of $t, e^{S_0} \to \infty$ while keeping τ fixed.

To see, how this limit relates to the universal limit of the spectral two-point function studied in section 2.4, we rewrite the SFF as

$$\begin{aligned}\kappa_\beta(e^{-S_0}\tau,\beta) &= \int_{-\infty}^{\infty} \mathrm{d}E\, e^{-2\beta E} \int_{-2E}^{2E} \mathrm{d}\Delta\, e^{-e^{S_0}\Delta\tau} \left\langle \rho\left(E+\frac{\Delta}{2}\right)\rho\left(E-\frac{\Delta}{2}\right)\right\rangle_c \\ &= e^{-S_0} \int_{-\infty}^{\infty} \mathrm{d}E\, e^{-2\beta E} \int_{-2Ee^{S_0}}^{2Ee^{S_0}} \mathrm{d}\Delta\, e^{-ix\tau} \left\langle \rho\left(E+\frac{e^{-S_0}x}{2}\right)\rho\left(E-\frac{e^{-S_0}x}{2}\right)\right\rangle_c .\end{aligned} \tag{4.1.4}$$

Here one can observe that in the limit of large e^{S_0} this integral explicitly only depends on the behaviour of the spectral two-point function for energy differences of the order of $\frac{1}{\langle\rho\rangle}$ which, as we discussed in section 2.4, is exactly the regime where the universal RMT expression for the spectral two-point function is valid. The validity of this expression in a matrix model, which is what we study here, was additionally given only in the limit of $N \to \infty$. Here, since this limit of N was already taken during the double-scaling procedure, the role of N is taken by e^{S_0} and hence what we denote as the universal RMT limit of the spectral two-point function is given by

$$\lim_{e^{S_0}\to\infty} \frac{\left\langle \rho\left(E+\frac{\Delta}{2}\right)\rho\left(E-\frac{\Delta}{2}\right)\right\rangle_c}{\langle\rho(E)\rangle^2} = \delta(\Delta\,\langle\rho(E)\rangle) - \mathrm{Y}^\beta(\Delta\,\langle\rho(E)\rangle), \tag{4.1.5}$$

which is the expression of eq. (2.4.18) adapted for double-scaled theories. Furthermore, it is important to note that for the one-point function it holds that

$$\langle\rho(x)\rangle = \sum_{g=0}^{\infty} \frac{\rho_g(x)}{(e^{S_0})^{2g-1}} = e^{S_0}\rho_0(x) + (\text{subleading}), \tag{4.1.6}$$

and hence in the limit $e^{S_0} \to \infty$ one can replace $\langle\rho(x)\rangle \to e^{S_0}\rho_0(x)$. Hence, for the actual two-point correlation function of energies we are interested in, it holds that

$$\begin{aligned}\lim_{e^{S_0}\to\infty} e^{-2S_0} \left\langle \rho\left(E+\frac{e^{-S_0}x}{2}\right)\rho\left(E-\frac{e^{-S_0}x}{2}\right)\right\rangle_c =& (\rho_0(E))^2\delta(x\rho_0(E)) \\ &- (\rho_0(E))^2\mathrm{Y}^\beta(x\rho_0(x)).\end{aligned} \tag{4.1.7}$$

Motivated by this, we define the τ-scaled spectral form factor as

$$\kappa^s_\beta(\tau,\beta) := \lim_{\substack{e^{S_0},t\to\infty \\ \tau:=e^{-S_0}t \text{ fixed}}} e^{-S_0}\kappa_\beta(t=e^{S_0}\tau,\beta), \tag{4.1.8}$$

which, using the universal result, can directly be worked out as

$$\begin{aligned}\kappa^{s}_{\beta}(\tau,\beta) &= \int_0^\infty \mathrm{d}E\, e^{-2\beta E}\int_{-\infty}^{\infty}\mathrm{d}x\, e^{-ix\tau}(\rho_0(E))^2\Big[\delta(x\rho_0(E)) - \mathrm{Y}^{\beta}(x\rho_0(x))\Big] \\ &= \int_0^\infty \mathrm{d}E e^{-2\beta E}\rho_0(E) - \int_0^\infty \mathrm{d}E e^{-2\beta E}\rho_0(E) b^{\beta}\left(\frac{\tau}{2\pi\rho_0(E)}\right) \\ &= \int_0^\infty \mathrm{d}E e^{-2\beta E}\rho_0(E)\left[1 - b^{\beta}\left(\frac{\tau}{2\pi\rho_0(E)}\right)\right],\end{aligned} \tag{4.1.9}$$

where the expression in square brackets in the last line is seen to be the universal limit of the microcanonical spectral form factor. To see this, one has to note that

$$\frac{\tau}{2\pi\rho_0(E)} = \frac{t}{2\pi e^{S_0}\rho_0(E)} = \lim_{e^{S_0}\to\infty}\frac{t}{T_{\mathrm{H}}}. \tag{4.1.10}$$

Hence, it holds that

$$\kappa^{s}_{\beta}(\tau,\beta) = \mathcal{L}\left[\rho_0(E)\kappa^{s}_{\beta}\left(\frac{\tau}{2\pi\rho_0(E)}\right), E, 2\beta\right], \tag{4.1.11}$$

which is a direct link between the universal limits of the canonical and the microcanonical variant of the SFF.

In fact, one could have obtained this result in an equivalent way by introducing, as in section 2.4, the Heisenberg time T_{H} as a unit of time and performing the universal limit in the way demonstrated there with replacing $N \to e^{S_0}$. It is, however, important to note that in both ways to derive the expression, the relevant scaling of the time variable for the SFF is indeed that introduced as τ-scaling. This holds, since the energy dependence of the unit of time as introduced by the Heisenberg time is lost upon performing the energy integral and hence only the "pure" scaling of the time such that it probes the late-time regime of the SFF remains.

Notably, the expression for $\kappa^{s}_{\beta}(\tau,\beta)$ depends only on the leading order density of states, which resonates with the fact that the perturbative expansion of double-scaled matrix models has the same property. In the following, we will now consider the universal predictions for the case of the matrix models dual to topological/JT gravity in the different symmetry classes.

Before going into this discussion, we would like to give some details on how to implement the steps to find the universal contributions in the geometric setting, where we only have perturbative information, i.e. the topological expansion of the SFF. Using this, defined in eq. (4.1.1), we can write out the universal limit explicitly as

$$\begin{aligned}\kappa^{s}_{\beta}(\tau,\beta) &= \lim_{e^{S_0},t\to\infty} e^{-S_0} \sum_{g=0,\frac{1}{2},1,\ldots} \frac{\kappa^{g}_{\beta}(t,\beta)}{(e^{S_0})^{2g}} = \lim_{e^{S_0},t\to\infty} \sum_{g=0,\frac{1}{2},1,\ldots} \frac{\kappa^{g}_{\beta}(t,\beta)\tau^{2g+1}}{(e^{S_0})^{2g+1}\tau^{2g+1}} \\ &= \lim_{t\to\infty} \sum_{g=0,\frac{1}{2},1,\ldots} \underbrace{\frac{\kappa^{g}_{\beta}(t,\beta)}{t^{2g+1}}}_{:=\kappa^{s,g}_{\beta}(t,\beta)} \tau^{2g+1},\end{aligned} \tag{4.1.12}$$

where the inclusion of the factors of τ in the second step is motivated by the explicit form of the series expansion of the universal results for the gravitational theories, which we shall discuss in the following. Also the fact that we could interchange the limit in e^{S_0}, while we did not do so for the limit in t, will only arise in the explicit discussion of the different cases. Having now fixed the notation for the universal limit for all symmetry classes (in fact, values of β), we can proceed with this study for the individual symmetry classes.

As expected from the discussion in chapter 3, the easiest case to deal with is that of the unitary symmetry class/orientable manifolds with which we start off this discussion.

4.2 THE ORIENTABLE CASE

This section is largely based on [1] and [6].
Naturally, we begin this study by deriving the universal RMT prediction for the gravitational theories we are interested in. In the following, in section 4.2.1 we will then compare to the geometric results from JT/topological gravity and verify the chaoticity of these theories by establishing them to show universal RMT behaviour. Building on this, we will then derive constraints on the orientable WP volumes from universality in section 4.2.2. These constraints, in addition to the matching of the scaled SFF evaluated from the WP volumes with the universal prediction, are precisely the implications of RMT universality in orientable JT/topological gravity, which are the main results of [1].

THE UNIVERSAL RMT PREDICTION FOR THE SFF. In the orientable case, we are in the fortunate situation of the universal RMT prediction being quite simple. Plugging eq. (2.4.13) into eq. (4.1.9), we find

$$\begin{aligned}\kappa_2^s(\tau,\beta) &= \int_0^\infty \mathrm{d}E e^{-2\beta E}\rho_0(E)\left[1-\max\left(1-\frac{\tau}{2\pi\rho_0(E)},0\right)\right]\\ &= \int_0^\infty \mathrm{d}E e^{-2\beta E}\left[\rho_0(E)-\max\left(\rho_0(E)-\frac{\tau}{2\pi},0\right)\right]\\ &= \int_0^\infty \mathrm{d}E e^{-2\beta E}\min\left(\rho_0(E),\frac{\tau}{2\pi}\right).\end{aligned} \tag{4.2.1}$$

While this integral can be evaluated in a more general case, we assume that ρ_0 is monotonously increasing and fulfils $\rho_0(0)=0$, which is valid for the JT/topological gravity densities of states. In this case, one can define E^* via the condition $\rho_0(E^*)=\frac{\tau}{2\pi}$ and is guaranteed to find exactly one solution for all positive values of τ. Using this, one can evaluate the integral further:

$$\begin{aligned}\kappa_2^s(\tau,\beta) &= \int_0^{E^*} \mathrm{d}E\, e^{-2\beta E}\rho_0(E)+\int_{E^*}^\infty \mathrm{d}E\, e^{-2\beta E}\frac{t}{2\pi}\\ &= \frac{1}{2\beta}\underbrace{\frac{t}{2\pi}}_{=\rho_0(E^*)}e^{-2\beta E^*}-\frac{1}{2\beta}\int_0^{E^*}\mathrm{d}E\rho_0(E)\frac{\partial e^{-2\beta E}}{\partial E}\\ &= \frac{1}{2\beta}\left[\rho_0(E)e^{-2\beta E}\right]_0^{E^*}-\frac{1}{2\beta}\int_0^{E^*}\mathrm{d}E\rho_0(E)\frac{\partial e^{-2\beta E}}{\partial E}\\ &= \frac{1}{2\beta}\int_0^{E^*}\mathrm{d}E e^{-2\beta E}\partial_E\rho_0(E).\end{aligned} \tag{4.2.2}$$

At this point, it is convenient to work out the integral for the two densities of states we are interested in: $\rho_0^{\mathrm{Airy}}(E)=\frac{\sqrt{E}}{2\pi}$ (eq. (2.3.31)) and $\rho_0^{\mathrm{JT}}(E)=\frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right)$ (eq. (2.3.1)).

For topological gravity one directly finds $E^*=\tau^2$ and hence

$$\begin{aligned}\kappa_2^{s,\mathrm{Airy}}(\tau,\beta) &= \frac{1}{4\pi\beta}\int_0^{\tau^2}\mathrm{d}E e^{-2\beta E}\frac{1}{2\sqrt{E}}=\frac{1}{4\pi\beta}\int_0^\tau \mathrm{d}x e^{-2\beta x^2}\\ &= \frac{1}{4\pi\beta\sqrt{2\beta}}\frac{\sqrt{\pi}}{2}\,\mathrm{Erf}\left(\sqrt{2\beta}\tau\right)=\frac{1}{2\pi}\frac{\sqrt{\pi}}{2^{\frac{5}{2}}\beta^{\frac{3}{2}}}\,\mathrm{Erf}\left(\sqrt{2\beta}\tau\right)\\ &= \frac{1}{2\pi}\sum_{n=0}^\infty\frac{(-1)^n(2\beta)^{n-1}}{n!(2n+1)}\tau^{2n+1},\end{aligned} \tag{4.2.3}$$

where Erf denotes the error function, whose (convergent) series expansion we wrote down in the second to last line. For JT gravity, one finds $E^*=$

$\frac{1}{4\pi^2}\operatorname{arsinh}^2(2\pi\tau)$, using which one can evaluate the integral like that for topological gravity to find

$$\kappa_2^{s,\text{JT}}(\tau,\beta) = \frac{e^{\frac{\pi^2}{2\beta}}}{16\sqrt{2\pi}\beta^{\frac{3}{2}}}\left[\text{Erf}\left(\frac{\pi + \frac{\beta}{\pi}\operatorname{arsinh}(2\pi\tau)}{\sqrt{2\beta}}\right) - \text{Erf}\left(\frac{\pi - \frac{\beta}{\pi}\operatorname{arsinh}(2\pi\tau)}{\sqrt{2\beta}}\right)\right], \tag{4.2.4}$$

where we omitted the intermediate steps. These two results were first reported to us in private communication by D. Stanford and later published in [140][3]. To introduce standard nomenclature of the field, we would like to point out that both universal results have a linear leading-order behaviour in τ, as we will see momentarily, which is denoted as the "ramp". For large times, however, both results converge to a β dependent constant, the so-called "plateau".

While the universal results for JT/topological gravity can be presented in a closed form, our approach to compute quantities geometrically, i.e. by using the WP volumes, in these theories is genuinely perturbative. Hence, to compare the universal result to the gravitational one, arising from the (Airy) WP volumes, we have to expand both results in a series in τ and β. For topological gravity we already worked out the full series expansion, which we now explicitly give up to $\mathcal{O}(\tau^{13})$, the order up to which we have computed gravitational results to compare, as

$$\kappa_2^{s,\text{Airy}}(\tau,\beta) = \frac{\tau}{4\pi\beta} - \frac{\tau^3}{6\pi} + \frac{\beta\tau^5}{10\pi} - \frac{\beta^2\tau^7}{21\pi} + \frac{\beta^3\tau^9}{54\pi} - \frac{\beta^4\tau^{11}}{165\pi} + \mathcal{O}\left(\tau^{13}\right). \tag{4.2.5}$$

For JT gravity we do not explicitly work out the coefficients' general expression but rather directly present the expanded result, given by

$$\begin{aligned}\kappa_2^{s,\text{JT}}(\tau,\beta) =& \frac{\tau}{4\pi\beta} - \frac{\tau^3}{6\pi} + \left(\frac{\beta}{10\pi} + \frac{2\pi}{15}\right)\tau^5 - \frac{\left(15\beta^2 + 60\pi^2\beta + 64\pi^4\right)\tau^7}{315\pi} + \\ &+ \frac{\left(35\beta^3 + 280\pi^2\beta^2 + 784\pi^4\beta + 768\pi^6\right)\tau^9}{1890\pi} \\ &- \frac{\left(315\beta^4 + 4200\pi^2\beta^3 + 21840\pi^4\beta^2 + 52480\pi^6\beta + 49152\pi^8\right)\tau^{11}}{51975\pi} \\ &+ \mathcal{O}\left(\tau^{13}\right).\end{aligned} \tag{4.2.6}$$

What one can directly see when comparing the two expressions, is the agreement of the topological gravity result with the terms of order π^0 of the JT gravity result. Of course, this is expected, since those are precisely the terms originating from the topological gravity part of the WP volumes.

3 Note that the result for topological gravity, eq. (4.2.3), was previously shown in [146, 147].

4.2.1 *A perturbative plateau*

Having worked out the universal RMT prediction for JT/topological gravity, we can compare to the geometric results. Specific to the orientable case is the knowledge of the full general structure of the WP volumes, given in theorem 2.2. This enables us to first compute the SFF for an arbitrary choice of coefficients and second to put in the actual coefficients of the orientable WP volumes. For the computation of the orientable WP volumes we used topological recursion, reviewed in section 2.2, with an example computation given in example 2.6. Using this procedure, we could find exact results for the WP volumes with $n = 2$ boundaries up to $g = 5$. We refer to [1][App. D] for the explicit results. It has to be noted at the beginning of this discussion that, as we mentioned already above, our approach to JT gravity from the geometric side produces the topological expansion of the SFF order-by-order. Hence, by construction, we can compare only order-by-order to the universal RMT predictions derived above.

THE SFF FOR ARBITRARY COEFFICIENTS. For the generic computation, we recall the structure of the orientable WP volumes for $g > 0$ to be given by (theorem 2.2):

$$V^2_{g,n}(\vec{b}) = \sum_{\vec{a}}^{\|\vec{a}\|_1 \leq k(g,n)} C^{\vec{a}}_{g,n} \prod_{i=1}^{n} b_i^{2a_i}, \tag{4.2.7}$$

where the coefficients are defined as in the theorem and we only renamed the collection of the n boundary lengths to $\vec{b}$ and included a superscript of $\beta = 2$ to indicate that we are dealing with the orientable case. For the specific case of two boundaries, this implies

$$V^2_{g,2}(b_1, b_2) = \sum_{\vec{\alpha} \in \mathbb{N}_0^2}^{\|\vec{\alpha}_1\| \leq 3g-1} C^{\vec{\alpha}}_g b_1^{2\alpha_1} b_2^{2\alpha_2}, \tag{4.2.8}$$

with $C^{\vec{\alpha}}_g \in \pi^{6g-2-2\|\vec{\alpha}\|_1} \cdot \mathbb{Q}_{\geq 0}$, where we dropped the reference to the number of considered boundaries to simplify notation. Using this, we can now explicitly work out the contributions to the topological expansion of the SFF.

For the special case of $g = 0$, we already worked out the result for the two-point correlation function of partition functions, determining the SFF, in eq. (2.1.39). Note that this result actually holds for all (unitary) double-scaled matrix models due to the expression for the corresponding contribution to the two-point correlation function ($R^2_0(x_1, x_2)$) being the same for all such matrix models (eq. (2.2.52)). In fact, using this result we can directly perform the generalisation to arbitrary β and determine the contribution to the SFF at genus 0 for any double-scaled matrix model (and hence all theories dual to such a matrix model) and symmetry class by putting $\beta_1 = \beta + it$, $\beta_2 = \beta - it$ into eq. (2.1.39) and adding the known dependence on β. This yields

$$\kappa^0_{\mathrm{\beta}}(t, \beta) = \frac{1}{\mathrm{\beta}\pi} \frac{\sqrt{\beta^2 + t^2}}{2\beta}. \tag{4.2.9}$$

We can now directly compute the τ-scaling limit for this expression as

$$\lim_{e^{S_0}, t \to \infty} e^{-S_0} \kappa^0_{\mathrm{\beta}}(e^{S_0}\tau, \beta) = \frac{\tau}{2\pi \mathrm{\beta} \beta}. \tag{4.2.10}$$

For β = 2 this matches with the universal prediction for JT/topological gravity. This was first noted in [68].

Coming back to the case of $g > 0$, we give the generic result for the contribution to the scaled SFF in the following proposition.

Proposition 4.1 (General result for the scaled SFF for β = 2). *For $g > 0$, it holds that*

$$\kappa^{s,g}_2(t, \beta) = \frac{(-1)^g 2^{2g}}{2\pi\gamma^{2g+1}} \sum_{a=0}^{g-1} \frac{2^a}{\gamma^a} \sum_{m=0}^{\lfloor \frac{a}{2} \rfloor} t^{2m} \beta^{a-2m} (-1)^m \sqrt{1 + \frac{\beta^2}{t^2}} k(g, a + 2g, 2m + 2g), \tag{4.2.11}$$

with γ defined in section 2.1 (in the rest of this work, $\gamma = \frac{1}{2}$) and

$$k(g, a, b) = \sum_{\vec{\alpha} \in \mathbb{N}_0^2}^{\alpha_1 + \alpha_2 = a} C_g^{\vec{\alpha}} \alpha_1! \alpha_2! \sum_{(j,k) \in \mathbb{N}_0}^{j+k=b} \binom{\alpha_2}{j} \binom{\alpha_1}{k} (-1)^j, \tag{4.2.12}$$

where the $C_g^{\vec{\alpha}}$ are the coefficients of the orientable WP volumes in the notation of eq. (4.2.8).

Proof of proposition 4.1. To find the SFF, we first work out the contribution to the two-point function of partition functions, using eq. (2.1.43), to be

$$\begin{aligned} Z^2_{g,2}(\beta_1, \beta_2) &= \int_0^\infty b_1 \, \mathrm{d}b_1 \int_0^\infty b_2 \, \mathrm{d}b_2 \, Z^t(\beta_1, b_1) Z^t(\beta_2, b_2) V^2_{g,2}(b_1, b_2) \\ &= \sum_{\vec{\alpha} \in \mathbb{N}_0^2}^{\alpha_1 + \alpha_2 \le 3g-1} C_g^{\vec{\alpha}} \frac{\sqrt{\beta_1 \beta_2}}{\pi} \frac{\alpha_1! \alpha_2! 2^{\alpha_1 + \alpha_2 - 1}}{\gamma^{\alpha_1 + \alpha_2 + 1}} \beta_1^{\alpha_1} \beta_2^{\alpha_2}, \end{aligned} \tag{4.2.13}$$

where we used the γ dependent version of the trumpet partition function, eq. (2.1.31). We included this to make the dependence of the SFF on γ explicit. Using this result, we can now work out the contributions to the SFF directly by putting $\beta_1 = \beta + it$, $\beta_2 = \beta_1^*$. Note that the resulting expression is necessarily real. This can be seen from first observing $Z^2_{g,2}(\beta_1, \beta_1^*)^* = Z^2_{g,2}(\beta_1^*, \beta_1)$ via

$\underset{z\in\mathbb{C}}{\forall}\underset{k\in\mathbb{N}_0}{\forall}(z^*)^k = (z^k)^*$ and then noting that $Z^2_{g,2}(\beta_1,\beta_2) = Z^2_{g,2}(\beta_2,\beta_1)$ by the symmetry of the WP volumes, which implies $Z^2_{g,2}(\beta_1^*,\beta_1) = Z^2_{g,2}(\beta_1,\beta_1^*)$. Putting in the required values for the complex temperatures then yields (using the definition in eq. (4.1.1))

$$\begin{aligned}
\kappa_2^g(t,\beta) &= \sum_{\vec{\alpha}\in\mathbb{N}_0^2}^{\alpha_1+\alpha_2\leq 3g-1} C_g^{\vec{\alpha}}\frac{\sqrt{\beta^2+t^2}}{\pi}\frac{\alpha_1!\alpha_2!2^{\alpha_1+\alpha_2-1}}{\gamma^{\alpha_1+\alpha_2+1}}\times \\
&\sum_{k=0}^{\alpha_1}\sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\binom{\alpha_1}{k} i^{j+k}(-1)^j\beta^{\alpha_2+\alpha_1-k-j}t^{j+k} \\
&= \frac{1}{2\pi}\sum_{a=0}^{3g-1}\frac{2^a}{\gamma^{a+1}}\sum_{b=0}^{a} t^{b+1}\beta^{a-b}i^b\sqrt{1+\frac{\beta^2}{t^2}}\sum_{\vec{\alpha}\in\mathbb{N}_0^2}^{\alpha_1+\alpha_2=a} C_g^{\vec{\alpha}}\alpha_1!\alpha_2!\times \\
&\sum_{(j,k)\in\mathbb{N}_0}^{j+k=b}\binom{\alpha_2}{j}\binom{\alpha_1}{k}(-1)^j \\
&=: \frac{1}{2\pi}\sum_{a=0}^{3g-1}\frac{2^a}{\gamma^{a+1}}\sum_{b=0}^{a} t^{b+1}\beta^{a-b}i^b\sqrt{1+\frac{\beta^2}{t^2}}k(g,a,b),
\end{aligned} \tag{4.2.14}$$

where we collected the sums of the summation indices in new summation variables. Due to the reality of the contribution to the SFF, proven above, we can infer that $k(g,a,b)$ has to vanish for all odd values of b for all choices of g and a[4]. Making this explicit, we rewrite the above expression as

$$\kappa_2^g(t,\beta) = \frac{1}{2\pi}\sum_{a=0}^{3g-1}\frac{2^a}{\gamma^{a+1}}\sum_{m=0}^{\lfloor\frac{a}{2}\rfloor} t^{2m+1}\beta^{a-2m}(-1)^m\sqrt{1+\frac{\beta^2}{t^2}}k(g,a,2m). \tag{4.2.15}$$

So far, we took into account all possible terms occurring in the SFF for JT gravity. However, we are interested only in the terms contributing in the universal limit, explained for the SFF in a double-scaled theory in section 4.1. To do this, we could directly use the definition of $\kappa_2^{s,g}(t,\beta)$ in eq. (4.1.12). However, we would like to give more intuition for this expression in the first case where we use it and hence we will work out the scaled SFF directly.

To perform this scaling explicitly, means that we have to perform the τ-scaling. As a first step for this, we rewrite $t = e^{S_0}\tau$ in eq. (4.2.15) yielding

$$\kappa_2^g(e^{S_0}\tau,\beta) = \frac{1}{2\pi}\sum_{a=0}^{3g-1}\frac{2^a}{\gamma^{a+1}}\sum_{m=0}^{\lfloor\frac{a}{2}\rfloor}\tau^{2m+1}{e^{S_0}}^{2m+1}\beta^{a-2m}(-1)^m\sqrt{1+\frac{\beta^2}{e^{2S_0}\tau^2}}k(g,a,2m). \tag{4.2.16}$$

4 One can also directly prove this from the explicit form of $k(g,a,b)$ using the symmetry of the coefficients.

Using this, we can directly evaluate the corresponding contribution to $\kappa_2^s(\tau,\beta)$ including the factors of e^{S_0} arising from the topological expansion as

$$\begin{aligned}
&\frac{1}{2\pi}\sum_{a=0}^{3g-1}\frac{2^a}{\gamma^{a+1}}\sum_{m=0}^{\lfloor\frac{a}{2}\rfloor}\tau^{2m+1}{e^{2S_0}}^{m-g}\beta^{a-2m}(-1)^m\sqrt{1+\frac{\beta^2}{e^{2S_0}\tau^2}}k(g,a,2m)\\
&=\tau^{2g+1}\frac{1}{2\pi}\sum_{a=2g}^{3g-1}\frac{2^a}{\gamma^{a+1}}\sum_{m=g}^{\lfloor\frac{a}{2}\rfloor}t^{2(m-g)}\beta^{a-2m}(-1)^m\sqrt{1+\frac{\beta^2}{t^2}}k(g,a,2m)+\mathcal{O}\left(e^{-S_0}\right)\\
&=\tau^{2g+1}\kappa_2^{s,g}(t,\beta)+\mathcal{O}\left(e^{-S_0}\right),
\end{aligned} \tag{4.2.17}$$

where to the second line we dropped contributions that can not contribute in the universal limit as they vanish upon $e^{S_0}\to\infty$ and to the third line recognised the remaining expression as $\kappa_2^{s,g}(t,\beta)$. This observation, of course, motivated its definition in eq. (4.1.12). It is illustrative to rewrite $\kappa_2^{s,g}(t,\beta)$ further, shifting the summation indices, as

$$\kappa_2^{s,g}(t,\beta)=\frac{(-1)^g2^{2g}}{2\pi\gamma^{2g+1}}\sum_{a=0}^{g-1}\frac{2^a}{\gamma^a}\sum_{m=0}^{\lfloor\frac{a}{2}\rfloor}t^{2m}\beta^{a-2m}(-1)^m\sqrt{1+\frac{\beta^2}{t^2}}k(g,a+2g,2m+2g), \tag{4.2.18}$$

yielding the claimed result. □

For the following discussion, it will be sometimes convenient so slightly modify the proven expression to

$$\kappa_2^{s,g}(t,\beta)=\frac{(-1)^g2^{2g}}{2\pi\gamma^{2g+1}}\sum_{a=0}^{g-1}\frac{2^a}{\gamma^a}\sum_{b=0}^{a}t^b\beta^{a-b}(i)^b\sqrt{1+\frac{\beta^2}{t^2}}k(g,a+2g,b+2g), \tag{4.2.19}$$

which reduces to the expression above upon noting that, as proven in the proof above,

$$\underset{g\in\mathbb{N}}{\forall}\ \underset{\substack{a\in\mathbb{N}_0\\ a\leq g-1}}{\forall}\ \underset{\substack{b\in\mathbb{N}\\ b\leq a\\ b\text{ odd}}}{\forall}\ k(g,a+2g,b+2g)=0. \tag{4.2.20}$$

COMPARISON TO THE UNIVERSAL RESULT. Having now found the generic form of $\kappa_2^{s,g}(t,\beta)$ for JT gravity, we now only have to take the limit of $t\to\infty$, presently in front of the whole sum computing $\kappa_2^s(t,\beta)$, to find the result for the JT gravity SFF upon inserting the coefficients of the WP volumes. Before following through with this, it is advisable to briefly contemplate eq. (4.2.19) to assess, whether this can match to the universal result for JT gravity at all.

For this, it is useful to recall that in the universal result for JT gravity, given in eq. (4.2.6), there is no additional dependence of the coefficients of the series expansion in τ on t. This is an observation valid for all symmetry classes and follows directly from eq. (4.1.9), where e^{S_0} does not appear and hence the functional dependence on time is given only by a dependence on the scaled time τ. A posteriori, this can be seen as the reason why the τ-scaling produces a sensible result.

From this observation, we can now infer that for our gravitational result to match with that from universal RMT it has to hold that for a given g all terms in $\kappa_2^{s,g}(t,\beta)$ except that for $m=0/b=0$ vanish. For the orientable case, these are precisely the constraints on the moduli space volumes arising from universality. We will discuss them in the next section and for the rest of this computation assume that they are fulfilled, i.e.

$$\kappa_2^{s,g}(t,\beta) = \frac{(-1)^g 2^{2g}}{2\pi\gamma^{2g+1}} \sum_{a=0}^{g-1} \frac{2^a}{\gamma^a}\beta^a \sqrt{1+\frac{\beta^2}{t^2}}\, k(g, a+2g, 2g). \tag{4.2.21}$$

Now we can take the limit on t explicitly inside the sum, leading to to the vanishing of all but the leading-order contributions from the square root. Hence, we can write the whole τ-scaled SFF from the topological expansion of JT gravity as

$$\kappa_2^{s}(\tau,\beta) = \frac{1}{4\pi} \sum_{g=0}^{\infty} (-1)^g \left(\frac{2\tau}{\gamma}\right)^{2g+1} \sum_{a=0}^{g-1} \left(\frac{2\beta}{\gamma}\right)^a k(g, a+2g, 2g). \tag{4.2.22}$$

We now work out the explicit form of these contributions, using the definition of $k(g, a+2g, 2g)$, for some exemplary values of the genus.

For $g=1$ we find

$$-\tau^3 \frac{2}{\pi\gamma^3}\left[4C_1^{2,0} - C_1^{1,1}\right] = -\tau^3 \frac{2}{\pi\gamma^3}\frac{1}{96} = \frac{1}{(2\gamma)^3}\frac{-\tau^3}{6\pi}, \tag{4.2.23}$$

where to the last line we put the values of the coefficients, which one can read off from eq. (2.3.29). Putting now $\gamma = \frac{1}{2}$ in this expression, or restoring the γ dependence in the universal results above (eqs. (4.2.5) and (4.2.6)), one observes the (expected) agreement of the two universal results with the contribution arising from the WP volume. Notably, in this case only those terms of the WP volume contributed which fulfilled $\alpha_1 + \alpha_2 = 2g = 3g - 1$ and hence the JT gravity and the topological gravity result coincide.

For $g = 2$ one can reason similarly to find for the contribution arising at this genus

$$\frac{32}{\pi}\left[\frac{2^5\tau^5}{(2\gamma)^5}\left(C_2^{2,2} - 3C_2^{3,1} + 12C_2^{4,0}\right) + \frac{12\times 2^6\tau^5\beta}{(2\gamma)^6}\left(C_2^{3,2} - 6C_2^{4,1} + 50C_2^{5,0}\right)\right]$$
$$= \left[\frac{2\pi}{15} + \frac{1}{10\pi}\frac{\beta}{2\gamma}\right]\left(\frac{\tau}{2\gamma}\right)^5, \tag{4.2.24}$$

where we took the explicit results for the coefficients from [1][App. D]. They are given by

$$C_2^{5,0} = \frac{1}{4423680}, C_2^{4,1} = \frac{1}{294912}, C_2^{3,2} = \frac{29}{2211840}, \tag{4.2.25}$$
$$C_2^{4,0} = \frac{7\pi^2}{7680}, C_2^{3,1} = \frac{29\pi^2}{69120}, C_2^{2,2} = \frac{11\pi^2}{276480}. \tag{4.2.26}$$

Note that to restore γ in the universal result, one has to scale every factor of β, in addition to the factors of τ, by $\frac{1}{2\gamma}$. This is due to the fact that one is actually scaling the *complex* β and hence it is necessary to scale both its imaginary and real part. Doing this, both terms agree with the universal result for JT gravity.

Up to the order to which we have checked ($g = 5$), this agreement persisted. Hence, we have very substantial evidence for the persistence of random matrix universality as induced by the matrix model dual in JT/topological gravity.

WHY THIS IS SURPRISING. This agreement is of course expected from our reasoning in section 4.1, leading to the universal RMT prediction for the gravitational theories we are interested in. Hence, one could argue that what we did up to now was merely a sanity check[5]. Indeed, the author is convinced of the persistence of random matrix universality in the way we present it here and so far is not aware of a counterexample. However, we would like to argue that the matching of the universal RMT prediction with the "perturbative" result arising from the WP volumes is quite surprising, already for topological gravity, from a perspective we shall explain momentarily.

This perspective is obtained by considering not correlation functions of partition functions but rather of densities of states. The behaviour of their two-point function in the universal limit, given in eq. (4.1.7), actually was the basis for the universal prediction for the scaled SFF. Consequently, it is plausible that the topological expansion of the spectral two-point function in orientable topological gravity as arising from the WP volumes should coincide with this prediction.

5 Note that even if taking this perspective, the constraints we prove below are a non-trivial statement. In fact, an even stronger one.

As we discussed in section 2.2, the contribution to this two-point function of densities of states at genus g is given by (eq. (2.3.42))

$$\rho_{g,2}(x_1,x_2) = \frac{-1}{4\pi^2} \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left[R(++) + R(--) - R(-+) - R(+-)\right], \quad (4.2.27)$$

where $R(\pm,\mp) := R(x_1 \pm \epsilon_1, x_2 \mp \epsilon_2)$. To illustrate this, we consider the following example.

Example 4.1 (Genus 0 contribution to $\langle\rho(x_1)\rho(x_2)\rangle$)**.** *We will perform this computation for an arbitrary value of the Dyson index for later use. Hence, our starting point is (eq. (2.2.52))*

$$R^{\beta}_{0,2}(x_1,x_2) = \frac{1}{2\beta}\frac{1}{\left(\sqrt{-x_1}+\sqrt{-x_2}\right)^2\sqrt{-x_1}\sqrt{-x_2}}. \quad (4.2.28)$$

Using the branch-cut structure of the square root, eq. (2.2.57), we find that in the limit of $\epsilon_i \to 0$ one can compute the above expression by replacing

$$+ \to -i\sqrt{E} \quad - \to i\sqrt{E}. \quad (4.2.29)$$

Hence, using the above expression for $\rho_{0,2}(x_1,x_2)$ we can deduce

$$\begin{aligned}\rho_{0,2}(x_1,x_2) &= \frac{2}{(2\pi i)^2}\frac{1}{2\beta}\left[\frac{1}{i^2\sqrt{E_1}\sqrt{E_2}i^2\left(\sqrt{E_1}+\sqrt{E_2}\right)^2} - \frac{1}{-i^2\sqrt{E_1}\sqrt{E_2}i^2\left(\sqrt{E_1}-\sqrt{E_2}\right)^2}\right]\\ &= \frac{-1}{2\pi^2}\frac{1}{\beta}\frac{E_1+E_2}{\sqrt{E_1}\sqrt{E_2}(E_1-E_2)^2}.\end{aligned} \quad (4.2.30)$$

To find the contribution in the universal limit from this expression, one has to evaluate

$$\lim_{e^{S_0}\to\infty} e^{-2S_0}\rho_{0,2}\left(E+\frac{e^{-S_0}x}{2}, E-\frac{e^{-S_0}x}{2}\right) = \frac{-1}{\beta\pi^2}\frac{1}{x^2}. \quad (4.2.31)$$

From this example, one can already see that the term that is singular in x as $x \to 0$ arises purely from the term in the denominator of $R^{\beta}_{0,2}(x_1,x_2)$ that is a sum of the two square roots. In double-cover coordinates this corresponds to a term of (z_1+z_2) in the denominator of the contribution to the two-point function of resolvents. As we derived in eq. (3.3.83), for the orientable case the resolvents for $g > 0$ lack such terms. Hence, the contributions to the two-point function of energy densities are non-singular for $g > 0$ as it was remarked already in [68] (for $g = 1$). Using eq. (3.3.83) explicitly, one can derive

$$\rho_{g,2}(x_1,x_2) \propto \frac{1}{(x_1x_2)^{6g+1}}\mathcal{P}_g(\sqrt{E_1},\sqrt{E_2}), \quad (4.2.32)$$

with a polynomial $P_g(\sqrt{E_1}, \sqrt{E_2})$ of combined degree at most $2k(g) = 6g - 2$. Consequently, putting $E_1 = E + \frac{e^{-S_0}x}{2}$ and $E_2 = E - \frac{e^{-S_0}x}{2}$ and multiplying with e^{-2S_0}, as required for the universal limit in eq. (4.1.7), we see that the resulting expression vanishes in the limit of $e^{S_0} \to \infty$.

In contrast to this, the universal prediction is given by

$$\lim_{e^{S_0}\to\infty} e^{-2S_0} \left\langle \rho\left(E + \frac{e^{-S_0}x}{2}\right)\rho\left(E - \frac{e^{-S_0}x}{2}\right)\right\rangle_c = (\rho_0(E))^2\delta(x\rho_0(E)) - (\rho_0(E))^2 Y^{\beta}(x\rho_0(x)), \tag{4.2.33}$$

where for $\beta = 2$ we have (eq. (2.4.9))

$$Y^2(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2} = \frac{1 - \cos(2\pi x)}{2\pi^2 x^2}. \tag{4.2.34}$$

Here, the term containing the δ distribution can be interpreted as arising as a contact-term and hence the topological expansion of the two-point function of spectral densities should reproduce

$$-(\rho_0(E))^2 Y^{\beta}(x\rho_0(x)) = -\frac{1}{2\pi^2 x^2} + \frac{\cos(2\pi x\rho_0(x))}{2\pi^2 x^2}. \tag{4.2.35}$$

The first term of this is exactly reproduced by the $g = 0$ contribution in the universal limit (eq. (4.2.31)), while the oscillatory term is not reproduced.

In fact, the non-reproducibility of the oscillatory term by a perturbative computation is not unexpected. Notably, one can reproduce this term in a JT gravity computation, but only in a non-perturbative approach like the computation of correlation functions of branes [68, 148] or the non-perturbative completion of JT gravity as a "universe field theory" of [149]. Consequently, it makes sense to speak of a decomposition of the universal result in a "perturbative" and a "non-perturbative" part, where the oscillatory contribution is precisely the non-perturbative part. Now the part we called perturbative directly corresponds to the "ramp" contribution of the scaled SFF, while consequently the "plateau" is a consequence of the non-perturbative part.

Thus, from this perspective, it is quite surprising that we observe the matching of the universal RMT predictions with the results from the WP volumes and hence a "perturbative plateau"[6]. Further comments on why this is the case can be found in [94, 140]. We will not go further into details here, as this would be outside of the scope of this thesis. A main goal of this thesis, however, is to discuss the implications this agreement with universal RMT has on the WP volumes, which will be explored in the next section.

6 We borrow this fitting alliteration, also as a title of this section, from [94].

4.2.2 *Constraints on orientable moduli space volumes from universality*

As we already remarked, the agreement of the universal prediction with the results from the WP volumes was possible only, since all the terms of non-zero order in t in eq. (4.2.19) dropped. In a formula, this means

$$\underset{g\in\mathbb{N}}{\forall}\ \underset{\substack{a\in\mathbb{N}_0\\ a\leq g-1}}{\forall}\ \underset{\substack{b\in\mathbb{N}\\ b\leq a}}{\forall} k(g, a+2g, b+2g) = 0, \tag{4.2.36}$$

which can be interpreted as a constraint on the coefficients of the WP volumes. Put in explicit contact with the expansion of the scaled SFF, this is just the coefficient of a term contributing to the scaled SFF $\kappa_2^s(\tau, \beta)$ before taking the limit in t as $\tau^{2g+1}t^b$, i.e. a divergent contribution to this. Since such a contribution would be inconsistent with the universal limit, its coefficient has to vanish.

Notably, for odd values of b this vanishing is exactly what we gave as eq. (4.2.20). Hence, we can interpret this as a constraint on the WP volumes, which is however "trivially" fulfilled (since it has to hold by the reality of the SFF or the symmetry of the coefficients).

EXPLICIT CHECKS OF THE CONSTRAINTS FOR JT GRAVITY. We will now first check, whether the other "non-trivial" constraints are actually fulfilled for the examples we have computed and then study them from the perspective of constraints on the coefficients imposed by universality.

For the cases of $g = 1$ and $g = 2$ we see that there is no constraint. Hence, the first ones appears for $g = 3$ where there is only one, given by $k(3, 8, 9) = 0$. Evaluating this and cancelling irrelevant global prefactors it is equivalent to

$$280C_3^{8,0} - 35C_3^{7,1} + 10C_3^{6,2} - 5C_3^{5,3} + 2C_3^{4,4} = 0. \tag{4.2.37}$$

To see that this is fulfilled, we read off the coefficients from the WP volume with two boundaries and $g = 3$ given in [1][App. D] as

$$\begin{aligned} C_3^{8,0} &= \frac{1}{856141332480}, \ C_3^{7,1} = \frac{1}{21403533312}, \ C_3^{6,2} = \frac{77}{152882380800}, \\ C_3^{5,3} &= \frac{503}{267544166400}, \ C_3^{4,4} = \frac{607}{214035333120}, \end{aligned} \tag{4.2.38}$$

and put them into the equation, finding that it is fulfilled. For the next higher genus, $g = 4$, we find two constraints, given by $k(4, 10, 10) = 0$ and $k(4, 11, 10) = 0$. The first of these constraints can be worked out as

$$-\frac{5}{56}C_4^{5,5} + \frac{3}{14}C_4^{6,4} - \frac{3}{8}C_4^{7,3} + C_4^{8,2} - \frac{9}{2}C_4^{9,1} + 45C_4^{10,0} = 0, \tag{4.2.39}$$

the second as

$$3630C_4^{11,0} - 270C_4^{10,1} + 42C_4^{9,2} - 10C_4^{8,3} + 3C_4^{7,4} - \frac{5}{7}C_4^{6,5} = 0. \qquad (4.2.40)$$

Again, we need to check the fulfilment of these constraints by the coefficients of the corresponding WP volume for $g = 4$. Reading them off from [1] as

$$C_4^{5,5} = \frac{533\pi^2}{7134511104000}, \; C_4^{4,6} = \frac{1081\pi^2}{20547391979520}, \; C_4^{3,7} = \frac{16243\pi^2}{898948399104000},$$

$$C_4^{2,8} = \frac{53\pi^2}{19025362944000}, \; C_4^{1,9} = \frac{149\pi^2}{924632639078400}, \; C_4^{0,10} = \frac{23\pi^2}{9246326390784000}, \qquad (4.2.41)$$

and

$$C_4^{11,0} = \frac{1}{650941377911193600}, \; C_4^{10,1} = \frac{1}{8453784128716800}, \; C_4^{9,2} = \frac{149}{59176488901017600},$$

$$C_4^{8,3} = \frac{947}{46026158034124800}, \; C_4^{6,5} = \frac{487}{3287582716723200}, \qquad (4.2.42)$$

one can check that the constraints are again explicitly fulfilled.

As one can see already from these examples, with increasing genus the constraints become quite non-trivial identities fulfilled by increasingly complicated coefficients. We checked them to be fulfilled up to $g = 5$ in [1], while later work ([150]) checked them up to $g = 10$ (!). The observed fulfilment of the constraints thus gives additional weight to the observation of RMT universality (and hence quantum chaos in the sense of the BGS conjecture) in JT/topological gravity.

CONSTRAINTS AS A CONSEQUENCE OF UNIVERSALITY. Turning this reasoning around, we can formulate our first main statement regarding the implications of RMT universality in JT/topological gravity, presented first in [1][7], as

Theorem 4.1 (Implications of RMT universality in orientable JT/topological gravity)**.** *Random matrix universality for the canonical SFF, in the form of eq.* (4.1.9) *for* $\beta = 2$*, implies for the WP volumes with* $n = 2$*, written as in eq.* (4.2.8)*:*

$$\underset{g\in\mathbb{N}}{\forall}\;\underset{\substack{a\in\mathbb{N}_0\\ a\leq g-1}}{\forall}\;\underset{\substack{l\in\mathbb{N}_0\\ l<a}}{\forall}\; K_g^l(a) = 0, \qquad (4.2.43)$$

7 To be precise, let us note that the constraints as presented in eq. (4.2.36) were simultaneously to and independently from us found by the authors of [94]. We were, however, the first to find the convenient form presented next.

with

$$K_g^l(a) = \sum_{n,m\in\mathbb{N}_0^2}^{n+m=a+2g} C_g^{n,m} n!m!(-1)^m(n-m)^l, \tag{4.2.44}$$

where the constraint is trivially fulfilled if $a-l$ is odd.

Proof. As we saw above, consistency of the computation of the scaled SFF from the WP volumes with the universal prediction necessitates

$$\underset{g\in\mathbb{N}}{\forall} \underset{\substack{a\in\mathbb{N}_0\\ a\leq g-1}}{\forall} \underset{\substack{b\in\mathbb{N}\\ b\leq a}}{\forall} k(g, a+2g, b+2g) = 0. \tag{4.2.45}$$

We can write out $k(g, a+2g, b+2g)$, using proposition 4.1, as

$$\sum_{(n,m)\in\mathbb{N}_0^2}^{n+m=a+2g} C_g^{n,m} n!m! \sum_{(j,k)\in\mathbb{N}_0}^{j+k=b+2g} \binom{n}{k}\binom{m}{j}(-1)^j. \tag{4.2.46}$$

The following computation turns out to be particularly simple if one defines $l = a - b$ and replaces the dependence on b by that on l. Using this, we find the constraints, written as above, to be equivalent to

$$\underset{g\in\mathbb{N}}{\forall} \underset{\substack{a\in\mathbb{N}_0\\ a\leq g-1}}{\forall} \underset{\substack{l\in\mathbb{N}_0\\ l<a}}{\forall} \tilde{K}_g^l(a) = 0, \tag{4.2.47}$$

where

$$\begin{aligned}
\tilde{K}_g^l(a) &= k(g, a+2g, a+2g-l)\\
&= \sum_{(n,m)\in\mathbb{N}_0^2}^{n+m=a+2g} C_g^{n,m} n!m! \sum_{(j,k)\in\mathbb{N}_0}^{j+k=a+2g-l} \binom{n}{k}\binom{m}{j}(-1)^j\\
&= \sum_{(n,m)\in\mathbb{N}_0^2}^{n+m=a+2g} C_g^{n,m} n!m! \sum_{(j,k)\in\mathbb{N}_0}^{j+k=n+m-l} \binom{n}{k}\binom{m}{j}(-1)^j\\
&=: \sum_{(n,m)\in\mathbb{N}_0^2}^{n+m=a+2g} C_g^{n,m} n!m! P_l(n,m).
\end{aligned} \tag{4.2.48}$$

Furthermore, we note that an odd value of $a-l$ is equivalent to an odd value of b and hence those constraints for which $a-l$ is odd are fulfilled "trivially" by eq. (4.2.20).

Hence, the proof has boiled down to showing that for admissible value for a and l the constraints $\tilde{K}_g^l(a)$ are equivalent to the $K_g^l(a)$.

To show this, we rewrite the $P_l(n,m)$ for given values of n,m and $l < (n+m)$ using the property $\binom{n}{m} = \binom{n}{n-m}$ of the binomial coefficient as

$$\begin{aligned} P_l(n,m) &= \sum_{(j,k)\in\mathbb{N}_0}^{j+k=n+m-l} \binom{n}{n-k}\binom{m}{m-j}(-1)^j \\ &= (-1)^m \sum_{\substack{\delta,\gamma=0\\ \delta+\gamma=l}} (-1)^\gamma \binom{m}{\gamma}\binom{n}{\delta}, \end{aligned} \tag{4.2.49}$$

with $\gamma = m - j$ and $\delta = n - k$.

As a next step, we use this to show that $P_l(n,m)$ is a polynomial of degree l in m, as well as in n. We do this by showing every occurring term in $P_l(n,m)$ is such a polynomial. For this, we use that for integers a and b with $a < b$, it holds that

$$\binom{b}{a} = \frac{b!}{a!(b-a)!} = \frac{1}{a!}\prod_{j=0}^{a-1}(b-j), \tag{4.2.50}$$

which is a polynomial of degree a in b. Since the maximal value of δ and γ is indeed l, we know that the maximal appearing degree of m and n in $P_l(n,m)$ is l. Consequently, we can write

$$\begin{aligned} P_l(n,m) &= (-1)^m\left[\binom{m}{0}\binom{n}{l} + (-1)^l\binom{m}{l}\binom{n}{0} + (\text{lower order in } n,m)\right] \\ &= (-1)^m\left[\frac{1}{l!}n^l + \frac{1}{l!}(-m)^l + (\text{lower order in } n,m)\right] \\ &= (-1)^m\left[\frac{1}{l!}(n-m)^l + (\text{lower order in } n,m)\right]. \end{aligned} \tag{4.2.51}$$

Now we notice that for a given l the lower-order terms in (n,m) can always be written as a sum of terms of the form $(n-m)^\alpha(n+m)^\beta$ with $\alpha < l$. Since for the constraints $n+m$ is fixed to be $a+2g$, such factors can be taken out of the constraints and consequently the lower order terms are already known to vanish by the constraints written as in the claim for $\tilde{l} < l$.

Consequently, we show the claim by induction. For the base case, $l = 0$, we have

$$\begin{aligned} P_0(n,m) &= \sum_{(j,k)\in\mathbb{N}_0}^{j+k=n+m} \binom{n}{k}\binom{m}{j}(-1)^j = \binom{n}{n}\binom{m}{n}(-1)^m \\ &= (-1)^m(n-m)^0, \end{aligned} \tag{4.2.52}$$

hence $\tilde{K}_g^0(a) = K_g^0(a)$. Then, by the above reasoning the claim follows for $l+1$ assuming its validity for l, which concludes the induction step and the proof. □

From this, it is evident that for a given genus g there can be constraints for $a \geq 2$ and hence there are none for $g \leq 2$, coinciding with our observation above. Furthermore, for a given even value of a there are $\frac{a}{2}$ constraints, while for an odd value of a there are $\frac{a-1}{2}$. This can be checked to be true for all the examples considered above.

As an application we study the non-trivial constraints for $g = 5$ which are easily computed using theorem 4.1. We find $a \leq 4$ and hence one constraint for $a = 2$ and $a = 4$ as well as two constraints for $a = 4$. Hence

$$K_5^0(2) = 0 \Leftrightarrow$$
$$0 = 27720C_5^{0,12} - 2310C_5^{1,11} + 420C_5^{2,10} - 126C_5^{3,9} + 56C_5^{4,8} - 35C_5^{5,7} + 15C_5^{6,6}, \quad (4.2.53)$$

$$K_5^1(3) = 0 \Leftrightarrow$$
$$0 = 22308C_5^{13,0} - 1452C_5^{12,1} + 198C_5^{11,2} - 42C_5^{10,3} + 12C_5^{9,4} - 4C_5^{8,5} + C_5^{7,6}, \quad (4.2.54)$$

$$K_5^0(4) = 0 \Leftrightarrow$$
$$0 = 48048C_5^{14,0} - 3432C_5^{13,1} + 528C_5^{12,2} - 132C_5^{11,3} + + 48C_5^{10,4} - 24C_5^{9,5} + 16C_5^{8,6} - 7C_5^{7,7}, \quad (4.2.55)$$

$$K_5^2(4) = 0 \Leftrightarrow$$
$$0 = 147147C_5^{14,0} - 7722C_5^{13,1} + 825C_5^{12,2} - 132C_5^{11,3} + 27C_5^{10,4} - 6C_5^{9,5} + C_5^{8,6}. \quad (4.2.56)$$

The fulfilment of these constraints can be checked by using the explicit result for the WP volume, given in [1][8]

It is important to remark that the presented constraints are not only valid for JT/topological gravity but any theory that has a matrix model dual and for which one can compute objects that have a polynomial form as the orientable WP volumes (theorem 2.2). Notable examples of such theories are the $(2, 2p+1)$ minimal string [142], which we already discussed above, and the Virasoro minimal string [150]. For the Virasoro minimal string, the constraints were already checked up to $g = 10$, while for the $(2, 2p+1)$ minimal string this is still to be done.

Hence, the constraints have applications beyond pure JT gravity and can be used, for example, to aid in explicit computations for any of these theories by providing a quite non-trivial check on the volume coefficients (or, in

8 Note that we mentioned and checked the latter 3 constraints in [1] but unintentionally did not include the first constraint presented here.

the general case, coefficient functions). Furthermore, they provide the first example of a class of interesting relation between moduli space volumes of hyperbolic manifolds and quantum chaos (defined via the BGS conjecture), of which we shall see more in the following.

Before discussing these, let us briefly emphasise aspects that are special to the orientable theory, being the arguably best understood variety of JT/topological gravity. First, it is notable that the orientable theory showed the expected universal behaviour in the τ-scaling limit without any additional considerations beyond taking said limit. This is due to it allowing for the interchange of the limit in t with the summation in eq. (4.1.12) due to the constraints, which we will find to be a peculiarity of the orientable theories/the unitary symmetry class. Another advantage of the orientable theory is, for the time being, the higher number of mathematical results known about its moduli space volumes as compared with the unorientable theories. Specifically notable is the connection with intersection theory, for the full WP volumes explicitly given by eq. (2.1.50). Using this relation, one can directly translate the constraints on the volumes into a statement about intersection numbers. This was pursued in [94] and resulted in the interesting observation that the constraints we and [94] observed are actually equivalent to a statement about intersection numbers recently found and proven in [151] by different means. Thus, for a theory obeying the orientable constraints we presented above or their analogues in the unorientable setting, discussed in section 4.3.2, finding the reason at the level of the respective theory why they are obeyed yields new insights into the theory or a new interpretation to existing results. This is particularly interesting for the unorientable setting since, as we discussed above, this theory is studied to a much smaller degree than the orientable theory and consequently there are more new results to be found concerning the appearing moduli space volumes.

Consequently, this concludes our discussion of the orientable setting and we now continue to the unorientable one.

4.3 THE UNORIENTABLE CASES

For the treatment of the two "unorientable" Wigner-Dyson classes, i.e. the orthogonal and symplectic one, we follow essentially the same route as for the unitary symmetry class. However, the corresponding computations will turn out to be quite more involved for the two symmetry classes we consider now and consequently we will base our discussion mostly on topological gravity. We will work out the corresponding universal result for both symmetry classes, complemented by a computation for full unorientable JT gravity for the orthogonal symmetry class. Comparing these results for the universal limit of the SFF with its topological expansion, as arising from the unorientable

(Airy) WP volumes, which we have computed in chapter 3, we will find that establishing the expected agreement of both results is not as direct as for the unitary symmetry class. Nevertheless, we provide in section 4.3.1 strong evidence for the matching of both results for topological gravity in the orthogonal and symplectic symmetry class by noticing the emergence of certain asymptotic/convergent series conspiring to make both results agree up to the order we consider. Likewise, we also demonstrate agreement for unorientable JT gravity in the orthogonal symmetry class.

As for the unitary symmetry class, this agreement is only possible due to multiple constraints. We explore these for topological gravity in section 4.3.2, where we find all constraints up to $g = \frac{7}{2}$ and the general form for constraints on particular, genuinely unorientable, contributions to the Airy WP volumes. Again, these constraints can be viewed as the implications of RMT universality on moduli space volumes, now on those for unorientable hyperbolic surfaces.

We would like to clarify again that for the orientable case the agreement of the universal RMT result with the geometric one for the unitary symmetry class was shown in [140], based on known results for the WP volumes, and the constraints were then observed by us in [1] and Blommaert and collaborators in [94] afterwards. In contrast to this, for the unorientable setting we were the first to compute the universal results, the relevant moduli space volumes and observe constraints. For the Airy model we published this in [2] for the orthogonal symmetry class and in [5][9] for the symplectic one, while we published the treatment of the orthogonal case for JT gravity in [3]. Consequently, the respective considerations for the symmetry classes and theories in the following are based on the corresponding publication. We will not explicitly say so, unless a result presented here is stronger/extends one of the published statements.

THE UNIVERSAL RMT PREDICTION FOR THE SFF FOR $\beta = 1$. We begin by considering the universal prediction for the orthogonal symmetry class. Plugging eq. (2.4.12) into eq. (4.1.9), we find for a given leading-order density of states $\rho_0(x)$:

$$\begin{aligned}\kappa_1^s(\tau,\beta) &= \int_0^{E^\star} \mathrm{d}E\, e^{-2\beta E}\left[2\rho_0(E) - \frac{\tau}{2\pi}\log\left(\frac{\frac{\tau}{\pi}+\rho_0(E)}{\frac{\tau}{\pi}-\rho_0(E)}\right)\right] \\ &\quad + \int_{E^\star}^{\infty} \mathrm{d}E\, e^{-2\beta E}\left[\frac{\tau}{\pi} - \frac{\tau}{2\pi}\log\left(1+\frac{\tau}{\pi\rho_0(E)}\right)\right] \\ &= 2\kappa_2^s(\tau,\beta) + \chi^1(\tau,\beta),\end{aligned} \tag{4.3.1}$$

9 As this thesis is written, the paper is accepted for publication, though not yet published.

with

$$\chi^1(\tau,\beta) = -\frac{\tau}{2\pi}\Big[\int_0^\infty dE\, e^{-2\beta E}\log\left(1+\frac{\tau}{\pi\rho_0(E)}\right) \\ -\int_0^{E^*} dE\, e^{-2\beta E}\log\left(-1+\frac{\tau}{\pi\rho_0(E)}\right)\Big], \tag{4.3.2}$$

and, as above, E^* defined via $\rho_0(E^*) = \frac{\tau}{2\pi}$.

We observe the convenient appearance of the result for the unitary symmetry class, which we evaluated already above for both theories we are interested in. Consequently, we only need to compute $\chi^1(\tau,\beta)$. We evaluate this expression further as[10]

$$\begin{aligned} -\frac{2\pi}{\tau}\chi^1(\tau,\beta) &= \int_0^\infty dE\, e^{-2\beta E}\log\left(\frac{\tau}{\pi}+\rho_0(E)\right)+ \\ &\quad -\int_0^{E^*} dEe^{-2\beta E}\log\left(\frac{\tau}{\pi}-\rho_0(E)\right) - \int_{E^*}^\infty dEe^{-2\beta E}\log\left(\rho_0(E)\right) \\ &= \frac{1}{2\beta}\left[\int_0^\infty e^{-2\beta E}\frac{\rho_0'(E)dE}{\left(\frac{\tau}{\pi}+\rho_0(E)\right)} + \int_0^{E^*} e^{-2\beta E}\frac{\rho_0'(E)dE}{\left(\frac{\tau}{\pi}-\rho_0(E)\right)} - \int_{E^*}^\infty e^{-2\beta E}\frac{\rho_0'(E)dE}{\rho_0(E)}\right] \\ &= \frac{1}{2\beta}\int_0^\infty dx\left[e^{-2\beta\rho_0^{-1}\left(\frac{x-\tau}{2\pi}\right)} - e^{-2\beta\rho_0^{-1}\left(\frac{x+\tau}{2\pi}\right)}\right]\frac{1}{x+\tau}, \end{aligned} \tag{4.3.3}$$

where to the second line a partial integration was performed and to the third line it was assumed that ρ_0 is invertible with the inverse function denoted as ρ_0^{-1}. For the densities of states we are considering, this is always true since they are monotonously increasing. Furthermore, we require that ρ_0^{-1} is symmetric, which is also always true for the densities of state we consider, due to the dependence on the energy via a square root. Now we use the translation operator

$$\hat{T}(a) := \exp\left(-\frac{i}{\hbar}a\hat{p}\right) = \exp\left(-a\partial_x\right), \tag{4.3.4}$$

which has the property $\hat{T}(a)f(x) = f(x-a)$[134]. Consequently, it holds that $\hat{T}(2\tau)f(x+\tau) = f(x-\tau)$ and we can write the above expression of $\chi^1(\tau,\beta)$, expanding the translation operator in its series expansion and noting that the $n=0$ term cancels against the subtracted term in the above expression, as

$$\begin{aligned} \chi^1(\tau,\beta) &= -\frac{\tau}{4\pi\beta}\sum_{n=1}^\infty \frac{(-2\tau)^n}{n!}\int_0^\infty dx\,\frac{1}{x+\tau}\frac{\partial^n}{\partial x^n}e^{-2\beta\rho_0^{-1}\left(\frac{x+\tau}{2\pi}\right)} \\ &= -\frac{\tau}{4\pi\beta}\sum_{n=1}^\infty \frac{(-2\tau)^n}{n!}\int_\tau^\infty dx\,\frac{1}{x}\frac{\partial^n}{\partial x^n}e^{-2\beta\rho_0^{-1}\left(\frac{x}{2\pi}\right)}. \end{aligned} \tag{4.3.5}$$

10 This reasoning follows the steps first written down by J.Tall in [2, 3], which we slightly generalise here.

Up to this point we have worked with a general density of states, now we will insert those in which we are actually interested.

First, we consider topological gravity, i.e. the theory given by $\rho_0(x) = \frac{\sqrt{x}}{2\pi}$ (eq. (2.3.31)). Consequently, $\rho_0^{-1} = 4\pi^2 x^2$ and it holds that

$$\chi^{1,\text{Airy}}(\tau,\beta) = -\frac{\tau}{4\pi\beta}\sum_{n=1}^{\infty}\frac{(-2\tau)^n}{n!}\int_{\tau}^{\infty}\mathrm{d}x\,\frac{1}{x}\frac{\partial^n}{\partial x^n}e^{-2\beta x^2}. \tag{4.3.6}$$

This can be evaluated further, the steps for this can be found in [2][App. E], to yield

$$\begin{aligned}\chi^{1,\text{Airy}}(\tau,\beta) = \frac{\tau}{8\pi\beta}\Bigg[&\Gamma(0,2\beta\tau^2)(e^{-8\beta\tau^2}-1) + e^{-8\beta\tau^2}\pi\,\text{Erfi}\left(\sqrt{8\beta\tau^2}\right)\\ &+ e^{-8\beta\tau^2}16\beta\tau^2\,{}_2F_2\left(1,1;\frac{3}{2},2;8\beta\tau^2\right)\\ &- e^{-8\beta\tau^2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(-1)^{n+m}(2)^{2m}}{(m)!(n-m)!(n-\frac{m}{2})}\right)\left(2\beta\tau^2\right)^n\Bigg],\end{aligned} \tag{4.3.7}$$

where Erfi denotes the imaginary Error function and ${}_2F_2(a,b;c,d;z)$ is the generalised hypergeometric function. Expanding this to the order up to which we can determine results from the Airy WP volumes, computed in section 3.3.3, yields (denoting by γ the Euler-Mascheroni constant)

$$\begin{aligned}\chi^{1,\text{Airy}}(\tau,\beta) = &-\frac{\tau^2}{\sqrt{2\pi}\sqrt{\beta}} + \frac{8}{3}\sqrt{\frac{2}{\pi}}\sqrt{\beta}\tau^4 - \frac{1}{15}128\sqrt{\frac{2}{\pi}}\beta^{3/2}\tau^6 + \frac{2048}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}\tau^8 +\\ &-\frac{\tau^3\left(\log\left(2\beta\tau^2\right)+\gamma\right)}{\pi} + \frac{2\beta\tau^5\left(6\log\left(2\beta\tau^2\right)+6\gamma-1\right)}{3\pi}\\ &-\frac{\beta^2\tau^7\left(160\log\left(2\beta\tau^2\right)+160\gamma-73\right)}{15\pi} + \mathcal{O}\left(\tau^9\right),\end{aligned} \tag{4.3.8}$$

where we already performed a split into the terms of even and odd order in τ. We will find in the following that odd orders arise from WP volumes at integer genus, while those of even order arise from half-integer genus. Consequently, the addition of the terms of even order in τ to the structure observed in the orientable theory is expected as the consequence of the addition of half-integer genus surfaces to the theory. Maybe more surprising, is the modification of the behaviour of the contributions at integer genus which in addition to the dependence on a power of β now also has a logarithmic dependence on β and τ.

Second, for JT gravity we have $\rho_0(x) = \frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{x}\right)$ and consequently $\rho_0^{-1}(x) = \frac{1}{4\pi^2}\,\text{arsinh}^2\left(4\pi^2 x\right)$. This then leads to

$$\chi^{1,\text{JT}}(\tau,\beta) = -\frac{\tau}{4\pi\beta}\sum_{n=1}^{\infty}\frac{(-2\tau)^n}{n!}\int_{\tau}^{\infty}\mathrm{d}x\,\frac{1}{x}\frac{\partial^n}{\partial x^n}e^{-\frac{2\beta}{4\pi^2}\text{arsinh}^2(2\pi x)}. \tag{4.3.9}$$

Unlike for the Airy case, we could not find a closed form solution for this expression. However, we could work out the contributions to the orders to which we will be able to compare to results on the geometric side. Those are the contributions up to $g = 1$ and hence those contributions with $n = 1$ and $n = 2$ in the presented expression. The explicit steps to work out these contributions are given in [3][App. C] and yield

$$\begin{aligned}\chi^{1,\mathrm{JT}}(\tau,\beta) &= \chi^{1,\mathrm{Airy}}(\tau,\beta) - \frac{2\tau^2}{\sqrt{2\pi\beta}}\sum_{k=1}^{\infty}(-1)^k\left(1 - k\sqrt{\frac{\beta\pi}{2}}e^{\frac{\beta k^2}{2}}\,\mathrm{Erfc}\left(k\sqrt{\frac{\beta}{2}}\right)\right)\\ &\quad + \frac{4\tau^3}{\pi}\sum_{k=1}^{\infty}\left(-1 + \frac{1}{2}e^{\frac{\beta k^2}{2}}(1+\beta k^2)E_1\left(\frac{\beta k^2}{2}\right)\right) + \mathcal{O}\left(\tau^4\right),\end{aligned} \tag{4.3.10}$$

with the complex Error function Erfc and the exponential integral E_1.

With this we have derived both universal results that we will compare to the SFF arising from the unorientable WP volumes (for β = 1) in the next section.

THE UNIVERSAL RMT PREDICTION FOR THE SFF FOR β = 4. Before doing this, we derive the universal result for the other symmetry class we are interested in, the symplectic one. For this, we again start with eq. (4.1.9), now putting in the result for $b^4(x)$ from eq. (2.4.14). Starting again with an arbitrarily chosen (monotonously increasing) ρ_0, this yields

$$\begin{aligned}\kappa_4^s(\tau,\beta) &= \int_0^{E^*}\mathrm{d}E\,e^{-2\beta E}\rho_0(E) + \int_{E^*}^{\infty}\mathrm{d}E\,e^{-2\beta E}\left(\frac{\tau}{4\pi} - \frac{\tau}{8\pi}\log\left(\left|1-\frac{\tau}{2\pi\rho_0(E)}\right|\right)\right)\\ &= \int_0^{\infty}\mathrm{d}E\,e^{-2\beta E}\min\left(\rho_0(E),\frac{\tau}{4\pi}\right) - \int_{E^*}^{\infty}\mathrm{d}E\,e^{-2\beta E}\frac{\tau}{8\pi}\log\left(\left|1-\frac{\tau}{2\pi\rho_0(E)}\right|\right)\\ &=: \kappa_2^s\left(\frac{\tau}{2},\beta\right) + \chi^4\left(\tau,\beta\right),\end{aligned} \tag{4.3.11}$$

where here we define E^* via $\rho_0(E^*) = \frac{\tau}{4\pi}$. Again, part of the result is conveniently given by the universal result for the unitary symmetry class. The computation of $\chi^4(\tau,\beta)$ can be pursued along similar lines as for $\chi^1(\tau,\beta)$[11], i.e.

$$\begin{aligned}
&\frac{-8\pi}{\tau}\chi^4(\tau,\beta) = \\
&= \lim_{\epsilon\to 0}\int_{E^*}^{E'-\epsilon} e^{-2\beta E}\log\left(\frac{\tau}{\sqrt{E}}-1\right)\mathrm{d}E + \int_{E'+\epsilon}^{\infty} e^{-2\beta E}\log\left(1-\frac{\tau}{\sqrt{E}}\right)\mathrm{d}E \\
&= \frac{-1}{2\beta}\lim_{\epsilon\to 0}\left[\int_{E^*}^{E'-\epsilon} e^{-2\beta E}\frac{\rho_0'(E)\,\mathrm{d}E}{\frac{\tau}{2\pi}-x} + \int_{E'+\epsilon}^{\infty} e^{-2\beta E}\frac{\rho_0'(E)\mathrm{d}E}{\frac{\tau}{2\pi}-E} + \int_{E^*}^{\infty}\mathrm{d}E\,\frac{\rho_0'(E)\,\mathrm{d}E}{\rho(E)}\right] \\
&= \frac{-1}{2\beta}\left[\int_{\frac{\tau}{2}}^{\tau-\epsilon} + \int_{\tau+\epsilon}^{\infty}\right]\mathrm{d}E\, e^{-2\beta\rho_0^{-1}\left(\frac{x}{2\pi}\right)}\left[\frac{1}{\tau-x}+x\right],
\end{aligned} \tag{4.3.12}$$

where we defined E' via $\rho_0(E') = \frac{\tau}{4\pi}$ and in each line proceeded analogously to the computation for $\beta = 1$ (eq. (4.3.3)). Using again the translation operator for the second integral and the geometric series for the first, one can rewrite this as

$$\begin{aligned}
\frac{-16\pi\beta}{\tau}\chi^4(\tau,\beta) = &-\int_{\frac{\tau}{2}}^{\infty}\mathrm{d}x\,\frac{e^{-2\beta\rho_0^{-1}\left(\frac{x}{2\pi}\right)}}{x} \\
&+\sum_{n=0}^{\infty}\left[\int_{\tau}^{\infty}\mathrm{d}x\, e^{-2\beta\rho_0^{-1}\left(\frac{x}{2\pi}\right)}\frac{\tau^n}{x^{n+1}} - \int_{\frac{\tau}{2}}^{\tau}\mathrm{d}x\, e^{-2\beta\rho_0^{-1}\left(\frac{x}{2\pi}\right)}\frac{x^n}{\tau^{n+1}}\right].
\end{aligned} \tag{4.3.13}$$

Let's now consider the model we are interested in, namely topological gravity. As above, we can do this by using $\rho_0^{-1}(x) = 4\pi^2x^2$. Then the integrals result in an incomplete Gamma function, in the case of the integral outside the summation, or a (generalised) exponential integral, for the integrals inside the summation. Consequently:

$$\begin{aligned}
\frac{-16\pi\beta}{\tau}\chi^{4,\mathrm{Airy}}(\tau,\beta) = &-\frac{1}{2}\Gamma\left(0,\beta\frac{\tau^2}{2}\right) + \\
&+\sum_{n=0}^{\infty}\left(\frac{1}{2}E_{\frac{n}{2}+1}\left(2\beta\tau^2\right) - \frac{1}{4}\left(2^{-n}E_{\frac{1}{2}-\frac{n}{2}}\left(\frac{\beta\tau^2}{2}\right) - 2E_{\frac{1}{2}-\frac{n}{2}}\left(2\beta\tau^2\right)\right)\right).
\end{aligned} \tag{4.3.14}$$

11 This reasoning follows the steps first written down by M. Lents in [5], which we slightly generalise here.

Using the series expansion of the occurring special functions, this can be simplified further, for the details we refer to [5][App. H], yielding

$$\begin{aligned}\chi^{4,\mathrm{Airy}}(\tau,\beta) =&\frac{\tau}{32\pi\beta}\left(-\gamma-\log\left(\beta\frac{\tau^2}{2}\right)-\sum_{n=1}^{\infty}\frac{\left(-\beta\frac{\tau^2}{2}\right)^n}{nn!}\right)\\ &-\sum_{n=0}^{\infty}\frac{\tau}{32\pi\beta}\frac{\left(-2\beta\tau^2\right)^n}{n!}\left(-\log 2\beta\tau^2+\psi(n+1)\right)\\ &-\sum_{n=0}^{\infty}\frac{\tau}{32\pi\beta}\Gamma\left(-\frac{2n+1}{2}\right)\left(2\beta\tau^2\right)^{\frac{2n+1}{2}}\\ &-\sum_{k=0}^{\infty}\frac{\tau}{64\pi\beta}\left(-\frac{\beta\tau^2}{2}\right)^k\frac{1}{k!}\frac{2\,{}_2F_1\left(1,2k+1;2k+2;\frac{1}{2}\right)}{2k+1},\end{aligned} \tag{4.3.15}$$

with the Digamma function ψ. Expanding this to the order up to which we have results to compare, we find

$$\begin{aligned}\chi^{4,\mathrm{Airy}}(\tau,\beta) =& \frac{1}{8}\frac{\tau^2}{\sqrt{2\pi}\sqrt{\beta}}-\frac{1}{12}\sqrt{\frac{2}{\pi}}\sqrt{\beta}\tau^4+\frac{1}{15}\sqrt{\frac{2}{\pi}}\beta^{3/2}\tau^6-\frac{4}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}\tau^8+\\ &-\frac{\tau^3\left(\log\left(2\beta\tau^2\right)+\gamma\right)}{16\pi}+\frac{\beta\tau^5\left(6\log\left(2\beta\tau^2\right)+6\gamma-1\right)}{96\pi}\\ &-\frac{\beta^2\tau^7\left(160\log\left(2\beta\tau^2\right)+160\gamma-73\right)}{3840\pi}+\mathcal{O}\left(\tau^9\right)\\ =&\frac{1}{4\times 2}\frac{\tau^2}{\sqrt{2\pi}\sqrt{\beta}}-\frac{1}{4\times 2^3}\frac{8}{3}\sqrt{\frac{2}{\pi}}\sqrt{\beta}\tau^4+\frac{1}{4\times 2^5}\frac{1}{15}128\sqrt{\frac{2}{\pi}}\beta^{3/2}\tau^6\\ &-\frac{1}{4\times 2^7}\frac{2048}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}\tau^8-\frac{1}{4\times 2^2}\frac{\tau^3\left(\log\left(2\beta\tau^2\right)+\gamma\right)}{\pi}+\\ &+\frac{1}{4\times 2^4}\frac{2\beta\tau^5\left(6\log\left(2\beta\tau^2\right)+6\gamma-1\right)}{3\pi}+\\ &-\frac{1}{4\times 2^6}\frac{\beta^2\tau^7\left(160\log\left(2\beta\tau^2\right)+160\gamma-73\right)}{15\pi}+\mathcal{O}\left(\tau^9\right),\end{aligned} \tag{4.3.16}$$

where to the second equation we pulled out factors of two, such that one can directly see that this is consistent with putting the factors that translate between $\beta = 4$ and $\beta = 1$ via theorem 3.2 to the expansion of $\chi^{1,\mathrm{Airy}}(\tau,\beta)$ in eq. (4.3.8). This is an important sanity check for the fidelity of the geometric with the universal results since the absence of this property would be a strong indicator of at least one of the variations of topological gravity not matching to the universal result.

While the evaluation of the integrals in eq. (4.3.13) for the JT gravity density of states seems possible, we did not pursue this computation in [5] and leave it for future work.

We have now derived the universal results for the scaled SFF for the theories we are interested in and can now continue with determining the same object from the geometric computation.

4.3.1 *Not entirely perturbative plateaus*

To perform this comparison, i.e. compare the universal results for the SFF with the geometric results for it, we have to compute first the τ-scaled SFF from the unorientable WP volumes. Due to the availability of results to a higher genus, we will focus first on computing the SFF for topological gravity. Although we have shown the general dependence of the volumes on the boundary lengths in theorem 3.4, we will focus on the explicit results for topological gravity here. This is due to the results being more complicated than those for the orientable theory and hence the comparison with the universal result will be more involved. In fact, we will observe that for the unorientable theories the limit and the sum in eq. (4.1.12) do not commute, i.e. the $\kappa_{\beta}^{s,g}(t,\beta)$ will indeed depend on t. We will give a mechanism enabling the cancellation of the t-dependent parts of certain contributions by the t-dependent parts of other contributions in such a way that we can provide strong evidence for the agreement of the universal with the geometric result up to $\mathcal{O}(\tau^4)$. This limitation is due to contributions of higher genus being required to show agreement to a higher order, as we will explain in more detail momentarily.

We will then compute the contributions to the JT gravity SFF from the full WP volumes we have given in section 3.3.4. Here, it will be apparent that the part of the WP volumes that produces "problematic" behaviour is precisely the topological gravity part of the WP volumes, while the "pure JT" part of the volumes produces contributions directly coinciding with the universal prediction for JT gravity. This, a posteriori, motivates the more in-depth discussion we give for topological gravity.

THE TOPOLOGICAL GRAVITY SFF FOR $\beta \neq 2$. To compute the canonical SFF, we have to work out the two-point function of partition functions in β topological gravity. Using eq. (3.1.8), we can write this as

$$Z_{g,2}^{\beta,\mathrm{Airy}}(\beta_1,\beta_2) = \int_0^\infty b_1\,\mathrm{d}b_1 \int_0^\infty b_2\,\mathrm{d}b_2\, Z^t(\beta_1,b_1)Z^t(\beta_2,b_2)V_{g,2}^{\beta,\mathrm{Airy}}(b_1,b_2). \tag{4.3.17}$$

From theorem 3.4 we recall

$$\mathcal{V}_{g,2}^{i,\mathrm{Airy}}(b_1,b_2) = \mathcal{V}_g^{i,>}(b_1,b_2)\theta(b_1-b_2) + \mathcal{V}_g^{i,>}(b_2,b_1)\theta(b_2-b_1), \quad (4.3.18)$$

with $V_g^{i,>}(b_1, b_2)$ a polynomial with rational coefficients in the lengths of combined order $6g-2$ and $V_{g,2}^{\beta,\text{Airy}}(b_1, b_2)$ arising from the $\mathcal{V}_{g,2}^{i,\text{Airy}}(b_1, b_2)$ via the general form of the β dependence for WP volumes, eq. (3.3.66). Consequently, to compute any contribution to $Z_{g,2}^{\beta,\text{Airy}}(\beta_1, \beta_2)$ it suffices to compute the integral for $b_1^\alpha b_2^\gamma \theta(b_1 - b_2)$, i.e. [12]

$$\begin{aligned} I_{\alpha,\gamma}(\beta_1, \beta_2) &:= \int_0^\infty \mathrm{d}b_1\, b_1 \int_0^\infty \mathrm{d}b_2\, b_2 \frac{e^{-\frac{b_1^2}{4\beta_1}}}{\sqrt{4\pi\beta_1}} \frac{e^{-\frac{b_2^2}{4\beta_2}}}{\sqrt{4\pi\beta_2}} b_1^\alpha b_2^\gamma \theta(b_1 - b_2) \\ &= \frac{1}{4\pi\sqrt{\beta_1\beta_2}} \int_0^\infty \mathrm{d}b_1\, e^{-\frac{b_1^2}{4\beta_2}} b_1^{\alpha+1} \int_0^{b_1} \mathrm{d}b_2\, e^{-\frac{b_2^2}{4\beta_2}} b_2^{\gamma+1}. \end{aligned} \tag{4.3.19}$$

This is the case, since the complementary contribution in eq. (4.3.18) ($b_2^\alpha b_1^\gamma \theta(b_2 - b_1)$) can be seen to be given by $I_{\alpha,\gamma}(\beta_2, \beta_1)$.

Here, we would like to recall briefly that the SFF is found from the two-point function of partition functions by setting $\beta_1 = \beta + it$ and $\beta_2 = \beta_1^*$. Additionally, it is useful to remark that the full volume, as well as its constituent parts in eq. (3.3.66), are symmetric in $b_1 \leftrightarrow b_2$. This statement, for the whole volume, is true as well for JT gravity (in fact, for every volume computed from a theory dual to a matrix model). Consequently,

$$\begin{aligned} \kappa_\beta(t, \beta) = Z_{g,2}^\beta(\beta_1, \beta_1^*) = \int_0^\infty b_1\, \mathrm{d}b_1 \int_0^\infty b_2\, \mathrm{d}b_2\, V_{g,2}^\beta(b_1, b_2) \frac{1}{2} \times \\ \left[Z^t(\beta_1, b_1) Z^t(\beta_1^*, b_2) + Z^t(\beta_1^*, b_1) Z^t(\beta_1, b_2) \right]. \end{aligned} \tag{4.3.20}$$

By a short computation it is easy to see that the expression in square brackets is necessarily real, which implies the SFF to be a real quantity in all theories dual to a matrix model.

Coming back to eq. (4.3.19), we present the evaluation of this integral in two ways. For the first way we employ differentiation under the integral sign. This has the major advantage of clarifying what functional dependence on the complex inverse temperatures can actually arise. We define $a := \frac{1}{4\beta_2}$ $b := \frac{1}{4\beta_1}$, using which one can work out the integrals as

$$\gamma \text{ even}, \alpha \text{ even} \quad I_{\alpha,\gamma}(\beta_1, \beta_2) = \frac{1}{16\pi\sqrt{\beta_1\beta_2}} \left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma}{2}} \left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha}{2}} \frac{1}{b(a+b)}, \tag{4.3.21}$$

12 Note that we did not include the dependence on γ (as defined in section 2.1, not to be distinguished with the index of the same name) and put $\gamma = \frac{1}{2}$. If one would like to restore this dependence, one can do so as discussed in section 2.1.

$$\gamma \text{ even}, \alpha \text{ odd} \quad I_{\alpha,\gamma}(\beta_1,\beta_2) = \frac{1}{16\sqrt{\pi}\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha+1}{2}}\frac{1}{a}\left[\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a+b}}\right], \tag{4.3.22}$$

$$\gamma \text{ odd}, \alpha \text{ even} \quad I_{\alpha,\gamma}(\beta_1,\beta_2) = \frac{1}{16\sqrt{\pi}\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma+1}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha}{2}}\frac{1}{b\sqrt{a+b}}, \tag{4.3.23}$$

$$\gamma \text{ odd}, \alpha \text{ odd} \quad I_{\alpha,\gamma}(\beta_1,\beta_2) = \frac{1}{8\pi\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma+1}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha+1}{2}}\frac{\arctan\left(\sqrt{\frac{a}{b}}\right)}{\sqrt{ab}}. \tag{4.3.24}$$

By observing these results, one can distinguish three cases for the terms following the common $\frac{1}{\sqrt{\beta_1\beta_2}}$ term (and potentially powers of $\frac{1}{\beta_1+\beta_2}$):

(i) γ, α both even: like the "orientable" contribution (cf. eq. (4.2.13)), i.e. purely polynomial in β_1, β_2.

(ii) γ, α containing one even and one odd number: polynomial terms that can be multiplied by $\sqrt{\beta_i}$ and $\sqrt{\beta_1+\beta_2}$.

(iii) γ, α both odd: terms of the type ii) with additional terms of a polynomial multiplying $\arctan\left(\frac{\beta_1}{\beta_2}\right)$.

From this consideration it is already apparent that most of the contributions to the SFF for the unorientable theories will have a quite similar form to those in the orientable case. A notable exception are the terms arising from *iii*) that contain an arctan. We will discuss momentarily, how these terms give rise to logarithmic behaviour in the large time limit. Let us already remark here that the possibility for a logarithmic term is given solely in the case of odd γ and α, for which hence $\alpha+\gamma$ is even. Due to theorem 3.4, for terms arising from the genus g contribution to the SFF this sum is given by $6g-2$. Consequently, contributions containing an arctan can only arise for integer genus. This conveniently resonates with the fact that in the universal result logarithmic terms also occur only at integer genus.

The other, equivalent, way to solve the integral, is to solve it as a finite sum. This is particularly useful for the consideration of constraints later on. We find that by using Hypergeometric functions one can express the integral for even α as

$$\begin{aligned} I_{\alpha,\gamma}(\beta_1,\beta_2) = & \frac{2^{\alpha+\gamma+1}\beta_1^{\frac{1}{2}(\alpha+\gamma+3)}\beta_2^{\frac{1}{2}(\alpha+\gamma+1)}\Gamma\left(\frac{1}{2}(\alpha+\gamma+4)\right)}{\pi(\gamma+2)\left(\beta_1+\beta_2\right)^{\left(\frac{\alpha+\gamma}{2}+1\right)}} \\ & \times \sum_{k=0}^{\frac{\alpha}{2}} \frac{\frac{\alpha}{2}!\left(\frac{\beta_1}{\beta_2}\right)^k\Gamma\left(\frac{\gamma}{2}+2\right)}{\left(\frac{\alpha}{2}-k\right)!\Gamma\left(k+\frac{\gamma}{2}+2\right)}. \end{aligned} \tag{4.3.25}$$

Conveniently, this also yields the case of odd α, even γ as one can use $1 = \theta(b_1 - b_2) + \theta(b_2 - b_1)$ to write

$$I_{\alpha,\gamma}(\beta_1, \beta_2) = -\tilde{I}_{\alpha,\gamma}(\beta_1, \beta_2) + R_{\alpha,\gamma}(\beta_1, \beta_2) = -I(\gamma, \alpha)(\beta_2, \beta_1) + R_{\alpha,\gamma}(\beta_1, \beta_2), \tag{4.3.26}$$

where $\tilde{I}_{\alpha,\gamma}$ denotes the analogue of $I_{\alpha,\gamma}$ with $\theta(b_1 - b_2)$ replaced by $\theta(b_2 - b_1)$ and we introduced

$$\begin{aligned} R_{a,b}(\beta_1, \beta_2) &:= \int_0^\infty \mathrm{d}b_1\, b_1 \int_0^\infty \mathrm{d}b_2\, b_2 Z^t(\beta_1, b_1) Z^t(\beta_2, b_2) b_1^a b_2^b \\ &= \frac{4\beta_1\beta_2}{4\pi\sqrt{\beta_1\beta_2}} 2^a \sqrt{\beta_1}^a \int_0^\infty \mathrm{d}x_1\, e^{-x_1} x_1^{\frac{a}{2}} 2^b \sqrt{\beta_2}^b \int_0^\infty \mathrm{d}x_2\, e^{-x_2} x_2^{\frac{b}{2}} \\ &= \frac{2^{a+b}\sqrt{\beta_1\beta_2}\beta_1^{\frac{a}{2}}\beta_2^{\frac{b}{2}}}{\pi} \Gamma\left(\frac{b}{2} + 1\right) \Gamma\left(\frac{a}{2} + 1\right). \end{aligned} \tag{4.3.27}$$

Thus, we only have to consider separately the case of both powers being odd. Setting $\alpha = 2a + 1, \gamma = 2b + 1$ we find

$$\begin{aligned} I_{2a+1,2b+1}(\beta_1, \beta_2) = & \frac{2^{2(a+b)+3} \beta_1^{a+b+\frac{5}{2}} \beta_2^{a+b+\frac{3}{2}} \Gamma(a+b+3)}{\pi(2b+3)\left(\beta_1 + \beta_2\right)^{a+b+2}} \\ & \times\, {}_2F_1\left(1, -a - \frac{1}{2}; b + \frac{5}{2}; -\frac{\beta_1}{\beta_2}\right), \end{aligned} \tag{4.3.28}$$

where the Hypergeometric function can explicitly be written as [87]

$$\begin{aligned} & {}_2F_1\left(1, -a - \frac{1}{2}; b + \frac{5}{2}; -\frac{\beta_1}{\beta_2}\right) = \\ & \frac{2\Gamma\left(a + \frac{3}{2}\right)\Gamma\left(b + \frac{5}{2}\right)}{\pi} \sum_{j=-a-1}^{b+1} \frac{\left(\frac{\beta_1}{\beta_2}\right)^{-j-\frac{1}{2}}}{(a+j+1)!(b-j+1)!} \\ & \left(-\sum_{k=0}^{j-1} \frac{(-1)^k \sqrt{\frac{\beta_1}{\beta_2}}^{2k+1}}{2k+1} + \sum_{k=j}^{-1} \frac{(-1)^k \sqrt{\frac{\beta_1}{\beta_2}}^{2k+1}}{2k+1} + \arctan\left(\sqrt{\frac{\beta_1}{\beta_2}}\right)\right), \end{aligned} \tag{4.3.29}$$

where we adopt the convention that sums where the upper limit is smaller than the lower limit vanish.

For the evaluation of the explicit contributions, we implemented the second way of solving the integral in a computer algebra system. Since the computations, even for small values of the genus, get quite tedious very fast, we work out explicitly only the simplest example:

Example 4.2 ($g = \frac{1}{2}$ contribution to the SFF)**.** *From eq.* (3.3.58) *we can read off*

$$C^{\frac{1}{2},1}_{1,0} = 1, \quad C^{\frac{1}{2},1}_{0,1} = 0. \tag{4.3.30}$$

Hence, we only need to find

$$\begin{aligned} I_{1,0}(\beta_1, \beta_2) &= \frac{-1}{16\sqrt{\pi}\sqrt{\beta_1\beta_2 a}} \frac{\partial}{\partial b}\left[\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a+b}}\right] \\ &= \frac{1}{32\sqrt{\pi}\sqrt{\beta_1\beta_2 a}}\left[\frac{1}{\sqrt{b}^3} - \frac{1}{\sqrt{a+b}^3}\right] \\ &= \frac{\sqrt{\beta_1\beta_2}}{\sqrt{\pi}}\left[\sqrt{\beta_1} - \frac{\sqrt{\beta_1\beta_2}\beta_2}{(\beta_1+\beta_2)^{\frac{3}{2}}}\right]. \end{aligned} \tag{4.3.31}$$

Consequently, using the β *dependence from eq.* (3.3.66)*:*

$$\begin{aligned} Z^{\beta,\text{Airy}}_{\frac{1}{2},2}(\beta_1, \beta_2) &= \frac{2-\beta}{\beta^2}[I_{1,0}(\beta_1,\beta_2) + I_{1,0}(\beta_2,\beta_1)] \\ &= \frac{2-\beta}{\beta^2}\frac{\beta_1\beta_2}{\sqrt{\pi}}\left[\frac{1}{\sqrt{\beta_1}} + \frac{1}{\sqrt{\beta_2}} - \frac{1}{\sqrt{\beta_1+\beta_2}}\right]. \end{aligned} \tag{4.3.32}$$

Putting into this the values for the inverse complex temperatures to compute the SFF, we find that the contribution at $t \to \infty$ that at least is $\mathcal{O}(t^{2g+1}) = \mathcal{O}(t^2)$ is given by

$$\kappa^{\frac{1}{2}}_{\beta}(t, \beta) = -\frac{2-\beta}{\beta^2}\frac{t^2}{\sqrt{2\pi\beta}} + \mathcal{O}\left(t^{\frac{3}{2}}\right). \tag{4.3.33}$$

From this example we can already deduce that the universal results for $\beta \in \{1, 4\}$, the relevant parts of which are written in eqs. (4.3.8) and (4.3.16), agree with the geometric one at $\mathcal{O}(\tau^2)$. We also see that upon putting the values of the arguments relevant for the SFF, terms which can not contribute at least to $\mathcal{O}(t^{2g+1})$ can be neglected. Extending this example, we now present the results we find by explicitly considering the $\beta = 1$ WP volumes, discussed above, and computing the contributions to the SFF that have the chance to contribute to the τ-scaled limit by neglecting said terms.

$$\kappa^0_1(t, \beta) = \frac{\sqrt{t^2+\beta^2}}{2\pi\beta}, \tag{4.3.34}$$

$$\kappa^{\frac{1}{2}}_1(t, \beta) = -\frac{t^2}{\sqrt{2\pi\beta}}, \tag{4.3.35}$$

$$\kappa^1_1(t, \beta) = \left[\frac{-10t^3}{3} - L_3(t, \beta)\right]\frac{1}{\pi}, \tag{4.3.36}$$

$$\kappa_1^{\frac{3}{2}}(t,\beta) = \frac{8\sqrt{2\pi\beta}}{3\pi}t^4 - \frac{it^4}{3\sqrt{\pi}}\left(\sqrt{\beta - it} - \sqrt{\beta + it}\right) \tag{4.3.37}$$

$$\kappa_1^2(t,\beta) = \frac{\beta}{\pi}\left[\frac{163t^5}{15} + 4L_5(t,\beta)\right], \tag{4.3.38}$$

$$\begin{aligned}\kappa_1^{\frac{5}{2}}(t,\beta) = &- \frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}t^6 + \frac{t^6\sqrt{t^2+\beta^2}}{30\sqrt{\pi}}\left(\sqrt{\beta - it} + \sqrt{\beta + it}\right) \\ &+ \frac{21it^5\sqrt{t^2+\beta^2}\beta}{5\sqrt{\pi}}\left(\sqrt{\beta - it} - \sqrt{\beta + it}\right),\end{aligned} \tag{4.3.39}$$

$$\kappa_1^3(t,\beta) = \frac{-2t^8\sqrt{\beta^2+t^2}}{45\pi} - \frac{1658\beta^2t^6\sqrt{\beta^2+t^2}}{63\pi} - \frac{32\beta^2L_7(t,\beta)}{3\pi}, \tag{4.3.40}$$

$$\begin{aligned}\kappa_\beta^{\frac{7}{2}}(t,\beta) = &\frac{it^7\sqrt{\beta^2+t^2}}{210\sqrt{\pi}}\left(\left(756\beta^2 - t^2\right)\left(\sqrt{\beta + it} - \sqrt{\beta - it}\right)\right. \\ &\left. +31i\beta t\left(\sqrt{\beta - it} + \sqrt{\beta + it}\right)\right) + \frac{2048}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}t^8,\end{aligned} \tag{4.3.41}$$

where we defined

$$L_n(t,\beta) := i\left[t^n \arctan\left(\sqrt{\frac{\beta + it}{\beta - it}}\right) + (-t)^n \arctan\left(\sqrt{\frac{\beta - it}{\beta + it}}\right)\right]. \tag{4.3.42}$$

As one can directly observe, the expansion of these contributions for large times is more involved than for the orientable case. In fact, there are two types of terms for which the expansion for large t is not immediately clear. First, the terms that contain a factor that can be written as $\sqrt{\beta_1} \pm \sqrt{\beta_2}$. For those, one uses the generalised binomial theorem (for $t > \beta$) to show

$$\sqrt{\beta_1} \pm \sqrt{\beta_2} = \sqrt{2t}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{\beta}{t}\right)^k\left[\cos\left(\frac{k\pi}{2}\right) \pm \sin\left(\frac{k\pi}{2}\right)\right]\begin{cases}1, & +,\\ i, & -.\end{cases} \tag{4.3.43}$$

Note that the factors of i for the "$-$" case conveniently cancel the appearances of the imaginary unit in the above results. Using this and the series expansion of $\sqrt{\beta^2+t^2} = t\sqrt{1+\left(\frac{\beta}{t}\right)^2}$ in $\frac{\beta}{t}$, one can work out the series expansion in $\frac{\beta}{t}$ of the whole term.

Second, we consider the terms containing a factor of $L_k(t,\beta)$. Those are of special interest, as they will result in the appearance of logarithmic dependence on the arguments of the SFF. Since we will need this statement again later on, for the discussion of constraints, we formulate it as

Proposition 4.2 (Logarithmic terms in the large time expansion)**.**

$$L_k(t,\beta) = \begin{cases} t^k\Big(\log\left(\frac{\beta}{2t}\right) + \sum_{m=2,4,\ldots} C(m)\left(\frac{\beta}{t}\right)^m\Big), & k \text{ odd} \\ t^k\frac{\pi}{2}, \quad k \text{ even}, \end{cases} \tag{4.3.44}$$

with $C(m)$ the coefficient of the $\mathcal{O}(x^m)$ term in

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} 2^n \left(\sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} x^{2k-2}\right)^n, \tag{4.3.45}$$

the first of which are given by $C(2) = -\frac{1}{4}$, $C(4) = \frac{3}{32}$, $C(6) = -\frac{5}{96}$.

Proof. To prove the expansion, it is useful to first put the definition of the arctan on $\mathbb{C}$, using the complex logarithm. This has the advantage of clarifying, among other things, the branch-cut structure of the functions as that arising from the complex logarithm, i.e. a branch-cut along the negative real axis. In the following, we shall always choose the principal branch, i.e.

$$\underset{z\in\mathbb{C}}{\forall} \log(z) = \log(|z|) + i\arg(z). \tag{4.3.46}$$

The definition is given by [87]

$$\arctan z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right). \tag{4.3.47}$$

For the sake of brevity, we define $z_\pm := \frac{\sqrt{\beta \pm it}}{\sqrt{\beta \mp it}}$. Using this and the definition, one finds after a small rewriting

$$\arctan(z_\pm) = \frac{1}{2i}\log\left(i\frac{\mp t + \sqrt{t^2+\beta^2}}{\beta}\right) = \frac{1}{2i}\left[\log\left(\frac{\mp t + \sqrt{t^2+\beta^2}}{\beta}\right) + i\frac{\pi}{2}\right], \tag{4.3.48}$$

where we used the definition of the principal branch to the second equation.

Now we expand this expression for large $\frac{t}{\beta} := \frac{1}{x}$, i.e. as a series in x, convergent for values of x that are small in a specific sense, defined momentarily. Writing the above logarithm in this form, starting with the $+$ case one finds

$$\begin{aligned}\log\left(-\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right) &= \log\left(-\frac{1}{x} + \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k} x^{2k-1}\right) \\ &= \log\left[\frac{x}{2}\left(1 + 2\sum_{k=2}^{\infty}\binom{\frac{1}{2}}{k} x^{2k-2}\right)\right] \\ &= \log\left(\frac{x}{2}\right) + \log\left(1 + 2\sum_{k=2}^{\infty}\binom{\frac{1}{2}}{k} x^{2k-2}\right) \\ &= \log\left(\frac{x}{2}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}2^n\left(\sum_{k=2}^{\infty}\binom{\frac{1}{2}}{k} x^{2k-2}\right)^n,\end{aligned} \tag{4.3.49}$$

where in the first line we explicitly use the considered regime of small x, more specifically $\frac{1}{x^2} > 1$, to use the generalised binomial series. Furthermore, in the last line we used that the absolute value of the sum, for $x < \infty$ is bounded above by 1 as one can see by resumming the series, which enables one then to use the Mercator series for the logarithm.

The final result, as one can see directly, takes the form of $\log\left(\frac{\beta}{2t}\right)$ with corrections in even powers of x. The coefficients of the corrections terms are elementary, though tediously, computable by using the multinomial theorem for the power of the sum and subsequently collecting powers of x. We define the coefficient of x^m as $C(m)$. The values of this function relevant for the present discussion are $C(2) = -\frac{1}{4}$, $C(4) = \frac{3}{32}$, $C(6) = -\frac{5}{96}$, which we have determined analytically and checked numerically. Explicitly, one can thus write

$$\log\left(\frac{-t+\sqrt{t^2+\beta^2}}{\beta}\right) = \log\left(\frac{\beta}{2t}\right) + \sum_{m=2,4,\ldots} C(m)\left(\frac{\beta}{t}\right)^m. \tag{4.3.50}$$

Using the same arguments for the $-$ case, one finds

$$\log\left(\frac{t+\sqrt{t^2+\beta^2}}{\beta}\right) = -\log\left(\frac{\beta}{2t}\right) - \sum_{m=2,4,\ldots} C(m)\left(\frac{\beta}{t}\right)^m. \tag{4.3.51}$$

Using this, we can now evaluate the combination of $L(t,\beta)$ with the factors we are interested in. First, let k be an odd integer. Then:

$$\begin{aligned} L_k(t,\beta) &= \frac{t^k}{2}\left[\log\left(\frac{-t+\sqrt{t^2+\beta^2}}{\beta}\right) + i\frac{\pi}{2} - \log\left(\frac{t+\sqrt{t^2+\beta^2}}{\beta}\right) - i\frac{\pi}{2}\right] \\ &= t^k\left(\log\left(\frac{\beta}{2t}\right) + \sum_{m=2,4,\ldots} C(m)\left(\frac{\beta}{t}\right)^m\right), \end{aligned} \tag{4.3.52}$$

where we used eqs. (4.3.50) and (4.3.51) to go to the second line. Next, let k be an even integer. Then, again by putting in the expansions in the second step:

$$\begin{aligned} L_k(t,\beta) &= \frac{t^k}{2}\left[\log\left(\frac{-t+\sqrt{t^2+\beta^2}}{\beta}\right) + i\frac{\pi}{2} + \log\left(\frac{t+\sqrt{t^2+\beta^2}}{\beta}\right) + i\frac{\pi}{2}\right] \\ &= t^k\frac{\pi}{2}. \end{aligned} \tag{4.3.53}$$

□

Using these results, one can now compute the expansion of the above results for the contributions to the SFF at large t, i.e. for each term take in front the leading order in t and expand the remainder of the term as a a series in $\frac{\beta}{t}$. Of the combination of said expansion with the leading-order factor we are again interested only in terms that are at least of $\mathcal{O}\left(t^{2g+1}\right)$. Doing this for the case of β = 1 topological gravity, we find [2]

$$\kappa_1^0(t,\beta) = \frac{t}{2\pi\beta}, \tag{4.3.54}$$

$$\kappa_1^{\frac{1}{2}}(t,\beta) = -\frac{t^2}{\sqrt{2\pi\beta}}, \tag{4.3.55}$$

$$\kappa_1^1(t,\beta) = -\left[\frac{10}{3} + \log\left(\frac{\beta}{2t}\right)\right]\frac{t^3}{\pi}, \tag{4.3.56}$$

$$\kappa_1^{\frac{3}{2}}(t,\beta) = -\frac{\sqrt{2\pi}t^{\frac{9}{2}}}{3\pi} + \frac{8\sqrt{2\pi\beta}}{3\pi}t^4, \tag{4.3.57}$$

$$\kappa_1^2(t,\beta) = \frac{\beta t^5}{\pi}\left[\frac{163}{15} + 4\log\left(\frac{\beta}{2t}\right)\right], \tag{4.3.58}$$

$$\kappa_1^{\frac{5}{2}}(t,\beta) = -\frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}t^6 + \frac{t^7\sqrt{t}}{15\sqrt{2\pi}} + \frac{17\beta t^6\sqrt{t}}{6\sqrt{2\pi}}, \tag{4.3.59}$$

$$\kappa_1^3(t,\beta) = \frac{-2t^9}{45\pi} - \frac{8297\beta^2 t^7}{315\pi} - \frac{32\beta^2 t^7\log\left(\frac{\beta}{2t}\right)}{3\pi}, \tag{4.3.60}$$

$$\kappa_1^{\frac{7}{2}}(t,\beta) = \frac{t^{10}\sqrt{t}}{105\sqrt{2\pi}} - \frac{3\beta t^9\sqrt{t}}{10\sqrt{2\pi}} - \frac{881\beta^2 t^8\sqrt{t}}{120\sqrt{2\pi}} + \frac{2048}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}t^8. \tag{4.3.61}$$

By τ-scaling of these results, we can now investigate whether they agree with the universal result for β = 1 topological gravity, derived above. Notably, the same reasoning that was applied for the case of β = 1 can be applied in the case of general β, using the β WP volumes instead of those for β = 1. There we find, again neglecting terms of smaller order than t^{2g+1} in each contribution and splitting the contributions according to the general form of the β dependence (theorem 3.3) [5]:

$$\kappa_\beta^0(t,\beta) = \frac{t}{2\pi\beta\beta}, \tag{4.3.62}$$

$$\kappa_\beta^{1/2}(t,\beta) = -\frac{2-\beta}{\beta^2}\frac{t^2}{\sqrt{2\pi}\sqrt{\beta}}, \tag{4.3.63}$$

$$\kappa_\beta^1(t,\beta) = \frac{t^3(2-\beta)^2\left(3\log\left(\frac{2t}{\beta}\right)-8\right)}{3\pi\beta^3} - \frac{2t^3}{3\pi\beta^2}, \tag{4.3.64}$$

$$\begin{aligned}\kappa_\beta^{3/2}(t,\beta) = &\frac{(1-\beta)(2-\beta)(4-\beta)}{\beta^4}\left(-\frac{59t^{9/2}}{60\sqrt{2\pi}} + 2\sqrt{\frac{2}{\pi}}\sqrt{\beta}t^4\right)\\ &+ \frac{(2-\beta)}{3\beta^3}\left(-\sqrt{\frac{2}{\pi}}t^{9/2} + 8\sqrt{\frac{2}{\pi}}\sqrt{\beta}t^4\right),\end{aligned} \tag{4.3.65}$$

$$\begin{aligned}
\kappa^{2}_{\beta}(t,\beta) =& \frac{3t^6(4-\beta)(2-\beta)^2(1-\beta)}{32\beta^5} \\
&- \frac{\beta t^5(1-\beta)(2-\beta)^2(4-\beta)\left(1890\log\left(\frac{2t}{\beta}\right)-3371\right)}{945\pi\beta^5} \\
&- \frac{\beta t^5(2-\beta)^2\left(60\log\left(\frac{2t}{\beta}\right)-151\right)}{15\pi\beta^4} + \frac{4\beta t^5}{5\pi\beta^3},
\end{aligned} \tag{4.3.66}$$

$$\begin{aligned}
\kappa^{5/2}_{\beta}(t,\beta) =& -\frac{t^6(1-\beta)^2(2-\beta)(4-\beta)^2\left(2838528\beta^{3/2}+31282t^{3/2}-966955\beta\sqrt{t}\right)}{532224\sqrt{2\pi}\beta^6} \\
&+ \frac{t^6(1-\beta)(2-\beta)(4-\beta)\left(-55824384\beta^{3/2}+43966t^{3/2}+14101571\beta\sqrt{t}\right)}{2661120\sqrt{2\pi}\beta^5} \\
&+ \frac{t^6(2-\beta)\left(-512\beta^{3/2}+2t^{3/2}+85\beta\sqrt{t}\right)}{30\sqrt{2\pi}\beta^4},
\end{aligned} \tag{4.3.67}$$

$$\begin{aligned}
\kappa^{3}_{\beta}(t,\beta) =& -\frac{16\beta^2 t^7}{21\pi\beta^4} \\
&+ \frac{t^7(4-\beta)^2(2-\beta)^2(1-\beta)^2}{145297152\pi\beta^7} \\
&\times\left(-419374208\beta^2+2338048t^2+290594304\beta^2\log\left(\frac{2t}{\beta}\right)-22891869\pi\beta t\right) \\
&- \frac{t^7(4-\beta)(2-\beta)^2(1-\beta)}{3632428800\pi\beta^6}\times\left(68734274048\beta^2+164049920t^2\right. \\
&\left.-36808611840\beta^2\log\left(\frac{2t}{\beta}\right)+1378872495\pi\beta t\right) \\
&- \frac{t^7(2-\beta)^2\left(1151\beta^2+2t^2-480\beta^2\log\left(\frac{2t}{\beta}\right)\right)}{45\pi\beta^5},
\end{aligned} \tag{4.3.68}$$

$$\begin{aligned}
\kappa^{7/2}_{\beta}(t,\beta) =& -\frac{t^8(4-\beta)^3(2-\beta)(1-\beta)^3}{28229160960\sqrt{2\pi}\beta^8}\times\left(-120444420096\beta^{5/2}-2521705676\beta t^{3/2}\right. \\
&\left.+61149640t^{5/2}+47197373065\beta^2\sqrt{t}\right) \\
&+ \frac{t^8(4-\beta)^2(2-\beta)(1-\beta)^2}{2258332876800\sqrt{2\pi}\beta^7}\times\left(707983777222144\beta^{5/2}+435487934100\beta t^{3/2}\right. \\
&\left.+38265437192t^{5/2}-22916260990543\beta^2\sqrt{t}\right) \\
&+ \frac{t^8(4-\beta)(2-\beta)(1-\beta)}{2258332876800\sqrt{2\pi}\beta^6}\times\left(146506298425344\beta^{5/2}-486701235820\beta t^{3/2}+\right. \\
&\left.+64361951752t^{5/2}-37542514725263\beta^2\sqrt{t}\right)
\end{aligned}$$

$$+\frac{t^8(2-\beta)\left(32768\beta^{5/2}-252\beta t^{3/2}+8t^{5/2}-6167\beta^2\sqrt{t}\right)}{840\sqrt{2\pi}\beta^5}. \tag{4.3.69}$$

As a quick sanity check of this result, one can insert $\beta = 1$ and observe that the contributions presented in eq. (4.3.54) to eq. (4.3.61) are reproduced. By putting $\beta = 4$ we can also derive the result for the symplectic symmetry class from this, most general result.

This concludes our derivation of the contributions relevant in the late-time limit of the canonical SFF. We proceed now by considering the cases of $\beta = 1$ and $\beta = 4$ separately when comparing them to the universal results.

COMPARISON OF THE GEOMETRIC AND THE UNIVERSAL RESULT FOR TOPOLOGICAL GRAVITY FOR $\beta = 1$. To find the τ-scaled SFF as defined in eq. (4.1.8), it remains to introduce $\tau = e^{-S_0}t$ and put the additional factors of e^{-S_0} before taking the limit of $t \to \infty$. We can directly do this using the results written in eq. (4.3.54) to eq. (4.3.61), noting the fact that this result derives from the Airy WP volumes by the superscript WP. We find (keeping in mind the to be performed limit of $t \to \infty$)

$$\begin{aligned}\kappa_1^{s,\mathrm{WP}}(\tau,\beta) &= \frac{\tau}{2\pi\beta} - \frac{\tau^2}{\sqrt{2\pi\beta}} + \frac{\tau^3}{\pi}\left[\frac{-10}{3} + \log\left(\frac{2t}{\beta}\right)\right] - \frac{\sqrt{2\pi\beta}\tau^4}{3\pi}\left(\frac{t}{\beta}\right)^{1/2} \\ &+ \frac{8\sqrt{2\pi\beta}}{3\pi}\tau^4 + \frac{\beta\tau^5}{\pi}\left[\frac{163}{15} - 4\log\left(\frac{2t}{\beta}\right)\right] - \frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}\tau^6 + \frac{17\tau^6\beta\sqrt{2\pi\beta}}{12\pi}\left(\frac{t}{\beta}\right)^{1/2} \\ &+ \frac{\tau^6(2\pi\beta)^{3/2}}{60\pi^2}\left(\frac{t}{\beta}\right)^{3/2} + \frac{\beta^2\tau^7}{\pi}\left[-\frac{8297}{315} + \frac{32}{3}\log\left(\frac{2t}{\beta}\right)\right] - \frac{2\tau^7\beta^2}{45\pi}\left(\frac{t}{\beta}\right)^2 \\ &+ \frac{\tau^8(2\pi\beta)^{\frac{5}{2}}}{840}\left(\frac{t}{\beta}\right)^{\frac{5}{2}} - \frac{3\tau^8\beta(2\pi\beta)^{\frac{3}{2}}}{40\pi^2}\left(\frac{t}{\beta}\right)^{\frac{3}{2}} - \frac{881\tau^8\beta^2\sqrt{2\pi\beta}}{240}\sqrt{\frac{t}{\beta}} + \frac{512(2\pi\beta)^{\frac{5}{2}}}{105\pi^3}\tau^8 \\ &+ \mathcal{O}\left(\tau^9\right).\end{aligned} \tag{4.3.70}$$

What maybe strikes the eye first, when contemplating this result, is the remaining dependence on t. Naively, this is unexpected, since the universal prediction for the τ-scaled SFF, already apparent from eq. (4.1.9), is independent on e^{S_0} and hence such terms would be inconsistent with the agreement of the geometric and the universal result for the SFF as $t = e^{S_0}\tau$. To perform a closer comparison, we combine the results found above in eqs. (4.2.5), (4.3.1) and (4.3.8) to write down the universal prediction for the SFF of topological gravity at $\beta = 1$ to the relevant order as

$$\kappa_1^s(\tau,\beta) = \frac{\tau}{2\pi\beta} - \frac{\tau^2}{\sqrt{2\pi\beta}} - \frac{\gamma + \log\left(2\beta\tau^2\right) + \frac{1}{3}}{\pi}\tau^3 + \frac{8\sqrt{2\pi\beta}}{3\pi}\tau^4 +$$

$$+\frac{\beta\left(4\gamma+4\log\left(2\beta\tau^2\right)-\frac{7}{15}\right)}{\pi}\tau^5-\frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}\tau^6+ \\ -\frac{\beta^2\tau^7\left(160\log\left(2\beta\tau^2\right)+160\gamma-73\right)}{15\pi}+\frac{512(2\pi\beta)^{\frac{5}{2}}}{105\pi^3}\tau^8+\mathcal{O}\left(\tau^9\right). \tag{4.3.71}$$

It is important to note that each of these terms is not only characterised by a certain order in τ but also by a specific order in β. Employing the same logic to collect terms also for the geometric result, we find

$$\kappa_1^{s,\mathrm{WP}}(\tau,\beta)=\lim_{t\to\infty}\Bigg\{-\frac{\tau^2}{\sqrt{2\pi\beta}}+\frac{8\sqrt{2\pi\beta}}{3\pi}\tau^4-\frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}\tau^6+\frac{512(2\pi\beta)^{5/2}}{105\pi^3}\tau^8+\dots \\ +\frac{\tau}{2\pi\beta} \\ +\frac{\tau^3}{\pi}\left[\frac{-10}{3}-\log\left(\frac{\beta}{2t}\right)-\frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2}+\frac{\sqrt{2\pi}}{30}\left(t\tau^2\right)^{3/2}-\frac{2\left(t\tau^2\right)^2}{45}+\frac{\sqrt{2\pi}}{210}(\tau^2t)^{5/2}+\dots\right. \\ +\frac{\tau^5\beta}{\pi}\left[\frac{163}{15}+4\log\left(\frac{\beta}{2t}\right)+\frac{17\sqrt{2\pi}}{12}\left(t\tau^2\right)^{1/2}-\frac{3\sqrt{2\pi}}{20}\left(t\tau^2\right)^{3/2}+\dots\right] \\ +\frac{\tau^7\beta^2}{\pi}\left[-\frac{8297}{315}-\frac{32}{3}\log\left(\frac{\beta}{2t}\right)-\frac{881\sqrt{2\pi}}{240}(\tau^2t)^{1/2}+\dots\right] \\ +\dots\Bigg\}, \tag{4.3.72}$$

where in the first line we collect terms of even order in τ and half-integer order in β that arise as the t-independent terms of half-integer contributions. In the subsequent lines we collected in each line the terms having the specific prefactor, $\tau^k\beta^m$ with k odd and m even, of an integer genus contribution to the universal SFF. Here, the second line corresponds to $g=0$, the third to $g=1$, the fourth to $g=2$, and the fifth to $g=3$. The dots in each line have the well-defined meaning of specific terms arising from contributions of higher genus, coinciding with the terms collected in the specific line or, in the case of the last line, additional contributions of the type of the "integer genus" lines. The additional contributions appearing after the third line can unambiguously be split into two types of terms. First, there are terms containing a dependence on $\sqrt{t}$, which arise from half-integer genus contributions. Secondly, there are terms that do not depend on $\sqrt{t}$ but on t, the only example of which is given by the fifth term in the second line of eq. (4.3.72), arising from the $g=3$ contribution. It is a well-motivated conjecture that this structure persists for the higher genus contributions in a way we will explain in more detail in section 4.3.2.

Comparing the ordered form of the geometric scaled SFF as written in eq. (4.3.72) with the universal result in eq. (4.3.71) we can already observe two things. First, the t-independent terms in the first line of the geometric result coincide precisely with the corresponding contributions in the universal result. Second, the coefficients of the logarithmic terms of both results are in agreement for all considered orders. At this point it has to be stressed that the computations leading to both results are technically completely independent, though their agreement is expected from RMT universality. The partial agreement observed so far, can hence already be seen as a quite non-trivial indicator for the presence of universality in the geometric result, despite the t-dependent terms considered momentarily.

We now consider the line with the prefactor $\tau^3\beta^0$ in eq. (4.3.72), which receives its first terms from the $g = 1$ contribution and then one from each higher genus contribution, except that for $g = 2$. We will now argue that the terms containing $\sqrt{t}$ arise as the asymptotic expansion for $t \to \infty$ of a certain function, whose definition as a power series with infinite radius of convergence is found by the t-dependent terms that depend on an integer power of t. To see this explicitly, we define the following function, using generalised hypergeometric functions:

$$f(t,\tau) := \frac{\left(t\tau^2\right)^2}{45}\left(6\,{}_2F_2\left(2,2;3,\frac{7}{2};-t\tau^2\right) - 4\,{}_1F_1\left(\frac{3}{2};\frac{7}{2};\frac{-t\tau^2}{2}\right)\right). \tag{4.3.73}$$

For this function, it holds that

$$f(t,\tau) = \frac{2\left(\tau^2 t\right)^2}{45} + \underbrace{\frac{\left(t\tau^2\right)^2}{45}\sum_{k=1}^{\infty} a_k (t\tau^2)^k}_{\mathcal{O}(\tau^6)} \tag{4.3.74}$$

$$\overset{t\to\infty}{=} \left(-\frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2} + \log\left(4t\tau^2\right) + \gamma - 3\right), \tag{4.3.75}$$

where the first line is the definition of the hypergeometric functions as a power series with coefficients a_k, which has an infinite radius of convergence, and the second line is its asymptotic expansion.

We now add zero in the form of $f(t,\tau) - f(t,\tau)$ to the terms in brackets of the third line of eq. (4.3.72). We then write out the defining power series for the first occurrence of the function, cancelling the $\mathcal{O}\left(t^2\right)$ term in the bracket, and then go to large times, where the second occurrence of the function can be replaced by its asymptotic expansion. Adding up all terms, this results for the whole line in

$$\begin{aligned}&\frac{\tau^3}{\pi}\left[\frac{-10}{3}-\log\left(\frac{\beta}{2t}\right)-\log\left(4t\tau^2\right)-\gamma+3\right]+\mathcal{O}\left(\tau^6\right)\\&=-\frac{\tau^3}{\pi}\left[\frac{1}{3}+\log\left(2\beta\tau^2\right)-\gamma\right]+\mathcal{O}\left(\tau^6\right),\end{aligned}\tag{4.3.76}$$

which is precisely the universal result at $\mathcal{O}(\tau^3\beta^0)$. Consequently, noting that the next order, $\mathcal{O}(\tau^4)$, did not contain any t-dependence we can use this "cancellation function" approach to show agreement up to $\mathcal{O}(\tau^4)$.

To go to higher orders, first considering the terms of higher order than τ^3 in the considered bracket, we have to add more functions that cancel the terms at the cost of adding another higher-order term. A good candidate for such a function would be

$$\begin{aligned}\left(t\tau^2\right)^2\left(t\tau^2\right)^n{}_1F_1\left(\frac{3}{2};\frac{5}{2};-\frac{3^{\frac{2}{3}}}{2}t\tau^2\right)&\overset{t\to\infty}{\longrightarrow}\sqrt{\frac{\pi}{2}}\left(\sqrt{t}\tau\right)^{2n+1},\\&=\left(t\tau^2\right)^n\left[\left(t\tau^2\right)^2-\frac{3}{10}3^{2/3}\left(t\tau^2\right)^3+\mathcal{O}\left(t^4\right)\right],\end{aligned}\tag{4.3.77}$$

which, for $n=1$ and $n=2$, can be modified to cancel the higher order terms in the bracket. However, the cost is the introduction of additional contributions of order $\left(t\tau^2\right)^{n+2}$, which have to cancel terms arising from contributions of higher genus than we have computed so far. For example, to cancel the term of $\left(\sqrt{t}\tau\right)^3$ one would need to know the term of third type of order $\left(t\tau^2\right)^3$ which appears earliest at genus 4. Consequently, to check whether this is the correct choice for the "cancellation function" or one needs to modify it, one has to compute the Airy WP volume for $g=4$.

Considering the brackets arising from the $g=2$ and higher contributions, we can conclude in a similar way that higher genus contributions than the ones we computed are necessary to find "cancellation functions" that show the full agreement of the geometric with the universal result.

This behaviour can be interpreted as the consequence of the limit in t and the sum over genus in eq. (4.1.12) not commuting in general. In fact, only the t-independent terms of the $\kappa_1^g(t,\beta)$[13] commute with the limit. For the other terms, our results suggest that the terms containing no $\sqrt{t}$ dependence have to be resummed to find a certain function which then, in the asymptotic limit cancels the terms that do depend on $\sqrt{t}$ and lead to the "correct" logarithms. In the computation of the SFF from WP volumes one is, by construction,

13 They are found by cancelling a factor of τ^{2g+1} in the contributions collected in eq. (4.3.54) to eq. (4.3.61), after introducing τ and the additional factor of e^{-S_0} from eq. (4.1.8).

limited to a computation order-by-order in τ. Unsurprisingly, the full resummation of the series would require results to all orders on the geometric side, i.e. in a sense "non-perturbative" information.

Consequently, we can conclude that there is a large amount of evidence suggesting the agreement of the universal prediction for the SFF and its geometric computation, expected from RMT universality, while to obtain full agreement another, non-perturbative, approach to the computation is necessary. We will comment further on this at the end of this chapter.

Before doing so, we continue our study with the version of topological gravity dual to a matrix model in the symplectic symmetry class.

COMPARISON OF THE GEOMETRIC AND THE UNIVERSAL RESULT FOR TOPOLOGICAL GRAVITY FOR $\beta = 4$. We can follow essentially the same route as for $\beta = 1$. We have determined the universal result already above, in eqs. (4.2.5), (4.3.11) and (4.3.16). Putting the terms together, we find

$$\begin{aligned}\kappa_4^s(\tau,\beta) =& \frac{\tau}{8\pi\beta} + \sqrt{\frac{2}{\pi}}\frac{\tau^2}{16\sqrt{\beta}} - \frac{\tau^3\left(3\log\left(\beta\frac{\tau^2}{2}\right)+3\gamma+1\right)}{48\pi} - \sqrt{\frac{2}{\pi}}\frac{\sqrt{\beta}\tau^4}{12} \\ &+ \frac{\beta\tau^5\left(60\log\left(\beta\frac{\tau^2}{2}\right)+60\gamma-7\right)}{960\pi} + \sqrt{\frac{2}{\pi}}\frac{\beta^{3/2}\tau^6}{15}+ \\ &- \frac{\beta^2\tau^7\left(1120\log\left(\beta\frac{\tau^2}{2}\right)+1120\gamma-501\right)}{26880\pi} - \sqrt{\frac{2}{\pi}}\frac{4\beta^{5/2}\tau^8}{105} + \mathcal{O}\left(\tau^9\right).\end{aligned} \tag{4.3.78}$$

The geometric result we can compute by putting $\beta = 4$ in eq. (4.3.62)-eq. (4.3.69). Doing this, and already performing the τ-scaling as defined in eq. (4.1.12), we find

$$\kappa_4^{s,0}(t,\beta) = \frac{1}{8\pi\beta}, \tag{4.3.79}$$

$$\kappa_4^{s,\frac{1}{2}}(t,\beta) = \frac{1}{8\sqrt{2\pi}\sqrt{\beta}}, \tag{4.3.80}$$

$$\kappa_4^{s,1}(t,\beta) = -\frac{-3\log\left(\frac{2t}{\beta}\right)+10}{48\pi}, \tag{4.3.81}$$

$$\kappa_4^{s,\frac{3}{2}}(t,\beta) = \frac{\sqrt{t}}{48\sqrt{2\pi}} - \frac{\sqrt{\beta}}{6\sqrt{2\pi}}, \tag{4.3.82}$$

$$\kappa_4^{s,2}(t,\beta) = \frac{\beta\left(-60\log\left(\frac{2t}{\beta}\right)+163\right)}{960\pi}, \tag{4.3.83}$$

$$\kappa_4^{s,\frac{5}{2}}(t,\beta) = \frac{1}{15}\sqrt{\frac{2}{\pi}}\beta^{3/2} - \frac{17\beta\sqrt{t}}{768\sqrt{2\pi}} - \frac{(\sqrt{t})^3}{1920\sqrt{2\pi}}, \tag{4.3.84}$$

$$\kappa_4^{s,3}(t,\beta) = -\frac{(\sqrt{t})^4}{5760\pi} - \frac{\beta^2\left(-3360\log\left(\frac{2t}{\beta}\right)+8297\right)}{80640\pi}, \tag{4.3.85}$$

$$\kappa_4^{s,\frac{7}{2}}(t,\beta) = -\frac{4}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2} + \frac{881\beta^2\sqrt{t}}{61440\sqrt{2\pi}} + \frac{3\beta\left(\sqrt{t}\right)^3}{5120\sqrt{2\pi}} - \frac{\left(\sqrt{t}\right)^5}{53760\sqrt{2\pi}}. \tag{4.3.86}$$

As expected, we observe the exact same behaviour as for β = 1 topological gravity, studied above. Grouping the terms as above, we find

$$\begin{aligned}\kappa_4^{s,\mathrm{WP}}(\tau,\beta) = \lim_{t\to\infty}\Bigg\{&\sqrt{\frac{2}{\pi}}\left(\frac{\tau^2}{16\sqrt{\beta}} - \frac{\sqrt{\beta}\tau^4}{12} + \frac{\beta^{3/2}\tau^6}{15} - \frac{4\beta^{5/2}\tau^8}{105} + \dots\right) + \\ &+ \frac{\tau}{8\pi\beta} + \\ &+ \frac{\tau^3\beta^0}{\pi}\left[-\frac{5}{24} - \frac{3}{48}\log\left(\frac{\beta}{2t}\right) + \sqrt{\frac{\pi}{2}}\frac{\sqrt{t}\tau}{48} - \sqrt{\frac{\pi}{2}}\frac{\left(\sqrt{t}\tau\right)^3}{1920} - \frac{t^2\tau^4}{5760} - \sqrt{\frac{\pi}{2}}\frac{\left(\sqrt{t}\tau\right)^5}{53760} + \dots\right] \\ &+ \frac{\tau^5\beta}{\pi}\left[\frac{163}{960} + \frac{1}{16}\log\left(\frac{\beta}{2t}\right) - \sqrt{\frac{\pi}{2}}\frac{17\sqrt{t}\tau}{768} + \sqrt{\frac{\pi}{2}}\frac{3\left(\sqrt{t}\tau\right)^3}{5120} + \dots\right] \\ &+ \frac{\tau^7\beta^2}{\pi}\left[-\frac{8297}{80640} - \frac{1}{24}\log\left(\frac{\beta}{2t}\right) + \sqrt{\frac{\pi}{2}}\frac{881\sqrt{t}\tau}{61440} + \dots\right] \\ &+ \dots\Bigg\},\end{aligned} \tag{4.3.87}$$

with the dots having the same interpretation as for the orthogonal symmetry class, already studied. Again, we observe the direct matching of the t-independent terms of the computed half-integer contributions, collected in the first line, and the prefactors of the logarithmic terms with the universal result.

To find full agreement up to $\mathcal{O}(\tau^4)$, we consider in more detail the contribution at $\mathcal{O}(\tau^3\beta^0)$, given by

$$\frac{\tau^3}{48\pi}\left[-\left(-3\log\left(\frac{2t}{\beta}\right)+10\right) + \sqrt{\frac{\pi}{2}}\sqrt{t}\tau - \sqrt{\frac{\pi}{2}}\frac{(\sqrt{t}\tau)^3}{40} - \frac{\left(t\tau^2\right)^2}{120} - \sqrt{\frac{\pi}{2}}\frac{\left(\tau\sqrt{t}\right)^5}{1120}\right]. \tag{4.3.88}$$

As above, we have to find "cancellation functions". To this end, we define

$$f_1(t,\tau) := \frac{1}{240}\left(t\tau^2\right)^2 {}_2F_2\left(2,2;\frac{5}{2},\frac{7}{2};-\frac{1}{16}t\tau^2\right), \tag{4.3.89}$$

$$f_2(t,\tau) := \frac{1}{240}\left(t\tau^2\right)^2 {}_1F_1\left(\frac{3}{2};\frac{5}{2};-\frac{1}{8\sqrt[3]{50}}t\tau^2\right). \tag{4.3.90}$$

The asymptotic expansions of these functions are given by

$$f_1(t,\tau) \overset{t\to\infty}{\longrightarrow} 3\log\left(t\tau^2\right) + 3\gamma - 9, \tag{4.3.91}$$

$$f_2(t,\tau) \overset{t\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{2}}\sqrt{t}\tau, \tag{4.3.92}$$

and their definitions as power series with infinite radius of convergence, are given by

$$f_1(t,\tau) = \left(t\tau^2\right)^2\left(\frac{1}{240} - \frac{t\tau^2}{8400} + \mathcal{O}\left(\tau^4\right)\right), \tag{4.3.93}$$

$$f_2(t,\tau) = \left(t\tau^2\right)^2\left(\frac{1}{240} - \frac{t\tau^2}{3200\sqrt[3]{100}} + \mathcal{O}\left(\tau^4\right)\right). \tag{4.3.94}$$

As done for the case of $\beta = 1$, we add and subtract the sum of both functions in the bracket of eq. (4.3.88). Writing now the added sum as its defining series, we see that the term of $\mathcal{O}(t^2\tau^4)$ in the bracket is cancelled. Now we take the large t limit, where the asymptotic expansion of the subtracted sum can be used. Subtracting 4.3.91 from the logarithmic term of 4.3.88 we find

$$3\log\left(\frac{2t}{\beta}\right) - 10 - 3\log\left(t\tau^2\right) - 3\gamma + 9 = -\left(3\log\left(\beta\frac{\tau^2}{2}\right) + 3\gamma + 1\right), \tag{4.3.95}$$

which is exactly the term obtained from the universal prediction (eq. (4.3.78)). Subtracting the asymptotic expansion of $f_2(t,\tau)$ cancels the $\mathcal{O}\left(\sqrt{t}\tau\right)$-term. Thus, by the present manipulation we have shown the agreement of the result from topological gravity and the universal RMT prediction up to $\mathcal{O}(\tau^3)$, in fact to $\mathcal{O}(\tau^4)$, also for the symplectic symmetry class.

As for the orthogonal symmetry class, establishing agreement to higher orders requires the computation of at least the $g = 4$ contribution to the SFF.

Before continuing by studying JT gravity in the orthogonal symmetry class, we would like to remark that, as for the unitary symmetry class, there are certain cancellations between coefficients of the unorientable Airy WP volumes that enable the matching of the geometric with the universal result, in the way we discussed it. These did not appear in the discussion so far, since we focused on the explicit results for topological gravity, leading of course to the vanishing of the constrained terms. We will discuss them further in section 4.3.2, making use of the details of the evaluation of the SFF for topological gravity given here.

COMPARISON OF THE GEOMETRIC AND THE UNIVERSAL RESULT FOR JT GRAVITY FOR $\beta = 1$. To finish the comparison of geometric and universal results for the SFF, we discuss now the case of JT gravity for $\beta = 1$. First, we already showed the universal result to be determined by eq. (4.3.1), requiring the input of eq. (4.2.6) and eq. (4.3.10). Since we have computed the WP volumes for unorientable JT gravity up to $g = 1$, we can only compare to terms up to $\mathcal{O}(\tau^3)$. Collecting the relevant terms, we find

$$\kappa_1^{s,\mathrm{JT}}(\tau,\beta) = \kappa_1^{s,\mathrm{Airy}}(\tau,\beta) - \frac{2\tau^2}{\sqrt{2\pi\beta}}\sum_{k=1}^{\infty}(-1)^k\left(1 - k\sqrt{\frac{\beta\pi}{2}}e^{\frac{\beta k^2}{2}}\operatorname{Erfc}\left(k\sqrt{\frac{\beta}{2}}\right)\right) + \frac{4\tau^3}{\pi}\sum_{k=1}^{\infty}\left(-1 + \frac{1}{2}e^{\frac{\beta k^2}{2}}(1+\beta k^2)E_1\left(\frac{\beta k^2}{2}\right)\right) + \mathcal{O}\left(\tau^4\right), \tag{4.3.96}$$

where we already noted that the universal results for topological gravity and JT gravity agree up to the order considered here and hence combined the contribution arising from the orientable part of the universal SFF with the "χ"-part of topological gravity to obtain the full universal SFF for topological gravity at $\beta = 1$. Hence, we know this part of the result up to $g = \frac{7}{2}$.

To compare the geometric result to this, we have to compute the contributions to the SFF from the full unorientable WP volumes with $n = 2$ for $g = \frac{1}{2}$ and $g = 1$, presented in section 3.3.4. Throughout this computation, we will use the p-regularisation, omitting the superscript that indicated this in the presentation above to simplify notation. We would like to recall that the main focus of this thesis is on topological gravity, for which we went into all details when discussing the computation of the SFF. Here, we shall not give all details but refer for the lengthier computations to [3][App. B]. The main reason for doing so, is that we will not pursue the study of cancellations, in the sense of sections 4.2.2 and 4.3.2, for JT gravity and will not need them for more than the derivation of the τ-scaled SFF. In this derivation we shall, however, note the vanishing of certain terms that are examples of constraints of RMT universality on unorientable JT gravity.

Before starting with the computation, we slightly simplify the unorientable WP volumes presented above. To directly read off, which part of the volumes corresponds to the Airy part of the WP volume, and has thus already been computed, we use the symmetry in $b_1 \leftrightarrow b_2$ to rewrite

$$V^{\beta}_{g,2}(b_1,b_2) = V^{\beta}_{g,2}(b_1,b_2)\theta(b_1-b_2) + V^{\beta}_{g,2}(b_2,b_1)\theta(b_2-b_1). \tag{4.3.97}$$

Using this, the contributions to the SFF can be computed, by the same reasoning as for topological gravity, as

$$\kappa^{g}_{\beta}(t,\beta) = \int_0^\infty\int_0^\infty b_1\,\mathrm{d}b_1\,b_2\,\mathrm{d}b_2\,Z^t(b_1,\beta_1)Z^t(b_2,\beta_2)V^{\beta}_{g,2}(b_1,b_2)\theta(b_1-b_2) \tag{4.3.98}$$

$$+ (\beta_1 \leftrightarrow \beta_2)\,,$$

with $\beta_1 = \beta + it$ and $\beta_2 = \beta - it$.

We begin by studying the contribution from $g = \frac{1}{2}$. For this, it is useful to rewrite the result for the WP volume, derived in eq. (3.3.100), as

$$V^{\beta}_{\frac{1}{2},2}(b_1, b_2) = 4\log(p) + b_1 + 2\log\left(1 + e^{\frac{b_2 - b_1}{2}}\right) + 2\log\left(1 + e^{\frac{-b_2 - b_1}{2}}\right). \tag{4.3.99}$$

Using the rewriting of eq. (4.3.97), we can observe that the polynomial non-divergent part of the WP volume is exactly given by the corresponding Airy WP volume (eq. (3.3.58)). Consequently, we can infer the contribution to the SFF from this part of the volume from our evaluation above.

Also the contribution from the divergent part can be read off from our reasoning above to be given by

$$4\log(p)[I_{0,0}(\beta_1, \beta_2) + I_{0,0}(\beta_2, \beta_1)]. \tag{4.3.100}$$

One can directly derive, using eq. (4.3.21), $I_{0,0}(\beta_1, \beta_2) = \frac{(\beta + it)\sqrt{\beta^2 + t^2}}{2\pi\beta}$. Consequently, the leading order ($t^{2g+1} = t^2$) vanishes, ensuring the reality of this contribution to the the SFF (this can be viewed as a "trivial" cancellation, using the nomenclature of section 4.2.2) and the next order contribution does not contribute in the τ-scaling limit. Consequently, the divergent part of the volume does not contribute to the τ-scaled SFF.

Hence, one is left with integrating the "genuinely JT" part of the finite part of the moduli space volume (the two logarithms). We will not spell out this computation in detail, it can be found in [3][App. B.1][14]. Essentially, one proceeds by expanding the logarithms in their series expansion, possible due to $b_1 > b_2$ as imposed by the Heaviside function, and then solving the integral for each summand. By this, one finds the contribution of the second logarithm to be vanishing, while the first contributes to the SFF as [3]

$$\frac{-2t^2}{\sqrt{2\beta\pi}} \sum_{k=1}^{\infty} (-1)^k \left(1 - k\sqrt{\frac{\beta\pi}{2}} e^{\frac{\beta k^2}{2}} \operatorname{Erfc}\left(\frac{\sqrt{\beta}k}{\sqrt{2}}\right)\right) + \mathcal{O}\left(t^{-\frac{1}{2}}\right). \tag{4.3.101}$$

Combining the results and performing τ-scaling, which can be done directly due to the absence of a dependence on t beyond order t^{2g+1}, we find

$$\kappa_1^{s,\frac{1}{2},\mathrm{JT}}(t, \beta) = \frac{-1}{\sqrt{2\pi\beta}} - \frac{2}{\sqrt{2\beta\pi}} \sum_{k=1}^{\infty} (-1)^k \left(1 - k\sqrt{\frac{\beta\pi}{2}} e^{\frac{\beta k^2}{2}} \operatorname{Erfc}\left(\frac{\sqrt{\beta}k}{\sqrt{2}}\right)\right). \tag{4.3.102}$$

14 We would like to note that this computation and the subsequent computations of the genuinely JT pars of the results were performed by J. Tall.

This coincides with the $\mathcal{O}(\tau^2)$ term of the universal RMT result (eq. (4.3.96)) without further modifications.

Continuing with $g = 1$, we study again first the divergent contributions to the WP volume. The contribution diverging as $\log(p)^2$, given in eq. (3.3.107), is just a symmetric polynomial in the lengths. Hence, we can use our computation for the orientable setting to evaluate the contribution arising from this volume as

$$\frac{8\sqrt{\beta_1}\sqrt{\beta_2}\,(\beta_1+\beta_2+\pi^2)}{\pi} = \frac{8\sqrt{\beta^2+t^2}}{\pi}\Big(2\beta+\pi^2\Big). \tag{4.3.103}$$

Consequently, to leading order this contributes at order t, which is smaller than the minimal order required to survive τ-scaling, given by $t^{2g+1} = t^3$. Note that the $\mathcal{O}(t^2)$ vanishes by the reality of the SFF.

For the $\log(p)$ contribution, given in eq. (3.3.106), we consider first the polynomial part. As one can infer from the reasoning for topological gravity[15], only terms that are at least of combined order 2 in the lengths can contribute to the τ-scaled SFF and hence we only have to consider this part of the polynomial contributions. For the non-polynomial part of the volume one proceeds analogously to the logarithmic term encountered at $g = \frac{1}{2}$ by expanding the occurring functions as a power series after using eq. (4.3.97) to introduce Heaviside functions. We will not write down this expansion for the occurring terms in the interest of brevity. Doing this, however, it turns out that only summands that are of the form $b_1 e^{\frac{k}{2}(b_2-b_1)}$, with k a summation index, can contribute to the τ-scaled SFF. Hence, the relevant contributions to the volumes to compute this are

$$v_{1,2,1}(b_1,b_2) \supset \frac{1}{3}\left(2b_1^3+3b_2^2b_1+b_2^3\right)+8\,(b_1-b_2)\,\mathrm{Li}_2\left(e^{\frac{1}{2}(b_2-b_1)}\right). \tag{4.3.104}$$

Starting with the polynomial part, let us suppose that we did not know the coefficients and use a generic polynomial of the expected form, i.e. $C_{3,0}b_1^3 + C_{2,1}b_1^2b_2 + C_{1,2}b_1b_2^2 + C_{0,3}b_2^3$. Using now our results from the discussion of topological gravity to compute the contribution to the SFF from such a volume, we find that there is only one contribution of higher than third order in t, namely a fourth order contribution proportional to

$$3C_{0,3} + C_{1,2} - C_{2,1} - C_{3,0}. \tag{4.3.105}$$

Since the universal result has no divergences in p, this has to vanish. It does so for the explicit coefficients of the considered part of the WP volume and this is the first example for a "non-trivial" constraint for unorientable JT

15 Extracting the leading order in t for a contribution arising from $b_1^{\alpha}b_2^{\gamma}\theta(b_1-b_2)$ and its complementary part (cf. theorem 3.4), one finds by the reasoning leading to eq. (4.3.123) that it is given by $\alpha+\gamma+1$.

gravity. For the non-polynomial part of the WP volume, the computation of the relevant integral is performed in [3][App. B.2] and also gives only subleading contributions. Consequently, as the contribution to the τ-scaled SFF from $g = \frac{1}{2}$, the contribution from $g = 1$ is non-divergent.

For the finite part we proceed analogously to the parts discussed up to here. It is more convenient not to use the symmetrised version of this part of the volume, presented in eq. (3.3.105) but rather $\tilde{v}_{1,2,0}$, given in eq. (3.3.108). The relevant parts of this are given by

$$\begin{aligned}\tilde{v}_{1,2,0}(b_1, b_2) \supset \frac{1}{12}\left(\frac{7}{8}b_1^4 + \frac{7}{4}b_1^2 b_2^2 + b_1 b_2^3 + \frac{3}{8}b_2^4\right) + \frac{\pi^2}{6}\left(2b_1 b_2 + 5b_1^2 + 3b_2^2\right)\\ + \left(2b_1^2 - 2b_2 b_1\right)\mathrm{Li}_2\left(e^{\frac{1}{2}(b_2-b_1)}\right) + 8b_1 \mathrm{Li}_3\left(e^{\frac{1}{2}(b_2-b_1)}\right)\\ + 8(b_2 - b_1)\mathrm{Li}_{2,1}\left(e^{\frac{1}{2}(b_2-b_1)}, 1\right).\end{aligned} \tag{4.3.106}$$

First, we would like to note that, as expected, the leading order terms of the polynomial part coincide, after using eq. (4.3.97), with the Airy WP volume for $g = 1$ and $n = 2$ found from eq. (3.3.70) by setting $\beta = 1$. Consequently, we already know the contribution to the scaled SFF arising from these terms. For the other terms we can again use the formulae derived above to find them producing a finite $\mathcal{O}(t^3)$ contribution, given by

$$\frac{2t^3}{\pi\beta}\frac{\pi^2}{6}(3-5) = -\frac{2t^3\pi}{3\beta}. \tag{4.3.107}$$

The non-polynomial terms can be integrated by the same methods as the previous contributions, the details of this can again be found in [3][App B.2]. They yield as a total contribution

$$\frac{2t^3\pi}{3\beta} + \frac{4t^3}{\pi}\sum_{k=1}^{\infty}\left(-1 + \frac{1}{2}\left(1+\beta k^2\right)e^{\frac{\beta k^2}{2}}E_1\left(\frac{\beta k^2}{2}\right)\right). \tag{4.3.108}$$

Combining this with the known contribution from the Airy part of the WP volume and performing the τ-scaling, we find

$$\begin{aligned}\kappa_1^{s,1,\mathrm{JT}}(t,\beta) = \frac{1}{\pi}\left(\frac{-10}{3} + \log\left(\frac{2t}{\beta}\right)\right) + \\ + \frac{4}{\pi}\sum_{k=1}^{\infty}\left(-1 + \frac{1}{2}\left(1+\beta k^2\right)e^{\frac{\beta k^2}{2}}E_1\left(\frac{\beta k^2}{2}\right)\right).\end{aligned} \tag{4.3.109}$$

We first notice the "genuinely JT" part of this to coincide precisely with the universal prediction for the JT gravity SFF for $\beta = 1$ (eq. (4.3.96)) without any additional considerations. By the reasoning pursued above, the topological gravity part of this result can quite non-trivially be argued to also agree with its universal prediction.

Consequently, we conclude that, up to the orders we considered, the universal and geometric results for JT gravity in the orthogonal symmetry class agree. However, the observation of this matching is only possible by studying first topological gravity, since otherwise we would not have been able to give an argument leading to agreement for the parts of the JT gravity SFF that do not directly match the universal result. Notably, the prime difference of the geometric and the universal result, one is divergent in p while the other does not know about p, was reconciled by the divergent terms in the geometric result not contributing in the τ-scaling limit. Hence, we provided a large set of non-trivial evidence for the persistence of random matrix universality also to full unorientable JT gravity.

As done for the orientable case, we will now take the point of view of taking universality as granted, with the caveat of its non-trivial manifestation in topological gravity, and study the implications this has on the gravitational theories in the form of constraints on their moduli space volumes.

4.3.2 *Constraints on unorientable moduli space volumes from universality*

In the last section we saw that for unorientable versions of topological gravity random matrix universality manifests itself in a more involved form than for the orientable case. Specifically the emergence of terms of higher than expected order in t, which are necessary to form the "cancellation functions" to obtain the logarithmic dependence expected from the universal result, is counter-intuitive with the intuition found from treating the orientable theory. Despite this, it is not true that all possible terms contribute to the τ-scaled SFF. Rather, as for the orientable case, there are certain cancellations leading to the result discussed above, which we will explore now.

For this, we will again focus on topological gravity, putting to use all the details of the computation of the SFF we have discussed above. For JT gravity we will not pursue this study to that extent due to time-constraints, but rather briefly note that for the finite part one could proceed mostly analogously to the treatment of topological gravity, which we will present momentarily[16]. It is however specific for the reasoning in JT gravity that RMT universality imposes the strict constraint of the contributions to the τ-scaled SFF from the divergent parts of the WP volumes all having to vanish. Of this type of cancellation we discussed several examples above and an extension of this to a generic result for the constraints is an interesting point for future work.

16 However, with the necessity to work out additionally the contributions from the non-polynomial parts of the WP volumes in more detail than we have presented so far. Hence, this is left for future work.

Our discussion for topological gravity is split into two parts. First, we discuss the maybe most notable part of the SFF for unorientable variants of the gravitational theory, the logarithmic part. For this, we derive the full set of constraints on the Airy WP volumes implied by the fidelity to the universal RMT result and show them to be closely related to those discussed for the orientable setting in theorem 4.1. Second, we continue this by exploring the full set of constraints, i.e. vanishing terms of the τ-scaled SFF up to $g = \frac{7}{2}$ and formulate a conjecture for their occurrence for arbitrary values of g. Though we will not pursue the explicit evaluation of the form of the additional constraints here[17], our results will be very helpful in the discussion of quantum chaos for topological gravity at non-Wigner-Dyson values of the Dyson index in the next section.

CONSTRAINTS ON AIRY WP VOLUMES: THE LOGARITHMIC PART. As we discussed already above, logarithmic terms arise only from contributions to the $V_g^{>}(b_1, b_2)$ (cf. theorem 3.4) which are of odd order in both lengths. It is important to note that two things are already apparent from this. First, as we remarked already above, such terms can only occur at integer genus, and second, they are genuine to the unorientable setting as orientable WP volumes contain only even orders in the lengths. Due to this, and our observation above that all other terms in the unorientable Airy WP volumes lead to quite similar contributions to the SFF as we had encountered in the orientable setting, the logarithmic contributions are of highest interest for the study of genuinely unorientable features of moduli space volumes. This motivates us focussing on them as giving rise to the prime examples for constraints on the unorientable Airy WP volumes.

As a first step towards them, let us briefly recall how the logarithms in the SFF arise from the mentioned type of contributions to the WP volumes. In eq. (4.3.24) we derived the contribution to the two-point function of partition functions from such a term. Since the derivative of an arctan is a rational function, it it is easy to see that the only part of this result which can give rise to a logarithm is that without any derivative, i.e. the contribution potentially leading to a logarithm from this expression can be isolated as

$$I^{\log}_{\alpha,\gamma}(\beta_1, \beta_2) := \frac{2^{6g-1}\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\gamma}{2}\right)}{\pi^2}\beta_1^{\frac{\alpha+1}{2}}\beta_2^{\frac{\gamma+1}{2}} \arctan\left(\sqrt{\frac{\beta_1}{\beta_2}}\right). \tag{4.3.110}$$

The contribution to the SFF from this can be evaluated by putting the by now well-known SFF values for the complex inverse temperatures ($\beta_1 = \beta + it$ and $\beta_2 = \beta_1^*$) and including also the complementary part of the Airy WP

17 This is under present scrutiny. The results will be included in [6].

volume (arising from the $\theta(b_2 - b_1)$ part (cf. theorem 3.4)). The "logarithmic" contribution to the SFF at integer genus g is hence given by

$$\kappa_g^{\log}(t,\beta) := e^{-2gS_0} \sum_{\substack{\alpha+\gamma=6g-2 \\ \alpha,\gamma \text{ odd}}} C^g_{\alpha,\gamma} \Big[I^{\log}_{\alpha,\gamma}(\beta_1,\beta_2) + I^{\log}_{\alpha,\gamma}(\beta_2,\beta_1) \Big], \tag{4.3.111}$$

where we collect the $C^{g,i}_{\alpha,\gamma}$ from theorem 3.4 into one coefficient, defined as the sum, including the β dependence, of the individual coefficients. We do this, since we are, at this stage, interested mainly in the unorientable Wigner-Dyson values of β and can simplify notation by this combination. Using the binomial theorem, the first part of this summation can be expanded as

$$e^{-2gS_0} i \arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right) \frac{t^{3g}}{\pi} \sum_{l=0}^{3g} C(g,l)\beta^l t^{-l} \underbrace{\sum_{\substack{\alpha+\gamma=6g-2 \\ \alpha,\gamma \text{ odd}}} C^g_{\alpha,\gamma} \frac{\Gamma(1+\frac{\alpha}{2})\Gamma(1+\frac{\gamma}{2})}{\pi} (-1)^{\frac{\gamma+1}{2}} \sum_{n+m=l} \binom{\frac{\alpha+1}{2}}{n} \binom{\frac{\gamma+1}{2}}{m} (-1)^m}_{:=K^{\text{un}}_g(l)}. \tag{4.3.112}$$

Here we defined $C(g,l)$ to be a non-vanishing function depending only on g and l and thus being irrelevant for the present discussion. Furthermore, we note that $K^{\beta=1}_g(l)$ is a rational number because the $C^g_{\alpha,\gamma} \in \mathbb{Q}_+$ by definition, the binomial coefficients are also natural numbers as α and γ are odd, and the factor of π in the denominator is cancelled by the two factors of $\sqrt{\pi}$ arising from the Γ-functions due to both indices being odd. Using that $\beta_1 \leftrightarrow \beta_2$ for the SFF values of the inverse temperatures is equivalent to $t \leftrightarrow -t$, we can use this rewriting to write the logarithmic contribution to the SFF as

$$\begin{aligned} \kappa_g^{\log}(t,\beta) &= e^{-2gS_0} \sum_{l=0}^{3g} \frac{C(g,l)}{\pi} \beta^l K^{\text{un}}_g(l) \times \\ &\quad i\left[t^{3g-l} \arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right) + (-t)^{3g-l} \arctan\left(\sqrt{\frac{\beta-it}{\beta+it}}\right)\right] + \mathcal{O}\left(t^{-1}\right) \\ &= e^{-2gS_0} \sum_{l=0}^{3g} \frac{C(g,l)}{\pi} \beta^l K^{\text{un}}_g(l) L_{3g-l}(t,\beta), \end{aligned} \tag{4.3.113}$$

where we used eq. (4.3.42) to the second line. We can now use proposition 4.2 to extract the relevant terms for the large t limit and then introduce τ. Doing this, while neglecting terms that vanish in the $t \to \infty$ limit, one finds

$$e^{-S_0}\kappa_g^{\log}(t,\beta) = \tau^{2g-1}\sum_{l=0}^{g-1}\frac{C(g,l)}{\pi}\beta^l t^{g-l-1}\times$$
$$K_g^{\mathrm{un}}(l)\times\begin{cases}\log\left(\frac{\beta}{2t}\right)+\sum_{k=0}^{\infty}C(2k)\left(\frac{\beta}{t}\right)^{2k} & \text{for } 3g-l \text{ odd},\\ \frac{\pi}{2} & \text{for } 3g-l \text{ even}.\end{cases} \tag{4.3.114}$$

For $l = g-1$, for which $3g-l = 2g+1$ is always odd (for integer g), this produces the logarithmic terms we observed in the discussion of the actual topological gravity SFFs for $\beta = 1$ and $\beta = 4$ as computed from the Airy WP volumes. For $g > 1$ there are however contributions that retain a t-dependence! Those contain either a logarithm or are of $\mathcal{O}(\pi^0)$.

Employing the cancellation functions approach, we could motivate that the logarithmic part without an additional t-dependence produces the logarithm expected from the universal result. Matching to the universal result actually requires all t-dependent terms to combine to the convergent or asymptotic expansion of the cancellation function and hence there are no additional terms left to perform a similar reasoning for the potential logarithmic terms with an additional dependence on t, arising from the above reasoning for $l \in [0, g-2]$ with $3g-l$ odd. Even if there were such terms, the procedure would only change the argument of the logarithm and the result would, due to the t-dependent factor, still be incompatible with RMT universality. Hence, universality implies all those terms to vanish. One can perform a similar reasoning for the terms for which $3g-l$ is even, arguing here by the dependence on π. This can be done by observing that even if there were such terms as to perform a reasoning akin to the treatment of e.g. the $t^4\sqrt{t}$ term for $\beta = 4$, this would give rise to terms that are of $\mathcal{O}\left(\pi^{\frac{1}{2}}\right)$ which are not found in the universal result.

Due to this reasoning, we infer

$$\underset{0\le l<(g-1)}{\forall} K_g^{\mathrm{un}}(l) = 0, \tag{4.3.115}$$

which is a statement about (a part of the) Airy WP volumes akin to theorem 4.1, which was valid for orientable WP volumes. We now give examples for these constraints actually being fulfilled. The first genus where they become relevant is $g = 2$, for which one can evaluate the constraint as

$$K_2^{\mathrm{un}}(0) \propto 21C_{1,9}^2 - 7C_{3,7}^2 + 5C_{5,5}^2 - 7C_{7,3}^2 + 21C_{9,1}^2. \tag{4.3.116}$$

For $g = 3$ there are two constraints, which can be evaluated as

$$K_3^{\text{un}}(0) \propto 715C_{1,15}^3 - 143C_{3,13}^3 + 55C_{5,11}^3 - 35C_{7,9}^3 + \\ + 35C_{9,7}^3 - 55C_{11,5}^3 + 143C_{13,3}^3 - 715C_{15,1}^3, \tag{4.3.117}$$

$$K_3^{\text{un}}(1) \propto 1001C_{1,15}^3 - 143C_{3,13}^3 + 33C_{5,11}^3 - 7C_{7,9}^3 \\ - 7C_{9,7}^3 + 33C_{11,5}^3 - 143C_{13,3}^3 + 1001C_{15,1}^3. \tag{4.3.118}$$

Explicitly putting the coefficients from the respective WP volumes for $\beta = 1$, which are given in [2] and the supplementary material, the reader can assure that they are indeed fulfilled. Also for the case of $\beta = 4$ we find that the constraints are fulfilled [5]. Of course, this is implied by the fulfilment for $\beta = 1$ via theorem 3.2.

It is interesting to note that these constraints are quite directly linked to the constraints for orientable topological gravity, given in theorem 4.1. In [2] we showed this by essentially rewriting[18] the orientable constraints for $a = g - 1$ (this is due to us considering only topological gravity) in their unsimplified form (eq. (4.2.46)) as

$$K_g^l = \sum_{\substack{\alpha+\gamma=6g-2 \\ \alpha,\gamma \text{ even}}} C_g^{\frac{\alpha}{2},\frac{\gamma}{2}} \Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{\gamma}{2}+1\right)(-1)^{\frac{\gamma}{2}} \sum_{k+j=l} \binom{\frac{\alpha+1}{2}}{k}\binom{\frac{\gamma+1}{2}}{j}(-1)^j, \tag{4.3.119}$$

which (up to a constant factor to make the result real) is the same as the expression we gave for $K_g^{\text{un}}(l)$, only with requiring α and γ to be even. Consequently, the constraints on the logarithmic terms can be thought of as the natural extension of the "orientable" constraints to the unorientable setting.

In fact, one can consequently use the reasoning in the proof of theorem 4.1 to put also the logarithmic constraints to a more convenient form. Following this reasoning, which extends the results of [2], we find the constraints to be equivalent to

$$\underset{0 \le l < g-1}{\forall} \tilde{K}_g^{\text{un}}(l) = 0, \tag{4.3.120}$$

with

$$\tilde{K}_g^{\text{un}}(l) = \sum_{\substack{\alpha+\gamma=6g-2 \\ \alpha,\gamma \text{ odd}}} C_{\alpha,\gamma}^g \frac{\Gamma(1+\frac{\alpha}{2})\Gamma(1+\frac{\gamma}{2})}{\pi}(-1)^{\frac{\gamma+1}{2}}(\gamma-\alpha)^l. \tag{4.3.121}$$

18 In fact, we start a step earlier and include the expansion of $\sqrt{\beta_1\beta_2}$ into the expansion of the whole result before power-counting as opposed to taking it out as a prefactor and expanding afterwards, as done above. The interested reader is referred to [2] for a full derivation.

It has to be noted that due to the absence of symmetry in the coefficients upon interchanging their labels, all of these constraints are "non-trivial" which is the reason why we find constraints in a greater number and starting at smaller genus than we did for the orientable setting.

This concludes the derivation and discussion of the constraints arising from RMT universality for the logarithmic parts of the SFF of unorientable topological gravity for now. We will continue with a more exploratory study of all constraints relevant for the contributions to the SFF for topological gravity.

CONSTRAINTS ON AIRY WP VOLUMES: THE NON-LOGARITHMIC PART. So far we were concerned specifically with the logarithmic contributions to the τ-scaled SFF of topological gravity. For this, we could argue for certain cancellations to occur, with their emergence having clear parallels to the orientable constraints, proven in theorem 4.1. For the non-logarithmic terms, we have to deal with the terms building up the cancellation functions which, as we already mentioned, would have to vanish, were the orientable reasoning applicable here. In other words one can say that not all terms that retain a t dependence after τ-scaling have to vanish. Here, we explore which of the terms that might occur do have to vanish and find a clear pattern of constraints giving further evidence to the mechanism of cancellation functions discussed above.

For the logarithmic cancellations we derived the full analytic form of the respective contributions to the SFF first and then derived from this the constraints. Here, we will proceed by first arguing in more detail than above, which contributions to the SFF we expect for a chosen value of the genus g in principle and then formulate a conjecture, based on our argumentation for agreement of the SFF with its universal prediction, which of these terms should cancel. We will then work out the explicit form of these constraints on the coefficients of the Airy WP volumes for all values of the genus we have computed and show them to be fulfilled. We do not pursue the evaluation of the general form of these constraints here due to time constraints and leave this to be treated in [6].

To see that generically there are contributions of larger order in t to the SFF than those presented in eq. (4.3.54) to eq. (4.3.61) or the equivalent result for $\beta = 4$, we study the evaluation of the contributions of a term of the form $b_1^{\alpha} b_2^{\gamma} \theta(b_1 - b_2)$ to the two-point function of partition functions as a finite sum. This was evaluated in eq. (4.3.25) and eq. (4.3.28). Putting in the SFF values of the complex inverse temperatures, we can see from both results that the contribution at the highest order in t is proportional to

$$(\beta_1 \beta_2)^{\frac{\alpha+\gamma+1}{2}} \beta_1 = \left(\beta^2 + t^2\right)^{\frac{6g-2+1}{2}} (\beta + it). \tag{4.3.122}$$

To compute the full contribution of this part of the Airy WP volume, one needs to take into account also the complementary term ($b_2^{\alpha} b_1^{\gamma} \theta(b_2 - b_1)$), whose contribution is found by exchanging $\beta_1 \leftrightarrow \beta_2$ in the above results. Consequently, the full contribution to the canonical SFF from this part of the WP volume is proportional to

$$\left(\beta^2 + t^2\right)^{\frac{6g-1}{2}} \propto t^{6g-1}, \tag{4.3.123}$$

which for $g \geq 1$ is of higher order than the maximal order compatible with universal RMT, i.e. $2g + 1$. The vanishing of the naive leading-order can also be explained by the requirement on the canonical SFF to be a real quantity. Employing this reasoning, one is led to conjecture that, generically, there is a contribution to every odd power $k \in [2g + 2, 6g - 1]$ of t for the integer genus and to every even power $k \in [2g + 2, 6g - 1]$ for the half-integer case. As we discussed above, for half-integer genus there can also be terms that contain an odd power of $\sqrt{t}$. For these we do not derive the maximal order here, but based on the results for $\beta = 1$ topological gravity we can conjecture that for every half-integer value of the genus g, there are contributions to the SFF at every half-integer power of t in $\left[2g + \frac{3}{2}, 3g\right]$.

The cancellation functions mechanism, as discussed above, combines all of the terms that depend in a specific way on β and τ to yield the universal result. It combines the terms dependent on a half-integer power of $\sqrt{t}$ and specific terms that retain a polynomial dependence on t after τ-scaling. Consequently, these terms should remain and not cancel out, though they retain a t dependence after τ-scaling. For the terms containing $\sqrt{t}$ this means that they should all remain, while for the other terms necessary to form the cancellation functions we only have the example of the t^9 term in eq. (4.3.60) and can hence base our reasoning only on this. We note that $g = 3$, where this additional term appears, is the first genus where we observed the cancellation of an additional logarithmic term. For $g = 2$ we also observed a cancellation of a term of logarithmic origin, though of the "$\mathcal{O}\left(\pi^0\right)$"-type (cf. eq. (4.3.114)). Notably, for $g = 2$ we do not observe an additional term and hence it is plausible that the occurrence of a non-cancelling t-dependent term after τ-scaling coincides with the cancellation of a logarithm, as discussed in the preceding paragraph. We thus conjecture the constraints on the unorientable Airy WP volumes to be of the following form:

Conjecture 4.1 (Constraints on unorientable Airy WP volumes from RMT universality)**.** *For a chosen genus g, those contributions to the canonical SFF are constrained to vanish by the matching to the universal RMT result which are of integer order larger than t^{2g+1} and*

- *of odd order, for integer g.*

- *of even order, for half-integer g.*

For integer genus there are, however, exceptions. They occur, whenever for the given genus there is a logarithmic term that is cancelled by a constraint à la eq. (4.3.115) *for $3g - l$ odd. Then, the term which depends on t as $t^{2g+1+g-l-1} = t^{3g-l}$ does not vanish. Any additional terms of higher integer power than $2g + 1$ vanish by the requirement of reality for the SFF without imposing any requirement on the coefficients of the unorientable Airy WP volumes. The non-triviality of this statement is guaranteed by the existence of contributions to the SFF of $\mathcal{O}\left(t^k\right)$ with $k \in [2g + 2, 6g - 1]$.*

Having put forward this conjecture, we put it to the test on the contributions to the SFF for which we have computed the unorientable Airy WP volumes, i.e. up to $g = \frac{7}{2}$. We do this, by considering the Airy WP volume for a given value of β as given by the structure of theorem 3.4 with the coefficients as variables, i.e.

$$V_g^{>}(b_1, b_2) = \sum_{\substack{\alpha_1,\alpha_2 \in \mathbb{N}_0 \\ \alpha_1+\alpha_2=6g-2}} C_{\alpha_1,\alpha_2} b_1^{\alpha_1} b_2^{\alpha_2}. \tag{4.3.124}$$

For a given value of g we then proceed to compute the general contribution to the SFF as arising from this volume by the methods discussed in section 4.3.1. This yields the coefficient of each occurring term as determined by a linear combination of the C_{α_1,α_2}.

Before presenting the results for this, let us derive from the conjecture what we expect. The smallest value of g for which $[2g + 2, 6g - 1]$ is not empty is $g = 1$. Then it contains one odd number and hence we expect one constraint, concerning the term of t^5, while the term of $\mathcal{O}(t^4)$ vanishes without the requirement of a constraint. The smallest half-integer value of g for which the set is not empty is $g = \frac{3}{2}$. The corresponding set of possible powers in t is thus $[5, 8]$ and we hence expect non-trivial cancellations for the terms of $\mathcal{O}(t^6)$ and $\mathcal{O}(t^8)$, while those of $\mathcal{O}(t^5)$ and $\mathcal{O}(t^7)$ vanish "trivially". By the same reasoning we expect 3 (non-trivial) constraints for $g = 2$ (there is no cancellation of a logarithmic term with $3g - l$ odd), 4 for $g = \frac{5}{2}$ and 6 for $g = \frac{7}{2}$. For $g = 3$ we would expect 5 (non-trivial) constraints, but there is a logarithmic cancellation occurring for $3g - l = 9$ and hence we expect a contribution at t^9 and thus only 4 constraints.

This is precisely what we find. We present the constraints for the two smallest values of g here, while we refer the interested reader for the collection of all of them to appendix E:

$$\begin{aligned} g = 1: \quad 0 =& 2C_{0,4} + C_{1,3} - C_{3,1} - 2C_{4,0}, \\ g = \frac{3}{2}: \quad 0 =& 7C_{0,7} + 5C_{1,6} + 3C_{2,5} + C_{3,4} \end{aligned} \tag{4.3.125}$$

$$- C_{4,3} - 3C_{5,2} - 5C_{6,1} - 7C_{7,0}, \tag{4.3.126}$$

$$0 = 35C_{0,7} + 35C_{1,6} + 25C_{2,5} + 9C_{3,4} - 9C_{4,3} - 25C_{5,2} - 35C_{6,1} - 35C_{7,0}. \tag{4.3.127}$$

It is important to note that all of the constraints we found can be written as

$$\sum_{i<j}^{i+j=6g-2} \mathcal{C}(i,j)\left[C_{i,j} - C_{j,i}\right], \tag{4.3.128}$$

with $\mathcal{C}(i,j) \in \mathbb{N}$, which we conjecture to be true in general.

Putting the explicit results for the Airy WP volumes for $\beta = 1$ and $\beta = 4$, one can verify all of the constraints to be fulfilled. Of course, due to theorem 3.2, one only has to check for one of the values of the Dyson index. Actually, it is more economical to use eq. (3.3.66) and to notice that for integer genus g all contributions of higher order than t^{2g+1} arising from $\mathcal{V}_{g,2}^{0,\text{Airy}}$ vanish by the orientable cancellations[19]. Thus, one has to check the cancellations only for the coefficients of $\mathcal{V}_{g,2}^{g,\text{Airy}}$(for integer g), or $\mathcal{V}_{g,2}^{g+\frac{1}{2},\text{Airy}}$ (for half-integer g).

Consequently, we have shown that there is a substantial set of constraints extending the logarithmic constraints discussed in the preceding paragraph. It is worthwhile to study these cancellation in as much detail as the logarithmic ones, which, among other things, would enable the proof of all the conjectures regarding their form put forward in our discussion. Furthermore, it would be very interesting to evaluate unorientable WP volumes to higher genus, at least $g = 4$, since this would enable the test of conjecture 4.1 on a value of g we did not know when formulating it. For this potential future test, let us already remark that for $g = 4$ we naively expect 7 non-trivial constraints, one of which (that for the $\mathcal{O}(t^{11}\beta^0)$ term) is not fulfilled, giving rise to an additional term of the type of the t^9 term from $g = 3$.

This concludes our discussion of the constraints imposed by RMT universality on the unorientable (Airy) WP volumes for the theories associated with a Wigner-Dyson class. We will extend further on the potential future application and extension of this study in section 4.5. Now we leave the setting of the Wigner-Dyson classes altogether and concern ourselves with extending the study, which we have now successfully pursued for all Wigner-Dyson values of the Dyson index, to the setting of general, non-Wigner-Dyson values of the index.

4.4 BEYOND THE DICHOTOMY OF SURFACES

The primary difference of considering arbitrary values of the Dyson index, as opposed to Wigner-Dyson values, is the unavailability of a full analytic

19 Also, they cancel due to the symmetry of the coefficients in the two indices by eq. (4.3.128).

result for the universal microcanonical SFF, i.e. $b^{\beta}(x)$ (cf. eq. (2.4.19)), which was the starting point for all of our discussions in the Wigner-Dyson classes. In fact, to our knowledge and as noted in [125], there is no full analytic result known for the microcanonical SFF of the Gaussian matrix model at the order of the Heisenberg time, which by analogy to the Wigner-Dyson classes one would expect to be the universal microcanonical SFF (after unfolding)[20]. Furthermore, the question emerges, whether RMT universality can actually be extended beyond the Wigner-Dyson classes. From the perspective of quantum chaos, this question is quite interesting: it is appealing to extend the definition of quantum chaos via the BGS conjecture to the setting of arbitrary Dyson index by the requirement of the "universal" observables of a theory matching to the predictions of the β Gaussian matrix model, in direct extension of the reasoning presented for the Wigner-Dyson classes in section 2.4. Doing this, the persistence of universality would imply that, in the specific sense of the definition via a generalised BGS conjecture, matrix models are quantum chaotic for arbitrary values of β.

In the following, we will combine the insights obtained throughout this thesis to work towards answering these questions for the specific setting of gravitational theories dual to matrix models.

First, in section 4.4.1, we will discuss the question of the fidelity of our results for β topological gravity to the general β Gaussian matrix model in the universal limit. Due to the unavailability of a full analytic result for the microcanonical SFF, we will first do this by comparing to a numerical evaluation of this quantity using the expression of the Gaussian matrix model at arbitrary Dyson index as an ensemble of tridiagonal matrices (cf. appendix D), implemented by the author and J. Dieplinger. Furthermore, we will compare to recent work on this ([152]) which provides an analytic ansatz for the microcanonical SFF of the arbitrary β Gaussian matrix model based on its numerical evaluation. Notably, this ansatz consists of two different functions for the ranges of $\beta \geq 2$ and $\beta \leq 2$. Due to the complications arising for $\beta \neq 2$ for contributions to the canonical SFF of larger order than t^2, we restrict our discussion here to contributions of smaller order than t^3 as only for these we can infer the contribution to the microcanonical SFF from that of the canonical SFF without any additional considerations. To these orders, we find our result to be in quite good agreement with the numerical results, which we interpret as a first positive sign for the persistence of RMT universality beyond the Wigner-Dyson classes.

Second, also in section 4.4.1, we extend this reasoning of checking necessary conditions for fidelity to universality by checking the constraints we found to be obeyed by the Airy WP volumes in the Wigner-Dyson classes in sec-

20 Of course, universality implies the unfolded microcanonical SFF in units of time given by the Heisenberg time to be the same for all matrix models. Though, if it is analytically computable in one of them, it will be the Gaussian model.

tions 4.2.2 and 4.3.2. We find that, unsurprisingly, the orientable constraints are obeyed by the "orientable" part of the Airy WP volumes (cf. theorem 3.3). Furthermore, also all non-logarithmic constraints following from conjecture 4.1 are obeyed by the full β Airy WP volumes, i.e. also by their non-Wigner-Dyson parts (cf. theorem 3.3). For the logarithmic constraints we observe the persistence of the cancellation of the actual logarithmic terms while the "$\mathcal{O}(\pi^0)$"-terms remain, for reasons we shall explain. We interpret this as another sign for the extension of universality to the arbitrary β setting.

While, of course, by studying one example we can not say anything about universality for a generic matrix model, this provides first important steps to establish the persistence of quantum chaos, as defined via the BGS conjecture, in β topological gravity beyond the purely orientable/unorientable setting, pointing towards its persistence for all related theories like JT gravity, the $(2, 2p+1)$ and Virasoro minimal strings.

A full check of this would require a full analytic result for the microcanonical SFF of the β Gaussian matrix model. Working towards this, we close the discussion of this chapter by first using theorem 3.3 to "uplift" the known universal results for the microcanonical SFF to the setting of arbitrary Dyson index to find an important part of the full result. Notably, our result covers the whole range of β without a split of domains as necessary for the ansatz of [152]. While the method we describe is a priori applicable without restricting the domain of times for which it can be used, for technical reasons we restrict the discussion here to the setting of times smaller than the Heisenberg time. In this range, we compare to the numerical evaluation of the microcanonical SFF for the β Gaussian matrix model and the ansatz of [152] and show that, up to deviations expected from us not including non-Wigner-Dyson terms in our discussion, our result is in good agreement with the numerical data and the ansatz. However, a more detailed study of the β dependence of the numerical result reveals our result to perform better than the ansatz by [152] in its regime of validity. We give an analytic argument motivating why this is expected, since the ansatz is at tension with theorem 3.2, valid also for the Gaussian matrix model. Furthermore, we compare to the highest order analytic results for the microcanonical SFF for the Gaussian matrix model for arbitrary Dyson index we have knowledge of, given in [153, 154]. We find full agreement with our results for the microcanonical SFF up to the orders to which we have full results and agreement of the Wigner-Dyson part of the results with our "uplifted" results for the remaining terms, establishing another strong sign for universal behaviour in β topological gravity.

4.4.1 *Is topological gravity quantum chaotic for a continuous Dyson index?*

Before going into the discussion of the fidelity of β topological gravity in the explained way, it is useful to briefly explain why we are dealing with the microcanonical SFF as our observable of interest here, while up to now we always considered the canonical SFF. The main reason for this is the unavailability, to our knowledge, of a full analytic result for the microcanonical SFF. Were one provided, we could proceed analogously to the discussion of the Wigner-Dyson classes. Since this is not the case, we have to evaluate the quantities for the Gaussian matrix model numerically, as already mentioned, and we hence have to compute first the actual universal quantity: the unfolded microcanonical SFF. Starting from this, one could in principle compute the numerical prediction for the canonical SFF for topological gravity by numerically computing the Laplace transform from microcanonical to canonical (eq. (4.1.11)) with inserting the density of states for topological gravity. As the computation starting form the WP volumes of the τ-scaled canonical SFF yields, by definition, a power-series, a comparison with the numerical result would then additionally require extracting the coefficient functions of the expected powers of τ and β. We do not pursue this route here, since it is more convenient and less cumbersome numerically to transform our analytic canonical results to microcanonical contributions and then directly compare to the unfolded microcanonical result from the numerical computation, as we will do now.

COMPARISON OF THE MICROCANONICAL SFFS. A complication of both approaches is the need to introduce cancellation functions starting with $g = 1$, i.e. $\mathcal{O}(\tau^3)$ to access the true universal limit. For the Wigner-Dyson parts of the contributions to the microcanonical SFF, i.e. the contributions originating in the part of the general structure (theorem 3.3) that does not vanish at the Wigner-Dyson values of the Dyson index, we can circumvent this by using RMT universality for those symmetry classes (cf. section 4.4.2), but for the non-Wigner Dyson parts of the result we can not do this due to the absence of a full analytic result for the microcanonical SFF for arbitrary Dyson index. Hence, for the discussion in this paragraph, we restrict to the terms up to $\mathcal{O}(\tau^2)$, for which the τ-scaling limit can be taken without any cancellation functions[21]. Consequently, we can perform an inverse Laplace transform of our results in eqs. (4.3.62) and (4.3.63) to find via eq. (4.1.11)

$$\kappa_{\beta}^{\text{Airy},s}(\tau, E) = \frac{\tau}{2\pi\rho_0^{\text{Airy}}}\frac{2}{\beta} - 2\frac{2-\beta}{\beta^2}\left(\frac{\tau}{2\pi\rho_0^{\text{Airy}}}\right)^2 + \mathcal{O}\left(\tau^3\right). \tag{4.4.1}$$

21 We will go beyond this order in the later discussion, going up to the contribution of $\mathcal{O}(\tau^8)$, when comparing to the available parts of the universal microcanonical SFF.

Assuming this to match the unfolded SFF of the Gaussian matrix model, we can read off the prediction from topological gravity for the expression of $b^\beta(x)$ for small values of $x = \frac{\tau}{2\pi\rho_0^{\text{Airy}}}$, using eq. (2.4.19), as

$$b^\beta(x) = 1 - \frac{2}{\beta}x + 2\frac{2-\beta}{\beta^2}x^2 + \mathcal{O}\left(x^3\right). \tag{4.4.2}$$

As a sanity check, we can compare this to the expansions for small x of the known results of $b^\beta(x)$ for the Wigner-Dyson values of β, given in eqs. (2.4.12) to (2.4.14). These expansions are given by

$$b^1(x) = 1 - 2x + 2x^2 + \mathcal{O}\left(x^4\right), \tag{4.4.3}$$

$$b^2(x) = 1 - x, \tag{4.4.4}$$

$$b^4(x) = 1 - \frac{x}{2} - \frac{x^2}{4} + \mathcal{O}\left(x^4\right), \tag{4.4.5}$$

which, as expected is matched by eq. (4.4.2) upon putting the respective values of the Dyson index.

We will now compare this to a numerical evaluation of the unfolded microcanonical SFF for the β Gaussian matrix model. The details of this computation are given in appendix D. Based on good agreement with the numerical results evaluated in this way, [152] provides an ansatz for the whole of b^β in the region of $\beta \in [1,2]$ as

$$\begin{aligned} b_\beta^{A,1}(x) &:= \begin{cases} 1 - \frac{2}{\beta}x + \frac{2-\beta}{\beta}x\log(1+2x) & \text{if } x < 1 \\ 1 - \frac{2}{\beta} + \frac{2-\beta}{\beta}x\log\left(\frac{2x+1}{2x-1}\right) & \text{if } x \geq 1 \end{cases} \\ &= \frac{2-\beta}{\beta}b_1(x) + 2\frac{\beta-1}{\beta}b_2(x), \end{aligned} \tag{4.4.6}$$

and for $x \leq 1$ in the region of $\beta \in [2,4]$ as

$$b_\beta^{A,2}(x) = 1 - \frac{2}{\beta}x + \frac{\beta-2}{2\beta}x\log(|1-x|). \tag{4.4.7}$$

Both ansätze coincide (in the range of their validity) with the analytic results for the Wigner-Dyson cases, given in eqs. (2.4.12) to (2.4.14), upon putting the respective value of the Dyson index. Furthermore, they can be expanded at $x = 0$ to yield

$$b_\beta^{A,1}(x) = 1 - \frac{2}{\beta}x + 2\frac{2-\beta}{\beta}x^2 + \mathcal{O}\left(x^3\right), \tag{4.4.8}$$

$$b_\beta^{A,2}(x) = 1 - \frac{2}{\beta}x + \frac{2-\beta}{2\beta}x^2 + \mathcal{O}\left(x^3\right). \tag{4.4.9}$$

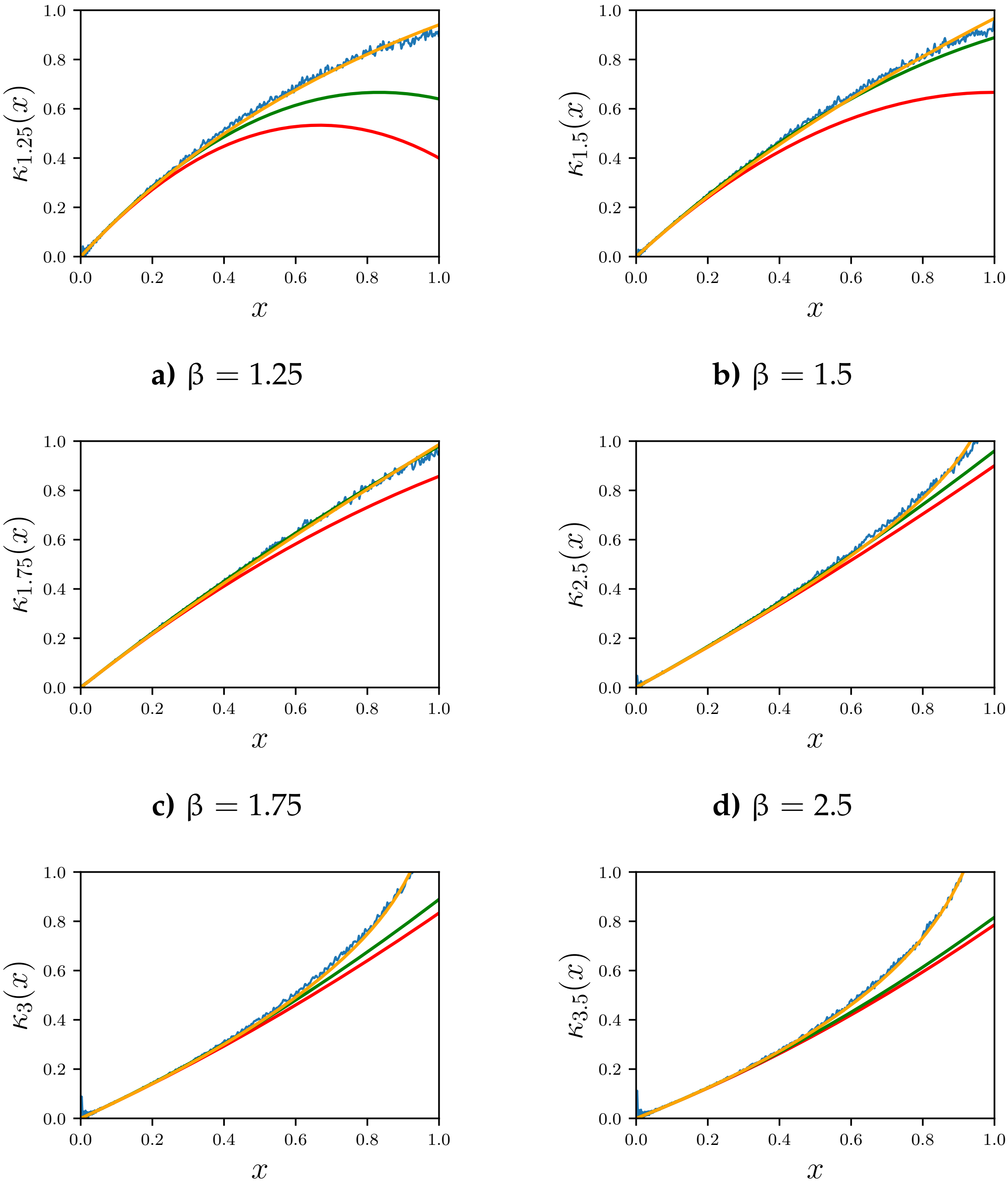
We first note that the $\mathcal{O}(x)$ term agrees with our result and thus we give an analytic argument for this numerical finding[22]. For the next order our result nearly, but not completely, agrees. In fact, the difference is an additional factor of β^{-1} in our result compared to the ansätze $b_\beta^{A,i}$. Notably, the ansätze also do not contain any non-Wigner-Dyson terms, at least in an obvious way. We will come back to investigate this further in section 4.4.2.

Before doing this, we present the comparison of our result eq. (4.4.2) with the numerical computation and the ansätze of [152], given in eqs. (4.4.6) and (4.4.7). This comparison, for various values of $\beta \in [1,4]$ can be found in fig. 4.4.1, where the numerical result for the unfolded microcanonical SFF is plotted as the blue line and our prediction for it in green. The whole ansätze of [152] are plotted as the orange line and their expansion up to second order, which is what can be compared to our result, is plotted in red. We observe that the orange curve for all values of β we consider here nicely follows the numerical line. However, we also see for all considered values of β that our prediction is following the numerical line with a smaller error than the ansatz appropriate for the respective regime expanded to second order. Notably, we do not have to change from one ansatz to another when switching from $\beta \in [1,2]$ to $\beta \in [2,4]$. From this we conclude that our result for the microcanonical SFF is a viable approximation to the microcanonical SFF for small times over the whole regime of $\beta \in [1,4]$ which, as expected, requires extension to higher orders to achieve an accuracy similar to the full $b_\beta^{A,i}$.

We interpret this as a first sign of β topological gravity agreeing with the β Gaussian matrix model for the unfolded microcanonical SFF, i.e. as an example for the persistence of RMT universality in the setting of arbitrary Dyson index. So far, we have, however, concerned ourselves only with terms up to $\mathcal{O}(\tau^2)$. To say something meaningful about terms of higher order, we continue our study by investigating whether the constraints that RMT universality imposed on the Airy WP volumes in the Wigner-Dyson classes persist in the setting of arbitrary Dyson index.

THE PERSISTENCE OF THE CONSTRAINTS IN β TOPOLOGICAL GRAVITY. Having studied the microcanonical SFF in the preceding paragraph, we now briefly turn to studying the canonical SFF. On this, random matrix universality would manifest itself as the τ-scaling limit of the topological gravity SFF being given by eq. (4.1.9), with an unknown function b^β. However, already from this one can infer the absence of a dependence on e^{S_0} in the prediction for the τ-scaled SFF. We studied how this precise statement implied constraints on the orientable WP volumes (theorem 4.1) and how its combination with the mechanism of cancellation functions gave rise to our conjecture for their

22 Likewise, an analytic argument for this is given in [153]



a) $\beta = 1.25$ **b)** $\beta = 1.5$

c) $\beta = 1.75$ **d)** $\beta = 2.5$

e) $\beta = 3$ **f)** $\beta = 3.5$

Figure 4.4.1: Comparison of the numerical evaluation of the unfolded microcanonical SFF for the general β Gaussian matrix model (blue line) with our prediction up to second order (green line), the prediction of [152] to all orders (orange line) and up to second order (red line) for different values of $\beta \in [1, 4]$. We used matrices of size $N_m = 200$, averaging over $N_r = 8000$ realisations. Figures adapted from [5].

unorientable generalisation (conjecture 4.1). The only geometric input needed for this was the general structure of the (Airy) WP volumes. Notably, the statement providing this (theorem 3.4) is valid also for the setting of arbitrary

Dyson index and is the same for all contributions to the volumes' general structure in terms of β (eq. (3.3.66))[23]. Consequently, while we already know that all constraints we have discussed above are fulfilled by the Wigner-Dyson parts of the β WP volumes, given by the $\mathcal{V}_{g,2}^{0,\text{Airy}}$ and $\mathcal{V}_{g,2}^{g,\text{Airy}}$ for integer and $\mathcal{V}_{g,2}^{g+\frac{1}{2},\text{Airy}}$ for half-integer genus, we can employ the same reasoning as for the unorientable WP volumes in the orthogonal or symplectic symmetry classes for the non-Wigner-Dyson parts.

This implies that a necessary condition for the persistence of universal behaviour in β topological gravity is given by the fulfilment of all the constraints as determined by conjecture 4.1 and eq. (4.3.115). Here, there is a small caveat. In fact, the vanishing of the "$\mathcal{O}(\pi^0)$" contributions as arising from the potentially logarithmic contributions to the SFF as determined in eq. (4.3.114) was not determined by the inability of our procedure to deal with terms retaining such a t-dependence after τ-scaling, but rather by the absence of terms of a certain order in π in the universal results for the orthogonal and symplectic symmetry classes. For general β, we can not exclude the existence of such terms, due to our ignorance of an analytic result for the whole of b^β, and hence breaking the corresponding constraints would not violate the conjectured universal behaviour but indicate the existence of additional terms in the universal prediction for the τ-scaled SFF for arbitrary Dyson index with a dependence on π differing from that observed in the terms occurring for Wigner-Dyson values of β.

We study first the non-logarithmic constraints as determined by conjecture 4.1. Putting the explicit results for the non-Wigner-Dyson parts of the β Airy WP volumes, given in the supplementary material of this thesis and of [5], we can verify that indeed all of the constraints presented in section 4.3.2 and appendix E are fulfilled by each and every contribution to the β WP volumes from $g = 1$ to $g = \frac{7}{2}$. For the logarithmic constraints, as determined by eq. (4.3.115), we know from the discussion above that there is one such constraint for $g = 2$ and two such constraints for $g = 3$. From those, only $K_3^{\text{un}}(0)$ constrains a genuine logarithm. Consequently, we expect this constraint to be fulfilled, while the others could be violated. We observe precisely this and hence we expect the additional terms explained above. The remarkable fulfilment of all of these constraints gives a very strong indication of the topological gravity result matching the predictions as arising from a (to be determined) universal prediction for it.

Both our arguments hence give evidence to the fidelity of β topological gravity to a universal result in the sense of the matching of the τ-scaled canonical SFF to a universal result as determined by eq. (4.1.9). Here, our argumentation via the constraints, with the caveat of the $\mathcal{O}(\pi^0)$ terms, is

23 Of course, for the "orientable part" the additional simplification of the $\mathcal{V}_g^{0,>}$ being symmetric polynomials occurs.

conveniently independent of the actual analytic form of b^β and hence we can convince ourselves of the plausibility of matching to a universal result to much higher order than we could check when explicitly comparing with the numerical result for the microcanonical SFF as we did above. Consequently, by using the constraints we provide a quite non-trivial check for chaos, in the sense of the BGS conjecture, in β topological gravity.

Building on this, we will now proceed with investigating how we can turn around the usual reasoning and infer analytic information about the unfolded SFF from our gravitational computations of chapter 3.

4.4.2 *Towards a universal SFF beyond the Wigner-Dyson classes*

As we have demonstrated in sections 4.2.1 and 4.3.1, β topological gravity matches the universal RMT prediction for the τ-scaled SFF for Wigner-Dyson values of the Dyson index. Consequently, computing from this the microcanonical SFF for all the Wigner-Dyson values of β, we expect to find the universal prediction for this quantity (eq. (2.4.19)). Thus, its expansion in the unfolded variable x obtains an interpretation as being induced by the topological expansion of the gravitational theory/the dual matrix model. Notably, for the generalisations of matrix models to arbitrary values of the Dyson index we have proven the general structure of the β dependence of the topological expansion of observables in a one-cut matrix model, such as JT/topological gravity or the Gaussian model, to be determined by theorem 3.3. One of the implications of this, was the determination of the full "Wigner-Dyson part" of all observables in an arbitrary β matrix model by knowing the results for $\beta = 1$ and $\beta = 2$ (cf. eqs. (3.3.44) to (3.3.46)). Making this explicit for the example of the WP volumes, we find

$$V_{g,n}^{\beta,\mathrm{WD}}(\vec{b}) = \frac{2^{g+n-1}}{\beta^{g+n-1}} V_{g,n}^{2}(\vec{b}) + \frac{(2-\beta)^2}{\beta^{g+n}} \left(V_{g,n}^{1}(\vec{b}) - 2^{g+|I|-1} V_{g,n}^{2}(\vec{b}) \right), \tag{4.4.10}$$

for integer genus and

$$V_{g,n}^{\beta,\mathrm{WD}}(\vec{b}) = \frac{(2-\beta)}{\beta^{g+n-\frac{1}{2}}} V_{g,n}^{1}(\vec{b}), \tag{4.4.11}$$

for half-integer genus. We will now use this to infer a statement about the Wigner-Dyson part of b^β, which we shall denote as b^β_{WD}. For this, we "only" need to find the expansion of b^1 and b^2 in x and then sum the terms again, with the correct β dependence as induced by theorem 3.3. However, this is more involved than one would expect, due to b^1 and b^2, given in eq. (2.4.12) and eq. (2.4.13), actually not being analytic at $x = 1$. This can be overcome by writing the functions as hyperfunctions, possessing a series expansion in a useful sense. Due to time-constraints, we will not pursue this computation here but present it in [6]. Here, we will restrict to the domain of x, where the functions are analytic in x, i.e. $x < 1$. Using the definition of x, we can rewrite this restriction as (cf. eq. (4.1.10))

$$\tau < 2\pi\rho_0^{\text{Airy}} \Leftrightarrow t < \lim_{e^{S_0}\to\infty} T_{\text{H}}, \tag{4.4.12}$$

and hence we can say that we restrict to times smaller than the Heisenberg time, defined in eq. (2.4.15).

We now use eqs. (2.4.12) and (2.4.13) to find the needed expansion of the results as a power series in x. We find

$$\underset{\substack{x\in\mathbb{R}_+\\ x<1}}{\forall} b^1(x) = 1 - 2x + \sum_{g\in\mathbb{N}_+} b_g^1(x) + \sum_{g\in\frac{\mathbb{N}_+}{2}} b_g^1(x), \tag{4.4.13}$$

with

$$b_g^1(x) = \frac{(-1)^{2g+1}2^{2g}}{2g}x^{2g+1}, \tag{4.4.14}$$

and

$$\underset{\substack{x\in\mathbb{R}_+\\ x<1}}{\forall} b^2(x) = 1 - x. \tag{4.4.15}$$

It is important to note that the microcanonical SFF is given by $1 - b^\beta$ (cf. eq. (2.4.19)) and hence only the part of the result subtracted from 1 has an interpretation as a topological expansion. To find now the uplift to general β of these results, one can proceed for every value of g, having the interpretation of the genus of the surface associated with the respective contribution, analogous to the reasoning for the WP volumes (eqs. (4.4.10) and (4.4.11)) to insert the dependence on β. Doing this, one finds

$$b_\beta^{\text{WD}}(x) = 1 - \frac{2}{\beta}x + (2-\beta)^2 \sum_{g\in\mathbb{N}_+} \frac{1}{\beta^{g+2}} b_g^1(x) + (2-\beta) \sum_{g\in\frac{\mathbb{N}_+}{2}} \frac{1}{\beta^{g+\frac{3}{2}}} b_g^1(x). \tag{4.4.16}$$

Inserting the coefficients, one can work this out explicitly as

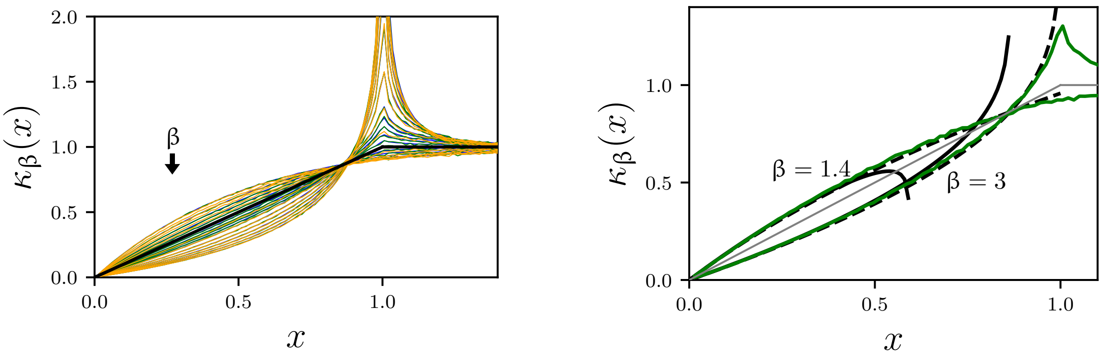
$$\begin{aligned} b_\beta^{\mathrm{WD}}(x) &= 1 - \frac{2}{\beta}x + \frac{(2-\beta)^2}{2\beta^2} x \sum_{g=1}^{\infty} \frac{(-1)^{g+1}}{g}\left(-\frac{4x^2}{\beta}\right)^g \\ &\quad + \frac{(2-\beta)}{\sqrt{\beta}^3} x \sum_{k=0}^{\infty} \frac{1}{2k+1}\left(\frac{2x}{\sqrt{\beta}}\right)^{2k+1} \\ &= 1 - \frac{2}{\beta}x + \frac{(2-\beta)^2}{2\beta^2} x \log\left(1 - \frac{4x^2}{\beta}\right) + \frac{(2-\beta)}{\sqrt{\beta}^3} x \operatorname{artanh}\left(\frac{2x}{\sqrt{\beta}}\right) \\ &= 1 - \frac{2}{\beta}x + \frac{(2-\beta)}{2\beta^2} x \left[\left(2-\beta+\sqrt{\beta}\right)\log\left(1+\frac{2x}{\sqrt{\beta}}\right)\right. \\ &\quad \left. + \left(2-\beta-\sqrt{\beta}\right)\log\left(\left|1-\frac{2x}{\sqrt{\beta}}\right|\right)\right], \end{aligned} \tag{4.4.17}$$

where, in principle, we have to restrict ourselves to the region of convergence of the two series, i.e. $|x| \leq \frac{1}{2}\sqrt{\beta}$. However, one can continue the result analytically beyond this region using the logarithm, as we have already rewritten the result in the last line.

To perform a quick sanity check of this result, we check that for the Wigner-Dyson values of β the expected results are reproduced. For $\beta = 2$ and $\beta = 1$ the found agreement is not surprising, as we started from theses results. For $\beta = 4$, we also find agreement with the result presented in eq. (2.4.14). This is of course expected from our reasoning in section 4.3.1. Nevertheless, one can note here that the combination of the GOE result with the dependence on β for the individual terms following from theorem 3.3 leads to this result directly with no additional considerations necessary. In contrast to this, the computation of b^4 by the same reasoning as applied for b^1, presented for example in [15], is quite tedious. Consequently, with the caveat of the necessity to treat also the post-Heisenberg time effects, our reasoning presents a shortcut to this result that has, to our knowledge, not been observed in the literature.

Having now found the Wigner-Dyson part of the unfolded microcanonical SFF in an analytic form, we proceed by comparing it to the numerical evaluation of this quantity and to the ansatz of [152].

As a first step towards this, we present the results for the unfolded microcanonical SFFs, computed in the way explained in appendix D, for various choices of the matrix size N_m and numbers of realisations averaged over (N_r) in fig. 4.4.2a. It is interesting to note here that we observe a divergence at $x = 1$ for all results with $\beta > 2$, while for all other values we observe a transition to the plateau without divergences.



a) **b)**

Figure 4.4.2: In **a)**: The connected part of unfolded microcanonical SFF of the Gaussian matrix model for values of the Dyson index between $\beta = 1$ and $\beta = 6$, computed from the tridiagonal matrix ensemble of [98]. The black line is the analytic result for $\beta = 2$, the colour of the other lines corresponds to the size of the matrices N_m, averaged over N_r realisations. Blue corresponds to $N_m = 200$, $N_r = 5000$, green to $N_m = 400$ $N_r = 1000$ and orange to $N_m = 1000$, $N_r = 200$. The plotted microcanonical SFFs are naturally ordered in the regime before intersecting the $\beta = 2$ curve, where a larger SFF corresponds to a smaller value of β. As one can see clearly for all $\beta > 2$, a divergence like that for the GSE case case appears at $x = 1$, while for all $\beta < 2$ the transition to the plateau is smooth.
In **b)**: Comparison of the connected part of the unfolded microcanonical SFF of the Gaussian matrix model for the values of $\beta = 1.4$ and $\beta = 3$ from the ensemble as in **a)** with $N_m = 1000$, $N_r = 200$ (green lines), with the various predictions, the black solid line ours (eq. (4.4.17)), the black broken one that from [152] (eqs. (4.4.6) and (4.4.7)). Figures adapted from [5], where they were prepared by J. Dieplinger.

This feature is not reproduced by our result (eq. (4.4.17)) which for all cases except $\beta \in \{1, 2\}$ diverges logarithmically at $x = \frac{1}{2}\sqrt{\beta}$, as it is immediately seen from eq. (4.4.17). This can also be seen directly in fig. 4.4.2b, where the numerical results for the values of $\beta = 1.4$ and $\beta = 3$ are depicted in green while our prediction is put as the black solid line. These lines diverge at the expected points, indicating that at this point latest, the effects from the non-Wigner-Dyson parts of the Airy WP volumes are crucial. Consequently, we can only expect good results from our prediction for all values of $\beta > 1$ if we restrict to a range of values of x smaller than $x = \frac{1}{2}$. Looking at the numerical curve we can, however, see very good agreement of our result with the numerical data before the onset of the divergence. In fact, the agreement in this range is slightly better than that with the predictions of [152], put as the black broken line, which however give a good approximation to the result over the whole considered range of x.

This warrants a closer comparison of our prediction of the SFF with that of [152]. Before doing this, we would like to comment briefly on the ansatz of [152] and show its small x expansion to be inconsistent with the structure of the perturbative expansion of correlation functions in one-cut β matrix models.

For this, we consider again only the region of $x < 1$ to avoid the complications of the discontinuity/divergence at the Heisenberg time. In this region, like the results for the Wigner-Dyson classes, we can expand the ansätze as a convergent power-series in x, the first terms of which we gave in eqs. (4.4.8) and (4.4.9). Notably, these results disagree with ours, starting with the second order, and have a "fixed" dependence on β, i.e. each term of higher order is multiplied with the same β dependent prefactor as that of the second order, as it is obvious from the definitions of the ansätze in eqs. (4.4.6) and (4.4.7). Forgetting for a second about universality, we note that these ansätze should be valid primarily for the β Gaussian matrix model. This model is a one-cut matrix model and hence its correlation functions need to transform under $\beta \to \frac{4}{\beta}$ as prescribed by theorem 3.2. To study whether the ansätze satisfy this condition, we choose $\beta \in [1, 2]$, the range of validity of $b_\beta^{A,1}$. Now, under the mapping this is sent to $\frac{4}{\beta} \in [2, 4]$, i.e. the range of validity of $b_\beta^{A,2}$. Consequently, using theorem 3.2, we infer that it has to hold that

$$b_\beta^{A,1,\frac{1}{2}} \overset{!}{=} (-1)\left(\frac{2}{\beta}\right)^3 b_{\frac{4}{\beta}}^{A,2,\frac{1}{2}}. \tag{4.4.18}$$

However, one finds

$$(-1)\left(\frac{2}{\beta}\right)^{2\left(\frac{1}{2}+1\right)} b_{\frac{4}{\beta}}^{A,2,\frac{1}{2}} = -\frac{8}{\beta^3}\frac{2-\frac{4}{\beta}}{2\frac{4}{\beta}} = -2\frac{\beta-2}{\beta^3} \neq b_\beta^{A,1,\frac{1}{2}}, \tag{4.4.19}$$

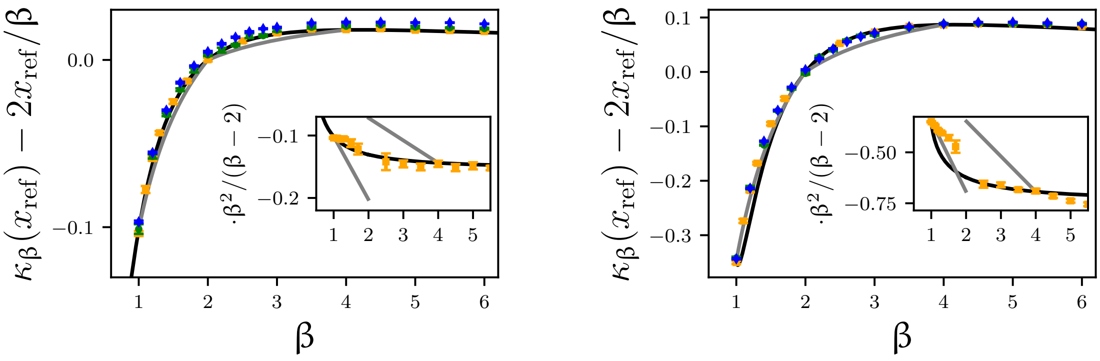
which shows that the $b_\beta^{A,i}$ are not compatible with each other (and also not with itself, but that was expected) under the reasonable assumption that each order in x can be associated to a specific contribution to the topological expansion of an observable in the Gaussian matrix model.

Consequently, it appears likely that the analytic expressions proposed in [152] require additional contributions of potentially non-Wigner-Dyson nature to achieve full agreement with the numerics and to be consistent with theorem 3.2 and consequently we should find a certain degree of disagreement of this proposal with the numerical evaluation.

Our result, meanwhile, is perfectly consistent with the requirements of theorem 3.2, though we acknowledge the fact that the ansatz of [152] is a much better approximation to the unfolded microcanonical SFF than ours for $x \geq \frac{1}{2}\sqrt{\beta}$, where our result diverges. To better compare specifically the dependence of the two predictions on β with that of the numerical results, we do so in fig. 4.4.3 for fixed values of x while varying β from $\beta = 1$ to $\beta = 6$. Here, we chose two examples for the reference values x_{ref} which exemplify the behaviour we observe generically and decided to present not the full microcanonical SFF but to subtract the arbitrary β generalisation of the "ramp" which is identical in our prediction and that of [152], as discussed above. We plot the numerical results for the respective x_{ref} as dots with error bars, where the choices of the matrix and ensemble sizes N_m and N_r are represented by the same choice of colours as in fig. 4.4.2a. Furthermore, we put our analytic prediction as the black and that of [152] as the grey solid line. In the inset, we plot the numerical result with $N_m = 1000$ and the analytic predictions, all scaled by $\frac{\beta^2}{\beta-2}$, which is a scaling suggested by our analytic result. Considering first fig. 4.4.3a, where we set $x_{\text{ref}} = 0.25$, we note that over the whole considered range of β our result agrees very well, nearly perfectly within the error bars, with the numerical result, while the prediction of [152] shows clear deviations. This becomes even more apparent when looking at the inset. In fig. 4.4.3b, where we put $x_{\text{ref}} = 0.5$, we can make the same observation, i.e. our predictions fitting nearly perfectly to the numerical results, for $\beta \geq 2$. Going to smaller values of β, we see that the numerical curve is increasingly well described by the prediction of [152]. This, we attribute to the increasing impact of the divergence at $x = \frac{1}{2}\sqrt{\beta}$ which for this range of β is progressively near to x_{ref}. From the inset, we can observe the described behaviour even better.

Consequently, in the range where it is applicable, i.e. for values of x not influenced by the divergence, we find better agreement of our analytic prediction (eq. (4.4.17)) with the numerical results as compared to that of the predictions of [152] (cf. eqs. (4.4.6) and (4.4.7)). This gives rise to the reasonable expectation that the extension of our prediction for $b_\beta(x)$ by the non-Wigner-Dyson contributions will lead to an even better approximation of the numerical result and possibly a full analytic result for the microcanonical SFF. What one can already say for certain about this extension, is that it has to encompass the cancellation of the divergence present in eq. (4.4.17) and replacing it by one at $x = 1$. This could potentially result in the final result of computing the universal microcanonical SFF from β topological gravity being actually of close similarity to that of [152], though its expansion in x has to fulfil the requirements of theorem 3.2.

This leads to our conclusion that we can already make a step forward in the determination of the unfolded SFF for the β Gaussian model by using "only" the information about the unfolded SFFs for the orthogonal and unitary symmetry classes. As we already commented, this is not the complete story, it has to be completed by first extending our reasoning beyond the Heisenberg



a) $x_{\text{ref}} = 0.25$ **b)** $x_{\text{ref}} = 0.5$

Figure 4.4.3: Comparison of the numerical results at a specific point x_{ref} for the unfolded microcanonical SFF, subtracting the "ramp", with our analytic result eq. (4.4.17) and the ansatz of [152]. In the main plot, the numerical results are put as the coloured dots with error bars, where the colours correspond to the same choices of N_m and N_r as in fig. 4.4.2a. Furthermore, our prediction is put as the solid black line, while that of [152] is put as the solid grey line. In the inset, all three results are scaled by $\frac{\beta^2}{(\beta-2)}$, where we only put the best converged numerical results, i.e. that for $N_m = 1000$, and retain their and the predictions' depiction from the main plot. Figures adapted from [5], where they were prepared by J. Dieplinger.

time and then also by including the non-Wigner-Dyson contributions to the SFF. This is left to be investigated in future work.

Another approach to computing the microcanonical SFF for the Gaussian matrix model at arbitrary Dyson index was proposed and evaluated in [153]. This resulted in the evaluation of $b^\beta(x)$ up to $\mathcal{O}(x^{10})$, which was extended in [154] to $\mathcal{O}(x^{11})$. To our knowledge, this is the highest order in a series expansion in x for which analytic results are known. Rewriting these results, for whose derivation we refer to to the original papers, into our notation, they can be presented as

$$1 - b^\beta(x) = \frac{2}{\beta}x - 2\frac{(2-\beta)}{\beta^2}x^2 + 2\frac{(2-\beta)^2}{\beta^3}x^3 + \frac{2(\beta-2)\left(\beta^2 - \frac{11\beta}{3} + 4\right)}{\beta^4}x^4 + \tag{4.4.20}$$

$$+ \frac{2(\beta-2)^2\left(\beta^2 - 3\beta + 4\right)}{\beta^5}x^5 + \frac{2(\beta-2)\left(\beta^4 - \frac{91\beta^3}{15} + \frac{248\beta^2}{15} - \frac{364\beta}{15} + 16\right)}{\beta^6}x^6 +$$

$$+ \frac{2(\beta-2)^2\left(\beta^4 - \frac{74\beta^3}{15} + 13\beta^2 - \frac{296\beta}{15} + 16\right)}{\beta^7}x^7 +$$

$$+ \frac{2(\beta-2)\left(\beta^6 - \frac{1607\beta^5}{210} + \frac{2011\beta^4}{70} - \frac{7288\beta^3}{105} + \frac{4022\beta^2}{35} - \frac{12856\beta}{105} + 64\right)}{\beta^8}x^8 +$$

$$
\begin{aligned}
&+\frac{2(\beta-2)^2\left(\beta^6-\frac{263\beta^5}{42}+\frac{6788\beta^4}{315}-\frac{6337\beta^3}{126}+\frac{27152\beta^2}{315}-\frac{2104\beta}{21}+64\right)}{\beta^9}x^9+\\
&+\frac{2(\beta-2)}{\beta^{10}}x^{10}\left(\beta^8-\frac{791\beta^7}{90}+\frac{73603\beta^6}{1890}-\frac{7355\beta^5}{63}+\frac{35696\beta^4}{135}\right.\\
&\left.-\frac{29420\beta^3}{63}+\frac{588824\beta^2}{945}-\frac{25312\beta}{45}+256\right)+\\
&+\frac{2(\beta-2)^2}{\beta^{11}}x^{11}\left(\beta^8-\frac{1523\beta^7}{210}+\frac{5058\beta^6}{175}-\frac{256189\beta^5}{3150}+\frac{284926\beta^4}{1575}\right.\\
&\left.-\frac{512378\beta^3}{1575}+\frac{80928\beta^2}{175}-\frac{48736\beta}{105}+256\right)\\
&+\mathcal{O}\left(x^{12}\right).
\end{aligned}
\tag{4.4.21}
$$

First we note that, up to $\mathcal{O}(x^2)$, this agrees with our result eq. (4.4.2), giving another strong sign for topological gravity agreeing with the Gaussian matrix model up to this order. To compare to our results also up to higher orders in x, it is useful to decompose the individual coefficient functions guided by their general structure, induced by theorem 3.3. Doing this, we find

$$
\begin{aligned}
1-b^\beta(x)=&\,x\frac{2}{\beta}-2x^2\frac{(2-\beta)}{\beta^2}+2x^3\frac{(2-\beta)^2}{\beta^3}+\\
&-x^4\left[\frac{2(1-\beta)(4-\beta)(2-\beta)}{\beta^4}+\frac{8(2-\beta)}{3\beta^3}\right]+\\
&+x^5\left[\frac{2(1-\beta)(4-\beta)(2-\beta)^2}{\beta^5}+\frac{4(2-\beta)^2}{\beta^4}\right]+\\
&-x^6\left[\frac{2(1-\beta)^2(2-\beta)(4-\beta)^2}{\beta^6}+\frac{118(1-\beta)(2-\beta)(4-\beta)}{15\beta^5}+\frac{32(2-\beta)}{5\beta^4}\right]\\
&+x^7\left[\frac{2(1-\beta)^2(4-\beta)^2(2-\beta)^2}{\beta^7}+\frac{152(1-\beta)(4-\beta)(2-\beta)^2}{15\beta^6}+\frac{32(2-\beta)^2}{3\beta^5}\right]+\\
&-x^8\left[\frac{2(1-\beta)^3(2-\beta)(4-\beta)^3}{\beta^8}+\frac{1543(1-\beta)^2(2-\beta)(4-\beta)^2}{105\beta^7}\right.\\
&\left.+\frac{3193(1-\beta)(2-\beta)(4-\beta)}{105\beta^6}+\frac{128(2-\beta)}{7\beta^5}\right]+\\
&+x^9\left[\frac{2(1-\beta)^3(2-\beta)^2(4-\beta)^3}{\beta^9}+\frac{367(1-\beta)^2(2-\beta)^2(4-\beta)^2}{21\beta^8}\right.\\
&\left.+\frac{13816(1-\beta)(2-\beta)^2(4-\beta)}{315\beta^7}+\frac{32(2-\beta)^2}{\beta^6}\right]+\\
&-x^{10}\left[\frac{2(1-\beta)^4(2-\beta)(4-\beta)^4}{\beta^{10}}+\frac{1009(1-\beta)^3(2-\beta)(4-\beta)^3}{45\beta^9}\right.\\
&\left.+\frac{77698(1-\beta)^2(2-\beta)(4-\beta)^2}{945\beta^8}+\frac{111487(1-\beta)(2-\beta)(4-\beta)}{945\beta^7}+\frac{512(2-\beta)}{9\beta^6}\right]+
\end{aligned}
$$

$$+ x^{11} \left[\frac{2(1-\beta)^4(2-\beta)^2(4-\beta)^4}{\beta^{11}} + \frac{2677(1-\beta)^3(2-\beta)^2(4-\beta)^3}{105\beta^{10}} \right.$$
$$\left. + \frac{18941(1-\beta)^2(2-\beta)^2(4-\beta)^2}{175\beta^9} + \frac{286016(1-\beta)(2-\beta)^2(4-\beta)}{1575\beta^8} + \frac{512(2-\beta)^2}{5\beta^7} \right]$$
$$+ \mathcal{O}\left(x^{12}\right). \tag{4.4.22}$$

By inspection of this result it is apparent that the Wigner-Dyson part of these contributions is in perfect agreement with our prediction, whose coefficient for the genus g contribution, i.e. that at $\mathcal{O}(x^{2g+1})$, is given by

$$-b_g^1(x) = \frac{(-1)^{2g} 2^{2g}}{2g}, \tag{4.4.23}$$

while the agreement of the β dependence can be seen from eq. (4.4.16). In addition to this, we can compare also the non-Wigner-Dyson parts of contributions to the microcanonical SFF of topological gravity as computed from those to the canonical SFF in eq. (4.3.62) to eq. (4.3.69). In our comparison with the numerical results above we stopped this comparison at $\mathcal{O}(x^2)$ due to the contributions at higher order requiring the introduction of cancellation functions. We will not give a full discussion of these for the whole of the arbitrary β results in this thesis, leaving this discussion for [6], but rather apply the knowledge we have about this mechanism to extract the terms from the canonical result that are relevant for the computation of the expansion of the microcanonical SFF that can then be compared to eq. (4.4.22). This essentially amounts to considering only those terms in eq. (4.3.62) to eq. (4.3.69) that retain no additional dependence on t after τ-scaling (i.e. those of $\mathcal{O}(t^{2g+1})$ for a given genus g) that is not logarithmic. For the contributions at $g \in \frac{\mathbb{N}}{2}$ this already suffices and we can write the generic form of the relevant contribution to $\kappa_\beta^{s,\mathrm{WP}}(\tau, \beta)$ for such a genus as

$$C_g(\beta)\tau^{2g+1}\beta^{g-1}, \tag{4.4.24}$$

with a coefficient function $C_g(\beta)$ that can be read off from the explicit results. For $g \in \mathbb{N}$, it is necessary to replace, anticipating the modifications due to the cancellation functions mechanism, $\log\left(\frac{\beta}{2t}\right) \to \log\left(2\beta\tau^2\right)$. Doing this, the generic relevant contribution is given by

$$C_g(\beta)\tau^{2g+1}\beta^{g-1}\log\left(2\beta\tau^2\right), \tag{4.4.25}$$

where the coefficient function $C_g(\beta)$ can again be read off from eq. (4.3.62) to eq. (4.3.69). To find the corresponding contributions to the microcanonical SFF for topological gravity we use eq. (4.1.11), i.e. need to perform inverse

Laplace transforms with respect to 2β and divide by ρ_0^{Airy}. The relevant transformations are given in [99] as

$$\mathcal{L}^{-1}\left[\frac{1}{x^{\alpha+1}}, x, t\right] = \theta(t)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{4.4.26}$$

and

$$\mathcal{L}^{-1}[x^m \log(x), x, t] = (-1)^{m+1} m! \frac{\theta(t)}{t^{m+1}} + \Psi(m+1)\delta^{(m)}(t), \tag{4.4.27}$$

where $\Psi(x)$ denotes the Euler Ψ function. For the comparison with eq. (4.4.22) we are interested only in the non-distributional terms as the distributional ones, as we argued above, contribute to the emergence of the post-Heisenberg time behaviour we are not considering here. Consequently, the contributions to the microcanonical SFF of topological gravity, expanded around $x = 0$, can be derived using the two formulae as

$$\sqrt{\pi} C_g(\beta) \frac{(2g-1)!(-1)^{g-\frac{1}{2}}}{\left(g-\frac{1}{2}\right)! 2^{3(g-1)}} x^{2g+1}, \tag{4.4.28}$$

for $g \in \frac{\mathbb{N}}{2}$ and

$$\pi C_g(\beta) \frac{(g-1)!(-1)^g}{2^{g-2}} x^{2g+1}, \tag{4.4.29}$$

for $g \in \mathbb{N}$. Note that we did not include the Heaviside functions as we can restrict to the regime of $x > 0$.

Now, reading off the coefficient functions from our explicit results, we can compare them to eq. (4.4.22). For $g = 0$ and $g = \frac{1}{2}$ we already saw agreement. Likewise for $g = 1$, as there is only a Wigner-Dyson contribution[24]. For the contributions of higher genus, we can read off the coefficient functions as

$$C_{\frac{3}{2}}(\beta) = \left[\frac{2(1-\beta)(4-\beta)(2-\beta)}{\beta^4} + \frac{8(2-\beta)}{3\beta^3}\right]\frac{\sqrt{2}}{\sqrt{\pi}}, \tag{4.4.30}$$

$$C_2(\beta) = \left[\frac{2(1-\beta)(4-\beta)(2-\beta)^2}{\beta^5} + \frac{4(2-\beta)^2}{\beta^4}\right]\frac{1}{\pi}, \tag{4.4.31}$$

$$C_{\frac{5}{2}}(\beta) = \left[\frac{2(1-\beta)^2(2-\beta)(4-\beta)^2}{\beta^6} + \frac{118(1-\beta)(2-\beta)(4-\beta)}{15\beta^5} + \frac{32(2-\beta)}{5\beta^4}\right]\frac{-8}{3\sqrt{2\pi}}, \tag{4.4.32}$$

$$C_3(\beta) = \left[\frac{2(1-\beta)^2(4-\beta)^2(2-\beta)^2}{\beta^7} + \frac{152(1-\beta)(4-\beta)(2-\beta)^2}{15\beta^6}\right.$$

24 Of course, one could find this also by using the explicit form of $C_1(\beta)$.

$$+\frac{32(2-\beta)^2}{3\beta^5}\Bigg]\frac{-1}{\pi}, \tag{4.4.33}$$

$$C_{\frac{7}{2}}(\beta)=\Bigg[\frac{2(1-\beta)^3(2-\beta)(4-\beta)^3}{\beta^8}+\frac{1543(1-\beta)^2(2-\beta)(4-\beta)^2}{105\beta^7} \\ +\frac{3193(1-\beta)(2-\beta)(4-\beta)}{105\beta^6}+\frac{128(2-\beta)}{7\beta^5}\Bigg]\frac{32}{15\sqrt{2\pi}}. \tag{4.4.34}$$

Using eq. (4.4.28) or eq. (4.4.29), we find agreement with eq. (4.4.22) for all orders for which we have results. This agreement with the results of [153, 154], whose derivation is completely independent from our reasoning here, is another very non-trivial sign for the persistence of universal behaviour in β topological gravity.

By this we conclude our discussion of the interplay of quantum chaos, as quantified via a generalisation of the BGS conjecture, with the gravitational results for arbitrary values of the Dyson index. Also, this finalises the whole of our discussion of this interplay.

4.5 CONCLUSION OF THE CHAOS RELATED PART

In this chapter we have studied how we can, indeed, use "random matrix universality as a tool in two-dimensional quantum gravity". Speaking first only about the Wigner-Dyson classes, we applied this tool in two ways:

First, by applying universality, the full results for the τ-scaled SFFs of topological/JT gravity in the three symmetry classes are given by the expressions evaluated from combining the respective universal unfolded microcanonical SFF with the respective leading order density of states. For topological gravity this expression is given in eq. (4.2.3) (for $\beta=2$), eqs. (4.3.1) and (4.3.15) (for $\beta=1$), and eqs. (4.3.11) and (4.3.15) (for $\beta=4$). For JT gravity we only evaluated the full result in a closed form for the case of $\beta=2$, as eq. (4.2.6), while for $\beta=1$ we restricted to the computation up to $g=1$, which is determined by eqs. (4.3.1) and (4.3.10). We established the explicit agreement of these results with the gravitational computation of the τ-scaled SFF as evaluated from the (Airy) WP volumes up to $g=5/\mathcal{O}(\tau^{11})$ (for $\beta=2$) and up to $\mathcal{O}(\tau^4)$ (for $\beta=1$) and to the same order for $\beta=4$, though we pursued this computation only for topological gravity in the latter case.

Second, this agreement implied several constraints on the geometric objects of the theories, the WP volumes. These constraints take the form of requiring certain linear combinations of specific subsets of the volumes' coefficients to vanish. For the orientable/$\beta=2$ setting, we give an analytic result for the form of all constraints on the WP volumes for $n=2$ in theorem 4.1 and prove them to be implied by universality. For the unorientable setting/$\beta\in\{1,4\}$, where we restrict our discussion mostly to topological gravity, we discover two kinds of constraints. First, a class of constraints relating to potential logarithmic contributions arising in the gravitational SFF that would contradict the universal behaviour. For these constraints we work out an explicit analytic form, given in eqs. (4.3.120) and (4.3.121). Second, the class of all other constraints, relating to potential terms in the gravitational SFF that are of too large order in time to be consistent with the universal result. Here, we did not work out an analytic result but give an explicit conjecture for the rules determining which potential terms should cancel in conjecture 4.1 and work out the constraints explicitly up to $g = \frac{7}{2}$, the value of genus for which moduli space volumes have been computed so far. We find all of the constraints to be fulfilled by the results for the WP volumes, whose computation we discussed in section 2.3 for the orientable and chapter 3 for the unorientable setting.

For non-Wigner-Dyson values of the Dyson index we could not apply our "tool" in the same way, since RMT universality is not an established statement beyond the Wigner-Dyson classes and not even a full analytic result for the microcanonical SFF for general β that could potentially be universal is known. Hence, the first way we applied universality in the Wigner-Dyson classes is (so far) not possible. The second way, studying constraints imposed as a necessary condition for universality on the β WP volumes, we can, however, follow even without an actual full analytic result for the microcanonical SFF. We could show all constraints imposed by this reasoning in the strict sense to be fulfilled. However, the constraints that required additional information about the analytic form of the SFF (cf. section 4.3.2) are not fulfilled, which does not contradict the presence of universal behaviour but rather gives additional information about the analytic form of the general β universal microcanonical SFF. This is a strong sign for universality in topological gravity. Another such sign we found by comparing our geometric results with a numerical evaluation of the microcanonical SFF for the β Gaussian model, which offers the easiest access to the putative universal microcanonical SFF. Additionally, the comparison with a certain well-established analytic result for the series expansion of a specific part of the microcanonical SFF reveals agreement with the corresponding contributions arising from our geometric results for all orders we considered. Together these results form a compelling body of evidence for the persistence of universal behaviour, and with this quantum chaos in the sense of the BGS conjecture, for β topological gravity. To present an idea for a reasoning arising from our considerations towards also being able to follow the first way of applying universality, as for the Wigner-Dyson classes, we showed how the Wigner-Dyson part of the (potentially) universal microcanonical SFF for arbitrary Dyson index arises from the universal results for the Wigner-Dyson classes. The result of this is given in eq. (4.4.17).

There are several important further directions of research arising from the results we have presented here. Most directly, those are given by all the further routes of investigation we pointed out in the text but did not pursue here, like the evaluation of all the constraints in unorientable JT gravity, the explicit evaluation of the non-logarithmic constraints in unorientable topological gravity etc. . We will present the results of many of these directions in [6].

In extension of this, it would be very useful to have all the routes of access to the universal part of observables in the unorientable setting as they are available in the orientable setting. By this, we specifically mean the computation of the two-point function of densities of states in orientable JT gravity via correlation functions of branes [68, 148] or the universe field theory [149]. The extension of this, first to the setting of unorientable topological gravity and then potentially even to to full JT gravity, would be very useful to establish universality in another, actually non-perturbative, way and could help to better understand the mechanism of cancellation functions we employed in this work. Furthermore, it is feasible that this extension would also extend directly to the setting of arbitrary Dyson index and hence enable another analytic approach to the universal SFF in this setting. Another way to approach this, would be to use the nonlinear σ-model approach (cf. [18]) which we already mentioned in section 2.4 as a way to study universal properties of matrix models in the Wigner-Dyson classes. This approach may also offer an analytic handle to approach the microcanonical SFF for general β in the Gaussian model and hence provide a completion of our result for this to a full expression for this quantity for arbitrary values of the Dyson index. Furthermore, it might be illustrating to revisit the "traditional" computation of the spectral two-point function in RMT using certain sets of orthogonal polynomials to express it in a way where one can apply the Christoffel-Darboux formula, as followed e.g. in [15]. For this approach we have, however, so far been unable to determine the correct set of orthogonal polynomials generalising the discussion of the Wigner-Dyson classes to arbitrary values of the Dyson index.

Also of special interest is the extension of the interpretation of the constraints on moduli space volumes as such on intersection numbers, mentioned for the orientable setting at the end of section 4.2.2, to the unorientable setting. Of course, this only makes sense provided an intersection theory interpretation of the unorientable (Airy) WP volumes in the first place and hence is a rather far fetched goal. However, due to the infancy stage of intersection theory for unorientable manifolds, the statements for them corresponding to the constraints is almost certainly new and unproven and hence the tool of RMT universality would, in the author's opinion, be the most convenient approach to prove this.

With these remarks for extensions of our studies in the realm of two-dimensional gravity and related subjects we close our discussions of the main part of this thesis and proceed to the conclusion, commenting especially on the possibility of applying the ideas pursued in our work presented up to now to a more broadly scoped setting.

5

CONCLUSION: WHAT CAN CHAOS TELL US ABOUT GEOMETRY?

The main aim of this thesis was to study the implications of one of the hallmarks of quantum chaos that is the universal RMT form of certain spectral correlation functions in a theory that is as chaotic as it can get (cf. [155]): quantum gravity. Specifically, we were concerned with applying this idea for the theory of JT gravity, a theory describing gravity in two euclidean dimensions. This theory is intimately connected with hyperbolic geometry in two dimensions as the topological expansion of the correlation functions in this theory, which we are interested in, are determined by moduli space volumes of hyperbolic surfaces (WP volumes). Also, it enjoys a direct connection with matrix models by the correlation functions, in the sense of a perturbative duality, also being computable from the topological expansion of a certain matrix model. Notably, as there are three classes of random matrices in the classical Wigner-Dyson classification of matrix models, characterised by the Dyson index $\beta \in \{1, 2, 4\}$, there are also three variants of JT gravity, each dual to a matrix model in one of the Wigner-Dyson classes. Geometrically, this choice of symmetry class is performed by selecting what types of surfaces to include in the gravitational path-integral defining correlation functions in JT gravity. Here, the choice of including only orientable surfaces corresponds to the unitary symmetry class ($\beta = 2$), while including also unorientable surfaces corresponds to the orthogonal ($\beta = 1$) or symplectic ($\beta = 4$) symmetry class, depending on the details of the inclusion.

The presence of the universal spectral correlation functions in matrix models, guaranteed by random matrix universality, enables the direct investigation of their implications in JT gravity by the intermediate step of the matrix model dual to this theory. While the orientable theory was studied to such a degree that we could directly apply this idea, the unorientable variants were not and hence a considerable part of this thesis dealt with extending our knowledge about these unorientable theories by investigating the moduli space volumes of unorientable hyperbolic surfaces (chapter 3).

GEOMETRIC RESULTS. Here, by using the established duality for matrix models in the three Wigner-Dyson classes with JT gravity, it was possible to study moduli space volumes by investigating the dual matrix model, mainly by the recursive approach via the so-called loop equations. Using this approach, we were able to fully classify the dependence of the contributions to the topological expansion of matrix model correlation functions on the Dyson index β in theorem 3.3. The realisation that only a small part of this full structure relates to contributions arising for Wigner-Dyson values of the Dyson index, led us to consider general positive real values of the Dyson index, possible without any problems in our approach. We then used this as a definition of generalised WP volumes depending on the Dyson index and allowing for an interpolation "between" orientable and unorientable moduli space volumes. We also provided a geometric interpretation for these intermediate moduli space volumes in theorem 3.5, clarifying that the surfaces themselves are only orientable or unorientable, as they have to, but the generalised WP volumes sum over the different ways one has to build a surface of genus g with a weight, in terms of β. This weight only depends on the number of occurrences of a certain, genuinely unorientable, building block of the surface in the respective way of building it. Our second group of results, now that we fully understood the dependence on the Dyson index, was concerned with the functional dependence of matrix model correlation functions/WP volumes for a fixed value of β. For this we did not study JT gravity structurally, but rather topological gravity, a theory emerging naturally from JT gravity by taking the limit of large boundary lengths of the WP volumes. While we had several other reasons to consider this theory, which are given in the main text, maybe the most important one is the existence of a different approach to topological gravity than the loop equations, given by Kontsevich diagrammatics. Using this, we could prove as theorem 3.4 the functional form of the arbitrary β WP volumes for $n = 2$ boundaries to be given by a direct generalisation of the structure for the orientable ($\beta = 2$) case. In fact, this is the most important statement for the investigation of the implications of quantum chaos in unorientable topological gravity. Furthermore, we give a well motivated conjecture for the form of generic contributions to matrix model correlation functions, i.e. arbitrary genus g and number of arguments n. Beyond the structural investigations, we also derived explicit results, well beyond the orders that had previously been known in the literature, for topological as well as JT gravity.

IMPLICATIONS OF QUANTUM CHAOS. Having found these results in unorientable JT/topological gravity, we could pursue in chapter 4 the investigation of implications of the universal form of spectral correlations, specifically for the two-point function, in JT gravity. We gave a detailed summary of this investigation and its results in section 4.5, which we will not repeat here. Let us only note that, for all values of the Dyson index we investigated, in topological and JT gravity the implications were always two-fold:

First, for the cases where an explicit universal result for the spectral two-point function is known (the Wigner-Dyson classes) universality implies an explicit result for the late-time behaviour of the so-called canonical spectral form factor (SFF) of the gravitation theory. We checked this to be matched by the perturbative results as arising from the explicit results for the WP volumes we had computed in our geometric discussion, though the establishment of this matching is in general non-trivial. For the non-Wigner-Dyson values of β we expect the same behaviour, though we cannot investigate it to the same extent, since there is no full analytic result for the universal spectral two-point function in this setting[1]. Evidence for this expectation could be drawn from the non-trivial agreement of our computations with certain analytic results for the series expansion of the microcanonical version of the "universal" SFF for arbitrary β for all orders we considered.

Second, for all values of β alike, universality implies certain constraints on the WP volumes for $n = 2$. This we found by observing certain, potentially occurring, contributions to the canonical SFF in the gravitation theories to be inconsistent with the universal prediction for this quantity. Consequently, these contributions have to vanish, which we could work out as explicit constraints on the WP volumes.

FURTHER DEVELOPMENTS AND CONCLUDING REMARKS. This concludes the recollection of the main results of this thesis. There are several interesting avenues for future research in the geometric part of this work, as well as in the investigation of the implications of universality. We collected a selection of these in the conclusions of the respective chapters, section 3.4 and section 4.5. Apart from these extensions, some of which we are actively pursuing at the moment, there have already been several applications/extensions of our reasoning by other groups, for which we give some important examples:

In this thesis, we have restricted our discussion to universal correlations in the spectral two-point function, having consequently implications for WP volumes with $n = 2$ boundaries. In fact, the spectral n-point function has a universal form. Consequently, one can perform an extension of our study also for a generalisation of the spectral form factor to an n-point quantity[2]. This was done for the orientable setting in [156], deriving an exact result for the topological gravity n-point SFF in the large time limit.

1 We do provide part of this universal result in section 4.4.2 and show it to be matched by the corresponding part of the WP volumes for arbitrary β. However, we essentially derive this result from our geometric computation and this is hence not surprising. Additionally it is noteworthy that, as we remark in the main text, RMT universality it unproven beyond the Wigner-Dyson classes and we thus need to treat universal behaviour in this setting as a well-motivated conjecture.

2 This arises from the n-point correlation function of partition functions with putting the arguments as $\beta + it_n$ with the constraint $\sum_{i=1}^{n} t_n = 0$.

We further restricted our discussion to the Wigner-Dyson classes or interpolations between them, corresponding to various choices of orientable or unorientable surfaces on the gravity side of the duality. However, as we pointed out in the main text, the full ten-fold Altland-Zirnbauer classification [135] of random matrices can be realised by variants of JT/topological gravity that introduce supersurfaces into the gravitational path-integral, i.e. by considering super-JT gravity [69]. Notably, also the non-Wigner-Dyson classes (in the sense of being one of the other seven Altland-Zirnbauer classes) obey random matrix universality and hence the precise analogue of our study in this setting is possible. For the supersymmetric analogue of the discussion for orientable "bosonic" JT gravity, i.e. including only orientable supersurfaces, which is dual to the so-called chiral unitary ensemble, this discussion was given in [157]. Notably, this discussion is more involved than for the bosonic case and requires the application of resurgence methods to the topological expansion of the canonical SFF from the gravitational side to find a match between it and the universal RMT prediction. In contrast to this, for the bosonic case this match was directly observable without any additional procedure necessary.

Furthermore, in a recent series of works [158–162], building on [163], it was proposed that one can "uplift" the matrix model dual of JT gravity to treat geometries of the topology $\Sigma_{0,n} \times S_1$, where $\Sigma_{0,n}$ denotes a hyperbolic surface of genus 0 with n geodesic boundaries and S_1 a circle, in pure AdS_3 gravity. This uplift, colloquially speaking, embeds a matrix model into a 2d CFT, i.e. in such a way that it respects modular and conformal symmetry. So far, this work has restricted itself to treating orientable surfaces, with the most worked-out example being the uplift of topological gravity for large times (which we treated here in section 4.3.1). However, in [162] the authors claim to include also unorientable manifolds into their upcoming work. Consequently, we expect our results for unorientable JT/topological gravity to be "uplifted" to 3d gravity soon.

This is quite exciting, as it nicely closes the loop to one of the motivations we gave in the introduction, why it is useful to study gravity in two dimensions: to then use this knowledge to subsequently increase the dimension and potentially be able to treat four-dimensional gravity in the future.

Finally, we close the discussion of this thesis by going back to the origin of quantum chaos and hence all the concepts we have used here to study hyperbolic geometry: classically chaotic systems. This is the idea of the other approach we[3] pursued to study JT gravity from a quantum chaos perspective, published in [4]. Specifically, we ask the question, whether one can find a classically chaotic system which upon quantisation is "dual" to JT gravity in the same sense as the matrix model, i.e. having the same correlation functions. Indeed, we propose such a system: the free motion of a particle

3 Here, the main part of the investigation was performed by F. Haneder and J.D. Urbina.

on a high-dimensional manifold of constant negative curvature, being one of the paradigmatic examples for classically chaotic systems (traditionally for two dimensions, cf. e.g. [164]). For the corresponding quantum theory we identify telltale signs for the existence of a duality in the explained sense by establishing full agreement of the leading order contributions to the one and two-point functions of partition functions with JT gravity for both the orientable and unorientable case. Notably, for the two-point function this necessitated the extension of the well-known periodic orbit theory approach to computing correlation functions in quantised classically chaotic systems semiclassically [111, 115] by introducing a coarse graining on phase space, i.e. essentially on the set of lengths of periodic orbits. The success of this for the leading order contribution to the two-point function warrants further study into this, extending the approach to treat higher-order contributions. This would be a substantial step, as it would imply that, beyond all the implications of quantum chaos on hyperbolic geometry we studied in this thesis, in fact the whole of the WP volumes could be determined by the semiclassical study of a classically chaotic system.

A

AN EXAMPLE FOR MIRZAKHANI'S RECURSION

To illustrate the recursion for the $V_{g,n}$ given in theorem 2.1 we give as an example the computation for $V_{1,2}$.

For the practical computation it is useful to define

$$F_{2k+1}(t) := \int_0^\infty x^{2k+1} H(x,t) \mathrm{d}x, \tag{A.1}$$

for which it holds that [70]

$$\int_0^\infty \int_0^\infty x^{2i+1} y^{2j+1} H(x+y,t) \mathrm{d}x \mathrm{d}y = \frac{(2i+1)!(2j+1)!}{(2i+2j+3)!} F_{2i+2j+3}(t), \tag{A.2}$$

and for which one can proof that

$$F_{2k+1}(t) = (2k+1)! \sum_{i=0}^{k+1} \zeta(2i) \left(2^{2i+1} - 4\right) \frac{t^{2k+2-2i}}{(2k+2-2i)!}, \tag{A.3}$$

i.e. the computation of the integrals occurring in the recursion can all be reduced to specific sums of values of the Riemann ζ-function. In essence, this is the origin of the factors of π^2 in the coefficients of the WP volumes due to the property of the ζ-function (e.g. [87][9.542.1]):

$$\underset{m \in \mathbb{N}}{\forall} \zeta(2m) = \frac{2^{2m-1}|B_{2m}|}{(2m)!} \pi^{2m}, \tag{A.4}$$

where B_m denotes the m-th Bernoulli number.

Using this result, we can now perform the computation for our case of interest, $(g,n) = (1,2)$. We start by giving the possible decompositions in fig. A.1 (one could alternatively just follow the word of the definition in theorem 2.1) Thus, one has for the connected part

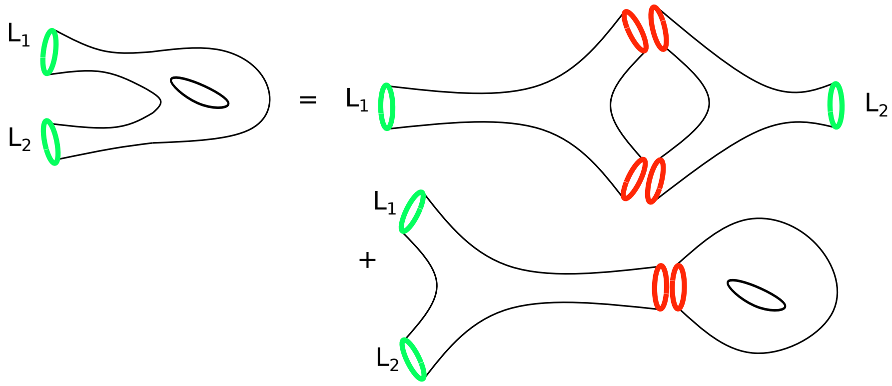


Figure A.1: Decomposition of a (1,2)-surface into the $\mathcal{A}^{\text{con}}_{1,2}$ and $\mathcal{B}_{1,2}$ part. Figure adapted from [77].

$$\begin{aligned}
\mathcal{A}^{\text{con}}_{1,2}(L_1,L_2) &:= \frac{1}{2}\int_0^\infty x\mathrm{d}x \int_0^\infty y\mathrm{d}y \frac{1}{2^{m(1-1,2+1)}} V_{1-1,2+1}(x,y,L_2)H(x+y,L_1) \\
&= \frac{1}{2}\int_0^\infty x\mathrm{d}x \int_0^\infty y\mathrm{d}y H(x+y,L_1) \\
&\overset{\text{eq. (A.2)}}{=} \frac{1}{2}\frac{(1)!(1)!}{(3)!}F_3(L_1) \\
&\overset{\text{eq. (A.3)}}{=} \frac{1}{2}\frac{1}{(3)!}(2+1)!\sum_{i=0}^{1+1}\zeta(2i)\left(2^{2i+1}-4\right)\frac{t^{2+2-2i}}{(2+2-2i)!} \\
&= -\zeta(0)\frac{L_1^4}{4!} + 2\zeta(2)\frac{t^2}{2} + 14\zeta(4)\frac{1}{0!} \\
&\overset{\text{eq. (A.4)}}{=} \frac{L_1^4}{2\cdot 4!} + \frac{\pi^2}{6}L_1^2 + 14\frac{\pi^4}{90} \\
&= \frac{L_1^4}{2\cdot 4!} + \frac{\pi^2}{6}L_1^2 + \frac{7\pi^4}{45}.
\end{aligned} \tag{A.5}$$

For the following computation it is useful to point out that we thus have derived

$$F_3(t) = \frac{t^4}{4} + 2\pi^2 t^2 + \frac{28\pi^4}{15}. \tag{A.6}$$

For the "extra boundary"-part one has to compute

$$\begin{aligned}
\mathcal{B}_{1,2}(L_1,L_2) &= \int_0^\infty x\mathrm{d}x \frac{1}{2^{m(1,2-1)}}\frac{1}{2}[H(x,L_1+L_2)+H(x,L_1-L_2)]V_{1,2-1}(x) \\
&:= \frac{1}{4}\int_0^\infty x\mathrm{d}x [H(x,L_1+L_2)+H(x,L_1-L_2)]\left[\frac{\pi^2}{6}+\frac{x^2}{24}\right] \\
&\overset{\text{eq. (A.1)}}{=} \frac{1}{4}\left[\frac{\pi^2}{6}(F_1(L_1+L_2)+F_1(L_1-L_2)) + \frac{1}{24}(F_3(L_1+L_2)+F_3(L_1-L_2))\right],
\end{aligned} \tag{A.7}$$

We proceed by computing the needed terms individually. For F_1 one finds

$$\begin{aligned} F_1(t) &\overset{\text{eq. (A.3)}}{=} (1)! \sum_{i=0}^{1} \zeta(2i) \left(2^{2i+1} - 4\right) \frac{t^{2-2i}}{(2-2i)!} \\ &= -2\zeta(0)\frac{t^2}{2} + 4\zeta(2)\frac{1}{0!} \\ &= \frac{t^2}{2} + \frac{2\pi^2}{3}. \end{aligned} \tag{A.8}$$

To simplify the computation of the next term, we first consider (for n even)

$$\begin{aligned} g_n(a,b) := (a+b)^n + (a-b)^n &= \sum_{k=0}^{n} \binom{n}{k} \left[a^{n-k}b^k + a^{n-k}(-b)^k\right] \\ &= 2\sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} a^{n-2i} b^{2i} \\ &= 2\sum_{i=0}^{\frac{n}{2}} \frac{n!}{(n-2i)!(2i)!} a^{n-2i} b^{2i}, \end{aligned} \tag{A.9}$$

which results in

$$g_2(L_1, L_2) = 2\left[L_1^2 + L_2^2\right], \tag{A.10}$$

$$g_4(L_1, L_2) = 2\left[L_1^4 + 6L_1^2L_2^2 + L_2^4\right]. \tag{A.11}$$

This can be used to rewrite the sought for expressions as

$$\begin{aligned} (F_3(L_1+L_2) + F_3(L_1-L_2)) &= \frac{1}{4} g_4(L_1,L_2) + 2\pi^2 g_2(L_1,L_2) + 2\frac{28\pi^4}{15} \\ &= \frac{1}{2}\left(L_1^4 + L_2^4\right) + 3L_1^2L_2^2 + 4\pi^2\left(L_1^2 + L_2^2\right) + 2\frac{28\pi^4}{15}, \end{aligned} \tag{A.12}$$

and

$$\begin{aligned} (F_1(L_1+L_2) + F_1(L_1-L_2)) &= \frac{1}{2} g_2(L_1,L_2) + \frac{4\pi^2}{3} \\ &= L_1^2 + L_2^2 + \frac{4\pi^2}{3}. \end{aligned} \tag{A.13}$$

Putting the pieces together, we find the full result for $\mathcal{B}_{1,2}$ as

$$\begin{aligned}\mathcal{B}_{1,2}(L_1,L_2) &= \frac{1}{4}\left[\frac{\pi^2}{6}(F_1(L_1+L_2)+F_1(L_1-L_2))+\frac{1}{24}(F_3(L_1+L_2)+F_3(L_1-L_2))\right]\\ &= \frac{1}{4\cdot 6}\left[\pi^2\left(L_1^2+L_2^2\right)+\frac{4\pi^4}{3}+\right.\\ &\left.+\frac{1}{4}\left(\frac{1}{2}\left(L_1^4+L_4^2\right)+3L_1^2L_2^2+4\pi^2\left(L_1^2+L_2^2\right)+2\frac{28\pi^4}{15}\right)\right]\\ &= \frac{1}{4\cdot 6}\left[\frac{1}{8}\left(L_1^4+L_2^4\right)+\frac{3}{4}L_1^2L_2^2+2\pi^2\left(L_1^2+L_2^2\right)+\pi^4\frac{4\cdot 5+14}{15}\right].\end{aligned} \tag{A.14}$$

Having found the individual contributions we can put all together to find

$$\begin{aligned}\mathcal{A}^{\mathrm{con}}_{1,2}(L_1,L_2)+\mathcal{B}_{1,2}(L_1,L_2) = &\frac{1}{192}\left(5L_1^4+L_2^4\right)+\frac{3}{96}L_1^2L_2^2+\\ &+\pi^2\frac{1}{12}\left(3L_1^2+L_2^2\right)+\pi^4\underbrace{\frac{17+28}{2^2\cdot 3^2\cdot 5}}_{:=\frac{1}{4}},\end{aligned} \tag{A.15}$$

and finally, by applying the recursion relation and inverting the derivative taken there, one obtains

$$V_{1,2}(L_1,L_2) = \frac{1}{192}\left(L_1^4+L_2^4\right)+\frac{1}{96}L_1^2L_2^2+\pi^2\frac{1}{12}\left(L_1^2+L_2^2\right)+\frac{\pi^4}{4}, \tag{A.16}$$

which is the final result for the WP volume of $\mathcal{M}_{1,2}(L_1,L_2)$. We have written the result in such a way that one can see general structure of theorem 2.2 directly.

One can go on now to compute higher volumes, which works analogously but becomes increasingly tedious. As an example, the decomposition of $V_{2,2}$ is shown in fig. A.2. In this example one can see one of the "problems" of the computation of higher volumes, the appearance of increasingly many lower ones (as it is of course expected for a recursion relation). E.g. here $V_{1,3}$ and $V_{2,1}$ appear which have to be computed before the computation of $V_{2,2}$. Hence, we will not write out further examples in the interest of brevity. Furthermore, the JT/MM duality offers a way to study the moduli space volumes using, in the orientable case, topological recursion which is, at least in our implementation, the faster way to compute the volumes on a computer.

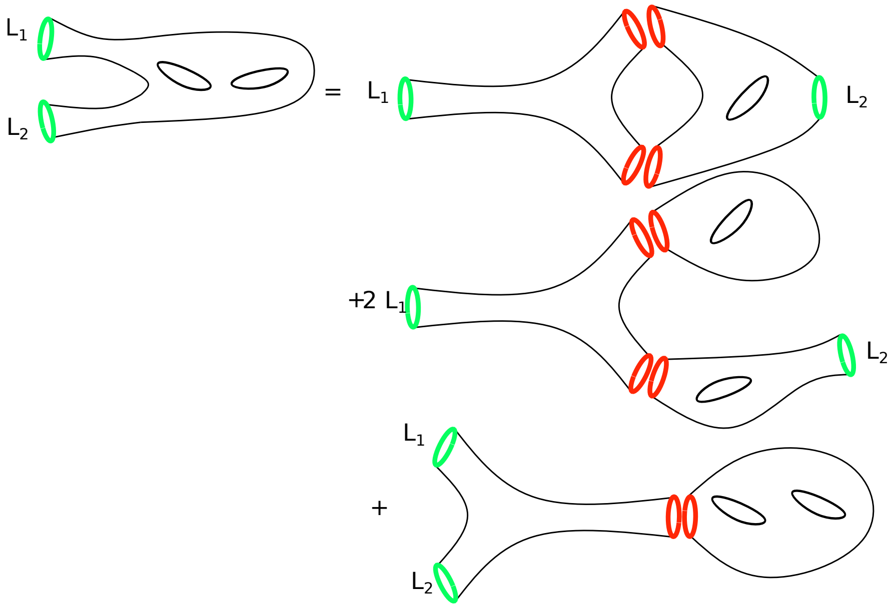


Figure A.2: Decomposition of a (2,2)-surface into the $\mathcal{A}^{\text{con}}_{2,2}.\mathcal{A}^{\text{dcon}}_{2,2}$ and $\mathcal{B}_{2,2}$ parts. Figure adapted from [77].

B

PROOF OF THE RELATIONS OF OBSERVABLES

First, the relation of $\rho(x)$ and the resolvent, eq. (2.2.13) is proven.

$$\begin{aligned}\lim_{\epsilon\to 0}\left[R(x+i\epsilon)-R(x-i\epsilon)\right] &= \lim_{\epsilon\to 0}\sum_{j=0}^{L}\left[\frac{1}{x-\lambda_j+i\epsilon}-\frac{1}{x-\lambda_j-i\epsilon}\right] \\ &= \sum_{i=1}^{L}\left[-i\pi\delta(x-\lambda_j)-i\pi\delta(x-\lambda_j)\right] = -2\pi i\rho(x),\end{aligned} \tag{B.1}$$

where the Sokhotski-Plemelj theorem was used in going to the second line, noting that the arising principal parts cancel due to the negative sign between the resolvents.

Next, we show eq. (2.2.14). We consider the case of a finite-dimensional diagonalisable matrix H with N eigenvalues λ_i which are ordered such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. Now we assume $x < \lambda_1$ and $\mathrm{Re}\{\beta\} > 0$ to compute

$$\begin{aligned}-\mathcal{L}[Z(\beta),\beta,-x] &= -\int_0^\infty \mathrm{d}\beta e^{\beta x} Z(\beta) = -\int_0^\infty \mathrm{d}\beta e^{\beta x}\,\mathrm{tr}\left(e^{-\beta H}\right) \\ &= -\sum_{i=1}^{N}\int_0^\infty \mathrm{d}\beta e^{-\beta(\lambda_i - x)} = -\sum_{i=1}^{N}\frac{-1}{\lambda_i - x}\left[e^{-\beta(\lambda_i-x)}\right]_0^\infty \\ &= \mathrm{tr}\left(\frac{1}{x-H}\right) = R(x),\end{aligned} \tag{B.2}$$

where the convergence of the integral was guaranteed by our choice of x since $x < \lambda_1 \implies \underset{i\in\mathbb{N}}{\forall}\, x < \lambda_i$ due to the ordering of the eigenvalues and hence $\underset{i\in\mathbb{N}}{\forall}\, \lambda_i - x > 0$. After doing this computation, one can analytically continue the resolvent to the whole complex plane. Doing this, the issue of the resolvent being actually defined on a two-sheeted cover of the complex plane with a branch-cut at $\mathrm{supp}(\rho)$ appears.

Let's illustrate both of these formulae with an example. For orientable topological gravity we know from e.g. the Ribbon graphs (example 2.2) that

$$V_{1,1}^{\mathrm{Airy}}(b) = \frac{b^2}{48}. \tag{B.3}$$

Hence, one can compute the corresponding contribution to the expectation value of the thermal partition function as

$$Z_{1,1}^{\text{Airy}}(\beta) = \int_0^\infty b\,\mathrm{d}b\, Z^t(\beta, b) V_{1,1}^{\text{Airy}}(b) = \frac{1}{12\sqrt{\pi}}\beta^{\frac{3}{2}}. \tag{B.4}$$

We now use the formula derived above to find the corresponding contribution to the expectation value of the resolvent as

$$\begin{aligned} R_1^{\text{Airy}}(x) &= -\frac{1}{12\sqrt{\pi}}\int_0^\infty \mathrm{d}\beta\, e^{\beta x}\beta^{\frac{3}{2}} \overset{y=\sqrt{\beta}}{=} -\frac{1}{6\sqrt{\pi}}\int_0^\infty \mathrm{d}y\, e^{y^2 x} y^4 \\ &= -\frac{1}{6\sqrt{\pi}}\frac{1}{2}\frac{\partial^2}{\partial x^2}\bigg|_{x=0}\sqrt{\frac{\pi}{-x}} = -\frac{1}{12}\frac{3}{4}\frac{1}{(-x)^{\frac{5}{2}}} \\ &= -\frac{1}{16}\frac{1}{(-x)^{\frac{5}{2}}}. \end{aligned} \tag{B.5}$$

It is now apparent that this function has a branch-cut through the positive real axis, which is precisely the support of ρ_0 for the Airy model.

This can now be used to find the corresponding contribution to the density of states via the other proven formula. In order to do this, we consider the general case of the expression leading to eq. (2.2.57), i.e.

$$\lim_{\epsilon\to 0}(-x \pm i\epsilon)^{n+\frac{1}{2}} = \left[(-1)^n \times \pm i\right]^{\theta(x)} |x|^{n+\frac{1}{2}}, \tag{B.6}$$

which derives from [99][eq. 1.96]. Using this, one can compute

$$\frac{1}{-2\pi i}\lim_{\epsilon\to 0}\left\{[-(x+i\epsilon)]^{n+\frac{1}{2}} - [-(x-i\epsilon)]^{n+\frac{1}{2}}\right\} = \theta(x)\frac{(-1)^n}{\pi x^{-n-\frac{1}{2}}}, \tag{B.7}$$

which for the present case of interest ($n = -3$) can be used to find

$$\begin{aligned} \rho_{1,1}^{\text{Airy}}(x) &= \frac{1}{-2\pi i}\lim_{\epsilon\to 0}\left[R_1(x+i\epsilon) - R_1(x-i\epsilon)\right] \\ &= -\frac{1}{16}\frac{\theta(x)(-1)^3}{\pi x^{\frac{5}{2}}} = \theta(x)\frac{1}{16\pi x^{\frac{5}{2}}}. \end{aligned} \tag{B.8}$$

Now one can go on to "close the loop" by computing from this again the contribution to the partition function via

$$\begin{aligned} Z_{1,1}^{\text{Airy}}(\beta) &= \mathcal{L}\left[\rho_{1,1}^{\text{Airy}}(x), x, \beta\right] = \frac{1}{16\pi}\mathcal{L}\left[x^{-\frac{5}{2}}, x, \beta\right] = \frac{7}{16\pi}\frac{\Gamma\left(-2+\frac{1}{2}\right)}{\beta^{-2+\frac{1}{2}}} \\ &= \frac{1}{16\pi}\frac{4\sqrt{\pi}}{3}\beta^{\frac{3}{2}} = \frac{\beta^{\frac{3}{2}}}{12\sqrt{\pi}}, \end{aligned} \tag{B.9}$$

which is indeed the result we started with. For more examples of changing between the different correlation functions see section 2.3, specifically the considerations of the Airy model there.

C

A (VERY) BRIEF INTRODUCTION TO HYPERFUNCTIONS

The aim of this appendix is to explain very concisely the concept of *hyperfunctions*. This, unfortunately not too well-known, notion of generalised functions offers an approach to mathematical objects like the Dirac δ, the Heaviside "function" and related objects alternative to the more standard notion of treating them as distributions. The reason why we are interested in hyperfunctions, is given by the relation of the density of states with the resolvent, given in eq. (2.2.13). The main goal of this appendix is to explain why this relation leads to the natural appearance of hyperfunctions as a useful tool to treat (the perturbative expansion of) expectation values of the density of states and to give some example of them, which we use in the main text.

Our exposition will be completely based on [99] and essentially is a abbreviated version of the introduction of hyperfunctions there, emphasising the points that are immediately necessary for our treatment in the main text.

We first introduce the notation of $\mathbb{C}_\pm$ for the upper/lower half-plane and give some basic definitions:

Definition C.1 (Complex neighbourhood)**.** *Let $I \subset \mathbb{R}$ be open. Then $\mathcal{D}(I) \subset \mathbb{C}$ is a complex neighbourhood of I iff $I \subset \mathcal{D}(I)$ is closed.*
Then, $\mathcal{D}_\pm(I) := \mathcal{D}(I) \cap \mathbb{C}_\pm$ are referred to as the upper/lower half-neighbourhoods.

For $\mathcal{D}$ an open subset of $\mathbb{C}$ we denote by $\mathcal{O}(\mathcal{D})$ the set (ring) of holomorphic functions on $\mathcal{D}$. Let $F(z) \in \mathcal{F}(I) := \mathcal{O}(\mathcal{D}(I))$ then:

$$F(z) = \begin{cases} F_+(z), & z \in \mathcal{D}_+(I), \\ F_-(z), & z \in \mathcal{D}_-(I), \end{cases} \tag{C.1}$$

with $F_\pm \in \mathcal{O}(\mathcal{D}_\pm(I))$, where one refers to $F_\pm$ as the upper/lower component of the function. On this set of function, one can define an equivalence relation as

Definition C.2. *Let $F, G \in \mathcal{F}(I)$, defined on complex neighbourhoods $\mathcal{D}_F(I)$ or $\mathcal{D}_G(I)$. Then*

$$F \sim G \Leftrightarrow \underset{\phi \in \mathcal{O}(\mathcal{D}(I))}{\exists} \underset{z \in \mathcal{D}_F(I) \cap \mathcal{D}_g(I)}{\forall} G(z) - F(z) = \phi(z).$$

It is easy to see that this is indeed an equivalence relation. The explicit steps are given in [99]. Using this equivalence relation, one can now define hyperfunctions:

Definition C.3 (Hyperfunctions). *The set of hyperfunctions over the interval $I \subset \mathbb{R}$, denoted as $\mathcal{B}(I)$, is defined as $\mathcal{F}(I)/\sim$. For $F(z) \in \mathcal{F}(I)$ we notate the associated equivalence class under $\sim$, i.e. the hyperfunction $f(x)$, as $f(x) = [F_+(z), F_-(z)]$. For the special case of $F_+(z) = F_-(z)$ we just write $f(x) = [F(z)]$.*

It is noteworthy that we use only the dependence on I in the definition of hyperfunctions, not on $\mathcal{D}(I)$. This is not due to imprecision but makes sense, since there is no preferred choice of complex neighbourhood and one can essentially define the same set of functions by "shrinking" the defining neighbourhood further and further (for details cf. [99]). Hence, to define a hyperfunction it is necessary to known only what happens slightly above and slightly below the interval I in the complex plane. By this reasoning one is led to study

$$f(x) := \lim_{\epsilon \to 0} \{F_+(x + i\epsilon) - F_-(x - i\epsilon)\}, \tag{C.2}$$

for $F \in \mathcal{F}(\mathcal{I})$ and $x \in I$. If this limit exists, f is just a "normal" function for $x \in I$. One says this is the "ordinary function" corresponding to F. Note that a function $G \sim F$ defines the same "ordinary function" and hence it depends only on the equivalence class, i.e. the hyperfunction defined by $F(z)$, $[F(z)]$. In fact, precisely the requirement for this equivalence can be seen as the motivation for the definition of the equivalence relation. If the limit does not exist, the only sensible thing to do, is to understand the result of the limit as the hyperfunction itself. In this precise sense, hyperfunctions can be understood as generalising analytic functions on $I \subset \mathbb{R}$.

From this reasoning it is apparent that we can interpret the relation of the density of states with the resolvent, or rather the relations of the topological expansions of their expectation values, given in eq. (2.3.40) as

$$\rho_{g,1}(x) = \frac{1}{-2\pi i} \lim_{\epsilon \to 0} \left[R_g(x + i\epsilon) - R_g(x - i\epsilon)\right], \tag{C.3}$$

in terms of hyperfunctions. In fact, $R_g(z)$ is defined on $\mathcal{C} \backslash [a_-, a_+]$ with $[a_-, a_+] \subset \mathbb{R}$. Hence, we can set $I = [a_-, a_+]$ and then the relation above can be written as

$$\rho_{g,1}(x) = \left[\frac{-1}{2\pi i} R_g(z)\right], \tag{C.4}$$

which we gave above as eq. (2.3.41)[1]. So far it is, however, not clear what has been achieved by referring to this object as a hyperfunction. To see this, let us discuss some examples of important hyperfunctions first.

Example C.1 (Dirac δ)**.** *Let us consider the hyperfunction defined by* $F(z) = -\frac{1}{2\pi i z}$ *with* $I = \mathbb{R}$*. For this, we compute*

$$F(x+i\epsilon) - F(x-i\epsilon) = \cdots = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}. \tag{C.5}$$

This clearly is the Lorentz-curve with the parameter ϵ*, which is a well-known example of a Dirac series. This implies*

$$\delta(x) = \left[-\frac{1}{2\pi i z}\right]. \tag{C.6}$$

Of course, it is also possible to see that this is indeed the Dirac δ *by its defining property that* $\int_{\mathbb{R}} \delta(x-a)f(x) = f(a)$ *(for a suitable class of functions). This reasoning would, however, require introducing integrals of hyperfunctions, which goes beyond the scope of this brief introduction. It can be found in [99],*

For the next example, we have to note that a derivative on a hyperfunction is defined just as the hyperfunction defined by the derivatives of the defining upper/lower components of the hyperfunction [99].

Example C.2 (Derivatives of the Dirac δ)**.** *To compute the derivatives of the Dirac* δ*, one just has to differentiate the function defining it as a hyperfunction. This directly implies*

$$\frac{\mathrm{d}^n\delta(x)}{\mathrm{d}x^n} =: \delta^{(n)}(x) = -\frac{1}{2\pi i}\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\frac{1}{z}\right] = -\frac{1}{2\pi i}\left[\frac{(-1)^n n!}{z^{n+1}}\right] = \left[\frac{n!}{2\pi i(-z)^{n+1}}\right]. \tag{C.7}$$

Consequently, by treating $\rho_{g,1}$ as a hyperfunction, we can directly read off the result from the functional form of $R_g(z)$ since all of the, potentially quite involved, reasoning with taking the limit in ϵ correctly etc. is taken care of by the formalism of hyperfunctions in a much more convenient way.

This is how we proceed in the main text, whenever computing the topological expansion of the density of states. For this computations we needed another result, whose proof can be found in [99][ch. 1.6.3]:

$$\theta(x)x^\alpha = \left[-\frac{1}{2i}\frac{(-z)^\alpha}{\sin(\alpha\pi)}\right], \tag{C.8}$$

1 Note that in the above presentation we renamed the variable z, denoting the complex argument of R_g, to y to avoid confusion with the double-cover coordinate, denoted as z above.

valid for $\alpha \in \mathbb{C}\backslash\mathbb{Z}$. Putting, for $n \in \mathbb{N}$, $\alpha = -\frac{2n+1}{2}$ this results in

$$\theta(x)x^{-\frac{2n+1}{2}} = \left[\frac{1}{2i}\frac{(-1)^n}{(-z)^{n+\frac{1}{2}}}\right], \tag{C.9}$$

which is the formula we cite in the main text.

This concludes our very brief introduction to hyperfunctions. For more information we refer to the excellent (and very accessible) introductory book about this subject, [99].

D

DETAILS OF THE NUMERICAL IMPLEMENTATION OF THE ARBITRARY β GAUSSIAN MATRIX MODEL

In this appendix we give details for the numerical implementation of the Gaussian matrix model for arbitrary Dyson index. We also use this algorithm to generate the results for the three Wigner-Dyson classes due to it being faster than generating genuine matrices from the three classes. It is based on the realisation of the arbitrary β Gaussian matrix model as an ensemble of tridiagonal matrices by Dumitriu and Edelman [98].

THE MATRIX ENSEMBLE. For the matrix size N the ensemble is given by matrices of the form

$$T = \frac{1}{\sqrt{\beta}} \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ 0 & \dots & 0 & b_{N-1} & a_N \end{pmatrix} , \tag{D.1}$$

with the entries a_i drawn from a normal distribution with $\mu = 0$ and $\sigma^2 = 2$ and the off-diagonal entries b_i drawn from χ_r distributions with $r = (N - i)\beta$. Note that in our implementation the probability density for the χ_r distribution is given by

$$\frac{1}{2^{\frac{r}{2}-1}\,\Gamma\left(\frac{r}{2}\right)}\, x^{r-1}\, e^{-\frac{x^2}{2}}\, \mathrm{d}x \,, \tag{D.2}$$

differing from the convention in [98], hence our normalisation of the matrix is different and coincides with that chosen in [152]. The proof that the probability distribution of the eigenvalues of matrices in this ensemble is exactly given by the integrand of eq. (2.2.7) with a Gaussian potential is given in [98] and proves the equivalence of the ensemble of tridiagonal matrices with the arbitrary β Gaussian matrix model.

To perform the numerical simulation, we chose to implement the matrix ensemble in python, having the advantage of directly providing functions implementing the drawing from the required probability distributions as numpy.random.normal() and scipy.stats.chi.rvs().

UNFOLDING. To evaluate the numerical data, we have to perform spectral unfolding as explained in section 2.4. For this, we need $\langle \rho(x) \rangle$ for the present matrix model. In [98] it is shown that the ensemble of tridiagonal matrices exhibits the so-called "semicircle-law" also present in the traditional Gaussian Wigner-Dyson ensembles [15]. By this, one means

$$\langle \rho(x) \rangle = \begin{cases} \frac{1}{2\pi}\sqrt{4N - x^2} & |x| \leq 2\sqrt{N}, \\ 0 & |x| > 2\sqrt{N}. \end{cases} \tag{D.3}$$

Hence,

$$\begin{aligned} e(x) &= \int_{-\infty}^{x} \mathrm{d}x' \, \langle \rho(x') \rangle = \frac{1}{2\pi} \int_{-2\sqrt{N}}^{x} \mathrm{d}x' \, \sqrt{4N - x'^2} \\ &= \cdots = \frac{N}{2} + \frac{N}{\pi} \arctan\left(\frac{x}{\sqrt{4N - x^2}} \right) + \frac{x}{4\pi}\sqrt{4N - x^2}, \end{aligned} \tag{D.4}$$

coinciding with the unfolding in [152].

For numerical reasons, it is better not to continue working with the density of states normalised as in the main text and up to now but rather with $\tilde{\rho} = \frac{1}{N}\rho$, which is normalised to unity. Another reason to do this, is the agreement with the convention of [152]. In the unfolded energies, it holds that $\langle \tilde{\rho} \rangle = \frac{1}{N}$.

EVALUATING THE SFF. Using $\tilde{\rho}$ to define the (microcanonical) SFF, it can be written as[1]

$$\begin{aligned} \kappa_\beta(t) &= \frac{1}{\langle \tilde{\rho} \rangle} \left[\frac{1}{N^2} \left\langle \sum_{i=1}^{N} \sum_{j=1}^{N} e^{it(e_i - e_j)} \right\rangle_\beta - \frac{1}{N^2} \left\langle \sum_{i=1}^{N} e^{ite_i} \right\rangle_\beta \left\langle \sum_{i=1}^{N} e^{-ite_i} \right\rangle_\beta \right] \\ &= \frac{1}{N} \left\langle \left| \sum_{i=1}^{N} e^{ite_i} \right|^2 \right\rangle_\beta - \frac{1}{N} \left| \left\langle \sum_{i=1}^{N} e^{ite_i} \right\rangle_\beta \right|^2, \end{aligned} \tag{D.5}$$

where $\langle \cdot \rangle_\beta$ denotes the average over the tridiagonal ensemble defined above. We implement this by averaging the argument over N_{avg} draws from the ensemble.

For the comparison to the universal RMT result one needs to write the time in units of the Heisenberg time $T_{\text{H}} = 2\pi \langle \rho(x) \rangle = 2\pi N \langle \tilde{\rho}(x) \rangle$ which in the unfolded energies is just given by 2π. Using this, the analytic prediction for the microcanonical SFF that was worked out in section 2.4 is given by

$$\kappa_\beta(t = 2\pi x) = 1 - b^\beta(x). \tag{D.6}$$

1 Note that in the unfolded variable the SFF is independent on the energy.

The comparison of the numerical results, evaluated in the way discussed here, with this analytic prediction is shown in fig. 2.4.1, displaying perfect agreement.

Beyond providing this numerical justification for the analytic formulae, the implementation of the Gaussian ensemble for arbitrary Dyson index can be used to find numerical predictions for the microcanonical SFF for the non-Wigner-Dyson values of β. This is extensively used in section 4.4.2. To give an illustration how these results look like, we close this chapter by presenting the numerical results for the microcanonical SFFs for the Wigner-Dyson and various intermediate values in fig. D.1.

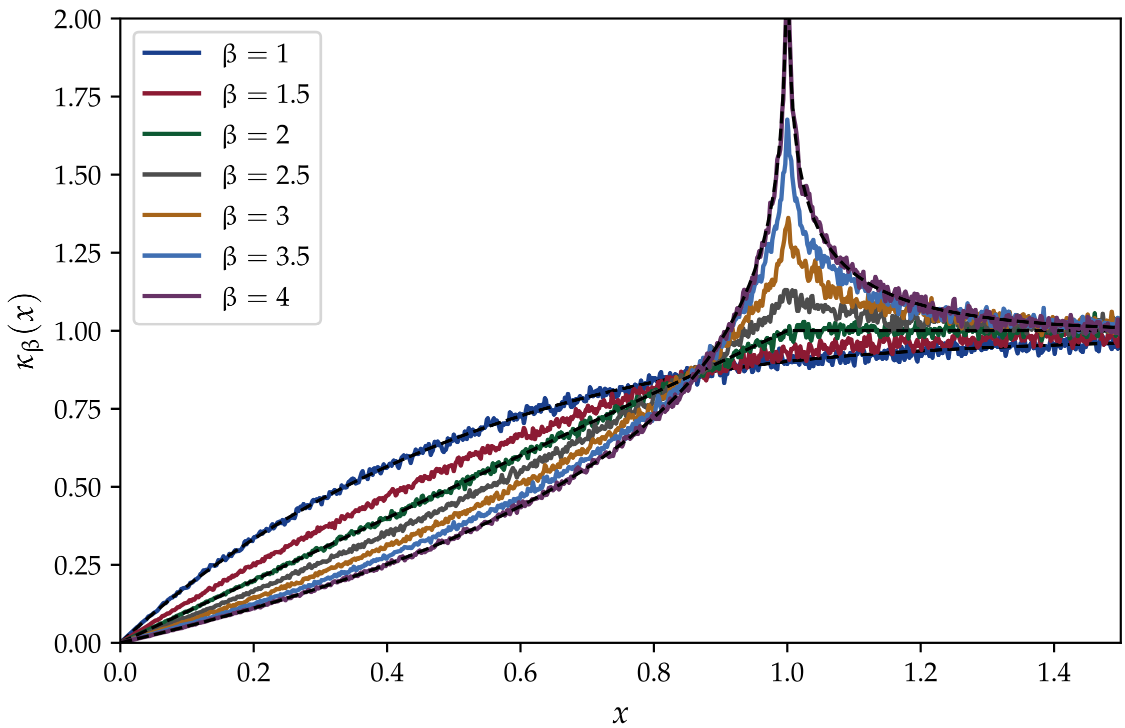


Figure D.1: Numerical results ($N = 300$ and $N_{\text{avg}} = 5000$) for the microcanonical SFF in the Gaussian matrix models for various values of the Dyson index β. For the Wigner-Dyson values of the index we also plot the analytic predictions as broken black lines. Note that we denote the microcanonical SFF as $\kappa_\beta(x)$ since it is independent on the energy with x in units of T_H.

E

COLLECTION OF NON-LOGARITHMIC CONSTRAINTS FOR UNORIENTABLE TOPOLOGICAL GRAVITY

In this appendix we collect all the non-logarithmic constraints for topological gravity which we did not put into the main text due to their length. A further reason for this is that not their form but their existence and number is relevant for the discussion of the main text. We provide now a list of them to illustrate this discussion. Note that we can not refer to [5] for this, since the constraints for $g = 3$ and $g = \frac{7}{2}$ are novel and so far only presented in this thesis (though we will include them in to [5] during the peer-review process).

$$
\begin{aligned}
g = 2 :& \\
0 =&5C_{0,10} + 4C_{1,9} + 3C_{2,8} + 2C_{3,7} + C_{4,6} \\
&- C_{6,4} - 2C_{7,3} - 3C_{8,2} - 4C_{9,1} - 5C_{10,0}, && \text{(E.1)} \\
0 =&55C_{0,10} + 50C_{1,9} + 41C_{2,8} + 29C_{3,7} + 15C_{4,6} \\
&- 15C_{6,4} - 29C_{7,3} - 41C_{8,2} - 50C_{9,1} - 55C_{10,0}, && \text{(E.2)} \\
0 =&495C_{0,10} + 420C_{1,9} + 441C_{2,8} + 394C_{3,7} + 235C_{4,6} \\
&- 235C_{6,4} - 394C_{7,3} - 441C_{8,2} - 420C_{9,1} - 495C_{10,0}, && \text{(E.3)} \\
g = \frac{5}{2} :& \\
0 =&13C_{0,13} + 11C_{1,12} + 9C_{2,11} + 7C_{3,10} + 5C_{4,9} + 3C_{5,8} + C_{6,7} \\
&- C_{7,6} - 3C_{8,5} - 5C_{9,4} - 7C_{10,3} - 9C_{11,2} - 11C_{12,1} - 13C_{13,0}, && \text{(E.4)} \\
0 =&3003C_{0,13} + 2717C_{1,12} + 2343C_{2,11} + 1897C_{3,10} + 1395C_{4,9} \\
&+ 853C_{5,8} + 287C_{6,7} - 287C_{7,6} - 853C_{8,5} - 1395C_{9,4} \\
&- 1897C_{10,3} - 2343C_{11,2} - 2717C_{12,1} - 3003C_{13,0}, && \text{(E.5)} \\
0 =&63063C_{0,13} + 59345C_{1,12} + 54747C_{2,11} + 47509C_{3,10} + 37023C_{4,9} \\
&+ 23577C_{5,8} + 8099C_{6,7} - 8099C_{7,6} - 23577C_{8,5} - 37023C_{9,4} \\
&- 47509C_{10,3} - 54747C_{11,2} - 59345C_{12,1} - 63063C_{13,0}, && \text{(E.6)} \\
0 =&105105C_{0,13} + 145431C_{1,12} + 100485C_{2,11} + 80619C_{3,10} + 79065C_{4,9} \\
&+ 64863C_{5,8} + 25613C_{6,7} - 25613C_{7,6} - 64863C_{8,5} - 79065C_{9,4} \\
&- 80619C_{10,3} - 100485C_{11,2} - 145431C_{12,1} - 105105C_{13,0}. && \text{(E.7)}
\end{aligned}
$$

$g = 3$:

$$\begin{aligned}
0 =& 8C_{0,16} + 7C_{1,15} + 6C_{2,14} + 5C_{3,13} + 4C_{4,12} + 3C_{5,11} \\
&+ 2C_{6,10} + C_{7,9} - C_{9,7} - 2C_{10,6} - 3C_{11,5} - 4C_{12,4} \\
&- 5C_{13,3} - 6C_{14,2} - 7C_{15,1} - 8C_{16,0}, \qquad \text{(E.8)}
\end{aligned}$$

$$\begin{aligned}
0 =& 476C_{0,16} + 434C_{1,15} + 385C_{2,14} + 330C_{3,13} + 270C_{4,12} \\
&+ 206C_{5,11} + 139C_{6,10} + 70C_{7,9} - 70C_{9,7} - 139C_{10,6} \\
&- 206C_{11,5} - 270C_{12,4} - 330C_{13,3} - 385C_{14,2} \\
&- 434C_{15,1} - 476C_{16,0}, \qquad \text{(E.9)}
\end{aligned}$$

$$\begin{aligned}
0 =& 7140C_{0,16} + 6727C_{1,15} + 6195C_{2,14} + 5509C_{3,13} \\
&+ 4658C_{4,12} + 3651C_{5,11} + 2513C_{6,10} + 1281C_{7,9} \\
&- 1281C_{9,7} - 2513C_{10,6} - 3651C_{11,5} - 4658C_{12,4} \\
&- 5509C_{13,3} - 6195C_{14,2} - 6727C_{15,1} - 7140C_{16,0}, \qquad \text{(E.10)}
\end{aligned}$$

$$\begin{aligned}
0 =& 30940C_{0,16} + 30394C_{1,15} + 28665C_{2,14} + 26390C_{3,13} \\
&+ 23374C_{4,12} + 19242C_{5,11} + 13811C_{6,10} + 7238C_{7,9} \\
&- 7238C_{9,7} - 13811C_{10,6} - 19242C_{11,5} - 23374C_{12,4} \\
&- 26390C_{13,3} - 28665C_{14,2} - 30394C_{15,1} - 30940C_{16,0}. \qquad \text{(E.11)}
\end{aligned}$$

$g = \frac{7}{2}$:

$$\begin{aligned}
0 =& 19C_{0,19} + 17C_{1,18} + 15C_{2,17} + 13C_{3,16} + 11C_{4,15} \\
&+ 9C_{5,14} + 7C_{6,13} + 5C_{7,12} + 3C_{8,11} + C_{9,10} - C_{10,9} \\
&- 3C_{11,8} - 5C_{12,7} - 7C_{13,6} - 9C_{14,5} - 11C_{15,4} - 13C_{16,3} \\
&- 15C_{17,2} - 17C_{18,1} - 19C_{19,0}, \qquad \text{(E.12)}
\end{aligned}$$

$$\begin{aligned}
0 =& 8075C_{0,19} + 7429C_{1,18} + 6715C_{2,17} + 5941C_{3,16} \\
&+ 5115C_{4,15} + 4245C_{5,14} + 3339C_{6,13} + 2405C_{7,12} \\
&+ 1451C_{8,11} + 485C_{9,10} - 485C_{10,9} - 1451C_{11,8} \\
&- 2405C_{12,7} - 3339C_{13,6} - 4245C_{14,5} - 5115C_{15,4} \\
&- 5941C_{16,3} - 6715C_{17,2} - 7429C_{18,1} - 8075C_{19,0}, \qquad \text{(E.13)}
\end{aligned}$$

$$\begin{aligned}
0 =& 10392525C_{0,19} + 9797559C_{1,18} + 9079785C_{2,17} + 8228051C_{3,16} \\
&+ 7240805C_{4,15} + 6124815C_{5,14} + 4893889C_{6,13} + 3567595C_{7,12} \\
&+ 2169981C_{8,11} + 728295C_{9,10} - 728295C_{10,9} - 2169981C_{11,8} \\
&- 3567595C_{12,7} - 4893889C_{13,6} - 6124815C_{14,5} - 7240805C_{15,4} \\
&- 8228051C_{16,3} - 9079785C_{17,2} - 9797559C_{18,1} - 10392525C_{19,0}, \qquad \text{(E.14)}
\end{aligned}$$

$$0 = 2078505C_{0,19} + 2006799C_{1,18} + 1903473C_{2,17} + 1768663C_{3,16}$$

$$
\begin{aligned}
&+ 1598025C_{4,15} + 1387215C_{5,14} + 1134705C_{6,13} + 843191C_{7,12}\\
&+ 519849C_{8,11} + 175695C_{9,10} - 175695C_{10,9} - 519849C_{11,8}\\
&- 843191C_{12,7} - 1134705C_{13,6} - 1387215C_{14,5} - 1598025C_{15,4}\\
&- 1768663C_{16,3} - 1903473C_{17,2} - 2006799C_{18,1} - 2078505C_{19,0},
\end{aligned} \tag{E.15}
$$

$$
\begin{aligned}
0 =&72747675C_{0,19} + 70404633C_{1,18} + 69684615C_{2,17} + 66775813C_{3,16}\\
&+ 61816755C_{4,15} + 55235505C_{5,14} + 46837791C_{6,13} + 36139805C_{7,12}\\
&+ 22982091C_{8,11} + 7903305C_{9,10} - 7903305C_{10,9} - 22982091C_{11,8}\\
&- 36139805C_{12,7} - 46837791C_{13,6} - 55235505C_{14,5} - 61816755C_{15,4}\\
&- 66775813C_{16,3} - 69684615C_{17,2} - 70404633C_{18,1} - 72747675C_{19,0},
\end{aligned} \tag{E.16}
$$

$$
\begin{aligned}
0 =&3968055C_{0,19} - 17673591C_{1,18} + 3968055C_{2,17} + 7078409C_{3,16}\\
&+ 3727815C_{4,15} + 2612025C_{5,14} + 3055143C_{6,13} + 2813785C_{7,12}\\
&+ 1690263C_{8,11} + 509865C_{9,10} - 509865C_{10,9} - 1690263C_{11,8}\\
&- 2813785C_{12,7} - 3055143C_{13,6} - 2612025C_{14,5} - 3727815C_{15,4}\\
&- 7078409C_{16,3} - 3968055C_{17,2} + 17673591C_{18,1} - 3968055C_{19,0}.
\end{aligned} \tag{E.17}
$$

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