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Hinich's model for Day convolution revisited

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Abstract

We prove that Hinich's construction of the Day convolution operad of two $\mathcal{O}$-monoidal ∞-categories is an exponential in the ∞-category of ∞-operads over $\mathcal{O}$, and use this to give an explicit description of the formation of algebras in the Day convolution operad as a bivariant functor.

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1 INTRODUCTION

Let $\mathcal{O}^{\otimes}$ be an ∞-operad and consider operad maps $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$. If it exists, the *Day convolution* of p and q is an exponential of q by p in the ∞-category of ∞-operads over $\mathcal{O}^{\otimes}$, ie a right adjoint object to q with respect to the product functor $- \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} : (\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes}} \to (\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes}}$. Using the combinatorial machinery of quasicategories, Lurie gives a very general construction of Day convolution operads in [8, Section 2.2.6]; see also [7, Section 2.8].

For practical purposes, it is often sufficient to know that the Day convolution operad exists in the case that $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are $\mathcal{O}$-monoidal categories. Hinich provided in [7] a rather straightforward description of the Day convolution operad for arbitrary $\mathcal{O}$-monoidal ∞-categories, and it is this model we will focus on in the following.

To facilitate Hinich's description, recall that there are equivalences of ∞-categories

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}_{\infty}) \xrightarrow{\sim} \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_{\infty}) \simeq \mathrm{Cocart}_{\otimes}(\mathcal{O}).$$

Here, $\mathrm{Cocart}_{\otimes}(\mathcal{O})$ denotes the ∞-category of cocartesian fibrations of ∞-operads over $\mathcal{O}^{\otimes}$. The first equivalence is given by composition with the cartesian structure $\mathrm{Cat}_{\infty}^{\times} \to \mathrm{Cat}_{\infty}$ [8, Proposition 2.4.2.5] and the second equivalence is induced by (un)straightening; this is implicit in [8] and explicitly spelled out in [6, Proposition A.2.1]. In the following, we will freely switch between these descriptions of $\mathcal{O}$-monoidal ∞-categories as necessary.

Denote by $\mathrm{Ar}^{\mathrm{opl}}$ the full subcategory of $(\mathrm{Cat}_\infty)_{/[1]}$ spanned by the cartesian fibrations and equip it with the cartesian symmetric monoidal structure. Restriction to $\{0\}$ and $\{1\}$ defines symmetric monoidal functors $t\colon \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat}_\infty$ and $s\colon \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat}_\infty$, where we also equip Cat_∞ with the cartesian symmetric monoidal structure.

Theorem 1.1 [7, Section 2.8.9]. *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{O}$-monoidal ∞-categories and denote by $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}$ the pullback*

$$\begin{array}{ccc} \mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}} & \longrightarrow & (\mathrm{Ar}^{\mathrm{opl}})^{\times} \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{O}^{\otimes} & \xrightarrow{(\mathcal{C},\mathcal{D})} & \mathrm{Cat}_\infty^{\times} \times_{\mathrm{Comm}^{\otimes}} \mathrm{Cat}_\infty^{\times} \end{array}$$

Then, $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}} \to \mathcal{O}^{\otimes}$ is the Day convolution of $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$.

The goal of this note is to give an alternative proof of this theorem. Instead of identifying $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}$ with another model for the Day convolution operad as in [7, Section 6.3.9], we verify directly that $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}$ possesses the correct universal property, also on the level of categories of algebras. In fact, we show first that the construction of $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}$ promotes the assignment $(\mathcal{C},\mathcal{D}) \mapsto \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Day}_{\mathcal{C},\mathcal{D}})$ to a bivariant functor and exhibit a natural equivalence

$$\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Day}_{\mathcal{C},\mathcal{D}}) \simeq \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}),$$

where the functoriality of the right-hand side is given by pre- and postcomposition with operad maps. The universal property of $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}$ follows easily from this. This is done in Section 2.

Section 3 comments on some variations of these statements for $\mathcal{O}$-monoidal ∞-categories living in certain suboperads of the cartesian symmetric monoidal structure on Cat_∞. Section 4 reproves the well-known statement that $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}} \to \mathcal{O}^{\otimes}$ defines an $\mathcal{O}$-monoidal ∞-category under suitable cocompleteness assumptions on $\mathcal{D}$.

Conventions

(1) In the remainder of this note, the word "category" means "∞-category". We write Cat for the category of small categories.
(2) The category of anima/spaces/∞-groupoids is denoted by An. The groupoid core $\iota\colon \mathrm{Cat} \to \mathrm{An}$ is the right adjoint to the inclusion of An into Cat.
(3) Given a category X, we denote by $\mathrm{Cocart}(X)$ the subcategory of $\mathrm{Cat}_{/X}$ given by the cocartesian fibrations over X and those functors over X which preserve cocartesian morphisms.
(4) We also drop the prefix "∞-" from related concepts. For example, "operad" means "∞-operad" from now on, and we denote the category of operads by Op.
(5) The notation concerning operads and categories of algebras follows the conventions of [8]. In particular, $\mathcal{C}^{\otimes}$ will denote an operad with underlying category $\mathcal{C}$, but the category of $\mathcal{O}$-algebras in an operad $\mathcal{C}^{\otimes}$ will be denoted by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, even though it depends on the

operads $\mathcal{O}^{\otimes}$ and $\mathcal{C}^{\otimes}$, not only their underlying categories. The symbol $\mathrm{Comm}^{\otimes}$ denotes the commutative operad (i.e., the category of pointed finite sets).

(6) Given an operad $\mathcal{A}^{\otimes}$, we denote by $\mathrm{Cocart}_{\otimes}(\mathcal{A})$ the subcategory of $\mathrm{Op}_{/\mathcal{A}^{\otimes}}$ given by the cocartesian fibrations of operads over $\mathcal{A}^{\otimes}$, that is, $\mathcal{A}$-monoidal categories, and those functors over $\mathcal{A}^{\otimes}$ which preserve cocartesian morphisms (corresponding to $\mathcal{A}$-monoidal functors).

2 | ALGEBRAS IN THE DAY CONVOLUTION OPERAD

Throughout this section, fix a base operad $\mathcal{O}^{\otimes}$ as well as $\mathcal{O}$-monoidal categories $\mathcal{C}$ and $\mathcal{D}$. Our goal is to prove the following strengthening of Theorem 1.1.

Theorem 2.1. *Let $\alpha : \mathcal{A}^{\otimes} \to \mathcal{O}^{\otimes}$ be an operad over $\mathcal{O}^{\otimes}$. Then, there exists a natural equivalence*

$$\mathrm{Alg}_{\mathcal{A}\times_{\mathcal{O}}\mathcal{C}/\mathcal{O}}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathrm{Day}_{\mathcal{C},\mathcal{D}}).$$

Note that Theorem 1.1 follows from this by passing to groupoid cores. The proof of this theorem is already implicit in [1, Remark 5.2.5], but it will be convenient to formulate the argument in terms of some concepts introduced in [3].

Definition 2.2 [3, Observation 2.3.2]. A functor $(p_1, p_2) : X \to Y \times Z$ is a *curved orthofibration* if

(1) $p_1 : X \to Y$ is a cartesian fibration;
(2) $p_2 : X \to Z$ is a cocartesian fibration;
(3) p_1-cartesian lifts of morphisms project to equivalences under p_2;
(4) p_2-cocartesian lifts of morphisms project to equivalences under p_1.

Denote by $\mathrm{Cocart}^{\mathrm{lax}}(Y)$ the full subcategory of $\mathrm{Cat}_{/Y}$ spanned by the cocartesian fibrations, and by $\mathrm{Cart}^{\mathrm{opl}}(Z)$ the full subcategory of $\mathrm{Cat}_{/Z}$ spanned by the cartesian fibrations. In addition, $\mathrm{CurvOrtho}(Y, Z)$ denotes the subcategory of $\mathrm{Cat}_{/Y\times Z}$ whose objects are curved orthofibrations and whose morphisms are functors over $Y \times Z$ preserving both p_1-cartesian and p_2-cocartesian morphisms. We will make use of the following description of $\mathrm{CurvOrtho}(Y, Z)$.

Proposition 2.3 [3, Corollary 2.3.4]. *Unstraightening over Y and Z induces equivalences*

$$\mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(Z))^{\mathrm{cart}} \simeq \mathrm{CurvOrtho}(Y, Z) \simeq \mathrm{Fun}(Z, \mathrm{Cart}^{\mathrm{opl}}(Y))^{\mathrm{cocart}}$$

which are natural in Y and Z; the superscripts cocart *and* cart *denote the wide subcategories on those natural transformations whose components all preserve (co)cartesian morphisms.*

We will also require the following statement about the interaction of cartesian and cocartesian morphisms in curved orthofibrations.

Lemma 2.4. *Let $(p_1, p_2) : X \to Y \times Z$ be a curved orthofibration, and let $g : y \to y'$ and $h : z \to z'$ be morphisms in Y and Z, respectively. Then, the following are equivalent:*

(1) *The cartesian transport* $g^* : p_1^{-1}(a') \to p_1^{-1}(a)$ *along* g *preserves* p_2*-cocartesian lifts of* h.
(2) *The cocartesian transport* $h_! : p_2^{-1}(b) \to p_2^{-1}(b')$ *along* h *preserves* p_1*-cartesian lifts of* g.

Proof. This follows by inspection of the proof of [3, Proposition 2.3.11]. □

Given Lemma 2.4, we can make the following definition, see [3, Definition 2.3.10].

Definition 2.5. A curved orthofibration

$$p = (p_1, p_2) : X \to Y \times Z$$

is an *orthofibration* if it satisfies the following equivalent conditions:

(1) the cartesian transport functor $g^* : p_1^{-1}(y') \to p_1^{-1}(y)$ preserves p_2-cocartesian morphisms for every morphism $g : y \to y'$ in Y;
(2) the cocartesian transport functor $h_! : p_2^{-1}(z) \to p_2^{-1}(x')$ preserves p_1-cartesian morphisms for every morphism $h : z \to z'$ in Z.

By [3, Corollary 2.5.6], the equivalences of Proposition 2.3 restrict to equivalences

$$\mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{Cocart}(Z)) \simeq \mathrm{Ortho}(Y, Z) \simeq \mathrm{Fun}(Z, \mathrm{Cart}(Y)), \tag{2.6}$$

where $\mathrm{Ortho}(Y, Z) \subseteq \mathrm{CurvOrtho}(Y, Z)$ denotes the full subcategory of orthofibrations. Consequently, every orthofibration straightens to a functor $Y^{\mathrm{op}} \times Z \to \mathrm{Cat}$, either by cocartesian straightening over Z or by cartesian straightening over Y. The resulting straightening functors $\mathrm{Ortho}(Y, Z) \to \mathrm{Fun}(Y^{\mathrm{op}} \times Z, \mathrm{Cat})$ are equivalent, see [3, Remark 2.5.7].

Since $(s, t) : \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat} \times \mathrm{Cat}$ is an orthofibration by [4, Proposition 7.9], combining [3, Theorem E] and [4, Theorem 7.21] shows that (s, t) straightens to the functor

$$\mathrm{Fun} : \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat} \to \mathrm{Cat}.$$

In fact, we will reprove this statement without recourse to the 2-categorical machinery of [4].

Theorem 2.7. *The functor*

$$(s_*, t_*) : \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}) \times \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})$$

is an orthofibration which straightens to the obvious functor

$$(\mathcal{X}, \mathcal{Y}) \mapsto \mathrm{Alg}_{\mathcal{X}/\mathcal{O}}(\mathcal{Y}).$$

Remark 2.8. By specializing Theorem 2.7 to the case that $\mathcal{O}^{\otimes}$ is the trivial operad, we recover the claim that the orthofibration $(s, t) : \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat} \times \mathrm{Cat}$ straightens to the functor $\mathrm{Fun} : \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat} \to \mathrm{Cat}$.

Assuming this statement, we can prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathcal{A}^\otimes$ be an operad and let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{A}$-monoidal categories. Writing $(\mathcal{X},\mathcal{Y}) \in \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat}) \times \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat})$ as the composite functor $* \xrightarrow{\mathrm{id}} \mathrm{Alg}_{\mathcal{A}}(\mathcal{A}) \xrightarrow{(\mathcal{X}_*,\mathcal{Y}_*)} \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat}) \times \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat})$ and using that $\mathrm{Alg}_{\mathcal{A}}(-)$ preserves limits, we obtain the pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{/\mathcal{A}}(\mathrm{Day}_{\mathcal{X},\mathcal{Y}}) & \longrightarrow & \mathrm{Alg}_{\mathcal{A}}(\mathrm{Ar}^{\mathrm{opl}}) \\ \big\downarrow & & \big\downarrow{\scriptstyle (s_*,t_*)} \\ * & \xrightarrow{(\mathcal{X},\mathcal{Y})} & \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat}) \times \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat}) \end{array}$$

exhibiting $\mathrm{Alg}_{/\mathcal{A}}(\mathrm{Day}_{\mathcal{X},\mathcal{Y}})$ as the fiber of (s_*, t_*) over $(\mathcal{X}, \mathcal{Y})$. Then, Theorem 2.7 yields an equivalence

$$\mathrm{Alg}_{/\mathcal{A}}(\mathrm{Day}_{\mathcal{X},\mathcal{Y}}) \simeq \mathrm{Alg}_{\mathcal{X}/\mathcal{A}}(\mathcal{Y}).$$

Therefore, we obtain equivalences

$$\begin{aligned} \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathrm{Day}_{\mathcal{C},\mathcal{D}}) &\simeq \mathrm{Alg}_{/\mathcal{A}}(\mathcal{A} \times_{\mathcal{O}} \mathrm{Day}_{\mathcal{C},\mathcal{D}}) \\ &\simeq \mathrm{Alg}_{/\mathcal{A}}(\mathrm{Day}_{\mathcal{A}\times_{\mathcal{O}}\mathcal{C},\mathcal{A}\times_{\mathcal{O}}\mathcal{D}}) \\ &\simeq \mathrm{Alg}_{\mathcal{A}\times_{\mathcal{O}}\mathcal{C}/\mathcal{A}}(\mathcal{A} \times_{\mathcal{O}} \mathcal{D}) \\ &\simeq \mathrm{Alg}_{\mathcal{A}\times_{\mathcal{O}}\mathcal{C}/\mathcal{O}}(\mathcal{D}). \end{aligned}$$

Specializing to the case $\mathcal{A}^\otimes = \mathrm{Day}^\otimes_{\mathcal{C},\mathcal{D}}$, this provides an evaluation map

$$\mathrm{Day}^\otimes_{\mathcal{C},\mathcal{D}} \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes \to \mathcal{D}^\otimes$$

over $\mathcal{O}^\otimes$ which induces the above equivalence, making it clear that this identification is natural. □

Before embarking on the proof of Theorem 2.7, let us record another easy consequence. We require some additional notation.

Definition 2.9. Define the operads $\mathrm{Cat}^\otimes_{\mathcal{C}/\!/}$ and $\mathrm{Cat}^\otimes_{/\!/\mathcal{D}}$ by the pullbacks

$$\begin{array}{ccc} \mathrm{Cat}^\otimes_{\mathcal{C}/\!/} & \longrightarrow & (\mathrm{Ar}^{\mathrm{opl}})^\times \\ \big\downarrow & & \big\downarrow{\scriptstyle (s,t)} \\ \mathcal{O}^\otimes \times_{\mathrm{Comm}^\otimes} \mathrm{Cat}^\times & \xrightarrow{\mathcal{C}\times\mathrm{id}} & \mathrm{Cat}^\times \times_{\mathrm{Comm}^\otimes} \mathrm{Cat}^\times \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Cat}^\otimes_{/\!/\mathcal{D}} & \longrightarrow & (\mathrm{Ar}^{\mathrm{opl}})^\times \\ \big\downarrow & & \big\downarrow{\scriptstyle (s,t)} \\ \mathrm{Cat}^\times \times_{\mathrm{Comm}^\otimes} \mathcal{O}^\otimes & \xrightarrow{\mathrm{id}\times\mathcal{D}} & \mathrm{Cat}^\times \times_{\mathrm{Comm}^\otimes} \mathrm{Cat}^\times \end{array}$$

Corollary 2.10. *Let* $\alpha : \mathcal{A}^{\otimes} \to \mathcal{O}^{\otimes}$ *be an operad over* $\mathcal{O}^{\otimes}$.

(1) *The functor*

$$t_C : \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathrm{Cat}_{C/\!/}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{O} \times \mathrm{Cat}) \simeq \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat})$$

is a cocartesian fibration which straightens to the functor

$$\mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat}) \to \mathrm{Cat}, \quad \mathcal{Y} \mapsto \mathrm{Alg}_{\mathcal{A} \times_{\mathcal{O}} C/\mathcal{A}}(\mathcal{Y})$$

whose functoriality is given by postcomposition with $\mathcal{A}$*-monoidal functors.*

(2) *The functor*

$$s_D : \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathrm{Cat}_{/\!/D}) \to \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathrm{Cat} \times \mathcal{O}) \simeq \mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat})$$

is a cartesian fibration which straightens to the functor

$$\mathrm{Alg}_{\mathcal{A}}(\mathrm{Cat})^{\mathrm{op}} \to \mathrm{Cat}, \quad \mathcal{X} \mapsto \mathrm{Alg}_{\mathcal{X}/\mathcal{O}}(\mathcal{D})$$

whose functoriality is given by precomposition with $\mathcal{A}$*-monoidal functors.*

Proof. This follows immediately from Theorem 2.7 applied to the base operad $\mathcal{A}^{\otimes}$ together with the naturality of unstraightening. □

The remainder of this section is concerned with the proof of Theorem 2.7. The key input for our argument is the existence of free cartesian fibrations.

Construction 2.11. Let I be a small category and let $f : X \to I$ be a functor. Define the *free cartesian fibration* $\mathrm{Fr}(f)$ on f as the pullback

$$\begin{array}{ccc} \mathrm{Fr}_{\mathrm{cart}}(f) & \longrightarrow & \mathrm{Fun}([1], I) \\ \downarrow & & \downarrow \mathrm{ev}_1 \\ X & \xrightarrow{\;f\;} & I \end{array}$$

together with the evaluation map

$$\mathrm{Fr}_{\mathrm{cart}}(f) \to \mathrm{Fun}([1], I) \xrightarrow{\mathrm{ev}_0} I.$$

Proposition 2.12 [2, Theorem 4.5]. *Let I be a category. The functor* $\mathrm{Cart}(I) \to \mathrm{Cat}_{/I}$ *admits a left adjoint*

$$\mathrm{Fr}_{\mathrm{cart}} : \mathrm{Cat}_{/I} \to \mathrm{Cart}(I)$$

which sends $f : X \to I$ to $\mathrm{Fr}_{\mathrm{cart}}(f)$.

Remark 2.13. Dualizing Proposition 2.12 shows that the pullback

$$\begin{array}{ccc} \mathrm{Fr}_{\mathrm{cocart}}(f) & \longrightarrow & \mathrm{Fun}([1], I) \\ \downarrow & & \downarrow \mathrm{ev}_0 \\ X & \xrightarrow{f} & I \end{array}$$

together with the evaluation map

$$\mathrm{Fr}_{\mathrm{cocart}}(f) \to \mathrm{Fun}([1], I) \xrightarrow{\mathrm{ev}_1} I$$

is the free cocartesian fibration on f.

Proposition 2.14.

(1) *The functor*

$$(s, t) : \mathrm{Cart}([1]) \to \mathrm{Cat} \times \mathrm{Cat}$$

is an orthofibration. A morphism $f : p \to q$ in Cart([1]) *is s-cartesian if and only if $t(f)$ is an equivalence, and f is t-cocartesian if and only if $s(f)$ is an equivalence.*

(2) *The functor*

$$(s, t) : \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat} \times \mathrm{Cat}$$

is an orthofibration. Moreover, the functor $\mathrm{Cart}([1]) \to \mathrm{Ar}^{\mathrm{opl}}$ *preserves both s-cartesian and t-cocartesian morphisms.*

Proof. By unstraightening, $\mathrm{Cart}([1]) \simeq \mathrm{Fun}([1]^{\mathrm{op}}, \mathrm{Cat})$, with s and t corresponding to evaluation at 1 and 0, respectively. By [9, Corollary 2.4.7.11], the evaluation functor at 0 is a cartesian fibration and the evaluation functor at 1 is a cocartesian fibration, and the characterisation of s-cartesian and t-cocartesian morphisms follows from [9, Lemma 2.4.7.5]. The explicit description of s-cartesian and t-cocartesian morphisms also implies that (s, t) is an orthofibration.

For assertion (2), it suffices to show that s-cartesian morphisms in Cart([1]) are also s-cartesian in $\mathrm{Ar}^{\mathrm{opl}}$, and that the same holds true for t-cocartesian morphisms.

So, let $f : p \to q$ be an s-cartesian morphism in Cart([1]), where $p : X \to [1]$ and $q : Y \to [1]$ are cartesian fibrations. For every cartesian fibration $r : Z \to [1]$, we have to show that the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Cat}_{/[1]}}(r, p) & \xrightarrow{f \circ -} & \mathrm{Hom}_{\mathrm{Cat}_{/[1]}}(r, q) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Cat}}(s(r), s(p)) & \xrightarrow{s(f) \circ -} & \mathrm{Hom}_{\mathrm{Cat}}(s(r), s(q)) \end{array}$$

is a pullback. Observing that $s(\mathrm{Fr}_{\mathrm{cart}}(r)) \simeq [1]_{1/} \times s(r) \simeq s(r)$, this square is identified via Proposition 2.12 with the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Cart}([1])}(\mathrm{Fr}_{\mathrm{cart}}(r), p) & \longrightarrow & \mathrm{Hom}_{\mathrm{Cart}([1])}(\mathrm{Fr}_{\mathrm{cart}}(r), q) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Cat}}(s(r), s(p)) & \xrightarrow{s(f) \circ -} & \mathrm{Hom}_{\mathrm{Cat}}(s(r), s(q)) \end{array}$$

which is a pullback by assumption.

Let now $f\colon\ p \to q$ be a t-cocartesian morphism in Cart([1]). Let $r\colon\ Z \to [1]$ be an arbitrary cartesian fibration. Note that $t(\mathrm{Fr}_{\mathrm{cart}}(p)) \simeq X$ and $t(\mathrm{Fr}_{\mathrm{cart}}(q)) \simeq Y$. Using Proposition 2.12 once more, it suffices to show that the outer square in the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Cart}([1])}(\mathrm{Fr}_{\mathrm{cart}}(q), r) & \xrightarrow{-\circ \mathrm{Fr}(f)} & \mathrm{Hom}_{\mathrm{Cart}([1])}(\mathrm{Fr}_{\mathrm{cart}}(p), r) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ \mathrm{Hom}_{\mathrm{Cat}}(Y, t(r)) & \xrightarrow{-\circ f} & \mathrm{Hom}_{\mathrm{Cat}}(X, t(r)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Cat}}(t(q), t(r)) & \xrightarrow{-\circ t(f)} & \mathrm{Hom}_{\mathrm{Cat}}(t(p), t(r)) \end{array} \tag{2.15}$$

is a pullback. Since $s(\mathrm{Fr}_{\mathrm{cart}}(f)) \simeq s(f)$ is an equivalence, the morphism $\mathrm{Fr}_{\mathrm{cart}}(f)$ is t-cocartesian, which means that the top square is a pullback. For the lower square, we use the explicit formula for cartesian unstraightening over $[1]^{\mathrm{op}}$ from [2, Proposition 3.1]: since the left square and outer square in the commutative diagram

$$\begin{array}{ccccc} s(p) & \xrightarrow{\mathrm{St}(p)} & t(p) & \xrightarrow{t(f)} & t(q) \\ {\scriptstyle 1\times\mathrm{id}}\downarrow & & \downarrow & & \downarrow \\ [1]^{\mathrm{op}} \times s(p) & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

are pushouts, so is the right square. This implies that the bottom square in (2.15) is a pullback. Consequently, the outer square in (2.15) is a pullback as required. □

Since we are considering the cartesian symmetric monoidal structures on $\mathrm{Ar}^{\mathrm{opl}}$ and Cat, we can bootstrap the analogous statements for categories of $\mathcal{O}$-algebras from this.

Lemma 2.16. *Let $p = (p_1, p_2)\colon\ X \to Y \times Z$ be a curved orthofibration/an orthofibration and let I be a small category. Then*

$$p_* = ((p_1)_*, (p_2)_*)\colon\ \mathrm{Fun}(I, X) \to \mathrm{Fun}(I, Y) \times \mathrm{Fun}(I, Z)$$

is also a curved orthofibration/an orthofibration. The relevant cartesian and cocartesian morphisms are given by those natural transformations, whose components are all cartesian or cocartesian, respectively.

Proof. This is immediate from [9, Corollary 3.2.2.12]. □

Lemma 2.17. *Let X, Y, and Z be categories with finite products. Suppose that $p = (p_1, p_2)\colon\ X \to Y \times Z$ is a functor such that*

(1) *p is a curved orthofibration;*
(2) *p preserves finite products;*
(3) *finite products of p_1-cartesian morphisms are p_1-cartesian;*
(4) *finite products of p_2-cocartesian morphisms are p_2-cocartesian.*

Then, the induced functor $p_* \colon \mathrm{Mon}_{\mathcal{O}}(X) \to \mathrm{Mon}_{\mathcal{O}}(Y) \times \mathrm{Mon}_{\mathcal{O}}(Z)$ *is a curved orthofibration. A morphism in* $\mathrm{Mon}_{\mathcal{O}}(X)$ *is* $(p_1)_*$*-cartesian or* $(p_2)_*$*-cocartesian if and only if it is* $(p_1)_*$*-cartesian or* $(p_2)_*$*-cocartesian in* $\mathrm{Fun}(\mathcal{O}^{\otimes}, X)$*.*

In particular, if p *is an orthofibration satisfying conditions (1)–(4), then* p_* *is an orthofibration.*

Proof. By Lemma 2.16, the induced functor

$$p_* = ((p_1)_*, (p_2)_*) \colon \mathrm{Fun}(\mathcal{O}^{\otimes}, X) \to \mathrm{Fun}(\mathcal{O}^{\otimes}, Y) \times \mathrm{Fun}(\mathcal{O}^{\otimes}, Z)$$

also satisfies properties (1)–(4).

Let $g \colon N \to N'$ be a morphism in $\mathrm{Mon}_{\mathcal{O}}(Y)$ and let M' be an $\mathcal{O}$-monoid in X lifting N'. Considering g as a morphism in $\mathrm{Fun}(\mathcal{O}^{\otimes}, Y)$, there exists a cartesian lift $f \colon M \to M'$ in $\mathrm{Fun}(\mathcal{O}^{\otimes}, X)$. We claim that M is also an $\mathcal{O}$-monoid in X. For $x = x_1 \boxtimes \cdots \boxtimes x_n$ in $\mathcal{O}^{\otimes}_{\langle n \rangle}$, the Segal maps of M and M' fit into a commutative square

$$\begin{array}{ccc} M(x) & \longrightarrow & \prod_{i=1}^n M(x_i) \\ \downarrow & & \downarrow \\ M'(x) & \xrightarrow{\simeq} & \prod_{i=1}^n M'(x_i) \end{array}$$

Since products of cartesian morphisms in X are cartesian, both vertical arrows are cartesian morphisms. By [9, Lemma 2.4.2.7], it follows that the top horizontal morphism is also cartesian. Since N is an $\mathcal{O}$-monoid, this morphism is a cartesian lift of an equivalence, and therefore itself an equivalence. It follows that $(p_1)_* \colon \mathrm{Mon}_{\mathcal{O}}(X) \to \mathrm{Mon}_{\mathcal{O}}(Y)$ is a cartesian fibration.

Since $(p_1)_*$-cartesian lifts in $\mathrm{Fun}(\mathcal{O}^{\otimes}, X)$ are characterized by being pointwise p_1-cartesian, it also follows that $(p_1)_*$-cartesian morphisms project to equivalences under $(p_2)_*$.

The argument for $(p_2)_*$ is completely analogous.

Since $(p_1)_*$-cartesian and $(p_2)_*$-cocartesian morphisms are detected in $\mathrm{Fun}(\mathcal{O}^{\otimes}, X)$, it follows from Lemma 2.16 that p_* is an orthofibration if p is additionally assumed to be an orthofibration. □

In particular, we obtain the following.

Proposition 2.18. *The functor*

$$(s_*, t_*) \colon \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}) \times \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})$$

is an orthofibration. Both s_**-cartesian and* t_**-cocartesian morphisms are detected by the functor*

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}) \xrightarrow{\sim} \mathrm{Mon}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}) \to \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Ar}^{\mathrm{opl}}).$$

Proof. Due to Proposition 2.14, the functor $(s, t) \colon \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat} \times \mathrm{Cat}$ is an orthofibration satisfying the assumptions of Lemma 2.17. The proposition follows. □

To determine the straightening of (s_*, t_*), we require some additional preparation. First, we observe that the existence of free (co)cartesian fibrations implies the existence of free orthofibrations.

Corollary 2.19. *The inclusion functor* $\mathrm{Ortho}(Y,Z) \to \mathrm{Cat}_{/Y\times Z}$ *admits a left adjoint*

$$\mathrm{Fr}\colon \mathrm{Cat}_{/Y\times Z} \to \mathrm{Ortho}(Y,Z).$$

Proof. We write $p_Z : Y \times Z \to Z$ for the projection functor. Since p_Z is a cartesian fibration, the equivalence $\mathrm{Cat}_{/Y\times Z} \simeq (\mathrm{Cat}_{/Y})/p_Z$ and Proposition 2.12 induce an adjunction

$$\mathrm{Cat}_{/Y\times Z} \rightleftarrows \mathrm{Cart}(Y)/p_Z.$$

After identifying

$$\mathrm{Cart}(Y)/p_Z \simeq \mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{Cat})_{/\,\mathrm{const}_Z} \simeq \mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{Cat}_{/Z}),$$

the existence of free cocartesian fibrations induces an adjunction

$$\mathrm{Cart}(Y)/p_Z \rightleftarrows \mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{Cocart}(Z)) \simeq \mathrm{Ortho}(Y,Z).$$

□

Remark 2.20. Unwinding the proof of Corollary 2.19, one finds that the free orthofibration on a functor $f : X \to Y \times Z$ is given by the pullback

$$\begin{array}{ccc} \mathrm{Fr}(f) & \longrightarrow & \mathrm{Fun}([1],Y)\times\mathrm{Fun}([1],Z) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_1\times\mathrm{ev}_0} \\ X & \xrightarrow{\;f\;} & Y\times Z \end{array}$$

together with the evaluation map

$$\mathrm{Fr}(f) \to \mathrm{Fun}([1],Y)\times\mathrm{Fun}([1],Z) \xrightarrow{\mathrm{ev}_0\times\mathrm{ev}_1} Y\times Z.$$

One can adapt the proof of [2, Theorem 4.5] to show directly that this yields a left adjoint to the functor $\mathrm{Ortho}(Y,Z) \to \mathrm{Cat}_{/Y\times Z}$.

We can apply this statement to identify the straightenings of cotensors of orthofibrations.

Lemma 2.21. *Let* $p : X \to Y \times Z$ *be an orthofibration. Consider the pullback*

$$\begin{array}{ccc} X^I & \longrightarrow & \mathrm{Fun}(I,X) \\ {\scriptstyle p^I}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y\times Z & \xrightarrow{\mathrm{const}} & \mathrm{Fun}(I, Y\times Z) \end{array}$$

Then, $p^I = (p_1^I, p_2^I)$ is an orthofibration such that both p_1^I-cartesian and p_2^I-cocartesian morphisms are detected componentwise in Fun(I, X)*. Moreover, p^I straightens to the functor*

$$\mathrm{Fun}(I, \mathrm{St}(p))\colon Y^{\mathrm{op}} \times Z \to \mathrm{Cat}.$$

Proof. The first part of the lemma is precisely Lemma 2.16, so we only have to prove the assertion about the straightening of p^I. For every functor $f\colon T \to Y \times Z$, there exist by Corollary 2.19 natural equivalences

$$\begin{aligned}
\mathrm{Hom}_{Y\times Z}(T, X^I) &\simeq \mathrm{Hom}_{Y\times Z}(T \times I, X)\\
&\simeq \mathrm{Hom}_{\mathrm{Ortho}(Y,Z)}(\mathrm{Fr}(T \times I \to Y \times Z), X)\\
&\simeq \mathrm{Hom}_{\mathrm{Ortho}(Y,Z)}(\mathrm{Fr}(f) \times I, X)\\
&\simeq \mathrm{Nat}(\mathrm{St}(\mathrm{Fr}(f)) \times I, \mathrm{St}(p))\\
&\simeq \mathrm{Nat}(\mathrm{St}(\mathrm{Fr}(f)), \mathrm{Fun}(I, \mathrm{St}(p)))\\
&\simeq \mathrm{Hom}_{\mathrm{Ortho}(Y,Z)}(\mathrm{Fr}(f), \mathrm{Un}(\mathrm{Fun}(I, \mathrm{St}(p))))\\
&\simeq \mathrm{Hom}_{Y\times Z}(T, \mathrm{Un}(\mathrm{Fun}(I, \mathrm{St}(p)))),
\end{aligned}$$

which implies the lemma. □

Recall the following definition from [3, Proposition 2.3.13].

Definition 2.22. A functor $p = (p_1, p_2)\colon X \to Y \times Z$ is a *bifibration* if the following conditions are satisfied:

(1) p_1 is a cartesian fibration such that a morphism in X is p_1-cartesian if and only if it projects to an equivalence under p_2;
(2) p_2 is a cocartesian fibration such that a morphism in X is p_2-cocartesian if and only if it projects to an equivalence under p_1.

By [3, Corollary 2.3.15], the equivalences from (2.6) restrict to equivalences

$$\mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{LFib}(Z)) \simeq \mathrm{Bifib}(Y, Z) \simeq \mathrm{Fun}(Z, \mathrm{RFib}(Y)),$$

where $\mathrm{Bifib}(Y, Z) \subseteq \mathrm{Ortho}(Y, Z)$ denotes the full subcategory of bifibrations, and $\mathrm{LFib}(Z)$ and $\mathrm{RFib}(Y)$ denote the categories of left fibrations over Z and right fibrations over Y, respectively.

Example 2.23. The functor $(\mathrm{ev}_1, \mathrm{ev}_0)\colon\ \mathrm{Fun}([1]^{\mathrm{op}}, X) \to X$ is a bifibration for every category X—the special case $X = \mathrm{Cat}$ was covered in Proposition 2.14. Moreover, this functor straightens to the functor

$$\mathrm{Hom}_X\colon X^{\mathrm{op}} \times X \to \mathrm{An}.$$

This follows for example from [5, Corollary A.2.5] because the cartesian unstraightening of Hom_X is the twisted arrow category.

Definition 2.24. Let $p = (p_1, p_2) : X \to Y \times Z$ be an orthofibration. Define X_{bicart} as the wide subcategory of X generated by the collections of p_1-cartesian and p_2-cocartesian morphisms.

Lemma 2.25. *Let $p = (p_1, p_2) : X \to Y \times Z$ be an orthofibration.*

(1) *The following are equivalent for a morphism f in X:*
   (a) *f lies in X_{bicart};*
   (b) *f is the composition of a p_1-cartesian morphism followed by a p_2-cocartesian morphism;*
   (c) *f is the composition of a p_2-cocartesian morphism followed by a p_1-cartesian morphism.*

(2) *The restriction $p_{\mathrm{bicart}} : X_{\mathrm{bicart}} \to Y \times Z$ of p is a bifibration. The inclusion functors $X_{\mathrm{bicart}} \to X$ assemble to the counit transformation of an adjunction*

$$\mathrm{inc} : \mathrm{Bifib}(Y, Z) \rightleftarrows \mathrm{Ortho}(Y, Z) \; : (-)_{\mathrm{bicart}}.$$

(3) *The bifibration p_{bicart} straightens to the functor*

$$Y^{\mathrm{op}} \times Z \xrightarrow{\mathrm{St}(p)} \mathrm{Cat} \xrightarrow{\iota} \mathrm{An}.$$

Proof. By [3, Definition 2.3.10], p_1-cartesian morphisms canonically commute with p_2-cocartesian morphisms, which shows assertion (1).

For assertion (2), let us first check that $(p_{\mathrm{bicart}})_1 : X_{\mathrm{bicart}} \to Y$ is a cartesian fibration. Every morphism in Y admits a p_1-cartesian lift, so this reduces to checking that for a p_1-cartesian morphism $\xi : x \to x'$ in X, an arbitrary morphism $\alpha : a \to x$ lies in X_{bicart} if and only if $\xi \circ \alpha$ lies in X_{bicart}. Writing α as the composite of a p_2-cocartesian morphism followed by a p_1-cartesian morphism, this is immediate from [9, Lemma 2.4.2.7]. In particular, every $(p_{\mathrm{bicart}})_1$-cartesian morphism projects to an equivalence under $(p_{\mathrm{bicart}})_2$.

Suppose now that $\xi : x \to x'$ is a morphism in X_{bicart} such that $p_2(\xi)$ is an equivalence. Writing $\xi = \xi_{\mathrm{cocart}} \circ \xi_{\mathrm{cart}}$ as a composition of a p_1-cartesian morphism followed by a p_2-cocartesian morphism, it follows that $p_2(\xi_{\mathrm{cocart}})$ is an equivalence. Hence, ξ_{cocart} is a p_2-cocartesian lift of an equivalence, and thus an equivalence. It follows that ξ is p_1-cartesian, and therefore also $(p_{\mathrm{bicart}})_1$-cartesian.

By dualizing, we see that $(p_{\mathrm{bicart}})_2 : X \to Z$ is a cocartesian fibration such that a morphism is $(p_{\mathrm{bicart}})_2$-cocartesian if and only if it projects to an equivalence under $(p_{\mathrm{bicart}})_1$. Hence, p_{bicart} is a bifibration.

Now, let $T \to Y \times Z$ be a bifibration. Note that $T_{\mathrm{bicart}} = T$ because every morphism (g, h) in $Y \times Z$ factors as $(g, \mathrm{id}) \circ (\mathrm{id}, h)$. Since morphisms in $\mathrm{Ortho}(Y, Z)$ preserve all relevant cartesian and cocartesian morphisms, it is now immediate that the inclusion $X_{\mathrm{bicart}} \to X$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Bifib}(Y,Z)}(T, X_{\mathrm{bicart}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ortho}(Y,Z)}(T, X).$$

Assertion (3) follows from the commutative diagram:

$$\begin{array}{ccccc} \mathrm{Bifib}(Y,Z) & \xrightarrow{\sim} & \mathrm{Fun}(Z,\mathrm{RFib}(Y)) & \xrightarrow{\sim} & \mathrm{Fun}(Z,\mathrm{Fun}(Y^{\mathrm{op}},\mathrm{An})) \\ \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow{\scriptstyle \mathrm{inc}} \\ \mathrm{Ortho}(Y,Z) & \xrightarrow{\sim} & \mathrm{Fun}(Z,\mathrm{Cart}(Y)) & \xrightarrow{\sim} & \mathrm{Fun}(Z,\mathrm{Fun}(Y^{\mathrm{op}},\mathrm{Cat})) \end{array}$$

by passing to right adjoints. □

Finally, recall that slice categories of Op are cotensored over Cat as follows. For I a small category and $\varphi: \mathcal{X}^{\otimes} \to \mathcal{B}^{\otimes}$ an operad map, define $\mathrm{Fun}(I,\mathcal{X})^{\otimes}$ as the pullback

$$\begin{array}{ccc} \mathrm{Fun}(I,\mathcal{X})^{\otimes} & \longrightarrow & \mathrm{Fun}(I,\mathcal{X}^{\otimes}) \\ \downarrow & & \downarrow{\scriptstyle \phi_*} \\ \mathcal{B}^{\otimes} & \xrightarrow{\mathrm{const}} & \mathrm{Fun}(I,\mathcal{B}^{\otimes}) \end{array}$$

This operad has the universal property that

$$\mathrm{Alg}_{\mathcal{A}/\mathcal{B}}(\mathrm{Fun}(I,\mathcal{X})) \simeq \mathrm{Fun}(I,\mathrm{Alg}_{\mathcal{A}/\mathcal{B}}(\mathcal{X}))$$

for every operad $\mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ over $\mathcal{B}^{\otimes}$ [8, Remark 2.1.3.4].

If $\pi: \mathcal{X}^{\otimes} \to \mathcal{X}$ is a cartesian structure, one checks directly that

$$\mathrm{Fun}(I,\mathcal{X})^{\otimes} \to \mathrm{Fun}(I,\mathcal{X}^{\otimes}) \xrightarrow{\pi\circ -} \mathrm{Fun}(I,\mathcal{X})$$

exhibits $\mathrm{Fun}(I,\mathcal{X})^{\otimes}$ as a cartesian structure on $\mathrm{Fun}(I,\mathcal{X})$. In particular, one obtains the cartesian symmetric monoidal structure

$$\mathrm{Fun}(I,\mathrm{Ar}^{\mathrm{opl}})^{\times} \to \mathrm{Comm}^{\otimes}$$

by applying this construction to the operad $(\mathrm{Ar}^{\mathrm{opl}})^{\times} \to \mathrm{Comm}^{\otimes}$.

Proof of Theorem 2.7

We can now finish the proof of our main result. By Proposition 2.18, the functor

$$(s_*,t_*): \ \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}) \times \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})$$

is an orthofibration. In combination with Proposition 2.14, we obtain a characterisation of the s_*-cartesian morphisms as those morphisms which map under

$$\mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}) \to \mathrm{Fun}(\mathcal{O}^{\otimes},\mathrm{Ar}^{\mathrm{opl}})$$

to a natural transformation whose components all preserve cartesian morphisms and project to an equivalence under $t: \ \mathrm{Ar}^{\mathrm{opl}} \to \mathrm{Cat}$. Analogously for t_*-cocartesian morphisms.

We will identify the composite of the straightening of (s_*, t_*) with the Yoneda embedding $よ : \mathrm{Cat} \to \mathcal{P}(\mathrm{Cat})$. By Lemma 2.21, the composite

$$\begin{aligned}\mathrm{Cat}^{\mathrm{op}} &\xrightarrow{(s_*,t_*)^{(-)}} \mathrm{Ortho}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}), \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}))\\ &\xrightarrow{\mathrm{St}} \mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})^{\mathrm{op}} \times \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat}), \mathrm{Cat})\\ &\xrightarrow{\iota\circ -} \mathrm{Fun}(\mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})^{\mathrm{op}}, \mathrm{An})\end{aligned}$$

corresponds to $よ \circ \mathrm{St}(s_*, t_*)$ after currying. By virtue of Lemma 2.25, the composite $(\iota\circ -)\circ \mathrm{St}$ is equivalent to the functor that first applies $(-)_{\mathrm{bicart}}$ and then straightens the resulting bifibration. This leaves us with identifying the bifibrations $(s_*, t_*)^I_{\mathrm{bicart}}$.

Consider the natural fully faithful functor

$$\begin{aligned}\Psi' : \mathrm{Fun}(I, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}})) &\simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Fun}(I, \mathrm{Ar}^{\mathrm{opl}}))\\ &\xrightarrow{\sim} \mathrm{Mon}_{\mathcal{O}}(\mathrm{Fun}(I, \mathrm{Ar}^{\mathrm{opl}}))\\ &\subseteq \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathrm{Fun}(I, \mathrm{Ar}^{\mathrm{opl}})) \simeq \mathrm{Fun}(\mathcal{O}^{\otimes} \times I, \mathrm{Ar}^{\mathrm{opl}}).\end{aligned}$$

Then $\mathrm{Fun}(I, s_*)$-cartesian morphisms in the domain correspond precisely to those natural transformations τ in the target with the property that for all $x \in \mathcal{O}^{\otimes}$ and $i \in I$, the functor $\tau(x, i)$ preserves cartesian morphisms over $[1]$ and $t(\tau(x, i))$ is an equivalence. From the analogous assertion for $\mathrm{Fun}(I, t_*)$-cocartesian morphisms, it follows that Ψ' restricts to a fully faithful functor

$$\mathrm{Fun}(I, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}))_{\mathrm{bicart}} \to \mathrm{Fun}(\mathcal{O}^{\otimes} \times I, \mathrm{Ar}^{\mathrm{opl}})^{\mathrm{cart}}.$$

Composing with the natural equivalence of Proposition 2.3, we obtain a fully faithful functor

$$\Psi : \mathrm{Fun}(I, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}))_{\mathrm{bicart}} \to \mathrm{Fun}([1]^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I))^{\mathrm{cocart}}.$$

Lemma 2.26. *The essential image of Ψ comprises of those functors*

$$E : [1]^{\mathrm{op}} \to \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I)$$

satisfying the following conditions:

(1) *for $k = 0, 1$ and $i \in I$, the functor $E(k)_i \to \mathcal{O}^{\otimes} \times \{i\}$ is a cocartesian fibration of operads;*
(2) *the functor $E(1)_i \to E(0)_i$ preserves inert morphisms for every $i \in I$.*

Proof. A functor $M : \mathcal{O}^{\otimes} \to \mathrm{Fun}(I, \mathrm{Ar}^{\mathrm{opl}})$ is an $\mathcal{O}$-monoid if and only if the evaluation $M_i : \mathcal{O}^{\otimes} \to \mathrm{Ar}^{\mathrm{opl}}$ is an $\mathcal{O}$-monoid for all $i \in I$. Recall that M_i being an $\mathcal{O}$-monoid means that for every $x = x_1 \boxtimes \cdots \boxtimes x_n \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, the appropriate inert morphisms induce equivalences

$$\rho : M_i(x) \xrightarrow{\sim} M_i(x_1) \underset{[1]}{\times} \cdots \underset{[1]}{\times} M_i(x_n)$$

of cartesian fibrations over [1]. This is the case if and only if each ρ preserves cartesian morphisms and induces fiberwise equivalences.

Denote by $E: [1]^{\mathrm{op}} \to \mathrm{Cocart}(\mathcal{O}^{\otimes} \times I)$ the image of M under Ψ, and let $E(k)_i$ be the restriction of $E(k)$ to $\mathcal{O}^{\otimes} \times \{i\}$. The cocartesian fibration $E(k)_i \to \mathcal{O}^{\otimes}$ is given by the unstraightening of the composite $\mathcal{O}^{\otimes} \to \mathrm{Fun}(I, \mathrm{Ar}^{\mathrm{opl}}) \xrightarrow{ev_i} \mathrm{Ar}^{\mathrm{opl}} \xrightarrow{(-)|_{\{k\}}} \mathrm{Cat}$, which is an $\mathcal{O}$-monoid because ev_i preserves products. Consequently, each map ρ is a fiberwise equivalence if and only if both $E(0)_i$ and $E(1)_i$ are cocartesian fibrations of operads [8, Example 2.4.2.4]. Since inert morphisms in $E(k)_i$ are precisely the cocartesian lifts of inert morphisms in $\mathcal{O}^{\otimes} \times \{i\}$, Lemma 2.4 shows that ρ preserves all cartesian morphisms if and only if $E(1)_i \to E(0)_i$ preserves all inert morphisms. □

Note that Ψ fits into a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}(I, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}))_{\mathrm{cart}} & \xrightarrow{\Psi} & \mathrm{Fun}([1]^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I))^{\mathrm{cocart}} \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (\mathrm{ev}_1, \mathrm{ev}_0)} \\ \mathrm{Fun}(I, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})) \times \mathrm{Fun}(I, \mathrm{Alg}_{\mathcal{O}}(\mathrm{Cat})) & \xrightarrow{\Phi} & \mathrm{Cocart}(\mathcal{O}^{\otimes} \times I) \times \mathrm{Cocart}(\mathcal{O}^{\otimes} \times I) \end{array}$$

with both Φ and Ψ fully faithful. The essential image of Φ comprises precisely of those pairs of functors whose restriction to $\mathcal{O}^{\otimes} \times \{i\}$ is a cocartesian fibration of operads for every $i \in I$. In particular, this induces a natural fully faithful functor Ψ_I from $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}})^I_{\mathrm{bicart}}$ to the pullback of

$$\begin{array}{ccc} & & \mathrm{Fun}([1]^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I))^{\mathrm{cocart}} \\ & & \downarrow{\scriptstyle (\mathrm{ev}_1, \mathrm{ev}_0) =: \varepsilon} \\ \mathrm{Cocart}(\mathcal{O}^{\otimes}) \times \mathrm{Cocart}(\mathcal{O}^{\otimes}) & \xrightarrow{(-\times I)\times(-\times I)} & \mathrm{Cocart}(\mathcal{O}^{\otimes} \times I) \times \mathrm{Cocart}(\mathcal{O}^{\otimes} \times I) \end{array}$$

The right vertical evaluation functor is the pullback of

$$(\mathrm{ev}_1, \mathrm{ev}_0): \ \mathrm{Fun}([1]^{\mathrm{op}}, \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I)) \to \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I) \times \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I),$$

along the inclusion functor

$$\mathrm{Cocart}(\mathcal{O}^{\otimes} \times I) \times \mathrm{Cocart}(\mathcal{O}^{\otimes} \times I) \to \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I) \times \mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I).$$

Consequently, the naturality of unstraightening together with Example 2.23 implies that this pullback of ε straightens to the functor

$$\mathrm{Hom}_{\mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I)}(- \times I, - \times I): \ \mathrm{Cocart}(\mathcal{O}^{\otimes}) \times \mathrm{Cocart}(\mathcal{O}^{\otimes}) \to \mathrm{An}.$$

Observe in addition that

$$\mathrm{Hom}_{\mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes} \times I)}(- \times I, - \times I) \simeq \mathrm{Hom}_{\mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes})}(- \times I, -).$$

It follows from Lemma 2.26 that Ψ_I identifies the straightening of $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}})^I_{\mathrm{bicart}}$ with the full subfunctor of $\mathrm{Hom}_{\mathrm{Cocart}^{\mathrm{lax}}(\mathcal{O}^{\otimes})}(- \times I, -)$ given by those functors $\mathcal{X}^{\otimes} \times I \to \mathcal{Y}^{\otimes}$ such that $\mathcal{X}^{\otimes} \times$

$\{i\} \to \mathcal{Y}^{\otimes}$ is an operad map for all $i \in I$. After currying, we obtain a natural equivalence

$$\mathrm{St}((s_*, t_*)^I_{\mathrm{bicart}}) \simeq \mathrm{Hom}_{\mathrm{Cat}}(I, \mathrm{Alg}_{\mathcal{X}/\mathcal{O}}(\mathcal{Y})),$$

which is precisely what we needed to show. Theorem 2.7 is now proved.

3 | VARIATION: DAY CONVOLUTION IN SUBOPERADS OF $\mathbf{Cat}^{\times}$

From the preceding results, one can deduce analogous assertions for certain symmetric monoidal categories which arise as suboperads of $\mathrm{Cat}^{\times}$. Consider a subcategory $\mathcal{U}$ of Cat which is closed under finite products. If U_1, U_2 and T are objects in $\mathcal{U}$, call a functor $F : U_1 \times U_2 \to T$ *$\mathcal{U}$-biexact* if both $F(u_1, -) : U_2 \to T$ and $F(-, u_2) : U_1 \to T$ are morphisms in $\mathcal{U}$ for all $u_1 \in U_1$ and $u_2 \in U_2$. There is an evident notion of a *$\mathcal{U}$-multiexact* functor for functors in more than two variables.

Assume that

(1) for each pair U_1 and U_2 of objects in $\mathcal{U}$, there exists an initial $\mathcal{U}$-biexact functor $U_1 \times U_2 \to U_1 \otimes U_2$;
(2) there exists a category $U \in \mathcal{U}$ and an object $u \in U$ such that evaluation at u induces an equivalence $\mathrm{Hom}_{\mathcal{U}}(U, T) \xrightarrow{\sim} \iota T$.

One example of a subcategory satisfying these conditions is the category $\mathrm{Cat}^{\mathrm{st}}$ of stable categories and exact functors.

Under these assumptions, $\mathcal{U}$ refines to a symmetric monoidal category by considering the suboperad $\mathcal{U}^{\otimes}$ of $\mathrm{Cat}^{\times}$ determined by the following conditions:

(1) the underlying category of $\mathcal{U}^{\otimes}$ is $\mathcal{U}$;
(2) morphisms $U_1 \boxtimes \cdots \boxtimes U_n \to T$ over the active map $\langle n \rangle \to \langle 1 \rangle$ correspond to $\mathcal{U}$-multiexact functors $U_1 \times \cdots \times U_n \to T$.

Observe that $\mathcal{O}$-algebras in $\mathcal{U}^{\otimes}$ correspond under unstraightening to cocartesian fibrations of operads over $\mathcal{O}^{\otimes}$ whose fibers lie in $\mathcal{U}$ and whose cocartesian transport functors are $\mathcal{U}$-multiexact.

Consider now the suboperad $(\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{U}})^{\otimes}$ of $(\mathrm{Ar}^{\mathrm{opl}})^{\times}$ determined by the following properties:

(1) objects in the underlying category $\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{U}}$ are given by cartesian fibrations $X \to [1]$ which straighten to functors $[1]^{\mathrm{op}} \to \mathcal{U}$;
(2) morphisms are precisely those morphisms in $(\mathrm{Ar}^{\mathrm{opl}})^{\times}$ which map to the suboperad $\mathcal{U}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{U}^{\otimes}$ under (s, t).

As before, given $\mathcal{O}$-algebras $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{U}$, we let $\mathcal{U}^{\otimes}_{\mathcal{C}/\!\!/}$ and $\mathcal{U}^{\otimes}_{/\!\!/\mathcal{D}}$ be given by the following pullbacks:

$$\begin{array}{ccc} \mathcal{U}^{\otimes}_{\mathcal{C}/\!\!/} & \longrightarrow & (\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{U}})^{\otimes} \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{O}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{U}^{\otimes} & \xrightarrow{\mathcal{C}\times\mathrm{id}} & \mathcal{U}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{U}^{\otimes} \end{array} \qquad \begin{array}{ccc} \mathcal{U}^{\otimes}_{/\!\!/\mathcal{D}} & \longrightarrow & (\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{U}})^{\otimes} \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{U}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{O}^{\otimes} & \xrightarrow{\mathrm{id}\times\mathcal{D}} & \mathcal{U}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{U}^{\otimes} \end{array}$$

As a final piece of notation, denote by $\mathrm{Alg}^{\mathcal{V}}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ the full subcategory of $\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$ spanned by those operad maps $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$ such that $\mathcal{C}^{\otimes}_x \to \mathcal{D}^{\otimes}_x$ is a morphism in $\mathcal{V}$ for every $x \in \mathcal{O}^{\otimes}$.

Proposition 3.1. *Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be cocartesian fibrations of operads corresponding to $\mathcal{O}$-algebras in $\mathcal{V}$.*

(1) *The functor*

$$(s_*, t_*) : \ \mathrm{Alg}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{V}}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

is an orthofibration which straightens to the functor

$$(\mathcal{X}, \mathcal{Y}) \mapsto \mathrm{Alg}^{\mathcal{V}}_{\mathcal{X}/\mathcal{O}}(\mathcal{Y}).$$

(2) *For every operad $\mathcal{A}^{\otimes} \to \mathcal{O}^{\otimes}$ over $\mathcal{O}^{\otimes}$, the functor*

$$t_{\mathcal{C}} : \ \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{V}_{\mathcal{C}/\!/}) \to \mathrm{Alg}_{\mathcal{A}}(\mathcal{V})$$

is a cocartesian fibration which straightens to the functor

$$\mathcal{Y} \mapsto \mathrm{Alg}^{\mathcal{V}}_{\mathcal{A} \times_{\mathcal{O}} \mathcal{C}/\mathcal{A}}(\mathcal{Y}).$$

(3) *For every operad $\mathcal{A}^{\otimes} \to \mathcal{O}^{\otimes}$ over $\mathcal{O}^{\otimes}$, the functor*

$$s_{\mathcal{D}} : \ \mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{V}_{/\!/\mathcal{D}}) \to \mathrm{Alg}_{\mathcal{A}}(\mathcal{V})$$

is a cartesian fibration which straightens to the functor

$$\mathcal{X} \mapsto \mathrm{Alg}^{\mathcal{V}}_{\mathcal{X}/\mathcal{O}}(\mathcal{D}).$$

Proof. The operad $(\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{V}})^{\otimes}$ can be constructed in two steps. Consider first the pullback

$$\begin{array}{ccc}
(\widetilde{\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{V}}})^{\otimes} & \longrightarrow & (\mathrm{Ar}^{\mathrm{opl}})^{\times} \\
\downarrow & & \downarrow{\scriptstyle (s,t)} \\
\mathcal{V}^{\otimes} \times_{\mathrm{Comm}^{\otimes}} \mathcal{V}^{\otimes} & \longrightarrow & \mathrm{Cat}^{\times} \times_{\mathrm{Comm}^{\otimes}} \mathrm{Cat}^{\times}
\end{array}$$

Then, $(\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{V}})^{\otimes}$ is the full subcategory of $(\widetilde{\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{V}}})^{\otimes}$ spanned by objects corresponding to tuples of cartesian fibrations over $[1]$, each of which straightens to a functor $[1]^{\mathrm{op}} \to \mathcal{V}$. Consequently, Theorem 2.7 implies that

$$(\widetilde{s}_*, \widetilde{t}_*) : \ \mathrm{Alg}_{\mathcal{O}}\left(\widetilde{\mathrm{Ar}^{\mathrm{opl}}_{\mathcal{V}}}\right) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \times \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

is an orthofibration which straightens to the functor

$$(\mathcal{X}, \mathcal{Y}) \mapsto \mathrm{Alg}_{\mathcal{X}/\mathcal{O}}(\mathcal{D}).$$

By construction, the fiber of $(\mathrm{Ar}_{\mathcal{V}}^{\mathrm{opl}})^{\otimes}$ over $(\mathcal{X}, \mathcal{Y})$ is precisely the full subcategory $\mathrm{Alg}_{\mathcal{X}/\mathcal{A}}^{\mathcal{V}}(\mathcal{Y})$, and both the $\widetilde{s}_*$-cartesian and $\widetilde{t}_*$-cocartesian transport functors along morphisms in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ preserve these full subcategories. This identifies $\mathrm{St}(s_*, t_*)$ as the correct subfunctor.

Assertions (2) and (3) follow as before from (1). □

4 | DAY CONVOLUTION AS AN $\mathcal{O}$-MONOIDAL CATEGORY

Fix a base operad $\mathcal{O}^{\otimes}$ as well as $\mathcal{O}$-monoidal categories $\mathcal{C}$ and $\mathcal{D}$. In this section, we reprove a well-known statement, see [8, Proposition 2.2.6.16], which is key for working with the Day convolution operad. We include a proof to demonstrate that $\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes}$ is a feasible description of Day convolution.

For every operation $\varphi \in \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_i, y)$, denote the associated tensor functors by $\otimes_{\varphi}^{\mathcal{C}} : \prod_i \mathcal{C}(x_i) \to \mathcal{C}(y)$ and $\otimes_{\varphi}^{\mathcal{D}} : \prod_i \mathcal{D}(x_i) \to \mathcal{D}(y)$.

Proposition 4.1. *For each $y \in \mathcal{O}$, consider the following collection of slice categories:*

$$\mathcal{K}(y) := \left\{ \otimes_{\psi}^{\mathcal{C}}/c \mid \psi \in \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_i, y),\ c \in \mathcal{C}(y) \right\}$$

Assume the following is true:

1. *for all $y \in \mathcal{O}$, the category $\mathcal{D}(y)$ admits all $\mathcal{K}(y)$-shaped colimits;*
2. *for every operation $\varphi \in \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_i, y)$ and every j, the associated tensor functor $\otimes_{\varphi}^{\mathcal{D}} : \prod_i \mathcal{D}(x_i) \to \mathcal{D}(y)$ preserves all $\mathcal{K}(x_j)$-shaped colimits in the jth component.*

Then, $\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes} \to \mathcal{O}^{\otimes}$ *is a cocartesian fibration of operads.*

Remark 4.2. The assumptions of Proposition 4.1 are for example satisfied if there exists some regular cardinal κ with the property that each $\mathcal{D}(y)$ is κ-cocomplete, every tensor functor of $\mathcal{D}$ preserves κ-small colimits in each variable, and each $\mathcal{C}(y)$ is κ-small, reproducing [8, Proposition 2.2.6.16].

Proof of Proposition 4.1. By construction, $\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes}$ is an operad, so [8, Proposition 2.1.2.12] shows that we only have to check that $\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration. As in [8, Section 2.2.6], the crucial part of the argument lies in identifying mapping anima in $\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes}$.

We require some notation. Let $\pi : \mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes} \to \mathcal{O}^{\otimes}$ and $u : \mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes} \to (\mathrm{Ar}^{\mathrm{opl}})^{\times}$ denote the projection functors, and abbreviate $\mathcal{X}^{\times} := \mathrm{Cat}^{\times} \times_{\mathrm{Comm}^{\otimes}} \mathrm{Cat}^{\times}$. Note that then $\mathcal{X} = \mathrm{Cat} \times \mathrm{Cat}$. Let $\varphi : x \to y$ be a morphism in $\mathcal{O}^{\otimes}$, let $F \in \pi^{-1}(x)$ and $G \in \pi^{-1}(y)$, and denote by $\mathrm{Hom}_{\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes}}^{\varphi}(F, G)$ the anima of morphisms lying over φ. Since $\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes}$ is defined as a pullback, we have a natural pullback square

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Day}_{\mathcal{C},\mathcal{D}}^{\otimes}}^{\phi}(F, G) & \longrightarrow & \mathrm{Hom}_{(\mathrm{Ar}^{\mathrm{opl}})^{\times}}(uF, uG) \\
\big\downarrow & & \big\downarrow \\
* & \xrightarrow{(\mathcal{C},\mathcal{D}) \circ \phi} & \mathrm{Hom}_{\mathcal{X}^{\times}}((s,t)F, (s,t)G)
\end{array}$$

In particular, the objects sF, tF, sG, and tG are identified with $\mathcal{C}(x)$, $\mathcal{D}(x)$, $\mathcal{C}(y)$, and $\mathcal{D}(y)$, respectively. Denoting by $\alpha : \langle k\rangle \to \langle l\rangle$ the image of φ in $\mathrm{Comm}^{\otimes}$, the anima $\mathrm{Hom}^{\varphi}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(F,G)$ sits in a natural fiber square

$$\begin{array}{ccc} \mathrm{Hom}^{\phi}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(F,G) & \longrightarrow & \mathrm{Hom}^{\alpha}_{(\mathrm{Ar}^{\mathrm{opl}})^{\times}}(uF,uG) \\ \downarrow & & \downarrow \\ * & \xrightarrow{(\mathcal{C},\mathcal{D})\circ\phi} & \mathrm{Hom}^{\alpha}_{\mathcal{X}^{\times}}((s,t)F,(s,t)G) \end{array}$$

Since both $(s,t) : (\mathrm{Ar}^{\mathrm{opl}})^{\times} \to \mathcal{X}^{\times}$ and $\mathcal{X}^{\times} \to \mathrm{Comm}^{\otimes}$ are cocartesian fibrations, the right vertical map is identified with the map

$$\mathrm{Hom}^{\mathrm{id}_{\langle l\rangle}}_{(\mathrm{Ar}^{\mathrm{opl}})^{\times}_{\langle l\rangle}}(\alpha_!(uF),uG) \to \mathrm{Hom}^{\mathrm{id}_{\langle l\rangle}}_{\mathcal{X}^{\times}_{\langle l\rangle}}(\alpha_!((s,t)F),(s,t)G)$$

induced by (s,t). Write $F = F_1 \boxtimes \ldots \boxtimes F_k$ and $G = G_1 \boxtimes \cdots \boxtimes G_l$. As both $\mathrm{Ar}^{\mathrm{opl}}$ and $\mathcal{X}$ carry the cartesian symmetric monoidal structure, this map is in turn identified with the map

$$\prod_{j=1}^{l} \mathrm{Hom}_{\mathrm{Ar}^{\mathrm{opl}}}\left(\prod_{i\in\alpha^{-1}(j)} F_i, G_j\right) \to \prod_{j=1}^{l} \mathrm{Hom}_{\mathcal{X}}\left(\prod_{i\in\alpha^{-1}(j)} (s,t)F_i, (s,t)G_j\right) \tag{4.3}$$

induced by (s,t). Consequently, it suffices to consider the case that $\alpha : \langle k\rangle \to \langle 1\rangle$ is an active morphism so that $\varphi \in \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_i, y)$.

With respect to the given identifications, the base point $(\mathcal{C},\mathcal{D})\circ\varphi$ now becomes the point in

$$\mathrm{Hom}_{\mathcal{X}}\left(\prod_{i=1}^{k}(\mathcal{C}(x_i),\mathcal{D}(x_i)),(\mathcal{C}(y),\mathcal{D}(y))\right)$$
$$\simeq \mathrm{Hom}_{\mathrm{Cat}}\left(\prod_{i=1}^{k}\mathcal{C}(x_i),\mathcal{C}(y)\right) \times \mathrm{Hom}_{\mathrm{Cat}}\left(\prod_{i=1}^{k}\mathcal{D}(x_i),\mathcal{D}(y)\right)$$

corresponding to the pair of multiplication functors $(\otimes^{\mathcal{C}}_{\varphi},\otimes^{\mathcal{D}}_{\varphi})$ of $\mathcal{C}$ and $\mathcal{D}$.

By Theorem 2.7 and Remark 2.8, the fiber of (4.3) is identified with the anima of natural transformations

$$\mathrm{Nat}\left(\otimes^{\mathcal{D}}_{\varphi}\circ\prod_{i=1}^{k}\mathrm{St}(F_i), \mathrm{St}(G)\circ\otimes^{\mathcal{C}}_{\varphi}\right).$$

Fix now $\varphi : x \to y$ and $F \in \pi^{-1}(x)$. As before, let $\alpha : \langle k\rangle \to \langle l\rangle$ be the image of φ in $\mathrm{Comm}^{\otimes}$ and let $F = F_1 \boxtimes \ldots \boxtimes F_k$ be the canonical decomposition of F with $F_i \in \mathrm{Ar}^{\mathrm{opl}}$. Denote by $\varphi_j \in \mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in\alpha^{-1}(j)}, y_j)$ the active morphisms determined by φ and α. Using assumption (1) and the pointwise formula for left Kan extensions, the composite

$$\prod_{i\in\alpha^{-1}(j)} \mathcal{C}(x_i) \xrightarrow{\prod_i \mathrm{St}(F_i)} \prod_{i\in\alpha^{-1}(j)} \mathcal{D}(x_i) \xrightarrow{\otimes^{\mathcal{D}}_{\varphi_j}} \mathcal{D}(y_j)$$

admits a left Kan extension G_j along $\otimes^{\mathcal{C}}_{\varphi_j} : \prod_{i\in\alpha^{-1}(j)} \mathcal{C}(x_i) \to \mathcal{C}(y_j)$ for each $j \in \langle l\rangle$. As we have seen, the unit transformations

$$\eta_j : \otimes^{\mathcal{D}}_{\varphi_j} \circ \prod_{i\in\alpha^{-1}(j)} \mathrm{St}(F_i) \Rightarrow \mathrm{St}(G_j)\circ \otimes^{\mathcal{C}}_{\varphi_j}$$

determine a point $\eta \in \mathrm{Hom}^{\varphi}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(F, G)$, where $G := G_1 \boxtimes \cdots \boxtimes G_l$. We claim that η is a cocartesian lift of φ.

This amounts to checking that for each $H \in \mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}$, the induced commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(G,H) & \xrightarrow{-\circ\eta} & \mathrm{Hom}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(F,H) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{O}^{\otimes}}(y,z) & \xrightarrow{-\circ\phi} & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(x,z) \end{array}$$

is a pullback, where we set $z := \pi(H)$. This is equivalent to the assertion that for each $\psi \in \mathrm{Hom}_{\mathcal{O}^{\otimes}}(y, z)$, the induced map on vertical fibers

$$-\circ\eta : \mathrm{Hom}^{\psi}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(G,H) \to \mathrm{Hom}^{\psi\varphi}_{\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}}}(F,H) \tag{4.4}$$

is an equivalence. Letting $\beta : \langle l\rangle \to \langle n\rangle$ denote the image of ψ in $\mathrm{Comm}^{\otimes}$, the preliminary discussion and Theorem 2.7 identify this map with the product of the maps

$$\begin{aligned} &\mathrm{Nat}\left(\otimes^{\mathcal{D}}_{\psi_m} \circ \prod_{j\in\beta^{-1}(m)} \left(\mathrm{St}(G_j)\circ\otimes^{\mathcal{C}}_{\varphi_j}\right), \mathrm{St}(H_m)\circ \otimes^{\mathcal{C}}_{\psi_m} \circ \prod_{j\in\beta^{-1}(m)} \otimes^{\mathcal{C}}_{\varphi_j}\right) \\ &\xrightarrow{\eta_j^*} \mathrm{Nat}\left(\otimes^{\mathcal{D}}_{\psi_m} \circ \prod_{j\in\beta^{-1}(m)} \left(\otimes^{\mathcal{D}}_{\varphi_j} \circ \prod_{i\in\alpha^{-1}(j)} \mathrm{St}(F_i)\right), \mathrm{St}(H_m)\circ\otimes^{\mathcal{C}}_{(\psi\varphi)_m}\right) \\ &\simeq \mathrm{Nat}\left(\otimes^{\mathcal{D}}_{(\varphi\psi)_m} \circ \prod_{i\in(\beta\alpha)^{-1}(m)} \mathrm{St}(F_i), \mathrm{St}(H_m)\circ\otimes^{\mathcal{C}}_{(\psi\varphi)_m}\right). \end{aligned}$$

Using assumption (2), the pointwise formula for left Kan extensions implies that the transformation

$$\otimes^{\mathcal{D}}_{\psi_m} \circ \prod_{j\in\beta^{-1}(m)} \left(\otimes^{\mathcal{D}}_{\varphi_j} \circ \prod_{i\in\alpha^{-1}(j)} \mathrm{St}(F_i)\right) \Rightarrow \otimes^{\mathcal{D}}_{\psi_m} \circ \prod_{j\in\beta^{-1}(m)} \left(\mathrm{St}(G_j)\circ\otimes^{\mathcal{C}}_{\varphi_j}\right)$$

induced by η_j also exhibits $\otimes^{\mathcal{D}}_{\psi_m} \circ \prod_{j\in\beta^{-1}(m)} \mathrm{St}(G_j)$ as a left Kan extension, so (4.4) is an equivalence. □

Remark 4.5. In the situation of Proposition 4.1, assume that $\mathcal{O}$ is a symmetric monoidal category, so that $\mathcal{C}$ and $\mathcal{D}$ correspond to lax symmetric monoidal functors $\mathcal{O} \to \mathrm{Cat}$. Unwinding the proof of Proposition 4.1, one obtains the following description of the lax symmetric monoidal functor $\mathcal{O} \to \mathrm{Cat}$ given by the straightening of $\mathrm{Day}^{\otimes}_{\mathcal{C},\mathcal{D}} \to \mathcal{O}^{\otimes}$:

(1) The underlying functor $\mathcal{O} \to \mathrm{Cat}$ sends $x \in \mathcal{O}$ to $\mathrm{Fun}(\mathcal{C}(x), \mathcal{D}(x))$ and a morphism $f : x \to x'$ to the composite

$$\mathrm{Fun}(\mathcal{C}(x), \mathcal{D}(x)) \xrightarrow{f\circ -} \mathrm{Fun}(\mathcal{C}(x), \mathcal{D}(x')) \xrightarrow{f_!} \mathrm{Fun}(\mathcal{C}(x'), \mathcal{D}(x')),$$

where $f_!$ denotes the left Kan extension functor.

(2) For $x, x' \in \mathcal{O}$, the lax monoidal structure map is given by the composite

$$\begin{aligned}\mathrm{Fun}(\mathcal{C}(x), \mathcal{D}(x)) \times \mathrm{Fun}(\mathcal{C}(x'), \mathcal{D}(x')) &\to \mathrm{Fun}(\mathcal{C}(x) \times \mathcal{C}(x'), \mathcal{D}(x) \times \mathcal{D}(x')) \\ &\xrightarrow{\otimes_{\mathcal{D}}\circ -} \mathrm{Fun}(\mathcal{C}(x) \times \mathcal{C}(x'), \mathcal{D}(x \otimes x')) \\ &\xrightarrow{(\otimes_{\mathcal{C}})_!} \mathrm{Fun}(\mathcal{C}(x \otimes x'), \mathcal{D}(x \otimes x')),\end{aligned}$$

where the first arrow arises from the lax symmetric monoidal structure on $\mathrm{Fun} : \mathrm{Cat}^{\mathrm{op}} \times \mathrm{Cat} \to \mathrm{Cat}$, and $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$ denote the tensor operations in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(3) The structure map associated with the monoidal unit is given by the object in $\mathrm{Fun}(\mathcal{C}(\mathbf{1}_{\mathcal{O}}), \mathcal{D}(\mathbf{1}_{\mathcal{O}}))$ which arises as the left Kan extension of $* \xrightarrow{\mathbf{1}_{\mathcal{D}}} \mathcal{D}(\mathbf{1}_{\mathcal{O}})$ along $* \xrightarrow{\mathbf{1}_{\mathcal{C}}} \mathcal{C}(\mathbf{1}_{\mathcal{O}})$.

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