A vanishing Theorem for Donaldson Invariants

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Felix Eberhart

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Die Arbeit wurde angeleitet von:

Erstbetreuer:	Prof. Dr. Raphael Zentner (University of Durham)
Zweitbetreuer:	Prof. Dr. Bernd Ammann (Universität Regensburg)

Prüfungsausschuss:

Vorsitzender:	(1)
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Summary

In this thesis, we investigate the conjecture that rational homology 3-spheres that admit taut foliations must also admit irreducible $SU(2)$-representations.

Rational homology 3-spheres with taut foliations embed into 4-manifolds with non-vanishing Donaldson invariants as separating hypersurface.

The main result of this thesis shows that if a 4-manifold is split by a rational homology 3-sphere whose perturbed $SU(2)$-representation variety (the critical set of the perturbed Chern-Simons functional) has only reducible points, then that 4-manifold's Donaldson invariants must vanish. This establishes the conjecture under an additional assumption about holonomy perturbations.

The proof is based on a compactness argument, a dimension count and the invariance of Donaldson's invariants under certain perturbations. The assumption on reducibility of critical points is reflected in a positive defect in a dimension sum formula, which forces the dimension of certain moduli spaces to be negative.

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1 Introduction

1.1 History and Motivation

The topology of a 3-dimensional manifold Y is essentially determined by its fundamental group. A great deal of progress in the theory of such three-manifold groups has been made by studying the group representations

$$\pi_1(Y) \to G$$

in Lie groups G such as S^1, $SU(2)$ or $Sl_2(\mathbb{C})$. It is a classical result that such representations, up to conjugation, are in one-to-one correspondence with *flat connections* on bundles with structure group G up to bundle automorphisms. These gauge equivalence classes of flat connections are in turn precisely the critical point of the *Chern-Simons functional cs*, defined on the space of connections on bundles over Y up to automorphisms.

From 1985, Andreas Floer noticed that the negative gradient flow of *cs* is closely related to the theory of *instantons* over cylinders of the form $Y \times \mathbb{R}$. These are defined as connections over 4-dimensional manifolds whose curvature is *anti-self-dual* and had been used to spectacular effect some years earlier by Simon Donaldson to prove his celebrated diagonalization theorem ([Don83]). The instanton equation over a cylinder in a certain gauge turns out to be essentially equivalent to the negative gradient flow of Chern-Simons, with the *energy* $\|F_A\|_{L^2}$, where F_A denotes the *curvature* 2-form, of a connection A playing the role of the length of a flow line. In particular, any finite energy instanton over a cylinder tends to flat connections over the ends of the cylinder.

This relation has since been studied extensively (leading to the development of *Floer homology* and motivating the study of *topological quantum field theories* among other developments). A particular climax in this study was reached in 2004, when Kronheimer and Mrowka published their proof of Property P ([KM04]) until then a conjecture borne out as an attempt to make progress towards the (then unproven) Poincaré conjecture: for $K \subset S^3$ a nontrivial knot, the fundamental group of the manifold $S^3_K(1)$ obtained by performing surgery with coefficient 1 along K on S^3, an *integral homology* 3*-sphere*, is always nontrivial. In particular, such a manifold can never be homotopy equivalent to the 3-sphere. Kronheimer and Mrowka in fact show a stronger statement: the fundamental group of $S^3_K(1)$ always admits a representation in $SU(2)$ with non-abelian image.

Their proof is based on the above relation between instantons and the gradient flow of Chern-Simons and a non-vanishing theorem for *Donaldson's polynomial invariants*. These had been introduced in the 1980's as a way to study differential structures on 4-manifolds and provide a systematic way of counting instantons.

To formally define these invariants, one associates to even dimensional homology classes in the base manifold classes in the cohomology of the space $\mathcal{B}(P)$ of connections on a bundle P modulo bundle automorphisms. One then evaluates these cohomology classes on *instanton moduli spaces*, which can equivalently be described as the algebraic count of intersection points of instanton moduli spaces and cycles representing the cohomology classes in $\mathcal{B}(P)$. In particular, these invariants can only be non-vanishing when some instanton moduli spaces are non empty.

Kronheimer-Mrowka's proof proceeds by showing that certain 3-manifolds Y (those admitting *taut foliations*) embed as a separating hypersurface into certain symplectic 4-manifolds. Using results by Feehan and Leness ([FL03]) and Taubes ([Tau94]) they showed that Donaldson's invariants do not vanish for certain symplectic 4-manifolds, among them the manifolds that Y embeds in. Now, using Floer's work, this can be interpreted as a non-trivial pairing between certain elements of the Floer homology of Y (which is generated by certain classes of *irreducible* flat connections) and so, in particular, there must always exist irreducible flat $SU(2)$-connections over Y, corresponding to the non-abelian representations from their theorem.

By contrast to the non-vanishing theorem for Donaldson's invariants that the proof of Property P is based on, Donaldson had shown earlier that his invariants do in fact vanish for manifolds X that are split by a 3-sphere (in other words, if X is a connected sum), $X = X_1 \cup_{S^3} X_2$, where the *self-dual second Betti numbers* $b_2^+(X_1), b_2^+(X_2)$ are both positive ([Don90]). His proof proceeds by *neck stretching* and a dimension count: one inserts longer and longer "necks" $S^3 \times [-L, L]$ into X around the separating S^3. Assuming the invariants did not vanish, one would get instantons for all neck lengths and then, by a compactness argument, a pair of instantons over two manifolds X_1 and X_2 with *cylindrical ends* $S^3 \times (0, \infty)$. Again, essentially (but not explicitly in Donaldsons orginal publication) by the relation between Chern-Simons and instantons over cylinders these limit solutions would have to tend to the only flat connection over S^3 up to gauge, the trivial connection d, over the ends. Now, there is a gluing formula relating the dimensions of the instanton moduli spaces (always denoted $\mathcal{M}$ in this thesis) over these two cylindrical end manifolds and the instanton moduli space on X:

$$\dim \mathcal{M}(X) = \dim \mathcal{M}(X_1) + \dim \mathcal{M}(X_2) + 3.$$

The extra summand 3 here reflects the fact that the action of the automorphism group on the space of connections over S^3 is not free at d: the stabilizer of d has dimension 3 and this is the 3 in the above formula. Using this formula, the definition of Donaldsons invariants by counting elements of moduli spaces and a counting argument that critically depends on the positivity of the stabilizer

summand, one derives a contradiction by showing that at least one of the above dimensions must be negative.

Combining Kronheimer-Mrowka's non-vanishing result for Donaldson invariants of manifolds split by 3-manifolds which admit taut foliations, the fact that flat $SU(2)$-connections with positive-dimensional stabilizer correspond precisely to those representations with abelian image and the dimension counting argument from Donaldson's vanishing theorem, we conjecture:

Conjecture 1.1.1. *Let Y be a rational homology 3-sphere that admits a taut foliation. Then there is a representation*

$$\pi_1(Y) \to SU(2)$$

with non-abelian image.

There are considerable technical difficulties to overcome when trying to prove this. One of the main problems is the failure of Chern-Simons to be Morse-Bott, which reflects the presence of certain singularities in the $SU(2)$-representation variety of Y. The usual method to solve this is to perturb the functional slightly using *holonomy perturbations*.

Over the 3-manifolds of interest in this thesis, there are always holonomy perturbations such that the perturbed Chern-Simons functional is Morse-Bott (i.e. such that the formal Hessian at each critical point becomes an invertible operator). Under additional topological assumptions on a 4-manifold X, there are then further *interior* holonomy perturbations over X such that the perturbed instanton moduli spaces are cut out transversally, that is, that these moduli spaces are smooth manifolds of the expected dimension. We call holonomy perturbations with these properties *regular*.

These perturbations introduce additional complications into the theory. In this thesis, we will prove the following Theorem (Theorem **??**):

Theorem 1.1.2 (Main Theorem)**.** *Let Y be a rational homology 3-sphere and π a regular holonomy perturbation such that all critical points of cs_π are reducible. If Y embeds into a closed, smooth 4-manifold X as a separating hypersurface,*

$$X = X_1 \cup_Y X_2,$$

where $b_1(X_1) = b_1(X_2) = 0$ and $b_2^+(X_1), b_2^+(X_2) > 0$, then the Donaldson invariants of X vanish identically.

Conjecture 1.1.1 does not follow immediately from this theorem together with the non-vanishing result of Kronheimer-Mrowka for 4-manifolds split by rational

homology spheres. For Y as in the Conjecture, there will always be holonomy perturbations such that all critical points of the perturbed Chern-Simons are non-degenerate and such that all instanton moduli spaces are regular enough for our purposes, as we will show in Sections 3 and 4. Additionally, for small enough perturbations, the reducible critical points of the perturbed and unperturbed Chern-Simons will be in one-to-one-correspondence with each other. However, perturbing the functional could introduce additional, necessarily irreducible, critical points in which case our dimension counting argument does not give the desired contradiction. For this reason, we have to make an additional assumption:

Assumption 1. *Let Y be a rational homology 3-sphere such that all representations $\pi_1(Y) \to SU(2)$ have abelian image. Assume that there are arbitrarily small holonomy perturbations for which all critical points of the perturbed Chern-Simons functional remain reducible.*

Under this assumption, using the results of Kronheimer-Mrowka, we would then get Conjecture 1.1.1 as a corollary from Theorem 1.1.2.

Assumption 1 holds for rational homology 3-spheres whose *universal abelian covering*, the normal covering space $\tilde{Y} \to Y$ corresponding to the commutator subgroup $[\pi_1(Y), \pi_1(Y)] \subset \pi_1(Y)$, is a rational homology 3-sphere. Hence, we get as a corollary of Theorem 1.1.2, appearing as Corollary 6.0.2 in the text:

Corollary 1.1.3. *Let Y be a rational homology 3-sphere that admits a taut foliation and whose universal abelian covering is a rational homology 3-sphere, as well.*

Then there is a representation

$$\pi_1(Y) \to SU(2)$$

with non-abelian image.

1.2 Outline of the thesis

In Section 2 we give a brief overview over the general theory of connections, describe the instanton equation and Chern-Simons functional and describe the relation between them. We also give an informal account of the basic analytic inputs involved.

In Section 3 the analysis begins in earnest. We describe the Sobolev completions we will be working in for the rest of the paper, describe the perturbations we are using and show that there are sufficiently many of them for our purposes. In particular, we explain why weighted Sobolev norms are needed to get a Fredholm deformation theory for the (perturbed) instanton equation when there are reducible critical points of our functional.

Section 4 deals with the moduli spaces of perturbed instantons. We give criteria when they are manifolds and enumerate reducibles when they are not, give the dimension formula and describe the compactness properties that these spaces enjoy.

In Section 5, we recollect the definition of Donaldson's invariants and Donaldson-Kronheimer's non-vanishing result for them. We also prove that these invariants can be computed with our perturbed moduli spaces (Theorem 5.1.7).

Finally, Section 6 contains the proof of our main Theorem.

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2 Bundles and Connections

Throughout this section, we will denote by Y a closed, oriented, Riemannian 3-manifold, by G one of the compact Lie groups $SU(2)$, $SO(3)$ or $U(2)$ with Lie algebra $\mathfrak{g}$ and by $P \to Y$ a smooth principal G-bundle over Y (our principal bundles will always be equipped with a *right* G-action). We also denote by

$$\mathfrak{g}(P) \coloneqq P \times_{G,Ad} \mathfrak{g}$$

the *adjoint bundle* (where $Ad\colon G \to \mathrm{Aut}(\mathfrak{g})$ denotes the adjoint action of G on its Lie algebra).

We will also sometimes denote the Lie algebras of $SU(2)$ and $U(2)$ by $\mathfrak{su}(2)$ and $\mathfrak{u}(2)$, respectively. Because the Lie algebra of $SO(3)$ is canonically isomorphic to $\mathfrak{su}(2)$, we do not introduce a new symbol for it and instead denote it by $\mathfrak{su}(2)$ as well. In keeping with this notation, we then also denote the adjoint bundle of principal $SU(2)$- and $SO(3)$-bundles P by $\mathfrak{su}(P)$.

Note that canonically $\mathfrak{u}(2) \cong \mathfrak{su}(2) \oplus i\mathbb{R}$ with $U(2)$ acting trivially on the second summand. We thus get an action $U(2) \to \mathrm{Aut}(\mathfrak{su}(2))$ which we will denote by Ad'. If Q is a principal $U(2)$-bundle, we then write

$$\mathfrak{su}(Q) \coloneqq Q \times_{U(2),Ad'} \mathfrak{su}(2).$$

Sections of this bundle can be identified with the traceless, skew-symmetric $\mathbb{C}$-linear endomorphisms of the associated complex plane bundle $Q \times_{U(2)} \mathbb{C}^2$.

The following is classical, cf. [HH58, Satz 2.5]

Remark 2.0.1. *i) Because Y has a structure as a 3-dimensional simplicial complex, such bundles are determined, up to smooth bundle isomorphism, by certain characteristic numbers:*

> *For $G = SU(2)$, P is trivial.*
>
> *For $G = SO(3)$, P is determined by its second Stiefel-Whitney class $w_2(P) \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$*
>
> *For $G = U(2)$, P is determined by its first Chern class, $c_1(P) \in H^2(Y; \mathbb{Z})$.*

ii) More generally, for a smooth closed manifold M of dimension at most 4, principal $U(2)$- and $SU(2)$-bundles $P \to M$ are determined by their first and second Chern classes $c_1(P)$ and $c_2(P)$ (with c_1 automatically trivial in the case of $SU(2)$-bundles), while principal $SO(3)$-bundles $Q \to M$ are determined by their first Pontryagin class $p_1(Q) \in H^4(M; \mathbb{Z})$, together with

their second Stiefel-Whitney class $w_2(Q) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$*, subject to the relation*

$$p_1(Q) = w_2(Q)^2 \mod 4.$$

The groups we are interested in are linear groups, i.e. they come equipped with canonical faithful representations in $\mathbb{C}^2$ in the case of $G \in \{U(2), SU(2)\}$, resp. in $\mathbb{R}^3$ in the case $G = SO(3)$. Therefore, to any such principal bundle P is canonically associated a vector bundle which we will always denote by E_P (a complex plane bundle for the unitary groups resp. a rank 3 real vector bundle for $SO(3)$).

Throughout this thesis, we will at several points want to have at our disposal the technical convenience of being able to vary the structure group of our bundles. We set this up as follows:

First, the adjoint action of $SU(2)$ on its Lie algebra $\mathfrak{su}(2)$ preserves orientations and the canonical metric defined by the Killing form, so induces a group homomorphism $Ad\colon SU(2) \to SO(\mathfrak{su}(2)) = SO(3)$. We can then define a canonical left action, that we will denote by Ad_*, by

$$\begin{aligned} Ad_*\colon SU(2) \times SO(3) &\to SO(3), \\ (u, g) &\mapsto Ad(u)g. \end{aligned}$$

Second, we have the canonical inclusion $SU(2) \hookrightarrow U(2)$ and consequently a canonical (left) action $m\colon SU(2) \to Aut(U(2))$ given by multiplication inside $U(2)$.

Thus, to an $SU(2)$-bundle P we can canonically associate an $SO(3)$-bundle $P \times_{SU(2),Ad_*} SO(3)$ and a $U(2)$-bundle $P \times_{SU(2),m} U(2)$. Over the three dimensional manifold Y, both of these constructions will necessarily yield trivial bundles.

Third, there is a surjective homomorphism

$$\begin{aligned} a\colon U(2) &\to SO(3) \\ u &\mapsto Ad\left(\frac{1}{\sqrt{det(u)}}u\right), \end{aligned}$$

where $Ad\colon SU(2) \to SO(3)$ is the homomorphism defined above.

Note that while there is a sign ambiguity in the choice of a branch of the square root in the above expression defining a, the overall expression is a well defined smooth homomorphism because $\ker(Ad) = \{\pm \mathrm{id}\}$.

Similarly to the above, a and left multiplication inside $SO(3)$ let us can define a $U(2)$-action on $SO(3)$ and we can associate to any $U(2)$-principal bundle P an $SO(3)$-principal bundle

$$\tilde{P} \coloneqq P \times_{U(2),a} SO(3).$$

An $SU(2)$ bundle $P \to X$ over a smooth closed manifold X of dimension 3 or 4 is determined by its second Chern class $c_2(P) \in H^4(X;\mathbb{Z})$ (in particular, such a bundle is automatically trivial in the case of a 3-manifold), while a $U(2)$-bundle is determined by its first and second Chern classes.

Similarly, associated to a $U(2)$-bundle Q are an $SU(2)$-bundle $Q \times_{U(2)} SU(2)$ and an $SO(3)$-bundle $Q \times_{U(2)} SO(3)$. The latter bundle can equivalently be described as the subbundle $\mathfrak{su}(E_Q) \subset \mathrm{End}(E_Q)$ of skew-hermitian, trace-free endomorphisms of E_Q.

Lemma 2.0.2. *Every $SO(3)$-bundle P over Y has a lift to a $U(2)$-bundle, i.e. arises in the above way as the associated $SO(3)$-bundle to a $U(2)$-bundle. The Stiefel-Whitney class of this $SO(3)$-bundle is $w_2(Q \times_{U(2)} SO(3)) = c_1(Q) \mod 2$.*

Proof. See for example [Don02, pp. 145-146]. □

In this work, our main object of study are connections on bundles. To define these, we fix some notation: for a smooth principal G-bundle $p\colon P \to M$ over some smooth manifold M, we denote by $\mathfrak{g} \coloneqq T_eG$ the Lie algebra of G and by TP and TM the respective tangent bundles. We will also write $V \subset TP$ for the *vertical subbundle* $V \coloneqq \ker dp$. For $g \in G$, we write $R_g\colon P \to P$ for the map given by the action of $g \in G$ on P and $(R_g)_*\colon TP \to TP$ for its derivative.

Sections of the subbundle $TP \supset V \to P$ are called *vertical*.

There is a linear map $(-)^{\#}\colon \mathfrak{g} \to \Gamma(V)$ defined by $\xi^{\#}(p) \coloneqq \frac{d}{dt}|_{t=0} p \cdot \exp_G(t\xi)$. Vector fields in the image of $(-)^{\#}$ are called *fundamental vector fields*.

Finally, we denote the locally trivial bundle $P \times_{G,conj} G$ by $\mathrm{Aut}(P)$ (note that this is in general not a principal bundle). The set of sections of $\mathrm{Aut}(P)$ is canonically identified with the group (under fibrewise multiplication) of bundle automorphisms of P covering the identity of the base. We denote this group by

$$\mathcal{G}(P) \coloneqq \Gamma(M, P \times_{G,conj} G)$$

and call it the *gauge group* of the bundle P. We note here that if $\rho\colon G \to \mathrm{Aut}(V)$ is any smooth representation of G, then the group $\mathcal{G}(P)$ canonically acts on the associated bundle $P \times_{G,\rho} X$ through its action on P. In particular, $\mathcal{G}(P)$ acts on the adjoint bundle $\mathfrak{g}(P)$. This latter action (considered as an action on the module of sections $\Gamma(\mathfrak{g}(P))$) is $\mathcal{C}^\infty$-linear and thus further induces an action on all (tensor) bundles of forms with values in $\mathfrak{g}(P)$, namely by acting on the last tensor factor in

$$\Omega^k(M, \mathfrak{g}(P)) = \Omega^k(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, \mathfrak{g}(P)).$$

Definition 2.0.3. Let M be a smooth manifold, $G \subset Gl_n(\mathbb{K})$ a Lie group, $P \to M$ a smooth principal G-bundle and let $E \coloneqq P \times_G \mathbb{K}^n$ the associated vector bundle. A *connection A on P* is given by one of the following equivalent sets of data:

- A smooth subbundle $H_A \subset TP$ such that $TP = V \oplus H_A$ and $(R_g)_* H_A = H_A$ for all $g \in G$
- A 1-form $\omega_A \in \Omega^1(P, \mathfrak{g})$ such that $R_g^* \omega = Ad(g^{-1}) \circ \omega$ for all $g \in G$ and $\omega(\xi^\#) = \xi$ for all $\xi \in \mathfrak{g}$.

  We call a form with these properties a *connection 1-form*.
- A $\mathbb{K}$-linear map $\nabla_A \colon \Gamma(M; E) \to \Gamma(M; T^*M \otimes E)$ such that for all smooth sections s of E and all smooth functions $f \colon M \to \mathbb{K}$ we have the Leibniz rule
$$\nabla_A(fs) = df \otimes s + f \nabla_A(s).$$
  The map ∇_A is called *covariant derivative*

We note here that there is a canonical action of the group $\mathcal{G}(P)$ on the set of connections on P: the pullback of a connection 1-form ω_A under an automorphism f of the bundle P is again a connection 1-form and we write

$$f^* A$$

for the connection corresponding to $f^*\omega_A$ (note that $\ker(f^*\omega_A) = f_*(\ker(\omega_A))$, so this action can equivalently described as carrying the horizontal subbundle H_A to the horizontal subbundle $f_* H_A$). In terms of covariant derivatives, we have

$$\nabla_{f^*A} = f \circ \nabla_A \circ f^{-1}.$$

Definition 2.0.4. We say two connections A and B on the same bundle P are *gauge equivalent* if there is $f \in \mathcal{G}(P)$ such that $B = f^* A$.

Remark 2.0.5. *(i) Given a connection A on a principal G-bundle $\pi \colon P \to M$, a smooth path $\gamma \colon I \to M$ and a point $p \in \pi^{-1}(\{\gamma(0)\})$, there is a unique* parallel lift $\tilde{\gamma}_p \colon I \to P$, *defined by the properties $\tilde{\gamma}_p(0) = p$, $\pi \circ \tilde{\gamma}_p = \gamma$ and $\frac{d}{dt}\tilde{\gamma}_p \in H_A$ for all $t \in I$. This defines an isomorphism of G-spaces $\pi^{-1}(\{\gamma(0)\}) \to \pi^{-1}(\{\gamma(1)\}) \colon p \mapsto \tilde{\gamma}_p(1)$, called* parallel transport along γ. *If we fix $p \in \pi^{-1}(\{x\})$, then for any curve γ with $\gamma(0) = x = \gamma(1)$, there is a unique $g_\gamma \in G$ such that the parallel transport of p along γ is the point pg_γ. The set $\mathrm{Hol}_A(p) \coloneqq \{g_\gamma \mid \gamma(0) = x = \gamma(1)\} \subset G$ is a Lie subgroup, the*

holonomy group at p. *The conjugacy class of this subgroup is independent of the choice of point p in the fibre over x, in fact if $g \in G$ then*

$$\mathrm{Hol}_A(pg) = g^{-1}\,\mathrm{Hol}_A(p)g.$$

Similarly, for $x \in M$, the set of all parallel transport automorphisms along closed curves γ with $\gamma(0) = \gamma(1) = x$ forms a group, the holonomy group of A at x. *This group is isomorphic to any of the* $\mathrm{Hol}_A(p)$, *$p \in \pi^{-1}(\{x\})$, though is not canonically a subgroup of G.*

The collection of holonomy automorphisms along all closed curves in M determines the connection gauge equivalence, cf. [Lew93, Theorem 8.2]

(ii) Let A, A' be two connections on the same G-bundle P with corresponding covariant derivatives ∇_A and $\nabla_{A'}$. Then the difference operator $\nabla_A - \nabla_{A'}$ is tensorial, i.e. $C^\infty(M)$-linear and given by multiplication by a 1-form a; we will denote this fact by $A - A' = a \in \Omega^1(M; \mathfrak{g}(P))$. Here, $\mathfrak{g}(P) \coloneqq P \times_{G,ad} \mathfrak{g}$ denotes the adjoint bundle.

(iii) Given a covariant derivative ∇_A on a vector bundle P, we can tensor the de Rham complex $(\Omega^\bullet(M), d)$ with E and make this into a sequence

$$\Omega^0(M; E) \overset{d_A}{\to} \Omega^1(M; E) \overset{d_A}{\to} \ldots \overset{d_A}{\to} \Omega^{\dim M}(M; E)$$

by defining $d_A(\omega \otimes s) \coloneqq d\omega \otimes s + \omega \wedge \nabla_A s$ for $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$.

The sequence $(\Omega^\bullet(M; E), d_A)$ is not a complex, instead we only have that $d_A \circ d_A$ is C^∞-linear. More concretely, on $\Omega^0(\mathfrak{g}(P))$ the composition $d_A \circ d_A$ is given by multiplication with a uniquely defined 2-form $F_A \in \Omega^2(M; \mathfrak{g}(E))$. This 2-form is called the *curvature form* of A.

We fix some notation: if ω, η are ordinary differential forms on Y and s, t are sections of a Lie algebra bundle E (with fibre $\mathfrak{g}$), then we write

$$(\omega \otimes s) \wedge (\eta \otimes t) \coloneqq (\omega \wedge \eta) \otimes [s, t],$$

where $[-,-]$ denotes the Lie bracket of $\mathfrak{g}$. This definition uniquely extends to an associative, *commutative*, $\mathcal{C}^\infty(Y)$-linear "wedge product"

$$- \wedge - \colon \Omega^k(Y, E) \otimes_{\mathcal{C}^\infty(Y)} \Omega^l(Y, E) \to \Omega^{k+l}(Y, E).$$

Using this notation, we then have:

if A is a connection and $a \in \Omega^1(\mathfrak{g}(P))$, then we can write the curvature form of $A + a$ as

$$F_{A+a} = F_A + d_A a + \frac{1}{2} a \wedge a$$

A connection is called *flat* if its curvature vanishes identically. In this case, the twisted deRham sequence defined above becomes a complex, and we will denote its cohomology groups by

$$H^\bullet(M; E).$$

There is an equivalent way of describing flat connections which allows for an alternative, more algebraic description of these groups:

Theorem 2.0.6. *There is a natural bijection*

$$\{\textit{flat } G\textit{-connections over } M\}/\textit{gauge} \cong \textit{Hom}(\pi_1(Y), G)/\textit{conj.}$$

If $G \subset \mathrm{Gl}(V)$ is a linear group, then under the above identification, if A is a flat connection and ρ is a representative of the conjugacy class of representations that A corresponds to, the cohomology groups of the twisted deRham complex defined by A are identified with the group cohomology groups

$$H^\bullet(\pi_1(Y); V_\rho),$$

(computed using the Bar resolution) which we will also denote as $H^\bullet(Y; V_\rho)$.

Definition 2.0.7. Let Y be a closed three dimensional manifold and let A be a connection on the trivial $SU(2)$-bundle over Y (note that $BSU(2) = \mathbb{H}P^\infty$ is 3-connected, hence every $SU(2)$-bundle over Y is trivial). The *Chern-Simons-invariant of* A is defined as follows:
Since all 3-manifolds are null-cobordant, there is a smooth, compact, oriented 4-manifold X s.t. $\partial X = Y$. We choose some $SU(2)$-bundle $P \to X$ with connection $\mathbf{A}$, extending A, i.e. s.t. $(Y = \partial X \hookrightarrow X)^* \mathbf{A} = A$ and define $cs(A)$ as

$$cs(A) := \int_X \mathrm{Tr}(F_{\mathbf{A}} \wedge F_{\mathbf{A}}).$$

Chern-Weil-theory implies that modulo $8\pi^2\mathbb{Z}$, this definition is independent of the choices of extensions, so we get a well defined function

$$cs \colon \mathcal{A}(Y) \to \mathbb{R}/8\pi^2\mathbb{Z}.$$

The most important formula for our purposes is:

Proposition 2.0.8. *Let A_0, A_1 be connections on the principal G-bundle $P \to Y$. We write $a := A_1 - A_0$ for the difference 1-form. Then we can compute the difference in Chern-Simons-invariants as an integral over the manifold Y:*

$$cs(A_1) - cs(A_0) = \int_Y \mathrm{Tr}\left(2a \wedge F_{A_0} + a \wedge d_{A_0} a + \frac{1}{3} a \wedge a \wedge a\right) \qquad \text{(relCS)}$$

Proof. Let $p\colon Y \times [0,1] \to Y$ denote the projection and $\mathbb{P} \coloneqq p^*P$. We define a connection $\mathbb{A}$ on $\mathbb{P}$ by $\mathbb{A} = p^*A_0 + ta$. Then

$$\begin{aligned} F_{\mathbb{A}} =& F_{p^*A_0} + d_{p^*A_0}(ta) + \frac{t^2}{2} a \wedge a \\ =& F_{p^*A_0} + dt \wedge a + t d_{p^*A_0} a + \frac{t^2}{2} a \wedge a \end{aligned}$$

All pairwise wedge products of terms in the last line above which do not involve $dt \wedge a$, are multiples of pullbacks of 4-forms on Y and hence vanish. We thus get

$$F_{\mathbb{A}} \wedge F_{\mathbb{A}} = 2dt \wedge \left(a \wedge F_{p^*A_0} + ta \wedge d_{p^*A_0} a + \frac{t^2}{2} a \wedge a \wedge a \right)$$

and then

$$\begin{aligned} cs(A_1) - cs(A_0) =& \int_{Y \times [0,1]} \operatorname{Tr}(F_{\mathbb{A}} \wedge F_{\mathbb{A}}) \\ =& 2 \int_0^1 \left(\int_Y \operatorname{tr}(a \wedge F_{A_0}) + t \int_Y \operatorname{tr}(a \wedge d_{A_0} a) + \frac{t^2}{2} \int_Y \operatorname{tr}(a \wedge a \wedge a) \right) dt \\ =& 2 \int_Y \operatorname{tr}(a \wedge F_{A_0}) + \frac{1}{2} \int_Y \operatorname{tr}(a \wedge d_{A_0} a) + \frac{1}{6} \int_Y \operatorname{tr}(a \wedge a \wedge a) \\ =& \int_Y \operatorname{Tr} \left(2a \wedge F_{A_0} + a \wedge d_{A_0} a + \frac{1}{3} a \wedge a \wedge a \right) \end{aligned}$$

as claimed. □

Given a G-bundle $p\colon P \to M$, let $\mathcal{G}_P \coloneqq \Gamma(M; P \times_{conj.} G)$ be the *gauge group*. Its elements are the bundle automorphisms of $P \to M$. There is also a natural identification

$$\mathcal{G}_P \overset{1:1}{\longleftrightarrow} \{ f \in C^\infty(P, G) \mid f(xu) = u^{-1} f(x) u \}$$

defined as follows: Given $\sigma \in \mathcal{G}_P$ and $x \in P$, we can write $\sigma(p(x)) = [(x, g_x)]$. It is then easy to see that f defined by $f(x) \coloneqq g_x$ is a well defined map with the claimed properties.

From now on, fix a Lie group G with a fixed faithful representation $\rho\colon G \to Gl_n(\mathbb{K})$, where $\mathbb{K}$ denotes one of $\mathbb{R}, \mathbb{C}$, and a smooth G-bundle $P \to M$. Let $\mathcal{A}_P$ be the set of connections on P. Throughout this thesis, we will identify the elements of $\mathcal{A}_P$ with covariant derivatives on the associated vector bundle $E \coloneqq P \times_\rho \mathbb{K}^n$ whenever convenient and without further comment. As we have seen in Remark 2.0.5, (ii), this set is an affine space over $\Omega^1(M; \mathfrak{g}(P))$.

Recall the action of $\mathcal{G}_P$ on this space defined by

$$u^*\nabla_A \coloneqq u \circ \nabla_A \circ u^{-1}.$$

If we fix a base connection A_0, and thus the identification $\Omega^1(M, \mathfrak{g}(P)) \to \mathcal{A}_P, a \mapsto A_0 + a$, this action can be expressed as

$$u^*(A_0 + a) = A_0 + uau^{-1} + ud_{A_0}u^{-1}.$$

Here, we have written $d_{A_0}u^{-1}$ for the section of $\Omega^2(M, \mathfrak{g}(E))$ defined by $(d_{A_0}u^{-1})a = u^{-1}(d_{A_0}a) - d_{A_0}(u^{-1}a)$.
Under this action, the curvature form F_A of the connection transforms as

$$F_{u^*A} = uF_Au^{-1}.$$

In particular, if the bundle P is equipped with a bundle metric with respect to which G acts isometrically, then the action of $\mathcal{G}_P$ on $\mathcal{A}_P$ preserves the induced L^2-norm of the curvature. For all three of our Lie groups, there is a natural choice for such a bundle metric and we will make heavy use of this fact.
The above action is among the most central objects of study in this thesis, so we record some of its properties. We are ultimately interested in the quotient space $\mathcal{A}_P/\mathcal{G}_P$. The structure of this space in a neighborhood of a point will turn out to depend most crucially on its stabilizer. We have:

Proposition 2.0.9. *Assume that the base M is connected and let $A \in \mathcal{A}_P$. Then the stabilizer subgroup* $\mathrm{Stab}(A) \subset \mathcal{G}(P)$ *is isomorphic to the centralizer $C_{\mathrm{Hol}_p(A)} \subset G$ of the holonomy subgroup of A at an arbitrarily chosen point $p \in P$.*

Proof. This Lemma 4.2.8 of [DK97]. □

Using this proposition, we will explicitly identify the possible isomorphism classes the stabilizers of the connections we are concerned with.

We note here that for all possible structure groups G, the center $Z(\mathcal{G}_P)$ acts trivially on $\mathcal{A}_P$. In particular, elements in the center of the gauge group are determined by their value over any one point. Since a central element of the gauge group necessarily induces a central automorphism on any given fiber, we get that $Z(\mathcal{G}_P)$ is isomorphic to $Z(G)$. Because the center subgroup is the unique representative of its conjugacy class, this isomorphism is even canonical.

We distinguish points according to whether the induced action of $\mathcal{G}_P/Z(\mathcal{G}_P)$ is free on their orbit:

Definition 2.0.10. Let $P \to M$ be a smooth principal G-bundle and let $A \in \mathcal{A}_P$. We call A *irreducible* if $\mathrm{Stab}(A) = Z(\mathcal{G}_P)$ and *reducible* otherwise.

The reason for this terminology is that reducibility in the above sense is equivalent to the bundle with connection (P, A) admitting a reduction to an H-bundle with connection (Q, B) where $H \coloneqq C_{\mathcal{G}_P}(\mathrm{Stab}(A))$ is the centralizer subgroup of $\mathrm{Stab}(A)$ in $\mathcal{G}_P$

2.1 The gauge actions for $G = SO(3)$ and $G = SU(2)$

We denote by $\mathcal{G}_P$ the group of bundle automorphisms of P (covering the identity on Y). This is the group of sections of the associated bundle $P \times_{SO(3),Ad} SO(3)$. As in the previous section, this group acts on the space $\mathcal{A}_P$ of connections on P. We would like to work in the *configuration space* $\mathcal{A}_P/\mathcal{G}_P$ however this space is far from being a manifold because of the existence of reducible connections (and thus the action not being free). To remedy this, we work with a space of *based connections*:

Definition 2.1.1. Let $y \in Y$ be any base point. Then the group $\mathcal{G}_P$ acts from the left on the fiber P_y via $(g, p) \mapsto g(y)p$. We call the space $\mathcal{A}_P \times E_y/\mathcal{G}_P$, where $\mathcal{G}_P$ acts diagonally on the product, the *framed configuration space*.

The framed configuration space carries an additional $SO(3)$-action induced from the (right-) action on the fiber. These actions commute and we get a double quotient $SO(3)\backslash\mathcal{A}_P \times P_y/\mathcal{G}_P$. This double quotient is (canonically isomorphic to) the configuration space $\mathcal{A}_P/\mathcal{G}_P$. More precisely, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}_P \times P_y/\mathcal{G}_P & & \\ \downarrow & \searrow & \\ SO(3)\backslash\mathcal{A}_P \times P_y/\mathcal{G}_P & \longrightarrow & \mathcal{A}_P/\mathcal{G}_P \end{array}$$

where the horizontal map is an isomorphism and the map on the right is induced from the projection onto the first coordinate $\mathcal{A}_P \times P_y \to \mathcal{A}_P$.

The advantage of this construction is that the gauge group action on the space of framed connections is free, in contrast to the action of the (full) gauge group on the space $\mathcal{A}_P$. This follows from Proposition 2.0.9: If an element g of the gauge group fixes a point (A, p), then $g(y)$ must be the identity of P_y (because the action of $SO(3)$ on any fiber of P is free) and must thus be the identity everywhere because g is A-parallel.

Having set this up, we next enumerate the possible stabilizers of connections on an $SO(3)$-bundle P. Here, we have to distinguish between the different possible gauge groups.

For $G = SO(3)$, the possible stabilizers are enumerated as follows:

Proposition 2.1.2. *If A is a connection on an $SO(3)$- bundle $P \to Y$, then the stabilizer $Stab(A)$ under the action of the gauge group $\mathcal{G}_P$ has the isomorphism type of one of the following subgroups of $SO(3)$:*

- $\{id\}$
- *The diagonal subgroup $V_4 \coloneqq \mathbb{Z}/2 \times \mathbb{Z}/2$ and its subgroups isomorphic to $\mathbb{Z}/2$*
- $SO(2)$
- $O(2)$
- $SO(3)$

For $G = SU(2)$, the enumeration of possible stabilizers is considerably simpler:

Proposition 2.1.3. *If A is a connection on an $SU(2)$- bundle $P \to Y$, then the stabilizer $Stab(A)$ under the action of the gauge group $\mathcal{G}_P$ has the isomorphism type of one of the following subgroups of $SU(2)$:*

- $\{id\}$
- $U(1)$
- $SU(2)$

Remark 2.1.4. *In both cases, we call a connection* fully reducible *if its stabilizer under the gauge action is all of G. Such connections can only occur if the underlying bundle is smoothly trivial and this is the main reason we want to be able to work with $SO(3)$- and $U(2)$-bundles: the nontriviality of an $U(2)$- or $SO(3)$-bundle follows from that of its first Chern-, resp. second Stiefel-Whitney-class. In contrast to the 4-dimensional classes c_2 and p_1, both of these characteristic classes are unaffected by our constructions later in this thesis and so by working on bundles with prescribed nontrivial c_1 or w_2, we can avoid having to deal with fully reducible connections, for which our technical methods break down, altogether.*

Proof of Propositions 2.1.2 and 2.1.3. By Proposition 2.0.9 it is enough to enumerate the possible stabilizers of points of $SO(3)$ and $SU(2)$ under the conjugation action of this group on itself. That for this action, the above lists are complete is seen by explicitly computing with matrices. □

2.2 The $U(2)$-picture

Throughout this section, we again denote by G a linear Lie group, equipped with a canonical representation $can\colon G \to \mathrm{Gl}_n(\mathbb{K})$ where $\mathbb{K}$ is one of the fields $\mathbb{R}$, $\mathbb{C}$. For a principal G-bundle P we will also denote by E_P the canonically associated vector bundle

$$E_P \coloneqq P \times_{G,can} \mathbb{K}.$$

In the dimensions we are working in, there is an equivalent way to set up the entire above theory with $U(2)$-bundles instead of $SO(3)$-bundles:

Let P be any $SO(3)$-bundle over Y. As previously stated in remark 2.0.1, such bundles are uniquely determined by their second Stiefel-Whitney class $w \coloneqq w_2(P) \in H^2(Y;\mathbb{Z}/2)$ and their first Pontrjagin class $p_1(P) \in H^4(Y;\mathbb{Z})$ (the latter of which being automatically trivial in dimensions ≤ 3). For a rational homology sphere, any such class w has an integral lift $c \in H^2(Y;\mathbb{Z})$ and we define $\tilde{P}$ to be the $U(2)$-bundle over Y with $c_1(\tilde{P}) = c$.

By Lemma 2.0.2 we can choose a $U(2)$-bundle $\tilde{P}$ over Y such that $\mathfrak{su}(\tilde{P})$ is isomorphic to E_P. We fix a base connection A_0 on the complex line bundle $\lambda \coloneqq \det E_P$ and define $\mathcal{A}_{E,\lambda}$ as the set of smooth $U(2)$-connections $\mathbf{A}$ on E_P which induce the connection A_0 on the determinant bundle, i.e. $\mathrm{tr}(\mathbf{A}) = A_0$. We also define a gauge group $\mathcal{G}_{E_P}^{\det}$ as the group of smooth vector bundle automorphisms of E_P with trivial determinant.

Lemma 2.2.1. *A $U(2)$-connection $\mathbf{A} \in \mathcal{A}_{E_P,\lambda}$ induces an $SO(3)$-connection $\overline{\mathbf{A}}$ on the bundle P and the resulting map*

$$\mathcal{A}_{E,\lambda} \to \mathcal{A}_P$$

is a bijection.

Proof. Let Ad be the adjoint action of $U(2)$ on its Lie algebra $\mathfrak{u}(2) \cong \mathfrak{su}(2) \times i\mathbb{R}$. This action is trivial on the second factor $i\mathbb{R}$ and induces a surjective Lie group homomorphism $\phi\colon U(2) \to SO(\mathfrak{su}(2)) = SO(3)$.
Then the claim of the Lemma can be seen by lifting horizontal distributions: In our situation, we have an $(Ad, \det)$-equivariant covering map $\phi\colon \tilde{P} \to Q$ of bundles where Q is the $SO(3) \times U(1)$-bundle over Y with fiber $P_y \times \det(\tilde{P})_y$ over the point y. Thinking of connections as G-invariant horizontal distributions, we can lift each $SO(3) \times U(1)$-distribution on Q uniquely to a $U(2)$-distribution along ϕ and conversely, any $U(2)$-distribution can be pushed down to a well defined $SO(3) \times U(1)$-distribution. A horizontal distribution on Q is uniquely determined by the distributions on P and $\det(\tilde{P})$ it induces and fixing A_0 fixes the latter. Thus, connections on $\tilde{P}$ with a fixed determinant are in canonical bijection with

connections on P. $\square$

Also, automorphisms of E induce automorphisms of the associated bundle $P = \mathfrak{su}(E)$, giving a homomorphism $\mathcal{G}_E \to \mathcal{G}_P$ which induces an injection

$$\mathcal{G}_E^{\det}/\{\pm\mathrm{id}\} \to \mathcal{G}_P.$$

The image of this injection is the subgroup of gauge transformations which lift to sections of the bundle $\mathrm{Aut}^{SU(2)}(P) \coloneqq P \times_{SO(3)} SU(2)$. The failure of a gauge transformation to lift to $SU(2)$ is detected by a map $\mathcal{G}_P \to H^1(Y;\mathbb{Z}/2)$, as can be seen by an obstruction theory argument: First of all, given a section u of $\mathrm{Aut}(P)$ and a lift of u over the 1-skeleton $Y^{(1)}$ of Y to a section of $\mathrm{Aut}^{SU(2)}(P)\big|_{Y^{(1)}}$, we automatically get lifts over all 2-cells and then over the three cells (because balls are contractible and $SU(2) \to SO(3)$ is a covering map), giving a globally defined lift $\tilde{u} \in \mathrm{Aut}^{SU(2)}(P)$. So, the only obstruction is lifting over the 1-skeleton over which the bundle P becomes trivial. Thus, the object becomes to lift a map $u\colon Y^{(1)} \to SO(3)$ to a map $\tilde{u}\colon Y^{(1)} \to SU(2)$ which is possible if and only if the induced map

$$\pi_1(u)\colon \pi_1(Y) = \pi_1(Y^{(1)}) \to \pi_1(SO(3)) = \mathbb{Z}/2$$

is trivial. We thus see that an $SO(3)$ gauge transform lifts if and only if this induced map $\pi_1(u) \in H^1(Y;\mathbb{Z}/2)$ is trivial.

Throughout this thesis, we will feel free to switch between these two pictures without further comment.

2.3 Configuration Spaces over Manifolds with Cylindrical Ends

Let X be a smooth, oriented, 4-dimensional manifold and $P \to X$ a smooth G-bundle.

Our ultimate argument hinges on a 'neck stretching' procedure which forces us to investigate moduli spaces over certain non-compact manifolds:

Definition 2.3.1. A *manifold with cylindrical ends* consists of the following data:

- A smooth 4-dimensional manifold X
- A smooth compact submanifold $K \subset X$ such that all connected components of $X \setminus K$ have noncompact closure in X.
- For each connected component $C \subset X \setminus K$ (C for 'cylinder') a diffeomorphism $\phi_C\colon C \to Y_C \times [0,\infty)$ for some closed, smooth, connected, 3-dimensional manifold Y_C

We will always denote the inclusion $Y_C \times \{0\} \hookrightarrow Y_C \times [0,\infty) \cong E$ by ι_C and the projection $C \to Y_C$ by p_C.

Remark 2.3.2. *If X and Y_C are oriented, as they will usually be in this thesis, then C and $Y_C \times [0,\infty)$ inherit orientations (the latter by requiring a frame of the form $(\partial_t, \partial_{y_1}, \partial_{y_2}, \partial_{y_3})$ to be oriented for any oriented local frame $(\partial_{y_1}, \partial_{y_2}, \partial_{y_3})$ of Y_C).*
In this situation, the diffeomorphism ϕ_C is either orientation preserving, in which case we call C a positive *end or orientation reversing, in which case we call C a* negative *end.*

The only examples of cylindrical end manifolds that will be relevant for the rest of this thesis are cylinders of the form $Y \times \mathbb{R}$ and manifolds with exactly one cylindrical end. We will

Definition 2.3.3. Let X be a manifold with cylindrical ends. An *adapted bundle* over X consists of data:

- A smooth bundle $P \to X$
- For each end C of X connections $\mathbf{A}_C$ on $P|_C$ and a bundle with connection (P_C, A_C) on Y_C such that $P|_C$ is identified with $p_C^* P_C$ and $p_C^* A_C$ is identified with $\mathbf{A}_C$ via ϕ_C.

Remark 2.3.4. *Since the space of connections on P is affine and hence convex, we can always find some connection $\mathbf{A}$ on P which interpolates between the above structures over the ends (every smooth bundle over a compact manifold carries some connection, so we can first choose one arbitrarily over the compact piece K, then interpolate between this connection and the given ones over the ends). If the A_C are all flat, then the curvature $F_{\mathbf{A}}$ has compact support and we can define a relative Chern number by*

$$c(\mathbf{A}) \coloneqq \int_X tr(F_{\mathbf{A}}^2).$$

This number does not depend on the choice of extension $\mathbf{A}$ but only on the adapted bundle on which it is defined and reduces modulo $\mathbb{Z}$ to the sum of Chern-Simons invariants of the limit connections A_C.
Later in this thesis, we will introduce perturbations which force us to work with non-flat limit connections. We will then define another invariant (of the adapted bundle together with perturbation data), the analytical energy *that will play the same role.*

Let X be a cylindrical end manifold and P an adapted bundle over X (for convenience, we suppress the fixed (perturbed) flat structures over the ends but recall that the data of such a bundle includes fixed trivializations and pullback connections over the ends). We let

$$\mathcal{A}(P)$$

be the set of connections on P that agree with the given configurations over the ends of X, that is, outside of some compact set $K \subset X$. This is an affine space over the vector space $\Omega^1_c(X; \mathrm{ad}(P))$ of 1-forms over X with compact support and coefficients in the adjoint bundle.

We want to investigate such connections up to gauge, so we also need to define an appropriate notion of gauge equivalence. We let

$$\mathcal{G}(P)$$

be the set of those smooth bundle isomorphisms σ of P such that for each end $E \cong Y_C \times [0, \infty)$ there is an element $\tau_C \in \mathrm{Stab}(A_C) \subset \mathcal{G}(Y_C)$ and some $T \geq 0$ such that

$$\sigma|_{Y \times (T, \infty)} = p_C^* \tau,$$

i.e. such that over each end, σ is eventually constant and equal to an element in the stabilizer of the end connection. We will sometimes call such sections *end parallel*.

2.4 The Instanton Equation

We fix (X, g) a smooth connected, oriented Riemannian 4-manifold, G as one of the linear groups $U(2)$, $SU(2)$ or $SO(3)$, $P \to X$ a principal G-bundle and E the vector bundle canonically associated to P via the corresponding inclusions $G \hookrightarrow Gl_n(\mathbb{K})$,

$$E \coloneqq P \times_G \mathbb{K}^n,$$

where $\mathbb{K} = \mathbb{C}$ and $n = 2$ if $G = U(2)$ or $SU(2)$, and $\mathbb{K} = \mathbb{R}$ and $n = 3$ if $G = SO(3)$. Note that for each of our possible structure groups G, the bundle E carries a canonical bundle metric. We denote the tensorial map induced from this metric pairing on E by

$$B \colon \Gamma(E) \times \Gamma(E) \to \mathbb{R}.$$

The metric structure on X induces a Hodge-*-operator

$$* \colon \Omega^\bullet(E) \to \Omega^{4-\bullet}(E).$$

This operator satisfies $*^2\omega = (-1)^{k(4-k)}$ for k-forms ω. With this, we also get a bilinear pairing on $\Omega^k(E) \times \Omega^{4-k}(E)$ via

$$\langle \alpha, \beta \rangle \, vol_g \coloneqq B(\alpha \wedge *\beta)$$

where the expression on the right hand side denotes the combination of wedge product on the form part and B on the coefficients of a k-form. In dimension 4, we get a bundle metric on 2-forms and an isometric involution

$$*\colon \Omega^2(E) \to \Omega^2(E).$$

This operator induces an L^2-orthogonal direct sum decomposition into eigenspaces corresponding to the eigenvalues 1 and -1, denoted $\Omega^2_+(E)$ and $\Omega^2_-(E)$, respectively, i.e.

$$\Omega^2(E) = \Omega^2_-(E) \oplus \Omega^2_+(E)$$

and we denote the projections of a 2-form ω to these eigenspaces by

$$\omega^- \text{ and } \omega^+,$$

respectively.

Definition 2.4.1. If A is a connection on E, we say that A is *anti self dual*, an *ASD-connection* or an *instanton*, if

$$*F_A = -F_A$$

or, equivalently,

$$F_A^+ = 0.$$

Note that this equation is invariant under the gauge group action, so there is no hope of ellipticity. Instead, we note that the invariance of the equation means that it defines a well defined subset of our (based) configuration space, the *instanton moduli space*

$$\mathcal{M}(P) \coloneqq \{[A] \in \mathcal{A}(P)/\mathcal{G}(P) \mid F_A^+ = 0\}.$$

It will turn out later that the instanton equation up to gauge is elliptic, after all, so that this moduli space becomes a finite dimensional orbifold (in fact a manifold in the cases of interest to us in this thesis).

Definition 2.4.2. Let $E \to M$ be a smooth $U(2)$-bundle over a smooth, oriented Riemannian 4-manifold M with $c_1(E) = c$. Then we say a connection $\mathbf{A}$ on E is a $U(2)$-*instanton* if the induced connection $\overline{\mathbf{A}}$ on the adjoint bundle $\mathfrak{su}(E)$ is an $SO(3)$-instanton in the above sense.

2.5 The Chern-Simons functional

The ultimate reason we are interested in instanton gauge theory is its connection with (infinite dimensional) Morse theory for the Chern-Simons functional in three dimensions, first developed by Andreas Floer. We give a brief overview here.

Let $P \to Y \times \mathbb{R}$ be a trivial G-bundle. Choosing some trivialization, a connection $\mathbf{A}$ on P can be written $\mathbf{A} = d + a$, where d denotes the trivial connection and a is a $\mathfrak{g}$-valued 1-form. Writing $a = a_0 dt + \sum_{i=1}^{3} a_i dy_i$, we say that $\mathbf{A}$ is in *temporal gauge* if $a_0 = 0$. We can always choose such a gauge by starting from a connection matrix for $\mathbf{A}\big|_{Y\times\{0\}}$ and extending it over the whole cylinder by $\mathbf{A}$-parallel transport along the t-direction.

Now let $\mathbf{A}$ be a connection on the trivial $SU(2)$-bundle $P = Y \times \mathbb{R} \times SU(2) \to Y \times \mathbb{R}$ in temporal gauge. We can then decompose the curvature tensor $F_{\mathbf{A}}$ in the following way:

$$F_{\mathbf{A}}(t, y) = F_{A_t} + \frac{\partial A_t}{\partial t} \wedge dt$$

where $A_t \coloneqq (\iota_t \colon Y \times \{t\} \hookrightarrow Y \times \mathbb{R})^* \mathbf{A}$ is the pulled back connection form.
Now let π be a perturbation as above. We compute:

$$\begin{aligned} 2F^+_{\mathbf{A}} =& F_{\mathbf{A}} + *_4 F_{\mathbf{A}} \\ =& F_{A_t} + *_3 \frac{\partial A_t}{\partial t} + \\ & dt \wedge \left(*_3 F_{A_t} + \frac{\partial A_t}{dt} \right) \end{aligned}$$

In particular, we see that the instanton equation $F^+_{\mathbf{A}} = 0$ is equivalent to the downward gradient flow equation for the functional cs, namely

$$\frac{\partial A_t}{dt} = - *_3 F_{A_t}$$

Having established this connection between flow lines of cs and instantons over the tube in temporal gauge, we now investigate the linearized instanton equation, albeit in a gauge invariant manner:

Let A_0 be some smooth connection on an adapted bundle over the cylinder, constant and equal to prescribed flat connections $\rho_\pm$ over the ends. Then, given a solution A to the instanton equation (at this stage, without the proper analytical/Sobolev setup, we mean by this any smooth connection without concern for integrability questions with anti self dual curvature), we write

$$a \coloneqq A - A_0 \in \Omega^1(\mathrm{ad}(P)).$$

On a cylinder, we can "separate the time variable", i.e. write

$$a = \alpha(t) + \beta(t)dt,$$

where α and β are smooth paths in $\Omega^1(Y; \mathrm{ad}(P|_Y))$ and $\Omega^0(Y; \mathrm{ad}(P|_Y))$, respectively. More formally, this defines an isomorphism

$$\Omega^1(Y \times \mathbb{R}; \mathrm{ad}(P)) \cong \Gamma(Y \times \mathbb{R}, p_Y^*(\Lambda^0(Y; \mathrm{ad}(P)) \oplus \Lambda^1(Y; \mathrm{ad}(P)))).$$

We get a similar identification for self dual 2-forms over a cylinder: Decomposing a 2-form $\omega = \phi + \psi \wedge dt$ as above and using the formula $*(\phi \wedge dt) = *_3\phi$ for a time-dependent form ϕ over Y and the definition of the Hodge-star, we get

$$(\phi + \psi \wedge dt)^+ = (*_3\phi + \psi) \wedge dt + *_3(*_3\phi + \psi).$$

We can thus identify a self dual 2-form over a cylinder with a single time-dependent 1-form $\eta := *_3\phi + \psi$ over Y by $\eta \mapsto \eta \wedge dt + *_3\eta$.

We will use these identifications frequently throughout the rest of this thesis, and so for brevity we introduce some notation for it:

For $q \geq 0$ we denote

$$\Lambda^q_Y := \Gamma(Y \times \mathbb{R}, p_Y^*\Lambda^q(Y; \mathrm{ad}(P))),$$

where $\Lambda^q(Y; \mathrm{ad}(P))$ denotes the bundle of q-forms over Y with values in the adjoint bundle $\mathrm{ad}(P)$, that is $\Lambda^q(Y; \mathrm{ad}(P)) = \bigwedge^q T^*Y \otimes \mathrm{ad}(P)$. We record the above discussion as the following Lemma:

Lemma 2.5.1. *There are canonical identifications*

$$\Omega^1(Y \times \mathbb{R}; ad(P)) \cong \Lambda^0_Y \oplus \Lambda^1_Y$$

and

$$\Omega^2_+(Y \times \mathbb{R}; ad(P)) \cong \Lambda^1_Y.$$

Finally, we note that

$$\Omega^0(Y \times \mathbb{R}; ad(P)) \cong \Lambda^0_Y$$

in the obvious way.

2.6 Tangent Spaces

We give a brief description of the tangent spaces to our (based) configuration spaces in the case of three dimensional base manifolds. We will later revisit this topic for four dimensional manifolds with cylindrical ends once we have set up our basic analytic machinery. Note that at this point, everything here is quite informal as we have not yet convinced ourselves that the spaces we are concerned with are in fact manifolds. The ideas and formulas given here will be made rigorous for the Sobolev completed configuration spaces that we will introduce in the next chapter.

The three dimensional case First of all, our space of adapted connections $\mathcal{A}(P)$ is an affine space over $\Omega^1(\mathrm{ad}(P))$, the space of smooth 1-forms with compact support. We can thus canonically identify this latter space of forms as the tangent space to $\mathcal{A}(P)$ at A_0,

$$T_{A_0}\mathcal{A}(P) \cong \Omega^1(\mathrm{ad}(P)).$$

It is also not hard to see that we can identify the tangent space of the gauge group $\mathcal{G}_0(P)$ as the space of 0-forms,

$$T_{\mathrm{id}}\mathcal{G}_0(P) \cong \Omega^0(\mathrm{ad}(P))$$

and the differential of the map $\mathcal{G}(P) \to \mathcal{A}(P), u \mapsto u^*A_0$ induced by the gauge group action is formally given by the operator

$$d_{A_0} \colon \Omega^0(\mathrm{ad}(P)) \to \Omega^1(\mathrm{ad}(P)).$$

We thus identify the tangent space to the orbit at A_0 with the image $\mathrm{im}(d_{A_0})$. A natural candidate for a slice to the gauge action, and thus for tangent space of the quotient $\mathcal{B}(P)$ at $[A_0]$, is the cokernel of this operator. This cokernel is given by

$$\ker d_{A_0}^*$$

where d_{A_0} is the formal L^2-adjoint. Unfortunately, these spaces do not define slices for the gauge action if $\mathcal{A}_0$ is reducible due to the failure of the action to be free, which implies that at reducible connections, the linearized action d_{A_0} is not injective. Precisely, we have:

Proposition 2.6.1. *The kernel of d_{A_0} is given by the Lie algebra to the stabilizer* $\mathrm{Stab}(A_0)$*,*

$$\ker d_{A_0} = \mathit{Lie}\,\mathrm{Stab}(A_0) \subset \Omega^0(\mathit{ad}(P)).$$

In particular, we see that this the operator d_{A_0} is injective iff the stabilizer $\mathrm{Stab}(A_0)$ *is discrete.*

Remark 2.6.2. *By our enumeration of the possible reducibles for $SU(2)$-bundles, the above Proposition immediately implies that a connection A_0 is irreducible iff d_{A_0} is injective, fully reducible iff $H^0(Y; \mathit{ad}(P)) = \ker d_{A_0}$ has dimension 3 and that in all other cases,* $\dim H^0(Y; \mathit{ad}(P)) = 1$*.*

The same is true for the action of the even *gauge group on the space of connections on an $SO(3)$-bundle, see [Eis23, p.16].*

The action on the space of framed connections, $\mathcal{A} \times E_b$, is free however. The differential of the action of the gauge group on the framed connections is given by

$$\tilde{d}_{A_0} \colon \Omega^0(\mathrm{ad}P) \to \Omega^1(\mathrm{ad}(P)) \times \mathfrak{g}_b, f \mapsto (d_{A_0}f, f(b)).$$

This introduces an additional difficulty: the image of the map $\tilde{d}_{A_0}$ is not L^2-closed, the issue being that evaluation is not a continuous map in the L^2-topology. We can still define slices for our action, however, given by the subspaces

$$\mathcal{K}_{A_0} \coloneqq \ker d^*_{A_0} \oplus \mathfrak{g}^{\perp}_{A_0}.$$

Here, $\mathfrak{g}_{A_0}$ is the subspace of $\mathfrak{g}_b$ of those elements that extend to A_0-parallel sections over all of M, and $\mathfrak{g}^{\perp}_{A_0}$ its orthogonal complement in $\mathfrak{g}_b$. We then have a (non-L^2-orthogonal!) decomposition

$$\Omega^1(\mathrm{ad}(P)) = \mathrm{Im}\tilde{d}_{A_0} \oplus \mathcal{K}_{A_0}$$

and in fact the three-dimensional framed configuration space $\tilde{\mathcal{B}} = \mathcal{A} \times E_b/\mathcal{G}$ will be locally diffeomorphic to $\mathcal{K}_{A_0}$ in suitable Sobolev completions. For details, we refer to [Eis23, sections 2.1 and 2.2].

The linearized ASD equation. As in the three dimensional case, the linearization at $\mathbf{A}$ of the action of the uncompleted gauge group $\mathcal{G}_0 \coloneqq \{\sigma \in \Gamma(\mathrm{Aut}(P)) \mid \sigma \equiv \mathrm{id} \text{ over some infinite portion of the end}\}$ is given in four dimensions by

$$d_{\mathbf{A}} \colon \Omega^0_c(\mathrm{ad}(P)) \to \colon \Omega^1_c(\mathrm{ad}(P)),$$

the only difference being that now we have spaces of forms with compact support. Again by Hodge theory, we have an L^2-orthogonal decomposition

$$\Omega^1_c = \mathrm{Im}d_{\mathbf{A}} \oplus \ker d^*_{\mathbf{A}}.$$

We will see later in section 4 how this leads to slice theorems and smooth structures on four dimensional configuration and moduli spaces.

For the moment, we want to get familiar with the linearized ASD equation on the configuration space (i.e. in Uhlenbeck gauge). It is given by the operator $d^*_{A_0} + d^+_{A_0}$. By making the above identifications, we can think of this as an operator that maps pairs of time-dependent 0- and 1-forms to itself. Using the identifications from Lemma 2.5.1, we write this as

$$d^*_{A_0} + d^+_{A_0} \colon \Lambda^0_Y \oplus \Lambda^1_Y \to \Lambda^0_Y \oplus \Lambda^1_Y$$

and we compute the effect of this operator on this space more explicitly next.

First,

$$\begin{aligned} d^*_{A_0}(\alpha(t) + \beta(t)dt) &= * \, d_{A_0} * (\alpha(t) + \beta(t)dt) \\ &= * \, (d_{A_0}(*_3\alpha(t) \wedge dt) + *_3\beta(t)) \\ &= d^*_{A_{0,t}}\alpha + *\frac{\partial \beta}{\partial t} \end{aligned}$$

where $d_{A_{0,t}}$ is the covariant derivative on three dimensional forms induced from the connection defined by restricting A_0 to $\{t\} \times Y$.

A similar calculation shows

$$d_{A_0}(\alpha + \beta dt) = d_{A_{0,t}}\alpha + (-d_{A_{0,t}}\beta + \frac{\partial \alpha}{\partial t}) \wedge dt,$$

whence, using the above formula for self dual 2-forms,

$$\begin{aligned} d^+_{A_0}(\alpha + \beta dt) =& (d_{A_{0,t}}\alpha + (-d_{A_{0,t}}\beta + \frac{\partial \alpha}{\partial t}) \wedge dt)^+ \\ =& (-d_{A_{0,t}}\beta + \frac{\partial \alpha}{\partial t} + *_3 d_{A_{0,t}}\alpha) \wedge dt \\ &+ *_3(-d_{A_{0,t}}\beta + \frac{\partial \alpha}{\partial t} + *_3 d_{A_{0,t}}\alpha). \end{aligned}$$

What we have computed above is that the ASD deformation operator can be written, over a cylinder, in the form

$$D_{A_0} = \frac{d}{dt} + L_{A_{0,t}},$$

where $L_{A_{0,t}}$ is the operator as in the following definition:

Definition 2.6.3. Given a smooth connection A on a bundle $Q \to Y$ and $k \leq 1$, we call the operator $L_A \colon \Omega^0_k(Y; ad(P)) \oplus \Omega^1_k(Y; ad(P)) \to \Omega^0_{k-1}(Y; ad(P)) \oplus \Omega^1_{k-1}(Y; ad(P))$ defined by

$$L_A \coloneqq \begin{pmatrix} 0 & -d^*_A \\ -d_A & *_3 d_A \end{pmatrix}$$

the *extended Hessian* operator.

This extended Hessian is essentially equivalent to the Hessian of *cs* on the Uhlenbeck slice but has the advantage of being more invariantly defined. In particular, the gauge fixing condition is "built in" in the sense that the formal Hessian $*_3 d_A$ at A is invertible up to gauge, i.e. on the slice $\ker d^*_A$, iff the extended Hessian is invertible without additional reference to a gauge fixing condition. We also record some convenient properties this operator has:

Lemma 2.6.4. *The operator L_A, acting on forms over the 3-manifold Y, is self-adjoint elliptic and thus Fredholm of index zero. The space $\Omega^0(Y) \oplus \Omega^1(Y)$ admits an L^2-orthogonal decomposition into eigenspaces of L_A.*

Its kernel is equal to the direct sum of the subspaces $\ker(*_3 d_A|_{\ker d^*_A}) \subset \Omega^1(Y)$ and $\ker d_A \subset \Omega^0(Y)$.

If A is flat, we can identify this kernel as the sum of the twisted cohomology groups,

$$H^0(Y; adA) \oplus H^1(Y; adA). \tag{6}$$

Proof. The square of the extended Hessian is given by the operator

$$L_A^2 = \begin{pmatrix} \Delta_A & - *_3 F_A \\ - *_3 F_A & \Delta_A \end{pmatrix} : \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^1.$$

This is the sum of an elliptic operator of order 2 and an operator of order 0 and hence is itself elliptic. At a flat connection A, where $F_A = 0$, it is in fact simply the Laplacian on $\Omega^0 \oplus \Omega^1$ and we can identify the kernel of the Laplacian on 0- resp. 1-forms as $H^0(Y; \mathrm{ad}A)$ resp. $H^1(Y; \mathrm{ad}A)$ by Hodge theory. The same then holds for L_A. For a more detailed discussion, see [Don02, section 2.5.4]. □

3 Banach space completions

Throughout this section, we fix a cylindrical end manifold X and a Riemannian metric g_X on X. We require that over each end of X the metric g_X is a product metric of the form $dt^2 + g_{Y_E}$. We further fix an adapted bundle $P \to X$ and a base connection $\mathbf{A}_0$ on P which is constant and equal to the fixed limit connections over each end. The specific choice of such a connection is immaterial, all constructions that follow will, up to canonical isomorphism, only depend on the adapted bundle and the metric structure.

The space of smooth connections $\mathcal{A}(P)$ on a bundle P is not a Banach manifold and the group $\mathcal{G}(P)$ is not a Banach Lie group. In order to bring analytical methods to bear in our moduli theory, we need to work with certain Sobolev-completions of these spaces:

Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The connection $\mathbf{A}_0$ induces a connection, which we will also denote by $\mathbf{A}_0$, on the space $\Gamma(X, ad(P))$ of sections of the vector bundle $ad(P)$. We also have the Levi-Civita connection ∇^{LC} determined by g_X and we get an induced connection, still called $\mathbf{A}_0$, on the spaces $\Omega^*(X, ad(P))$ of forms with coefficients in $ad(P)$.

We then set

$$\|\sigma\|_{p,k,\mathbf{A}_0} \coloneqq \|\sigma\|_{L^p_k,\mathbf{A}_0} \coloneqq \sum_{i=1}^{k} \left(\int_X |\nabla^{(i)}_{\mathbf{A}_0} \sigma|^p vol_{g_X} \right)^{\frac{1}{p}}.$$

As indicated by our notation, these norms depend not only on p and k but also on the choice of base connection. A different choice of base connection does not, however, change the equivalence class in which these norms lie:

Proposition 3.0.1. *Let $\mathbf{A}_0$ and $\mathbf{A}_1$ be two smooth base connections on the common adapted bundle P, both eventually constant and equal to the same pullback connection over each end of X. Then the norms $\|-\|_{p,k,\mathbf{A}_0}$ and $\|-\|_{p,k,\mathbf{A}_1}$ are equivalent.*

Because of this, we will henceforth abuse notation slightly and simply write $\|-\|_{p,k}$ for any choice of norm as above.

Proof of Proposition 3.0.1. This follows from Sobolev embedding theorems over cylindrical end manifolds and is proved in [Don02, Ch. 3].

□

For the moment, our goal is to develop a good enough analytic machinery for us to bring to bear elliptic and Fredholm techniques and results in our investigation of instanton moduli spaces over certain non compact manifolds. So, the natural first question to ask is under which conditions the deformation operator D_A is Fredholm. This is not a very interesting question over compact manifolds as this operator is elliptic and hence automatically Fredholm over compact domains. This no longer holds over non compact manifolds and so we begin the investigation with a result over cylinders, i.e. the model for all non-compactness we are concerned with here.

To motivate our setup, recall that with respect to an adapted connection $\mathbf{A}_0$ on an adapted bundle P over the cylinder, the linearized ASD operator in Coulomb gauge, $D_{\mathbf{A}_0} = d^+_{\mathbf{A}_0} + d^*_{\mathbf{A}_0}$, could be written in the form $\frac{d}{dt} + L_{\mathbf{A}_{0,t}}$ with a family of elliptic, self adjoint operators $L_{\mathbf{A}_{0,t}}$ converging over the ends to the extended Hessian operators $L_{\rho_\pm}$ of the Chern-Simons functional at the limit connections $\rho_\pm$.

In the simplest case where these limit connections are equal, $\rho_- = \rho_+ =: \rho$, we can take $\mathbf{A}_0$ to be the constant path of connections, equal to ρ for all times t. Since in this case, the family $L_{\mathbf{A}_{0,t}} \equiv L_\rho$ is constant in t, we can decompose the four dimensional deformation operator according to the spectrum of L_ρ,

$$D = \sum_{\lambda \in Spec(L_\rho)} \frac{d}{dt} + L_\lambda.$$

Now, under our assumptions on the critical points of Chern-Simons and formula 6, the operator L will have non-trivial kernel and the component corresponding to this kernel in the above decomposition is the operator

$$\frac{d}{dt} \colon L^2_1(\mathbb{R}, \ker L_\rho) \to L^2(\mathbb{R}, \ker L_\rho).$$

This is however not a Fredholm operator, and so the deformation operator cannot be Fredholm either.

This problem will occur whenever there are reducible ($H^0_\rho \neq \{0\}$) or *degenerate* ($H^1_\rho \neq \{0\}$) critical points. Dealing with these two issues is the object in most of the next chapters. We begin by dealing with the nontrivial stabilizers we always assume to exist.

For this, we start by recording a general criterion under which operators of the type above are Fredholm:

Theorem 3.0.2. *Let $L(t)$ be a smooth path of self adjoint elliptic operators on a vector bundle E over Y, let D be the operator over the cylinder $Y \times \mathbb{R}$,*

$$D = \frac{d}{dt} + L(t)$$

and let $p \geq 2$. Then D is a bounded, Fredholm operator $D\colon L^p_1 \to L^p$ if $L(t)$ converges in the operator norm to elliptic operators $L\pm$ over the ends for $t \to \pm\infty$ such that the spectra of both $L\pm$ are bounded away from zero.

We will abbreviate the condition of this theorem and say that the path L is invertible over the ends *in this case.*

Proof. This is proved for the specific case of L the extended Hessian in [Don02, Ch. 3]. The arguments there imply our slightly more general formulation without much change. Alternatively, this also follows from the results of [LMO85]. □

If the family $L(t)$ is the family of operators L_{A_t} defined by a connection over the cylinder which is constant and equal to flat connections $\rho_\pm$ over the ends, then the conditions of the above theorem are satisfied if and only if the limit connections $\rho_\pm$ are *acyclic*, i.e. if

$$H^0(Y; \mathrm{ad}\rho_\pm) = H^1(Y; \mathrm{ad}\rho_\pm) = \{0\}.$$

Thus, the ASD deformation operator is Fredholm if and only if these two cohomology groups vanish.

Note that under our assumption on the 3-manifold Y, this will never be the case: recall that we assumed that *all* G-representations of $\pi_1(Y)$ have abelian image, which we have seen (Proposition 2.6.1)to be equivalent to $H^0(Y; \mathrm{ad}\rho) \neq \{0\}$ for all such representations, i.e. for all flat connections over Y. We remedy this defect by working in different Sobolev norms.

3.1 Weighted Sobolev Spaces

Definition 3.1.1. We enumerate the ends of our fixed cylindrical end manifold X as

$$\phi_1 \colon Y_1 \times (0,\infty) \tilde{\to} C_1, \ldots, \phi_n \colon Y_n \times (0,\infty) \tilde{\to} C_n$$

and we denote by $K \coloneqq X \setminus \bigcup_{i=1}^n C_i$ the complement of the ends, a compact submanifold of X with boundary $\partial K = Y_1 \sqcup \cdots \sqcup Y_n$.

Let E be an adapted vector bundle (see Definition 2.3.3) over X with bundle norm $|-|$, $\mathbf{A}_0$ a fixed adapted connection with limit connections ρ_i on bundles $P_i \to Y_i$ and $\underline{\alpha} \coloneqq (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Given this data, we let $\sigma_{\underline{\alpha}} \colon X \to \mathbb{R}$ be a smooth function such that

$$\sigma_{\underline{\alpha}}(\phi_i(y,t)) = -\alpha_i t \text{ for all } t > T_0$$

(which condition we will sometimes abbreviate to "$\sigma_{\underline{\alpha}} = -\alpha_i t$ over C_i"). We now define our *weight function* W as

$$W \coloneqq \exp(\sigma_{\underline{\alpha}}).$$

Note that W is bounded away from zero on any compact interior domain in X. We define the *weighted Sobolev space* $L^2_{k,\underline{\alpha}}(Y \times \mathbb{R}, E)$ as the set of $L^2_{k,loc}$-sections s of E such that

$$\sum_{i=0}^{k} \int_{Y \times \mathbb{R}} W^2 |\nabla_A^{(i)} s|^2 dvol_Y dt < \infty.$$

The following is elementary but will be useful at several points throughout this thesis, so we record it as a remark:

Remark 3.1.2.

- *The map*

$$s \mapsto W^{-1} \cdot s,$$

  is a smooth isomorphism $L^2_k \to L^{2,\alpha}_k$ (this map will even be an isometry for $k = 0$ but is only bounded above and below for higher k). Conjugating with this map allows us to view any operator on weighted spaces as one on unweighted spaces.

- *These spaces depend on the adapted bundle, including the fixed limit structures ρ_i over the ends and on $\underline{\alpha}$, but not on the particular choices of $\sigma_{\underline{\alpha}}$ or $\mathbf{A}_0$.*

- *We will first only consider the Fredholm theory of the linearized instanton equations which goes through whwnever the weights we choose do not lie in the spectrum of the formal Hessian operators defined by our fixed end structures (see Definition 2.6.3). In fact, for certain index formulae, we will have to consider positive as well as negative weights. However, as soon as we apply this theory to actual instanton moduli spaces, we are forced to choose all weights strictly positive and we will then always choose them such that for all i, α_i lies strictly between zero and the smallest positive Eigenvalue of the formal Hessians operator defined by the limit connection ρ_i. The reason for this restriction is that the weighted Sobolev completed spaces of gauge transformations that play a central role in the the theory will only form a group for positive weights.*

- *Assume that we are given two adapted bundles over cylindrical end manifolds $\mathbf{P_1} \to X_1$ and $\mathbf{P_2} \to X_2$ with "matching end structures", i.e. a 3-manifold Y such that X_1 has an end E_+ modelled on $Y \times (0, \infty)$, X_2 has an end E_- modelled on $\overline{Y} \times (0, \infty)$ and such that the corresponding limit connections ρ_+ over E_+ and ρ_- over E_- agree. Then if we choose the corresponding weights $\alpha_- = -\alpha_+$, then the diffeomorphism*

  $$Y \times [-T_1, -T_0] \cong \overline{Y} \times [T_0, T_1]$$

  mapping $t \mapsto -t$, exactly identifies the deformation operators we will consider over the two ends. This will become important for proofs of gluing theorems.

Note that this weighted norm is induced by an inner product

$$\langle s, t \rangle_{\underline{\alpha}} \coloneqq \int W^2 \cdot (s, t),$$

where $(-,-)$ denotes the pointwise inner product of sections of our metric bundles. We thus have a notion of formal adjoints with respect to *weighted* L^2-norms, and we denote the formal adjoint of d_A with respect to this inner product by

$$d_A^{*,\alpha}.$$

This operator is more concretely given by

$$d_A^{*,\underline{\alpha}} = W^{-2} d_A^* W^2$$

and will give us the appropriate gauge condition for the based configuration spaces, as we will explain shortly. We can consider this adjoint as an operator on

unweighted spaces by conjugating it with the weight function as in the remark above.

In the following, we denote one end of our cylindrical end manifold X by $Y \times (0, \infty)$ (suppressing the fixed embedding of this cylinder into X in our notation) and we let W be a weight function as above with weight $\alpha \in \mathbb{R}$ corresponding to the end we consider. We also choose $T_0 > 0$ such that $W = \exp(-\alpha t)$ on $Y \times [T_0, \infty)$

Then the following is a simple calculation.

Lemma 3.1.3. *On $Y \times [T_0, \infty)$, we can write the operator $d_A^{*,\alpha}$, considered as an operator on unweighted spaces as above, as*

$$d_A^* + 2\alpha i_t,$$

where $i_t \colon \Omega^1(Y \times \mathbb{R}) \to \Omega^1(Y \times \mathbb{R})$ denotes the contraction with the vector $\frac{\partial}{\partial t}$.

We return to our deformation operators: recall that the operator $D_A \coloneqq d_A^* + d_A^+$ acting on $L_1^2\Omega^1(Y \times \mathbb{R})$ could be written

$$D_A = \frac{d}{dt} + L_{A_t}.$$

As before, an operator of this form over the tube is Fredholm if and only if L_{A_t} tends to invertible operators over each end (and in this case, it is itself, along with all restrictions to slices $\{t\} \times Y$, invertible when restricted over some infinite portion of the ends). Under our assumptions (which imply that the subspace $H^1(Y; \mathrm{ad}(\rho))$ of the kernel of L_∞ is at least one dimensional), this will, however, never be satisfied.

However, this operator, acting on *weighted* Sobolev spaces, *is* Fredholm: We identify it with an operator on unweighted spaces by conjugating with the weight function σ_α as in Remark 3.1.2 and compute over the ends:

$$e^{\alpha t} D_A e^{-\alpha t} = e^{\alpha t} \left(\frac{d}{dt} + L_{A_t} \right) e^{-\alpha t} = \frac{d}{dt} + (L_{A_t} - \alpha).$$

By Theorem 3.0.2, then, this operator will be Fredholm whenever $\alpha \notin Spec(L_{A_\infty})$.

3.2 Holonomy Perturbations

The above Fredholm theory is satisfactory from the point of view of analysis on 4-manifolds. However, the device of weighted Sobolev spaces is quite artificial and we need to make sure that the resulting Fredholm deformation theory captures all finite energy instantons over our cylindrical end maifolds. This amounts to

ensuring that all finite length flow lines of the Chern-Simons functional converge to critical points exponentially fast. As in finite dimensional Morse(-Bott) theory, this can only be expected if the critical points themselves are nondegenerate and for this, we need a further technical device: holonomy perturbations.

To set this up, we define the kinds of perturbations we will use throughout this work and recall some basic properties. Throughout this section, we fix a 2-form μ on D^2 with support in the interior and $\int_{D^2} \mu = 1$ and an $SU(2)$-bundle $P \to Y$.

Definition 3.2.1. Let $(\gamma_i \colon S^1 \times D^2 \to Y)_{i=1}^n$ be a collection of embedded solid tori. Given any conjugation invariant function $f \colon SU(2)^n \to \mathbb{R}$, we can define a function $\sigma_\gamma \colon \mathcal{B}_P \to \mathbb{R}$ by

$$\Phi_{\gamma,f}([A]) \coloneqq \int_{D^2} f\left(Hol_{\gamma_1(-,z)}(A), \dots, Hol_{\gamma_n(-,z)}(A)\right) \mu(z).$$

We call functions on $\mathcal{B}_P$ which arise in this way from a family of tori (γ_i) and a function f as above *cylinder functions*.

As a function on the space of connections $\mathcal{A}(Y)$, the formal gradient of a cylinder function $\Phi_{\gamma,f}$ at a point A is given by the formula

$$\operatorname{grad}_A \Phi_{\gamma,f} = \sum_{i=1}^{n} (\gamma_i)^* (\partial_i f)_\dagger (\operatorname{Hol}_{\gamma(-,z)}(A)\mu), \tag{7}$$

where $()_\dagger \colon \mathfrak{su}(2)^* \to \mathfrak{su}(2)$ is the isomorphism defined by the Killing form

$$(X, Y) \mapsto 2\operatorname{Tr}(XY)$$

of $SU(2)$ (this is equation (73) in [KM11]; for a proof, see Lemma 3 of [BD95]).

From this description of $\operatorname{grad} \pi$, we derive several useful properties. The following appears as Lemma 3.5 in [KM11]. Here, we denote the spaces of L^2_k-connections by $\mathcal{A}_k$ and their tangent bundles by $\mathcal{T}_k$.

Proposition 3.2.2. *Let Φ be a cylinder function. Then the following hold:*

- *For any $k \geq 1$, the formal gradient $\operatorname{grad} \pi$ defines a smooth section of the tangent bundle of the space of L^2_k-connections $\mathcal{A}(Y)$*
- *For any $j \leq k$, the derivative of the formal gradient, $\nabla \operatorname{grad} \pi$, on the L^2_k-configuration space extends to a smooth section*

$$\nabla \operatorname{grad} \pi \colon \mathcal{A}_k \to Hom(\mathcal{T}_j, \mathcal{T}_j)$$

- $\operatorname{grad}\pi$ *is uniformly* L^∞*-bounded by a constant which only depends on the function* f *and the form* μ *defining* Φ*:*

$$\|\operatorname{grad}\pi\|_{L^\infty} \leq C_{f,\mu}$$

- *For all* j*, there is a constant* K_j *such that for any base connection* A_0*,*

$$\|\operatorname{grad}_A \pi\|$$

- *For all* $p > 1$*, there is a constant* $C_p > 0$ *such that for all connections* $A_0, A_1 \in \mathcal{A}(Y)$ *we have*

$$\left\|\operatorname{grad}_{A_0}\Phi - \operatorname{grad}_{A_1}\Phi\right\|_{L^p} \leq C_p \left\|A_0 - A_1\right\|_{L^p}$$

In particular, these functions and their derivatives define smooth sections of $T\mathcal{A}$ *in any* L^p_k*-norm.*

Let (Φ_i) be a sequence of cylinder functions. We can then find a sequence of non zero constants (K_i) such that for any sequence (a_i) with the property that $\sum K_i|a_i|$ converges, the formula $\Phi \coloneqq \sum K_i\Phi_i$ defines a smooth function on $\mathcal{B}$. We call such functions *holonomy perturbations*. The set of sequences (a_i) such that $\sum K_i|a_i| < \infty$ is a Banach space, which we denote by Π. If such a choice of cylinder functions and constants is made, we will denote the holonomy perturbation function defined by $\pi \in \Pi$ by Φ_π. Functions of this kind behave in much the same way as cylinder functions. The following appears as Proposition 3.7 in [KM11]:

Lemma 3.2.3. *Let* Φ *be a cylinder function. Then for any* p, k*,* Φ *defines a smooth function on the space* $\mathcal{A}^p_k$*. Denoting its formal gradient, a section of its tangent bundle* $\mathcal{T}^p_k$*, we have the following:*

- Φ *is bounded on* $\mathcal{A}^p_k$
- $(\Phi, B) \mapsto V_\pi(B)$ *defines a smooth map* $V \in C^\infty(\Pi \times \mathcal{A}^p_k, \mathcal{T}^p_k)$
- $\|V_\pi(B)\|_{L\infty} \leq K \|\Phi\|_\Pi$*, uniformly in* B*, for some constant* K*.*
- *For* $l \leq k$*, the derivative in the* B*-variable defines a smooth map* $DV \in C^\infty(\Pi \times \mathcal{A}^p_k, Hom(\mathcal{T}^p_l, \mathcal{T}^p_l))$
- *For all* l*, there is a constant* K_l *such that*

$$\|V_\pi(B)\|_{L^2_{l,B_0}} \leq K_l \left(1 + \|B - B_0\|_{L^2_{l,B_0}}\right).$$

- *There is a constant C such that for all B, B' and all p, we have*

$$\|V_\pi(B) - V_\pi(B')\|_{L^p} \leq C \|B - B'\|_{L^p}.$$

Form now on, throughout the rest of this thesis, given a bundle $P \to Y$, we will always denote by $\mathcal{P}$ a choice of cylinder functions and Banach space Π with these properties.

Remark 3.2.4. *For bundles with structure group $SU(2)$, a more restricted family of perturbation functions has the same density property as the above proposition: Instead of allowing all functions $f\colon SU(2)^n \to \mathbb{R}$ that are invariant under the diagonal action by conjugation, we could instead start with any function $h\colon \mathbb{R}^n \to \mathbb{R}$. Using [Don02, Lemma 5.14], we can then choose a finite sequence of words $W_1, \dots, W_M$ such that the functions defined by the formula*

$$\Phi_{h,\gamma,W}([A]) \coloneqq \int_Y h(T_1(A), \dots, T_M(A))\mu,$$

where $T_i(A) \coloneqq trW_i(\mathrm{Hol}_{\gamma_1}(A), \dots, \mathrm{Hol}_{\gamma_n}(A))$, have the same genericity properties as the cylinder functions defined above.

The set of critical points of the unperturbed Chern Simons functional, the flat connections, is compact: by choosing a finite presentation of the fundamental group $\pi_1(Y)$, we can identify the set of flat connections up to gauge as a quotient of a closed subset in a finite product $SU(2)^K$. We will need an analogous compactness property for the critical points of the perturbed functional. In fact, a more general result is true:

Proposition 3.2.5. *Let Π be a Banach space of perturbations as above and let $\mathcal{C} \subset \Pi \times \tilde{\mathcal{B}}$ be the* parametrized critical set $\mathcal{C} \coloneqq \{(\pi, [(B, b)]) \mid \mathrm{grad}_B\, cs_\pi = 0\}$. *Then the projection map*

$$\mathcal{C} \to \Pi$$

is proper.

Proof. This is Lemma 3.3 in [Eis23]. The argument is unchanged from the corresponding statement in the unframed case, which appears as Lemma 3.8 in [KM11]. □

What we want to achieve by introducing these perturbations is to get a good Morse theory for the Chern-Simons functional $cs\colon \mathcal{B} \to \mathbb{R}/\mathbb{Z}$. We recall from Proposition 3.4 that, with respect to a fixed base connection A_0, this is given by

$$cs(A_0 + a) = cs(A_0) + \int_Y \mathrm{tr}(2a \wedge F_{A_0} + a \wedge d_{A_0} a + \frac{1}{3} a \wedge a \wedge a)$$

with formal gradndsdient

$$\operatorname{grad}_{A_0} cs = *_3 F_{A_0}$$

and formal Hessian $*_3 d_{A_0}$.

Definition 3.2.6. For a holonomy perturbation Φ as above, we call

$$cs_\pi := cs + \Phi$$

the Φ-*perturbed Chern-Simons functional*. We call critical points of cs_π Φ-*flat connections*.

This functional enjoys the same basic analytic properties as the unperturbed cs.

3.3 Linear Analysis

We have already seen that the Hessian of the unperturbed Chern-Simons functional on $\mathcal{A}_k^2(E)$ at A is given by the operator

$$*d_A \colon \Omega_k^1(Y; \operatorname{ad}(E)) \to \Omega_{k-1}^1(Y; \operatorname{ad}(E))$$

where the index indicates the Sobolev norms in which we complete these spaces of forms. At a flat connection A, where $d_A^* * d_A = \pm * d_A^2 = 0$, we can restrict this operator to the Coulomb slice through A to get an operator

$$H \colon \ker d_A^* \to \ker d_A^*$$

that we can identify with the Hessian of the Chern-Simons-functional on the configuration space $\mathcal{B}$ (i.e. up to gauge). At non-flat connections however, this description is not very convenient and this is where we bring the extended Hessian

$$L_A = \begin{pmatrix} 0 & -d_A^* \\ -d_A & *d_A \end{pmatrix} : \Omega^0(Y) \oplus \Omega^1(Y) \to \Omega^0(Y) \oplus \Omega^1(Y)$$

back in.

The formal Hessian of the perturbed Chern-Simons-functional cs_π is given by

$$\operatorname{Hess}_A(cs_\pi) = *d_A + D_A V_\pi \colon \Omega_k^1(Y; \operatorname{ad}(E)) \to \Omega_{k-1}^1(Y; \operatorname{ad}(E)),$$

where again V_π denotes the formal gradient of the holonomy perturbation function defined by π and $D_A V_\pi$ its derivative.

We ultimately want to study cs_π as a functional on our framed configuration spaces $\tilde{\mathcal{B}}$. First, we extend cs_π to the space of framed connections $\mathcal{A} \times E_b$ by

zero on the fiber summand and then use the invariance of cs and holonomy perturbations under the action of $\mathcal{G}$ to define it on the quotient. The tangent space to the quotient at $[A]$ can be identified with the subspace $\ker d_A^* \oplus \mathfrak{g}_A^\perp \subset \Omega^1(Y; \mathrm{ad}(E)) \times \mathfrak{g}_b$. We can then identify the formal Hessian of cs_π on the based configuration space as the operator

$$(*d_A + D_A V_\pi) \oplus 0$$

on $\ker d_A^* \oplus 0$.

We can again wrap up our gauge fixing condition with the formal Hessian by considering the extended *perturbed* Hessian operator

$$L_A^\pi = \begin{pmatrix} 0 & -d_A^* \\ -d_A & *d_A + D_A V_\pi \end{pmatrix} : \Omega^0(Y) \oplus \Omega^1(Y) \to \Omega^0(Y) \oplus \Omega^1(Y)$$

This operator differs from the unperturbed extended Hessian by the compact operator $*d_A + D_A V_\pi$ and is thus itself first order self adjoint elliptic. In particular, it's spectrum is real and discrete.

We have Hodge decompositions

$$\Omega^0 = \ker d_A \oplus \mathrm{Im} d_A^*$$

and

$$\Omega^1 = \mathrm{Im} d_A \oplus \ker d_A^*.$$

With respect to this decomposition, L_A^π can be written

$$L_A^\pi = 0 \oplus \mathrm{Hess}_A(cs + \pi) \oplus S_A,$$

where

$$S_A = \begin{pmatrix} 0 & -d_A^* \\ d_A & 0 \end{pmatrix}$$

is the signature operator on $\Omega^0 \oplus \mathrm{Im} d_A$ and $\mathrm{Hess}_A(cs+\pi)$ denotes the restriction of the Hessian to the Coulomb slice $\ker d_A^*$. As before, the kernel of L_A^π is identified with the kernel of the Laplacian on $\Omega^0 \oplus \Omega^1$. Note that in particular,

$$H_A^0 \coloneqq \ker(d_A \colon \Omega^0 \to \Omega^1) = Lie\, \mathrm{Stab}(A)$$

is always in the kernel of L_A^π.

Definition 3.3.1. We call a critical point A of $cs + \pi$ *nondegenerate* if the kernel of extended perturbed Hessian consists only of the space of A-harmonic 0-forms H_A^0.

Note that the nondegenerate critical points are thus equivalently those at which the Hessian is an isomorphism on the slice $\ker d_A^*$.

As alluded to in the introduction of section 4.2, we will use holonomy perturbations to make sure we can work with a functional whose critical points are all non-degenerate. That this is always possible is ensured by the following:

Proposition 3.3.2. *Let Y be a smooth, closed, oriented 3-manifold and let $E \to Y$ be a smooth $SO(3)$-bundle. Then we can choose a Banach space of holonomy perturbations Π as above in such a way that for every π in a dense open subset of Π, there are only finitely many critical orbits of cs_π, all of which are non-degenerate.*

Proof. This is Theorem 3.6 in [Eis23]. □

4 Moduli Spaces

Throughout this entire section, we let Φ be a holonomy perturbation over Y and X a smooth, connected, oriented Riemannian 4-manifold with cylindrical ends. We will denote its set of ends by $\mathcal{C}_X$. We will use the convention that that a closed manifold X is a manifold with cylindrical ends and $\mathcal{C}_X = \emptyset$.
We will also fix a class $c \in H^2_{comp}(X;\mathbb{Z})$ (where $H^\bullet_{comp}$ denotes cohomology with compact support). We require that this class restricts trivially on the ends. We also a smooth adapted $U(2)$-bundle $(P \to X, \eta, \mathbf{A}_0)$ with $c_1(P) = c$. Recall that an adapted bundle is a bundle together with a family $\eta = (\eta_C)_{E\in\mathcal{C}_X}$ of isomorphisms $\eta_C \colon P\big|_C \to p^*Q_C$, where $p_C \colon C \to Y$ is the projection and Q_C is a $U(2)$ bundle over Y, a family $\rho = (\rho_C)$ of Φ-flat connections ρ_C on Q_C and a connection $\mathbf{A}_0$ on P such that $(\eta_C)_*\mathbf{A}_0 = p_C^*\rho_C$.

4.1 Configuration Spaces over Manifolds with Cylindrical Ends

Definition 4.1.1. Let X be a cylindrical end manifold and π a holonomy perturbation. A *π-adapted bundle* over X consists of the following data:

- a smooth principal G-bundle $P \to X$
- for each end $C \cong Y_C \times [0,\infty)$ of X a fixed isomorphism of G-bundles $\psi_C \colon P\big|_E \to (p_C^*Q_C \to C)$ for some G-bundle $Q_C \to Y_C$ (and where $p_C \colon E \to Y_C$ is the projection induced from the fixed isomorphism $C \cong Y_C \times [0,\infty)$)
- a connection $\mathbf{A}_0$ on P such that for each end C, we have $\psi_C^*\mathbf{A}_0\big|_C = p_C^*A_C$ for some π-flat connection A_C on Q_C.

As we have seen in the case of flat limit connections, we can associate to such a bundle a relative Chern class $c(P)$. In the presence of a nontrivial perturbation π, the limit connections will no longer be flat and the Chern-Weil form $\mathrm{tr}(F_{\mathbf{A}}^2)$ that we used to define this class will no longer be integrable. We can instead associate to P an analytic energy term, which will play the same role as the relative Chern class did in the unperturbed situation, as follows:

Definition 4.1.2. Given a holonomy perturbation π, a π-adapted bundle P as above and an adapted connection $\mathbf{A}$ on P, we call

$$\mathcal{F}(\mathbf{A}) \coloneqq \int \left\| F_{\mathbf{A}} + \hat{V}_\pi(\mathbf{A}) \right\|^2$$

the *analytic energy* of $\mathbf{A}$.

Lemma 4.1.3. *The analytic energy depends only on the adapted bundle P.*

We now fix a cylindrical end manifold X with ends $C_1, \dots, C_n$ modelled on 3-manifolds $Y_1, \dots, Y_n$, a family $\pi = (\pi_1, \dots, \pi_n)$ of holonomy perturbations (where π_i is defined over Y_i) and a π-adapted bundle $P \to X$ with adapted connection $\mathbf{A}_0$, equal to the pullback of a π-flat connection ρ_i over the end C_i.

Our next task is to define and study moduli spaces for a π-perturbed instanton equation using the Fredholm theory developed in section 3. We thus fix weights $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ for each of the ends of X. Then Theorem 3.0.2 tells us that in order for the deformation operators arising as the linearization of our perturbed equation to be Fredholm, we have to choose each α_i in the complement of the spectrum of the formal Hessian operator defined by ρ_i. We will see shortly that in order to define a *group* of gauge transformations acting on weighted spaces of connection 1-forms, we will in addition have to choose all our weights to be strictly positive. For these reasons, we choose our weights $\underline{\alpha}$ such that for all $i \in \{1, \dots, n\}$

$$0 < \alpha_i < \min\{\lambda \in \operatorname{Spec} \operatorname{Hess}_{\rho_i} cs_\pi \mid \lambda > 0\}.$$

Definition 4.1.4. Let $\underline{\alpha}$ as above and let $p > 1$. We denote our spaces of connections

$$\mathcal{A}_k^{p,\underline{\alpha}}(P) \coloneqq \{\mathbf{A}_0 + a \mid a \in L_k^{p,\underline{\alpha}}(X, \Lambda^1(X; \operatorname{ad}(P)))\}$$

and

$$\tilde{\mathcal{A}}_k^{p,\underline{\alpha}}(P) \coloneqq \mathcal{A}_k^{p,\underline{\alpha}}(P) \times P_x,$$

where $x \in X$ is some base point and P_x the fiber of P over x.

$\mathcal{A}_k^{p,\underline{\alpha}}(P)$ is called the *configuration space*, $\tilde{\mathcal{A}}_k^{p,\underline{\alpha}}(P)$ is called the *framed configuration space*.

This definition is chosen so that our technical setup from the previous chapters applies to these configuration spaces. However, we will later deal with connections arising from compactness arguments, where we do not have immediate control of the Sobolev norms of connection forms, only of the analytic energy. To make sure that we can apply all results about the moduli spaces as defined above to such solutions, we note the following consequence of our decay results in chapter 5:

Lemma 4.1.5. *Let X be a cylindrical end manifold, π be a regular perturbation on X and let $\mathbf{A}$ be a π-ASD-connection of class $L^2_{2,loc}$. Assume in addition that $\|\mathcal{F}(\mathbf{A})\|_{L^2(X)} < \infty$. Then there is some adapted bundle $P \to X$ with adapted connection $\mathbf{A}_0$ and $\delta > 0$ such that*

$$\mathbf{A} \in \mathcal{A}_2^{2,\delta}(P).$$

Proof. The claim is that for each end of X, there is a critical point ρ of cs_π such that the 1-form $\mathbf{A} - pr^*\rho$ decays exponentially over the end. This follows immediately from theorem 4.5.4.

We can then choose P to be the adapted bundle whose underlying topological bundle is the one that $\mathbf{A}$ is defined on, with end structures given by the critical points ρ as above.

Note that in particular, if $\mathbf{A}_0$ is an adapted connection which is constant and equal to these critical points over the ends of X, then the form $\mathbf{A} - \mathbf{A}_0$ will be of class $L_2^{2,\delta}$ for some δ depending only on the spectra of the formal Hessian operators at these finitely many critical points. □

The appropriate gauge group in this context is slightly more complicated than in the more familiar case of irreducible limits over the ends, in which case one simply takes gauge transformations that tend sufficiently rapidly to the identity over each end. This is not quite sufficient in our case (as we assume that all possible limits are reducible, i.e. have positive dimensional stabilizer), instead we want to define a group of gauge transformations that tend to parallel sections over the ends. For this purpose, we say an automorphism u_0 of the smooth bundle P has *parallel ends* if the restriction of u_0 to each end E is the pullback of an element in the stabilizer $\mathrm{Stab}(A_E)$ along the projection $E \to Y_E$ and denote the set of automorphisms with parallel ends as $\mathcal{G}(P)_{par}$. We can then define

$$\mathcal{G}^{p,\alpha}_{k+1}(P) \coloneqq \{u \in \Gamma(AutP) \mid \exists u_0 \in \mathcal{G}(P)_{par} \colon u - u_0 \in L^{p,\alpha}_{k+1}(\mathrm{End}P)\}.$$

This contains the subgroup of those gauge transformations which tend to the identity over the ends and which we will denote by $\mathcal{G}^0_{k+1}(P)$, suppressing mention of the weights in our notation.

Proposition 4.1.6. *If $k + 1 - \frac{4}{p} > 0$, then $\mathcal{G}^{p,\alpha}_{k+1}(P)$ is a Banach Lie group modelled on $L(\mathcal{G}) \coloneqq \{\mathfrak{u} \in \Gamma(\mathfrak{g}(P)) \mid \nabla\mathfrak{u} \in L^{p,\alpha}_k(\mathfrak{g}(P))\}$. If in addition $p = 2$, then it is a Hilbert Lie group.*

Proof. This is Proposition 4.12 in [Don02]. □

Elements of $\mathcal{G}^{p,\alpha}_{k+1}(P)$ are continuous sections that tend to a limit in the stabilizer subgroups Γ_{ρ_C} over the end C. These evaluation maps $\mathcal{G}^{p,\alpha}_{k+1}(P) \to \mathrm{Stab}(\rho_C)$ are surjective continuous group homomorphisms and we get that

$$\mathcal{G}^{p,\alpha}_{k+1}(P)/\mathcal{G}^0_{k+1} \cong \prod_C \mathrm{Stab}(\rho_C). \tag{8}$$

This group acts on both $\mathcal{A}_k^{p,\alpha}(P)$ and $\tilde{\mathcal{A}}_k^{p,\alpha}(P)$ and we denote the quotients

$$\mathcal{B}_k^{p,\alpha} \coloneqq \mathcal{A}_k^{p,\alpha}(P)/\mathcal{G}_{k+1}^{p,\alpha}(P) \text{ and } \tilde{\mathcal{B}}_k^{p,\alpha} \coloneqq \tilde{\mathcal{A}}_k^{p,\alpha}(P)/\mathcal{G}_{k+1}^{p,\alpha}(P).$$

Proposition 4.1.7. *(i) The action of $\mathcal{G}_{k+1}^{p,\alpha}(P)$ on $\mathcal{A}_k^{p,\alpha}(P)$ is smooth.*

(ii) The action of $\mathcal{G}_{k+1}^{p,\alpha}(P)$ on $\tilde{\mathcal{A}}_k^{p,\alpha}(P)$ is smooth and free and the quotient $\tilde{\mathcal{B}}_k^{p,\alpha}$ is a smooth Banach manifold (Hilbert manifold if $p = 2$).

Proof. This is again Proposition 4.12 of [Don02]. □

Note that $L(\mathcal{G})$ is strictly larger than $L_{k+1}^{p,\alpha}(\mathfrak{g}(P))$ when the π-flat structures over the end are not irreducible. In fact, the latter space is precisely the Lie algebra of the subgroup $\mathcal{G}_{k+1}^0$:

Proposition 4.1.8. *Under the same hypothesis as the previous proposition, $\mathcal{G}_0(P)$ is a Banach Lie group modelled on $L_{k+1}^{p,\alpha}$.*

Note that the action by this subgroup is free on the unframed space of connections $\mathcal{A}$: a $\mathbf{A}$-parallel section $u \in \mathcal{G}_{k+1}^0$ for an adapted connection $\mathbf{A}$ is uniquely determined by its value at any one point. Now choose an open path γ running down one end of X and pull back P, $\mathbf{A}$ and u to $\mathbb{R}$ along this path. Then in a trivialization of the pulled back bundle wrt which $\mathbf{A}$ becomes the trivial connection d, the path u in the automorphisms of the pulled back bundle, is constant. OTOH, the path also converges to the identity at ∞, so it must be the constant path with value id. However, the equation $u(x_0) = \mathrm{id}$ is gauge invariant (as the identity transformation is), so $u(x_0) = \mathrm{id}$ for any point x_0 lying on the path. From this, we get $u = \mathrm{id}$.

We proceed with the linearized version of our perturbed gauge theory. First, we identify the tangent spaces to our configuration spaces:

As we have seen, the space of connections is affine over a space of 1-forms, so we can canonically identify

$$T_{\mathbf{A}}\mathcal{A}_k^{p,\alpha} \cong L_k^{p,\alpha}(X, \Omega^1(\mathrm{ad}(P))).$$

(For the rest of this section, we fix p, k such that $k + 1 - \frac{4}{p} > 0$ and write this latter space as simply Ω_k^1 for convenience). The differential of the (free) action of $\mathcal{G}_{k+1}^0(P)$ on $\mathcal{A}_k^{p,\alpha}$ at $\mathbf{A}$ is given by

$$d_{\mathbf{A}} \colon \Omega_k^0(X, \mathrm{ad}(P)) \to \Omega_{k-1}^1(X, \mathrm{ad}(P)).$$

For the weighted Sobolev completions we are working in, we have a Hodge decomposition

$$\Omega^1_k = \ker d^{*,\alpha}_{\mathbf{A}} \oplus \mathrm{Im} d_{\mathbf{A}},$$

and so the cokernel of $d_{\mathbf{A}}$ can be identified with $\ker d^{*,\alpha}_{\mathbf{A}}$. The quotient of the unframed space of connections enjoys the same regularity properties as the based configuration space. In fact, the action is free and smooth if $k - \frac{4}{p} > 1$ and we again have slices for this action:

Proposition 4.1.9 (Slice Theorem)**.** *Let $\varepsilon > 0$, $p \geq 2$, $k \in \mathbb{N}$, $\underline{\alpha}$ a tuple of weights for the ends of X and P as above and $\mathbf{A}$ a connection of class $L^{p,\underline{\alpha}}_k$. We define $d^{*,\underline{\alpha}}_{\mathbf{A}}$ as the $L^{2,\underline{\alpha}}$-adjoint of $d_{\mathbf{A}}$. Then, we have the following:*

- *$d^{*,\alpha}_{\mathbf{A}} \colon L^{p,\alpha}_k \to L^{p,\alpha}_{k-1}$ is well defined and continuous.*
- *The image of $d_{\mathbf{A}} \colon L^{p,\alpha}_1 \to L^{p,\alpha}$ is closed and we have an L^2-orthogonal decomposition $L^{p,\alpha}_1 = \ker d^{*,\alpha}_{\mathbf{A}} \oplus d_{\mathbf{A}}(L^{p.\alpha}_2)$*
- *A neighborhood of $[\mathbf{A}]$ in the quotient $\mathcal{A}^{p,\alpha}_1/\mathcal{G}^0_2(P)$ is diffeomorphic to a neighborhood of 0 in $\ker d^{*,\alpha}_{\mathbf{A}}$.*

Proof. The first two items are proved in Lemma 10.2.1 and Proposition 10.3.1 in [MM92]. For the third, see [Don02], p.90-91, below Proposition 4.12. □

The quotient $\mathcal{A}^{p,\alpha}_k(P)/\mathcal{G}^0_{k+1}$ is diffeomorphic to $\tilde{\mathcal{B}}^{p,\alpha}_k$ only in the case where the limit connections over the ends are fully reducible. If these limit connections are instead irreducible, the quotient by $\mathcal{G}^0_{k+1}$ is instead diffeomorphic to the unframed configuration space $\mathcal{B}$. For a reducible, but not fully reducible limit connection, we think of this quotient as an intermediate step between the framed and unframed configuration spaces.

This space is of interest to us as our dimension formulas are most conveniently stated and proved for the configuration space based at infinity.

4.2 Perturbations in four dimensions

As said in the introduction, our ultimate goal is to define a type of Morse-homology for the perturbed functional. This is not possible naively because:
the gradient flow equation does not even satisfy short time existence for arbitrary initial data, a consequence of the non-linearity of the ASD-equation.

For this reason, we want to reinterpret this flow equation as an equation for 4-dimensional connections as in section 2.5.

We first define the appropriate perturbations we need over 4-dimensional manifolds. Our exposition here is quite brief and we refer to section 4.1 in [Eis23] and section 3(i) in [Kro05] for more details.

Definition 4.2.1. Let π be a holonomy perturbation over Y. On a cylinder $Y \times \mathbb{R}$, we define $\hat{V}_\pi \colon \mathcal{A}(P) \to \Omega^2_+(X; \mathrm{ad}(P))$ over X by

$$\hat{V}_\pi(\mathbf{A}) \coloneqq (dt \wedge V_\pi(\mathbf{A}_t))^+.$$

On cylinders $Y \times \mathbb{R}$, we define perturbations to be simply $\hat{V}_\pi$. For X a cylindrical end manifold that is not diffeomorphic to a cylinder with ends modelled on $Y_e \times (0, \infty)$, we have to interpolate between different perturbations on the different ends of X. For this, let $\pi \coloneqq (\pi_e)$ be a family of holonomy perturbations over the Y_e and $P \to X$ a bundle. We then choose a smooth function β such that $\beta \geq 0$ on all of X and

$$\beta|_{\bigcup_e Y_e \times (2,\infty)} \equiv 1 \text{ and } \beta_{X \setminus \bigcup_e (Y_e \times [1,\infty))} \equiv 0.$$

We now define the *end perturbation* on X defined π as

$$\beta \hat{V}_{\pi_e}$$

on the end $Y_e \times (0, \infty)$ of X. Abusing notation, and hoping that it will be clear from context that the term means slightly different things over cylinders and non-cylinder cylindrical end manifolds, we write the above simply as

$$\hat{V}_\pi$$

over all of X.

There is a further type of perturbations that one needs to make. This is only needed for the regularity theorems for our moduli spaces, which we state but not prove. We will thus only give a very coarse overview for these:

Let X be a cylindrical end manifold and let $X_0 \subset X$ be the non-empty compact complement of the ends of X. We then choose an open ball $B \subset X_0$ and a finite collection of submersions

$$\underline{\gamma} \coloneqq (\gamma_i \colon B \times S^1 \to X_0)_{i=1}^N,$$

all of which restrict to the inclusion $B \to X_0$ on $B \times \{1\}$ and to immersed curves $\{b\} \times S^1 \to X_0$ for all $b \in B$. The holonomy automorphisms along the curves $\gamma_i(b, -)$ of a connection $\mathbf{A}$ on $E \to X$ define sections of the automorphism bundle

of E over B. We can collect these into one section of the bundle $\mathrm{Aut}(E)^N\big|_B$ which we call $\mathrm{Hol}_{\underline{\gamma}}(\mathbf{A})$. We now choose a smooth gauge-equivariant bundle map

$$r\colon \mathrm{Aut}(E)^N \to \mathfrak{g}(E)$$

and a smooth, self-dual 2-form μ on B and define the *interior* or *inner holonomy perturbation* as the map $\mathcal{A}(E) \to \Omega^2_+(X;\mathfrak{g}(E))$ defined by

$$r(\mathrm{Hol}_{\underline{\gamma}}(\mathbf{A})) \otimes \mu,$$

smoothly extended by zero to the complement of the support B of μ.

Now, over a cylindrical end manifold X that is not a cylinder, by a *perturbation* we mean a family of end perturbations over all ends of X and an interior perturbation on the complement of the ends.

Again, perturbations of this type can be collected into sufficiently large Banach spaces which we will write $\Pi = \Pi_{inn} \oplus \bigoplus_e \Pi^e_{end}$.

Having chosen such a perturbation $\pi \in \Pi$, in the interest of notational brevity we will always write the perturbation terms, evaluated on $\mathbf{A}$ as

$$\hat{V}_\pi(\mathbf{A}).$$

These perturbations enjoy analogous analytic properties as our 3-dimensional perturbations. This is the content of Proposition 4.4 from [Eis23].

4.3 The perturbed Instanton Equation and Moduli Spaces

As before, the theory of perturbed instantons over cylinders is intricately connected with the Morse theory of the perturbed Chern-Simons. Explicitly, let $\mathbf{A}$ be a connection on the trivial $SU(2)$-bundle $P = Y \times \mathbb{R} \times SU(2) \to Y \times \mathbb{R}$ in temporal gauge. We can then decompose the curvature tensor $F_{\mathbf{A}}$ in the following way:

$$F_{\mathbf{A}}(t,y) = F_{A_t} + \frac{\partial A_t}{\partial t} \wedge dt,$$

where $A_t \coloneqq (\iota_t\colon Y \times \{t\} \hookrightarrow Y \times \mathbb{R})^*\mathbf{A}$ is the pulled back connection form.

Now let π be a perturbation as above. Writing $\hat{V}_\pi(\mathbf{A}) \coloneqq (dt \wedge \mathrm{grad}_{A_t}\, \pi)^+$ for the four dimensional perturbation term and

$$\mathcal{F}(\mathbf{A}) \coloneqq F_{\mathbf{A}} + \hat{V}_\pi(\mathbf{A})$$

for the perturbed curvature, we compute:

$$\begin{aligned}2\mathcal{F}(\mathbf{A})^+ =&\mathcal{F}(\mathbf{A}) + *_4\mathcal{F}(\mathbf{A})\\ =&F_{A_t} + *_3\frac{\partial A_t}{\partial t} + *_3 \operatorname{grad}_{A_t}\pi +\\ &dt\wedge\left(*_3F_{A_t} + \frac{\partial A_t}{dt} + \operatorname{grad}_{A_t}\pi\right).\end{aligned}$$

In particular, we see that the *π-perturbed instanton equation* $\mathcal{F}(\mathbf{A})^+ = 0$ is equivalent to the downward gradient flow equation for the perturbed functional cs_π, namely

$$\frac{\partial A_t}{dt} = -*_3 F_{A_t} - \operatorname{grad}_{A_t}\pi.$$

Our configuration spaces come equipped with bundles $\mathcal{S}\to\tilde{\mathcal{B}}^{p,\alpha}_k$ with fiber (the $L^{p,\alpha}_k$-completions of) Ω^2_+: The trivial bundles with fiber Ω^2_+ over $\mathcal{A}^{p,\alpha}_k(P)$ and $\tilde{\mathcal{A}}^{p,\alpha}_k(P)$ are $\mathcal{G}^{p,\alpha}_k(P)$- resp. $\mathcal{G}_0$-equivariant , so descend to bundles over the configuration spaces. These bundles admit smooth sections, given by

$$s\colon [\mathbf{A}]\mapsto F^+_{\mathbf{A}} + \hat{V}_\pi(\mathbf{A}).$$

We define the *framed moduli space of Φ-ASD connections on P* as

$$\tilde{\mathcal{M}}_\pi(P) \coloneqq s^{-1}(0)\subset\tilde{\mathcal{B}}^{p,\alpha}(P).$$

The linearization of the perturbed ASD-equation at $\mathbf{A}$ is given by

$$d^+_{\mathbf{A}} + \nabla_{\mathbf{A}}\hat{V}_\pi\colon \Omega^1\to\Omega^2_+.$$

Just as in the unperturbed case we wrap up the linearized ASD-operator with our slice condition and consider the *perturbed deformation operator*

$$D^\pi_{\mathbf{A}} \coloneqq d^{*,\alpha}_{\mathbf{A}} + d^+_{\mathbf{A}} + \nabla_{\mathbf{A}}\hat{V}_\pi\colon L^{p,\alpha}_k\Omega^1\to L^{p,\alpha}_{k-1}\Omega^0\oplus L^{p,\alpha}_{k-1}\Omega^2_+.$$

This operator is the sum of the elliptic unperturbed deformation operator and an operator of order 0, so the perturbed deformation operator is elliptic, as well. However, as we are working on non-compact manifolds, it is not automatically Fredholm. However, we can again proceed as in section **??**: For a cylinder $X = Y\times\mathbb{R}$, we can write

$$\Omega^1(X) = \Gamma(X; pr^*(\Lambda^0_Y\oplus\Lambda^1_Y)). \tag{9}$$

Abusing notation, we will from now on write this latter space of sections as $\Omega^0_Y\oplus\Omega^1_Y$. Again using the formulas

$$d^*_{\mathbf{A}_0}(\alpha(t)+\beta(t)dt) = d^*_{\mathbf{A}_{0,t}}\alpha + *_3\frac{\partial\beta}{\partial t}$$

and

$$d^+_{\mathbf{A}_0}(\alpha + \beta dt) = (-d_{\mathbf{A}_{0,t}}\beta + \frac{\partial \alpha}{\partial t} + *_3 d_{\mathbf{A}_{0,t}}\alpha) \wedge dt + *_3(-d_{\mathbf{A}_{0,t}}\beta + \frac{\partial \alpha}{\partial t} + *_3 d_{\mathbf{A}_{0,t}}\alpha).$$

we see that we can write this operator with respect to the composition 9 in the form

$$D^\pi_{\mathbf{A}_0} = \frac{d}{dt} + L^\pi_{\mathbf{A}_{0,t}}, \tag{10}$$

where $L^\pi_{\mathbf{A}_{0,t}}$ is the perturbed extended Hessian operator.

Using this, one can prove:

Proposition 4.3.1. *Let X be a smooth oriented 4-manifold with cylindrical end, P an adapted bundle over X and $\mathbf{A}$ any adapted connection on P. Then the deformation operator $d^*_{\mathbf{A}} + d^+_{\mathbf{A}} + \nabla_{\mathbf{A}}\hat{V}_\pi \colon L^{p,\alpha}_1 \to L^{p,\alpha}$ is Fredholm for any $p \geq 2$, regular perturbation Φ and weight α not in the spectrum of all limiting connections of $\mathbf{A}$.*

Proof. This is proved in [Don02, section 3.3.3]. □

The index of the deformation operator depends on the adapted bundle that $\mathbf{A}$ is defined on and on the perturbation π, but not on the specific choice of adapted connection. So, attached to an adapted bundle P and perturbation π, w have an index $\operatorname{ind} D^\pi_{\mathbf{A}}$. This is the *expected dimension* of the moduli space $\mathcal{M}_\pi(P)$.

We can then, for a sufficiently large Banach space of perturbations Π, always find $\pi \in \Pi$ such that the corresponding framed moduli spaces behave as expected from the case of closed manifolds:

Proposition 4.3.2. *Given regular end perturbations, there are interior perturbations arbitrarily close to 0 such that the framed moduli space is a smooth, finite dimensional manifold. In fact, the set of such interior perturbations is residual dense in an open neighborhood of 0 in the space of perturbations agreeing with the chosen end perturbations over the ends.*

Proof. For closed X, this is Uhlenbeck-Freed's generic metrics theorem. For X with cylindrical ends, this is the content of [Eis23, section 4.6]. In the closed case, we can take the perturbation to be 0, in the non-closed case, a non-vanishing perturbation is generally necessary. □

Corollary 4.3.3. *For a perturbation π as in the above Proposition, the dimension of the framed moduli space is given by*

$$\dim \tilde{\mathcal{M}}_\pi(P) = \operatorname{ind} D^\pi_{\mathbf{A}},$$

where $[\mathbf{A}]$ is any element of $\mathcal{M}_\pi(P)$ or an adapted base connection on P.

There is an additional action by the group $\Gamma \coloneqq \mathcal{G}^{p,\alpha}_k(P)/\mathcal{G}_0$ and we call the quotient $\mathcal{M}_\pi(P)$. The action by Γ is free on the subset of irreducible π-ASD-connections. The dimension of the irreducible part $\mathcal{M}_\pi(P)^*$ is then given by

$$\dim \mathcal{M}_\pi(P)^* = \operatorname{ind} D^\pi_{\mathbf{A}} - \dim \Gamma. \tag{11}$$

and by formula 8 it's then clear that

$$\dim \mathcal{M}_\pi(P)^* = \operatorname{ind} D^\pi_{\mathbf{A}} - \sum_e \dim \operatorname{Stab}(\rho_e). \tag{12}$$

where $[\rho_e] \in \mathcal{B}(Y_e)$ is the π-flat end structure of the adapted bundle P over the end e.

Definition 4.3.4. We call

$$\dim_f \mathcal{M}_\pi(P) \coloneqq \operatorname{ind} D^\pi_{\mathbf{A}} - \dim \Gamma$$

the *formal dimension* of the moduli space $\mathcal{M}_\pi(P)$

Remark 4.3.5. *Note that if the perturbation can be chosen such that irreducible part $\mathcal{M}_\pi(P)^*$ of the moduli space is a manifold (i.e. such that the π-perturbed deformation operator is surjective at any π-perturbed instanton), then the formal dimension agrees with the dimension of this manifold. This is the reason for the terminology; we expect the moduli spaces to be generically manifolds of dimension $\dim_f \mathcal{M}_\pi(P)$.*

If X is a cylindrical end manifold with ends modelled on rational homology 3-spheres, $b_1(X) = 0$ and $b^+_2(X) > 0$, and P is non-trivial, we can always adjust the Riemannian metric and perturbation slightly such that $\mathcal{M}_\pi(P)^*$ becomes a manifold (this is Theorem 4.37 in [Eis23]).

In particular, if for X, P as in the previous paragraph the irreducible part of the moduli space is non empty, then the formal dimension must be non negative.

Due to the possible presence of reducible solutions, we do not expect the whole unframed moduli space to be a manifold in general. However, under certain topological assumptions on the base manifold we have:

Proposition 4.3.6. *Let X^4 have two cylindrical ends modelled on Y_1 and Y_2 and $b_1(X) = 0$ and $b_2^+(X) > 0$. Let $\mathbb{E} \to X$ be a smooth G-bundle over X. If $\mathbb{E}$ is non-trivial, then for any pair (Φ_1, Φ_2) of regular perturbations over the ends, there is a category 2 set of interior holonomy perturbations such that the moduli space of perturbed instantons on $\mathbb{E}$ does not contain any reducible connections.*

Proof. This is Proposition 4.16 from [Eis23]. □

Remark 4.3.7. *If we use $U(2)$-bundles with fixed nontrivial c_1, the above Lemma covers the general case: because $c_1 \neq 0$ and Chern-Weil theory, the corresponding moduli spaces never contain flat connections. This theory subsumes the $SO(3)$ and $SU(2)$- Donaldson invariants by [Morgan-Mrowka, A Note On Donaldson's Polynomial Invariants]. Alternatively, the vanishing theorem of Kronheimer-Mrowka holds for $U(2)$-invariants, so no blow-ups are necessary at all.*

In our situation, we will only deal with manifolds X with at most one cylindrical end $b_1(X) = 0$ and $b_2^+(X) > 0$ and cylinders $Y \times \mathbb{R}$. To see that the Proposition applies to the former, we simply delete one point x from X and reparametrize the complement of x in a small ball around x as $S^3 \times (0, \infty)$ to get a manifold with two cylindrical ends, one of them a cylinder over a 3-sphere. Note that we can choose such a reparametrization as a conformal equivalence, so the corresponding Hodge star operators and thus the ASD-equation remain unchanged.

Unfortunately, in the latter case of cylinders over rational homology 3-spheres, the above Propsition is not applicable and in fact its conclusion does not hold in this case, as the constant (in the time variable) connection, equal to a reducible critical point of cs_π for all times, will always be a solution. However, up to gauge, these examples are in fact the only possible reducibles over cylinders:

Lemma 4.3.8. *Let Y be a rational homology 3-sphere.*
Then for π ranging in a small neighborhood of 0 in Π, the only reducible π-perturbed instantons over $Y \times \mathbb{R}$ with finite analytic energy are gauge equivalent to the pullback of a reducible critical point of cs_π to the cylinder.

Proof. Let $\mathbf{A}$ be a reducible π-instanton on P with $\|\mathcal{F}_A\|^2_{L^2(C)} < \infty$. Being reducible, $\mathbf{A}$ is induced from a π-perturbed $U(1)$-instanton A on a $U(1)$-bundle $Q \to C$ with

$$Q \times_{U(1) \hookrightarrow SU(2)} SU(2) \cong P \quad (*).$$

Since the inclusion $Y \times \{0\} \hookrightarrow C$ is a homotopy equivalence, the bundle Q is pulled back from a $U(1)$-bundle $Q' \to Y$. In particular, all restrictions $Q|_{Y \times \{t\}}$ are isomorphic to Q' as $U(1)$-bundles over Y.

We claim that if $\eta \to Y$ is a line bundle, then there exists $\varepsilon > 0$ such that for all η-regular holonomy perturbations Φ with $\|\Phi\| < \varepsilon$, η supports at most one gauge equivalence class of critical points of $cs + \Phi$ (and in fact, if η supports a flat connection, then there will be exactly one critical point for all such perturbations). We show this in two steps: first, there is at most one gauge equivalence class of flat connections on η. Second, if for a connected set of η-regular perturbations $\mathcal{P}_\delta$ there exists any critical point for a perturbation in $\mathcal{P}_\delta$, then the projection map

$$f\colon \mathcal{P}_\delta \mathrm{Crit} \to \mathcal{P}_\delta$$

from the parametrized critical set to the space of perturbations is a smooth, proper submersion. In particular, in a connected neighborhood of any regular perturbation for which there exist critical points, the map f will be a locally trivial fibration by Ehresmann's Lemma. We then note that for a line bundle over a rational homology sphere, the critical points of the unperturbed Chern Simons functional for a line bundle are always nondegenerate. So the above claims together prove that there exists a small ball containing 0 in $\mathcal{P}_\delta$ such that $cs + \Phi$ has exactly one critical point for any Φ in this ball.

To prove the first claim, that there is a unique gauge equivalence class of flat connections on Q', let ρ, ρ' be two flat connections on Q' and write $\rho' = \rho + i\omega$, where $\omega \in \Omega^1(Y; \mathbb{R})$. Because both ρ and ρ' are flat, we have

$$id\omega = F_{\rho'} - F_\rho = 0,$$

so ω is a closed 1-form. But $H^1(Y; \mathbb{R}) = \{0\}$, and so there is $\eta\colon Y \to \mathbb{R}$ such that

$$\omega = d\eta.$$

We now define a gauge transformation u of Q' by $u \coloneqq \exp(i\eta)\colon Y \to U(1)$. Then a simple calculation shows

$$\rho' = \rho + id\eta = \rho + u^{-1}du,$$

i.e. $u^*\rho = \rho'$.

For our second claim, we assume that Φ is some regular perturbation and that A represents a critical point of $cs + \Phi$. Note that nondegenerate critical points are isolated. By Lemma 3.8 in [KM11], there are thus at most finitely many critical points for any given regular perturbation and we can assume that the absolute values of the eigenvalues of the Hessians at these critical points are uniformly bounded below by some $\delta > 0$. By continuity, we can choose $\delta > 0$ so that this remains true for critical points for nearby perturbations.

Now, by definition a perturbation Φ is regular if and only if

$$\operatorname{grad}(cs + \Phi)\colon \mathcal{B} \to \Omega^1$$

is transverse to zero, i.e. if and only if the Hessian is onto at all critical points. But by the same exact argument as in our proof of 5.1.8, for solutions A of $\operatorname{grad}_A(cs+\Phi) = 0$, the cokernels of $Df|_A$ and of $D\operatorname{grad}(cs+\Phi)|_A$ are isomorphic, so df must be onto as well. This proves that f is a submersion and we already know that it's smooth and proper, proving the second claim.

Hence we see that for any small enough perturbation Φ, any two critical points of $cs+\Phi$ on η are gauge equivalent. As before, we get for any two representatives of this unique critical point ρ, ρ':

$$\rho' = \rho + u^{-1}du.$$

Because A is of finite energy, its restrictions $A\big|_{Y\times\{t\}}$ tend to Φ-flat connections on Q' over both ends of C. Call these limiting connections ρ_- and ρ_+. As we have just seen, there is a gauge transformation $u\colon Y \to U(1)$ such that $u^*\rho_+ = \rho_-$. Now, because $\pi_1(Y)$ is finite, we have

$$[Y, U(1)] \cong \operatorname{Hom}(\pi_1(Y), \mathbb{Z}) = \{0\},$$

whence it follows that the set of gauge transformations of a line bundle over Y, which is equal to the set of maps $Y \to U(1)$, is contractible. In particular, the gauge group of such a bundle is path connected and so we can pick a map

$$\gamma\colon \mathbb{R} \to \mathcal{G}(\eta)$$

with $\gamma(t) \equiv \mathrm{id}$ for $t << 0$ and $\gamma(t) \equiv u$ for $t >> 0$. But then the formula $g(y,t) \coloneqq \gamma(t)(y)$ defines a gauge transformation over the entire cylinder, putting A in a gauge g^*A for which $g^*A_{-\infty} = \rho = g^*A_\infty$. We will assume this from now on and write A for g^*A.

Now let $A_0 := pr_1^*\rho$ where $pr_1\colon C \to Y$ is the projection. Let $a := A - A_0$. Putting A in Uhlenbeck gauge with respect to A_0 gives a representative in the kernel of the operator $D := d^*_{A_0} + d^+_{A_0}$. On the weighted Sobolev spaces of interest, we can write D in the form $d/dt + L$ where L is the pullback to C of a self adjoint invertible elliptic operator

$$L\colon H := \Omega^0(Y;\mathfrak{su}(2)) \oplus \Omega^1(Y;\mathfrak{su}(2)) \to \Omega^0(Y;\mathfrak{su}(2)) \oplus \Omega^1(Y;\mathfrak{su}(2)).$$

Being self adjoint elliptic, there is an orthonormal basis $\{e_\lambda \mid \lambda \in \Lambda\}$ of H consisting of Eigenvectors of L. We can then write a in the form

$$a = \sum_{\lambda\in\Lambda} a_\lambda e_\lambda$$

and the equation $Da = 0$ becomes the system of ordinary differential equations

$$\frac{da_\lambda}{dt} + \lambda a_\lambda = 0, \lambda \in \Lambda.$$

This means that $a_\lambda = c_\lambda \exp(-\lambda t)$. Because $a \in L^2(C; \mathfrak{su}(2))$ we now get that $a = 0$, so $A = A_0$, as claimed. □

The proof of our main Theorem is essentially a dimension counting argument. For this, we need to be able to estimate the virtual dimension of moduli spaces over cylinders that contain reducibles which the above Lemma enables.

To set this up, let ρ be a flat connection on the $SU(2)$-bundle $E \to Y$ and A_0 the pullback $\pi^*\rho$ on the pullback bundle $\pi^*E \to C$. We also denote the three dimensional deformation operator

$$L_\rho := \begin{pmatrix} 0 & -d_\rho^* \\ -d_\rho & *d_\rho \end{pmatrix}.$$

Being self adjoint elliptic over the closed manifold Y, this operator has real discrete spectrum. We set $0 < \alpha := 1/2 \min\{|x| \mid x \in Spec(L_\rho) \setminus \{0\}\}$ and $\underline{\alpha} := (-\alpha, \alpha)$. The four dimensional deformation operator $D_{A_0} = d/dt + L_\rho$ is then Fredholm as an operator

$$L^2_{k,\underline{\alpha}}(C; ad(\pi^*E)) \to L^2_{k-1,\underline{\alpha}}(C; ad(\pi^*E))$$

with index not depending on $k \geq 1$. We compute this index:

Lemma 4.3.9. *If ρ is reducible, non trivial and non degenerate, then the index of D_{A_0} is -1.*

Proof. This follows from Proposition 4.26 in [Eis23]. We include an explicit calculation for completeness:

we denote by

$$H := \Gamma(Y \times \mathbb{R}; \pi^*\Omega^0(Y; \mathfrak{su}(2)) \oplus \pi^*\Omega^1(Y; \mathfrak{su}(2))).$$

The operator in question is the operator

$$\frac{d}{dt} + L \colon H \to H.$$

We identify the weighted Sobolev space $L^2_{1,\underline{\alpha}}$ with L^2_1 via $s \mapsto \exp(-\sigma_{\underline{\alpha}}) \cdot s$. Under this identification, D_{A_0} corresponds to the operator $\frac{d}{dt} + L_\rho - \sigma'_{\underline{\alpha}}$ (which is also

the main observation needed to prove that it's Fredholm). By our choices of $\underline{\alpha}$ and $\sigma_{\underline{\alpha}}$, the operator $L_\rho - \sigma'_{\underline{\alpha}}$ is invertible and induces the same orthonormal Eigenspace decomposition $\{e_\lambda \mid \lambda \in \Lambda\}$ on both ends of C. We again write an L^2_1-section s of π^*E as

$$s = \sum_{\lambda \in \Lambda} s_\lambda e_\lambda$$

with s_λ constant in the Y-direction and e_λ constant in the $\mathbb{R}$-direction. Then

$$D_{A_0} s = \sum_{\lambda \in \Lambda} \left(\frac{ds_\lambda}{dt} + (\lambda - \sigma'_{\underline{\alpha}}) s_\lambda \right) e_\lambda$$

and, as L_ρ is self-adjoint,

$$D^*_{A_0} s = \sum_{\lambda \in \Lambda} \left(-\frac{ds_\lambda}{dt} + (\lambda - \sigma'_{\underline{\alpha}}) s_\lambda \right) e_\lambda.$$

Thus, the kernel of D_{A_0} contains exactly those sections s such that for all λ, the coefficient s_λ is an L^2- solution of the ordinary differential equation

$$\frac{ds_\lambda}{dt} + (\lambda - \sigma'_{\underline{\alpha}}) s_\lambda = 0.$$

The general solution to this equation is

$$s_\lambda(t) = C \exp\left(- \int_0^t \lambda - \sigma'_{\underline{\alpha}}(\tau) d\tau \right)$$

where $C \in \mathbb{R}$. This function lies in $L^2(\mathbb{R})$ if and only if $C = 0$ or $\lambda - \sigma'_{\underline{\alpha}}(\tau) = \lambda + \alpha < 0$ for $\tau << 0$ and $\lambda - \sigma'_{\underline{\alpha}}(\tau) = \lambda - \alpha > 0$ for $\tau >> 0$. Because $Spec(L_\rho) \cap [-\alpha, \alpha] = \{0\}$, the latter condition is equivalent to $\lambda < 0$ and $\lambda > 0$, which is impossible. Hence all $s_\lambda = 0$ and so $\ker D_{A_0} = \{0\}$.
Similarly, we identify $\ker D^*_{A_0}$ as those L^2-sections $s = \sum_{\lambda \in \Lambda} s_\lambda e_\lambda$ for which

$$s_\lambda(t) = C \exp\left(\int_0^t \lambda - \sigma'_{\underline{\alpha}}(\tau) d\tau \right) \quad (*).$$

The L^2 condition now forces $C = 0$ or $\lambda = 0$. This means that the restriction of s to any $\{t\} \times Y$ is an element of $\ker L_\rho$ and, because of $(*)$, s is uniquely determined by any one such restriction. This immediately implies that $\dim \ker D^*_{A_0} = \dim \ker L_\rho$.
Altogether, we obtain

$$\operatorname{ind}(D_{A_0}) = \dim \ker D_{A_0} - \dim \ker D^*_{A_0} = - \dim \ker L_\rho,$$

which by our assumption on ρ equals -1. $\square$

4.4 Regularity

Proposition 4.4.1. *Let X be a smooth oriented Riemannian 4-manifold, let π be a regular holonomy perturbation and let $\mathbf{A}$ a π-ASD connection on a bundle $P \to X$ of class $L^p_{k,loc}$ (i.e. such that any restriction of $\mathbf{A}$ to a compact submanifold $K \subset X$ is of class L^p_k over K.) with respect to a smooth base connection $\mathbf{A}_0$. Then the following hold:*
If $k - \frac{4}{p} > 0$ (in particular if $p > 4$ and $k \geq 1$), then there exists a gauge transformation u of class $L^p_{k+1,loc}$ such that $u^*\mathbf{A}$ is smooth.

Proof. This follows from a bootstrapping argument:

because the smooth 1-forms are dense in $L^{p,\alpha}_k$, we can choose a smooth base connection $\mathbf{B}$ such that $\|\mathbf{A} - \mathbf{B}\|_n$ is arbitrarily small. By the slice Theorem, we know that there is a gauge transformation u of class L^p_{k+1} such that $d^*_{\mathbf{A}}(u^*\mathbf{A} - \mathbf{B}) = 0$. We write $a \coloneqq u^*\mathbf{A} - \mathbf{B}$. Since $d^*_{\mathbf{A}} = d^*_{\mathbf{B}} + [-, a]$ for an antisymmetric bilinear form $[-,-]$, we also have

$$d^*_{\mathbf{B}} a = 0.$$

We can then write the π-ASD equation for $u^*\mathbf{A}$ as

$$(d^*_{\mathbf{B}} + d^+_{\mathbf{B}})a = -F^+_{\mathbf{B}} - (a \wedge a)^+ - \hat{V}_\pi(\mathbf{A}) \tag{13}$$

We now restrict our entire analysis to a compact submanifold.

Because $k - \frac{4}{p} > 0$, there is a bounded multiplication map $L^p_k \times L^p_k \to L^p_k$, and so the quadratic term $(a \wedge a)^+$ is bounded in L^p_k. Now, because B is smooth and thus contained in $L^p_{k,loc}$ for all p, k, we have that the difference $F^+_{\mathbf{B}} - \hat{V}_\pi(\mathbf{A}) = F^+_{\mathbf{B}} - F^+_{\mathbf{A}}$ is in L^p_k (note here also that by the Φ-ASD equation and the properties of our perturbations, $F^+_{\mathbf{A}}$ is in L^p_k rather than only in L^p_{k-1}.). The operator $d^*_{\mathbf{B}} + d^+_{\mathbf{B}}$ is elliptic with smooth coefficients, and so a is of class L^p_{k+1}, as required. □

It is an easy corollary that if the connections $\mathbf{A}$ and $\mathbf{B}$ in the above Proposition are exponentially decaying of class $L^{p,\alpha}_k(X)$, then the connection $u^*\mathbf{A}$ in Coulomb gauge will be, as well, and the gauge transformation can be taken to be of class $L^{p,\alpha}_{k+1}$.

Corollary 4.4.2. *The moduli spaces defined above do not depend, up to diffeomorphism, on the choice of Sobolev completion.*

4.5 Compactness

The proof of our main theorem hinges on certain compactness properties of ASD-moduli spaces, which are well known in the case of the unperturbed equations over closed manifolds. In the presence of perturbations and over the kind of non compact manifolds we are considering, some adaptions have to be made. We have the following:

Theorem 4.5.1. *Let M be a smooth, oriented Riemannian 4-manifold with cylindrical ends, π a regular perturbation and $P \to M$ a π-adapted bundle. Let $\mathbf{A}_n$ be a sequence of π-ASD connections of regularity $L^2_{k,loc}$, $k \geq 3$, with uniformly bounded analytic energy, i.e. there is some constant $C > 0$ such that*

$$\|\mathcal{F}(\mathbf{A}_n)\|_{L^2} < C$$

for all n. Then for every compact submanifold $K \subset M$ there is a subsequence $(A_{n_k})_k$ of $(\mathbf{A}_n)$ that converges in the following sense:

There exist

- *a finite collection $\mathbf{x}$ of points in M*
- *a π-adapted bundle $P' \to M$ with an isomorphism $\phi\colon P'|_{M\setminus\mathbf{x}} \to P|_{M\setminus\mathbf{x}}$*
- *a sequence of gauge transformations u_n of P of class $L^2_{k+1,loc}$ and*
- *a π-ASD connection $\mathbf{A}$ on P' of class $L^2_{k,loc}$ and with $\|\mathcal{F}(\mathbf{A})\|_{L^2} < C$*

such that

$$u^*_{n_k}\mathbf{A}_{n_k} \to \mathbf{A}$$

in $L^p_1(\tilde{K})$ for all $p \geq 2$ over every compact submanifold $\tilde{K} \subset K \setminus \mathbf{x}$.

We can in fact choose the bundle P' and the limit connection $\mathbf{A}$ independently from K.

The crucial difference to the unperturbed case is that we cannot expect the convergence to be in C^∞. This is due to the perturbation terms which are non-local: the perturbation terms on a ball depend on the curvature along the entire thickened loop on which the perturbation is supported. If this loop contains a point at which the curvatures blow up (a bubbling point), we lose control of the L^p_1-norms of the perturbation along the entire loop. Holonomy perturbations are locally uniformly L^p-bounded for all p, however, and that is how we get L^p_1-convergence of connection forms.

In order to get a limit that is even locally L^2_3, we then must invoke our regularity results.

For the proof of this compactness theorem, we will need the following Lemma, which is proved in [Uhl82].

Lemma 4.5.2. *There exist constants ε, C such that any L^2_k $U(2)$-connection A over a closed euclidean 4-ball $B \subset \mathbb{R}^4$ with energy*

$$\|F_A\|_{L^2} < \varepsilon$$

is gauge equivalent to a connection $\tilde{A} = d_{triv} + a$ that satisfies

$$d^*_{triv} a = 0$$

and

$$\|a\|_{L^p_1} \leq C \left\|F_{\tilde{A}}\right\|_{L^p} \tag{14}$$

for all $p \geq 2$.

Remark 4.5.3. *The collection of points $\underline{x}$ is the set of "bubbling points", i.e. the points with the following property: for every neighborhood U of x there are $\varepsilon > 0$ and a subsequence $(F_{A_{n_k}})$ auch that*

$$\left\|F_{A_{n_k}}\right\|_{L^2} > \varepsilon.$$

(In fact, we can choose ε uniformly in x and U in this case.) That there are only finitely many of these follows from Uhlenbeck's removable singularity theorem.

Proof of Theorem 4.5.1. Given some compact submanifold $K \subset X$, we can estimate

$$\|F_{A_n}\|^2_{L^2(K)} \leq \|\mathcal{F}(A_n)\|^2_{L^2(K)} + \left\|\hat{V}_\pi(A_n)\right\|^2.$$

Now, the first term on the right side is bounded by the analytic energy of A_n over all of X, which we assumed to be uniformly bounded in n. Because the holonomy perturbation term $\hat{V}_\pi(A_n)$ is uniformly L^∞-bounded, the second term in the above inequality is also bounded, by a constant that only depends on K but not on n. Thus, the curvatures of the A_n are uniformly bounded in L^2 over any compact submanifold $K \subset X$.

By the Banach-Alaoglu theorem, there is a subsequence, call it A_n again, such that the curvature densities $|F_{A_n}|^2$ converge weakly to a measure μ. We then find a finite collection of points $\mathbf{x}$ such that every point $x \in X \setminus \mathbf{x}$ is contained in a geodesic ball B_x such that $\mu(x) < \varepsilon^2$. But then we must also have

$$\|F_{A_n}\|_{L^2(D_x)} < \varepsilon \tag{15}$$

for almost all n and we can apply Lemma 4.5.2 to get

$$\|A_n\|_{L^2_1(D_x)} < C \|F_{A_n}\|_{L^2(D_x)}. \tag{16}$$

We write the Φ-perturbed instanton equation as

$$d^+ A_n = -(A_n \wedge A_n)^+ - \hat{V}_\pi(A_n).$$

Then, choosing a slightly smaller ball $D \subset B_x$ and a smooth cutoff function $\psi\colon B_x \to [0,1]$ that vanishes in a neighborhood of the boundary and is identically equal to 1 on D, we can extend the cut off connection matrix ψA_n to a connection matrix on the trivial bundle over a 4-sphere and we have

$$(d^* + d^+)(\psi A_n) = *(d\psi \wedge *A_n) + (d\psi \wedge A_n)^+ - ((\psi A_n) \wedge A_n)^+ - \psi \hat{V}_\pi(A_n). \quad (17)$$

Note that on S^4, the operator $d^* + d^+\colon L^p_1 \to L^p$ is invertible for all p (the anti self dual deRham complex $\Omega^0(S^4) \xrightarrow{d} \Omega^1(S^4) \xrightarrow{d^+} \Omega^2_+(S^4)$ is acyclic), so we have

$$\|B\|_{L^p_1} \leq C_p \left\|(d^* + d^+)B\right\|_{L^p}$$

for some constant C_p that only depends on p and all 1-forms B.

By the Sobolev embedding theorems, we have for $2 \leq p < 4$ and $q \coloneqq \frac{4p}{4-p}$ bounded inclusions $L^p_1 \hookrightarrow L^q$ and bounded multiplication maps $L^4 \times L^q \to L^p$. From this and 17, we get a bound

$$\|\psi A_n\|_{L^p_1} \leq C \left(\|d\psi\|_{L^p_1} \|A_n\|_{L^4} + \|\psi A_n\|_{L^p_1} \|A_n\|_{L^4} + \left\|\hat{V}_\pi\right\|_{L^p} \right)$$

By 16 and 15, the first and third term on the right hand side are uniformly bounded, by a constant C_1, say.

Using the embedding $L^p_1 \hookrightarrow L^4$ for $p \geq 2$ in dimension 4, 16 and 15 again and rearranging, we get

$$(1 - C\varepsilon) \|\psi A_n\|_{L^p_1} \leq 2CC_1.$$

In particular, if we choose $\varepsilon < \frac{1}{2C}$, we get from this

$$\|\psi A_n\|_{L^p_1(S^4)} < 4CC_1,$$

uniformly in n. From this and the trivial observation $\|A_n\|_{L^p_1(D_x)} \leq \|\psi A_n\|_{L^p_1(S^4)}$, we immediately get a uniform bound on $\|A_n\|_{L^p_1(D_x)}$ for $2 \leq p < 4$.

To get such a bound for $p \geq 4$ also, we first note that we get bounds

$$\|A_n \wedge A_n\|_{L^6} \leq \|A_n\|^2_{L^{12}} \leq C_2 \|A_n\|^2_{L^3_1},$$

by Hölder's inequality and the embedding $L^3_1 \hookrightarrow L^{12}$ in dimension four. So, since we already know that the L^3_1-norm is uniformly bounded, we get that the L^6-norm of the quadratic term in 17 is uniformly bounded (without using multiplication

theorems). Again by the invertibility of $d^* + d^+$ and 17, we get that $\|A_n\|_{L^6_1}$ is uniformly bounded.

But this means that A_n is in fact pointwise uniformly bounded because of the embedding $L^6_1 \hookrightarrow L^\infty$. From this, we finally get uniform boundedness over balls in all L^p_1-norms.

Just as in the unperturbed situation, we can now patch gauge transformations to get that the sequence A_n is uniformly L^p_1-bounded on compact submanifolds of $M \setminus \mathbf{x}$. The reason this works is that by what we have proved above, we have pointwise control over the A_n on balls.

We can now invoke Lemma 10 from [Kro05], giving us a subsequence of the A_n such that the corresponding sequence of perturbations $\hat{V}(A_n)$ is Cauchy in L^p, again over compact submanifolds of $M \setminus \mathbf{x}$.

The rest of the argument is standard: we restrict to balls again, put our connection forms in Coulomb gauge and use elliptic regularity, 17 and what we have just shown to get a subsequence that converges in L^p_1 over balls in the interior of $M \setminus \mathbf{x}$ for all $p \geq 2$. Again patching gauge transformations and Uhlenbeck's removable singularities theorem then give the bundle P' and the limit connection A on P. That A is π-ASD follows from the fact that the equation

$$F^+_A + \hat{V}_\pi(A) = 0$$

is the pointwise limit of the (gauge invariant) equations

$$F^+_{A_n} + \hat{V}_\pi(A) = 0.$$

The same observation, together with the dominated convergence theorem and our assumption that the analytic energies of the A_n are uniformly bounded over M, proves that the analytic energy of the limit connection A is bounded by the same constant. This finishes the proof. □

Exponential Decay Our first application of this compactness result is the exponential decay of π-ASD connections of bounded analytic energy over cylinders for regular perturbations π.

The decay result is:

Theorem 4.5.4. *Let $\pi \in \mathcal{P}$ be any perturbation such that all critical points of cs_π are nondegenerate. Then, up to some gauge transformation, any π-perturbed instanton $\mathbf{A}$ over a tube $Y \times (0, \infty)$ with bounded analytic energy, $\|\mathcal{F}(\mathbf{A})\|_{L^2} < \infty$, tends exponentially to some critical point ρ of cs_π.*

Proof. Let ρ be some critical point of cs_π. We first note that by our assumption on the perturbation π (namely, that the Hessian of the perturbed functional cs_π is invertible at every critical point of cs_π) and the implicit function theorem, the restriction of the formal gradient map

$$\begin{aligned} L^2_1(\Omega^1(Y;\mathfrak{su}(2))) &\to L^2(\Omega^1(Y;\mathfrak{su}(2))) \\ a &\mapsto \nabla_{\rho+a} cs_\pi \end{aligned}$$

to a sufficiently small neighborhood of ρ is smoothly invertible (the notation $L^p_k(\Omega^n)$ is used to make explicit the Sobolev norms with respect to which we complete the respective spaces of forms over Y). There are thus constants $c, C > 0$ such that

$$c \left\| \nabla_{\rho+a} cs_\pi \right\|^2_{L^2(Y)} \leq \|a\|^2_{L^2_1(Y)} \leq C \left\| \nabla_{\rho+a} cs_\pi \right\|^2_{L^2(Y)}$$

[Compare the proofs of Lemma 13.4.3. and 13.5.1. in [KronMrow - Monopoles...]].

Because cs_π is of class C^1 as a function $L^2_1(Y) \to \mathbb{R}/8\pi\mathbb{Z}$ and the critical point ρ is non degenerate, there is additionally a neighborhood U of the critical point ρ and a constant $C' > 0$ such that for all $A \in U$

$$|cs_\pi(a) - cs_\pi(\rho)| \leq C' \left\| A - \rho \right\|_{L^2_1}.$$

Two remarks are in order concerning this last statement: first, this uses the infinite dimensional version of the well known Morse Lemma, due to Palais, see [Pal63, Morse Lemma, p.9]. Second, note that the difference on the left hand side of the inequality is a well defined element of the real numbers $\mathbb{R}$, instead of only an element of the quotient circle $\mathbb{R}/8\pi\mathbb{Z}$, once A is sufficiently L^2_1-close to ρ.

With these observations in place, the rest of the standard proof of exponential convergence towards a critical point once the energy of a path falls below a certain uniform threshold ε from finite dimensional Morse theory goes through without any further changes, see e.g. [MW21, Prop 1.2.11 and Thm 1.2.14]

Compare also [Don02, pp. 4.1–4.2].

□

As in [Don02], this theorem gives us precise control over connection forms up to gauge of perturbed ASD-connections:

Proposition 4.5.5. *If π is a regular perturbation, then after applying an appropriate gauge transformation, any π-perturbed instanton $\mathbf{A}$ on a bundle E over a cylinder tends to a critical point ρ of cs_π exponentially fast. Precisely, there are*

a bundle isomorphism $u\colon E_0 \to E$*, constants* $C_k, \delta > 0$ *not depending on* $\mathbf{A}$ *and* t_0 *such that for* $a \coloneqq u^*\mathbf{A} - \rho$ *we have*

$$\forall t > t_0\colon \ \|a\|_{L^p_k(Y\times(t,\infty))} \leq C_k \exp(-\delta t)$$

whenever the Sobolev norm on the left hand side is defined. In particular, we get pointwise exponential bounds for connection 1-forms if $k - \frac{4}{p} > 0$*.*

Proof. The proof is identical to that of Proposition 4.3 in [Don02]. □

We also gain control over connection 1-forms on finite cylinders once the analytic energy is sufficiently small as in Proposition 4.4 in [Don02].

Remark 4.5.6. *This theorem shows that all solutions of the perturbed instanton equations with bounded analytic energy are contained in moduli spaces modelled on weighted Sobolev spaces, compare Lemma 4.1.5.*

Over cylindrical end manifolds, there is an additional type of (global) non-compactness, corresponding to "energy sliding down the ends". This results in the limiting connections of P' *over the ends of* X *in general not being gauge equivalent to those of* P*. This phenomenon is the reason why one must incorporate additional cylindrical pieces into the* geometric limits *which are the appropriate limits in the context of the result on compactness under neck stretching that we describe next.*

Global Convergence We now want to apply the compactness property from Theorem 4.5.1, together with our control of Φ-ASD connections over cylinders, to show that a "neck stretching" procedure always produces geometric limits when applied to our moduli spaces. We begin by describing this neck stretching:

Let X be an oriented, smooth, closed, connected 4-manifold that contains Y as a smoothly embedded separating hypersurface,

$$X = X_1 \cup_Y X_2.$$

We pick a closed tubular neighborhood $\nu(Y) \cong Y \times [-\varepsilon, \varepsilon]$ in X and a smooth Riemannian metric g_X on X that restricts to the product metric $g_Y + dt^2$ on this tube. Now, given $L > 0$, we define a Riemannian manifold $(X(L), g_L)$ by inserting a neck of length $2L$ around Y, i.e. we set

$$X(L) \coloneqq X_1 \cup_Y (Y \times [-L, L]) \cup_Y X_2$$

with metric g_L equal to g on X_1 and X_2 and equal to the product metric on $Y \times [-L, L]$. Of course, the underlying smooth manifold is our original X for all L and so any smooth bundle P over X canonically induces bundles (smoothly bundle isomorphic to P) $P(L) \to X(L)$ for all L.

However, these manifolds are only locally isometric for different L, so to ultimately define a notion of convergence for metric dependent geometric data (for instance, instantons) we need to additionally make a choice of a set of base points, arbitrary at this stage, and our notion of limit will depend on this choice. In our application of this theory later, there will however be (almost) canonical choices for these base points, giving us well defined limits.

To get to our definition of convergence, then, assume we are given a bundle $P \to X$, a sequence $L_n \overset{n\to\infty}{\longrightarrow} \infty$, a sequence of k-tuples of points in $X(L_n)$, $(x_n^{(1)}, \dots, x_n^{(k)})$ with $d_{X(L_n)}(x_n^{(i)}, x_n^{(i+1)}) \to \infty$, and for each n, a connection A_n on $P(L_n)$. Then a *geometric limit* of such a sequence of data then consists of the following:

- a smooth, oriented, complete Riemannian 4-manifold X_∞ and points $x^{(1)}, \dots, x^{(k)} \in X_\infty$ such that for all $t \geq 1$ and all sufficiently large n there are compact submanifolds $K^\infty \subset X^\infty$ and $K_n^{(i)} \subset X(L_n)$ containing the balls of radius t around $x^{(i)} \in K_\infty^{(i)}$ and $x_n^{(i)} \in K_n$ and isometries
$$\phi_n^{(i)} \colon K_\infty^{(i)} \to K_n^{(i)}$$
- a smooth bundle $P_\infty \to X_\infty$ with connection A^∞ and
- smooth bundle isomorphisms
$$\Phi_n^{(i)} \colon P_\infty\big|_{K_\infty^{(i)}} \to P(L_n)\big|_{K_n^{(i)}}$$

such that
$$lim_{n\to\infty}(\Phi_n^{(i)})^* A_n = A$$
in the L_1^p-topology on $K_\infty^{(i)}$ for all $p \geq 2$.

We can now state our compactness theorem for sequences of perturbed instantons under neck stretching:

Theorem 4.5.7. *Let $X, Y, P \to X$ and $P(L_n) \to X(L_n)$ be as above and let π be a regular perturbation.*

If $(\mathbf{A}_n)_n$ is a sequence of L_3^2 π-ASD connections on $P(L_n) \to X(L_n)$ such that there is $C > 0$ with
$$\|\mathcal{F}(\mathbf{A_n})\|_{L^2(X(L_n))} < C,$$

then there is a sequence of base points $(x_n^{(1)}, \dots, x_n^{(k)})$ *with* $d_{X(L_n)}(x_n^{(i)}, x_n^{(i+1)}) \to \infty$ *(k independent of n) and a subsequence that has a geometric limit* $(X_\infty, P_\infty, \mathbf{A}_\infty, (x^{(1)}, \dots, x^{(k)}))$ *with the following properties:*

- X_∞ *is the cylindrical end manifold*

$$\begin{aligned} X_\infty =& (X_1 \cup_Y (Y \times [0,\infty)) \\ & \sqcup \left(\bigsqcup_{i=1}^{k_1} Y \times \mathbb{R} \right) \\ & \sqcup (X_2 \cup_Y (Y \times [0,\infty)) \sqcup \left(\bigsqcup_{i=1}^{k_2} S^4 \right) \end{aligned}$$

  with $k = k_1 + k_2 + 2$ *components.*

- P_∞ *can be equipped with the structure of a* π*-adapted bundle*

- $\mathbf{A}_\infty$ *is* π*-ASD of regularity* $L_3^{2,\alpha}$*. Denoting the limiting value of* $\mathbf{A}_\infty$ *over the end of* $\tilde{X}_1 \cup_Y (Y \times [0,\infty))$ *by* ρ_0^+ *and the limiting value of* $\mathbf{A}_2$ *over the end of* $\tilde{X}_2 \cup_Y (Y \times [0,\infty))$ *by* ρ_{s+1}^- *and similarly by* ρ_i^+ *resp.* ρ_i^- *the limiting values of* $\mathbf{A}_\infty$ *over the positive resp. negative end of* $Y \times \mathbb{R}$*, we have, up to reordering*

$$\rho_i^+ = \rho_{i+1}^-.$$

Proof of Theorem 4.5.7. Given the inputs of Theorems 4.5.1 and 4.5.4, the proof of this theorem is essentially identical to the proof of Theorem 5.4 in [Don02].

We give an extremely rough sketch here so we can refer to the basic constructions from this proof later:

By the Banach-Alaoglu theorem, the sequence of measures

$$|\mathcal{F}(\mathbf{A}_n)|^2 \, dvol$$

converges over any compact submanifold in the weak-*-topology. Using elliptic regularity, the uniform boundedness of the analytic energy and Uhlenbeck's results on removable singularities for instantons with bounds on curvature, one constructs the limit connection $\mathbf{A}_\infty$ and shows that the limit of the sequence of measures $|\mathcal{F}(\mathbf{A}_n)|^2 \, dvol$ is the measure

$$|\mathcal{F}(\mathbf{A}_\infty)|^2 \, dvol + \sum_{i=1}^{k_2} \delta_{x^{(i)}}$$

and that the sequence $\mathbf{A}_n$ converges in $L^2_3(K)$ for any compact $K \subset X_\infty \setminus \{x^{(1)}, \ldots, x^{(k_2)}\}$.

In particular, the analytic energy of the limit is bounded above by the analytic energy of the $\mathbf{A}_n$. To account for any analytic energy that might be lost, one then chooses a sequence of disjoint subcylinders $C_n^{(1)}, \ldots, C_n^{(k_1)} \subset X(L_n)$ in such a way that

- $length(C_n^{(i)}) \to \infty$ as $n \to \infty$ for all i
- the analytic energy of the restrictions $\mathbf{A}_n\big|_{C_n^{(i)}}$ remains uniformly bounded by some small ε and
- if $C_n \subset X(L_n)$ is any sequence of subcylinders such that C_n is not contained in one of the $C_n^{(i)}$ for all large n and such that the analytic energy $\mathcal{E}(\mathbf{A}_n\big|_{C_n})$ is uniformly bounded by the same ε from the previous item, then uniformly

$$length(C_n) < K$$

for some $K < \infty$.

Now, $(Y \times [-L_n, L_n]) \setminus \bigcup_i C_n^{(i)}$ is a disjoint union of cylinders and using the now familiar compactness arguments for each of these cylinders lets one construct the k_1 "cylindrical parts" of our limit connection, $\mathbf{A}_\infty\big|_{Y\times\mathbb{R}}$ in such a way that the analytic energy of $\mathbf{A}_\infty$ is equal to (instead of strictly smaller than) the analytic energy of the $\mathbf{A}_n$ for large n.

Finally, one shows that when the analytic energy is preserved entirely in the geometric limit, the limit must satisfy the statement of the last item of the theorem.

For details on this argument, also compare chapter 6 of [MMR94] □

Remark 4.5.8. *We will refer to the last property of our geometric limits listed in Theorem 4.5.7 as the limit having* compatible limit values.

As indicated in the proof sketch here, this property of our geometric limits will be guaranteed when the analytic energy is preserved in the limit. We will later always implicitly assume that we have indeed chosen any geometric limit such that this holds and say that no energy is lost *in our limit.*

4.6 The Gluing Formula

Let X_1, X_2 be two cylindrical end manifolds such that X_1 has a positive end $Y \times [0, \infty)$ and X_2 has a negative end $Y \times (-\infty]$. We call their disjoint union (a disconnected cylindrical end manifold)

$$X \coloneqq X_1 \sqcup X_2.$$

Then, for $T > 0$, we can eliminate the portions $\{t_1 > T+1\}$ resp. $\{t_2 < -T-1\}$ from these two ends, giving us manifolds with boundary $X_1(T+1)$ and $X_2(T+1)$, and glue the resulting manifolds by identifying the bands $Y \times (T, T+1)$ and $Y \times (-T-1, -T)$ in the ends in the obvious way to obtain the cylindrical end manifold X_T with two fewer ends and one fewer connected component than X. By the *common region* we refer to the manifold $X_1(T) \sqcup X_2(T)$ that is canonically embedded in both X and X_T.

We would like to compare the π-instanton moduli spaces over X_1, X_2 and X_T, at least for large T. There is a plethora of results in this direction, among them gluing theorems giving "coordinates" for moduli spaces on X_T in terms of the moduli spaces on X_1 and X_2. See for example [Eis23, section 4.7], [Don02, section 4.4].

For our purposes here, it suffices to be able to compare the *dimensions* of these moduli spaces, which we will do by relating the respective deformation operators and thus their tangent spaces.

For this purpose, let π be a regular holonomy perturbation and $P_1 \to X_1$ and $P_2 \to X_2$ be two π-adapted bundles. In order to be able to glue, we assume that the π-flat structures over the common ends of X_1 and X_2, ρ_1 and ρ_2, are gauge equivalent. Such a gauge transformation is unique up to an element of the stabilizer of ρ_2 (equivalently up to an element of $\mathrm{Stab}(\rho_1)$) and we make such a choice, u_0 say, arbitrarily (the glued up bundle will depend on this choice and the dependence on this "gluing parameter", which is reflected in an extra summand $\dim \mathrm{Stab}\, \rho$ in our gluing formula, is precisely what we hope to exploit in our ultimate argument). We can now glue up P_1 and P_2 by trivializing both bundles over a common base point in the band arbitrarily, extending these to trivializations over the entire gluing region ρ_i-parallely and gluing up in the now obvious way. We call the resulting bundle

$$P_T \to X_T.$$

We choose some smooth cutoff function $\beta_T \colon [0, \infty) \to [0, 1]$ with $\beta_T(t) = 1$ for $t < 1$, $\beta_T(t) = 0$ for $t > T$ and with a uniform pointwise bound $|d\beta_T/dt| < 2/(T-1)$. Now, given connections A_1 on P_1 and A_2 on P_2 with common Φ-flat

structure ρ over the common ends, we can write $A_i = \rho + a_i$. We define new, nearby connections by

$$A_i(T) \coloneqq \rho + \beta_T((-1)^{i+1}t)a_i$$

and we define $A(T)$ to be the connection on P_T that is equal to $A_i(T)$ over $X_i(T+1)$.

With this, we get Fredholm deformation operators $D^\pi_{A_i(T)}$ on $L^{2,\alpha_i}_k(X_i)$ with indices $\mathrm{ind}(D^\pi_{A_i}, \alpha_i)$ and $D^\pi_{A(T)}$ on $L^2_k(X_T)$ with index $\mathrm{ind}(D^\pi_{A(T)})$ and our dimension formula states that the latter is equal to the sum of the first two:

Theorem 4.6.1. *In the above situation, we have an equality*

$$\mathrm{ind}(D^\pi_{A(T)}) + \dim \mathrm{Stab}(\rho) = \mathrm{ind}(D^\pi_{A_1}, \alpha) + \mathrm{ind}(D^\pi_{A_2}, \alpha).$$

where $\alpha > 0$ is any sufficiently small, positive weight.

Proof. This follows from combining formula (3.2), Proposition 3.10 and Proposition 3.19 from [Don02] with the fact that by definition, for a regular perturbation π,

$$\ker L^\pi_\rho = \ker(d_A \colon \Omega^0 \to \Omega^1)$$

for A a critical point of cs_π.

Our setup here is slightly more general than is stated in that reference; there, no deformations are present and the formula is only directly stated for connections that are constant over the ends. However, the proof given there does not depend on the absence of perturbations at all.

The only slight modification necessary in our situation is the realization that the indices of the deformation operators corresponding to the original connections A_i and the cut off connections $A_i(T)$ are identical: $a_i \coloneqq A_i - A_i(T)$ is of class $L^{2,\alpha}_k$ and the difference operator

$$D_{A_i} - D_{A_i(T)} \colon L^{2,\alpha}_k \to L^{2,\alpha}_{k-1}$$

is supported on the common end and arbitrarily small in the $L^{2,\alpha}_k \to L^{2,\alpha}_{k-1}$ operator norm if we choose T large enough. As in [Don02], p.52 directly below Corollary 3.7, this means that because $D_{A_i(T)}$ is invertible as an operator over the two-sided cylinder, D_{A_i} is as well. In particular, both operators, considered over a doubly infinite cylinder, have identical indices, namely zero. Using a gluing procedure and the parametrix method as in *loc. cit.*, proof of Proposition 3.6, (ii), we now see that the operators on our cylindrical end manifold X, one with restriction $D_{A_i(T)}$, one with restriction D_{A_i} over the end, have identical indices as well.

The rest of the proof goes through verbatim and we get the stated equality with the indices $D_{A_i(T)}$ in place of the indices of the D_{A_i}. But as we have just seen, these indices are the same. □

Combining the above index formula with Corollary 4.3.3 and 12, we get the following dimension formula:

Corollary 4.6.2. *In the above situation, we have*

$$\dim_f \mathcal{M}_\pi(P_T) = \dim_f \mathcal{M}_\pi(P_1) + \dim_f \mathcal{M}_\pi(P_2) + \dim \mathrm{Stab}(\rho),$$

where $\dim_f$ *is the formal dimension defined in Definition 4.3.4.*

4.7 Unique continuation

We will frequently need the following unique continuation results:

Proposition 4.7.1. *Let X be a closed, connected, smooth, oriented Riemannian 4-manifold, $P \to X$ a smooth $SU(2)$- or $SO(3)$-bundle and $\mathbf{A}$ an ASD connection on P. If $\mathbf{A}$ is reducible on an open subset $U \subset X$, then $\mathbf{A}$ is locally reducible everywhere. If the induced map $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is surjective, then $\mathbf{A}$ is reducible over all of X.*

The proof of Proposition 4.7.1 proceeds in two parts, first for the compact complement of the ends in the next Lemma and then for cylindrical ends in Lemma 4.7.3:

Lemma 4.7.2. *Let X be a cylindrical end manifold, Φ a regular perturbation and P a Φ-adapted bundle over X. Then Φ-ASD-connections on P have the following unique continuation property:*

Let $U \subset X$ be an open subset such that the induced map $\pi_1(U, x_0) \to \pi_1(X, x_0)$ is surjective. Then, if a Φ-ASD connection $\mathbf{A}$ on P is reducible over U, then $\mathbf{A}$ is reducible over all of X, i.e. there exists $u \in \mathcal{G}_{k,\alpha}(P)$ with $d_{\mathbf{A}} u = 0$.

Proof. The proof is essentially identical to the proof of Lemma 7.5.1 in [MMR94]. The only necessary change we have to make being to replace the unique continuation property for unperturbed ASD-connections on balls with the following Lemma: □

Lemma 4.7.3. *Let Y be a smooth, closed, oriented 3-manifold and $Q \times [0,\infty) \to Y \times [0,\infty)$ a principal G-bundle. Let $\mathbf{A}$ be a Φ-ASD-connection on this bundle. Then if $\mathbf{A}$ is reducible over $Y \times [0,\varepsilon]$ for any $\varepsilon > 0$, then $\mathbf{A}$ is reducible over all of $Y \times [0,\infty)$*

Proof. The proof is along the same lines as [DK97, Lemma 4.3.21]: The assumption that $\mathbf{A}$ is reducible over $Y \times (0,1)$ means that there is an $\mathbf{A}$-parallel section of $\mathcal{G}(P)$ over $Y \times (0,1)$. We put $\mathbf{A}$ in temporal gauge and note that then

$$0 = \left\langle d_{\mathbf{A}} u, \frac{\partial}{\partial t} \right\rangle = \frac{\partial u}{\partial t}.$$

We can thus extend u trivially in the t-direction to get a smooth gauge transformation over the entire cylinder which is $\mathbf{A}$-parallel for $t < \varepsilon$. We now set

$$a \coloneqq \mathbf{A} - u^* \mathbf{A}$$

and claim that a vanishes over all of $Y \times [0,\infty)$. For this, we use a result by Agmon and Nirenberg (see [ADN64]): If we can show that a satisfies a differential inequality of the form

$$\left\| \frac{\partial a}{\partial t} - L(t)a \right\| \leq K \|a\| \tag{18}$$

for some smooth family of self adjoint differential operators $L(t)$ that satisfy

$$\left| \left\langle \frac{\partial L(t)}{\partial t} v, v \right\rangle \right| \leq C \|L(t)a\| \, \|a\| \tag{19}$$

for all v in the common domain of all the $L(t)$, then from the vanishing of a over $Y \times [0,\varepsilon)$, we can conclude that a in fact vanishes for all times t.

We choose the family $L(t)$ constant in t as the three dimensional deformation operator

$$L_\pi \coloneqq \begin{pmatrix} 0 & -d_\rho^* \\ -d_\rho & *d_\rho + \hat{V}_\pi \end{pmatrix}.$$

We have already seen that this operator is self adjoint and the inequality 19 is obviously satisfied as $\frac{\partial L_\Phi}{\partial t} = 0$. It remains to check 18: for this, we first notice that we can put into Coulomb gauge with respect to ρ, $d_\rho^* a = 0$, and compute

$$\frac{da_t}{dt} = *(d_\rho a_t + [A_t, a_t] + [a_t, A_t'] + V_\pi(A_t) - V_\pi(A_t'))$$

where A resp. A' are the connection 1-forms of $\mathbf{A}_t$ resp. $u^*\mathbf{A}_t$ with respect to the base connection ρ. Since V_π is smooth in the L^2-topology, $V_\pi(A_t) - V_\pi(A_t'))$ is bounded by some multiple of the L^2-norm of a_t, as are $[A_t, a_t]$ and $[a_t, A_t']$, so 18 is now clear. □

5 Review of Donaldson Invariants and Non-Vanishing Theorems

5.1 Polynomial Invariants

We give here a brief overview of Donaldsons polynomial invariants in the way we will actually use them later on and indicate all changes and additions which must be made to the theory for our purposes. This material is mostly standard, standard references for this section are [Kro05, Ch. 7], [DK97, Ch. 5].
Let X be a closed, smooth, oriented 4-manifold with Riemannian metric g_X and corresponding Hodge-star-operator $*\colon \Omega^\bullet(X) \to \Omega^{4-\bullet}(X)$. Let $E \to X$ be a smooth, non trivial G-vector bundle and let $\mathcal{M}(E)$ be the moduli space of ASD-connections on E (for $G = U(2)$, we take $\mathcal{M}(E)$ to be the moduli space of ASD connections on the $SO(3)$-bundle $\mathfrak{su}(E)$ of traceless, skew adjoint endomorphisms of E), sitting inside the configuration space $\mathcal{B}(E)$. We denote

$$\dim \mathcal{M}(P) = 2d$$

and we let $\mathcal{B}^*(E)$ be the open subset of irreducible configurations (we always implicitly work in Sobolev completions of these spaces as before, but suppress mention of this the notation here for convenience). This space is a smooth Banach manifold. The product $X \times \mathcal{B}^*(E)$ carries an $SO(3)$ bundle

$$\mathbb{P} \to X \times \mathcal{B}^*(E),$$

equipped with a universal family of connections. This bundle can be defined as a quotient of $\mathrm{ad}(P) \times \mathcal{A}^*(E) \to X \times \mathcal{A}^*(E)$ by the diagonal action of the gauge group $\mathcal{G}(E)$. We denote

$$\mathbf{c} \coloneqq -\frac{1}{4} p_1(\mathbb{P}).$$

Using the slant product

$$/\colon H^d(X \times \mathcal{B}^*(E); \mathbb{Q}) \times H_l(X; \mathbb{Q}) \to H^{d-l}(\mathcal{B}^*(E); \mathbb{Q}),$$

we can then define the map

$$\mu\colon H_k(X; \mathbb{Q}) \to H^{4-k}(\mathcal{B}^*(E), \mathbb{Q}), \quad \mu(\alpha) \coloneqq \mathbf{c}/\alpha.$$

We now let $\mathbf{A}(X)$ be the free symmetric algebra over the rational homology of X. Note that in the case $b_1(X) = 0$, the only case relevant for us here, all odd dimensional rational homology classes on X vanish, so $\mathbf{A}(X)$ is equivalently

generated by the even dimensional homology classes. We can then extend the map μ to a ringhomomorphism

$$\mu\colon \mathbf{A}(X) \to H^\bullet(\mathcal{B}^*(E)).$$

We would like to now define a $\mathbb{Q}$-valued polynomial map $\mathcal{D}_E\colon H_2(X;\mathbb{Q})^{\otimes d} \to \mathbb{Q}$ by

$$\mathcal{D}_E(\alpha_1 \otimes \cdots \otimes \alpha_d) \coloneqq \langle \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_d), [\mathcal{M}(E)] \rangle .$$

There are several problems with this, most importantly that the ASD moduli space is in general not compact and so we a priori do not have a fundamental class against which to evaluate. To get around this, we give a geometric construction:

Let $\alpha_1, \ldots, \alpha_d \in H_2(X;\mathbb{Z})$. We can choose smoothly embedded surfaces

$$\Sigma_1, \ldots, \Sigma_n \subset X$$

representing these classes. As above, the spaces $X \times \mathcal{B}^*(E|_{\Sigma_i})$ carry $SO(3)$-bundles equipped with a universal family of connections. By coupling the connections in this family with the Dirac operators on Σ_i corresponding to a complex structure, we obtain a family of elliptic and hence Fredholm operators parametrized by $\mathcal{B}^*(E|_{\Sigma_i})$. There is a notion of index for such a family. Finally, the determinant of this virtual index bundle is represented by an actual line bundle

$$\mathcal{L}_{\Sigma_i} \to \mathcal{B}^*(E|_{\Sigma_i}).$$

Now, we do not quite have a restriction map $\mathcal{B}^*(E) \to \mathcal{B}^*(E|_{\Sigma_i})$, however we can still pull back $\mathcal{L}_{\Sigma_i}$ to the space $\mathcal{B}^{**}(E)$ of those connections on E that remain irreducible when restricted to Σ_i and this pullback extends to a line bundle $\mathcal{L}_{\Sigma_i} \to \mathcal{B}^*(E)$ by Lemma 5.2.6 of [DK97] (where we suppressed the restriction in our notation). We have then

Proposition 5.1.1. $c_1(\mathcal{L}_{\Sigma_i}) = \mu([\Sigma_i])$.

Proof. This is Corollary 5.2.7 in [DK97]. □

To evaluate our classes $\mu([\Sigma_i])$ on the moduli spaces, we now use the following:

Lemma 5.1.2. *Let $p > 2$ be an even integer. There is a Banach space B and a continuous linear map $s\colon B \to \Gamma(\mathcal{B}^*(E), \mathcal{L}_{\Sigma_i})$ such that*

(i) For all $b \in B$, $s(b)$ is smooth and the evaluation map $B \times \mathcal{B}^*(E) \to \mathcal{L}_{\Sigma_i}$ defined by s is a smooth map of Banach manifolds

(ii) For each $[A] \in \mathcal{B}^*(E)$, there is $b \in B$ such that $s(b)([A]) \neq 0$

(iii) For every $b \in B$ the section $s(b)\colon \mathcal{B}^*(E) \to \mathcal{L}_{\Sigma_i}$ is continuous with respect to the L^p_1-topologies.

Proof. This is Lemma 7.2 in [Kro05]. □

Next we record [Kro05, Corollary 7.3]:

Corollary 5.1.3. *If M is a finite dimensional smooth manifold and $f\colon M \to \mathcal{B}^*(E)$ a smooth map, then there exists a section s of $\mathcal{L}_{\Sigma_i}$ that is transverse to f.*

We write V_{Σ_i} for the zero set of such a section. To be able to apply this to our moduli spaces, we need to make sure that there are well defined restriction maps from the moduli spaces defined by the bundles we are working with to the irreducible part of the configuration spaces on the representatives Σ. In order to get this, we need one further technical assumption:

Lemma 5.1.4. *We can choose the representing surfaces Σ_i such that all restriction maps $H_1(\Sigma_i;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ are surjective.*

Proof. Since $H_1(X;\mathbb{Z})$ is finitely generated, there is a finite set of smoothly embedded disjoint loops in X representing a generating set of $H_1(X;\mathbb{Z})$. Take small disjoint tubular neighborhoods of these loops and replace each Σ_i by the connected sum of Σ_i and the boundaries of these neighborhoods. Because each such boundary is homologically trivial in X, we do not change the homology class represented by the Σ_i in this way. Because the boundaries of the attached tubular neighborhoods contain isotopic copies of the generators, for each such class the induced restriction map on H_1 will be surjective.

Note that by general position in codimension ≥ 2, for a finite set of homology classes $[\Sigma_1], \ldots, [\Sigma_n]$, we can choose the tori above such that all n inclusions induce surjections on H_1 while keeping condition gp intact. □

Using the previous Lemma, we can prove the following:

Proposition 5.1.5. *If Σ_i are chosen as above, then there are well defined induced restriction maps $res_i\colon \mathcal{M}(P) \to \mathcal{B}^*(\Sigma_i)$.*

Proof. With the preceding Lemma, this now follows immediately from Proposition 4.7.2. See [MMR94, Lemma 7.5.3]: This follows from unique continuation and compactness. We first assume that the interior part of the perturbation vanishes. Then, if a Φ-ASD connection A becomes reducible after restricting to an open tubular neighborhood of Σ, then this connection is everywhere locally reducible on the compact submanifold $X_0 \coloneqq X \setminus Y \times [1, \infty)$. This follows from the usual unique continuation property of ASD connections because over this compact interior, $\mathbf{A}$ is unperturbed ASD.

This means that, given an A-parallel gauge transformation u over $\nu\Sigma$, we can extend u via a sequence of balls covering some path from $\nu\Sigma$ to a neighborhood of any point of X, A-parallely.

For the restriction map from the claim to be well defined, we however need the stronger conclusion that the so defined local system of local gauge transformations is in fact single valued, i.e. defines an A-parallel gauge transformation over all of X. For this, we use the assumption that the map $\pi_1(\Sigma) \to \pi_1(X)$, induced from the inclusion, is surjective: First of all, note that the system of u's locally defined by parallel transport along paths, defines an $O(2)$-reduction of P: the only possible ambiguity when extending u along two different paths in X between the same points is an overall sign.

We want to show that the first Stiefel-Whitney class of this $O(2)$- bundle vanishes, in which case it defines a further reduction of our bundle-with-connection to $SO(2)$ which would then prove our assertion.

But to prove this, we simply have to note that the induced map $H^1(\Sigma; \mathbb{Z}) \to H^1(X; \mathbb{Z})$ is surjective: from this, we get by left exactness of $\mathrm{Hom}((-), \mathbb{Z}/2)$ that the map

$$\iota^* \colon H^1(X; \mathbb{Z}/2) = \mathrm{Hom}(H_1(X), \mathbb{Z}/2) \to \mathrm{Hom}(H_1(\Sigma), \mathbb{Z}/2) = H^1(\Sigma; \mathbb{Z}/2)$$

must be *in*jective, so if w_1 of the $O(2)$-reduction over X (call it Q) were non-vanishing, this would also have to be true for the reduction over $\nu\Sigma$ because

$$0 \neq \iota^* w_1(Q) = w_1(\iota^* Q).$$

We already know, however, that the reduction to Σ further reduces to $SO(2)$, so this is impossible.

We need to show that this unique continuation property holds over all of X. By what we have just proved, the restriction of $\mathbf{A}$ to the band $Y \times (0, 1)$ is reducible. But by Lemma 4.7.3, $\mathbf{A}$ must then also be reducible over the entire end, finishing the proof if the only non-zero perturbations present are end-perturbations.

Finally, in the general case where there are non vanishing inner and end perturbations present, we can use thee same compactness argument as in [Kro05,

Lemma 36] together with the fact that the projection map from the parametrized moduli space to the space of perturbations is proper. □

We apply the above to the inclusion of the (perturbed) moduli space into the configuration space and write

$$\mathcal{M}_\pi(E) \cap V_{\Sigma_i}$$

for the zero set of a transverse section in $\mathcal{M}_\pi(E)$. It is a smooth codimension 2 submanifold of the moduli space. An easy generalization of the above shows that we can in fact choose sections s_i of $\mathcal{L}_{\Sigma_i}$ such that all multiple intersections

$$\mathcal{M}_\pi(E) \cap V_{\Sigma_{i_1}} \cap \cdots \cap V_{\Sigma_{i_k}},$$

$\{i_1 \ldots, i_k\} \subset \{1, \ldots, d\}$, are simultaneously transverse. In particular, because $2d = \dim \mathcal{M}_\pi(E)$, $\mathcal{M}_\pi(E) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$ is a discrete set of points. To finally define the Donaldson invariant, we need to know that this intersection is also compact. For this, we choose small tubular neighborhoods $\nu\Sigma_1, \ldots, \nu\Sigma_d$ and assume that all triple intersections are empty, i.e. whenever i, j, k are pairwise distinct, then

$$\nu\Sigma_i \cap \nu\Sigma_j \cap \nu\Sigma_k = \emptyset. \tag{gp}$$

In fact, it is easy to see that in the situation of the Corollary, we can choose sections s_i of $\mathcal{L}_{\Sigma_i}$ that are simultaneously transverse to the inclusions of all moduli spaces defined by a finite set of bundles. We will use this for representatives of the finite set of isomorphism classes of bundles of Pontrjagin number bounded by some constant.

We then have

Lemma 5.1.6. *Let E be an $SU(2)$-bundle over the smooth 4-manifold M. Let Φ be a regular perturbation and $\mathcal{M}_\pi(E)$ be the moduli space of Φ-perturbed instantons on E. Let $d \coloneqq \frac{1}{2}\dim \mathcal{M}_\pi(E)$. Then for any family $\alpha_1, \ldots, \alpha_d \in H_2(X; \mathbb{Z})$ we can find smoothly embedded surfaces $\Sigma_1, \ldots, \Sigma_d$ such that $[\Sigma_i] = \alpha_i$ and such that the intersection*

$$\mathcal{M}_\pi(E) \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d} \tag{cup}$$

is transverse and compact, i.e. a finite set of points.

Proof. This is proved by the same dimension counting argument as in [DK97, p. 351, Lemma 9.2.9]. We simply have to note that the V_Σ are not only closed under C^∞-limits, but are in fact also closed under L^p_1-limits by Lemma 5.1.2. □

We take formula cup as our definition of polynomial invariants *computed from the π-perturbed moduli spaces*. We need to check that these invariants are independent of the choice of regular perturbation, and more precisely are equal to Donaldsons polynomial invariant defined using unperturbed moduli spaces:

Theorem 5.1.7. *Let $X = X_1 \cup_Y X_2$ with $b_1(X) = 0$, $b^+(X) > 2$ and $b^+(X_i) > 0$, $i = 1, 2$. Then for any regular perturbation π in some non empty neighborhood of 0 in the parametrizing Banach space of perturbations, the polynomial invariants computed from the π-perturbed moduli spaces as in Lemma 5.1.6 are equal to Donaldsons polynomial invariants.*

Recall that we call a perturbation Φ *regular* if all critical points of $cs + \Phi$ are nondegenerate.

For this proof, we need the following

Lemma 5.1.8. *Let Z be a manifold with precisely one cylindrical end $Y \times [0, \infty)$ where Y is a $\mathbb{Q}HS^3$ and $b_2^+(Z) - b_1(Z) \geq 2$. Let $E \to Z$ be a non trivial adapted $U(2)$-bundle. Then there is some $\varepsilon > 0$ with the following significance:*

For a generic path of perturbations Φ_t with $\Phi_0 = 0$ and $\|\Phi_t\| < \varepsilon$ uniformly in t, none of the moduli spaces $\mathcal{M}_{\Phi_t}(E, \rho)$ (where ρ is any reducible critical point of $cs + \Phi_t$) contains reducible connections.

Proof of Lemma 5.1.8. This is a modest generalization of [Eis23, Proposition 4.16]. We replace the assumption $b^+(Z) > b_1(Z)$ by the stronger one $b_2^+(Z) - b_1(Z) \geq 2$ and claim the stronger conclusion that, first, the permissible set of perturbations $\mathcal{P}_\delta$ there contains a small ball around 0 and that, second, the conclusion from the reference holds for generic paths of perturbations.

Our strategy for proving this is essentially identical to the one used in the cited paper and we only indicate the modifications we have made:

Recall that a reducible $[\mathbf{A}]$ in the configuration space for E induces a reduction of the bundle-with connection $(E, \mathbf{A})$ to a line-bundle-with-connection (η, A). We will show that for ε and a generic path Φ_t as in our stated conclusion, the parametrized moduli space $\mathcal{M}_{\Phi_t}(\eta)$ is empty for all t. For a single line bundle, we will see that the family of paths for which this holds is in fact open. Intersecting these open dense sets of paths over the at most countably infinite family of (isomorphism classes of) line bundles over Z, gives a residual set of paths with the stated property.

First, we want to make sure that the results from [Eis23] are applicable for all small enough perturbations, in particular for $\Phi = 0$. We have to check that for $\|\Phi\|$ sufficiently small, the perturbed extended Hessian operator

$$L_A^\Phi \coloneqq \begin{pmatrix} 0 & -d_A^* \\ -d_A & *d_A + D_A \nabla_\pi \end{pmatrix}.$$

for any critical point A of $cs+\Phi$ on η has no eigenvalues of absolute value $< \delta$ for some $\delta > 0$, independent of A. We first note that the adjoint bundle $\mathrm{ad}(\eta) \cong \underline{i\mathbb{R}}$ is trivial and the operator $d_A = d\colon \Omega^i(Y; i\mathbb{R})$ is the usual deRham differential, independent of A. In particular, in the case $\Phi = 0$, the operator L_A is a root of the Hodge-Laplace operator on Y which is self adjoint elliptic and invertible on a rational homology 3-sphere. The same is thus true for L_A and in particular, L_A has no Eigenvalues of absolute value less than some $\varepsilon_0 > 0$. Note that this constant depends only on Y but not on E, η or A.

We are only interested in the spectra of these Hessian operators at the reducible critical points of the perturbed Chern-Simons. For a small enough perturbation, there are only finitely many of these; in fact, for every small enough perturbation they will be in bijection with the reducible critical points of the unperturbed Chern Simons and we have already seen that there are only finitely many (corresponding to the finitely many flat $U(1)$-bundles over Y). This is the content of Lemma 3.4 in [Eis23].

By Proposition 3.2.2, the operator norms of the perturbation terms $D_A\nabla_\pi$ is bounded, with a bound that depend continuously on A, and it is obvious that they depend continuously on Φ. Because $D_A\nabla_0 = 0$, we can thus find some $\varepsilon > 0$ such that for all perturbations Φ with $\|\Phi\| < \varepsilon$ and all critical points A of cs_π the operator norm

$$\|D_A\nabla_\pi\| < \frac{\varepsilon_0}{4} =: \delta.$$

Now, let Φ be such a perturbation and $\lambda < \delta$. Then we have

$$\left\|(L_A^\Phi - \lambda)\omega\right\| > \|L_A\omega\| - (\delta + \lambda)\|\omega\| > \varepsilon_0\|\omega\| - \frac{\varepsilon_0}{2}\|\omega\| = \frac{\varepsilon_0}{2}\|\omega\|.$$

In particular, we see that λ is not in the spectrum of L_A^Φ and thus $\Phi \in \mathcal{P}_\delta$ in the notation of [Eis23].

Now, as in (the proof of) [Eis23, Lemma 4.21], there is a smooth Banach manifold $\mathcal{P}_\delta\mathcal{B}_\eta$, parametrizing pairs $([A], \Phi)$ where Φ is a perturbation with the eigenvalue restriction as above and $[A]$ is a configuration on η with limiting value A_∞ over the end a critical point of $cs + \Phi$. This comes equipped with a smooth Fredholm map $s\colon \mathcal{P}_\delta\mathcal{B}_\eta \to \Omega^2_+(i\mathbb{R})$ whose zero set, the parametrized perturbed ASD moduli space $\mathcal{P}_\delta\mathcal{M}_\eta$, is a smooth submanifold. We restrict the projection map $p\colon \mathcal{P}_\delta\mathcal{B}_\eta \to \mathcal{P}_\delta$ to this submanifold to obtain a smooth map

$$f\colon \mathcal{P}_\delta\mathcal{M}_\eta \to \mathcal{P}_\delta.$$

The tangent space to the parametrized configuration space at $([A], \Phi)$ decomposes as $\Omega^1_\delta(i\mathbb{R}) \oplus T_\pi\mathcal{P}_\delta$ and accordingly, we can decompose the differential of a smooth map F with domain $\mathcal{P}_\delta\mathcal{B}_\eta$ as $dF = d_1F + d_2F$.

We write points in the tangent space of the parametrized configuration space as (a, ϕ). It is clear that $dp(a, \phi) = \phi$ and that at the point $([A], \Phi)$, we have $d_1 s = D_A^\Phi$, the Φ-perturbed linearized ASD-operator on the line bundle η. In particular, $d_1 s$ is Fredholm of index $b_1 - b^+$. Also note that

$$T_{([A],\Phi)} \mathcal{P}_\delta \mathcal{M}_\eta = \ker ds = \{(a, \phi) \mid d_1 s(a) = -d_2 s(\phi)\}$$

and

$$df(a, \phi) = \phi.$$

We now claim that the index of f is equal to the index of $d_1 s$: If $(a, \phi) \in \ker ds \cap \ker dp$, then $d_1 s(a) + d_2 s(\phi) = 0$ and $dp(a, \phi) = \phi = 0$, so $a \in \ker d_1 s$ and $\phi = 0$. Since the converse is obvious, we get an isomorphism $\ker df \to \ker d_1 s$.

Also, for every $([A], \Phi) \in \mathcal{P}_\delta \mathcal{M}_\eta$, the map $d_2 s \colon T_\pi \mathcal{P}_\delta \to \Omega^2_+(i\mathbb{R})$ induces an isomorphism

$$\overline{c} \colon : \operatorname{coker} df = T_\pi \mathcal{P}_\delta / \operatorname{im} df \to \Omega^2_+(i\mathbb{R}) / \operatorname{im} d_1 s = \operatorname{coker} d_1 s :$$

Because ds is surjective at every point of the parametrized moduli space (see [Eis23, Lemma 4.21], we can write an arbitrary $\omega \in \Omega^2_+$ as $\omega = d_1 s(a) + d_2 s(\phi)$. This means precisely that the map $d_2 s$ induces a surjection $c \colon T_\pi \mathcal{P}_\delta \to \operatorname{coker} d_1 s$. Now, $d_2 s(\phi) \in \operatorname{im} d_1 s$ if and only if there is some a such that $d_2 s(\phi) = d_1 s(a)$ and this is equivalent to $(-a, \phi) \in T_{([A],\Phi)} \mathcal{P}_\delta \mathcal{M}_\eta$. But this means precisely that $\phi \in \operatorname{im} df$, so $\ker c = \operatorname{im} df$. This proves that $\overline{c}$ as above is well defined and injective and we have already seen that it is surjective, too.

To finish our proof, let $\gamma \colon [0, 1] \to \mathcal{P}_\delta$ be any path with $\gamma(0) = 0$ and $\gamma(1)$ a regular perturbation for the bundle E. Note that any such path is automatically transverse to f at 0 and 1 and so by general transversality theory, we can find arbitrarily close smooth paths with the same endpoints, that are transverse to f.

But because the index of f is smaller or equal -2 by our assumption, this means that at all points, $\dim \operatorname{coker} df \geq 2$. So, because $\dim \operatorname{im} d\gamma \leq 1$, the images of these differentials cannot generate the entire tangent space of $\mathcal{P}_\delta$, at *any* $([A], \phi) \in \mathcal{P}_\delta \mathcal{M}_\eta$. So, for a path that is transverse to f, the space of $\gamma(t)$-perturbed ASD-configurations on η must in fact be empty for all t. □

Now we are able to prove Theorem 5.1.7:

Proof of Theorem 5.1.7. The proof is based on [Zen18, proof of Proposition 6.11]. Donaldsons polynomial invariants, where defined, are independent of the choice of Riemannian metric. So we assume that the metric on X is equal to the canonical product metric on a small tube $Y \times (-\varepsilon, \varepsilon)$ around $Y \hookrightarrow X$.

We again define, for $L > 0$, the manifold X_L as X with a collar neighborhood $Y \times [-L, L]$ inserted around Y and Riemannian metric g_L equal to the product metric on the cylinder. What we'd like to show, then, is that there is a smooth cobordism inside the configuration space $\mathcal{B}(P)$ between $\mathcal{M}(P, g_L)$ and $\mathcal{M}_\pi(P, g_L)$ for the kind of perturbation Φ that we have used above.

We recall that Φ has an inner component Φ_{inn}, compactly supported on $X \setminus Y \times (-\varepsilon, \varepsilon)$ and a constant, end component, Φ_{end}, supported entirely on the tube $Y \times (-L, L) \subset X_L$.

By [DK97, Corollary 4.3.15], we can adjust the metric on X in a small ball disjoint from the collar around Y to achieve that the unperturbed moduli space $\mathcal{M}(P)$ contains no reducible connections and is a smooth submanifold $\mathcal{M}(P) \subset \mathcal{B}^*(P)$. Since the component Φ_{inn} has compact support independent of L, we see by a compactness argument that there is $\varepsilon_{inn} > 0$ such that this conclusion remains true for all moduli spaces $\mathcal{M}_{\Phi_{inn}}(P)$ with $\|\Phi_{inn}\| < \varepsilon_{inn}$ and the invariants computed from intersecting our cycles V_Σ with Φ_{inn}-ASD moduli spaces are equal to the invariants computed from unperturbed moduli spaces, compare [Kro05, Proposition 43].

It remains to check that there is a smooth cobordism between $\mathcal{M}_{\Phi_{inn}}(P, g_L)$ and $\mathcal{M}_\Phi(P, g_L)$ inside $\mathcal{B}^*(P)$, for all large enough L.

First, we need to make sure that $\mathcal{M}_\Phi(P, g_L)$ contains no reducibles. Assume for a contradiction that there were a sequence $L_n \to \infty$ and reducibles $[A_n] \in \mathcal{M}_\Phi(P, g_{L_n})$ for all n. Then, by our compactness results, we would get a reducible Φ-perturbed instanton A_∞ on some adapted $U(2)$-bundle $P_1 \to X_1 \cup_Y (Y \times [0, \infty))$ with non-trivial c_1. But we have already seen that there we can choose Φ such that $\Phi_{inn} < \varepsilon_{inn}$ and such that $\mathcal{M}_\pi(P_1)$ does not contain any reducibles. So, for large enough L and a generic choice of Φ, $\mathcal{M}_\Phi(P, g_L)$ contains no reducibles whatsoever.

By the argument from the proof of Theorem 4.37 of [Eis23] (albeit in a simplified form as our base manifold X_L here is closed), we can then slightly adjust Φ_{inn} again to make sure that all the other properties we used above are satisfied and in addition, $\mathcal{M}_\Phi(P, g_L)$ is a smooth submanifold of $\mathcal{B}^*(P)$.

Finally, we can now use the argument from the proof of [Zen18, Proposition 6.11], to show that $\mathcal{M}_{\Phi_{inn}}(P, g_L)$ and $\mathcal{M}_\Phi(P, g_L)$ are in fact cobordant inside $\mathcal{B}^*(P)$. □

5.2 A Nonvanishing Theorem

Having now defined Donaldson's polynomial invariants, we state here the most important known property for our purposes:

Theorem 5.2.1 (5.9)**.** *[Eliashberg-Thurston, Kronheimer-Mrowka, Taubes] Let Y be a rational homology sphere which admits a taut foliation. Then Y embeds as a smooth separating hypersurface into a closed, smooth, symplectic 4-manifold X that satisfies the following:*

- $b_1(X) = 0$
- $b_2^+(X) > 1$ *is odd*
- *The $SO(3)$-Donaldson invariants $\mathcal{D}_X^w$ do not vanish identically, for any class $w \in H^2(X; \mathbb{Z}/2\mathbb{Z})$.*

Proof. This is essentially the argument in [KM04, 3.2 Proof of Theorem 2]. By Proposition 15 of that paper, a rational homology sphere (a closed, oriented 3-manifold distinct from $S^1 \times S^2$) that admits a taut foliation embeds into a symplectic 4-manifold X that satisfies the stated assumptions on Betti numbers and in addition satisfies all assumptions of [KM04, Corollary 7] and of the main theorem of [Tau94]. The cited corollary implies that the Donaldson invariants of such an X do not vanish if and only if its Seiberg-Witten invariants do not vanish, while the latter (that the Seiberg Witten invariants of X do not vanish) is the statement of Taubes' theorem. □

Combining the above two theorems, we get

Corollary 5.2.2. *If the rational homology 3-sphere Y admits a taut foliation, then Y embeds smoothly as a separating hypersurface into a closed smooth 4-manifold X with non-vanishing Donaldson invariants.*

5.3 Avoiding flat connections by Blow-Ups

We are ultimately interested in $SU(2)$-Donaldson invariants. However, working with moduli spaces of $SU(2)$-instantons, one cannot avoid flat connections (because the weak compactness properties of $SU(2)$-instanton moduli spaces only guarantee convergence of subsequences up to some loss of energy) which are always singular in their moduli spaces and cannot be perturbed away with the class of perturbations we will consider.
This is the reason for considering $U(2)$-instantons with fixed, non-trivial first Chern class instead: Whereas the topology of the bundle on which the connections are defined can change under our limits, this only affects the second Chern class (which in the case of $G = SU(2)$ determines all of the topology in our dimensions). By contrast, the *first* Chern class is *un*changed under Uhlenbeck-limits

because the first Chern class of a $U(2)$-bundle is determined by its restriction to the 2-skeleton of any CW-structureon X. We can always perturb the CW-structure slightly so that the 2-skeleton is disjoint from any given finite set of points. Hence c_1 of such a bundle only depends on the restriction of the bundle to the complement of any finite set of points and we can apply this to the set of bubbling points. Hence limits of sequences of instantons with non-trivial first Chern class will never be flat by Chern-Weil theory.
To make use of this in our situation, we need a way to compute $SU(2)$-invariants from $U(2)$-invariants. This is afforded to us by the following construction:
Let X be a closed, smooth, oriented 4-dimensional manifold. The *blow up* of X is the connected sum $X\#\overline{\mathbb{C}P}^2$. Up to diffeomorphism, the blow up is independent of the point at which $\overline{\mathbb{C}P}^2$ is attached, so we will leave this point implicit in our notation.
Let $L \to \overline{\mathbb{C}P}^2$ be the line bundle with $c \coloneqq c_1(L)$ the Poincaré dual of the standard generator of $H_2(\overline{\mathbb{C}P}^2;\mathbb{Z})$, $E \coloneqq \underline{\mathbb{C}} \oplus L \to \overline{\mathbb{C}P}^2$ and we will denote the $U(2)$-bundle of unitary frames of E by P_0. Now, given any $SU(2)$-bundle $Q \to X$ we can consider the $U(2)$-bundle $P \to X$ that is associated to Q via the inclusion $SU(2) \hookrightarrow U(2)$ (this is the $U(2)$-bundle with $c_2(P) = c_2(Q)$ and $c_1(P) = 0$) and then form the connected sum-bundle $P\#P_0 \to X\#\overline{\mathbb{C}P}^2$ over the blow up. Then $c_1(P\#P_0) = c$ and $c_2(P\#P_0) = c_2(Q)$. In this way, we can assign to any $SU(2)$-bundle over X a $U(2)$-bundle over the blow up and we get moduli spaces $\mathcal{M}(P)$ and $\mathcal{M}_c(P_0\#P)$ of $SU(2)$- and $U(2)$-instantons, respectively.
Now, let $2d \coloneqq \dim \mathcal{M}_P$ and $\alpha \in H_2(X)^{\otimes d}$. Then we have $\dim \mathcal{M}_c(P_0\#P) = 2d+2$ and we get invariants $\mathcal{D}_P(\alpha)$ and, in the "stable range", i.e. for d large enough, $\mathcal{D}_{P_0\#P}(c \otimes \alpha)$. These invariants are related by

Proposition 5.3.1. $\mathcal{D}_{P_0\#P}(c \otimes \alpha) = 2\mathcal{D}_P(\alpha)$.

Proof. This is [MM92, Corollary 3.4]. □

6 The main Theorem

Using our knowledge from the previous sections, we are finally able to prove our main Theorem:

Theorem 6.0.1 (main Theorem)**.** *Let Y be a rational homology 3-sphere and π a holonomy perturbation such that all critical points of cs_π are reducible and the normal Hessian is invertible at each critical point. If Y embeds into a closed, smooth 4-manifold X as a separating hypersurface,*

$$X = X_1 \cup_Y X_2,$$

where $b_1(X_1) = b_1(X_2) = 0$ and $b_2^+(X_1), b_2^+(X_2) > 0$, then the Donaldson invariants of X vanish identically.

Proof. Assume that $\mathcal{D}_X$ does not vanish identically. Our strategy is to deduce a contradiction by a neck stretching argument as follows: we choose a sequence of Riemannian metrics g_n on X such that a tubular neighborhood of Y in X is isometric to

$$Y \times [-n, n]$$

with its canonical product metric. As the value of the invariants is independent of the Riemannian metric, we know that the Donaldson invariants of the manifolds (X, g_n) are also non vanishing for all n, in particular, the corresponding moduli spaces cannot be empty. By our compactness results, we get a geometric limit, i.e. a tuple of elements in the moduli spaces corresponding to a finite set of cylindrical end manifolds; this means in particular that all of these moduli spaces must be non-empty and have dimension ≥ 0. On the other hand, however, using our assumption that all critical points of Chern-Simons are reducible and the dimension formulas, we then show that at least one of these moduli spaces must have negative dimension and hence be empty after all.

To make this argument precise, first of all we have to make sure that the moduli spaces in question have well defined dimensions. For this, we need to avoid encountering reducible solutions in the limit. To deal with fully reducible solutions, which cannot be perturbed away, we use Remark 2.1.4 and blow ups as in section 5.3. We choose points $x_i \in X_i \subset X$, $i = 1, 2$, in the complement of collar neighborhoods of Y and replace the X_i by the blowups

$$\tilde{X}_i \coloneqq X_i \# \overline{\mathbb{C}P^2}$$

along the points x_i and $X = X_1 \cup_Y X_2$ by the double blow up

$$\tilde{X} \coloneqq \tilde{X}_1 \cup_Y \tilde{X}_2.$$

Let h_1, h_2 be the generators of the cohomology rings of the $\overline{\mathbb{C}P^2}$-summands. We now consider the $U(2)$-Donaldson invariants $\mathcal{D}^c_{\tilde{X}}$ defined by the non-vanishing class

$$c \coloneqq h_1 + h_2 \in H^2(\tilde{X}).$$

By Proposition 5.3.1, our assumption on X that $\mathcal{D}_X$ does not vanish identically implies that the invariants $\mathcal{D}^c_{\tilde{X}}$ also cannot vanish identically.

To get rid of the remaining possible reducibles, we make use of the regularity results for the perturbed ASD-moduli spaces: we first note that blowing up does not affect the first and positive second Betti numbers of X, so our topological assumptions on X imply for $\tilde{X}$ that

$$b_1(\tilde{X}) = 0 \text{ and } b_2^+(\tilde{X}) > 0.$$

By Proposition 4.3.2, we can then choose a regular perturbation π such that the moduli spaces corresponding to $U(2)$-bundles over $\tilde{X}$ with first Chern class c do not contain any reducibles. The same will also be true for the moduli spaces corresponding to $U(2)$-bundles over the $\tilde{X}_i$ with first Chern class h_i, for the same reason.

Now, we apply the neck stretching argument to $\tilde{X}$: as the Donaldson invariants of $\tilde{X}$ do not depend on the choice of Riemannian metric on $\tilde{X}$, for a sequence $(g_n)_n$ of metrics on $\tilde{X}$ as in Theorem 4.5.7 we know the following:

there are a $U(2)$-bundle $P \to \tilde{X}$ and $d = \frac{1}{2}\dim \mathcal{M}(P, g_n)$ embedded surfaces $\Sigma_1, \dots, \Sigma_d \subset \tilde{X}$ such that for all n the finite set

$$\mathcal{M}(P, g_n) \cap V_{\Sigma_1} \cap \dots \cap V_{\Sigma_d}$$

is non-empty (because the algebraic intersection number is non-zero). Because $\mathcal{D}^c_{\tilde{X}}$ vanishes on torsion classes (as is the case for any multilinear map with values in the integers), we can assume every surface Σ_i is contained in one of $\tilde{X}_1 \setminus \partial(\tilde{X}_1)$ or $\tilde{X}_2 \setminus \partial(\tilde{X}_2)$ and thus, after renumbering if necessary, we can assume that

$$\Sigma_1, \dots, \Sigma_{d_1} \subset \tilde{X}_1 \setminus \partial(\tilde{X}_1) \text{ and } \Sigma_{d_1+1}, \dots, \Sigma_d \subset \tilde{X}_2 \setminus \partial(\tilde{X}_2).$$

Choosing any $[\mathbf{A}_n] \in \mathcal{M}(P, g_n)$, we get a sequence satisfying the assumptions of Theorem 4.5.7, so in the limit we get an idealized π-ASD connection

$$(\mathbf{A}_1, \omega_1, \dots, \omega_s, \mathbf{A}_2, x_1, \dots, x_r)$$

over $\tilde{X}_1^\infty \sqcup T_1 \sqcup \dots \sqcup T_s \sqcup \tilde{X}_2^\infty$ with compatible boundary values (i.e. the last item of Theorem 4.5.7 is satisfied) and without loss of energy (compare the remark after Theorem 4.5.7). We denote the limiting value of $\mathbf{A}_1$ over the end of $\tilde{X}_1^\infty$

by ρ_0^+ and the limiting value of $\mathbf{A}_2$ over the end of $\tilde{X}_2^\infty$ by ρ_{s+1}^-. Similarly, let ρ_i^+ resp. ρ_i^- be the limiting values of ω_i over the positive resp. negative end of T_i. Note that because no energy is lost in our limit, we have for $i = 0, \ldots, s$:

$$\rho_i^+ = \rho_{i+1}^-. \tag{20}$$

We denote by r_1 the number of points x_i lying in $\tilde{X}_1^\infty$ and by r_2 the number of points lying in $\tilde{X}_2^\infty$. Then

$$r_1 + r_2 \leq r.$$

Let $P_i \to \tilde{X}_i^\infty$ denote the adapted bundles on which the $\mathbf{A}_i$ are defined. Assuming, as we do, that each of the x_i is contained in at most two of the $\nu(\Sigma_j)$, and using that the V_{Σ_j} are closed under L_1^p-limits by Lemma 5.1.2, (iii), we get

$$[\mathbf{A}_1] \in \mathcal{M}_\pi(P_1) \cap V_{\Sigma_{j_1}} \cap \cdots \cap V_{\Sigma_{j_{l_1}}}, \;\; l_1 \geq d_1 - 2r_1,$$

$$[\mathbf{A}_2] \in \mathcal{M}_\pi(P_2) \cap V_{\Sigma_{j_2}} \cap \cdots \cap V_{\Sigma_{j_{l_2}}}, \;\; l_2 \geq d_2 - 2r_2,$$

where d_1 is the number of surfaces in $\tilde{X}_1$ and $d_2 := d - d_1$ is the number of surfaces in $\tilde{X}_2$.

We can also assume that the moduli spaces and the V_{Σ_i} are smooth manifolds and are in general position to each other. Each V_{Σ_i} is of codimension 2 in its respective configuration space $\mathcal{B}^*$, so from general position and $l_i \geq d_i - 2r_i$, we get

$$\dim(\mathcal{M}_\pi(P_1) \cap V_{\Sigma_{j_1}} \cap \cdots \cap V_{\Sigma_{j_{l_1}}}) \leq \dim(\mathcal{M}_\pi(P_1)) - 2d_1 + 4r_1$$

and

$$\dim(\mathcal{M}_\pi(P_2) \cap V_{\Sigma_{j_2}} \cap \cdots \cap V_{\Sigma_{j_{l_2}}}) \leq \dim(\mathcal{M}_\pi(P_2)) - 2d_2 + 4r_2.$$

Because these intersections are nonempty, we get from this

$$\dim(\mathcal{M}_\pi(P_2)) \geq 2d_1 - 4r_1 \tag{21}$$

and

$$\dim(\mathcal{M}_\pi(P_2)) \geq 2d_2 - 4r_2.$$

Adding these inequalities, we get

$$\dim(\mathcal{M}_\pi(P_1)) + \dim(\mathcal{M}_\pi(P_2)) \geq d - 4r = \dim \mathcal{M}_\pi(P, g_n) - 4r.$$

Denote the adapted bundles carrying the solutions ω_i by $Q_i \to T_i$. Because the limit has compatible boundary values (see equation 20), similarly as in section 4.6 we can glue the bundles $P_1, Q_1, \ldots, Q_s, P_2$ to obtain a new $U(2)$-bundle

$$P' := P_1 \cup_Y Q_1 \cup_Y \cdots \cup_Y Q_s \cup_Y P_2$$

over $\tilde{X}$. As in the unperturbed case, each of the points x_i corresponds to the "bubbling off" of a solution $\mathbf{B}_i$ to the π-perturbed ASD-equations on a non-trivial bundle $P_{S^4,i}$ over the 4-sphere S^4. Our dimension formula 12 implies that

$$\dim \mathcal{M}_\pi(P') \leq \dim \mathcal{M}_\pi(P) - \sum_{i=1}^{r} \dim \mathcal{M}_\pi(P_{S^4,i}) - \dim \mathrm{Stab}(\theta), \qquad (22)$$

where θ is some connection on a bundle over S^3 that is a critical point for a perturbed Chern-Simons functional over the three-sphere. In the unperturbed case, θ would necessarily be gauge equivalent to a trivial connection with $\dim \mathrm{Stab}(\theta) = 3$, considerably strengthening our estimate (meaning that the dimension of the unperturbed moduli spaces drops by 8 for every quantum of energy lost via bubbling). In our perturbed case, it is not immediately obvious (although almost certainly true), however, that we can take θ to be a trivial reducible because in the presence of perturbations, there might be more (a priori even irreducible!) critical points of the functional involved. This stronger, but unproven variant of inequality 22 is ultimately unnecessary for our argument here, however, so we forego further investigation.

(It is probably true that as in the unperturbed case, the dimension of the perturbed moduli spaces drops by)

To make use of this, we have to estimate the negative summands on the right side of inequality refbubblingIneq. These dimensions are equal to the indices of the corresponding deformation operators

$$D^\pi_{\mathbf{B}} = d^*_{\mathbf{B}_i} + d^+_{\mathbf{B}_i} + \nabla^2_{\mathbf{B}_i} \hat{V}_\pi(\mathbf{B}_i).$$

Over the closed manifold S^4, this is a *compact* perturbation of the unperturbed operator $D_{\mathbf{B}} = d^*_{\mathbf{B}} + d^+_{\mathbf{B}}$, so we can just as well compute these dimensions with the unperturbed operators. Using the Atiyah-Singer index theorem and the non-triviality of P, one sees

$$\mathrm{ind}\, D_{\mathbf{B}_i} = 8c_2(P) - 3 \geq 5$$

(compare [DK97, Formula 4.2.21]).

Together, we obtain

$$\dim \mathcal{M}_\pi(P_{S^4,i}) = \mathrm{ind}\, D^\pi_{\mathbf{B}} \geq 5. \qquad (23)$$

Plugging this into inequality 22 gives

$$\dim \mathcal{M}_\pi(P') \leq \dim \mathcal{M}_\pi(P) - 5r. \qquad (24)$$

We can now finish our proof by estimating

$$\begin{aligned}
\dim(\mathcal{M}_\pi(P_1)) + \dim(\mathcal{M}_\pi(P_2)) &\overset{21}{\geq} \dim \mathcal{M}_\pi(P) - 4r \\
&\geq \dim \mathcal{M}_\pi(P) - 5r \\
&\overset{24}{\geq} \dim \mathcal{M}_\pi(P') \\
&\overset{4.6.2}{=} \dim \mathcal{M}_\pi(P_1) + \dim \mathrm{Stab}(\rho_0^+) + \\
&\quad \sum_{i=1}^{s} (\dim_f \mathcal{M}_\pi(Q_i) + \dim \mathrm{Stab}(\rho_i^+)) + \\
&\quad \dim \mathcal{M}_\pi(P_2) \\
&> \dim(\mathcal{M}_\pi(P_1)) + \dim(\mathcal{M}_\pi(P_2)),
\end{aligned}$$

where the last inequality is a result of our assumption that flat connections ρ over Y are all reducible, hence $\dim \mathrm{Stab}(\rho) \geq 1$ and of the index computation 4.3.9, which together imply that $\dim \mathrm{Stab}(\rho_0)$ is positive and each summand $\dim_f \mathcal{M}_\pi(Q_i) + \dim \mathrm{Stab}(\rho_i^+)$ is non-negative. Finally, this inequality is our desired contradiction. □

From this Theorem, we get the following corollary:

Corollary 6.0.2. *Let Y be a rational homology 3-sphere that admits a taut foliation and whose universal abelian covering is a rational homology 3-sphere, as well.*

Then there is a representation

$$\pi_1(Y) \to SU(2)$$

with non-abelian image.

Proof. Assume for a contradiction that all representations

$$\rho \colon \pi_1(Y) \to SU(2)$$

have abelian image.

Then, by Proposition 4.5 and the paragraph following it in [BS18], all critical points of the unperturbed Chern-Simons functional cs_Y are non-degenerate. This means that the perturbation $\pi = 0$ is a regular perturbation on Y.

Now, because Y admits a taut foliation, by Corollary 5.2.2 Y embeds as a separating hypersurface into a 4-manifold X,

$$X = X_1 \cup_Y X_2,$$

with $b_1(X) = 0$ and $b_2^+(X) > 0$ with non-vanishing Donaldson invariants.

By Theorem 6.0.1, however, the Donaldson invariants of X would have to vanish under the assumption that all $SU(2)$-representations of $\pi_1(Y)$ are reducible. This is a contradiction. Thus, there must be a representation

$$\pi_1(Y) \to SU(2)$$

with non-abelian image. $\square$

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Selbstständigkeitserklärung

Ich, Felix Eberhart, habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Ausserdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die elektronische Version identisch sind und dass ich von vorgesehenen Rechtsfolgen Kenntnis habe.

Ort, Datum

Unterschrift