Letter

Phase-space integrals through Mellin-Barnes representation

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This letter introduces a novel analytical approach to calculating phase-space integrals, crucial for precision in particle physics. We develop a method to compute angular components using multifold Mellin-Barnes integrals, yielding results in terms of Goncharov polylogarithms for integrals involving three denominators. Our results include expressions for massless momenta to $\mathcal{O}(\epsilon^2)$ and for massive momentum to $\mathcal{O}(\epsilon)$. We derive recursion relations that reduce integrals with higher powers of denominators to simpler ones. We detail how to combine the angular part with the radial one which requires a careful handling of singularities.

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Phase-space (PS) integrals are essential for achieving precise theoretical predictions in perturbative quantum field theory owing to their appearance through real-emission Feynman diagrams. Calculating an observable beyond the leading perturbative order requires consideration of both virtual and real emission diagrams. Over the past few decades, a plethora of work has been performed in handling Feynman integrals analytically arising in virtual diagrams. In contrast, the analytical computation of PS integrals has not received comparable attention, despite being a crucial component. In this letter, we present a method based on the Mellin-Barnes (MB) representation [1] to perform an efficient analytical computation of PS integrals within the framework of dimensional regularization.

Phase-space integrals can be decomposed into radial and angular parts in an appropriately chosen reference frame [2,3]. In $d = 4 - 2\epsilon$ space-time dimensions, the latter with n denominators can be expressed as

$$\Omega_{j_1,\ldots,j_n} \equiv \int \mathrm{d}\Omega_{d-1}(q)\frac{1}{(p_1\cdot q)^{j_1}\ldots(p_n\cdot q)^{j_n}}, \qquad (1)$$

where $\{p_i^\mu\}$ are a set of fixed vectors and $\mathrm{d}\Omega_{d-1}(q)$ is a rotationally invariant angular measure for the massless vector q^μ. The $\{j_i\}$ are integers whose values depend on the scattering process and the perturbative order of interest. For instance, for the double real emission in semi-inclusive deep inelastic scattering at next-to-next-to-leading order, we require a maximum of two denominators, i.e., $n = 2$. When $n = 1$ and the momentum p_1 is massless ($p_1^2 = 0$), the integral is straightforward to express in terms of the total volume element. For massive momentum ($p_1^2 \neq 0$), the angular integral can be solved in terms of the ${}_2F_1$ hypergeometric function [2,4]. For the case of two denominators with massless momenta ($n = 2$ and $p_1^2 = p_2^2 = 0$), the angular integral can again be solved using the ${}_2F_1$ hypergeometric function [2,4]. However, when one or both momenta are massive, the result requires Appell functions of the first kind and Lauricella functions, respectively [2,4,5]. The latter is a three-variable hypergeometric function. Integrals with three or more denominators can be expressed in terms of the H function of several variables as an all-order expression in ϵ. However, the evaluation of this function in powers of ϵ is a highly nontrivial task. In this letter, we demonstrate a method based on the MB representation to solve these integrals as a Laurent series in the dimensional regulator. Using this method, we solve the angular integrals of the three denominators and present their results up to $\mathcal{O}(\epsilon^2)$ for massless case and to $\mathcal{O}(\epsilon)$ for massive cases for the first time.

A key feature of our work is the expression of all results in terms of Goncharov polylogarithms (GPLs) [6], which offers clear advantages for both symbolic manipulation and numerical evaluation. Furthermore, should it be necessary to integrate over the angular part—such as in convolution with a radial component—the iterative structure of GPLs makes them particularly well suited for such operations. Since the method is algorithmic, it can be extended to evaluate angular integrals involving a larger number of denominators. This would require handling higher-fold MB integrals, which our approach is well suited to manage without introducing any new conceptual difficulties [7]. Recently, small-mass asymptotic behaviors of angular integrals of three and four denominators have become available [8].

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Mellin-Barnes representation. The angular integral (1) exhibits the following MB representation:

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\},\epsilon)=\frac{2^{2-j-2\epsilon}\pi^{1-\epsilon}}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}\times\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\left[\prod_{k=1}^{n}\prod_{l=k}^{n}\frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}}\Gamma(-z_{kl})(v_{kl})^{z_{kl}}\right]\times\left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right]\Gamma(1-j-\epsilon-z). \quad (2)$$

The scalar products among p_i^μ's are defined through

$$v_{kl}\equiv\begin{cases}\frac{p_k\cdot p_l}{2} & \text{for } k\neq l\\ \frac{p_k^2}{4} & \text{for } k=l\end{cases}. \quad (3)$$

The MB integrals are over the variables z_{kl} from which we further define

$$z=\sum_{k=1}^{n}\sum_{l=k}^{n}z_{kl} \quad \text{and} \quad z_k=\sum_{l=1}^{k}z_{lk}+\sum_{l=k}^{n}z_{kl}. \quad (4)$$

In words, z represents the total sum of all MB variables. For each index k, z_k denotes the sum of all variables involving k as one of their indices, with z_{kk} being counted twice. Specifically, this can be expressed as $z_k=z_{1k}+\cdots+z_{k-1k}+2z_{kk}+z_{kk+1}+\cdots+z_{kn}$. The variable j is the sum of all j_k's, i.e., $j=\sum_{k=1}^{n}j_k$. In deriving the MB representation, it is implicitly assumed that the indices, j_k's, are positive integers. In case of negative integers, the result can be obtained through analytic continuation. Moreover, in (2), we also assume $v_{kl}\neq 0$. If some of v_{kl} are zero, which could happen for massless momentum p_i for which $v_{ii}=0$, then the corresponding integral over z_{kl} should be omitted and z_{kl} should be put to zero in the rest of the expression.

Except in very limited cases, these MB integrals can not be evaluated in a closed form. Since our goal is to solve the integrals in a Laurent series expansion of ϵ rather than in terms of an all-order expression, we follow the following strategy. We expand the MB integral around $\epsilon\to 0$ and then evaluate the coefficients of this expansion, which generally also contain multifold MB integrals. To perform such an expansion safely, it is essential to ensure that, as the integrand approaches the limit $\epsilon\to 0$, the integration contour does not cross any of the poles. This occurs when the real parts of the arguments of the gamma functions in the integrand are not all positive. In such scenarios, an analytic continuation to a region around $\epsilon=0$ becomes inevitable. To accomplish this, we must deform the contour [9] and account for the corresponding residues when the contour crosses any of the poles during this deformation. In the MB representation, the originally chosen integration contours are usually parallel to the imaginary axis, so that all the poles originating from the gamma function of the type $\Gamma(a+z)$ are on the left and all the poles of gamma functions of the type $\Gamma(a-z)$ are on the right. Therefore, we have to effectively shift the contour along the real axis and pick up the contributions of the residues when the contour crosses the poles. Alternatively, following the method in [10,11], we can maintain a fixed integration contour and track the movement of the gamma function poles as ϵ varies. Each time a pole crosses an integration contour, its residue is added to the right-hand side of (2) for the corresponding variable. The residue terms themselves may contain poles in the remaining variables, which might also cross their respective contours as ϵ changes. These instances must be handled similarly. Ultimately, this process yields a collection of residue terms in addition to the original multifold MB integral. The final integrand can then be safely expanded, as the contours remain clear of poles as $\epsilon\to 0$. This process effectively decomposes the original MB integral into several MB integrals. After the ϵ expansion of each integral is complete, we collect all the terms of all the integrals that are of the same order in ϵ and obtain a total ϵ expansion for the original MB integral. The primary task then is to calculate the resulting MB integrals that appear in the coefficients of the expansion. In the following Sec. II, we outline the method for evaluating these MB integrals and represent the results in terms of multiple polylogarithms or iterated integrals. However, when dealing with multifold MB integrals, applying this general approach to fully solve them becomes highly challenging. We illustrate these complexities in Sec. III by addressing the case of three denominators.

Analytic integration from MB representation. To calculate the MB integrals that appear in the coefficients of the ϵ expansion, we observe that the integrals are balanced. We say that a MB integral,

$$\int_{-i\infty}^{+i\infty}\frac{dz_j}{2\pi i}\prod_{k=1}^{n_+}\Gamma(a_k+z_j)^{\alpha_k}\prod_{l=1}^{n_-}\Gamma(a_l-z_j)^{\beta_l}, \quad \alpha_k,\beta_l\in\mathbb{Z}, \quad (5)$$

is balanced in the variable z_j if the condition $\sum_{k=1}^{n_+}\alpha_k=\sum_{l=1}^{n_-}\beta_l$ is fulfilled. A multidimensional MB integral is considered balanced if each integration variable satisfies this condition individually. For a balanced MB integral, the integrand can be expressed as a product of beta functions. This is made possible by ensuring that the gamma functions have positive real parts, which is typically achieved after performing analytic continuation and expanding around $\epsilon=0$. In such cases, all gamma functions can be combined into beta functions, ensuring that none of the remaining gamma functions contain integration variables. We then replace the beta functions with their integral representations, selecting the appropriate one based on the specific context,

$$B(a,b)=\begin{cases}\int_0^1 dx x^{a-1} & (1-x)^{b-1},\\ \int_0^\infty dx x^{a-1} & (1+x)^{-a-b},\end{cases} \quad (6)$$

where $\mathcal{R}(a) > 0$ and $\mathcal{R}(b) > 0$. Since the resulting integral is convergent, we can safely interchange the order of the MB integration with the real integrations arising from the beta functions. This allows us to recast the integral into

$$\int_0^\kappa \left(\prod_{i=1}^K dx_i\right) R_0(\mathbf{x},\mathbf{u}) \int_{-i\infty}^{+i\infty} \prod_{l=1}^L \frac{d\tilde{z}_l}{2\pi i} [R_l(\mathbf{x},\mathbf{u})]^{\tilde{z}_l}, \quad (7)$$

where $\mathbf{x} = (x_1, \ldots, x_K)$ are K real integration variables originating from the beta functions, $\mathbf{u} = (u_1, \ldots, u_N)$ are the parameters other than the integration variables present in the original MB integral and the $R_0, \ldots, R_l$ are ratios of products of x_i, $(1 \pm x_i)$, and u_j. The upper limit, κ, is either 1 or ∞. The MB integral in (7) can now be readily evaluated, reducing the problem to solving a few integrals over real parameters, $\mathbf{x}$. The residual integrand typically consists of rational functions and logarithms of polynomials in the integration variables. Ideally, if these integrals can be performed iteratively, the result can be expressed in terms of multiple polylogarithms (MPL) or iterated integrals. However, if the integrand involves more complex structures, it may not be possible to represent the result in terms of MPLs. In the case of multifold MB integrals, this process involves significant difficulties. In the next Sec. III, we illustrate how, by following this approach, we successfully solve the case of three propagator angular phase-space integrals in terms of MPLs.

Massless integrals with three denominators. We consider the case of three denominators with massless p_i's for which the MB representation (2) becomes

$$\begin{aligned}\Omega_{j_1,j_2,j_3}(\{v_{ij}\},\epsilon) = &\frac{2^{2-j-2\epsilon}\pi^{1-\epsilon}}{\Gamma(j_1)\Gamma(j_2)\Gamma(j_3)\Gamma(2-j-2\epsilon)} \\ &\times \int_{-i\infty}^{+i\infty} \frac{dz_{12}dz_{13}dz_{23}}{(2\pi i)^3} \Gamma(-z_{12})\Gamma(-z_{13}) \\ &\times \Gamma(-z_{23})\Gamma(j_1+z_{12}+z_{13})\Gamma(j_2+z_{12}+z_{23}) \\ &\times \Gamma(j_3+z_{13}+z_{23})\Gamma(1-j-\epsilon-z) \\ &\times (v_{12})^{z_{12}}(v_{13})^{z_{13}}(v_{23})^{z_{23}} \qquad (8)\end{aligned}$$

with $z = z_{12} + z_{13} + z_{23}$, $j = j_1 + j_2 + j_3$, and $\{v_{ij}\} = \{v_{12}, v_{13}, v_{23}\}$. For convenience, we include a normalization factor and calculate

$$I^{(0)}_{j_1,j_2,j_3} = C_\epsilon \Omega_{j_1,j_2,j_3}(\{v_{ij}\},\epsilon) \quad (9)$$

with $C_\epsilon = 2^{-1+2\epsilon}\pi^\epsilon \frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}$. The superscript (0) indicates that the integral is massless. We start by setting $j_i = 1$, and in Sec. V, we extend the discussion to the case of general j_i's. As outlined in Sec. I, when the integration contours cross the poles as $\epsilon \to 0$, analytic continuation of the integrand becomes necessary. We then expand the integrand, retaining terms up to $\mathcal{O}(\epsilon^2)$. This procedure yields four distinct types of MB integrals at $\mathcal{O}(\epsilon)$ and eight at $\mathcal{O}(\epsilon^2)$. Of the latter, only four are genuinely new types compared to the former. These integrals share the same arguments but are multiplied by additional polygamma functions. Up to this order, we do not encounter any square roots over the integration variables. Following the methods outlined in Sec. II, we rewrite the resulting MB integrals in the form presented in (7). By utilizing the identity

$$\int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} A^z = \delta(1-A), \qquad A > 0, \qquad (10)$$

we evaluate the remaining MB integral, ultimately reducing it to a integration over real variables.

$$\int_0^1 \left(\prod_{i=1}^K dx_i\right) R_0(\mathbf{x},\mathbf{u}) \prod_{l=1}^L \delta[1 - R_l(\mathbf{x},\mathbf{u})]. \qquad (11)$$

Since the integrand consists of rational functions and logarithms of polynomials, we expect iterated integrals to yield GPLs. However, in practice, this process is far from straightforward due to the complexity of the integrands. Once a GPL appears in our expressions, we systematically shift the integration variables to the rightmost argument of the GPL in order to perform the remaining integrals and write the final results in terms of GPLs. For simpler cases involving linear arguments, we use the fibration basis feature implemented in PolyLogTools [12]. However, in many cases, the integration variables appear in the weights of the GPLs as nonlinear rational functions, making their manipulation significantly more challenging. Our approach relies on expressing the first-appearing GPLs in terms of lower-weight ones via their integral representation. We then shift the integration variable to the right and reconstruct the original expression through iterative integration [13–16].

To illustrate, consider a GPL of the form $G(\vec{a}_n(z); 1)$, where $\vec{a}_n(z) = \{a_1(z), a_2(z), \ldots, a_n(z)\}$ is an n-tuple of rational functions of the variable z [17]. Using the integral representation, we write

$$\begin{aligned}G(\vec{a}_n(z);1) = &G(\vec{a}_n(z');1) \\ &+ \int_{z'}^z dz \underbrace{\int_0^1 dt_1 \frac{\partial}{\partial z}\left[\frac{1}{t_1 - a_1(z)} G(\vec{a}_{n-1}(z);t_1)\right]}_{I_1}.\end{aligned}$$

The boundary point z' is chosen such that the GPL remains regular. The term I_1 evaluates to a linear combination of rational functions times GPLs of weight $(n-1)$. In simple cases, the fibration basis allows us to move z to the rightmost position in all GPLs. However, when nonlinear rational functions prevent this, we restructure the expression to isolate z at the rightmost argument of each GPL. This ultimately allows us to recast $G(\vec{a}_n(z); 1)$ as $G(\vec{b}_n; z)$, where the entries $\vec{b}_n$ are independent of z, thereby enabling the evaluation of subsequent MB integrals. During this process, appropriate regularization is required to manage spurious singularities arising in the intermediate GPL representations.

Using in-house algorithms, we automate this process and obtain the final result in terms of GPLs, which we then verify numerically. We provide the analytic result of $I^{(0)}_{1,1,1}$ to $\mathcal{O}(\epsilon^2)$ in the Supplemental Material [18] <cofs0mass.m>. Extending the evaluation to $\mathcal{O}(\epsilon^2)$ introduces additional fold integrals, thereby considerably complicating the manipulations. As expected, there is only a single pole in ϵ for this case. We provide our results up to $\mathcal{O}(\epsilon^2)$, as evaluating angular integrals to higher orders in ϵ may be essential for computing physical observables at higher orders in perturbation theory [19]. In Ref. [20], a similar integral has been solved using the MB technique up to $\mathcal{O}(\epsilon^0)$.

Massive integrals with three denominators. The MB representation of the angular integral with one of the $\{p_i\}$'s massive, say p_1, and $j_i = 1$ takes the form

$$\Omega_{1,1,1}(v_{11},\{v_{ij}\},\epsilon) = \frac{2^{-1-2\epsilon}\pi^{1-\epsilon}}{\Gamma(-1-2\epsilon)}\int_{-i\infty}^{+i\infty}\frac{dz_{11}dz_{12}dz_{13}dz_{23}}{(2\pi i)^4}\Gamma(-z_{11})\Gamma(-z_{12})\Gamma(-z_{13}) \times\Gamma(-z_{23})\Gamma(1+2z_{11}+z_{12}+z_{13})\Gamma(1+z_{12}+z_{23}) \times\Gamma(1+z_{13}+z_{23})\Gamma(-2-\epsilon-z_{11}-z_{12}-z_{13}-z_{23})(v_{11})^{z_{11}}(v_{12})^{z_{12}}(v_{13})^{z_{13}}(v_{23})^{z_{23}}. \tag{12}$$

Similar to the massless case, we include the same normalization factor and define $I^{(1)}_{1,1,1} = C_\epsilon\Omega_{1,1,1}(v_{11},\{v_{ij}\},\epsilon)$, where the superscript (1) indicates that one of the denominators involves massive momentum. After performing an analytic continuation and expanding to $\mathcal{O}(\epsilon)$, we obtain 11 distinct MB integrals. Ten of these are evaluated using a method similar to that used in the massless case, with no square roots over the integration variables. However, one MB integral involves a square root over a quadratic function of the integration variable. To address this, we linearize the square root by applying a variable transformation, such that the arguments and coefficients of the resulting GPLs, as well as the Jacobian of the transformation, become rational functions of the new variable. For example, any general quadratic function of x, such as $\sqrt{ax^2+bx+c}$, can be rationalized using the transformation $x \to \frac{b+2c\eta}{c\eta^2-a}$. This transformation satisfies the necessary conditions mentioned above. It is important to note that the transformation is not unique. This procedure allows us to shift the integration variable to the rightmost argument of the GPLs, enabling us to perform all the iterated integrals and express the final results in terms of GPLs. We provide the results of $I^{(1)}_{1,1,1}$ in the Supplemental Material [18] <cofs1mass.m>.

By applying partial fraction decomposition [5] to the propagators, angular integrals with multiple massive momenta can be reduced to integrals involving a single massive momentum. The fact that angular integrals are linear in parameter space when expressed in a suitable coordinate system enables us to perform this decomposition. For double-massive integral, this leads to

$$I^{(2)}_{j,k,l}(v_{11},v_{22},\{v_{ij}\}) = \sum_{n=0}^{j-1}\Lambda_{k,n,\lambda_\pm} I^{(1)}_{j-n,k+n,l}\left(v_{11},\frac{v^{\pm}_{13}}{2},v_{13},\frac{v^{\pm}_{33}}{2}\right) + \{j\leftrightarrow k,\lambda_\pm\leftrightarrow\lambda^c_\pm,p_1\leftrightarrow p_2\} \tag{13}$$

with $\Lambda_{k,n,\lambda_\pm} = \binom{k-1+n}{k-1}\lambda^k_\pm(\lambda^c_\pm)^n$, $\lambda^c_\pm = 1-\lambda_\pm$ and

$$\lambda_\pm = \frac{2v_{12}-4v_{11}\pm\sqrt{4v_{12}^2-16v_{11}v_{22}}}{4v_{12}-4v_{11}-4v_{22}},$$
$$v^{\pm}_{13} = \lambda^c_\pm 4v_{11}+\lambda_\pm 2v_{12}, \qquad v^{\pm}_{23} = \lambda^c_\pm 2v_{12}+\lambda_\pm 4v_{22},$$
$$v^{\pm}_{33} = \lambda^c_\pm 2v_{13}+\lambda_\pm 2v_{23}. \tag{14}$$

The second term on the right-hand side can be derived from the first term by applying the specified transformation rules. Similarly one can do it for a triple-massive integral as well. This implies that integrals involving only massless momenta or a single massive momentum are sufficient to determine all other integrals. The file, <main.m>, determines the results of massive integrals.

Recursion relations. The angular integrals with higher powers of denominators can be related to lower ones through a set of recursion relations, which follow from the equations:

$$\frac{\partial}{\partial v_{kl}} I^{(n)}_{j_1,j_2,j_3} = \sum_{i_1,i_2,i_3} C^{(i_1 i_2 i_3)}_{kl} I^{(n)}_{i_1,i_2,i_3}. \tag{15}$$

Here, the coefficients $C^{(i_1 i_2 i_3)}_{kl}$ are functions of v_{ij},ϵ,i_1,i_2 and i_3. By utilizing this set of equations, we can express all angular integrals with higher powers of denominators in terms of a finite set of integrals, known as master integrals. These relations are analogous to integration-by-parts identities [21,22]. In Ref. [5], these relations are explicitly derived for angular integrals with two denominators. For readers' convenience, we provide the complete set of recursion relations, consisting of six linear equations, in the Supplemental Material, [18] <recursion.m>.

Singularity resolution and phase-space integrals. The angular integrals must be combined with the corresponding radial part to compute the complete phase-space integral.

While the functional form of the radial part depends on the specific process, the angular part can be considered process independent, allowing the result of the angular integration to be applied across various cases. The final phase-space result comes from performing a single parametric integration over the angular part convoluted with the radial part. The integration parameter is typically constructed from dimensionless ratios of the Mandelstam variables.

If the parametric integration contains singularities along the integration path, it is not possible to naively expand the integrand in ϵ to perform the integration. Therefore, the use of the ϵ-expanded angular integrals requires careful treatment. The singularities from the angular integration are generally collinear in nature and are regularized through the ϵ parameter. Any additional singularities arising from the parametric integral are typically soft. Consequently, if the ϵ-expanded angular part does not introduce singularities along the integration path, it is safe to use the expanded form order by order.

We observe that for massive angular integrals, the coefficients in the ϵ expansion do not introduce singularities along the parametric integration path, meaning that the soft singularities arise only from the radial part. However, for massless angular integrals, these parametric singularities can also appear within the individual ϵ-expansion coefficients of the angular part. Since these singularities are soft in nature, we conjecture that they can be resummed to all orders and factorized from the full ϵ expansion of the angular part. This behavior is similar to the general principle of soft divergence factorization through exponentiation in QCD [23–25]. We have validated this for the two-propagator case in Ref. [19]. A similar observation has been made in Refs. [26,27].

By correctly factoring out the soft singular part from the angular integral and regulating it using ϵ through pole subtraction, the full ϵ-expanded expression can be safely used order by order to compute the complete phase-space integrals. The final stage of the phase-space integration reduces to a one-dimensional integral over an analytic expression. In cases where this last integral cannot be evaluated analytically, one can reliably resort to high-precision numerical integration. Since the integral is only onefold and all singularities have been resolved beforehand, such numerical evaluation is straightforward and fully adequate for all phenomenological applications.

Integrals with higher number of denominators. As the number of propagators increases, which could be due to multiple emissions, the MB representation becomes higher dimensional due to more scalar products v_{kl}. While this introduces more gamma functions and real integrations, many are resolved by delta functions [see (10) and (11)], keeping the complexity manageable. Despite larger integrands, the integrals remain tractable. We believe the structural simplicity of our method enables it to scale to more complex integrals. Our upcoming work [7] applies it to a four-propagator case, briefly summarized here.

For four propagators, the number of v_{kl} increases to 6 (massless) and 7 (single-massive), resulting in 10 and 14 real MB integrations after expanding in ϵ. We solve both cases up to $\mathcal{O}(\epsilon^0)$, without encountering square roots in the integration variables. The method remains the same as in the three-propagator case; only the integrands grow due to more gamma functions, requiring no new techniques. The results containing only GPLs have weight 2 at $\mathcal{O}(\epsilon^0)$ for both configurations, compared to weights 1, 2, and 3 at $\mathcal{O}(\epsilon^0)$, $\mathcal{O}(\epsilon^1)$, and $\mathcal{O}(\epsilon^2)$ in the three-propagator case. Square roots in GPL arguments arise only after integration. Notably, the complexity at $\mathcal{O}(\epsilon^0)$ for the four-denominator case is lower than that at $\mathcal{O}(\epsilon^2)$ for the three-denominator case.

Conclusion. Phase-space integrals are crucial for achieving precise calculations of observables in particle physics. In this letter, we introduce a novel method for calculating these integrals analytically. First, we present a technique for computing the angular component based on the evaluation of multifold Mellin-Barnes integrals, which commonly arise in various physics scenarios but pose significant challenges for analytic evaluation. For the first time, we provide the analytical results for angular phase-space integrals involving three denominators.

We present results for integrals with all massless momenta up to $\mathcal{O}(\epsilon^2)$ and with massive momentum up to $\mathcal{O}(\epsilon)$, expressed in terms of GPLs. Additionally, we derive a set of recursion relations that reduce integrals with higher powers of denominators to those with lower powers. Utilizing partial fraction decomposition of the propagators, we also introduce a set of linear relations that allows the calculation of angular integrals involving two or more massive momenta. We provide our findings as Supplemental Material [18].

To combine the angular and radial components, a careful treatment of singularities is required, which we detail in this letter. An important advantage of expressing angular integrals in terms of GPLs is that it facilitates representing the final results of complete phase-space integrals, following radial integration, in terms of well-defined analytic functions or at most onefold numerical integrals. This powerful method not only handles phase-space integrals for massless external particles but also extends to those with massive particles. These integrals are essential for advancing beyond next-to-next-to-leading order accuracy in many scattering processes, such as semi-inclusive deep inelastic scattering.

An independent calculation of the angular part is carried out in Ref. [28] to $\mathcal{O}(\epsilon)$ and we have a perfect numerical agreement.

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Data availability. No data were created or analyzed in this study.

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