INCOMPLETENESS OF THE MODULI SPACE OF STRUCTURED RIEMANNIAN METRICS ON THE KUMMER $K3$ SURFACE

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Chapter 1

Introduction

1.1 Motivation

A Riemannian metric on a smooth manifold M makes it possible to define different geometric properties on M. Some examples of these notions are length, area, volume, as well as higher dimensional analogues, each one related to a definition of curvature.

A fundamental area of research in Riemannian geometry is primarily focused on the matter of existence. However, a manifold may admit several Riemannian metrics exhibiting different curvature characteristics. Thus, it arises the problem of classification: Is it possible to continuously deform a given metric into another one? How do the geometric properties of the manifold change (or are preserved) under such deformations?

To rigorously address these questions, a natural approach is to equip the space of metrics with a suitable topology and ask about its topological properties.

Denote by $\mathcal{R}(M)$ the space of all complete Riemannian metrics on M. In practice, this space is too big to study. Therefore, mathematicians often work with moduli spaces instead. The topology of moduli spaces of Riemannian metrics satisfying curvature conditions (positive scalar curvature, positive Ricci curvature, non-negative sectional curvature) has been the topic of fruitful research in the past years. For an exposition of some classic results, see [40].

In this work we will study the moduli space of structured Riemannian metrics, defined as follows.

Let $\mathrm{Diff}_0(M)$ be the connected component of the group of diffeomorphisms of M containing the identity. This group acts on $\mathcal{R}(M)$ by pull back:

$$\begin{aligned} \mathrm{Diff}_0(M) \times \mathcal{R}(M) &\longrightarrow \mathcal{R}(M) \\ (\psi, g) &\longmapsto \psi^*(g). \end{aligned}$$

The quotient $\mathcal{R}(M)/\mathrm{Diff}_0(M)$ is known as the moduli space of complete Riemannian metrics on M.

Now, consider M to be compact, connected, oriented and without boundary. Let $\pi : \widetilde{M} \longrightarrow M$ be its universal covering, with $\widetilde{M}$ spin and take the pullback metric $\widetilde{g} = \pi^* g$ on $\widetilde{M}$.

We define the moduli space of structured Riemannian metrics as:

$$\begin{aligned} \mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M) = \{ g \in \mathcal{M}(M) \,|\, (\widetilde{M}, \tilde{g}) \text{ carries a nonzero parallel spinor,} \\ \text{and } \mathrm{Vol}(M, g) = 1 \}. \end{aligned}$$

Proposition 92 shows that this quotient is a finite dimensional smooth manifold. Its tangent bundle at $[g]$ can and will be identified with the space of symmetric 2-tensors that are trace and divergence free with respect to g. On such tensors, the L^2-scalar product defines a scalar product given by

$$(h,k)_g = \int_M \langle h, k \rangle_g \mathrm{dVol}(M,g) = \int_M \mathrm{tr}_g(hk) \mathrm{dVol}(M,g),$$

where $\langle h, k \rangle_g$ is defined as

$$\langle h, k \rangle_g := \sum_{i,j} h(e_i, e_j) k(e_i, e_j),$$

and (e_i) is a locally defined orthonormal frame. This turns $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ into a Riemannian manifold.

The main result of the thesis is the following.

Theorem 1. Let S be a Kummer $K3$ surface. Then the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ is not complete.

The idea behind the proof is to find a non-convergent Cauchy sequence in the moduli space. The main difficulty when constructing such a sequence comes from the fact that, because S is compact, the metrics in $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ are not explicitly known. They correspond to Ricci-flat metrics whose existence is a consequence of the Calabi-Yau theorem, but the proof of said theorem does not provide information on how these metrics look like.

Therefore, we rely on the classic approach of finding a priori metric estimates.

The chosen topics are deeply related with many important areas of research in differential geometry such as manifolds with special holonomy as well as with other areas of Mathematics like algebraic geometry. In addition, several concepts involved are of great value in modern physics, as e.g. in string theory or general relativity.

1.2 Outline of the thesis

The present work is structured as follows. In Chapter 2 we will cover basic material about Kähler geometry, with the aim of explaining the celebrated Calabi-Yau Theorem. We are interested in one particular case of the aformentioned result: Let us consider (M, g) a compact Kähler manifold with vanishing first Chern class $c_1(M) = 0$ and complex dimension m. Then, the Calabi-Yau Theorem guarantees the existence of a Kähler metric $\widetilde{g}$ on M with zero Ricci form.

However, we will see that the proof of the Calabi-Yau Theorem is not constructive. It requires solving the equation $(\omega + i\partial\bar{\partial}u)^m = Ae^f\omega^m$, which is a nonlinear, elliptic, second-order partial differential equation of Monge-Ampère type. Such equations are very difficult to solve, often the preferred technique to tackle the problem consists of finding suitable a priori estimates for the solution u, and then applying what is know as the continuity method.

This complexity is what makes the study of the moduli space $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ so difficult: it requires working with approximate Ricci-flat metrics, which should then be deformed into actual Ricci-flat ones.

Chapter 3 deals with the concept of holonomy, which happens to be relevant in the study of the moduli space $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$, because a manifold M carries a parallel spinor on its universal covering if and only if the reduced holonomy group is a product of the so called special holonomy groups.

Here we will state that for a compact Riemannian manifold (M, g) whose universal covering is spin and carries a parallel spinor, the holonomy group is rigid, that is, if g_t, $t \in [0, T]$ is a smooth family of Ricci-flat metrics with $g_0 = 0$ then $\mathrm{Hol}(M, g_t)$ is conjugate to $\mathrm{Hol}(M, g)$ in $GL(n, \mathbb{R})$.

After all this general background, we begin to focus in the specific case we want to assess. In Chapter 4 we introduce the concept of $K3$ surface and explain how to construct one explicitly trough a process known as the Kummer construction. Roughly speaking, this procedure consists of first taking a compact complex orbifold arising from a complex 4-torus modulo an involution, then observing that each of the 16 singular points can be modeled as a singularity of type A_1, which can be resolved by blow up. Therefore, we can resolve each of them and obtain a smooth compact complex surface S which is simply connected and has trivial canonical bundle, i.e., it is a $K3$ surface.

Afterwards, in Chapter 5, we will compute the a priori estimates necessary to solve the Calabi-Yau equation for S, which in this case reads as

$$(\omega_a + i\partial\bar{\partial}u_a)^2 = \exp(G_a)\omega_a^2,$$

where $a = (a_i)_{i=1}^{16} \in \mathbb{R}^{16}$ is a parameter measuring the volume of the exceptional divisors arising from the resolutions of the singular points of S, ω_a is the Kähler form of an approximately Ricci-flat metric g_a, and $G_a = \log(\frac{\eta\wedge\bar{\eta}}{\omega_a^2})$, for η a holomorphic 2-form in $\mathbb{C}^2$.

Additionally, we will find bounds for the norm of functions on S in terms of the Laplacian. This Chapter is the most technically challenging, as it requires heavy machinery from PDE theory.

Finally, we arrive to Chapter 6, which we start by formally defining the moduli space of structured Riemannian metrics $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$, and stating some of its fundamental properties. In particular, as a consequence of the holonomy rigidity reviewed in Chapter 3, $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ is a finite dimensional Riemannian manifold, with the L^2-metric. Having this additional structure, we are in conditions of proving the main result of the thesis: Theorem 95, which states that the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ is not a complete manifold. The proof of this theorem requires us to exhibit a non convergent Cauchy sequence, which we do as follows: by taking each $a_i = t$, we have a family of metrics g_t, constructed as in Chapter 5.

By the Calabi-Yau theorem we get a map

$$\begin{aligned} \mathcal{R}(S) &\longrightarrow \mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S) \\ g_t &\longmapsto [\widetilde{g}_t], \end{aligned} \tag{1.1}$$

which yields an associated path of classes of Ricci-flat metrics $[\widetilde{g}_t]$. We will see that g_t defines a Cauchy sequence and that the map (1.1) is a Lipschitz map. Then $[\widetilde{g}_t]$ is a Cauchy sequence that is not convergent, because in the limit the metric becomes singular.

Chapter 2

Kähler Geometry and the Calabi Conjecture

The present chapter aims to provide a background on Kähler geometry in order to study the construction of Ricci-flat metrics on compact manifolds. Namely, in Section 2.6 we will present some aspects of the celebrated Calabi-Yau Theorem (Theorem 39), which in particular tells us that for any compact manifold M with $c_1(M) = 0$ there exists a Kähler metric with zero Ricci form.

This material will be relevant in the thesis because there is a strong relationship between Ricci-flat metrics and metrics with parallel spinors, given by the following result:

Proposition 2. [6] A simply connected, irreducible, compact, Ricci-flat manifold of dimension n either has holonomy $SO(n)$, or it admits a non-zero parallel spinor.

Moreover, all known examples of simply connected, compact, Ricci-flat manifolds have special holonomy and hence admit parallel spinors.

Many results are well known and therefore stated without proof. Some suitable references for the content of this chapter are [32] and [4].

2.1 Complexified bundles

Definition 3. A $(1, 1)$-tensor J on a smooth manifold M satisfying $J^2 = -\mathrm{Id}$ is called an almost complex structure. The pair (M, J) is then called an almost complex manifold.

Let (M, J) be an almost complex manifold. We define the complexified tangent space as

$$T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}.$$

The real endomorphisms and differential operators from TM can be extended to $T_{\mathbb{C}}M$ by $\mathbb{C}$-linearity. In particular, we extend J to the complexified tangent bundle

Let $T^{1,0}M$ denote the eigenbundle of $T_{\mathbb{C}}M$ associated with the eigenvalue i of the endomorphism J, and $T^{0,1}M$ be the one associated with the eigenvalue $-i$.

Then the complexified tangent space is decomposed as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

with

$$T^{1,0}M = \{X - iJX \mid X \in TM\}, \qquad T^{0,1}M = \{X + iJX \mid X \in TM\}.$$

We denote the exterior bundles as $\Lambda^*M := \Lambda^*T^*M$. Then, similarly as before, we define the complexified exterior bundles

$$\Lambda^*_{\mathbb{C}}M := \Lambda^*M \otimes_{\mathbb{R}} \mathbb{C},$$

as well as the following sub-bundles of $\Lambda^1_{\mathbb{C}}M$:

$$\begin{aligned} \Lambda^{1,0}M &:= \{\xi \in \Lambda^1_{\mathbb{C}}M \mid \xi(Z) = 0 \quad \forall Z \in T^{0,1}M\} \\ \Lambda^{0,1}M &:= \{\xi \in \Lambda^1_{\mathbb{C}}M \mid \xi(Z) = 0 \quad \forall Z \in T^{1,0}M\}. \end{aligned}$$

Observe that the complexified exterior bundle $\Lambda^1_{\mathbb{C}}M$ is decomposed as

$$\Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M, \tag{2.1}$$

with

$$\begin{aligned} \Lambda^{1,0}M &= \{\omega - i\omega \circ J \mid \omega \in \Lambda^1M\} \\ \Lambda^{0,1}M &= \{\omega + i\omega \circ J \mid \omega \in \Lambda^1M\}. \end{aligned}$$

Now, let us denote the k-th exterior power of $\Lambda^{1,0}$ (resp. $\Lambda^{0,1}$) by $\Lambda^{k,0}$ (resp. $\Lambda^{0,k}$), and let $\Lambda^{p,q}$ denote the tensor product $\Lambda^{p,0} \otimes \Lambda^{0,q}$.

Consider vector spaces E and F. Then, we have an isomorphism

$$\Lambda^k(E \oplus F) \cong \bigoplus_{i=0}^{k} \Lambda^i E \otimes \Lambda^{k-i} F,$$

given by the map

$$x_1 \wedge \cdots \wedge x_i \wedge y_1 \wedge \cdots \wedge y_{k-i} \hookleftarrow (x_1 \wedge \cdots \wedge x_i) \otimes (y_1 \wedge \cdots \wedge y_{k-i}),$$

for $x_j \in E$ and $y_j \in F$.

By making use of Equation (2.1), we also get the decomposition

$$\Lambda^k_{\mathbb{C}} M \cong \bigoplus_{p+q=k} \Lambda^{p,q} M.$$

Sections of $\Lambda^{p,q} M$ are called forms of type (p, q).

Definition 4. Let J be an almost complex structure on a smooth manifold M, i.e. a bundle endomorphism $J : TM \to TM$ with $J^2 = -\mathrm{Id}$. The Nijenhuis tensor of J is defined as:

$$N^J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Definition 5. A complex manifold M is a smooth manifold with a complex structure, that is, an atlas of charts to the open unit disc in $\mathbb{C}^n$, such that the transition maps are holomorphic functions.

Clearly, complex multiplication on $\mathbb{C}^n$ defines an almost complex structure on any complex manifold. Furthermore, it is easy to check that for such an almost complex structure, the Nijenhuis tensor vanishes. The converse is also true in the following sense:

Theorem 6. [28] If J is an almost-complex structure on M whose Nijenhuis tensor vanishes, then M carries a complex structure that induces the almost complex structure J. In this case we say that J is integrable.

From now on we will assume that M is a complex manifold.

For every fixed (p, q), we define the differential operators

$$\partial : C^\infty(\Lambda^{p,q}M) \longrightarrow C^\infty(\Lambda^{p+1,q}M)$$
$$\bar{\partial} : C^\infty(\Lambda^{p,q}M) \longrightarrow C^\infty(\Lambda^{p,q+1}M)$$

by the relation

$$\mathrm{d} = \partial + \bar{\partial}.$$

Where d denotes the exterior derivative.

Lemma 7. The following identities hold:

$$\partial^2 = 0, \qquad \bar{\partial}^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proof. Observe that

$$0 = \mathrm{d}^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2.$$

If α is a form of type (p, q), then $\partial^2\alpha$ is of type $(p + 2, q)$, $(\partial\bar{\partial} + \bar{\partial}\partial)\alpha$ is of type $(p + 1, q + 1)$ and $\bar{\partial}^2\alpha$ is of type $(p, q + 2)$. Therefore $\mathrm{d}^2\alpha = 0$ implies that

$$\partial^2\alpha = (\partial\bar{\partial} + \bar{\partial}\partial)\alpha = \bar{\partial}^2\alpha = 0$$

□

If f is a function (i.e. a $(0, 0)$-form) and we consider local coordinates $z_k = x_k + iy_k$, then the operators ∂ and $\bar{\partial}$ can be written as:

$$\partial f = \sum \left(\frac{\partial f}{\partial z_k}\right) \mathrm{d}z_k,$$
$$\bar{\partial} f = \sum \left(\frac{\partial f}{\partial \bar{z}_k}\right) \mathrm{d}\bar{z}_k,$$

where $\frac{\partial f}{\partial z_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\right)$ and $\frac{\partial f}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right)$.

More generally, we consider a (p, q)-form α, written in local coordinates as

$$\alpha = \sum_{|I|=p,|J|=q} \alpha_{I,J} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J,$$

where I, J are multi-indices and $\alpha_{I,J}$ are holomorphic functions.
Then the operators ∂ and $\bar{\partial}$ take the following form:

$$\partial\alpha = \sum_{|I|,|J|}\sum_{k}\frac{\partial\alpha_{IJ}}{\partial z_k}\mathrm{d}z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J,$$

$$\bar{\partial}\alpha = \sum_{|I|,|J|}\sum_{k}\frac{\partial\alpha_{IJ}}{\partial \bar{z}_k}\mathrm{d}\bar{z}_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J.$$

Definition 8. A form $\alpha \in \Lambda^{p,0}$ is holomorphic if $\bar{\partial}\alpha = 0$.

Observation 9. A $(p, 0)$-form α is holomorphic if and only if it can locally be written as

$$\alpha = \sum_{|I|=p}\alpha_I \mathrm{d}z_I,$$

with each coefficient α_I a holomorphic function. Then, we have

$$\alpha \wedge \bar{\alpha} = \sum_{I,J}\alpha_I\bar{\alpha}_J \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

Definition 10. If $Z = X \otimes z \in T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$, then we define its conjugate as $\bar{Z} = X \otimes \bar{z}$.

Consider $T_{\mathbb{C}}M$ to be the complexification of the tangent space of M. It carries two distinct almost complex structures; one being multiplication by i and the other the complex linear extension of J. If g is a J-Hermitian metric, then the complex bilinear extension with respect to i on $T_{\mathbb{C}}M \times T_{\mathbb{C}}M$ (also denoted by g) is Hermitian and thus satisfies the following conditions:

$$\begin{aligned} g(\bar{Z}_1, \bar{Z}_2) &= \overline{g(Z_1, Z_2)} \\ g(Z_1, Z_2) &= 0 \qquad \text{for } Z_1, Z_2 \in T^{1,0}M \\ g(\bar{Z}, Z) &> 0 \qquad \text{for } Z \neq 0. \end{aligned}$$

Conversely, a Hermitian complex bilinear form on $T_{\mathbb{C}}M$ satisfying the previous conditions is the complex bilinear extension of a Hermitian metric on M.

Definition 11. Let M be a complex manifold with complex structure J and Hermitian metric g. The alternating 2-form

$$\omega(X,Y) := g(JX,Y), \qquad X,Y \in TM,$$

acting on real tangent vector fields is called the associated Hermitian form.

Observe that in turn, we can retrieve g from ω using the formula

$$g(X,Y) = \omega(X,JY), \qquad X,Y \in TM.$$

2.2 Natural operators on manifolds

Consider an oriented n-dimensional Riemannian manifold (M,g), with volume form dVol . We denote by $\{e_1,\cdots,e_n\}$ a local orthonormal frame on M and identify vectors and 1-forms via the metric g. Then we can write $\mathrm{dVol} = e_1 \wedge \cdots \wedge e_n$.

We adopt the standard convention in differential geometry for the wedge product of 1-forms: given $\alpha,\beta \in \Omega^1(M)$, their wedge product is defined as the skew-symmetrization

$$(\alpha\wedge\beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$

without any normalization factor.

Nevertheless, when defining an inner product on the bundle $\Lambda^k T^*M$, we view it as a subbundle of the full tensor bundle $T^*M^{\otimes k}$ consisting of alternating tensors. Under this identification, we define a pointwise inner product by pulling back the canonical inner product on $T^*M^{\otimes k}$ via the inclusion map $\varphi : \Lambda^k T^*M \hookrightarrow T^*M^{\otimes k}$. Explicitly, for $\alpha,\beta \in \Lambda^k T^*M$, we define:

$$\langle \alpha,\beta\rangle := \frac{1}{k!}\, g(\varphi(\alpha),\varphi(\beta)),$$

where the normalization factor $\frac{1}{k!}$ compensates for overcounting due to permutations in the symmetric inner product on the full tensor product.

Observe that, with respect to this scalar product, the interior and exterior products are adjoint operators. Namely,

$$\langle X \lrcorner\, \alpha,\beta\rangle = \langle \alpha, X^\flat \wedge \beta\rangle, \qquad \forall X \in TM,\ \alpha \in \Lambda^k M,\ \beta \in \Lambda^{k-1}M,$$

where $X^\flat = g(X,\cdot)$.

Definition 12. Let (M, g) be an oriented Riemannian manifold, of real dimension n. We define the Hodge $*$-operator as

$$\begin{aligned} * : \Lambda^k M &\longrightarrow \Lambda^{n-k} M \\ \alpha \wedge *\beta &:= \langle \alpha, \beta \rangle \mathrm{dVol}, \qquad \forall \alpha, \beta \in \Lambda^k M. \end{aligned}$$

Now, let d be the exterior derivative. We define its adjoint operator d^* (also called codifferential) as

$$\mathrm{d}^* := (-1)^{n(k+1)} * \circ \mathrm{d} \circ * : C^\infty(\Lambda^{k+1} M) \longrightarrow C^\infty(\Lambda^k M).$$

The Laplace operator is given by

$$\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*.$$

We observe that, for a scalar function f, the Laplacian is just $\Delta f := \mathrm{d}^*\mathrm{d}f$. If the dimension of M is even, then $\mathrm{d}^* = + * \circ \mathrm{d} \circ *$.

Definition 13. If (X, g) is an Hermitian manifold, then

$$\partial^* := - * \circ \bar{\partial} \circ * \quad \text{and} \quad \bar{\partial}^* = - * \circ \partial \circ *.$$

In this case, we define the Laplace operators

$$\Delta^{\partial} := \partial\partial^* + \partial^*\partial \qquad \text{and} \qquad \Delta^{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Lemma 14. ([41], Lemma 5.8) The operators ∂^* and $\bar{\partial}^*$ are formal adjoints of ∂ and $\bar{\partial}$, respectively.

Later we will see that for the special case of a Kähler manifold the Laplacians Δ^{∂} and $\Delta^{\bar{\partial}}$ are a multiple of Δ.

For the moment being, consider a Hermitian manifold (M^m, g, J), where $m = n/2$ is the complex dimension.

The aforementioned operators can be extended by $\mathbb{C}$-linearity to the complexified bundles $\Lambda^k_{\mathbb{C}} M$.

2.3 Kähler manifolds

Definition 15. Let M be a smooth manifold, and ω a nondegenerate differential 2-form. If ω is closed, that is $\mathrm{d}\omega = 0$, then ω is said to be a symplectic form and (M, ω) a symplectic manifold.

Definition 16. Let (M, J) be a complex manifold, and g a Hermitian metric on M with Hermitian form ω. We say that g is a Kähler metric if ω is closed. In this case we call ω the Kähler form and the triple (M, J, g) is a Kähler manifold.

Therefore, a manifold is Kähler when it possesses mutually compatible Riemannian, symplectic and complex structures.

In particular, as a consequence of the symplectic structure, if z_α are holomorphic coordinates on a Kähler manifold (M, J, g), and the coefficients of the metric tensor in these local coordinates are

$$g_{\alpha\bar{\beta}} := g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right),$$

then locally the associated Kähler form is given by

$$\omega = i \sum_{\alpha,\beta=1}^{m} g_{\alpha\bar{\beta}} \mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\beta \tag{2.2}$$

(see for example [10], Theorem 8.1).

Definition 17. Let M be a Kähler manifold. We define the twisted differential by

$$\mathrm{d}^c = i(\bar{\partial} - \partial),$$

together with its adjoint

$$\delta^c = i(\partial^* - \bar{\partial}^*).$$

Lemma 18. We have the following relations between the differentials:

$$(\mathrm{d}\mathrm{d}^c)^2 = 0, \quad \mathrm{d}\mathrm{d}^c + \mathrm{d}^c\mathrm{d} = 0, \quad \partial = \frac{1}{2}(\mathrm{d} + i\mathrm{d}^c), \quad \bar{\partial} = \frac{1}{2}(\mathrm{d} - i\mathrm{d}^c), \quad \text{and} \quad \mathrm{d}\mathrm{d}^c = 2i\partial\bar{\partial}.$$

Proof. Straightforward computation, omitted here. □

Definition 19. Let M be a Kähler manifold with Kähler form ω. We define the Lefschetz operator as

$$\begin{aligned} L : \Lambda^k M &\longrightarrow \Lambda^{k+2} M \\ \alpha &\longmapsto \alpha \wedge \omega, \end{aligned}$$

which is an operator of degree 2.

Lemma 20. The adjoint of the Lefschetz operator is

$$\Lambda := *^{-1} \circ L \circ * : \Lambda^k M \longrightarrow \Lambda^{k-2} M, \tag{2.3}$$

which is an operator of degree -2.

Proof. Let α be a k-form and β be a $k+2$-form. By properties of the Hodge star operator, we have

$$\langle L\alpha, \beta \rangle \mathrm{dVol} = L\alpha \wedge *\beta = \alpha \wedge \omega \wedge *\beta = \langle \alpha, *^{-1}(\omega \wedge *\beta) \rangle \mathrm{dVol}.$$

□

Lemma 21. ([32], Lemma 8.4) The following relations hold.

1. The Hodge $*$-operator maps (p, q)-forms to $(m-q, m-p)$-forms . We have the following commutative diagram:

$$\begin{array}{ccc} C^\infty(\Lambda^{p,q}M) & \xrightarrow{\partial^*} & C^\infty(\Lambda^{p-1,q}M) \\ {\scriptstyle *}\big\downarrow & & \big\uparrow{\scriptstyle -*} \\ C^\infty(\Lambda^{n-q,n-p}M) & \xrightarrow{\bar\partial} & C^\infty(\Lambda^{n-q,n-p+1}M) \end{array}$$

2. Consider $[\cdot, \cdot]$ to be the Lie bracket. Then we have $[X \lrcorner\, , \Lambda] = 0$, and $[X \lrcorner\, , L] = JX \wedge \cdot$.

Lemma 22. ([32] Equation (14.15)) In a Kähler manifold, the following relation holds:

$$[\Lambda, \partial] = i\bar\partial^*.$$

Theorem 23. ([32], Theorem 14.6) The Laplacian on a Kähler manifold satisfies

$$\Delta = 2\Delta^{\partial} = 2\Delta^{\bar\partial}.$$

Lemma 24. For a Kähler form ω and a smooth function u on a manifold M, we have

$$\mathrm{dd}^c u \wedge \omega^{m-1} = -\frac{1}{m}\, \Delta u\, \omega^m.$$

Proof. We know from Lemma 18 that $\mathrm{dd}^c = 2i\partial\bar{\partial}$. Then we use Lemma 22 and standard properties of the Lie bracket to compute the following:

$$\begin{aligned} m\frac{2i\partial\bar{\partial}u \wedge \omega^{m-1}}{\omega^m} &= \Lambda(2i\partial\bar{\partial}u) = 2i\Lambda(\partial\bar{\partial}u) = i\left(\partial\Lambda\bar{\partial}u - [\partial,\Lambda]\bar{\partial}u\right) \\ &= 2i\left(i\bar{\partial}^*\bar{\partial}u\right) = -2\bar{\partial}^*\bar{\partial}u = -2\Delta^{\bar{\partial}}u. \end{aligned}$$

□

Let u be a smooth real function on M. Then $\mathrm{dd}^c u$ is a closed and exact 2-form which is also a $(1,1)$-form.

Lemma 25 (The local dd^c- Lemma)**.** ([4], Lemma 5.50) Let $\omega \in \Lambda^{1,1}M \cap \Lambda^2 M$ be a real 2-form of type $(1,1)$-form on a complex manifold M. Then ω is closed if and only if every point $x \in M$ has an open neighborhood U such that

$$\omega|_U = \mathrm{dd}^c u,$$

for some real function u on U.

It follows that if g is a Kähler metric on M with Kähler form ω, then locally in M we may write $\omega = \mathrm{dd}^c u$ for some real function u. Such function u is called the Kähler potential for the metric g.

Lemma 26 (Global dd^c-Lemma)**.** ([41], Lemma 6.17) Let M be a compact Kähler manifold. Then every real, exact $(1,1)$-form ω on M satisfies

$$\omega = \mathrm{dd}^c u$$

for some smooth real-valued function u on M.

Example 27. The complex plane $\mathbb{C}^2$ is a Kähler manifold with the flat Euclidean metric g_{Euc} associated with the potential $f_{Euc}(z) = |z|^2$.

A consequence of the dd^c- Lemma is the following.

Proposition 28. Let M be a compact, complex manifold, and let g, g' be Kähler metrics on M with Kähler forms ω and ω'. Suppose that $[\omega] = [\omega'] \in H^2(M,\mathbb{R})$. Then there exists a smooth, real function u on M such that $\omega' = \omega + \mathrm{dd}^c u$. This function u is unique up to the addition of a constant.

Proof. Since $[\omega] = [\omega']$, we have that $\omega' - \omega$ is an exact, real $(1,1)$- form. By the global dd^c- Lemma, there exists a function u such that $\omega' - \omega = \mathrm{dd}^c u$, as we want. If we have two solutions u_1 and u_2, then we have $\mathrm{dd}^c(u_1 - u_2) = 0$ on M. As M is compact, this implies that the difference $u_1 - u_2$ is constant. Therefore, the solution u is unique up to a constant. □

Observation 29. Proposition 28 also tells us how to express g' in terms of g and u. Namely, we know that in local holomorphic coordinates

$$g_{\alpha\bar{\beta}} = -i\omega_{\alpha\bar{\beta}}, \qquad g_{\bar{\alpha}\beta} = i\omega_{\bar{\alpha}\beta}, \qquad g'_{\alpha\bar{\beta}} = -i\omega'_{\alpha\bar{\beta}}, \qquad g'_{\bar{\alpha}\beta} = i\omega'_{\bar{\alpha}\beta}.$$

As a consequence, we have

$$g'_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \partial_\alpha\partial_{\bar{\beta}}u, \quad \text{and} \quad g'_{\bar{\alpha}\beta} = g_{\bar{\alpha}\beta} + \partial_{\bar{\alpha}}\partial_\beta u.$$

2.4 The Ricci curvature of Kähler manifolds

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and denote by R its curvature tensor

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for all $X, Y, Z \in C^\infty(TM)$.

The Riemann curvature tensor can also be seen as a (0,4)-tensor via

$$R(X,Y,Z,T) := g(R(X,Y)Z,T),$$

where $R(X,Y)$ is the usual curvature endomorphism.

The Ricci tensor on M is defined as

$$\mathrm{Ric}(X,Y) := \mathrm{Tr}\{V \longmapsto R(V,X)Y\},$$

or equivalently

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{2m} R(e_i, X, Y, e_i),$$

where $\{e_i\}$ is a local orthonormal basis of TM.

Definition 30. A Riemannian manifold is said to be Einstein if the Ricci tensor is proportional to the metric tensor g at each point $x \in M$:

$$\mathrm{Ric}(X,Y) = \lambda g(X,Y), \qquad \forall X, Y \in T_xM.$$

Additionally, if the Ricci tensor vanishes identically, the manifold is said to be Ricci-flat.

Proposition 31 ([32], Theorem 5.5)**.** A Hermitian metric g on a complex manifold (M, J) is Kähler if and only if J is parallel with respect to the Levi-Civita connection of g.

Now, suppose that (M, J, g) is a Kähler manifold.
Since J is parallel with respect to the Levi-Civita connection ∇, we have:

$$R(X,Y)JZ = JR(X,Y)Z.$$

Taking the inner product with JT, we obtain:

$$g(R(X,Y)JZ, JT) = g(JR(X,Y)Z, JT) = g(R(X,Y)Z, T),$$

so

$$R(X,Y,JZ,JT) = R(X,Y,Z,T).$$

Using the block symmetry of the curvature tensor,

$$R(X,Y,Z,T) = R(Z,T,X,Y),$$

we obtain

$$R(JX,JY,Z,T) = R(Z,T,JX,JY) = R(Z,T,X,Y) = R(X,Y,Z,T).$$

Hence,

$$\mathrm{Ric}(JX,JY) = \sum_{i=1}^{2m} R(e_i, JX, JY, e_i) = \sum_{i=1}^{2m} R(Je_i, X, Y, Je_i) = \mathrm{Ric}(X,Y).$$

This equation motivates the following

Definition 32. The Ricci form ρ of a Kähler manifold is defined by

$$\rho(X,Y) := \mathrm{Ric}(JX,Y), \qquad \forall X, Y \in TM.$$

Now we revise some properties of the Ricci form.

Lemma 33. ([32], Proposition 6.2) Let (M, J) be a compact, complex manifold and g a Kähler metric on M, with Ricci form ρ. Then ρ is a closed $(1,1)$-form, which represents the first Chern class of M, namely, $[\rho] = 2\pi c_1(M) \in H^2(M, \mathbb{R})$.

Lemma 34. ([22], §4.6) Let (M^{2m}, J, g) be a Kähler manifold and z_α be a system of local holomorphic coordinates. Consider the following local basis of the complexified tangent space

$$Z_\alpha := \frac{\partial}{\partial z_\alpha}, \qquad Z_{\bar{\alpha}} := \frac{\partial}{\partial \bar{z}_\alpha}, \qquad 1 \leq \alpha \leq m$$

Denote the components of the Kähler metric in these coordinates by

$$g_{AB} := g(Z_A, Z_B).$$

Where we use the indices $A, B \in \{1, \bar{1}, \ldots, m, \bar{m}\}$ to denote both holomorphic and anti-holomorphic directions. That is, each Z_A is either $\frac{\partial}{\partial z_\alpha}$ or $\frac{\partial}{\partial \bar{z}_\alpha}$, for $1 \leq \alpha \leq m$.

Then we can express the determinant of the metric g, $\det(g_{\alpha\bar{\beta}})$, as

$$\omega^m = i^m\, m!\, \det(g_{\alpha\bar{\beta}}) \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \wedge \cdots \wedge \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m. \tag{2.4}$$

The Ricci tensor can be written locally as

$$\mathrm{Ric}_{\alpha\bar{\beta}} = -\frac{\partial^2 \log \det(g_{\alpha\bar{\beta}})}{\partial z_\alpha \partial \bar{z}_\beta},$$

and therefore the Ricci form is

$$\rho = -i\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}). \tag{2.5}$$

Observation 35. For a metric g to be Ricci-flat, we require that the Ricci form ρ vanishes identically. Clearly this happens when, in local coordinates, $\det(g_{\alpha\bar{\beta}}) = 1$.

On the other hand, if the metric g is Kähler, we know that $\mathrm{Hol}(g) \subseteq U(n)$ (for more details on holonomy groups, we refer to Chapter 3). It can be checked that $U(n)$ acting on $\mathbb{C}^n$ preserves a $\binom{0}{n}$-tensor, which implies that there exists a nowhere vanishing holomorphic m-form on the $2m$ dimensional manifold M. Without loss of generality, take this form to be η such that $\eta\wedge\bar{\eta}$ induces the volume form of the flat metric. Because of Observation 9, we know that locally

$$\eta \wedge \bar{\eta} = \sum_{I,J} \eta_I \bar{\eta}_J \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J,$$

where the multi-indices are interpreted as

$$\mathrm{d}z_I = \mathrm{d}z_{i_1 \cdots i_m} = \mathrm{d}z_{i_1} \wedge \mathrm{d}z_{i_m}.$$

This, together with Equation (2.4) implies that the condition of being Ricci flat can be also stated in the following form

$$\omega^m = \lambda \eta \wedge \bar{\eta}, \tag{2.6}$$

for a constant λ.

2.5 Kähler form and Volume

By Equation (2.4), we have that $\omega^m = m!\mathrm{dVol}(M,g)$, where $\mathrm{dVol}(M,g)$ is the Riemannian volume form.

The volume of the manifold is then defined as

$$\mathrm{Vol}(M,g) := \int_M \mathrm{dVol}(M,g).$$

Theorem 36. Let M be a Kähler manifold of complex dimension m with Kähler form ω. Then,

1. If M is closed, then the cohomology class of ω^k in $H^{2k}(M,\mathbb{R})$ is non-zero for $0 \leq k \leq m$. In particular, $H^{2k}(M,\mathbb{R}) \neq 0$ for such k.

2. If $N \subset M$ is a compact complex submanifold without boundary of complex dimension k, then the cohomology class of ω^k in $H^{2k}(M,\mathbb{R})$ and the homology class of N on $H_{2k}(M,\mathbb{R})$ are non-zero.

Proof. The first assertion comes from the fact

$$\int_M \omega^m = m!\mathrm{Vol}(M, g) \neq 0.$$

For the second part, observe that N together with the induced metric is a Kähler manifold whose Kähler form is the restriction of ω to N. Therefore, integrating ω^k over N we obtain $k!\,\mathrm{Vol}(N, g|_N)$. □

Theorem 37. Let M be as above. Let $N \subset M$ be a compact complex submanifold of real dimension $2k$, with boundary ∂N (possibly empty). Let $P \subset M$ be an oriented submanifold of dimension $2k$ and boundary $\partial P = \partial N$. If $N - P$ is the boundary of a real singular chain, then $\mathrm{Vol}(N, g|_N) \leq \mathrm{Vol}(P, g|_P)$, with equality if P is also a complex submanifold.

Proof. Can be found in [4], Theorem 4.27. □

2.6 The Calabi-Yau Theorem

In this section we go deeper in the study of Ricci-flat metrics on compact manifolds. The presentation of the material in this section is inspired by [22].

Let (M, J) be a compact, complex manifold and g a Kähler metric on M, with Ricci form ρ. We know from Lemma 33 that ρ is a closed $(1, 1)$-form and $[\rho] = 2\pi c_1(M) \in H^2(M, \mathbb{R})$. Calabi asked himself which closed $(1, 1)$-forms can be the Ricci forms of a Kähler metric on M, and formulated the following conjecture.

Conjecture 38 (Calabi)**.** Let (M, J) be a compact, complex manifold and g a Kähler metric on M, with Kähler form ω. Suppose that ρ' is a real, closed $(1, 1)$-form on M with $[\rho'] = 2\pi c_1(M)$. Then there exist a unique Kähler metric g' on M with Kähler form ω', such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and the Ricci form of g' is ρ'.

This conjecture was affirmatively solved by Shing-Tung Yau in the late 1970s, for which he was awarded the Fields Medal.

The importance of this result in the present work comes from the following. Suppose that M is a compact Kähler manifold with $c_1(M) = 0$. Then we may choose the 2-form ρ' to be zero and and Yau's solution guarantees the existence of a Kähler metric g' on M with zero Ricci form. Thus, on such a compact complex manifolds there exist a unique Ricci-flat Kähler metric g' with $[\omega'] = [\omega]$.

In Chapter 3 we will address in more detail the relationship between holonomy and Ricci-flatness, but for the moment let us state without proof that if g is a Ricci-flat Kähler metric on M, then $\mathrm{Hol}_0(M, g) \subset \mathrm{SU}(m)$.

Furthermore, if $\mathrm{Hol}_0(M, g)$ is irreducible, then either $\mathrm{Hol}_0(M, g) = \mathrm{SU}(m)$ or $m = 2k$ and $\mathrm{Hol}_0(g) = \mathrm{Sp}(k)$.

Therefore, the solution of the Calabi conjecture yields examples of compact Riemannian manifolds with holonomy contained in $\mathrm{SU}(m)$ or $\mathrm{Sp}(k)$. These manifolds are called Calabi-Yau and hyperkähler, respectively.

It is possible to reformulate the Calabi conjecture in terms of a partial differential equation as follows.

Let (M, J) be a compact, complex manifold, g a Kähler metric on M with Kähler form ω, and ρ the Ricci form of g. Let ρ' be a real, closed $(1, 1)$-form on M with $[\rho'] = 2\pi c_1(M)$. To solve the Calabi conjecture we must find a Kähler metric g' with Kähler form ω' such that $[\omega] = [\omega']$ and g' has Ricci form ρ'.

Since $[\rho'] = 2\pi c_1(M) = [\rho]$, we have that $[\rho' - \rho] = 0$ in $H^2(M, \mathbb{R})$. By the proof of Proposition 28, there exist a smooth real function f on M, unique up to the addition of a constant, such that

$$\rho' = \rho - \frac{1}{2}\mathrm{dd}^c f.$$

This is already a simplification of the original problem, as we have replaced the Ricci form ρ', which depends on the second derivatives of the metric, by the function f, which depends only on ω'.

We define a smooth, positive function F on M by $(\omega')^m = F \cdot \omega^m$. Using equations (2.4) and (2.5), we can deduce that

$$\frac{1}{2}\mathrm{dd}^c(\log F) = \rho - \rho' = \frac{1}{2}\mathrm{dd}^c f.$$

Therefore, $\mathrm{dd}^c(f - \log F) = 0$, and $f - \log F$ must be constant on M. Let us label this constant as $f - \log F = -\log A$, for $A > 0$. Then $F = Ae^f$ and g' must satisfy

$$(\omega')^m = Ae^f \omega^m. \tag{2.7}$$

As $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and M is compact, we have that

$$\int_M (\omega')^m = \int_M \omega^m.$$

Using equation (2.7), we deduce

$$A \int_M e^f \mathrm{dVol}(M, g) = \int_M \mathrm{dVol}(M, g) = \mathrm{Vol}(M, g).$$

From the previous discussion, the Calabi conjecture can be restated in the following form.

Theorem 39 (Calabi-Yau , [45]). Let (M, J) be a compact, complex manifold and g a Kähler metric on M, with Kähler form ω. Let f be a smooth real function on M and define $A > 0$ by $A \int_M e^f \ \mathrm{dVol}(M, g) = \mathrm{Vol}(M, g)$. Then there exist a unique smooth real function u such that

1. $\omega + \mathrm{dd}^c u$ is a positive $(1,1)$-form,
2. $\int_M u \ \mathrm{dVol}(M, g) = 0$, and
3. $(\omega + \mathrm{dd}^c u)^m = A e^f \omega^m$ on M.

   Moreover, part 3 is equivalent to the following

4. Choose holomorphic coordinates $z_1, ..., z_m$ on an open set U in M. Then $g_{\alpha\bar{\beta}}$ may be interpreted as an $m \times m$ Hermitian matrix indexed by $\alpha, \bar{\beta} = 1, ..., m$ in U. The condition on u is

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}}\right) = A e^f \det(g_{\alpha\bar{\beta}}). \tag{2.8}$$

Equation (2.8) is a nonlinear, elliptic, second-order partial differential equation in u of Monge-Ampère type. Solving the Calabi conjecture is now equivalent to showing that this equation has a unique, smooth solution.

A portion of this program (the continuity method reviewed below) was carried out by Calabi himself. The rest, which required heavy machinery on PDE theory, was completed by Yau (although significant advances were made before by Aubin).

The continuity method

The proof of the Calabi-Yau Theorem 39 builds upon finding a priori estimates for the solution of (2.8), and applying a technique known as the continuity method, which we explain here.

We need to prove that the nonlinear equation

$$(\omega + \mathrm{dd}^c u)^m = Ae^f \omega^m \tag{2.9}$$

admits a solution u.

To start, we consider a similar equation, which we already know admits a solution. Namely, we choose

$$(\omega + \mathrm{dd}^c u)^m = \omega^m, \tag{2.10}$$

with the trivial solution $u = 0$.

Next, we consider the family of equations depending continuously on a parameter $t \in [0, 1]$

$$(\omega + \mathrm{dd}^c u)^m = A_t e^{tf} \omega^m. \tag{2.11}$$

It has the property that for $t = 0$ it becomes Equation (2.10), which we already know has a solution, and for $t = 1$ it corresponds to Equation (2.9), the one we want to solve.

Consider the subset Sol consisting of all the $t \in [0, 1]$ for which Equation (2.11) has a solution u_t .

The last step of the proof is showing that is both open and closed in $[0, 1]$. This suffices for the following reason: since $[0, 1]$ is connected, this would imply that either $\mathrm{Sol} = \emptyset$ or $\mathrm{Sol} = [0, 1]$. As for $t = 0$ Equation (2.10) has a solution, Sol cannot be empty and must therefore be the whole interval, in particular containing $t = 1$, which provides a solution for (2.9).

To show that Sol is open, we start with a $t' \in S$, such that a solution $u_{t'}$ exists and we wish to show that for $t \in \mathrm{Sol}$ close to t' in $[0, 1]$, there is a solution u_t which is close to $u_{t'}$ in some Banach norm. This is done by considering the linearization of the equation about $u_{t'}$.

In order to prove that Sol is closed, we need to show that it contains all its limit points. Take $\{t_j\}_{j=0}^{\infty}$ to be a sequence in Sol, which converges to t'. Then there is a corresponding sequence of solutions $\{u_{t_j}\}_{j=0}^{\infty}$. By establishing a priori bounds on all solutions u_t in some Banach norm, it may be possible to show that they lie in some compact subset. In this case, the sequence $\{u_{t_j}\}_{j=0}^{\infty}$ contains a convergent subsequence, whose limit can be shown to be a solution $u_{t'}$, for $t = t'$. This would imply that $t' \in \mathrm{Sol}$, and therefore S is closed. By the analysis explained before, this gives a solution for Equation (2.9).

Such a priori bounds were obtained by Yau in [44], [45]. Because of this work, he was awarded the Fields Medal in 1982.

Let us note that the proof of the Calabi-Yau Theorem is not constructive. In particular, the Ricci-flat metrics on compact manifolds do not admit an explicit algebraic description. This is the reason why the study of the moduli space $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ is so difficult: it requires working with approximate Ricci-flat metrics, which should then be deformed into actual Ricci-flat ones.

In our specific case, the a priori bounds on the solutions of the Calabi-Yau equation for a $K3$ surface S will be recasted as the building blocks for proving the incompleteness of the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$. The computation of said bounds will be performed in detail in Chapter 5.

Chapter 3

Holonomy and Parallel spinors

The concept of holonomy is important when it comes to study the moduli space $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$, because it is known that a manifold M carries a parallel spinor on its universal covering if and only if the reduced holonomy group is a product of the special holonomy groups $\{1\}$, $SU(k)$, $Sp(k)$, G_2 and Spin(7).

This chapter starts by reviewing the basics on Riemannian holonomy groups, stating without proof several important results on the topic. All this with the purpose of understanding the following crucial fact: the moduli space $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ is a finite dimensional Riemannian manifold. All the work conducted in the rest of the thesis relies on having this additional structure.

3.1 Holonomy

Definition 40. Let M be a connected manifold, E a vector bundle over M and ∇^E a connection on E. Fix a point $x \in M$. We say that γ is a loop based at x if $\gamma : [0,1] \to M$ is a piecewise-smooth path with $\gamma(0) = \gamma(1) = x$. If γ is a loop based at x, then the parallel transport map $P_\gamma : E_x \to E_x$ is an invertible linear map, so that P_γ lies in $GL(E_x)$, the group of invertible transformations of the E_x. Define the holonomy group $\mathrm{Hol}_x(\nabla^E)$ of ∇^E based at x to be

$$\mathrm{Hol}_x(\nabla^E) = \{P_\gamma : \gamma \text{ is a loop based at } x\} \subset GL(E_x).$$

The holonomy group $\mathrm{Hol}_x(\nabla^E)$ may be regarded as a subgroup of $GL(E_x)$

defined up to conjugation, and in this sense it is independent of the base point x. Therefore, we will denote it by just $\mathrm{Hol}(\nabla^E)$.

Similarly, the reduced holonomy group is defined by only considering null-homotopic loops, and is denoted by $\mathrm{Hol}_0(\nabla^E)$.

Definition 41. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Define the holonomy group of (M, g) to be $\mathrm{Hol}(M, g) := \mathrm{Hol}(\nabla)$.

Definition 42. A tensor S on M, that is, $S \in C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$ is called a parallel tensor if $\nabla S = 0$.

The following result shows that the space of all parallel tensors on M is entirely determined by the holonomy group.

Proposition 43 ([22], Proposition 2.5.2)**.** Let M be a manifold and ∇ a connection on TM. Let E be the vector bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ over M. Then the connection ∇ induces a connection ∇^E on E and $\mathrm{Hol}_x(\nabla^E)$ has a natural representation on the fiber E_x of E at $x \in M$.

Suppose $S \in C^\infty(E)$ is a parallel tensor, so that $\nabla^E S = 0$. Then $S|_x$ is fixed by the action of $\mathrm{Hol}_x(\nabla)$ on E_x. Conversely, if $S_x \in E_x$ is fixed by the action of $\mathrm{Hol}_x(\nabla)$, then there exist a unique tensor $S \in C^\infty(E)$ such that $\nabla^E S = 0$ and $S|_x = S_x$.

If (M, g) is a Riemannian manifold with Levi-Civita connection ∇, then by the fundamental theorem of Riemannian Geometry we have $\nabla g = 0$, and therefore g is a parallel tensor. By Proposition 43, if $x \in M$ then the action of $\mathrm{Hol}_x(\nabla)$ on T_xM preserves the metric $g|_x$ on T_xM. But the group of transformations of T_xM preserving $g|_x$ is $O(n)$. Therefore, the holonomy group $\mathrm{Hol}(\nabla)$ is a subgroup of $O(n)$.

In addition, if (M, g) is oriented, then the Levi-Civita connection ∇ preserves the metric and the orientation, thus it must also preserve the volume form Vol. Therefore, the action of $\mathrm{Hol}_x(\nabla)$ on $\Lambda^n T_x^*M$ must also preserve Vol_x. The stabilizer of the volume form is $SL(n)$, and from this we deduce that $\mathrm{Hol}(\nabla)$ is contained in $O(n) \cap SL(n) \cong SO(n)$.

Under certain conditions, it is possible to find manifolds whose holonomy groups are strictly contained in $SO(n)$. Such manifolds are said to have special holonomy and exhibit interesting curvature characteristics. We will give a classification of them, but before it is necessary to study the reducibility of Riemannian manifolds.

3.2 Reducible Riemannian manifolds

Let (M_1, g_1), (M_2, g_2) be Riemannian manifolds, and $M_1 \times M_2$ the product manifold. Then at each point (p_1, p_2) of $M_1 \times M_2$, we have $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$ and the product metric $g_1 \times g_2$ defined point wise by $g_1 \times g_2|_{p_1,p_2} = g_1|_{p_1} + g_2|_{p_2}$ is a Riemannian metric on $M_1 \times M_2$.

We call $(M_1 \times M_2, g_1 \times g_2)$ a Riemannian product.

Definition 44. A Riemannian manifold (M, g) is said to be reducible if it is isometric to a Riemannian product. Also, (M, g) is said to be locally reducible if every point has a reducible open neighborhood. If (M, g) is not locally reducible, it shall be called irreducible.

The following propositions show that the reducibility of a Riemannian manifold has consequences in the structure of the holonomy group.

Proposition 45 ([22], Proposition 3.2.1)**.** Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. Then the product metric $g_1 \times g_2$ has holonomy $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$.

Proposition 46 ([22], Corollary 3.2.5)**.** Let M be an n-dimensional manifold and g an irreducible Riemannian metric on M. Then the representations of $\mathrm{Hol}(g)$ and $\mathrm{Hol}_0(g)$ on $\mathbb{R}^n$ are irreducible.

Moreover, if the manifold M is simply connected and the metric g is geodesically complete, then we have a global product structure. This is the content of the following result, known as the de Rham Splitting Theorem.

Theorem 47 ([22], Corollary 3.2.7)**.** Let (M, g) be a complete, simply-connected Riemannian manifold. Then there exist complete, simply-connected Riemannian manifolds $(M_1, g_1), \cdots, (M_k, g_k)$, such that the holonomy representation of $\mathrm{Hol}(g_j)$ is irreducible, (M, g) is isometric to the product $(M_1 \times \cdots \times M_k, g_1 \times \cdots \times g_k)$, and $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \cdots \times \mathrm{Hol}(g_k)$.

3.3 The classification of Riemannian holonomy groups

As mentioned before, a generic oriented manifold has holonomy $O(n)$. Therefore the following natural question arises: which subgroups of $O(n)$ are good candidates to be holonomy groups of a Riemannian n-manifold (M, g)?

In order to simplify the problem, we consider simply connected manifolds. This is equivalent to study the reduced holonomy group $\mathrm{Hol}_0(g)$.

Additionally, we suppose that g is irreducible and not locally symmetric (the classification of holonomy groups for locally symmetric spaces is independently studied, see for example [6] §10.K).

Then we have the more refined question: which subgroups of $SO(n)$ can be the holonomy groups of an irreducible, non-symmetric Riemannian metric g on a simply connected n-manifold M?

Using the classification of Lie groups and their representations, as well as the symmetry properties of the curvature tensor, Berger proved the following result.

Theorem 48. (Berger's Theorem [5]) Suppose M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, which is irreducible and non-symmetric. Then exactly one of the following cases holds.

1. $\mathrm{Hol}(g) = SO(n)$,
2. $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}(g) = U(m)$ in $SO(2m)$,
3. $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}(g) = SU(m)$ in $SO(2m)$,
4. $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}(g) = Sp(m)$ in $SO(4m)$,
5. $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}(g) = Sp(m)Sp(1)$ in $SO(4m)$,
6. $n = 7$ and $\mathrm{Hol}(g) = G_2$ in $SO(7)$,
7. $n = 8$ and $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ in $SO(8)$.

Berger proposed that the groups in the list were the only possibilities (although originally the list included also the Spin(9) case, which was later proven to be symmetric through independent work by D. Alekseevski and Brown–Gray), but he did not show that all of these groups do occur as holonomy groups of Riemannian manifolds. Only later it was proven that this is in fact the case, by constructions by Calabi, Yau, Joyce, and others.

Let us make some remarks about each group appearing on Berger's list.

1. A generic Riemannian metric has holonomy $SO(n)$.
2. Riemannian metrics g with $\mathrm{Hol}(g) \subseteq U(m)$ are Kähler.

3. Riemannian metrics g with $\mathrm{Hol}(g) = SU(m)$ are called Calabi-Yau metrics, which are Kähler and Ricci-flat.

4. Metrics g with $\mathrm{Hol}(g) = Sp(m)$ are called hyperkähler metrics.

5. Metric g with $\mathrm{Hol}(g) = Sp(m)Sp(1)$, for $m \geq 2$, are called quaternionic Kähler metrics. They are Einstein, but not Ricci-flat.

6. The holonomy groups G_2 and $\mathrm{Spin}(7)$ are called exceptional holonomy groups. There are explicit examples of complete metrics with exceptional holonomy, as well as metrics on compact manifolds (see for example [22]).

The existence of metrics with holonomy $SU(m)$ on compact manifolds is a consequence of the Calabi-Yau Theorem 39. A particular case of this is the $K3$ surface, a central object of study in this thesis, which will be introduced in Chapter 4.

In the following, we will explore the relationship between manifolds with special holonomy and manifolds with parallel spinors.

3.4 Parallel spinors and holonomy groups

In order to fix notation, let us give a very brief exposition of spin geometry. A suitable reference for details is [27].

For each $n \geq 3$, the Lie group $SO(n)$ is connected and has fundamental group $\pi_1(SO(n)) = \mathbb{Z}_2$. Therefore, there exists a double cover $\pi : \mathrm{Spin}(n) \longrightarrow SO(n)$; where $\mathrm{Spin}(n)$ is a compact, connected and simply connected Lie group and π is a Lie group homomorphism.

The group $\mathrm{Spin}(n)$ admits a natural representation Δ^n, called the spin representation, with the following properties:

- For $n = 2m$, Δ^{2m} is a complex representation of $\mathrm{Spin}(2m)$, with complex dimension 2^m. It splits into a direct sum $\Delta^{2m} = \Delta^{2m}_+ \oplus \Delta^{2m}_-$, where $\Delta^{2m}_\pm$ are irreducible representations of $\mathrm{Spin}(n)$, of complex dimension 2^{m-1}.

- For $n = 2m + 1$, Δ^{2m+1} is an irreducible complex representation of $\mathrm{Spin}(2m + 1)$, of complex dimension 2^m.

- For $n = 8k-1,\ 8k$ or $8k+1$, $\Delta^n = \Delta^n_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, where $\Delta^n_{\mathbb{R}}$ is a real representation of $\mathrm{Spin}(n)$.

Definition 49. Let (M,g) be an oriented Riemannian n-manifold. The metric and orientation on M induce a unique $SO(n)$-structure P on M. A spin structure $(\tilde{P},\pi)$ on M is a principal bundle $\tilde{P}$ over M with fiber $\mathrm{Spin}(n)$, together with a bundle map $\pi : \tilde{P} \longrightarrow P$, which is locally modelled by the projection $\pi : \mathrm{Spin}(n) \longrightarrow SO(n)$.

Spin structures exist on an oriented Riemannian manifold (M,g) if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M,\mathbb{Z}_2)$ vanishes, $w_2 = 0$.

A manifold which admits a spin structure shall be called a spin manifold.

Definition 50. Let (M,g) be an oriented, Riemannian, spin manifold of dimension n, with spin structure $(\tilde{P},\pi)$. We define the spin bundle to be the complex vector bundle $S \longrightarrow M$ with total space $S = \tilde{P} \times_{\mathrm{Spin}(n)} \Delta^n$, and fiber Δ^n. Sections $\sigma \in C^\infty(S)$ are called spinors.

If $n = 2m$, we have seen that the spin representation splits as $\Delta^n = \Delta^n_+ \oplus \Delta^n_-$. This induces a splitting of the spin bundle as $S = S_+ \oplus S_-$, where each $S_\pm$ is a vector subbundle of S, with fiber $\Delta^n_\pm$. The sections of S_+, S_- are called positive and negative spinors, respectively.

The $SO(n)$-bundle P over M is naturally endowed with the Levi-Civita connection ∇ of g. Since $\pi : \tilde{P} \longrightarrow P$ is a local isomorphism, it is possible to lift ∇ to $\tilde{P}$. Therefore, we have an induced connection $\nabla^S : C^\infty(S) \longrightarrow C^\infty(T^* \otimes S)$, called the spin connection.

Definition 51. Clifford multiplication provides a natural, linear map $T^*M \otimes S \longrightarrow S$. The composition of this map with ∇^S results in a first order, linear, partial differential operator $D : C^\infty(S) \longrightarrow C^\infty(S)$, called the Dirac operator.

The Dirac operator is self-adjoint and elliptic. In even dimensions, the splitting of the spin bundle induces a splitting of the Dirac operator as $D = D_+ \oplus D_-$, where $D_+ : C^\infty(S_+) \longrightarrow C^\infty(S_-)$, and $D_- : C^\infty(S_-) \longrightarrow C^\infty(S_+)$. Both D_+ and D_- are first-order linear elliptic operators, and one is the formal adjoint of the other.

Consider an oriented Riemannian spin manifold (M,g) with spin bundle S and spin connection ∇^S. Then the holonomy group $\mathrm{Hol}(\nabla^S)$ is a subgroup of Spin(n). Moreover, under the covering map $\pi : \mathrm{Spin}(n) \longrightarrow SO(n)$, the image of $\mathrm{Hol}(\nabla^S)$ is exactly $\mathrm{Hol}(g)$.

Depending on the choice of the spin structure, the projection $\pi : \mathrm{Hol}(\nabla^S) \longrightarrow \mathrm{Hol}(g)$ may be either an isomorphism or a double cover. However, if M is simply connected, then $\mathrm{Hol}(g)$ and $\mathrm{Hol}(\nabla^S)$ are connected, which forces $\mathrm{Hol}(\nabla^S)$ to be the connected component of the identity of $\pi^{-1}(\mathrm{Hol}(g))$ in Spin(n). As a result, the classification of holonomy groups of spin connections for simply-connected spin manifolds follows from that of Riemannian holonomy groups.

Definition 52. Let (M, g) be a Riemannian spin manifold with spin bundle S. A spinor $\sigma \in C^\infty(S)$ is said to be parallel if $\nabla^S \sigma = 0$.

One important property of manifolds with parallel spinors is the followig.

Lemma 53 ([19])**.** The Ricci tensor of a Riemannian spin manifold admitting a parallel spinor vanishes.

In fact, all known closed Ricci-flat manifolds admit a finite covering with a non-trivial parallel spinor.

There is a 1-1 correspondence between parallel spinors and elements of Δ^n, which are invariant under the action of $\mathrm{Hol}(\nabla^S)$. Using this fact, together with Berger's classification of Riemannian holonomy groups, it is possible to classify the holonomy groups of metrics with parallel spinors. This has been done by Wang, by proving the following result.

Theorem 54. (M. Wang, [43]) Let (M, g) be a complete, simply connected, irreducible Riemannian spin manifold of dimension n. Let N denote the dimension of the space of parallel spinors. If $N > 0$, then one of the following holds:

(a) $n = 2m, m \geq 2$, the holonomy representation is $SU(m)$ and $N = 2$,

(b) $n = 4m, m \geq 2$, the holonomy representation is $Sp(m)$ and $N = m + 1$,

(c) $n = 8$, the holonomy representation is Spin(7) and $N = 1$,

(d) $n = 7$, the holonomy representation is G_2 and $N = 1$.

Conversely, if the holonomy representation is one of the above, then N must assume the given value.

For this reason, the concept of holonomy is very important in the process of constructing manifolds with parallel spinors.

Additionally, the reduced holonomy group of a manifold (M, g) is related to the existence of parallel spinor on its universal covering $(\widetilde{M}, \tilde{g})$:

Theorem 55 ([19], [43]). $(\widetilde{M}, \tilde{g})$ carries a parallel spinor if and only if $\mathrm{Hol}_0(M, g)$ is a product of: $\{1\}$, $SU(k)$, $Sp(k)$, G_2, $\mathrm{Spin}(7)$.

Furthermore, we have the following result describing the behavior of the holonomy group of a metric with parallel spinors, under Ricci-flat deformations (known before for simply connected manifolds with irreducible holonomy).

Theorem 56. (Rigidity of the holonomy group [2]). Let (M, g) be a compact Riemannian manifold whose universal covering is spin and carries a parallel spinor. If g_t, $t \in I := [0, T]$ is a smooth family of Ricci-flat metrics such that $g_0 = g$, then $\mathrm{Hol}(M, g_t)$ is conjugate to $\mathrm{Hol}(M, g)$ in $GL(n, \mathbb{R})$.

The importance of the previous theorem comes from the fact that it has an important corollary (which will be reviewed in Proposition 92): the moduli space of structured Riemannian metrics $\mathcal{M}(M)/\mathrm{Diff}_0(M)$ is a smooth manifold of finite dimension.

Having this additional structure will allow us to define a metric on it (the L^2-metric) and study its completeness.

Chapter 4

The Kummer $K3$ surface S

By the analysis performed by the end of Chapter 2, the key ingredient needed to solve the Calabi-Yau equation $(\omega + \mathrm{dd}^c u)^m = Ae^f \omega^m$ are the a priori bounds for the solution u. These are in general difficult to obtain, however for the well studied $K3$ surface there are useful results (namely, the bounds presented in Chapter 5), which fit our situation well.

In this chapter we introduce the concept of $K3$ surface, present some of its basic properties and construct one explicit example S, via a process known as the Kummer construction. This construction requires gluing together a compact complex orbifold arising from a complex 4-torus together with 16 ALE spaces known as Eguchi-Hanson spaces which "repair" the orbifold singularities.

Afterwards, we will endow this resulting $K3$ surface with Ricci-flat metrics arising from the solution of the Calabi-Yau equation.

4.1 Introduction to $K3$ surfaces

The following definition arises as a generalization of that of an elliptic curve, to complex dimension two.

Definition 57. A $K3$ surface X is a connected, simply connected, non singular compact complex surface X, whose canonical bundle K_X is trivial as a holomorphic bundle.

Observation 58. Let us note that the condition of having a trivial canonical bundle K_X is equivalent to the existence of a non-vanishing holomorphic 2-

form on X (this is because K_X is precisely the line bundle of holomorphic 2-forms, and the fact that it is trivial means that it admits a nowhere vanishing global section).

Now we list several important results regarding $K3$ surfaces.

From the definition, and as a consequence of Chern-Weil theory, the first Chern class $c_1(X)$ vanishes in $H^2(X;\mathbb{Z})$. This implies that the second Stiefel-Whitney class $w_2(X) \in H^2(X, \mathbb{Z}_2)$ vanishes as well, and hence X is a spin manifold. Moreover, we have the following celebrated theorem.

Theorem 59 ([36])**.** Every $K3$ surface admits a Kähler metric.

Therefore, we also have a particular case of the Calabi-Yau Theorem.

Theorem 60. Every Kähler class of a $K3$ surface contains a unique Ricci-flat Kähler form.

In real dimensions 2 and 3, it is well known that Ricci flatness implies that the whole Riemann tensor vanishes.

However, in real dimension 4, any $K3$ surface carries Ricci-flat metrics and thus they provide examples of simply-connected, closed, Ricci-flat, but non-flat Riemannian manifolds. Conversely, one can show that any any simply-connected, closed, Ricci-flat manifold of dimension at most 4 is isometric to such an example.

Additionally, $K3$ surfaces are the only compact 4-manifolds carrying metrics with holonomy $SU(2) = Sp(1)$.

Example 61. (i) Any non-singular quartic surface in $\mathbb{CP}^3$ is a $K3$ surface (the proof can be found in [20], Example 1.3).

(ii) The so called Fermat quartic

$$X_0 := \{[X : Y : Z : W] \in \mathbb{CP}^3 \mid X^4 + Y^4 + Z^4 + W^4 = 0\}$$

is non-singular, hence provides a concrete example of a K3 surface.

Kodaira proved the following result.

Theorem 62 ([25])**.** Every K3 surface is diffeomorphic to the quartic surface X_0 above.

Now, as a special case of Theorem 54, we have:

Theorem 63. $K3$ surfaces admit non-trivial parallel spinors.

Therefore, it does makes sense to study the moduli space $\mathcal{M}_{\|}(X)/\mathrm{Diff}_0(X)$.
Later in the chapter, we will present an explicit way of obtaining a $K3$ surface, known as the Kummer construction. After this, we shall show how to endow this surface with a family of Ricci-flat metrics, which will be non-convergent in the moduli space.

4.2 Rotationally symmetric Kähler metrics

Before we go to our explicit construction of a $K3$ surface, we need a few preliminaries. Here we present how to find an explicit expression of a Ricci-flat Kähler metric on $\mathbb{C}^2 - \{0\}$. Part of the computations here appeared in the PhD thesis [29].

Definition 64. A Kähler metric on $\mathbb{C}^n$ is said to be rotationally symmetric if its Kähler form can be written as $\omega = \mathrm{dd}^c f(z)$, where $f(z)$ is a smooth function depending only on the square of the radius function $u = |z|^2$.

Lemma 65. For $n \geq 2$ let g be a Kähler metric on $\mathbb{C}^n$ associated with the Kähler form $\omega = \mathrm{dd}^c f(z)$, which is rotationally symmetric . Then in local coordinates the metric takes the form

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta} f'(u) + \bar{z}_\alpha z_\beta f''(u), \tag{4.1}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and $f'(u)$, $f''(u)$ are the derivatives of f with respect to u.

Proof. We prove the statement for $n = 2$, the higher dimensional cases are analogous. Because of Equation (2.2) and the Local dd^c- Lemma 25, we have that, in local holomorphic coordinates, $g_{\alpha,\bar{\beta}} = -i\omega_{\alpha,\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} f(z)$.

Since f is a function of $u = |z_1|^2 + |z_2|^2 = z_1\bar{z}_1 + z_2\bar{z}_2$, we apply the chain rule:

$$\bar{\partial} f(u) = f'(u) \sum z_\alpha \mathrm{d}\bar{z}_\alpha, \quad \partial f(u) = f'(u) \sum \bar{z}_\alpha \mathrm{d}z_\alpha \tag{4.2}$$

and then compute the metric directly:

$$\begin{aligned}
g = \partial\bar{\partial} f &= \partial\left(f'(u) z_1 \mathrm{d}\bar{z}_1 + f'(u) z_2 \mathrm{d}\bar{z}_2\right) \\
&= \left(f''(u)\bar{z}_1 z_1 + f'(u)\right) \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + f''(u)\bar{z}_1 z_2 \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_2 \\
&+ f''(u)\bar{z}_2 z_1 \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_1 + \left(f''(u)\bar{z}_2 z_2 + f'(u)\right) \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \\
&= \begin{pmatrix} \mathrm{d}z_1 & \mathrm{d}z_2 \end{pmatrix} \begin{pmatrix} f' + |z_1|^2 f'' & z_1\bar{z}_2 f'' \\ \bar{z}_1 z_2 f'' & f' + |z_2|^2 f'' \end{pmatrix} \begin{pmatrix} \mathrm{d}\bar{z}_1 \\ \mathrm{d}\bar{z}_2 \end{pmatrix},
\end{aligned}$$

which yields formula (4.1).

□

Lemma 66. Assume g is a Kähler metric with rotationally invariant Kähler potential f, defined on $\mathbb{C}^n - \{0\}$. Then the determinant is given by

$$\det(g) = f'(u)^{n-1}(uf'(u))'.$$

Proof. According to Lemma 65, the metric takes the form

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta} f'(u) + \bar{z}_\alpha z_\beta f''(u).$$

From this formula, we have

$$\det(g) = f'(u)^n \det\left(\delta_{\alpha\bar{\beta}} + \frac{f''(u)}{f'(u)}\bar{z}_\alpha z_\beta\right)$$

The matrix $\bar{z}_\alpha z_\beta$ can be seen as the outer product $\bar{z} \otimes z = \bar{z} z^{\mathrm{T}}$. Therefore, making use of the Matrix Determinant Lemma 131 we obtain

$$\det\left(\delta_{\alpha\bar{\beta}} + \frac{f''(u)}{f'(u)}\bar{z}_\alpha z_\beta\right) = \det\left(\mathbb{I} + \frac{f''(u)}{f'(u)}\bar{z} z^{\mathrm{T}}\right) = 1 + \frac{f''(u)}{f'(u)} z^{\mathrm{T}}\bar{z} = 1 + \frac{f''(u)}{f'(u)} u.$$

Substituting, we finally get

$$\begin{aligned}
\det(g) &= f'(u)^n \left(1 + \frac{f''(u)}{f'(u)} u\right) \\
&= f'(u)^{n-1}\left(f'(u) + f''(u) u\right) \\
&= f'(u)^{n-1}(uf'(u))'.
\end{aligned}$$

□

Lemma 67. Assume g is a Kähler metric with rotationally invariant Kähler potential f, defined on $\mathbb{C}^n - \{0\}$. If g is Ricci-flat, then locally the metric g takes the form

$$g_{\alpha\bar{\beta}} = \left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}} \left(\delta_{\alpha\bar{\beta}} - \frac{a^n}{a^n + u^n}\frac{\bar{z}_\alpha z_\beta}{u}\right)$$

for $a \geq 0$.

Proof. By Observation 35, the metric is Ricci-flat if and only if $\det(g) = 1$. Substituting this into the formula given by Lemma 66, we obtain the following ODE:

$$\det(g) = f'(u)^{n-1}(uf'(u))' = 1. \tag{4.3}$$

Multiplying by u^{n-1} and integrating in u we obtain

$$(uf'(u))^n = u^n + a^n,$$

where a^n is a constant of integration. From this we have

$$\begin{aligned} f'(u) &= \left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}}, \\ f''(u) &= -\left(\frac{a^n}{u^{n+1}}\right)\left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}-1}. \end{aligned}$$

Now we apply Lemma 65 and simplify:

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \delta_{\alpha\beta} f'(u) + \bar{z}_\alpha z_\beta f''(u) \\ &= \delta_{\alpha\beta}\left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}} - \bar{z}_\alpha z_\beta \left(\frac{a^n}{u^{n+1}}\right)\left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}-1} \\ &= \left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}}\left[\delta_{\alpha\beta} - \bar{z}_\alpha z_\beta \left(1 + \left(\frac{a}{u}\right)^n\right)^{-1}\left(\frac{a^n}{u^{n+1}}\right)\right] \\ &= \left(1 + \left(\frac{a}{u}\right)^n\right)^{\frac{1}{n}}\left(\delta_{\alpha\beta} - \frac{a^n}{a^n + u^n}\frac{\bar{z}_\alpha z_\beta}{u}\right). \end{aligned}$$

□

Because the metric varies with respect to the parameter a, we will write it as g_a.

4.3 The Eguchi-Hanson space

In this section we present the first ingredient necessary to build a $K3$ surface.

It is the first example of an ALE space, asymptotic to $\mathbb{H}/\{\pm 1\}$ and was found by Eguchi and Hanson in [12]. For this reason, it is known as the Eguchi-Hanson space. Now we give its explicit description, making use of the material reviewed in Section 4.2.

For more details about ALE spaces, please see the Appendix A.

Example 68 (The Eguchi-Hanson space)**.** The image of the map

$$\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$$
$$\phi(z_1, z_2) = (z_1^2, z_2^2, z_1 z_2)$$

is the variety $\mathcal{V} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 = z_3^2\}$, also known as the quadratic cone. Observe that it has a singularity at the origin.

The singularity occurring in $\mathcal{V}$ corresponds to an ordinary double point A_1 in the Du Val classification of singularities (for some characterizations of said singularities see [11]). A key feature of the Du Val singularities is that they all admit a resolution which can be computed via a series of blow ups (for more details on the blow up construction, see Appendix C).

Consider $\mathbb{C}^2$ with complex coordinates (z_1, z_2), acted upon by $\{\pm 1\}$ (the group of isometries generated by the involution $-1 : (z_1, z_2) \longmapsto (-z_1, -z_2)$).

It is a classic result by Klein that the map ϕ induces a homeomorphism $\mathbb{C}^2/\{\pm 1\} \longrightarrow \mathcal{V}$.

Observation 69. The regular part of the cone is, topologically $\mathbb{C}^2 - \{0\}/\{\pm 1\} \cong \mathbb{R}P^3 \times (0, \infty)$.

Let (X, π) be the minimal resolution of $\mathbb{C}^2/\{\pm 1\}$ (considered as in Appendix C), where $\pi : X = T^*\mathbb{C}P^1 \longrightarrow \mathbb{C}^2/\{\pm 1\}$ is the blow up map at 0, and denote by E the exceptional divisor $\pi^{-1}(0)$.

We seek to equip X with a rotationally symmetric Ricci-flat Kähler metric.

By Equation (4.3), the problem is reduced to solving the ODE:

$$f'(f' + uf'') = 1, \tag{4.4}$$

where f is the rotationally invariant Kähler potential associated with the Kähler metric.

We perform the variable change $f'(u) = g(u)$ in (4.4), to get

$$\begin{aligned} \frac{1}{2u}\frac{\mathrm{d}(u^2 g(u)^2)}{\mathrm{d}u} &= 1 \\ u^2 g(u)^2 &= \int 2u\mathrm{d}u = u^2 + a^2 \\ g(u)^2 &= 1 + \frac{a^2}{u^2} \\ g(u) = f'(u) &= \sqrt{1 + \frac{a^2}{u^2}}\ , \end{aligned}$$

which further integrates to

$$f_{EH,a}(u) = \sqrt{u^2 + a^2} + a \log \frac{u}{\sqrt{u^2 + a^2} + a} \tag{4.5}$$

$$= \sqrt{a^2 + u^2} - a \operatorname{arsinh}\left(\frac{a}{u}\right). \tag{4.6}$$

The function $f_{EH,a}$ behaves like u at infinity and like $a \log u$ near the vertex of the quadratic cone.

We then define a 2-form on $X - \pi^{-1}(0)$ by $\omega_{EH,a} = i\partial\bar{\partial} f_{EH,a}$, for $a > 0$.

Lemma 70 ([30], Lemma 28)**.** The Kähler form $\omega_{EH,a}$ extends smoothly across E and defines a complete, Ricci-flat asymptotically locally Euclidean metric on the whole X.

This is the Eguchi-Hanson metric, and X together with this metric $g_{EH,a}$ is called the Eguchi-Hanson space.

4.4 The Kummer construction

Now we are ready to explain the gluing procedure used to obtain a $K3$ surface, by making use of the previously defined Eguchi-Hanson spaces.

Example 71 (The Kummer surface)**.** First consider a lattice Λ in $\mathbb{C}^2$, so that $\Lambda \cong \mathbb{Z}^4$. Then $\mathbb{C}^2/\Lambda$ is a complex 4-torus T^4. Define the map

$$\begin{aligned} \sigma : T^4 &\longrightarrow T^4 \\ (z_1, z_2) + \Lambda &\longmapsto (-z_1, -z_2) + \Lambda, \end{aligned}$$

which fixes the 16 points

$$\{p_i\} = \left\{(z_1, z_2) + \Lambda : (z_1, z_2) \in \frac{1}{2}\Lambda\right\}.$$

Thus, $T^4/\langle\sigma\rangle$ is a complex orbifold with 16 singular points, each modelled on $\mathbb{C}^2/\{\pm 1\}$. We denote these points by Sing.

Each point in Sing is a singularity of type A_1 and can be resolved by blowing up, as before. Let $\pi : S \longrightarrow T^4/\langle\sigma\rangle$ be the resolution of $T^4/\langle\sigma\rangle$ computed in that way and E be its exceptional divisor.

The manifold S obtained this way is a closed complex surface without singularities. We call this construction of a complex surface the Kummer construction. Note that S belongs to a family of complex surfaces called Kummer surfaces.

It has the following property:

Lemma 72 ([38])**.** The Kummer surface S is simply connected.

As a consequence, we have the following important result.

Proposition 73. The Kummer surface S is a $K3$ surface.

Proof. Lemma 72 says that S is simply connected. It only remains to prove that it has trivial canonical bundle.

We observe that a non-vanishing holomorphic 2-form on the torus T^4 projects down to $T^4/\langle\sigma\rangle - \text{Sing}$, which then lifts to $S - E$ and extends to a nowhere vanishing holomorphic 2-form on S. From Observation 58, this implies that S has trivial canonical bundle K_S.

Therefore, S is a $K3$ surface. □

Theorem 74. The second cohomology group of the Kummer $K3$ surface S is $H^2(S) \cong \mathbb{Z}^{22}$.

Proof. We have seen that S can be covered by the interiors of $\frac{T^4 - \{p_i\}}{\langle\sigma\rangle}$ and 16 copies of Eguchi-Hanson spaces X_i. Additionally, Observation 69 tells us that $\frac{T^4 - \{p_i\}}{\langle\sigma\rangle} \cap X_i \cong \mathbb{R}P^3$. Then the Mayer-Vietoris exact sequence for the de Rham cohomology is:

$$\cdots \longrightarrow H^1\left(\bigsqcup_{16}\mathbb{R}P^3\right) \longrightarrow H^2\left(S\right) \longrightarrow H^2\left(\frac{T^4-\{p_i\}}{\langle\sigma\rangle}\right)\bigoplus H^2\left(\bigsqcup_{i=1}^{16}X_i\right)$$
$$\longrightarrow H^2\left(\bigsqcup_{16}\mathbb{R}P^3\right) \longrightarrow \cdots \tag{4.7}$$

For an n-torus T^n we have $H^k(T^n) \cong \mathbb{Z}^{\binom{n}{k}}$, so $H^2(T^4) = \mathbb{Z}^6$.

Observe that the involution σ acts as -1 on H^1 and hence as $(-1)^2 = 1$ on H^2. Therefore, $H^2(T^4/\langle\sigma\rangle) = \mathbb{Z}^6$.

Now, it is a well known fact that the cohomology groups of the real projective spaces $\mathbb{R}P^n$ are:

$$H^k(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = n \text{ if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Additionally, we know that, as a smooth manifold, the Eguchi Hanson space X is $T^*\mathbb{C}P^1$, and therefore there is only one cohomology class in $H^2(X)$.

Putting this together, the sequence (4.7) yields an isomorphism

$$H^2(S) \cong H^2\left(\frac{T^4-\{p_i\}}{\langle\sigma\rangle}\right)\bigoplus H^2\left(\bigsqcup_{16}X_i\right) \cong \mathbb{Z}^{22}.$$

□

Now, we seek to equip S with Ricci-flat metrics. By the discussion at the end of Chapter 2, we know that there is no hope of writing them down explicitly. However, there is a method for obtaining approximate Ricci-flat metrics on S, described in the Example below.

Example 75. Observe that for each singular point $\{p_i\}$, there exist a neighborhood U_i isomorphic to $B_{1+\delta}/\{\pm 1\}$, for some positive number $\delta << 1$, where B_t denotes the metric ball in $\mathbb{C}^2$ of radius t. We assume moreover that $U_i \cap U_j = \emptyset$, if $i \neq j$.

Then every U_i can be endowed with an Eguchi-Hanson metric, as in Section 4.3. In the complement of the union of U_i's, we interpolate with the Euclidean flat metric, as follows.

Let $u = |z_1|^2 + |z_2|^2$ be the square of the radius function in $\mathbb{C}^2$ and define the real valued cut-off function function $t(u)$ as

$$t(u) = \begin{cases} 1 & u < 1 - \delta \\ 0 & u > 1, \end{cases} \tag{4.9}$$

such that $|t'(u)| \leq \frac{2}{\delta}$.

For a positive number a_i sufficiently small, the function

$$f_{a_i}(u) = f_{Euc}(u) + t(u)\left(f_{EH,a_i}(u) - f_{Euc}(u)\right) \tag{4.10}$$

is a Kähler potential on the minimal resolution $\widetilde{U}_i$ of U_i (considered as in Appendix C). The Kähler metric $i\partial\bar{\partial} f_{a_i}$ coincides with the Eguchi-Hanson metric on $\widetilde{U}_i$, for $u < 1 - \delta$; and with the Euclidean metric on T^4, for $u > 1$.

Therefore, we can glue 16 Eguchi-Hanson metrics $\omega_{a_i} = i\partial\bar{\partial} f_{a_i}$, defined on small neighborhoods of p_i with an appropriate flat orbifold metric $\omega_{Euc} = i\partial\bar{\partial} f_{Euc}$ on S to construct an approximate Ricci-flat Kähler metric ω_a, where $a = \{a_i\}_{i=1}^{16}$ is a tuple of sufficiently small positive numbers. The Kähler metric ω_a is not Ricci-flat only on the union of the neck regions

$$N_i \cong \{z \in (\mathbb{C}^2 - \{0\})/\{\pm 1\} \,\big|\, 1 \leq |z|^2_{\mathbb{C}^2} \leq 1 + \delta\}.$$

Our aim is to deform this family of approximate Ricci-flat metrics to one of actual Ricci-flat ones.

As we have seen in Chapter 2, this is equivalent to solving the Calabi-Yau equation, which can be done if we have suitable a priori bounds on the solution. The problem was solved by Kobayashi in [24].

In Chapter 5, we go trough the computation of said bounds. Afterwards, we will explain how they constitute a crucial ingredient to prove our main result: the incompleteness of the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$.

Chapter 5

Estimates for the solution of the Calabi-Yau Equation on S

In this chapter we will consider the Kummer $K3$ surface S described before and compute several a priori estimates, which will be used later in the thesis to prove that the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ is not complete.

First, we will follow the approach of Kobayashi in [24], (who in turn adapted Yau's original methods from [45], taking advantage of the fact that the $K3$ surface is a relatively simple case to obtain better bounds), which consist of the following: a flat background orbifold metric and 16 Eguchi-Hanson spaces are glued together on the Kummer surface S to construct a family of approximately Ricci-flat metrics ω_a, where $a \in \mathbb{R}^{16}$. We aim to deform it into a family of "actual" Ricci-flat metrics $\widetilde{\omega}_a = \omega_a + i\partial\bar{\partial}u_a$. Namely, we will find the following a priori bounds for the solution u_a:

$$\|u_a\|_{L^\infty} \leq C|a|^2, \tag{5.1}$$
$$\|u_a\|_{C^{k,\alpha}} \leq C_k|a|^2, \tag{5.2}$$
$$\|u_a\|_{C^k} \leq C'|a|^{2-\frac{k}{2}}; \tag{5.3}$$

for constants C, C_k and C', independent of a, and $|a|$ is the Euclidean norm. The inequality (5.2) holds on the complement of a neighborhood of exceptional divisor, whereas (5.1) and (5.3) hold everywhere.

Secondly, we will adapt results found by Biquard-Minerbe in [7] to obtain a bound over the norm of functions on S, in terms of the Laplacian.

In order to do it, we need to consider an alternative Kummer construction, which glues a rescaled Eguchi Hanson metric and results in a new family of approximately Ricci-flat metrics ω_ε on S.

Then, we will prove that for $0 < \beta < 2$, there exist a constant c such that for every small ε and every function $f \in C^{2,\alpha}_{\varepsilon,\beta}$, the following inequality holds:

$$\|f\|_{C^{2,\alpha}_{\varepsilon,\beta}} \leq c\|\Delta_\varepsilon f\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} + \|f\|_{C^0},$$

where $C^{k,\alpha}_{\varepsilon,\beta}$ are suitable Banach spaces, which will be defined in Section 5.4.

At the end of the chapter, we will show that difference of the Kähler potentials defining the metrics ω_a and ω_ε can be made arbitrarily small. Therefore, all the aforementioned estimates are compatible and the families of associated Ricci-flat metrics $\widetilde{\omega}_a$ and $\widetilde{\omega}_\varepsilon$ are the same.

5.1 Preliminary estimates

Consider the metric ω_a on S, as before. Yau's solution of the Calabi conjecture tells us that there exists a smooth function u_a on S such that $\widetilde{\omega}_a = \omega_a + i\partial\bar{\partial}u_a$ is a Ricci-flat Kähler form on S, and such u_a is unique up to an additive constant.

Let $\eta = \sqrt{2}\mathrm{d}z_1 \wedge \mathrm{d}z_2$ be a holomorphic 2-form in $\mathbb{C}^2$. As we have stated in Observation 35, any Ricci-flat volume form on S is a constant multiple of $\eta \wedge \bar{\eta}$ (after lifting and extending, as before).

Consider a real valued function G_a, defined by the relation:

$$G_a = \log\left(\frac{\eta \wedge \bar{\eta}}{\omega_a^2}\right)$$

Then Yau's theorem is equivalent to saying that the following equation (5.4), together with a normalization condition (5.5), has a unique solution

$$(\omega_a + i\partial\bar{\partial}u_a)^2 = \exp(G_a)\omega_a^2 \tag{5.4}$$
$$\int_S u_a\,\omega_a^2 = 0. \tag{5.5}$$

In the following we will find a priori bounds for the solution u_a. We start by computing a norm bound for G_a and its Laplacian.

Lemma 76. The function G_a has the following bound

$$\|G_a\|_\infty \leq C|a|^2, \tag{5.6}$$

for some constant C independent of a, where $|a|^2 = \sum_{i=1}^{16} a_i^2$.

Proof. In local coordinates, Equation (2.4) implies that $\frac{\eta\wedge\bar\eta}{\omega_a^2} = \frac{1}{\det(g_a)}$.

By Lemma 70, we have that $\det(g_a) = 1$ both in EH and in Euclidean parts. Then $G_a = 0$ also in those parts.

Now we analyse what happens on the gluing or neck regions. Remember that locally, one of such necks can written as

$$N_i = \{z \in (\mathbb{C}^2 - \{0\})/\{\pm 1\} \,\big|\, 1 \leq |z|^2 \leq 1+\delta\}.$$

Remember that $u = |z_1|^2 + |z_2|^2$. In N_i, the metric g_a is given by the Kähler potential (4.10), which is rotationally symmetric because the Kähler potentials

$$\begin{aligned} f_{EH,a}(u) &= \sqrt{a^2+u^2} - a\cdot \operatorname{arsinh}\left(\frac{a}{u}\right),\\ f_{Eucl}(u) &= u. \end{aligned}$$

are.

Observe that we have chosen to drop the dependence on i from the notation, in order to make it simpler. However one should keep in mind that we are working at a single neck at a time.

Consider the difference

$$\begin{aligned} f_{EH,a} - f_{Eucl} &= \sqrt{u^2+a^2} - u - a\cdot \operatorname{arsinh}\left(\frac{a}{u}\right)\\ &= u\left(\sqrt{1+\frac{a^2}{u^2}} - 1\right) - a\cdot \operatorname{arsinh}\left(\frac{a}{u}\right) \end{aligned}$$

We perform a change of variable $x = \frac{a}{u}$, and observe that the function $\left(\sqrt{1+x^2}-1\right)$ has a series expansion $\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} \cdots$.

Similarly, the function arsinh(x) has the expansion $x - \frac{x^3}{6} + \frac{3x^5}{40} \cdots$.
Together, the two last expressions give

$$f_{EH,a} - f_{Eucl} = \frac{a^2}{u} + \mathcal{O}\left(\frac{a^4}{u^3}\right) = a^2 \cdot h_a, \tag{5.7}$$

where $h_a(u)$ and its derivatives are smooth regular functions, as long as u is bounded from below.

Then we can write the Kähler potential (4.10) as

$$\begin{aligned} f_{a_i}(u) &= f_{Eucl} + t(u)\left(f_{EH,a_i} - f_{Eucl}\right) \\ &= u + a^2 t(u) h_a = u + a^2 \tilde{h}_a(u), \end{aligned}$$

where by construction the function $\tilde{h}_a(u) := t(u)h_a$ and its derivatives are regular.

Now, from Equation (4.1) we know that the metric takes the form

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \delta_{\alpha\beta} f'(u) + \bar{z}_\alpha z_\beta f''(u) \\ &= \delta_{\alpha\beta}\left(1 + a^2 \tilde{h}'_a(u)\right) + \bar{z}_\alpha z_\beta a^2 \tilde{h}''_a(u) \\ &= \mathbb{I} + a^2\left(\delta_{\alpha\beta}\tilde{h}'_a(u) + \bar{z}_\alpha z_\beta \tilde{h}''_a(u)\right) \\ &= \mathbb{I} + a^2 H_a(u), \end{aligned} \tag{5.8}$$

where we have defined $H_a(u) := \delta_{\alpha\beta}\tilde{h}'_a(u) + \bar{z}_\alpha z_\beta \tilde{h}''_a(u)$.

Given the form of $g_{\alpha\bar{\beta}}$, we can apply Lemma 131 to obtain

$$\det(g_{\alpha\bar{\beta}})(u) = 1 + a^2 \operatorname{Tr}\left(H_a(u)\right) + \mathcal{O}(a^4).$$

By shrinking a, we can arrange $|a^2 \operatorname{Tr}(H_a(u)) + \mathcal{O}(a^4)| < 1$, and using the series expansion of the logarithm we get

$$G_a(u) = \frac{1}{\ln(\det(g_a)(u))} = -\ln\left(1 + a^2 \operatorname{Tr}\left(H_a(u)\right) + \mathcal{O}(a^4)\right) \approx -a^2\left(\operatorname{Tr}\left(H_a(u)\right) + \mathcal{O}(a^2)\right).$$

From this we get that it is possible to write $G_a(u) = a^2 \hat{h}_a(u)$, where $\hat{h}_a := -\operatorname{Tr}\left(H_a(u)\right) + \mathcal{O}(a^2)$ is regular as $a \longrightarrow 0$.

Summarizing, we have

$$G_a = \begin{cases} 0 & \text{in EH and Eucl,} \\ a^2 \cdot \hat{h}_a(u) & \text{in neck region.} \end{cases}$$

Therefore,

$$\|G_a\|_\infty \leq C|a|^2 \tag{5.9}$$

as we wanted.

□

Observation 77. Equation (5.8) implies that

$$\|g_a - g_{Eucl}\|_{C^k(N, g_{Eucl})} \leq C_k a^2, \tag{5.10}$$

for a constant C_k independent of a.

From this we obtain bounds for the Riemann curvature tensor

$$\|Rm\| \leq Ca^2,$$

and therefore also for its components. In particular, we have for every point x away from the exceptional divisor

$$|\text{sectional curvature at } x| \leq C|a|^2.$$

Lemma 78. Assume the setup of the Kummer construction of the K3 surface S. Let Δ be the Laplacian associated to the metric g_a. Then

$$\|\Delta G_a\|_{L^\infty(S, g_a)} \leq C|a|^2.$$

Proof. We know that $G_a = 0$ outside of the neck regions. Therefore we consider the situation at a single neck, as before. We shall write a for a_i, to simplify notation.

Using the local expression of the Laplacian acting on functions, as well as the form of the metric $g_{\alpha\bar{\beta}}$ given by Equation (5.8), we have

$$\Delta G_a = a_i^2 \mathrm{Tr}\left(g^{-1}_{\alpha\bar{\beta}} \nabla^2 \hat{h}_a(u)\right). \tag{5.11}$$

Using Lemma 132, we get that the inverse of $g_{\alpha\bar{\beta}}$ is of the form

$$g^{-1}_{\alpha\bar{\beta}} = \mathbb{I} - a^2\nabla^2\hat{h}_a(u) + \mathcal{O}(a^2) =: \mathbb{I} + a^2\bar{h}_a,$$

once again, with $\bar{h}_{a_i}$ regular as $a \longrightarrow 0$. Inserting this expression into the Laplacian (5.11), we get the norm bound:

$$\|\Delta G_a\|_{L^\infty(N_i, g_a)} \leq Ca^2.$$

Performing the same analysis on the other necks, and taking $a = \max_{i=1\cdots16}\{a_i\}$, we get the result. ☐

5.2 Derivation of the C^0-estimate

Our goal in this section is to attain the estimate

$$\|u_a\|_\infty \leq C|a|^2.$$

We will need several lemmas, which can be found in Appendix D, to perform a process sometimes called Moser iteration. It consists of finding first an L^2 estimate, then we apply successively a series of inequalities (all carefully stated) to build better L^p bounds, and finally taking the limit $p \longrightarrow \infty$.

Proposition 79. For each $p \geq 2$, there exist a constant C such that

$$\|u_a\|_p \leq C|a|^2. \tag{5.12}$$

Proof. Let us observe that for $0 < |x| < 1$, we have

$$|e^x - 1| < \frac{7}{4}|x|.$$

Applying this to Equation (D.2) from Lemma 146, we get

$$\int_M \left|\nabla|u_a|^{p/2}\right|^2 \mathrm{dVol}_g \leq Cp\int_M |G_a|\,|u_a|^{p-1}\mathrm{dVol}_g. \tag{5.13}$$

For $p = 2$, this equation further simplifies to

$$\|\nabla u_a\|_2^2 \leq C|a|^2\|u_a\|_1.$$

Hölder's inequality (A.2) implies $\|u_a\|_1 \leq C_1\mathrm{Vol}_g(M)^{1/2}\|u_a\|_2$. On the other hand, as $\int_M u_a \mathrm{dVol}_g = 0$, Lemma 147 shows that $\|u_a\|_2 \leq C_2\|\nabla u_a\|_2$.

Combining all of these we have

$$\|\nabla u_a\|_2^2 \leq C|a|^2\mathrm{Vol}_g(M)^{1/2} \cdot \|\nabla u_a\|_2,$$

which implies

$$\|\nabla u_a\|_2 < C|a|^2.$$

Therefore

$$\|u_a\|_2 \leq C|a|^2. \tag{5.14}$$

Now we consider the case $p \in [2, 4]$. By Lemma 147, we get that for $\epsilon = m/(m+1) = 2/3$:

$$\|u_a\|_{2\epsilon} \leq C|a|^2. \tag{5.15}$$

Applying Hölder's inequality to equations (5.14) and (5.15) we conclude that the estimate

$$\|u_a\|_p \leq C|a|^2 \tag{5.16}$$

holds for every $p \in [2, 4]$.

Now let consider a general $p \leq 2$. We apply Li's Sobolev inequality to the function $|u_a|^{\frac{p}{2}}$ to obtain

$$\left\| u_a^{\frac{p}{2}} \right\|_2^4 \leq C \left\| \mathrm{d}|u_a|^{\frac{p}{2}} \right\|_2^2 + C' \left\| u_a^{\frac{p}{2}} \right\|_2^2. \tag{5.17}$$

We observe that the following equivalences hold:

$$\begin{aligned}\left\| u_a^{\frac{p}{2}} \right\|_4^2 &= \left(\left(\int_X (|u_a|^{\frac{p}{2}})^4 \mathrm{dVol}_g \right)^{\frac{1}{4}} \right)^2 = \|u_a^p\|_2 \\ &= \left(\int_X |u_a|^{2p} \mathrm{dVol}_g \right)^{\frac{1}{2}} = \left(\left(\int_X |u_a|^{2p} \mathrm{dVol}_g \right)^{\frac{1}{2p}} \right)^p = \|u_a\|_{2p}^p .\end{aligned}$$

Therefore, Equation (5.17) becomes

$$\|u_a\|_{2p}^p \leq C \left\| \mathrm{d}|u_a|^{\frac{p}{2}} \right\|_2^2 + C' \left\| u_a^{\frac{p}{2}} \right\|_2^2. \tag{5.18}$$

By definition, $\left\| \mathrm{d}|u_a|^{\frac{p}{2}} \right\|_2^2 = \int_M \left|\nabla |u_a|^{p/2}\right|^2 \mathrm{dVol}_g$, so we can use Equation (5.13), together with the Hölder's inequality as follows. Since $1 = \frac{p-1}{p} + \frac{1}{p}$, we have

$$\left\| \mathrm{d}|u_a|^{\frac{p}{2}} \right\|_2^2 \leq \|G_a\|_{\frac{p}{p-1}} \|u_a\|_p^{p-1}$$

Also, $\left\| u_a^{\frac{p}{2}} \right\|_2^2 = \left(\left(\int_X u^p \mathrm{dVol}_g\right)^{1/2} \right)^2 = \|u_a\|_p^p$, so inequalities (5.18) and (5.6) imply

$$\begin{aligned} \|u_a\|_{2p}^p &\leq Cp \|G_a\|_{\frac{p}{p-1}} \|u_a\|_p^{p-1} + C'\|u_a\|_p^p \\ &\leq \left(\frac{C|a|^2 p}{\|u_a\|_p} + C' \right) \|u_a\|_p^p, \end{aligned} \tag{5.19}$$

or

$$\|u_a\|_{2p}^p \leq \left(C' + \|u_a\|_p + Cp|a|^2\right) \|u_a\|^{p-1}. \tag{5.20}$$

Now, in order to prove (5.12), we put $p_n = 2^n p_0$, with $p_0 = 4$, and proceed by induction over n. For $n = 0$, Equation (5.16) shows that the property holds.

Next, suppose that

$$\|u_a\|_{p_n} \leq C_n |a|^2.$$

We will show that $\|u_a\|_{p_{n+1}} \leq C_{n+1}|a|^2$, making use or inequalities (5.19) and (5.20):

$$\begin{aligned} \|u_a\|_{p_{n+1}}^{p_n} &\leq \left(C' + \frac{C|a|^2 p_n}{\|u_a\|_{p_n}} \right) \|u_a\|_{p_n}^{p_n} \\ &= \left(C'\|u_a\|_{p_n} + Cp_n|a|^2\right) \|u_a\|_{p_n}^{p_n - 1} \\ &\leq \left(C'C_n|a|^2 + Cp_n|a|^2\right) (C_n|a|^2)^{p_n - 1} \\ &= \left(C' + \frac{Cp_n}{C_n} \right) (C_n|a|^2)^{p_n}. \end{aligned}$$

After computing the p_n-th root we obtain

$$\|u_a\|_{p_{n+1}} \leq C_{n+1}|a|^2,$$

where $C_{n+1} = \left(C' + \frac{Cp_n}{C_n}\right)^{1/p_n} C_n$.

Finally, we observe that the constant C_{n+1} obeys the following inequalities

$$C_{n+1} \leq (C' + 2^n C)^{1/2^n} C_n, \quad \text{if} \quad C_n \geq 1;$$
$$C_{n+1} \leq (C' + 2^n C)^{1/2^n}, \quad \text{if} \quad C_n \leq 1.$$

From this we conclude that there exists a constant C, independent of a and n, such that $C_n \leq C$. □

By taking the limit, we have proven the following:

Proposition 80. There exist a constant C, independent of a, such that

$$\|u_a\|_{L^\infty} = \|u_a\|_{C^0} \leq C|a|^2. \tag{5.21}$$

5.3 Derivation of C^2- estimate

Let Δ_a and $\tilde{\Delta}_a$ be respectively the Laplacians of the Kähler metric ω_a and $\widetilde{\omega}_a = \omega_a + i\partial\bar{\partial}u_a$.

In local Darboux coordinates, the Monge-Ampère equation (5.4) becomes $\det(g_a + \nabla^2 u_a) = 1$, which holds if and only if

$$2(\exp(G_a) - 1) = 2\Delta_a u_a + (\Delta_a u_a)^2 - |\nabla^2 u_a|^2_{g_a}.$$

Then, a bound for the Laplacian also yields a bound on the complex Hessian $\nabla^2 u_a$. Now, the complex Hessian is the (1,1)-part of the real Hessian so, in principle, this does not control the full real second derivative tensor. However, it is possible to follow the strategy of Siu and Błocki, to obtain bounds on $\|u_a\|_{C^2}$. This will be the content of Proposition 82.

Now, we proceed to find bounds for $\Delta_a u_a$. In order to do this, we need the following result, proven by Yau in [45].

Proposition 81. Let M be a Kähler manifold of complex dimension n and Rm its Riemannian curvature tensor. Given two J-invariant (holomorphic) planes σ and σ' in $T_x(M)$, we define the holomorphic bisectional curvature as

$$H(\sigma, \sigma') = Rm(X, JX, Y, JY),$$

where X is a unit vector in σ and Y a unit vector in σ'. In tensor index notation we denote it by $R_{i\bar{i}j\bar{j}}$.

Let R_a be the maximum of the holomorphic bisectional curvature of the Kähler metric ω. We set $c_a = 2R_a$. Then we have:

$$e^{c_a u_a}\tilde{\Delta}_a(e^{-c_a u_a}(2+\Delta_a u_a))(x) \tag{5.22}$$
$$\geq A(x) + B(x)(2+\Delta_a u_a) + C(x)(2+\Delta_a u_a)^2, \tag{5.23}$$

where the functions $A(x)$, $B(x)$ and $C(x)$ are defined by

$$\begin{aligned} A(x) &= \Delta_a G_a(x) - 4\inf_{i\neq j}(-R_{i\bar{i}j\bar{j}}), \\ B(x) &= -2c_a, \\ C(x) &= (c_a + \inf_{i\neq j}(-R_{i\bar{i}j\bar{j}})(x))e^{G_a(x)}. \end{aligned}$$

Here we have taken the subscripts i and j with respect to any local unitary frame for the holomorphic tangent bundle of X.

We are going to use Equation (5.22) to obtain a priori estimates for $\Delta_a u_a$.

First, let us observe that it is possible to choose local coordinates such that the metric is of the form

$$g_{\mu\bar{\nu}}(p) = \delta_{\mu\nu}.$$

The Ricci flatness condition (which is attained outside of the neck region) is then

$$0 = \mathrm{Ric}_{\mu\bar{\nu}} = \mathrm{R}_{\mu\bar{\nu}\alpha\bar{\beta}}g^{\bar{\beta}\alpha}.$$

Therefore, in complex dimension $n = 2$, we have

$$\begin{aligned} &\mathrm{R}_{\mu\bar{\nu}1\bar{1}} + \mathrm{R}_{\mu\bar{\nu}2\bar{2}} = 0 \\ &\implies \mathrm{R}_{\mu\bar{\nu}1\bar{1}} = -\mathrm{R}_{\mu\bar{\nu}2\bar{2}} \\ &\implies \mathrm{R}_{1\bar{1}1\bar{1}} = -\mathrm{R}_{1\bar{1}2\bar{2}}, \\ &\qquad\ \mathrm{R}_{2\bar{2}2\bar{2}} = -\mathrm{R}_{2\bar{2}1\bar{1}}. \end{aligned}$$

We recognize the terms $\mathrm{R}_{i\bar{i}i\bar{i}}$ as the holomorphic sectional curvature. From this we conclude that, in our case, considering the holomorphic sectional and bisectional curvatures is equivalent.

Define the continuous function

$$k(x) = -\inf_{i\neq j}(-R_{i\bar{i}j\bar{j}})(x)/R_a,$$

where inf is taken for all unitary frames at x. From the definition we have that $|k(x)| \leq 1$

Suppose that $e^{-c_a u_a}(2 + \Delta_a u_a)$ attains its maximum at $x \in X$. Then Equation (5.22) gives

$$\begin{aligned} 0 &\geq e^{c_a u_a}\tilde{\Delta}_a(e^{-c_a u_a}(2 + \Delta_a u_a))(x) \\ &= (\Delta_a G_a(x) + 4k(x)R_a) - 4R_a(2 + \Delta_a u_a)(x) + (2 - k(x))R_a e^{G_a}(2 + \Delta_a u_a)^2(x) \\ &= e^{G_a}(2 - k(x))R_a \left\{ \left((2 + \Delta_a u_a)(x) - \frac{2e^{-G_a}(x)}{2 - k(x)} \right)^2 \right. \\ &\left. - \left(\frac{2e^{-G_a}(x)}{2 - k(x)} \right)^2 + \frac{e^{-G_a}(x)(\Delta_a G_a + 4R_a k)(x)}{(2 - k(x))R_a} \right\}. \end{aligned}$$

Since $e^{G_a}(2 - k(x))R_a \geq 0$, we have that

$$\left((2 + \Delta_a u_a)(x) - \frac{2e^{-G_a}(x)}{2 - k(x)} \right)^2 - \left(\frac{2e^{-G_a}(x)}{2 - k(x)} \right)^2 + \frac{e^{-G_a}(x)(\Delta_a G_a + 4R_a k)(x)}{(2 - k(x))R_a} \leq 0,$$

which implies

$$\left|(2+\Delta_a u_a)(x) - \frac{2e^{-G_a}(x)}{2-k(x)}\right| \leq \left|\left(\frac{2e^{-G_a}(x)}{2-k(x)}\right)^2 - \frac{e^{-G_a}(x)(\Delta_a G_a + 4R_a k)(x)}{(2-k(x))R_a}\right|^{\frac{1}{2}}. \tag{5.24}$$

If x lies outside of the neck regions, then $G_a(x) = 0$ and therefore $\Delta_a G_a(x) = 0$. In this case, equation (5.24) becomes:

$$2 + \Delta_a u_a(x) \leq \frac{2}{2-k(x)} + \left|\left(\frac{2}{2-k(x)}\right)^2 - \frac{4k(x)}{2-k(x)}\right|^{\frac{1}{2}} = 2 \tag{5.25}$$

and then $\Delta_a u_a(x) \leq 0$.

On the other hand, if x is in a neck region, Lemmas 76, 78 and Observation 77 yield the following inequalities:

$$\begin{aligned} |G_a(x)| &\leq C|a|^2, \\ |\text{sectional curvature at } x| &\leq C|a|^2, \\ |\Delta_a G_a(x)| &\leq C|a|^2, \\ |k(x)| &\leq C|a|^3. \end{aligned}$$

We substitute those estimates, together with $\frac{1}{R_a} = \min_{1\leq i\leq 16}\{a_i\} \leq |a|$ in equation (5.24) to get

$$(2+\Delta_a u_a)(x) \leq \frac{2(1+C|a|^2)}{2-C|a|^3} + \left|\left(\frac{2(1+C|a|^2)}{2-C|a|^3}\right)^2 - \frac{C(1+C|a|^2)|a|^3}{2-C|a|^3}\right|^{\frac{1}{2}} \leq 2 + C|a|^2 \tag{5.26}$$

It follows from equations (5.25) and (5.26) that

$$\begin{aligned} 2 + \Delta_a u_a &= e^{c_a u_a} e^{-c_a u_a}(2+\Delta_a u_a) && (5.27) \\ &\leq e^{c_a u_a} e^{-c_a u_a(x)}(2+\Delta_a u_a)(x) && (5.28) \\ &\leq e^{c_a(\sup u_a - \inf u_a)}(2 + C|a|^2). && (5.29) \end{aligned}$$

Putting

$$r_a = \frac{\max\{a_i\}}{\min\{a_i\}},$$

and using the fact that

$$R_a = \max\{a_i^{-1}\} = \frac{1}{\min\{a_i\}} = \frac{r_a}{\max\{a_i\}},$$

we have the following inequalities:

$$\begin{aligned}
\max\{a_i\} \leq |a| \leq 16 \max\{a_i\} \\
\frac{1}{\max\{a_i\}} \geq \frac{1}{|a|} \geq \frac{1}{16 \max\{a_i\}} \\
\frac{r_a}{\max\{a_i\}} \geq \frac{r_a}{|a|} \geq \frac{r_a}{16 \max\{a_i\}} \\
R_a \geq \frac{r_a}{|a|} \geq \frac{R_a}{16}.
\end{aligned}$$

In particular, we get

$$c_a = 2R_a \leq \frac{32 r_a}{|a|}. \tag{5.30}$$

As a consequence of (5.30), we obtain

$$c_a(\sup u_a - \inf u_a) \leq C r_a |a|$$

and therefore we have proved the upper bound for the Laplacian

$$\begin{aligned}
2 + \Delta_a u_a &\leq e^{C r_a |a|}(2 + C|a|^2) \\
&\leq 2 + C r_a |a|. \qquad (5.31)
\end{aligned}$$

This gives an a priori C^2-estimate for u_a. Now, consider the following proposition:

Proposition 82. There are constants $C > 0$ and $0 < \alpha < 1$, independent of a such that

$$\|u_a\|_{C^{2,\alpha}} \leq C$$

holds for all small enough values of a.

Proof. We refer the reader to [31], Proposition 6.8 for the full proof of this result.

The general idea is that, as we saw in the proof of proposition 76, in a local holomorphic chart V_i we can arrange that $\det(g_{\alpha\bar{\beta}})(u) = F_a$ for a smooth and uniformly bounded function F_a, for a sufficiently small.

Hence, u_a satisfies a uniformly elliptic nonlinear PDE. By applying the strategy of Siu ([37]) and Błocki ([8], [9]), whose methods in turn stem from the work from Evans ([13], [14]) Krylov ([26]) and Trudinger ([39]); we conclude that u_a is bounded in $C^{2,\alpha}(V_i)$, for some $\alpha > 0$, with constants independent of a.

Finally, by patching together the estimates on a finite cover, we get a global bound, for a small enough. □

It is possible to obtain a better C^2- estimate in the following way. Consider the equation, containing a parameter $t \in [0, 1]$, together with a normalization condition:

$$(\omega_a + i\partial\bar{\partial}u_{a,t})^2 = (1 + t(e^{G_a} - 1))\omega_a^2, \tag{5.32}$$

$$\int_X u_{a,t}\omega_a^2 = 0. \tag{5.33}$$

We observe that for $t = 1$ it coincides with the Calabi Yau equation for a Ricci-flat Kähler metric. We differentiate equation (5.32) with respect to t.

$$2(\omega_a + i\partial\bar{\partial}u_{a,t}) \wedge \partial\bar{\partial}\left(\frac{\partial}{\partial t}u_{a,t}\right) = (e^{G_a} - 1)\omega_a^2$$

$$\Delta\frac{\partial}{\partial t}u_{a,t}(\omega_a + i\partial\bar{\partial}u_{a,t})^2 = (e^{G_a} - 1)\omega_a^2$$

$$\Delta\frac{\partial}{\partial t}u_{a,t} = \frac{e^{G_a} - 1}{1 + t(e^{G_a} - 1)}$$

where the last equality is true from Lemma 24. Applying estimate (5.9) to the right side of the last equation, we have

$$\Delta \frac{\partial}{\partial t} u_{a,t} \leq C|a|^2. \tag{5.34}$$

From this, we can attain a bound for the oscillation of $\frac{\partial}{\partial t} u_{a,t}$, as follows.

The Laplacian is not an invertible operator (constant functions are always in its kernel). However, we may consider π to be the orthogonal projection to its kernel, and construct an elliptic, pseudo-differential operator of order -2:

$$(\Delta + \pi)^{-1} : W^{0,p} \longrightarrow W^{2,p}.$$

Composing it with suitable inclusions, we obtain

$$C^0 \hookrightarrow L^p = W^{0,p} \longrightarrow W^{2,p} \hookrightarrow C^0.$$

Now, consider $\fint u$ to be the average of the function u. Then we have:

$$\begin{aligned}
\text{Osc } u &= u_{max} - u_{min} \\
&= (u_{max} - \fint u) - (u_{min} - \fint u) \\
&= |u_{max} - \fint u| - |u_{min} - \fint u| \\
&\leq 2\|u - \fint u\|_{C^0} \\
&= 2\|(1-\pi)u\|_{C^0} \\
&= 2\|(\Delta + \pi)^{-1}\Delta((1-\pi)(u))\|_{C^0} \\
&\leq 2C\|\Delta u\|_{C^0}
\end{aligned}$$

From this, and using (5.34), we obtain an estimate which is independent from a and t.

$$\text{Osc}\left(\frac{\partial u_{a,t}}{\partial t}\right) \leq C|a|^2. \tag{5.35}$$

The interior Schauder estimates read in our case as follows:

$$\left\| \frac{\partial u_{a,t}}{\partial t} \right\|_{C^{2,\alpha}} \leq C \left(\left\| \frac{\partial u_{a,t}}{\partial t} \right\|_{C^0} + \left\| \Delta \frac{\partial u_{a,t}}{\partial t} \right\|_{C^{0,\alpha}} \right). \tag{5.36}$$

As functions whose oscillation decay at a fixed rate are Hölder continuous (please see [16], Theorem 8.22), we can use estimates (5.36), together with (5.21), (5.34) and (5.35) to obtain the following a priori $C^{2,\alpha}$ estimate:

$$\left\| \frac{\partial u_{a,t}}{\partial t} \right\|_{C^{2,\alpha}} \leq C|a|^2, \tag{5.37}$$

which is independent of a and t, on any relatively compact subdomain in the complement of the 16 exceptional divisors.

Integrating we get the a priori $C^{2,\alpha}$ estimate

$$\|u_a\|_{C^{2,\alpha}} \leq C|a|^2. \tag{5.38}$$

By using bootstrapping argument, we also have $C^{k,\alpha}$ estimates:

$$\|u_a\|_{C^{k,\alpha}} \leq C_k|a|^2. \tag{5.39}$$

In a neighborhood of the exceptional divisor, the curvature of the manifold blows up. For this reason, we cannot apply elliptic regularity results directly and therefore we do not have the estimate (5.37). However, it is still possible to obtain weaker bounds, as follows.

For $\alpha > 0$, consider the map

$$\begin{aligned} \phi_\alpha : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ z &\longmapsto \sqrt{\alpha} z \end{aligned}$$

Which induces a new lattice $\Lambda_\alpha = \phi_\alpha(\Lambda)$, giving rise to a $K3$ surface S_α, which is the resolution via blow-up of $(\mathbb{C}^2/\Lambda_\alpha)/\langle\sigma\rangle$.

Similarly as we did before, on each singularity of $(\mathbb{C}^2/\Lambda_\alpha)/\langle\sigma\rangle$, we can glue a scaled Eguchi-Hanson space. To do this we must consider the parameter changes

$$\begin{aligned} a &\longmapsto \alpha a \\ u &\longmapsto \alpha u. \end{aligned}$$

As $u(z) = |z|^2$, we observe that

$$u(\sqrt{\alpha}z) = |\sqrt{\alpha}z|^2 = \alpha|z|^2 = \alpha u(z) \tag{5.40}$$

Observation 83. The Eguchi-Hanson Kähler potential (4.5) has the following behavior under multiplication by a parameter $\alpha \in \mathbb{R}$.

$$\begin{aligned} f_{EH,\alpha a}(\alpha u) &= \sqrt{\alpha^2 u^2 + \alpha^2 a^2} + \alpha a \log(\alpha u) - \alpha a \log(\sqrt{\alpha^2 u^2 + \alpha^2 a^2} + \alpha a) \\ &= \alpha \left(\sqrt{u^2 + a^2} + a \log(\alpha) + a \log(u) - a \log(\alpha) - a \log(\sqrt{u^2 + a^2} + a) \right) \\ &= \alpha\ f_{EH,a}(u). \end{aligned}$$

This translates to the metric scaling as

$$\omega_{\alpha a} = i\partial\bar{\partial} f_{\alpha a}(\alpha u) = i\partial\bar{\partial}(\alpha f_a(u)) = \alpha\omega_a.$$

The volumes of the metrics are related by

$$\mathrm{Vol}_{g_{\alpha a}} = \frac{1}{2}\omega_{\alpha a}^2(\alpha u) = \frac{1}{2}\alpha^2\omega_a^2(u) = \alpha^2 \mathrm{Vol}_{g_a}.$$

We scale the holomorphic 2-form on $\mathbb{C}^2$ as $\eta_\alpha = \alpha\sqrt{2}\mathrm{d}z_1 \wedge \mathrm{d}z_2$. Then we have

$$\exp(G_{\alpha a}) = \frac{\eta_\alpha \wedge \bar{\eta}_\alpha}{\omega_{\alpha a}^2} = \frac{\alpha^2 \eta \wedge \bar{\eta}}{\alpha^2 \omega_a^2} = \exp(G_a).$$

For the Monge-Ampère equation, we have

$$\begin{aligned} (\omega_{\alpha a} + i\partial\bar{\partial}u_{\alpha a})^2(\alpha u) &= \exp(G_{\alpha a})(\omega_{\alpha a})^2(\alpha u) \\ &= \exp(G_a)\alpha^2\omega_a(u) = \alpha^2(\omega_a + i\partial\bar{\partial}u_a)^2(u). \end{aligned}$$

By uniqueness of solution, we conclude that

$$u_{\alpha a}(\alpha u) = \alpha^2 u_a(u).$$

Now we analyze the behavior of the following norms:

$$\|f\|^2_{L^2(\alpha g)} = \int |f|^2 \mathrm{dVol}_{\alpha g} = \alpha^2 \|f\|^2_{L^2(g)} \tag{5.41}$$

$$\|f\|^2_{L^4(\alpha g)} = \left(\left(\int |f|^4 \alpha^2 \mathrm{dVol}_g \right)^{1/4} \right)^2 = \alpha \|f\|^2_{L^4(g)}. \tag{5.42}$$

Proceeding as in Proposition 79, we take

$$\hat{\omega} := \frac{1}{|a|}\omega_a \quad , \quad \hat{u}_{a,t} := \frac{1}{|a|}u_{a,t}.$$

Then the estimate (5.34) translates to

$$\mathrm{Osc}\left(\frac{\partial \hat{u}_{a,t}}{\partial t}\right) \leq C|a|.$$

In this new rescaled picture, we are "away" from the exceptional divisor (in particular, Schauder estimates still apply). Therefore we can proceed similarly as before and get the bound

$$\left\| \Delta \frac{\partial}{\partial t} \hat{u}_{a,t} \right\|_{C^{0,\alpha}} \leq C|a|.$$

Then, once again we use the fact that the Laplacian is elliptic to obtain:

$$\left\| \frac{\partial}{\partial t} \hat{u}_{a,t} \right\|_{C^{2,\alpha}(K,\hat{\omega})} \leq C|a|,$$

which holds in a compact domain K of the Eguchi-Hanson part of the manifold, which remains fixed in normal coordinates around the exceptional divisor. Now we perform integration as before:

$$\|\hat{u}_a\|_{C^{2,\alpha}(K,\hat{\omega})} \leq C|a|$$

Applying the process iteratively, we have the following bound for all $k \in \mathbb{N}_0$:

$$\|u_a\|_{C^k} \leq C|a|^{2-\frac{k}{2}}.$$

5.4 Additional estimates: An alternative Kummer construction

In addition to the C^2-estimates already found, the proof of our main result requires to have a bound over functions on S, in terms of the Laplacian.

In this section we adapt results obtained by Biquard-Minerbe in [7] to gain the required estimate. This will be done by carefully defining convenient Banach spaces.

This approach is different from the one followed in the rest of the chapter, because it requires a modification of the Kummer construction considered before.

At the end, the goal is to unify this two pictures. This is done by observing that their difference is controlled. The precise argument will be explained by the end of the chapter.

An alternative Kummer construction

Here we introduce an alternative version of the Kummer construction.

Similarly as before, we start by defining a Kähler metric on $T^*\mathbb{C}P^1$, with coordinates $\tilde{z}$.

We first modify the cut-off function (4.9) to be

$$t_\varepsilon(\tilde{z}) := t(\sqrt{\varepsilon}|\tilde{z}|),$$

where ε is a small positive parameter. Now we define

$$\phi_\varepsilon(\tilde{z}) = t_\varepsilon(\tilde{z}) f_{EH,1}(\tilde{z}) + (1 - t_\varepsilon(\tilde{z})) f_{Euc}(\tilde{z}). \tag{5.43}$$

It follows that $\mathrm{dd}^c\phi_\varepsilon$ is a $(1,1)$-form on $T^*\mathbb{C}P^1$, which coincides with the Eguchi-Hanson Kähler form for $|\tilde{z}| \leq \frac{1-\delta}{\sqrt{\varepsilon}}$, and with the Euclidean form on $\mathbb{C}^2/\{\pm 1\}$ for $|\tilde{z}| \geq \frac{1}{\sqrt{\varepsilon}}$.

Now, consider the K3 surface S as before.

Take ρ to be the distance to the singular points in $T^4/\langle\sigma\rangle$, which can also be pulled back to the distance to the exceptional divisors on S.

Consider $V := \left\{z \in S \mid \rho(z) < \frac{1}{16}\right\} = \bigsqcup_{j=1}^{16} V_j$, where V_j are the 16 connected components. We identify each V_j with $\widetilde{V}_j(\varepsilon) := \{|\tilde{z}| \leq \frac{1}{16\varepsilon}\} \subset T^*\mathbb{C}P^1$ as follows.

Let us fix one j, so we are focusing around one component of the exceptional divisor, and choose local coordinates such that $\rho(z) = |z|$.

Then we take $z = \varepsilon\tilde{z}$, which implies that $|\tilde{z}| < \frac{1}{16\varepsilon} \Longleftrightarrow |z| < \frac{1}{16}$.

Using this identification, we split S in regions $\widetilde{V}_j(\varepsilon)$ (neighborhoods of the 16 exceptional divisors) and a remaining part $W := S \setminus \bigsqcup_{j=1}^{16} \widetilde{V}_j(\varepsilon)$.

Now, we define an approximate Ricci-flat Kähler metric on S as

$$\omega_\varepsilon = \begin{cases} \mathrm{dd}^c(\varepsilon^2\phi_\varepsilon) & ,\rho \leq (1-\delta)\sqrt{\varepsilon} \\ \omega_0 & ,\rho \geq \sqrt{\varepsilon} \end{cases} \tag{5.44}$$

where ω_0 denotes the flat Euclidean Kähler form.

Observation 84. If we make the parameter ε go to zero, the gluing region given by (5.43) in $T^*\mathbb{C}P^1$ goes away from the exceptional divisor. However, due to the coordinate rescaling and gluing described by (5.44), all this geometry gets smaller and smaller until it collapses to a point in S.

Now we construct the norm on the space of functions which will define the desired Banach spaces.

Start by choosing a smooth positive function $r_\varepsilon : S \longrightarrow \mathbb{R}$ with the following properties:

- $r_\varepsilon = \varepsilon$, for $\rho \leq \varepsilon$
- r_ε is non-decreasing on ρ in $\varepsilon \leq \rho \leq 2\varepsilon$
- $r_\varepsilon = \rho$, for $2\varepsilon \leq \rho$

The function r_ε shall act as a weight for the norm of our Banach spaces.

Consider k to be a nonnegative integer and α a number in $(0,1)$. For a positive real number β, we define the Banach space $C^{k,\alpha}_{\varepsilon,\beta}$ as the set of $C^{k,\alpha}$ functions u on S such that the following norm is finite:

$$\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}} := \sum_{i=0}^{k} \sup |r_\varepsilon^{i+\beta}\nabla_\varepsilon^i u|_\varepsilon + \sup_{\mathrm{d}_\varepsilon(x,y)<\mathrm{inj}_\varepsilon} \Big| \min\left(r_\varepsilon^{\beta+k+\alpha}(x), r_\varepsilon^{\beta+k+\alpha}(y)\right) \cdot \frac{\nabla_\varepsilon^k u(x) - \nabla_\varepsilon^k u(y)}{\mathrm{d}_\varepsilon(x,y)^\alpha} \Big|_\varepsilon .$$

The subscript ε means that everything is computed with respect to the metric g_ε, corresponding to ω_ε, and inj_ε denotes the injectivity radius of g_ε. The parameter ε only matters near the exceptional divisors.

Observe that a $C^{k,\alpha}_{\varepsilon,\beta}$ bound on u means it decays like $r_\varepsilon^{-\beta}$.

Since each neighborhood V_j is identified with the domain $\widetilde{V}_j(\varepsilon) := \{|\tilde{z}| \leq \frac{1}{16\varepsilon}\}$ in $T^*\mathbb{C}P^1$, we may pull back any function u on V_j to a function u_ε on $\widetilde{V}_j(\varepsilon)$, via the rescaling map $u_\varepsilon(z) \longrightarrow u(\varepsilon\tilde{z})$. Then, locally, a $C^0_{\varepsilon,\beta}$ bound on u means $|u_\varepsilon| \leq c\varepsilon^{-\beta}|\tilde{z}|^{-\beta}$, for $1 \leq |\tilde{z}| \leq \frac{1}{16\varepsilon}$; and $|u_\varepsilon| \leq c\varepsilon^{-\beta}$, where $|\tilde{z}| \leq 1$.

In this setting, we have the standard Schauder estimates (derived as in [16], Corollary 6.7)

$$\|u\|_{C^{k+2,\alpha}_{\varepsilon,\beta}} \leq c(k,\varepsilon)\left(\|u\|_{C^0_{\varepsilon,\beta}} + \|\Delta_\varepsilon u\|_{C^{k,\alpha}_{\varepsilon,\beta+2}}\right). \tag{5.45}$$

However, such estimates depend on the gluing parameter ε. What we want are estimates uniform in ε, resembling the form

$$\|u\| \leq c\|\Delta u\|,$$

for a constant c, independent of u.

We observe that, as it is written, the inequality cannot be attained since the nonzero constant functions already yield counterexamples. To fix this, we need to consider a suitable additional term to the right hand side of the inequality.

Finding such estimates will be the content of Proposition 86, but before it is necessary to state and prove a required norm equivalence.

The norm $\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}}$ can be decomposed in two parts, the part that takes all suprema over $S\backslash V$, $V := \bigcup_j V_j$ which we denote from now on as $\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(S\backslash V)}$ and in the parts that takes all suprema over V_j which we denote from now on as $\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(V_j)}$.

Clearly, we have

$$\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}} = \max\left\{\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(S\backslash V)}, \|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(V_1)}, \cdots, \|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(V_{16})}\right\}$$

Proposition 85 (Equivalence of Hölder norms under rescaling)**.** Fix one component $V_j \subset S$ and identify it with

$$\widetilde{V}_j(\varepsilon) = \left\{|\tilde{z}| \leq \tfrac{1}{16\varepsilon}\right\} \subset T^*\mathbb{C}P^1$$

via $z = \varepsilon\tilde{z}$. Let g_ε be the Riemannian metric associated to ω_ε, and define the rescaled metric $G_\varepsilon := \varepsilon^{-2} g_\varepsilon$. For u defined on V_j, set

$$U(\tilde{z}) := \varepsilon^\beta \, u(\varepsilon\tilde{z}).$$

Then for every $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\beta \in \mathbb{R}$, there exists a constant $C > 0$, independent of ε such that

$$C^{-1} \|U\|_{C^{k,\alpha}_\beta\left(\widetilde{V}_j(\varepsilon),\, g_{EH}\right)} \;\le\; \|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(V_j)} \;\le\; C\,\|U\|_{C^{k,\alpha}_\beta\left(\widetilde{V}_j(\varepsilon),\, g_{EH}\right)}. \tag{5.46}$$

Proof. We work in local coordinates, with $\rho(z) = |z|$ and $z = \varepsilon\tilde{z}$, so that $\rho(z) = \varepsilon|\tilde{z}|$.

Recall that $r_\varepsilon : S \to \mathbb{R}$ satisfies

$$r_\varepsilon = \begin{cases} \varepsilon & \rho \le \varepsilon, \\ \rho & \rho \ge 2\varepsilon, \end{cases}$$

with smooth monotone interpolation in between. By a suitable choice of these functions r_ε, we may even achieve

$$r_\varepsilon(z) = \varepsilon\tilde{r}(\tilde{z}) \tag{5.47}$$

for a function $\tilde{r}(\tilde{z})$ independent of ε and satisfying

$$\tilde{r}(\tilde{z}) = \begin{cases} 1 & |\tilde{z}| \le 1, \\ |\tilde{z}| & |\tilde{z}| \ge 2, \end{cases}$$

so that $\tilde{r} \simeq \max(1, |\tilde{z}|)$.

Now, we claim that there exists a constant $\Lambda > 0$, independent of ε, such that

$$\Lambda^{-1} \, g_{EH} \;\le\; G_\varepsilon \;\le\; \Lambda \, g_{EH} \tag{5.48}$$

on all of $\widetilde{V}_j(\varepsilon)$, and similarly for all covariant derivatives of the metrics up to order k.

To see this, we track the three regions determined by (5.44). On the inner region $\rho \le (1-\delta)\sqrt{\varepsilon}$, equivalently $|\tilde{z}| \le (1-\delta)/\sqrt{\varepsilon}$, the metric $\omega_\varepsilon = dd^c(\varepsilon^2\phi_\varepsilon)$ is exactly the Eguchi-Hanson potential rescaled, so $G_\varepsilon = \varepsilon^{-2} g_\varepsilon = g_{EH}$ exactly, and (5.48) holds with $\Lambda = 1$.

On the gluing region $(1-\delta)\sqrt{\varepsilon} \leq \rho \leq \sqrt{\varepsilon}$, i.e. $(1-\delta)/\sqrt{\varepsilon} \leq |\tilde{z}| \leq 1/\sqrt{\varepsilon}$ in the rescaled coordinates, the metric G_ε interpolates between g_{EH} and the flat metric. As $\varepsilon \to 0$, this region moves to infinity in the $\tilde{z}$-coordinates, so for any fixed compact subset $K \subset T^*\mathbb{C}P^1$ and all sufficiently small ε the gluing region does not intersect K, and $G_\varepsilon = g_{EH}$ on K.

On the outer region $\rho \geq \sqrt{\varepsilon}$, equivalently $|\tilde{z}| \geq 1/\sqrt{\varepsilon}$, the metric ω_ε is the flat metric ω_0, which after rescaling by ε^{-2} gives the flat metric on $\mathbb{C}^2$. This is precisely the asymptotic model of g_{EH} at infinity, and the two are uniformly equivalent there with constants depending only on the Eguchi-Hanson geometry.

Combining these three regions, the pointed Riemannian manifolds (V_j, G_ε) converge in to $(T^*\mathbb{C}P^1, g_{EH})$, which yields (5.48) with Λ independent of ε.

It remains to prove that

$$\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(V_j)} = \varepsilon^\beta \, \|U\|_{C^{k,\alpha}_\beta\left(\widetilde{V}_j(\varepsilon),\, G_\varepsilon\right)}, \tag{5.49}$$

where the right-hand side uses the metric G_ε throughout.

Since $G_\varepsilon = \varepsilon^{-2} g_\varepsilon$, distances and tensor norms scale as

$$d_{G_\varepsilon} = \varepsilon^{-1}\, d_{g_\varepsilon}, \qquad |T|_{G_\varepsilon} = \varepsilon^p \, |T|_{g_\varepsilon} \quad \text{for a } p\text{-tensor } T,$$

and the Levi-Civita connections of g_ε and G_ε coincide since they differ only by a constant conformal factor. By the chain rule, the ℓ-th covariant derivative satisfies

$$\nabla^\ell_{g_\varepsilon} u(z) = \varepsilon^{-\ell}\, \nabla^\ell_{G_\varepsilon} U(\tilde{z})$$

Derivative terms. For $0 \leq \ell \leq k$, using (5.47) and the tensor norm scaling $|T|_{g_\varepsilon} = \varepsilon^{-\ell} |T|_{G_\varepsilon}$ for an ℓ-tensor:

$$\begin{aligned} r_\varepsilon^{\ell+\beta}(z)\left|\nabla^\ell_{g_\varepsilon} u(z)\right|_{g_\varepsilon} &= \left(\varepsilon\, \tilde{r}(\tilde{z})\right)^{\ell+\beta} \cdot \varepsilon^{-\ell} \left|\nabla^\ell_{G_\varepsilon} U(\tilde{z})\right|_{G_\varepsilon} \\ &= \varepsilon^\beta\, \tilde{r}^{\ell+\beta}(\tilde{z}) \left|\nabla^\ell_{G_\varepsilon} U(\tilde{z})\right|_{G_\varepsilon}, \end{aligned}$$

Taking the supremum over V_j ($\widetilde{V}_j(\varepsilon)$) we get:

$$\sup_{V_j} r_\varepsilon^{\ell+\beta} \left|\nabla^\ell_{g_\varepsilon} u\right|_{g_\varepsilon} = \varepsilon^\beta \sup_{\widetilde{V}_j(\varepsilon)} \tilde{r}^{\ell+\beta} \left|\nabla^\ell_{G_\varepsilon} U\right|_{G_\varepsilon}.$$

Hölder seminorm. Using the same scaling relations:

$$\min\big(r_\varepsilon^{k+\alpha+\beta}(z_1),\, r_\varepsilon^{k+\alpha+\beta}(z_2)\big)\frac{\big|\nabla^k_{g_\varepsilon} u(z_1)-\nabla^k_{g_\varepsilon} u(z_2)\big|_{g_\varepsilon}}{d_{g_\varepsilon}(z_1,z_2)^\alpha}$$
$$=\varepsilon^{k+\alpha+\beta}\min\big(\tilde r^{k+\alpha+\beta}(\tilde z_1),\,\tilde r^{k+\alpha+\beta}(\tilde z_2)\big)\cdot\frac{\varepsilon^{-k}\big|\nabla^k_{G_\varepsilon}U(\tilde z_1)-\nabla^k_{G_\varepsilon}U(\tilde z_2)\big|_{G_\varepsilon}}{\varepsilon^\alpha\, d_{G_\varepsilon}(\tilde z_1,\tilde z_2)^\alpha}$$
$$=\varepsilon^{\beta}\,\min\big(\tilde r^{k+\alpha+\beta}(\tilde z_1),\,\tilde r^{k+\alpha+\beta}(\tilde z_2)\big)\frac{\big|\nabla^k_{G_\varepsilon}U(\tilde z_1)-\nabla^k_{G_\varepsilon}U(\tilde z_2)\big|_{G_\varepsilon}}{d_{G_\varepsilon}(\tilde z_1,\tilde z_2)^\alpha},$$

Taking the supremum yields the same factor ε^β in the Hölder seminorm. Summing all terms establishes (5.49).

From (5.49) and the uniform metric equivalence (5.48), which implies that the norms defined with respect to G_ε and g_{EH} are uniformly equivalent with constants independent of ε, we obtain

$$\|u\|_{C^{k,\alpha}_{\varepsilon,\beta}(V_j)}=\varepsilon^\beta\,\|U\|_{C^{k,\alpha}_\beta(\widetilde V_j(\varepsilon),\,G_\varepsilon)}\;\simeq\;\varepsilon^\beta\,\|U\|_{C^{k,\alpha}_\beta(\widetilde V_j(\varepsilon),\,g_{EH})},$$

which is precisely (5.46), with constant C depending only on k, α, β, and Λ from (5.48), but not on ε. □

Proposition 86 (Adaptation of Lemma 1.2 [7])**.** Let β be a positive number, such that $\beta<2$. Then there exists a constant c such that for every small ε and every $u\in C^{2,\alpha}_{\varepsilon,\beta}$, the following inequality holds:

$$\|u\|_{C^{2,\alpha}_{\varepsilon,\beta}}\le c\|\Delta_\varepsilon u\|_{C^{0,\alpha}_{\varepsilon,\beta+2}}+\|u\|_{C^0}.$$

Proof. We proceed by contradiction, assuming the statement is false. Then there exist a sequence of positive numbers $\varepsilon_i\longrightarrow 0$ and a sequence of functions u_i such that $\|u_i\|_{C^{2,\alpha}_{\varepsilon_i,\beta}}=1$, but $\|\Delta_{\varepsilon_i}u_i\|_{C^{0,\alpha}_{\varepsilon_i,\beta+2}}+\|u_i\|_{C^0}\longrightarrow 0$.

In the following, we will write $\|u_i\|_{C^{2,\alpha}_{\varepsilon_i,\beta}(K)}$ for the $C^{2,\alpha}_{\varepsilon_i,\beta}(K)$-norm of a function u evaluated only over the subset $K\subset S$.

We will now show for an arbitrary compact subset K of the complement of the exceptional divisor that $\|u_i\|_{C^{2,\alpha}_{\varepsilon_i,\beta}(K)}\longrightarrow 0$.

Let $A_l, l\in\mathbb{N}$, A_l contained in the interior of A_{l+1}, be a sequence of compact sets, exhausting the regular part of $T^4/\langle\sigma\rangle$. Since $(u_i)_{i\in\mathbb{N}}$ is a bounded sequence in $C^{2,\alpha}_{\varepsilon_i,\beta}$, we obtain that $(u_i|_{A_l})_{i\in\mathbb{N}}$ is bounded in $C^{2,\alpha}(A_l)$ for any l.

After passing to a subsequence, $(u_i|_{A_l})_{i\in\mathbb{N}}$ converges in $C^{2,\alpha}(A_l)$ to some limit $u_{\infty,l}$. By a diagonal sequence argument we may choose this subsequence in such a way that we have $u_{\infty,l}|A_m = u_{\infty,m}$, for $m < l$. The functions $u_{\infty,l}$ thus fit together to a function u_∞, which is a well-defined harmonic $C^{2,\alpha}$-function on the regular part of $T^4/\langle\sigma\rangle$.

As $(u_i)_{i\in\mathbb{N}}$ has norm 1 in $C^{2,\alpha}_{\varepsilon_i,\beta}$, it follows from the $C^{2,\alpha}$-convergence on A_l that u_∞ is bounded by $\rho^{-\beta}$ near each singular point.

Moreover, we see that the function $\rho^{-\beta}$ is in L^2_{loc}, given the fact that $\beta < 2$. Therefore, taking $n = k = 4$, we can apply Theorem 148 to get that u_∞ can be lifted into a smooth harmonic function $\tilde{u}_\infty$ on the whole T^4.

Since T^4 is compact, $\tilde{u}_\infty$ is constant, and constant functions are always in the kernel of the Laplacian. Additionally, we have by assumption that $\|u_i\|_{C^0} \longrightarrow 0$, and therefore $\tilde{u}_\infty$ must vanish.

Then we have that $\|u_\infty\|_{C^0(K)} = 0$ on every compact set K outside of the exceptional divisor.

Now we additionally assume that $W := S \setminus \bigcup_{j=1}^{16} V_j \subset A_1 \subset A_l$, where V_j are defined on Page 69. Furthermore, we write S_{ne} for the non-exceptional part of S, i.e. the region where t_ε is not constant is the region where we have $(1-\delta)\sqrt{\varepsilon} \leq |z| \leq \sqrt{\varepsilon}$. Then, for any $l \in \mathbb{N}$ there is an $\eta_l > 0$ such that for any $\varepsilon \in (0, \eta_l)$ the function t_ε is constant 1 on A_l. In particular, the Riemannian metric in A_l is the standard flat one for $\varepsilon \in (0, \eta_l)$.

Thus, the standard Schauder estimates (5.45) give us for $\varepsilon \in (0, \eta_{l+1})$

$$\begin{aligned} \|u_i\|_{C^{2,\alpha}_{\varepsilon_i,\beta}(A_l)} &\leq c_l \left(\|u_i\|_{C^0(A_{l+1})} + \|u_i\|_{C^{0,\alpha}_{\varepsilon_i,\beta}(A_{l+1})} \right) \\ &\leq c_l \left(\|u_i\|_{C^0(S_{ne})} + \|u_i\|_{C^{0,\alpha}_{\varepsilon_i,\beta}(S_{ne})} \right), \end{aligned}$$

for a constant c_l independent of ε, but possibly depending on l. This implies for any $l \in \mathbb{N}$ that

$$\lim_{i\to\infty} \|u_i\|_{C^{2,\alpha}_{\varepsilon_i,\beta}(A_l)} = 0.$$

As every compact subset K of the complement of the exceptional divisors is contained in A_l, for l sufficiently large, the claimed statement follows.

As a second step of the proof, we investigate what happens around the j-th exceptional divisor, where a rescaling argument is necessary.

Recall that at V_j we can pull back any function $u(z)$ to a function $u_\varepsilon(\tilde{z}) := u(\varepsilon z)$, on $\widetilde{V}_j(\varepsilon)$.

We consider the rescaled pulled back functions $U_i(\tilde{z}) := \varepsilon_i^{\beta}(u_i)_{\varepsilon_i}(\varepsilon\tilde{z})$. Note that U_i is again defined on $\widetilde{V}_j(\varepsilon_j) \subset T^*\mathbb{C}P^1$.

Then, we can apply Propositon 85 to get the norm equivalence

$$\|u_i\|_{C^{k,\alpha}_{\varepsilon_i,\beta}(V_j)} \cong \|U_i\|_{C^{k,\alpha}_{\beta}(V_j)}$$

for some constant independent of i, and for all natural numbers k. Taking $k = 0, 2$, since $\varepsilon_i \longrightarrow 0$ we have $\|U_i\|_{C^{2,\alpha}_{\beta}} \longrightarrow 0$ which implies $\|u_i\|_{C^{2,\alpha}_{\varepsilon_i,\beta}(V_j)} \longrightarrow 0$.

Using the same argument around each component of the exceptional divisor we obtain a contradiction. □

5.5 Unifying the metrics ω_a and ω_ε

In principle, rescaling a Kähler metric changes the Kähler class. However, as we have seen in Theorem 74, near one blown up point of S there is only one Kähler class (up to a constant) and therefore the metric induced by (5.43) and (4.10) are in the same class.

Moreover, we have the following result.

Proposition 87. Take $a = \varepsilon^2$. Then the paths of Ricci-flat metrics $\widetilde{\omega}_a$ and $\widetilde{\omega}_\varepsilon$, associated with ω_a and ω_ε, respectively, are the same.

Proof. We will show that the metrics ω_a and ω_ε define the same Kähler class on the $K3$ surface S. To do this, we consider its difference around one component of the exceptional divisor, at the level of Kähler potentials.

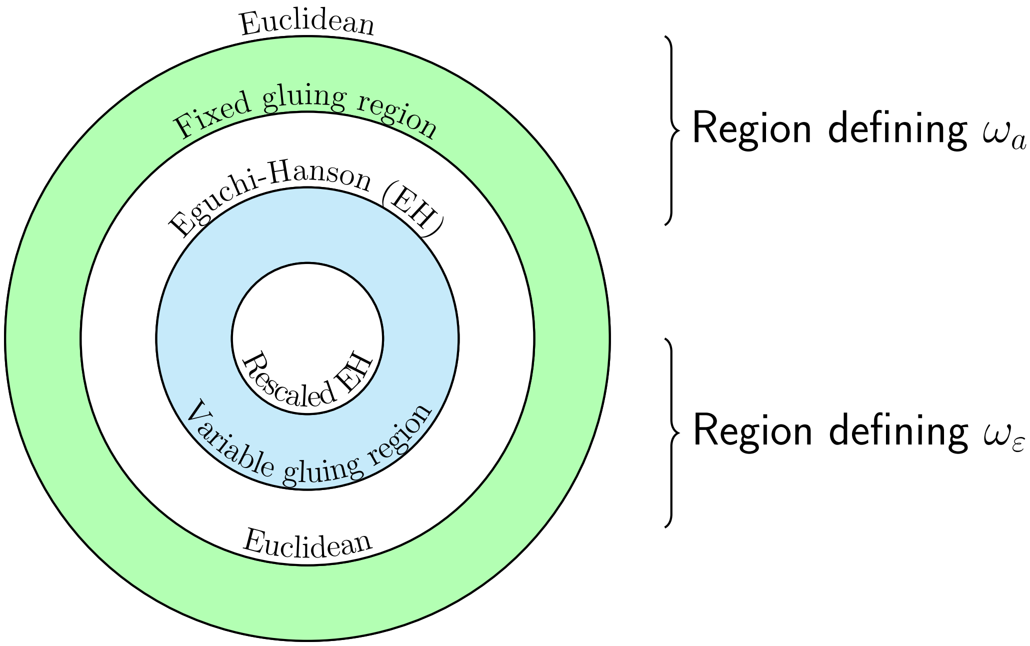
First observe that the gluing region considered to define f_a is fixed as

$$N = \{z \in (\mathbb{C}^2 - \{0\})/\{\pm 1\} \mid 1 \leq |z|^2 \leq 1 + \delta\},$$

whereas the one used to define ω_ε,

$$N_\varepsilon = \{z \in (\mathbb{C}^2 - \{0\})/\{\pm 1\} \mid (1 - \delta)\sqrt{\varepsilon} \leq |z| \leq \sqrt{\varepsilon}\},$$

varies with each value of ε. In particular, by Observation 84, if we make ε small enough, we can arrange that N_ε is completely contained in the "Eguchi-Hanson part" of the first cut-off, as schematically shown in the following picture:



Now, we need to consider the difference $f_a(z) - \varepsilon^2\phi_\varepsilon(\tilde{z})$. In order to make this comparison, we need to unify the coordinates z of S and $\tilde{z}$ of $T^*\mathbb{C}P^1$, as well as the parameters a and ε. We proceed with the coordinates first.

Notice that, because of the rescaling behavior of the Eguchi-Hanson potential exhibited in Observation 83, we have

$$\begin{aligned}\varepsilon^2\phi_\varepsilon(\tilde{z}) &= \varepsilon^2\left(t_\varepsilon(\tilde{z})f_1(\tilde{z}) + (1 - t_\varepsilon(\tilde{z}))f_{Euc}(\tilde{z})\right)\\ &= t_\varepsilon(\tilde{z})f_{\varepsilon^2}(\varepsilon\tilde{z}) + (1 - t_\varepsilon(\tilde{z}))|\varepsilon\tilde{z}|^2\\ &= t\left(\frac{z}{\sqrt{\varepsilon}}\right)f_{\varepsilon^2}(z) + \left(1 - t\left(\frac{z}{\sqrt{\varepsilon}}\right)\right)|z|^2.\end{aligned}$$

Now, we take $a = \varepsilon^2$. Because we have made N_ε small enough, we have

$$
\begin{aligned}
f_{\varepsilon^2}(z) - \varepsilon^2 \phi_\varepsilon(\tilde{z}) &= f_{\varepsilon^2}(z)\left(1 - t\left(\frac{z}{\sqrt{\varepsilon}}\right)\right) + \left(1 - t\left(\frac{z}{\sqrt{\varepsilon}}\right)\right)|z|^2 \\
&= \left(1 - t\left(\frac{z}{\sqrt{\varepsilon}}\right)\right)\left(f_{\varepsilon^2}(z) - |z|^2\right).
\end{aligned}
$$

The first factor of this product is a number in $(0, 1)$, and from the proof of Lemma 76, we know that the second factor is bounded by $\varepsilon^4 \cdot h_{\varepsilon^2}(z)$, where $h_{\varepsilon^2}(z)$ and its derivatives are smooth regular functions.

Then, by shrinking the parameter ε, we can make this difference as small as we want.

As a consequence, the paths of Ricci-flat metrics $\widetilde{\omega}_a$ and $\widetilde{\omega}_\varepsilon$, obtained by varying the parameters ε and a, are the same. □

Chapter 6

The Moduli space of structured Riemannian metrics

We start this chapter by introducing formally the main object of study in the present work: the moduli space of structured Riemannian metrics $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$, for a manifold M.

After presenting some definitions, we enunciate Proposition 92, which states that the aformetioned moduli space is in fact a finite dimensional Riemannnian manifold.

As a consequence, we are able to define the L^2-metric on $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$, and give an explicit description of the tangent space.

Having covered this material, we will be in good conditions to present the main result of the thesis: Theorem 95, which states that if S is the Kummer $K3$ surface we constructed before, then $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$, together with the L^2-metric, is not a complete manifold. The proof will be carried by the end of the chapter.

6.1 The moduli space $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$

Let us start by considering an n-dimensional smooth manifold M. We write S^2T^*M for the second symmetric power of its cotangent bundle, and $C^\infty(M, S^2T^*M)$ for the real vector space of smooth symmetric $(0,2)$-tensor fields on M.

Definition 88. The space $\mathcal{R}(M)$ of all complete Riemannian metrics on M is the subspace of $C^\infty(M, S^2T^*M)$ consisting of all sections which are complete Riemannian metrics on M, equipped with the smooth topology of uniform convergence on compact subsets.

Let $\mathrm{Diff}(M)$ be the Lie group of self diffeomorphisms of M and $\mathrm{Diff}_0(M)$ be the connected component of the identity. This group acts on $\mathcal{R}(M)$ by pulling back metrics:

$$\begin{aligned} \mathrm{Diff}_0(M) \times \mathcal{R}(M) &\longrightarrow \mathcal{R}(M) \\ (\psi, g) &\longmapsto \psi^*(g). \end{aligned}$$

Definition 89. The moduli space $\mathcal{M}(M)$ of complete Riemannian metrics on M is the quotient of $\mathcal{R}(M)$ by the above action of $\mathrm{Diff}_0(M)$. An element in $\mathcal{M}(M)$ is called a Riemannian structure.

Definition 90. Let M be a compact, connected, oriented manifold without boundary, and let $\pi : \widetilde{M} \longrightarrow M$ be its universal covering. We assume that $\widetilde{M}$ is spin and consider the pullback metric $\tilde{g} := \pi^* g$ on $\widetilde{M}$. Then we define the moduli space of structured Riemannian metrics as

$$\begin{aligned} \mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M) = \{g \in \mathcal{M}(M) \,|\, &(\widetilde{M}, \tilde{g}) \text{ carries a nonzero parallel spinor,} \\ &\text{and } \mathrm{Vol}(M, g) = 1\}. \end{aligned} \tag{6.1}$$

As we saw in Chapter 3, there is a strong relationship between holonomy and parallel spinors.

Additionally, we have the next property:

Proposition 91. Let $[g] \in \mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$. Then g is Ricci flat.

Therefore, Theorem 56 yields the following structural result about the moduli space.

Proposition 92 ([2], Corollary 4)**.** Assume that M is a compact spin manifold. Then $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ is a smooth manifold of finite dimension. Furthermore, $\mathcal{M}_{\|}(M)$ is a smooth infinite dimensional submanifold in the space of all metrics on M.

For the moment being, let us assume this as a fact and equip the manifold $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ with a suitable metric.

6.1.1 The L^2-metric on $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$

A Riemannian metric g on M induces a Riemannian metric on any bundle obtained from TM by taking tensor products, dual, symmetrization, etc. In particular, there is an induced Riemannian metric on S^2T^*M. To see this, first we notice that the scalar product induced by the Riemannian metric g on the cotangent space T_x^*M is given in local coordinates by

$$g(x)(\alpha,\beta) = g^{ij}\alpha_i\beta_j.$$

On the tensor product $T_x^*M \otimes T_x^*M$, the scalar product on elements of the form $\alpha \otimes \beta$ reads

$$g(x)(\alpha\otimes\beta,\gamma\otimes\delta) = g(x)(\alpha,\gamma)\cdot g(x)(\beta,\delta) = g^{ij}\alpha_i\gamma_j g^{kl}\beta_k\delta_l.$$

By bilinearity, it can be extended to arbitrary elements $h,k \in T_x^*M\otimes T_x^*M$ by

$$g(x)(h,k) = g^{ij}h_{il}g^{lm}k_{jm}.$$

Let us restrict to symmetric tensors $h,k \in S^2T_x^*M$ and let H and K be the $(1,1)$-tensors obtained from h and k, respectively, by rising an index with g. Then we define the g-trace of hk as

$$g(x)(h,k) = g^{ij}h_{il}g^{lm}k_{jm} = g^{ij}h_{il}g^{lm}k_{mj} = H_l^jK_j^l = \mathrm{tr}(HK) =: \mathrm{tr}_g(hk).$$

Similarly, we define the g-trace of $h \in S^2T_x^*M$ as

$$\mathrm{tr}_g h := \mathrm{tr}H = \mathrm{tr}(g^{-1}h) = g^{ij}h_{ji}.$$

It is also possible to define the g-trace of sections of the bundle $S^2T_x^*M$ by simply taking the g-trace at each point.

Moreover, we can use what we have described to define a metric on $\mathcal{M}(M)$ by integrating.

Definition 93. Let (M,g) be a Riemannian manifold. The L^2-metric on the moduli space $\mathcal{M}(M)$ is defined to be

$$(h,k)_g := \int_M \mathrm{tr}_g(hk)\mathrm{dVol}(M,g). \tag{6.2}$$

6.1.2 The tangent space of $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$

The tangent space of $\mathcal{M}(M)$ at g is

$$T_g(\mathcal{M}(M)) = \left\{ h \in C^\infty(M, S^2T^*M) : \int \mathrm{tr}_g h \, \mathrm{dVol}(M,g) = 0 \ , \ \mathrm{Vol}(M,g) = 1 \right\}.$$

Then, the tangent space of $\mathcal{M}(M)/\mathrm{Diff}_0(M)$ at g is:

$$\begin{aligned} T_g(\mathcal{M}(M)/\mathrm{Diff}_0(M)) &\cong T_g(\mathcal{M}(M))/T_g(\mathrm{Diff}_0(M)) \\ &\cong \{X \in T_g(\mathcal{M}(M)) : X \perp T_g(\mathrm{Diff}_0(M))\} . \end{aligned}$$

For a vector field X in M, let ϕ_t be its one-parameter subgroup. Then $\phi_t(m)$ is the integral curve of X starting at $m \in M$, which implies

$$\frac{d}{dt}\phi_t(m)|_{t=0} = X(m).$$

Conversely, given a path ϕ_t in the diffeomorphism group with $\phi_0 = \mathrm{Id}$, we have for every $m \in M$

$$\frac{d}{dt}\phi_t(m)|_{t=0} \in T_m M.$$

Thus, the tangent space to the diffeomorphism group at the identity may be viewed as the Lie algebra of vector fields on M.

It is natural to ask: what are the symmetric $(0,2)$-tensor fields that are tangent to the orbit of $g \in \mathcal{R}(M)$ under the action of Diff_0? Let X be one of such tensor fields and ϕ_t be its one-parameter subgroup, then

$$\frac{d}{dt}\phi_t^* g = \mathcal{L}_X g,$$

which is the Lie derivative of g in the direction X. Therefore the tangent space to the orbit of g consists of tensor fields of the form $\mathcal{L}_X g$.

If ω is the covariant form of X and h is another tensor field, we have

$$(\mathcal{L}_X g, h)_g = -2\int_M (\nabla_i S^{ij}) X_j \mathrm{dVol}(M,g) = 2(\omega, \mathrm{div} h),$$

where $\mathrm{div} h$ denotes the divergence of h.

We then have that a symmetric tensor field $h \in T_g\mathcal{R}(M)$ is orthogonal to the orbit at g under $\mathrm{Diff}_0(M)$ if and only if $\mathrm{div}h = 0$, thus

$$T_g(\mathcal{M}(M)/\mathrm{Diff}_0(M)) \cong \left\{ h \in C^\infty(M, S^2T^*M) : \int \mathrm{tr}_g h \, \mathrm{dVol}(M,g) = 0, \mathrm{div}h = 0 \right\}.$$

The additional condition of admiting a parallel spinor on the universal covering implies that the metric considered is Ricci-flat.

It is known that Ricci-flat metrics correspond to stationary points of the Einstein-Hilbert functional (see for example [6] p.4)

$$\begin{aligned} \varepsilon_M : \mathcal{M}(M) &\longrightarrow \mathbb{R} \\ g &\longmapsto \int_M \mathrm{scal}^g \, \mathrm{dvol}(M,g). \end{aligned}$$

Therefore we have

$$\begin{aligned} &T_g(\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)) = \\ &\ker(\mathrm{Hess}_g \varepsilon_M) \cap \left\{ h \in C^\infty(M, S^2T^*M) : \int \mathrm{tr}_g h = 0, \mathrm{div}h = 0 \right\}. \end{aligned}$$

The scalar product given by the L^2-metric (6.2) descends to the quotient

$$T_g(\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)).$$

Observation 94. Let us notice that given a path in $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ parametrising a family of equivalence classes of metrics $\{[g_L]\}$, we may have different paths in $\mathcal{M}_{\|}(M)$ parametrising the family of metrics $\{g_L\}$, which not necessarily have the same length. Therefore, in order to compute uniquely the length of a curve in the quotient space, we choose the path of smallest length parametrising $\{g_L\}$. This path is the one consisting of sections whose derivative is divergence free.

Then, $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$ together with the distance function induced by the L^2-metric is a Riemannian manifold.

Now we are in conditions to present the main result of the present work:

Theorem 95. Let S be the Kummer $K3$ surface constructed in Example 71. Then the moduli space of structured Riemannian metrics $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$, together with its L^2-metric, is not complete.

Theorem 95 was anticipated by the experts (see for example Wang's paper [42]). However, as far as we know, it has not been proven explicitly before.

Our method of choice for proving it consists in exhibiting an example of a non-convergent Cauchy sequence in the moduli space.

6.1.3 The Lipschitz map

As we have now seen, the main difficulty when it comes to study the moduli space of Ricci-flat metrics on a compact manifold, comes from the fact that it consists of metrics which do not admit an explicit algebraic description.

However, we can still grasp some information about the behavior of such metrics, in the following way.

Consider a path ω_t of approximate Ricci-flat Kähler metrics on a manifold M. Then, by the solution of the Calabi conjecture, we have a map

$$\begin{aligned} \{\text{Kähler classes}\} &\longrightarrow \{\text{Solution of the Monge-Ampère equation in this class}\} \\ \omega_t &\longmapsto \omega_t + \mathrm{dd}^c\phi_t =: \widetilde{\omega}_t, \end{aligned} \tag{6.3}$$

which gives us a path $[\widetilde{g}_t]$ in $\mathcal{M}_{\|}(M)/\mathrm{Diff}_0(M)$, consisting of classes of Ricci-flat metrics associated with $\widetilde{\omega}_t$.

We need to get information about $\widetilde{\omega}_t$, in terms of what we know about ω_t. In order to do this, it is necessary to have some notion of regularity on the map (6.3). Consider the following definition.

Definition 96. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \longrightarrow Y$ is said to be Lipschitz continuous if for any $a \neq b$ in X, there exist a constant $k \geq 0$ such that

$$\sup_{\substack{a,b\in X \\ a\neq b}} \left(\frac{d_Y(f(a), f(b))}{d_X(a, b)} \right) \leq k$$

One calls k a Lipschitz constant of f.

The importance of Lipschitz maps in the present context comes from the next result.

Lemma 97. Let $f : X \longrightarrow Y$ be a Lipschitz function. If (x_n) is a Cauchy sequence on X, then $f(x_n)$ is a Cauchy sequence on Y.

Proof. Take k to be the Lipschitz constant of f and consider $\epsilon > 0$. As (x_n) is Cauchy, there exists an $N \in \mathbb{N}$ such that for $n, m > N$, $d_X(x_n, x_m) < \frac{\epsilon}{k}$. Then for every $n, m > N$ we have

$$d_Y(f(x_n), f(x_m)) \leq k\, d_X(x_n, x_m) < k \cdot \frac{\epsilon}{k} = \epsilon,$$

and therefore $f(x_n)$ is a Cauchy sequence on Y. □

Recall that, for a tuple $a := \{a_i\}_{i=1}^{16}$, we have constructed the approximate Kähler metric ω_a on S. By the Calabi–Yau theorem, there exists a Ricci-flat Kähler metric $\widetilde{\omega}_a := \omega_a + i\partial\bar{\partial}u_a$ in the same Kähler class, for each a consisting of sufficiently small a_i's.

Now consider a real positive parameter t and take each $a_i = t$. This gives us a family of approximately Ricci-flat metrics g_t on S, with associated Kähler forms ω_t, and corresponding Ricci-flat metrics $\widetilde{g}_t$ induced by $\widetilde{\omega}_t = \omega_t + \mathrm{dd}^c u_t$.

To show that the path $[\widetilde{g}_t]$ in the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ contains a non-convergent Cauchy sequence, we apply Lemma 97, which requires the map

$$g_t \mapsto \omega_t \mapsto \widetilde{\omega}_t \mapsto \widetilde{g}_t \tag{6.4}$$

to be Lipschitz with respect to the L^2-metric on the moduli space.

To verify this, we consider the intermediate path of Ricci-flat Kähler forms $\widetilde{\omega}_t$, and observe that Lipschitz continuity of the map (6.4) follows from bounding the derivative

$$\|\widetilde{\omega}_t'\|_{L^2(\widetilde{g}_t)},$$

where the subscript indicates the L^2-norm with respect to the Ricci-flat metric $\widetilde{g}_t$.

This is a challenging task, because the metrics $\widetilde{g}_t$ are not known explicitly. Thus, we should first find a way to estimate $\|\cdot\|_{L^2(\widetilde{g}_t)}$, and then derive uniform bounds for $\|\widetilde{g}_t'\|_{L^2(\widetilde{g}_t)}$.

This will be the content of the rest of the chapter. The core technical ingredient is an elliptic estimate for the derivatives u_t', derived in Proposition 86, together with bounds for the inverses of $\widetilde{\omega}_t$ stated in Lemma 102.

6.2 Incompleteness of $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$

In this section, we will prove that the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ is not complete, by exhibiting a non-convergent Cauchy sequence. We start gathering some results, which are a consequence of the norm bounds we computed in Chapter 5.

Proposition 98. By taking $t_n = \frac{1}{n}$, the sequence of approximately Ricci-flat metrics $(g_{t_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{M}_{\|}(S)$.

Proof. Let us recall that the L^2-metric on $\mathcal{M}_{\|}(S)$ is defined as:

$$(h,k)_g := \int_S \mathrm{tr}_g\,(hk)\;\mathrm{dVol}(M,g).$$

Now, we fix $m, n \in \mathbb{N}$, take g_{t_m}, g_{t_n}, where $t_m := \frac{1}{m}$, $t_n := \frac{1}{n}$, and assume $n > m$ so that $t_n < t_m$.

By construction, each g_{t_n} coincides with a flat metric away from the neck regions N_i, and near the exceptional divisors it coincides with the Eguchi–Hanson metric. Then the difference $g_{t_n} - g_{t_m}$ is non-zero only at $N := \bigcup_i N_i$. On this region, we have by Equation (5.10):

$$\|g_{t_n} - g_{t_m}\|_{C^k(N, g_{\mathrm{Eucl}})} \leq C_k(t_n^2 + t_m^2),$$

for all $k \in \mathbb{N}$, with a constant C_k independent of n and m.

As a consequence, the following holds:

$$\mathrm{tr}_{g_{t_n}}(g_{t_n} - g_{t_m})^2 \leq C'(t_n^2 + t_m^2)^2.$$

And therefore,

$$(g_{t_n} - g_{t_m})_{g_{t_n}} = \int_S \mathrm{tr}_{g_{t_n}}(g_{t_n} - g_{t_m})^2\,\mathrm{dVol}_{g_{t_n}} \leq C''(t_n^2 + t_m^2)^2 = C''\left(\frac{1}{n^2} + \frac{1}{m^2}\right)^2.$$

Then, for any $\delta > 0$, we may choose $P \in \mathbb{N}$ large enough so that for all $n, m > P$, the L^2-distance between g_{t_n} and g_{t_m} is smaller than δ. This proves that (g_{t_n}) is a Cauchy sequence in $\mathcal{M}_{\|}(S)$ with respect to the L^2-metric. □

Proposition 99. Let S be the $K3$-surface, viewed as smooth manifold without fixing its complex structure. The moduli space

$$\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S) \tag{6.5}$$

contains a non convergent Cauchy sequence.

Proof. Recall that for a tuple $a := \{a_i\}_{i=1}^{16}$ we have constructed the Kähler metric ω_a on S.

It coincides with an Eguchi Hanson metric near each exceptional divisor, with a flat metric in a neighborhood of its complement, and it is an interpolation of both at the gluing or neck regions N_i.

By the Calabi-Yau Theorem, we know that there exist a Ricci-flat Kähler metric $\widetilde{\omega}_a := \omega_a + i\partial\bar{\partial}u_a$, for each a consisting of sufficiently small a_i' s.

Consider a real positive parameter t and take each $a_i = t$.

Then we have a family of metrics g_t coming from ω_t, and a corresponding family of classes of Ricci-flat metrics $[\widetilde{g}_t]$ in $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$.

Now, suppose that there exists a sequence $([\widetilde{g}_{t_n}])_{n\in\mathbb{N}}$ in the path $[\widetilde{g}_t]$ which does converge in the moduli space $([\widetilde{g}_{t_n}])_{n\in\mathbb{N}} \longrightarrow [g_\infty]$, as $t_n \longrightarrow 0$. This means that for each t_n, there exist a diffeomorphism $\varphi_{t_n} \in \mathrm{Diff}_0(S)$ such that $\varphi_{t_n}^*(\widetilde{g}_{t_n}) \longrightarrow g_\infty$.

Let us focus in one component of the exceptional divisor on S. By construction, this component is a copy of the projective space $\mathbb{C}P^1$, and defines a non-trivial class in homology, by the inclusions

$$[\mathbb{C}P^1] \in \pi_2(S) \hookrightarrow H_2(S,\mathbb{Z}) \hookrightarrow H_2(S,\mathbb{R}).$$

In particular, we can take the restriction of the metric as

$$\varphi_{t_n}^*(\widetilde{g}_{t_n})|_{\mathbb{C}P^1} \longrightarrow g_\infty|_{\mathbb{C}P^1}.$$

For $[\alpha] = \left[\sum_{i=1}^k c_i\, f_i\right] \in H_2(S,\mathbb{R})$, we define the stable norm (for a more general definition and details, please see [17], [15]):

$$\|\alpha\|_{\mathrm{st},g} = \inf\left\{\sum_{i=1}^k |c_i|\mathrm{Vol}(f_i)\right\}.$$

Where g appears on the notation because the volume of a simplex $f_i : \Delta^n \longrightarrow S$ depends on the metric:

$$\mathrm{Vol}(f_i) = \int_{\Delta^n} \sqrt{\det(g_{ij})}\mathrm{d}\mu.$$

Being a norm, it is non degenerate and therefore we have that

$$\|[\mathbb{C}P^1]\|_{\mathrm{st},\varphi_{t_n}^*(\tilde{g}_{t_n})} \longrightarrow \|[\mathbb{C}P^1]\|_{\mathrm{st},g_\infty} > 0, \tag{6.6}$$

where we are considering the metric restricted to $\mathbb{C}P^1$, but did not write it explicitly to simplify notation.

However, because the parameter t_n is a measure for the volume of the exceptional divisors, we have that $\mathrm{Vol}(\mathbb{C}P^1, \widetilde{g}_{t_n}) \longrightarrow 0$ as $t_n \longrightarrow 0$. Then we have

$$\|[\mathbb{C}P^1]\|_{st,\tilde{g}_{t_n}} \longrightarrow 0,$$

which contradicts Equation (6.6).

From this we conclude that no sequence $([\widetilde{g}_{t_n}])_{n\in\mathbb{N}}$ in the path $[\widetilde{g}_t]$ converges in the moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$, as $t_n \longrightarrow 0$.

In particular, the sequence presented in Proposition 98 is a non convergent Cauchy sequence in $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$.

□

Proposition 100. Consider the Kummer $K3$ surface S. For sufficiently small $|a|$, the L^2-norms on $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ induced by g_a and $\widetilde{g}_a$ are comparable. More precisely, we can estimate its quotient from above and below by constants that converge to 1 as $|a| \to 0$.

Proof. Recall that, in local coordinates, the Calabi-Yau equation $\widetilde{\omega}_a := \omega_a + i\partial\bar{\partial}u_a$ can be written as

$$\widetilde{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} + \frac{\partial^2 u_a}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}},$$

where for simplicity the parameter a does not appear in the notation of g.

By Equation (5.8), we already know that locally (away from the exceptional divisor) $g_{\alpha\bar{\beta}} = \mathbb{I} + a^2 H_a(u)$, where $\mathbb{I}$ is the identity matrix and the function $H_a(u)$, together with its derivatives, is regular.

Additionally, by elliptic regularity, estimates (5.38) and (5.41) imply that we have estimates for the second derivatives of u_a, namely

$$\left\| \frac{\partial^2 u_a}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}} \right\|_{C^{0,\alpha}} \leq C|a|^2, \tag{6.7}$$

$$\left\| \frac{\partial^2 u_a}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}} \right\|_{C^0} \leq C|a|, \tag{6.8}$$

where (6.7) holds in a neighborhood of the complement of the exceptional divisor, and (6.8) holds everywhere.

Then, locally we have

$$\widetilde{g}_{\alpha\bar{\beta}} = \begin{cases} \mathbb{I} + a^2 \widetilde{H}_a(u), & \text{away from the exceptional divisor} \\ \mathbb{I} + a \widehat{H}_a(u), & \text{everywhere,} \end{cases}$$

where the functions $\widetilde{H}_a(u)$, $\widehat{H}_a(u)$ and their respective derivatives are regular.

In addition, Lemma 132 says that the inverse of the matrix $\widetilde{g}_{\alpha\bar{\beta}}$ is given by

$$\widetilde{g}^{\alpha\bar{\beta}} = \begin{cases} \mathbb{I} - a^2\ \widetilde{H}_a(u) + \mathcal{O}(a^4), & \text{away from the exceptional divisor} \\ \mathbb{I} - a\ \widehat{H}_a(u) + \mathcal{O}(a^2), & \text{everywhere.} \end{cases}$$

From this we see that, for sufficiently small a, we have

$$1 - \epsilon < \frac{\mathrm{tr}_{g_a}(hk)}{\mathrm{tr}_{\widetilde{g}_a}(hk)} < 1 + \epsilon$$

for an ϵ independent of a.

Therefore, we can perform our computations with respect to g_a instead of $\widetilde{g}_a$. □

Now it remains to prove the bound for $\|\widetilde{\omega}'_t\|_{L^2(g_t)}$.

The situation can be simplified in the following way. First, observe that by definition $\widetilde{\omega}'_a = \omega'_a + i\partial\bar{\partial}u'_a$. The difficult part to bound is $i\partial\bar{\partial}u'_a$, because it involves the derivatives of the solution of the Calabi-Yau equation

$$(\omega_a + i\partial\bar{\partial}u_a)^2 = e^{G_a}\omega_a^2.$$

Consider the exterior derivative

$$2(\omega_a + i\partial\bar{\partial}u_a) \wedge (\omega'_a + i\partial\bar{\partial}u'_a) = G'_a e^{G_a}\omega_a^2 + 2e^{G_a}\omega'_a \wedge \omega_a.$$

From this we obtain

$$\begin{aligned} &G'_a e^{G_a}\omega_a^2 + 2e^{G_a}\omega'_a \wedge \omega_a - 2(\omega_a + i\partial\bar{\partial}u_a) \wedge \omega'_a \\ &= 2(\omega_a + i\partial\bar{\partial}u_a) \wedge i\partial\bar{\partial}u'_a \\ &= \Delta(u'_a) \cdot (\omega_a + i\partial\bar{\partial}u_a)^2. \end{aligned}$$

Where the last equality is true from Lemma 24. Then we use the Calabi-Yau equation once again to simplify:

$$G'_a + 2\frac{\omega'_a \wedge \omega_a}{\omega_a^2} - 2\frac{(\omega_a + i\partial\bar{\partial}u_a) \wedge \omega'_a}{(\omega_a + i\partial\bar{\partial}u_a)^2} = \Delta(u'_a). \tag{6.9}$$

This expression for the Laplacian will be very useful in our strategy. As a preliminary step, we will prove that the third summand of Equation (6.9) admits a useful characterization.

Lemma 101. For all $p \in S$, the map

$$\frac{\widetilde{\omega}_t \wedge \cdot}{\widetilde{\omega}_t^2} : \Lambda^2 T_p^* S \longrightarrow \mathbb{R}$$

is the same as $\widetilde{\omega}_t^{-1}$.

Proof. Let p be a point in S. We may choose holomorphic coordinates z_1, z_2 on S near p such that

$$\left\{\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j}\right\}_{j=1,2}$$

is a local frame for the complexified tangent bundle,

$$\left\{\mathrm{d}\bar{z}^j, \mathrm{d}z^j\right\}_{j=1,2} = \{f_1, \cdots, f_4\}$$

is a basis for the differential 1-forms and

$$\widetilde{\omega}_t = \sum_{j=1}^{2} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$$

is a local expression for the Kähler form. Since it is non degenerate, we can invert it to get

$$\widetilde{\omega}_t^{-1} = \sum_{j=1}^{2} \mathrm{d}\bar{z}_j \wedge \mathrm{d}z_j = \sum_{j=1}^{2} f_{2j-1} \wedge f_{2j}.$$

Let us observe that

$$\widetilde{\omega}_t \in \Omega^2(S) \subset \Gamma(T^*S \otimes T^*S) = \Gamma(\mathrm{Hom}(TS, T^*S)).$$

Therefore,

$$\widetilde{\omega}_t^{-1} \in \Gamma(\mathrm{Hom}(T^*S, TS)) = \Gamma(\Lambda^2 TS) = \Gamma(\mathrm{Hom}(\Lambda^2 T^*S, \mathbb{R})).$$

Where the last equality comes from the fact that since $\Lambda^k(V^*) = (\Lambda^k V)^*$, we have an isomorphism $\Lambda^2 TS \cong \mathrm{Hom}(\Lambda^2 T^*S, \mathbb{R})$.

Then the inverse form $\widetilde{\omega}_t^{-1}$ can be seen as a map

$$\widetilde{\omega}_t^{-1} : S \longrightarrow \mathrm{Hom}(\Lambda^2 T^*S, \mathbb{R})$$

and its norm is equivalent (up to a constant) to the operator norm.

Any other 2-form on S can be written as

$$\alpha = \sum_{i<j} \alpha_{ij} f_i \wedge f_j.$$

Observe that the wedge product has the following property:

$$\begin{aligned}\widetilde{\omega}_t \wedge \alpha &= \left(\sum_{j=1}^{2} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j\right) \wedge \left(\sum_{i<j} \alpha_{ij} f_i \wedge f_j\right)\\ &= c\left(\sum_{l=1}^{2} \alpha_{2l-1,2l}\right) \cdot \widetilde{\omega}_t^2\end{aligned}$$

where c is a constant.

Then the map

$$\begin{aligned}\frac{\widetilde{\omega}_t \wedge \cdot}{\widetilde{\omega}_t^2} &: \Lambda^2 T_p^* S \longrightarrow \mathbb{R}\\ \sum_{i<j} \alpha_{ij} f_i \wedge f_j &\longmapsto \sum_{l=1}^{m} \alpha_{2l-1,2l}\end{aligned}$$

is the same as $\widetilde{\omega}_t^{-1}$. □

Lemma 102. The map $\widetilde{\omega}_t^{-1}$ is bounded in the C^0 and $C^{0,\alpha}$ norms.

Proof. Corollary 130 of Appendix B states that for a bounded linear operator A between Banach spaces, if $\|A-\mathcal{I}\| \leq q < 1$, then $\|A^{-1}\| \leq (1-q)^{-1}$.

Therefore, to bound $\widetilde{\omega}_t^{-1}$ we need to prove that the composition $(\widetilde{\omega}_t \circ \omega_t^{-1})$ is sufficiently close to the identity. Let us compute the difference

$$\begin{aligned} \widetilde{\omega}_t \circ \omega_t^{-1} - \mathcal{I} &= (\omega_t + \mathrm{dd}^c u_t) \circ \omega_t^{-1} - \mathcal{I} \\ &= \mathcal{I} + \mathrm{dd}^c u_t \circ \omega_t^{-1} - \mathcal{I} \\ &= \mathrm{dd}^c u_t \circ \omega_t^{-1} \end{aligned}$$

From this we see that the problem is reduced to finding C^2- bounds for the solution u_t, which are given by Equations (5.38) and (5.41). Then, taking $|a| = t$ we have

$$\left\|\widetilde{\omega}_t^{-1}\right\|_{C^{0,\alpha}} \leq \frac{1}{1 - C|a|^2} \tag{6.10}$$

away from the exceptional divisor, and

$$\left\|\widetilde{\omega}_t^{-1}\right\|_{C^0} \leq \frac{1}{1 - C|a|} \tag{6.11}$$

everywhere.

□

Proposition 103. Consider the path of metrics ω_t, as constructed in the proof of Proposition 99. Then, the map (6.3) is Lipschitz.

Proof. The problem is reduced to find a bound for the derivative $\|\widetilde{\omega}_t'\|_{L^2(\omega_t)} = \|\omega_t' + \mathrm{dd}^c u_t'\|_{L^2(\omega_t)}$.

In Proposition 86, we have proven the following elliptic estimate:

$$\|u_a'\|_{C^{2,\alpha}_{\varepsilon,\beta}} \leq c \left(\|\Delta u_a'\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} + \|u_a'\|_{C^0} \right),$$

where c is a constant independent of a.

Using the expression for the Laplacian given by Equation (6.9), we have

$$\|u_a'\|_{C^{2,\alpha}_{\varepsilon,\beta}} \leq c\left(\left\|G_a' + 2\frac{\omega_a'\wedge\omega_a}{\omega_a^2} - 2\frac{(\omega_a + \mathrm{dd}^c u_a)\wedge\omega_a'}{(\omega_a+\mathrm{dd}^c u_a)^2}\right\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} + \|u_a'\|_{C^0}\right),$$

which, by the triangle inequality, implies

$$\|u_a'\|_{C^{2,\alpha}_{\varepsilon,\beta}} \leq c\left(\|G_a'\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} + 2\left\|\frac{\omega_a'\wedge\omega_a}{\omega_a^2}\right\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} + 2\left\|\frac{(\omega_a + \mathrm{dd}^c u_a)\wedge\omega_a'}{(\omega_a+\mathrm{dd}^c u_a)^2}\right\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} + \|u_a'\|_{C^0}\right).$$

In suitable holomorphic coordinates, the Calabi-Yau equation becomes $\det(g_a + \nabla^2 u_a) = 1$, which holds if and only if

$$2(\exp(G_a) - 1) = 2\Delta_a u_a + (\Delta_a u_a)^2 - |\nabla^2 u_a|^2_{g_a}.$$

For this reason, the bounds for the Laplacian (5.31), also yields bounds on the complex Hessian $\nabla^2 u_a$. We now wish to obtain bounds for the real Hessian as well. Let us recall the following result:

Proposition 82. There are constants $C > 0$ and $0 < \alpha < 1$, independent of a such that

$$\|u_a\|_{C^{2,\alpha}} \leq C$$

holds for all small enough values of a.

This justifies that while $\nabla^2 u_a$ only controls the complex Hessian, the full real C^2-regularity of u_a follows from elliptic regularity for the complex Monge–Ampère equation.

For this reason, finding bounds for the Laplacian also yields bounds on $\nabla^2 u_a$ and therefore on $\|u_a\|_{C^2}$.

This in turn implies the bound $\|u_a'\|_{C^0} \leq C|a|$.

By Lemma 102, the third summand is bounded $C^{0,\alpha}$-norm, computed with respect to the metric ω_α. Because of Proposition 87, this implies that it is also bounded in the $C^{0,\alpha}$-norm with respect of the metric ω_ε.

Therefore, considering the fact that S is compact, it remains bounded in the weighted norm $C^{0,\alpha}_{\varepsilon,\beta+2}$, as

$$\left\| \frac{(\omega_a + \mathrm{dd}^c u_a) \wedge \omega_a'}{(\omega_a + \mathrm{dd}^c u_a)^2} \right\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} \leq C' \left(\frac{1}{1 - C|a|^2} \right). \tag{6.12}$$

Following the same logic, Lemma 102 also implies that

$$\left\| \frac{\omega_a' \wedge \omega_a}{\omega_a^2} \right\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} \leq C'' \left(\frac{1}{1 - C|a|^2} \right).$$

Lastly, as a consequence of the estimate (5.38), we obtain that

$$\|G_a'\|_{C^{0,\alpha}_{\varepsilon,\beta+2}} \leq C'''|a|^2.$$

Putting all these estimates together, we have found a bound

$$\|u_a'\|_{C^{2,\alpha}_{\varepsilon,\beta}} \leq c \left(F(|a|^2, |a|) \right).$$

By elliptic regularity, this implies that $\mathrm{dd}^c u_t'$ is bounded in the $C^{0,\alpha}$-norm. Thus, we have that

$$\|\omega_t' + \mathrm{dd}^c u_t'\|_{L^2(\omega_t)} \leq cF(|a|^2, |a|),$$

and therefore the map (6.3) is Lipschitz, as desired. □

Theorem 104. The moduli space $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ is not complete.

Proof. We have proven in Proposition 103 that the map (6.3) is Lipschitz.

Additionally, Observation 98 says that for $t_n = \frac{1}{n}$, the sequence $(g_{t_n})_{n\in\mathbb{N}}$ is Cauchy.

Then, Lemma 97 implies that the sequence $([\widetilde{g}_{t_n}])_{n\in\mathbb{N}}$ is also Cauchy, and from Proposition 99 it is not convergent.

Therefore, $\mathcal{M}_{\|}(S)/\mathrm{Diff}_0(S)$ is not complete.

□

Appendix A

Analysis

A.1 Spaces of functions on Riemannian manifolds

To start this section, we review some spaces of functions which are very useful when it comes to performing analysis on Riemannian manifolds.

Let (M, g) be a smooth Riemannian manifold. For an integer k and a smooth function $f : M \longrightarrow \mathbb{R}$, we denote by $\nabla^k f$ the k-th covariant derivative of f and $|\nabla^k f|$ the norm of $\nabla^k u$, defined in a local chart by

$$|\nabla^k f|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k f)_{i_1 \cdots i_k} (\nabla^k f)_{j_1 \cdots j_k}. \tag{A.1}$$

Recall that $(\nabla f)_i = \partial_i f$, while

$$(\nabla^2 f)_{ij} = \partial_{ij} f - \Gamma^k_{ij} \partial_k f.$$

Definition 105. Let M be a Riemannian manifold with metric g. For $q \geq 1$, define the Lebesgue space $L^q(M)$ to be the set of locally integrable functions f on M for which the norm

$$\|f\|_{L^q} = \left(\int_M |f|^q \mathrm{dV}_g \right)^{1/q}$$

is finite, where dV_g is the volume form of the metric g.

Proposition 106. Suppose that $r, s, t \geq 1$ and that $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$. If $f \in L^s(M)$, and $g \in L^t(M)$, then $fg \in L^r(M)$ and

$$\|fg\|_{L^r} \leq \|f\|_{L^s}\|g\|_{L^t}. \tag{A.2}$$

This is known as Hölder's inequality and is a classic result proven in [21].

Definition 107. Let $q \geq 1$ and let k be a nonnegative integer. Define the Sobolev space $L^q_k(M)$ to be the set of $f \in L^q(M)$ such that f is k-times weakly differentiable and $|\nabla^j f| \in L^q(M)$ for $j \leq k$. Define the Sobolev norm on $L^q_k(M)$ to be

$$\|f\|_{L^q_k} = \left(\sum_{j=0}^{k} \int_M |\nabla^j f|^q \mathrm{dV}_g\right)^{1/q}.$$

Then $L^q_k(M)$ is a Banach space with respect to the Sobolev norm.

The spaces $L^q(M)$ and $L^q_k(M)$ are vector spaces of real functions on M. However we can generalize them to vector spaces of sections of a vector bundle over M. Let $V \longrightarrow M$ be a vector bundle on M, equipped with Euclidean metrics on its fibres. Let $\hat{\nabla}$ be a connection on V preserving these metrics. Then for $q \geq 1$, the Lebesgue space $L^q(V)$ is the set of locally integrable sections v of V for which the norm

$$\|v\|_{L^q} = \left(\int_M |v|^q \mathrm{dV}_g\right)^{1/q}$$

is finite, and the Sobolev space $L^q_k(V)$ to be the set of $v \in L^q(V)$ such that f is k-times weakly differentiable and $|\nabla^j v| \in L^q(V)$ for $j \leq k$, with the analogous Sobolev norm.

Definition 108. Let M be a Riemannian manifold with metric g. For each integer $k \geq 0$, define $C^k(M)$ to be the space of continuous, bounded functions f on M which have k continuous, bounded derivatives. Define the norm on $C^k(M)$ by

$$\|f\|_{C^k} = \sum_{j=0}^{k} \sup_M |\nabla^j f|,$$

where ∇ is the Levi-Civita connection.

Definition 109. Let $d(x,y)$ be the distance between x and y calculated using g, and let $\alpha \in (0,1)$. Then a function f on M is said to be Hölder continuous with exponent α if

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

is finite. We denote by $C^{0,\alpha}$ the vector space of continuous, bounded functions on M which are Hölder continuous with exponent α, and the norm defined by

$$\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_\alpha.$$

In the same way as before, we extend this notion to spaces of sections v of a vector bundle V over M. Let $\delta(g)$ be the injectivity radius of the metric g on M, which we assume is positive, and set

$$[v]_\alpha = \sup_{\substack{x \neq y \in M \\ d(x,y) < \delta(g)}} \frac{|v(x) - v(y)|}{d(x,y)^\alpha},$$

whenever the supremum exist.

Observe that $v(x)$ and $v(y)$ lie in different vector spaces. In order to make sense of the expression $|v(x) - v(y)|$ we observe that if $x \neq y$ and $d(x,y) < \delta(g)$, there is a unique geodesic γ of length $d(x,y)$ joining x and y in M. Then parallel transport along γ using $\hat{\nabla}$ identifies the fibres of V over x and y, and the metrics on the fibres and $|v(x) - v(y)|$ is well defined.

Now, we define the Hölder space $C^{k,\alpha}(M)$ to be the set of f in $C^k(M)$ for which the supremum $[\nabla^k f]_\alpha$ exist, (in the vector bundle $\otimes^k T^*M$ with its natural metric and connection). The Hölder norm on $C^{k,\alpha}(M)$ is

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha,$$

and makes $C^{k,\alpha}$ a Banach space.

Given a vector bundle V over M, we can once again generalize the definition above to give Banach spaces $C^k(V)$ and $C^{k,\alpha}(V)$.

The embedding theorems

Theorem 110 (Sobolev Embedding Theorem)**.** Suppose M is a compact Riemannian n-manifold, k. l are integers with $k \geq l \geq 0$, q, r are real numbers with $q, r \geq 1$, and $\alpha \in (0,1)$. If

$$\frac{1}{q} \leq \frac{1}{r} + \frac{k-l}{n},$$

then $L^q_k(M)$ is continuously embedded in $L^r_l(M)$ by inclusion. If

$$\frac{1}{q} \leq \frac{k-l-\alpha}{n},$$

then $L^q_k(M)$ is continuously embedded in $C^{l,\alpha}(M)$, by inclusion.

Proof. May be found in [3], Theorem 2.30. □

Definition 111. Let U_1, U_2 be Banach spaces, and let $\psi : U_1 \longrightarrow U_2$ be a continuous linear map. Let $B_1 = \{u \in U_1 \ : \ \|u\|_{U_1} \leq 1\}$ be the unit ball in U_1. We call ψ a compact linear map if the image $\psi(B_1)$ is a precompact subset of U_2, that is, if the closure $\overline{\psi(B_1)}$ is a compact subspace of U_2.

Theorem 112 (Kondrakov Theorem)**.** Suppose M is a compact Riemannian n-manifols, k, l are integers with $k \geq l \geq 0$, q, r are real numbers with $q, r \geq 1$ and $\alpha \in (0,1)$. If

$$\frac{1}{q} \leq \frac{1}{r} + \frac{k-l}{n},$$

then the embedding $L^q_k(M) \hookrightarrow L^r_l(M)$ is compact. If

$$\frac{1}{q} \leq \frac{k-l-\alpha}{n},$$

then $L^q_k(M) \hookrightarrow C^{l,\alpha}(M)$ is compact. Also, $C^{k,\alpha}(M) \hookrightarrow C^k(M)$ is compact.

Proof. May be found in [3], Theorem 2.34. □

Differential operators on functions and vector bundles

Let M be a manifold, and ∇ a connection on the tangent bundle of M. Let u be a smooth function on M. Then the k^{th} derivative of u is $\nabla^k u$.

Definition 113. A partial differential operator P on M of order k is an operator taking real functions u on M to real functions on M that depends on u and its first k derivatives $\nabla u, \cdots, \nabla^k u$

$$(Pu)(x) = Q(x,\ u(x),\ \nabla u(x),\ \cdots,\nabla^k u(x)).$$

It is often useful to regard a differential operator as a mapping between some vector spaces of functions. The following examples exhibit that this is possible:

- If P is a differential operator of order k and $u \in C^{k+l}(M)$, then $Pu \in C^l(M)$.
- If P is a linear differential operator of order k whose coefficients are bounded, then $P : L^q_{k+l}(M) \longrightarrow L^q_l(M)$ is a linear map.
- If P is a linear differential operator of order k whose coefficients are at least $C^{l,\alpha}$, then $P : C^{k+l,\alpha}(M) \longrightarrow C^{l,\alpha}(M)$ is also a linear map .

Let P be a differential operator of order $2m$ defined on an m-dimensional Riemannian manifold M. A solution to $P(u) = f$ is a function $u \in C^{2m}(M)$ such that the equation is satisfied pointwise. However, there are other natural ways to define a natural notion of solution:

Definition 114. Suppose P is a linear partial differential operator on a manifold. If u and f are locally integrable functions on M, we say that u is a weak solution to $P(u) = f$ if for every smooth, compactly supported function φ

$$\int_M uP^*\varphi \mathrm{dV}_g = \int_M f\varphi \mathrm{dV}_g,$$

where P^* is the formal adjoint of P.

Definition 115. Let P be a differential operator of order k. Let u be a function with k derivatives. We define the linearization L_uP of P at u to be the derivative of $P(v)$ with respect to v at u, that is:

$$L_uPv = \lim_{\alpha \to 0} \left(\frac{P(u + \alpha v) - P(u)}{\alpha} \right). \tag{A.3}$$

Then L_uP is a linear differential operator of order k. If P is linear, then $L_uP = P$.

Now we define differential operators on vector bundles. Let M be a manifold, and let V be a vector bundle over M. Let ∇ be some connection on TM and let ∇^V be a connection on V. Let v be a section of V. By coupling the connections ∇ and ∇^V, one can form repeated derivatives of v. We will write $\nabla^V_{a_1 a_2 \cdots a_k} v$ for the k^{th} derivative of v defined in this way.

Definition 116. A differential operator P of order k taking sections of V to sections of W is an operator that depends on v and its first k derivatives

$$(Pv)(x) = Q(x, v(x), \nabla^V_{a_1} v(x), \cdots, \nabla^V_{a_1 \cdots a_k} v(x)) \in W_x$$

for $x \in M$. If Q is a smooth function of its arguments, then P is called smooth and if Pv is linear in v, then P is called linear. We can define the linearization using equation (A.3) for sections.

Consider another vector bundle W, and let P be a linear differential operator of order k from V to W. Then, in index notation, we write

$$Pv = A^{i_1 \cdots i_k} \nabla_{i_1 \cdots i_k} v + B^{i_1 \cdots i_{k-1}} \nabla_{i_1 \cdots i_{k-1}} v + \cdots + K^{i_1} \nabla_{i_1} v + Lv.$$

Here $A^{i_1 \cdots i_k}$, $B^{i_1 \cdots i_{k-1}}$, ... are tensors taking values in $V^* \otimes W$, so that if ξ_i is a 1-form at $x \in M$, then $A^{i_1 \cdots i_k}(x) \xi_{i_1} \cdots \xi_{i_k}$ is an element of $V^*_x \otimes W_x$ (equivalently, a linear map $V_x \to W_x$). We call $A^{i_1 \cdots i_k}, \cdots, L$ the coefficients of P.

Definition 117. Let P be a linear differential operator of order k, mapping sections of V to sections of W. For each point $x \in M$ and each $\xi \in T^*_x M$, define $\sigma_\xi(P; x) := A^{i_1 \cdots i_k} \xi_{i_1} \cdots \xi_{i_k}$. Then $\sigma_\xi(P; x)$ is a linear map $V_x \to W_x$. Consider the bundle map

$$\begin{aligned} \sigma(P) : T^* \times V &\longrightarrow W \\ \sigma(P)(\xi, v) &= \sigma_\xi(P; x) v \in W_x \end{aligned}$$

whenever $x \in M$, $\xi \in T^*_x M$ and $v \in V_x$, Then $\sigma(P)$ is called the principal symbol of P and $\sigma(P)(\xi, v)$ is homogeneous of degree k in ξ and linear in v.

Definition 118. Let V, W be vector bundles over a manifold M, and let P be a linear differential operator of degree k from V to W. We say P is an elliptic operator for each $x \in M$ and each nonzero $\xi \in T^*_x M$, the linear map $\sigma_\xi(P; x) : V_x \to W_x$ is invertible, where $\sigma(P)$ is the principal symbol of P.

Consider the equation $Pv = w$. If $v \in C^{k+l}(V)$, then $w \in C^l(W)$. It is natural to ask whether the converse holds: if $w \in C^l(W)$, is it true that $v \in C^{k+l}(V)$? In general, this is false. However, for $\alpha \in (1, 0)$, if $w \in C^{l,\alpha}(W)$, then $v \in C^{k+l,\alpha}(V)$ and for $p > 1$, if $w \in L^p_l(W)$ then $v \in L^p_{k+l}(V)$. This property, which is the content of the Theorem below, is called elliptic regularity and is the main reason that Hölder and Sobolev are used instead of C^k spaces.

Theorem 119. Suppose M is a compact Riemannian manifold , V, W are vector bundles over M of the same dimension , and P is a smooth, linear, elliptic differential operator of order k from V to W. Let $\alpha \in (0,1)$, $p > 1$, and $l \geq 0$ be an integer. Suppose that $P(v) = w$ holds weakly, with $v \in L^1(V)$ and $w \in L^1(W)$. If $w \in C^\infty(W)$, then $v \in C^\infty(V)$. If $w \in L^p_l(W)$, then $v \in L^p_{k+l}(V)$, and

$$\|v\|_{L^p_{k+l}} \leq C(\|w\|_{L^p_l} + \|v\|_{L^1}), \tag{A.4}$$

for some $C > 0$ independent of v, w. If $w \in C^{l,\alpha}(W)$, then $v \in C^{k+l,\alpha}(V)$, and

$$\|v\|_{C^{k+l,\alpha}} \leq C(\|w\|_{C^{l,\alpha}} + \|v\|_{C^0}), \tag{A.5}$$

for some $C > 0$ independent of v, w.

Proof. May be found in [33], Theorem 6.4.8. □

The estimates (A.4) and (A.5) are called the L^p estimates and Schauder estimates for P, respectively.

A.2 ALE manifolds

We saw in Chapter 3 that if (M, g) is a Riemannian 4-manifold and $\mathrm{Hol}(g) \subset Sp(m)$ then g is called a hyperkähler metric.

Since $Sp(m)$ is a subgroup of $SU(2m)$, we have g is Kähler and Ricci-flat.

Moreover, g is Kähler with respect to complex structures J_1, J_2, J_3 on M, with the property $J_1J_2 = J_3$. If $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1^2 + a_2^2 + a_3^2 = 1$, then $a_1J_1 + a_2J_2 + a_3J_3$ is also a complex structure on M with respect to which g is Kähler. Therefore, g is Kähler with respect to a whole 2-sphere $\mathcal{S}^2$ of complex structures.

In this situation, we call (J_1, J_2, J_3, g) a hyperkähler structure on M, with respect to which M becomes a hyperkähler manifold.

Example 120. The quaternions

$$\mathbb{H} = \{x_0 + x_1i_1 + x_2i_2 + x_3i_3 \mid x_j \in \mathbb{R}\} \cong \mathbb{R}^4$$

endowed with the flat Euclidean metric, is a hyperkähler manifold. To see this, we define the metric g and 2-forms $\omega_1, \omega_2, \omega_3$ on $\mathbb{H}$ by the formulas

$$
\begin{aligned}
g &= \sum_{k=0}^{3} (\mathrm{d}x_k)^2, \\
\omega_1 &= \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\
\omega_2 &= \mathrm{d}x_0 \wedge \mathrm{d}x_2 - \mathrm{d}x_1 \wedge \mathrm{d}x_3 \\
\omega_3 &= \mathrm{d}x_0 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2.
\end{aligned}
$$

We recognize that g is the Euclidean metric on $\mathbb{R}^4$. Let J_1, J_2 and J_3 be the complex structures on $\mathbb{R}^4$ corresponding to left multiplication by i_1, i_2 and i_3, respectively. Then g is Kähler with respect to each J_i, with Kähler form ω_i.

Analogously, each space $\mathbb{H}^n$ of n-tuples of quaternions is also hyperkähler.

There is a special class of noncompact, complete Ricci-flat Riemannian 4-manifolds, known as ALE manifolds. All known examples in dimension four are in fact hyperkähler, although hyperkähler structure is not part of the definition in general. Historically, the term was introduced to describe noncompact hyperkähler 4-manifolds M with one end, which asymptotically resemble $\mathbb{H}/G$ for a finite subgroup $G \subset SU(2)$ acting freely at infinity.

We will consider the following more general setting: take $\mathbb{R}^n$ with the standard Euclidean metric g_0. Suppose that G is a finite subgroup of $SO(n)$, acting freely on $\mathbb{R}^n \setminus \{0\}$. Then g_0 is preserved by G and therefore descends to $\mathbb{R}^n/G$. The quotient $\mathbb{R}^n/G$ has an isolated quotient singularity at the origin 0. Let $r(x)$ be the distance from 0 to x, computed using g_0.

Definition 121. Let X be a noncompact manifold of dimension n and g a Riemannian metric on X. We say that (X, g) is an Asymptotically Locally Euclidean manifold (ALE for short), asymptotic to $\mathbb{R}^n/G$, and we say that g is an ALE metric if the following conditions hold:

- There exists a compact subset $S \subset X$ and a map $\pi : X \setminus S \longrightarrow \mathbb{R}^n/G$, which is a diffeomorphism between $X \setminus S$ and $\{z \in \mathbb{R}^n/G \mid r(z) > R\}$, for some fixed $R > 0$.

- Under this diffeomorphism, the push-forward metric $\pi_*(g)$ satisfies

$$\nabla^k(\pi_*(g) - g_0) = O(r^{-n-k}), \tag{A.6}$$

  on $\{z \in \mathbb{R}^n/G \mid r(z) > R\}$, for all $k \geq 0$, where ∇ is the Levi-Civita connection of h_0, and $T = O(r^{-j})$ if $|T| \leq K r^{-j}$ for some $K > 0$.

Observe that Equation A.6 says that towards infinity, the metric g on X and its derivatives converge to the Euclidean metric on $\mathbb{R}^n/G$, with a given rate of decay.

Definition 122. Let (X, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$. A smooth function $\rho : X \longrightarrow [1, \infty)$ is called a radius function on X if given any asymptotic coordinate system $\pi : X \setminus S \longrightarrow \mathbb{R}^n/G$ we have

$$\nabla^k(\pi_*(\rho) - r) = O(r^{1-n-k}), \tag{A.7}$$

for all $k \geq 0$.

Equation A.7 means that ρ approximates r on $\mathbb{R}^n/G$ near infinity.

Now we consider metrics which are both Kähler and ALE. Suppose G is a finite subgroup of $U(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$. Similarly as before, $\mathbb{C}^m/G$ has an isolated singularity at the origin 0, and the standard Hermitian metric g_0 on $\mathbb{C}^m$ descends to the quotient.

Definition 123. Let (X, π) be a resolution of $\mathbb{C}^m/G$, with complex structure J, and let g be a Kähler metric on X. We say that (X, J, g) is an ALE manifold asymptotic to $\mathbb{C}^m/G$, and that g is an ALE Kähler metric, if for some $R > 0$

$$\nabla^k(\pi_*(g) - g_0) = O(r^{-2m-k}),$$

on $\{z \in \mathbb{C}^m/G \mid r(z) > R\}$, for all $k \geq 0$. We say that a smooth function $\rho : X \longrightarrow [1, \infty)$ is a radius function on X if $\rho = \pi^*(r)$ on the subset $\{x \in X \mid \pi^*(r) \geq 2\}$.

The resolution map $\pi : X \longrightarrow \mathbb{C}^m/G$ gives a natural asymptotic coordinate system for X, with respect to which both the metric g and the complex structure J are asymptotic to the metric and complex structure of $\mathbb{C}^m/G$.

Suppose that (M, g) is an ALE manifold asymptotic to $\mathbb{R}^n/G$. Then the Schauder and Hölder estimates from Theorem 119 do not hold since M is noncompact. Therefore, the spaces $L^q_k(M)$ and $C^{k,\alpha}(M)$ are not suitable for studying operators on ALE manifolds. A radius function will be used as a weight to define new Hölder and Sobolev spaces, needed to perform analysis on ALE manifolds.

Definition 124. Let (M, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$, and let ρ be a radius function on M. For $q \geq 1$, $\beta \in \mathbb{R}$ and k a nonnegative integer, define the weighted Sobolev space $L^q_{k,\beta}(M)$ to be the set of functions f on M that are locally integrable and k times differentiable, and for which the norm

$$\|f\|_{L^q_{k,\beta}} = \left(\sum_{j=0}^{k} \int_X |\rho^{j-\beta}\nabla^j f|^q \rho^{-n} \mathrm{dVol}(M,g) \right)^{1/q}$$

is finite. Then $L^q_{k,\beta}(M)$ is a Banach space.

Definition 125. Let (M,g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$, and let ρ be a radius function on M. For $q \geq 1$, $\beta \in \mathbb{R}$ and k a nonnegative integer, define the space $C^k_\beta(M)$ to be the set of functions f on M with k continuous derivatives, such that $\rho^{j-\beta}|\nabla^j f|$ is bounded on M for $j = 0, \cdots, k$. Define the norm on $C^k_\beta(M)$ by

$$\|f\|_{C^k_\beta} = \sum_{j=0}^{k} \sup_M \|\rho^{j-\beta}\nabla^j f\|.$$

Let $\delta(g)$ be the injectivity radius of g and write $d(x,y)$ for the distance between x and y in M. For T a tensor field on M and $\alpha, \gamma \in \mathbb{R}$, define

$$[T]_{\alpha,\gamma} = \sup_{\substack{x \neq y \in M \\ d(x,y) < \delta(g)}} \left[\min(\rho(x),\rho(y))^{-\gamma} \cdot \frac{|T(x)-T(y)|}{d(x,y)^\alpha} \right].$$

Here we interpret $|T(x)-T(y)|$ using parallel translation along the unique geodesic of length $d(x,y)$ joining x and y.

For $\beta \in \mathbb{R}$, k a nonnegative integer and $\alpha \in (0,1)$, define the weighted Hölder space $C^{k,\alpha}_\beta(M)$ to be the set of $f \in C^k_\beta(M)$ for which the norm

$$\|f\|_{C^{k,\alpha}_\beta} = \|f\|_{C^k_\beta} + [\nabla^k f]_{\alpha,\beta-k-\alpha}$$

is finite.

In this notation, the index β should be interpreted as the order of growth: a function f in $L^q_{k,\beta}$, C^k_β, $C^{k,\alpha}_\beta$ grows at most like ρ^β as $\rho \to \infty$. Similarly the derivatives $\nabla^j f$ grow at most like $\rho^{\beta-j}$ for $j = 1, \cdots, k$.

As vector spaces of functions, the spaces $L^q_{k,\beta}(X)$, $C^k_\beta(X)$ and $C^{k,\alpha}_\beta(X)$ are independent of the choice of radius function ρ. Even though the norms on these spaces do depend on the choice of ρ, all choices yield equivalent norms.

Next we review some elliptic estimates of the Laplacian on ALE manifolds. First, consider the following weighted version of the Schauder estimates on $\mathbb{R}^n$.

Theorem 126. Let $n > 2$ and $k \geq 0$ be integers and $\alpha \in (0,1)$. Then

- Suppose $\beta \in (-n,-2)$. Then for each $f \in C^{k,\alpha}_{\beta}(\mathbb{R}^n)$ there is a unique $u \in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ with $\Delta u = f$.
- Suppose $\beta \in (-1-n,-n)$. Then for each $f \in C^{k,\alpha}_{\beta}(\mathbb{R}^n)$ there is a unique $u \in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ with $\Delta u = f$ if and only if $\int_{\mathbb{R}^n} f \mathrm{dV} = 0$.

In each case, $\|u\|_{C^{k+2,\alpha}_{\beta+2}} \leq C\|f\|_{C^{k,\alpha}_{\beta}}$ for some $C > 0$ depending only on $n,\ k,\ \alpha$ and β.

Proof. May be found in [22], p. 180. □

Now we extend the result to ALE manifolds.

Theorem 127. Suppose (X,g) is an ALE manifold asymptotic to $\mathbb{R}^n/G$ for $n > 2$ and ρ is a radius function on X. Let $k \geq 0$ be an integer and $\alpha \in (0,1)$. Then

- Let $\beta \in (-n,-2)$. Then there exists $C > 0$ such that for each $f \in C^{k,\alpha}_{\beta}(X)$ there is a unique $u \in C^{k+2,\alpha}_{\beta+2}(X)$ with $\Delta u = f$, which satisfies
$$\|u\|_{C^{k+2,\alpha}_{\beta+2}} \leq C\|f\|_{C^{k,\alpha}_{\beta}}.$$
- Let $\beta \in (-1-n,-n)$. Then there exists $C_1, C_2 > 0$ such that for each $f \in C^{k,\alpha}_{\beta}(X)$ there is a unique $u \in C^{k+2,\alpha}_{2-n}(X)$ with $\Delta u = f$. Moreover, $u = A\rho^{2-n} + v$, where
$$A = \frac{|G|}{(n-2)\Omega_{n-1}} \cdot \int_X f \mathrm{dV}_g,$$
and $v \in C^{k+2,\alpha}_{\beta+2}(X)$ satisfy
$$|A| \leq C_1\|f\|_{C^0_{\beta}},$$
$$\|v\|_{C^{k+2,\alpha}_{\beta+2}} \leq C_2\|f\|_{C^{k,\alpha}_{\beta}}.$$
Here Ω_{n-1} is the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$.

Proof. May be found in [22], p. 182. □

The following result shows that we can modify any ALE Kähler metric to be flat outside a compact set.

Proposition 128. Let $\mathbb{C}^m/G$ have an isolated singularity at 0 (for $m > 1$), with a resolution (X, π) admiting ALE Kähler metrics, and let ρ be a radius function on X. Then in each Kähler class there exists an ALE Kähler metric g' on X such that $g' = \pi^*(g_0)$ on the subset $\{x \in X \mid \rho(x) > R\}$, where g_0 is the Hermitian metric on $\mathbb{C}^m/G$ and $R > 0$ is a constant.

Proof. May be found in [22], p. 185. □

Appendix B

Some useful linear algebra results

B.1 Bounding of inverse of a map using Von Neumann series

Theorem 129. If A is a bounded linear operator in a Banach space E and $\|A\| < |\lambda|$, then $A_\lambda = (A - \lambda\mathcal{I})^{-1}$ is a bounded operator,

$$A_\lambda = -\sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}, \tag{B.1}$$

and

$$\|A_\lambda\| \leq \frac{1}{|\lambda| - \|A\|}. \tag{B.2}$$

Proof. Since $\|A/\lambda\| < 1$, we have

$$\sum_{n=0}^{\infty} \left\| \frac{A^n}{\lambda^n} \right\| \leq \left\| \frac{A}{\lambda} \right\|^n < \infty.$$

Therefore, by the completeness of $\mathcal{B}(E, E)$, there exist a bounded linear operator B on E such that

$$B = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^n}.$$

Moreover,

$$(A - \lambda\mathcal{I})B = (A - \lambda\mathcal{I})\left(\sum_{n=0}^{\infty}\frac{A^n}{\lambda^n}\right) = \sum_{n=0}^{\infty}(A - \lambda\mathcal{I})\frac{A^n}{\lambda^n}$$
$$= \sum_{n=0}^{\infty}\frac{A^{n-1} - \lambda A^n}{\lambda^n} = \lambda\sum_{n=0}^{\infty}\left(\frac{A^{n+1}}{\lambda^{n+1}} - \frac{A^n}{\lambda^n}\right) = -\lambda\mathcal{I}.$$

Similarly, $B(A - \lambda\mathcal{I}) = -\lambda\mathcal{I}$. Thus

$$A_\lambda = (A - \lambda\mathcal{I})^{-1} = -\frac{B}{\lambda} = -\sum_{n=0}^{\infty}\frac{A^n}{\lambda^{n+1}}.$$

To prove (B.2), we observe that

$$\|A_\lambda\| \leq \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\|\frac{A}{\lambda}\|^n = \frac{1}{|\lambda|}\frac{1}{1 - \|A/\lambda\|} = \frac{1}{|\lambda| - \|A\|}.$$

□

Corollary 130. Let A be a bounded linear operator in a Banach space E. If $\|A - \mathcal{I}\| \leq q < 1$, then $\|A^{-1}\| \leq (1-q)^{-1}$.

Proof. We have

$$\|((A - \mathcal{I}) - \lambda\mathcal{I})^{-1}\| = \|(A - \mathcal{I}(1 + \lambda))^{-1}\|$$

Taking $\lambda = -1$ we have

$$\|A^{-1}\| \leq \frac{1}{1 - \|A - \mathcal{I}\|} \leq (1-q)^{-1}$$

□

B.2 Generalization of matrix determinant lemma

Lemma 131. Let $A, B \in GL(n, \mathbb{R})$, $t \in \mathbb{R}$. Then

$$\det(A + tB) = \det(A)\left(1 + t\,\mathrm{Tr}(A^{-1}B) + \mathcal{O}(t^2)\right).$$

Proof. First consider the case $A = I$. Let $\{h_{ij}\}$ (resp. $\{b_{ij}\}$)be the coefficients of the matrix $I + tB$ (resp. B). From the definition of the determinant we have the following.

$$\begin{aligned}\det(I + tB) &= \sum_{\sigma \in S_n} d_{1\sigma(1)} \cdots d_{n\sigma(n)} \\ &= (1 - tb_{11}) \cdots (1 + t\, b_{nn}) + \mathcal{O}(t^2) \\ &= 1 + t\,\mathrm{Tr}(B) + \mathcal{O}(t^2).\end{aligned}$$

Now take $A \in GL(n, \mathbb{R})$ with coefficients $\{a_{ij}\}$. Then

$$\begin{aligned}\det(A + tB) &= \det(A)\det(I + A^{-1}tB) \\ &= \det(A)\left(1 + t\,\mathrm{Tr}(A^{-1}B) + \mathcal{O}(t^2)\right).\end{aligned}$$

□

B.3 Generalization of Sherman-Morrison formula

Lemma 132. Let $A, B \in GL(n, \mathbb{R})$, $t \in \mathbb{R}$. Then $(A + tB)^{-1} = A^{-1} - tA^{-1}BA^{-1} + \mathcal{O}(t^2)$.

Proof. Let us start by considering the case $A = I$. By making use of the Von Neumann series we have

$$(I + tB)^{-1} = I - tB + \mathcal{O}(t^2).$$

Now, for a general A, by the Woodbury matrix identity we obtain

$$\begin{aligned}(A + tB)^{-1} &= A^{-1} - A^{-1}\left(I + tBA^{-1}\right)^{-1} tBA^{-1} \\ &= A^{-1} - A^{-1}\left(I - tBA^{-1} + \mathcal{O}(t^2)\right) tBA^{-1} \\ &= A^{-1} - A^{-1}\left(tBA^{-1} - tBA^{-1}tBA^{-1} + \mathcal{O}(t^2)\, tBA^{-1}\right) \\ &= A^{-1} - tA^{-1}BA^{-1} + \mathcal{O}(t^2).\end{aligned}$$

□

Appendix C

Resolution of singularities

Definition 133. Let $\mathcal{V}$ be an algebraic variety, with singular points $\mathrm{Sing}(\mathcal{V})$. A resolution of $\mathcal{V}$ is a smooth variety $\widetilde{\mathcal{V}}$, together with a birational proper function $\pi : \widetilde{\mathcal{V}} \longrightarrow \mathcal{V}$ such that π is an isomorphism over the regular points $\mathcal{V} - \{\mathrm{Sing}(\mathcal{V})\}$.

The set $\pi^{-1}(\mathrm{Sing}(\mathcal{V}))$ is called the exceptional divisor of the resolution.

Hironaka proved in [18] that every algebraic variety over a field of characteristic zero admits a resolution.

Definition 134. A resolution $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ is called minimal if, given any other resolution $\pi' : \widetilde{\mathcal{V}}_1 \to \mathcal{V}$, there is a rational function ϕ such that $\pi' = \pi \circ \phi$.

Theorem 135 ([35], Theorem 5.7)**.** Let V be a complex surface with an isolated singular point. Then, up to isomorphism, V admits a unique minimal resolution.

One method for resolving singularities is the blow up, which is described as follows.

Definition 136. The blow up at the origin $\bar{0}$ in $\mathbb{C}^n$ is the following algebraic variety in $\mathbb{C}^n \times \mathbb{C}P^{n-1}$:

$$Bl_0(\mathbb{C}^n) = \left\{ ((x_1, \cdots, x_n), [\xi_1, \cdots \xi_n]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \,|\, x_i \xi_j = x_j \xi_i \right\},$$

together with the projection

$$\pi : Bl_0(\mathbb{C}^n) \longrightarrow \mathbb{C}^n$$
$$\pi\left((x_1, \cdots, x_n), [\xi_1, \cdots \xi_n]\right) = (x_1, \cdots x_n).$$

Definition 137. Let $\mathcal{V} \subset \mathbb{C}^n$ be an affine variety containing the origin $\bar{0}$. Let $\pi : Bl_0(\mathbb{C}^n) \longrightarrow \mathbb{C}^n$ be the blow up at $\bar{0}$ in $\mathbb{C}^n$. The blow up of $\bar{0}$ at $\mathcal{V}$ is defined by

$$\widetilde{\mathcal{V}} = \overline{\pi^{-1}(\mathcal{V} - \bar{0})} \subset Bl_0(\mathbb{C}^n).$$

Lemma 138. The restriction of the function $\pi : \widetilde{\mathcal{V}} \to \mathcal{V}$ defines an isomorphism of algebraic varieties between $\mathcal{V} - \bar{0}$ and $\widetilde{\mathcal{V}} - \pi^{-1}(\bar{0})$.

Definition 139. Let $\mathcal{V} \subset \mathbb{C}^n$ be an affine algebraic variety, $Bl_0(\mathbb{C}^n)$ the blow up at the origin in $\mathbb{C}^n$ and $\widetilde{\mathcal{V}}$ the blow up at the origin in $\mathcal{V}$. The exceptional divisor of the blow up is the set $E = \pi^{-1}(\bar{0}) \cap \widetilde{\mathcal{V}}$.

C.1 Quotient singularities

Given Γ a finite subgroup of $SL(2, \mathbb{C})$, we can define a right action of Γ on $\mathbb{C}[z_1, z_2]$ as follows.

If $z = (z_1, z_2) \in \mathbb{C}^2$, $g \in \Gamma$, $f \in \mathbb{C}[z_1, z_2]$, then

$$(fg)(z) := f(g(z)).$$

Theorem 140. Let Γ be a finite subgroup of $SU(2)$. Then the set of Γ-invariant polynomials S^Γ is generated by three homogeneous invariant polynomials $f_1, f_2, f_3 \in \mathbb{C}[z_1, z_2]$, which satisfy a polynomial equation $h(f_1, f_2, f_3) = 0$, showed in the following table.

Table C.1:

Group Γ	**Equation h**
Cyclic of order n	$f_1^2 + f_2^2 + f_3^n = 0$
Binary dihedral	$f_1^2 + f_2^2 f_3 + f_3^{n-1} = 0$
Binary tetrahedral	$f_1^2 + f_2^3 + f_3^4 = 0$
Binary octahedral	$f_1^2 + f_2^3 + f_2 f_3^3 = 0$
Binary icosahedral	$f_1^2 + f_2^3 + f_3^5 = 0$

Proof. May be found in [23], p.50. □

Proposition 141 (Klein's Theorem)**.** The map

$$\begin{aligned} F : \mathbb{C}^2 &\longrightarrow \mathbb{C}^3 \\ (z_1, z_2) &\longmapsto (f_1(z_1, z_2), f_2(z_1, z_2), f_3(z_1, z_2)) \end{aligned}$$

induces a homeomorphism $\widetilde{F} : \mathbb{C}^2/\Gamma \longrightarrow \mathcal{V}$ between the orbit space $\mathbb{C}^2/\Gamma$ and the affine variety $\mathcal{V} := F(\mathbb{C}^2)$.

Therefore, $\mathbb{C}^2/\Gamma$ is isomorphic to $h^{-1}(0)$, where h is the corresponging polynomial in the second column of Table C.1.

In particular, consider $\{\pm 1\}$ to be the group of isometries generated by the involution

$$\begin{aligned} -1 : \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (z_1, z_2) &\longmapsto (-z_1, -z_2). \end{aligned}$$

Clearly, $\{\pm 1\}$ is the cyclic group of order 2 and therefore the quotient $\mathbb{C}^2/\{\pm 1\}$ corresponds to the surface $\{x^3 + y^2 + z^2 = 0\}$.

The action of $\{\pm 1\}$ on $\mathbb{C}^2$ extends to $Bl_0(\mathbb{C}^2)$ as:

$$g \cdot ((z_1, z_2), [\xi_1, \xi_2]) = (g \cdot (z_1, z_2), g \cdot [\xi_1, \xi_2]) = (g \cdot (z_1, z_2), [\xi_1, \xi_2])\,.$$

Therefore we have the following commutative diagram:

$$\begin{array}{ccc} Bl_0(\mathbb{C}^2) & \xrightarrow{p'} & Bl_0(\mathbb{C}^2)/\{\pm 1\} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathbb{C}^2 & \xrightarrow{\quad p \quad} & \mathbb{C}^2/\{\pm 1\}. \end{array} \tag{C.1}$$

C.2 A characterization of the blow up

Definition 142. The tautological line bundle, denoted $\mathcal{O}(-1)$ is a vector bundle over the complex projective space $\mathbb{C}P^1$ whose total space is

$$\mathcal{O}(-1) := \{((z_1, z_2), [\xi_1, \xi_2]) \mid z_1\xi_2 = z_2\xi_1\} \subset \mathbb{C}^2 \times \mathbb{C}P^1,$$

together with the natural projection onto the second component.

Observe that the total space of $\mathcal{O}(-1)$ is the total space of the blow up at the origin in $\mathbb{C}^2$.

Now, take the set

$$\mathcal{O}(-2) := \mathcal{O}(-1) \otimes \mathcal{O}(-1) = \left\{ ((z_1, z_2), [\xi_1, \xi_2]) \mid z_1\xi_2^2 = z_2\xi_1^2 \right\} \subset \mathbb{C}^2 \times \mathbb{C}P^1.$$

The following result holds.

Lemma 143. The map

$$f : \mathcal{O}(-1) \longrightarrow \mathcal{O}(-2)$$
$$f\left((z_1, z_2), [\xi_1, \xi_2]\right) = \left((z_1^2, z_2^2), [\xi_1, \xi_2]\right)$$

induces a bijection

$$\frac{\mathcal{O}(-1)}{\{\pm 1\}} \longrightarrow \mathcal{O}(-2). \tag{C.2}$$

Proof. Surjectivity is clear. Let us prove injectivity, observing that the map does not do anything on the second component.

Take two elements with the same image, as follows

$$\begin{aligned}
&f((z_1, z_2), [\xi_1, \xi_2]) = f((w_1, w_2), [\xi_1, \xi_2]) \\
&\left((z_1^2, z_2^2), [\xi_1, \xi_2]\right) = \left((w_1^2, w_2^2), [\xi_1, \xi_2]\right) \\
&\Longrightarrow z_1^2 = w_1^2 \ , \ z_2^2 = w_2^2 \\
&\Longrightarrow z_1 = \lambda_1 w_1 \ , \ z_2 = \lambda_2 w_2 \ , \ \lambda_i \in \{\pm 1\}.
\end{aligned}$$

On the other hand, by the definition of $\mathcal{O}(-2)$, we have

$$\begin{aligned}
z_1^2\xi_2^2 = z_2^2\xi_1^2 \ , \ w_1^2\xi_2^2 = w_2^2\xi_1^2 \\
\Longrightarrow z_1\xi_2 = z_2\xi_1 \ , \ w_1\xi_2 = w_2\xi_1.
\end{aligned}$$

Putting this together we obtain

$$(\lambda_1 w_1)\xi_2 = (\lambda_2 w_2)\xi_1$$

and therefore $\lambda_1 = \lambda_2$. This implies that the map C.2 is injective (and therefore bijective). □

Observation 144. As a consequence of Lemma 143 and the commutative diagram (C.1), we have that the resolution of the quotient singularity $\mathbb{C}^2/\{\pm 1\}$ is precisely $\pi' : \mathcal{O}(-2) \longrightarrow \mathbb{C}^2/\{\pm 1\}$.

Appendix D

Additional auxiliary lemmas

In this appendix, we gather several results that were used in the proof of the estimates found in Chapter 5. They appear either with proof or with reference.

D.1 Results used in the proof of the C^0-estimate

Lemma 145. Let (M, J) be a compact, complex manifold and g a Kähler metric on M, with Kähler form ω. Let $f \in C^0(M)$, $\phi \in C^2(M)$ and $A > 0$. Set $\tilde{\omega} = \omega + \mathrm{dd}^c\phi$, suppose that $\tilde{\omega}^m = Ae^f\omega^m$, and let $\tilde{g}$ be the Kähler metric with Kähler form $\tilde{\omega}$. Then

$$\begin{aligned} \mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-1} &= \frac{1}{m}|\nabla\phi|_g^2\,\omega^m, \qquad \text{and} \\ \mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-j-1} \wedge \tilde{\omega}^j &= F_j\,\omega^m \end{aligned}$$

for $j = 1, 2, \cdots, m-1$, where F_j is a nonnegative real function on M.

Proof. By properties of the Hodge star, we have $\mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-1} = (\mathrm{d}\phi \wedge \mathrm{d}^c\phi, *(\omega^{m-1}))\mathrm{dVol}_g$.

We also know that $*(\omega^{m-1}) = (m-1)!\omega$, and $\omega^m = m!\mathrm{dVol}_g$, therefore

$$\mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-1} = \frac{(m-1)!}{m!}(\mathrm{d}\phi \wedge \mathrm{d}^c\phi, \omega) \cdot \omega^m. \tag{D.1}$$

Now, in tensor notation and by properties of the complex structure, we have $(\mathrm{d}^c\phi)_a = -J_a^b(\mathrm{d}\phi)_b$, and $-J_b^e\omega_{cd}g^{ac}g^{bd} = g^{ae}$. As a consequence, applying the definition of the pointwise inner product we have $(\mathrm{d}\phi \wedge \mathrm{d}^c\phi, \omega) =$

$-(\mathrm{d}\phi)_a\, J_b^e\, (\mathrm{d}\phi)_e\, \omega_{cd}\, g^{ac} g^{bd} = (\mathrm{d}\phi)_a g^{ae} = |\nabla\phi|_g^2$. Substituting this into (D.1) gives the first equality.

Analogously,

$$\begin{aligned}
\mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-j-1} \wedge \tilde{\omega}^j &= \big(\mathrm{d}\phi \wedge \mathrm{d}^c\phi, *(\omega^{m-j-1} \wedge \tilde{\omega}^j)\big)\, \mathrm{dVol}_g \\
&= \frac{1}{m!}\big(\mathrm{d}\phi \wedge \mathrm{d}^c\phi, *(\omega^{m-j-1} \wedge \tilde{\omega}^j)\big) \cdot \omega^m \\
&= F_j\, \omega^m,
\end{aligned}$$

where F_j is nonnegative due to the fact that $*(\omega^{m-j-1} \wedge \tilde{\omega}^j)$ is a positive $(1,1)$-form. □

Lemma 146. Let $p > 1$ be a real number. Then in the situation of the previous lemma,

$$\int_M \big|\nabla|\phi|^{p/2}\big|_g^2 \mathrm{dVol}_g \leq \frac{mp^2}{4(p-1)} \int_M (1 - e^f)\phi|\phi|^{p-2} \mathrm{dVol}_g. \tag{D.2}$$

Proof. From equation (5.4), and since $\omega_a - \tilde{\omega}_a = -\mathrm{d}\mathrm{d}^c\phi$, we have

$$(1 - e^f)\omega^m = \omega^m - \tilde{\omega}^2 = -\mathrm{d}\mathrm{d}^c\phi \wedge (\omega^{m-1} + \cdots + \tilde{\omega}^{m-1}) \tag{D.3}$$

Since M is compact, Stokes' Theorem shows that

$$\int_M \mathrm{d}\left[\phi|\phi|^{p-2}\mathrm{d}^c\phi \wedge \big(\omega^{m-1} + \omega^{m-2} \wedge \tilde{\omega} + \cdots + \tilde{\omega}^{m-1}\big)\right] = 0 \tag{D.4}$$

We multiply (D.3) by $|\phi|^{p-2}\phi$, with $p > 1$, and apply (D.4) to it, remembering that $\mathrm{d}\omega = \mathrm{d}\tilde{\omega} = 0$, and $\mathrm{d}(\phi|\phi|^{p-2}) = (p-1)|\phi|^{p-2}\mathrm{d}\phi$. We obtain:

$$\int_M \phi|\phi|^{p-2}(1 - e^f)\omega^m = (p-1)\int_M |\phi|^{p-2}\mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge (\omega^{m-1} + \omega^{m-2} \wedge \tilde{\omega} + \cdots + \tilde{\omega}^{m-1}). \tag{D.5}$$

By Lemma 145, we have expressions for $\mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-1}$ and $\mathrm{d}\phi \wedge \mathrm{d}^c\phi \wedge \omega^{m-j-1} \wedge \tilde{\omega}^j$. Substituting them in (D.5) gives

$$\int_M \phi|\phi|^{p-2}(1-e^f)\omega^m = \frac{p-1}{m}\int_M |\phi|^{p-2}\left(|\nabla\phi|_g^2 + F_1 + \cdots + F_{m-1}\right)\omega^m$$

where $F_1, \cdots, F_{m-1}$ are nonnegative functions on M.

Now, using the fact that $\omega^m = m!\mathrm{dVol}_g$, we obtain

$$\int_M |\phi|^{p-2}\left(|\nabla\phi|_g^2 + F_1 + \cdots + F_{m-1}\right)\mathrm{dVol}_g = \frac{m}{p-1}\int_M \phi|\phi|^{p-2}(1-e^f)\omega^m.$$

Combining this with equation $\frac{1}{4}p^2|\phi|^{p-2}|\nabla\phi|_g^2 = |\nabla|\phi|^{p/2}|_g^2$, we finally obtain Equation (D.2) .

□

Lemma 147. Let $\epsilon = \frac{m}{(m-1)}$. There are constants C_1, C_2, depending on M and g, such that if $\psi \in L_1^2(M)$, then

$$\|\psi\|_{L^{2\epsilon}}^2 \leq C_1\left(\|\nabla\psi\|_{L^2}^2 + \|\psi\|_{L^2}^2\right).$$

If, in addition, $\int_M \psi \mathrm{dVol}_g = 0$, then

$$\|\psi\|_{L^2} \leq C_2\|\nabla\psi\|_{L^2}.$$

Proof. Please see [22], Lemma 5.4.2. □

D.2 Results used in the proof of the additional estimates

Theorem 148. ([1], Theorem 1.91) Let (M,g) be a Riemannian manifold (not necessarily complete) of dimension $n \geq 3$. Let N be a compact $(n-k)$-dimensional submanifold, $3 \geq k \geq n$. Given $u \in L^p_{\mathrm{loc}}(M)$, $p \geq \frac{k}{k-2}$, such that

$$\Delta u = \rho \tag{D.6}$$

over $M - N$, in the weak sense, where $\rho \in L^1_{\mathrm{loc}}(M)$, then (D.6) holds also over M in the weak sense. If, in addition, $\rho \in C^{0,\alpha}$ for $0 < \alpha < 1$, then $u \in C^{2,\alpha}(M)$ and (D.6) holds in the classical sense.

Proposition 149. [34] For $\beta \in (0,4)$, the kernels of the Laplacian Δ_{EH} on the Eguchi-Hanson space, acting on forms in $C^{2,\alpha}_\beta$ of degree $p \neq 2$ are

$$\mathrm{Ker}\left(\Delta_{EH} : C^{2,\alpha}_\beta(\Lambda^p(X_{EH})) \longrightarrow C^{0,\alpha}_{\beta-2}(\Lambda^p(X_{EH}))\right) = 0.$$

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