The Two-Phase Periodic Stokes Flow in Two Dimensions: Well-Posedness and Stability

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Abstract

In this thesis, we study the two-phase horizontally periodic quasistationary Stokes flow in two dimensions. This system models the evolution of a sharp interface, given by the graph of a periodic function, separating two immiscible Newtonian fluids with possibly different densities and viscosities. The interface dynamics are driven by surface tension effects and we may incorporate a gravitational force as well. Such two-phase flows are of great significance in many industrial processes and life sciences.

The first part of this thesis focuses on the analysis of stationary two-phase Stokes systems with a fixed interface. We derive the horizontally periodic Stokeslet and use potential theory to define and analyze the hydrodynamic single- and double-layer potentials. Moreover, the invertibility of the hydrodynamic double-layer potential operator is extensively studied by means of a new class of (singular) integral operators. Using these layer potentials, we are able to construct the velocity field and the pressure of the fluids for any sufficiently regular interface.

In the second part, we use the formula obtained for the velocity field together with the kinematic boundary condition to reformulate the two-phase Stokes system as a fully nonlinear and nonlocal evolution equation for the function parametrizing the interface. We show that this evolution equation is of parabolic type and employ abstract parabolic theory to obtain local well-posedness in subcritical Sobolev spaces whose exponent is arbitrarily close to a certain critical value. Additionally, we establish, by means of a classical parameter trick, a parabolic smoothing property. Furthermore, we present a full overview of equilibrium solutions to the two-phase Stokes system. We then study the stability properties of these equilibria and show that flat interfaces are exponentially stable in the Rayleigh–Taylor stable regime. Moreover, we show that finger-shaped equilibria occur when the denser fluid lies on top of the less dense fluid and prove that these equilibrium solutions are unstable due to the onset of a Rayleigh–Taylor instability pattern.

Zusammenfassung

In dieser Arbeit studieren wir den horizontal periodischen, quasistationären zweiphasigen Stokes-Fluss in zwei Dimensionen. Dieses System modelliert die Evolution einer scharfen Grenzschicht, gegeben durch den Graphen einer periodischen Funktion, welche zwei nicht mischbare, Newtonsche Flüssigkeiten mit möglicherweise verschiedenen Dichten und Viskositäten trennt. Die Dynamik der Grenzschicht wird durch Oberflächenspannung hervorgerufen, zudem kann die Gravitation modelliert werden. Zweiphasige Flüsse dieser Art sind von großer Bedeutung in verschiedenen Industriezweigen und Biowissenschaften.

Der erste Teil dieser Arbeit beschäftigt sich mit stationären zweiphasigen Stokes-Problemen mit einer vorgegebenen Grenzschicht. Wir leiten die Fundamentallösung der horizontal periodischen Stokes-Gleichungen in zwei Dimensionen her und benutzen Potentialtheorie, um die hydrodynamischen Potentiale einfacher und doppelter Flächenbelegung zu studieren. Darüber hinaus analysieren wir die Invertierbarkeit des hydrodynamischen Potentials doppelter Flächenbelegung, indem wir eine neue Klasse (singulärer) Integraloperatoren einführen. Mithilfe dieser Potentiale konstruieren wir das Geschwindigkeitsfeld und den Druck der Flüssigkeiten für Grenzschichten, die regulär genug sind.

Im zweiten Teil benutzen wir die hergeleitete Darstellung des Geschwindigkeitsfeldes zusammen mit der kinematischen Randbedingung, um das zweiphasige Stokes-System in eine voll nichtlineare und nichtlokale Evolutionsgleichung umzuformulieren, welche die Evolution der Funktion beschreibt, die die Grenzschicht parametrisiert. Wir zeigen, dass diese Evolutionsgleichung parabolisch ist, und nutzen abstrakte parabolische Theorie, um lokale Wohlgestelltheit in subkritischen Sobolev-Räumen zu erhalten. Der Exponent dieser Räume kann dabei beliebig nah an einem gewissen kritischen Exponenten liegen. Zusätzlich zeigen wir eine parabolische Glättungseigenschaft, welche mithilfe eines klassischen Parametertricks bewiesen wird. Des Weiteren präsentieren wir eine vollständige Liste der Gleichgewichtslösungen des zweiphasigen Stokes-System und studieren die Stabilitätseigenschaften dieser Lösungen: Wir zeigen, dass flache Grenzschichten im stabilen Rayleigh–Taylor-Szenario exponentiell stabil sind und dass fingerförmige Gleichgewichtslösungen im instabilen Rayleigh–Taylor-Szenario existieren. Dies tritt auf, wenn die dichtere Flüssigkeit über der leichteren Flüssigkeit liegt, und wir zeigen, dass die fingerförmigen Gleichgewichtslösungen instabil sind.

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Contents

Chapter 1

Introduction

1.1. Two-phase flows

Two-phase flows are ubiquitous in nature and appear whenever two immiscible fluids interact. This happens in everyday scenarios, for example when pouring cream into coffee before stirring or when steam bubbles rise in boiling water. But two-phase flows are also relevant in industrial and biological contexts. In metallurgy, for instance, some molten metals do not mix and this must be accounted for when producing alloys to prevent phase separation or the formation of droplets in the hardened product. Other applications include the flow of air through the lungs which are coated with mucus or the extraction of oil from an oil reservoir by injecting water.

From a mathematical point of view, the natural approach to model two-phase flows is to use systems of partial differential equations, whose analysis often poses significant challenges. Within the framework of partial differential equations, there are numerous possibilities to model such flows. One modeling choice concerns the interface separating the two fluids, which can either be a sharp interface, meaning that the fluids are separated by a well-defined boundary, or a diffuse interface where a smooth transition between phases is allowed in a small interfacial layer. Next, the individual physical properties of the fluids, such as density and viscosity, have to be taken into account since these properties determine the governing equations in each fluid phase. Lastly, the interaction of the two fluids at the interface where various phenomena, such as surface tension effects, elasticity, or Marangoni forces may manifest, has to be modeled appropriately. In addition, external forces like gravity may be included in the model.

As discussed above, an essential aspect of the modeling is the representation of the interface between the two fluid phases. The diffuse interface setting typically considers one domain that contains both fluids. To distinguish the two fluid phases, an order parameter that describes the mass or volume fraction of each fluid at any point in the domain is introduced. At the interface, this order parameter makes a smooth transition from one phase to the other in a thin region, such that the interface is modeled as a small layer where both fluids are present. This leads to an evolution equation for the order parameter, usually given by the Cahn–Hilliard equation, which is then coupled to equations modeling the fluid dynamics. We refer to the (review) articles [3, 91] and the references therein for an overview of different phase-field models. In contrast, the sharp interface description relies on the fact that at each time the interface is given by a hypersurface that mathematically acts as a boundary for the fluid phases. Forces emanating from or acting on the interface can then be incorporated as jump conditions across the interface for the governing equations of each fluid phase. Physically, the interface is transported with the flow of the fluids, which turns the problem into a moving boundary problem. Such problems pose another mathematical challenge since now the domains of the fluids are a priori unknown and must be determined along with the fluid velocities and pressures. We refer to the monograph [76] and the references therein for a variety of two-phase flow models treated in the sharp interface setting. Both approaches are well established, however, in this thesis we consider the sharp interface setting.

A fundamental component in describing a two-phase flow is the system of partial differential equations that models the motion of the fluid. One of the most versatile systems is encompassed by the incompressible Navier–Stokes equations. They describe the motion of a Newtonian fluid where both inertial and viscous forces play a role. The ratio between inertial and viscous forces is called the Reynolds number. For large Reynolds numbers, where inertial forces dominate the viscous forces, the flow can be approximated by the Euler equations describing the flow of an inviscid fluid. Typical situations where the Euler equations work best include flows of gases at high speed such as the flow of air around the wing of an airplane but also the motion of water waves. Conversely, at very small Reynolds numbers, viscous forces dominate the inertial forces. Then, the Stokes equations describing creeping flow – that is, flows with negligible inertia – are a good approximation. Cases where the Stokes equations apply include the flow of honey, molten metals, or, more generally, slow viscous flows. For a broad overview of all three models we refer to the classical monograph [7]. A key mathematical advantage of the Stokes equations lies in their linear structure. In contrast, the Navier–Stokes equations and Euler equations are nonlinear due to the convective term which, together with the variation of the velocity, describes the inertia. This property makes the Stokes system particularly appealing to be used as the governing equations for modeling two-phase flows, and our approach in the analysis of the two-phase Stokes flow relies heavily on the linearity of the Stokes equations.

The two-phase Stokes system is completed by prescribing suitable coupling conditions for the fluids at the interface. For instance, one can allow for phase transition, which means that mass can be transported from one phase to the other across the interface. In this case, the fluid velocities are generally not continuous across the interface. Instead, additional balance laws concerning the transfer of mass and energy must be imposed. Physically, this often happens when temperature differences are present and phase transition corresponds to melting or solidification of one phase. A prominent example modeling such behavior is the Stefan problem, see [76] for this and other related problems. In the absence of phase transition, the velocities are assumed to be continuous which implies that there is no transfer of mass across the interface. The interface is then transported by the flow. Next, surface tension is modeled as a jump condition for the stresses of the fluids across the interface. Other interactions include elastic effects, as in the Peskin problem where an elastic string is immersed in a fluid, or Marangoni effects where the gradient of the surface tension exhibits a jump across the interface. Additionally, an external body force may be introduced, typically modeling gravity. In our case, we consider both surface tension effects and gravitational force.

From a physical perspective, two-phase flows are often described by complex models, such as the three-dimensional two-phase Navier–Stokes equations. However, from a mathematical point of view, these problems are often inaccessible: even the global well-posedness of the Navier–Stokes equations in three dimensions in a fixed domain is a major open problem.

In this thesis, we consider a two-dimensional, mathematically tractable model that retains essential physical mechanisms, including interfacial motion, surface tension, and gravitational effects with the dynamics in the bulk described by the Stokes equations. We present our two-phase Stokes model in the following section.

1.2. The mathematical model

We now introduce the two-phase periodic quasistationary Stokes flow whose analysis is the focus of this thesis. We begin by describing the geometric setting. Let

$$\mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$$

denote the unit circle and interpret functions defined on it as 2π-periodic on $\mathbb{R}$. We assume that the two fluid phases are separated by a sharp interface $\Gamma(t)$ which at each time $t \geq 0$ is given by the graph of a periodic function $f(t) : \mathbb{S} \to \mathbb{R}$, that is

$$\Gamma(t) := \{x = (x_1, x_2) \in \mathbb{S} \times \mathbb{R} : x_2 = f(t, x_1)\}, \qquad t \geq 0.$$

In order to focus on the dynamics close to the interface that separates the fluids, which is of particular interest in this thesis, we assume that the flow fills the entire periodic plane $\mathbb{S}\times\mathbb{R}$. We denote the fluid phase above the interface by $\Omega^+(t)$ and the one below by $\Omega^-(t)$. This means that the domains $\Omega^\pm(t)$ are defined by

$$\Omega^\pm(t) := \{x = (x_1, x_2) \in \mathbb{S} \times \mathbb{R} : x_2 \gtrless f(t, x_1)\}, \qquad t \geq 0.$$

In particular, we have

$$\partial\Omega^+(t) = \partial\Omega^-(t) = \Gamma(t) \qquad \text{and} \qquad \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t) = \mathbb{S} \times \mathbb{R}, \qquad t \geq 0.$$

We denote by $\tilde{\nu}(t)$ the unit normal exterior to $\Omega^-(t)$. The complete setting is depicted in Figure 1.1 below.

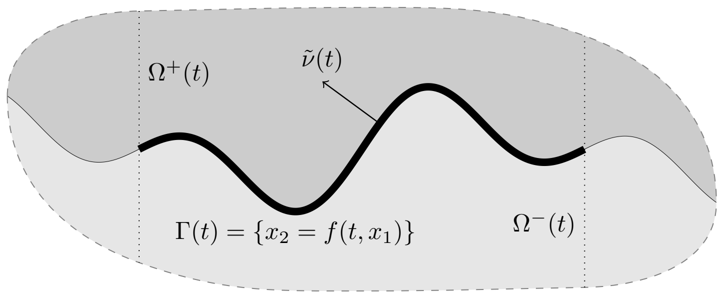


Figure 1.1: The geometric setting of the two-phase flow.

Next, we introduce the governing equations for each fluid phase. We impose the stationary Stokes equations for each phase in its corresponding domain, that is

$$\left.\begin{aligned} \mu^\pm \Delta v^\pm(t) - \nabla p^\pm(t) &= (0, g\rho^\pm) \text{ in } \Omega^\pm(t), \\ \operatorname{div} v^\pm(t) &= 0 \qquad\quad\ \text{ in } \Omega^\pm(t) \end{aligned}\right\} \qquad \text{for } t > 0.$$

Here, $v^\pm = v^\pm(t) : \Omega^\pm(t) \to \mathbb{R}^2$ denotes the velocity field of the fluid located in the phase $\Omega^\pm(t)$, while $p^\pm = p^\pm(t) : \Omega^\pm(t) \to \mathbb{R}$ represents the pressure in $\Omega^\pm(t)$. The constants $\mu^\pm > 0$ and $\rho^\pm > 0$ describe the viscosity and density of the fluid occupying $\Omega^\pm(t)$, respectively. Lastly, the constant $g \geq 0$ models Earth's gravity. We recall that these equations describe the flow of an incompressible Newtonian fluid at low Reynolds numbers, i.e., in the regime where viscous forces dominate inertial forces, with the inclusion of a gravitational force, when $g > 0$, acting in the vertical direction.

We proceed by defining the jump relations for the fluid velocities and stresses across the interface and assume that the fluid velocities are continuous across the interface such that no phase transition occurs and the interface is transported by the flow. Moreover, we model surface tension by introducing a jump for the stress proportional to the local curvature. This leads to us to

$$\left.\begin{aligned} [v(t)] &= 0 && \text{on } \Gamma(t), \\ [T_\mu(v(t),p(t))]\tilde{\nu}(t) &= -\sigma\tilde{\kappa}(t)\tilde{\nu}(t) && \text{on } \Gamma(t) \end{aligned}\right\} \quad \text{for } t > 0,$$

where $[v(t)]$ and $[T_\mu(v(t),p(t))]$ is the jump of the velocity and the stress tensor across the interface, respectively, see the definition (3.1.3) below. Furthermore, the stress tensor of the fluid in $\Omega^\pm(t)$ is defined by

$$T_{\mu^\pm}(v^\pm(t),p^\pm(t)) := -p^\pm(t)I_2 + \mu^\pm(\nabla v^\pm(t) + (\nabla v^\pm)^\top(t)) \tag{1.2.1}$$

where $I_n \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, is the identity matrix and, for $w \in \mathrm{C}^1(\Omega,\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$ open, we denote by ∇w the matrix with entries

$$(\nabla w)_{ij} := \partial_j w_i, \qquad 1 \le i,j \le n.$$

Lastly, $\sigma > 0$ is the surface tension coefficient and $\tilde{\kappa}(t) : \Gamma(t) \to \mathbb{R}$ is the signed curvature of the interface $\Gamma(t)$.

Since we work in a vertically unbounded domain, we need to consider the far-field behavior of both the fluid velocities and pressures. To do so, we introduce the following asymptotic condition:

$$(v^\pm(t), p^\pm(t) + g\rho^\pm x_2)(x) \to \Big(\pm\frac{c_{1,\Gamma(t)}}{\mu^\pm}, \pm\frac{c_{2,\Gamma(t)}}{\mu^\pm}, \pm c_{3,\Gamma(t)}\Big) \text{ for } x_2 \to \pm\infty$$

uniformly in $x_1 \in \mathbb{S}$, where $c_{\Gamma(t)} := (c_{1,\Gamma(t)}, c_{2,\Gamma(t)}, c_{3,\Gamma(t)}) \in \mathbb{R}^3$ is a (spatially) constant vector that is a priori unknown and, together with $v^\pm(t)$ and $p^\pm(t)$, constitutes part of the solution.

To complete our model, we introduce the kinematic boundary condition

$$V_n(t) = v(t)\cdot\tilde{\nu}(t) \qquad \text{on } \Gamma(t),$$

where $V_n(t)$ is the normal velocity of the interface. This condition ensures that the interface is transported with the flow.

Using the substitution

$$q^\pm(t,x) := p^\pm(t,x) + g\rho^\pm x_2, \qquad x \in \Omega^\pm(t),$$

we can combine the governing equations for the Stokes flow into the following system:

$$\left.\begin{aligned} \mu^\pm\Delta v^\pm(t) - \nabla q^\pm(t) &= 0 && \text{in } \Omega^\pm(t), \\ \operatorname{div} v^\pm(t) &= 0 && \text{in } \Omega^\pm(t), \\ [v(t)] &= 0 && \text{on } \Gamma(t), \\ [T_\mu(v(t),q(t))]\tilde{\nu}(t) &= (\Theta x_2 - \sigma\tilde{\kappa}(t))\tilde{\nu}(t) && \text{on } \Gamma(t), \\ (v^\pm(t),q^\pm(t))(x) &\to \Big(\pm\tfrac{c_{1,\Gamma(t)}}{\mu^\pm}, \pm\tfrac{c_{2,\Gamma(t)}}{\mu^\pm}, \pm c_{3,\Gamma(t)}\Big) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \\ V_n(t) &= v(t)\cdot\tilde{\nu}(t) && \text{on } \Gamma(t) \end{aligned}\right\} \tag{1.2.2a}$$

for $t > 0$, and supplement them with the initial condition

$$f(0) = f_0. \tag{1.2.2b}$$

In (1.2.2a), the constant $\Theta \in \mathbb{R}$ is defined by

$$\Theta := -g[\rho] = -g(\rho^+ - \rho^-) \in \mathbb{R}, \tag{1.2.3}$$

and describes a weighted difference of the densities of the fluids.

Summarizing, the system (1.2.2) describes the motion of two immiscible Newtonian fluids separated by a sharp interface in the periodic plane. The flow is assumed to be in the low Reynolds numbers regime, such that viscous forces dominate inertial forces and a description via the Stokes equations is justified. Additionally, the fluids are subject to surface tension effects at the interface and we may incorporate an external body force acting in the vertical direction which models gravity. The flow is quasistationary which is reflected by the fact that the Stokes equations themselves are stationary while the domain on which they are defined changes over time as the interface is transported by the flow due to the kinematic boundary condition.

1.3. Discussion of related literature

Free boundary problems arising in fluid dynamics have been studied extensively over the past decades and remain an active field of research to this date. Early contributions in this area include work on the one-phase Navier–Stokes equations [8,9,84–86], though the list is far from being complete. In these seminal works, the case of a sufficiently smooth evolving domain as well as a horizontally unbounded fluid phase modeling the ocean are considered and analytic results include local well-posedness and global existence for small initial data in fractional Sobolev spaces and Hölder spaces. More recent results are contained, e.g., in [80, 83] using the approach of maximal regularity and establishing new well-posedness results in certain Besov spaces. The case of a moving interface separating two fluid phases subject to the Navier–Stokes equations was studied later on, e.g., in [22–24, 75, 96]. Here, one fluid phase either takes the form of a drop enclosed by the other fluid or the two phases form non-enclosing domains separated by a hypersurface.

Research on quasistationary Stokes flow also has its origins in the study of single-phase fluids evolving in sufficiently smooth domains $\Omega(t) \subset \mathbb{R}^n$, $n \geq 2$. In [48], well-posedness near smooth and strictly star-shaped configurations and exponential stability of balls were established. Alternative proofs of exponential stability, based on power series techniques, were later provided in dimensions $n \in \{2, 3\}$ in [34, 35]. More recently, exponential stability of balls was also shown in the planar two-phase situation, where one fluid encloses the other and the interface dynamics are governed by surface tension [17]. In a different direction, the quasistationary Stokes equations arise as the singular zero-Reynolds number limit of the Navier–Stokes system, see [87, 88]. Lastly, in [45], the nonstationary two-phase Stokes flow in a bounded rectangle is studied.

The two-phase version of the problem in bounded geometries, including possible phase transitions and in arbitrary dimension $n \geq 2$, is treated systematically in the monograph [76] while stabilization of balls using a feedback operator acting on the interface was studied in [19].

The non-periodic counterpart of (1.2.2) with gravity effects excluded and general viscosities was first studied in [5] where existence of solutions was shown for initial data which are small in a certain space of Fourier transforms of bounded measures. More recently, in [64,65], local well-posedness was obtained for initial data in $\mathrm{H}^r(\mathbb{R})$ for exponents $r \in (3/2, 2)$ arbitrarily close to the critical value $r = 3/2$, cf. [65, Remark 1.2]. The singular limit of the two-phase problem, when the viscosity of the upper fluid tends to zero, was studied in [66] and it was shown therein that it converges to the one-phase problem.

When surface tension is neglected ($\sigma = 0$), the problem reduces to an ODE formulation, yielding local [37] and global well-posedness [38] in the setting of Hölder spaces. Additionally, [36] presents stability analysis of this system along with corresponding numerical simulations. Importantly, the case $\sigma = 0$ relates to the transport Stokes system, a model for sedimentation of rigid particles in viscous fluids [47,52,57,68,69].

There also have been intensive studies on the numerical analysis of two-phase Stokes flow. We refer to [6,43,50] and the references therein.

A related line of research concerns the Peskin problem for an elastic string immersed in a viscous fluid, which shares similar equations for the dynamics in the bulk as (1.2.2) (with $g = 0$) but features different boundary conditions at the interface. Recent results include [14,16,39–42,59,70].

Lastly, we want to mention results concerning the stability of equilibrium solutions of related two-phase problems. For the Muskat problem in 2D, exponential stability of flat equilibria and Rayleigh–Taylor instability of finger-shaped stationary solutions is established in [29,63]. In the context of the Verigin problem [77], Rayleigh–Taylor instability is shown for flat equilibria. For the two-phase Navier–Stokes equations, several results regarding the Rayleigh–Taylor instability of flat equilibria are available, see, e.g., [49,54,74,94,97].

1.4. Main results and contributions

This thesis is the first work studying the two-phase periodic quasistationary Stokes flow in the periodic plane $\mathbb{S} \times \mathbb{R}$ where both surface tension and gravity effects are incorporated. In addition, a comprehensive study of equilibrium solutions to (1.2.2) is provided.

The closest relative to our model studied in the literature is the system (1.2.2) driven solely by gravitational forces, that is $\sigma = 0$. In this case, the system is hyperbolic, whereas, in contrast, the presence of surface tension turns it into a parabolic problem. In [37,38], this hyperbolic system is analyzed in detail in the case of equal viscosities and global well-posedness is shown for the evolution of the interface if it is initially given by a curve in $\mathrm{C}^{1+\alpha}$, $\alpha \in (0,1)$. Similar to our approach, the references use potential theory to construct explicit representations for the velocity and the pressure in the bulk. The case of different viscosities is not considered, since these explicit representations, which are fundamental to their analysis, are not available, see Chapter 5.

Another closely related model to ours is the one studied in [64,65]. Here, the geometry is non-periodic and the fluids fill the whole plane $\mathbb{R}^2$. Moreover, only surface tension effects were considered, since the logarithmic term of the Stokeslet (a feature specific to the 2D setting), see (3.2.22), makes the inclusion of gravity terms difficult to handle in the unbounded graph framework. Nonetheless, we still pursue the same strategy as therein for solving (1.2.2). Specifically, at each time step $t > 0$, we determine the velocity, the pressure, and the constant $c_\Gamma(t)$ from (1.2.2) assuming that $f(t) \in \mathrm{H}^3(\mathbb{S})$. This approach relies on potential theory to construct the solution in the bulk, which is applicable because the Stokes equations are linear. Similar approaches were also considered in the analysis of the Mullins–Sekerka problem [28] or the Muskat problem [18].

A substantial difference between the periodic and non-periodic cases lies in the far-field boundary condition $(1.2.2)_5$. In the non-periodic case, the velocity converges to zero at infinity. Surprisingly, in the periodic setting, the velocity profile becomes asymptotically horizontal as $x_2 \to \pm\infty$, with its magnitude determined by the average vorticity in $\mathbb{S}\times\mathbb{R}$ and the profiles $+\infty$ and $-\infty$ having opposite signs. Similarly, for $x_2 \to \pm\infty$, the pressure differs from the hydrostatic pressure by a constant depending on f taking opposite signs at $\pm\infty$. The precise relation is given by

$$c_{\Gamma(t)} = \Big(- \frac{\mu^+\mu^-}{2\pi(\mu^+ + \mu^-)} \,\mathrm{PV}\!\int_{\mathbb{S}\times\mathbb{R}} \operatorname{curl} v(t)\,\mathrm{d}x,\ 0,\ -\frac{\Theta}{2}\langle f(t)\rangle \Big), \tag{1.4.1}$$

where the vorticity $\operatorname{curl} v(t)$ of the velocity field $v(t) = (v_1(t), v_2(t))$ is given by

$$\operatorname{curl} v(t) := \partial_1 v_2(t) - \partial_2 v_1(t), \tag{1.4.2}$$

and $\langle g\rangle$ denotes the integral mean of a periodic function $g : \mathbb{S} \to \mathbb{R}$ defined by

$$\langle g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\,\mathrm{d}\xi. \tag{1.4.3}$$

Using the kinematic boundary condition $(1.2.2)_6$, we may then reformulate (1.2.2) as a single nonlocal, fully nonlinear evolution equation for f, the corresponding evolution operator being well-defined for $f \in \mathrm{H}^r(\mathbb{S})$ with $r > 3/2$. Exploiting mapping properties established for the first time in this thesis for the underlying singular integral operators, we prove that the resulting evolution equation is parabolic in the sense of Proposition 6.3.1. Applying the abstract parabolic theory due to Lunardi [60], we finally obtain local well-posedness of (1.2.2) together with a parabolic smoothing property, which states that the interface, though initially lying in $\mathrm{H}^r(\mathbb{S})$ with $r \in (3/2, 2)$, becomes smooth instantaneously. The results are summarized in the following first main theorem.

Theorem 1.4.1. *Let* $\Theta \in \mathbb{R}$, $\sigma, \mu^+, \mu^- \in (0,\infty)$, $r \in (3/2, 2)$, *and* $f_0 \in \mathrm{H}^r(\mathbb{S})$. *Then, the following hold:*

(i) (Well-posedness) *There exists a unique maximal solution* $(f, v^\pm, q^\pm, c_\Gamma)$ *to* (1.2.2) *with*

$$f := f(\cdot, f_0) \in \mathrm{C}([0,T_+), \mathrm{H}^r(\mathbb{S})) \cap \mathrm{C}^1([0,T_+), \mathrm{H}^{r-1}(\mathbb{S})),$$

and such that for each $t \in (0, T_+)$, *we have* $f(t) \in \mathrm{H}^3(\mathbb{S})$ *and*

$$v^\pm(t) \in \mathrm{C}^2(\Omega^\pm(t), \mathbb{R}^2) \cap \mathrm{C}^1(\overline{\Omega^\pm(t)}, \mathbb{R}^2), \qquad q^\pm(t) \in \mathrm{C}^1(\Omega^\pm(t)) \cap \mathrm{C}(\overline{\Omega^\pm(t)}),$$

where c_Γ *is given by* (1.4.1) *and* $T_+ = T_+(f_0) \in (0, \infty]$.

(ii) (Parabolic smoothing) *The function* f *is smooth on* $(0, T_+) \times \mathbb{S}$, *that is,*

$$[(t,\xi) \mapsto f(t,\xi)] \in \mathrm{C}^\infty((0,T_+) \times \mathbb{S}, \mathbb{R}).$$

(iii) (Global existence) *The solution is global, that is* $T_+(f_0) = \infty$, *provided that for each* $T > 0$,

$$\sup_{t\in[0,T]\cap[0,T_+(f_0))} \|f(t)\|_{\mathrm{H}^r} < \infty.$$

We make the following remarks concerning Theorem 1.4.1.

Remark 1.4.2. (i) In the non-periodic setting, the resulting evolution equation has a certain scaling invariance which identifies $\mathrm{H}^{3/2}(\mathbb{R})$ as a critical space, see [65, Remark 1.2]. Naturally, it follows that $\mathrm{H}^{3/2}(\mathbb{S})$ is a critical space for the periodic Stokes flow described by (1.2.2). In this sense, the local well-posedness result established in Theorem 1.4.1 is (close to) optimal as it covers all subcritical spaces $\mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, with r arbitrarily close to the critical exponent $3/2$.

(ii) The proof of Theorem 1.4.1 relies on the abstract parabolic theory developed by Lunardi [60, Chapter 8], in particular on the result formulated in Theorem 2.3.9. In this theorem, existence and uniqueness is established for strict solutions lying in certain weighted Hölder spaces. When proving Theorem 1.4.1 in Chapter 6, we provide a refinement of this uniqueness outside the framework of weighted Hölder spaces. This refinement is essential when establishing the parabolic smoothing property, which relies on a classical parameter trick that fails in the setting of weighted Hölder spaces, since these spaces are not invariant under the transformation involved in the parameter trick.

(iii) Since we only assume $f_0 \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, the curvature $\tilde{\kappa}$ may not be well-defined in a classical sense. However, due to the parabolic smoothing property, it is defined in a classical sense after the initial time. In comparison, the classical results in [34, 35, 48, 76] all consider initial interfaces which are at least in C^2.

Before presenting our results on the existence and stability properties of equilibrium solutions to (1.2.2), see Theorem 1.4.3 and Theorem 1.4.4, we note that the set of solutions to (1.2.2) is invariant under horizontal and vertical translations. Indeed, if $(f, v^\pm, q^\pm, c_\Gamma)$ is a solution to (1.2.2) with initial data f_0, then $(\tilde{f}, \tilde{v}^\pm, \tilde{q}^\pm, c_{\tilde{\Gamma}})$ given by

$$\begin{aligned} \tilde{f}(t,\xi) &= f(t,\xi - a) + c, \\ \tilde{v}^\pm(t,x) &= v^\pm(t, x - (a,c)), \\ \tilde{q}^\pm(t,x) &= q^\pm(t, x-(a,c)) \mp \frac{c\Theta}{2}, \\ c_{\tilde{\Gamma}} &= c_\Gamma - (0,0,c\Theta/2), \end{aligned} \tag{1.4.4}$$

is also a solution to (1.2.2) with initial data $\tilde{f}_0 = f_0(\cdot - a) + c$ for a, $c \in \mathbb{R}$. Furthermore, the integral mean $\langle f \rangle$, see (1.4.3), is preserved by the flow since, by $(1.2.2)_2$, $(1.2.2)_6$, and (1.4.1), we have

$$\frac{\mathrm{d}\langle f\rangle}{\mathrm{d}t}(t) = \int_{\Gamma(t)} v(t)\cdot\tilde{\nu}(t)\,\mathrm{d}\sigma = \int_{\Omega^\pm(t)} \operatorname{div} v^\pm(t)\,\mathrm{d}x = 0, \qquad t \in (0, T_+). \tag{1.4.5}$$

The properties (1.4.4)–(1.4.5) will be considered when analyzing the stability of equilibrium solutions. In particular, we focus on equilibrium solutions with zero integral mean and examine their stability under perturbations that also have zero integral mean. To this end, we define

$$\widehat{\mathrm{H}}^s(\mathbb{S}) := \{ f \in \mathrm{H}^s(\mathbb{S}) : \langle f \rangle = 0\}, \qquad s \geq 0.$$

Equilibrium solutions to (1.2.2) with zero integral mean are characterized by the fact that $f \in \mathrm{C}^\infty(\mathbb{S})$ solves the capillary equation

$$\kappa(f) + \lambda f = 0, \qquad \text{where } \lambda := \frac{g[\rho]}{\sigma}, \tag{1.4.6}$$

see (1.2.3), and where $\kappa(f)$ is the signed curvature of f defined by

$$\kappa(f) = \left(\frac{f'}{(1+f'^2)^{1/2}}\right)' = \frac{f''}{(1+f'^2)^{3/2}}.$$

The complete picture of the equilibrium solutions to (1.2.2) is provided in Chapter 7.

In Theorem 1.4.3, we show that the trivial equilibrium $f = 0$ is exponentially stable if $g[\rho]/\sigma < 1$, which happens either when the fluid below is denser than the fluid above, or when $[\rho] > 0$ is smaller than σ/g. If, however, $g[\rho]/\sigma > 1$, the equilibrium $f = 0$ is unstable in the sense that there exists a neighborhood of zero in $\widehat{\mathrm{H}}^r(\mathbb{S})$ and solutions starting arbitrarily close to 0 that eventually leave this neighborhood. This reflects the fact that gravity amplifies small disturbances when $\rho^+ > \rho^-$, a phenomenon known as the Rayleigh–Taylor instability, see [78, 90].

Theorem 1.4.3 (Exponential stability)**.** *Let* $\Theta \in \mathbb{R}$ *and* σ, μ^+, $\mu^- \in (0,\infty)$*, as well as* $r \in (3/2, 2)$ *be given, and introduce the constant*

$$\vartheta_0 := \frac{\sigma+\Theta}{2(\mu^+ + \mu^-)}\mathbf{1}_{[0,\infty)}(\sigma-\Theta) + \frac{\sqrt{\sigma\Theta}}{\mu^+ + \mu^-}\mathbf{1}_{(0,\infty)}(\Theta-\sigma). \tag{1.4.7}$$

(a) (Exponential stability) *Assume* $\sigma + \Theta > 0$ *and choose* $\vartheta \in (0, \vartheta_0)$*. Then, there exist constants* $\delta > 0$ *and* $M > 0$*, such that for any* $f_0 \in \widehat{\mathrm{H}}^r(\mathbb{S})$ *satisfying* $\|f_0\|_{\mathrm{H}^r} < \delta$*, the solution to* (1.2.2) *exists globally, i.e.* $T_+(f_0) = \infty$ *and*

$$\|f(t)\|_{\mathrm{H}^r} + \left\|\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right\|_{\mathrm{H}^{r-1}} \leq M e^{-\vartheta t}\|f_0\|_{\mathrm{H}^r} \qquad \textit{for all } t \geq 0.$$

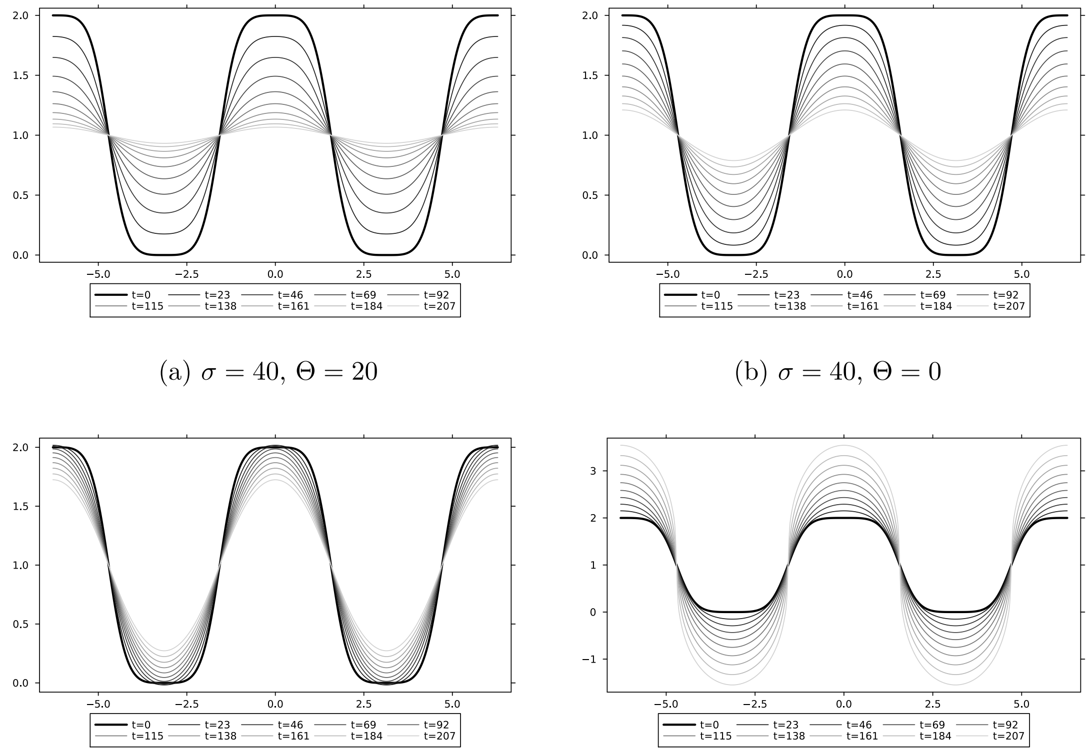
(b) (Rayleigh–Taylor instability) *If* $\sigma + \Theta < 0$*, then the zero solution is unstable.*

In Figure 1.2, the evolution of the interface initially given by the smooth function

$$f_0(\xi) := \frac{2}{1 + \tan(\xi/2)^4}$$

is simulated for different physical parameters. The simulation is done in Julia [10] for the case of equal viscosities $\mu^+ = \mu^- = 1$, by calculating the trace of the velocity on the interface according to (3.3.51) at every timestep and the evolution equation is advanced in time by using the explicit Euler method. It is important to mention that this formula is only available because we work with equal viscosities and the initial interface is at least in $\mathrm{H}^3(\mathbb{S})$ (in fact, it is smooth). Note that while the integrands involved have a singularity, they are still integrable. To compute these integrals, we use the rectangle rule and a first-order Taylor approximation of the integrand at the singularity. Differentiation in space is done with the `interpolations.jl` package. We choose a time step size of $\Delta t = 0.001$ and an equispaced grid of size $\Delta x = 2\pi/2000$. These simulations support Theorem 1.4.3 in the sense that we observe convergence to the flat interface in the Rayleigh–Taylor stable regime in Figure 1.2 (a)–1.2 (b) while, in Figure 1.2 (c), convergence is still visible but is much slower since the fluid above is more dense than the fluid below, but the small density jump is dominated by surface tension. However, in the Rayleigh–Taylor unstable scenario where surface tension cannot balance the positive density jump, the numerical simulations in Figure 1.2 (d) show that the gradient of the solution blows up in finite time.

If $g[\rho]/\sigma$ exceeds a threshold value $\lambda_* \in (0,1)$, see (7.3.2), the Stokes flow (1.2.2) may also possess finger-shaped equilibria which are located on global bifurcation branches



(a) $\sigma = 40$, $\Theta = 20$

(b) $\sigma = 40$, $\Theta = 0$

(c) $\sigma = 40$, $\Theta = -20$

(d) $\sigma = 40$, $\Theta = -45$

Figure 1.2: The evolution of the interface initially given by $f_0(\xi) = 2/(1 + \tan(\xi/2)^4)$ for different physical parameters.

(of solutions to (1.4.6)). These describe configurations where the heavier fluid positioned above intrudes into the lighter fluid below, forming characteristic finger-like patterns. We completely classify these finger-shaped equilibria in Section 7.3 in view of the fact that any stationary solution to (1.2.2) is a horizontal and/or vertical translation of the solutions determined in Theorem 7.3.1. In Theorem 1.4.4 we prove, by applying the exchange of stability principle due to Crandall and Rabinowitz [21], that the small finger-shaped equilibrium solutions, which have a cosine-shaped profile, also feature the Rayleigh–Taylor instability property. The stability properties of finger-shaped equilibria with large slopes (which become unbounded as one approaches the end of the global bifurcation branches) are not addressed by Theorem 1.4.4, and this remains an open problem.

Theorem 1.4.4 (Rayleigh–Taylor instability of small finger-shaped equilibria)**.** *Again, let* $\Theta \in \mathbb{R}$*,* $\sigma, \mu^+, \mu^- \in (0,\infty)$*, and* $r \in (3/2, 2)$ *be given. For any* $\ell \in \mathbb{N}$*, there exists a smooth bifurcation curve* $(\lambda_\ell, f_\ell) : (-\delta_\ell, \delta_\ell) \to \mathbb{R} \times \widehat{\mathrm{H}}^r(\mathbb{S})$*,* $\delta_\ell > 0$*, such that*

$$\begin{cases} \lambda_\ell(s) := \ell^2 - \dfrac{3\ell^4}{8}s^2 + \mathcal{O}(s^4) & \text{in } \mathbb{R}, \\ f_\ell(s) := s\cos(\ell\cdot) + \mathcal{O}(s^2) & \text{in } \widehat{\mathrm{H}}^r(\mathbb{S}), \end{cases} \qquad \text{for } s \to 0,$$

with $f_\ell(s)$ *being an even equilibrium to* (1.2.2) *if* $\Theta = -\sigma\lambda_\ell(s)$*. The equilibrium* $f_\ell(s)$ *is unstable if* $0 \le |s| < \delta_\ell$ *and* δ_ℓ *is sufficiently small.*

In the case of equal viscosities, Theorem 1.4.1 and Theorem 1.4.3 are published in

[12] D. Böhme and B.-V. Matioc, *Well-posedness and stability for the two-phase*

periodic quasistationary Stokes flow, Interfaces Free Bound., 27 (2025), pp. 659–701.

In the case of different viscosities, the results in Theorem 1.4.1 and Theorem 1.4.3–Theorem 1.4.4 build on those established in [12] and appear in the preprint

[11] D. Böhme and B.-V. Matioc, *Well-posedness and Rayleigh-Taylor instability of the two-phase periodic quasistationary Stokes flow*, 2025. arXiv:2508.15502.

1.5. Outline

We now give a brief overview of the content of this thesis.

In Chapter 2, we define some notation and review the basic mathematical concepts used throughout this thesis.

In Part I, we study stationary two-phase Stokes problems on fixed domains with the ultimate goal of solving $(1.2.2)_{1-5}$ for a prescribed interface. These correspond to boundary value problems described by the Stokes equations in the bulk supplemented with boundary conditions of transmission type. To this end, Chapter 3 introduces the method of layer potentials, which we use to solve two transmission boundary value problems related to (1.2.2) in Section 3.3. The results of Subsection 3.3.1 are contained in [12], while the results of Subsection 3.3.2 appear in [11]. Chapter 4 focuses on the invertibility of the hydrodynamic double-layer potential operator. In particular, we introduce a new class of (singular) integral operators $B^{p,q}_{n,m}$ whose mapping properties are essential when proving that all real numbers $\lambda \in \mathbb{R}$ with $|\lambda| > 1/2$ are contained in the resolvent set of the double-layer potential operator. Finally, in Chapter 5, we show the unique solvability of the fixed time problem associated with (1.2.2), building on the results from the previous two chapters. The results of Chapter 4 and Chapter 5 are contained in [11].

In Part II, we reformulate the two-phase quasistationary Stokes problem (1.2.2) as an evolution equation for the function that parametrizes the interface and prove our main theorems from Section 1.4. More precisely, in Section 6.2, we use the solution to the fixed time problem of (1.2.2) obtained in Chapter 5 and the kinematic boundary condition $(1.2.2)_6$ to reformulate (1.2.2) as a single nonlocal, fully nonlinear evolution equation for the function $f(t)$ whose graph represents the interface $\Gamma(t)$. In Section 6.3, we analyze the evolution operator arising from this evolution equation in detail and show that its Fréchet derivative is the generator of a strongly continuous analytic semigroup. This property allows us to use abstract parabolic theory due to Lunardi [60] that is essential in the proof of Theorem 1.4.1 which we present in Section 6.4. Lastly, in Chapter 7, we give a comprehensive overview of the equilibrium solutions to (1.2.2) and analyze their stability properties, culminating in the proofs of Theorem 1.4.3 and Theorem 1.4.4. The contents of these chapters appear in [11], while similar analysis was already carried out in the case of equal viscosities in [12].

In Appendix A we study the integral operators $B^{p,q}_{n,m}$ introduced in Chapter 4 and other related integral operators. The results obtained in Appendix A may be of independent interest as they seem to be applicable when tackling other two-phase problems in the periodic plane such as the Mullins–Sekerka problem. Moreover, we present a localization result which is of particular importance in Section 6.4. The results of this chapter are based on [11, 12].

Chapter 2

Preliminaries

In this chapter, we state some preliminary results and fix basic notation. Throughout this thesis we define $\mathbb{N}$ as the set of positive integers, that is

$$\mathbb{N} := \{1, 2, 3, \dots\},$$

and define $\mathbb{N}_0$ as the set of non-negative integers, that is

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Moreover, we denote by C a positive constant which may change value from line to line. If it is important that C depends only on certain quantities, we make this clear by listing them.

2.1. Resolvent and spectrum

Given a Banach space X, we denote its norm by $\|\cdot\|_X$ and given an additional Banach space Y, we denote by $\mathcal{L}(X,Y)$ the set of all linear and bounded operators $T : X \to Y$. In particular, we set $\mathcal{L}(X) := \mathcal{L}(X,X)$. Moreover, we introduce the set of isomorphisms from X to Y as

$$\mathcal{L}_{\mathrm{Iso}}(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is bijective}\}.$$

Let $E_1, \dots, E_n$, E, F be Banach spaces, with $n \in \mathbb{N}$. We write $\mathcal{L}^n(\prod_{i=1}^n E_i, F)$ to denote the Banach space of bounded n-linear maps from $\prod_{i=1}^n E_i$ into F. In the special case when $E_i = E$ for all $1 \leq i \leq n$, we abbreviate this space by $\mathcal{L}^n(E,F)$, and let $\mathcal{L}^n_{\mathrm{sym}}(E,F)$ denote the subspace of symmetric operators. Lastly, given linear operators A, B, C, $D \in \mathcal{L}(E,F)$, we use the notation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} := \begin{pmatrix} A[f] + B[g] \\ C[f] + D[g] \end{pmatrix}, \qquad f,\, g \in E, \tag{2.1.1}$$

to distinguish from the usual matrix-vector product.

Next, we introduce the notion of a closed operator.

Definition 2.1.1. Let X, Y be Banach spaces and let $D \subset X$ be a subspace of X. We call a linear operator $T : D \to Y$ *closed* if for all sequences $(x_n) \subset D$ with $x_n \to x \in X$ and $Tx_n \to y \in Y$, we have $x \in D$ and $Tx = y$. If D is dense in X, we call T *densely defined.*

We call $\operatorname{dom}(T) = D$ the *domain* of T and write $T : \operatorname{dom}(T) \subset X \to Y$. Note that any operator $T \in \mathcal{L}(X, Y)$ is a closed operator.

We next introduce the resolvent set and spectrum of a linear operator.

Definition 2.1.2. Let $T : \operatorname{dom}(T) \subset X \to X$ be a linear operator. We call

$$\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T : \operatorname{dom}(T) \subset X \to X \text{ bijective and } (\lambda - T)^{-1} \in \mathcal{L}(X)\}$$

the *resolvent set* of T. Moreover, we define the *spectrum* of T as

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

The map $R(\cdot, T) : \rho(T) \to \mathcal{L}(X)$, defined by

$$R(\lambda, T) := (\lambda - T)^{-1},$$

is called the *resolvent*.

In the next theorem we recall useful properties of the resolvent set and the spectrum, see e.g. [60, 95].

Theorem 2.1.3. *Let $T : \operatorname{dom}(T) \subset X \to X$ be a densely defined operator. Then,*

(i) *the resolvent set $\rho(T)$ is open.*

(ii) *the spectrum $\sigma(T)$ is closed.*

(iii) *if additionally $T \in \mathcal{L}(X)$, the spectrum $\sigma(T)$ is compact.*

We finish this section by stating the method of continuity, see [2, Proposition I.1.1.1], which is used several times in our analysis.

Theorem 2.1.4. *Let M be a connected metric space, X, Y Banach spaces, and let $B : M \to \mathcal{L}(X, Y)$ be a continuous mapping satisfying*

$$\|B(t)[x]\|_Y \geq \beta \|x\|_X, \qquad x \in X, \quad t \in M,$$

for some constant $\beta > 0$. Then, if

$$B(M) \cap \mathcal{L}_{\text{Iso}}(X, Y) \neq \emptyset,$$

we have

$$B(M) \subset \mathcal{L}_{\text{Iso}}(X, Y).$$

2.2. Spaces of continuous functions

Given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $\mathrm{C}^k(\mathbb{S})$, $k \in \mathbb{N}_0$, the Banach space of continuous and $\mathbb{K}$-valued 2π-periodic functions on $\mathbb{R}$ with continuous derivatives up to order k, which is endowed with the standard norm

$$\|f\|_{\mathrm{C}^k} := \|f\|_\infty + \sum_{i=1}^{k} \|f^{(i)}\|_\infty, \qquad f \in \mathrm{C}^k(\mathbb{S}),$$

where $\|\cdot\|_\infty$ is the supremum-norm. Moreover, we set

$$\mathcal{D}(\mathbb{S}) := \mathrm{C}^\infty(\mathbb{S}) := \bigcap_{k \in \mathbb{N}} \mathrm{C}^k(\mathbb{S}).$$

We define the topological dual $\mathcal{D}'(\mathbb{S})$ of $\mathcal{D}(\mathbb{S})$ as the space of all linear functionals $g : \mathcal{D}(\mathbb{S}) \to \mathbb{K}$ that are continuous with respect to the canonical LF-topology. Similarly, we define $\mathrm{C}^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, and denote by $\mathcal{D}(\mathbb{R}^n) = \mathrm{C}_c^\infty(\mathbb{R}^n)$ the subspace of functions that additionally have compact support on $\mathbb{R}^n$. Then, we define the topological dual $\mathcal{D}'(\mathbb{R}^n)$ of $\mathcal{D}(\mathbb{R}^n)$ as the space of all linear functionals $g : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$ that are continuous with respect to the canonical LF-topology. We call the elements of $\mathcal{D}'(\mathbb{S})$ and $\mathcal{D}'(\mathbb{R}^n)$ *distributions.*

Given $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$, we denote by $\mathrm{C}^{k+\alpha}(\mathbb{S})$ the subspace of $\mathrm{C}^k(\mathbb{S})$ consisting of functions whose k'th derivative is α-Hölder continuous, that is

$$\mathrm{C}^{k+\alpha}(\mathbb{S}) := \Big\{ f \in \mathrm{C}^k(\mathbb{S}) : [f^{(k)}]_\alpha := \sup_{\substack{x,y\in\mathbb{S}\\ x\neq y}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} < \infty \Big\}.$$

This space is a Banach space equipped with the norm

$$\|f\|_{\mathrm{C}^{k+\alpha}} := \|f\|_{\mathrm{C}^k} + [f^{(k)}]_\alpha.$$

Now, let X, Y be Banach spaces and let $\mathcal{O} \subset X$ be an open set. Then, we denote by $\mathrm{C}(\mathcal{O}, Y)$ the space of continuous functions from $\mathcal{O}$ to Y. Furthermore, we denote by $\mathrm{C}^{1-}(\mathcal{O}, Y)$ the space of continuous functions $f : \mathcal{O} \to Y$ that satisfy a local Lipschitz condition. That is, for each $x_0 \in \mathcal{O}$, there exist constants $L(x_0) > 0$ and $r(x_0) > 0$ such that

$$\|f(x_1) - f(x_2)\|_Y \le L(x_0)\|x_1 - x_2\|_X$$

for all x_1, $x_2 \in B_{r(x_0)}(x_0) \subset \mathcal{O}$. Here,

$$B_r(x_0) := \{x \in X : \|x - x_0\|_X < r)\} \subset \mathcal{O}, \qquad r > 0,$$

is the open ball of radius $r > 0$ and center x_0 in $\mathcal{O}$. A function $f : \mathcal{O} \to Y$ is called *Fréchet differentiable* in $x_0 \in \mathcal{O}$ if there exists a linear operator $A(x_0) \in \mathcal{L}(X, Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|f(x_0 + h) - f(x_0) - A(x_0)[h]\|_Y}{\|h\|_X} = 0.$$

If such an operator exists, it is unique and we call $\partial f(x_0) := A(x_0)$ the Fréchet derivative of f in x_0 and denote by $\mathrm{C}^1(\mathcal{O}, Y)$ the set of functions $f : X \to Y$ that are Fréchet differentiable in $\mathcal{O}$ and posses a continuous Fréchet derivative $\partial f : \mathcal{O} \to \mathcal{L}(X, Y)$. We recursively define the space $\mathrm{C}^{k+1}(\mathcal{O}, Y)$, $k \in \mathbb{N}$, as the set of functions $f \in \mathrm{C}^k(\mathcal{O}, Y)$ such that

$$\frac{\|\partial^k f(x_0 + h) - \partial^k f(x_0) - \partial^{k+1} f(x_0)[h]\|_{\mathcal{L}^k(X,Y)}}{\|h\|_X} \to 0$$

as $\|h\|_X \to 0$, for all $x_0 \in \mathcal{O}$, and such that $\partial^{k+1} : \mathcal{O} \to \mathcal{L}^{k+1}(X, Y)$ is continuous. We call $\partial^k f(x_0) \in \mathcal{L}^k(X, Y)$ the k'th Fréchet derivative of f in x_0 and set

$$\mathrm{C}^\infty(\mathcal{O}, Y) := \bigcap_{k\in\mathbb{N}} \mathrm{C}^k(\mathcal{O}, Y).$$

Given any interval $I \subset \mathbb{R}$ and $k \in \mathbb{N}_0$, we denote by $\mathrm{C}^k(I, X)$ the space of continuous functions $f : I \to X$ with continuous derivatives up to order k. Note that these functions are not necessarily bounded if I is open. Moreover, we define for $\alpha \in (0,1)$ the Banach space $\mathrm{C}^\alpha(I, X)$ of Hölder continuous functions of order α by

$$\mathrm{C}^\alpha(I, X) := \Big\{ f : I \to X \text{ bounded} : [f]_{\mathrm{C}^\alpha(I,X)} := \sup_{\substack{t_1,t_2\in I\\ t_1\neq t_2}} \frac{\|f(t_1) - f(t_2)\|_X}{|t_1 - t_2|^\alpha} < \infty \Big\},$$

equipped with the norm

$$\|f\|_{\mathrm{C}^\alpha(I,X)} := \sup_{t\in I} \|f(t)\|_X + [f]_{\mathrm{C}^\alpha(I,X)}.$$

2.3. Analytic semigroups

Let X be a Banach space. Consider the abstract Cauchy problem

$$u'(t) = Au(t), \quad t > 0, \qquad u(0) = u_0 \in \operatorname{dom}(A), \tag{2.3.1}$$

for a linear operator $A : \operatorname{dom}(A) \subset X \to X$. We are looking for a *strict solution* to (2.3.1), by which we mean a solution u satisfying (2.3.1) pointwise with regularity

$$u \in \mathrm{C}([0,T], \operatorname{dom}(A)) \cap \mathrm{C}^1([0,T], X) \tag{2.3.2}$$

for some $T > 0$. If $T = \infty$, the solution is called a global strict solution. The simplest setting one can consider for this problem is that when A is a matrix, that is $A \in \mathbb{R}^{n\times n}$, $n \geq 1$, and it is well known that in this case there exists a unique global strict solution to (2.3.1) which is given by the matrix-exponential function

$$u(t) := e^{tA}u_0 := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} u_0, \qquad t \geq 0. \tag{2.3.3}$$

Similarly, if $A \in \mathcal{L}(X)$, one can construct a global strict solution via formula (2.3.3). However, if A is not bounded but only closed, formula (2.3.3) is not feasible anymore, since the domain of A^k, $k \geq 1$, may be trivial. Semigroup theory provides a framework to overcome this difficulty. The key idea is to associate to a suitable closed operator A a family of bounded operators $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$, called a *semigroup*, see Definition 2.3.1, which describes the evolution of (2.3.1). This approach allows to define solutions even when the initial value u_0 does not belong to $\operatorname{dom}(A)$ and no simple exponential formula exists, while also providing a systematic way to study existence, uniqueness, and continuity of solutions with respect to initial data.

As we will only work with a special class of semigroups, namely the so-called *strongly continuous analytic semigroups*, we will lay our focus on them and start with the definition of a *(strongly continuous) semigroup.*

Definition 2.3.1. A family of bounded operators $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a *semigroup* if it satisfies the semigroup property

$$\begin{cases} T(t)T(s) = T(t+s), & t, s \geq 0, \\ \quad T(0) = \mathrm{id}_X. \end{cases} \tag{2.3.4}$$

If, in addition,

$$\lim_{t\to 0} \|T(t)x - x\|_X = 0$$

holds for all $x \in X$, we call $\{T(t)\}_{t\geq 0}$ a *strongly continuous semigroup.*

Remark 2.3.2. The name "semigroup" comes from the fact that $(\{T(t)\}_{t\geq 0}, \circ)$ is a semigroup in the algebraic sense and the mapping $T : (\mathbb{R}_{\geq 0}, +) \to (\{T(t)\}_{t\geq 0}, \circ)$ is a semigroup homomorphism.

Next, we introduce the infinitesimal generator of a semigroup.

Definition 2.3.3. Let $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ be a semigroup. We define its *infinitesimal generator* $A : \operatorname{dom}(A) \subset X \to X$ by setting

$$\operatorname{dom}(A) := \left\{ x \in X : \lim_{h\to 0} \frac{T(h)x - x}{h} \text{ exists in } X \right\}$$

and

$$Ax := \lim_{h\to 0} \frac{T(h)x - x}{h}, \qquad x \in \operatorname{dom}(A).$$

The next result is the crucial piece in linking a strongly continuous semigroup generated by the operator A to the abstract initial value problem (2.3.1) and shows that $t \mapsto T(t)u_0$ is a global strict solution to it whenever $u_0 \in \operatorname{dom}(A)$. The results are taken from [26, Chapter II.1].

Theorem 2.3.4. *Given a strongly continuous semigroup $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ with generator A, we have:*

(i) *A is a densely defined closed operator which determines the semigroup uniquely.*

(ii) *If $x \in \operatorname{dom}(A)$, then $T(t)x \in \operatorname{dom}(A)$ for all $t \geq 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)x) = T(t)Ax = AT(t)x, \qquad t \geq 0, \qquad T(t)x\big|_{t=0} = x.$$

This naturally leads to the question of which additional conditions a densely defined closed operator must satisfy in order to generate a strongly continuous semigroup. This question was first partially answered by Hille [51] and Yosida [98] and it turns out that the resolvent set of A must contain the right half-plane and the resolvent has to fulfill certain decay conditions, see [26, Chapter II.3] for the precise statement and related results.

We now focus on a special class of linear operators fulfilling the requirements of Hille and Yosida (if densely defined), called sectorial operators. To start, we define for $\omega \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ the open sector

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega \text{ and } |\arg(\lambda - \omega)| < \theta\} \subset \mathbb{C}.$$

Definition 2.3.5. A closed linear operator A on X is called *sectorial* if there exist $\omega \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supset S_{\theta,\omega}$, and if there exists $M > 0$ such that

$$\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega|}, \qquad \lambda \in S_{\theta,\omega}.$$

Given a sectorial operator A and $\theta' \in (0, \theta - \pi/2)$, we may define for any $z \in S_{\theta',0}$ the operator $T(z)$ on X by

$$T(z) := \frac{1}{2\pi i} \int_{\omega + \gamma_{r,\eta}} e^{z\lambda} R(\lambda, A)\, \mathrm{d}\lambda, \qquad z \in S_{\theta',0}, \tag{2.3.5}$$

where, for $r > 0$ and $\eta \in (\pi/2, \theta)$, we define the counterclockwise oriented curve $\gamma_{r,\eta}$ by

$$\gamma_{r,\eta} := \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \eta \text{ and } r \leq |\lambda|\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \eta \text{ and } r = |\lambda|\},$$

see Figure 2.1. Moreover, we set $T(0) := \mathrm{id}_X$. The integral representation is motivated by the Cauchy integral formula and it can be shown that the operator $T(z) \in \mathcal{L}(X)$ is well-defined for $z \in S_{\theta',0}$, that the mapping $z \mapsto T(z)$ is analytic in $S_{\theta-\pi/2,0}$, and that the definition (2.3.5) is independent of the particular choice of the integration path, that is, of $r > 0$ and $\eta \in (\pi/2, \theta)$, see, e.g., [26, Proposition II.4.3]. We therefore make the following definition:

Definition 2.3.6. A family of operators $\{T(z)\}_{z\in S_{\theta-\pi/2,0}\cup\{0\}} \subset \mathcal{L}(X)$ is called an *analytic semigroup* if $T(0) = \mathrm{id}_X$,

$$T(z_1)T(z_2) = T(z_1 + z_2), \qquad z_1,\, z_2 \in S_{\theta-\pi/2,0},$$

and if the mapping $[z \mapsto T(z)] : S_{\theta-\pi/2,0} \to \mathcal{L}(X)$ is analytic. We call an analytic semigroup *bounded*, if $z \mapsto \|T(z)\|_{\mathcal{L}(X)}$ is bounded in $S_{\theta-\pi/2-\varepsilon,0}$ for all $0 < \varepsilon < \theta - \pi/2$.

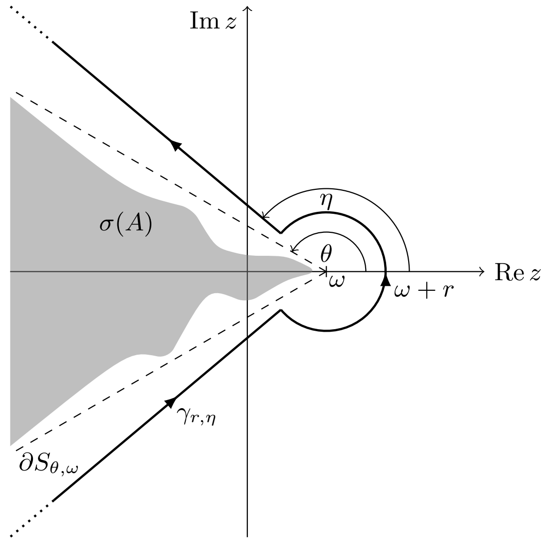


Figure 2.1: The path of integration around the spectrum.

In the next theorem we will connect analytic semigroups to sectorial operators and see that the family of operators defined in (2.3.5) is a bounded analytic semigroup with generator A, and, vice versa, that the generator of any bounded analytic semigroup is a sectorial operator.

Theorem 2.3.7. *Given a densely defined linear operator $A : \operatorname{dom}(A) \subset X \to X$, the following are equivalent:*

(i) *A is a sectorial operator.*

(ii) *A generates a bounded analytic semigroup $\{T(z)\}_{z \in S_{\pi/2-\theta,0} \cup \{0\}}$ on X which is given by* (2.3.5)*.*

In this case, the mapping $z \mapsto T(z)$ is strongly continuous in $S_{\theta-\pi/2-\varepsilon,0} \cup \{0\}$.

We refer to [26, Chapter II.4.a] and [60, Chapter 2.1] for proofs of this result and further properties of analytic semigroups. If $\{T(z)\}_{z \in S_{\theta-\pi/2,0} \cup \{0\}}$ is a strongly continuous analytic semigroup for some $\theta \in (\pi/2, \pi)$, we also call its restriction to the non-negative real numbers $\{T(t)\}_{t \geq 0}$ a strongly continuous analytic semigroup.

We now give a further characterization for operators that generate strongly continuous analytic semigroups, inspired by [2, Section I.1.2]. To start, let E_0, E_1 be Banach spaces such that E_1 is densely embedded in E_0, that is E_1 is a dense subset of E_0 and

$$\|x\|_{E_0} \leq C\|x\|_{E_1}, \qquad x \in E_1.$$

Next, we define

$$\mathcal{H}(E_1, E_0) \subset \mathcal{L}(E_1, E_0)$$

as the set of all operators $A \in \mathcal{L}(E_1, E_0)$ that, viewed as a linear operator in E_0 with domain E_1, are generators of strongly continuous analytic semigroups in $\mathcal{L}(E_0)$. Then, the following characterization of this set holds:

Theorem 2.3.8. *An operator $A \in \mathcal{L}(E_1, E_0)$ belongs to $\mathcal{H}(E_1, E_0)$ if and only if there exist constants $\kappa \geq 1$ and $\omega > 0$ such that*

$$\omega - A \in \mathcal{L}_{\mathrm{Iso}}(E_1, E_0),$$

and

$$\kappa\|(\lambda - A)[x]\|_{E_0} \geq |\lambda|\|x\|_{E_0} + \|x\|_{E_1}, \qquad x \in E_1, \quad \operatorname{Re}\lambda \geq \omega.$$

So far, we only considered linear operators but the evolution operator introduced in Chapter 6 that describes the system (1.2.2) is fully nonlinear. We therefore now turn our attention to problems of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = F(u(t)), \quad t > 0, \qquad u(0) = u_0, \tag{2.3.6}$$

where $F : \mathcal{O} \to X$ is a nonlinear function, D is a Banach space that is continuously and densely embedded in the Banach space X, $\mathcal{O} \subset D$ is an open subset, and $u_0 \in D$. To solve this problem, we first recall the variation of constants formula for a system of inhomogeneous linear ordinary differential equations. Given $A \in \mathbb{R}^{n\times n}$, $y_0 \in \mathbb{R}^n$, and a continuous function $f : \mathbb{R} \to \mathbb{R}^n$, the ordinary differential equation

$$y'(t) = Ay(t) + f(t), \qquad y(0) = y_0,$$

has a unique solution $y : [0, \infty) \to \mathbb{R}^n$ given by

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A} f(s)\,\mathrm{d}s. \tag{2.3.7}$$

This formula is a key ingredient when studying the solvability of (2.3.6). Indeed, if $u \in \mathrm{C}([0,T], D) \cap \mathrm{C}^1([0,T], X)$ is a strict solution to (2.3.6) for some $T > 0$, then, assuming that F is Fréchet differentiable, we have

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t}(t) &= \partial F(u_0)[u(t)] + (F(u(t)) - \partial F(u_0)[u(t)])\\ &= Au(t) + f(t), \qquad t \in [0,T],\end{aligned}$$

where $A := \partial F(u_0) \in \mathcal{L}(D, X)$ and $f(t) := F(u(t)) - \partial F(u_0)[u(t)]$ for $t \in [0,T]$. Inspired by (2.3.7), we formally write

$$\begin{aligned}u(t) &= e^{tA}u_0 + \int_0^t e^{(t-s)A} f(s)\,\mathrm{d}s\\ &= e^{tA}u_0 + \int_0^t e^{(t-s)A}(F(u(s)) - \partial F(u_0)[u(s)])\,\mathrm{d}s, \qquad t \in [0,T].\end{aligned} \tag{2.3.8}$$

To make sense of (2.3.8), we require that A is the generator of a strongly continuous analytic semigroup $\{T(t)\}_{t\geq 0} := \{e^{tA}\}_{t\geq 0}$ in $\mathcal{L}(X)$ such that the function $t \mapsto e^{tA}$ is well-defined. Concerning the integral term, additional regularity assumptions must be imposed on f and therefore implicitly on the nonlinear operator F to obtain a local well-posedness result.

Before stating a variant of this result due to Lunardi [60, Theorem 8.1.1], we define for $\alpha \in (0,1)$, $T > 0$, and a Banach space X the weighted Hölder space $\mathrm{C}^\alpha_\alpha((0,T], X)$ as

$$\mathrm{C}^\alpha_\alpha((0,T], X) := \left\{ f : (0,T] \to X \text{ bounded} : \sup_{0<t<s\leq T} \frac{\|t^\alpha f(t) - s^\alpha f(s)\|_X}{|t-s|^\alpha} < \infty \right\}, \tag{2.3.9}$$

and equip it with the norm

$$\|f\|_{\mathrm{C}^\alpha_\alpha} := \sup_{t\in(0,T]} \|f(t)\|_X + \sup_{0<t<s\leq T} \frac{\|t^\alpha f(t) - s^\alpha f(s)\|_X}{|t-s|^\alpha}.$$

We have the important embedding property $\mathrm{C}^\alpha([0,T],X) \hookrightarrow \mathrm{C}^\alpha_\alpha((0,T],X)$ since

$$\begin{aligned} \|f\|_{\mathrm{C}^\alpha_\alpha} &\le \sup_{t\in[0,T]} \|f(t)\|_X + \sup_{0\le t<s\le T} \left(\frac{|t|^\alpha \|f(t)-f(s)\|_X + \|f(s)\|_X |t^\alpha - s^\alpha|}{|t-s|^\alpha} \right) \\ &\le C\|f\|_{\mathrm{C}^\alpha([0,T],X)}, \qquad f \in \mathrm{C}^\alpha([0,T],X). \end{aligned} \tag{2.3.10}$$

We are now in a position to formulate the theorem which we will use to conclude local existence and uniqueness for the evolution problem (6.2.4), encoding (1.2.2), in Section 6.4.

Theorem 2.3.9. *Let D and X be Banach spaces such that D is densely embedded in X, let $\mathcal{O} \subset D$ be an open subset, and let $F : \mathcal{O} \to X$ be a continuous mapping which is Fréchet differentiable. Assume that $\partial F(u) : D \subset X \to X$ is a sectorial operator for all $u \in \mathcal{O}$. Moreover, assume that for each $u \in \mathcal{O}$ there exist constants $R = R(u) > 0$, $L = L(u) > 0$ such that*

$$\|\partial F(v) - \partial F(w)\|_{\mathcal{L}(D,X)} \le L\|v-w\|_X$$

for all v, $w \in B_R(u) \subset \mathcal{O}$. Then, given $\bar{u} \in \mathcal{O}$, there exist constants $r = r(\bar{u}) > 0$ and $T = T(\bar{u}) > 0$ such that for every $u_0 \in B_r(\bar{u}) \subset \mathcal{O}$, there exists a strict solution

$$u \in \mathrm{C}^\alpha_\alpha((0,T],D) \cap \mathrm{C}([0,T],D) \cap \mathrm{C}^1([0,T],X)$$

for all $\alpha \in (0,1)$ which solves (2.3.6) pointwise in $[0,T]$. Furthermore, this solution is unique in the set

$$\bigcup_{\alpha\in(0,1)} \mathrm{C}^\alpha_\alpha((0,T],D) \cap \mathrm{C}([0,T],D) \cap \mathrm{C}^1([0,T],X).$$

Remark 2.3.10. (i) This theorem is [60, Theorem 8.1.1] with stronger assumptions. To be precise, [60, Theorem 8.1.1] allows for non-autonomous problems and a Banach space $D \hookrightarrow X$ which is not necessarily dense in X. In the non-autonomous case, it is additionally assumed that for given $T > 0$, the evolution operator $F : [0,T] \times \mathcal{O} \to X$ and its Fréchet derivative with respect to the second variable are α-Hölder continuous in the first variable for some $\alpha \in (0,1)$. This is why we get a solution for all $\alpha \in (0,1)$ in Theorem 2.3.9. If D is not dense in X, one may solve (2.3.6) only for initial data u_0 such that $F(u_0)$ belongs to the closure of D in X (and also $F(\bar{u})$) needs to be in that closure).

(ii) The proof of [60, Theorem 8.1.1] relies on a fixed-point argument for the operator Γ defined by

$$\Gamma v(t) := e^{tA}u_0 + \int_0^t e^{(t-s)A}\big(F(u(s)) - \partial F(\bar{u})[u(s)]\big)\,\mathrm{d}s, \qquad 0 \le t \le T,$$

see (2.3.8), defined on the complete metric space

$$M_T := \left\{ u \in \mathrm{C}^\alpha_\alpha((0,T],D) \cap \mathrm{C}([0,T],D) \,\middle|\, \begin{aligned}&u(0) = u_0,\\ &\|u(\cdot) - \bar{u}\|_{\mathrm{C}^\alpha_\alpha((0,T],D)} \le \rho \end{aligned} \right\},$$

where $T > 0$, $\rho > 0$ are suitable constants, and M_T is equipped with the metric induced by the norm of $\mathrm{C}^\alpha_\alpha((0,T],D)$. Moreover, the *continuous maximal regularity property* of the generator of a strongly continuous analytic semigroup is a crucial ingredient in this approach. This property states that, given $A \in \mathcal{H}(D,X)$, we have that the mapping

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A,\, u \mapsto u(0)\right) : \mathbb{X}_T \to \mathbb{Y}_T \times D, \qquad T > 0,$$

is an isomorphism, where the Banach spaces $\mathbb{X}_T$ and $\mathbb{Y}_T$ are defined by

$$\begin{aligned}\mathbb{X}_T &:= \{u \in \mathrm{C}^\alpha_\alpha((0,T],D) \cap \mathrm{C}([0,T],D) \cap \mathrm{C}^1([0,T],X) : u' \in \mathrm{C}^\alpha_\alpha((0,T],X)\},\\ \mathbb{Y}_T &:= \mathrm{C}([0,T],X) \cap \mathrm{C}^\alpha_\alpha((0,T],X).\end{aligned}$$

Furthermore, it holds that

$$\left\| \left(\frac{\mathrm{d}}{\mathrm{d}t} - A,\, u \mapsto u(0) \right)^{-1} \right\|_{\mathcal{L}(\mathbb{Y}_T \times D, \mathbb{X}_T)} \le C(1+T),$$

where $C > 0$ is a constant independent of T.

2.4. Sobolev spaces

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We begin by introducing the $\mathbb{K}$-valued Lebesgue spaces on $\mathbb{R}^n$ and $\mathbb{S}$. Let $\mathbb{X} \in \{\mathbb{S}, \mathbb{R}^n\}$, $n \in \mathbb{N}$, and $1 \le p \le \infty$ and define

$$\mathrm{L}^p(\mathbb{X}) := \left\{ f : \mathbb{X} \to \mathbb{K} \text{ measurable} : \int_{\mathbb{X}} |f(x)|^p \,\mathrm{d}x < \infty \right\} / \sim, \qquad 1 \le p < \infty,$$

and

$$\mathrm{L}^\infty(\mathbb{X}) := \{ f : \mathbb{X} \to \mathbb{K} \text{ measurable} : \operatorname*{ess\,sup}_{x \in \mathbb{X}} |f(x)| < \infty \} / \sim,$$

where $f \sim g$ if $f = g$ almost everywhere on $\mathbb{X}$. We equip these spaces with the usual norms

$$\begin{aligned}\|f\|_{\mathrm{L}^p(\mathbb{X})} &:= \left(\int_{\mathbb{X}} |f(x)|^p \,\mathrm{d}x \right)^{1/p}, \qquad 1 \le p < \infty,\\ \|f\|_{\mathrm{L}^\infty(\mathbb{X})} &:= \operatorname*{ess\,sup}_{x \in \mathbb{X}} |f(x)|.\end{aligned}$$

If $\mathbb{X} = \mathbb{S}$, we write $\|\cdot\|_p$ instead of $\|\cdot\|_{\mathrm{L}^p(\mathbb{S})}$, $1 \le p \le \infty$.

It is clear that any $g \in \mathrm{L}^p(\mathbb{X})$, $1 \le p \le \infty$ defines a distribution via

$$g(f) := \int_{\mathbb{X}} g(x) f(x) \,\mathrm{d}x, \qquad f \in \mathcal{D}(\mathbb{X}). \tag{2.4.1}$$

This means that $\mathrm{L}^p(\mathbb{X}) \subset \mathcal{D}'(\mathbb{X})$, $1 \le p \le \infty$.

Let $f \in \mathrm{L}^1(\mathbb{S})$. We say that f has a weak derivative $g \in \mathrm{L}^1(\mathbb{S})$ if

$$\int_{\mathbb{S}} f(\xi) \varphi'(\xi) \,\mathrm{d}\xi = - \int_{\mathbb{S}} g(\xi) \varphi(\xi) \,\mathrm{d}\xi$$

holds for all $\varphi \in \mathrm{C}^\infty(\mathbb{S})$. If a weak derivative exists, it is unique and we write $f' := g$. Given $1 \le p \le \infty$ and $k \in \mathbb{N}_0$, we define the periodic Sobolev space $\mathrm{W}^{k,p}(\mathbb{S})$ by

$$\mathrm{W}^{k,p}(\mathbb{S}) := \{ f \in \mathrm{L}^p(\mathbb{S}) : \|f^{(i)}\|_p < \infty, \quad 0 \le i \le k \},$$

where the derivatives have to be interpreted in the weak sense. Equipped with the norm

$$\|f\|^p_{\mathrm{W}^{k,p}} := \|f\|^p_{\mathrm{W}^{k,p}(\mathbb{S})} := \sum_{i=0}^{k} \|f^{(i)}\|_p^p, \qquad f \in \mathrm{W}^{k,p}(\mathbb{S}),$$

the Sobolev space $\mathrm{W}^{k,p}(\mathbb{S})$ is a Banach space. If $p = 2$, we set

$$\mathrm{H}^k(\mathbb{S}) := \mathrm{W}^{k,2}(\mathbb{S}), \qquad k \in \mathbb{N}. \tag{2.4.2}$$

Next, we introduce the Fourier transform on $\mathbb{R}^n$. To start, we denote by $\mathcal{S}(\mathbb{R}^n)$ the set of $\mathbb{K}$-valued Schwartz functions, that is the set of all functions $f \in \mathrm{C}^\infty(\mathbb{R}^n)$ that additionally satisfy

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < C_{\alpha,\beta}$$

for all multi-indices α, $\beta \in \mathbb{N}_0^n$ and some constant $C_{\alpha,\beta} > 0$. Then, given a function $f \in \mathcal{S}(\mathbb{R}^n)$, we define its Fourier transform by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n,$$

and its inverse Fourier transform by

$$\mathcal{F}^{-1}(f)(x) := \check{f}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi}\,\mathrm{d}\xi, \qquad x \in \mathbb{R}^n.$$

We now list a few important properties of the Fourier transform. These and further properties can be found in [46, Section 2.2].

Proposition 2.4.1. *Given a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, we have:*

(i) $\hat{f}$, $\check{f} \in \mathcal{S}(\mathbb{R}^n)$.

(ii) $\mathcal{F}^{-1}(\hat{f}) = \mathcal{F}(\check{f}) = f$.

(iii) (Plancherel's identity) $\|f\|_{\mathrm{L}^2(\mathbb{R}^n)} = \|\hat{f}\|_{\mathrm{L}^2(\mathbb{R}^n)} = \|\check{f}\|_{\mathrm{L}^2(\mathbb{R}^n)}$.

(iv) $\widehat{\partial^\beta f}(\xi) = (i\xi)^\beta \hat{f}(\xi)$, *for all* $\beta \in \mathbb{N}_0^n$ *and* $\xi \in \mathbb{R}^n$.

It is worth mentioning, that the Fourier transform is well-defined if $f \in \mathrm{L}^1(\mathbb{R}^n)$. By a density argument one can then extend the domain of definition from $\mathrm{L}^1(\mathbb{R}^n) \cap \mathrm{L}^2(\mathbb{R}^n)$ to $\mathrm{L}^2(\mathbb{R}^n)$ and it follows that Proposition 2.4.1 (ii)–(iii) also hold for $f \in \mathrm{L}^2(\mathbb{S})$. Moreover, Proposition 2.4.1 (iv) also holds if f lies in a suitable Sobolev space.

Next, we introduce the concept of Fourier series. Given $f \in \mathrm{L}^2(\mathbb{S})$, we denote by

$$\mathcal{F}(f)(k) := \hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\,\mathrm{d}x, \qquad k \in \mathbb{Z},$$

the k'th Fourier coefficient of f. We associate to $f \in \mathrm{L}^1(\mathbb{S})$ its Fourier series given by

$$\sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ik\xi}, \qquad \xi \in \mathbb{S}.$$

We denote by $\tau_y f := f(\cdot + y)$, $y \in \mathbb{R}$, the left shift operator and state some basic properties which can be found in [46, Chapter 3].

Proposition 2.4.2. *Given $f \in \mathrm{L}^2(\mathbb{S})$, we have:*

(i) $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,\mathrm{d}\xi = \langle f \rangle$.

(ii) $\sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ik\cdot} = f$ *in* $\mathrm{L}^2(\mathbb{S})$.

(iii) (Plancherel's identity) $\|f\|_2^2 = 2\pi \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$.

(iv) $\widehat{\tau_y f}(k) = e^{iky}\hat{f}(k)$ *for all* $y \in \mathbb{S}$ *and* $k \in \mathbb{Z}$.

(v) $\widehat{f^{(n)}}(k) = (ik)^n \hat{f}(k)$ *if* $f \in \mathrm{C}^n(\mathbb{S})$, $n \in \mathbb{N}$, *and* $k \in \mathbb{Z}$.

Given a distribution $g \in \mathcal{D}'(\mathbb{S})$, we define its k'th Fourier coefficient by

$$\hat{g}(k) := \frac{1}{2\pi} g(e^{-ik\cdot}), \qquad k \in \mathbb{Z}.$$

Notice that this definition is consistent with the one for $\mathrm{L}^1(\mathbb{S})$ by (2.4.1).

We are now in a position to define fractional order periodic Sobolev spaces. Following [82], we set for given $-\infty < s < \infty$

$$\mathrm{H}^s(\mathbb{S}) := \Big\{ f \in \mathcal{D}'(\mathbb{S}) : \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |\hat{f}(k)|^2 < \infty \Big\},$$

and equip it with the norm

$$\|f\|_{\mathrm{H}^s}^2 := \|f\|_{\mathrm{H}^s(\mathbb{S})}^2 := 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |\hat{f}(k)|^2. \tag{2.4.3}$$

We state important properties of this space which can be found in [82, Chapter 3]. To start, this space is a Banach space and for $s = m \in \mathbb{N}_0$ it coincides with the usual Sobolev space defined in (2.4.2). Next, we have the following embedding properties:

$$\begin{aligned} \mathrm{H}^{s_1}(\mathbb{S}) &\hookrightarrow \mathrm{H}^{s_0}(\mathbb{S}), && -\infty < s_0 < s_1 < \infty, \\ \mathrm{H}^{k+s+\frac{1}{2}}(\mathbb{S}) &\hookrightarrow \mathrm{C}^{k+s}(\mathbb{S}), && k \in \mathbb{N}_0, \quad s \in (0,1), \end{aligned}$$

where the first embedding is compact and dense, see, e.g., [53, Proposition 3.193]. Moreover, we have the interpolation property

$$\big[\mathrm{H}^{s_0}(\mathbb{S}), \mathrm{H}^{s_1}(\mathbb{S})\big]_\theta = \mathrm{H}^{(1-\theta)s_0+\theta s_1}(\mathbb{S}), \qquad \theta \in (0,1), \quad -\infty < s_0 \le s_1 < \infty, \tag{2.4.4a}$$

where $[\cdot,\cdot]_\theta$ is the complex interpolation functor of exponent θ. Consequently, the interpolation inequality

$$\|f\|_{\mathrm{H}^{(1-\theta)s_0+\theta s_1}} \le \|f\|_{\mathrm{H}^{s_0}}^{1-\theta} \|f\|_{\mathrm{H}^{s_1}}^{\theta} \tag{2.4.4b}$$

holds for all $f \in \mathrm{H}^{s_1}(\mathbb{S})$.

Let us also note that $\mathrm{H}^s(\mathbb{S})$ is a Banach algebra for all $s > 1/2$, that is, there exists a constant $C_s > 0$ such that

$$\|fg\|_{\mathrm{H}^s} \le C_s \|f\|_{\mathrm{H}^s} \|g\|_{\mathrm{H}^s}, \qquad f,\, g \in \mathrm{H}^s(\mathbb{S}).$$

A proof can be found in [53, Theorem 3.200].

Next, we introduce for $s \in \mathbb{R}_{\ge 0} \setminus \mathbb{N}_0$ the space

$$\mathrm{W}^{s,2}(\mathbb{S}) := \{ f \in \mathrm{W}^{[s],2}(\mathbb{S}) : [f]_{\mathrm{W}^{s,2}} < \infty \}, \qquad s = [s] + \{s\}, \quad [s] \in \mathbb{N}_0,\ \{s\} \in (0,1),$$

where

$$[f]_{\mathrm{W}^{s,2}}^2 := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f^{([s])}(\xi+y) - f^{([s])}(\xi)|^2}{|y|^{1+2\{s\}}} \,\mathrm{d}\xi \,\mathrm{d}y = \int_{-\pi}^{\pi} \frac{\|\tau_y f^{([s])} - f^{([s])}\|_2^2}{|y|^{1+2\{s\}}} \,\mathrm{d}y$$

and $\tau_y f := f(\cdot + y)$ is the left shift operator. The space $\mathrm{W}^{s,2}(\mathbb{S})$ is equipped with the norm

$$\|f\|_{\mathrm{W}^{s,2}}^2 := \|f\|_{\mathrm{W}^{[s],2}}^2 + [f]_{\mathrm{W}^{s,2}}^2.$$

We then have the following Lemma:

Lemma 2.4.3. *Given $s \in \mathbb{R}_{\geq 0} \setminus \mathbb{N}_0$, the norms $\|\cdot\|_{\mathrm{W}^{s,2}}$ and $\|\cdot\|_{\mathrm{H}^s}$ are equivalent.*

Proof. Given $f \in \mathrm{C}^\infty(\mathbb{S})$, we compute using Proposition 2.4.2 (iii)–(v) that

$$\begin{aligned}
[f]^2_{\mathrm{W}^{s,2}} &= \int_{-\pi}^{\pi} \frac{\|\tau_y f^{([s])} - f^{([s])}\|_2^2}{|y|^{1+2\{s\}}} \,\mathrm{d}y = 2\pi \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} |k|^{2[s]} \frac{|e^{iky}\hat{f}(k) - \hat{f}(k)|^2}{|y|^{1+2\{s\}}} \,\mathrm{d}y \\
&= 2\pi \sum_{k\in\mathbb{Z}} |k|^{2[s]} |\hat{f}(k)|^2 \int_{-\pi}^{\pi} \frac{|e^{iky} - 1|^2}{|y|^{1+2\{s\}}} \,\mathrm{d}y.
\end{aligned}$$

Performing a change of variables, we find that

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{|e^{iky} - 1|^2}{|y|^{1+2\{s\}}} \,\mathrm{d}y &= \int_{-\pi}^{\pi} \frac{2(1-\cos(ky))}{|y|^{1+2\{s\}}} \,\mathrm{d}y \\
&= |k|^{2\{s\}} \int_{-|k|\pi}^{|k|\pi} \frac{2(1-\cos(t))}{|t|^{1+2\{s\}}} \,\mathrm{d}t := |k|^{2\{s\}} c_k, \qquad k \in \mathbb{Z},
\end{aligned} \tag{2.4.5}$$

where c_k satisfies

$$\begin{aligned}
c_k &= 4 \int_0^{|k|\pi} \frac{1-\cos(t)}{t^{1+2\{s\}}} \,\mathrm{d}t \leq 2 \int_0^1 \frac{t^2}{t^{1+2\{s\}}} \,\mathrm{d}t + 4 \int_1^\infty \frac{2}{t^{1+2\{s\}}} \,\mathrm{d}t \\
&= \frac{4 - 3\{s\}}{\{s\}(1-\{s\})} := C_1,
\end{aligned} \tag{2.4.6}$$

and $0 < c_1 \leq c_k$ for $k \neq 0$. This means, we have

$$c_1 \sum_{k\in\mathbb{Z}} |k|^{2s} |\hat{f}(k)|^2 \leq [f]^2_{\mathrm{W}^{s,2}} = \sum_{k\in\mathbb{Z}} c_k |k|^{2s} |\hat{f}(k)|^2 \leq C_1 \sum_{k\in\mathbb{Z}} |k|^{2s} |\hat{f}(k)|^2.$$

Thus, appealing again to Proposition 2.4.2, we find constants c, $C > 0$ such that

$$\begin{aligned}
\|f\|^2_{\mathrm{H}^s} &= 2\pi |\hat{f}(0)|^2 + 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} (1+|k|^2)^s |\hat{f}(k)|^2 \\
&\leq 2\pi \|f\|_2^2 + 2\pi 2^s \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2s} |\hat{f}(k)|^2 \leq C \|f\|^2_{\mathrm{W}^{s,2}},
\end{aligned}$$

and

$$\begin{aligned}
\|f\|^2_{\mathrm{H}^s} &\geq \pi \Big(\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2 \Big) + \pi \Big(\sum_{k\in\mathbb{Z}} |k|^{2s} |\hat{f}(k)|^2 \Big) \\
&\geq c \Big(\sum_{\ell=0}^{[s]} \sum_{k\in\mathbb{Z}} |k|^{2\ell} |\hat{f}(k)|^2 \Big) + c [f]^2_{\mathrm{W}^{s,2}} \\
&\geq c \|f\|^2_{\mathrm{W}^{s,2}},
\end{aligned}$$

and the proof is complete. □

Using the equivalent norm of $\mathrm{W}^{s,2}(\mathbb{S})$, we can show the following lemma:

Lemma 2.4.4. *Given $s \in (1/2, 1)$, there exists a constant $C > 0$ such that*

$$\|fg\|_{\mathrm{H}^s} \leq C(\|f\|_\infty \|g\|_{\mathrm{H}^s} + \|g\|_\infty \|f\|_{\mathrm{H}^s}), \qquad f,\, g \in \mathrm{H}^s(\mathbb{S}).$$

Proof. Given f, $g \in \mathrm{H}^s(\mathbb{S})$, $s \in (1/2, 1)$ we use Lemma 2.4.3 to calculate

$$\begin{aligned}
\|fg\|^2_{\mathrm{H}^s} &\leq C \|fg\|^2_{\mathrm{W}^{s,2}} = C \|fg\|_2^2 + C [fg]^2_{\mathrm{W}^{s,2}} \\
&\leq C \|fg\|_2^2 + C \int_{-\pi}^{\pi} \frac{\|(\tau_y f - f)\tau_y g\|_2^2 + \|(\tau_y g - g) f\|_2^2}{|y|^{1+2s}} \,\mathrm{d}y \\
&\leq C \|fg\|_2^2 + C \|g\|_\infty [f]_{\mathrm{W}^{s,2}} + C \|f\|_\infty [g]_{\mathrm{W}^{s,2}} \\
&\leq C(\|f\|_\infty \|g\|_{\mathrm{H}^s} + \|g\|_\infty \|f\|_{\mathrm{H}^s})^2.
\end{aligned}$$

□

Part I

Stationary two-phase Stokes problems

Chapter 3

Some boundary value problems with transmission type boundary conditions

3.1. Introduction

In this chapter, we lay the groundwork for solving the fixed time problem associated with (1.2.2a). By fixed time problem, we mean the stationary system of equations $(1.2.2a)_{1-5}$ at any fixed time stamp $t > 0$ under the assumption that $f(t)$ is regular enough. To be precise, we require that the interface Γ is the graph of a function $f \in \mathrm{H}^3(\mathbb{S})$, that is

$$\Gamma := \{(\xi, f(\xi)) \in \mathbb{S} \times \mathbb{R} : \xi \in \mathbb{S}\},$$

and we further define

$$\Omega^\pm := \{x = (x_1, x_2) \in \mathbb{S} \times \mathbb{R} : x_2 \gtrless f(x_1)\}.$$

Moreover, we will use the following notation throughout the thesis and set

$$\begin{gathered} \omega := \omega(f) := (1 + f'^2)^{1/2}, \qquad \nu := \nu(f) := \omega^{-1}(-f', 1)^\top, \\ \tau := \tau(f) := \omega^{-1}(1, f')^\top. \end{gathered} \tag{3.1.1}$$

Here, ν is the pull-back of the unit normal $\tilde{\nu}$ exterior to Ω^- under the $\mathrm{C}^{5/2}$-diffeomorphism $\Xi := \Xi_f : \mathbb{S} \to \Gamma$, $\Xi := (\mathrm{id}_{\mathbb{S}}, f)$, that maps the x_1-axis onto Γ, that is $\nu = \tilde{\nu} \circ \Xi$, while τ is the pull-back of the positively oriented unit tangent vector $\tilde{\tau}$ along Γ under Ξ and ω is the length of the (non-unit) normal vector and tangent vector, respectively.

Given a function $w : (\mathbb{S} \times \mathbb{R}) \setminus \Gamma \to \mathbb{R}^n$, $n \in \mathbb{N}$, we set $w^\pm := w|_{\Omega^\pm}$ and denote by

$$\{w\}^\pm(x_0) := \lim_{\Omega^\pm \ni x \to x_0} w(x), \qquad x_0 \in \Gamma, \tag{3.1.2}$$

the one-sided limits of w in $x_0 \in \Gamma$, provided that these limits exist. Moreover,

$$[w](x_0) := \{w\}^+(x_0) - \{w\}^-(x_0) \tag{3.1.3}$$

is the jump of w across Γ in $x_0 \in \Gamma$.

Vice versa, we associate to functions $w^\pm : \Omega^\pm \to \mathbb{R}^n$, $n \in \mathbb{N}$, the mapping

$$w := \mathbf{1}_{\Omega^+} w^+ + \mathbf{1}_{\Omega^-} w^-,$$

which is defined a.e. in $\mathbb{S} \times \mathbb{R}$.

Then, the fixed time problem corresponding to (1.2.2a) is given by

$$\left.\begin{aligned} \mu^\pm \Delta v^\pm - \nabla q^\pm &= 0 && \text{in } \Omega^\pm, \\ \operatorname{div} v^\pm &= 0 && \text{in } \Omega^\pm, \\ [v] &= 0 && \text{on } \Gamma, \\ [T_\mu(v,q)]\tilde{\nu} &= (\Theta x_2 - \sigma\tilde{\kappa})\tilde{\nu} && \text{on } \Gamma, \\ (v^\pm, q^\pm)(x) &\to \pm\Big(\tfrac{c_{1,\Gamma}}{\mu^\pm}, \tfrac{c_{2,\Gamma}}{\mu^\pm}, c_{3,\Gamma}\Big) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \end{aligned}\right\} \tag{3.1.4}$$

where the constant $c_\Gamma = (c_{1,\Gamma}, c_{2,\Gamma}, c_{3,\Gamma}) \in \mathbb{R}^3$ is an unknown of the problem. We call a solution (v, q) to (3.1.4) (and to the other problems of transmission type considered in this section) classical, if $(v, q) \in X_f$, where

$$X_f := \left\{ (v,q) : (\mathbb{S} \times \mathbb{R}) \setminus \Gamma \to \mathbb{R}^2 \times \mathbb{R} \;\middle|\; \begin{aligned} & v^\pm \in \mathrm{C}^2(\Omega^\pm, \mathbb{R}^2) \cap \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2), \\ & q^\pm \in \mathrm{C}^1(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm}) \end{aligned} \right\}. \tag{3.1.5}$$

To solve the transmission boundary value problem (3.1.4) in Chapter 5, we consider two related boundary value problems of transmission type and construct the solution to (3.1.4) as a linear combination of the solutions to these two problems, see (5.3.7). The first one is (3.1.4) in the case of equal viscosities, that is, we have $\mu := \mu^+ = \mu^-$. Then, the problem reads as

$$\left.\begin{aligned} \mu \Delta v_s^\pm - \nabla q_s^\pm &= 0 && \text{in } \Omega^\pm, \\ \operatorname{div} v_s^\pm &= 0 && \text{in } \Omega^\pm, \\ [v_s] &= 0 && \text{on } \Gamma, \\ [T_\mu(v_s,q_s)]\tilde{\nu} &= (\Theta x_2 - \sigma\tilde{\kappa})\tilde{\nu} && \text{on } \Gamma, \\ (v_s^\pm, q_s^\pm)(x) &\to \pm(c_{1,s}, c_{2,s}, c_{3,s}) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \end{aligned}\right\} \tag{3.1.6}$$

where the constant $c_s = (c_{1,s}, c_{2,s}, c_{3,s}) \in \mathbb{R}^3$ is a priori unknown and must be determined as part of the solution. The second problem we consider prescribes a jump of the velocity across the interface rather than a jump of the stress tensor. That means, given a function $\beta \in \mathrm{H}^2(\mathbb{S})$, we seek solutions to

$$\left.\begin{aligned} \Delta v_d^\pm - \nabla q_d^\pm &= 0 && \text{in } \Omega^\pm, \\ \operatorname{div} v_d^\pm &= 0 && \text{in } \Omega^\pm, \\ [v_d] &= \beta \circ \Xi^{-1} && \text{on } \Gamma, \\ [T_1(v_d,q_d)]\tilde{\nu} &= 0 && \text{on } \Gamma, \\ (v_d^\pm, q_d^\pm)(x) &\to \pm(c_{1,d}, c_{2,d}, c_{3,d}) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \end{aligned}\right\} \tag{3.1.7}$$

where $c_d = (c_{1,d}, c_{2,d}, c_{3,d}) \in \mathbb{R}^3$ is again part of the solution.

We construct solutions to these problems using methods from potential theory. Potential theory has its origins in the discovery of a solution to what is now called Poisson's equation by Isaac Newton: given $f \in \mathrm{L}^1(\mathbb{R}^3)$, find $u : \mathbb{R}^3 \to \mathbb{R}$ such that

$$-\Delta u = f \qquad \text{in } \mathbb{R}^3. \tag{3.1.8}$$

One modern approach to find a particular solution to (3.1.8) uses two elements. First, a fundamental solution of the Laplace equation, which is a distribution Φ that satisfies

$$-\Delta \Phi = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^3),$$

where δ_0 is given by $\delta_0(\varphi) = \varphi(0)$, $\varphi \in \mathcal{D}(\mathbb{R}^3)$, and second, the property of the delta distribution that it acts as the identity element for convolutions, i.e. $\delta_0 * f = f$. Then, $u := \Phi * f$ is formally a solution to (3.1.8), since

$$-\Delta u = -\Delta(\Phi * f) = (-\Delta\Phi) * f = \delta_0 * f = f$$

due to the linearity of Δ. The term

$$(\Phi * f)(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y)\,\mathrm{d}y$$

is also called *Newtonian potential* or *volume potential.* When working in a bounded domain Ω, the above formula still produces a particular solution but boundary conditions are not incorporated. For this purpose, the single- and double-layer potentials are needed. The single-layer potential generated by the surface density $g : \partial\Omega \to \mathbb{R}$ is given by

$$S[g](x) := -\int_{\partial\Omega} \Phi(x-y) g(y)\,\mathrm{d}\sigma,$$

while the double-layer potential is given by

$$D[g](x) := -\int_{\partial\Omega} \partial_{\tilde{\nu}_y} \Phi(x-y) g(y)\,\mathrm{d}\sigma,$$

where $\tilde{\nu}_y$ is the outward unit normal of $\partial\Omega$. These layer potentials exhibit characteristic jump relations when approaching the boundary from inside or outside the domain. For an in-depth treatment of these topics we refer to [33, 67]. We will use the Stokes analogues of these layer potentials to solve the transmission boundary value problems (3.1.6)–(3.1.7).

This chapter is structured as follows: in Section 3.2, we derive the fundamental solution of our periodic Stokes problem. To do so, we first use standard methods to find the fundamental solution of the Stokes equation in $\mathbb{R}^n$, $n \geq 2$, in terms of the fundamental solution of the Laplace equation (and the biharmonic equation), see (3.2.6) and (3.2.11) below. Then, in Theorem 3.2.2, we present the fundamental solution of the Laplace equation in $\mathbb{S} \times \mathbb{R}$ which enables us to construct the fundamental solution of the Stokes equations in $\mathbb{S} \times \mathbb{R}$. In Section 3.3, we first follow the early work of Odqvist [72] and construct solutions to the inhomogeneous Stokes equation in $\mathbb{R}^n$, $n \geq 2$, using volume and hydrodynamic layer potentials, cf. (3.3.9) and (3.3.10). Next, we study the single-layer potentials corresponding to our situation and show in Theorem 3.3.2 that the unique solution to (3.1.6) is given by the single-layer potentials generated by a density corresponding to the right-hand side of $(3.1.6)_4$. Afterwards, we turn our attention to the double-layer potentials adapted to our setting. We finish this section with Theorem 3.3.9 which states that the unique solution to (3.1.7) is given by the double-layer potentials generated by $-\beta \circ \Xi^{-1}$.

3.2. Derivation of the horizontally periodic Stokeslet

The goal of this section is to find a horizontally periodic fundamental solution of the two-dimensional Stokes equation. To be precise, we look for a solution pair $(\mathcal{U}, \mathcal{P})$, where

$$\mathcal{U} := (\mathcal{U}^1, \mathcal{U}^2) : (\mathbb{S} \times \mathbb{R}) \to \mathbb{R}^{2\times 2}, \qquad \mathcal{P} := (\mathcal{P}^1, \mathcal{P}^2)^\top : (\mathbb{S} \times \mathbb{R}) \to \mathbb{R}^2,$$

that solves

$$\left.\begin{aligned} \mu\Delta\mathcal{U}^k - \nabla\mathcal{P}^k &= \delta_0 e^k \\ \operatorname{div}\mathcal{U}^k &= 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}'(\mathbb{S} \times \mathbb{R}) \tag{3.2.1}$$

for $1 \leq k \leq 2$, where $e^1 := (1,0)$ and $e^2 = (0,1)$. To do so, we follow the approach of [56, Chapter 3.1] and first derive the fundamental solution of the (non-periodic) Stokes equation in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, using Fourier analysis. As the fundamental solutions of the Laplace and the biharmonic equations naturally occur, we first define distributions G, H as solutions of

$$\begin{aligned} -\Delta G &= \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \\ -\Delta^2 H &= \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \end{aligned} \tag{3.2.2}$$

such that

$$\Delta H = G \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{3.2.3}$$

Since the Fourier transform of δ_0 is constant, it follows easily that G and H are formally given by

$$\begin{aligned} G(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{|\xi|^2}\,\mathrm{d}\xi, \\ H(x) &= -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{|\xi|^4}\,\mathrm{d}\xi, \qquad x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{3.2.4}$$

Applying the Fourier transform to (3.2.1), we get

$$\left.\begin{aligned} -\mu|\xi|^2\hat{u}^k(\xi) - i\xi\hat{p}^k(\xi) &= \tfrac{1}{(2\pi)^{n/2}}e^k \\ \xi\cdot\hat{u}^k(\xi) &= 0 \end{aligned}\right\}, \qquad 1 \leq k \leq n, \tag{3.2.5}$$

in Fourier space. Multiplying the first equation of (3.2.5) by ξ, the incompressibility condition yields

$$\hat{p}^k(\xi) = \frac{1}{(2\pi)^{n/2}}\frac{i\xi_k}{|\xi|^2}, \qquad 1 \leq k \leq n.$$

By plugging this expression back into (3.2.5), we immediately get

$$\hat{u}^k(\xi) = \frac{1}{(2\pi)^{n/2}\mu}\Big(-\frac{e_k}{|\xi|^2} + \frac{\xi\xi_k}{|\xi|^4}\Big), \qquad 1 \leq k \leq n.$$

The formulas (3.2.4) enable us to conclude that, after applying the inverse Fourier transform, we have

$$\begin{aligned} u_j^k(x) &= \frac{1}{\mu}\big(-\delta_{jk}G(x) + \partial_j\partial_k H(x)\big), \\ p^k(x) &= \partial_k G(x), \qquad x \in \mathbb{R}^n \setminus \{0\}, \quad 1 \leq j,\, k \leq n. \end{aligned} \tag{3.2.6}$$

To proceed, it is convenient to have explicit formulas for G and H in order to obtain explicit expressions for u and p from (3.2.6). While such formulas are readily available in $\mathbb{R}^n$ (see, e.g., [32, Chapter 2.2.1] and [44, Chapter 4.2]), deriving them in $\mathbb{S} \times \mathbb{R}$ is more involved, as we will see when computing the fundamental solution of the Laplace equation in $\mathbb{S} \times \mathbb{R}$. It is worth mentioning, that in [37, Section 3], the fundamental solution of the biharmonic equation in $\mathbb{S} \times \mathbb{R}$ is derived and expressed as a rather involved Fourier series in x_1. However, for the moment we do not want to work with Fourier series. To avoid this complexity, we now present a representation for the term $\partial_j\partial_k H$ in (3.2.6) that depends solely on the function G.

Given $1 \leq j,\, k \leq n$, we claim that

$$2\partial_j\partial_k H = x_j\partial_k G + \delta_{jk}G + R \qquad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{3.2.7}$$

where

$$\Delta R = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{3.2.8}$$

Due to (3.2.8), it suffices to show that

$$2\partial_j\partial_k\Delta H = \Delta(x_j\partial_k G) + \delta_{jk}\Delta G \qquad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Considering the left-hand side, we infer from (3.2.3) that

$$\partial_j\partial_k\Delta H = \partial_j\partial_k G \qquad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{3.2.9}$$

To estimate the right-hand side, we fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and use (3.2.2) to calculate

$$\begin{aligned}\langle\Delta(x_j\partial_k G),\varphi\rangle + \delta_{jk}\langle\Delta G,\varphi\rangle &= -\langle G, x_j\partial_k\Delta\varphi\rangle - \delta_{jk}\langle G,\Delta\varphi\rangle + \delta_{jk}\langle G,\Delta\varphi\rangle\\ &= -\langle x_j\Delta G,\partial_k\varphi\rangle - 2\langle\partial_j G,\partial_k\varphi\rangle\\ &= 2\langle\partial_j\partial_k G,\varphi\rangle + \langle\delta_0, x_j\partial_k\varphi\rangle\\ &= 2\langle\partial_j\partial_k G,\varphi\rangle.\end{aligned} \tag{3.2.10}$$

We thus can conclude from (3.2.9)–(3.2.10) that (3.2.7)–(3.2.8) indeed holds.

Note that (3.2.6) holds for any H satisfying (3.2.2)–(3.2.3). Thus, we can assume $R = 0$ in (3.2.7) (since we can define $\tilde{H} = H - R$ which satisfies (3.2.7) with $R = 0$). Plugging formula (3.2.7) into (3.2.6) gives us the following expression for u which now solely depends on G:

$$\begin{aligned}u_j^k(x) &= \frac{1}{2\mu}\big(x_j\partial_k G(x) - \delta_{jk}G(x)\big),\\ p^k(x) &= \partial_k G(x), \qquad x \in \mathbb{R}^n\setminus\{0\}, \quad 1 \le j,\, k \le n.\end{aligned} \tag{3.2.11}$$

Assuming that (3.2.11) in some sense also holds in the periodic case, i.e., in $\mathbb{S}\times\mathbb{R}$, we now focus on finding a fundamental solution of the Laplace equation in $\mathbb{S}\times\mathbb{R}$. Our approach starts with an explicit solution of (3.2.2) in two dimensions, where we take the usual radial symmetric solution

$$G(x) = -\frac{1}{2\pi}\ln\left(|x|\right), \qquad x \in \mathbb{R}^2\setminus\{0\}, \tag{3.2.12}$$

see, e.g., [32, Chapter 2.2.1] for a derivation. Given a function $F : \mathbb{R}^2 \to \mathbb{R}$, one can formally associate to F a function which is periodic in the first variable, namely the function $F_\pi : \mathbb{R}^2 \to \mathbb{R}$, called the *periodization* of F, cf. [89, Chapter 5.3], defined by

$$F_\pi(x) = \sum_{k\in\mathbb{Z}} F(x_1 + 2\pi k, x_2), \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

But this approach needs at least some decay as $|x_1| \to \infty$ which is not given in our case due to the logarithm. To work around this problem, we now take the gradient of G which has enough decay so that the summation process may be made rigorously. Differentiating (3.2.12) gives

$$\nabla G(x) = -\frac{1}{2\pi}\frac{x}{|x|^2}, \qquad x \in \mathbb{R}^2\setminus\{0\}, \tag{3.2.13}$$

and we may define functions $\widetilde{G_1}, \widetilde{G_2} : \mathbb{R}^2 \setminus \big(2\pi\mathbb{Z}\times\{0\}\big) \to \mathbb{R}$ by

$$\begin{aligned}\widetilde{G_1}(x) &:= \lim_{N\to\infty}\sum_{k=-N}^{N}\partial_1 G(x_1 + 2k\pi, x_2) = -\frac{1}{2\pi}\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{x_1 + 2k\pi}{(x_1 + 2k\pi)^2 + x_2^2},\\ \widetilde{G_2}(x) &:= \lim_{N\to\infty}\sum_{k=-N}^{N}\partial_2 G(x_1 + 2k\pi, x_2) = -\frac{1}{2\pi}\lim_{N\to\infty}\sum_{k=-N}^{N}\frac{x_2}{(x_1 + 2k\pi)^2 + x_2^2},\end{aligned} \tag{3.2.14}$$

The following lemma establishes closed-form expressions for $\widetilde{G_1}$ and $\widetilde{G_2}$.

Lemma 3.2.1. *Given* $x \in \mathbb{R}^2 \setminus (2\pi\mathbb{Z} \times \{0\})$, *we have*

$$\begin{aligned} \widetilde{G_1}(x) &= -\frac{1}{4\pi}\frac{\tan(x_1/2)(1-\tanh^2(x_2/2))}{\tan^2(x_1/2)+\tanh^2(x_2/2)}, \\ \widetilde{G_2}(x) &= -\frac{1}{4\pi}\frac{\tanh(x_2/2)(1+\tan^2(x_1/2))}{\tan^2(x_1/2)+\tanh^2(x_2/2)}. \end{aligned} \tag{3.2.15}$$

Proof. The proof is based on the partial fraction development of the cotangent. Namely, it holds that

$$\pi\cot(\pi z) = \lim_{N\to\infty}\sum_{k=-N}^{N}\frac{1}{z+k} \tag{3.2.16}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$. We refer to [79, Chapter 11.2] for a detailed discussion.

Let $a \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $b \in \mathbb{R} \setminus \{0\}$ and define $f_1,\ f_2 : \mathbb{Z} \to \mathbb{R}$ by

$$f_1(k) = \frac{a+2\pi k}{(a+2\pi k)^2+b^2}, \qquad f_2(k) = \frac{b}{(a+2\pi k)^2+b^2}.$$

Then, we may rewrite

$$f_1(k) = \mathrm{Re}\left(\frac{1}{a+2\pi k - ib}\right), \qquad f_2(k) = \mathrm{Im}\left(\frac{1}{a+2\pi k - ib}\right).$$

Summing over $k \in \mathbb{Z}$ and using (3.2.16), we have

$$\sum_{k\in\mathbb{Z}} f_1(k) = \frac{1}{2}\mathrm{Re}\left(\cot\left(\tfrac{a-ib}{2}\right)\right), \qquad \sum_{k\in\mathbb{Z}} f_2(k) = \frac{1}{2}\mathrm{Im}\left(\cot\left(\tfrac{a-ib}{2}\right)\right). \tag{3.2.17}$$

Next, we use (complex) trigonometric identities (found, e.g., in [79, Chapter 5.1]) to calculate

$$\begin{aligned} \cot(a-ib) &= \frac{\cos(a)\cosh(b) + i\sin(a)\sinh(b)}{\sin(a)\cosh(b) - i\cos(a)\sinh(b)} \\ &= \frac{\sin(a)\cos(a) + i\sinh(b)\cosh(b)}{\sin^2(a)\cosh^2(b) + \cos^2(a)\sinh^2(b)} \\ &= \frac{\sin(a)\cos(a) + i\sinh(b)\cosh(b)}{\sin^2(a)+\sinh^2(b)} \\ &= \frac{\tan(a)(1-\tanh^2(b)) + i\tanh(b)(1+\tan^2(a))}{\tan^2(a)+\tanh^2(b)}. \end{aligned} \tag{3.2.18}$$

The claim (3.2.15) now follows from (3.2.14) and (3.2.17)–(3.2.18). □

To get the desired periodization of G as defined in (3.2.12), we need to integrate $\widetilde{G_1}$ with respect to x_1 and $\widetilde{G_2}$ with respect to x_2. For the first integral we use the substitution $y_1 = \tan^2(x_1/2)$ and for the second $y_2 = \tanh^2(x_2/2)$ and arrive at

$$\begin{aligned} \int \widetilde{G_1}(x)\,\mathrm{d}x_1 &= -\frac{1}{4\pi}\ln\left(\frac{\tan^2(x_1/2)+\tanh^2(x_2/2)}{1+\tan^2(x_1/2)}\right) + R_1(x_2), \\ \int \widetilde{G_2}(x)\,\mathrm{d}x_2 &= -\frac{1}{4\pi}\ln\left(\frac{\tan^2(x_1/2)+\tanh^2(x_2/2)}{1-\tanh^2(x_2/2)}\right) + R_2(x_1), \end{aligned}$$

where again $x \in \mathbb{R}^2 \setminus (2\pi\mathbb{Z} \times \{0\})$. The choices

$$R_1(x_2) = -\frac{1}{4\pi}\ln\left(\frac{1}{1-\tanh^2(x_2/2)}\right), \qquad R_2(x_1) = -\frac{1}{4\pi}\ln\left(\frac{1}{1+\tan^2(x_1/2)}\right)$$

lead us to our candidate $G_\pi : (\mathbb{S}\times\mathbb{R})\setminus\{0\}\to\mathbb{R}$ defined by

$$G_\pi(x) := -\frac{1}{4\pi}\ln\left(\frac{\tan^2(x_1/2)+\tanh^2(x_2/2)}{(1+\tan^2(x_1/2))(1-\tanh^2(x_2/2))}\right). \tag{3.2.19}$$

That G_π actually is a fundamental solution of the Laplace equation in $\mathbb{S}\times\mathbb{R}$ follows from the next theorem.

Theorem 3.2.2. *Let $G_\pi : (\mathbb{S}\times\mathbb{R})\setminus\{0\}\to\mathbb{R}$ be as defined in* (3.2.19)*. Then G_π satisfies*

$$-\Delta G_\pi = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{S}\times\mathbb{R}).$$

Proof. Going on, we will use the shorthand notation

$$\begin{aligned} t_{[x_1]} &:= \tan\left(\frac{x_1}{2}\right), && x_1\in\mathbb{R}\setminus(\pi+2\pi\mathbb{Z}),\\ T_{[x_2]} &:= \tanh\left(\frac{x_2}{2}\right), && x_2\in\mathbb{R}. \end{aligned} \tag{3.2.20}$$

Next, we check that G_π actually is a fundamental solution of the Laplace equation in $\mathbb{S}\times\mathbb{R}$. To start, we calculate for $x\in(\mathbb{S}\times\mathbb{R})\setminus\{0\}$ that

$$\begin{aligned} \partial_1^2 G_\pi(x) &= \frac{1}{8\pi}\frac{(1+t_{[x_1]}^2)(1-T_{[x_2]}^2)(t_{[x_1]}^2-T_{[x_2]}^2)}{(t_{[x_1]}^2+T_{[x_2]}^2)^2},\\ \partial_2^2 G_\pi(x) &= -\frac{1}{8\pi}\frac{(1+t_{[x_1]}^2)(1-T_{[x_2]}^2)(t_{[x_1]}^2-T_{[x_2]}^2)}{(t_{[x_1]}^2+T_{[x_2]}^2)^2}, \end{aligned}$$

and it immediately follows that $\Delta G_\pi = 0$ in $(\mathbb{S}\times\mathbb{R})\setminus\{0\}$. It remains to show that indeed $-\Delta G_\pi=\delta_0$ in $\mathcal{D}'(\mathbb{S}\times\mathbb{R})$. To do so, we first rewrite G_π as

$$G_\pi(x) = -\frac{1}{4\pi}\ln\left(\sin^2\left(\frac{x_1}{2}\right)+\sinh^2\left(\frac{x_2}{2}\right)\right), \qquad x\in(\mathbb{S}\times\mathbb{R})\setminus\{0\}. \tag{3.2.21}$$

Notice that for $|x|$ small, $G_\pi(x)$ behaves like $G(x)+\ln(2)/(2\pi)$. Given $\varphi\in\mathcal{D}(\mathbb{S}\times\mathbb{R})$, we use the divergence theorem to calculate

$$\begin{aligned} \langle -\Delta G_\pi,\varphi\rangle &= -\lim_{\varepsilon\to0}\int_{B_\varepsilon(0)^c} G_\pi\Delta\varphi\,\mathrm{d}x\\ &= -\lim_{\varepsilon\to0}\left(\int_{\partial B_\varepsilon(0)^c}(G_\pi\nabla\varphi\cdot\nu-\varphi\nabla G_\pi\cdot\nu)\,\mathrm{d}\sigma+\int_{B_\varepsilon(0)^c}\varphi\Delta G_\pi\,\mathrm{d}x\right). \end{aligned}$$

The volume integral vanishes since $\Delta G_\pi=0$ in $\mathbb{S}\times\mathbb{R}$. Using the first order approximation of G_π from above, we infer

$$\begin{aligned} &-\lim_{\varepsilon\to0}\int_{\partial B_\varepsilon(0)^c} G_\pi\nabla\varphi\cdot\nu\,\mathrm{d}\sigma\\ &\qquad= -\frac{1}{2\pi}\lim_{\varepsilon\to0}\int_0^{2\pi}\varepsilon\ln(\varepsilon/2)\nabla\varphi(\varepsilon\cos(t),\varepsilon\sin(t))\cdot(\cos(t),\sin(t))^\top\,\mathrm{d}t, \end{aligned}$$

and thus this integral also vanishes since $\varepsilon\ln(\varepsilon/2)\to0$ for $\varepsilon\to0$ and $\nabla\varphi$ is bounded. To estimate the remaining term, we again use the first order approximation and recall (3.2.13) to calculate

$$\begin{aligned} \lim_{\varepsilon\to0}\int_{\partial B_\varepsilon(0)^c}\varphi\nabla G_\pi\cdot\nu\,\mathrm{d}\sigma &= \lim_{\varepsilon\to0}\frac{1}{2\pi}\int_0^{2\pi}\varphi(\varepsilon\cos(t),\varepsilon\sin(t))\frac{\varepsilon(\cos^2(t)+\sin^2(t))}{\varepsilon^2}\varepsilon\,\mathrm{d}t\\ &= \varphi(0). \end{aligned}$$

This shows that indeed $-\Delta G_\pi=\delta_0$ in $\mathcal{D}'(\mathbb{S}\times\mathbb{R})$. □

We are now in a position to write down our fundamental solution of the x_1-periodic Stokes system (3.2.1). To do so, we use the formula (3.2.11) which we derived for the Stokes system in $\mathbb{R}^n$ and use G_π instead of G. However, the resulting function is not periodic in x_1 due to the factor x_1 in the first term. To resolve this problem, we recall (3.2.2) and (3.2.10) and infer

$$\Delta(x_1\partial_1 G_\pi) = -\Delta(x_2\partial_2 G_\pi) \qquad \text{and} \qquad \Delta(x_1\partial_2 G_\pi) = \Delta(x_2\partial_1 G_\pi) \qquad \text{in } \mathcal{D}'(\mathbb{S}\times\mathbb{R}).$$

These identities enable us to manipulate (3.2.11) in such a way that only x_2 occurs in the terms $x_j\partial_k G_\pi$. To be precise, we define

$$\begin{aligned} \mathcal{U}(x) &:= \frac{1}{2\mu}\begin{pmatrix} -G_\pi(x) - x_2\partial_2 G_\pi(x) & x_2\partial_1 G_\pi(x) \\ x_2\partial_1 G_\pi(x) & -G_\pi(x) + x_2\partial_2 G_\pi(x) \end{pmatrix}, \\ \mathcal{P}(x) &:= \big(\partial_1 G_\pi(x),\, \partial_2 G_\pi(x)\big)^\top, \qquad x \in (\mathbb{S}\times\mathbb{R})\setminus\{0\}. \end{aligned} \tag{3.2.22}$$

Next, we show that $(\mathcal{U},\mathcal{P})$ indeed satisfies (3.2.1). To start, we recall (3.2.10) which leads us to

$$\begin{aligned} \mu\Delta\mathcal{U}_1^1 - \partial_1\mathcal{P}^1 &= -\frac{1}{2}(\Delta G_\pi + \Delta(x_2\partial_2 G_\pi)) - \partial_1^2 G_\pi = -\partial_2^2 G_\pi - \partial_1^2 G_\pi = \delta_0, \\ \mu\Delta\mathcal{U}_2^1 - \partial_2\mathcal{P}^1 &= \mu\Delta\mathcal{U}_1^2 - \partial_1\mathcal{P}^2 = \frac{1}{2}(\Delta(x_2\partial_1 G_\pi)) - \partial_1\partial_2 G_\pi = 0, \\ \mu\Delta\mathcal{U}_2^2 - \partial_2\mathcal{P}^2 &= \frac{1}{2}\big(-\Delta G_\pi + \Delta(x_2\partial_2 G_\pi)\big) - \partial_2^2 G_\pi = \frac{\delta_0}{2} - \frac{\Delta G_\pi}{2} = \delta_0 \end{aligned}$$

in $\mathcal{D}'(\mathbb{S}\times\mathbb{R})$ and, given $\varphi \in \mathcal{D}(\mathbb{S}\times\mathbb{R})$,

$$\begin{aligned} 2\mu\,\operatorname{div}\mathcal{U}^1 &= -\partial_1 G_\pi - x_2\partial_1\partial_2 G_\pi + \partial_2(x_2\partial_1 G_\pi) = 0, \\ 2\mu\langle\operatorname{div}\mathcal{U}^2,\varphi\rangle &= \langle x_2\partial_1^2 G_\pi - \partial_2 G_\pi + \partial_2(x_2\partial_2 G_\pi),\varphi\rangle = \langle\Delta G_\pi, x_2\varphi\rangle = 0. \end{aligned}$$

This shows that $(\mathcal{U},\mathcal{P})$ as defined in (3.2.22) is a fundamental solution of the x_1-periodic Stokes system (3.2.1).

3.3. Hydrodynamic layer potentials

The theory of hydrodynamic layer potentials dates back to the late 1920s, when the seminal works by Lichtenstein [58] and Odqvist [72] independently studied the boundary behavior of solutions to the stationary non-homogeneous Stokes problem (3.3.1). Given a regular enough bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the task was to solve

$$\left.\begin{aligned} \mu\Delta v - \nabla q &= F \text{ in } \Omega, \\ \operatorname{div} v &= 0 \;\text{ in } \Omega, \end{aligned}\right\} \tag{3.3.1}$$

where $F : \Omega \to \mathbb{R}^n$ is a regular enough given function and study the boundary behavior of v and the stress tensor $T_\mu(v,q)$. We formally construct solutions to (3.3.1) along the lines of [56] which itself follows [72] closely. The natural appearance of layer potentials will provide us a good starting point when investigating the transmission boundary value problems (3.1.6)–(3.1.7). We assume that any appearing functions are regular enough and calculate for solenoidal u, $v : \Omega \to \mathbb{R}^n$ (that is, $\operatorname{div} u = \operatorname{div} v = 0$ in Ω) and $q : \Omega \to \mathbb{R}$ the identity

$$\partial_j\big((T_\mu(v,q))_{ij}u_i\big) = (\mu\Delta v_i - \partial_i q)u_i + \frac{\mu}{2}(\partial_j u_i + \partial_i u_j)(\partial_j v_i + \partial_i v_j). \tag{3.3.2}$$

Integration over Ω and an application of the divergence theorem yields

$$\int_\Omega \partial_j\big((T_\mu(v,q))_{ij}u_i\big)\,\mathrm{d}x = \int_{\partial\Omega} \big((T_\mu(v,q))_{ij}u_i\big)\tilde{\nu}_j\,\mathrm{d}\sigma. \tag{3.3.3}$$

By interchanging u and v and replacing q by any other function $-p$ in (3.3.2) and subtracting the resulting equation from (3.3.2), we obtain, after integrating and using (3.3.3), the identity

$$\begin{aligned}&\int_\Omega (\mu\Delta v_i - \partial_i q)u_i - (\mu\Delta u_i + \partial_i p)v_i\,\mathrm{d}x\\ &= \int_{\partial\Omega} \big((T_\mu(v,q))_{ij}u_i - (T'_\mu(u,p))_{ij}v_i\big)\tilde{\nu}_j\,\mathrm{d}\sigma,\end{aligned} \tag{3.3.4}$$

where

$$(T'_\mu(u,p))_{ij} = p\delta_{ij} + \mu(\partial_j u_i + \partial_i u_j).$$

To proceed, we use our fundamental solution (u^k, p^k) of the Stokes equation defined in (3.2.11) and define

$$\tilde{u}_i^k(x,y) := u_i^k(x-y), \qquad \tilde{p}^k(x,y) := p^k(x-y).$$

Then, $(\tilde{u}^k, \tilde{p}^k)$ satisfies

$$\left.\begin{aligned}\mu\Delta_y\tilde{u}^k(x,y) + \nabla_y\tilde{p}^k(x,y) = \delta(x-y)e^k\\ \operatorname{div}_y \tilde{u}^k(x,y) = 0\end{aligned}\right\} \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{3.3.5}$$

Assuming further that (v,q) is a solution of (3.3.1), we formally get from (3.3.4) that

$$\begin{aligned}v_k(x) &= \lim_{\varepsilon\to 0}\int_{B_\varepsilon(x)} v_i(y)\delta_{ik}\delta(x-y)\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}\int_{B_\varepsilon(x)} v_i(y)\big(\mu\Delta_y\tilde{u}_i^k(x,y) + \partial_{y_i}\tilde{p}^k(x,y)\big)\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} \big((T'_\mu(\tilde{u}^k,\tilde{p}^k)_\eta)_{ij}(x,\eta)v_i(\eta) - (T_\mu(v,q))_{ij}(\eta)\tilde{u}_i^k(x,\eta)\big)\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma\\ &\quad + \lim_{\varepsilon\to 0}\int_{B_\varepsilon(x)} \tilde{u}^k(x,y)\big(\mu\Delta v_i(y) - \partial_i q(y)\big)\,\mathrm{d}y,\end{aligned}$$

where subscripts after differential operators (or $T'_\mu(\tilde{u}^k,\tilde{p}^k)$) are used to denote that differentiation has to be done with respect to the corresponding variable. Let us note that the volume integral in the last line vanishes, since the singularity at 0 of the fundamental solution is integrable as it is of order $\ln(|x|)$ in two dimensions and of order $2-n$ in dimension $n \geq 3$, cf. [32, Chapter 2.2.1]. Thus, we have

$$\begin{aligned}&v_k(x)\\ &= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} \big((T'_\mu(\tilde{u}^k,\tilde{p}^k)_\eta)_{ij}(x,\eta)v_i(\eta) - (T_\mu(v,q))_{ij}(\eta)\tilde{u}_i^k(x,\eta)\big)\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma.\end{aligned} \tag{3.3.6}$$

From (3.3.5), we infer that

$$\begin{aligned}&\int_{\Omega_\varepsilon(x)} (\mu\Delta v_i(y) - \partial_i q(y))\tilde{u}_i^k(x,y) - (\mu\Delta_y\tilde{u}_i^k(x,y) + \partial_{y_i}\tilde{p}^k(x,y))v_i(y)\,\mathrm{d}y\\ &= \int_{\Omega_\varepsilon(x)} \tilde{u}_i^k(x,y)(\mu\Delta v_i(y) - \partial_i q(y))\,\mathrm{d}y,\end{aligned} \tag{3.3.7}$$

where $\Omega_\varepsilon(x) := \Omega \setminus B_\varepsilon(x)$. On the other hand, appealing to (3.3.4), we have

$$\begin{aligned}&\int_{\Omega_\varepsilon(x)} (\mu\Delta v_i(y) - \partial_i q(y))\tilde{u}_i^k(x,y) - (\mu\Delta_y \tilde{u}_i^k(x,y) + \partial_{y_i}\tilde{p}^k(x,y))v_i(y)\,\mathrm{d}y \\ &= \int_{\partial\Omega_\varepsilon(x)} ((T_\mu(v,q))_{ij}(\eta)\tilde{u}_i^k(x,\eta) - (T'_\mu(\tilde{u}^k,\tilde{p}^k)_\eta)_{ij}(x,\eta)v_i(\eta))\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma.\end{aligned} \tag{3.3.8}$$

Splitting the boundary $\partial\Omega_\varepsilon(x) = \partial\Omega \cup \partial B_\varepsilon(x)$ in (3.3.8) and using (3.3.6), we infer from (3.3.7)–(3.3.8), after letting $\varepsilon \to 0$, that

$$\begin{aligned}v_k(x) = &\int_\Omega u_i^k(x,y)F_i(y)\,\mathrm{d}y - \int_{\partial\Omega} (T_\mu(u^k,p^k))_{ij}(x-\eta)v_i(\eta)\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma \\ &- \int_{\partial\Omega} u_i^k(x-\eta)(T_\mu(v,q))_{ij}(\eta)\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma,\end{aligned} \tag{3.3.9}$$

where we used that $(T'_\mu(\tilde{u}^k,\tilde{p}^k)_\eta)_{ij}(x,\eta) = -(T_\mu(u^k,p^k))_{ij}(x-\eta)$. To obtain a similar formula for the pressure, we first note that due to (3.2.11) we have $\Delta p^k = 0$ in $\mathbb{R}^n \setminus \{0\}$. Using this identity, it is easy to verify that

$$-\Delta(T_\mu(u^k,p^k))_{ij}(x-y) = -2\partial_i\partial_j p^k(x-y)$$

holds for all $x \neq y \in \mathbb{R}^n$. After formally applying the Laplacian to (3.3.9), we use this identity, (3.3.1), and (3.3.5) to obtain, after integrating, that

$$\begin{aligned}q(x) = &\int_\Omega p^k(x-y)F_k(y)\,\mathrm{d}y - 2\mu\int_{\partial\Omega} \partial_i p^k(x-\eta)v_i(\eta)\tilde{\nu}_k(\eta)\,\mathrm{d}\sigma \\ &- \int_{\partial\Omega} p^k(x-\eta)(T_\mu(v,q))_{kj}(\eta)\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma + q_0,\end{aligned} \tag{3.3.10}$$

where $q_0 \in \mathbb{R}$ is some constant.

Adopting [56, 72], we call the volume integrals

$$U_i(x) = \int_\Omega u_i^k(x-y)F_k(y)\,\mathrm{d}y, \qquad P(x) = \int_\Omega p^k(x-y)F_k(y)\,\mathrm{d}y \tag{3.3.11}$$

appearing in (3.3.9)–(3.3.10) the *hydrodynamic volume potentials generated by the density F*. Note that we made use of the identity $u_i^k = u_k^i$, cf. (3.2.6). Moreover, the formulas (3.3.9)–(3.3.10) suggest to define the *hydrodynamic single-layer potentials generated by the density ϕ* by

$$V_i(x) = -\int_{\partial\Omega} u_i^k(x-\eta)\phi_k(\eta)\,\mathrm{d}\sigma, \qquad Q(x) = -\int_{\partial\Omega} p^k(x-\eta)\phi_k(\eta)\,\mathrm{d}\sigma, \tag{3.3.12}$$

and the *hydrodynamic double-layer potentials generated by the density ψ* by

$$\begin{aligned}W_k(x) &= -\int_{\partial\Omega} (T_\mu(u^k,p^k))_{ij}(x-\eta)\psi_i(\eta)\tilde{\nu}_j(\eta)\,\mathrm{d}\sigma, \\ \Pi(x) &= -2\mu\int_{\partial\Omega} \partial_i p^k(x-\eta)\psi_i(\eta)\tilde{\nu}_k(\eta)\,\mathrm{d}\sigma,\end{aligned} \tag{3.3.13}$$

where ϕ, $\psi : \partial\Omega \to \mathbb{R}^n$.

It is shown in [72] that the hydrodynamic single-layer potentials satisfy the homogeneous Stokes equation (that is, (3.3.1) with $F = 0$) and that the velocity is continuous across $\partial\Omega$, that is

$$V_i(\xi) = \lim_{\Omega\ni x\to\xi} V_i(x) = \lim_{(\mathbb{R}^n\setminus\overline{\Omega})\ni x\to\xi} V_i(x), \qquad \xi \in \partial\Omega, \tag{3.3.14}$$

provided that the density generating the potential is continuous itself. Moreover, the stress tensor of the single-layer potentials satisfies the following jump relation at the boundary:

$$\begin{aligned} \lim_{\Omega \ni x \to \xi} (T_\mu(V,Q))_{ij}(x)\tilde{\nu}_j(x) &= \frac{\phi_i}{2}(\xi) + (T_\mu(V,Q))_{ij}(\xi)\tilde{\nu}_j(\xi), \\ \lim_{(\mathbb{R}^n \setminus \overline{\Omega}) \ni x \to \xi} (T_\mu(V,Q))_{ij}(x)\tilde{\nu}_j(x) &= -\frac{\phi_i}{2}(\xi) + (T_\mu(V,Q))_{ij}(\xi)\tilde{\nu}_j(\xi), \end{aligned} \tag{3.3.15}$$

where $\tilde{\nu}(x) := \tilde{\nu}(\bar{\xi})$ and $\bar{\xi} \in \partial\Omega$ is the closest point to x in $\partial\Omega$ (which is unique provided that x is close enough to $\partial\Omega$ and $\partial\Omega$ is smooth enough) and the term $(T_\mu(V,Q))_{ij}(\xi)$ has to be evaluated using the principal value in a suitable sense. A similar jump relation holds for the velocity of the hydrodynamic double-layer potential which satisfies

$$\begin{aligned} \lim_{\Omega \ni x \to \xi} W_k(x) &= \frac{\psi_k}{2}(\xi) + W_k(\xi), \\ \lim_{(\mathbb{R}^n \setminus \overline{\Omega}) \ni x \to \xi} W_k(x) &= -\frac{\psi_k}{2}(\xi) + W_k(\xi), \end{aligned} \tag{3.3.16}$$

where the term $W_k(\xi)$ also has to be evaluated using the principal value.

3.3.1. The hydrodynamic single-layer potential

We now turn our attention to the system of equations (3.1.6) and consider the more general problem

$$\left.\begin{aligned} \mu\Delta v^\pm - \nabla q^\pm &= 0 && \text{in } \Omega^\pm, \\ \operatorname{div} v^\pm &= 0 && \text{in } \Omega^\pm, \\ [v] &= 0 && \text{on } \Gamma, \\ [T_\mu(v,q)]\tilde{\nu} &= (\omega^{-1}G)\circ\Xi^{-1} && \text{on } \Gamma, \\ (v^\pm, q^\pm)(x) &\to \pm(c_1,c_2,c_3) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \end{aligned}\right\} \tag{3.3.17}$$

where $G := (G_1, G_2) \in \mathrm{H}^1(\mathbb{S})^2$ is a given function that satisfies $\langle G_1 \rangle = 0$ and the constant $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ is an unknown of the problem. We use the notation introduced in Section 3.1 and recall that $\Gamma = \operatorname{graph}(f)$ for a given $f \in \mathrm{H}^3(\mathbb{S})$.

Remark 3.3.1. We note that in the particular case when

$$G := G(f) := \Theta(-ff', f) - \sigma(\omega^{-1} - 1, \omega^{-1}f')', \tag{3.3.18}$$

we have $G = (G_1, G_2) \in \mathrm{H}^1(\mathbb{S})^2$, $\langle G_1 \rangle = 0$, and

$$(\omega^{-1}G)\circ\Xi^{-1} = (\Theta x_2 - \sigma\widetilde{\kappa})\tilde{\nu}.$$

Consequently, the right-hand sides of $(3.3.17)_4$ and $(3.1.6)_4$ coincide in this case.

Recalling (3.3.14)–(3.3.15), we are led to take the educated guess that the solution to (3.3.17) is given by the hydrodynamic single-layer potentials generated by the density function $-(\omega^{-1}G)\circ\Xi^{-1}$. In what follows, we will pursue this approach and define and analyze the hydrodynamic single-layer potential adapted to our situation. Naturally, the main ingredient is the x_1-periodic Stokeslet $(\mathcal{U}, \mathcal{P})$ with $\mathcal{U} = (\mathcal{U}_j^k)_{1\le j,k\le 2}$ and $\mathcal{P} = (\mathcal{P}^1, \mathcal{P}^2)^\top$ as defined in (3.2.22). Recalling the notation introduced in (3.2.20), we first define the function

$$z_0(x) := \ln\Big(\frac{t_{[x_1]}^2 + T_{[x_2]}^2}{(1+t_{[x_1]}^2)(1-T_{[x_2]}^2)}\Big) = \ln\Big(\sin^2\Big(\frac{x_1}{2}\Big) + \sinh^2\Big(\frac{x_2}{2}\Big)\Big), \tag{3.3.19}$$

where $x = (x_1, x_2) \in (\mathbb{S} \times \mathbb{R}) \setminus \{0\}$. Related to it, we further introduce

$$\begin{aligned} z_1(x) &:= \frac{t_{[x_1]}(1 - T_{[x_2]}^2)}{t_{[x_1]}^2 + T_{[x_2]}^2}, \qquad & z_7(x) &:= \frac{(1 + t_{[x_1]}^2)(1 - T_{[x_2]}^2)(t_{[x_1]}^2 - T_{[x_2]}^2)}{2(t_{[x_1]}^2 + T_{[x_2]}^2)^2}, \\ z_2(x) &:= \frac{(1 + t_{[x_1]}^2)T_{[x_2]}}{t_{[x_1]}^2 + T_{[x_2]}^2}, \qquad & z_8(x) &:= \frac{t_{[x_1]}T_{[x_2]}(1 + t_{[x_1]}^2)(1 - T_{[x_2]}^2)}{2(t_{[x_1]}^2 + T_{[x_2]}^2)^2}, \end{aligned} \tag{3.3.20}$$

and

$$\begin{aligned} z_3(x) &:= x_2 z_7(x), \qquad & z_5(x) &:= x_2 z_1(x), \\ z_4(x) &:= x_2 z_8(x), \qquad & z_6(x) &:= x_2 z_2(x). \end{aligned} \tag{3.3.21}$$

It is clear that

$$z_i \in \mathrm{C}^\infty((\mathbb{S} \times \mathbb{R}) \setminus \{0\}), \qquad 1 \le i \le 8, \tag{3.3.22}$$

and we note the relations

$$\nabla z_0 = (z_1, z_2)^\top, \qquad \nabla z_1 = (-z_7, -2z_8)^\top, \qquad \nabla z_2 = (-2z_8, z_7)^\top \tag{3.3.23}$$

in $(\mathbb{S} \times \mathbb{R}) \setminus \{0\}$. Recalling (3.2.19), we have $-4\pi G_\pi = z_0$ in $(\mathbb{S} \times \mathbb{R}) \setminus \{0\}$ and (3.2.22) together with (3.3.23) enable us to write $(\mathcal{U}, \mathcal{P})$ as

$$\mathcal{U}(x) = \frac{1}{8\pi\mu} \begin{pmatrix} z_0(x) + z_6(x) & -z_5(x) \\ -z_5(x) & z_0(x) - z_6(x) \end{pmatrix}, \qquad \mathcal{P} = -\frac{1}{4\pi}(z_1, z_2)^\top \tag{3.3.24}$$

for $x \in (\mathbb{S} \times \mathbb{R}) \setminus \{0\}$. Using (3.3.12), our candidate solution to the transmission boundary value problem (3.3.17) is given by

$$\begin{aligned} v_G^\pm(x) &:= \int_\Gamma \mathcal{U}^k(x - \eta)(\omega^{-1}G_k) \circ \Xi^{-1}(\eta)\, \mathrm{d}\sigma = \int_{-\pi}^{\pi} \mathcal{U}^k(r(x,s))G_k(s)\, \mathrm{d}s, \\ q_G^\pm(x) &:= \int_\Gamma \mathcal{P}^k(x - \eta)(\omega^{-1}G_k) \circ \Xi^{-1}(\eta)\, \mathrm{d}\sigma = \int_{-\pi}^{\pi} \mathcal{P}^k(r(x,s))G_k(s)\, \mathrm{d}s, \end{aligned} \tag{3.3.25}$$

where $x \in \Omega^\pm$ and $r = (r_1, r_2)$ is defined by

$$r := r(x,s) := x - (s, f(s)), \qquad x \in \Omega^\pm, \quad s \in \mathbb{R}. \tag{3.3.26}$$

In accordance with (3.3.12), we call the functions defined in (3.3.25) the hydrodynamic single-layer potentials generated by the density $-(\omega^{-1}G) \circ \Xi^{-1}$.

The next theorem states that the unique solution to (3.3.17) in X_f is precisely given by (3.3.25) with a constant added to the second component of the velocity field.

Theorem 3.3.2. *Given $f \in \mathrm{H}^3(\mathbb{S})$ and $G \in \mathrm{H}^1(\mathbb{S})^2$ with $\langle G_1 \rangle = 0$, the transmission boundary value problem* (3.3.17) *has a solution $(v^\pm, q^\pm) \in X_f$ if and only if the constant $c = (c_1, c_2, c_3) \in \mathbb{R}^3$ in* $(3.3.17)_5$ *is given by*

$$(c_1, c_2, c_3) = \Big(-\frac{\langle fG_1 \rangle}{2\mu}, 0, -\frac{\langle G_2 \rangle}{2} \Big). \tag{3.3.27}$$

If $c = (c_1, c_2, c_3)$ is defined by (3.3.27)*, then the solution $(v^\pm, q^\pm)$ is unique and is given by the formula*

$$v^\pm := v_G^\pm + \Big(0, \frac{\langle G_2 \rangle \ln 4}{4\mu}\Big) \qquad \text{and} \qquad q^\pm := q_G^\pm, \tag{3.3.28}$$

where $(v_G^\pm, q_G^\pm)$ is defined in (3.3.25)*.*

We split the proof into four parts. First, we show that (3.3.17) admits at most one solution and then that $(v^\pm, q^\pm)$ defined in (3.3.28) satisfies $(3.3.17)_{1-2}$. In the third part, we establish that the same pair $(v^\pm, q^\pm)$ from (3.3.28) also satisfies the boundary conditions $(3.3.17)_{3-4}$ and as a last step we show that the far-field condition $(3.3.17)_5$ is fulfilled as well.

Proof. Uniqueness. To prove the uniqueness statement, we assume that there exist two solutions (v, q) and $(\tilde{v}, \tilde{q}) \in X_f$. Then, their difference $(\mathsf{v}, \mathsf{q}) := (v - \tilde{v}, q - \tilde{q})$ also lies in X_f and solves the boundary value problem

$$\left.\begin{array}{rcll} \mu\Delta \mathsf{v}^\pm - \nabla \mathsf{q}^\pm &=& 0 & \text{in } \Omega^\pm, \\ \operatorname{div} \mathsf{v}^\pm &=& 0 & \text{in } \Omega^\pm, \\ [\mathsf{v}] &=& 0 & \text{on } \Gamma, \\ [T_\mu(\mathsf{v}, \mathsf{q})]\tilde{\nu} &=& 0 & \text{on } \Gamma, \\ (\mathsf{v}^\pm, \mathsf{q}^\pm)(x) &\to& \pm(\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3) & \begin{array}{l}\text{for } x_2 \to \pm\infty \text{ uni-} \\ \text{formly in } x_1 \in \mathbb{S}, \end{array} \end{array}\right\} \tag{3.3.29}$$

for some $(\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3) \in \mathbb{R}^3$. Then, we need to show that actually $(\mathsf{v}^\pm, \mathsf{q}^\pm) = (0, 0)$ in $\Omega^\pm$ and $\mathsf{c}_1 = \mathsf{c}_2 = \mathsf{c}_3 = 0$. We first note, in view of $(3.3.29)_2$, that

$$T_\mu(\mathsf{v}^\pm, \mathsf{q}^\pm)\tilde{\nu} = -\mathsf{q}^\pm \tilde{\nu} + \mu \begin{pmatrix} \partial_{\tilde{\nu}} \mathsf{v}_1^\pm + \partial_{\tilde{\tau}} \mathsf{v}_2^\pm \\ \partial_{\tilde{\nu}} \mathsf{v}_2^\pm - \partial_{\tilde{\tau}} \mathsf{v}_1^\pm \end{pmatrix},$$

and, since $[\partial_{\tilde{\tau}} \mathsf{v}] = 0$ as a consequence of $(\mathsf{v}^\pm, \mathsf{q}^\pm) \in X_f$ and $(3.3.29)_3$, we arrive together with $(3.3.29)_4$ at

$$\mu[\partial_{\tilde{\nu}} \mathsf{v}] - [\mathsf{q}]\tilde{\nu} = [T_\mu(\mathsf{v}, \mathsf{q})]\tilde{\nu} = 0. \tag{3.3.30}$$

Set $(\mathsf{v}, \mathsf{q}) := \mathbf{1}_{\Omega^+}(\mathsf{v}^+, \mathsf{q}^+) + \mathbf{1}_{\Omega^-}(\mathsf{v}^-, \mathsf{q}^-) \in \mathrm{L}^\infty(\mathbb{S} \times \mathbb{R}, \mathbb{R}^2 \times \mathbb{R})$. We then compute, in light of $(\mathsf{v}, \mathsf{q}) \in X_f$, $(3.3.29)_{1-3}$, and (3.3.30), that

$$\left.\begin{array}{rcl} \mu\Delta \mathsf{v} - \nabla \mathsf{q} &=& 0, \\ \operatorname{div} \mathsf{v} &=& 0 \end{array}\right\} \quad \text{in } \mathcal{D}'(\mathbb{S} \times \mathbb{R}).$$

In particular, taking the divergence of the first equation yields $\Delta \mathsf{q} = 0$, hence, q is a harmonic function in $\mathbb{S} \times \mathbb{R}$. Since q is bounded, Liouville's theorem and $(3.3.29)_5$ now yield $\mathsf{q} = 0$ in $\mathbb{R}^2$. This in turn means that v_1 and v_2 are harmonic in $\mathbb{S} \times \mathbb{R}$, and, since v is bounded, we conclude together with $(3.3.29)_5$ that $\mathsf{v} = 0$ and $\mathsf{c}_1 = \mathsf{c}_2 = \mathsf{c}_3 = 0$, which proves the uniqueness claim.

Solution of the Stokes equations. To prove that $(v^\pm, q^\pm)$ actually solves the homogeneous Stokes equation, we fix $x_0 \in \Omega^\pm$ and choose $\varepsilon > 0$ such that the closed ball $\overline{B}_\varepsilon(x_0)$ is contained in $\Omega^\pm$. Since $(\mathcal{U}, \mathcal{P})(\cdot - (s, f(s))) \in \mathrm{C}^\infty(\Omega^\pm, \mathbb{R}^{2\times 2} \times \mathbb{R}^2)$ for each fixed $s \in \mathbb{S}$, cf. (3.3.22), we infer that the partial derivatives $\partial_x^\alpha \mathcal{U}_j^k(\cdot - (s, f(s)))$ and $\partial_x^\alpha \mathcal{P}^k(\cdot - (s, f(s)))$, $\alpha \in \mathbb{N}^2$, are bounded in $\overline{B}_\varepsilon(x_0)$ uniformly in $s \in \mathbb{S}$. Therefore, the function $(v^\pm, q^\pm)$ is well-defined in (3.3.28) and we may interchange differentiation with respect to x and the integral sign in (3.3.25). Recalling that $(\mathcal{U}^k, \mathcal{P}^k)$, $1 \le k \le 2$, solve the Stokes equations (3.2.1) pointwise in $(\mathbb{S} \times \mathbb{R}) \setminus \{0\}$, it follows immediately that $(v^\pm, q^\pm)$ solves $(3.3.17)_{1-2}$ in $\Omega^\pm$.

Boundary conditions and regularity. From Lemma 3.3.6 and Lemma 3.3.8 below it follows that our solution (3.3.28) satisfies $(3.3.17)_{3-4}$ after observing that

$$G = (G \cdot \nu)\nu + (G \cdot \tau)\tau. \tag{3.3.31}$$

Moreover, these lemmas also show that $(v, q) \in X_f$.

Far-field conditions. We again combine Lemma 3.3.6 and Lemma 3.3.8, to find that $(v^\pm, q^\pm)$ from (3.3.28) satisfies $(3.3.17)_5$ provided that the constant $c = (c_1, c_2, c_3)$ is given by (3.3.27). □

Next, we establish some results concerning the boundary and far-field behavior of some integral operators related to (3.3.25) which were used in the proof of Theorem 3.3.2. The analogues of these results for non-periodic operators were established in [64].

We start by computing the first order partial derivatives of $\mathcal{U}$ since the gradient $\nabla v^\pm$ is determined by simply differentiating under the integral sign in (3.3.28) (see the first part of the proof of Theorem 3.3.2). From the relations (3.3.21)–(3.3.23) and formula (3.3.24) we infer, that, for given $x \in (\mathbb{S} \times \mathbb{R}) \setminus \{0\}$, these derivatives are given by the following expressions

$$\begin{aligned} \partial_1 \mathcal{U} &= \frac{1}{8\pi\mu} \begin{pmatrix} z_1 - 2z_4 & z_3 \\ z_3 & z_1 + 2z_4 \end{pmatrix}, & \partial_1 \mathcal{P} &= \frac{1}{4\pi}(z_7, 2z_8)^\top, \\ \partial_2 \mathcal{U} &= \frac{1}{8\pi\mu} \begin{pmatrix} 2z_2 + z_3 & -z_1 + 2z_4 \\ -z_1 + 2z_4 & -z_3 \end{pmatrix}, & \partial_2 \mathcal{P} &= \frac{1}{4\pi}(2z_8, -z_7)^\top. \end{aligned} \tag{3.3.32}$$

This motivates us to first establish the following preparatory result.

Lemma 3.3.3. *Given $\varphi \in \mathrm{L}^2(\mathbb{S})$, let $Z_n(f)[\varphi] : (\mathbb{S} \times \mathbb{R}) \setminus \Gamma \to \mathbb{R}$, $1 \le n \le 6$, be defined by*

$$Z_n(f)[\varphi] := \frac{1}{2\pi} \int_{-\pi}^{\pi} z_n(r)\varphi(s)\,\mathrm{d}s, \tag{3.3.33}$$

where z_n, $1 \le n \le 6$, were defined in (3.3.20)–(3.3.21) and where we use the notation introduced in (3.3.26). For $1 \le n \le 6$, we further set

$$B_n(f)[\varphi](\xi) := \mathrm{PV}\frac{1}{2\pi} \int_{-\pi}^{\pi} z_n(\xi - s, f(\xi) - f(s))\varphi(s)\,\mathrm{d}s, \qquad \xi \in \mathbb{S}. \tag{3.3.34}$$

Then, $Z_n(f)[\varphi]^\pm \in \mathrm{C}^\infty(\Omega^\pm)$, $1 \le n \le 6$, and $Z_n(f)[\varphi] \in \mathrm{C}(\mathbb{S} \times \mathbb{R})$, $n = 5, 6$, with

$$\{Z_n(f)[\varphi]\}^\pm \circ \Xi = B_n(f)[\varphi], \qquad n = 5, 6. \tag{3.3.35}$$

Moreover, if additionally $\varphi \in \mathrm{H}^1(\mathbb{S})$, then $Z_n(f)[\varphi]^\pm \in \mathrm{C}(\overline{\Omega^\pm})$, $1 \le n \le 4$, and

$$\left\{ \begin{pmatrix} Z_1(f)[\varphi] \\ Z_2(f)[\varphi] \\ Z_3(f)[\varphi] \\ Z_4(f)[\varphi] \end{pmatrix} \right\}^\pm \circ \Xi = \begin{pmatrix} B_1(f)[\varphi] \\ B_2(f)[\varphi] \\ B_3(f)[\varphi] \\ B_4(f)[\varphi] \end{pmatrix} \pm \frac{1}{\omega^2} \begin{pmatrix} -f' \\ 1 \\ -\frac{2f'^2}{\omega^2} \\ \frac{f' - f'^3}{2\omega^2} \end{pmatrix} \varphi. \tag{3.3.36}$$

Related to definition (3.3.34), we observe that we may evaluate the integrals (3.3.33) at $\Xi(\xi)$ with $\xi \in \mathbb{S}$, provided that we interpret some of the integrals as being singular, see Lemma A.1.3 and Lemma A.1.4, since the operators $B_n(f)$, $1 \le n \le 6$, can be represented as linear combinations of the operators $B^{p,q}_{n,m}(f)$, n, m, p, $q \in \mathbb{N}_0$, with $1 \le p \le n + q + 1$, defined in (4.1.6), see (4.1.7). In fact, Lemma A.1.3 ensures that $B_n(f)[\varphi] \in \mathrm{C}(\mathbb{S})$, $n = 5, 6$, while, for $\varphi \in \mathrm{H}^1(\mathbb{S})$, we also have $B_n(f)[\varphi] \in \mathrm{C}(\mathbb{S})$, for $1 \le n \le 4$, cf. Lemma A.1.3 and Lemma A.1.6 (i).

Proof of Lemma 3.3.3. Arguing as in the first part of the proof of Theorem 3.3.2, it immediately follows that the functions $Z_n(f)[\varphi]^\pm$ belong to $\mathrm{C}^\infty(\Omega^\pm)$ for $1 \le n \le 6$. Moreover, Lebesgue's dominated convergence theorem leads to

$$\{Z_n(f)[\varphi]\}^\pm \circ \Xi = B_n(f)[\varphi] \in \mathrm{C}(\mathbb{S}), \qquad n = 5, 6,$$

so that $Z_n(f)[\varphi] \in \mathrm{C}(\mathbb{S}\times\mathbb{R})$ for $n = 5, 6$. This proves (3.3.35).

For the remainder of this proof, we assume that $\varphi \in \mathrm{H}^1(\mathbb{S})$. It is shown in [63, Lemma 2.2] that

$$\begin{aligned} \{Z_1(f)[\varphi]\}^\pm \circ \Xi &= B_1(f)[\varphi] \mp \frac{f'}{\omega^2}\varphi, \\ \{Z_2(f)[\varphi]\}^\pm \circ \Xi &= B_2(f)[\varphi] \pm \frac{1}{\omega^2}\varphi, \end{aligned} \tag{3.3.37}$$

and since $B_n(f)[\varphi] \in \mathrm{C}(\mathbb{S})$, $n = 1, 2$, we conclude that $Z_n(f)[\varphi]^\pm \in \mathrm{C}(\overline{\Omega^\pm})$ for $n = 1, 2$. In order to derive similar properties for $Z_n(f)[\varphi]$, $n = 3, 4$, we use integration by parts to deduce that

$$\left.\begin{aligned} Z_5(f)[\varphi'] &= Z_1(f)[f'\varphi] - Z_3(f)[\varphi] - 2Z_4(f)[f'\varphi], \\ Z_6(f)[\varphi'] &= Z_2(f)[f'\varphi] + Z_3(f)[f'\varphi] - 2Z_4(f)[\varphi] \end{aligned}\right\} \quad \text{in } (\mathbb{S}\times\mathbb{R})\setminus\Gamma,$$

and

$$\left.\begin{aligned} B_5(f)[\varphi'] &= B_1(f)[f'\varphi] - B_3(f)[\varphi] - 2B_4(f)[f'\varphi], \\ B_6(f)[\varphi'] &= B_2(f)[f'\varphi] + B_3(f)[f'\varphi] - 2B_4(f)[\varphi] \end{aligned}\right\} \quad \text{in } \mathrm{C}(\mathbb{S}),$$

respectively. Since $Z_n(f)[\varphi'] \in \mathrm{C}(\mathbb{S}\times\mathbb{R})$, $n = 5, 6$, the latter formulas combined with (3.3.35) and (3.3.37) (with φ replaced by $f'\varphi$) yield

$$\begin{aligned} \{Z_3(f)[\varphi] + 2Z_4(f)[f'\varphi]\}^\pm \circ \Xi &= B_3(f)[\varphi] + 2B_4(f)[f'\varphi] \mp \frac{f'^2}{\omega^2}\varphi, \\ \{Z_3(f)[f'\varphi] - 2Z_4(f)[\varphi]\}^\pm \circ \Xi &= B_3(f)[f'\varphi] - 2B_4(f)[\varphi] \mp \frac{f'}{\omega^2}\varphi. \end{aligned} \tag{3.3.38}$$

We now replace φ by φ/ω^2 in $(3.3.38)_1$ and by $(f'\varphi)/\omega^2$ in $(3.3.38)_2$ to obtain, after taking the sum of the two relations, that

$$\begin{aligned} \{Z_3(f)[\varphi]\}^\pm \circ \Xi &= B_3(f)[\varphi] \mp \frac{2f'^2}{\omega^4}\varphi, \\ \{Z_4(f)[\varphi]\}^\pm \circ \Xi &= B_4(f)[\varphi] \pm \frac{f' - f'^3}{2\omega^4}\varphi, \end{aligned} \tag{3.3.39}$$

with $(3.3.39)_2$ being a consequence of $(3.3.39)_1$ and $(3.3.38)_2$. This shows (3.3.36) and completes the proof. □

As a further preparatory result, we establish the following lemma which is related to the logarithmic term in $\mathcal{U}$, see (3.3.19) and (3.3.24).

Lemma 3.3.4. *Given $\varphi \in \mathrm{L}^2(\mathbb{S})$, let $Z_0(f)[\varphi] : (\mathbb{S}\times\mathbb{R})\setminus\Gamma \to \mathbb{R}$ be given by*

$$Z_0(f)[\varphi](x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\Big(\sin^2\Big(\frac{r_1}{2}\Big) + \sinh^2\Big(\frac{r_2}{2}\Big)\Big)\varphi(s)\,\mathrm{d}s. \tag{3.3.40}$$

Then, $Z_0(f)[\varphi] \in \mathrm{C}^\infty((\mathbb{S}\times\mathbb{R})\setminus\Gamma)$ and $\nabla(Z_0(f)[\varphi]) = (Z_1(f)[\varphi], Z_2(f)[\varphi])$. Additionally, if $\varphi \in \mathrm{H}^1(\mathbb{S})$, we have $Z_0(f)[\varphi] \in \mathrm{C}(\mathbb{S}\times\mathbb{R})$ and $Z_0(f)[\varphi]^\pm \in \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$, with

$$\{Z_0(f)[\varphi]\}^\pm \circ \Xi = B_0(f)[\varphi], \tag{3.3.41}$$

where

$$B_0(f)[\varphi](\xi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\Big(\sin^2\Big(\frac{s}{2}\Big) + \sinh^2\Big(\frac{f(\xi) - f(\xi - s)}{2}\Big)\Big)\varphi(\xi - s)\,\mathrm{d}s \tag{3.3.42}$$

for $\xi \in \mathbb{S}$.

Proof. Arguing along the lines of the first part of the proof of Theorem 3.3.2, we obtain that $Z_0(f)[\varphi]^\pm \in \mathrm{C}^\infty(\Omega^\pm)$ with gradient $\nabla(Z_0(f)[\varphi]) = (Z_1(f)[\varphi],\, Z_2(f)[\varphi])$. Since $Z_n(f)[\varphi]^\pm \in \mathrm{C}^1(\overline{\Omega^\pm})$, $n = 1,\, 2$, for $\varphi \in \mathrm{H}^1(\mathbb{S})$, cf. Lemma 3.3.3, we deduce that $Z_0(f)[\varphi]^\pm \in \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$. Additionally, Lebesgue's dominated convergence theorem ensures that both one-sided limits of $Z_0(f)[\varphi]$ in $\Xi(\xi)$ exist for all $\xi \in \mathbb{S}$ and coincide with $B_0(f)[\varphi](\xi)$ (which exists as an improper integral). This proves (3.3.41) and the continuity property $Z_0(f)[\varphi] \in \mathrm{C}(\mathbb{S} \times \mathbb{R})$. □

Related to the asymptotic behavior of the operators defined above, we establish the following lemma.

Lemma 3.3.5. *Given $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$Z_i(f)[\varphi]^\pm \longrightarrow 0, \qquad i \in \{1,3,4,5\} \tag{3.3.43}$$

$$Z_2(f)[\varphi]^\pm \longrightarrow \pm\langle\varphi\rangle, \tag{3.3.44}$$

$$Z_6(f)[\varphi]^\pm \mp x_2\langle\varphi\rangle \longrightarrow \mp\langle f\varphi\rangle, \tag{3.3.45}$$

$$Z_0(f)[\varphi]^\pm \mp x_2\langle\varphi\rangle \longrightarrow \mp\langle f\varphi\rangle - \langle\varphi\rangle \ln 4 \tag{3.3.46}$$

for $x_2 \to \pm\infty$ uniformly in $x_1 \in \mathbb{S}$.

Proof. The property (3.3.43) is a simple consequence of Lebesgue's dominated convergence theorem, which implies, via

$$Z_2(f)[\varphi]^\pm(x) \mp \langle\varphi\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 \mp T_{[r_2]})(T_{[r_2]} \mp t^2_{[r_1]})}{t^2_{[r_1]} + T^2_{[r_2]}}\varphi(s)\,\mathrm{d}s, \qquad x \in \Omega^\pm,$$

and

$$Z_6(f)[\varphi]^\pm(x) \mp x_2\langle\varphi\rangle \pm \langle f\varphi\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_2(1 \mp T_{[r_2]})\frac{T_{[r_2]} \mp t^2_{[r_1]}}{t^2_{[r_1]} + T^2_{[r_2]}}\varphi(s)\,\mathrm{d}s, \qquad x \in \Omega^\pm,$$

also (3.3.44)–(3.3.45). Finally, with respect to (3.3.46), we note that since

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(4^{\mp 1}e^{r_2})\varphi\,\mathrm{d}s = \langle f\varphi\rangle + (\pm\ln 4 - x_2)\langle\varphi\rangle, \qquad x \in \Omega^\pm,$$

Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
&Z_0(f)[\varphi]^\pm(x) \pm [\langle f\varphi\rangle + (\pm\ln 4 - x_2)\langle\varphi\rangle] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\ln\left(\sin^2\left(\frac{r_1}{2}\right) + \frac{e^{r_2} - 2 + e^{-r_2}}{4}\right) \mp \ln(4^{\mp 1}e^{r_2})\right]\varphi(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(4e^{\mp r_2}\sin^2\left(\frac{r_1}{2}\right) + e^{\mp 2r_2} - 2e^{\mp r_2} + 1\right)\varphi(s)\,\mathrm{d}s \underset{x_2\to\pm\infty}{\longrightarrow} 0.
\end{aligned}$$

□

We are now in a position to study the behavior of the velocity v defined in (3.3.28) close to the interface and in the far-field (under the assumptions of Theorem 3.3.2).

Lemma 3.3.6. *We have $v \in \mathrm{C}(\mathbb{S} \times \mathbb{R}, \mathbb{R}^2)$, $v^\pm \in \mathrm{C}^\infty(\Omega^\pm, \mathbb{R}^2) \cap \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$, and*

$$\mu([\nabla v + (\nabla v)^\top]\tilde{\nu}) \circ \Xi = \omega^{-1}(G \cdot \tau)\tau \qquad \textit{on } \mathbb{S}, \tag{3.3.47}$$

$$v^\pm(x) \longrightarrow \left(\mp\frac{\langle fG_1\rangle}{2\mu}, 0\right) \qquad \textit{for } x_2 \to \pm\infty \textit{ uniformly in } x_1 \in \mathbb{S}, \tag{3.3.48}$$

$$\nabla v^\pm(x) \longrightarrow 0 \qquad \textit{for } x_2 \to \pm\infty \textit{ uniformly in } x_1 \in \mathbb{S}. \tag{3.3.49}$$

Proof. Recalling (3.3.25), we write

$$v_G = \frac{1}{4\mu}\begin{pmatrix}(Z_0(f)+Z_6(f))[G_1]-Z_5(f)[G_2]\\(Z_0(f)-Z_6(f))[G_2]-Z_5(f)[G_1]\end{pmatrix}^{\top} \qquad \text{in } (\mathbb{S}\times\mathbb{R})\setminus\Gamma, \tag{3.3.50}$$

and Lemma 3.3.3 and Lemma 3.3.4 ensure that indeed $v_G \in \mathrm{C}(\mathbb{S}\times\mathbb{R},\mathbb{R}^2)$ and

$$\{v_G\}^{\pm}\circ\Xi = \frac{1}{4\mu}\begin{pmatrix}(B_0(f)+B_6(f))[G_1]-B_5(f)[G_2]\\(B_0(f)-B_6(f))[G_2]-B_5(f)[G_1]\end{pmatrix}^{\top}. \tag{3.3.51}$$

Noticing also that

$$\left.\begin{aligned}\nabla(Z_5(f)[\varphi]) &= (-Z_3(f)[\varphi], Z_1(f)[\varphi]-2Z_4(f)[\varphi])\\ \nabla(Z_6(f)[\varphi]) &= (-2Z_4(f)[\varphi], Z_2(f)[\varphi]+Z_3(f)[\varphi])\end{aligned}\right\} \qquad \text{in } \mathbb{S}\times\mathbb{R}\setminus\Gamma, \tag{3.3.52}$$

we infer from Lemma 3.3.3 and Lemma 3.3.4 that $v_G^{\pm} \in \mathrm{C}^\infty(\Omega^\pm,\mathbb{R}^2)\cap \mathrm{C}^1(\overline{\Omega^\pm},\mathbb{R}^2)$ and the formula (3.3.36) leads us to

$$[\nabla v_G]\circ\Xi = \begin{bmatrix}\partial_1 v_{G,1} & \partial_2 v_{G,1}\\ \partial_1 v_{G,2} & \partial_2 v_{G,2}\end{bmatrix}\circ\Xi = \frac{G\cdot\tau}{\mu\omega^3}\begin{pmatrix}-f' & 1\\ -f'^2 & f'\end{pmatrix},$$

hence

$$\mu([\nabla v_G + (\nabla v_G)^{\top}]\widetilde{\nu})\circ\Xi = \omega^{-1}(G\cdot\tau)\tau,$$

and (3.3.47) follows.

Moreover, in view of Lemma 3.3.5, we have

$$v_G^{\pm}(x) \longrightarrow \Big(\mp\frac{\langle fG_1\rangle}{2\mu}, -\frac{\langle G_2\rangle \ln 4}{4\mu}\Big) \qquad \text{for } x_2\to\pm\infty \text{ uniformly in } x_1\in\mathbb{S},$$

which proves (3.3.48).

Using Lemma 3.3.4 and (3.3.52) to differentiate (3.3.50), the claim (3.3.49) follows from (3.3.28) and Lemma 3.3.5. ☐

The following observation, together with (3.3.51), is used when formulating the Stokes problem (1.2.2) as an evolution problem for f, as it provides an expression for the trace of v_G on Γ, in the particular case when $G=F'$ for some function $F=(F_1,F_2)$, which involves the function F (and not its derivative), see (3.3.54) below.

Remark 3.3.7. Assume that $G=F'$ for some function $F=(F_1,F_2)\in \mathrm{H}^2(\mathbb{S})^2$. Then, observing that $[s\mapsto \mathcal{U}(x-(s,f(s)))]:\mathbb{S}\to\mathbb{R}^{2\times 2}$ is continuously differentiable, integration by parts in (3.3.25) leads to the following representation

$$v_G^{\pm}(x) = \int_{-\pi}^{\pi} F(s)\left(\partial_1\begin{pmatrix}\mathcal{U}^1\\ \mathcal{U}^2\end{pmatrix}(r) + f'(s)\partial_2\begin{pmatrix}\mathcal{U}^1\\ \mathcal{U}^2\end{pmatrix}(r)\right)\mathrm{d}s, \qquad x\in\Omega^\pm. \tag{3.3.53}$$

In view of (3.3.32) and (3.3.33), we conclude from Lemma 3.3.3 that

$$\begin{aligned}&\{v_G\}^{\pm}\circ\Xi\\ &= \frac{1}{4\mu}\begin{pmatrix}(B_1-2B_4)(f)[F_1-f'F_2]+(2B_2+B_3)(f)[f'F_1]+B_3(f)[F_2]\\ B_1(f)[F_2-f'F_1]+B_3(f)[F_1-f'F_2]+2B_4(f)[f'F_1+F_2],\end{pmatrix}^{\top}.\end{aligned} \tag{3.3.54}$$

Finally, we consider the pressure q.

Lemma 3.3.8. *We have* $q^\pm \in \mathrm{C}^\infty(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm})$ *and*

$$[q] \circ \Xi = -\omega^{-1} G \cdot \nu \qquad \text{on } \mathbb{S},$$

$$q^\pm(x) \longrightarrow \mp \frac{\langle G_2 \rangle}{2} \qquad \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}.$$

Proof. Since

$$q_G = -\frac{Z_1(f)[G_1] + Z_2(f)[G_2]}{2},$$

Lemma 3.3.3 yields $q^\pm \in \mathrm{C}^\infty(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm})$ together with $[q_G] \circ \Xi = -\omega^{-1} G \cdot \nu$. Moreover, Lemma 3.3.5 shows that

$$q^\pm(x) \longrightarrow \mp \frac{\langle G_2 \rangle}{2} \qquad \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S},$$

which completes the proof. □

3.3.2. The hydrodynamic double-layer potential

In this section, we study the transmission boundary value problem (3.1.7). At first sight, this system may not appear closely related to (3.1.4) due to the jump condition imposed on the velocity across the interface, whereas the velocity is continuous across the interface in (3.1.4). But our first step in Chapter 5 when constructing solutions to (3.1.4) is to use the substitution $\tilde{v}^\pm := \mu^\pm v^\pm$. Then, the jump condition for $\tilde{v}^\pm$ becomes

$$\mu^- \{\tilde{v}\}^+ = \mu^+ \{\tilde{v}\}^- \qquad \text{on } \Gamma. \tag{3.3.55}$$

This means that $\tilde{v}$ is discontinuous across the interface (whenever $\mu^+ \neq \mu^-$) and to solve this problem we use a solution to (3.1.7) with a specific β.

Henceforth, we set $\mu = 1$. Recalling (3.3.16), we notice that the velocity of the hydrodynamic double-layer potential exhibits the desired jump behavior. Thus, the natural conjecture for a solution $(v_d^\pm, q_d^\pm)$ is, that it is given by the double-layer potentials generated by the density $-\beta \circ \Xi^{-1}$ which we write as

$$\begin{aligned} v_{d,j}^\pm(x) := v_{d,j}^\pm[\beta](x) &:= \int_\Gamma \mathcal{W}_j^{i,k}(x-\eta) \beta_k \circ \Xi^{-1}(\eta) \tilde{\nu}_i(\eta)\, \mathrm{d}\sigma \\ &= \int_{-\pi}^{\pi} \mathcal{W}_j^{i,k}(r) \nu_i(s) \beta_k(s) \omega(s)\, \mathrm{d}s, \qquad 1 \le j \le 2, \\ q_d^\pm(x) := q_d^\pm[\beta](x) &:= \int_\Gamma \mathcal{Q}^{i,k}(x-\eta) \beta_k \circ \Xi^{-1}(\eta) \tilde{\nu}_i(\eta)\, \mathrm{d}\sigma \\ &= \int_{-\pi}^{\pi} \mathcal{Q}^{i,k}(r) \nu_i(s) \beta_k(s) \omega(s)\, \mathrm{d}s, \end{aligned} \tag{3.3.56}$$

where $x \in \Omega^\pm$, $r = r(x,s)$ was defined in (3.3.26) and where we define the kernels $(\mathcal{W}^{i,k}, \mathcal{Q}^{i,k}) : (\mathbb{S} \times \mathbb{R}) \setminus \{0\} \to \mathbb{R}^2 \times \mathbb{R}$ by

$$\begin{aligned} \mathcal{W}_j^{i,k} &:= (T_1(\mathcal{U}^j, \mathcal{P}^j))_{ki} = -\mathcal{P}^j \delta_{ik} + \partial_i \mathcal{U}_k^j + \partial_k \mathcal{U}_i^j, \qquad 1 \le j \le 2, \\ \mathcal{Q}^{i,k} &:= 2\partial_k \mathcal{P}^i \end{aligned} \tag{3.3.57}$$

for $1 \le i,\, k \le 2$. Setting $\mathcal{W}_j := (\mathcal{W}_j^{i,k})_{1 \le i,k \le 2}$, $1 \le j \le 2$, and $\mathcal{Q} := (\mathcal{Q}^{i,k})_{1 \le i,k \le 2}$, we infer from (3.3.24) and (3.3.32) that

$$\begin{aligned} \mathcal{W}_1 &= \frac{1}{4\pi} \begin{pmatrix} 2z_1 - 2z_4 & z_2 + z_3 \\ z_2 + z_3 & 2z_4 \end{pmatrix}, \qquad \mathcal{W}_2 = \frac{1}{4\pi} \begin{pmatrix} z_2 + z_3 & 2z_4 \\ 2z_4 & z_2 - z_3 \end{pmatrix}, \\ \mathcal{Q} &= \frac{1}{2\pi} \begin{pmatrix} z_7 & 2z_8 \\ 2z_8 & -z_7 \end{pmatrix}. \end{aligned} \tag{3.3.58}$$

Recalling the integral operators $Z_i := Z_i(f)$ defined in (3.3.33), we write

$$v_d^\pm = v_d^\pm[\beta] = \frac{1}{2}\begin{bmatrix} Z_2+Z_3 & 2Z_4 \\ 2Z_4 & Z_2-Z_3 \end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2Z_1-2Z_4 & Z_2+Z_3 \\ Z_2+Z_3 & 2Z_4 \end{bmatrix}\begin{bmatrix} f'\beta_1 \\ f'\beta_2\end{bmatrix}, \tag{3.3.59}$$

where we use the notation introduced in (2.1.1). To derive a similar formula for $\mathcal{Q}$, we infer from (3.3.32) and (3.3.58) that

$$\mathcal{Q}^{i,k}(r)\nu_i(s)\beta_k(s)\omega(s) = 2\partial_s(\mathcal{P}^1(r))\beta_2(s) - 2\partial_s(\mathcal{P}^2(r))\beta_1(s), \qquad s\in\mathbb{S}, \quad x\in\Omega^\pm.$$

The formula (3.3.24) together with integration by parts in (3.3.56) lead us to

$$q_d^\pm = q_d^\pm[\beta] = Z_1(f)[\beta_2'] - Z_2(f)[\beta_1']. \tag{3.3.60}$$

In the next theorem we show that the pair (v_d, q_d) defined in (3.3.56) is the unique solution in X_f to the transmission boundary value problem (3.1.7).

Theorem 3.3.9. *Given $f\in \mathrm{H}^3(\mathbb{S})$ and $\beta\in\mathrm{H}^2(\mathbb{S})^2$, the transmission boundary value problem* (3.1.7) *has a solution $(v,q)\in X_f$ if and only if the constant $c=(c_{1,d},c_{3,d},c_{3,d})$ in* $(3.1.7)_5$ *is given by*

$$(c_{1,d},c_{3,d},c_{3,d}) = \frac{1}{2}(\langle\beta_1 - f'\beta_2\rangle, \langle\beta_2 - f'\beta_1\rangle, 0). \tag{3.3.61}$$

Assuming (3.3.61)*, the solution (v,q) is unique and is given by* (3.3.56) *(or the explicit formulas* (3.3.59)–(3.3.60)*).*

Moreover, we have

$$T_1(v_d^\pm, q_d^\pm)(x)\to 0 \qquad \text{for } x_2\to\pm\infty \text{ uniformly in } x_1\in\mathbb{S}. \tag{3.3.62}$$

We postpone the proof to the end of this section and first show some results regarding the boundary and far-field behavior of $(v_d^\pm, q_d^\pm)$. We use the results in [65] concerning the operators appearing in the non-periodic problem as an orientation.

Lemma 3.3.10. *We have $v_d^\pm\in \mathrm{C}^1(\overline{\Omega^\pm},\mathbb{R}^2)$ and $q_d^\pm\in\mathrm{C}(\overline{\Omega^\pm})$. Furthermore, it holds that*

$$\begin{aligned}\{v_d\}^\pm &= \pm\frac{1}{2}\beta\circ\Xi^{-1} + \frac{1}{2}\begin{bmatrix} B_2+B_3 & 2B_4 \\ 2B_4 & B_2-B_3\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\circ\Xi^{-1} \\ &\quad - \frac{1}{2}\begin{bmatrix} 2B_1-2B_4 & B_2+B_3 \\ B_2+B_3 & 2B_4\end{bmatrix}\begin{bmatrix} f'\beta_1\\ f'\beta_2\end{bmatrix}\circ\Xi^{-1} \qquad \text{on } \Gamma, \end{aligned}\tag{3.3.63}$$

$$\begin{aligned}\{T_1(v_d,q_d)\}^\pm &= \begin{pmatrix} (2B_2+B_3) & (2B_4-B_1) \\ (2B_4-B_1) & -B_3\end{pmatrix}[\beta_1']\circ\Xi^{-1} \\ &\quad + \begin{pmatrix} (2B_4-B_1) & -B_3 \\ -B_3 & -(B_1+2B_4)\end{pmatrix}[\beta_2']\circ\Xi^{-1} \qquad \text{on } \Gamma, \end{aligned}\tag{3.3.64}$$

where we use the shorthand notation $B_n := B_n(f)$, $1\le n\le 4$, for the operators introduced in (3.3.34).

Proof. Since $\beta\in\mathrm{H}^2(\mathbb{S})$, the regularity properties $v_d^\pm\in\mathrm{C}(\overline{\Omega^\pm},\mathbb{R}^2)$ and $q_d^\pm\in\mathrm{C}(\overline{\Omega^\pm})$ follow from Lemma 3.3.3 and the formulas (3.3.59)–(3.3.60). Moreover, the identity (3.3.63) follows from (3.3.36) by straightforward calculations in view of (3.3.59).

Next, given a function $\mathcal{Z} \in \mathrm{C}^1((\mathbb{S}\times\mathbb{R})\setminus\{0\})$, we define the vector field

$$\operatorname{rot}\mathcal{Z} := (\operatorname{rot}^1\mathcal{Z}, \operatorname{rot}^2\mathcal{Z}) := (-\partial_2\mathcal{Z}, \partial_1\mathcal{Z}) \in \mathrm{C}((\mathbb{S}\times\mathbb{R})\setminus\{0\}, \mathbb{R}^2)$$

and compute for $1 \leq j \leq 4$ and $\varphi \in \mathrm{H}^1(\mathbb{S})$, using integration by parts,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(\operatorname{rot}^i z_j)(r)\nu_i\varphi\omega\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\pi}^{\pi}(f'\partial_2 z_j(r) + \partial_1 z_j(r))\varphi\,\mathrm{d}s = Z_j(f)[\varphi']. \quad (3.3.65)$$

Defining

$$(\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3) := (z_1 - z_4, z_2 + z_3, z_4, z_2 - z_3) \qquad \text{in } (\mathbb{S}\times\mathbb{R})\setminus\{0\}, \qquad (3.3.66)$$

we infer from (3.3.58) that

$$\mathcal{W}_1 = \frac{1}{4\pi}\begin{pmatrix} 2\mathcal{Z}_0 & \mathcal{Z}_1 \\ \mathcal{Z}_1 & 2\mathcal{Z}_2 \end{pmatrix} \qquad \text{and} \qquad \mathcal{W}_2 = \frac{1}{4\pi}\begin{pmatrix} \mathcal{Z}_1 & 2\mathcal{Z}_2 \\ 2\mathcal{Z}_2 & \mathcal{Z}_3 \end{pmatrix},$$

and the identities (3.3.20)–(3.3.23) lead us to

$$\begin{aligned}
\operatorname{rot}^i\mathcal{Z}_0 &= -2\pi\partial_2\mathcal{W}_1^{i,1}, \\
\operatorname{rot}^i\mathcal{Z}_1 &= 4\pi\partial_1\mathcal{W}_1^{i,1} = -4\pi\partial_2\mathcal{W}_2^{i,1} = -4\pi\partial_2\mathcal{W}_1^{i,2}, \\
\operatorname{rot}^i\mathcal{Z}_2 &= 2\pi\partial_1\mathcal{W}_1^{i,2} = 2\pi\partial_1\mathcal{W}_2^{i,1} = -2\pi\partial_2\mathcal{W}_2^{i,2}, \\
\operatorname{rot}^i\mathcal{Z}_3 &= 4\pi\partial_1\mathcal{W}_2^{i,2}, \qquad 1 \leq i \leq 2.
\end{aligned} \qquad (3.3.67)$$

Moreover, for $1 \leq i \leq 2$, we obtain by combining (3.3.23), (3.3.58), and (3.3.66) the following identities

$$\operatorname{rot}^i(\mathcal{Z}_0 + \mathcal{Z}_2) = 2\pi\mathcal{Q}^{i,2} \qquad \text{and} \quad \operatorname{rot}^i(\mathcal{Z}_1 + \mathcal{Z}_3) = -4\pi\mathcal{Q}^{i,1}. \qquad (3.3.68)$$

Given $x \in (\mathbb{S}\times\mathbb{R})\setminus\Gamma$ and $1 \leq j, \ell \leq 2$, the definitions (1.2.1) and (3.3.56) yield, after interchanging in $(3.3.56)_1$ partial differentiation with respect to x_1 and x_2 and integration with respect to s,

$$(T_1(v_d^\pm, q_d^\pm))_{j\ell}(x) = \int_{-\pi}^{\pi}\big(-\delta_{j\ell}\mathcal{Q}^{i,k} + \partial_\ell\mathcal{W}_j^{i,k} + \partial_j\mathcal{W}_\ell^{i,k}\big)(r)\nu_i(s)\beta_k(s)\omega(s)\,\mathrm{d}s,$$

and together with (3.3.65)–(3.3.68) we arrive at

$$\begin{aligned}
&T_1(v_d^\pm, q_d^\pm) \\
&= \begin{pmatrix} (2Z_2 + Z_3)[\beta_1'] + (2Z_4 - Z_1)[\beta_2'] & (2Z_4 - Z_1)[\beta_1'] - Z_3[\beta_2'] \\ (2Z_4 - Z_1)[\beta_1'] - Z_3[\beta_2'] & -Z_3[\beta_1'] - (Z_1 + 2Z_4)[\beta_2'] \end{pmatrix}
\end{aligned} \qquad (3.3.69)$$

in $(\mathbb{S}\times\mathbb{R})\setminus\Gamma$, where we again used the shorthand $Z_n = Z_n(f)$, $1 \leq n \leq 4$. Invoking (3.3.36), the boundary condition (3.3.64) can be easily verified. Moreover, Lemma 3.3.3 implies that $T_1(v_d, q_d) \in \mathrm{C}(\overline{\Omega^\pm}, \mathbb{R}^{2\times 2})$. Then, since $q_d^\pm \in \mathrm{C}(\overline{\Omega^\pm})$, we can deduce that $\nabla v_d + (\nabla v_d)^\top \in \mathrm{C}(\overline{\Omega^\pm}, \mathbb{R}^{2\times 2})$. Calculating

$$\partial_1 v_{d,2} = Z_4(f)[\beta_1'] + \frac{1}{2}(Z_2 - Z_3)(f)[\beta_2'] \in \mathrm{C}(\overline{\Omega^\pm}, \mathbb{R}^2),$$

we may also conclude that $v_d^\pm \in \mathrm{C}^1(\overline{\Omega^\pm}, \mathbb{R}^2)$, and the proof is complete. □

Proof of Theorem 3.3.9. That our solutions satisfy the equations $(3.1.7)_{1-2}$ and are unique follows as in the proof of Theorem 3.3.2. That (v_d, q_d) lies in X_f and satisfies the boundary conditions $(3.1.7)_{3-4}$ is established in Lemma 3.3.10 while the claims (3.3.61)–(3.3.62) follow by applying Lemma 3.3.5 to the representations of $(v_d^\pm, q_d^\pm)$ found in (3.3.59)–(3.3.60) and of $T_1(v_d^\pm, q_d^\pm)$ found in (3.3.69), respectively. □

Chapter 4

The resolvent of the hydrodynamic double-layer potential operator

4.1. Introduction

In this chapter, we study the resolvent of the hydrodynamic double-layer potential operator, defined in (4.1.1) below, in different function spaces. Recalling that the solution to problem (3.1.7) is given by the hydrodynamic double-layer potentials, formula (3.3.63), which describes the trace of the velocity on the boundary, enables us to define for given $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$ and $\beta \in \mathrm{L}^2(\mathbb{S})^2$ the *hydrodynamic double-layer potential operator* $\mathbb{D}(f)$ by

$$\mathbb{D}(f)[\beta] := -\frac{1}{2}\begin{bmatrix} B_2 + B_3 & 2B_4 \\ 2B_4 & B_2 - B_3 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2B_1 - 2B_4 & B_2 + B_3 \\ B_2 + B_3 & 2B_4 \end{bmatrix}\begin{bmatrix} f'\beta_1 \\ f'\beta_2 \end{bmatrix}, \quad (4.1.1)$$

where $B_i := B_i(f)$, $1 \leq i \leq 4$, are defined in (3.3.34) and we use the notation introduced in (2.1.1).

Given $f \in \mathrm{H}^3(\mathbb{S})$, the main goal of this chapter is to show that for $\lambda \in \mathbb{R}$ with $|\lambda| > 1/2$, the operator $\lambda - \mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^2(\mathbb{S})^2)$ is an isomorphism. This goal is motivated in Chapter 5, when solving the fixed time problem (3.1.4) associated to the two-phase Stokes problem (1.2.2a). It is shown there, that finding a solution to (3.1.4) boils down to finding a solution $\beta \in \mathrm{H}^2(\mathbb{S})^2$ to

$$(\lambda - \mathbb{D}(f))[\beta] = \phi, \quad (4.1.2)$$

where $\phi \in \mathrm{H}^2(\mathbb{S})$ is a given function and λ is a parameter depending on the viscosities μ^+ and μ^-, see Theorem 5.3.1.

To handle the analysis of the hydrodynamic double-layer potential operator $\mathbb{D}(f)$ in an efficient way, we notice that it is a linear combination of the operators $B_i(f)$, $1 \leq i \leq 4$ introduced in (3.3.34). Moreover, we can break these operators down to a linear combination of their building blocks, namely to the operators $B^{p,q}_{n,m}$ which we introduce in (4.1.3) below, see (4.1.7) for the precise relation. This means that the operator $\mathbb{D}(f)$ is itself a linear combination of the operators $B^{p,q}_{n,m}$ and we are able to limit ourselves to showing mapping properties for these operators which then carry over to $\mathbb{D}(f)$ due to linearity. To start, we define for given integers m, n, p, $q \in \mathbb{N}_0$ satisfying the relation $p \leq n+q+1$, and Lipschitz continuous mappings

$$\mathbf{a} = (a_1, \ldots, a_m) : \mathbb{R} \to \mathbb{R}^m, \quad \mathbf{b} = (b_1, \ldots, b_n) : \mathbb{R} \to \mathbb{R}^n, \quad \mathbf{c} = (c_1, \ldots, c_q) : \mathbb{R} \to \mathbb{R}^q,$$

the integral operator

$$B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi](\xi) := \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{\prod_{i=1}^{n} \frac{T_{[\xi,s]}b_i}{t_{[s]}} \prod_{i=1}^{q} \frac{\delta_{[\xi,s]}c_i/2}{t_{[s]}}}{\prod_{i=1}^{m}\left[1+\left(\frac{T_{[\xi,s]}a_i}{t_{[s]}}\right)^2\right]} \frac{\varphi(\xi-s)}{t_{[s]}} t_{[s]}^p \,\mathrm{d}s, \tag{4.1.3}$$

where $\varphi \in \mathrm{L}^2(\mathbb{S})$ and $\xi \in \mathbb{R}$. We use the notation introduced in (3.2.20) together with the shorthand

$$\delta_{[\xi,s]}g := g(\xi) - g(\xi - s) \qquad \text{and} \qquad T_{[\xi,s]}g := \tanh\left(\frac{\delta_{[\xi,s]}g}{2}\right), \qquad \xi,\, s \in \mathbb{R}, \tag{4.1.4}$$

with $g : \mathbb{R} \to \mathbb{R}$. We point out that the principal value is not needed in (4.1.3) if $p \geq 1$ as shown in Lemma A.1.3. Let us also mention that the operator $B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is 2π-periodic whenever all the coordinate functions $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are 2π-periodic. Moreover, we have the following relation:

$$B_{0,0}^{0,0}[\varphi](\xi) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi} \frac{\varphi(\xi-s)}{t_{[s]}}\,\mathrm{d}s = \mathsf{H}[\varphi](\xi), \qquad \xi \in \mathbb{R}, \tag{4.1.5}$$

where H is the periodic Hilbert transform, see, e.g., [13, 92].

If all coordinate functions of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are identical to a given function $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$, we set

$$B_{n,m}^{p,q}(f) := B_{n,m}^{p,q}(f,\ldots,f|f,\ldots,f)[f,\ldots,f,\cdot], \qquad 0 \leq p \leq n+q+1. \tag{4.1.6}$$

This allows us to rewrite the operators $B_i := B_i(f)$, $1 \leq i \leq 6$, introduced in (3.3.34) as linear combinations of the operators $B_{n,m}^{p,q}(f)$. The precise relations are given by

$$\begin{aligned}
B_1(f) &:= B_{0,1}^{0,0}(f) - B_{2,1}^{2,0}(f),\\
B_2(f) &:= B_{1,1}^{0,0}(f) + B_{1,1}^{2,0}(f),\\
B_3(f) &:= B_{0,2}^{0,1}(f) + B_{0,2}^{2,1}(f) - B_{2,2}^{0,1}(f) - 2B_{2,2}^{2,1}(f)\\
&\quad - B_{2,2}^{4,1}(f) + B_{4,2}^{2,1}(f) + B_{4,2}^{4,1}(f),\\
B_4(f) &:= B_{1,2}^{0,1}(f) + B_{1,2}^{2,1}(f) - B_{3,2}^{2,1}(f) - B_{3,2}^{4,1}(f),\\
B_5(f) &:= 2\big(B_{0,1}^{1,1}(f) - B_{2,1}^{3,1}(f)\big),\\
B_6(f) &:= 2\big(B_{1,1}^{1,1}(f) + B_{1,1}^{3,1}(f)\big).
\end{aligned} \tag{4.1.7}$$

The structure of this chapter is as follows: in Section 4.2, we state important mapping properties of the operators $B_{n,m}^{p,q}$ in $\mathrm{L}^2(\mathbb{S})$ and establish in Theorem 4.2.3 that the resolvent set of $\mathbb{D}(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S})^2)$, $f \in \mathrm{C}^1(\mathbb{S})$, contains all $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$. Afterwards, in Section 4.3, we consider the operator $B_{n,m}^{p,q}$ in $\mathrm{H}^{r-1}(\mathbb{S})$, $r \in (3/2, 2)$, and use Theorem 4.2.3 to study the resolvent set of $\mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})^2)$, $f \in \mathrm{H}^r(\mathbb{S})$, in Theorem 4.3.2. Finally, in Section 4.4, we use the results obtained in the previous sections to establish the invertibility of $\lambda - \mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^2(\mathbb{S})^2)$, $f \in \mathrm{H}^3(\mathbb{S})$, whenever $|\lambda| > \frac{1}{2}$.

4.2. Invertibility in $\mathrm{L}^2(\mathbb{S})^2$

We start this section with a result on the $\mathrm{L}^2(\mathbb{S})$-boundedeness of the operator $B^{p,q}_{n,m}$.

Lemma 4.2.1. *Let n, m, p, $q \in \mathbb{N}_0$ satisfying $p \leq n+q+1$, and $(\mathbf{a},\mathbf{b}) \in \mathrm{W}^{1,\infty}(\mathbb{S})^{m+n}$ be given. Then, there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a}',\mathbf{b}')\|_\infty$ such that for all $\mathbf{c} \in \mathrm{W}^{1,\infty}(\mathbb{S})^q$ and $\varphi \in \mathrm{L}^2(\mathbb{S})$ we have*

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_2 \leq C\|\varphi\|_2 \prod_{i=1}^{q} \|c_i'\|_\infty. \tag{4.2.1}$$

Furthermore,

$$B^{p,q}_{n,m} \in \mathrm{C}^{1-}(\mathrm{W}^{1,\infty}(\mathbb{S})^{m+n}, \mathcal{L}^q_{\mathrm{sym}}(\mathrm{W}^{1,\infty}(\mathbb{S}), \mathcal{L}(\mathrm{L}^2(\mathbb{S})))). \tag{4.2.2}$$

We postpone the proof to Appendix A but want to make the following remark.

Remark 4.2.2. The result (4.2.1) is only optimal if $p = 0$. If $p \geq 1$, we can improve the regularity of $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ whilst lowering the required regularity for φ. This is due to the fact that the operator is no longer singular if $p \geq 1$, see Lemma A.1.3 for details.

It is also convenient for us to work with the $\mathrm{L}^2(\mathbb{S})^2$-adjoint of the hydrodynamic double-layer potential. To this end, we notice that the $\mathrm{L}^2(\mathbb{S})$-adjoint $(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot])^*$ of $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot]$ is given by the relation

$$(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot])^* = (-1)^{p+1} B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot]. \tag{4.2.3}$$

Using (4.2.3), it can be easily verified that the $\mathrm{L}^2(\mathbb{S})^2$-adjoint of $\mathbb{D}(f)$ is given by

$$\mathbb{D}(f)^*[\beta] := \frac{1}{2}\begin{bmatrix} B_2 + B_3 & 2B_4 \\ 2B_4 & B_2 - B_3 \end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} - \frac{f'}{2}\begin{bmatrix} 2B_1 - 2B_4 & B_2 + B_3 \\ B_2 + B_3 & 2B_4 \end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix}. \tag{4.2.4}$$

The next theorem is the main result of this section and it states that the resolvent set of the hydrodynamic double-layer potential operator contains all real numbers with modulus greater than 1/2. We point out that this result gives rise to the same spectral properties as the result in [15] concerning the double-layer potential of the Stokes equation in bounded Lipschitz domains. Moreover, a similar result is obtained in [65] for the double-layer potential operator in the non-periodic setting.

Theorem 4.2.3. *Given $\delta \in (0,1)$, there exists a constant $C = C(\delta) > 0$ with the property that for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1/2 + \delta$ and $f \in \mathrm{C}^1(\mathbb{S})$ with $\|f'\|_\infty \leq 1/\delta$ we have*

$$\|(\lambda - \mathbb{D}(f)^*)[\beta]\|_2 \geq C\|\beta\|_2, \qquad \beta \in \mathrm{L}^2(\mathbb{S})^2. \tag{4.2.5}$$

Moreover, $\lambda - \mathbb{D}(f)^*$, $\lambda - \mathbb{D}(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S})^2)$ are isomorphisms for all $\lambda \in \mathbb{R}$ satisfying $|\lambda| > 1/2$ and all $f \in \mathrm{C}^1(\mathbb{S})$.

The proof of Theorem 4.2.3 will be presented at the end of this section, as it requires some preparatory work. To start, we introduce the singular integral operators $\mathbb{B}_1(f)$, $\mathbb{B}_2(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S}))$, which also appear in the analysis of the periodic Muskat problem, see [63], by setting for $\varphi \in \mathrm{L}^2(\mathbb{S})$

$$\begin{aligned} \mathbb{B}_1(f)[\varphi] &:= -B_1(f)[f'\varphi] + B_2(f)[\varphi], \\ \mathbb{B}_2(f)[\varphi] &:= B_1(f)[\varphi] + B_2(f)[f'\varphi]. \end{aligned} \tag{4.2.6}$$

One of the key ingredients in the proof of Theorem 4.2.3 is the following result.

Lemma 4.2.4. *Given $K>0$, there exists a constant $C=C(K)>0$ with the property that for all $\beta\in \mathrm{L}^2(\mathbb{S})^2$, $\lambda\in[-K,K]$, and $f\in \mathrm{C}^1(\mathbb{S})$ satisfying $\|f'\|_\infty\leq K$ we have*

$$\begin{aligned} C\big\|(\lambda-\mathbb{D}(f)^*)[\beta]\big\|_2\|\beta\|_2 \geq m(\lambda)\|\omega^{-1}\beta\cdot\tau\|_2^2+\langle\beta_1\rangle^2 \\ +\big\|(\lambda-\tfrac{1}{2}\mathbb{B}_1(f))[\omega^{-1}\beta\cdot\nu]-\tfrac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta\cdot\tau]\big\|_2^2, \end{aligned} \tag{4.2.7}$$

where $m(\lambda):=\max\{(\lambda+\frac{1}{2})(\lambda-\frac{3}{2}),(\lambda-\frac{1}{2})(\lambda+\frac{3}{2})\}$.

Proof. Recalling (4.1.7) and (4.2.1), it suffices to establish (4.2.7) for $f\in \mathrm{C}^\infty(\mathbb{S})$ satisfying $\|f'\|_\infty\leq K$ and $\beta\in \mathrm{C}^\infty(\mathbb{S})^2$. To proceed, we define the hydrodynamic single-layer potentials $(u,p)=(u,p)[\beta]$, generated by the density $-\beta=-(\beta_1,\beta_2)^\top$, by setting for $x\in(\mathbb{S}\times\mathbb{R})\setminus\Gamma$

$$\begin{aligned} u(x)&:=\int_{-\pi}^{\pi}\mathcal{U}^k(x-(s,f(s)))\beta_k(s)\,\mathrm{d}s,\\ p(x)&:=\int_{-\pi}^{\pi}\mathcal{P}^k(x-(s,f(s)))\beta_k(s)\,\mathrm{d}s, \end{aligned}$$

with $(\mathcal{U},\mathcal{P})$ defined in (3.3.24) and $\mu=1$ therein. From Lemma 3.3.6 and Lemma 3.3.8, we may deduce that the mappings $u^\pm\in \mathrm{C}^\infty(\Omega^\pm,\mathbb{R}^2)$, $p^\pm\in \mathrm{C}^\infty(\Omega^\pm)$ have extensions $u^\pm\in \mathrm{C}^1(\overline{\Omega^\pm},\mathbb{R}^2)$, and $p^\pm\in \mathrm{C}(\overline{\Omega^\pm})$, being solutions to the homogeneous Stokes system

$$\left.\begin{aligned} \Delta u^\pm-\nabla p^\pm=0,\\ \operatorname{div}u^\pm=0 \end{aligned}\right\}\qquad\text{in }\Omega^\pm. \tag{4.2.8}$$

Using (3.3.24) and (3.3.32) together with the formula (3.3.36), we have, relying on notation (2.1.1), that

$$\begin{aligned} \{\partial_1 u\}^\pm\circ\Xi&=\frac{1}{4}\begin{bmatrix} B_1-2B_4 & B_3\\ B_3 & B_1+2B_4\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\pm\frac{\nu_1(\beta\cdot\tau)\tau}{2\omega},\\ \{\partial_2 u\}^\pm\circ\Xi&=\frac{1}{4}\begin{bmatrix} 2B_2+B_3 & -B_1+2B_4\\ -B_1+2B_4 & -B_3\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}\pm\frac{\nu_2(\beta\cdot\tau)\tau}{2\omega},\\ \{p\}^\pm\circ\Xi&=-\frac{1}{2}(B_1[\beta_1]+B_2[\beta_2])\mp\frac{\beta\cdot\nu}{2\omega}, \end{aligned} \tag{4.2.9}$$

with $\nu=(\nu_1,\nu_2)$, τ, ω defined in (3.1.1) and $B_i:=B_i(f)$, $1\leq i\leq 4$, defined in (4.1.7). We also note that, if we discard the jump terms in (4.2.9) and replace for $1\leq i\leq 4$ the operator $B_i(f)$ by $Z_i(f)$, then (4.2.9) provides a formula for $\partial_i u$, $1\leq i\leq 2$, and p in $(\mathbb{S}\times\mathbb{R})\setminus\Gamma$. Moreover, letting $\mathbb{T}(f)\in\mathcal{L}(\mathrm{L}^2(\mathbb{S})^2)$ be the integral operator in the formula (4.2.9) for $\{\partial_2 u\}^\pm\circ\Xi$, that is,

$$\mathbb{T}(f)[\beta]:=\frac{1}{4}\begin{bmatrix} 2B_2+B_3 & -B_1+2B_4\\ -B_1+2B_4 & -B_3\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix},$$

then, due to (4.2.3), we deduce that $\mathbb{T}(f)^*=-\mathbb{T}(f)$, and therefore

$$\langle\mathbb{T}(f)[\beta]\,|\,\beta\rangle_2=0,\qquad \beta\in \mathrm{L}^2(\mathbb{S})^2, \tag{4.2.10}$$

where $\langle\cdot\,|\,\cdot\rangle_2$ denotes here and below the $\mathrm{L}^2(\mathbb{S})^2$-inner product. Using (4.2.9), we find for the normal stress at the interface that

$$\omega(\{T_1(u,p)\}^\pm\circ\Xi)\nu=\Big(\pm\frac{1}{2}+\mathbb{D}(f)^*\Big)[\beta]. \tag{4.2.11}$$

Since $(u^\pm, p^\pm)$ are solutions to (4.2.8), a straightforward computation leads us to the identity

$$\operatorname{div}(T_1(u,p)\partial_2 u) = \frac{1}{4}\partial_2|pI_2 + T_1(u,p)|^2 \qquad \text{in } (\mathbb{S}\times\mathbb{R})\setminus\Gamma. \tag{4.2.12}$$

Recalling Lemma 3.3.5, we infer from (4.2.9) (and the discussion following it) that for $x_2 \to \pm\infty$, we have

$$\begin{aligned}(T_1(u^\pm, p^\pm)\partial_2 u^\pm)(x) &\to \frac{\langle\beta_1\rangle}{4}\begin{pmatrix}\langle\beta_2\rangle\\ \langle\beta_1\rangle\end{pmatrix},\\ |p^\pm I_2 + T_1(u^\pm, p^\pm)|^2(x) &\to \frac{\langle\beta_1\rangle^2}{2}\end{aligned} \tag{4.2.13}$$

uniformly with respect to $x_1 \in \mathbb{S}$. Integrating the relation (4.2.12) over

$$\Omega_n^\pm := \Omega^\pm \cap \{x = (x_1, x_2) \in \mathbb{S}\times\mathbb{R} \,:\, |x_2| < n\}, \qquad n > \|f\|_\infty, \tag{4.2.14}$$

and using Stokes' theorem, we obtain in virtue of (4.2.13), after letting $n \to \infty$,

$$\begin{aligned}&4\langle \omega(\{T_1(u,p)\}^\pm \circ \Xi)\nu \,|\, \{\partial_2 u\}^\pm \circ \Xi\rangle_2\\ &= \int_{-\pi}^{\pi} \big|(\{p\}^\pm \circ \Xi I_2 + \{T_1(u,p)\}^\pm \circ \Xi)\big|^2 \,\mathrm{d}s + \pi\langle\beta_1\rangle^2.\end{aligned} \tag{4.2.15}$$

The relations (4.2.9)–(4.2.11) together with the property that $\mathbb{T}(f) \in \mathcal{L}(L^2(\mathbb{S})^2)$ ensure that there exists a constant $C = C(K) > 0$ such that the left-hand side of (4.2.15) satisfies

$$\begin{aligned}&4\langle \omega(\{T_1(u,p)\}^\pm \circ \Xi)\nu \,|\, \{\partial_2 u\}^\pm \circ \Xi\rangle_2\\ &= 4\Big\langle\Big(\pm\frac{1}{2} + \mathbb{D}(f)^*\Big)[\beta]\,\Big|\,\mathbb{T}(f)[\beta] \pm \frac{\nu_2(\beta\cdot\tau)\tau}{2\omega}\Big\rangle_2\\ &= 4\Big\langle\Big(\lambda \pm \frac{1}{2}\Big)\beta - (\lambda - \mathbb{D}(f)^*)[\beta]\,\Big|\,\mathbb{T}(f)[\beta] \pm \frac{\nu_2(\beta\cdot\tau)\tau}{2\omega}\Big\rangle_2\\ &\leq C\|(\lambda - \mathbb{D}(f)^*)[\beta]\|_2\|\beta\|_2 \pm 2\Big(\lambda \pm \frac{1}{2}\Big)\|\omega^{-1}\beta\cdot\tau\|_2^2.\end{aligned}$$

Concerning the integral term on the right-hand side of (4.2.15), we note that due to (4.2.11), we have

$$\begin{aligned}I &:= \int_{-\pi}^{\pi} \big|(\{p\}^\pm \circ \Xi)I_2 + \{T_1(u,p)\}^\pm \circ \Xi\big|^2 \,\mathrm{d}s\\ &\geq \int_{-\pi}^{\pi} \big|(\{p\}^\pm \circ \Xi)\nu + (\{T_1(u,p)\}^\pm \circ \Xi)\nu\big|^2 \,\mathrm{d}s\\ &= \Big\|(\{p\}^\pm \circ \Xi)\nu + \frac{1}{\omega}\Big(\pm\frac{1}{2} + \mathbb{D}(f)^*\Big)[\beta]\Big\|_2^2.\end{aligned}$$

Using (3.3.31) and observing that

$$B_1(f)[\beta_1] + B_2(f)[\beta_2] = \mathbb{B}_1(f)[\omega^{-1}\beta\cdot\nu] + \mathbb{B}_2(f)[\omega^{-1}\beta\cdot\tau],$$

Next, Hölder's inequality, the uniform bounds $|\lambda| \leq K$ and $\|\mathbb{B}_i(f)\|_{\mathcal{L}(L^2(\mathbb{S}))} \leq C(K)$, with $1 \leq i \leq 2$ (see (4.1.7) and (4.2.1)), and the formulas (4.2.9) and (4.2.11) lead us to

$$
\begin{aligned}
I &\geq \Big\| (\{p\}^{\pm} \circ \Xi)\nu + \frac{1}{\omega}\Big(\lambda \pm \frac{1}{2}\Big)\beta - \frac{1}{\omega}(\lambda - \mathbb{D}(f)^*)[\beta] \Big\|_2^2 \\
&= \Big\| \Big(\frac{1}{2}(\mathbb{B}_1(f)[\omega^{-1}\beta \cdot \nu] + \mathbb{B}_2(f)[\omega^{-1}\beta \cdot \tau]) \pm \frac{\beta \cdot \nu}{2\omega} \Big)\nu \\
&\qquad - \frac{1}{\omega}\Big(\lambda \pm \frac{1}{2}\Big)\beta + \frac{1}{\omega}(\lambda - \mathbb{D}(f)^*)[\beta] \Big\|_2^2 \\
&\geq \Big\| \Big(\frac{1}{2}(\mathbb{B}_1(f)[\omega^{-1}\beta \cdot \nu] + \mathbb{B}_2(f)[\omega^{-1}\beta \cdot \tau]) \pm \frac{\beta \cdot \nu}{2\omega} \Big)\nu - \frac{1}{\omega}\Big(\lambda \pm \frac{1}{2}\Big)\beta \Big\|_2^2 \\
&\qquad + \Big\| \frac{1}{\omega}(\lambda - \mathbb{D}(f)^*)[\beta] \Big\|_2^2 - C(K)\big\|(\lambda - \mathbb{D}(f)^*)[\beta]\big\|_2 \|\beta\|_2 \\
&\geq \Big\| - \frac{1}{\omega}\Big(\lambda \pm \frac{1}{2}\Big)(\beta \cdot \tau)\tau + \Big(\frac{1}{2}\mathbb{B}_1(f) - \lambda\Big)[\omega^{-1}\beta \cdot \nu]\nu + \frac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta \cdot \tau]\nu \Big\|_2^2 \\
&\qquad - C\big\|(\lambda - \mathbb{D}(f)^*)[\beta]\big\|_2 \|\beta\|_2 \\
&= \Big(\lambda \pm \frac{1}{2}\Big)^2 \|\omega^{-1}\beta \cdot \tau\|_2^2 + \Big\| \Big(\frac{1}{2}\mathbb{B}_1(f) - \lambda\Big)[\omega^{-1}\beta \cdot \nu] + \frac{1}{2}\mathbb{B}_2(f)[\omega^{-1}\beta \cdot \tau] \Big\|_2^2 \\
&\qquad - C\big\|(\lambda - \mathbb{D}(f)^*)[\beta]\big\|_2 \|\beta\|_2.
\end{aligned}
$$

The desired estimate (4.2.7) now follows directly from the inequalities previously derived for both sides of (4.2.15). □

As a further preparation for the proof of Theorem 4.2.3, we infer from (4.1.7), (4.2.3), and (4.2.6), after a straightforward computation, that the L^2-adjoint

$$\mathbb{A}(f) := \mathbb{B}_1(f)^* \in \mathcal{L}(\mathrm{L}^2(\mathbb{S}))$$

satisfies the formula

$$\mathbb{A}(f) = f' B_1(f) - B_2(f). \tag{4.2.16}$$

Related to $\mathbb{A}(f)$, we introduce the operator

$$\mathbb{B}(f) := B_1(f) + f' B_2(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S})) \tag{4.2.17}$$

and establish the following result.

Lemma 4.2.5. *Given $K > 0$, there exists a constant $C = C(K) > 0$ with the property that for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ and $f \in \mathrm{C}^1(\mathbb{S})$ with $\|f'\|_\infty \leq K$ we have*

$$\|\varphi\|_2 \leq C\|(\lambda - \mathbb{A}(f))[\varphi]\|_2, \qquad \varphi \in \mathrm{L}^2(\mathbb{S}). \tag{4.2.18}$$

Moreover, $\lambda - \mathbb{A}(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S}))$ is an isomorphism for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$ and $f \in \mathrm{C}^1(\mathbb{S})$.

Proof. Thanks to formulas (4.1.7), (4.2.1), and (4.2.16), it is sufficient to establish (4.2.18) for $f \in \mathrm{C}^\infty(\mathbb{S})$ satisfying $\|f'\|_\infty \leq K$ and $\varphi \in \mathrm{C}^\infty(\mathbb{S})$. We begin by introducing the vector field $V = (V_1, V_2)^\top$ by

$$V := \big(- Z_2(f)[\varphi],\, Z_1(f)[\varphi]\big)^\top : (\mathbb{S} \times \mathbb{R}) \setminus \Gamma \to \mathbb{R}^2,$$

see (3.3.33). Recalling Lemma 3.3.3 and Lemma 3.3.5, it follows straightforwardly that $V^\pm := V|_{\Omega^\pm}$ belongs to $\mathrm{C}^\infty(\Omega^\pm) \cap \mathrm{C}(\overline{\Omega^\pm})$ and satisfies

$$\operatorname{div} V^\pm = \operatorname{curl} V^\pm = 0 \qquad \text{in } \Omega^\pm$$

by (1.4.2) and (3.3.23). Hence, we have

$$\operatorname{div}\begin{pmatrix} 2V_1^\pm V_2^\pm \\ (V_2^\pm)^2-(V_1^\pm)^2 \end{pmatrix} = 0 \qquad \text{in } \Omega^\pm, \tag{4.2.19}$$

and

$$\begin{pmatrix} 2V_1V_2 \\ (V_2)^2-(V_1)^2 \end{pmatrix} \underset{|x_2|\to\infty}{\longrightarrow} \begin{pmatrix} 0 \\ -\langle\varphi\rangle^2 \end{pmatrix} \qquad \text{uniformly in } x_1, \tag{4.2.20}$$

while

$$\{V\}^\pm\circ\Xi = \big(-B_2(f)[\varphi],\, B_1(f)[\varphi]\big)^\top \mp \frac{\tau}{\omega}\varphi \qquad \text{on } \mathbb{S}. \tag{4.2.21}$$

Integrating (4.2.19) over $\Omega_n^\pm$ with $n>\|f\|_\infty$ (recalling (4.2.14)), it follows from Stokes' theorem, and (4.2.19)–(4.2.21), after letting $n\to\infty$, that

$$\int_{-\pi}^{\pi} \omega\nu\cdot\begin{pmatrix} 2\{V_1\}^\pm\{V_2\}^\pm \\ \{(V_2^\pm)\}^2-(\{V_1\}^\pm)^2 \end{pmatrix}\circ\Xi\,\mathrm{d}s = -2\pi\langle\varphi\rangle^2.$$

Equivalently, noticing from (4.2.16)–(4.2.17) and (4.2.21) that

$$\{V\}^\pm\circ\Xi = \frac{1}{\omega}\big((\mp1+\mathbb{A}(f))[\varphi]\tau+\mathbb{B}(f)[\varphi]\nu\big),$$

we have

$$\begin{aligned}&\int_{-\pi}^{\pi}\frac{1}{\omega^2}\big[|\mathbb{B}(f)[\varphi]|^2+2f'\mathbb{B}(f)[\varphi](\mp1+\mathbb{A}(f))[\varphi]-|(\mp1+\mathbb{A}(f))[\varphi]|^2\big]\,\mathrm{d}s\\&=-2\pi\langle\varphi\rangle^2.\end{aligned} \tag{4.2.22}$$

Using Young's inequality, we find a constant $C(K)>0$ such that

$$\|(\mp1+\mathbb{A}(f))[\varphi]\|_2\le C\big(\|\mathbb{B}(f)[\varphi]\|_2+|\langle\varphi\rangle|\big),$$

hence

$$\|\varphi\|_2=\frac{1}{2}\|(1+\mathbb{A}(f))[\varphi]-(-1+\mathbb{A}(f))[\varphi]\|_2\le C\big(\|\mathbb{B}(f)[\varphi]\|_2+|\langle\varphi\rangle|\big). \tag{4.2.23}$$

Noticing for $|\lambda|\ge1$ that

$$(\mp1+\mathbb{A}(f))[\varphi]=(\lambda\mp1)\varphi-(\lambda-\mathbb{A}(f))[\varphi],$$

we infer from (4.2.22), after plugging in this expression and adding the equation with sign $-$ multiplied by $\lambda+1$ to the equation with sign $+$ multiplied by $1-\lambda$, that

$$\begin{aligned}&\int_{-\pi}^{\pi}\frac{1}{\omega^2}\big[(\lambda^2-1)|\varphi|^2+|\mathbb{B}(f)[\varphi]|^2-2f'\mathbb{B}(f)[\varphi](\lambda-\mathbb{A}(f))[\varphi]-|(\lambda-\mathbb{A}(f))[\varphi]|^2\big]\,\mathrm{d}s\\&=-2\pi\langle\varphi\rangle^2.\end{aligned}$$

With Young's inequality, we again find a constant $C(K)>0$ such that

$$(\lambda^2-1)\|\varphi\|_2+\|\mathbb{B}(f)[\varphi]\|_2+|\langle\varphi\rangle|\le C\|(\lambda-\mathbb{A}(f))[\varphi]\|_2.$$

The desired claim (4.2.18) is now a straightforward consequence of this estimate and (4.2.23).

Finally, since $\mathbb{A}(f)\in\mathcal{L}(\mathrm{L}^2(\mathbb{S}))$, its spectrum is compact and the method of continuity, see Theorem 2.1.4, together with (4.2.18) yields the isomorphism property. □

We are now in a position to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. We use a proof by contradiction. Assuming that the claim is false, we find sequences $(\lambda_k) \subset \mathbb{R}$, $(f_k) \subset \mathrm{C}^1(\mathbb{S})$, and $(\beta_k) \subset \mathrm{L}^2(\mathbb{S})^2$ satisfying for all $k \in \mathbb{N}$

$$|\lambda_k| \geq \frac{1}{2} + \delta, \qquad \|f_k'\|_\infty \leq \frac{1}{\delta}, \qquad \|\beta_k\|_2 = 1,$$

and such that

$$(\lambda_k - \mathbb{D}(f_k)^*)[\beta_k] \to 0 \qquad \text{in } \mathrm{L}^2(\mathbb{S})^2. \tag{4.2.24}$$

Recalling (3.1.1), we define $\nu_k := \nu(f_k)$, $\tau_k := \tau(f_k)$, and $\omega_k := \omega(f_k)$ and note from (4.1.7), (4.2.4), and Lemma 4.2.1 (ii) that there exists a constant $M > 0$ such that $\|\mathbb{D}(f_k)^*\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{S}))} \leq M$ for all $k \in \mathbb{N}$. This bound, together with (4.2.24), implies the boundedness of the sequence (λ_k) in $\mathbb{R}$. Observing that the constant $m = m(\lambda)$ in Lemma 4.2.4 satisfies $m(\lambda_k) \geq \delta(2+\delta) > 0$ for all $k \in \mathbb{N}$, we deduce from (4.2.7) that

$$\omega_k^{-1}\beta_k \cdot \tau_k \to 0 \quad \text{and} \quad (\lambda_k - \tfrac{1}{2}\mathbb{B}_1(f_k))[\omega_k^{-1}\beta_k \cdot \nu_k] - \tfrac{1}{2}\mathbb{B}_2(f_k)[\omega_k^{-1}\beta_k \cdot \tau_k] \to 0 \tag{4.2.25}$$

in $\mathrm{L}^2(\mathbb{S})$. In view of the boundedness of the sequence $(\mathbb{B}_2(f_k))_k$ in $\mathcal{L}(\mathrm{L}^2(\mathbb{S}))$, which is a consequence of (4.1.7), (4.2.1), and (4.2.6), we infer from (4.2.25) that

$$(\lambda_k - \tfrac{1}{2}\mathbb{B}_1(f_k))[\omega_k^{-1}\beta_k \cdot \nu_k] \to 0 \qquad \text{in } \mathrm{L}^2(\mathbb{S}).$$

Since $\mathbb{A}(f_k) = \mathbb{B}_1(f_k)^*$, $k \in \mathbb{N}$, Lemma 4.2.5 implies in turn that

$$\omega_k^{-1}\beta_k \cdot \nu_k \to 0 \qquad \text{in } \mathrm{L}^2(\mathbb{S}). \tag{4.2.26}$$

Combining (4.2.25) and (4.2.26), we have $\beta_k \to 0$ in $\mathrm{L}^2(\mathbb{S})^2$, which contradicts the fact that $\|\beta_k\|_2 = 1$ for all $k \in \mathbb{N}$. This proves (4.2.5).

The isomorphism property of $\lambda - \mathbb{D}(f)^* \in \mathcal{L}(\mathrm{L}^2(\mathbb{S})^2)$ with $|\lambda| > 1/2$ and $f \in \mathrm{C}^1(\mathbb{S})$ is now a straightforward consequence of (4.2.5), the compactness of the spectrum of $\mathbb{D}(f)^* \in \mathcal{L}(\mathrm{L}^2(\mathbb{S})^2)$, and the method of continuity, see Theorem 2.1.4, the invertibility of its adjoint $\lambda - \mathbb{D}(f)$ following at once. ☐

4.3. Invertibility in $\mathrm{H}^{r-1}(\mathbb{S})^2$

In this section, we assume that $f \in \mathrm{H}^r(\mathbb{S})$ with $r \in (3/2, 2)$, and we show that the resolvent set of $\mathbb{D}(f)$, viewed as an operator in $\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})^2)$, contains all real numbers λ such that $|\lambda| > 1/2$, see Theorem 4.3.2. To start, we state some mapping properties of the operator $B^{p,q}_{n,m}$ which we use throughout this section.

Lemma 4.3.1. *Let n, m, p, $q \in \mathbb{N}_0$ satisfying $p \leq n + q + 1$, and $(\mathbf{a}, \mathbf{b}) \in \mathrm{H}^r(\mathbb{S})^{m+n}$, with $r \in (3/2, 2)$, be given.*

(i) *There exists a constant $C > 0$ that depends only on n, m, p, q, r, and $\|(\mathbf{a}, \mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})^q$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ we have*

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{q} \|c_i\|_{\mathrm{H}^r}. \tag{4.3.1}$$

(ii) *Let $q \geq 1$. Then, there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a}, \mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ and $\mathbf{c} \in \mathrm{H}^1(\mathbb{S}) \times \mathrm{H}^r(\mathbb{S})^{q-1}$ we have*

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\|_2 \leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{q} \|c_i\|_{\mathrm{H}^r}. \tag{4.3.2}$$

Moreover, $B^{p,q}_{n,m} \in \mathrm{C}^{1-}(\mathrm{H}^r(\mathbb{S})^{m+n}, \mathcal{L}^{q+1}(\mathrm{H}^1(\mathbb{S}) \times \mathrm{H}^r(\mathbb{S})^{q-1} \times \mathrm{H}^{r-1}(\mathbb{S}), \mathrm{L}^2(\mathbb{S})))$.

(iii) *Let* $d, \tilde{d} \in \mathrm{H}^1(\mathbb{S})$ *and* $p \leq n+q+2$ *be given. Then, there exists a constant* $C > 0$ *that depends only on* n, m, p, q, $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$, *and* $\|(d,\tilde{d})\|_{\mathrm{H}^1}$ *such that for all* $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ *and* $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})$ *we have*

$$\begin{aligned}&\|B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))[\mathbf{c},\varphi] - B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi]\|_2\\ &\leq C\|(d-\tilde{d})'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^{q}\|c_i\|_{\mathrm{H}^r}.\end{aligned} \tag{4.3.3}$$

The results of Lemma 4.3.1 are contained in Lemma A.1.6 and Lemma A.1.8, where we distinguish the cases $p=0$ and $p \geq 1$.

From (4.1.1) and (4.1.7), we conclude, using (4.3.1), that $\mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})^2)$ for all $f \in \mathrm{H}^r(\mathbb{S})$. We now establish with Theorem 4.3.2 a counterpart of Theorem 4.2.3 in $\mathrm{H}^{r-1}(\mathbb{S})^2$. A similar result was also obtained in the non-periodic case in [65].

Theorem 4.3.2. *Given* $\delta \in (0,1)$, *there exists a constant* $C = C(\delta, r) > 0$, *such that for all* $\lambda \in \mathbb{R}$ *with* $|\lambda| \geq 1/2+\delta$, $f \in \mathrm{H}^r(\mathbb{S})$ *satisfying* $\|f\|_{\mathrm{H}^r} \leq 1/\delta$, *and* $\beta \in \mathrm{H}^{r-1}(\mathbb{S})^2$ *we have*

$$\|(\lambda - \mathbb{D}(f))[\beta]\|_{\mathrm{H}^{r-1}} \geq C\|\beta\|_{\mathrm{H}^{r-1}}. \tag{4.3.4}$$

Moreover, $\lambda - \mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})^2)$ *is an isomorphism for all* $\lambda \in \mathbb{R}$ *with* $|\lambda| > 1/2$ *and* $f \in \mathrm{H}^r(\mathbb{S})$.

Proof. Let $\delta \in (0,1)$ be fixed. Recalling Theorem 4.2.3 and using the embedding property $\mathrm{H}^r(\mathbb{S}) \hookrightarrow \mathrm{C}^1(\mathbb{S})$, we find a positive constant $C = C(\delta)$ such that we have $\|(\lambda - \mathbb{D}(\tau_\zeta f))^{-1}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{S})^2)} \leq C$ for all $f \in \mathrm{H}^r(\mathbb{S})$ with $\|f\|_{\mathrm{H}^r} \leq 1/\delta$, $\zeta \in \mathbb{S}$, and $\lambda \in \mathbb{R}$ satisfying $|\lambda| \geq 1/2+\delta$. Here, $\tau_\zeta f := f(\zeta + \cdot)$ is the left-shift operator. Using Lemma 2.4.3, we now estimate

$$\begin{aligned}[\beta]^2_{\mathrm{W}^{r-1,2}} &\leq C\int_{-\pi}^{\pi}\frac{\|(\lambda - \mathbb{D}(\tau_\zeta f))[\tau_\zeta\beta - \beta]\|_2^2}{|\zeta|^{2r-1}}\,\mathrm{d}\zeta\\ &\leq C\int_{-\pi}^{\pi}\frac{\|\tau_\zeta((\lambda-\mathbb{D}(f))[\beta]) - (\lambda - \mathbb{D}(f))[\beta]\|_2^2 + \|(\mathbb{D}(\tau_\zeta f) - \mathbb{D}(f))[\beta]\|_2^2}{|\zeta|^{2r-1}}\,\mathrm{d}\zeta\\ &\leq C[(\lambda - \mathbb{D}(f))[\beta]]^2_{\mathrm{W}^{r-1,2}} + C\int_{-\pi}^{\pi}\frac{\|(\mathbb{D}(\tau_\zeta f) - \mathbb{D}(f))[\beta]\|_2^2}{|\zeta|^{2r-1}}\,\mathrm{d}\zeta.\end{aligned} \tag{4.3.5}$$

Concerning the second term in the last line of (4.3.5), we fix an arbitrary $r' \in (3/2, r)$ and show below that there exists a constant $C = C(\delta, r') > 0$ such that we have

$$\|(\mathbb{D}(\tau_\zeta f) - \mathbb{D}(f))[\beta]\|_2 \leq C\|\tau_\zeta f' - f'\|_2\|\beta\|_{\mathrm{H}^{r'-1}} \tag{4.3.6}$$

for all $\zeta \in \mathbb{S}$, $f \in \mathrm{H}^r(\mathbb{S})$ with $\|f\|_{\mathrm{H}^r} \leq 1/\delta$, and $\beta \in \mathrm{H}^{r-1}(\mathbb{S})^2$. Recalling the representations (4.1.1) and (4.1.7), it suffices to consider terms of the form

$$(B^{p,q}_{n,m}(\tau_\zeta f) - B^{p,q}_{n,m}(f))[\beta_i] \qquad \text{and} \qquad B^{p,q}_{n,m}(\tau_\zeta f)[(\tau_\zeta f)'\beta_i] - B^{p,q}_{n,m}(f)[f'\beta_i], \tag{4.3.7}$$

with n, m, p, $q \in \mathbb{N}_0$ with $p \leq n+q+1$, and $1 \leq i =\leq 2$. Since

$$\begin{aligned}&B^{p,q}_{n,m}(\tau_\zeta f)[(\tau_\zeta f)'\beta_i] - B^{p,q}_{n,m}(f)[f'\beta_i]\\ &= (B^{p,q}_{n,m}(\tau_\zeta f) - B^{p,q}_{n,m}(f))[(\tau_\zeta f)'\beta_i] + B^{p,q}_{n,m}(f)[(\tau_\zeta f' - f')\beta_i],\end{aligned}$$

the last term on the right-hand side being estimated in view of (4.2.1) by

$$\|B^{p,q}_{n,m}(f)[(\tau_\zeta f' - f')\beta_i]\|_2 \leq C\|(\tau_\zeta f' - f')\beta_i\|_2 \leq C\|(\tau_\zeta f' - f')\|_2\|\beta\|_{\mathrm{H}^{r'-1}},$$

it remains to estimate the first term in (4.3.7) according to (4.3.6). To this end, we infer from the formula (A.1.37) that the operator $B^{p,q}_{n,m}(\tau_\zeta f) - B^{p,q}_{n,m}(f)$ can be written as a linear combination of terms of the form

$$T_1 := B^{\bar p,\bar q}_{\bar n,\bar m}(\tau_\zeta f,\dots,\tau_\zeta f|\tau_\zeta f,\dots,\tau_\zeta f)[\underbrace{f,\dots,f}_{i-1 \text{ times}},\tau_\zeta f - f,\tau_\zeta f,\dots,\tau_\zeta f,\cdot]$$

and

$$\begin{aligned} T_2 := (&B^{\bar p,\bar q}_{\bar n,\bar m}(\underbrace{f,\dots,f}_{j \text{ times}},\tau_\zeta f,\dots,\tau_\zeta f|\underbrace{f,\dots,f}_{\ell-1 \text{ times}}\tau_\zeta f,\dots\tau_\zeta f)\\ &- B^{\bar p,\bar q}_{\bar n,\bar m}(\underbrace{f,\dots,f}_{j \text{ times}},\tau_\zeta f,\dots,\tau_\zeta f|\underbrace{f,\dots,f}_{\ell \text{ times}}\tau_\zeta f,\dots\tau_\zeta f))[f,\dots,f,\cdot]\end{aligned}$$

where $\bar n$, $\bar m$, $\bar p$, $\bar q \in \mathbb{N}_0$ with $\bar p \le \bar n + \bar q + 1$, $1 \le i \le \bar q$, $0 \le j \le \bar m - 1$, and $1 \le \ell \le \bar n$. Applying (4.3.2) to estimate T_1 (with r replaced by r'), and (4.3.3) for T_2 (again with r replaced by r'), we obtain

$$\|B^{p,q}_{n,m}(\tau_\zeta f) - B^{p,q}_{n,m}(f)\|_{\mathcal{L}(\mathrm{H}^{r'-1}(\mathbb{S}),\mathrm{L}^2(\mathbb{S}))} \le C\|\tau_\zeta f' - f'\|_2,$$

from which (4.3.6) readily follows.

Plugging (4.3.6) into (4.3.5) immediately yields

$$[\beta]^2_{\mathrm{W}^{r-1,2}} \le C(\|\beta\|^2_{\mathrm{H}^{r'-1}} + [(\lambda - \mathbb{D}(f))[\beta]]^2_{\mathrm{W}^{r-1,2}}).$$

Using the interpolation property (2.4.4), Young's inequality, and Theorem 4.2.3, we get

$$\begin{aligned}\|\beta\|^2_{\mathrm{H}^{r-1}} &\le C\Big(\|\beta\|^2_2 + \|\beta\|^2_{\mathrm{H}^{r'-1}} + [(\lambda - \mathbb{D}(f))[\beta]]^2_{\mathrm{W}^{r-1,2}}\Big)\\ &\le \frac{1}{2}\|\beta\|^2_{\mathrm{H}^{r-1}} + C\|(\lambda - \mathbb{D}(f))[\beta]\|^2_{\mathrm{H}^{r-1}},\end{aligned}$$

which proves (4.3.4). The remaining isomorphism property follows from (4.3.4) by the same continuity argument used in the proof of Theorem 4.2.3. □

4.4. Invertibility in $\mathrm{H}^2(\mathbb{S})^2$

We start this section by establishing mapping properties for the operator $B^{p,q}_{n,m}$ in higher order Sobolev spaces in Lemma 4.4.1 and Corollary 4.4.2 below. First, we introduce for given $\mathbf{z} = (z_1,\dots,z_n) \in X^n$, where X is a Banach space, the notation

$$\mathbf{z}_i := (z_1,\dots,z_{i-1},z_{i+1},\dots,z_n) \in X^{n-1}, \qquad 1 \le i \le n, \quad n \in \mathbb{N}, \tag{4.4.1}$$

which will be used throughout the rest of this thesis.

Lemma 4.4.1. *Let n, m, p, $q \in \mathbb{N}_0$ with $p \le n + q + 1$, $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathrm{H}^2(\mathbb{S})^{m+n+q}$, and $\varphi \in \mathrm{H}^1(\mathbb{S})$ be given. Then, $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot] \in \mathcal{L}(\mathrm{H}^1(\mathbb{S}))$ with*

$$\begin{aligned}&(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi])'\\ &= B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi'] + \sum_{j=1}^{q} B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[c_1,\dots,c_{j-1},c_j',c_{j+1},\dots,c_q,\varphi]\\ &\quad + \sum_{j=1}^{n}(B^{p,q+1}_{n-1,m}(\mathbf{a}|\mathbf{b}_j)[\mathbf{c},b_j',\varphi] - B^{p+2,q+1}_{n+1,m}(\mathbf{a}|\mathbf{b},b_j)[\mathbf{c},b_j',\varphi])\\ &\quad - 2\sum_{j=1}^{m}(B^{p,q+1}_{n+1,m+1}(\mathbf{a},a_j|\mathbf{b},a_j)[\mathbf{c},a_j',\varphi]\\ &\qquad\qquad - B^{p+2,q+1}_{n+3,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j,a_j)[\mathbf{c},a_j',\varphi]).\end{aligned} \tag{4.4.2}$$

Moreover, there exists a constant $C > 0$ depending only on n, m, p, q, and $\|(\mathbf{a}, \mathbf{b})\|_{\mathrm{H}^2}$ such that

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\|_{\mathrm{H}^1} \leq C\|\varphi\|_{\mathrm{H}^1} \prod_{i=1}^{q} \|c_i\|_{\mathrm{H}^2}. \tag{4.4.3}$$

In addition, $B^{p,q}_{n,m} \in \mathrm{C}^{1-}\big(\mathrm{H}^2(\mathbb{S})^{m+n}, \mathcal{L}^q_{\mathrm{sym}}(\mathrm{H}^2(\mathbb{S}), \mathcal{L}(\mathrm{H}^1(\mathbb{S})))\big)$.

The proof is again postponed to Appendix A, where we show the result separately in the case $p = 0$ and, in an improved form, in the case $p \geq 1$ in Lemma A.1.9 and Lemma A.1.10, respectively.

As a straightforward consequence of Lemma 4.4.1, especially of the formula (4.4.2), we conclude via induction the following result.

Corollary 4.4.2. *Given n, m, p, $q \in \mathbb{N}_0$ with $p \leq n + q + 1$ and $k \in \mathbb{N}$, we have*

$$B^{p,q}_{n,m} \in \mathrm{C}^{1-}\big(\mathrm{H}^{k+1}(\mathbb{S})^{m+n}, \mathcal{L}^q_{\mathrm{sym}}(\mathrm{H}^{k+1}(\mathbb{S}), \mathcal{L}(\mathrm{H}^k(\mathbb{S})))\big).$$

In particular, Corollary 4.4.2 enables us to conclude, together with (4.1.1) and (4.1.7), that $\mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^2(\mathbb{S})^2)$ whenever $f \in \mathrm{H}^3(\mathbb{S})$. The final result in this chapter is the following theorem which deals with the resolvent set of $\mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^2(\mathbb{S})^2)$ and is crucial when solving (4.1.2) in Chapter 5.

Theorem 4.4.3. *Let $\lambda \in \mathbb{R}$ with $|\lambda| > 1/2$ and $f \in \mathrm{H}^3(\mathbb{S})$ be given. Then, the operator $\lambda - \mathbb{D}(f) \in \mathcal{L}(\mathrm{H}^2(\mathbb{S})^2)$ is an isomorphism.*

Proof. Arguing as in the proofs of Theorem 4.2.3 and Theorem 4.3.2, it suffices to show that given $\delta \in (0, 1)$, there exists a constant $C = C(\delta) > 0$ such that for all $|\lambda| \geq 1/2 + \delta$ and $\|f\|_{\mathrm{H}^3} \leq 1/\delta$ we have

$$C\|(\lambda - \mathbb{D}(f))[\beta]\|_{\mathrm{H}^2} \geq \|\beta\|_{\mathrm{H}^2}, \qquad \beta \in \mathrm{H}^2(\mathbb{S})^2. \tag{4.4.4}$$

To this end, we recall the formulas (4.1.1), (4.1.7), and (4.4.2), and express the difference $(\mathbb{D}(f)[\beta])'' - \mathbb{D}(f)[\beta''] =: T_{\mathrm{lot}}[\beta]$ as a linear combination of terms of the form

$$\begin{aligned}
&B^{p,q}_{n,m}(f, \ldots, f|f, \ldots, f)[f'', \ldots, f, (f')^k\beta_i],\\
&B^{p,q}_{n,m}(f, \ldots, f|f, \ldots, f)[f', f, \ldots, f, ((f')^k\beta_i)'],\\
&B^{p,q}_{n,m}(f, \ldots, f|f, \ldots, f)[f', f', \ldots, f, (f')^k\beta_i],\\
&B^{p,q}_{n,m}(f)[f'''\beta_i + 2f''\beta_i'],
\end{aligned}$$

where $k \in \{0, 1\}$, $i \in \{1, 2\}$, and n, m, p, $q \in \mathbb{N}_0$ satisfy $p \leq n + q + 1$. Using (4.2.1) to estimate the last three terms above and (4.3.2) for the first term, we find a constant $C > 0$ that depends only on n, m, p, q and $\|f\|_{\mathrm{H}^3}$ such that

$$\|T_{\mathrm{lot}}[\beta]\|_2 \leq C\|\beta\|_{\mathrm{H}^1}, \qquad \beta \in \mathrm{H}^2(\mathbb{S})^2.$$

This estimate, the interpolation property (2.4.4) together with Young's inequality, and the observation that $\|(\lambda - \mathbb{D}(f))^{-1}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{S}))} \leq C$ by Theorem 4.2.3 show that for $|\lambda| \geq 1/2 + \delta$ and $\|f\|_{\mathrm{H}^3} \leq 1/\delta$, we have

$$\begin{aligned}
\|\beta\|_{\mathrm{H}^2} &\leq C(\|\beta\|_2 + \|\beta''\|_2) \leq C(\|\beta\|_2 + \|(\lambda - \mathbb{D}(f))[\beta'']\|_2)\\
&\leq C(\|\beta\|_2 + \|T_{\mathrm{lot}}[\beta]\|_2 + \|((\lambda - \mathbb{D}(f))[\beta])''\|_2)\\
&\leq C(\|\beta\|_{\mathrm{H}^1} + \|((\lambda - \mathbb{D}(f))[\beta])''\|_2)\\
&\leq \frac{1}{2}\|\beta\|_{\mathrm{H}^2} + C(\|\beta\|_2 + \|((\lambda - \mathbb{D}(f))[\beta])''\|_2)\\
&\leq \frac{1}{2}\|\beta\|_{\mathrm{H}^2} + C\|(\lambda - \mathbb{D}(f))[\beta]\|_{\mathrm{H}^2},
\end{aligned}$$

which proves (4.4.4). ☐

Chapter 5

The fixed time problem associated to (1.2.2)

5.1. Introduction

In this chapter, we use the same notation as in Chapter 3 to study the fixed time problem (3.1.4) associated to (1.2.2). We assume that $f \in \mathrm{H}^3(\mathbb{S})$ and we solve the more general transmission boundary value problem

$$\left.\begin{array}{rll} \mu^\pm \Delta v^\pm - \nabla q^\pm &= 0 & \text{in } \Omega^\pm, \\ \operatorname{div} v^\pm &= 0 & \text{in } \Omega^\pm, \\ [v] &= 0 & \text{on } \Gamma, \\ [T_\mu(v,q)]\tilde{\nu} &= (\omega^{-1}(G+H')) \circ \Xi^{-1} & \text{on } \Gamma, \\ (v^\pm, q^\pm)(x) &\to \pm\Big(\frac{c_{1,\Gamma}}{\mu^\pm}, \frac{c_{2,\Gamma}}{\mu^\pm}, c_{3,\Gamma}\Big) & \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \end{array}\right\} \tag{5.1.1}$$

where $G = (G_1, G_2) \in \mathrm{H}^1(\mathbb{S})^2$ and $H = (H_1, H_2) \in \mathrm{H}^2(\mathbb{S})^2$ are given functions satisfying $\langle G_1 \rangle = 0$, and the constant $c_\Gamma = (c_{1,\Gamma}, c_{2,\Gamma}, c_{3,\Gamma}) \in \mathbb{R}^3$ is an unknown of the system. As in Chapter 3, we look for a solution $(v,q) \in X_f$, see (3.1.5). Recalling Remark 3.3.1, we notice that (3.1.4) is just a special case of (5.1.1) for the particular choice

$$G := G(f) := \Theta(-ff', f) \qquad \text{and} \qquad H := H(f) := -\sigma(\omega^{-1} - 1, \omega^{-1} f'), \tag{5.1.2}$$

see (3.3.18).

To find a solution $(v,q) \in X_f$ to (5.1.1), we first employ the substitution $\tilde{v}^\pm := \mu^\pm v^\pm$. Then, finding a solution to (5.1.1) becomes equivalent to finding a solution $(\tilde{v}, \tilde{q}) \in X_f$ to

$$\left.\begin{array}{rll} \Delta \tilde{v}^\pm - \nabla \tilde{q}^\pm &= 0 & \text{in } \Omega^\pm, \\ \operatorname{div} \tilde{v}^\pm &= 0 & \text{in } \Omega^\pm, \\ \mu^-\{\tilde{v}\}^+ - \mu^+\{\tilde{v}\}^- &= 0 & \text{on } \Gamma, \\ [T_1(\tilde{v}, \tilde{q})]\tilde{\nu} &= (\omega^{-1}(G+H')) \circ \Xi^{-1} & \text{on } \Gamma, \\ (\tilde{v}^\pm, \tilde{q}^\pm)(x) &\to \pm(c_{1,\Gamma}, c_{2,\Gamma}, c_{3,\Gamma}) & \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}. \end{array}\right\} \tag{5.1.3}$$

The transmission type condition $(5.1.3)_3$ together with formula (3.3.16) suggest that a solution involves the hydrodynamic double-layer potential, while the jump condition for the stress tensor $(5.1.3)_4$ provides evidence that also the hydrodynamic single-layer potential is part of the solution, see (3.3.15). Recalling (3.3.9), we use the natural approach to write $(\tilde{v}, \tilde{q})$ as a sum of singe-layer and double-layer potentials. To be precise, straightforward calculations show that $(\tilde{v}_s + \tilde{v}_d, \tilde{q}_s + \tilde{q}_d) \in X_f$ is a solution to (5.1.3), if $(\tilde{v}_s, \tilde{q}_s) \in X_f$ solves

$$\left.\begin{aligned} \Delta \tilde{v}_s^{\pm} - \nabla \tilde{q}_s^{\pm} &= 0 && \text{in } \Omega^{\pm}, \\ \operatorname{div} \tilde{v}_s^{\pm} &= 0 && \text{in } \Omega^{\pm}, \\ [\tilde{v}_s] &= 0 && \text{on } \Gamma, \\ [T_1(\tilde{v}_s, \tilde{q}_s)]\tilde{\nu} &= (\omega^{-1}(G + H')) \circ \Xi^{-1} && \text{on } \Gamma, \\ (\tilde{v}_s^{\pm}, \tilde{q}_s^{\pm})(x) &\to \pm(c_{1,s}, c_{2,s}, c_{3,s}) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}, \end{aligned}\right\} \tag{5.1.4}$$

and $(\tilde{v}_d, \tilde{q}_d) \in X_f$ is a solution to

$$\left.\begin{aligned} \Delta \tilde{v}_d^{\pm} - \nabla \tilde{q}_d^{\pm} &= 0 && \text{in } \Omega^{\pm}, \\ \operatorname{div} \tilde{v}_d^{\pm} &= 0 && \text{in } \Omega^{\pm}, \\ \mu^-\{\tilde{v}_d\}^+ - \mu^+\{\tilde{v}_d\}^- &= (\mu^+ - \mu^-)\tilde{v}_s|_\Gamma && \text{on } \Gamma, \\ [T_1(\tilde{v}_d, \tilde{q}_d)]\tilde{\nu} &= 0 && \text{on } \Gamma, \\ (\tilde{v}_d^{\pm}, \tilde{q}_d^{\pm})(x) &\to \pm(c_{1,d}, c_{2,d}, c_{3,d}) && \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}. \end{aligned}\right\} \tag{5.1.5}$$

The solution $(\tilde{v}_s, \tilde{q}_s) \in X_f$ to (5.1.4) is given by the hydrodynamic single-layer potentials, see Theorem 3.3.2. Following the discussion above, we expect that the hydrodynamic double-layer potentials generated by a specific density provide a solution to (5.1.5). That this expectation is correct is shown in the proof of Theorem 5.3.1 below.

We now introduce some further notation used throughout this chapter. To start, we define the function

$$\mathcal{V}(f)[(G,H)] := (\mathcal{V}_1(f)[(G,H)], \mathcal{V}_2(f)[(G,H)])$$

by

$$\begin{aligned} \mathcal{V}_1(f)[(G,H)] :=& \frac{1}{4}((B_0(f) + B_6(f))[G_1] - B_5(f)[G_2]) \\ &+ \frac{1}{4}((B_1 - 2B_4)(f)[H_1 - f'H_2] + (2B_2 + B_3)(f)[f'H_1] \\ &\quad + B_3(f)[H_2]), \\ \mathcal{V}_2(f)[(G,H)] :=& \frac{1}{4}((B_0(f) - B_6(f))[G_2] - B_5(f)[G_1] + \langle G_2 \rangle \ln 4) \\ &+ \frac{1}{4}(B_1(f)[H_2 - f'H_1] + B_3(f)[H_1 - f'H_2] \\ &\quad + 2B_4(f)[f'H_1 + H_2]), \end{aligned} \tag{5.1.6}$$

see (3.3.51) and (3.3.54). It follows from (4.1.7), Corollary 4.4.2, and Lemma A.1.12 that $\mathcal{V}(f)[(G,H)] \in \mathrm{H}^2(\mathbb{S})^2$. Moreover, we set

$$a_\mu := \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} \in (-1, 1), \tag{5.1.7}$$

and infer from Theorem 4.4.3 that there exists a unique solution

$$\beta(f)[(G,H)] = (\beta_1(f)[(G,H)], \beta_2(f)[(G,H)]) \in \mathrm{H}^2(\mathbb{S})^2$$

to

$$(1 + 2a_\mu \mathbb{D}(f)) [\beta(f)[(G,H)]] = \mathcal{V}(f)[(G,H)]. \tag{5.1.8}$$

The structure of this chapter is as follows: In Section 5.2 we consider a homogeneous boundary value problem and prove that it has a unique solution which lies in X_f. Subsequently, in Section 5.3, we show in Theorem 5.3.1 the unique solvability of problem (5.1.1) using the result from Section 5.2.

5.2. A homogeneous boundary value problem

To establish the unique solvability of the boundary value problem (5.1.1) in Theorem 5.3.1 below, we consider the homogeneous boundary value problem

$$\left.\begin{aligned} \Delta w^\pm - \nabla q^\pm &= 0 \quad \text{in } \Omega^\pm,\\ \operatorname{div} w^\pm &= 0 \quad \text{in } \Omega^\pm,\\ \mu^-\{w\}^+ - \mu^+\{w\}^- &= 0 \quad \text{on } \Gamma,\\ [T_1(w,q)]\tilde\nu &= 0 \quad \text{on } \Gamma,\\ (\mu^\mp w^\pm, q^\pm)(x) &\to \pm\tilde c \ \begin{array}{l}\text{for } x_2\to\pm\infty \text{ uni-}\\ \text{formly in } x_1\in\mathbb{S},\end{array} \end{aligned}\right\} \tag{5.2.1}$$

with $\tilde c \in \mathbb{R}^3$, and prove that (5.2.1) has a solution $(w,q)\in X_f$ if and only if $\tilde c = 0$, the solution being moreover unique. Note that (5.2.1) and (5.1.1) (with $G = H = 0$ in $(5.1.1)_4$) are equivalent problems. This can be seen by applying the substitutions

$$w^\pm := \mu^\pm\Big(v^\pm + \frac{\mu^+-\mu^-}{2\mu^+\mu^-}(c_{1,\Gamma}, c_{2,\Gamma})^\top\Big), \qquad \tilde c = \Big(\frac{\mu^++\mu^-}{2}c_{1,\Gamma}, \frac{\mu^++\mu^-}{2}c_{2,\Gamma}, c_{3,\Gamma}\Big).$$

Proposition 5.2.1. *Given $f\in \mathrm{C}^1(\mathbb{S})$ and $\tilde c\in\mathbb{R}^3$, the transmission boundary value problem* (5.2.1) *has a solution $(w,q)\in X_f$ if and only if $\tilde c = 0$. In this case, the solution is unique (and trivial).*

Proof. To start, we assume that (5.2.1) has a solution $(w,q)\in X_f$ for some constant $\tilde c = (\tilde c_1, \tilde c_2, \tilde c_3)\in\mathbb{R}^3$. Then, in view of $(5.2.1)_3$, we have

$$\psi := \mu^- w^+ \mathbf{1}_{\Omega^+} + \mu^+ w^- \mathbf{1}_{\Omega^-} \in \mathrm{C}(\mathbb{S}\times\mathbb{R})^2 \cap \mathrm{H}^1_{\mathrm{loc}}(\mathbb{S}\times\mathbb{R})^2.$$

Given $\phi\in \mathrm{C}(\mathbb{S}\times\mathbb{R})^2\cap \mathrm{H}^1(\mathbb{S}\times\mathbb{R})^2$ with compact support, we compute, using Stokes' theorem and $(5.2.1)_2$, that

$$\begin{aligned} &\int_{\Omega^\pm} \phi^\pm\cdot(\Delta w^\pm - \nabla q^\pm)\,\mathrm{d}x \pm \int_\Gamma {\phi^\pm}^\top\big(-q^\pm I_2 + \nabla w^\pm + (\nabla w^\pm)^\top\big)\tilde\nu\,\mathrm{d}\sigma\\ &= -\frac12\int_{\Omega^\pm}\big(\nabla\phi^\pm + (\nabla\phi^\pm)^\top\big) : \big(\nabla w^\pm + (\nabla w^\pm)^\top\big)\,\mathrm{d}x + \int_{\Omega^\pm} q^\pm \operatorname{div}\phi^\pm\,\mathrm{d}x, \end{aligned}$$

where $A : B$ denotes the Frobenius inner product of two matrices $A, B\in\mathbb{R}^{2\times2}$, with $|A| := \sqrt{A:A}$ being the associated norm. Summing up, we obtain in view of $(5.2.1)_1$ and $(5.2.1)_4$, using also the continuity of ϕ, that

$$\begin{aligned} &\int_{\mathbb{S}\times\mathbb{R}}\Big(\frac{1}{\mu^-}\mathbf{1}_{\Omega^+} + \frac{1}{\mu^+}\mathbf{1}_{\Omega^-}\Big)\big(\nabla\psi + (\nabla\psi)^\top\big) : \big(\nabla\phi + (\nabla\phi)^\top\big)\,\mathrm{d}x\\ &= 2\int_{\mathbb{S}\times\mathbb{R}} q\operatorname{div}\phi\,\mathrm{d}x. \end{aligned} \tag{5.2.2}$$

Let $u\in \mathrm{C}^\infty(\mathbb{R},[0,1])$ be an even function with $u(t) = 1$ for $|t|\le 1$ and $u(t) = 0$ for $|t|\ge 2$. For each integer $n > \|f\|_\infty$, we define the even function $u_n\in \mathrm{C}^\infty(\mathbb{R},[0,1])$ by

$$u_n(t) := \begin{cases} 1, & |t|\le n,\\ u(t-n+1), & t\in[n, n+1],\\ 0, & |t|\ge n+1.\end{cases} \tag{5.2.3}$$

Then, setting

$$\phi_n(x) := u_n(x_2)\psi(x), \qquad x\in\mathbb{S}\times\mathbb{R},$$

we conclude that $\phi_n \in \mathrm{C}(\mathbb{S}\times\mathbb{R})^2 \cap \mathrm{H}^1(\mathbb{S}\times\mathbb{R})^2$ has compact support.

Using ϕ_n as a test function in (5.2.2) and noticing by $(5.2.1)_2$ that for a.e. $x \in \mathbb{S}\times\mathbb{R}$

$$\begin{aligned} \operatorname{div}\phi_n(x) &= u_n'(x_2)\psi_2(x),\\ (\nabla\phi_n + (\nabla\phi_n)^\top)(x) &= u_n(x_2)(\nabla\psi + (\nabla\psi)^\top)(x) + u_n'(x_2)\begin{pmatrix} 0 & \psi_1 \\ \psi_1 & 2\psi_2 \end{pmatrix}(x), \end{aligned}$$

we arrive at

$$\int_{\mathbb{S}\times\mathbb{R}} \mathbf{1}_{\{|x_2|<n\}} \left|\nabla\psi + (\nabla\psi)^\top\right|^2 \mathrm{d}x \le \max\{\mu^+,\mu^-\}R_n, \tag{5.2.4}$$

where $R_n := R_n^I + R_n^{II}$ with

$$\begin{aligned} R_n^I &:= 2\int_{\mathbb{S}\times\mathbb{R}} u_n'(x_2)(q\psi_2)(x)\,\mathrm{d}x,\\ R_n^{II} &:= -\int_{\mathbb{S}\times\mathbb{R}} \left(\frac{1}{\mu^-}\mathbf{1}_{\Omega^+} + \frac{1}{\mu^+}\mathbf{1}_{\Omega^-}\right) u_n'(x_2)(\nabla\psi + (\nabla\psi)^\top)(x) : \begin{pmatrix} 0 & \psi_1 \\ \psi_1 & 2\psi_2 \end{pmatrix}(x)\,\mathrm{d}x. \end{aligned}$$

Since $(w,q) \in X_f$, it is straightforward to infer from $(5.2.1)_2$ and $(5.2.1)_5$ that (R_n) is a bounded (actually convergent) sequence, implying that

$$\nabla\psi + (\nabla\psi)^\top \in \mathrm{L}^2(\mathbb{S}\times\mathbb{R}, \mathbb{R}^{2\times2}). \tag{5.2.5}$$

Moreover, it holds that

$$\int_{\mathbb{S}\times\mathbb{R}} |\nabla\phi_n|^2\,\mathrm{d}x + \int_{\mathbb{S}\times\mathbb{R}} |\operatorname{div}\phi_n|^2\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{S}\times\mathbb{R}} |\nabla\phi_n + (\nabla\phi_n)^\top|^2\,\mathrm{d}x. \tag{5.2.6}$$

Since (5.2.5) ensures that the right-hand side of (5.2.6) defines a bounded sequence in $\mathbb{R}$, we deduce that $\nabla\psi \in \mathrm{L}^2(\mathbb{S}\times\mathbb{R}, \mathbb{R}^{2\times2})$. Exploiting this property, the oddness of u', and $(5.2.1)_5$, we easily obtain that $R_n \to 0$ for $n \to \infty$. Consequently, we may infer from (5.2.4) that $\nabla\psi + (\nabla\psi)^\top = 0$ in $\mathrm{L}^2(\mathbb{S}\times\mathbb{R}, \mathbb{R}^{2\times2})$. In view of this property, the identity (5.2.6) may be reformulated as

$$\int_{\mathbb{S}\times\mathbb{R}} u_n^2(x_2)|\nabla\psi|^2(x)\,\mathrm{d}x + \int_{\mathbb{S}\times\mathbb{R}} (u_n^2)'(x_2)(\psi\cdot\partial_2\psi)(x)\,\mathrm{d}x = 0.$$

Note that the second term on the left-hand side converges to 0 as $n \to \infty$ due to $(5.2.1)_5$, $(w,q) \in X_f$, and $\nabla\psi \in \mathrm{L}^2(\mathbb{S}\times\mathbb{R}, \mathbb{R}^{2\times2})$. This implies that $\nabla\psi = 0$ and together with $(5.2.1)_5$ we conclude that $\psi = 0$ and $(\tilde c_1, \tilde c_2) = 0$. Since $(5.2.1)_1$ and $(5.2.1)_5$ ensure that $q = 0$ and $\tilde c_3 = 0$, this completes the proof. ☐

5.3. The solution of the transmission boundary value problem (5.1.1)

This section is devoted entirely to the result in Theorem 5.3.1 which provides the unique solvability of problem (5.1.1) in X_f.

Theorem 5.3.1. *Given $f \in \mathrm{H}^3(\mathbb{S})$, the transmission boundary value problem* (5.1.1) *has a solution $(v,q) \in X_f$ if and only if the constant $c_\Gamma = (c_{1,\Gamma}, c_{2,\Gamma}, c_{3,\Gamma}) \in \mathbb{R}^3$ is given by*

$$c_\Gamma = \left(a_\mu\langle \beta_1(f)[(G,H)] - f'\beta_2(f)[(G,H)]\rangle - \frac{\langle fG_1 - f'H_1\rangle}{2}, 0, \frac{-\langle G_2\rangle}{2}\right), \tag{5.3.1}$$

where $\beta(f)[(G,H)] \in \mathrm{H}^2(\mathbb{S})^2$ *is defined in* (5.1.8). *For this choice of* c_Γ, *the solution is also unique and satisfies*

$$\begin{aligned}\{v\}^\pm \circ \Xi &= \frac{1}{\mu^\pm}\Big[\mathcal{V}(f)[(G,H)] + 2a_\mu\Big(\pm\frac{1}{2} - \mathbb{D}(f)\Big)[\beta(f)[(G,H)]]\Big] \\ &= \frac{2}{\mu^+ + \mu^-}\beta(f)[(G,H)].\end{aligned} \tag{5.3.2}$$

Moreover, the constant $c_{1,\Gamma}$ *can be expressed as*

$$\begin{aligned}c_{1,\Gamma} &= -\frac{\mu^+\mu^-}{2\pi(\mu^+ + \mu^-)}\mathrm{PV}\int_{\mathbb{S}\times\mathbb{R}} \operatorname{curl} v \,\mathrm{d}x \\ &:= -\frac{\mu^+\mu^-}{2\pi(\mu^+ + \mu^-)}\lim_{n\to\infty}\int_{\{|x_2|<n\}} \operatorname{curl} v \,\mathrm{d}x,\end{aligned} \tag{5.3.3}$$

and it additionally holds that

$$T_{\mu^\pm}(v^\pm, q^\pm)(x) \to \pm\frac{\langle G_2\rangle}{2} I_2 \qquad \textit{for } x_2 \to \pm\infty \textit{ uniformly in } x_1 \in \mathbb{S}. \tag{5.3.4}$$

Proof. We divide the proof into several steps.

Uniqueness. Let $c_\Gamma := (c_{1,\Gamma}, c_{2,\Gamma}, c_{3,\Gamma}) \in \mathbb{R}^3$ be arbitrary and assume that $(v,q) \in X_f$ is a solution to the homogeneous problem associated to (5.1.1) (that is with the right-hand side of $(5.1.1)_4$ replaced by 0). Since this problem is equivalent to problem (5.2.1), Proposition 5.2.1 ensures that $c_\Gamma = 0$ and $(v,q) = (0,0)$, which establishes the uniqueness claim.

Identification of $c_{2,\Gamma}$**.** If $(v,q) \in X_f$ is a solution to (5.1.1), we may integrate equation $(5.1.1)_2$ over $\Omega_n^\pm$ with $n \geq \|f\|_\infty$, recalling (4.2.14), to deduce from Stokes' theorem and $(5.1.1)_3$ that

$$\int_{-\pi}^{\pi} v_2^+(x_1, n)\,\mathrm{d}x_1 = \int_{-\pi}^{\pi} v_2^-(x_1, -n)\,\mathrm{d}x_1.$$

Taking the limit $n \to \infty$, it follows from $(5.1.1)_5$ that indeed $c_{2,\Gamma} = 0$.

Existence. We use Theorem 3.3.2 to construct the unique solution $(\tilde v_s, \tilde q_s) \in X_f$ to (5.1.4) and get

$$c_s = \frac{1}{2}(-\langle fG_1\rangle, 0, -\langle G_2\rangle).$$

Moreover, we infer from (3.3.28), (3.3.51), and Remark 3.3.7 that the trace of $\tilde v_s$ on Γ is given by the relation

$$\{\tilde v_s\}^\pm \circ \Xi = \mathcal{V}(f)[(G,H)], \tag{5.3.5}$$

with $\mathcal{V}(f)$ defined in (5.1.6). Furthermore, it follows from (3.3.49) and Lemma 3.3.8 that

$$T_1(v_s^\pm, q_s^\pm)(x) \to \pm\frac{\langle G_2\rangle}{2} I_2 \qquad \text{for } x_2 \to \pm\infty \text{ uniformly in } x_1 \in \mathbb{S}. \tag{5.3.6}$$

Let

$$(\tilde v_d, \tilde q_d) := \big(v_d[2a_\mu\beta(f)[(G,H)]], q_d[2a_\mu\beta(f)[(G,H)]]\big) \in X_f$$

denote the unique solution to (3.1.7) determined in Theorem 3.3.9 with β therein replaced by $2a_\mu\beta(f)[(G,H)] \in \mathrm{H}^2(\mathbb{S})^2$ (and $\beta(f)[(G,H)]$ defined in (5.1.8)). Then, if c_d in (5.1.5) is given by

$$c_d = a_\mu\big(\langle\beta_1(f)[(G,H)] - f'\beta_2(f)[(G,H)]\rangle, \langle\beta_2(f)[(G,H)] - f'\beta_1(f)[(G,H)]\rangle, 0\big),$$

it follows that $(\tilde{v}_d, \tilde{q}_d)$ is a solution to $(5.1.5)_{1-2}$ and $(5.1.5)_{4-5}$. By using (3.3.63) and (4.1.1), we have

$$\mu^-\{\tilde{v}_d\}^+ - \mu^+\{\tilde{v}_d\}^- = (\mu^+ - \mu^-)\big(1 + 2a_\mu \mathbb{D}(f)\big)\,[\beta(f)[(G,H)]] \circ \Xi^{-1} \qquad \text{on } \Gamma,$$

and since $\beta(f)[(G,H)]$ is defined as the solution to (5.1.8), $(5.1.5)_3$ is also satisfied. Setting

$$v^\pm := \frac{1}{\mu^\pm}(\tilde{v}_s^\pm + \tilde{v}_d^\pm) \qquad \text{and} \qquad q^\pm := \tilde{q}_s^\pm + \tilde{q}_d^\pm, \tag{5.3.7}$$

it follows from the previous arguments that $(v,q) \in X_f$ is a solution to (5.1.1), see also the discussion preceding (5.1.4). The relation (5.3.4) is a direct consequence of (5.3.6), (5.3.7), and (3.3.62).

Identification of $c_{1,\Gamma}$. Let $(v,q) \in X_f$ be the unique solution to (5.1.1). Integrating $\operatorname{curl} v$ over the domain $\{|x_2| < n\}$ with $n > \|f\|_\infty$, Stokes' theorem together with $(5.1.1)_3$ and $(5.1.1)_5$ yields

$$\begin{aligned}\int_{\{|x_2|<n\}} \operatorname{curl} v \,\mathrm{d}x &= \int_{-\pi}^{\pi} v_1^-(x_1,-n) - v_1^+(x_1,n)\,\mathrm{d}x_1 + \int_\Gamma \big[(-v_2,v_1)^\top\big] \cdot \tilde{\nu}\,\mathrm{d}\sigma \\ &\underset{n\to\infty}{\longrightarrow} -2\pi c_{1,\Gamma} \frac{\mu^+ + \mu^-}{\mu^+\mu^-},\end{aligned}$$

and (5.3.3) follows. □

Part II

The evolution problem

Chapter 6

Reformulation and well-posedness of the two-phase Stokes problem

6.1. Introduction

The main goal of this chapter is to reformulate the two-phase Stokes system (1.2.2) as a single evolution equation for the function $f(t)$ parametrizing the interface $\Gamma(t)$ and prove its local well-posedness as stated in Theorem 1.4.1.

The unique solvability of the fixed time problem (5.1.1) associated to (1.2.2a) for any $f \in \mathrm{H}^3(\mathbb{S})$ in the previous chapter, cf. Theorem 5.3.1, together with the formula for the trace of the velocity on the interface, see (5.3.2), the kinematic boundary condition $(1.2.2\mathrm{a})_6$, and the results from Chapter 4 allow us to recast (1.2.2) as an evolution problem for f of the form

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \Psi(f(t)), \quad t \geq 0, \qquad f(0) = f_0,$$

see (6.2.4). Using the properties established for the hydrodynamic double-layer potential operator in Chapter 4, together with the mapping properties of the operator $B^{p,q}_{n,m}$, we are able to show that the evolution operator Ψ is well-defined for functions in $\mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$. Due to the low regularity assumed for f, the evolution problem (6.2.4) is not only nonlocal, but must also be treated as fully nonlinear. This is evident, for example, when considering the term $H(f)$, defined in (6.2.1) below, which depends fully nonlinearly on f and appears in the definition of the nonlinearities in the evolution problem (6.2.4), as seen through (6.2.1)–(6.2.6).

As we want to use the abstract parabolic theory provided in [60], we need to show that the evolution operator Ψ, interpreted as a mapping from $\mathrm{H}^r(\mathbb{S})$ to $\mathrm{H}^{r-1}(\mathbb{S})$, is continuously differentiable with Lipschitz continuous Fréchet derivative and that the Fréchet derivative $\partial\Psi(f_0)$ generates a strongly continuous analytic semigroup in $\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))$ for all $f_0 \in \mathrm{H}^r(\mathbb{S})$ when viewed as a linear operator on $\mathrm{H}^{r-1}(\mathbb{S})$ with domain $\mathrm{H}^r(\mathbb{S})$, see the discussion preceding Theorem 2.3.9. In fact, we will show that Ψ is smooth, see (6.2.12).

The structure of this chapter is as follows: in Section 6.2, we use the results from Part I to reformulate the two-phase Stokes system as an evolution equation for the function parametrizing the interface between the two fluids and establish the smoothness of the evolution operator Ψ. In Section 6.3, we calculate the Fréchet derivative $\partial\Psi(f_0)$

of Ψ and show that the Fréchet derivative $\partial\Psi(f_0)$ actually generates a strongly continuous analytic semigroup by locally approximating $\partial\Psi(f_0)$ by certain Fourier multipliers related to the Hilbert transform, which themselves generate strongly continuous analytic semigroups, see Proposition 6.3.4 below. Lastly, in Section 6.4, we prove Theorem 1.4.1. The local well-posedness of the two-phase Stokes flow is obtained by applying abstract results for fully nonlinear parabolic problems established by Lunardi in [Chapter 8] [60]. Afterwards, we apply a parameter trick to get the parabolic smoothing property.

6.2. Reformulation of (1.2.2)

To start this section, we define the trace of the velocity (5.1.6), coming from the hydrodynamic single-layer potential solving (5.1.4), in the special case when the functions G and H are given by (5.1.2). Given $f \in \mathrm{H}^3(\mathbb{S})$, we recall that

$$H = H(f) = (H_1(f), H_2(f)) = ((\omega(f))^{-1} - 1, f'(\omega(f))^{-1}) \in \mathrm{H}^2(\mathbb{S})^2 \tag{6.2.1}$$

and define $\mathcal{V} := \mathcal{V}(f) := (\mathcal{V}_1(f), \mathcal{V}_2(f))$ by setting

$$\mathcal{V}(f) := \mathcal{V}(f)[(G(f), H(f))],$$

see (5.1.6), that is

$$\begin{aligned} \mathcal{V}_1(f) &:= -\frac{\Theta}{4}\mathcal{W}_1(f) - \frac{\sigma}{4}\mathcal{W}_2(f), \\ \mathcal{V}_2(f) &:= \frac{\Theta}{4}(\mathcal{W}_3(f) + \langle f\rangle \ln 4) - \frac{\sigma}{4}\mathcal{W}_4(f), \end{aligned} \tag{6.2.2}$$

where

$$\begin{aligned} \mathcal{W}_1(f) &:= (B_0 + B_6)(f)[ff'] + B_5(f)[f], \\ \mathcal{W}_2(f) &:= (B_1 - 2B_4)(f)[H_1(f) - f'H_2(f)] + (2B_2 + B_3)(f)[f'H_1(f)] \\ &\quad + B_3(f)[H_2(f)], \\ \mathcal{W}_3(f) &:= (B_0 - B_6)(f)[f] + B_5(f)[ff'], \\ \mathcal{W}_4(f) &:= B_1(f)[H_2(f) - f'H_1(f)] + B_3(f)[H_1(f) - f'H_2(f)] \\ &\quad + 2B_4(f)[f'H_1(f) + H_2(f)]. \end{aligned} \tag{6.2.3}$$

We recall that $\mathcal{V}(f) \in \mathrm{H}^2(\mathbb{S})^2$.

Using Theorem 5.3.1, we can now reformulate problem (1.2.2) as an evolution problem for f. Indeed, if $(f, v^\pm, q^\pm, c_\Gamma)$ is a solution to (1.2.2) as defined in Theorem 1.4.1 (i), the kinematic boundary condition $(1.2.2\mathrm{a})_6$, according to which the normal velocity satisfies

$$V_n(t) = \tilde{\nu}_2(t)\frac{\mathrm{d}f}{\mathrm{d}t}(t) \circ \Xi^{-1}(t) \qquad \text{on } \Gamma(t),$$

where $\Xi(t) : \mathbb{S} \to \Gamma(t)$ is given by $\Xi(t) := (\mathrm{id}_{\mathbb{S}}, f(t))$, together with (5.3.2) implies that f solves the evolution problem

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \Psi(f(t)), \quad t \geq 0, \qquad f(0) = f_0, \tag{6.2.4}$$

where

$$\Psi(f) := \frac{2}{\mu^+ + \mu^-}(-f', 1) \cdot \beta(f), \tag{6.2.5}$$

and $\beta(f) := \beta(f)[(G,H)] := (\beta_1(f)[(G,H)], \beta_2(f)[(G,H)]) \in \mathrm{H}^2(\mathbb{S})^2$ is the unique solution to

$$(1 + 2a_\mu \mathbb{D}(f))[\beta(f)] = \mathcal{V}(f), \tag{6.2.6}$$

see (5.1.7)–(5.1.8), and (5.3.2).

To formulate problem (6.2.4) within a suitable functional-analytic framework, we fix $r \in (3/2, 2)$ and infer by arguing similarly as in [64, Lemma 3.5], that the mapping H introduced in (6.2.1) satisfies

$$[f \mapsto H(f)] \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2). \tag{6.2.7}$$

Moreover, we infer from Corollary A.1.13 that for given $r \in (3/2, 2)$, $f \in \mathrm{H}^r(\mathbb{S})$, and n, m, p, $q \in \mathbb{N}_0$ with $1 \le p \le n + q + 1$, we have

$$\begin{aligned} [f \mapsto B^{0,q}_{n,m}(f)] &\in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))), \\ [f \mapsto B_0(f)],\ [f \mapsto B^{p,q}_{n,m}(f)] &\in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))). \end{aligned} \tag{6.2.8}$$

It follows immediately from the definition (4.1.1) of the hydrodynamic double-layer potential operator $\mathbb{D}(f)$, (4.1.7), and (6.2.8) that

$$[f \mapsto \mathbb{D}(f)] \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})^2)). \tag{6.2.9}$$

The property (6.2.7), together with the relations (4.1.7), (6.2.2)–(6.2.3), and (6.2.8), shows that the mapping $\mathcal{V}$ defined in (6.2.2) satisfies

$$[f \mapsto \mathcal{V}(f)] \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2). \tag{6.2.10}$$

Moreover, H and $\mathcal{V}$ both map bounded sets in $\mathrm{H}^r(\mathbb{S})$ to bounded sets in $\mathrm{H}^{r-1}(\mathbb{S})^2$. Indeed, for H this follows by a straightforward computation, while the corresponding property for $\mathcal{V}$ relies additionally on Lemma 4.3.1 (i) and Lemma A.1.12.

Recalling (6.2.9) and using the smoothness of the mapping which associates to an invertible bounded operator in $\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})^2)$ its inverse operator, we conclude together with (6.2.10) and Theorem 4.3.2 that the mapping β, defined in (6.2.6), satisfies

$$[f \mapsto \beta(f)] \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2). \tag{6.2.11}$$

Moreover, β inherits the property to map bounded sets in $\mathrm{H}^r(\mathbb{S})$ to bounded sets in $\mathrm{H}^{r-1}(\mathbb{S})^2$ from $\mathcal{V}$ in view of the estimate (4.3.4). We now directly infer from (6.2.5) and (6.2.11) that

$$[f \mapsto \Psi(f)] \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})), \tag{6.2.12}$$

and that Ψ maps bounded sets in $\mathrm{H}^r(\mathbb{S})$ to bounded sets in $\mathrm{H}^{r-1}(\mathbb{S})$.

6.3. Analysis of the evolution operator

Having established the smoothness of Ψ in (6.2.12), the only ingredient missing to apply Theorem 2.3.9 to our evolution operator Ψ is that the Fréchet derivative $\partial\Psi(f_0)$ generates a strongly continuous analytic semigroup in $\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))$. This property is exactly the result of the following proposition where we use the notation introduced in Section 2.3.

Proposition 6.3.1. *Given $f_0 \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, we have*

$$\partial\Psi(f_0) \in \mathcal{H}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})). \tag{6.3.1}$$

The proof of this proposition needs some preparation which is done in the following two subsections. We start by introducing for given integers n, $m \in \mathbb{N}_0$, and Lipschitz continuous mappings

$$\mathbf{a} = (a_1, \dots, a_m) : \mathbb{R} \to \mathbb{R}^m \qquad \text{and} \qquad \mathbf{b} = (b_1, \dots, b_n) : \mathbb{R} \to \mathbb{R}^n$$

the singular integral operator

$$C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi](\xi) := \frac{1}{\pi} \mathrm{PV}\int_{-\pi}^{\pi} \frac{\prod_{i=1}^{n} \frac{\delta_{[\xi,s]} b_i}{s}}{\prod_{i=1}^{m} \left[1 + \left(\frac{\delta_{[\xi,s]} a_i}{s}\right)^2\right]} \frac{\varphi(\xi - s)}{s} \, \mathrm{d}s, \tag{6.3.2}$$

where $\varphi \in \mathrm{L}^2(\mathbb{S})$, $\xi \in \mathbb{R}$, and we use the notation introduced in (3.2.20) and (4.1.4). We work with this operator instead of $B^{p,q}_{n,m}$ since they retain the singular part of the singular integral operator $B^{0,q}_{n,m}$ in a suitable sense while being easier to handle. Similarly as for the operator $B^{p,q}_{n,m}$, the function $C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi]$ is 2π-periodic whenever all the coordinate functions $\mathbf{a}$ and $\mathbf{b}$ are 2π-periodic. If all coordinate functions of $\mathbf{a}$ and $\mathbf{b}$ are identical to a given function $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$, we set

$$C^0_{n,m}(f) := C_{n,m}(f, \dots, f)[f, \dots, f, \cdot]. \tag{6.3.3}$$

We collect some mapping properties for the operator $C_{n,m}$ which we need throughout this chapter in the following lemma.

Lemma 6.3.2. *Let n, $m \in \mathbb{N}_0$ be given.*

(i) *Given $r \in (3/2, 2)$ and $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^m$, there exists a constant $C > 0$ that depends only on n, m, r, and $\|\mathbf{a}\|_{\mathrm{H}^r}$ such that for all $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi]\|_{\mathrm{H}^{r-1}} \le C \|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{n} \|b_i'\|_{\mathrm{H}^{r-1}}. \tag{6.3.4}$$

(ii) *Let $n \in \mathbb{N}$, $3/2 < r' < r < 2$, and $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^m$ be given. Then, there exists a constant $C > 0$ that depends only on n, m, r, r', and $\|\mathbf{a}\|_{\mathrm{H}^r}$ such that for all $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\begin{aligned} &\|C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi] - \varphi C_{n-1,m}(\mathbf{a})[(b_2, \dots, b_n), b_1']\|_{\mathrm{H}^{r-1}} \\ &\qquad \le C \|b_1\|_{\mathrm{H}^{r'}} \|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{n} \|b_i\|_{\mathrm{H}^r}. \end{aligned} \tag{6.3.5}$$

The results of this lemma are established in Lemma A.1.1.

6.3.1. The Fréchet derivative

We start by computing the Fréchet derivative $\partial\Psi(f_0)$ and identifying its highest-order terms. For the remainder of this section, we fix $f_0 \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, and $r' \in (3/2, r)$. Then, it is straightforward to infer from (6.2.5) that

$$\partial\Psi(f_0)[f] = \frac{2}{\mu^+ + \mu^-}\big((-f_0', 1) \cdot \partial\beta(f_0)[f] - f'\beta_1(f_0)\big), \qquad f \in \mathrm{H}^r(\mathbb{S}). \tag{6.3.6}$$

Moreover, differentiating the relation (6.2.6), we get

$$(1 + 2a_\mu \mathbb{D}(f_0))\,[\partial\beta(f_0)[f]] = \partial\mathcal{V}(f_0)[f] - 2a_\mu \partial\mathbb{D}(f_0)[f][\beta(f_0)], \qquad f \in \mathrm{H}^r(\mathbb{S}). \tag{6.3.7}$$

In order to handle these derivatives in an efficient way, we infer from (A.1.48) that

$$\|B^{p,q}_{n,m}(f_0)[\varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r'-1}}, \qquad 1 \leq p \leq n+q+1, \quad \varphi \in \mathrm{H}^{r-1}(\mathbb{S}). \tag{6.3.8}$$

Furthermore, in view of (A.1.2) and Lemma A.1.3, we also have for $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ that

$$B^{0,q}_{n,m}(f_0)[\varphi] = C^0_{n+q,m}(f_0)[\varphi] + R[\varphi], \qquad \|R[\varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{6.3.9}$$

The constant C in (6.3.8) and (6.3.9) depends only on $\|f_0\|_{\mathrm{H}^r}$ and on n, m, p, q. Additionally, we define for n, m, p, $q \in \mathbb{N}_0$ with $p \leq n+q+2$ the operators

$$\begin{aligned} &B^{0,q,1}_{n,m} : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))), \\ &B^{p,q,1}_{n,m} : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))), \qquad 1 \leq p \leq n+q+2, \end{aligned} \tag{6.3.10a}$$

by setting for f_0, $f \in \mathrm{H}^r(\mathbb{S})$,

$$B^{p,q,1}_{n,m}(f_0)[f] := B^{p,q+1}_{n,m}(f_0,\ldots,f_0|f_0,\ldots,f_0)[f_0,\ldots,f_0,f,\cdot] \in \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})), \tag{6.3.10b}$$

see also (A.1.59). For some fixed $\varphi_0 \in \mathrm{H}^{r-1}(\mathbb{S})$ we again infer from (A.1.2) and Lemma A.1.3 that

$$B^{0,q,1}_{n,m}(f_0)[f][\varphi_0] = C_{n+q+1,m}(f_0,\ldots,f_0)[f_0,\ldots,f_0,f,\varphi_0] + \bar{R}_1[f], \tag{6.3.11}$$

and from (A.1.51) that

$$B^{p,q,1}_{n,m}(f_0)[f][\varphi_0] = \bar{R}_2[f], \qquad 1 \leq p \leq n+q+2, \tag{6.3.12}$$

where, for a positive constant C depending only on $\|f_0\|_{\mathrm{H}^r}$, $\|\varphi_0\|_{\mathrm{H}^{r-1}}$, and n, m, p, q, we have

$$\|\bar{R}_1[f]\|_{\mathrm{H}^{r-1}} + \|\bar{R}_2[f]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \qquad f \in \mathrm{H}^r(\mathbb{S}). \tag{6.3.13}$$

From (6.3.2) and (6.3.3) we next deduce the identity

$$C^0_{n,m}(f_0) + C^0_{n+2,m}(f_0) = C^0_{n,m-1}(f_0), \qquad m \geq 1, \tag{6.3.14}$$

and, together with (4.1.7), (6.3.8), (6.3.9), we find

$$\begin{aligned} B_1(f_0)[\varphi] &= (C^0_{0,2} + C^0_{2,2})(f_0)[\varphi] + R_1[\varphi], \\ B_2(f_0)[\varphi] &= (C^0_{1,2} + C^0_{3,2})(f_0)[\varphi] + R_2[\varphi], \\ B_3(f_0)[\varphi] &= (C^0_{1,2} - C^0_{3,2})(f_0)[\varphi] + R_3[\varphi], \\ B_4(f_0)[\varphi] &= C^0_{2,2}(f_0)[\varphi] + R_4[\varphi], \end{aligned} \tag{6.3.15}$$

where, with a positive constant C depending only on $\|f_0\|_{\mathrm{H}^r}$, it holds that

$$\|R_i[\varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r'-1}}, \qquad \varphi \in \mathrm{H}^{r-1}(\mathbb{S}), \quad 1 \leq i \leq 4. \tag{6.3.16}$$

Moreover, from (4.1.7) and (6.3.8), we may infer

$$\|B_5(f_0)[\varphi]\|_{\mathrm{H}^{r-1}} + \|B_6(f_0)[\varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r'-1}}, \qquad \varphi \in \mathrm{H}^{r-1}(\mathbb{S}). \tag{6.3.17}$$

Recalling (4.1.1), it follows from (6.3.15) that

$$\mathbb{D}(f_0)[\beta] = -\begin{bmatrix} C^0_{1,2} & C^0_{2,2} \\ C^0_{2,2} & C^0_{3,2} \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} C^0_{0,2} & C^0_{1,2} \\ C^0_{1,2} & C^0_{2,2} \end{bmatrix}\begin{bmatrix} f_0'\beta_1 \\ f_0'\beta_2 \end{bmatrix} + R_5[\beta], \tag{6.3.18}$$

where by (6.3.16), for a positive constant C depending only on $\|f_0\|_{\mathrm{H}^r}$, we have

$$\|R_5[\beta]\|_{\mathrm{H}^{r-1}} \leq C\|\beta\|_{\mathrm{H}^{r'-1}}, \qquad \beta \in \mathrm{H}^{r-1}(\mathbb{S})^2. \tag{6.3.19}$$

Using Lemma A.1.14 and (6.3.10), we may express the Fréchet derivative $\partial B^{p,q}_{n,m}(f_0)$ in the compact form

$$\begin{aligned}\partial B^{p,q}_{n,m}(f_0)[f] &= n(B^{p,q,1}_{n-1,m}(f_0) - B^{p+2,q,1}_{n+1,m}(f_0))[f] \\ &\quad + 2m(B^{p+2,q,1}_{n+3,m+1}(f_0) - B^{p,q,1}_{n+1,m+1}(f_0))[f] \\ &\quad + qB^{p,q-1,1}_{n,m}(f_0)[f],\end{aligned} \tag{6.3.20}$$

where terms with negative indices have to be neglected. Moreover, we can infer from Lemma A.1.15 and (6.3.12)–(6.3.13) that

$$\|\partial B_0(f_0)[f][\varphi_0]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \qquad f \in \mathrm{H}^r(\mathbb{S}), \quad \varphi_0 \in \mathrm{H}^{r-1}(\mathbb{S}), \tag{6.3.21}$$

where the constant C depends only on $\|f_0\|_{\mathrm{H}^r}$ and $\|\varphi_0\|_{\mathrm{H}^{r-1}}$. Recalling (4.1.7), we conclude from (6.3.5), (6.3.10)–(6.3.14), and (6.3.20) that

$$\begin{aligned}\partial B_1(f_0)[f][\varphi_0] &= -2\varphi_0(C^0_{1,3} + C^0_{3,3})(f_0)[f'] + R_6[f], \\ \partial B_2(f_0)[f][\varphi_0] &= \varphi_0(C^0_{0,3} - C^0_{4,3})(f_0)[f'] + R_7[f], \\ \partial B_3(f_0)[f][\varphi_0] &= \varphi_0(C^0_{0,3} - 6C^0_{2,3} + C^0_{4,3})(f_0)[f'] + R_8[f], \\ \partial B_4(f_0)[f][\varphi_0] &= 2\varphi_0(C^0_{1,3} - C^0_{3,3})(f_0)[f'] + R_9[f],\end{aligned} \tag{6.3.22}$$

for $\varphi_0 \in \mathrm{H}^{r-1}(\mathbb{S})$ and a positive constant C depending only on $\|f_0\|_{\mathrm{H}^r}$ and $\|\varphi_0\|_{\mathrm{H}^{r-1}}$ satisfying

$$\|R_i[f]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \qquad f \in \mathrm{H}^r(\mathbb{S}), \quad 6 \leq i \leq 9. \tag{6.3.23}$$

Furthermore, (4.1.7) together with (6.2.8) yields

$$\|\partial B_5(f_0)[f][\varphi_0]\|_{\mathrm{H}^{r-1}} + \|\partial B_6(f_0)[f][\varphi_0]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \qquad f \in \mathrm{H}^r(\mathbb{S}), \tag{6.3.24}$$

for $\varphi_0 \in \mathrm{H}^{r-1}(\mathbb{S})$ and a positive constant C depending only on $\|f_0\|_{\mathrm{H}^r}$ and $\|\varphi_0\|_{\mathrm{H}^{r-1}}$.
Similarly as in [64, Lemma 3.5], the Fréchet derivative of the coordinate H_i, $1 \leq i \leq 2$, of the mapping H, introduced in (6.2.1), is given by

$$\partial H_i(f_0) = a_i(f_0)\frac{\mathrm{d}}{\mathrm{d}\xi} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})), \qquad 1 \leq i \leq 2, \tag{6.3.25}$$

with $a_i(f_0) \in \mathrm{H}^{r-1}(\mathbb{S})$, $1 \leq i \leq 2$, defined by

$$a_1(f_0) := -\frac{f_0'}{(1+f_0'^2)^{3/2}} \qquad \text{and} \qquad a_2(f_0) := \frac{1}{(1+f_0'^2)^{3/2}}. \tag{6.3.26}$$

We set for the sake of brevity

$$a_i := a_i(f_0) \qquad \text{and} \qquad H_i := H_i(f_0), \qquad 1 \leq i \leq 2. \tag{6.3.27}$$

Using (6.3.20), (A.1.65), and Lemma A.1.16 to differentiate (6.2.3), we get, after applying (6.3.15)–(6.3.17), (6.3.21), and (6.3.22)–(6.3.24), that

$$\partial\mathcal{W}_1(f_0)[f] + \partial\mathcal{W}_3(f_0)[f] = R_{10}[f], \tag{6.3.28}$$

and

$$\begin{aligned}
\partial\mathcal{W}_2(f_0)[f] &= (C^0_{0,2} - C^0_{2,2})(f_0)[(a_1 - H_2 - f_0'a_2)f'] \\
&\quad + C^0_{1,2}(f_0)[(3(H_1 + f_0'a_1) + a_2)f'] \\
&\quad + C^0_{3,2}(f_0)[(H_1 + f_0'a_1 - a_2)f'] \\
&\quad + H_1(3f_0'C^0_{0,3} - 6C^0_{1,3} - 6f_0'C^0_{2,3} + 2C^0_{3,3} - f_0'C^0_{4,3})(f_0)[f'] \\
&\quad + H_2(C^0_{0,3} + 6f_0'C^0_{1,3} - 6C^0_{2,3} - 2f_0'C^0_{3,3} + C^0_{4,3})(f_0)[f'] \\
&\quad + R_{11}[f], \\
\partial\mathcal{W}_4(f_0)[f] &= (C^0_{1,2} - C^0_{3,2})(f_0)[(a_1 - H_2 - f_0'a_2)f'] \\
&\quad - C^0_{0,2}(f_0)[(H_1 + f_0'a_1 - a_2)f'] \\
&\quad + C^0_{2,2}(f_0)[(H_1 + f_0'a_1 + 3a_2)f'] \\
&\quad + H_1(C^0_{0,3} + 6f_0'C^0_{1,3} - 6C^0_{2,3} - 2f_0'C^0_{3,3} + C^0_{4,3})(f_0)[f'] \\
&\quad - H_2(f_0'C^0_{0,3} - 2C^0_{1,3} - 6f_0'C^0_{2,3} + 6C^0_{3,3} + f_0'C^0_{4,3})(f_0)[f'] \\
&\quad + R_{12}[f],
\end{aligned} \tag{6.3.29}$$

where, for a positive constant C depending only on $\|f_0\|_{\mathrm{H}^r}$, it holds that

$$\|R_i[f]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \qquad f \in \mathrm{H}^r(\mathbb{S}), \quad 10 \leq i \leq 12. \tag{6.3.30}$$

Lastly, we set

$$\beta_0 := (\beta_{0,1}, \beta_{0,2}) := \beta(f_0) \in \mathrm{H}^{r-1}(\mathbb{S})^2, \tag{6.3.31}$$

and use (6.3.20) to differentiate (4.1.1). After applying (6.3.15)–(6.3.16) as well as (6.3.22)–(6.3.23), we arrive at

$$\begin{aligned}
&\partial\mathbb{D}(f_0)[f][\beta_0] \\
&= \begin{bmatrix} C^0_{0,2}(f_0) & C^0_{1,2}(f_0) \\ C^0_{1,2}(f_0) & C^0_{2,2}(f_0) \end{bmatrix} \begin{bmatrix} f'\beta_{0,1} \\ f'\beta_{0,2} \end{bmatrix} \\
&\quad - \begin{pmatrix} \beta_{0,1}(C^0_{0,3} - 3C^0_{2,3})(f_0)[f'] + 2\beta_{0,2}(C^0_{1,3} - C^0_{3,3})(f_0)[f'] \\ 2\beta_{0,1}(C^0_{1,3} - C^0_{3,3})(f_0)[f'] + \beta_{0,2}(3C^0_{2,3} - C^0_{4,3})(f_0)[f'] \end{pmatrix} \\
&\quad + \begin{pmatrix} -4f_0'\beta_{0,1}C^0_{1,3}(f_0)[f'] + f_0'\beta_{0,2}(C^0_{0,3} - 3C^0_{2,3})(f_0)[f'] \\ f_0'\beta_{0,1}(C^0_{0,3} - 3C^0_{2,3})(f_0)[f'] + 2f_0'\beta_{0,2}(C^0_{1,3} - C^0_{3,3})(f_0)[f'] \end{pmatrix} + R_{12}[f],
\end{aligned} \tag{6.3.32}$$

where, for some constant C depending only on $\|f_0\|_{\mathrm{H}^r}$ and $\|\beta_0\|_{\mathrm{H}^{r-1}}$, we have

$$\|R_{12}[f]\|_{\mathrm{H}^{r-1}} \leq C\|f\|_{\mathrm{H}^{r'}}, \qquad f \in \mathrm{H}^r(\mathbb{S}). \tag{6.3.33}$$

6.3.2. Localization of the Fréchet derivative

In this section we establish Proposition 6.3.1. This is achieved by adopting the strategy employed in [65], see also [27, 30, 62]. The main step consists of locally approximating the Fréchet derivative $\partial\Psi(f_0)$ by certain Fourier multipliers, which themselves generate strongly continuous analytic semigroups, see Proposition 6.3.4.

To start, for each $\varepsilon \in (0,1)$, we fix a set of smooth functions

$$\{\pi^\varepsilon_j : 1 \leq j \leq N\} \subset \mathrm{C}^\infty(\mathbb{S}, [0,1]),$$

where the integer $N = N(\varepsilon)$ is sufficiently large, such that

$$\begin{aligned} &\bullet\ \operatorname{supp}\pi_j^\varepsilon = I_j^\varepsilon + 2\pi\mathbb{Z} \text{ with } I_j^\varepsilon := [\xi_j^\varepsilon - \varepsilon, \xi_j^\varepsilon + \varepsilon] \text{ and } \xi_j^\varepsilon := j\varepsilon;\\ &\bullet\ \sum_{j=1}^N \pi_j^\varepsilon = 1 \text{ in } \mathrm{C}^\infty(\mathbb{S}). \end{aligned} \tag{6.3.34}$$

We call $\{\pi_j^\varepsilon : 1 \le j \le N\}$ an ε-partition of unity. To a given ε-partition of unity, we associate a further set

$$\{\chi_j^\varepsilon : 1 \le j \le N\} \subset \mathrm{C}^\infty(\mathbb{S}, [0,1])$$

with

$$\begin{aligned} &\bullet\ \operatorname{supp}\chi_j^\varepsilon = J_j^\varepsilon + 2\pi\mathbb{Z} \text{ with } J_j^\varepsilon = [\xi_j^\varepsilon - 2\varepsilon, \xi_j^\varepsilon + 2\varepsilon];\\ &\bullet\ \chi_j^\varepsilon = 1 \text{ on } \operatorname{supp}\pi_j^\varepsilon. \end{aligned} \tag{6.3.35}$$

We associate to such an ε-partition of unity a new norm on $\mathrm{H}^s(\mathbb{S})$, $s \ge 0$, via the following lemma.

Lemma 6.3.3. *Given $\varepsilon \in (0,1)$, $s \ge 0$, and an ε-partition of unity $\{\pi_j^\varepsilon : 1 \le j \le N\}$, the mapping*

$$\Big[f \mapsto \sum_{j=1}^N \|\pi_j^\varepsilon f\|_{\mathrm{H}^s}\Big] : \mathrm{H}^s(\mathbb{S}) \to \mathbb{R}$$

is equivalent to the standard norm on $\mathrm{H}^s(\mathbb{S})$ defined in (2.4.3) in the sense that there exists a constant $c = c(\varepsilon, s) \in (0,1)$ such that we have

$$c\|f\|_{\mathrm{H}^s} \le \sum_{j=1}^N \|\pi_j^\varepsilon f\|_{\mathrm{H}^s} \le c^{-1}\|f\|_{\mathrm{H}^s}, \qquad f \in \mathrm{H}^s(\mathbb{S}). \tag{6.3.36}$$

Proof. Let $s = [s] + \{s\}$ where $[s] \in \mathbb{N}_0$ and $\{s\} \in (0,1)$. Then, given $\alpha \in \mathrm{C}^{1+[s]}(\mathbb{S})$, we have

$$\|\alpha f\|_{\mathrm{H}^s} \le C\|\alpha\|_{\mathrm{C}^{1+[s]}}\|f\|_{\mathrm{H}^s}, \qquad f \in \mathrm{H}^s(\mathbb{S}),$$

see, e.g., [93, Section 2.8.2]. It follows that

$$\sum_{j=1}^N \|\pi_j^\varepsilon f\|_{\mathrm{H}^s} \le C\Big(\sum_{j=1}^N \|\pi_j^\varepsilon\|_{\mathrm{C}^{1+[s]}}\Big)\|f\|_{\mathrm{H}^s}.$$

The reverse inequality follows from a simple application of the triangle inequality of the $\mathrm{H}^s(\mathbb{S})$ norm. □

Similarly to the non-periodic case [65], we define $\Phi : [0,1] \to \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S}))$ as the continuous path given by

$$\Phi(\tau)[f] := \frac{2}{\mu^+ + \mu^-}\big(-\tau f'\beta_{0,1} + (-\tau f_0', 1)\cdot \mathcal{B}(\tau)[f]\big), \qquad \tau \in [0,1], \tag{6.3.37}$$

where $\beta_0 = (\beta_{0,1}, \beta_{0,2})$ was introduced in (6.3.31), $f \in \mathrm{H}^r(\mathbb{S})$, and we define the mapping $\mathcal{B}(\tau)[f] \in \mathrm{H}^{r-1}(\mathbb{S})^2$, $\tau \in [0,1]$, as the unique solution to

$$\big(1 + 2\tau a_\mu \mathbb{D}(f_0)\big)[\mathcal{B}(\tau)[f]] = \mathscr{V}(\tau)[f] - 2\tau a_\mu \partial\mathbb{D}(f_0)[f][\beta_0], \tag{6.3.38}$$

which exists due to Theorem 4.3.2. Here, $\mathscr{V} := (\mathscr{V}_1, \mathscr{V}_2) : [0,1] \to \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2)$ denotes the continuous mapping

$$\begin{aligned} \mathscr{V}_1(\tau) &:= -\tau\frac{\Theta}{4}\partial\mathcal{W}_1(f_0) - \frac{\sigma}{4}\mathcal{W}_2(\tau f_0), \\ \mathscr{V}_2(\tau) &:= \tau\frac{\Theta}{4}(\partial\mathcal{W}_3(f_0) + \ln(4)\langle\cdot\rangle) - \frac{\sigma}{4}\mathcal{W}_4(\tau f_0) \end{aligned} \tag{6.3.39}$$

for $\tau \in [0,1]$. Note that $\mathcal{B} : [0,1] \to \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2)$ is continuous as well and, using Theorem 4.3.2, (6.2.9), and (6.2.10), we may find a constant $C > 0$ such that

$$\|\mathcal{B}(\tau)[f]\|_{\mathrm{H}^{r-1}} \le C\|f\|_{\mathrm{H}^r}, \qquad f \in \mathrm{H}^r(\mathbb{S}), \quad \tau \in [0,1]. \tag{6.3.40}$$

Moreover, we observe that $\Phi(1) = \partial\Psi(f_0)$ and that

$$\mathscr{V}(0) = \Big(0, -\frac{\sigma}{4}\mathsf{H} \circ \frac{\mathrm{d}}{\mathrm{d}\xi}\Big)^{\top},$$

where $\mathsf{H} = B^{0,0}_{0,0}$ is the periodic Hilbert transform, c.f. (4.1.5). It follows that

$$\Phi(0) = -\frac{\sigma}{2(\mu^+ + \mu^-)}\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2}, \tag{6.3.41}$$

since the symbol of the periodic Hilbert transform is given by $(-i\,\mathrm{sign}(k))_{k\in\mathbb{Z}}$. The invertibility of the operator $\lambda - \Phi(0)$ for $\lambda > 0$ will later be used to conclude the invertibility of $\lambda - \Phi(1)$ for $\mathrm{Re}\,\lambda$ large enough.

In Proposition 6.3.4, we approximate not only $\partial\Psi(f_0) = \Phi(1)$ by Fourier multipliers but also the entire path $\Phi(\tau)$, $\tau \in [0,1]$.

Proposition 6.3.4. *Given $\gamma > 0$, there exist $\varepsilon \in (0,1)$, a constant $K = K(\varepsilon) > 0$, an ε-partition of unity $\{\pi_j^\varepsilon : 1 \le j \le N\}$, and bounded operators*

$$\mathbb{A}_{j,\tau} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})), \qquad 1 \le j \le N, \quad \tau \in [0,1],$$

such that

$$\big\|\pi_j^\varepsilon\Phi(\tau)[f] - \mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\big\|_{\mathrm{H}^{r-1}} \le \gamma\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \tag{6.3.42}$$

for all $1 \le j \le N$, $f \in \mathrm{H}^r(\mathbb{S})$, and $\tau \in [0,1]$. The operators $\mathbb{A}_{j,\tau}$ are defined by

$$\mathbb{A}_{j,\tau} := -\alpha_\tau(\xi_j^\varepsilon)\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2} + \vartheta_\tau(\xi_j^\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad 1 \le j \le N, \quad \tau \in [0,1], \tag{6.3.43}$$

where the functions α_τ and ϑ_τ are given, recalling (3.1.1) *and* (6.3.31)*, by*

$$\alpha_\tau := \frac{\sigma}{2(\mu^+ + \mu^-)}(\omega(\tau f_0))^{-1} \qquad \text{and} \qquad \vartheta_\tau := -\frac{2\tau}{\mu^+ + \mu^-}\beta_{0,1}. \tag{6.3.44}$$

The proof of this proposition uses the following commutator-type result for the operators $C_{n,m}$.

Lemma 6.3.5. *Given n, $m \in \mathbb{N}_0$ and a, $f \in \mathrm{C}^1(\mathbb{S})$, there exists a constant $C > 0$ that depends only on n, m, and $\|(a,f)\|_{\mathrm{C}^1}$ such that for all $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$\big\|aC^0_{n,m}(f)[\varphi] - C^0_{n,m}(f)[a\varphi]\big\|_{\mathrm{H}^1} \le C\|\varphi\|_2. \tag{6.3.45}$$

The main ingredient in the proof of Proposition 6.3.4 is the next lemma, which describes how to “freeze the kernel” of the operators $C_{n,m}$.

Lemma 6.3.6. *Let $n, m \in \mathbb{N}_0$, $r' \in (3/2, r)$, $f \in \mathrm{H}^r(\mathbb{S})$, $a, b \in \mathrm{H}^{r-1}(\mathbb{S})$, and $\eta > 0$ be given. Then, for any sufficiently small $\varepsilon \in (0,1)$, there exists a constant $K > 0$ that depends only on ε, n, m, $\|f\|_{\mathrm{H}^r}$, and $\|(a,b)\|_{\mathrm{H}^{r-1}}$ such that for all $1 \le j \le N$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\left\| \pi_j^\varepsilon a C_{n,m}^0(f)[b\varphi] - \frac{abf'^m}{(1+f'^2)^m}(\xi_j^\varepsilon)\mathsf{H}[\pi_j^\varepsilon \varphi] \right\|_{\mathrm{H}^{r-1}} \le \eta \|\pi_j^\varepsilon \varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \quad (6.3.46)$$

We present the proofs of Lemma 6.3.5 and Lemma 6.3.6 in Appendix A.2.

We now prove Proposition 6.3.4, following closely the proof of [65, Theorem 6.2].

Proof of Proposition 6.3.4. Let $\gamma > 0$ be fixed, and let $\varepsilon \in (0,1)$ (to be chosen later on), together with an associated ε-partition of unity $\{\pi_j^\varepsilon : 1 \le j \le N\}$ and the corresponding family $\{\chi_j^\varepsilon : 1 \le j \le N\}$, satisfying (6.3.34) and (6.3.35), respectively, be given. We also recall the Banach algebra-type inequality provided in Lemma 2.4.4, where for a constant $C = C(s) > 0$, we have

$$\|fg\|_{\mathrm{H}^s} \le C(\|f\|_\infty \|g\|_{\mathrm{H}^s} + \|g\|_\infty \|f\|_{\mathrm{H}^s}), \qquad f, g \in \mathrm{H}^s(\mathbb{S}), \quad s \in (1/2, 1). \quad (6.3.47)$$

In what follows, the symbol C is used for constants that are independent of ε, while K denotes constants that depend on ε. In view of definition (6.3.37), we need to approximate the two operators

$$\left[f \mapsto \frac{2}{\mu^+ + \mu^-}(\mathcal{B}_2(\tau)[f] - \tau f_0' \mathcal{B}_1(\tau)[f]) \right] \qquad \text{and} \qquad \left[f \mapsto \frac{-2\tau}{\mu^+ + \mu^-} f' \beta_{0,1} \right]. \quad (6.3.48)$$

We will use the following identity frequently

$$\chi_j^\varepsilon \pi_j^\varepsilon = \pi_j^\varepsilon, \qquad 1 \le j \le N, \quad (6.3.49)$$

which, in particular, leads us to

$$\begin{aligned} &\|a\pi_j^\varepsilon f' - a(\xi_j^\varepsilon)(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} \\ &\le \|\chi_j^\varepsilon (a - a(\xi_j^\varepsilon))(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} + \|(\pi_j^\varepsilon)'\chi_j^\varepsilon a f + a(\xi_j^\varepsilon)(\chi_j^\varepsilon)'\pi_j^\varepsilon f\|_{\mathrm{H}^{r-1}} \\ &\le \|\chi_j^\varepsilon (a - a(\xi_j^\varepsilon))(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} + K\|f\|_{\mathrm{H}^{r'}}, \qquad a \in \mathrm{H}^{r-1}(\mathbb{S}), \quad 1 \le j \le N. \end{aligned} \quad (6.3.50)$$

Using (6.3.47) together with (6.3.49)–(6.3.50) for the second operator in (6.3.48), we get

$$\begin{aligned} &\left\| \frac{-2\tau}{\mu^+ + \mu^-} \pi_j^\varepsilon \beta_{0,1} f' - \vartheta_\tau(\xi_j^\varepsilon)(\pi_j^\varepsilon f)' \right\|_{\mathrm{H}^{r-1}} \\ &\le C\|\chi_j^\varepsilon(\beta_{0,1} - \beta_{0,1}(\xi_j^\varepsilon))\|_\infty \|(\pi_j^\varepsilon f)'\|_{\mathrm{H}^{r-1}} + K\|f\|_{\mathrm{H}^{r'}} \\ &\le \frac{\gamma}{3}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \end{aligned} \quad (6.3.51)$$

for all $1 \le j \le N$ and $f \in \mathrm{H}^r(\mathbb{S})$, provided that ε is small enough. The estimate in the last line of (6.3.51) follows from (6.3.35) in view of $\beta_{0,1} \in \mathrm{H}^{r-1}(\mathbb{S}) \hookrightarrow \mathrm{C}^{r-3/2}(\mathbb{S})$.

Next, we show that

$$\|\pi_j^\varepsilon \mathcal{B}(\tau)[f]\|_{\mathrm{H}^{r-1}} \le C_{\mathcal{B}} \|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}, \quad (6.3.52)$$

with $C_{\mathcal{B}}$ independent of $\varepsilon \in (0,1)$, $\tau \in [0,1]$, $1 \le j \le N$, and $f \in \mathrm{H}^r(\mathbb{S})$. To this end, we infer from (6.3.38), after multiplying this relation by π_j^ε, that

$$\begin{aligned} (1 + 2\tau a_\mu \mathbb{D}(f_0))[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] &= 2\tau a_\mu (\mathbb{D}(f_0)[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] - \pi_j^\varepsilon \mathbb{D}(f_0)[\mathcal{B}(\tau)[f]]) \\ &\quad + \pi_j^\varepsilon \mathscr{V}(\tau)[f] - 2\tau a_\mu \pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0]. \end{aligned} \quad (6.3.53)$$

The terms on the second line of the right-hand side of (6.3.53) can be estimated according to

$$\|\pi_j^\varepsilon \mathscr{V}(\tau)[f]\|_{\mathrm{H}^{r-1}} + \|\pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0]\|_{\mathrm{H}^{r-1}} \leq C\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}. \tag{6.3.54}$$

Indeed, we first recall (6.3.28) and (6.3.30) to infer

$$\|\pi_j^\varepsilon \partial\mathcal{W}_1(f_0)[f]\|_{\mathrm{H}^{r-1}} + \|\pi_j^\varepsilon \partial\mathcal{W}_3(f_0)[f]\|_{\mathrm{H}^{r-1}} + \|\langle f\rangle\|_{\mathrm{H}^{r-1}} \leq K\|f\|_{\mathrm{H}^{r'}}. \tag{6.3.55}$$

The other terms, corresponding to $\partial\mathcal{W}_i(f_0)$, $i = 2, 4$, and to $\partial\mathbb{D}(f_0)[f][\beta_0]$ are estimated using (6.3.29)–(6.3.30), (6.3.32)–(6.3.33), Lemma 6.3.2 (i), and Lemma 6.3.5. Moreover, combining (6.3.18), Lemma 6.3.5 and (6.3.40) (with $r = r'$) we get

$$\big\|\mathbb{D}(f_0)\big[\pi_j^\varepsilon \mathcal{B}(\tau)[f]\big] - \pi_j^\varepsilon \mathbb{D}(f_0)\big[\mathcal{B}(\tau)[f]\big]\big\|_{\mathrm{H}^{r-1}} \leq K\|\mathcal{B}(\tau)[f]\|_{\mathrm{H}^{r'-1}} \leq K\|f\|_{\mathrm{H}^{r'}}, \tag{6.3.56}$$

the desired claim (6.3.52) following now from Theorem 4.3.2 and (6.3.53)–(6.3.56).

With the estimate (6.3.52) at hand, we next approximate the operator

$$[f \mapsto \mathcal{B}(\tau)[f]].$$

To begin, we first define the Fourier multipliers $\mathbb{B}_{j,\tau} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})^2)$, $1 \leq j \leq N$ and $\tau \in [0,1]$, by

$$\mathbb{B}_{j,\tau} := -\frac{\sigma}{4}\begin{pmatrix} a_1(\tau f_0)(\xi_j^\varepsilon)\mathsf{H} \circ (\mathrm{d}/\mathrm{d}\xi) \\ a_2(\tau f_0)(\xi_j^\varepsilon)\mathsf{H} \circ (\mathrm{d}/\mathrm{d}\xi) \end{pmatrix}.$$

Given $\tilde{\gamma} > 0$, we then show that if $\varepsilon \in (0,1)$ is small enough, that

$$\|\pi_j^\varepsilon \mathcal{B}(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{\mathrm{H}^{r-1}} \leq \tilde{\gamma}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \tag{6.3.57}$$

for all $\tau \in [0,1]$, $1 \leq j \leq N$, and $f \in \mathrm{H}^r(\mathbb{S})$. To this end, we multiply (6.3.38) by π_j^ε and arrive at

$$\pi_j^\varepsilon \mathcal{B}(\tau)[f] = \pi_j^\varepsilon \mathscr{V}(\tau)[f] - 2\tau a_\mu \pi_j^\varepsilon\big(\partial\mathbb{D}(f_0)[f][\beta_0] + \mathbb{D}(f_0)\big[\mathcal{B}(\tau)[f]\big]\big). \tag{6.3.58}$$

Recalling (6.3.39) and (6.3.29)–(6.3.30), the estimate (6.3.55) and repeated use of Lemma 6.3.6 in the context of (6.3.29) yield

$$\|\pi_j^\varepsilon \mathscr{V}(\tau)[f] - \mathbb{B}_{j,\tau}[\pi_j^\varepsilon f]\|_{\mathrm{H}^{r-1}} \leq \frac{\tilde{\gamma}}{3}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \tag{6.3.59}$$

for all $\tau \in [0,1]$, $1 \leq j \leq N$, and $f \in \mathrm{H}^r(\mathbb{S})$, provided that ε is small enough. Moreover, using the relations (6.3.32)–(6.3.33), we get, by repeatedly applying Lemma 6.3.6, that

$$\begin{aligned}
&\|\pi_j^\varepsilon \partial\mathbb{D}(f_0)[f][\beta_0]\|_{\mathrm{H}^{r-1}} \\
&\leq \left\|\pi_j^\varepsilon \begin{bmatrix} C_{0,2}^0 & C_{1,2}^0 \\ C_{1,2}^0 & C_{2,2}^0 \end{bmatrix}\begin{bmatrix} f'\beta_{0,1} \\ f'\beta_{0,2} \end{bmatrix}\right. \\
&\qquad\left. - \frac{1}{(1+f_0'(\xi_j^\varepsilon)^2)^2}\begin{pmatrix} 1 & f_0'(\xi_j^\varepsilon) \\ f_0'(\xi_j^\varepsilon) & f_0'(\xi_j^\varepsilon)^2 \end{pmatrix}\begin{pmatrix} \beta_{0,1}(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)'] \\ \beta_{0,2}(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)'] \end{pmatrix}\right\|_{\mathrm{H}^{r-1}} \\
&\quad + \left\| -\pi_j^\varepsilon \begin{pmatrix} \beta_{0,1}(C_{0,3}^0 - 3C_{2,3}^0)(f_0)[f'] + 2\beta_{0,2}(C_{1,3}^0 - C_{3,3}^0)(f_0)[f'] \\ 2\beta_{0,1}(C_{1,3}^0 - C_{3,3}^0)(f_0)[f'] + \beta_{0,2}(3C_{2,3}^0 - C_{4,3}^0)(f_0)[f'] \end{pmatrix}\right. \\
&\qquad + \pi_j^\varepsilon \begin{pmatrix} -4f_0'\beta_{0,1}C_{1,3}^0(f_0)[f'] + f_0'\beta_{0,2}(C_{0,3}^0 - 3C_{2,3}^0)(f_0)[f'] \\ f_0'\beta_{0,1}(C_{0,3}^0 - 3C_{2,3}^0)(f_0)[f'] + 2f_0'\beta_{0,2}(C_{1,3}^0 - C_{3,3}^0)(f_0)[f'] \end{pmatrix} \\
&\qquad \left. + \frac{1}{(1+f_0'(\xi_j^\varepsilon)^2)^2}\begin{pmatrix} 1 & f_0'(\xi_j^\varepsilon) \\ f_0'(\xi_j^\varepsilon) & f_0'(\xi_j^\varepsilon)^2 \end{pmatrix}\begin{pmatrix} \beta_{0,1}(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)'] \\ \beta_{0,2}(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)'] \end{pmatrix}\right\|_{\mathrm{H}^{r-1}} \\
&\quad + K\|f\|_{\mathrm{H}^{r'}} \\
&\leq \frac{\tilde{\gamma}}{6|a_\mu|+1}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}
\end{aligned} \tag{6.3.60}$$

for all $1 \leq j \leq N$ and $f \in \mathrm{H}^r(\mathbb{S})$, provided that ε is small enough. Similarly, we get, by repeatedly applying Lemma 6.3.6 to (6.3.18) and using (6.3.40) and (6.3.52), that

$$\begin{aligned}
&\|\pi_j^\varepsilon \mathbb{D}(f_0)[\mathcal{B}(\tau)[f]]\|_{\mathrm{H}^{r-1}} \\
&\leq \Bigg\|\pi_j^\varepsilon \begin{bmatrix} C_{0,2}^0 & C_{1,2}^0 \\ C_{1,2}^0 & C_{2,2}^0 \end{bmatrix} \begin{bmatrix} f_0'\mathcal{B}(\tau)[f] \\ f_0'\mathcal{B}(\tau)[f] \end{bmatrix} \\
&\qquad - \frac{f_0'(\xi_j^\varepsilon)}{(1+f_0'(\xi_j^\varepsilon)^2)^2} \begin{pmatrix} 1 & f_0'(\xi_j^\varepsilon) \\ f_0'(\xi_j^\varepsilon) & f_0'(\xi_j^\varepsilon)^2 \end{pmatrix} \begin{pmatrix} \mathsf{H}[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] \\ \mathsf{H}[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] \end{pmatrix} \Bigg\|_{\mathrm{H}^{r-1}} \\
&\quad + \Bigg\|\pi_j^\varepsilon \begin{bmatrix} C_{1,2}^0 & C_{2,2}^0 \\ C_{2,2}^0 & C_{3,2}^0 \end{bmatrix} \begin{bmatrix} \mathcal{B}(\tau)[f] \\ \mathcal{B}(\tau)[f] \end{bmatrix} \\
&\qquad - \frac{f_0'(\xi_j^\varepsilon)}{(1+f_0'(\xi_j^\varepsilon)^2)^2} \begin{pmatrix} 1 & f_0'(\xi_j^\varepsilon) \\ f_0'(\xi_j^\varepsilon) & f_0'(\xi_j^\varepsilon)^2 \end{pmatrix} \begin{pmatrix} \mathsf{H}[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] \\ \mathsf{H}[\pi_j^\varepsilon \mathcal{B}(\tau)[f]] \end{pmatrix} \Bigg\|_{\mathrm{H}^{r-1}} \\
&\quad + K\|f\|_{\mathrm{H}^{r'}} \\
&\leq \frac{\tilde{\gamma}}{(6|a_\mu|+1)C_{\mathcal{B}}} \|\pi_j^\varepsilon \mathcal{B}(\tau)[f]\|_{\mathrm{H}^{r-1}} + K\|f\|_{\mathrm{H}^{r'}} \leq \frac{\tilde{\gamma}}{(6|a_\mu|+1)} \|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}
\end{aligned} \tag{6.3.61}$$

for all $\tau \in [0,1]$, $1 \leq j \leq N$ and $f \in \mathrm{H}^r(\mathbb{S})$, provided that ε is small enough. Combining (6.3.59)–(6.3.61) then proves, in view of (6.3.58), the desired claim (6.3.57).

To approximate the first operator in (6.3.48), we first infer from (6.3.57), that for ε small enough, we have

$$\Big\|\pi_j^\varepsilon \mathcal{B}_2(\tau)[f] + \frac{\sigma}{4} a_2(\tau f_0)(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)']\Big\|_{\mathrm{H}^{r-1}} \leq \frac{\gamma(\mu^+ + \mu^-)}{6}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}} \tag{6.3.62}$$

for all $\tau \in [0,1]$, $1 \leq j \leq N$, and $f \in \mathrm{H}^r(\mathbb{S})$. Moreover, using the property (6.3.47), the identity (6.3.49), the regularity of $f_0 \in \mathrm{H}^r(\mathbb{S}) \hookrightarrow \mathrm{C}^{r-1/2}(\mathbb{S})$, and the estimates (6.3.40) (with $r = r'$), (6.3.52), and (6.3.57), we obtain

$$\begin{aligned}
&\Big\|\pi_j^\varepsilon f_0' \mathcal{B}_1(\tau)[f] + \frac{\sigma}{4} f_0'(\xi_j^\varepsilon) a_1(\tau f_0)(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)']\Big\|_{\mathrm{H}^{r-1}} \\
&\leq C\|\chi_j^\varepsilon(f_0' - f_0'(\xi_j^\varepsilon))\|_\infty \|\pi_j^\varepsilon \mathcal{B}_1(\tau)[f]\|_{\mathrm{H}^{r-1}} + K\|\pi_j^\varepsilon \mathcal{B}_1(\tau)[f]\|_\infty \\
&\quad + C\Big\|\pi_j^\varepsilon \mathcal{B}_1(\tau)[f] + \frac{\sigma}{4} a_1(\tau f_0)(\xi_j^\varepsilon)\mathsf{H}[(\pi_j^\varepsilon f)']\Big\|_{\mathrm{H}^{r-1}} \\
&\leq \frac{\gamma(\mu^+ + \mu^-)}{6}\|\pi_j^\varepsilon f\|_{\mathrm{H}^r} + K\|f\|_{\mathrm{H}^{r'}}
\end{aligned} \tag{6.3.63}$$

for all $\tau \in [0,1]$, $1 \leq j \leq N$, and $f \in \mathrm{H}^r(\mathbb{S})$, provided that ε is small enough.

Recalling (6.3.26), we have

$$a_2(\tau f_0) - \tau f_0' a_1(\tau f_0) = (\omega(\tau f_0))^{-1}$$

and the desired claim (6.3.42) follows from (6.3.37), (6.3.51), (6.3.62), and (6.3.63). □

Since $f_0 \in \mathrm{H}^r(\mathbb{S}) \hookrightarrow \mathrm{C}^{r-1/2}(\mathbb{S})$, it follows from (3.1.1) and (6.3.31) that there exists a constant $\eta \in (0,1)$ such that the functions defined in (6.3.44) satisfy

$$\eta \leq \alpha_\tau \leq \eta^{-1} \qquad \text{and} \qquad |\vartheta_\tau| \leq \eta^{-1}, \qquad \tau \in [0,1].$$

Next, we introduce for $\alpha \in [\eta, \eta^{-1}]$ and $\vartheta \in [-\eta^{-1}, \eta^{-1}]$ the Fourier multiplier $\mathbb{A}_{\alpha,\vartheta}$ by

$$\mathbb{A}_{\alpha,\vartheta} := -\alpha\Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big)^{1/2} + \vartheta\frac{\mathrm{d}}{\mathrm{d}\xi} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})).$$

Concerning this operator, we have the following lemma.

Lemma 6.3.7. *Given $\eta \in (0,1)$, $\alpha \in [\eta, \eta^{-1}]$, and $|\vartheta| \le \eta^{-1}$, we have that*

$$\lambda - \mathbb{A}_{\alpha,\vartheta} \in \mathcal{L}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})) \text{ is an isomorphism for all } \operatorname{Re}\lambda \ge 1. \tag{6.3.64}$$

Proof. We first note that the symbol of the operator $\mathbb{A}_{\alpha,\vartheta}$ is given by

$$m_{\alpha,\vartheta}(k) := -\alpha|k| + i\vartheta k, \qquad k \in \mathbb{Z}.$$

Now, let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 1$ be given and define the Fourier multiplier $R(\lambda, \mathbb{A}_{\alpha,\vartheta})$ by

$$R(\lambda, \mathbb{A}_{\alpha,\vartheta})[\varphi](\xi) := \sum_{k\in\mathbb{Z}} \frac{1}{\lambda - m_{\alpha,\vartheta}(k)} \hat{\varphi}(k) e^{ik\xi}, \qquad \varphi \in \mathrm{H}^{r-1}(\mathbb{S}).$$

Then, $R(\lambda, \mathbb{A}_{\alpha,\vartheta}) \in \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))$ since for $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have

$$\begin{aligned}\|R(\lambda, \mathbb{A}_{\alpha,\vartheta})[\varphi]\|_{\mathrm{H}^r}^2 &= 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^r |\mathcal{F}(R(\lambda, \mathbb{A}_{\alpha,\vartheta})[\varphi])(k)|^2 \\ &= 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^r \frac{1}{|\lambda - m_{\alpha,\vartheta}(k)|^2} |\hat{\varphi}(k)|^2 \\ &\le 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^r \frac{1}{\eta^2(1+|k|^2)} |\hat{\varphi}(k)|^2 = \eta^{-2}\|\varphi\|_{\mathrm{H}^{r-1}}^2,\end{aligned}$$

where we used

$$|\lambda - m_{\alpha,\vartheta}(k)|^2 \ge \eta^2(1+|k|^2), \qquad \operatorname{Re}\lambda \ge 1. \tag{6.3.65}$$

It is immediate that $R(\lambda, \mathbb{A}_{\alpha,\vartheta})$ and $\lambda - \mathbb{A}_{\alpha,\vartheta}$, $\operatorname{Re}\lambda \ge 1$, are inverses of each other and the claim follows. ☐

Together with Lemma 6.3.7, the next lemma establishes that

$$\mathbb{A}_{\alpha,\vartheta} \in \mathcal{H}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})).$$

Lemma 6.3.8. *Given $\eta \in (0,1)$, there exists a constant $\kappa_0 = \kappa_0(\eta) \ge 1$ such that for all $\alpha \in [\eta, \eta^{-1}]$, and $|\vartheta| \le \eta^{-1}$, we have*

$$\kappa_0 \|(\lambda - \mathbb{A}_{\alpha,\vartheta})[f]\|_{\mathrm{H}^{r-1}} \ge |\lambda|\,\|f\|_{\mathrm{H}^{r-1}} + \|f\|_{\mathrm{H}^r}, \qquad f \in \mathrm{H}^r(\mathbb{S}), \quad \operatorname{Re}\lambda \ge 1. \tag{6.3.66}$$

Proof. For $\operatorname{Re}\lambda \ge 1$ we have

$$\begin{aligned}\frac{|\lambda|^2}{|\lambda - m_{\alpha,\vartheta}(k)|^2} &= \frac{(\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2}{(\operatorname{Re}\lambda + \alpha|k|)^2 + (\operatorname{Im}\lambda - \vartheta k)^2} \\ &\le 1 + \frac{(\operatorname{Im}\lambda)^2 + (\operatorname{Im}\lambda - 2\vartheta k)^2}{(\operatorname{Re}\lambda + \alpha|k|)^2 + (\operatorname{Im}\lambda - \vartheta k)^2} \\ &= 1 + \frac{2(\operatorname{Im}\lambda - \vartheta k)^2 + 2\vartheta^2 k^2}{(\operatorname{Re}\lambda + \alpha|k|)^2 + (\operatorname{Im}\lambda - \vartheta k)^2} \\ &\le 1 + 2\left(1 + \frac{\vartheta^2}{\alpha^2}\right) \le 5\eta^{-4}.\end{aligned}$$

Therefore, we infer by using also (6.3.65), that

$$\begin{aligned}
\frac{\sqrt{12}}{\eta^2}\|(\lambda-\mathbb{A}_{\alpha,\vartheta})[f]\|_{\mathrm{H}^{r-1}} &\geq \sqrt{2}\big((\eta^{-2}+5\eta^{-4})\|(\lambda-\mathbb{A}_{\alpha,\vartheta})[f]\|^2_{\mathrm{H}^{r-1}}\big)^{1/2}\\
&\geq \sqrt{2}\Big(2\pi\sum_{k\in\mathbb{Z}}(1+|k|^2)^{r-1}(\eta^{-2}+5\eta^{-4})|\lambda-m_{\alpha,\vartheta}(k)|^2|\hat{f}(k)|^2\Big)^{1/2}\\
&\geq \sqrt{2}\Big(2\pi\sum_{k\in\mathbb{Z}}(1+|k|^2)^{r-1}((1+|k|^2)+|\lambda|^2)|\hat{f}(k)|^2\Big)^{1/2}\\
&= \sqrt{2}\big(|\lambda|^2\|f\|^2_{\mathrm{H}^{r-1}}+\|f\|^2_{\mathrm{H}^r}\big)^{1/2} \geq |\lambda|\|f\|_{\mathrm{H}^{r-1}}+\|f\|_{\mathrm{H}^r}, \qquad f\in\mathrm{H}^r(\mathbb{S}),
\end{aligned}$$

and the claim follows. □

By combining (6.3.64) and (6.3.66) with Proposition 6.3.4 and the interpolation property (2.4.4), we are now in a position to prove Proposition 6.3.1.

Proof of Proposition 6.3.1. We begin by fixing $r'\in(3/2,r)$ and letting $\kappa_0\geq 1$ denote the constant from estimate (6.3.66). Applying Proposition 6.3.4 with $\gamma=1/(2\kappa_0)$ we find $\varepsilon\in(0,1)$, an ε-partition of unity $\{\pi_j^\varepsilon : 1\leq j\leq N\}$, a constant $K=K(\varepsilon)>0$, and operators

$$\mathbb{A}_{j,\tau}\in\mathcal{L}(\mathrm{H}^r(\mathbb{S}),\mathrm{H}^{r-1}(\mathbb{S})), \qquad 1\leq j\leq N, \quad \tau\in[0,1],$$

such that for all $\tau\in[0,1]$, $1\leq j\leq N$, and $f\in\mathrm{H}^r(\mathbb{S})$, we have

$$2\kappa_0\big\|\pi_j^\varepsilon\Phi(\tau)[f]-\mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\big\|_{\mathrm{H}^{r-1}}\leq\|\pi_j^\varepsilon f\|_{\mathrm{H}^r}+2\kappa_0K\|f\|_{\mathrm{H}^{r'}}.$$

Additionally, we infer from (6.3.43)–(6.3.44) and (6.3.66) that

$$2\kappa_0\big\|(\lambda-\mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\big\|_{\mathrm{H}^{r-1}}\geq 2|\lambda|\,\|\pi_j^\varepsilon f\|_{\mathrm{H}^{r-1}}+2\|\pi_j^\varepsilon f\|_{\mathrm{H}^r}$$

for all $1\leq j\leq N$, $\tau\in[0,1]$, $\operatorname{Re}\lambda\geq 1$, and $f\in\mathrm{H}^r(\mathbb{S})$.

By applying the reverse triangle inequality together with the above estimates, we obtain

$$\begin{aligned}
&2\kappa_0\big\|\pi_j^\varepsilon(\lambda-\Phi(\tau))[f]\big\|_{\mathrm{H}^{r-1}}\\
&\geq 2\kappa_0\big\|(\lambda-\mathbb{A}_{j,\tau})[\pi_j^\varepsilon f]\big\|_{\mathrm{H}^{r-1}}-2\kappa_0\big\|\pi_j^\varepsilon\Phi(\tau)[f]-\mathbb{A}_{j,\tau}[\pi_j^\varepsilon f]\big\|_{\mathrm{H}^{r-1}}\\
&\geq 2|\lambda|\,\|\pi_j^\varepsilon f\|_{\mathrm{H}^{r-1}}+\|\pi_j^\varepsilon f\|_{\mathrm{H}^r}-2\kappa_0K\|f\|_{\mathrm{H}^{r'}}.
\end{aligned}$$

Summing over $1\leq j\leq N$, and using the interpolation property (2.4.4), the equivalence of norms (6.3.36), and Young's inequality, we find constants $\kappa\geq 1$ and $\omega>1$ such that

$$\kappa\big\|(\lambda-\Phi(\tau))[f]\big\|_{\mathrm{H}^{r-1}}\geq|\lambda|\,\|f\|_{\mathrm{H}^{r-1}}+\|f\|_{\mathrm{H}^r} \tag{6.3.67}$$

for all $\tau\in[0,1]$, $\operatorname{Re}\lambda\geq\omega$, and $f\in\mathrm{H}^r(\mathbb{S})$.

Moreover, by (6.3.41) and (6.3.64), the operator $\omega-\Phi(0)$ is an isomorphism. The method of continuity, cf. Theorem 2.1.4, together with the estimate (6.3.67) imply now that $\omega-\Phi(1)=\omega-\partial\Psi(f_0)$ is an isomorphism too. This property, together with (6.3.67) (with $\tau=1$), shows that $\partial\Psi(f_0)$ generates a strongly continuous analytic semigroup, see Theorem 2.3.8. Thus, the proof is complete. □

6.4. Local well-posedness

In this section we prove the local well-posedness result stated in Theorem 1.4.1.

Proof of Theorem 1.4.1. Well-posedness: To start, we fix $3/2 < r' < r < 2$. The properties (6.2.12) and (6.3.1) enable us to apply the abstract parabolic theory from [60, Chapter 8] to our evolution problem (6.2.4). To be precise, the assumptions of Theorem 2.3.9 are fulfilled which yields, for each $f_0 \in \mathrm{H}^r(\mathbb{S})$, the existence of a positive time $T > 0$ and a solution $f = f(\cdot; f_0)$ to (6.2.4) such that

$$f \in \mathrm{C}\big([0,T], \mathrm{H}^r(\mathbb{S})\big) \cap \mathrm{C}^1\big([0,T], \mathrm{H}^{r-1}(\mathbb{S})\big) \cap \mathrm{C}^\alpha_\alpha\big((0,T], \mathrm{H}^r(\mathbb{S})\big)$$

for all $\alpha \in (0,1)$, where the weighted Hölder space $\mathrm{C}^\alpha_\alpha\big((0,T], \mathrm{H}^{r-1}(\mathbb{S})\big)$ was defined in (2.3.9). Furthermore, this solution is unique within the set

$$\bigcup_{\alpha \in (0,1)} \mathrm{C}^\alpha_\alpha\big((0,T], \mathrm{H}^r(\mathbb{S})\big) \cap \mathrm{C}\big([0,T], \mathrm{H}^r(\mathbb{S})\big) \cap \mathrm{C}^1\big([0,T], \mathrm{H}^{r-1}(\mathbb{S})\big).$$

We improve this uniqueness statement and show that the solution is unique within

$$Y := \mathrm{C}\big([0,T], \mathrm{H}^r(\mathbb{S})\big) \cap \mathrm{C}^1\big([0,T], \mathrm{H}^{r-1}(\mathbb{S})\big).$$

To do so, we choose any solution $f \in Y$ to (6.2.4) and set $\alpha = r - r' \in (0,1)$. Using (2.4.4) with $\theta = 1 - \alpha$, we get

$$\mathrm{H}^{r'}(\mathbb{S}) = \big[\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S})\big]_{1-\alpha},$$

and thus find a constant $C > 0$ such that

$$\|f(t_1) - f(t_2)\|_{\mathrm{H}^{r'}} \le C \|f(t_1) - f(t_2)\|^\alpha_{\mathrm{H}^{r-1}} \|f(t_1) - f(t_2)\|^{1-\alpha}_{\mathrm{H}^r}, \qquad t_1,\, t_2 \in [0,T].$$

Applying the fundamental theorem of calculus, we have

$$\begin{aligned} \|f(t_1) - f(t_2)\|_{\mathrm{H}^{r'}} &\le C \max_{0 \le t \le T} \left(\|f(t)\|^{1-\alpha}_{\mathrm{H}^r} \left\| \frac{\mathrm{d}f}{\mathrm{d}t}(t) \right\|^\alpha_{\mathrm{H}^{r-1}} \right) |t_1 - t_2|^\alpha \\ &\le C |t_1 - t_2|^\alpha, \qquad t_1,\, t_2 \in [0,T]. \end{aligned} \tag{6.4.1}$$

This means that $f \in \mathrm{C}^\alpha\big([0,T], \mathrm{H}^{r'}(\mathbb{S})\big)$, and since

$$\mathrm{C}^\alpha\big([0,T], \mathrm{H}^{r'}(\mathbb{S})\big) \hookrightarrow \mathrm{C}^\alpha_\alpha\big((0,T], \mathrm{H}^{r'}(\mathbb{S})\big)$$

by (2.3.10), the uniqueness statement from Theorem 2.3.9 applied to (6.2.4) with r replaced by r' shows that f coincides with this unique solution in $[0,T]$. Since $f \in Y$ was chosen arbitrarily, the claim follows. This unique solution can then be extended to a maximal existence time $T_+ = T_+(f_0)$, see [60, Section 8.2].

Parabolic smoothing: We make use of a parameter trick dating back to [4] which has already been successfully applied to multiple other problems, see, e.g., [1,31,61,73]. To this end, let $\lambda = (\lambda_1, \lambda_2) \in (0,\infty) \times \mathbb{R}$ and let $f = f(\cdot; f_0)$ be a maximal solution to (6.2.4) with maximal existence time $T_+ = T_+(f_0)$. We set

$$f_\lambda(t,\xi) := f(\lambda_1 t, \xi + \lambda_2 t), \qquad \xi \in \mathbb{S}, \quad 0 \le t \le T_{+,\lambda} := T_+/\lambda_1.$$

It follows that $f_\lambda \in \mathrm{C}\big([0,T_{+,\lambda}), \mathrm{H}^r(\mathbb{S})\big) \cap \mathrm{C}^1\big([0,T_{+,\lambda}), \mathrm{H}^{r-1}(\mathbb{S})\big)$ is a solution to

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \widetilde{\Psi}(f,\lambda), \quad t \ge 0, \qquad f(0) = f_0, \tag{6.4.2}$$

where $\widetilde{\Psi} : \mathrm{H}^r(\mathbb{S}) \times (0,\infty) \times \mathbb{R} \to \mathrm{H}^{r-1}(\mathbb{S})$ is given by

$$\widetilde{\Psi}(f,\lambda) = \lambda_1 \Psi(f) + \lambda_2 \frac{\mathrm{d}f}{\mathrm{d}\xi}.$$

From (6.2.12), we immediately infer

$$\widetilde{\Psi} \in \mathrm{C}^\infty(\mathrm{H}^r(\mathbb{S}) \times (0,\infty) \times \mathbb{R}, \mathrm{H}^{r-1}(\mathbb{S})).$$

For given $f_0 \in \mathrm{H}^r(\mathbb{S})$ and $\lambda \in (0,\infty) \times \mathbb{R}$, we can furthermore compute that the Fréchet derivative of $\widetilde{\Psi}$ with respect to f is given by

$$\partial_f \widetilde{\Psi}(f_0,\lambda) = \lambda_1 \partial\Psi(f_0) + \lambda_2 \frac{\mathrm{d}}{\mathrm{d}\xi}.$$

Since the term $\mathrm{d}/\mathrm{d}\xi$ is a Fourier multiplier with symbol $m(k) = ik$, $k \in \mathbb{Z}$, we can easily adapt the arguments in the proof of Proposition 6.3.1 to get

$$-\partial_f \widetilde{\Psi}(f_0,\lambda) \in \mathcal{H}(\mathrm{H}^r(\mathbb{S}), \mathrm{H}^{r-1}(\mathbb{S})), \qquad (f_0,\lambda) \in \mathrm{H}^r(\mathbb{S}) \times (0,\infty) \times \mathbb{R}.$$

Using Theorem 2.3.9 and arguing as in the proof of part (i), we conclude that for each $(f_0,\lambda) \in \mathrm{H}^r(\mathbb{S}) \times (0,\infty) \times \mathbb{R}$, there is a unique maximal solution f to (6.4.2) satisfying

$$f = f(\cdot; f_0,\lambda) \in \mathrm{C}([0,\tilde{T}_+), \mathrm{H}^r(\mathbb{S})) \cap \mathrm{C}^1([0,\tilde{T}_+), \mathrm{H}^{r-1}(\mathbb{S})),$$

where $\tilde{T}_+ = T_+(f_0,\lambda) > 0$ is the maximal existence time. By [60, Corollary 8.3.8], we further conclude that the set

$$\mathcal{O} := \{(t, f_0, \lambda) : (f_0,\lambda) \in \mathrm{H}^r(\mathbb{S}) \times (0,\infty) \times \mathbb{R},\ 0 < t < T_+(f_0,\lambda)\}$$

is open and that

$$[(t,f_0,\lambda) \mapsto f(t; f_0,\lambda)] \in \mathrm{C}^\infty(\mathcal{O}, \mathrm{H}^r(\mathbb{S})).$$

The uniqueness of the solution to (6.4.2) now yields

$$T_+(f_0,\lambda) = T_+(f_0)/\lambda_1 \qquad \text{and} \qquad f(t; f_0,\lambda) = f_\lambda(t), \qquad 0 \le t < T_+(f_0)/\lambda_1.$$

Given $0 < t_0 < T_+(f_0)$, we find $\delta > 0$ such that $t_0 < T_+(f_0,\lambda)$ for all λ belonging to the ball $B_\delta((1,0)) \subset \mathbb{R}^2$. It follows that

$$[\lambda \mapsto f_\lambda(t_0)] \in \mathrm{C}^\infty(B_\delta((1,0)), \mathrm{H}^r(\mathbb{S})).$$

For fixed $\xi_0 \in \mathbb{S}$, the map

$$[\lambda \mapsto f(\lambda_1 t_0, \xi_0 + \lambda_2 t_0)] : B_\delta((1,0)) \to \mathbb{R}$$

is smooth as well. Elementary methods show that for small enough $\varepsilon > 0$, the function $\varphi : B_\varepsilon((t_0,\xi_0)) \to B_\delta((1,0))$ defined by

$$\varphi(t,x) = \left(\frac{t}{t_0}, \frac{\xi - \xi_0}{t_0}\right)$$

is smooth too. Composing φ with the previous function yields the smoothness of

$$[(t,\xi) \mapsto f(t,\xi)] : B_\varepsilon((t_0,\xi_0)) \to \mathbb{R},$$

and our desired result is shown.

Global existence: We use a contradiction argument similar to the one used in [65] to prove the statement. Assume there exists a maximal solution

$$f = f(\cdot; f_0) \in \mathrm{C}\big([0, T_+), \mathrm{H}^r(\mathbb{S})\big) \cap \mathrm{C}^1\big([0, T_+), \mathrm{H}^{r-1}(\mathbb{S})\big)$$

to (6.2.4) with maximal existence time $T_+ < \infty$ that further satisfies

$$\sup_{t\in[0,T_+)} \|f(t)\|_{\mathrm{H}^r} < \infty. \tag{6.4.3}$$

Since Ψ maps bounded sets in $\mathrm{H}^r(\mathbb{S})$ to bounded sets in $\mathrm{H}^{r-1}(\mathbb{S})$, cf. Section 6.2, it follows that

$$\sup_{t\in[0,T_+)} \left\|\frac{\mathrm{d}f}{\mathrm{d}t}(t)\right\|_{\mathrm{H}^{r-1}} = \sup_{t\in[0,T_+)} \|\Psi(f(t))\|_{\mathrm{H}^{r-1}} < \infty. \tag{6.4.4}$$

Choosing $r' \in (3/2, r)$, (6.4.3) and (6.4.4) enable us to argue as in (6.4.1) to get for some $\alpha \in (0,1)$ that $f \in \mathrm{C}^\alpha\big([0, T_+), \mathrm{H}^{r'}(\mathbb{S})\big)$. In particular, $f : [0, T_+) \to \mathrm{H}^{r'}(\mathbb{S})$ is uniformly continuous. We now may apply [60, Proposition 8.2.1] to (6.2.4) with

$$\Psi \in \mathrm{C}^\infty\big(\mathrm{H}^{r'}(\mathbb{S}), \mathrm{H}^{r'-1}(\mathbb{S})\big)$$

to extend the solution f to a time interval $[0, T_+')$ with $T_+ < T_+'$ such that

$$f \in \mathrm{C}\big([0, T_+'), \mathrm{H}^{r'}(\mathbb{S})\big) \cap \mathrm{C}^1\big([0, T_+'), \mathrm{H}^{r'-1}(\mathbb{S})\big).$$

The parabolic smoothing property (ii) now shows $f \in \mathrm{C}^1\big((0, T_+'), \mathrm{H}^r(\mathbb{S})\big)$ which contradicts the assumption that f is a maximal solution. With this, the proof is now complete. □

Chapter 7

Stability analysis

7.1. Introduction

In this chapter, we identify all equilibrium solutions to the two-phase Stokes problem (1.2.2) and study their stability properties. By an equilibrium solution or stationary solution to (1.2.2), we mean a solution $(f, v^{\pm}, q^{\pm}, c_{\Gamma})$ in the sense of Theorem 1.4.1 (i) to a given initial value $f_0 \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, where the interface does not change over time, that is, we have

$$f(t) = f_0, \qquad t \geq 0.$$

Since a stationary solution is fully characterized by its corresponding initial value, we will also simply refer to f as a stationary solution to (1.2.2). By the parabolic smoothing property, cf. Theorem 1.4.1 (ii), we can directly deduce that any stationary solution $f \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, to (1.2.2) actually belongs to $\mathrm{C}^{\infty}(\mathbb{S})$. This is important when constructing the velocity and pressure corresponding to the interface parametrized by f, since we need a regularity of at least $f \in \mathrm{H}^3(\mathbb{S})$ to apply Theorem 5.3.1. It follows that these quantities also do not change over time and we have

$$v^{\pm}(t) = v^{\pm}(0), \qquad q^{\pm}(t) = q^{\pm}(0), \qquad c_{\Gamma(t)} = c_{\Gamma(0)}$$

for all $t \geq 0$ and they are given by (5.3.1) and (5.3.7). It is worth noting that due to the reformulation of (1.2.2) in Section 6.2, $f \in \mathrm{H}^r(\mathbb{S})$, $r \in (3/2, 2)$, is a stationary solution to (1.2.2) if and only if f is an element of the kernel of Ψ, that is

$$\Psi(f) = 0,$$

see (6.2.4) and (6.2.5).

It is clear from the definition (6.2.5) of Ψ that the constant zero function $f = 0$ is a stationary solution to (1.2.2). In particular, this holds for all configurations of the physical parameters $\Theta \in \mathbb{R}$ and $\sigma > 0$. By introducing the parameter $\lambda = -\Theta/\sigma$, we then look for solution pairs $(\lambda, f) \in \mathbb{R} \times \mathrm{C}^{\infty}(\mathbb{S})$. Specifically, we are interested in finding bifurcation points $(\bar{\lambda}, 0)$ where a branch of nontrivial stationary solutions, that is with f being not a constant, emerges from the trivial solution, see Theorem 7.3.1 and Figure 7.1 below.

A second point of interest besides identifying stationary solutions is to consider their stability properties. We divide equilibrium solutions into two different categories. The first one contains the so-called stable solutions. An equilibrium solution $\bar{f} \in \mathrm{H}^r(\mathbb{S})$ is stable, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f_0 \in \mathrm{H}^r(\mathbb{S})$ with

$$\|f_0 - \bar{f}\|_{\mathrm{H}^r} \leq \delta,$$

we have that the solution emerging from the initial value f_0 exists globally and that it satisfies

$$\|f(t; f_0) - \bar{f}\|_{\mathrm{H}^r} \leq \varepsilon, \qquad t > 0.$$

Loosely speaking, a stable equilibrium solution is therefore characterized by the fact that any small perturbation of it is again the origin of a global solution which never leaves a small neighborhood of the stationary solution. If an equilibrium solution is not stable, we call it unstable. An important subclass of the stable stationary solutions are the so called exponentially stable stationary solutions. A stationary solution $\bar{f} \in \mathrm{H}^r(\mathbb{S})$ is called exponentially stable, if it is stable and if there exist constants $\delta > 0$, $\vartheta > 0$, and $M > 0$ such that for all $f_0 \in \mathrm{H}^r(\mathbb{S})$ with

$$\|f_0 - \bar{f}\|_{\mathrm{H}^r} \leq \delta,$$

we have

$$\|f(t; f_0) - \bar{f}\|_{\mathrm{H}^r} \leq M e^{-\vartheta t} \|f_0 - \bar{f}\|_{\mathrm{H}^r}, \qquad t > 0.$$

The structure of this chapter is as follows: in Section 7.2, we show in Lemma 7.2.1 that the equilibrium solutions to (1.2.2) are exactly the functions $f \in \mathrm{C}^\infty(\mathbb{S})$ that solve the capillary equation

$$\Theta(f - \langle f \rangle) - \sigma\kappa(f) = 0. \tag{7.1.1}$$

In Section 7.3, we then turn (7.1.1) into a bifurcation problem with $\lambda = -\Theta/\sigma$ as bifurcation parameter and collect in Theorem 7.3.1 all bifurcation points and describe the bifurcation branches emerging from them. In Section 7.4, we prove Theorem 1.4.3 using the principle of linearized stability. Lastly, in Section 7.5, we show Theorem 1.4.4 by using the principle of exchange of stability due to Crandall and Rabinowitz and again the principle of linearized stability.

7.2. Identification of the stationary solutions

In this section, we show that the stationary solutions of (1.2.2) are exactly the solutions to the capillary equation (7.1.1). This is shown in the following lemma:

Lemma 7.2.1. *A function $f \in \mathrm{C}^\infty(\mathbb{S})$ is an equilibrium solution to* (1.2.2) *if and only if it solves* (7.1.1).

Proof. Let $f \in \mathrm{C}^\infty(\mathbb{S})$ be an equilibrium solution to (1.2.2) and let v and q be the associated velocity and pressure fields. By the kinematic boundary condition $(1.2.2)_6$ we then have $v \cdot \tilde{\nu} = 0$ on Γ. With u_n, $n > \|f\|_\infty$, defined in (5.2.3), we then compute, using the boundary conditions $(1.2.2)_{3-4}$,

$$\begin{aligned}\int_{\mathbb{S}\times\mathbb{R}} \operatorname{div}\left(u_n(x_2) T_\mu(v,q)(x) v(x)\right) \mathrm{d}x &= -\int_\Gamma v \cdot [T_\mu(v,q)]\tilde{\nu} \,\mathrm{d}\sigma \\ &= -\int_\Gamma (\Theta x_2 - \sigma\tilde{\kappa}) v \cdot \tilde{\nu} \,\mathrm{d}\sigma = 0.\end{aligned}$$

Consequently, since

$$2 \operatorname{div}\left(T_\mu(v,q)v\right) = \left|\nabla v + (\nabla v)^\top\right|^2 \qquad \text{in } (\mathbb{S}\times\mathbb{R}) \setminus \Gamma,$$

by $(1.2.2)_{1-2}$, we infer from $(1.2.2)_5$, (1.4.1), and (5.3.4) that

$$\frac{1}{2}\int_{\{|x_2|<n\}} |\nabla v + (\nabla v)^\top|^2 \,\mathrm{d}x \le -\int_{\{n<|x_2|<n+1\}} \nabla u_n(x_2)\cdot (T_\mu(v,q)(x)v(x))\,\mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0.$$

Thus, $\nabla v + (\nabla v)^\top = 0$ in $\mathrm{L}^2(\mathbb{S}\times\mathbb{R},\mathbb{R}^{2\times 2})$ and, arguing similarly as in the final part of the proof of Proposition 5.2.1, we obtain that $\nabla v = 0$ in $\mathrm{L}^2(\mathbb{S}\times\mathbb{R},\mathbb{R}^{2\times 2})$. By using $(v,q)\in X_f$, $(1.2.2)_1$, and $(1.2.2)_5$, it immediately follows that $v=0$ in $\mathbb{S}\times\mathbb{R}$ and $q^\pm = \mp\Theta\langle f\rangle/2$ in $\Omega^\pm$, and the relation (7.1.1) is simply a reformulation of $(1.2.2)_4$.

For the converse implication, it suffices to show that if $f\in \mathrm{C}^\infty(\mathbb{S})$ is a solution to (7.1.1) with zero integral mean, then $(f,v^\pm,q^\pm,c_\Gamma) = (f,0,0,0)$ is an equilibrium solution to (1.2.2), since the solution set to (1.2.2) is invariant under horizontal and vertical translations, see the discussion preceding (1.4.4). This is, however, a straightforward consequence of the uniqueness result in Theorem 5.3.1, and the proof is complete. □

7.3. A bifurcation problem

A complete description of the solution set to (7.1.1) has been provided in [25, 81] in the context of the Muskat problem. It is shown there, in particular, that any solution to (7.1.1) is a horizontal and/or vertical translation of a (distributional) solution $f\in \widehat{\mathrm{H}}^r_e(\mathbb{S})$, with $r\in(3/2,2)$, to

$$\Big(\frac{f'}{(1+f'^2)^{1/2}}\Big)' + \lambda f = 0, \qquad \text{where } \lambda := -\frac{\Theta}{\sigma} = \frac{g[\rho]}{\sigma}, \tag{7.3.1}$$

which we view as an equation in $\widehat{\mathrm{H}}^{r-2}_e(\mathbb{S})$. We define, for $s\in\mathbb{R}$,

$$\widehat{\mathrm{H}}^s_e(\mathbb{S}) := \{f\in \mathrm{H}^s(\mathbb{S}) \,:\, \langle f,1\rangle = 0 \text{ and } \langle f, e^{ik\cdot}\rangle = \langle f, e^{-ik\cdot}\rangle \text{ for all } k\in\mathbb{Z}\},$$

where $\langle\cdot,\cdot\rangle$ is the canonical duality pairing between $\mathcal{D}'(\mathbb{S})$ and $\mathcal{D}(\mathbb{S}) = \mathrm{C}^\infty(\mathbb{S})$. A simple bootstrap argument shows that any solution $f\in\widehat{\mathrm{H}}^r_e(\mathbb{S})$, with $r\in(3/2,2)$, to (7.3.1) actually belongs to $\mathrm{C}^\infty(\mathbb{S})$. Moreover, testing the equation against f gives

$$\langle (f'/(1+f'^2)^{1/2})', f\rangle + \lambda\langle f, f\rangle = -\|f'(1+f'^2)^{1/4}\|_2^2 + \lambda\|f\|_2^2 = 0,$$

and therefore, if $\lambda\le 0$, we have $f=0$ as the unique solution in this case. However, if $\lambda>0$, which corresponds to the scenario where the denser fluid lies above the less dense one, equation (7.3.1) may also admit finger-shaped solutions. These are nontrivial periodic solutions to (1.2.2), characterized by the fact that the interface between the fluids forms repeating upward and downward fingers.

In Theorem 7.3.1, we summarize results from [25], adapted to our functional-analytic setting, concerning the solution set of (7.3.1) in $\widehat{\mathrm{H}}^r_e(\mathbb{S})$, regarded now as a bifurcation problem with bifurcation parameter λ and solution $(\lambda,f)\in\mathbb{R}\times\widehat{\mathrm{H}}^r_e(\mathbb{S})$. Theorem 7.3.1 shows that the solutions to (7.3.1) lie on global bifurcation branches $\mathcal{C}_\ell$, $\ell\in\mathbb{N}$. Each branch $\mathcal{C}_\ell$ consists of equilibrium solutions of minimal period $2\pi/\ell$ and intersects the trivial branch of solutions only once. Along each branch, the flat equilibrium deforms into a cosine-shaped function and the slope at $\xi = \pi/(2\ell)$ blows up monotonically as one approaches the end of the branch.

Theorem 7.3.1. *Let*

$$\lambda_* := \frac{1}{2\pi^2} B\Big(\frac{3}{4},\frac{1}{2}\Big)^2 \approx 0.2909, \tag{7.3.2}$$

where B is the beta function. Then, the following holds true:

(i) *If* $\lambda \le \lambda_*$*, then equation* (7.3.1) *only has the trivial solution* $(\lambda, f) = (\lambda, 0)$.

(ii) *Let* $\lambda > \lambda_*$.

(a) *Equation* (7.3.1) *admits solutions* $(\lambda, f) \in \mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S})$ *of minimal period* 2π *if and only if* $\lambda_* < \lambda < 1$. *More precisely, for each* $\lambda \in (\lambda_*, 1)$*, there exist exactly two even solutions* $(\lambda, \pm f_\lambda)$ *to* (7.3.1) *of minimal period* 2π. *Moreover, we have* $|f_{\lambda_1}| \le |f_{\lambda_2}|$ *for* $\lambda_2 < \lambda_1$*, and* $\|f_\lambda\|_\infty \to 0$ *as* $\lambda \to 1$*, while*

$$\|f_\lambda\|_\infty = |f_\lambda(0)| \to \sqrt{2/\lambda_*}, \quad \|f'_\lambda\|_\infty = |f'_\lambda(\pi/2)| \to \infty \qquad \text{as } \lambda \to \lambda_*.$$

(b) *Equation* (7.3.1) *admits solutions* $(\lambda, f) \in \mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S})$ *of minimal period* $2\pi/\ell$*, where* $2 \le \ell \in \mathbb{N}$*, if and only if* $\ell^2\lambda_* < \lambda < \ell^2$. *To be precise, for every* $\lambda \in (\ell^2\lambda_*, \ell^2)$*, there exist exactly two even solutions* $(\lambda, \pm f_\lambda)$ *of minimal period* $2\pi/\ell$ *given by*

$$f_\lambda = \ell^{-1} f_{\lambda\ell^{-2}}(\ell\cdot),$$

where $f_{\lambda\ell^{-2}}(\ell\cdot)$ *is the function identified in* (a).

(iii) *If we consider* (7.3.1) *as an abstract bifurcation problem in* $\mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S})$*, the global bifurcation branch* $\mathcal{C}_\ell$ *arising at* $(\ell^2, 0)$*,* $\ell \in \mathbb{N}$*, described in* (ii)*, admits in a small neighborhood of* $(\ell^2, 0)$ *a smooth parametrization*

$$(\lambda_\ell, f_\ell) \in \mathrm{C}^\infty\big((-\delta_\ell, \delta_\ell), \mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S})\big), \qquad \delta_\ell > 0,$$

satisfying

$$\begin{cases} \lambda_\ell(s) := \ell^2 - \dfrac{3\ell^4}{8} s^2 + \mathcal{O}(s^4) & \text{in } \mathbb{R}, \\ f_\ell(s) := s\cos(\ell\cdot) + \mathcal{O}(s^2) & \text{in } \widehat{\mathrm{H}}^r_e(\mathbb{S}), \end{cases} \qquad \text{for } s \to 0. \tag{7.3.3}$$

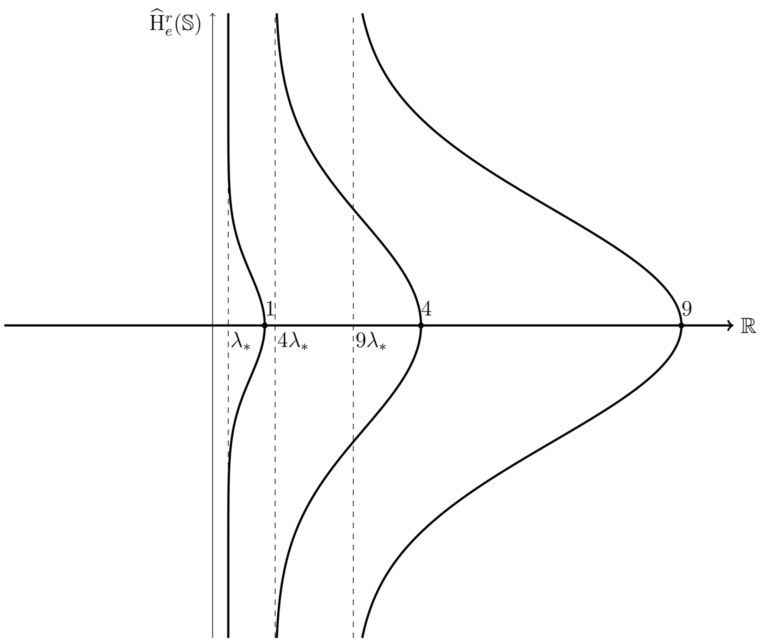


Figure 7.1: The bifurcation branches of (7.3.1) described in Theorem 7.3.1

Proof. The claims (i) and (ii) are established in [25]. The proof of (iii) follows the strategy pursued in [29], without differentiating (7.3.1). To this end, recalling (3.1.1), we define $F : \mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S}) \to \widehat{\mathrm{H}}^{r-2}_e(\mathbb{S})$ by

$$F(\lambda, f) = (f'(\omega(f))^{-1})' + \lambda f, \tag{7.3.4}$$

and reformulate (7.3.1) as the bifurcation problem

$$F(\lambda, f) = 0. \tag{7.3.5}$$

Recalling the definition of $H_2 = f'(\omega(f))^{-1}$ in (6.2.1), we infer from (6.2.7) that

$$F \in \mathrm{C}^\infty(\mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S}), \widehat{\mathrm{H}}^{r-2}_e(\mathbb{S}))$$

and moreover $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Representing functions in $\widehat{\mathrm{H}}^r_e(\mathbb{S})$ by their cosine series $\sum_{k=1}^\infty a_k \cos(k\cdot)$, it is easy to compute that the Fréchet derivative $\partial_f F(\lambda, 0)$ of F with respect to the second variable is the Fourier multiplier

$$\partial_f F(\lambda, 0)\left[\sum_{k=1}^\infty a_k \cos(k\cdot)\right] = \sum_{k=1}^\infty (\lambda - k^2) a_k \cos(k\cdot).$$

It is now obvious that the assumptions of the theorem on bifurcation from simple eigenvalues due to Crandall and Rabinowitz [20] are fulfilled in $(\ell^2, 0)$, $\ell \in \mathbb{N}$. Consequently, for each $\ell \in \mathbb{N}$, there exists a smooth local bifurcation curve

$$(\lambda_\ell, f_\ell) : (-\delta_\ell, \delta_\ell) \to \mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S})$$

such that, for $s \to 0$,

$$\begin{cases} \lambda_\ell(s) := \ell^2 + \mathcal{O}(s) & \text{in } \mathbb{R}, \\ f_\ell(s) := s\cos(\ell\cdot) + \tau_\ell(s) & \text{in } \widehat{\mathrm{H}}^r_e(\mathbb{S}), \end{cases} \tag{7.3.6}$$

with $\tau_\ell \in \mathrm{C}^\infty((-\delta_\ell, \delta_\ell), \widehat{\mathrm{H}}^r_e(\mathbb{S}))$ satisfying $\tau_\ell(0) = \tau_\ell'(0) = 0$ and $\langle \tau_\ell(s), \cos(\ell\cdot)\rangle = 0$ for all $|s| < \delta_\ell$, consisting only of equilibrium solutions to (1.2.2). Moreover, in a small neighborhood of $(\ell^2, 0)$ in $\mathbb{R} \times \widehat{\mathrm{H}}^r_e(\mathbb{S})$, the solutions to (7.3.1) are either trivial or belong to the local bifurcation curve (λ_ℓ, f_ℓ). Noticing that

$$F(\lambda_\ell(s), -f_\ell(s)) = 0 \qquad \text{for } |s| < \delta_\ell,$$

we deduce from (7.3.6) and (ii), after making δ_ℓ smaller if necessary, that

$$\lambda_\ell(-s) = \lambda_\ell(s) \qquad \text{and} \qquad f_\ell(-s) = -f_\ell(s) \qquad \text{for all } |s| < \delta_\ell.$$

In particular, we have $\lambda_\ell'(0) = \lambda_\ell'''(0) = 0$. In order to show that $\lambda_\ell''(0) = -3\ell^4/4$, and prove in this way (7.3.3), we differentiate the identity $F(\lambda_\ell(s), f_\ell(s)) = 0$ three times with respect to s and evaluate the resulting identity at $s = 0$ to arrive at

$$\lambda_\ell''(0) f_\ell'(0) - \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\left(\frac{\mathrm{d}}{\mathrm{d}\xi} f_\ell'(0)\right)^3\right) + \frac{1}{3}\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}(f_\ell'''(0)) + \ell^2 f_\ell'''(0)\right) = 0 \tag{7.3.7}$$

in $\widehat{\mathrm{H}}^{r-2}_e(\mathbb{S})$. Since $f_\ell'''(0) = \tau_\ell'''(0)$ with $\langle \tau_\ell'''(0), \cos(\ell\cdot)\rangle = 0$ and $f_\ell'(0) = \cos(\ell\cdot)$, testing (7.3.7) with $\cos(\ell\cdot)$ leads to

$$\lambda_\ell''(0) = \frac{\left\langle \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\cos(\ell\cdot)\right)^3\right), \cos(\ell\cdot)\right\rangle}{\langle \cos(\ell\cdot), \cos(\ell\cdot)\rangle} = -\frac{3\ell^4}{4},$$

which completes the proof. □

7.4. Stability analysis of flat equilibria

In this section, we prove Theorem 1.4.3. Recall that this states that the flat equilibria are exponentially stable if $\sigma + \Theta > 0$ and unstable if the reverse inequality holds, see also Figure 1.2.

Proof of Theorem 1.4.3. To establish claim (i), we first infer from the evolution equation (6.2.4) that $\Psi(f) \in \widehat{\mathrm{H}}^{r-1}(\mathbb{S})$ for each $f \in \widehat{\mathrm{H}}^{r}(\mathbb{S})$, since the integral mean $\langle f \rangle$ is preserved, see (1.4.5). Recalling (6.2.12), we thus have $\Psi \in \mathrm{C}^\infty(\widehat{\mathrm{H}}^{r}(\mathbb{S}), \widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$. Moreover, combining (4.1.1), (4.1.7), (6.2.1), (6.2.3), (6.2.6), (6.3.6), and (6.3.7) we compute that the Fréchet derivative $\partial\Psi(0) \in \mathcal{L}(\widehat{\mathrm{H}}^{r}(\mathbb{S}), \widehat{\mathrm{H}}^{r-1}(\mathbb{S})$ is the Fourier multiplier

$$\partial\Psi(0) := \frac{\Theta}{2(\mu^+ + \mu^-)} B_0(0) - \frac{\sigma}{2(\mu^+ + \mu^-)} \Big(-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \Big)^{\frac{1}{2}},$$

with $B_0(0) = H \circ S$, where H is the Hilbert transform and $S \in \mathcal{L}(\widehat{\mathrm{H}}^{r}(\mathbb{S}))$ is the operator which associates to each function $f \in \widehat{\mathrm{H}}^{r}(\mathbb{S})$ its antiderivative

$$S[f](\xi) := \int_0^\xi f(s)\,\mathrm{d}s + \frac{1}{2\pi} \int_0^{2\pi} s f(s)\,\mathrm{d}s, \qquad \xi \in \mathbb{S}. \tag{7.4.1}$$

In particular, $\partial\Psi(0) \in \mathcal{L}(\widehat{\mathrm{H}}^{r}(\mathbb{S}), \widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$ generates a strongly continuous analytic semigroup in $\mathcal{L}(\widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$, cf. (6.3.64)–(6.3.66), and its spectrum is given by

$$\sigma(\partial\Psi(0)) = \Big\{ -\frac{\Theta + \sigma k^2}{2(\mu^+ + \mu^-)k} : k \in \mathbb{N} \Big\}, \tag{7.4.2}$$

consisting only of eigenvalues with finite multiplicity.

Hence, assuming $\sigma + \Theta > 0$, we have

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\partial\Psi(0))\} \le -\vartheta_0 < 0,$$

and we may apply the principle of linearized stability [60, Theorem 9.1.2] in the context of (6.2.4) (regarded as an evolution problem in $\widehat{\mathrm{H}}^{r-1}(\mathbb{S})$) to obtain (a).

Conversely, if $\sigma + \Theta < 0$, the spectrum of $\partial\Psi(0) \in \mathcal{L}(\widehat{\mathrm{H}}^{r}(\mathbb{S}), \widehat{\mathrm{H}}^{r-1}(\mathbb{S}))$ contains a finite number of positive eigenvalues. More precisely, we have

$$\begin{cases} -\dfrac{\sigma + \Theta}{2(\mu^+ + \mu^-)} \in \sigma_+(\partial\Psi(0)) := \sigma(\partial\Psi(0)) \cap \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\,\lambda > 0\}, \\ \inf\{\mathrm{Re}\,\lambda \,:\, \lambda \in \sigma_+(\partial\Psi(0))\} > 0. \end{cases}$$

Claim (b) now follows directly from an application of [60, Theorem 9.1.3] in the context of (6.2.4) (regarded again as an evolution problem in $\widehat{\mathrm{H}}^{r-1}(\mathbb{S})$). □

7.5. The Rayleigh–Taylor instability of small finger-shaped equilibria

In this section, we show Theorem 1.4.4, which states that all equilibrium solutions located on the bifurcation curve (7.3.3) are unstable.

Proof of Theorem 1.4.4. To start, we first formulate (6.2.4) as an evolution problem in $\widehat{\mathrm{H}}^{r-1}_e(\mathbb{S})$ depending on the parameter λ introduced in (7.3.1) (by eliminating the parameter Θ). Recalling (6.2.2) and (6.2.6), for each $f \in \widehat{\mathrm{H}}^r_e(\mathbb{S})$ and $\lambda \in \mathbb{R}$, we denote by $\beta = \beta(f,\lambda) \in \mathrm{H}^{r-1}(\mathbb{S})^2$ the unique solution to

$$(1 + 2a_\mu \mathbb{D}(f))[\beta] = -\frac{\sigma}{4}\begin{pmatrix} -\lambda \mathcal{W}_1(f) + \mathcal{W}_3(f) \\ \lambda \mathcal{W}_3(f) + \lambda \ln 4\langle f\rangle + \mathcal{W}_4(f) \end{pmatrix}. \tag{7.5.1}$$

Note that the solution $\beta(f)$ defined in (6.2.6) coincides with $\beta(f,\lambda)$ when $\lambda = -\Theta/\sigma$. Related to (6.2.4), we consider the evolution problem

$$\frac{\mathrm{d}f}{\mathrm{d}t}(t) = \Psi(f(t),\lambda), \quad t \geq 0, \qquad f(0) = f_0, \tag{7.5.2}$$

where

$$\Psi(f,\lambda) := \frac{2}{\mu^+ + \mu^-}(-f',1)^\top \cdot \beta(f,\lambda). \tag{7.5.3}$$

The mappings defined in (7.5.1) and (7.5.3) depend linearly on λ, which implies that

$$\Psi \in \mathrm{C}^\infty(\widehat{\mathrm{H}}^r_e(\mathbb{S}) \times \mathbb{R}, \widehat{\mathrm{H}}^{r-1}(\mathbb{S})).$$

To show that $\Psi(f,\lambda)$ is also an even function, we introduce the reflection operator $[\varphi \mapsto \check{\varphi}] \in \mathcal{L}(\mathrm{C}(\mathbb{S}))$ by setting

$$\check{\varphi}(\xi) := \varphi(-\xi), \qquad \xi \in \mathbb{S},$$

and observe that

$$(B^{p,q}_{n,m}(f)[\varphi])^{\check{}} = (-1)^{n+p+q+1} B^{p,q}_{n,m}(f)[\check{\varphi}].$$

Since

$$H_i(f)^{\check{}} = (-1)^{i+1} H_i(f), \qquad 1 \leq i \leq 2,$$

and

$$\begin{aligned}(B_i(f)[\varphi])^{\check{}} &= -B_i(f)[\check{\varphi}], \quad i \in \{1, 4, 5\},\\ (B_i(f)[\varphi])^{\check{}} &= B_i(f)[\check{\varphi}], \quad i \in \{0, 2, 3, 6\},\end{aligned}$$

respectively, by (3.3.42), (4.1.7), and (6.2.1), we infer from (6.2.3) that

$$\mathcal{W}_i(f)^{\check{}} = -\mathcal{W}_i(f), \quad 1 \leq i \leq 2, \qquad \text{and} \qquad \mathcal{W}_i(f)^{\check{}} = \mathcal{W}_i(f), \quad 3 \leq i \leq 4. \tag{7.5.4}$$

Recalling (4.1.1), we deduce from (7.5.1), by using (7.5.4), that

$$(1 + 2a_\mu \mathbb{D}(f))\begin{bmatrix} -\check{\beta}_1 \\ \check{\beta}_2 \end{bmatrix} = -\frac{\sigma}{4}\begin{pmatrix} -\lambda \mathcal{W}_1(f) + \mathcal{W}_3(f) \\ \lambda \mathcal{W}_3(f) + \lambda \ln 4\langle f\rangle + \mathcal{W}_4(f) \end{pmatrix},$$

and Theorem 4.3.2 ensures that

$$\begin{bmatrix} \beta_1(f,\lambda)^{\check{}} \\ \beta_2(f,\lambda)^{\check{}} \end{bmatrix} = \begin{bmatrix} -\beta_1(f,\lambda) \\ \beta_2(f,\lambda) \end{bmatrix}, \qquad f \in \widehat{\mathrm{H}}^r_e(\mathbb{S}), \quad \lambda \in \mathbb{R}. \tag{7.5.5}$$

It is straightforward to infer from this relation together with (7.5.3) that

$$\Psi \in \mathrm{C}^\infty(\widehat{\mathrm{H}}^r_e(\mathbb{S}) \times \mathbb{R}, \widehat{\mathrm{H}}^{r-1}_e(\mathbb{S})). \tag{7.5.6}$$

Moreover, the Fréchet derivative $\partial_f \Psi(0,\lambda)$ generates a strongly continuous analytic semigroup in $\mathcal{L}(\widehat{\mathrm{H}}^{r-1}_e(\mathbb{S}))$, being the Fourier multiplier

$$\partial_f \Psi(0,\lambda)\Big[\sum_{k=1}^\infty a_k \cos(k\cdot)\Big] = \sum_{k=1}^\infty \frac{\sigma(\lambda - k^2)}{2(\mu^+ + \mu^-)k} a_k \cos(k\cdot).$$

We now study the stability properties of the equilibrium solution $f_\ell(s)$, with $\ell \in \mathbb{N}$ and $|s| < \delta_\ell$, to (7.5.2) (for $\lambda = \lambda_\ell(s)$). To this end, we observe that if δ_ℓ is sufficiently small, the perturbation result [2, Theorem I.1.3.1 (i)] ensures that $\partial_f \Psi(f_\ell(s), \lambda_\ell(s))$ generates a strongly continuous analytic semigroup in $\widehat{\mathrm{H}}_e^{r-1}(\mathbb{S})$ for all $|s| < \delta_\ell$. Moreover, since the embedding $\widehat{\mathrm{H}}_e^r(\mathbb{S}) \hookrightarrow \widehat{\mathrm{H}}_e^{r-1}(\mathbb{S})$ is compact, we infer from [55, Theorem III.6.29] and the generator property of $\partial_f \Psi(f_\ell(s), \lambda_\ell(s))$ that the spectrum of $\partial_f \Psi(f_\ell(s), \lambda_\ell(s))$ consists entirely of isolated eigenvalues with finite algebraic multiplicities which do not accumulate at 0.

Assume now that $\ell \geq 2$. Then, $\partial_f \Psi(0, \ell^2)$ has exactly $(\ell - 1)$ positive simple eigenvalues. Since a finite set of simple eigenvalues of $\partial_f \Psi(f_\ell(s), \lambda_\ell(s))$ depends continuously on s as s varies in $(-\delta_\ell, \delta_\ell)$, cf. [55, Chapter IV.3.5], we conclude that

$$\sigma_+(\partial_f \Psi(f_\ell(s), \lambda_\ell(s))) \neq \emptyset \quad \text{and} \quad \inf\{\operatorname{Re}\lambda \,:\, \lambda \in \sigma_+(\partial_f \Psi(f_\ell(s), \lambda_\ell(s)))\} > 0 \tag{7.5.7}$$

for all $|s| < \delta_\ell$, provided that δ_ℓ is sufficiently small. Therefore, we can apply [60, Theorem 9.1.3] to deduce that $f_\ell(s)$ is an unstable equilibrium.

Finally, we consider the more intricate case $\ell = 1$, in which the eigenvalues of $\partial_f \Psi(0, 1)$ are all negative, except for the simple eigenvalue zero. Let

$$\gamma(\lambda) := \frac{\sigma(\lambda - 1)}{2(\mu^+ + \mu^-)}$$

denote the eigenvalue of $\partial_f \Psi(0, \lambda)$ that satisfies $\gamma(\lambda) = 0$ for $\lambda = 1$. Assuming that δ_1 is small, the operator $\partial_f \Psi(f_1(s), \lambda_1(s))$ has, for each $|s| < \delta_1$, a simple eigenvalue $z(s)$ in the vicinity of 0. According to the principle of exchange of stability due to Crandall and Rabinowitz, cf. [21, Theorem 1.16], it holds that

$$\lim_{s \to 0} \frac{-s\lambda_1'(s)\gamma'(0)}{z(s)} = 1.$$

Since $s\lambda_1'(s) < 0$ for all $0 < |s| < \delta_1$ by (7.3.3) and $\gamma'(0) > 0$, we deduce that $z(s)$ is a positive simple eigenvalue of $\partial_f \Psi(f_1(s), \lambda_1(s))$ for each $0 < |s| < \delta_1$. Hence, (7.5.7) also holds true for $\ell = 1$ and $0 < |s| < \delta_1$, and therefore $f_1(s)$ is an unstable equilibrium solution to (7.5.2) by the same abstract result [60, Theorem 9.1.3]. □

Part III

Appendices

Appendix A

Analysis and localization of some (singular) integral operators

In this appendix, we establish several useful properties of the (singular) integral operators $B^{p,q}_{n,m}$ and $C_{n,m}$ introduced in (4.1.3) and (6.3.2), respectively. In Section A.1, we prove the mapping properties we used throughout Chapter 4 when we analyzed the resolvent of the hydrodynamic double-layer potential operator. Moreover, we show that the mapping $[f \mapsto B^{p,q}_{n,m}(f)]$ is smooth when defined on suitable function spaces, and we provide a formula for the Fréchet derivative, see Lemma A.1.14 below. This result was crucial in Chapter 6, where we studied the evolution operator associated to (1.2.2). The second part of this appendix deals with localization results for the operator $C_{n,m}$ which we used throughout the localization procedure in Chapter 6, see Section A.2.

A.1. Analysis

We start this section by relating the two families of singular integral operators $B^{0,q}_{n,m}$ and $C_{n,m}$. To this end, we define for integers $m,\, n,\, q \in \mathbb{N}_0$, $\ell \in \{1,2\}$, and Lipschitz continuous mappings $\mathbf{a} = (a_1,\dots,a_m) : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{b} = (b_1,\dots,b_n) : \mathbb{R} \to \mathbb{R}^n$, and $\mathbf{c} = (c_1,\dots,c_q) : \mathbb{R} \to \mathbb{R}^q$ the integral operator

$$
\begin{aligned}
A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi](\xi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \Bigg[& \frac{\prod_{i=1}^{n} \frac{T_{[\xi,s]}b_i}{t_{[s]}} \prod_{i=1}^{q} \frac{\delta_{[\xi,s]}c_i/2}{t_{[s]}}}{\prod_{i=1}^{m}\Big[1+\Big(\frac{T_{[\xi,s]}a_i}{t_{[s]}}\Big)^2\Big]} \frac{1}{t^{\ell}_{[s]}} \\
& - \frac{\prod_{i=1}^{n} \frac{\delta_{[\xi,s]}b_i/2}{s/2} \prod_{i=1}^{q} \frac{\delta_{[\xi,s]}c_i/2}{s/2}}{\prod_{i=1}^{m}\Big[1+\Big(\frac{\delta_{[\xi,s]}a_i/2}{s/2}\Big)^2\Big]} \frac{1}{(s/2)^{\ell}} \Bigg] \varphi(\xi - s)\,\mathrm{d}s,
\end{aligned}
\tag{A.1.1}
$$

where $\varphi \in \mathrm{L}^2(\mathbb{S})$ and $\xi \in \mathbb{R}$, and where we use the notation (3.2.20) and (4.1.4). The relation

$$
B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi] = A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi] + C_{n+q,m}(\mathbf{a})[(\mathbf{b},\mathbf{c}),\varphi], \qquad m,\, n,\, q \in \mathbb{N}_0, \tag{A.1.2}
$$

and the fact that the operators $A^{\ell,q}_{n,m}$ are regularizing, see Lemma A.1.3 below, are the crucial elements in transferring mapping properties of the operators $C_{n,m}$ to the operators $B^{0,q}_{n,m}$ (which have kernels with a higher degree of nonlinearity than the former), see also Remark 4.2.2. We note that $A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is 2π-periodic if the mappings $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ have this property.

Next, we recall the elementary inequalities

$$\begin{aligned} |\tanh(\xi)| \le |\xi| \qquad &\text{and} \qquad |\tanh(\xi)-\xi| \le |\xi \tanh^2(\xi)|,\\ |\zeta| \le |\tan(\zeta)| \qquad &\text{and} \qquad |\tan(\zeta)-\zeta| \le |\zeta^2\tan(\zeta)|, \end{aligned} \tag{A.1.3}$$

and

$$\Big|\frac{T_{[\xi,s]}d}{t_{[s]}}\Big| \le \Big|\frac{\delta_{[\xi,s]}d/2}{t_{[s]}}\Big| \le \Big|\frac{\delta_{[\xi,s]}d}{s}\Big|\Big|\frac{s/2}{t_{[s]}}\Big| \le \|d'\|_\infty\Big|\frac{s/2}{t_{[s]}}\Big| \le \|d'\|_\infty, \quad d\in \mathrm{W}^{1,\infty}(\mathbb{S}), \tag{A.1.4}$$

which hold for all $\xi\in\mathbb{R}$, $\zeta\in(-\pi/2,\pi/2)$, and $0\neq s\in(-\pi,\pi)$. These inequalities immediately yield for all $0\neq s\in(-\pi,\pi)$ and $\xi\in\mathbb{R}$ that

$$\begin{aligned} \Big|\frac{1}{t^\ell_{[s]}}-\frac{1}{(s/2)^\ell}\Big| &\le 2|s|^{2-\ell}, \qquad \ell\in\{1,\,2\},\\ \Big|\frac{T_{[\xi,s]}d}{t_{[s]}}-\frac{\delta_{[\xi,s]}d}{s}\Big| + \Big|\frac{\delta_{[\xi,s]}d/2}{t_{[s]}}-\frac{\delta_{[\xi,s]}d}{s}\Big| &\le C|\delta_{[\xi,s]}d|\,|s| \le C|s|^2, \end{aligned} \tag{A.1.5}$$

with C depending only on $\|d'\|_\infty$. We also note the following inequalities

$$\frac{|T_{[\xi,s]}d|}{|s/2|^{1/2}} \le \frac{|\delta_{[\xi,s]}d|}{|s|^{1/2}} \le \|d'\|_2, \qquad d\in \mathrm{H}^1(\mathbb{S}), \quad 0\neq s\in(-\pi,\pi), \quad \xi\in\mathbb{R}, \tag{A.1.6}$$

and

$$\begin{aligned} |\tanh(\xi)-\tanh(\zeta)| &\le |\xi-\zeta|\\ |(\xi-\tanh(\xi))-(\zeta-\tanh(\zeta))| &\le (\xi^2+\zeta^2)|\xi-\zeta| \end{aligned} \tag{A.1.7}$$

for ξ, $\zeta\in\mathbb{R}$, which follow from (A.1.3) via the fundamental theorem of calculus.

A.1.1. Mapping properties

We first consider the operator $C_{n,m}$ and show various mapping properties in the next lemma.

Lemma A.1.1. *Let n, $m\in\mathbb{N}_0$ be given.*

(i) *Given $\mathbf{a}\in \mathrm{W}^{1,\infty}(\mathbb{R})^m$, there exists a constant $C>0$ depending only on n, m, and $\|\mathbf{a}'\|_\infty$ such that for all $\mathbf{b}\in \mathrm{W}^{1,\infty}(\mathbb{R})^n$ and $\theta\in\mathbb{R}$, we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b},\cdot]\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{S}),\mathrm{L}^2((\theta-\pi,\theta+\pi)))} \le C\prod_{i=1}^n \|b_i'\|_{\mathrm{L}^\infty(\mathbb{R})}. \tag{A.1.8}$$

Moreover, $C_{n,m}\in \mathrm{C}^{1-}\big(\mathrm{W}^{1,\infty}(\mathbb{S})^m, \mathcal{L}^n_{\mathrm{sym}}(\mathrm{W}^{1,\infty}(\mathbb{S}),\mathcal{L}(\mathrm{L}^2(\mathbb{S})))\big)$.

(ii) *Given $n\in\mathbb{N}$, $r\in(3/2,2)$, $\bar r\in(5/2-r,1)$, and $\mathbf{a}\in \mathrm{H}^r(\mathbb{S})^m$, there exists a constant $C>0$ that depends only on n, m, r, and $\|\mathbf{a}\|_{\mathrm{H}^r}$ (and on $\bar r$ in* (A.1.10)*), such that for all $\mathbf{b}\in \mathrm{H}^1(\mathbb{S})\times \mathrm{H}^r(\mathbb{S})^{n-1}$ and $\varphi\in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]\|_2 \le C\|b_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=2}^n \|b_i'\|_{\mathrm{H}^{r-1}} \tag{A.1.9}$$

and

$$\begin{aligned} &\|C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]-\varphi C_{n-1,m}(\mathbf{a})[(b_2,\dots,b_n),b_1']\|_2\\ &\qquad \le C\|b_1\|_{\mathrm{H}^{\bar r}}\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=2}^n \|b_i'\|_{\mathrm{H}^{r-1}}. \end{aligned} \tag{A.1.10}$$

(iii) *Given* $r \in (3/2, 2)$ *and* $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^m$*, there exists a constant* $C > 0$ *that depends only on* n*,* m*,* r*, and* $\|\mathbf{a}\|_{\mathrm{H}^r}$ *such that for all* $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ *and* $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$*, we have*

$$\|C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{n} \|b_i'\|_{\mathrm{H}^{r-1}}. \tag{A.1.11}$$

(iv) *Let* $n \in \mathbb{N}$*,* $3/2 < r' < r < 2$*, and* $\mathbf{a} \in \mathrm{H}^r(\mathbb{S})^m$ *be given. Then, there exists a constant* $C > 0$ *that depends only on* n*,* m*,* r*,* r'*, and* $\|\mathbf{a}\|_{\mathrm{H}^r}$ *such that for all* $\mathbf{b} \in \mathrm{H}^r(\mathbb{S})^n$ *and* $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$*, we have*

$$\begin{aligned}&\|C_{n,m}(\mathbf{a})[\mathbf{b},\varphi] - \varphi C_{n-1,m}(\mathbf{a})[(b_2,\ldots,b_n),b_1']\|_{\mathrm{H}^{r-1}}\\ &\qquad \leq C\|b_1\|_{\mathrm{H}^{r'}}\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{n} \|b_i'\|_{\mathrm{H}^{r-1}}.\end{aligned} \tag{A.1.12}$$

Before proving Lemma A.1.1, we make the following remark.

Remark A.1.2. The result (A.1.8) concerning the operator $C_{n,m}$ has first been proven in [63] in the context of the periodic Muskat problem. The proof therein is based on mapping properties of the non-periodic analogue of the operator $C_{n,m}$, namely

$$B_{n,m}(\mathbf{a})[\mathbf{b},\varphi](\xi) := \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{\prod_{i=1}^{n} \frac{\delta_{[\xi,s]}b_i}{s}}{\prod_{i=1}^{m}\Big[1+\Big(\frac{\delta_{[\xi,s]}a_i}{s}\Big)^2\Big]} \frac{\varphi(\xi-s)}{s}\,\mathrm{d}s, \qquad \xi \in \mathbb{R},$$

where $\mathbf{a} \in \mathrm{W}^{1,\infty}(\mathbb{R})^{m+n}$ and $\varphi \in \mathrm{L}^2(\mathbb{R})$. This operator has been introduced in [62] in the study of the Muskat problem in $\mathbb{R}^2$. It has a similar connection to the Hilbert transform as the operator $B^{p,q}_{n,m}$ has to the periodic Hilbert transform, see (4.1.5), and it is shown in [62] that $B_{n,m}(\mathbf{a})[\mathbf{b},\cdot] \in \mathcal{L}(\mathrm{L}^2(\mathbb{R}))$. This property relies on a result in [71] concerning the $\mathrm{L}^2(\mathbb{R})$-boundedness of the singular integral operator

$$\varphi \mapsto \Big[x \mapsto \mathrm{PV}\int_{\mathbb{R}} \frac{\varphi(y)}{x-y}\exp\Big(i\frac{a(x)-a(y)}{x-y}\Big)\,\mathrm{d}y\Big],$$

for a locally Lipschitz continuous function $a : \mathbb{R} \to \mathbb{R}$.

Proof. *Claim* (i). We prove (i) along the lines of [63, Lemma A.1] in the case $\theta = 0$. To start, fix $u \in \mathrm{C}_0^\infty(\mathbb{R},[0,1])$ such that $u(\xi) = 1$ if $|\xi| < 2\pi$ and $u(\xi) = 0$ if $|\xi| > 4\pi$. Then, it holds that

$$\|\varphi\|_2 \leq \|u\varphi\|_{\mathrm{L}^2(\mathbb{R})} \leq 4\|\varphi\|_2, \qquad \varphi \in \mathrm{L}^2(\mathbb{S}). \tag{A.1.13}$$

Given $\xi \in (-\pi,\pi)$ and $\varphi \in \mathrm{L}^2(\mathbb{S})$, we may decompose $C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]$ as

$$\begin{aligned}C_{n,m}(\mathbf{a})[\mathbf{b},\varphi](\xi) &= B_{n,m}(\mathbf{a})[\mathbf{b},u\varphi](\xi)\\ &\qquad - \frac{1}{\pi}\int_{\pi<|s|<5\pi} \frac{\prod_{i=1}^{n} \frac{\delta_{[\xi,s]}b_i}{s}}{\prod_{i=1}^{m}\Big[1+\Big(\frac{\delta_{[\xi,s]}a_i}{s}\Big)^2\Big]} \frac{(u\varphi)(\xi-s)}{s}\,\mathrm{d}s,\end{aligned}$$

see Remark A.1.2 for the definition of $B_{n,m}$ and recall from there and [62, Lemma 3.1 and Remark 3.3] that $B_{n,m}(\mathbf{a}) \in \mathcal{L}^n_{\mathrm{sym}}(\mathrm{W}^{1,\infty}(\mathbb{R}),\mathcal{L}(\mathrm{L}^2(\mathbb{R})))$. This property, together with (A.1.13), enables us to conclude that

$$\|B_{n,m}(\mathbf{a})[\mathbf{b},u\varphi]\|_{\mathrm{L}^2((-\pi,\pi))} \leq C\|u\varphi\|_{\mathrm{L}^2(\mathbb{R})} \prod_{i=1}^{n}\|b_i'\|_{\mathrm{L}^\infty(\mathbb{R})} \leq C\|\varphi\|_2 \prod_{i=1}^{n}\|b_i'\|_{\mathrm{L}^\infty(\mathbb{R})}.$$

Furthermore, we use (A.1.4) to compute

$$\left\| \frac{1}{\pi} \int_{\pi<|s|<5\pi} \frac{\prod_{i=1}^{n} \frac{\delta_{[\cdot,s]} b_i}{s}}{\prod_{i=1}^{m} \left[1 + \left(\frac{\delta_{[\cdot,s]} a_i}{s}\right)^2\right]} \frac{(u\varphi)(\cdot - s)}{s} \, \mathrm{d}s \right\|_{\mathrm{L}^\infty((-\pi,\pi))} \leq C \|\varphi\|_2 \prod_{i=1}^{n} \|b_i'\|_{\mathrm{L}^\infty(\mathbb{R})},$$

from which the estimate (A.1.8) immediately follows. The result for $\theta \neq 0$ is obtained from the result for $\theta = 0$ via the identity

$$C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi](\xi) = C_{n,m}(\tau_\theta \mathbf{a})[\tau_\theta \mathbf{b}, \tau_\theta \varphi](\xi - \theta), \qquad \xi, \theta \in \mathbb{R}.$$

The local Lipschitz continuity property follows from (A.1.8) in view of the decomposition

$$\begin{aligned} &\big(C_{n,m}(\tilde{\mathbf{a}}) - C_{n,m}(\mathbf{a})\big)[\mathbf{b}, \varphi] \\ &= \sum_{i=1}^{m} C_{n+2,m+1}(\tilde{a_1}, \ldots, \tilde{a_i}, a_i, \ldots, a_m)[(\mathbf{b}, a_i - \tilde{a_i}, a_i + \tilde{a_i}), \varphi], \end{aligned} \tag{A.1.14}$$

for $(\mathbf{a}, \tilde{\mathbf{a}}, \mathbf{b}) \in \mathrm{W}^{1,\infty}(\mathbb{S})^{2m+n}$.

Claim (ii). We argue as in [1, Lemma 4] where this claim is proven for the non-periodic operator $B_{n,m}$. We use the identities

$$\frac{\partial}{\partial s}\left(\frac{\delta_{[\xi,s]} b_1}{s}\right) = \frac{b_1'(\xi - s)}{s} - \frac{\delta_{[\xi,s]} b_1}{s^2} \qquad \text{and} \qquad \varphi'(\xi - s) = \frac{\partial}{\partial s}(\varphi(\xi) - \varphi(\xi - s))$$

and integration by parts to find

$$\begin{aligned} C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi](\xi) &= \varphi(\xi) C_{n-1,m}(\mathbf{a})[(b_2, \ldots, b_n), b_1'](\xi) - K_B(\xi) \\ &\quad - \sum_{j=2}^{n} \int_{-\pi}^{\pi} K_{1,j}(\xi, s) \, \mathrm{d}s + 2 \sum_{j=1}^{m} \int_{-\pi}^{\pi} K_{2,j}(\xi, s) \, \mathrm{d}s - \int_{-\pi}^{\pi} K(\xi, s) \, \mathrm{d}s, \end{aligned}$$

where, given $\xi \in \mathbb{S}$ and $s \neq 0$, we set

$$\begin{aligned} K(\xi, s) &:= \frac{1}{\pi} \frac{\prod_{i=1}^{n} (\delta_{[\xi,s]} b_i / s)}{\prod_{i=1}^{m} \left[1 + (\delta_{[\xi,s]} a_i / s)^2\right]} \frac{\delta_{[\xi,s]} \varphi}{s}, \\ K_{1,j}(\xi, s) &:= \frac{1}{\pi} \frac{\prod_{i=1, i\neq j}^{n} (\delta_{[\xi,s]} b_i / s)}{\prod_{i=1}^{m} \left[1 + (\delta_{[\xi,s]} a_i / s)^2\right]} \frac{\delta_{[\xi,s]} b_j - s b_j'(\xi - s)}{s^2} \varphi(\xi), \\ K_{2,j}(\xi, s) &:= \frac{1}{\pi} \frac{(\delta_{[\xi,s]} a_j / s) \prod_{i=1}^{n} (\delta_{[\xi,s]} b_i / s)}{\left[1 + (\delta_{[\xi,s]} a_j / s)^2\right] \prod_{i=1}^{m} \left[1 + (\delta_{[\xi,s]} a_i / s)^2\right]} \frac{\delta_{[\xi,s]} a_j - s a_j'(\xi - s)}{s^2} \varphi(\xi), \\ K_B(\xi) &:= \frac{1 - (-1)^n}{\pi} \frac{\prod_{i=1}^{n} (\delta_{[\xi,\pi]} b_i / \pi)}{\prod_{i=1}^{m} \left[1 + (\delta_{[\xi,\pi]} a_i / \pi)^2\right]} \varphi(\xi). \end{aligned}$$

In view of (i), we get

$$\|\varphi C_{n-1,m}(\mathbf{a})[(b_2, \ldots, b_n), b_1']\|_2 \leq C \|b_1'\|_2 \|\varphi\|_\infty \prod_{i=2}^{n} \|b_i'\|_\infty. \tag{A.1.15}$$

Choose $\alpha \in \{\bar{r}, 1\}$. Since $\alpha > 1/2$, we have $b_1 \in \mathrm{H}^\alpha(\mathbb{S}) \hookrightarrow \mathrm{C}^{\alpha-1/2}(\mathbb{S})$ and it is immediate that $\|\delta_{[\cdot,\pi]} b_1\|_\infty \leq \pi [b_1]_{\alpha-1/2}$, while $\|\delta_{[\cdot,\pi]} b_i\|_\infty \leq \pi \|b_i'\|_\infty$ for $2 \leq i \leq n$. Hence,

$$\|K_B\|_2 \leq C \|K_B\|_\infty \leq C \|\varphi\|_\infty \prod_{i=1}^{n} \|\delta_{[\cdot,\pi]} b_i\|_\infty \leq C [b_1]_{\alpha-1/2} \|\varphi\|_\infty \prod_{i=2}^{n} \|b_i'\|_\infty. \tag{A.1.16}$$

Using Minkowski's integral inequality, we have

$$\left(\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}K_{1,j}(\xi,s)\,\mathrm{d}s\right|^2\mathrm{d}\xi\right)^{1/2}\leq\|\varphi\|_\infty[b_1]_{\alpha-1/2}\int_{-\pi}^{\pi}\frac{\|\tau_sb_j-b_j-sb_j'\|_2}{|s|^{7/2-\alpha}}\,\mathrm{d}s\prod_{\substack{i=2\\i\neq j}}^{n}\|b_i'\|_\infty$$

for $2\leq j\leq n$. Fubini's theorem, Minkowski's integral inequality, Hölder's inequality, Lemma 2.4.3, a change of variables, and the fact that $r+\alpha-3>-1/2$ lead us to

$$\begin{aligned}
&\int_{-\pi}^{\pi}\frac{\|\tau_sb_j-b_j-sb_j'\|_2}{|s|^{7/2-\alpha}}\,\mathrm{d}s\\
&\leq\int_{-\pi}^{\pi}\frac{1}{|s|^{5/2-\alpha}}\left(\int_{-\pi}^{\pi}\left|\int_0^1 b_j'(\xi+ts)-b_j'(\xi)\,\mathrm{d}t\right|^2\mathrm{d}\xi\right)^{1/2}\mathrm{d}s\\
&\leq\int_0^1\left[\int_{-\pi}^{\pi}\frac{1}{|s|^{5/2-\alpha}}\left(\int_{-\pi}^{\pi}|b_j'(\xi+ts)-b_j'(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\mathrm{d}s\right]\mathrm{d}t\\
&=\int_0^1\int_{-\pi}^{\pi}\frac{\|\tau_{ts}b_j'-b_j'\|_2}{|s|^{5/2-\alpha}}\,\mathrm{d}s\,\mathrm{d}t\\
&=\int_0^1\int_{-t\pi}^{t\pi}\frac{\|\tau_sb_j'-b_j'\|_2}{|s|^{5/2-\alpha}}t^{3/2-\alpha}\,\mathrm{d}s\,\mathrm{d}t\\
&\leq\int_{-\pi}^{\pi}\frac{\|\tau_sb_j'-b_j'\|_2}{|s|^{r-1/2}}|s|^{r+\alpha-3}\,\mathrm{d}s\leq\|b_j'\|_{\mathrm{H}^{r-1}}\left(\int_{-\pi}^{\pi}|s|^{2(r+\alpha-3)}\,\mathrm{d}s\right)^{1/2}\\
&\leq C\|b_j'\|_{\mathrm{H}^{r-1}}.
\end{aligned}$$

Consequently, given $2\leq j\leq n$, we get

$$\left\|\int_{-\pi}^{\pi}K_{1,j}(\cdot,s)\,\mathrm{d}s\right\|_2\leq C[b_1]_{\alpha-1/2}\|\varphi\|_\infty\prod_{i=2}^{n}\|b_i'\|_{\mathrm{H}^{r-1}},\tag{A.1.17}$$

and, by similar arguments,

$$\left\|\int_{-\pi}^{\pi}K_{2,j}(\cdot,s)\,\mathrm{d}s\right\|_2\leq C[b_1]_{\alpha-1/2}\|\varphi\|_\infty\prod_{i=2}^{n}\|b_i'\|_\infty,\qquad 1\leq j\leq m.\tag{A.1.18}$$

Moreover, arguing along the same lines, we infer that

$$\begin{aligned}
&\left(\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}K(\xi,s)\,\mathrm{d}s\right|^2\mathrm{d}\xi\right)^{1/2}\\
&\leq\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|K(\xi,s)|^2\,\mathrm{d}\xi\right)^{1/2}\mathrm{d}s\\
&\leq[b_1]_{\alpha-1/2}\left(\prod_{i=2}^{n}\|b_i'\|_\infty\right)\int_{-\pi}^{\pi}\frac{\|\tau_s\varphi-\varphi\|_2}{|s|^{5/2-\alpha}}\,\mathrm{d}s\\
&\leq[b_1]_{\alpha-1/2}\|\varphi\|_{\mathrm{H}^{r-1}}\left(\prod_{i=2}^{n}\|b_i'\|_\infty\right)\left(\int_{-\pi}^{\pi}|s|^{2(r+\alpha-3)}\,\mathrm{d}s\right)^{1/2}\\
&\leq C[b_1]_{\alpha-1/2}\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=2}^{n}\|b_i'\|_\infty.
\end{aligned}\tag{A.1.19}$$

The estimates (A.1.15)–(A.1.19) with $\alpha=1$ and the relation $[b_1]_{1/2}\leq\|b_1'\|_2$ enable us to conclude (A.1.9). Moreover, (A.1.10) follows from (A.1.16)–(A.1.19) with $\alpha=\bar{r}$. This finishes the proof of (ii).

Claim (iii). We follow the proof of [1, Lemma 5] where this property is established for the non-periodic operator $B_{n,m}$. Recalling Lemma 2.4.3 and (A.1.8), it suffices to show

$$\begin{aligned}[C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]]^2_{\mathrm{W}^{r-1,2}} &= \int_{-\pi}^{\pi} \frac{\|\tau_y(C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]) - C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y \\ &\leq C\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^n \|b_i'\|_{\mathrm{H}^{r-1}}.\end{aligned}$$

To do so, we infer from (A.1.14) that for $\xi \in \mathbb{S}$ and $y \in (-\pi,\pi)\setminus\{0\}$ we have

$$(\tau_y(C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]) - C_{n,m}(\mathbf{a})[\mathbf{b},\varphi])(\xi) = T_1(\xi,y) + T_2(\xi,y) - T_3(\xi,y),$$

where

$$\begin{aligned} T_1(\xi,y) &:= C_{n,m}(\tau_y\mathbf{a})[\tau_y\mathbf{b}, \tau_y\varphi - \varphi](\xi), \\ T_2(\xi,y) &:= \sum_{i=1}^n C_{n,m}(\tau_y\mathbf{a})[(\tau_y b_1,\ldots,\tau_y b_{i-1},\tau_y b_i - b_i, b_{i+1},\ldots,b_n),\varphi](\xi), \\ T_3(\xi,y) &:= \sum_{i=1}^m C_{n+2,m+1}(\tau_y a_1,\ldots,\tau_y a_i, a_i,\ldots,a_m)[(\mathbf{b},\tau_y a_i - a_i, \tau_y a_i + a_i),\varphi](\xi). \end{aligned}$$

We thus get

$$[C_{n,m}(\mathbf{a})[\mathbf{b},\varphi]]^2_{\mathrm{W}^{r-1,2}} \leq C\sum_{k=1}^3 \int_{-\pi}^{\pi} \frac{\|T_k(\cdot,y)\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y.$$

Using (A.1.8), we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\|T_1(\cdot,y)\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y &\leq C^2\prod_{i=1}^n \|b_i'\|_\infty^2 \int_{-\pi}^{\pi} \frac{\|\tau_y\varphi - \varphi\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y \\ &\leq \Big(C\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^n \|b_i'\|_\infty\Big)^2.\end{aligned}$$

In view of (A.1.9), we have

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\|T_2(\cdot,y)\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y &\leq C^2\|\varphi\|^2_{\mathrm{H}^{r-1}}\sum_{j=1}^n\Bigg(\Big(\prod_{\substack{i=1\\ i\neq j}}^n \|b_i'\|^2_{\mathrm{H}^{r-1}}\Big)\int_{-\pi}^{\pi} \frac{\|\tau_y b_j' - b_j'\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y\Bigg) \\ &\leq \Big(C\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^n \|b_i'\|_{\mathrm{H}^{r-1}}\Big)^2,\end{aligned}$$

and by a similar calculation

$$\int_{-\pi}^{\pi} \frac{\|T_3(\cdot,y)\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y \leq \Big(C\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^n \|b_i'\|_{\mathrm{H}^{r-1}}\Big)^2.$$

The previous estimates combined yield (A.1.11).

Claim (iv). This claim is shown for the non-periodic operator $B_{n,m}$ in [1, Lemma 6] and we adapt the proof to our situation. We first set

$$T := C_{n,m}(\mathbf{a})[\mathbf{b},\varphi] - \varphi C_{n-1,m}(\mathbf{a})[(b_2,\ldots,b_n),b_1'],$$

and, by applying (A.1.10), it remains to consider the term

$$[T]^2_{\mathrm{W}^{r-1,2}} = \int_{-\pi}^{\pi} \frac{\|\tau_y T - T\|_2^2}{|y|^{1+2(r-1)}}\,\mathrm{d}y$$

in view of Lemma 2.4.3. To this end, we write for $\xi \in \mathbb{R}$ and $y \in (-\pi, \pi) \setminus \{0\}$,

$$\tau_y T - T = T_1(\xi, y) + T_2(\xi, y) + T_3(\xi, y) + T_4(\xi, y),$$

where

$$\begin{aligned} T_1(\xi, y) &:= C_{n,m}(\tau_y \mathbf{a})[\tau_y \mathbf{b}, \tau_y \varphi - \varphi](\xi) \\ &\quad - (\tau_y \varphi - \varphi) C_{n-1,m}(\tau_y \mathbf{a})[(\tau_y b_2, \ldots, \tau_y b_n), \tau_y b_1'](\xi), \\ T_2(\xi, y) &:= C_{n,m}(\tau_y \mathbf{a})[(\tau_y b_1 - b_1, \tau_y b_2, \ldots, \tau_y b_n), \varphi](\xi) \\ &\quad - \varphi C_{n-1,m}(\tau_y \mathbf{a})[(\tau_y b_2, \ldots, \tau_y b_n), \tau_y b_1' - b_1'](\xi), \\ T_3(\xi, y) &:= C_{n,m}(\tau_y \mathbf{a})[(b_1, \tau_y b_2, \ldots, \tau_y b_n), \varphi](\xi) \\ &\quad - C_{n,m}(\mathbf{a})[\mathbf{b}, \varphi](\xi), \\ T_4(\xi, y) &:= \varphi C_{n-1,m}(\mathbf{a})[(b_2, \ldots, b_n), b_1'](\xi) \\ &\quad - \varphi C_{n-1,m}(\tau_y \mathbf{a})[(\tau_y b_2, \ldots, \tau_y b_n), b_1'](\xi). \end{aligned}$$

To estimate T_1, we use (A.1.8), (A.1.11) with r replaced by r', and the embedding $\mathrm{H}^{r'-1}(\mathbb{S}) \hookrightarrow \mathrm{C}(\mathbb{S})$ to obtain

$$\begin{aligned} \|T_1(\cdot, y)\|_2 &\le \|C_{n,m}(\tau_y \mathbf{a})[\tau_y \mathbf{b}, \tau_y \varphi - \varphi]\|_2 \\ &\quad + C\|\tau_y \varphi - \varphi\|_2 \|C_{n-1,m}(\tau_y \mathbf{a})[(\tau_y b_2, \ldots, \tau_y b_n), \tau_y b_1']\|_{\mathrm{H}^{r'-1}} \\ &\le C\|\tau_y \varphi - \varphi\|_2 \|b_1'\|_{\mathrm{H}^{r'-1}} \prod_{i=2}^n \|b_i'\|_{\mathrm{H}^{r-1}}. \end{aligned} \tag{A.1.20}$$

Considering the second term, we apply (A.1.10) with $\bar{r} = r' - r + 1 \in (5/2 - r, 1)$ to infer

$$\|T_2(\cdot, y)\|_2 \le C\|\tau_y b_1 - b_1\|_{\mathrm{H}^{r'-r+1}} \|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^n \|b_i'\|_{\mathrm{H}^{r-1}}. \tag{A.1.21}$$

To handle the remaining two terms, we first use (A.1.14) to write

$$\begin{aligned} T_3(\cdot, y) &= \sum_{i=2}^n C_{n,m}(\tau_y \mathbf{a})[(b_1, \ldots, b_{i-1}, \tau_y b_i - b_i, \tau_y b_{i+1}, \ldots, \tau_y b_n), \varphi] \\ &\quad - \sum_{i=1}^m C_{n+2,m+1}(\tau_y a_1, \ldots, \tau_y a_i, a_i, \ldots, a_m)[(\mathbf{b}, \tau_y a_i - a_i, \tau_y a_i + a_i), \varphi], \\ T_4(\cdot, y) &= \varphi \sum_{i=2}^n C_{n-1,m}(\mathbf{a})[(\tau_y b_2, \ldots, \tau_y b_{i-1}, b_i - \tau_y b_i, b_{i+1}, \ldots, b_n), b_1'] \\ &\quad + \varphi \sum_{i=1}^m C_{n+1,m+1}(a_1, \ldots, a_i, \tau_y a_i, \ldots \tau_y a_m)[(\mathbf{b}_1, \tau_y a_i - a_i, \tau_y a_i + a_i), b_1'], \end{aligned}$$

where $\mathbf{b}_1 = (b_2, \ldots, b_n)$, see (4.4.1). Applying (A.1.9) with r replaced by r' to every term above, we get

$$\begin{aligned} \|T_3(\cdot, y)\|_2 + \|T_4(\cdot, y)\|_2 &\le C \sum_{i=2}^n \Bigg(\|\tau_y b_i' - b_i'\|_2 \|\varphi\|_{\mathrm{H}^{r'-1}} \prod_{\substack{j=1 \\ j \ne i}} \|b_j'\|_{\mathrm{H}^{r'-1}} \Bigg) \\ &\quad + C \sum_{i=1}^m \Bigg(\|\tau_y a_i' - a_i'\|_2 \|\varphi\|_{\mathrm{H}^{r'-1}} \prod_{j=1} \|b_j'\|_{\mathrm{H}^{r'-1}} \Bigg). \end{aligned} \tag{A.1.22}$$

Combining (A.1.20)–(A.1.22), we arrive at

$$\begin{aligned} &[T]_{\mathrm{W}^{r-1,2}} \\ &\le C\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^n \|b_i'\|_{\mathrm{H}^{r-1}} \left[\|b_1\|_{\mathrm{H}^{r'}} + \left(\int_{-\pi}^{\pi} \frac{\|\tau_y b_1 - b_1\|_{\mathrm{H}^{r'-r+1}}^2}{|y|^{1+2(r-1)}} \, \mathrm{d}y \right)^{1/2} \right]. \end{aligned} \tag{A.1.23}$$

Concerning the integral term, we claim that, given $1 < s' < s < 2$, we have

$$\int_{-\pi}^{\pi} \frac{\|\tau_y f - f\|_{\mathrm{H}^{s'-s+1}}^2}{|y|^{1+2(s-1)}}\,\mathrm{d}y \le C\|f\|_{\mathrm{H}^{s'}}^2 \tag{A.1.24}$$

for all $f \in \mathrm{H}^{s'}(\mathbb{S})$. To show this, we first recall Proposition 2.4.2 (iv) to obtain

$$\|\tau_y f - f\|_{\mathrm{H}^{s'-s+1}}^2 = 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s'-s+1}|e^{iky}-1|^2|\hat{f}(k)|^2, \qquad f \in \mathrm{H}^{s'}(\mathbb{S}), \quad y \in \mathbb{R}.$$

Next, we use (2.4.5)–(2.4.6) to obtain

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\|\tau_y f - f\|_{\mathrm{H}^{s'-s+1}}^2}{|y|^{1+2(s-1)}}\,\mathrm{d}y &= 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s'-s+1}|\hat{f}(k)|^2 \int_{-\pi}^{\pi} \frac{|e^{iky}-1|^2}{|y|^{1+2(s-1)}}\,\mathrm{d}y \\ &\le C2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s'-s+1}|k|^{2(s-1)}|\hat{f}(k)|^2 \\ &\le C\|f\|_{\mathrm{H}^{s'}}^2, \qquad f \in \mathrm{H}^{s'},\end{aligned}$$

which establishes the claim (A.1.24). Applying (A.1.24) with $s' = r'$ and $s = r$ to (A.1.23) yields the desired result (A.1.12). ☐

Next, we consider the operators $A_{n,m}^{\ell,q}$ and $B_{n,m}^{p,q}$, $p \ge 1$, and show that they are regularizing.

Lemma A.1.3. *Let n, m, p, $q \in \mathbb{N}_0$ with $1 \le p \le n+q+1$, $\ell \in \{1,2\}$, $r \in (3/2,2)$, and let $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathrm{W}^{1,\infty}(\mathbb{S})^{m+n+q}$ be given. Then,*

$$A_{n,m}^{\ell,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot],\ B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot] \in \mathcal{L}(\mathrm{L}^1(\mathbb{S}),\mathrm{C}(\mathbb{S})), \tag{A.1.25}$$

and there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a}',\mathbf{b}')\|_\infty$ such that for all $\varphi \in \mathrm{L}^1(\mathbb{S})$, we have

$$\|A_{n,m}^{\ell,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_\infty + \|B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_\infty \le C\|\varphi\|_1 \prod_{i=1}^{q}\|c_i'\|_\infty. \tag{A.1.26}$$

Moreover, if $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$, there exists a constant $C > 0$ depending only on n, m, q, and $\|(\mathbf{a}',\mathbf{b}')\|_\infty$ such that for all $\varphi \in \mathrm{C}(\mathbb{S})$, we have

$$\|A_{n,m}^{1,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_{\mathrm{C}^1} \le C\|\varphi\|_\infty \prod_{i=1}^{q}\|c_i\|_{\mathrm{C}^1}. \tag{A.1.27}$$

Proof. To start, we fix $\varphi \in \mathrm{C}(\mathbb{S})$ and denote the kernels of the integral operators $A_{n,m}^{\ell,q}$ and $B_{n,m}^{p,q}$ by K_A and K_B, respectively, that is,

$$\begin{aligned}A_{n,m}^{\ell,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi](\xi) &= \int_{-\pi}^{\pi} K_A(\xi,s)\varphi(\xi-s)\,\mathrm{d}s, \\ B_{n,m}^{p,q}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi](\xi) &= \int_{-\pi}^{\pi} K_B(\xi,s)\varphi(\xi-s)\,\mathrm{d}s, \qquad \xi \in \mathbb{R}.\end{aligned} \tag{A.1.28}$$

We begin by establishing (A.1.26) for $B_{n,m}^{p,q}$. Since $p \ge 1$ and $n+q+1 \ge p$, we infer from (A.1.4) that

$$|K_B(\xi,s)| \le \left(\prod_{i=1}^{n}\|b_i'\|_\infty\right)\left(\prod_{i=1}^{q}\|c_i'\|_\infty\right)|s|^{p-1}\Big|\frac{s/2}{t_{[s]}}\Big|^{n+q+1-p} \le C\prod_{i=1}^{q}\|c_i'\|_\infty$$

for $\xi \in \mathbb{R}$ and $0 \neq s \in (-\pi, \pi)$, which proves (A.1.26) for $B^{p,q}_{n,m}$.

Since $A^{\ell,q}_{n,m}$ is linear in c_i, $1 \leq i \leq q$, it suffices to establish the estimate (A.1.26) for $A^{\ell,q}_{n,m}$ under the assumption that $\|\mathbf{c}'\|_\infty \leq 1$. Let $F : \mathbb{R}^{n+q+m} \to \mathbb{R}$ be the locally Lipschitz continuous function defined by

$$F(x, y, z) = \frac{1}{2\pi} \frac{\Big(\prod_{i=1}^{n} x_i\Big)\Big(\prod_{i=1}^{q} y_i\Big)}{\prod_{i=1}^{m}(1 + z_i^2)}, \qquad (x, y, z) \in \mathbb{R}^{n+q+m}. \tag{A.1.29}$$

Given $\xi \in \mathbb{R}$, $s \neq 0$, and $\mathbf{d} = (d_1, \ldots, d_l) \in \mathrm{W}^{1,\infty}(\mathbb{S})^l$, $l \in \mathbb{N}$, we introduce the shorthand notation

$$\frac{T_{[\xi,s]}\mathbf{d}}{t_{[s]}} := \Big(\frac{T_{[\xi,s]}d_1}{t_{[s]}}, \ldots, \frac{T_{[\xi,s]}d_l}{t_{[s]}}\Big). \tag{A.1.30}$$

Together with (A.1.4), we may now estimate for $\xi \in \mathbb{R}$ and $0 \neq s \in (-\pi, \pi)$,

$$\begin{aligned} |K_A(\xi, s)| \leq & C\Big|\frac{1}{t^\ell_{[s]}} - \frac{1}{(s/2)^\ell}\Big| \\ & + \Big|\frac{1}{(s/2)^\ell}\Big| \Big| F\Big(\frac{T_{[\xi,s]}\mathbf{b}}{t_{[s]}}, \frac{\delta_{[\xi,s]}\mathbf{c}/2}{t_{[s]}}, \frac{T_{[\xi,s]}\mathbf{a}}{t_{[s]}}\Big) \\ & \qquad - F\Big(\frac{\delta_{[\xi,s]}\mathbf{b}/2}{s/2}, \frac{\delta_{[\xi,s]}\mathbf{c}/2}{s/2}, \frac{\delta_{[\xi,s]}\mathbf{a}/2}{s/2}\Big)\Big|. \end{aligned} \tag{A.1.31}$$

The estimates (A.1.5) applied to (A.1.31) immediately imply that

$$|K_A(\xi, s)| \leq C|s|^{2-\ell}, \qquad 0 \neq s \in (-\pi, \pi), \quad \xi \in \mathbb{R}, \quad \ell \in \{1, 2\}, \tag{A.1.32}$$

and the desired estimate (A.1.26) for $A^{\ell,q}_{n,m}$ follows.

Since $\varphi \in \mathrm{C}(\mathbb{S})$, the continuity of parameter integrals implies that both $A^{\ell,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]$ and $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]$ belong to $\mathrm{C}(\mathbb{S})$ and thus the density of $\mathrm{C}(\mathbb{S})$ in $\mathrm{L}^1(\mathbb{S})$ leads us to (A.1.25).

It remains to establish (A.1.27). To this end, we now let $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$ and assume $\varphi \in \mathrm{C}^1(\mathbb{S})$. Since

$$\begin{aligned} &K_A(\cdot, s)\varphi(\cdot - s) \in \mathrm{C}^1(\mathbb{S}), \qquad 0 \neq s \in (-\pi, \pi), \\ &K_A(\xi, \cdot)\varphi(\xi - \cdot) \in \mathrm{C}^1([-\pi, \pi]), \qquad \xi \in \mathbb{R}, \end{aligned}$$

Fubini's theorem and integration by parts imply that $A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]$ is weakly differentiable with

$$\begin{aligned} &\big(A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}, \varphi]\big)'(\xi) \\ &= (K_A(\xi, -\pi) - K_A(\xi, \pi))\varphi(\xi - \pi) + \int_{-\pi}^{\pi} (\partial_\xi K_A + \partial_s K_A)(\xi, s)\varphi(\xi - s)\,\mathrm{d}s \end{aligned}$$

for $\xi \in \mathbb{R}$, hence we have

$$\begin{aligned}
&(A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi])' \\
&= 2\frac{1-(-1)^{n+q+1}}{2\pi}\frac{\prod_{i=1}^{n}(\delta_{[\cdot,\pi]}b_i/\pi)\prod_{i=1}^{q}(\delta_{[\cdot,\pi]}c_i/\pi)}{\prod_{i=1}^{m}\left[1+(\delta_{[\cdot,\pi]}a_i/\pi)^2\right]}\frac{\varphi(\cdot-\pi)}{\pi} \\
&\quad+\sum_{j=1}^{n}\frac{b_j'}{2}\Big(A^{2,q}_{n-1,m}(\mathbf{a}|\mathbf{b}_j)[\mathbf{c},\varphi]-B^{1,q}_{n+1,m}(\mathbf{a}|\mathbf{b},b_j)[\mathbf{c},\varphi]\Big) \\
&\quad+\sum_{j=1}^{q}\frac{c_j'}{2}A^{2,q-1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}_j,\varphi] \\
&\quad-\frac{n+q+1}{2}\Big(A^{2,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]+B^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\Big) \\
&\quad+\sum_{j=1}^{m}\Big[A^{2,q}_{n+2,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j)[\mathbf{c},\varphi] \\
&\qquad+B^{1,q}_{n+2,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j)[\mathbf{c},\varphi] \\
&\qquad-a_j'A^{2,q}_{n+1,m+1}(\mathbf{a},a_j|\mathbf{b},a_j)[\mathbf{c},\varphi] \\
&\qquad+a_j'B^{1,q}_{n+3,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j,a_j)[\mathbf{c},\varphi]\Big],
\end{aligned} \tag{A.1.33}$$

where we use the notation introduced in (4.4.1).

The remaining claim (A.1.27) follows now from the previous relation and (A.1.25) in view of the density of $C^1(\mathbb{S})$ in $C(\mathbb{S})$. □

Using the results obtained for the operators $C_{n,m}$ in Lemma A.1.1 and $A^{\ell,q}_{n,m}$ in Lemma A.1.3, we next study the singular integral operator $B^{0,q}_{n,m}$ viewed as an operator acting on $L^2(\mathbb{S})$.

Lemma A.1.4. *Let n, m, $q \in \mathbb{N}_0$ and let $(\mathbf{a},\mathbf{b}) \in W^{1,\infty}(\mathbb{S})^{m+n}$ be given. Then, there exists a constant $C>0$ that depends only on n, m, q, and $\|(\mathbf{a}',\mathbf{b}')\|_\infty$ such that for all $\mathbf{c} \in W^{1,\infty}(\mathbb{S})^q$ and $\varphi \in L^2(\mathbb{S})$, we have*

$$\|B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_2 \leq C\|\varphi\|_2\prod_{i=1}^{q}\|c_i'\|_\infty. \tag{A.1.34}$$

Proof. Using (A.1.2), the claim follows immediately by applying Lemma A.1.1 (i) to $C_{n+q,m}(\mathbf{a})[(\mathbf{b},\mathbf{c}),\varphi]$, and (A.1.26) to $A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$. □

The next result shows that the (singular) integral operators $B^{p,q}_{n,m}$ are locally Lipschitz continuous with respect to $(\mathbf{a},\mathbf{b},\mathbf{c})$.

Lemma A.1.5. *Given n, m, $q \in \mathbb{N}_0$, and $p \in \mathbb{N}$ with $1 \leq p \leq n+q+1$, we have*

$$B^{0,q}_{n,m} \in C^{1-}(W^{1,\infty}(\mathbb{S})^{m+n}, \mathcal{L}^q_{\mathrm{sym}}(W^{1,\infty}(\mathbb{S}), \mathcal{L}(L^2(\mathbb{S})))), \tag{A.1.35}$$

$$B^{p,q}_{n,m} \in C^{1-}(W^{1,\infty}(\mathbb{S})^{m+n}, \mathcal{L}^q_{\mathrm{sym}}(W^{1,\infty}(\mathbb{S}), \mathcal{L}(L^1(\mathbb{S}), C(\mathbb{S})))). \tag{A.1.36}$$

Proof. Given $(\mathbf{a},\mathbf{b},\mathbf{c}),(\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}})\in \mathrm{W}^{1,\infty}(\mathbb{S})^{m+n+q}$, $\varphi\in \mathrm{C}^\infty(\mathbb{S})$, and $p\in\mathbb{N}_0$, we have

$$\begin{aligned}
&B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]-B^{p,q}_{n,m}(\tilde{\mathbf{a}}|\tilde{\mathbf{b}})[\tilde{\mathbf{c}},\varphi]\\
&=\sum_{i=1}^{q}B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[(\tilde c_1,\ldots,\tilde c_{i-1},c_i-\tilde c_i,c_{i+1},\ldots,c_q),\varphi]\\
&\quad+\sum_{i=1}^{n}\big(B^{p,q}_{n,m}(\mathbf{a}|\tilde b_1,\ldots,\tilde b_{i-1},b_i,\ldots,b_n)\\
&\qquad\qquad-B^{p,q}_{n,m}(\mathbf{a}|\tilde b_1,\ldots,\tilde b_i,b_{i+1},\ldots,b_n)\big)[\tilde{\mathbf{c}},\varphi]\\
&\quad+\sum_{i=1}^{m}\big(B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},\tilde a_i,\tilde a_i)\\
&\qquad\qquad-B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},\tilde a_i,a_i)\big)[\tilde{\mathbf{c}},\varphi]\\
&\quad+\sum_{i=1}^{m}\big(B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},\tilde a_i,a_i)\\
&\qquad\qquad-B^{p,q}_{n+2,m+1}(\tilde a_1,\ldots,\tilde a_i,a_i,\ldots,a_m|\tilde{\mathbf{b}},a_i,a_i)\big)[\tilde{\mathbf{c}},\varphi].
\end{aligned}\tag{A.1.37}$$

The first term on the right-hand side may be estimated by using Lemma A.1.4 if $p=0$ and Lemma A.1.3 whenever $p\geq 1$. For the remaining terms, it thus suffices to show that, given $d,\tilde d\in \mathrm{W}^{1,\infty}(\mathbb{S})$, we have

$$\begin{aligned}
&\|B^{0,q}_{n+1,m}(\mathbf{a}|\mathbf{b},d)[\mathbf{c},\varphi]-B^{0,q}_{n+1,m}(\mathbf{a}|\mathbf{b},\tilde d)[\mathbf{c},\varphi]\|_2\\
&\leq C\|d-\tilde d\|_{\mathrm{W}^{1,\infty}}\|\varphi\|_2
\end{aligned}\tag{A.1.38}$$

for (A.1.35), and

$$\begin{aligned}
&\|B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))[\mathbf{c},\varphi]-B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde d))[\mathbf{c},\varphi]\|_\infty\\
&\leq C\|d-\tilde d\|_{\mathrm{W}^{1,\infty}}\|\varphi\|_1,\qquad p\geq 1,
\end{aligned}\tag{A.1.39}$$

for (A.1.36), respectively, with a constant C depending only on $\|(\mathbf{a},\mathbf{b},\mathbf{c},d,\tilde d)\|_{\mathrm{W}^{1,\infty}}$ and n, m, p, q.

With $F:\mathbb{R}^{n+q+m}\to\mathbb{R}$ denoting the smooth function defined in (A.1.29), we compute, by using also the notation (A.1.30), that

$$\begin{aligned}
&\big(B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))[\mathbf{c},\varphi]-B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde d))[\mathbf{c},\varphi]\big)(\xi)\\
&=\mathrm{PV}\!\int_{-\pi}^{\pi}F\Big(\frac{T_{[\xi,s]}\mathbf{b}}{t_{[s]}},\frac{\delta_{[\xi,s]}\mathbf{c}/2}{t_{[s]}},\frac{T_{[\xi,s]}\mathbf{a}}{t_{[s]}}\Big)\frac{T_{[\xi,s]}d-T_{[\xi,s]}\tilde d}{t_{[s]}}\frac{\varphi(\xi-s)}{t_{[s]}^{1-p}}\,\mathrm{d}s\\
&=B^{p,q+1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},d-\tilde d,\varphi](\xi)-\int_{-\pi}^{\pi}K(\xi,s)\varphi(\xi-s)\,\mathrm{d}s
\end{aligned}\tag{A.1.40}$$

for $\xi\in\mathbb{R}$ and $p\in\mathbb{N}_0$, where, given $\xi\in\mathbb{R}$ and $0\neq s\in(-\pi,\pi)$, we set

$$\begin{aligned}
K(\xi,s):=&F\Big(\frac{T_{[\xi,s]}\mathbf{b}}{t_{[s]}},\frac{\delta_{[\xi,s]}\mathbf{c}/2}{t_{[s]}},\frac{T_{[\xi,s]}\mathbf{a}}{t_{[s]}}\Big)\\
&\times\frac{(\delta_{[\xi,s]}d/2-T_{[\xi,s]}d)-(\delta_{[\xi,s]}\tilde d/2-T_{[\xi,s]}\tilde d)}{t_{[s]}^{2-p}}.
\end{aligned}\tag{A.1.41}$$

The function $B^{p,q+1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},d-\tilde d,\varphi]$ may be estimated by using Lemma A.1.4 if $p=0$ and Lemma A.1.3 for $p\geq 1$, and we are left to estimate the integral term. To this end, we rely on (A.1.4) and (A.1.7) to obtain that $|K(\xi,s)| \leq C\|d-\tilde{d}\|_\infty$ for all $\xi \in \mathbb{R}$ and $0 \neq s \in (-\pi,\pi)$, and therefore

$$\begin{aligned}\Big|\int_{-\pi}^{\pi} K(\xi,s)\varphi(\xi-s)\,\mathrm{d}s\Big| &\leq C\|d-\tilde{d}\|_\infty \|\varphi\|_1\\ &\leq C\|d-\tilde{d}\|_\infty\|\varphi\|_2, \qquad \xi\in\mathbb{R}.\end{aligned}$$

This completes the proof. □

As a next step, we consider the operator $B^{0,q}_{n,m}$ viewed as an operator acting on $\mathrm{H}^{r-1}(\mathbb{S})$, where $r \in (3/2, 2)$. The results of the lemma below are used in Section 4.3 when studying the hydrodynamic double-layer potential operator in $\mathrm{H}^{r-1}(\mathbb{S})$.

Lemma A.1.6. *Let n, m, $q \in \mathbb{N}_0$, and $(\mathbf{a},\mathbf{b}) \in \mathrm{H}^r(\mathbb{S})^{m+n}$, with $r \in (3/2,2)$, be given.*

(i) *There exists a constant $C > 0$ that depends only on n, m, q, r, and $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})^q$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\big\|B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big\|_{\mathrm{H}^{r-1}} \leq C\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^{q}\|c_i\|_{\mathrm{H}^r}. \tag{A.1.42}$$

(ii) *Let $q \geq 1$. Then, there exists a constant $C > 0$ that depends only on n, m, q, and $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ and $\mathbf{c} \in \mathrm{H}^1(\mathbb{S}) \times \mathrm{H}^r(\mathbb{S})^{q-1}$, we have*

$$\big\|B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big\|_2 \leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=2}^{q}\|c_i\|_{\mathrm{H}^r}. \tag{A.1.43}$$

Moreover, $B^{0,q}_{n,m} \in \mathrm{C}^{1-}(\mathrm{H}^r(\mathbb{S})^{m+n}, \mathcal{L}^{q+1}(\mathrm{H}^1(\mathbb{S}) \times \mathrm{H}^r(\mathbb{S})^{q-1} \times \mathrm{H}^{r-1}(\mathbb{S}), \mathrm{L}^2(\mathbb{S})))$.

(iii) *Let d, $\tilde{d} \in \mathrm{H}^1(\mathbb{S})$ be given. Then, there exists a constant $C > 0$ that depends only on n, m, q, $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$, and $\|(d,\tilde{d})\|_{\mathrm{H}^1}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ and $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})$, we have*

$$\begin{aligned}&\big\|B^{0,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))[\mathbf{c},\varphi] - B^{0,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi]\big\|_2\\ &\leq C\|(d-\tilde{d})'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^{q}\|c_i\|_{\mathrm{H}^r}.\end{aligned} \tag{A.1.44}$$

Proof. *Claim* (i). The claim follows immediately from (A.1.2), Lemma A.1.1 (iii), and (A.1.27).

Claim (ii). We use (A.1.2) and then (A.1.9) to estimate the term $C_{n+q,m}(\mathbf{a})[(\mathbf{b},\mathbf{c}),\varphi]$ by the right-hand side of (A.1.43). We are thus left with the term $A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$. Using a telescoping sum argument together with (A.1.5), it is easy to see that the kernel K_A, see (A.1.28), of $A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\cdot]$ satisfies

$$|K_A(\xi,s)| \leq C|\delta_{[\xi,s]}c_1|\prod_{i=2}^{q}\|c_i'\|_\infty, \tag{A.1.45}$$

where C depends only on $\|(\mathbf{a}',\mathbf{b}')\|_\infty$. Applying (A.1.6) to (A.1.45), we have

$$\begin{aligned}\|A^{1,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_\infty &\leq C\|c_1'\|_2\|\varphi\|_\infty\Big(\prod_{i=2}^{q}\|c_i'\|_\infty\Big)\int_{-\pi}^{\pi}|s|^{1/2}\,\mathrm{d}s\\ &\leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=2}^{q}\|c_i\|_{\mathrm{H}^r},\end{aligned}$$

which proves (A.1.43).

Recalling (A.1.37), the local Lipschitz continuity property stated in (ii) follows by showing that for $q \geq 1$, we have

$$\|(B^{0,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d)) - B^{0,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi]\|_2 \leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^{q}\|c_i\|_{\mathrm{H}^r}$$

for all $(\mathbf{a},\mathbf{b}) \in \mathrm{H}^r(\mathbb{S})^{m+n}$, $c_1 \in \mathrm{H}^1(\mathbb{S})$, $c_2,\ldots,c_q,d,\tilde{d} \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, with a constant $C > 0$ that depends only on $\|(\mathbf{a},\mathbf{b},d,\tilde{d})\|_{\mathrm{H}^r}$. To show this, we recall (A.1.40) (with $p = 0$) and estimate the term $B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},d-\tilde{d},\varphi]$ by (A.1.43). Concerning the integral term, using (A.1.3)–(A.1.7), we have

$$|K(\xi,s)| \leq C\frac{|\delta_{[\xi,s]}c_1|}{|s|}\frac{|\delta_{[\xi,s]}(d-\tilde{d})|}{|s|}\Big(\prod_{i=2}^{q}\|c_i'\|_\infty\Big)|s|^{p+1}, \tag{A.1.46}$$

and therefore

$$\Big\|\int_{-\pi}^{\pi}K(\cdot,s)\varphi(\cdot-s)\,\mathrm{d}s\Big\|_\infty \leq C\|c_1'\|_2\|\varphi\|_\infty\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^{q}\|c_i\|_{\mathrm{H}^r}. \tag{A.1.47}$$

Therewith, the proof of the local Lipschitz continuity property is complete.

Claim (iii). After using (A.1.2) and (A.1.9), it remains to estimate the term

$$A^{1,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))[\mathbf{c},\varphi] - A^{1,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi],$$

for which, appealing to (A.1.3)–(A.1.7), we obtain

$$\begin{aligned}
&\|A^{1,q}_{n+1,m}(\mathbf{a}|\mathbf{b},d)[\mathbf{c},\varphi] - A^{1,q}_{n+1,m}(\mathbf{a}|\mathbf{b},\tilde{d})[\mathbf{c},\varphi]\|_\infty\\
&\leq C\|(d-\tilde{d})'\|_2\|\varphi\|_\infty\Big(\prod_{i=1}^{q}\|c_i'\|_\infty\Big)\int_{-\pi}^{\pi}|s|^{1/2}\,\mathrm{d}s\\
&\leq C\|(d-\tilde{d})'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^{q}\|c_i\|_{\mathrm{H}^r},
\end{aligned}$$

which proves (A.1.44). □

Next, we provide an analogue lemma to Lemma A.1.4 in the case $p \geq 1$ and show that the operator $B^{p,q}_{n,m}$ has a regularizing effect.

Lemma A.1.7. *Let n, m, p, $q \in \mathbb{N}_0$, $1 \leq p \leq n+q+1$, and $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$. Then, there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a}',\mathbf{b}')\|_\infty$ such that for all $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_{\mathrm{H}^1} \leq C\|\varphi\|_2\prod_{i=1}^{q}\|c_i'\|_\infty. \tag{A.1.48}$$

Proof. We first assume that $(\mathbf{a},\mathbf{b},\mathbf{c},\varphi) \in \mathrm{C}^\infty(\mathbb{S})^{m+n+q+1}$. Recalling the notation introduced in (A.1.28), the theorem on the differentiation of parameter integrals ensures that $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is continuously differentiable with

$$(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi])'(\xi) = \int_{-\pi}^{\pi}(\partial_\xi K_B(\xi,s))\varphi(\xi-s) - K_B(\xi,s)\partial_s(\varphi(\xi-s))\,\mathrm{d}s, \qquad \xi \in \mathbb{R}.$$

Using integration by parts, we then get

$$\begin{aligned}
&(B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi])' \\
&= \frac{1}{2}\sum_{j=1}^{n} b_j'(B^{p-1,q}_{n-1,m}(\mathbf{a}|\mathbf{b}_j) - B^{p+1,q}_{n+1,m}(\mathbf{a}|\mathbf{b},b_j))[\mathbf{c},\varphi] \\
&\quad + \frac{1}{2}\sum_{j=1}^{q} c_j' B^{p-1,q-1}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c}_j,\varphi] \\
&\quad + \sum_{j=1}^{m} a_j'(B^{p+1,q}_{n+3,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j,a_j) - B^{p-1,q}_{n+1,m+1}(\mathbf{a},a_j|\mathbf{b},a_j))[\mathbf{c},\varphi] \\
&\quad + \sum_{j=1}^{m} (B^{p-1,q}_{n+2,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j) + B^{p+1,q}_{n+2,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j))[\mathbf{c},\varphi] \\
&\quad + \frac{p-n-q-1}{2}(B^{p-1,q}_{n,m}(\mathbf{a}|\mathbf{b}) + B^{p+1,q}_{n,m}(\mathbf{a}|\mathbf{b}))[\mathbf{c},\varphi],
\end{aligned} \tag{A.1.49}$$

where we used notation (4.4.1) and make the observation that the last term is meaningful only if $1 \le p \le n+q$, otherwise it is not present in the formula above. A standard density argument together with Lemma A.1.5 ensures that $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi] \in \mathrm{H}^1(\mathbb{S})$ for all $(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathrm{C}^1(\mathbb{S})^{m+n+q}$ and $\varphi \in \mathrm{L}^2(\mathbb{S})$, while the estimate (A.1.48) is a direct consequence of Lemma A.1.3 and Lemma A.1.4. □

The next lemma is the counterpart to Lemma A.1.6 for the operator $B^{p,q}_{n,m}$, if $p \ge 1$. While the regularity properties established for the operators $B^{0,q}_{n,m}$ in Lemma A.1.6 would suffice for $B^{p,q}_{n,m}$ in our analysis in Section 4.3, we still include the stronger regularity properties for the sake of completeness in the following lemma.

Lemma A.1.8. *Let n, m, p, $q \in \mathbb{N}_0$, with $1 \le p \le n+q+1$, and $(\mathbf{a},\mathbf{b}) \in \mathrm{H}^r(\mathbb{S})^{m+n}$, with $r \in (3/2, 2)$, be given.*

(i) *There exists a constant $C > 0$ that depends only on n, m, p, q, r, and $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})^q$ and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_{\mathrm{H}^r} \le C\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{q} \|c_i\|_{\mathrm{H}^r}. \tag{A.1.50}$$

(ii) *Let $q \ge 1$. Then, there exists a constant $C > 0$ that depends only on n, m, p, q, and $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ and $\mathbf{c} \in \mathrm{H}^1(\mathbb{S}) \times \mathrm{H}^r(\mathbb{S})^{q-1}$, we have*

$$\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_{\mathrm{H}^1} \le C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^{q} \|c_i\|_{\mathrm{H}^r}. \tag{A.1.51}$$

Moreover, $B^{p,q}_{n,m} \in \mathrm{C}^{1-}(\mathrm{H}^r(\mathbb{S})^{m+n}, \mathcal{L}^{q+1}(\mathrm{H}^1(\mathbb{S}) \times \mathrm{H}^r(\mathbb{S})^{q-1} \times \mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^1(\mathbb{S})))$.

(iii) *Let d, $\tilde{d} \in \mathrm{H}^1(\mathbb{S})$ be given. Then, there exists a constant $C > 0$ that depends only on n, m, p, q, $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^r}$, and $\|(d,\tilde{d})\|_{\mathrm{H}^1}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ and $\mathbf{c} \in \mathrm{H}^r(\mathbb{S})$, we have*

$$\begin{aligned}
&\|B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))[\mathbf{c},\varphi] - B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi]\|_{\mathrm{H}^1} \\
&\le C\|(d-\tilde{d})'\|_2\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=1}^{q} \|c_i\|_{\mathrm{H}^r}.
\end{aligned} \tag{A.1.52}$$

Proof. *Claim* (i). The estimate (A.1.50) follows by applying the estimate (A.1.48) to $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ and the terms on the right-hand side of (A.1.49) with $p \geq 1$ while the terms with $p = 0$ are estimated with the help of (A.1.42).

Claim (ii). Using (A.1.6), we get

$$\begin{aligned} \|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\|_2 &\leq C\|\varphi\|_\infty \Big(\prod_{i=2}^q \|c_i'\|_\infty\Big) \int_{-\pi}^{\pi} \frac{|T_{[\xi,s]}c_1|}{|s/2|^{1/2}}|s|^{p-3/2}\,\mathrm{d}s \\ &\leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}} \prod_{i=2}^q \|c_i\|_{\mathrm{H}^r}. \end{aligned} \tag{A.1.53}$$

After applying (A.1.53) to the terms on the right-hand side of (A.1.49) with $p \geq 1$ and (A.1.43) to the terms with $p = 0$, the claim (A.1.51) follows.

Due to (A.1.37), the local Lipschitz continuity follows by showing that for $q \geq 1$, we have

$$\|(B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d)) - B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi]\|_{\mathrm{H}^1} \leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^q\|c_i\|_{\mathrm{H}^r}$$

for all $(\mathbf{a},\mathbf{b}) \in \mathrm{H}^r(\mathbb{S})^{m+n}$, $c_1 \in \mathrm{H}^1(\mathbb{S})$, $c_2,\ldots,c_q,d,\tilde{d} \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, with a constant $C > 0$ that depends only on $\|(\mathbf{a},\mathbf{b},d,\tilde{d})\|_{\mathrm{H}^r}$. To do so, we use (A.1.49) to infer that $(B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d)) - B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi])'$ is a linear combination of terms of the form

$$\begin{aligned} T_1 &= (d-\tilde{d})'B^{\bar{p},q}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[\mathbf{c},\varphi], \\ T_2 &= d'B^{\bar{p},q}_{\bar{n}+2,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},d,d))[\mathbf{c},\varphi] - \tilde{d}'B^{\bar{p},q}_{\bar{n}+2,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},\tilde{d},\tilde{d}))[\mathbf{c},\varphi], \\ T_3 &= h(B^{\bar{p},q}_{\bar{n}+1,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},d)) - B^{\bar{p},q}_{\bar{n}+1,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},\tilde{d})))[\mathbf{c},\varphi], \\ T_4 &= c_1'(B^{\bar{p},q-1}_{\bar{n}+1,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},d)) - B^{\bar{p},q-1}_{\bar{n}+1,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},\tilde{d})))[\mathbf{c}_1,\varphi], \\ T_5 &= c_i'(B^{\bar{p},q-1}_{\bar{n}+1,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},d)) - B^{\bar{p},q-1}_{\bar{n}+1,\bar{m}}(\bar{\mathbf{a}}|(\bar{\mathbf{b}},\tilde{d})))[\mathbf{c}_i,\varphi], \qquad 2 \leq i \leq q, \end{aligned}$$

where $\bar{n}$, $\bar{m}$, $\bar{p} \in \mathbb{N}_0$ satisfy $\bar{p} \leq \bar{n}+q+1$ and may change from line to line, $h \in \mathrm{H}^{r-1}(\mathbb{S})$, and $(\bar{\mathbf{a}},\bar{\mathbf{b}}) \in \mathrm{H}^r(\mathbb{S})^{\bar{m}+\bar{n}}$. However, we have

$$\|h\|_{\mathrm{H}^{r-1}} + \|(\bar{\mathbf{a}},\bar{\mathbf{b}})\|_{\mathrm{H}^r} \leq C$$

with $C > 0$ only depending on $\|(\mathbf{a},\mathbf{b},d,\tilde{d})\|_{\mathrm{H}^r}$. Using (A.1.43) if $\bar{p} = 0$ and (A.1.53) if $\bar{p} \geq 1$, we can handle T_1 by

$$\begin{aligned} \|T_1\|_2 &\leq \|(d-\tilde{d})'\|_\infty\|B^{\bar{p},q}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[\mathbf{c},\varphi]\|_2 \\ &\leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^q\|c_i\|_{\mathrm{H}^r}. \end{aligned}$$

It is easy to see that T_2 can be decomposed into a sum involving terms of the form T_1 and T_3. Concerning T_3, we first recall (A.1.40)–(A.1.41) and estimate the term $hB^{\bar{p},q+1}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[\mathbf{c},d-\tilde{d},\varphi]$ with the help of (A.1.43) if $p = 0$ and (A.1.53) if $p \geq 1$. Using (A.1.47), we can also handle the integral term to get

$$\|T_3\|_2 \leq C\|c_1'\|_2\|\varphi\|_\infty\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^q\|c_i\|_{\mathrm{H}^r}.$$

We are thus left with T_4 and T_5. To estimate T_4, we use (A.1.40)–(A.1.41), (A.1.47) to estimate the integral term, and (A.1.42) if $\bar{p}=0$ and (A.1.50) if $\bar{p}\geq 1$, to obtain

$$\begin{aligned}\|T_4\|_2 &\leq C\|c_1'\|_2\Big(\|B^{\bar{p},q}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[(\mathbf{c}_1,d-\tilde{d}),\varphi]\|_{\mathrm{H}^{r-1}}+\Big\|\int_{-\pi}^{\pi}K(\cdot,s)\varphi(\cdot-s)\,\mathrm{d}s\Big\|_\infty\Big)\\ &\leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^q\|c_i\|_{\mathrm{H}^r}.\end{aligned}$$

Finally, we estimate T_5 by (A.1.40)–(A.1.41) but rely on (A.1.43) if $\bar{p}=0$ and (A.1.51) if $\bar{p}\geq 1$, to get

$$\begin{aligned}\|T_5\|_2 &\leq C\|c_j'\|_\infty\Big(\|B^{\bar{p},q}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[(\mathbf{c}_j,d-\tilde{d}),\varphi]\|_2+\Big\|\int_{-\pi}^{\pi}K(\cdot,s)\varphi(\cdot-s)\,\mathrm{d}s\Big\|_2\Big)\\ &\leq C\|c_1'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\|d-\tilde{d}\|_{\mathrm{H}^r}\prod_{i=2}^q\|c_i\|_{\mathrm{H}^r},\qquad 2\leq j\leq q.\end{aligned}$$

Since the term $(B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},d))-B^{p,q}_{n+1,m}(\mathbf{a}|(\mathbf{b},\tilde{d}))[\mathbf{c},\varphi]$ is given by T_3 with $h=1$, the local Lipschitz continuity follows.

Claim (iii). Similarly as before, we only need to handle the terms T_1, T_3, T_4, T_5. To estimate T_1, we use (A.1.42) if $\bar{p}=0$ and (A.1.26) if $\bar{p}\geq 1$ to get

$$\begin{aligned}\|T_1\|_2 &\leq \|(d-\tilde{d})'\|_2\|B^{\bar{p},q}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[\mathbf{c},\varphi]\|_\infty\\ &\leq C\|(d-\tilde{d})'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^q\|c_i\|_{\mathrm{H}^r}.\end{aligned}$$

The terms T_3, T_4, T_5 are all estimated the same way and we do it exemplary for T_3. We use the decomposition (A.1.40)–(A.1.41), (A.1.47) to estimate the integral term, and (A.1.43) if $\bar{p}=0$ and (A.1.51) if $\bar{p}\geq 1$ to estimate $B^{\bar{p},q+1}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[\mathbf{c},d-\tilde{d},\varphi]$ and obtain

$$\begin{aligned}\|T_3\|_2 &\leq C\|h\|_\infty\Big(\|B^{\bar{p},q+1}_{\bar{n},\bar{m}}(\bar{\mathbf{a}}|\bar{\mathbf{b}})[(\mathbf{c},d-\tilde{d}),\varphi]\|_2+\Big\|\int_{-\pi}^{\pi}K(\cdot,s)\varphi(\cdot-s)\,\mathrm{d}s\Big\|_2\Big)\\ &\leq C\|(d-\tilde{d})'\|_2\|\varphi\|_{\mathrm{H}^{r-1}}\prod_{i=1}^q\|c_i\|_{\mathrm{H}^r}.\end{aligned}$$

□

The next lemma deals with the boundedness of the operator $B^{0,q}_{n,m}$ in higher order Sobolev spaces.

Lemma A.1.9. *Let n, m, $q\in\mathbb{N}_0$, $(\mathbf{a},\mathbf{b},\mathbf{c})\in\mathrm{H}^2(\mathbb{S})^{m+n+q}$, and $\varphi\in\mathrm{H}^1(\mathbb{S})$ be given. Then, there exists a constant $C>0$ that depends only on n, m, q, and $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^2}$ such that*

$$\big\|B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big\|_{\mathrm{H}^1}\leq C\|\varphi\|_{\mathrm{H}^1}\prod_{i=1}^q\|c_i\|_{\mathrm{H}^2}. \tag{A.1.54}$$

In addition, $B^{0,q}_{n,m}\in\mathrm{C}^{1-}(\mathrm{H}^2(\mathbb{S})^{m+n},\mathcal{L}^q_{\mathrm{sym}}(\mathrm{H}^2(\mathbb{S}),\mathcal{L}(\mathrm{H}^1(\mathbb{S}))))$.

Proof. Assume first that $(\mathbf{a},\mathbf{b},\mathbf{c})\in\mathrm{C}^\infty(\mathbb{S})^{m+n+q}$ and $\varphi\in\mathrm{C}^\infty(\mathbb{S})$. Then, by rewriting the principal value integral as a Riemann integral over the interval $[0,\pi]$, the theorem on the differentiation of parameter-dependent integrals ensures that $B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is continuously differentiable with its derivative given by

$$\begin{aligned}
&\big(B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big)' \\
&= B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi'] + \sum_{j=1}^{q} B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[(c_1,\ldots,c_{j-1},c_j',c_{j+1},\ldots,c_q),\varphi] \\
&\quad + \sum_{j=1}^{n} \big(B^{0,q+1}_{n-1,m}(\mathbf{a}|\mathbf{b}_j)[(\mathbf{c},b_j'),\varphi] - B^{2,q+1}_{n+1,m}(\mathbf{a}|\mathbf{b},b_j)[(\mathbf{c},b_j'),\varphi]\big) \qquad \text{(A.1.55)} \\
&\quad - 2\sum_{j=1}^{m} \big(B^{0,q+1}_{n+1,m+1}(\mathbf{a},a_j|\mathbf{b},a_j)[(\mathbf{c},a_j'),\varphi] \\
&\qquad\qquad - B^{2,q+1}_{n+3,m+1}(\mathbf{a},a_j|\mathbf{b},a_j,a_j,a_j)[(\mathbf{c},a_j'),\varphi]\big).
\end{aligned}$$

A standard density argument together with the local Lipschitz continuity properties established in Lemma A.1.5 and Lemma A.1.6 (ii) show that $B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ is an element of $\mathrm{H}^1(\mathbb{S})$ for all $\varphi\in\mathrm{H}^1(\mathbb{S})$ and $(\mathbf{a},\mathbf{b},\mathbf{c})\in\mathrm{H}^2(\mathbb{S})^{m+n+q}$.

The estimate (A.1.54) is obtained by applying (A.1.34) to $B^{0,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ and to the first term on the right-hand side of (A.1.55), and by applying (A.1.43) to the remaining terms on the right-hand side of (A.1.55). Furthermore, the local Lipschitz continuity follows from the corresponding local Lipschitz continuity properties stated in Lemma A.1.5 and Lemma A.1.6 (ii). □

As before, we obtain stronger estimates for the operator $B^{p,q}_{n,m}$, $p\geq 1$, than for the operator $B^{0,q}_{n,m}$.

Lemma A.1.10. *Let n, m, p, $q\in\mathbb{N}_0$ with $1\leq p\leq n+q+1$, $(\mathbf{a},\mathbf{b},\mathbf{c})\in\mathrm{H}^2(\mathbb{S})^{m+n+q}$, and $\varphi\in\mathrm{H}^1(\mathbb{S})$ be given. Then, there exists a constant $C>0$ that depends only on n, m, p, q, and $\|(\mathbf{a},\mathbf{b})\|_{\mathrm{H}^2}$ such that*

$$\big\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big\|_{\mathrm{H}^2} \leq C\|\varphi\|_{\mathrm{H}^1}\prod_{i=1}^{q}\|c_i\|_{\mathrm{H}^2}. \qquad \text{(A.1.56)}$$

In addition, $B^{p,q}_{n,m}\in\mathrm{C}^{1-}\big(\mathrm{H}^2(\mathbb{S})^{m+n},\mathcal{L}^q_{\mathrm{sym}}(\mathrm{H}^2(\mathbb{S}),\mathcal{L}(\mathrm{H}^1(\mathbb{S}),\mathrm{H}^2(\mathbb{S})))\big)$.

Proof. To start, the estimate

$$\big\|B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]\big\|_2 \leq C\|\varphi\|_{\mathrm{H}^1}\prod_{i=1}^{q}\|c_i\|_{\mathrm{H}^2}$$

follows from (A.1.26). Therefore, it remains to estimate the term $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]'$ in the $\mathrm{H}^1(\mathbb{S})$ norm. To do so, we use (A.1.49) to differentiate $B^{p,q}_{n,m}(\mathbf{a}|\mathbf{b})[\mathbf{c},\varphi]$ and we then apply (A.1.54) to all the terms on the right-hand side with $p=0$ and (A.1.48) to all the terms on the right-hand side with $p\geq 1$ and the claim (A.1.56) follows. Moreover, the local Lipschitz continuity property follows from (A.1.36), (A.1.52), and Lemma A.1.7.

□

As a straightforward consequence of Lemma A.1.10, especially of the formulas (A.1.49) and (A.1.55), we conclude via induction the following result which improves Corollary 4.4.2 if $p\geq 1$.

Corollary A.1.11. *Given n, m, p, $q \in \mathbb{N}_0$ with $1 \le p \le n+q+1$ and $k \in \mathbb{N}$, we have*

$$B^{p,q}_{n,m} \in \mathrm{C}^{1-}\big(\mathrm{H}^{k+1}(\mathbb{S})^{m+n}, \mathcal{L}^q_{\mathrm{sym}}(\mathrm{H}^{k+1}(\mathbb{S}), \mathcal{L}(\mathrm{H}^k(\mathbb{S}), \mathrm{H}^{k+1}(\mathbb{S})))\big).$$

As a last result in this section, we consider the operator B_0 introduced in (3.3.42) and show in the following lemma that it has similar mapping properties as the operator $B^{p,q}_{n,m}$, $p \ge 1$.

Lemma A.1.12. *Let $r \in (3/2, 2)$. Given $f \in \mathrm{H}^r(\mathbb{S})$, there exists a constant $C > 0$ that depends only on $\|f\|_{\mathrm{H}^r}$ such that for all $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, we have*

$$\|B_0(f)[\varphi]\|_{\mathrm{H}^r} \le C\|\varphi\|_{\mathrm{H}^{r-1}}. \tag{A.1.57}$$

Moreover, given $f \in \mathrm{H}^2(\mathbb{S})$, there exists a constant $C > 0$ that depends only on $\|f\|_{\mathrm{H}^2}$ such that for all $\varphi \in \mathrm{H}^1(\mathbb{S})$, we have

$$\|B_0(f)[\varphi]\|_{\mathrm{H}^2} \le C\|\varphi\|_{\mathrm{H}^1}. \tag{A.1.58}$$

Proof. We first prove that $B_0(f) \in \mathcal{L}(\mathrm{L}^2(\mathbb{S}), \mathrm{L}^\infty(\mathbb{S}))$ if $f \in \mathrm{W}^{1,\infty}(\mathbb{S})$. Indeed, using the fact that $\ln(\sin(\cdot/2)^2) \in \mathrm{L}^2(\mathbb{S})$, we deduce, in view of the inequality

$$\big|\ln\big(\sin^2(s/2)+\sinh^2(\delta_{[\xi,s]}f/2)\big)\big| \le \big|\ln\big(\sin^2(s/2)\big)\big| + \ln\big(1+\sinh^2(\pi\|f'\|_\infty)\big), \quad \xi, s \in \mathbb{S},$$

that for $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have

$$\begin{aligned} |B_0(f)[\varphi](\xi)| &\le \int_{-\pi}^{\pi} \big[\big|\ln\big(\sin^2(s/2)\big)\big| + \ln\big(1+\sinh^2(\pi\|f'\|_\infty)\big)\big]|\varphi(\xi - s)|\,\mathrm{d}s \\ &\le C\|\varphi\|_2. \end{aligned}$$

We now assume that $f \in \mathrm{H}^r(\mathbb{S})$ and $\varphi \in \mathrm{C}^\infty(\mathbb{S})$. Using the theorem on the differentiation of parameter integrals and subsequently integration by parts, we find that $B_0(f)[\varphi]$ is continuously differentiable and its derivative is given by

$$(B_0(f)[\varphi])' = f'B_2(f)[\varphi] + B_1(f)[\varphi] \in \mathrm{H}^{r-1}(\mathbb{S}),$$

cf. (3.3.23), (4.1.7), Lemma A.1.6 (i) and Lemma A.1.8 (i). The claim (A.1.57) follows now by a standard density argument in view of Lemma A.1.6 (i) and Lemma A.1.8 (i). Given $f \in \mathrm{H}^2(\mathbb{S})$ and $\varphi \in \mathrm{H}^1(\mathbb{S})$, the claim (A.1.58) is now a direct consequence of (4.1.7), Lemma A.1.9, and Lemma A.1.10. □

A.1.2. Fréchet differentiability

This section is devoted to establishing the following result.

Corollary A.1.13. *Given $r \in (3/2, 2)$ and n, m, p, $q \in \mathbb{N}_0$ with $1 \le p \le n+q+1$, the mappings*

$$\begin{gathered} [f \mapsto B^{0,q}_{n,m}(f)] : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S})), \\ [f \mapsto B_0(f)],\ [f \mapsto B^{p,q}_{n,m}(f)] : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S})), \end{gathered}$$

are smooth.

The proof of Corollary A.1.13 is presented at the end of this section, as it requires some preparation. Let us first note that Lemma A.1.6 (i), Lemma A.1.8 (i), and Lemma A.1.12 ensure that the mappings defined above are well-defined. In order to establish the smoothness of these mappings, we further introduce the operators

$$\begin{aligned}
&B^{0,q,k}_{n,m} : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}^k_{\mathrm{sym}}(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))),\\
&B^{p,q,k}_{n,m} : \mathrm{H}^r(\mathbb{S}) \to \mathcal{L}^k_{\mathrm{sym}}(\mathrm{H}^r(\mathbb{S}), \mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}), \mathrm{H}^r(\mathbb{S}))), \quad 1 \le p \le n+q+k+1,
\end{aligned} \tag{A.1.59}$$

by

$$B^{p,q,k}_{n,m}(f)[f_1,\ldots,f_k][\cdot] := B^{p,q+k}_{n,m}(f,\ldots,f|f,\ldots,f)[(f,\ldots,f,f_1,\ldots,f_k),\cdot].$$

Let us note that $B^{p,q}_{n,m}(f) = B^{p,q,0}_{n,m}(f)$. The next lemma is the main step towards proving the smoothness property for the operator $B^{p,q}_{n,m}$.

Lemma A.1.14. *The mappings* (A.1.59) *are Fréchet differentiable. Furthermore, the Fréchet derivative $\partial B^{p,q,k}_{n,m}(f_0)$ is given by*

$$\begin{aligned}
\partial B^{p,q,k}_{n,m}(f_0)[f][f_1,\ldots,f_k] = \Big(&n(B^{p,q,k+1}_{n-1,m}(f_0) - B^{p+2,q,k+1}_{n+1,m}(f_0))\\
&+ 2m(B^{p+2,q,k+1}_{n+3,m+1}(f_0) - B^{p,q,k+1}_{n+1,m+1}(f_0))\\
&+ qB^{p,q-1,k+1}_{n,m}(f_0)\Big)[f_1,\ldots,f_k,f]
\end{aligned} \tag{A.1.60}$$

for f_0, f, $f_1,\ldots,f_k \in \mathrm{H}^r(\mathbb{S})$, where terms with negative indices are to be neglected.

Proof. Defining $\phi := \phi^{p,q}_{n,m}$ by the formula

$$\phi(\eta,s) := \frac{1}{2\pi}\frac{\left(\frac{\tanh(\eta)}{t_{[s]}}\right)^n\left(\frac{\eta}{t_{[s]}}\right)^q}{\left[1+\left(\frac{\tanh(\eta)}{t_{[s]}}\right)^2\right]^m} t^p_{[s]}, \qquad \eta\in\mathbb{R}, \quad 0\neq s\in(-\pi,\pi),$$

we have for $\xi \in \mathbb{R}$, f, $f_1,\ldots,f_k \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$,

$$B^{p,q,k}_{n,m}(f)[f_1,\ldots,f_k][\varphi](\xi) = \mathrm{PV}\int_{-\pi}^{\pi}\Big(\prod_{i=1}^k \frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)\phi(\delta_{[\xi,s]}f/2,s)\frac{\varphi(\xi-s)}{t_{[s]}}\,\mathrm{d}s,$$

the PV being needed only when $p=0$. Our goal is to prove that

$$\begin{aligned}
&\partial B^{p,q,k}_{n,m}(f_0)[f][f_1,\ldots,f_k][\varphi](\xi)\\
&\quad= \mathrm{PV}\int_{-\pi}^{\pi}\Big(\prod_{i=1}^k \frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)(\delta_{[\xi,s]}f/2)\partial_\eta\phi(\delta_{[\xi,s]}f_0/2,s)\frac{\varphi(\xi-s)}{t_{[s]}}\,\mathrm{d}s
\end{aligned} \tag{A.1.61}$$

for $\xi \in \mathbb{R}$, f_0, f, $f_1,\ldots,f_k \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$, as straightforward computations show that the formulas (A.1.60) and (A.1.61) are equivalent.

Using Taylor's formula, we compute

$$\begin{aligned}
&(B^{p,q,k}_{n,m}(f_0+f) - B^{p,q,k}_{n,m}(f_0) - \partial B^{p,q,k}_{n,m}(f_0)[f])[f_1,\ldots,f_k][\varphi](\xi)\\
&= \mathrm{PV}\int_{-\pi}^{\pi}\bigg\{\Big[\int_0^1(1-\tau)\partial^2_\eta\phi(\delta_{[\xi,s]}f_\tau/2,s)\,\mathrm{d}\tau\Big]\\
&\qquad\times\Big(\prod_{i=1}^k\frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)(\delta_{[\xi,s]}f/2)^2\frac{\varphi(\xi-s)}{t_{[s]}}\bigg\}\,\mathrm{d}s,
\end{aligned} \tag{A.1.62}$$

where $f_\tau := f_0 + \tau f$ for $\tau \in [0,1]$, and $\partial_\xi^2 \phi = \partial_\xi^2 \phi^{p,q}_{n,m}$ is given by

$$\begin{aligned}\partial_\eta^2 \phi^{p,q}_{n,m} = \frac{1}{t_{[s]}^2}\Big\{ & n(n-1)\phi^{p,q}_{n-2,m} + 2nq\phi^{p,q-1}_{n-1,m} + q(q-1)\phi^{p,q-2}_{n,m} \\ & - 2m(2n+1)\phi^{p,q}_{n,m+1} - 2nq\phi^{p+2,q-1}_{n+1,m} - 2n^2\phi^{p+2,q}_{n,m} \\ & + 8m(n+1)\phi^{p+2,q}_{n+2,m+1} + n(n+1)\phi^{p+4,q}_{n+2,m} \\ & - 2m(2n+3)\phi^{p+4,q}_{n+4,m+1} + 4mq\phi^{p+2,q-1}_{n+3,m+1} - 4mq\phi^{p,q-1}_{n+1,m+1} \\ & + 4m(m+1)\phi^{p+4,q}_{n+6,m+2} - 8m(m+1)\phi^{p+2,q}_{n+4,m+2} \\ & + 4m(m+1)\phi^{p,q}_{n+2,m+2}\Big\}\end{aligned} \tag{A.1.63}$$

in $\mathbb{R} \times ((-\pi,\pi) \setminus \{0\})$ and for all $0 \le p \le n+q+k+1$. Recalling (A.1.4), in all the terms on the right-hand side of (A.1.62) where $\phi^{p,q}_{n,m}$ with $p \ge 1$ appears, the PV is not needed and we may interchange the order of integration by using Fubini's theorem.

Assume first that $p \ge 1$. We then infer from (A.1.62) and (A.1.63), after interchanging the order of integration in the last line of (A.1.62), that

$$\begin{aligned}&(B^{p,q,k}_{n,m}(f_0+f) - B^{p,q,k}_{n,m}(f_0) - \partial B^{p,q,k}_{n,m}(f_0)[f])[f_1,\dots,f_k][\varphi] \\ &= \int_0^1 (1-\tau)\Big\{ n(n-1)B^{p,q,k+2}_{n-2,m} + 2nqB^{p,q-1,k+2}_{n-1,m} + q(q-1)B^{p,q-2,k+2}_{n,m} \\ &\qquad - 2m(2n+1)B^{p,q,k+2}_{n,m+1} - 2nqB^{p+2,q-1,k+2}_{n+1,m} - 2n^2B^{p+2,q,k+2}_{n,m} \\ &\qquad + 8m(n+1)B^{p+2,q,k+2}_{n+2,m+1} + n(n+1)B^{p+4,q,k+2}_{n+2,m} \\ &\qquad - 2m(2n+3)B^{p+4,q,k+2}_{n+4,m+1} + 4mqB^{p+2,q-1,k+2}_{n+3,m+1} \\ &\qquad - 4mqB^{p,q-1,k+2}_{n+1,m+1} + 4m(m+1)B^{p+4,q,k+2}_{n+6,m+2} \\ &\qquad - 8m(m+1)B^{p+2,q,k+2}_{n+4,m+2} \\ &\qquad + 4m(m+1)B^{p,q,k+2}_{n+2,m+2}\Big\}(f_\tau)[f_1,\dots,f_k,f,f][\varphi]\,\mathrm{d}\tau.\end{aligned} \tag{A.1.64}$$

Moreover, Lemma A.1.8 (i) implies that there exists a constant $C > 0$ such that for all $\|f\|_{\mathrm{H}^r} \le 1$, we have

$$\begin{aligned}&\|(B^{p,q,k}_{n,m}(f_0+f) - B^{p,q,k}_{n,m}(f_0) - \partial B^{p,q,k}_{n,m}(f_0)[f])[f_1,\dots,f_k]\|_{\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}),\mathrm{H}^r(\mathbb{S}))} \\ &\qquad \le C\|f\|^2_{\mathrm{H}^r}\prod_{i=1}^k \|f_i\|_{\mathrm{H}^r},\end{aligned}$$

which proves (A.1.60) for $p \ge 1$.

Let now $p = 0$. In this case, the formula (A.1.64) is still valid (and defines a function in $\mathrm{H}^{r-1}(\mathbb{S})$). This formula is obtained again by interchanging the order of integration in (A.1.62) via (A.1.63), but slightly more subtle arguments are needed when considering the terms of (A.1.63) with $p = 0$ as the PV symbol appears in front of the first integral in (A.1.62). More precisely, letting

$$I(\xi,s,\tau) := \Big(\prod_{i=1}^k \frac{\delta_{[\xi,s]}f_i/2}{t_{[s]}}\Big)(\delta_{[\xi,s]}f/2)^2(1-\tau)\partial_\eta^2\phi(\delta_{[\xi,s]}f_\tau/2,s)\frac{\varphi(\xi-s)}{t_{[s]}}$$

denote the integrand in (A.1.62), it holds that

$$\begin{aligned}
&\mathrm{PV}\int_{-\pi}^{\pi}\Big(\int_0^1 I(\xi,s,\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&=\int_0^{\pi}\Big(\int_0^1 I(\xi,s,\tau)+I(\xi,-s,\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&=\int_0^{1}\Big(\int_0^{\pi} I(\xi,s,\tau)+I(\xi,-s,\tau)\,\mathrm{d}s\Big)\,\mathrm{d}\tau\\
&=\int_0^{1}\Big(\mathrm{PV}\int_{-\pi}^{\pi} I(\xi,s,\tau)\,\mathrm{d}s\Big)\,\mathrm{d}\tau
\end{aligned}$$

by Fubini's theorem and in view of the estimate

$$\big|I(\xi,s,\tau)+I(\xi,-s,\tau)\big|\leq\frac{C}{|s|^{5/2-r}},\qquad \xi\in\mathbb{R},\quad 0\neq s\in(-\pi,\pi),\quad \tau\in[0,1].$$

Applying Lemma A.1.6 (i) and Lemma A.1.8 (i), we conclude from (A.1.64) that there exists a constant $C>0$ such that for all $\|f\|_{\mathrm{H}^r}\leq 1$, we have

$$\begin{aligned}
&\big\|(B^{p,q,k}_{n,m}(f_0+f)-B^{p,q,k}_{n,m}(f_0)-\partial B^{p,q,k}_{n,m}(f_0)[f])[f_1,\ldots,f_k]\big\|_{\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}))}\\
&\quad\leq C\|f\|^2_{\mathrm{H}^r}\prod_{i=1}^{k}\|f_i\|_{\mathrm{H}^r},
\end{aligned}$$

which proves the claim for $p=0$. □

We now show the Fréchet differentiability of the operator B_0 defined in (3.3.42).

Lemma A.1.15. *Given $r\in(3/2,2)$, the mapping $B_0:\mathrm{H}^r(\mathbb{S})\to\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}),\mathrm{H}^r(\mathbb{S}))$ is Fréchet differentiable and the Fréchet derivative $\partial B_0(f_0)$ is given by*

$$\partial B_0(f_0)[f]=2B^{1,0,1}_{1,1}(f_0)[f]+2B^{3,0,1}_{1,1}(f_0)[f],\qquad f_0,\ f\in\mathrm{H}^r(\mathbb{S}).\tag{A.1.65}$$

Proof. We apply the same strategy as in the proof of Lemma A.1.14. Defining ϕ by the formula

$$\phi(\eta,s):=\frac{1}{2\pi}\ln\Big(\frac{t^2_{[s]}+\tanh^2(\eta)}{(1+t^2_{[s]})(1-\tanh^2(\eta))}\Big),\qquad 0\neq\eta\in\mathbb{R},\quad s\in(-\pi,\pi),$$

we have

$$\begin{aligned}
\partial_\eta\phi(\eta,s)&=\frac{1}{\pi}\frac{(1+t^2_{[s]})\tanh(\eta)}{t^2_{[s]}+\tanh^2(\eta)},\\
\partial^2_\eta\phi(\eta,s)&=\frac{1}{\pi}\frac{(1+t^2_{[s]})(1-\tanh^2(\eta))(t^2_{[s]}-\tanh^2(\eta))}{(t^2_{[s]}+\tanh^2(\eta))^2}.
\end{aligned}\tag{A.1.66}$$

We prove that

$$\partial B_0(f_0)[f][\varphi](\xi)=\int_{-\pi}^{\pi}(\delta_{[\xi,s]}f/2)\partial_\eta\phi(\delta_{[\xi,s]}f_0/2,s)\varphi(\xi-s)\,\mathrm{d}s,\tag{A.1.67}$$

since straightforward calculations show that (A.1.65) and (A.1.67) coincide. Using Taylor's formula, Fubini's theorem, and the formulas (A.1.66), (A.1.67), we compute for $\xi \in \mathbb{R}$, f_0, $f \in \mathrm{H}^r(\mathbb{S})$, and $\varphi \in \mathrm{H}^{r-1}(\mathbb{S})$ that

$$\begin{aligned}
&B_0(f_0+f)[\varphi](\xi) - B_0(f_0)[\varphi](\xi) - \partial B_0(f_0)[f][\varphi](\xi)\\
&= \int_{-\pi}^{\pi} (\delta_{[\xi,s]}f/2)^2 \int_0^1 (1-\tau)\partial_\eta^2\phi(\delta_{[\xi,s]}f_\tau/2, s)\,\mathrm{d}\tau\,\varphi(\xi-s)\,\mathrm{d}s\\
&= \int_0^1 (1-\tau)\int_{-\pi}^{\pi} (\delta_{[\xi,s]}f/2)^2\partial_\eta^2\phi(\delta_{[\xi,s]}f_\tau/2, s)\varphi(\xi-s)\,\mathrm{d}s\,\mathrm{d}\tau\\
&= 2\int_0^1 (1-\tau)\Big\{B_{0,2}^{1,0,2} + B_{0,2}^{3,0,2} - B_{2,2}^{1,0,2} - 2B_{2,2}^{3,0,2}\\
&\qquad\qquad - B_{2,2}^{5,0,2} + B_{4,2}^{3,0,2} + B_{4,2}^{5,0,2}\Big\}(f_\tau)[f,f][\varphi]\,\mathrm{d}\tau,
\end{aligned}$$

where $f_\tau = f_0 + \tau f$. Using (A.1.50), we thus find a constant $C > 0$ such that for all $f \in \mathrm{H}^r(\mathbb{S})$ with $\|f\|_{\mathrm{H}^r} \le 1$, we have

$$\|B_0(f_0+f) - B_0(f_0) - \partial B_0(f_0)[f]\|_{\mathcal{L}(\mathrm{H}^{r-1}(\mathbb{S}),\mathrm{H}^r(\mathbb{S}))} \le C\|f\|_{\mathrm{H}^r}^2,$$

which proves the claim. □

We are now in a position to establish Corollary A.1.13.

Proof of Corollary A.1.13. Recalling that $B_{n,m}^{p,q}(f) = B_{n,m}^{p,q,0}(f)$ for $f \in \mathrm{H}^r(\mathbb{S})$, the assertion is a direct consequence of Lemma A.1.14 and Lemma A.1.15. □

As the last result in this section, we present a formula for the derivative of a non-standard composition type mapping, which we apply to the operators $B_{n,m}^{p,q}$ and B_0 in Chapter 6.

Lemma A.1.16. *Let E_0, E_1, F be Banach spaces and let $\mathcal{O} \subset F$ be an open set. Moreover, let $\phi \in \mathrm{C}^1(\mathcal{O}, E_1)$, and $B \in \mathrm{C}^1(\mathcal{O}, \mathcal{L}(E_1, E_0))$. Then,*

$$[f \mapsto B(f)[\phi(f)]] \in \mathrm{C}^1(\mathcal{O}, E_0),$$

and the Fréchet derivative at $f_0 \in \mathcal{O}$ is given by

$$\partial(B(f)[\phi(f)])\big|_{f_0}[h] = \partial B(f_0)[h][\phi(f_0)] + B(f_0)\big[\partial\phi(f_0)[h]\big], \qquad h \in F. \tag{A.1.68}$$

Proof. Let $h \in F$ such that $f_0 + h \in \mathcal{O}$. We then have

$$\begin{aligned}
&\frac{\big\|B(f_0+h)[\phi(f_0+h)] - B(f_0)[\phi(f_0)] - \partial B(f_0)[h][\phi(f_0)] - B(f_0)\big[\partial\phi(f_0)[h]\big]\big\|_{E_0}}{\|h\|_F}\\
&\le \frac{\big\|B(f_0+h)\big[\phi(f_0+h) - \phi(f_0) - \partial\phi(f_0)[h]\big]\big\|_{E_0}}{\|h\|_F}\\
&\quad + \frac{\big\|\big(B(f_0+h) - B(f_0) - \partial B(f_0)[h]\big)[\phi(f_0)]\big\|_{E_0}}{\|h\|_F}\\
&\quad + \frac{\big\|\big(B(f_0+h) - B(f_0)\big)\big[\partial\phi(f_0)[h]\big]\big\|_{E_0}}{\|h\|_F}\\
&\le \|B(f_0+h)\|_{\mathcal{L}(E_1,E_0)}\frac{\|\phi(f_0+h) - \phi(f_0) - \partial\phi(f_0)[h]\|_{E_1}}{\|h\|_F}\\
&\quad + \|\phi(f_0)\|_{E_1}\frac{\|B(f_0+h) - B(f_0) - \partial B(f_0)[h]\|_{\mathcal{L}(E_1,E_0)}}{\|h\|_F}\\
&\quad + \|B(f_0+h) - B(f_0)\|_{\mathcal{L}(E_1,E_0)}\|\partial\phi(f_0)\|_{\mathcal{L}(F,E_1)}\\
&\longrightarrow 0
\end{aligned}$$

for $\|h\|_F \to 0$ formula (A.1.68) follows. Since the continuity of the Fréchet derivative of the mapping $f \mapsto B(f)[\phi(f)]$ is a straightforward consequence of (A.1.68), this proves the claim. □

A.2. Localization

In this section, we show that the singular integral operators $C^0_{n,m}$ defined in (6.3.3) can be locally approximated by Fourier multipliers, see Lemma 6.3.6 for the precise statement. As a starting point, we infer for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ from (6.3.2) the following algebraic relation

$$\begin{aligned}&dC_{n,m}(\mathbf{a})[\mathbf{b},\varphi] - C_{n,m}(\mathbf{a})[\mathbf{b}, d\varphi]\\ &= b_1 C_{n,m}(\mathbf{a})[(b_2,\ldots,b_n,d),\varphi] - C_{n,m}(\mathbf{a})[(b_2,\ldots,b_n,d), b_1\varphi],\end{aligned} \tag{A.2.1}$$

which holds for all $\mathbf{a} \in \mathrm{W}^{1,\infty}(\mathbb{S})^m$, $\mathbf{b} \in \mathrm{W}^{1,\infty}(\mathbb{S})^n$, $d \in \mathrm{W}^{1,\infty}(\mathbb{S})$, and $\varphi \in \mathrm{L}^2(\mathbb{S})$.

Next, we prove the commutator property Lemma 6.3.5 by following [1, Lemma 12] which provides a similar result in a non-periodic setting.

Proof of Lemma 6.3.5. First, recall that a, $f \in \mathrm{C}^1(\mathbb{S})$. Then, assuming $\varphi \in \mathrm{C}^\infty(\mathbb{S})$, we set

$$T := aC^0_{n,m}(f)[\varphi] - C^0_{n,m}(f)[a\varphi].$$

From (A.1.8) we may conclude

$$\|T\|_2 \le C\|\varphi\|_2. \tag{A.2.2}$$

Now, given $0 \ne y \in (-\pi,\pi)$, we have

$$\frac{\tau_y T - T}{y} = T_1 + T_2 + T_3 + T_4 + T_5,$$

where, by (A.1.14) and (A.2.1), we have

$$\begin{aligned}
T_1 &:= \frac{\tau_y a - a}{y} C^0_{n,m}(\tau_y f)[\tau_y\varphi],\\
T_2 &:= aC^0_{n,m}(\tau_y f)\Big[\frac{\tau_y\varphi - \varphi}{y}\Big],\\
T_3 &:= -C^0_{n,m}(\tau_y f)\Big[\frac{\tau_y(a\varphi) - (a\varphi)}{y}\Big],\\
T_4 &:= \frac{\tau_y f - f}{y}\sum_{i=1}^n C_{n,m}(\tau_y f,\ldots,\tau_y f)[(f_1,\ldots,f_{i-1},a,\tau_y f,\ldots,\tau_y f),\varphi]\\
&\quad - \sum_{i=1}^n C_{n,m}(\tau_y f,\ldots,\tau_y f)\Big[(f_1,\ldots,f_{i-1},a,\tau_y f,\ldots,\tau_y f), \frac{\tau_y f - f}{y}\varphi\Big],\\
T_5 &:= -\frac{\tau_y f - f}{y}\sum_{i=1}^m C_{n+2,m+1}(\tau_y f_1,\ldots,\tau_y f_i, f,\ldots,f)[(f,\ldots,f,\tau_y f + f, a),\varphi]\\
&\quad + \sum_{i=1}^m C_{n+2,m+1}(\tau_y f_1,\ldots,\tau_y f_i, f,\ldots,f)\Big[(f,\ldots,f,\tau_y f + f,a), \frac{\tau_y f - f}{y}\varphi\Big],
\end{aligned}$$

where $f_j := f$, $1 \le j \le i$. Lemma A.1.1 (i) implies that the limit

$$T' = \lim_{y\to 0}\frac{\tau_y T - T}{y}$$

exists in $\mathrm{L}^2(\mathbb{S})$ and therefore $T \in \mathrm{H}^1(\mathbb{S})$ with

$$
\begin{aligned}
T' &= a' C^0_{n,m}(f)[\varphi] + a C^0_{n,m}(f)[\varphi'] - C^0_{n,m}(f)[a'\varphi] - C^0_{n,m}(f)[a\varphi'] \\
&\quad + n f' C_{n,m}(f,\dots,f)[(f,\dots,f,a),\varphi] - n C_{n,m}(f,\dots,f)[(f,\dots,f,a) f'\varphi] \\
&\quad - 2m f' C_{n+2,m+1}(f,\dots,f)[(f,\dots,f,a),\varphi] \\
&\quad + 2m C_{n+2,m+1}(f,\dots,f)[(f,\dots,f,a), f'\varphi].
\end{aligned}
$$

Lemma A.1.1 (i) yields

$$
\|T' - a C^0_{n,m}(f)[\varphi'] + C^0_{n,m}(f)[a\varphi']\|_2 \le C\|\varphi\|_2. \tag{A.2.3}
$$

We are thus left with the term

$$
T_6 := a C^0_{n,m}(f)[\varphi'] - C^0_{n,m}(f)[a\varphi'].
$$

To this end, we write

$$
T_6(\xi) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{(\delta_{[\xi,s]}f/s)^n}{\big[1+(\delta_{[\xi,s]}f/s)^2\big]^m} \frac{\delta_{[\xi,s]}a}{s} \frac{\mathrm{d}}{\mathrm{d}s}(-\varphi(\xi-s))\,\mathrm{d}s, \qquad \xi \in \mathbb{R},
$$

and integration by parts gives for $\xi \in \mathbb{R}$,

$$
\begin{aligned}
T_6(\xi) = {} & \frac{(-1)^{n+1}-1}{\pi} \frac{(\delta_{[\xi,\pi]}f/\pi)^n}{\big[1+(\delta_{[\xi,\pi]}f/\pi)^2\big]^m} \frac{\delta_{[\xi,\pi]}a}{\pi}\varphi(\xi-\pi) + C^0_{n,m}(f)[a'\varphi](\xi) \\
& + n C_{n,m}(f,\dots,f)[(f,\dots,f,a), f'\varphi](\xi) \\
& - (n+1) C_{n+1,m}(f,\dots,f)[(f,\dots,f,a),\varphi](\xi) \\
& - 2m C_{n+2,m+1}(f,\dots,f)[(f,\dots,f,a), f'\varphi](\xi) \\
& + 2m C_{n+3,m+1}(f,\dots,f)[(f,\dots,f,a),\varphi](\xi).
\end{aligned}
$$

We then use Lemma A.1.1 (i) in order to obtain

$$
\|T_6\|_2 \le C\|\varphi\|_2. \tag{A.2.4}
$$

The estimates (A.2.2)–(A.2.4) imply

$$
\|a C^0_{n,m}(f)[\varphi] - C^0_{n,m}(f)[a\varphi]\|_{\mathrm{H}^1} \le C\|\varphi\|_2,
$$

and a standard density argument due to the local Lipschitz continuity property stated in Lemma A.1.1 (i) completes the proof. □

Let us now recall the definition of an ε-partition of unity from Section 6.3. Before presenting the proof of Lemma 6.3.6 we establish the following lemma which is used in a key step in the proof of Lemma 6.3.6 later on.

Lemma A.2.1. *Given $n, m \in \mathbb{N}_0$, $3/2 < r < 2$, $\eta \in (0,\infty)$, and $f \in \mathrm{H}^r(\mathbb{S})$, for sufficiently small $\varepsilon \in (0,1)$ and all $1 \le j \le N$, $|y| \le \varepsilon$, and $\varphi \in \mathrm{L}^2(\mathbb{S})$, we have*

$$
\big\|T^\varepsilon_j(f)[\tau_y(\pi^\varepsilon_j\varphi) - \pi^\varepsilon_j\varphi]\big\|_2 \le \eta\|\tau_y(\pi^\varepsilon_j\varphi) - \pi^\varepsilon_j\varphi\|_2, \tag{A.2.5}
$$

where

$$
T^\varepsilon_j(f) := \chi^\varepsilon_j C_{n+1,m}(f,\dots,f)[(f,\dots,f,f-f'(\xi^\varepsilon_j)\mathrm{id}_{\mathbb{R}}),\cdot].
$$

Proof. Let $\varepsilon \in (0,1)$. Since

$$T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi] = \chi_j^\varepsilon(C_{n+1,m}^0(f)-f'(\xi_j^\varepsilon)C_{n,m}^0(f))[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi] \in \mathrm{L}^2(\mathbb{S}),$$

we have by Lemma A.1.1 (i) that

$$\|T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi]\|_2 = \|T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi]\|_{\mathrm{L}^2((\xi_j^\varepsilon-\pi,\xi_j^\varepsilon+\pi))}. \tag{A.2.6}$$

We now introduce the Lipschitz continuous function $F_j : \mathbb{R}\to\mathbb{R}$ that satisfies $F_j = f$ on J_j^ε and $F_j' = f'(\xi_j^\varepsilon)$ on $\mathbb{R}\setminus J_j^\varepsilon$. Given $\xi \in (\xi_j^\varepsilon-\pi, \xi_j^\varepsilon+\pi)$, we then have

$$\begin{aligned}
&T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi](\xi)\\
&= \chi_j^\varepsilon(\xi)\frac{1}{\pi}\mathrm{PV}\int_{-\pi}^{\pi}\phi\Big(\frac{\delta_{[\xi,s]}f}{s}\Big)\frac{\delta_{[\xi,s]}(f-f'(\xi_j^\varepsilon)\mathrm{id}_\mathbb{R})}{s}\frac{(\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi)(\xi-s)}{s}\,ds\\
&= \chi_j^\varepsilon(\xi)\frac{1}{\pi}\mathrm{PV}\int_{-\pi}^{\pi}\phi\Big(\frac{\delta_{[\xi,s]}f}{s}\Big)\frac{\delta_{[\xi,s]}(F_j-f'(\xi_j^\varepsilon)\mathrm{id}_\mathbb{R})}{s}\frac{(\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi)(\xi-s)}{s}\,ds\\
&= (\chi_j^\varepsilon C_{n+1,m}(f,\ldots,f)[(f,\ldots,f,F_j-f'(\xi_j^\varepsilon)\mathrm{id}_\mathbb{R}),\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi])(\xi),
\end{aligned} \tag{A.2.7}$$

where

$$\phi(x) = x^n(1+x^2)^{-m}, \qquad x\in\mathbb{R}.$$

Indeed, if on the one hand $\xi \in (\xi_j^\varepsilon-\pi,\xi_j^\varepsilon+\pi)\setminus J_j^\varepsilon$, this is a consequence of $\chi_j^\varepsilon(\xi)=0$. If on the other hand $\xi\in J_j^\varepsilon$, then $f(\xi)=F_j(\xi)$ by the definition of F_j. Moreover, since $|s|<\pi$, for sufficiently small ε, we have $\xi-s\in(\xi_j^\varepsilon-3\pi/2,\xi_j^\varepsilon+3\pi/2)$, while $\operatorname{supp}\pi_j^\varepsilon\cap(\xi_j^\varepsilon-3\pi/2,\xi_j^\varepsilon+3\pi/2)=I_j^\varepsilon$. Therefore, for $\xi-s\notin J_j^\varepsilon$, it holds $\xi-s+y\notin I_j^\varepsilon$ for all $|y|\le\varepsilon$, hence $\pi_j^\varepsilon\varphi(\xi-s)=\tau_y(\pi_j^\varepsilon\varphi)(\xi-s)=0$. Consequently, the integrand is not zero at most when $\xi-s\in J_j^\varepsilon$, and in this case we also have $f(\xi-s)=F_j(\xi-s)$. This proves (A.2.7).

Lemma A.1.1 (i) together with (A.2.6), (A.2.7), and the definition of F_j enables us to deduce that there exists a constant $C>0$ such that for all $1\le j\le N$, $|y|\le\varepsilon$, and $\varphi\in\mathrm{L}^2(\mathbb{S})$, we have

$$\|T_j^\varepsilon(f)[\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi]\|_2 \le C\|f'-f'(\xi_j^\varepsilon)\|_{\mathrm{L}^\infty(J_j^\varepsilon)}\|\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi\|_2.$$

The estimate (A.2.5) follows by choosing $\varepsilon\in(0,1)$ sufficiently small in view of the embedding $f'\in\mathrm{H}^{r-1}(\mathbb{S})\hookrightarrow\mathrm{C}^{r-3/2}(\mathbb{S})$. □

We are now in a position to establish Lemma 6.3.6.

Proof of Lemma 6.3.6. In the following, we denote constants that do not depend on ε by C and constants that depend on ε by K.

Recalling that $\mathsf{H}=B_{0,0}^{0,0}$, cf. (4.1.5), the relation $\mathsf{H}=A_{0,0}^{1,0}+C_{0,0}$, cf. (A.1.2), together with Lemma A.1.3 yield

$$\|(\mathsf{H}-C_{0,0})[\pi_j^\varepsilon\varphi]\|_{\mathrm{H}^{r-1}} \le C\|A_{0,0}^{1,0}[\pi_j^\varepsilon\varphi]\|_{\mathrm{C}^1} \le C\|\pi_j^\varepsilon\varphi\|_\infty \le K\|\varphi\|_{\mathrm{H}^{r'-1}},$$

and therefore

$$\begin{aligned}
&\Big\|\pi_j^\varepsilon aC_{n,m}^0(f)[b\varphi]-\frac{a(\xi_j^\varepsilon)b(\xi_j^\varepsilon)(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m}\mathsf{H}[\pi_j^\varepsilon\varphi]\Big\|_{\mathrm{H}^{r-1}}\\
&\le \Big\|\pi_j^\varepsilon aC_{n,m}^0(f)[b\varphi]-\frac{a(\xi_j^\varepsilon)b(\xi_j^\varepsilon)(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m}C_{0,0}[\pi_j^\varepsilon\varphi]\Big\|_{\mathrm{H}^{r-1}}+K\|\varphi\|_{\mathrm{H}^{r'-1}}.
\end{aligned}$$

To estimate the first term on the left-hand side of the latter inequality, we write

$$\pi_j^\varepsilon a C^0_{n,m}(f)[b\varphi] - \frac{a(\xi_j^\varepsilon)b(\xi_j^\varepsilon)(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m} C_{0,0}[\pi_j^\varepsilon\varphi] = a(T_1+T_2) + b(\xi_j^\varepsilon)(T_3 + a(\xi_j^\varepsilon)T_4),$$

where

$$\begin{aligned}
T_1 &:= \pi_j^\varepsilon C^0_{n,m}(f)[(b-b(\xi_j^\varepsilon))\varphi] - C^0_{n,m}(f)[\pi_j^\varepsilon(b-b(\xi_j^\varepsilon))\varphi],\\
T_2 &:= C^0_{n,m}(f)[\pi_j^\varepsilon(b-b(\xi_j^\varepsilon))\varphi],\\
T_3 &:= \pi_j^\varepsilon a C^0_{n,m}(f)[\varphi] - a(\xi_j^\varepsilon)C^0_{n,m}(f)[\pi_j^\varepsilon\varphi],\\
T_4 &:= C^0_{n,m}(f)[\pi_j^\varepsilon\varphi] - \frac{(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m} C_{0,0}[\pi_j^\varepsilon\varphi].
\end{aligned}$$

We consider these terms successively.

The term aT_1. In view of Lemma 6.3.5 and the Banach algebra property of $\mathrm{H}^{r-1}(\mathbb{S})$, we have

$$\|aT_1\|_{\mathrm{H}^{r-1}} \leq K\|(b-b(\xi_j^\varepsilon))\varphi\|_2 \leq K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{A.2.8}$$

The term aT_2. We use Lemma A.1.1 (iii), (6.3.47), (6.3.49), and the Banach algebra property of $\mathrm{H}^{r-1}(\mathbb{S})$ to obtain, in view of $b \in \mathrm{H}^{r-1}(\mathbb{S}) \hookrightarrow \mathrm{C}^{r-3/2}(\mathbb{S})$, that

$$\begin{aligned}
\|aT_2\|_{\mathrm{H}^{r-1}} &\leq C\|\pi_j^\varepsilon(b-b(\xi_j^\varepsilon))\varphi\|_{\mathrm{H}^{r-1}}\\
&\leq C\|\chi_j^\varepsilon(b-b(\xi_j^\varepsilon))\|_\infty\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}\\
&\leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}},
\end{aligned} \tag{A.2.9}$$

provided that $\varepsilon \in (0,1)$ is sufficiently small.

The term $b(\xi_j^\varepsilon)T_3$. By (6.3.49) we have

$$T_3 = T_{3,1} + T_{3,2} + T_{3,3},$$

where

$$\begin{aligned}
T_{3,1} &:= (\chi_j^\varepsilon a)\big(\pi_j^\varepsilon C^0_{n,m}(f)[\varphi] - C^0_{n,m}(f)[\pi_j^\varepsilon\varphi]\big),\\
T_{3,2} &:= \chi_j^\varepsilon(a-a(\xi_j^\varepsilon))C^0_{n,m}(f)[\pi_j^\varepsilon\varphi],\\
T_{3,3} &:= a(\xi_j^\varepsilon)\big(\chi_j^\varepsilon C^0_{n,m}(f)[\pi_j^\varepsilon\varphi] - C^0_{n,m}(f)[\chi_j^\varepsilon(\pi_j^\varepsilon\varphi)]\big),
\end{aligned}$$

and Lemma 6.3.5 yields

$$\|b(\xi_j^\varepsilon)T_{3,1}\|_{\mathrm{H}^{r-1}} + \|b(\xi_j^\varepsilon)T_{3,3}\|_{\mathrm{H}^{r-1}} \leq K\|\varphi\|_{\mathrm{H}^{r'-1}}.$$

Moreover, (6.3.47), Lemma A.1.1 (iii), and the property $a \in \mathrm{H}^{r-1}(\mathbb{S}) \hookrightarrow \mathrm{C}^{r-3/2}(\mathbb{S})$ lead us to

$$\begin{aligned}
\|b(\xi_j^\varepsilon)T_{3,2}\|_{\mathrm{H}^{r-1}} &\leq C\|\chi_j^\varepsilon(a-a(\xi_j^\varepsilon))\|_\infty\|C^0_{n,m}(f)[\pi_j^\varepsilon\varphi]\|_{\mathrm{H}^{r-1}} + K\|C^0_{n,m}(f)[\pi_j^\varepsilon\varphi]\|_{\mathrm{H}^{r'-1}}\\
&\leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}},
\end{aligned}$$

provided that $\varepsilon \in (0,1)$ is small enough, and therefore

$$\|b(\xi_j^\varepsilon)T_3\|_{\mathrm{H}^{r-1}} \leq (\eta/3)\|\pi_j^\varepsilon\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{A.2.10}$$

The term $(ab)(\xi_j^\varepsilon)T_4$. Using again the relation (6.3.49), we have

$$T_4 = T_{4,1} + T_{4,2},$$

where

$$T_{4,1} := \frac{(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m}\big(\chi_j^\varepsilon C_{0,0}[\pi_j^\varepsilon\varphi] - C_{0,0}[\chi_j^\varepsilon(\pi_j^\varepsilon\varphi)]\big)$$
$$\qquad - \big(\chi_j^\varepsilon C_{n,m}^0(f)[\pi_j^\varepsilon\varphi] - C_{n,m}^0(f)[\chi_j^\varepsilon(\pi_j^\varepsilon\varphi)]\big),$$
$$T_{4,2} := \chi_j^\varepsilon\Big(C_{n,m}^0(f)[\pi_j^\varepsilon\varphi] - \frac{(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^m}C_{0,0}[\pi_j^\varepsilon\varphi]\Big),$$

and, by Lemma 6.3.5,

$$\|T_{4,1}\|_{\mathrm{H}^{r-1}} \leq K\|\varphi\|_2. \tag{A.2.11}$$

It remains to estimate the term $T_{4,2}$ for which we first use Lemma A.1.1 (i) to deduce that

$$\|T_{4,2}\|_2 \leq K\|\varphi\|_2. \tag{A.2.12}$$

In order to estimate the seminorm $[T_{4,2}]_{\mathrm{W}^{r-1,2}}$, we note, by using (A.1.14) together with the identity $f'(\xi_j^\varepsilon) = \delta_{[\xi,s]}(f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}})/s$, that

$$T_{4,2} = \sum_{k=0}^{n-1}(f'(\xi_j^\varepsilon))^{n-k-1}\chi_j^\varepsilon C_{k+1,m}(f,\ldots,f)[(f,\ldots,f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\pi_j^\varepsilon\varphi]$$
$$\quad - \sum_{k=0}^{m-1}\frac{(f'(\xi_j^\varepsilon))^n}{[1+(f'(\xi_j^\varepsilon))^2]^{m-k}}\chi_j^\varepsilon C_{2,k+1}(f,\ldots,f)[(f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\pi_j^\varepsilon\varphi]$$
$$\quad - \sum_{k=0}^{m-1}\frac{(f'(\xi_j^\varepsilon))^{n+1}}{[1+(f'(\xi_j^\varepsilon))^2]^{m-k}}\chi_j^\varepsilon C_{1,k+1}(f,\ldots,f)[f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi].$$

Consequently,

$$\begin{aligned}
&[T_{4,2}]_{\mathrm{W}^{r-1,2}}\\
&\leq C_0\Big(\sum_{k=0}^{n-1}\big[\chi_j^\varepsilon C_{k+1,m}(f,\ldots,f)[(f,\ldots,f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\pi_j^\varepsilon\varphi]\big]_{\mathrm{W}^{r-1,2}}\\
&\qquad + \sum_{k=0}^{m-1}\big[\chi_j^\varepsilon C_{2,k+1}(f,\ldots,f)[(f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\pi_j^\varepsilon\varphi]\big]_{\mathrm{W}^{r-1,2}}\\
&\qquad + \sum_{k=0}^{m-1}\big[\chi_j^\varepsilon C_{1,k+1}(f,\ldots,f)[f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\pi_j^\varepsilon\varphi]\big]_{\mathrm{W}^{r-1,2}}\Big).
\end{aligned} \tag{A.2.13}$$

Set

$$S_k := \chi_j^\varepsilon C_{k+1,m}(f,\ldots,f)[(f,\ldots,f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\pi_j^\varepsilon\varphi], \qquad 0\leq k\leq n-1.$$

In order to estimate the $\mathrm{W}^{r-1,2}$-seminorm of S_k, we write for $y\in(-\pi,\pi)$

$$\tau_y S_k - S_k = S_{k,1} + S_{k,2} + \chi_j^\varepsilon S_{k,3},$$

where, using again (A.1.14), we have

$$S_{k,1} := (\tau_y\chi_j^\varepsilon - \chi_j^\varepsilon)\tau_y C_{k+1,m}(f,\ldots,f)[(f,\ldots,f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\pi_j^\varepsilon\varphi],$$
$$S_{k,2} := \chi_j^\varepsilon C_{k+1,m}(f,\ldots,f)[(f,\ldots,f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\tau_y(\pi_j^\varepsilon\varphi)-\pi_j^\varepsilon\varphi],$$
$$S_{k,3} := \sum_{i=1}^{k}C_{k+1,m}(f,\ldots,f)[(\underbrace{f,\ldots,f}_{i-1},\tau_y f-f,\tau_y f,\ldots,\tau_y f,f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}}),\tau_y(\pi_j^\varepsilon\varphi)]$$
$$\quad + C_{k+1,m}(f,\ldots,f)[(\tau_y f,\ldots,\tau_y f,\tau_y f-f),\tau_y(\pi_j^\varepsilon\varphi)]$$
$$\quad - \sum_{i=1}^{m}C_{k+3,m+1}^i[(\tau_y f,\ldots,\tau_y f,\tau_y f-f'(\xi_j^\varepsilon)\mathrm{id}_{\mathbb{R}},\tau_y f+f,\tau_y f-f),\tau_y(\pi_j^\varepsilon\varphi)],$$

and

$$C^i_{k+3,m+1} := C_{k+3,m+1}(\underbrace{f,\dots,f}_{i},\tau_y f,\dots,\tau_y f).$$

Lemma A.1.1 (iii) (with $r = r'$) yields

$$\|S_{k,1}\|_2 \le K\|\tau_y\chi^\varepsilon_j - \chi^\varepsilon_j\|_2\|\varphi\|_{\mathrm{H}^{r'-1}}.$$

To estimate $S_{k,2}$, we consider two cases. If $|y| > \varepsilon$, we use Lemma A.1.1 (i) and obtain

$$\|S_{k,2}\|_2 \le K\|\varphi\|_2.$$

If $|y| \le \varepsilon$, we use (A.2.5) which gives

$$\|S_{k,2}\|_2 \le (\eta/C_1)\|\tau_y(\pi^\varepsilon_j\varphi) - \pi^\varepsilon_j\varphi\|_2,$$

provided that $\varepsilon \in (0,1)$ is small enough, with a positive constant C_1 which we fix below. Finally, Lemma A.1.1 (ii) (with $r = r'$) produces

$$\|\chi^\varepsilon_j S_{k,3}\|_2 \le K\|\tau_y f' - f'\|_2\|\varphi\|_{\mathrm{H}^{r'-1}}.$$

Combining the above estimates, we have

$$[S_k]_{\mathrm{W}^{r-1,2}} \le (\eta/C_1)\|\pi^\varepsilon_j\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{A.2.14}$$

It is now obvious that actually all the terms on the right-hand side of (A.2.13) can be estimated by the right-hand side of (A.2.14), provided that $\varepsilon \in (0,1)$ is sufficiently small. From (A.2.12)–(A.2.14) we then deduce, after choosing

$$C_1 := 3CC_0(n+2m)(1+\|ab\|_\infty),$$

where C is the constant from Lemma 2.4.3, that

$$\begin{aligned}\|T_{4,2}\|_{\mathrm{H}^{r-1}} &\le C(\|T_{4,2}\|_2 + [T_{4,2}]_{\mathrm{W}^{r-1,2}})\\ &\le \frac{CC_0(n+2m)\eta}{C_1}\|\pi^\varepsilon_j\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}\\ &\le \frac{\eta}{3(1+\|ab\|_\infty)}\|\pi^\varepsilon_j\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}},\end{aligned}$$

and together with (A.2.11) we get

$$\|(ab)(\xi^\varepsilon_j)T_4\|_{\mathrm{H}^{r-1}} \le (\eta/3)\|\pi^\varepsilon_j\varphi\|_{\mathrm{H}^{r-1}} + K\|\varphi\|_{\mathrm{H}^{r'-1}}. \tag{A.2.15}$$

Gathering (A.2.8)–(A.2.10) and (A.2.15), we obtain (6.3.46), and the proof is complete. □

Bibliography

[1] H. Abels and B.-V. Matioc, *Well-posedness of the Muskat problem in subcritical L_p-Sobolev spaces*, European J. Appl. Math., 33 (2022), pp. 224–266.

[2] H. Amann, *Linear and Quasilinear Parabolic Problems. Vol. I*, vol. 89 of Monographs in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.

[3] D. M. Anderson, G. B. McFadden, and A. A. Wheeler, *Diffuse-interface methods in fluid mechanics*, in Annual review of fluid mechanics, Vol. 30, vol. 30 of Annu. Rev. Fluid Mech., Annual Reviews, Palo Alto, CA, 1998, pp. 139–165.

[4] S. B. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh Sect. A, 115 (1990), pp. 91–107.

[5] A. Badea and J. Duchon, *Capillary driven evolution of an interface between viscous fluids*, Nonlinear Anal., 31 (1998), pp. 385–403.

[6] J. W. Barrett, H. Garcke, and R. Nürnberg, *Eliminating spurious velocities with a stable approximation of viscous incompressible two-phase Stokes flow*, Comput. Methods Appl. Mech. Engrg., 267 (2013), pp. 511–530.

[7] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2000.

[8] J. T. Beale, *Large-time regularity of viscous surface waves*, Arch. Rational Mech. Anal., 84 (1983/84), pp. 307–352.

[9] J. T. Beale and T. Nishida, *Large-time behavior of viscous surface waves*, in Recent topics in nonlinear PDE, II (Sendai, 1984), vol. 128 of North-Holland Math. Stud., North-Holland, Amsterdam, 1985, pp. 1–14.

[10] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, *Julia: A fresh approach to numerical computing*, SIAM Review, 59 (2017), pp. 65–98.

[11] D. Böhme and B.-V. Matioc, *Well-posedness and Rayleigh–Taylor instability of the two-phase periodic quasistationary Stokes flow*, 2025. arXiv:2508.15502.

[12] ———, *Well-posedness and stability for the two-phase periodic quasistationary Stokes flow*, Interfaces Free Bound., 27 (2025), pp. 659–701.

[13] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, vol. Vol. 40 of Pure and Applied Mathematics, Academic Press, New York-London, 1971. Volume 1: One-dimensional theory.

[14] S. Cameron and R. M. Strain, *Critical local well-posedness for the fully nonlinear Peskin problem*, Comm. Pure Appl. Math., 77 (2024), pp. 901–989.

[15] T. K. Chang and D. H. Pahk, *Spectral properties for layer potentials associated to the Stokes equation in Lipschitz domains*, Manuscripta Math., 130 (2009), pp. 359–373.

[16] K. Chen and Q.-H. Nguyen, *The Peskin problem with $\dot{B}^1_{\infty,\infty}$ initial data*, SIAM J. Math. Anal., 55 (2023), pp. 6262–6304.

[17] J. H. Choi, *Stability of a two-phase Stokes problem with surface tension*, 2024. arXiv:2406.08417.

[18] A. Córdoba, D. Córdoba, and F. Gancedo, *Interface evolution: the Hele-Shaw and Muskat problems*, Ann. of Math. (2), 173 (2011), pp. 477–542.

[19] S. Court, *Feedback stabilization of a two-fluid surface tension system modeling the motion of a soap bubble at low Reynolds number: the two-dimensional case*, J. Math. Fluid Mech., 26 (2024), pp. Paper No. 7, 33.

[20] M. G. Crandall and P. H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. Functional Analysis, 8 (1971), pp. 321–340.

[21] ———, *Bifurcation, perturbation of simple eigenvalues and linearized stability*, Arch. Rational Mech. Anal., 52 (1973), pp. 161–180.

[22] I. V. Denisova, *Problem of the motion of two viscous incompressible fluids separated by a closed free interface*, in Mathematical problems for Navier-Stokes equations (Centro, 1993), vol. 37, 1994, pp. 31–40.

[23] I. V. Denisova and V. A. Solonnikov, *Classical solvability of the problem of the motion of two viscous incompressible fluids*, Algebra i Analiz, 7 (1995), pp. 101–142. translation in St. Petersburg Math. J. 7 (1996), no. 5, 755–786.

[24] ———, *Motion of a Drop in an Incompressible Fluid*, Advances in Mathematical Fluid Mechanics, Birkhäuser/Springer, Cham, 2021. Lecture Notes in Mathematical Fluid Mechanics.

[25] M. Ehrnström, J. Escher, and B.-V. Matioc, *Steady-state fingering patterns for a periodic Muskat problem*, Methods Appl. Anal., 20 (2013), pp. 33–46.

[26] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[27] J. Escher, *The Dirichlet-Neumann operator on continuous functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 21 (1994), pp. 235–266.

[28] J. Escher, A.-V. Matioc, and B.-V. Matioc, *The Mullins-Sekerka problem via the method of potentials*, Math. Nachr., 297 (2024), pp. 1960–1977.

[29] J. Escher and B.-V. Matioc, *On the parabolicity of the Muskat problem: well-posedness, fingering, and stability results*, Z. Anal. Anwend., 30 (2011), pp. 193–218.

[30] J. Escher and G. Simonett, *Maximal regularity for a free boundary problem*, NoDEA Nonlinear Differential Equations Appl., 2 (1995), pp. 463–510.

[31] ———, *Analyticity of the interface in a free boundary problem*, Math. Ann., 305 (1996), pp. 439–459.

[32] L. C. Evans, *Partial Differential Equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2010.

[33] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, second ed., 1995.

[34] A. Friedman and F. Reitich, *Quasi-static motion of a capillary drop. II. The three-dimensional case*, J. Differential Equations, 186 (2002), pp. 509–557.

[35] ———, *Quasistatic motion of a capillary drop. I. The two-dimensional case*, J. Differential Equations, 178 (2002), pp. 212–263.

[36] F. Gancedo, R. Granero-Belinchón, Z. Hu, E. Salguero, and Y. Yao, *Unstable interface dynamics for gravity stokes flow*, 2026. arXiv:2601.18460.

[37] F. Gancedo, R. Granero-Belinchón, and E. Salguero, *Long time interface dynamics for gravity Stokes flow*, SIAM J. Math. Anal., 57 (2025), pp. 1680–1724.

[38] ———, *On the global well-posedness of interface dynamics for gravity Stokes flow*, J. Differential Equations, 428 (2025), pp. 654–687.

[39] F. Gancedo, R. Granero-Belinchón, and S. Scrobogna, *Global existence in the Lipschitz class for the N-Peskin problem*, Indiana Univ. Math. J., 72 (2023), pp. 553–602.

[40] E. García-Juárez and S. V. Haziot, *Critical well-posedness for the 2D Peskin problem with general tension*, Adv. Math., 460 (2025), pp. Paper No. 110047, 46.

[41] E. García-Juárez, P.-C. Kuo, Y. Mori, and R. M. Strain, *Well-posedness of the 3D Peskin problem*, Math. Models Methods Appl. Sci., 35 (2025), pp. 113–216.

[42] E. García-Juárez, Y. Mori, and R. M. Strain, *The Peskin problem with viscosity contrast*, Anal. PDE, 16 (2023), pp. 785–838.

[43] H. Garcke, D. Trautwein, and G. Zhang, *Structure-preserving parametric finite element methods for two-phase Stokes flow based on Lagrange multiplier approaches*, 2025. arXiv:2508.12326.

[44] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. 1*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1964. Properties and Operations, Translated from the Russian by Eugene Saletan.

[45] Y. Giga and S. Takahashi, *On global weak solutions of the nonstationary two-phase Stokes flow*, SIAM J. Math. Anal., 25 (1994), pp. 876–893.

[46] L. Grafakos, *Classical Fourier Analysis*, vol. 249 of Graduate Texts in Mathematics, Springer, New York, third ed., 2014.

[47] H. Grayer, II, *Dynamics of density patches in infinite Prandtl number convection*, Arch. Ration. Mech. Anal., 247 (2023), pp. Paper No. 69, 29.

[48] M. Günther and G. Prokert, *Existence results for the quasistationary motion of a free capillary liquid drop*, Z. Anal. Anwendungen, 16 (1997), pp. 311–348.

[49] Y. Guo and I. Tice, *Compressible, inviscid Rayleigh-Taylor instability*, Indiana Univ. Math. J., 60 (2011), pp. 677–711.

[50] P. Hansbo, M. G. Larson, and S. Zahedi, *A cut finite element method for a Stokes interface problem*, Appl. Numer. Math., 85 (2014), pp. 90–114.

[51] E. Hille, *Functional Analysis and Semi-Groups*, vol. Vol. 31 of American Mathematical Society Colloquium Publications, American Mathematical Society, New York, 1948.

[52] R. M. Höfer, *Sedimentation of inertialess particles in Stokes flows*, Comm. Math. Phys., 360 (2018), pp. 55–101.

[53] R. J. Iorio, Jr. and V. d. M. a. Iorio, *Fourier Analysis and Partial Differential Equations*, vol. 70 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2001.

[54] J. Jang, I. Tice, and Y. Wang, *The compressible viscous surface-internal wave problem: nonlinear Rayleigh-Taylor instability*, Arch. Ration. Mech. Anal., 221 (2016), pp. 215–272.

[55] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[56] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, vol. Vol. 2 of Mathematics and its Applications, Gordon and Breach Science Publishers, New York-London-Paris, english ed., 1969. Translated from the Russian by Richard A. Silverman and John Chu.

[57] A. Leblond, *Well-posedness of the Stokes-transport system in bounded domains and in the infinite strip*, J. Math. Pures Appl. (9), 158 (2022), pp. 120–143.

[58] L. Lichtenstein, *Über einige Existenzprobleme der Hydrodynamik*, Math Z, 28 (1928), pp. 387–415.

[59] F.-H. Lin and J. Tong, *Solvability of the Stokes immersed boundary problem in two dimensions*, Comm. Pure Appl. Math., 72 (2019), pp. 159–226.

[60] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1995. 2013 reprint of the 1995 original.

[61] B.-V. Matioc, *Viscous displacement in porous media: the Muskat problem in 2D*, Trans. Amer. Math. Soc., 370 (2018), pp. 7511–7556.

[62] ———, *The Muskat problem in two dimensions: equivalence of formulations, well-posedness, and regularity results*, Anal. PDE, 12 (2019), pp. 281–332.

[63] ———, *Well-posedness and stability results for some periodic Muskat problems*, J. Math. Fluid Mech., 22 (2020), pp. Paper No. 31, 45.

[64] B.-V. Matioc and G. Prokert, *Two-phase Stokes flow by capillarity in full 2D space: an approach via hydrodynamic potentials*, Proc. Roy. Soc. Edinburgh Sect. A, 151 (2021), pp. 1815–1845.

[65] ———, *Two-phase Stokes flow by capillarity in the plane: the case of different viscosities*, NoDEA Nonlinear Differential Equations Appl., 29 (2022), pp. Paper No. 54, 34.

[66] ———, *Capillarity-driven Stokes flow: the one-phase problem as small viscosity limit*, Z. Angew. Math. Phys., 74 (2023), pp. Paper No. 212, 24.

[67] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[68] A. Mecherbet, *Sedimentation of particles in Stokes flow*, Kinet. Relat. Models, 12 (2019), pp. 995–1044.

[69] A. Mecherbet and F. Sueur, *A few remarks on the transport-Stokes system*, Ann. H. Lebesgue, 7 (2024), pp. 1367–1408.

[70] Y. Mori, A. Rodenberg, and D. Spirn, *Well-posedness and global behavior of the Peskin problem of an immersed elastic filament in Stokes flow*, Comm. Pure Appl. Math., 72 (2019), pp. 887–980.

[71] T. Murai, *Boundedness of singular integral operators of Calderón type. VI*, Nagoya Math. J., 102 (1986), pp. 127–133.

[72] F. K. G. Odqvist, *Über die Randwertaufgaben der Hydrodynamik zäher Flüssigkeiten*, Math Z, 32 (1930), pp. 329–375.

[73] J. Prüss, Y. Shao, and G. Simonett, *On the regularity of the interface of a thermodynamically consistent two-phase Stefan problem with surface tension*, Interfaces Free Bound., 17 (2015), pp. 555–600.

[74] J. Prüss and G. Simonett, *On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations*, Indiana Univ. Math. J., 59 (2010), pp. 1853–1871.

[75] ———, *On the two-phase Navier-Stokes equations with surface tension*, Interfaces Free Bound., 12 (2010), pp. 311–345.

[76] ———, *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, vol. 105 of Monographs in Mathematics, Birkhäuser/Springer, [Cham], 2016.

[77] J. Prüss, G. Simonett, and M. Wilke, *The Rayleigh-Taylor instability for the Verigin problem with and without phase transition*, NoDEA Nonlinear Differential Equations Appl., 26 (2019), pp. Paper No. 18, 35.

[78] L. Rayleigh, *Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density*, Proc. Lond. Math. Soc., 14 (1882/83), pp. 170–176.

[79] R. Remmert, *Theory of Complex Functions*, vol. 122 of Graduate Texts in Mathematics, Springer-Verlag, New York, english ed., 1991. Translated from the second German edition by Robert B. Burckel.

[80] H. Saito and Y. Shibata, *On the global wellposedness of free boundary problem for the Navier-Stokes system with surface tension*, J. Differential Equations, 384 (2024), pp. 1–92.

[81] O. Sánchez, *Steady-state solutions for the Muskat problem*, Collect. Math., 74 (2023), pp. 313–321.

[82] H.-J. Schmeisser and H. Triebel, *Topics in Fourier Analysis and Function Spaces*, A Wiley-Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1987.

[83] Y. Shibata and S. Shimizu, *On a free boundary problem for the Navier-Stokes equations*, Differential Integral Equations, 20 (2007), pp. 241–276.

[84] V. A. Solonnikov, *Solvability of a problem on the motion of a viscous incompressible fluid bounded by a free surface*, Izv. Akad. Nauk SSSR Ser. Mat., 41 (1977), pp. 1388–1424, 1448. English translation: Math. USSR-Izv. 11 (1977), no. 6, 1323–1358 (1978).

[85] ———, *Free boundary problems and problems in noncompact domains for the Navier-Stokes equations*, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 1113–1122.

[86] ———, *Unsteady motion of an isolated volume of a viscous incompressible fluid*, Izv. Akad. Nauk SSSR Ser. Mat., 51 (1987), pp. 1065–1087, 1118. translation in Math. USSR-Izv. 31 (1988), no. 2, 381–405.

[87] ———, *On quasistationary approximation in the problem of motion of a capillary drop*, in Topics in Nonlinear Analysis, vol. 35 of Progr. Nonlinear Differential Equations Appl., Birkhäuser, Basel, 1999, pp. 643–671.

[88] ———, *On the justification of the quasistationary approximation in the problem of motion of a viscous capillary drop*, Interfaces Free Bound., 1 (1999), pp. 125–173.

[89] E. M. Stein and R. Shakarchi, *Fourier Analysis*, vol. 1 of Princeton Lectures in Analysis, Princeton University Press, Princeton, NJ, 2003. An introduction.

[90] G. Taylor, *The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I*, Proc. Roy. Soc. London Ser. A, 201 (1950), pp. 192–196.

[91] M. F. P. ten Eikelder, K. G. van der Zee, I. Akkerman, and D. Schillinger, *A unified framework for Navier-Stokes Cahn-Hilliard models with non-matching densities*, Math. Models Methods Appl. Sci., 33 (2023), pp. 175–221.

[92] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, vol. 123 of Pure and Applied Mathematics, Academic Press, Inc., Orlando, FL, 1986.

[93] H. Triebel, *Theory of Function Spaces*, vol. 78 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1983.

[94] Y. Wang and I. Tice, *The viscous surface-internal wave problem: nonlinear Rayleigh-Taylor instability*, Comm. Partial Differential Equations, 37 (2012), pp. 1967–2028.

[95] D. Werner, *Funktionalanalysis*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2018.

[96] M. Wilke, *The two-phase Navier-Stokes equations with surface tension in cylindrical domains*, Pure Appl. Funct. Anal., 5 (2020), pp. 121–201.

[97] ———, *On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations in cylindrical domains*, Interfaces Free Bound., 24 (2022), pp. 487–531.

[98] K. Yosida, *On the differentiability and the representation of one-parameter semi-group of linear operators*, Journal of the Mathematical Society of Japan, 1 (1948).