Exploring Quantum Gravity Through the Lens of Quantum Chaos – Jackiw-Teitelboim Gravity, Matrix Models and the Selberg Trace Formula

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Abstract

Jackiw-Teitelboim (JT) gravity is an exactly solvable model of two-dimensional quantum gravity that has found remarkable application in the study of holography in recent years. In this thesis, I develop methods informed by quantum chaos to address open problems within JT gravity. The perturbative duality to a random matrix model is used to make nonperturbative predictions about the spectral form factor of both orientable JT gravity and unorientable topological gravity (JT gravity's low energy limit), obtaining intricate relations obeyed by the volumes of moduli spaces appearing in the respective path integrals. Furthermore, I address the factorisation problem in quantum gravity posed by the duality of JT gravity to an ensemble of random matrices by introducing a single quantum chaotic system, a high-dimensional variant of the Hadamard-Gutzwiller model, that can nevertheless reproduce the leading-topology one- and two-point correlation functions of JT gravity in the infinite-dimensional limit in a semiclassical, but exact calculation based on Selberg's trace formula. I also show that this system, upon accounting for correlations in the spectrum of lengths of classical periodic orbits, gives the correct first topology correction to the two-point function of unorientable topological gravity. Finally, I compute the quantum Lyapunov exponent of the system and show that in the infinite-dimensional limit, it saturates the Maldacena-Shenker-Stanford bound on chaos, further supporting a possible duality.

Author's note

In this thesis, I present mostly material that has been published, or is in preparation for being published soon after the time of writing. Part II is based on the following two publications:

1. T. Weber, F. Haneder, K. Richter, and J. D. Urbina, "Constraining Weil–Petersson volumes by universal random matrix correlations in low-dimensional quantum gravity", Journal of Physics A: Mathematical and Theoretical **56**, 205206 (2023)

2. T. Weber, J. Tall, F. Haneder, J. D. Urbina, and K. Richter, "Unorientable topological gravity and orthogonal random matrix universality", Journal of High Energy Physics **2024**, 267 (2024)

Part III is mostly based on the following two publications, as well as unpublished work extending the studies undertaken therein:

1. F. Haneder, J. D. Urbina, C. Moreno, T. Weber, and K. Richter, "Beyond the ensemble paradigm in low-dimensional quantum gravity: Schwarzian density, quantum chaos, and wormhole contributions", Physical Review D **111**, 126015 (2025)

2. F. Haneder, G. Caspari, J. D. Urbina, and K. Richter, "The relation between classical and quantum Lyapunov exponent and the bound on chaos in classically chaotic quantum systems", arXiv:2512.19869 [nlin] (2025)

The contents of the latter publication were obtained in close collaboration with Gerrit Caspari in the course of his Master's thesis, which I co-supervised. Other work that I have supervised has not directly led to any publications, but has nevertheless played a role in the development of some of the work presented. This includes:

1. The Bachelor's thesis of Stefan Miedaner and a research project conducted by Daniel Riese, on the systematics of contributions to the correlation functions of Jackiw-Teitelboim gravity, in patricular the Weil-Petersson volumes, and their algorithmic computation.

2. The Bachelor's thesis of Jakob Minár and a research project conducted by David Horn, on the practical evaluation of Selberg's trace formula and pushing it out of its realm of applicability.

The results I present in this thesis were produced collaboratively, and I have aimed throughout the text to highlight the contributions of my collaborators.

When reporting these results, I use the plural pronoun "we" to reflect the collaborative nature of their being obtained. I also use "we" when presenting background material, mostly in part I, while I use the singular "I" to indicate authorial choices or intent in the text.

The work I present requires background material from various different fields, some of which I introduce in the first part of the thesis. In order not to shift the focus too far away from our original work however, I must assume the reader to possess a certain familiarity with classical and quantum mechanics and field theories, in a formulation that occasionally has perhaps a stronger focus on mathematical structure than what might be presented in a standard Physics curriculum. I have kept mostly to natural units, but not necessarily the usual natural units familiar in the field. When discussing the quantum mechanical system perturbatively dual to JT gravity, I use units such that Planck's constant $\hbar = 1$, the mass of the particle $m = 2$, the Ricci curvature on the manifold the particle is moving on is $R = -2$ and correspondingly, the curvature radius $L = 1$. These units give a simple form for many semiclassical results, but require somewhat unusual units in JT gravity. I use the common energy scale $\gamma = 1/2$, but pick different conventions for the lengths of gluing geodesics in different chapters. In the standard convention, gluing geodesics are normalised to be of length b_{standard}, and I apply this convention in part II, where I solely work in the well-established context of JT/matrix model duality. In chapter 9, where I aim to make contact with semiclassics, I use the convention we employed in [3], i.e.,

$$2b_{\text{alternative}} = b_{\text{standard}}.$$

This leads to different trumpet partition functions, Weil-Petersson volumes and a different Weil-Petersson measure, but it leaves the disk partition function unchanged. I will explain the convention in some more detail in chapter 4.

Finally, in chapter 10, where I perform a small $\hbar$ expansion, I need to reinsert physical units. Additionally, the Einstein summation convention, i.e., summation over repeated indices, is used throughout the text, with raised and lowered indices being used as appropriate.

Für Oma

Contents

Chapter 1

Introduction

"Let a hundred flowers bloom;

let a hundred schools of thought contend."

The above is not just a political slogan used by Chairman Mao to launch the Communist Party of China's Hundred Flowers Campaign [5], but also a quotation of the late James Hartle at the end of an online talk he gave for the series "The Quantum Information Structure of Spacetime" in 2021 [6]. Hartle used the citation after a particularly lively discussion on his 2015 essay [7] on taking the concept of superposition in quantum mechanics seriously on a macroscopic, indeed cosmological scale.

A hundred schools of thought contending is, in my view, quite an apt description of the state of research on quantum gravity, a field that owes much to Hartle's work [8]. Unlike the three other fundamental forces realised in nature – the electromagnetic, strong and weak nuclear forces – gravity does not lend itself to canonical quantisation and treatment as an ordinary quantum field theory (QFT) [9]. I will touch on why this is the case in chapter 4, but in a nutshell, gravity leads to a so-called *non-renormalisable* field theory [9, 10], meaning that it is not afforded the same luxury as other QFTs of pushing back its divergent predictions to a further "UV-complete"[1] theory. Quantum gravity must itself be the UV-finite theory.

This theoretical difficulty has lead to a plethora of distinct research programmes, the most active of which at the moment is indubitably string theory [11–13]. String theory replaces point-like particles in its formalism with the eponymous strings, objects of finite length. This smooths out the point-interactions that introduce divergences in ordinary QFT, leading to a perturbatively finite theory [9]. The price to be paid is quite steep however – superstring theories that allow for fermions live in ten dimensions rather than the four realised in nature [12]. They also predict supersymmetric partners for every species of particle that have not been found [14, 15], and the schemes used to reduce the required ten dimensions down to four, as well as the symmetry breaking mechanisms that would eventually yield the standard model and nothing more at energies we are able to realise in laboratory settings, introduce a veritable "swampland" of inequivalent string theories that may or may not be consistent at all energies[2].

[1] Finite in the ultraviolet, i.e., at arbitrarily large energies.

[2] The "landscape" of consistent string theories is estimated to contain at least on the order of $10^{272,000}$ inequivalent theories [16]. The swampland, comprised also of low-energy effective theories that are not consistent theories of quantum gravity, is much larger still [17].

An alternative approach is to canonically quantise gravity in its Hamiltonian formulation due to Arnowitt, Deser and Misner (ADM) [18], and extending the resulting quantum theory by an internal gauge symmetry. This programme leads to loop quantum gravity (LQG) [19, 20], a background-independent, manifestly finite quantum theory. The mathematical complexity of this theory however is such that while it has been shown that there are kinematical coherent states peaked around classical geometries [21, 22], and while simplified models like cosmology (e.g. [23–28]) and black holes (e.g. [29–32]) behave in a desirable way, it is not clear that LQG reduces to general relativity (GR) in the semiclassical limit.

Various further attempts at defining a consistent theory of quantum gravity, such as asymptotic safety [33], causal dynamical triangulations [34] or stochastic gravity [35], have emerged, begging the question of how to decide for one over the others. In other fields of physics, this arbitration process would be carried out by conducting experiments. In quantum gravity however, we have to deal with twin difficulties: falsifiable predictions are hard to come by due to the complexity of the available models and the sheer range of energy scales across which they have to be applicable. At the same time, actual experiments are prohibitively expensive, on top of being fairly challenging to carry out even in the absence of resource constraints. As an example of a highly sophisticated experiment that still only probes classical general relativity, consider the gravitational wave interferometry conducted at the LIGO and VIRGO collaborations that received the 2017 Nobel prize in physics for the experimental verification of gravitational waves [36, 37] – a prediction made by Albert Einstein some 100 years prior [38]! Only three years later, in 2020, Reinhard Genzel and Andrea Ghez received a share of the Nobel prize for the discovery of the supermassive black hole at the centre of our galaxy [39] – yet another long-standing prediction of classical GR based on Roger Penrose's singularity theorem [40]. In the meantime, so-called *tabletop quantum gravity experiments* attempt to probe whether gravity even is quantum at all [41, 42]. This is challenging due to how weak gravity is compared to other forces; even Newton's $1/r^2$ law is verified only down to a scale of tens of microns [43]. One can imagine then that actually probing the Planck scale, i.e., the scale at which we expect quantum gravitational effects to rival ordinary quantum effects, is an unfathomably hard undertaking. With current state-of-the-art plasma wakefield acceleration [44], the energy transfer rate achievable in particle accelerators is on the order of $\sim 100\,\mathrm{GeV/m}$. In order to probe the Planck energy,

$$E_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G}} = 1.2 \times 10^{28}\,\mathrm{eV}, \tag{1.1}$$

where $\hbar$ is the reduced Planck constant, c the speed of light and G Newton's constant, we would require a linear accelerator that is

$$\frac{E_{\mathrm{P}}}{100\,\mathrm{GeV/m}} \approx 13 \text{ light years} \tag{1.2}$$

long, even with the best technology available[3].

In the absence of realistic experiments we can conduct to gather information about quantum gravity then, the only tools we have are mathematical consistency of the theory, and the study of features that we expect every theory of quantum gravity to exhibit. A central such feature, and the one that will be the focus of this thesis, is the amenability of the theory to a *holographic* description. Famously, Bekenstein [45] and Hawking [46] have computed the entropy of a black hole,

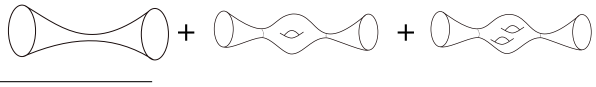
$$S_{\mathrm{BH}} = \frac{k_{\mathrm{B}} c^3 A}{4\hbar G}, \tag{1.3}$$

with k_{B} Boltzmann's constant, to be determined by the *area* A of its event horizon, rather than the volume enclosed by it. Equation (1.3) heuristically suggests that (quantum) gravitational degrees of freedom are redundant, and can be non-locally encoded on a lower-dimensional submanifold of the total system, at least at the level of macroscopic quantities like the entropy. Importantly, the derivation of eq. (1.3) [45, 46] does not rely on a particular UV-complete theory of quantum gravity, but only on semiclassical gravity and unitarity, hence justifying the expectation that *any* theory of quantum gravity should be able to reproduce the black hole entropy.

This argument alone is not sufficient to prove a stronger notion of holography, namely the *AdS/CFT correspondence* [47], or more generally, *gauge/gravity duality* [48]. These proposals suggest that *any and all physical information* in a gravitational bulk theory – such as superstring theory on some suitable Anti-de Sitter (AdS) space – can be equivalently described in terms of the degrees of freedom of a non-gravitational gauge theory – such as a conformal field theory (CFT) – that lives on the boundary of the "gravitational" region.

The original programme of Maldacena [47] has been generalised to many more-or-less-well established such dualities, i.e., pairs of gravitational bulk- and non-gravitational boundary theories, of varying degree of complexity (e.g. [49–51]). A particularly important, and simultaneously very atypical duality is that between a two-dimensional dilaton[4] gravity, namely Jackiw-Teitelboim (JT) gravity [52, 53], and a random matrix model[5] [54–56] that can be viewed not as an individual boundary dual, but rather as an *ensemble* of them.

Pictorially, and without going into too much technical detail at this point, we can imagine JT gravity to be the theory that teaches us how to perform for instance the following computation:

$\mathcal{Z}(\beta_1, \beta_2) =$  $+ \dots . \qquad (1.4)$

[3]The question as to why not to simply build a circular accelerator instead is left as an exercise to the reader.

[4]Dilaton is the name given to the scalar field of the theory because it can be seen to describe volume fluctuations ("dilations") in a higher-dimensional theory. The dilaton is the remnant of these fluctuations upon symmetry reduction to the lower number (here: two) of dimensions described by the theory of interest.

[5]A matrix model is an integral over a vector space of matrices, e.g., Hermitean matrices, together with a matrix potential that specifies the measure of the integral. It can be viewed as a zero-dimensional QFT with matrix degrees of freedom. See chapter 5 for a brief introduction.

The symbol $\mathcal{Z}(\beta_1, \beta_2)$ denotes the gravitational path integral with some boundary conditions, e.g., that the manifolds we integrate over possess two boundaries with lengths β_1, β_2. The path integral then instructs us to integrate over all possible geometries that fill in these boundary conditions, in full analogy to the more familiar quantum mechanical [57] or quantum field theoretical [58] path integrals. As alluded to above, this is not a well-defined operation in gravity in arbitrary dimension, but two dimensions are a special case: due to the Gauß-Bonnet theorem [59], there are no propagating gravitational degrees of freedom, rendering two-dimensional gravity topological[6]. This feature is what allows the topological expansion indicated on the right-hand side of eq. (1.4): the path integral decomposes into distinct topological sectors that can be evaluated separately. Indeed, these sectors can in turn be split up into two independent problems: solving the path integral over the boundary dilaton mode (the so-called *trumpet* path integral, indicated by the outer cylindrical portions of the geometries in eq. (1.4)), and the integral over the moduli space of inequivalent bulk geometries (depicted in eq. (1.4) as the convex core with "handles"). The resulting objects of this latter integral – the so-called Weil-Petersson (WP) volumes [60] – are well-studied objects in branches of mathematics such as Teichmüller theory [60] and intersection theory [61, 62].

Due to its simplicity, JT gravity has been fruitfully applied in the study of questions that are otherwise (too) difficult to address in quantum gravity[7], such as the inclusion of nontrivial topologies in the path integral [65], the black hole information problem [66–69], and quantum information and scrambling more generally [70–73].

Nontrivial topologies, such as the ones depicted in eq. (1.4) address the so-called *factorisation problem* [74]: correlation functions of operators that are localised very far away from each other – for instance, on distinct parts of a disconnected boundary, such as the two-component boundary depicted in eq. (1.4) – are expected to factorise in any ordinary QFT [75]. In a holographic picture, however, they manifestly do not factorise in the gravitational bulk theory. In two dimensions, this nonfactorisation can be explained by invoking the dual matrix ensemble [76–79], since an ensemble average can correlate distinct boundary segments. However, in higher dimensions, such an ensemble is not available[8], inspiring attempts to restore factorisation in JT gravity as well [80–82]. Solving this problem is of particular interest because these precise nontrivial topologies are in turn used to address the black hole information problem [83, 84].

Information scrambling – i.e., the spreading of initially localised quantum information in the form of, e.g., correlation functions, over a system – in the context of JT gravity is of particular interest since black holes have been conjectured to be the fastest scramblers appearing in nature [85–87], going back to a thought experiment due to Hayden and Preskill [88]. This conjecture has inspired work regarding the possibility of teleporting quantum information across topologies like the ones depicted in eq. (1.4) [70–73], and more generally, a renewed interest in the study of out-of-time-ordered commutators (OTOCs),

[6] The dilaton in JT gravity meanwhile only retains a dynamical boundary mode [55].
[7] See also [63, 64].
[8] See section 4.2 for an argument as to why this is the case.

which were first considered by Larkin and Ovchinnikov in the context of the physics of condensed matter [89].

OTOCs, and in particular, OTOCs that initially[9] exhibit a regime of exponential growth, have long been viewed as a key discriminant of quantum chaotic systems[10] [90]. While there is no universally agreed-upon definition of what quantum chaos *is*, two notions stand out and are of particular importance to this thesis [93, 94]: first, and most restrictively, we might define a quantum chaotic system as any quantum system that has a chaotic classical limit. If a chaotic classical limit is available, we may use the rich structure that is present in such a system as a tool to construct (and understand) a quantisation of the system; a key manifestation of this concept is Gutzwiller's trace formula [95],

$$\rho(E) \simeq \rho_0(E) + \sum_\gamma A_\gamma e^{\frac{i}{\hbar}S_\gamma(E) - i\frac{\pi}{2}\mu_\gamma}, \tag{1.5}$$

with the density of states,

$$\rho(E) = \sum_n \delta(E - E_n), \tag{1.6}$$

on the left-hand side and the smoothed spectral density $\rho_0(E)$ and a sum over periodic orbits γ of the classical dynamics, weighted by further quantities that only depend on classical information, on the right-hand side. The symbol $\simeq$ indicates that the two sides are only asymptotically equal. Remarkably, this formula relates the exact quantum energy spectrum with a function of only classical quantities, but still in a way that accounts for interference and similar quantum effects [94]. While Gutzwiller's trace formula as written in eq. (1.5) is only valid for single-particle systems, generalisations for bosonic many-body systems and field theories exist [96, 97].

The semiclassical formalism arising from this method [97–100] therefore bodes well for a study of quantum gravity as an instance of quantum chaos in general [87, 101], but two-dimensional gravity is special, as observed above: the boundary dilaton mode is subject to the so-called Schwarzian action [102], and this action classically only describes one degree of freedom. Consequently, it is necessarily integrable [103], not chaotic. If we still intend to call JT gravity quantum chaotic (and we better, seeing as how JT gravity has been shown to be *maximally* quantum chaotic, in the sense of exhibiting the fastest allowed growth of the OTOC [86, 87, 104]), we need another, more permissive notion of quantum chaos. This notion is provided by the Bohigas-Giannoni-Schmit (BGS) conjecture [105], which states that any classically chaotic system, upon quantisation, obeys energy level statistics predicted by random matrix theory (RMT). Conversely, we may take RMT level statistics *as the definition* of quantum chaos. This definition has the marked advantage that it encompasses both systems with an integrable classical limit, and systems that have no classical limit at all, such as fermionic

[9]Meaning for times that are larger than the ergodic time, but shorter than the scrambling time [90].

[10]See however [91, 92] for examples of integrable systems with exponentially growing OTOCs near instabilities of the classical dynamics.

systems [106], spin chains [107] and in particular, the Sachdev-Ye-Kitaev (SYK) model [104, 108–110], whose low-energy sector is dual to JT gravity.

Indeed, we have already learned that JT gravity is dual to a random matrix model, which does not admit a classical limit, but does follow RMT statistics [1, 2, 111–113], suggesting that this is the (or at least, *a*) notion of quantum chaos that is particularly relevant for JT gravity. Luckily, there is a connection between the very strict and the rather loose notions of quantum chaos introduced above, in the sense that the emergence of RMT statistics is by now well-understood semiclassically [97–100].

The above exposition may serve to impart to the reader that perhaps I did not choose the initial quotation wholly inappropriately, in particular as concerns a hundred schools of thought contending in the context of quantum gravity. The goal of this thesis now is to elaborate on how my collaborators and I were able to use these contending schools of thought to bring maybe not a hundred, but at least a few flowers to bloom: we were able to make forays into the study of the nonperturbative extension of the JT/matrix model duality of [55, 56], both in the setting of orientable manifolds presented above [1], and in the much more complex *unorientable* setting [2]. We were able to identify an individual quantum mechanical system whose semiclassical limit partially describes the topological expansion of JT gravity, thereby providing a putative boundary dual with a well-defined, chaotic *classical* limit [3]. And finally, we were able to show that this system is maximally *quantum* chaotic in the sense of [87], developing a formalism for computing the growth rate of the OTOC in a much more general class of systems with a classical limit along the way [4].

In order to situate our work in its context within the literature, I will use part I of the thesis to introduce much (though unfortunately not all) of the theoretical background needed to appreciate our results. I will begin with an exposition of classical chaos in contrast to the more familiar, but less typical classical integrability in chapter 2, before proceeding to describe how to quantise such classically chaotic systems. The primary tool used to perform this quantisation is Gutzwiller's trace formula (1.5), of which I will sketch a brief derivation in section 2.1. Departing from eq. (1.5), I will use section 2.2 to introduce the so-called *self-encounter formalism* of [99, 100] that makes use of classical structures, such as periodic orbits, to practically evaluate spectral correlations from Gutzwiller's trace formula, and to understand, e.g., the emergence of RMT level statistics.

I will then, in chapter 3, move on to a more specific application of the semiclassical methods developed up this point: geodesic motion on compact hyperbolic manifolds. This type of motion makes for prototypical examples of chaotic systems. Many of the requirements for the classical definition I will give for chaos in chapter 2 are typically only *effectively* fulfilled, but they can be shown with mathematical rigour in this context. In particular, Gutzwiller's trace formula – usually only an approximation – is *exactly* true for a particle sliding freely on a compact hyperbolic manifold [114]. The identity is called the Selberg trace formula (STF) in this context, and I will give a brief derivation in section 3.1 in order to provide an understanding for both the smooth term and the (oscillatory) periodic orbit term appearing in the trace formula. Both of these terms will play a crucial role in the remainder of the thesis, in particular in the aim of establishing a duality between JT gravity and the type of system described by the STF. I will further discuss how to break time reversal invariance in the system by introducing magnetic fluxes in section 3.2, and motivate the STF as a physical, rather than mathematical statement by discussing a concrete realisation in section 3.3. The Selberg trace formula was first mentioned in the context of JT gravity in [115], but only in [3] were we able to provide the first strong arguments for a duality between JT gravity and a system described by the STF.

These tools established, I will return to a more detailed exposition of low-dimensional quantum gravity in chapter 4, where I will begin by discussing the difficulty of quantising gravity in some more detail, before moving on to a proper introduction to JT gravity in section 4.1. In particular, I will discuss how to account for the boundary dilaton mode in section 4.1.1 and for the bulk moduli space in section 4.1.2, before motivating the problem of allowing unorientability in section 4.1.3. I will close the section by introducing the low-energy limit of JT gravity, known as (Witten-Kontsevich) topological gravity, or the Airy model [61] in section 4.1.4. This limit will be particularly useful in the unorientable case, where many computations that are doable in orientable JT gravity will not carry over straightforwardly. Finally, I will give a brief overview of the previously noted factorisation problem, in order to motivate one of the major aims of the thesis that we addressed in [3].

The last bit of theoretical context needed is a brief introduction into matrix models that I will provide in chapter 5. This introduction should serve to motivate why such matrix models – and thereby, also JT gravity – obey predictions from RMT in a particular late-time limit, the subject of part II. I will begin in chapter 6 by reviewing a well-known result about the microcanonical spectral form factor (SFF), that is, the correlation function of spectral densities evaluated at two different energies: in the so-called *universal limit*, where energy levels become infinitely dense, and simultaneously the separation of the arguments of the correlation function vanishingly small, the microcanonical SFF takes a universal form depending only on the symmetries of the matrices integrated over.

I will then argue in chapter 7 that upon Laplace transforming to the canonical ensemble, this universality has ramifications in the canonical SFF when evaluated at late times, in a sense to be made precise later on[11]. In orientable JT gravity in particular, we can compute the canonical SFF in this late-time limit by evaluating its topological expansion. It is not trivially clear however, that this computation should agree with the SFF computed from Laplace transforming the universal microcanonical SFF, both because the notion of universal limit is somewhat more subtle in the JT matrix model (as we will see below), and because the duality between JT gravity and said matrix model was initially only rigorously proven at the level of the topological expansion of JT gravity, and the corresponding perturbative expansion of the matrix model [55]. The universal limit of the SFF however is a nonperturbative result from the matrix model

[11]The canonical SFF of JT gravity was first considered in this limit in [116, 117].

perspective [94]. What we were able to show in [1][12] is that from demanding that the two computations nevertheless agree, we can predict highly nontrivial relations that the Weil-Petersson volumes, i.e., the volumes of the moduli space of manifolds appearing in JT gravity, must obey; and indeed, all the relations that have been checked so far *are* obeyed by the WP volumes.

We set out to perform the same calculation in the unorientable setting in [2], where however the staggeringly increased complexity of the computation forced us to confine ourselves to the Airy model, i.e., the low-energy limit of JT gravity. There, the problem of divergent moduli space volumes (cf. sections 4.1.3 and 4.1.4) can be sidestepped, allowing for a calculation without introducing any additional regularisation à la [119]. I will report our computation in chapter 8, noting however that my personal involvement in this part of the project was much more limited than in the contents of the other chapters. Nevertheless, the complexity of the computation forces me to address it at some length. I have aimed to reformulate certain parts in a way that is hopefully a bit easier to digest. In section 8.1, I will perform the RMT calculation of the canonical SFF of the unorientable Airy model, and repeat the computation on the gravity side in section 8.2. As I alluded to above, agreement of these calculations is highly nontrivial, and indeed, we will see at this point that the two really do not agree naively. I will argue in section 8.3 that they *can* be brought to agreement in an asymptotic sense however; in [2], we interpreted this as a manifestation of the difference between a nonperturbative calculation on the one side and a perturbative one on the other. Finally, in section 8.4, I will briefly discuss that in the unorientable case, we can find nontrivial relations very much like the ones reported in the orientable case in [1] and reviewed in section 7.2. Their being fulfilled to all orders that we have checked lends some credence to our approach in section 8.3.

While in part II, I show how to use random matrix theory – in a sense, the most generic tool available to describe quantum chaotic systems – to gather nonperturbative information about the bulk moduli of JT gravity, I turn to a much more specific system in part III, setting out initially, in chapter 9 to study the complementary sector: the boundary dilaton mode that is responsible for the so-called disk and trumpet partition functions in JT gravity.

The disk describes the only geometry allowed in classical JT gravity and encodes the smoothed spectral density of the theory. Importantly, this spectral density grows exponentially with the (square root of the) energy – a feature expected from a quantum theory describing black holes [120]. In the JT matrix model, the smoothed spectral density recursively determines the topological expansion of *all other correlation functions*. It can therefore be viewed as arguably the most important feature of JT gravity that any proposed dual theory must get right. I will discuss in section 9.1 how in [3], we were able to derive, rather than require, the JT gravity disk, as well as the corresponding smoothed spectral density, by taking the limit of the configuration space dimension going to infinity in the system introduced in chapter 3. In particular, I will show that this agreement is not simply a coincidence of two completely unrelated functions, but that JT gravity and the Selberg trace formula are really performing the

[12]See also [117, 118] for related work.

same computation by reviewing said computation in the first-order formulation of JT gravity in section 9.1.1.

Moving on to the trumpet partition function, we encounter the fundamental building block that needs to be combined with the recursively determined Weil-Petersson volumes to obtain actual physical correlation functions in JT gravity. Reproducing the trumpet partition function turns out to be equivalent to identifying the correct operator to insert into the Selberg trace formula – I identify this operator in section 9.2 before showing our computation in [3] of the so-called *double trumpet* partition function[13], i.e., the result of combining two trumpets without inserting a regular geometry with a well-defined moduli space in between. This partition function is not simply determined recursively, but it *is* fairly universal: it can be computed in the JT matrix model, but equally well in a very large class of other matrix models as well. Correctly computing the double trumpet therefore serves as an important consistency check for our proposed dual theory. I will show that in order to reproduce the double trumpet in any parameter regime, we need to introduce a novel way of evaluating the Selberg trace formula based on correlations between the lengths of geodesics entering the STF.

Finally, in section 9.3, I will show that using the aforementioned length correlations, it is possible to systematically define and compute an expansion of correlators of the STF and to match this expansion with a quantum gravitational topological expansion. I restrict myself to topological gravity in this section, corresponding to the three-dimensional version of the model introduced in chapter 3. Currently ongoing work [121] aims at capturing different topological sectors in this formalism, and extending it to full JT gravity.

In the last chapter of this thesis, I will shift my focus from late- to early-time signatures of quantum chaos, turning once more to OTOCs. Having established strong arguments for the duality between JT gravity and the geodesic motion on high-dimensional hyperbolic manifolds, I will use chapter 10 to report our calculation of the OTOC in this system in [4]. In the course of this introduction, I have already remarked on there being a notion of a *maximally* quantum chaotic system. This notion is provided by the Maldacena-Shenker-Stanford (MSS) bound on the so-called quantum Lyapunov exponent, i.e., the growth rate of the OTOC [87]. It has been shown that JT gravity is maximally chaotic in this sense [86, 104], and saturation of the MSS bound is expected more generally of systems that describe black holes [85]. It therefore stands to reason that if the system I studied in chapter 9 is dual to JT gravity, it ought to saturate the MSS bound as well. I will briefly review the bound in section 10.1 before introducing the formalism of Wigner-Moyal phase space quantisation in section 10.2. This formalism is an important semiclassical tool somewhat complementary to periodic orbit theory as used in chapter 9, and allows for a systematic computation of the OTOC as an expansion in small $\hbar$, allowing for a straightforward identification of the initial regime of exponential growth, as well as for the extraction of the growth exponent. To make sense of this expansion, I will have to restore physical units in section 10.3, in contrast to the natural units I use throughout the thesis up to this point. I will then

[13]Depicted as the first geometry in eq. (1.4).

report our computation of the leading-order quantum Lyapunov exponent in section 10.4, noting in particular the point up to which our method can be used to compute the quantum Lyapunov exponent in arbitrary classically chaotic (and even classically integrable, but unstable) systems, before showing that high-dimensional hyperbolic motion does indeed saturate the MSS bound in the limit of the configuration space dimension going to infinity, i.e., the limit in which we were also able to recover the disk partition function of JT gravity. Furthermore, we were able to identify a temperature regime in which the bound is approached even at finite dimension – indeed, the quantum Lyapunov exponent seems to scale only very simply with the configuration space dimension.

Finally, in section 10.5, I will report on our attempts to go even further: systems that saturate the MSS bound necessarily acquire subleading corrections to the initial exponential growth [122, 123], and these corrections obey bounds on their growth in turn. Using the Wigner-Moyal formalism, we were able to find a formal expression for the first correction to the leading Lyapunov growth. Evaluating it exactly, or even estimating its growth rate, has proved challenging however, and is the target of ongoing work.

Part I

Theoretical Background

Chapter 2

Semiclassics and periodic orbit theory

I introduce basic concepts of semiclassical and periodic orbit theory needed in the thesis, starting with a definition of classical integrable systems, then moving to classically chaotic and eventually quantum chaotic systems. I give a brief derivation of Gutzwiller's trace formula in section 2.1, before discussing Berry's diagonal approximation and the self-encounter formalism used to evaluate the trace formula in practice in section 2.2.

Many of the systems one first learns to describe in theoretical mechanics are *integrable*. A Hamiltonian system with f degrees of freedom is called integrable (or Liouville-Arnol'd integrable) if it possesses f functionally independent, Poisson-commuting constants of motion $F_1, \dots, F_f$, i.e., [103],

$$dF_1 \wedge \dots \wedge dF_f \neq 0, \tag{2.1}$$

$$\{F_i, F_j\} = 0, \tag{2.2}$$

where d is the exterior derivative, $\wedge$ the exterior product and $\{\cdot, \cdot\}$ the Poisson bracket. If these conditions hold, one can, at least formally, integrate the equations of motion and find an exact solution. Every trajectory of the system can then be fully specified by giving the values of the constants F_i, and, following the Liouville-Arnol'd theorem [103], the system's phase space is foliated by f-dimensional tori, the surfaces of constant F_i. Each trajectory as a curve in phase space is then confined to one of these tori, making the motion quasiperiodic. For an illustration of this type of foliation, see fig. 2.1. This simple structure of the phase space is very powerful, allowing for a fairly detailed description of the system's dynamics, and is in some sense very generic. It is, however, not the type of system that is of interest in the study of quantum gravity that I aim to perform in this thesis. Neither is it particularly typical, as even small nonlinear perturbations of integrable systems lead to a loss of integrability [93].

This is because perturbative corrections are weighted by the inverse of linear combinations of the frequencies of the above-mentioned quasiperiodic motion [93]. To understand this, let an invariant f-torus of an unperturbed integrable system be parametrised by angles φ_i, with the radii given by the particular values of the constants of motion F_i. For a given trajectory on the torus $\varphi(t)$,

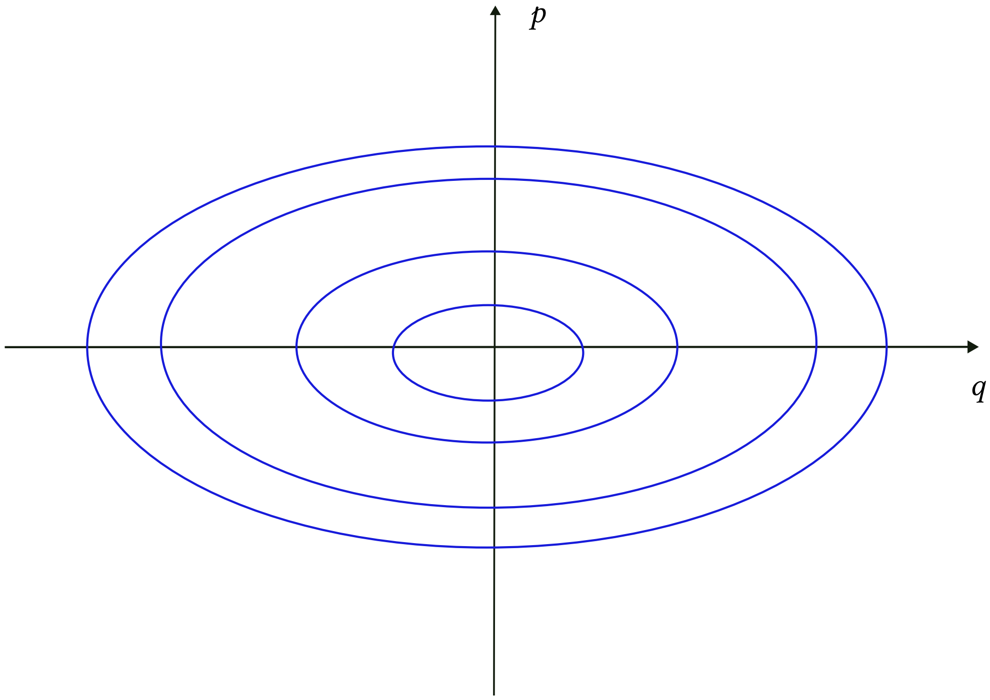


FIGURE 2.1: Tori of constant energy foliating the phase space of a one-dimensional harmonic oscillator $H = p^2/2m + m\omega^2 q^2/2$.

we can compute the frequencies

$$\omega_i = \frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \text{const.} \tag{2.3}$$

If we now perturb the system with the Hamiltonian H_0 by some nonlinearity ϵH_1, we can compute corrections to the angles and the constants of motion due to the perturbation. The transformation to the new, perturbed angles and constants of motion will be mediated by a generating function which, in a self-consistent manner, will in turn be corrected by the perturbation. If we call the new angles and constants θ_i, J_i, the generating function $S(\vec{\theta}, \vec{J})$ can be expanded as

$$S(\vec{\theta}, \vec{J}) = \theta_i J_i + \epsilon S_1(\vec{\theta}, \vec{J}) + \mathcal{O}\left(\epsilon^2\right), \tag{2.4}$$

with the first order correction given by

$$S_1(\vec{\theta}, \vec{J}) = \frac{H_{1,\vec{k}}(\vec{J})}{k_i \omega_i}, \tag{2.5}$$

where $H_{1,\vec{k}}(\vec{J})$ are the Fourier components of the nonlinearity H_1. In eq. (2.5), we can see a manifestation of the so-called small denominator problem of classical perturbation theory. If two frequencies of the unperturbed motion are rational multiples of each other, the denominator in eq. (2.5) can vanish, and it can become arbitrarily small even for "insufficiently irrational" [93] multiples. The Poincaré-Birkhoff [124] and Kolmogorov-Arnol'd-Moser (KAM) theorems [103]

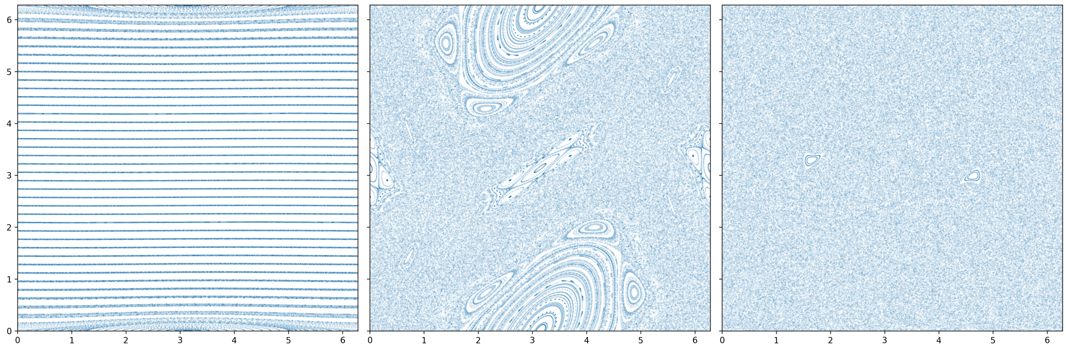


FIGURE 2.2: The Poincaré surface of section for the standard map (2.6) for $K = 0.01$ (left), $K = 1.5$ (middle) and $K = 6$ (right). In the near integrable case $K = 0.01$, the system is clearly restricted to particular allowed trajectories in the phase space. In the soft-chaotic case $K = 1.5$, the integrable structure starts to disappear, being replaced by stable islands, and a quasi-uniformly filled phase space in between. As the kicking strength increases, $K = 6$, the stable islands seem to disappear, eventually leaving only a uniformly filled phase space.

quantify this notion and describe the fate of invariant tori under nonlinear perturbations: tori that are close to resonance are destroyed by the small denominator problem to be replaced by pairs of stable and unstable fixpoints, while non-resonant tori survive, albeit deformed.

As the strength of this perturbation increases, this "soft chaos" in the terminology of Gutzwiller [93] is replaced by hard chaos: all the invariant tori are eventually destroyed, leaving only a dense set of periodic orbits in the phase space. The transition from the (near-)integrable regime to soft and eventually hard chaos is depicted in the Poincaré surfaces of section in fig. 2.2 for the standard map,

$$p_{n+1} = p_n + K \sin x_n, \quad x_{n+1} = x_n + p_{n+1} \mod 2\pi. \tag{2.6}$$

Since the hard-chaotic case is the one of relevance for quantum gravity, I will use the rest of this chapter to introduce the notions and tools required to describe both classical and quantum systems that exhibit chaos.

For a system to be called chaotic, we typically apply three criteria [94]:

1. The system possesses a dense set of periodic orbits.

2. The Hamiltonian flow of the system is mixing/topologically transitive.

3. The system displays exponential sensitivity to initial conditions.

Condition 1 is met by any hard-chaotic system, as per the discussion above. Meanwhile, the Hamiltonian flow ϕ_t of the system is called topologically transitive if for any two open sets A, B in the phase space, there exists t such that

$$\phi_t(A) \cap B \neq \emptyset, \tag{2.7}$$

i.e., any open set will, after some finite time, intersect any other open set. This notion, while already quite difficult to prove in practice, is only topological

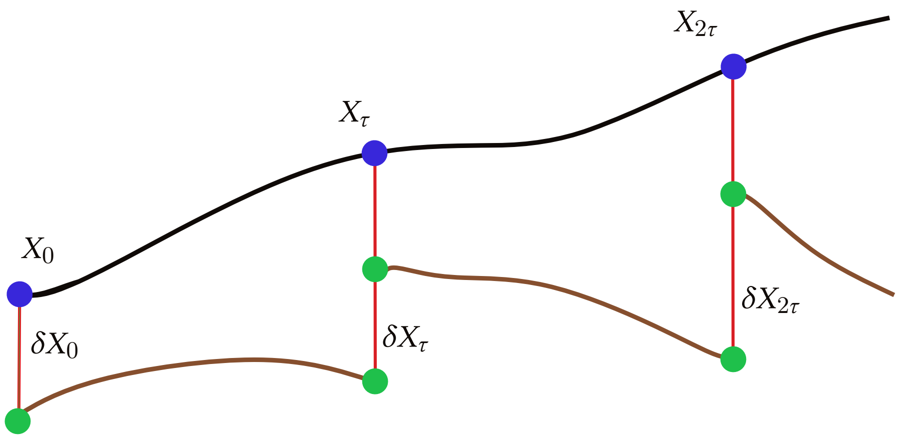


FIGURE 2.3: Time evolution of two nearby trajectories starting at X_0 (black) and $X_0 + \delta X_0$ (brown) by some discrete time steps τ. The separation δX_τ after evolution by a time step is recorded and renormalised by δX_0 to account for the boundedness of the flow required for it to be mixing. Iterating this process allows the definition of the Lyapunov exponent via the average of the logarithms of the stretching factors $\chi_{n\tau} = \left|\frac{\delta X_{n\tau}}{\delta X_0}\right|$ according to eq. (2.12). Adapted from [130].

and does not say anything about the statistics of the mixing of phase space regions. Mixing [125] is a similar-in-spirit measure theoretical notion that neither implies, nor is implied by topological transitivity, though there are some (much) stronger notions that are not typically required for chaos that speak towards the statistical properties of the associated flows. Kolmogorov systems (or K-systems) [125] are systems with positive Kolmogorov-Sinai entropy, i.e., the flow in the system actually produces entropy, which implies mixing [126]. Bernoulli shifts [127] describe flows that are *as random as a coin toss*, i.e., they generate entropy at the maximal rate allowed by their Kolmogorov-Sinai entropy – hence, the Bernoulli property implies the K-property and thereby, strong mixing. The type of flow I will be considering in chapter 3 and the rest of the thesis has been shown to be Bernoullian [128].

Finally, exponential sensitivity to initial conditions means that trajectories that are initially "nearby"[1] in phase space diverge at an exponential rate, as depicted in fig. 2.3. If we consider an initial condition X_0 and its time evolution

$$X_t = \phi_t(X_0), \tag{2.8}$$

then a slight initial deviation δX_0 will evolve according to [94]

$$\delta X_t = \frac{\partial \phi_t(X_0)}{\partial X_0} \delta X_0 =: M_t(X_0) \delta X_0, \tag{2.9}$$

[1]In chapter 3, I will briefly discuss so-called Anosov systems that allow for a foliation of the phase space along each trajectory into exponentially contracting and expanding submanifolds, and a one-dimensional manifold along the direction of the flow. These systems are defined on Riemannian manifolds, and as such, the phase space carries a natural notion of distance given by the Sasaki metric [129] that is induced by any Riemannian metric in the configuration space. The Sasaki distance is simply the sum of squared position and momentum deviations, and hence offers a particularly intuitive notion of "nearby" initial conditions.

defining the monodromy matrix $M_t(X_0)$. The monodromy matrix can be understood as transporting initial deviations in the tangent space,

$$M_t(X_0)e_i(X_0) = \chi_i(t, X_0)e_i(X_t), \tag{2.10}$$

where $e_i(X_0)$ is some vector in the tangent space to X_0 that is neither in the direction of the Hamiltonian flow, nor orthogonal to the energy shell X_0 is on. Under the assumption that this vector does not grow exponentially in time,

$$\lim_{t\to\infty} \frac{1}{t} \log \|e_i(X_t)\| = 0, \tag{2.11}$$

the so-called *stretching factors* $\chi_i(t, X_0)$ determine the system's Lyapunov exponents according to [94]

$$\lambda_i(X) = \lim_{t\to\infty} \frac{1}{t} \log \|\chi_i(t, X)\|. \tag{2.12}$$

In general, a system with f configuration space degrees of freedom will have $2f-2$ (possibly different) Lyapunov exponents that appear in pairs $\pm\lambda$, a manifestation of Liouville's theorem [131]. Typically, one would try to determine the largest positive Lyapunov exponent and denote it as *the* Lyapunov exponent of the system, noting that the exponents may not only differ from direction to direction, but also from trajectory to trajectory[2]. In the kind of system I will be considering in the rest of this thesis however, all positive Lyapunov exponents are in fact equal [132], allowing me to sidestep this complication. I should also note at this stage that the Lyapunov exponent defined in this way is a microcanonical object, as evident from the reference to energy shells in the above discussion. There is a related notion of "quantum" Lyapunov exponents that I will discuss in chapter 10, and that is in fact canonical, i.e., temperature-dependent.

Everything I have discussed so far pertains mainly to classical chaotic systems. In order to pursue the aim of this thesis and describe quantum gravity however, I will need a way to quantise such chaotic systems. To do so, I will return my focus to the periodic orbits discussed above; they are the prime objects used in semiclassical quantisation, particularly in the Gutzwiller trace formula and the self-encounter formalism for computing correlators of Gutzwiller traces that I will discuss in the rest of this chapter.

2.1 Gutzwiller's trace formula

In this section, I will briefly sketch a derivation of the Gutzwiller trace formula compiled from [93, 94, 96], except where otherwise indicated. As we have seen above, classically chaotic systems possess a rich structure in terms of periodic orbits, as well as the classical phase space Poisson algebra, but a priori, it is not clear how and whether these structures can be carried over to the quantum realm.

[2]In systems that are ergodic, i.e., a weaker condition than the mixing required for chaotic systems, infinitely long trajectories all have the same Lyapunov exponents [94], although (short) periodic orbits may exhibit their own exponents.

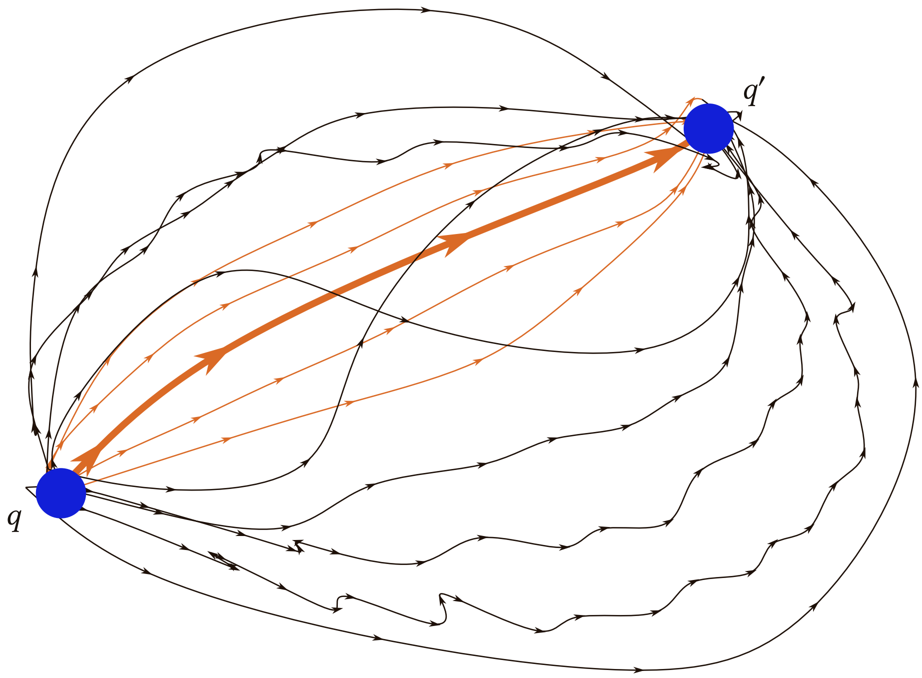


FIGURE 2.4: Some of the paths from q to q' contributing to the path integral (2.14). The path integral includes all paths, with typical paths being continuous, but nowhere differentiable. Most paths (black) interfere destructively, but near stationary points in the path integral (thick orange path), paths can interfere constructively (thin orange paths), justifying the evaluation of the path integral by the saddle point approximation (2.17) [57].

I will briefly discuss how Moyal extended the Poisson algebra to account for quantum effects in chapter 10; this extension is crucial for the computation of out-of-time-ordered commutators we performed in [4]. Understanding the emergence of classical periodic orbits in quantum systems is a somewhat complementary, but likewise very useful semiclassical method. We consider a quantum system with the Hamiltonian

$$\hat{H} = \hat{H}(\hat{x}, \hat{p}), \tag{2.13}$$

where $\hat{x} = (\hat{x}^1, \dots, \hat{x}^f)$ and $\hat{p} = (\hat{p}_1, \dots, \hat{p}_f)$ are conjugate position and momentum operators. $\hat{H}$ should be self-adjoint and bounded from below [133]. Position space transition amplitudes between states $|q\rangle, |q'\rangle$ can be computed using the *propagator* [57],

$$K(q, q', t) = \langle q'|e^{-i\hat{H}t}|q\rangle = \int_{x(0)=q}^{x(t)=q'} \mathcal{D}[x(s)]e^{i\int p\mathrm{d}x - H(x,p)\mathrm{d}t}, \tag{2.14}$$

where the measure $\mathcal{D}[x(s)]$, together with the integral "boundaries", indicates the integration over all possible paths connecting the initial position $x(0) = q$ to the final position $x(t) = q'$ in time t. In the phase of eq. (2.14), we can identify the classical action functional of the path $x(s)$,

$$S[x(s), \dot{x}(s), t] = \int p\mathrm{d}x - H(x, p)\mathrm{d}t =: \int_0^t \mathrm{d}s\, L[x(s), \dot{x}(s)], \tag{2.15}$$

with the Lagrangian $L[x(s), \dot{x}(s)]$. This action will in general cause many of the paths contributing to the path integral, illustrated in fig. 2.4, to interfere destructively. However, constructive interference can occur near stationary points of the path integral, i.e., the solutions of

$$\frac{\delta}{\delta x(s)} S[x(s), \dot{x}(s), t] = \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \dot{x}(s)} - \frac{\partial L}{\partial x(s)} = 0, \tag{2.16}$$

together with the appropriate boundary conditions[3]. Equation (2.16) are the classical Euler-Lagrange equations, meaning that the stationary points in the path integral are precisely the *classical* trajectories from q to q' in time t. Since eq. (2.16) specifies a boundary value problem, there will in general be many possible solutions [134]. Evaluating the path integral (2.14) in a stationary phase approximation[4] entails summing over all stationary paths, eventually yielding the well-known semiclassical approximation to the quantum propagator, the van Vleck-Gutzwiller propagator [93, 135],

$$K_{\mathrm{sc}}(q, q', t) = \sum_{x(t)} A_{x(t)}(q, q', t) e^{iS_{x(t)}(q, q', t) + \frac{i\pi}{2}\nu_{x(t)}}, \tag{2.17}$$

where

$$A_{x(t)}(q, q', t) = \frac{1}{(2\pi i)^f} \sqrt{\left| \det \frac{\partial^2 S_{x(t)}(q, q', t)}{\partial q \partial q'} \right|} \tag{2.18}$$

are called *stability amplitudes* of the classical path $x(t)$, since they measure the deviation of the final position due to initially different momenta. $S_{x(t)}(q, q', t)$ is Hamilton's principal function, i.e., the action functional (2.15) evaluated along $x(t)$, and $\nu_{x(t)}$ is the Morse index related to the number of focal points of $x(t)$, i.e., points where an initially infinitesimally close bunch of trajectories, after spreading out, reconverges again.

In many systems of interest, it is much easier to find closed trajectories, rather than solving the general boundary value problem [136]. Hence, it is useful to compute the trace of eq. (2.17), although not before taking the Fourier transform from time to energy[5], yielding the semiclassical Green's function [93],

$$G_{\mathrm{sc}}(q, q', E) = \sum_{x(E)} A_{x(E)}(q, q', E) e^{iS_{x(E)}(q, q', E) - \frac{i\pi}{2}\mu_{x(E)}}. \tag{2.19}$$

I am slightly abusing notation by denoting the stabilities and actions by the same symbols as for the time-dependent case, which is somewhat unusual in the semiclassics literature. The sum is now over the infinite set of trajectories with energy E connecting q and q', and $\mu_{x(E)}$ counts the number of conjugate points

[3] Derivatives with respect to vector-valued quantities, such as $x(s)$ and $\dot{x}(s)$, are to be understood in the obvious componentwise way.

[4] The stationary phase approximation is valid up to corrections of $O(\hbar)$ when reinserting physical units [93]. As we have just seen, the stationary paths are classical, such that their actions $S_{x(t)} \gg \hbar$, justifying the approximation.

[5] Recall that the discussion in the introduction to this chapter defined many of the objects relevant for classical chaos on fixed-energy shells.

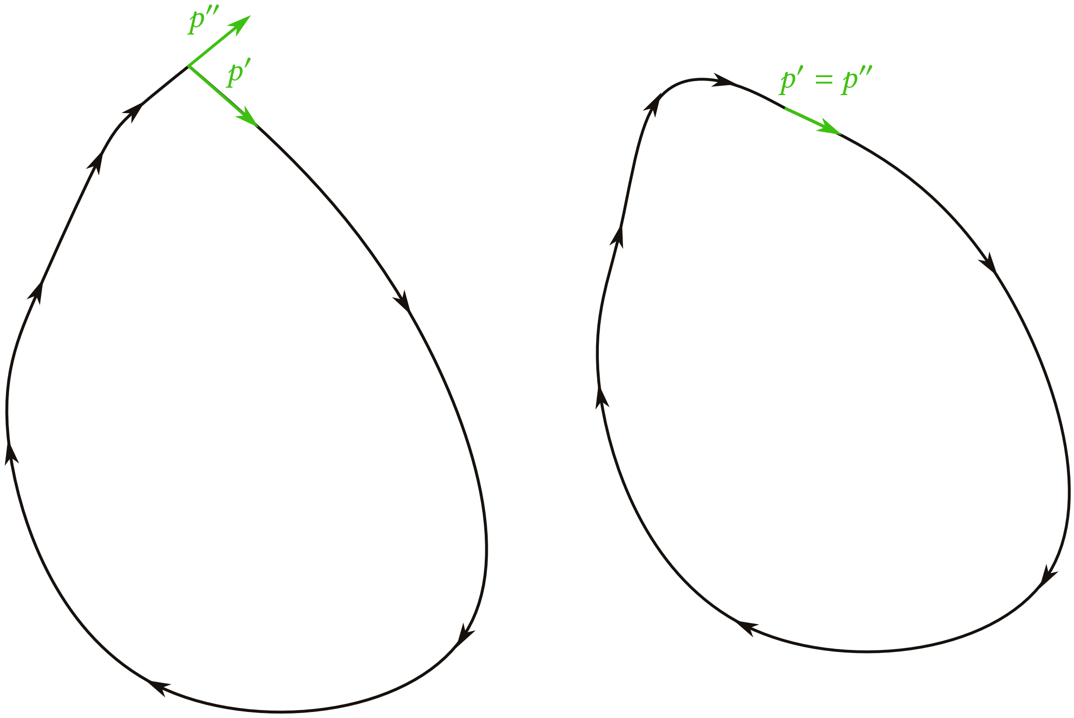


FIGURE 2.5: Left: a typical closed path contributing to the trace of Green's function, with differing initial and final momenta. Right: a periodic orbit with equal initial and final momentum obeying eq. (2.21) and hence contributing to Gutzwiller's trace formula (2.22) [57].

obtained by varying a given trajectory at constant energy.

Computing the trace now consists of setting $q = q'$ in eq. (2.19) and integrating over the base point q,

$$g_{\mathrm{sc}}(E) = \int \mathrm{d}q\, G_{\mathrm{sc}}(q, q, E). \tag{2.20}$$

To ensure consistency with the initial stationary phase approximation used to derive eq. (2.19) in the first place, we should evaluate the trace in a stationary phase approximation as well. The stationarity condition then reads [93]

$$\frac{\partial S(q, q, E)}{\partial q} = \left(\frac{\partial S(q', q'', E)}{\partial q'} + \frac{\partial S(q', q'', E)}{\partial q''}\right)_{q''=q'=q} = p' - p'' = 0, \tag{2.21}$$

where I denote the first (second) end point of the trajectory by a single (double) prime. The stationary phase condition therefore assures that the trajectories in the trace (2.20) not only close, but close *smoothly*, as illustrated in fig. 2.5. In other words, eq. (2.21) selects precisely the classical periodic orbits of the system as the primary contributors to the trace of the Green's function! Evaluating the integral now yields Gutzwiller's trace formula for the oscillatory contribution to the trace of the Green's function [93],

$$g_{\mathrm{sc}}(E) = \sum_{\gamma(E)} A_\gamma e^{iS_\gamma}, \tag{2.22}$$

where I omit the phase factor originating from the Morse index, as it will be irrelevant to the rest of the thesis[6]. I will denote periodic orbits as γ from now on; the sum is now over all periodic orbits with energy E, *regardless* of their location in configuration space. A_γ is the stability amplitude associated to γ, and S_γ the action functional evaluated along γ.

There is a caveat to using eq. (2.22) as an approximation to the trace of the Green's function, i.e.,

$$g(E) = \int \mathrm{d}q\, G(q,q,E) = \int \mathrm{d}q \int \frac{\mathrm{d}t}{2\pi} e^{-itE} K(q,q,t). \tag{2.23}$$

Namely, when evaluating eq. (2.23) in the stationary phase approximation, we need not only to account for stationary points inside the integration range, but also for enhanced contributions from the boundaries of the integral [94]. To appreciate this, consider an integral of the form

$$\int_a^b \mathrm{d}t A(t) e^{i\nu\phi(t)} = \frac{A(b)e^{i\nu\phi(b)}}{i\nu\phi'(b)} - \frac{A(a)e^{i\nu\phi(a)}}{i\nu\phi'(a)} + O\left(1/\nu^2\right), \tag{2.24}$$

where $A(t), \phi(t)$ are analytic functions of t without stationary points in $[a,b]$ and $\nu \to \infty$. This can easily be seen by writing [137]

$$e^{i\nu\phi(t)} = \frac{1}{i\nu\phi'(t)} \frac{\mathrm{d}}{\mathrm{d}x} e^{i\nu\phi(t)} \tag{2.25}$$

and subsequently integrating by parts. If there were stationary points of $\phi(t)$ in the integration interval, we would simply pick up the $O\big(1/\sqrt{\nu}\big)$ stationary phase contributions in addition to the $O(1/\nu)$ boundary terms. With this in mind, eq. (2.24) easily maps onto eq. (2.23) by identifying $a = 0, b \to \infty$ and $\nu = 1/\hbar$, which I have set equal to unity for notational convenience; however, it is best to recall that small $\hbar$ is needed to justify the stationary phase approximation to begin with.

We therefore obtain a contribution to eq. (2.23) coming from the lower limit $t = 0$ of the t integral, which we can interpret as the contribution of "orbits of period 0" [138]. In the rest of the thesis, where in the system of interest, I will typically be more interested in the lengths of periodic orbits, rather than their period, I will also refer to this contribution as originating from zero-length or point orbits. For the trace of Green's function, the zero-length contribution is given by Weyl's law, and for this reason also often referred to as the Weyl term

[6]This is because a non-vanishing Morse index requires a refocussing of trajectories – a situation which never occurs in manifolds with constant negative curvature, i.e., the subject of the rest of the thesis; instead, trajectories diverge exponentially everywhere, cf. chapter 3.

[94],

$$
\begin{aligned}
g_0(E) := g(E) - g_{\mathrm{sc}}(E) &\sim \lim_{t\to 0} \int \mathrm{d}q\,\mathrm{d}q' \,\frac{\delta(q-q')}{E - H(q,q',t)} K_{\mathrm{sc}}(q,q',t) \\
&\sim \lim_{t\to 0} \frac{1}{(2\pi)^f} \int \mathrm{d}q\,\mathrm{d}q'\,\mathrm{d}p \,\frac{e^{-ip(q-q')}}{E - H(q,p)} \underbrace{K_{\mathrm{sc}}(q,q',t)}_{\xrightarrow{t\to 0}\delta(q-q')} \\
&\sim \frac{1}{(2\pi)^f} \int \mathrm{d}q\,\mathrm{d}p \,\frac{1}{E - H(q,p)}.
\end{aligned} \tag{2.26}
$$

In the rest of the thesis, I will be more interested in the density of states,

$$
\rho(E) = \sum_n \delta(E - E_n), \tag{2.27}
$$

where E_n are the eigenenergies of $\hat{H}$. Using the standard relation between the density of states and Green's function,

$$
\rho(E) = -\frac{1}{\pi} \operatorname{Im}\{g(E)\}, \tag{2.28}
$$

we arrive at the well-known Thomas-Fermi approximation to the density of states, or what I will refer to throughout the thesis as the smooth part of the density of states,

$$
\rho_0(E) \sim \frac{1}{(2\pi)^f} \int \mathrm{d}q\,\mathrm{d}p\, \delta(E - H(q,p)), \tag{2.29}
$$

i.e., the density of states is approximated by the number of Planck cells of volume $(2\pi)^f$ in the classical energy shell.

2.2 Berry's diagonal approximation and the encounter formalism

In the remainder of the thesis, I will be interested not only in the density of states itself, but also in correlation functions of the density evaluated at two different energies under a notion of semiclassical average, as well as Fourier transforms thereof, such as the heat kernel,

$$
\operatorname{tr} e^{-\beta\hat{H}} = \int_0^\infty \mathrm{d}E e^{-\beta E} \rho(E). \tag{2.30}
$$

This is because even with all the simplifications undertaken up to now compared to the full quantum propagator, it is still not feasible in most cases to actually evaluate Gutzwiller's trace formula, relying as it does on knowledge about periodic orbits of arbitrary length and complexity. To be able to work with limited knowledge about the set of periodic orbits, we define the semiclassical average $\langle\cdot\rangle_{\mathrm{sc}}$ in the following way [100]: Consider any observable $f(z)$ evaluated on a point z of a periodic orbit γ, as well as a reference point $z_0 \in \gamma$, together with its time evolution $\phi_t^\gamma(z_0)$. Then the average of f along an orbit γ of period T is

$$[f]_\gamma = \frac{1}{T}\int_0^T \mathrm{d}t f(\phi_t^\gamma(z_0)). \tag{2.31}$$

In addition, one can perform an ensemble average over the ergodically distributed orbits inside a small time window ΔT, weighted by the stability amplitudes. Doing so yields an average that, by the equidistribution theorem due to Parry and Pollicott [139], is equivalent to the energy shell average[7]:

$$\frac{1}{T}\left\langle \sum_\gamma |A_\gamma|^2 \delta(T - T_\gamma)[f]_\gamma \right\rangle_{\Delta T,\mathrm{sc}} = \int \frac{\mathrm{d}\mu(z)}{\Omega} f(z). \tag{2.32}$$

Here, $\mu(z)$ is the Liouville measure and Ω the energy shell volume. A particularly useful consequence of this equivalence is obtained by setting $f = 1$, yielding the sum rule due to Hannay and Ozorio de Almeida [140],

$$\left\langle \sum_\gamma |A_\gamma|^2 \delta(T - T_\gamma) \right\rangle_{\Delta T,\mathrm{sc}} = T, \tag{2.33}$$

which allows the evaluation of correlation functions relevant, e.g., for the spectral form factor that I will describe presently[8]. The (microcanonical) spectral form factor (SFF) is defined as the Laplace transform of the correlator

$$K(\tau) = \left\langle \int \frac{\mathrm{d}\epsilon}{\rho_0(E)} e^{i\epsilon\tau T_H} \rho_{\mathrm{osc}}\left(E + \frac{\epsilon}{2}\right) \rho_{\mathrm{osc}}\left(E - \frac{\epsilon}{2}\right) \right\rangle_{\mathrm{sc}}, \tag{2.34}$$

where $\rho_{\mathrm{osc}}(E)$ is the oscillatory (periodic orbit) part of the density of states as computed by Gutzwiller's trace formula, and the periods τ are measured in units of the Heisenberg time,

$$T_H = 2\pi\rho_0(E). \tag{2.35}$$

The microcanonical spectral form factor exhibits a particular universal form [94] depending only on the symmetry class of the system, as well as the smoothed spectral density[9] ρ_0, cf. chapter 6. Like many other spectral measures in

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[7]While the equidistribution theorem is often used for generic chaotic systems, e.g., [140], ergodicity or even mixing of the flow is not enough: ergodic or mixing flows can have non-dense sets of periodic orbits, and need not satisfy the shadowing or closing lemmas [141] required by [139]. However, such counterexamples are somewhat pathological and most chaotic systems "effectively" satisfy the equidistribution theorem (recall, e.g., that we *required* the set of periodic orbits to be dense), even though it cannot be proved that they do. The fully rigorous proof of [139] requires the system to have the Anosov property, which the model I will introduce in chapter 3 does.

[8]From now on, I will drop the subscript ΔT in the average, as this is only a technical detail and the precise implementation of the semiclassical average will not be very important for what I want to show.

[9]One is often interested in computing the spectral form factor deep in the middle of the spectrum, where the density of states is roughly constant, so that eq. (2.34) has a truly universal form as a function of τ that depends only on the discrete symmetries of the system.

quantum chaotic systems, it agrees with the analogous quantities computed in random matrix theory [105]. The integral over the energy separation ϵ can be evaluated to find [100]

$$K(\tau) = \frac{1}{T_H} \left\langle \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'} e^{i(S_\gamma - S_{\gamma'})} \delta\left(\tau T_H - \frac{T_\gamma + T'_\gamma}{2}\right) \right\rangle_{\mathrm{sc}} . \tag{2.36}$$

The phase of contributions to eq. (2.36) oscillates rapidly for most pairs γ, γ', cf. the discussion above, unless the action difference of the orbits γ and γ' is very small. Berry [98] argued that the most obvious way of ensuring a small action difference – namely by selecting only pairs where $\gamma = \gamma'$ (up to time reversal, in time-reversal invariant systems) – produces the leading order term of the universal spectral form factor,

$$K(\tau) \approx \frac{1}{T_H} \left\langle \sum_{\gamma} |A_\gamma|^2 \delta(\tau T_H - T_\gamma) \right\rangle_{\mathrm{sc}} = (2)\tau, \tag{2.37}$$

where the factor 2 appears in time-reversal invariant systems. This is Berry's celebrated diagonal approximation. It should be noted that eq. (2.37) is not a trivial consequence of eq. (2.33), which can be cast as a purely classical statement; rather, the diagonal approximation requires genuine quantum mechanical input to reduce the double to a single sum. Indeed, at subleading order in $\tau = T/T_H$, eq. (2.37) acquires corrections from pairs of periodic orbits that have small, but non-vanishing action differences. For time-reversal invariant systems, it was shown in [99] that the first correction to eq. (2.37) originates from pairs of periodic orbits where one orbit crosses itself at a small angle, while its partner narrowly avoids the self-crossing. Such a Sieber-Richter pair is schematically depicted in fig. 2.6, where I also illustrate the lift of this configuration space picture to its modern phase space formulation due to [100] that I want to briefly review in the following.

This formulation is based on so-called *self-encounters* of the orbits: consider for simplicity a two-dimensional Anosov flow[10] (cf. chapter 3) and the associated four-dimensional phase space. A Poincaré surface of section $\mathcal{P}$ within the phase space is a two-dimensional subset of the phase space based at a point $x = (q, p)$, consisting of points in the same energy shell as x, parametrised by

$$x + \delta x = (q + \delta q, p + \delta p), \tag{2.38}$$

such that $\delta q \perp p$. For Anosov flows, we can exactly decompose the tangent space to a point into stable, unstable and trivial submanifolds (cf. chapter 3), and projecting these onto $\mathcal{P}$, we can define a stable and an unstable direction in the Poincaré surface of section. Denoting the coordinates in these directions by s, u and allowing their time evolution by considering a comoving Poincaré

In JT gravity however (cf. chapter 4), we are interested in the situation at one of the edges of the spectrum.

[10]Although the picture lifts straightforwardly to more dimensions, and is also effectively true for non-Anosov, merely chaotic flows.

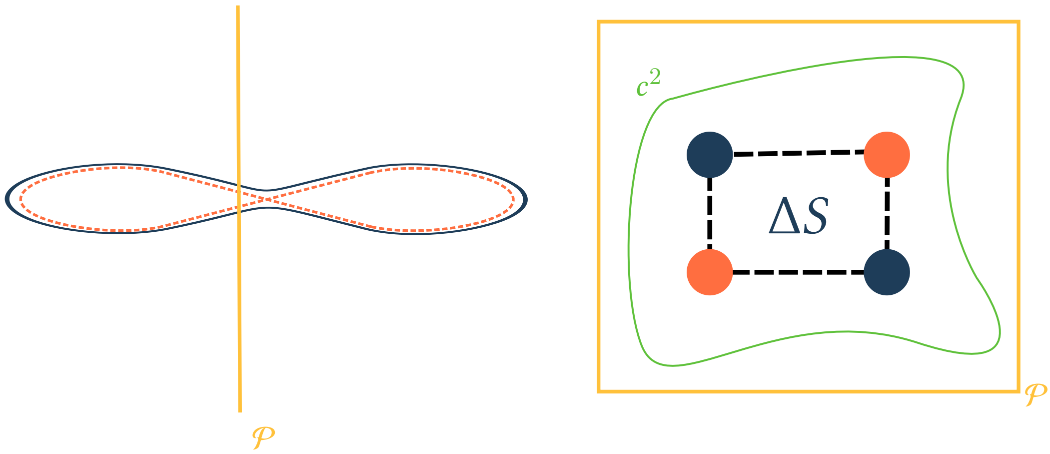


FIGURE 2.6: Left: a Sieber-Richter pair in configuration space, composed of one periodic orbit crossing itself with a small angle (orange) and a partner orbit narrowly avoiding the self-crossing (blue). The loops between the stretches contributing to the self-encounter are depicted in a strongly simplified manner, and the orbit difference is vastly exaggerated. The yellow line represents a Poincaré surface of section $\mathcal{P}$, placed here at one "end" of the encounter. Right: a projection of the two periodic orbits on the left as phase space curves onto the Poincaré surface of section $\mathcal{P}$. The piercing points of the crossing orbit (orange) and the crossing-avoiding orbit (blue) span a parallelogram in phase space whose symplectic area ΔS is the action difference between the two orbits. If this area is smaller than the linearisation scale c^2 (green), the orbit pair contributes to the spectral form factor at next-to-leading order in T/T_H. Adapted from [100].

section, we are interested in the regime in which the time dependence of the coordinates can be linearised, i.e.,

$$s(t) = \chi(x,t)^{-1}s(0), \quad u(t) = \chi(x,t)u(0), \tag{2.39}$$

where $\chi(x,t) \sim e^{\lambda(x)t}$ is the stretching factor encoding the local Lyapunov exponent $\lambda(x)$. This regime is typically defined by setting an action scale c^2 describing a symplectic area in the phase space within which the orbit γ must pierce $\mathcal{P}$ twice to constitute a self-encounter, cf. fig. 2.6. The exponential stability of the dynamics [94] then guarantees the existence of a partner orbit γ' closely following γ everywhere except in the encounter region; there, γ' pierces $\mathcal{P}$ at different points, and the area enclosed by the piercing points is precisely the action difference $\Delta S = S_\gamma - S_{\gamma'}$. The above description can be straightforwardly generalised to so-called l-encounters, where l stretches of each partner orbit are involved in the encounter, as illustrated in fig. 2.7.

The action scale c^2 is of classical origin and in the usual semiclassical treatment, its precise value is not important. It should however be set such that the encounter duration

$$t_{\text{enc}} = \frac{1}{\lambda}\log\frac{c^2}{\max\{s\}\max\{u\}} \tag{2.40}$$

is roughly of the order of the Ehrenfest time $t_E \sim \log(\text{const.}/\hbar)$ (in physical units) [100], since this ensures that the relevant action differences are small in

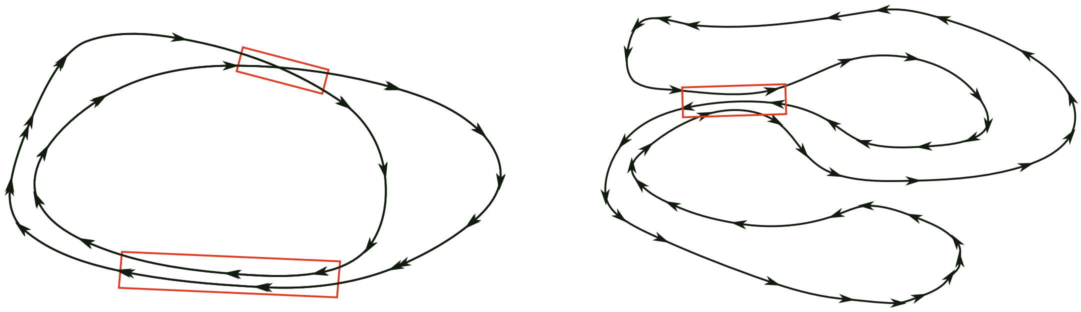


FIGURE 2.7: Examples of periodic orbits contributing at the same order to the periodic orbit double sum (2.41) in the time-reversal invariant case [100]. Left: an orbit with two 2-encounters (red rectangles), $L - V = 4 - 2 = 2$. Right: an orbit with a single 3-encounter, $L - V = 3 - 1 = 2$. The right orbit can contribute to eq. (2.41) at this order only in the time-reversal invariant case, since one of the encounter stretches runs in a different direction to the other two, rendering it far away in phase space, unless TRI is present.

the sense discussed above, i.e., c^2 is a scale of classical origin[11]. The notation $\max\{s, u\}$ signifies the maximal separations in stable and unstable directions in a given encounter's Poincaré surface of section. The authors of [100] have systematically evaluated contributions coming from all orders of encounters and shown that the spectral form factor in chaotic quantum systems takes the form

$$K(\tau) = \kappa\tau + \kappa \sum_{\vec{v}} N(\vec{v}) h(\vec{v}) \tau^{L-V+1} =: \kappa\tau + \kappa \sum_{n\geq 2} K_n \tau^n, \tag{2.41}$$

where $\vec{v} = (v_2, v_3, \ldots)$ is the number of $2, 3, \ldots$-encounters, V is the total number of encounters, L the number of orbit stretches involved in the encounters, $h(\vec{v})$ some combinatorial function and $N(\vec{v})$ encodes the statistics of the occurrence of orbit pairs depending on the system's symmetry class. The index κ denotes the absence ($\kappa = 1$) or presence ($\kappa = 2$) of time-reversal invariance (TRI). Interestingly, the symmetry class determines the allowed powers of τ in eq. (2.41), with only odd powers occurring in the broken TRI case, while all integer powers are allowed if TRI is preserved. Comparing this expansion to the genus expansion of the SFF in JT gravity (cf. chapter 4) suggests a tantalising connection of the encounter formalism to genus expansions in quantum gravity that has been studied, e.g., in [79, 117]. However, as shown in [100], for broken TRI,

$$K_n = 0, \quad n \geq 2, \tag{2.42}$$

i.e., highly non-trivially, *all* encounter contributions in this case have to cancel each other. This is in agreement with the perturbative form of the universal behaviour of the microcanonical SFF as computable in RMT [94, 105], elucidating how random matrix universality emerges in quantum chaotic systems with a classical limit. Meanwhile, the SFF of JT gravity in the orientable case,

[11] It should be noted that this is in stark contrast to the formalism of [3] that I will extend in chapter 9. There, I will argue that contributions are enhanced on the basis of *length differences* of periodic orbits that are small compared to the classical encounter scale. This is – remarkably – a purely classical statement.

corresponding to broken TRI in the boundary dual, has non-vanishing higher-order-in τ contributions. This means that if there is a connection between the encounter formalism and higher topologies, it is at least not fully straightforward. I will discuss a possibility of connecting the two in section 9.3, where I will also exemplarily show how to compute (simple) encounter integrals needed in the derivation of eq. (2.41) in the specific system introduced in chapter 3.

In the time-reversal invariant case, [100] also evaluates the combinatorial problem of determining the $N(\vec{v})$ and finds

$$K_n = \frac{(-2)^{n-1}}{n-1}, \tag{2.43}$$

i.e., the SFF again agrees with the RMT prediction [94].

Chapter 3

The Selberg trace formula

I introduce the Selberg trace formula and interpret it as a version of Gutzwiller's trace formula introduced in chapter 2. I begin with a description of motion on hyperbolic space in arbitrary dimension, then discuss the compactification of the space by quotienting out a discrete subgroup of isometries. This procedure allows the derivation of Selberg's pretrace formula in section 3.1.1 and trace formula in section 3.1.2. I discuss how to include magnetic fluxes in order to break time reversal invariance in section 3.2, and finally introduce the Hadamard-Gutzwiller model as a concrete application of the Selberg trace formula in a physical system.

In this chapter, I will introduce the central mathematical identity used for the semiclassical study of the quantum systems that I aim to convince the reader possess gravitational duals – the Selberg trace formula (STF) [142]. This identity is a remarkable achievement for both mathematics and physics, relating the eigenvalue spectrum of the Laplace-Beltrami operator on certain spaces to the spectrum of lengths of closed geodesics on the same space, exploiting its homotopy group structure. Physically, this is equivalent to relating the eigenenergies of a particle sliding freely on some manifold to the length spectrum of periodic orbits of the flow generated by the free-particle Hamiltonian on the manifold – precisely the same as Gutzwiller's trace formula [95]! The STF therefore spans the bridge from hyperbolic geometry to group theory, from group theory to periodic orbit theory, and from periodic orbit theory to quantum mechanics; and it achieves this with *mathematical rigour*, involving no approximations. One goal of this thesis is to begin extending this bridge to the field of quantum gravity.

In a very transparent form that I will be using throughout this thesis, the Selberg trace formula reads [143]

$$\sum_{n=0}^{\infty} h(p_n) = \mathcal{V} \int_0^{\infty} h(p)\Phi_f(p)\mathrm{d}p + \sum_{\gamma\in\Gamma^*}\sum_{m=1}^{\infty} \frac{\chi_\gamma^m l_\gamma \tilde{h}(ml_\gamma)}{S_f(m,l_\gamma)}. \tag{3.1}$$

I will explain all the symbols appearing in due course, but for now, I want to focus on the interpretation of the individual terms. On the left-hand side, $h(p)$ is a function on the spectrum of the Laplace-Beltrami operator on some manifold, whose eigenvalues are parametrised as $p_n^2 + \frac{f-1}{4}$. This function, which I will refer to as the *spectral function*, must satisfy some technical conditions that I will detail below; physically, the left-hand side is typically going to be something like the trace of the heat kernel, or a similarly interesting function of the Hamiltonian, and the sum is simply one over the eigenenergies. On the right-hand side, mathematically, we see essentially a sum over the elements of the homotopy group of the manifold, with the integral term corresponding to the identity element, and the non-identity elements appearing in the sum. The physical meaning of the right-hand side is a sum over the system's periodic orbits (here already grouped into primitive orbits and their repetitions), with the integral corresponding to zero-length (or point) orbits. The quantity $\Phi_f(p)$ is called the *Plancherel measure*, and plays the role of the spectral density of the system. For this reason, in analogy to the terminology common in periodic orbit theory, I will refer to the integral part as the *Weyl term*.

Originally devised by Selberg as an identity on manifolds that can be written as $\mathbb{H}^2/\Gamma$, where $\mathbb{H}^2$ is the hyperbolic plane and Γ a Fuchsian group, the STF has been generalised to higher-dimensional settings [143–145], to include the Dirac operator [146], magnetic fluxes [147, 148] and fields [149], manifolds with boundaries [150], and to many other contexts too numerous to list. For a concise introduction to the STF, I refer the reader to [151], and for a fairly complete mathematical treatment, especially in the two-dimensional case, to [152, 153]. A more field-theoretic view, as well as the application of the STF to spacetime manifolds, i.e., pseudo-Riemannian manifolds with a timelike direction, is available in [154], where the reader may also find some references regarding the application of the STF to string theory.

In the rest of this chapter, I will sketch a derivation of the STF with a focus on several aspects that are going to be relevant in the comparison to quantum gravity in section 3.1. I will also discuss how to deal with magnetic fluxes in section 3.2, as this will be necessary to break time-reversal invariance in the system. Finally, I will review some work related to the more concrete model that is the object of study in this thesis in section 3.3.

3.1 Derivation of the Selberg trace formula in arbitrary dimension

In this section, I will sketch a brief derivation of the Selberg trace formula taken mostly from [143], unless otherwise indicated. The goal is to find a trace formula for the geodesic flow on compact hyperbolic manifolds in f dimensions, i.e., manifolds with constant negative Gauß curvature $K = -1$. Geodesics on such manifolds diverge exponentially, as can be easily seen by solving the Jacobi equation [155],

$$D_t^2 J(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0, \tag{3.2}$$

where D_t is the covariant derivative along some curve $\gamma(t)$ and $R(X, Y)Z$ is the Riemann curvature tensor applied to vector fields X, Y, Z on the manifold. A Jacobi field J satisfying eq. (3.2) describes the difference between γ and a "nearby" geodesic, and hence $J(t)$ gives insights into the rate of divergence between such nearby geodesics, as illustrated in fig. 3.1. There are two trivial Jacobi fields – $\dot{\gamma}(t)$ and $t\dot{\gamma}(t)$ – which merely reparametrise the geodesic γ. Excluding those, we can make the ansatz

$$J(t) = u(t)E(t), \tag{3.3}$$

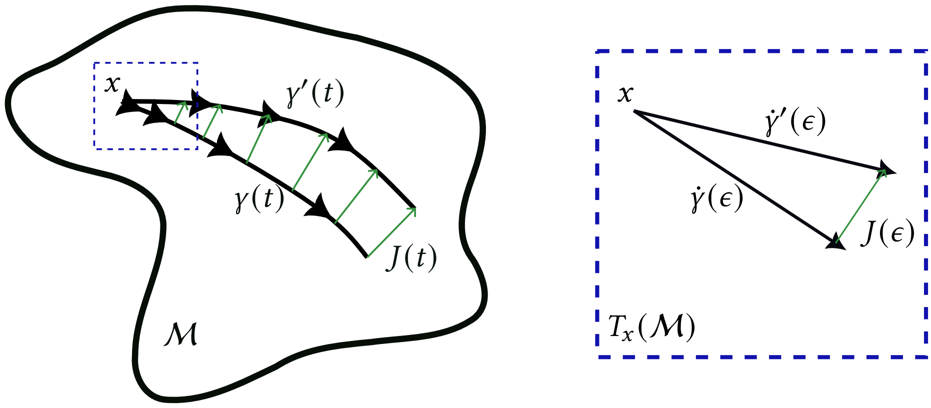


FIGURE 3.1: Left: two nearby geodesics $\gamma(t)$ and $\gamma'(t)$ on a manifold $\mathcal{M}$, originating at the same base point $x \in \mathcal{M}$. The Jacobi field $J(t)$ describes the divergence of the two geodesics. Right: The tangent vectors $\dot\gamma$ and $\dot\gamma'$ in the tangent space $T_x(\mathcal{M})$, at some small time $\epsilon > 0$. The Jacobi field $J(\epsilon)$ is the instantaneous deviation between the two tangent vectors [155].

where $E(t)$ is a parallel normal vector field along γ, i.e.,

$$D_t E(t) = 0, \tag{3.4}$$

and hence, $\{\dot\gamma(t), E(t)\}$ form an orthonormal basis of the tangent space to $\gamma(t)$ for all times t. On a manifold with constant negative curvature, the Riemann tensor simplifies to

$$R(X,Y)Z = (X,Z)Y - (Y,Z)X, \tag{3.5}$$

where (X,Y) is the inner product. Plugging eqs. (3.3) and (3.5) into eq. (3.2), we find

$$\begin{aligned} 0 = D_t^2 J + R(J,\dot\gamma)\dot\gamma &= D_t^2 J + \underbrace{(J,\dot\gamma)}_{0}\,\dot\gamma - \underbrace{(\dot\gamma,\dot\gamma)}_{1}\,J \\ &= D_t^2(uE) - uE = (\ddot u - u)E, \end{aligned} \tag{3.6}$$

where we used multiple times that $J \propto E$ is normal to $\dot\gamma$ by construction (3.4), as well as that $\dot\gamma$ is normalised. The resulting equation is solved by

$$u(t) = e^{\pm t}, \tag{3.7}$$

i.e., the geodesics diverge exponentially[1].

For the study of chaotic systems, such exponential divergence of nearby trajectories is a good starting point, but as we saw in chapter 2, it is not enough [94]; there must be a dense set of periodic orbits, and the Hamiltonian flow must be “mixing” or “topologically transitive”. The in some sense most generic kind of flows satisfying all these conditions are Anosov flows [156], and it has been shown that the geodesic flows on compact hyperbolic manifolds are precisely Anosov flows. In a nutshell, a flow on a manifold $\mathcal{M}$ is called Anosov if the tangent bundle $T\mathcal{M}$ can be decomposed into a stable and unstable subbundle

[1] From here, it may also be appreciated why we do not need to account for Morse indices in the trace formula (3.1). Nonzero Morse indices correspond to focal points, which would require a vanishing Jacobi field. Evidently, the solutions (3.7) to eq. (3.2) do not vanish anywhere.

that are uniformly exponentially contracted/expanded by the flow, as well as a submanifold that is tangential to the flow. In this sense, Anosov flows are prototypical examples of chaotic flows, and their extraordinary amenability to a precise mathematical description makes them a worthwhile target for study both out of an interest for (quantum) chaos in general, as well as with a view towards quantum gravity, as will be the focus of this thesis.

3.1.1 The pretrace formula

I will begin by deriving what is called the pretrace formula, in a manner mostly taken from [143] as already mentioned. In section 3.1.2, we will obtain the STF by – unsurprisingly – taking the trace of the final result obtained in this subsection.

We consider the Laplace-Beltrami operator on the hyperbolic space of dimension f, denoted $\mathbb{H}^f$. This operator can be given for the half-space model, i.e., using coordinates $x = (\vec{x} = (x_1, \ldots, x_{f-1}), t)$ with $\vec{x} \in \mathbb{R}^{f-1}, t > 0$, as

$$\Delta = t^2 \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{f-1}^2} + \frac{\partial^2}{\partial t^2} \right) - (f-2) t \frac{\partial}{\partial t}, \tag{3.8}$$

or alternatively, in coordinates that are radial about some point $p \in \mathbb{H}^f$ as

$$\Delta = \frac{\partial^2}{\partial r^2} + (f-1) \coth(r) \frac{\partial}{\partial r} + \Delta_{S(p;r)}, \tag{3.9}$$

with $\Delta_{S(p;r)}$ the Laplacian on the geodesic sphere of radius r around p.

A different useful model, in particular for gaining some geometric intuition, is the Poincaré ball model $y = (y_1, \ldots, y_f)$ with $|y| < 1$, for which the Laplacian reads

$$\Delta = \frac{\left(1 - |y|^2\right)^2}{4} \left(\frac{\partial^2}{\partial y_1^2} + \ldots + \frac{\partial^2}{\partial y_f^2} \right). \tag{3.10}$$

The boundary $\partial\mathbb{H}^f$ of $\mathbb{H}^f$ is located at $t = 0$ and $|y| = 1$ in these models, respectively.

With this in hand, we can define *point-pair invariants* $k(x, y) : \mathbb{H}^f \times \mathbb{H}^f \to \mathbb{C}$ by considering functions $k(\rho)$ that are even and $C^\infty(\mathbb{R})$ and setting

$$k(x, y) := k(d(x, y)), \tag{3.11}$$

where $d(x, y)$ is the distance between x and y on $\mathbb{H}^f$. Now if $\varphi_\lambda(x)$ is some eigenfunction of the Laplacian,

$$\Delta \varphi_\lambda(x) = \lambda \varphi_\lambda(x), \tag{3.12}$$

one can show [143] that

$$\int_{\mathbb{H}^f} \mathrm{d}y \, k(x, y) \varphi_\lambda(y) = h(\lambda) \varphi_\lambda(x), \tag{3.13}$$

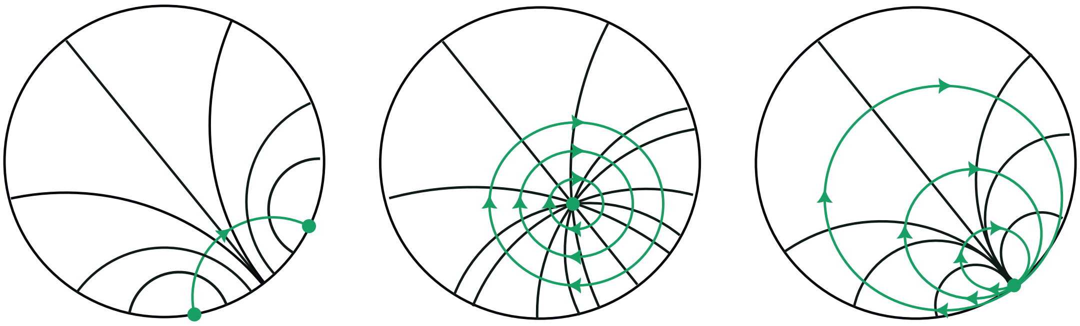


FIGURE 3.2: Illustration of the action of hyperbolic, elliptic and parabolic isometries (from left to right) on the Poincaré disk. The fixpoints of the isometries are depicted as green dots, the black arcs are geodesics acted upon by the isometries. Left: a hyperbolic isometry fixes two points on the boundary of $\mathbb{H}^2$ and moves other points along a geodesic (green), i.e., a circular arc intersecting the boundary perpendicularly. Middle: an elliptic isometry fixes a point in the interior of $\mathbb{H}^2$ and rotates other points along Euclidean circles centred at the fixpoint (green). Right: a parabolic isometry fixes one point on the boundary of $\mathbb{H}^2$ and rotates other points along horocycles, i.e., Euclidean circles tangent to the boundary at the fixpoint [152].

where $h(\lambda)$ is a function on the spectrum of Δ that is *independent* of φ_λ! It is precisely this function that appears in the STF (3.1). Indeed, it can be given in terms of the point-pair invariant $k(x, y)$ as

$$h(p) = 2c_{f-2}\int_0^\infty \cos(pu)\,\mathrm{d}u \int_u^\infty k(\rho)(z(\rho) - z(u))^{(f-3)/2}\sinh\rho\,\mathrm{d}\rho\,, \qquad (3.14)$$

where $\lambda = ((f-1)/2)^2 + p^2$, c_{f-2} is the area of the unit sphere in $\mathbb{R}^{f-1}$ and $z(u) = (2\sinh u/2)^2$.

We can use eq. (3.13) on a compact hyperbolic manifold $\mathcal{M}$ if we recall that, by the Killing-Hopf theorem [157, 158], we can write such a manifold as $\mathcal{M} = \mathbb{H}^f/\Gamma$, where Γ is a discrete subgroup of the isometry group of $\mathbb{H}^f$. An isometry $\gamma \in \Gamma$ is called [152]

hyperbolic if	$\mathrm{tr}(\gamma)^2 > 4$,
elliptic if	$\mathrm{tr}(\gamma) \in (-2, 2)$,
parabolic if	$\mathrm{tr}(\gamma) = \pm 2$.

These three types of isometries differ in the amount and location of fixpoints they possess in $\mathbb{H}^f \cup \partial\mathbb{H}^f$, as illustrated in fig. 3.2 for $f = 2$. After projecting onto the quotient manifold $\mathcal{M}$, elliptic isometries produce conical singularities, while parabolic isometries lead to cusps. In this thesis, I will restrict to compact, non-singular manifolds and hence to dealing with hyperbolic isometries, although generalisations of the STF to include parabolic and elliptic isometries are available (e.g. [152, 153]). With this restriction, the group Γ is isomorphic to the fundamental group[2] $\pi_1(\mathcal{M})$, i.e., the group of geodesic loops on $\mathcal{M}$ [159].

[2]While this construction may seem somewhat abstract, I ask the reader for some patience until section 3.3, where I will introduce a more concrete example of the construction in two dimensions.

Taking then an isometry subgroup Γ with only hyperbolic isometries, we can tessellate the hyperbolic space by some fundamental domain F for Γ, i.e.,

$$\mathbb{H}^f = \Gamma F = \bigcup_{\gamma \in \Gamma} \gamma F. \tag{3.15}$$

This allows us to rewrite integrals like the one in eq. (3.13) as

$$\int_{\mathbb{H}^f} k(x,y) f(y) \mathrm{d}y = \int_F \underbrace{\left(\sum_{\gamma \in \Gamma} k(x, \gamma y) \right)}_{=: K(x,y)} f(y) \mathrm{d}y. \tag{3.16}$$

The function $K(x,y)$ induces a Hilbert-Schmidt operator [143]

$$f \to \int_{\mathcal{M}} K(x,y) f(y) \mathrm{d}y. \tag{3.17}$$

Acting with this operator on a complete set of eigenfunctions $\varphi_0, \varphi_1, \ldots$ of the Laplacian, we conclude

$$\int_{\mathcal{M}} K(x,y) \varphi_n(y) \mathrm{d}y \overset{(3.13)}{=} h(p_n) \varphi_n(x), \tag{3.18}$$

where p_n is defined by the eigenvalues of the Laplacian $\lambda_n = ((f-1)/2)^2 + p_n^2$. Hence, the eigenfunctions of the Laplacian are also eigenfunctions of the integral operator induced by the kernel $K(x,y)$, which can therefore be expanded in terms of these eigenfunctions by Mercer's theorem [160]. This expansion is known as Selberg's pretrace formula,

$$\sum_{\gamma \in \Gamma} k(x, \gamma y) = \frac{1}{2} \sum_n h(p_n) \varphi_n(x) \varphi_n^*(y), \tag{3.19}$$

with the factor $1/2$ accounting for there being two roots p_n per eigenvalue λ_n. Taking the trace of eq. (3.19) will yield the Selberg trace formula, but the result will still need a bit of massaging to arrive at the form (3.1). This will be the subject of the next subsection.

3.1.2 The trace formula

In the spirit of trace formulas known from semiclassics, we now trace (3.19) over a fundamental domain of Γ. For notational convenience, we define

$$k(d(x,y)) =: L(\sinh^2(d/2)) \tag{3.20}$$

and set $x = y$ before integrating. This yields the trace formula

$$\sum_{n=0}^{\infty} h(p_n) = \mathcal{V} L(0) + 2 \sum_{\gamma}{}' \int_F L(x, \gamma x) \mathrm{d}x, \tag{3.21}$$

where the primed sum means we exclude the identity $e \in \Gamma$. Obviously, $d(x, ex) = 0$, so integrating $L(0)$ over the fundamental domain F simply means multiplying with the volume $\mathcal{V}$ of the manifold $\mathcal{M}$. At this point, it is also quite transparent why it was desirable to restrict to hyperbolic isometries γ. As illustrated in fig. 3.2, elliptic isometries fix a point in the interior of $\mathbb{H}^f$. This makes the integral on the RHS of eq. (3.21) fairly difficult to deal with; in particular, we could not have simply pulled the sum over γ out of the integral. The only isometry for which this works without major issue is the identity e, which fixes all points.

We proceed with the evaluation of the sum over nontrivial isometries. We can group the sum in eq. (3.21) into conjugacy classes $\{\gamma\}$. Since $\Gamma \simeq \pi_1(\mathcal{M})$ is simply the fundamental group of $\mathcal{M}$ [159], all isometries in a given conjugacy class correspond to the same closed geodesic on $\mathcal{M}$. Likewise, each conjugacy class corresponds to either a *prime geodesic*, i.e., one that is traversed exactly once, or to a multiple of a prime geodesic. We call the set of conjugacy classes corresponding to prime geodesics the *primitive* conjugacy classes Γ^*, and finally group the sum in eq. (3.21) into primitive conjugacy classes and their multiples. If we pick a representative $\gamma \in \Gamma^*$ for each primitive conjugacy class, we can write $\gamma = KS$, where $K \in O(f-1)$ is a rotation acting on a point $x \in \mathbb{H}^f$ as

$$K(\vec{x}, t) = (K\vec{x}, t), \tag{3.22}$$

and S is a dilation,

$$S(x_1, \dots, t) = (N_\gamma x_1, \dots, N_\gamma t), \tag{3.23}$$

with the stretching factor N_γ. We define

$$l_\gamma = \log N_\gamma \tag{3.24}$$

and call it the *length*[3] of γ. Skipping some technical details, the trace formula can be expressed after this regrouping as

$$\sum_n h(p_n) = \mathcal{V}L(0) + \sum_{\gamma\in\Gamma^*} \sum_{m=1}^{\infty} \frac{l_\gamma \tilde{h}(ml_\gamma)}{S_f(m, l_\gamma)}, \tag{3.25}$$

where we introduced the *stability amplitudes*[4] $S_f(m, l_\gamma)^{-1} = N_\gamma^{m(f-1)/2} |\det(e - K^m S^m)|^{-1}$ and included only one of the two possible roots r_n, eliminating the factors of 2. Here, $\tilde{h}$ is the cosine transform of h,

$$\tilde{h}(l) = \frac{1}{\pi} \int_0^\infty \cos(pl) h(p) \mathrm{d}p. \tag{3.26}$$

In two dimensions, K is an $O(1)$ rotation, which is to say, $K = 1$. Hence, the stability is exactly

$$S_2(m, l_\gamma) = 2\sinh\big(ml_\gamma/2\big). \tag{3.27}$$

In higher dimensions, this is no longer true, but we can still use a similar

[3] Indeed, l_γ *is* the length of the geodesic on $\mathcal{M}$ corresponding to γ.

[4] I will often refer to them as *stabilities* for brevity.

approximation following this simple argument: The rotation K has a bounded spectrum [161], and the largest eigenvalue of S is e^{l_γ}. Hence, the dominant behaviour of the total dilation S^m, and thereby the total isometry γ^m, will be governed by e^{ml_γ}, which will contribute with a multiplicity of $f-1$ due to the determinant. Including the prefactor then, the stability in f dimensions will be roughly given by

$$S_f(m, l_\gamma) \sim e^{(f-1)ml_\gamma/2} \tag{3.28}$$

for sufficiently long periodic orbits. This estimate will be important in the discussion of the results we obtained in [3], conducted in chapter 9.

It remains to evaluate the term $\mathcal{V}L(0)$ in the trace formula, corresponding to the identity isometry $e \in \Gamma$. To do so, we first rewrite eq. (3.14) as

$$h(p) = c_{f-2} \int_0^\infty \cos(pu)\,\mathrm{d}u \int_{z(u)}^\infty L(\rho)(\rho - z(u))^{(f-3)/2}\mathrm{d}\rho. \tag{3.29}$$

This can be inverted, most easily in terms of the cosine transform of h, to find [143]

$$L(u) = -\frac{b_f}{\pi} \int_u^\infty \left(\frac{\tilde{h}(\delta(\rho))}{c_{f-2}} \right)^{f/2} (\rho - u)^{-1/2}\mathrm{d}\rho, \tag{3.30}$$

where b_f are constants whose value I will omit for the time being. Plugging in the definition of the cosine transform to get back to the original spectral function $h(r)$ and switching the integrations, we are able to compute

$$L(0) = \int_0^\infty \Phi_f(p)h(p)\mathrm{d}p, \tag{3.31}$$

with the Plancherel measure $\Phi_f(p)$ given by

$$\Phi_f(p) = \begin{cases} \dfrac{b_f}{\pi c_{f-2}} \dfrac{\partial^{(f-1)/2}}{\partial u^{(f-1)/2}} \cos(p\delta(u))|_{u=0} & : f \text{ odd}, \\ -\dfrac{b_f}{\pi c_{f-2}} \displaystyle\int_0^\infty \rho^{-1/2} \dfrac{\partial^{f/2}}{\partial \rho^{f/2}} \cos(p\delta(\rho))\mathrm{d}\rho & : f \text{ even}. \end{cases} \tag{3.32}$$

It would be entirely possible, if tedious, to explicitly compute the Plancherel measure for a given dimension f from eq. (3.32). However, in a slightly simpler approach [145, 154], the measure is understood as the density of eigenfunctions of the Laplacian,

$$\int_{\mathbb{H}^f} \varphi_p(x)\varphi_{p'}(x)\mathrm{d}x = \frac{1}{\Phi_f(p)}\delta(p - p'), \tag{3.33}$$

and then computed by evaluating the scalar product of eigenfunctions of the continuous part of the spectrum of Δ. The result is the form of the Plancherel measure that I will be using in the remainder of the thesis,

$$\Phi_f(p) = \begin{cases} \dfrac{p\tanh(\pi p)}{(2\pi)^{f/2}(f-2)!!}\displaystyle\prod_{k=0}^{\frac{f-4}{2}}\left(p^2+\left(k+\frac{1}{2}\right)^2\right) & : f \text{ even}, \\ \dfrac{1}{2^{(f-1)/2}\pi^{(f+1)/2}(f-2)!!}\displaystyle\prod_{k=0}^{\frac{f-3}{2}}(p^2+k^2) & : f \text{ odd}. \end{cases} \tag{3.34}$$

This approach to the Plancherel measure will be important later in the thesis in section 9.1.1, where a very similar computation will appear in JT gravity. With $\Phi_f(p)$ determined, we finally arrive at the celebrated Selberg trace formula,

$$\sum_{n=0}^{\infty} h(p_n) = \mathcal{V}\int_0^{\infty} h(p)\Phi_f(p)dp + \sum_{\gamma\in\Gamma^*}\sum_{m=1}^{\infty}\frac{l_\gamma \tilde{h}(ml_\gamma)}{S_f(m,l_\gamma)}. \tag{3.35}$$

Comparing this to, e.g., Gutzwiller's trace formula [95], we identify the integral term with the Weyl term, named so because in the Gutzwiller trace formula, it is simply the density of states as determined by Weyl's law [94]. This is usually only an approximation, and a remarkable advantage of working with the Selberg trace formula is the fact that we know the *exact* density of states $\Phi_f(p)$, including all quantum corrections[5]. As the above construction makes transparent, the density of states is determined by the identity in the isometry group of $\mathcal{M}$, which is to say, by periodic orbits of length 0, as claimed in the introduction to this chapter.

Inspecting eq. (3.35), the reader will notice that we have eliminated any reference to the functions $k(x,y)$ and $L(x,y)$ from the trace formula, which can now be understood as an identity relating the spectrum p_n of the Laplace-Beltrami operator on $\mathcal{M}$ to the spectrum l_γ of closed geodesics on $\mathcal{M}$. This relation is mediated by the spectral function $h(p)$. In order for both sides of eq. (3.35) to converge absolutely, the spectral function must satisfy certain conditions, namely [143]

1. $h(p)$ is even.
2. $h(p)$ is analytic at least in a strip of width $f-1$ around the real axis.
3. $h(p) = \mathcal{O}\left(p^{-(f+\epsilon)}\right)$ in the strip as $|p| \to \infty$.

The analyticity condition is illustrated in fig. 3.3 and interestingly, the analytic structure of out-of-time ordered correlators (OTOCs) demands a similar (although not equivalent) condition, which eventually leads to the Maldacena-Shenker-Stanford bound [87]. This remarkable coincidence motivates the study of OTOCs in systems like the Hadamard-Gutzwiller model, which I will introduce

[5]Weyl's law usually gives a volume law density of states, with quantum corrections given by, e.g., a curvature expansion [162–164], taking the form of lower degree polynomials in the square root of the energy. Equation (3.34) gives not only the top degree monomial, i.e., the volume law, but all possible lower orders (important at low energies) as well, meaning it accounts for all quantum corrections.

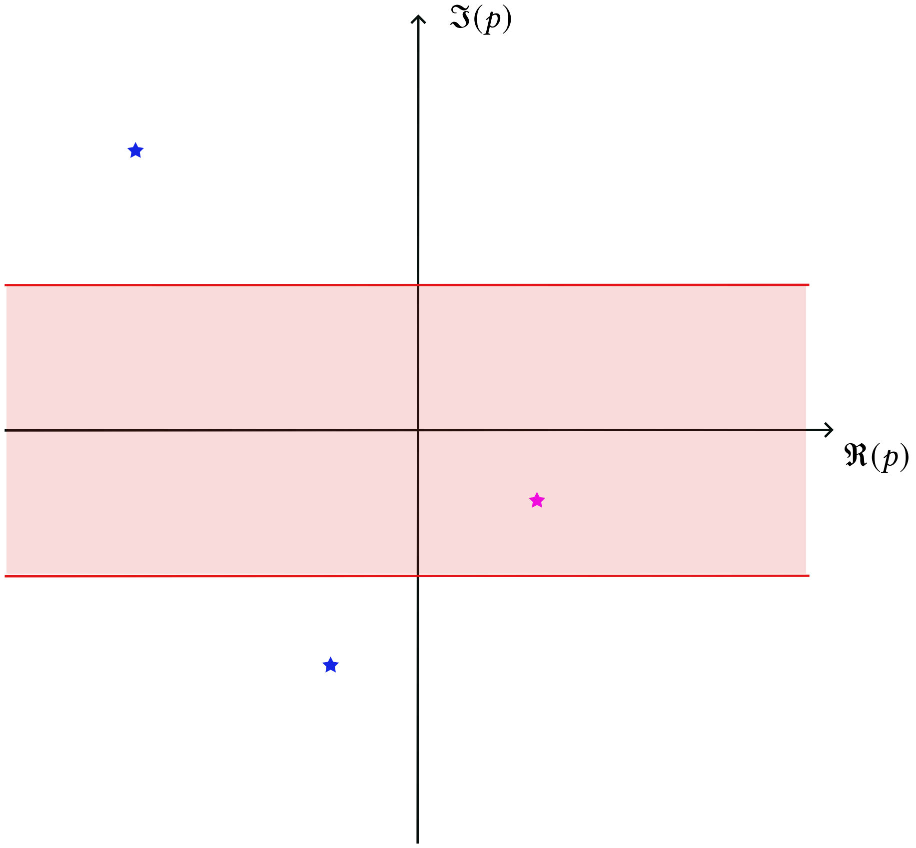


FIGURE 3.3: The complex p plane. The red shaded strip marks the region in which the spectral function $h(p)$ must be analytic. If all poles of $h(p)$ are outside the strip (e.g. blue stars), the Selberg trace formula still converges absolutely, whereas this is no longer guaranteed if there are singularities inside the strip (pink star).

in section 3.3, or more generally, the system we introduced in [3] and used in [4]. This study will be the subject of chapter 10.

Finally, the attentive reader will have noticed the missing factors χ_γ^m in eq. (3.35) compared to eq. (3.1). These factors describe phases collected by the periodic orbit γ from possible Aharonov-Bohm fluxes, which are necessary if breaking time-reversal invariance in the system is desired. I will discuss how to incorporate them in the next section.

3.2 Magnetic fluxes

The Selberg trace formula as introduced in section 3.1 describes a time-reversal invariant dynamical system by construction. This is because the periodic orbits γ and γ^{-1}, while not part of the same conjugacy class, contribute exactly in the same way to the STF. The only information about an orbit γ that enters eq. (3.35) is its length, and clearly $l_\gamma = l_{\gamma^{-1}}$. In the setting of JT gravity however, I am interested in describing dual systems both with and without TRI, corresponding respectively to the inclusion and exclusion of unorientable geometries in the JT gravity path integral. In order to break TRI then, I require a modification of the STF that distinguishes geodesics by their orientation. One way to achieve this is

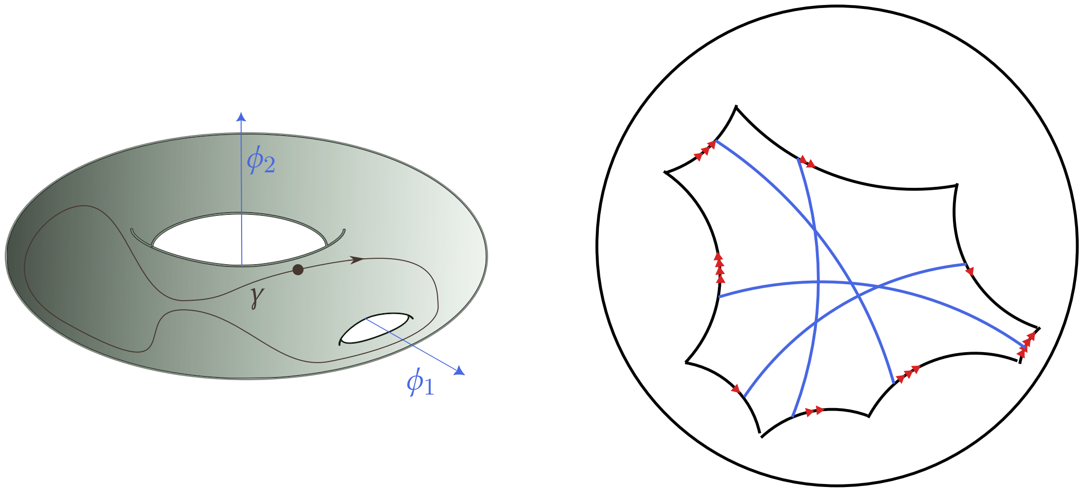


FIGURE 3.4: Left: a double torus with Aharonov-Bohm fluxes $\phi_{1,2}$ threading the handles, but chosen in such a way that the magnetic field vanishes everywhere on the double torus. An orbit γ will collect a phase depending on its winding number around each flux, eq. (3.36). The pictured fluxes only account for half of all possible holonomies. The other holonomies would correspond, e.g., to magnetic fluxes running through the interior of the double torus. Adapted from [3]. Right: a possible non-symmetric fundamental domain for a double torus like the one on the left. The boundaries of the octagon are geodesic segments, and opposite boundaries are to be identified in the orientation indicated by the red arrows. The blue lines in the interior of the octagon are closed geodesics on the double torus and correspond to the holonomies (3.39).

by endowing the particle with a charge $q = 1$ and adding Aharonov-Bohm fluxes in a precise manner, detailed in the following. As described in [147, 148], in two dimensions, the STF describes surfaces of genus[6] g that can be endowed with $2g$ Aharonov-Bohm fluxes threading the handles of the surface, illustrated for example in fig. 3.4. If none of the fluxes pierce the manifold, the magnetic field will vanish everywhere on the manifold, and the only effects of this procedure in the STF will be that the momentum is minimally coupled to the vector potential corresponding to the fluxes, or equivalently, the Laplacian replaced by the Bochner-Laplace operator [165], and that the terms in the sum on the RHS of eq. (3.35) are multiplied by

$$\chi_\gamma = \exp\left(i \sum_{j=1}^{2g} \phi_j n_j(\gamma)\right), \tag{3.36}$$

where ϕ_j is the phase collected from winding around the jth flux and $n_j(\gamma)$ is the winding number of the periodic orbit γ around the jth flux. This construction can be lifted to arbitrary dimension in a very precise sense by realising that the Aharonov-Bohm phases are nothing but the holonomies of the vector potential A around all non-contractible loops of the manifold $\mathcal{M}$. If the manifold has a non-trivial first homology group, such loops exist and correspond to closed differential forms. In particular, by Hodge's theorem [166], there is a basis of harmonic 1-forms ω^i, and the vector potential can be decomposed in terms of

[6] I exclude the possibility of cusps, as discussed above.

this basis,

$$A = \phi_i \omega^i. \tag{3.37}$$

Now since the ω^i are harmonic, they are in particular closed, such that $dA = 0$, i.e., the magnetic field derived from A vanishes everywhere on the manifold $\mathcal{M}$. After choosing a vector potential in this manner, we can construct the Hamiltonian,

$$H = -(\nabla + iA)^2, \tag{3.38}$$

repeat the construction of the STF, and obtain eq. (3.1), where

$$\chi_\gamma = \exp\left(i \oint_\gamma A\right), \tag{3.39}$$

while leaving the energy levels, as well as the length spectrum, otherwise unaffected [147, 148].

3.3 The Hadamard-Gutzwiller model

The Selberg trace formula as discussed so far is in principle a mathematical result. It was discovered by Gutzwiller however [114] that the semiclassical trace formula he developed [95] is exact for the free motion of a particle on a hyperbolic Riemann surface, and indeed equivalent to the STF. In this section, I will collect some useful results obtained by Aurich and Steiner [167–169] in their study of the *Hadamard-Gutzwiller model*, where the two-dimensional version of the Selberg trace formula will come into play. This will serve the purpose of making the fairly abstract presentation given in this chapter so far more transparent to physical intuition. All material presented in this section is taken from [167–169], unless otherwise indicated.

As a dynamical system, consider a particle sliding freely on the Bolza surface $\mathcal{B}$ [170], i.e., the genus 2 surface with the highest possible degree of symmetry[7]. Its fundamental domain in the Poincaré disk $\mathbb{H}^2$ is the regular hyperbolic octagon, depicted in fig. 3.5. Working in units where the mass of the particle is $m = 2$, the curvature radius $L = 1$ and $\hbar = 1$, the system's Hamilton function is

$$H = \frac{1}{4} p_i g^{ij} p_j, \tag{3.40}$$

where p_i are the canonical momenta and g_{ij} is the hyperbolic metric

$$g_{ij} = \frac{4}{\left(1 - x_1^2 - x_2^2\right)^2} \delta_{ij}. \tag{3.41}$$

The microcanonical Lyapunov exponent at energy E is given by

$$\lambda = \sqrt{E}, \tag{3.42}$$

[7] In the sense that the order of its automorphism group, i.e., the group of bijections of $\mathcal{B}$ onto itself, is maximal.

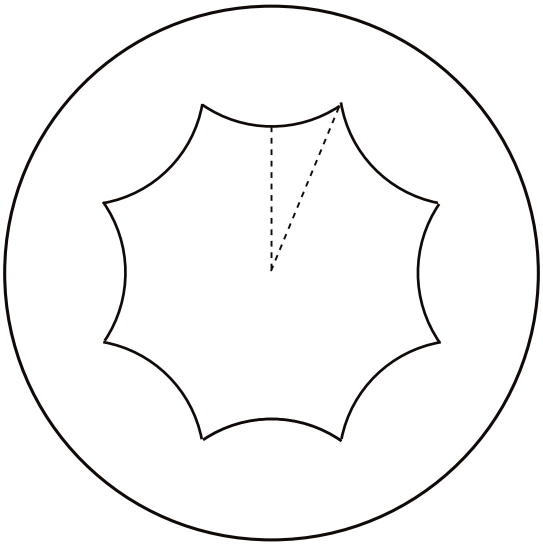


FIGURE 3.5: The regular hyperbolic octagon, i.e., the fundamental domain of the Bolza surface $\mathcal{B}$. The triangle indicated by the dashed lines can be understood as the fundamental domain of one symmetry sector of the Bolza surface [169]. The whole octagon can be recovered only from the triangle by applying reflections and rotations. Adapted from [167].

and the area of $\mathcal{B}$ is 4π. I will briefly review how to construct the spectrum of classical periodic orbits of the flow generated by eq. (3.40), before discussing the quantum energy spectrum of the corresponding Hamiltonian, and specialising the STF to the heat kernel, which I will be particularly interested in throughout the remainder of the thesis.

The set of periodic orbits Γ, or equivalently, the fundamental group Γ of $\mathcal{B} = \mathbb{H}^2/\Gamma$ has the presentation

$$\Gamma = \left\langle b_0, b_1, b_2, b_3 \middle| b_0 b_1^{-1} b_2 b_3^{-1} b_0^{-1} b_1 b_2^{-1} b_3 \right\rangle . \tag{3.43}$$

This notation means that the group is generated by building words with the letters (also called generators for this reason) b_k and their inverses, while observing the relation

$$b_0 b_1^{-1} b_2 b_3^{-1} b_0^{-1} b_1 b_2^{-1} b_3 = 1. \tag{3.44}$$

All group elements can be represented by $\mathrm{PSL}(2, \mathbb{R})$ matrices

$$T = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1, \tag{3.45}$$

which act on the Poincaré disk as Möbius transformations,

$$z \mapsto Tz = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}. \tag{3.46}$$

For the Bolza surface, the generators are given by

$$b_k = \begin{pmatrix} \cosh\left(\frac{l_0}{2}\right) & e^{i(k\pi/4)}\sinh\left(\frac{l_0}{2}\right) \\ e^{-i(k\pi/4)}\sinh\left(\frac{l_0}{2}\right) & \cosh\left(\frac{l_0}{2}\right) \end{pmatrix}, \tag{3.47}$$

where

$$\cosh\left(\frac{l_0}{2}\right) = \cot\left(\frac{\pi}{8}\right). \tag{3.48}$$

Applying the generators (3.47), or more generally any group element

$$\gamma = b_{i_1} \dots b_{i_n} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \tag{3.49}$$

to a point on the Poincaré disk according to eq. (3.46), it is clear that the point is moved by a total distance of

$$l_\gamma = 2\,\mathrm{arcosh}\,|\mathrm{Re}\,\alpha|, \tag{3.50}$$

where Re denotes the real part. For this reason, l_γ is called the *length* of the periodic orbit γ, and consequently, l_0 is the length of the shortest periodic orbit of the system[8].

The generators are precisely the transformations that pairwise identify the sides of the fundamental polygon of $\mathcal{B}$. More generally, for a generic genus-g surface, one would construct a $4g$-sided polygon tessellating the Poincaré disk, and determine the generators by identifying the image of one corner with a corner on the opposite edge [169]. A fairly similar construction obtains in 3 dimensions [171], and in practice is often simpler due to a reduced number of necessary generators[9].

On a generic manifold, it is expected that the length spectrum is almost non-degenerate [172], but on the Bolza surface, we can already see from the form of the generators, as well as the shape of its fundamental domain (cf. fig. 3.5), that there will be a large number of different orbits with the same length. Hence, it is necessary to determine not only the lengths in the spectrum, but also the multiplicities of the lengths, in order to apply the STF. The lengths themselves are easily determined by an arithmetical rule [167],

$$\cosh\left(\frac{l_{n-1}}{2}\right) = m + \sqrt{2}n, \tag{3.51}$$

where l_n is the nth length appearing in the spectrum and m is the odd natural number minimising $\left|m/n - \sqrt{2}\right|$. The multiplicities g_n of the nth length however have to be determined by generating all group elements, sorting them into conjugacy classes and counting the orbits with a given length, by brute force.

[8]It is also clear from this definition that the identity corresponds to periodic orbits of length 0.

[9]For instance, the fundamental group of a handlebody of genus g is generated by $2g$ $\mathrm{PSL}(2,\mathbb{C})$ generators, as opposed to the $4g$ $\mathrm{PSL}(2,\mathbb{R})$ generators needed for the surface of the same handlebody [171].

This has been carried out and tabulated for the first 206 million periodic orbits of the system in [167]. In his Bachelor's thesis [173], Jakob Minár used both the tabulated orbits, as well as extrapolated to longer orbits using the asymptotic multiplicity

$$\hat{g}(l) \sim 8\sqrt{2}\frac{e^{l/2}}{l}, \tag{3.52}$$

which is chosen so as to combine with the density of lengths determined by eq. (3.51) to yield Huber's law for the exponential proliferation of periodic orbits [174],

$$N(l) \xrightarrow{l\to\infty} \frac{e^l}{l}, \tag{3.53}$$

where

$$N(l) := \sum_{l_n \leq l} g_n \tag{3.54}$$

is the number of periodic orbits of length less than l. The approximation (3.52) is in general very bad at predicting the exact multiplicity of a given length, but it is quite adequate for the mean multiplicity in a length interval [167]. This fact has been used in [173] to examine the convergence of the STF in regimes where it is not strictly applicable, and hence absolute convergence not guaranteed.

In order to quantise the theory, the classical Hamilton function (3.40) is replaced by the quantum Hamiltonian,

$$\hat{H} = -\frac{1}{4}\Delta, \tag{3.55}$$

where Δ is the Laplacian on $\mathcal{B}$. Energy levels can be obtained by solving the Schrödinger equation,

$$\hat{H}\psi_n = E_n\psi_n, \tag{3.56}$$

as, e.g. in [168, 175]. With this information in hand, one can turn to the study of interesting spectral functions $h(p)$. I will mention just two examples: first, the spectral representation of the time evolution operator,

$$U_T(p) = e^{-iTp^2/4}, \tag{3.57}$$

with the complex time $T = t - i\beta$. The cosine transform of $U_T(p)$ entering the STF in the periodic orbit sum is

$$\tilde{U}_T(l) = \frac{e^{-\frac{l^2}{iT}}}{\sqrt{\pi i T}.} \tag{3.58}$$

Correlation functions of $U_T(p)$ are studied in chapter 9 and shown to agree with partition functions of JT gravity with certain natural boundary conditions. For this choice of spectral function, the Selberg trace formula is well studied [168], at least in the two-dimensional case. A slightly more exotic choice is the spectral representation of the grand canonical partition function,

$$F(p;\beta,\mu) = \frac{1}{\beta}\log\left(1 + \mathrm{e}^{-\beta(p^2-\mu)}\right), \tag{3.59}$$

with the chemical potential μ. This function allows for controlling its analytic structure by changing μ, which was exploited to tune the STF out of its regime of validity in [173].

As a final note, when studying the energy spectrum of the Laplacian on the Bolza surface, one quickly notices that eigenvalues are often degenerate [175] due to the high degree of symmetry of the surface. A way to mitigate this is to desymmetrise the fundamental domain of $\mathscr{B}$, as described in [168] and indicated in fig. 3.5. In this way, one obtains a spectrum that nicely agrees with the prediction of random matrix theory. Alternatively, a fundamental domain picked at random is not expected to exhibit quite as many symmetries. Collecting all possible genus-g manifolds in an ensemble and averaging over this ensemble, one can also recover RMT statistics [169].

Chapter 4

Gravity and holography in low dimensions

I motivate the study of two-dimensional Jackiw-Teitelboim gravity and review the evaluation of the gravitational path integral by splitting into a boundary path integral and an integral over bulk moduli. In section 4.1.1, I evaluate the boundary mode path integral in the two relevant cases of the disk and trumpet geometries. In section 4.1.2, I briefly review Mirzakhani's recursion for the bulk moduli space volumes in the orientable case before discussing its extension to unorientable manifolds in section 4.1.3. The low energy limit of JT gravity, i.e., the Airy model, is introduced in section 4.1.4. Finally, in section 4.2, I review the factorisation problem in holography that is posed by theories like JT gravity with an ensemble of dual theories.

Studying gravity, both classical and quantum, in the 4 spacetime dimensions realised in nature, or in more dimensions, i.e., the case of interest for string theory and supergravity, is a task of considerable difficulty, due not least to the mathematical complexity of the classical theory. Trying to quantise this theory is further complicated by the fact that the coupling constant, which is to say Newton's constant G_d in d spacetime dimensions, is dimensionful if $d > 2$. To see this, consider the Einstein-Hilbert action (in units where $\hbar = c = 1$),

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_d} \int \mathrm{d}^d x \sqrt{-g} R, \tag{4.1}$$

where g is the determinant of the metric and R the Ricci scalar. In these units, the action needs to be dimensionless, while

$$[\mathrm{d}^d x] = (\text{mass})^{-d},\ [R] = (\text{mass})^2,\ [g] = 1 \to [G_d] \overset{!}{=} (\text{mass})^{2-d}. \tag{4.2}$$

Field theories with couplings of negative mass dimension are not renormalisable in the Wilsonian sense, since one cannot integrate out UV modes without the coupling introducing infinitely many unsuppressed operators from loop corrections, and taming them requires infinitely many counter terms, each with their own coupling, rendering the theory non-predictive [9]. In gravity, such loop contributions would, e.g., read

$$R^2, R_{\mu\nu}R^{\mu\nu}, R^3, \nabla^2 R^2, \ldots, \tag{4.3}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor and ∇ the covariant derivative. Conventionally quantised gravity can still serve as an effective field theory at energies much lower than the chosen UV cutoff [10], but the theory is not UV complete.

The difficulty of dealing with (particularly quantum) gravity in 4 or more dimensions suggests an alternative path: studying gravity in 2 or 3 dimensions. This programme comes with its own problems however, that are revealed when trying to count the dynamical degrees of freedom of the theory: the metric $g_{\mu\nu}$, being a symmetric rank 2 tensor, has $d(d+1)/2$ independent components, but the d diffeomorphism constraints, as well as the gauge symmetries they generate [176], remove $2d$ degrees of freedom [177], leaving in total

$$f(d) = \frac{d(d+1)}{2} - 2d = \frac{d(d-3)}{2} \tag{4.4}$$

physical degrees of freedom. This means that in 3 dimensions,

$$f(3) = 0, \tag{4.5}$$

i.e., three-dimensional gravity is merely topological and has no dynamical degrees of freedom. Even worse, in 2 dimensions,

$$f(2) = -1, \tag{4.6}$$

meaning that two-dimensional gravity is in some sense "overconstrained". Indeed, the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu} \equiv 0, \tag{4.7}$$

simply vanishes identically in two dimensions. For this reason, it was unknown how to even describe gravity in 2 dimensions, until Bunster (then still Teitelboim) [52] and Jackiw [53] introduced an action whose path integral quantisation is nowadays referred to as Jackiw-Teitelboim (JT) gravity. Jackiw arrived there by writing down the only geometrical second order differential equation for the metric in two dimensions,

$$R + 2\Lambda = 0, \tag{4.8}$$

and then searching for an action whose variation yields eq. (4.8)[1]. Introducing a dilaton field ϕ that acts as a Lagrange multiplier does the trick, and furthermore sidesteps the problem of not having any degrees of freedom available in pure 2-dimensional gravity,

$$S = \int_\Sigma \mathrm{d}^2x\, \sqrt{-g}\phi(R + 2\Lambda), \tag{4.9}$$

[1]Note that Λ is *not* the cosmological constant in eq. (4.8), but merely a parameter setting the curvature. In two dimensions, due to the vanishing of the Einstein tensor, special care must be taken when introducing a cosmological constant and trying to describe its effect on the curvature of the spacetime.

where we integrate over some spacetime Σ. In its modern formulation, JT gravity is described by the total action [55]

$$S_{\mathrm{JT}} = -\underbrace{\frac{S_0}{2\pi}\left(\frac{1}{2}\int_\Sigma \sqrt{g}R + \int_{\partial\Sigma}\sqrt{h}K\right)}_{S_0\chi(\Sigma)} - \left(\frac{1}{2}\int_\Sigma \sqrt{g}\phi(R+2) + \int_{\partial\Sigma}\sqrt{h}\phi_\partial(K-1)\right). \tag{4.10}$$

This action is quite a bit more complicated than Jackiw's proposal (4.9) and warrants some explanation. Compared to eq. (4.9), we have performed a Wick rotation to Euclidean signature (and absorbed the measure d^2x into the metric factor for notational convenience) and set $\Lambda = 1$. The term in the first parenthesis is the Einstein-Hilbert action for the bulk spacetime Σ, together with the Gibbons-Hawking-York term for the boundary $\partial\Sigma$ [176]. If we have coordinates x^α on Σ and y^a on $\partial\Sigma$, we can define tangent vectors

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}, \tag{4.11}$$

use them to pull the metric back to the boundary,

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b, \tag{4.12}$$

and from it, compute its determinant,

$$h = \det(h_{ab}), \tag{4.13}$$

which provides the measure for the boundary integrals. Meanwhile, K is the trace of the second fundamental form and describes the extrinsic curvature on the boundary. If we have a unit normal vector field n^α to the boundary, the extrinsic curvature is given by

$$K = \nabla_\alpha n^\alpha. \tag{4.14}$$

In two dimensions, the first two terms in eq. (4.10) are a topological invariant due to the Gauß-Bonnet theorem [59],

$$\frac{1}{2}\int_\Sigma \sqrt{g}R + \int_{\partial\Sigma}\sqrt{h}K = 2\pi\chi(\Sigma) = 2\pi(2-2g-n), \tag{4.15}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, which we will parametrise by its genus g and the number of circular boundaries n, detailed below.

The second parenthesis in eq. (4.10) contains the bulk action (4.9) introduced by Jackiw, as well as a boundary term needed to obtain a well-defined action principle. The bulk term serves only to set the curvature

$$R = -2, \tag{4.16}$$

meaning that Σ is locally isometric to the two-dimensional anti-de Sitter space AdS_2. The boundary term, as we will see in section 4.1.1, is responsible for the actual dynamics of the system and in particular, the Schwarzian action, of interest due to its describing the low energy sector of the Sachdev-Ye-Kitaev (SYK) model [108, 109, 178].

The JT gravity action emerges quite universally from the symmetry reduction of higher-dimensional near-extremal Reissner-Nordström black holes, see, e.g., [179] for an example reduction from $d = 3$ dimensions. Essentially, the near-horizon geometry of such black holes contains an AdS_2 factor,

$$\Sigma_d = \mathrm{AdS}_2 \times \mathcal{K}_{d-2}, \tag{4.17}$$

and the dilaton of JT gravity describes the codimension 2 volume of the transverse part of the spacetime. This volume[2] is precisely the Bekenstein-Hawking entropy S_{BH} of the black hole [46],

$$S_{\mathrm{BH}} = \frac{\mathrm{Vol}(\mathcal{K}_{d-2})}{4G_d}, \tag{4.18}$$

meaning that the dilaton equivalently measures the entropy of the excitations of the black hole away from extremality. The parameter S_0 meanwhile sets the extremal ground state entropy of the higher-dimensional black hole. In JT gravity, it can be identified with the inverse two-dimensional Newton constant,

$$S_0 = \frac{1}{8G_2}, \tag{4.19}$$

and will simply play the role of a large parameter punishing nontrivial topologies in the remainder of the thesis.

The theory is quantised by means of a path integral over the metric and dilaton, with the latter restricting the path integral to be over manifolds of constant curvature $R = -2$, as discussed above. Such manifolds[3], by the uniformisation theorem [180], are isometric to $\mathbb{H}^2/\Gamma$ with some Fuchsian group Γ, and hence are completely classified by their genus g and their number of circular boundaries n. Thanks to this classification and the appearance of the Einstein-Hilbert topological term in the action, the path integral therefore decomposes into a so-called *genus expansion*, taking the form [55]

$$\mathcal{Z}_n(\beta_1, \ldots, \beta_n) \equiv \langle Z(\beta_1) \ldots Z(\beta_n) \rangle = \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1, \ldots, \beta_n)}{e^{-(2-2g-n)S_0}}. \tag{4.20}$$

This equation should be read in the following way: the left-hand side instructs us to compute the so-called *correlation functions* by performing the gravitational path integral over the action (4.10) for spacetimes with n boundaries, on which we set the boundary conditions β_j. In the case of interest for this thesis, the boundaries will be such that the metric and dilaton restricted to the boundary are

$$g_{uu}|_\partial = \frac{1}{\epsilon^2}, \quad \phi_\partial = \frac{\gamma}{\epsilon}, \quad \epsilon \to 0, \tag{4.21}$$

[2]The area of the event horizon in 4 spacetime dimensions.

[3]Recall also chapter 3, where I discuss this construction in arbitrary dimension.

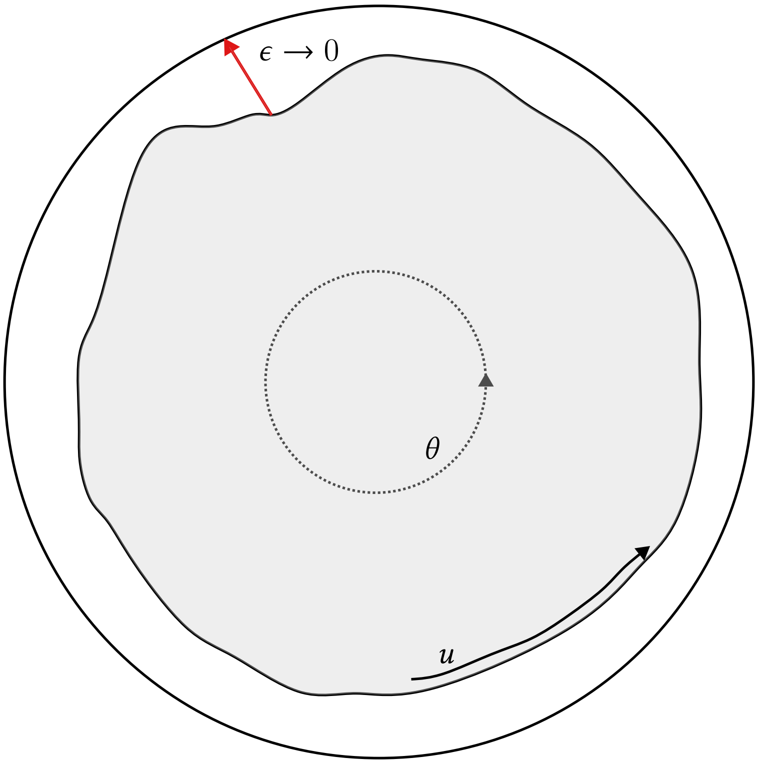


FIGURE 4.1: The two-dimensional Anti-de Sitter space AdS_2 with the boundary circle at infinity, and a manifold Σ with a "wiggly" boundary in grey. The angle θ runs from 0 to 2π, while u measures the proper length along the wiggly boundary of Σ; this boundary is not quite at the AdS_2 boundary, but approaches it in the limit $\epsilon \to 0$, indicated by the red arrow. Adapted from [55].

where $u \in [0, \beta]$ is a coordinate measuring the proper length along the boundary up to rescaling by a UV cutoff ϵ that is illustrated in fig. 4.1, and γ sets the energy scale of the system. In the JT literature, one often works with $\gamma = 1/2$, and I will adopt this convention hereinafter. Connecting this to eq. (4.20), the total rescaled proper length of the jth boundary is β_j. Other choices of boundary conditions are possible and exhaustively discussed in [181].

The right-hand side of eq. (4.20) sums up the genus-g contributions to the path integral with the aforementioned boundary conditions. They can be computed exactly by means of evaluating the integral over the boundary modes, which is one-loop exact [56], and then computing the volume of the moduli space of hyperbolic manifolds of genus g with n geodesics boundaries. Whether these volumes are well-defined objects hinges on the question of whether only orientable manifolds are allowed in the path integral, or if unorientable manifolds need to be considered as well. In the former case, we can use the remarkable recursion due to Mirzakhani [62] to compute these volumes. In the unorientable case, the volumes are all divergent, and special care needs to be taken to regularise them. I will briefly review all of these computations in section 4.1.

The middle expression in eq. (4.20) meanwhile is written suggestively as some average over quantities $Z(\beta)$, to be defined in chapter 5. This is no accident and hints at the duality of orientable JT gravity with a matrix model, proven in [55]. It has long been conjectured that multiple variants of 2-dimensional dilaton gravity are dual to matrix models with different symmetry properties, see [56] for a comprehensive account. These initially perturbative dualities have sparked wide-ranging interest around extending them nonperturbatively (see, e.g., [117, 118], the universe field theory approach of [79, 182, 183], but also the prolific work of Johnson on the topic [63, 184–191]). The duality of JT gravity to a matrix ensemble in particular also raises the question of how to interpret ensemble holography, particularly in higher dimensions, and how to deal with the so-called *factorisation problem* that I will discuss in section 4.2. Addressing this problem is particularly important because the multi-boundary connected geometries that pose the factorisation problem have been shown to be important ingredients in the use of JT gravity to address scrambling in the context of quantum teleportation [70–73, 85, 88] and the black hole information problem [66, 67].

I will review our work on all of these areas – the nonperturbative sector of the dual matrix models [1, 2], ensemble holography and the factorisation problem [3], and scrambling, in particular as regards to the out-of-time ordered commutator [4], in the rest of this thesis.

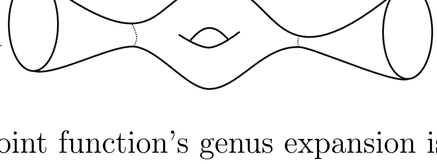
A crucial tool for some of this work is the low-energy limit of JT gravity, known as the Airy model. This model serves to alleviate some of the difficulties that come with allowing unorientable geometries. I will briefly review it in section 4.1.4.

4.1 Jackiw-Teitelboim gravity

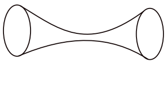
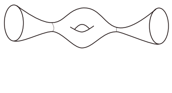
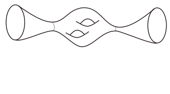
In this section, I want to introduce and review some of the key concepts relevant to our study of JT gravity. First, I will show how to compute the gravitational correlation functions (4.20) genus by genus. At a given g, the contributions are [55]

$$Z_{g,n}(\beta_1, \dots, \beta_n) = \int \mathrm{d}(\text{bulk moduli}) \int \mathcal{D}[\text{boundary fluctuations}] e^{\int_{\partial\Sigma} \sqrt{h}\phi_\partial (K-1)}. \tag{4.22}$$

To arrive there, we need to integrate out the bulk dilaton as discussed above, leaving the integrals over the possible boundary shapes, as well as over the moduli of the bulk spacetime. I will review the evaluation of the former integral in section 4.1.1, and the latter (in the orientable case) in section 4.1.2. Putting these results together, we can pictorially represent the genus-g contribution to the n-point correlation function as, e.g.,

$Z_{g=1,n=2}(\beta_1, \beta_2) =$ β_1  β_2 , (4.23)

and consequently, the entire n-point function's genus expansion is given by, e.g.,

$\mathcal{Z}_2(\beta_1, \beta_2) =$  $+ e^{-2S_0}$  $+ e^{-4S_0}$  $+ \dots .$ (4.24)

On top of these computations, we may want to account for unorientable manifolds in the gravitational path integral, corresponding to a time-reversal invariant boundary dual. I will discuss the inclusion of such geometries, and in particular the arising difficulties, in section 4.1.3. Finally, in section 4.1.4, I will discuss the low-energy limit of JT gravity, which makes some of the aforementioned difficulties easier to deal with.

4.1.1 The disk and trumpet partition functions

In order to perform the path integral over the boundary fluctuations, we first need to parametrise the boundary according to eq. (4.21), and introduce a bulk coordinate r that measures the distance to the boundary. Allowing for boundary fluctuations, the metric on the manifold Σ then reads [55]

$$ds^2 = dr^2 + \left(\frac{1}{4}e^{2r} - \left\{\tan\left(\frac{\theta(u)}{2}\right), u\right\} + \ldots\right), \tag{4.25}$$

where $\{\cdot,\cdot\}$ denotes the Schwarzian derivative and $\theta(u)$ is the angle in the hyperbolic plane $\mathbb{H}^2$ at boundary length coordinate u, recalling that Σ is described asymptotically by $\mathrm{AdS}_2 \cong \mathbb{H}^2$. Computing the action from this metric yields the well-known *Schwarzian action*

$$S = -\frac{1}{2}\int \mathrm{d}u \left\{\tan\left(\frac{\theta(u)}{2}\right), u\right\} =: -\frac{1}{2}\int \mathrm{d}u\, \mathrm{Sch}(u). \tag{4.26}$$

This action can also be derived from the first order *BF* theory formulation of JT gravity that I briefly discuss in section 9.1.1. The *BF* formulation further allows the derivation of the symplectic form that induces the path integral measure,

$$\Omega = \frac{1}{2}\int_0^\beta \mathrm{d}u \left[\mathrm{d}\epsilon'(u) \wedge \mathrm{d}\epsilon''(u) - 2\mathrm{Sch}(u)\mathrm{d}\epsilon(u) \wedge \mathrm{d}\epsilon'(u)\right], \tag{4.27}$$

where $\epsilon(u)$ describes an infinitesimal reparametrisation of the boundary length coordinate,

$$u \longrightarrow u + \epsilon(u). \tag{4.28}$$

There are two different geometries for which we need to evaluate the boundary path integral: the disk and the trumpet, depicted in figs. 4.1 and 4.2. The disk is a special contribution to the one-point function[4], while the trumpet, as evident from eq. (4.24), suffices to account for the boundary contribution of all remaining geometries appearing in the path integral. Starting with the disk, the path integral we need to compute is

$$Z_{\mathrm{d(isk)}}(\beta) = e^{S_0}\int \frac{\mathrm{d}\mu[\theta]}{\mathrm{SL}(2,\mathbb{R})} \exp\left[-\frac{1}{4}\int_0^\beta \mathrm{d}u \left(\frac{\theta''^2}{\theta'^2} - \theta'^2\right)\right], \tag{4.29}$$

[4] Indeed, as we will see in chapter 5, it can be understood as determining the entire genus expansion by serving as a recursion input.

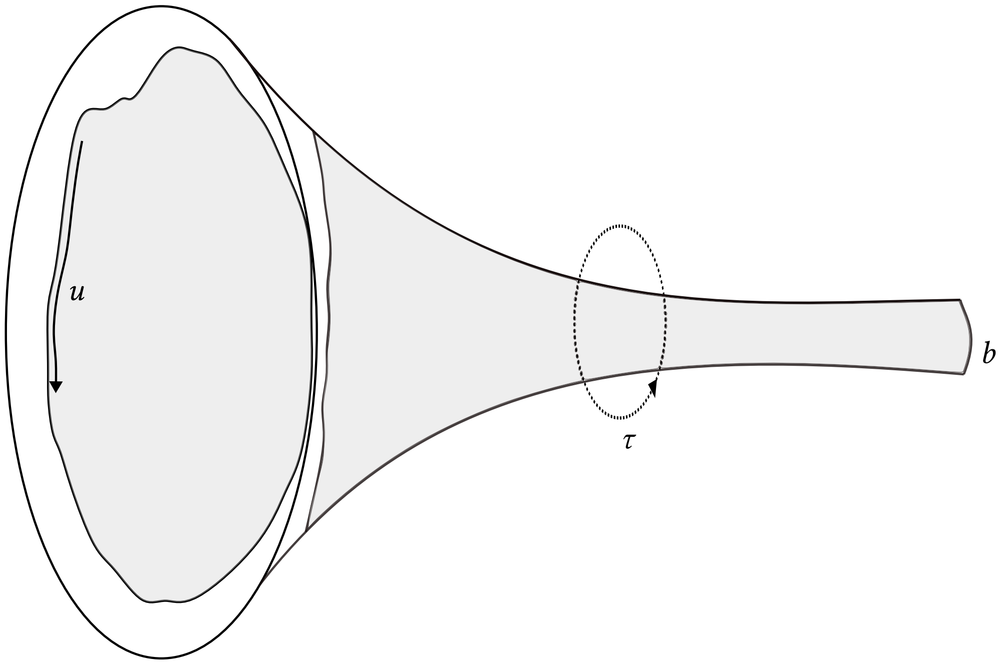


FIGURE 4.2: The trumpet geometry with a fluctuating boundary that asymptotes to AdS_2 and a geodesic boundary of length b. Dashed circle: the b-periodic twist parameter τ. Adapted from [55].

where $\mu[\theta]$ is the measure induced by eq. (4.27) and $\mathrm{SL}(2,\mathbb{R})$ is the volume of the $\mathrm{SL}(2,\mathbb{R})$ isometry group of the disk. We can Fourier-decompose the reparametrisations (4.28), describing small fluctuations about the saddle point

$$\theta(u) = \frac{2\pi}{\beta}(u + \epsilon(u)), \tag{4.30}$$

according to[5]

$$\epsilon(u) = \sum_{|n|\geq 2} e^{-\frac{2\pi}{\beta} inu} \left(\epsilon_n^{(R)} + i\epsilon_n^{(I)}\right), \tag{4.31}$$

with real/imaginary parts $\epsilon_n^{(R/I)}$, which gives the symplectic form

$$\Omega = 2\frac{(2\pi)^3}{\beta^2} \sum_{n\geq 2} (n^3 - n)\, \mathrm{d}\epsilon_n^{(R)} \wedge \mathrm{d}\epsilon_n^{(I)}\,. \tag{4.32}$$

This allows the explicit computation of the disk path integral [55],

$$\begin{aligned} Z_{\mathrm{d}}(\beta) =& e^{S_0} e^{\frac{\pi^2}{\beta}} \prod_{n\geq 2} 2\frac{(2\pi)^3}{\beta^2}(n^3 - n) \\ & \times \int \mathrm{d}\epsilon_n^{(R)}\, \mathrm{d}\epsilon_n^{(I)} \exp\left\{-(2\pi)^4 \frac{(\epsilon_n^{(R)})^2 + (\epsilon_n^{(I)})^2}{2\beta^3}(n^4 - n^2)\right\} \\ =& e^{S_0} e^{\frac{\pi^2}{\beta}} \prod_{n\geq 2} \frac{\beta}{2n} = \frac{1}{(2\pi)^{1/2}(2\beta)^{3/2}} e^{\frac{\pi^2}{\beta}}, \end{aligned} \tag{4.33}$$

[5] Excluding the modes with $n = 0, \pm 1$ precisely implements the $\mathrm{SL}(2,\mathbb{R})$ quotient.

where the evaluation of the infinite product requires, e.g., a zeta function regularisation. Importantly, the disk partition function is nothing but the Laplace transform of the density of states at leading order in e^{S_0},

$$Z_{\mathrm{d}}(\beta) = e^{S_0} \int_0^\infty \mathrm{d}E\, e^{-\beta E} \rho_0(E), \quad \rho_0(E) = \frac{1}{4\pi^2} \sinh\left(2\pi\sqrt{E}\right). \tag{4.34}$$

The density of states inherits a genus expansion via Laplace transforming $\mathcal{Z}_1(\beta)$ according to eq. (4.20), and $e^{S_0}\rho_0(E)$ is the genus-zero part of this expansion. Equivalently, one could compute the path integral with different boundary conditions than (4.21); fixing both the dilaton and its normal derivative on the boundary gives a path integral that computes the genus expansion of the density of states directly [181].

A similar derivation to the one above can be performed for the trumpet geometry, which however possesses a parameter b describing the length of the additional geodesic boundary, cf. fig. 4.2. In appropriate coordinates, the metric for this geometry reads

$$ds^2 = d\sigma^2 + \cosh^2(\sigma) d\tau^2, \quad \tau \sim \tau + b, \tag{4.35}$$

where τ is connected to the familiar angle θ in the hyperbolic plane by [55]

$$\cos(\theta) = \tanh(\tau). \tag{4.36}$$

Accounting for the fact that the geodesic boundary breaks the $\mathrm{SL}(2,\mathbb{R})$ reparametrisation symmetry of the disk's boundary down to $\mathrm{U}(1) \cong \{e^{i\phi} | \phi \in \mathbb{R} \mod 2\pi\mathbb{Z}\}$, i.e., simple rotations, the trumpet path integral reads

$$Z_{\mathrm{t(rumpet)}}(\beta, b) = \int \frac{\mathrm{d}\mu[\tau]}{\mathrm{U}(1)} \exp\left[-\frac{1}{4}\int_0^\beta \mathrm{d}u \left(\frac{\tau''^2}{\tau'^2} + \tau'^2\right)\right]. \tag{4.37}$$

Expanding once again in small fluctuations,

$$\tau = \frac{b}{\beta}(u + \epsilon(u)), \tag{4.38}$$

we can compute the trumpet partition function as before,

$$\begin{aligned} Z_{\mathrm{t}}(\beta, b) =& e^{-\frac{b^2}{4\beta}} \prod_{n\geq 1} 2\frac{(2\pi)^3}{\beta^2}\left(n^3 + \frac{b^2}{(2\pi)^2}n\right) \\ &\times \int \mathrm{d}\epsilon_n^{(R)}\, \mathrm{d}\epsilon_n^{(I)} \exp\left\{-(2\pi)^4 \frac{(\epsilon_n^{(R)})^2 + (\epsilon_n^{(I)})^2}{2\beta^3}\left(n^4 + \frac{b^2}{(2\pi)^2}n^2\right)\right\} \\ =& e^{-\frac{b^2}{4\beta}} \prod_{n\geq 1} \frac{\beta}{2n} = \frac{1}{(4\pi\beta)^{1/2}} e^{-\frac{b^2}{4\beta}}. \end{aligned} \tag{4.39}$$

Recovering eqs. (4.33) and (4.39) from our proposed perturbative dual will be a central goal in the main part of the thesis.

4.1.2 Weil-Petersson volumes and Mirzakhani's recursion

To be able to evaluate contributions like eq. (4.23), we need not only the boundary trumpet contribution (4.39), but also the integral over the bulk moduli. These moduli are determined by splitting a given bulk manifold X or "convex core" into three-holed spheres and then parametrising these basic building blocks (often referred to as topological "pairs of pants") by their boundary lengths $\tilde{b}_i$ and the relative twists between two adjacent boundaries, τ_i, the so-called Fenchel-Nielsen coordinates [60]. The volume of the moduli space of X, i.e., the so-called *Weil-Petersson (WP) volume* of a given topological sector, is then given by integrating over the moduli and dividing by the volume of the mapping class group[6] $\mathrm{MCG}(X)$,

$$V_{g,n}(b_1,\ldots,b_n) = \int_0^\infty \mathrm{d}\tilde{b}_1 \ldots \mathrm{d}\tilde{b}_k \int_0^{\tilde{b}_1} \mathrm{d}\tau_1 \ldots \int_0^{\tilde{b}_k} \mathrm{d}\tau_k \frac{1}{\mathrm{MCG}(X)}. \tag{4.40}$$

Following [119], I use tildes to distinguish the internal gluing boundaries from the external boundaries that are going to be attached to trumpets, as illustrated in the pants decomposition in fig. 4.3. It can be shown that [60]

$$k = 3g - 3 + n. \tag{4.41}$$

An important consequence of the form of eq. (4.40) is that in the gravitational path integral, the gluing boundaries inherit a measure $b\mathrm{d}b$ from the Weil-Petersson measure [60]

$$\omega = \mathrm{d}b\mathrm{d}\tau. \tag{4.42}$$

Performing integrals like eq. (4.40) is impractically difficult, but Mirzakhani [62] has provided a better, recursive method to determine the WP volumes. In the following, I will very briefly review the quite pedagogical derivation of the recursion given in [119].

To this end, it is helpful to consider not the WP volume directly, but multiply by the length of a particular boundary b,

$$bV_{g,n}(\underbrace{b, b_2,\ldots,b_n}_{B}) = \int_0^b \mathrm{d}p \int \frac{\mathrm{d}(\text{moduli of } X)}{\mathrm{MCG}(X)}. \tag{4.43}$$

There is now a unique geodesic γ_p orthogonal to this boundary at a given point p. Following γ_p until it either reaches another (or the same) external boundary or self-intersects, called the *fate* f of γ_p in [119], we can determine a decomposition of X into a particular three-holed sphere Λ containing γ_p, and the rest of the manifold Y, as illustrated for example in fig. 4.4. One can then organise the

[6] The mapping class group is the group of orientation-preserving diffeomorphisms up to smooth deformations, i.e., $\mathrm{MCG}(X) = \mathrm{Diff}(X)/\mathrm{Diff}_0(X)$, where $\mathrm{Diff}_0(X)$ is the component connected to the identity. In physical language, it is the group of large diffeomorphisms. For all manifolds appearing in Mirzakhani's recursion, the mapping class group is an infinite, non-abelian group, making the quotient hard to compute in practice [60].

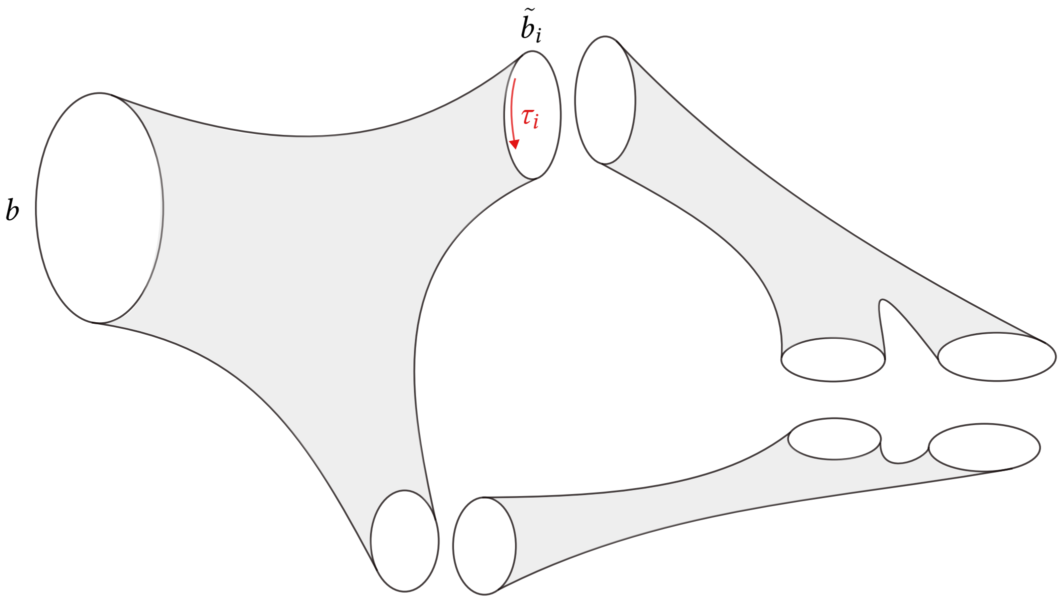


FIGURE 4.3: "Pair-of-pants" decomposition of a hyperbolic surface of genus 2 with one boundary into three-holed spheres or pairs of pants, i.e., hyperbolic surfaces of genus 0 with three boundaries. The boundaries with lengths $\tilde{b}_i$ can be glued with a relative twist τ_i, depicted in red. One boundary of length b remains, to be glued to a trumpet [62].

integrals in eq. (4.43) in a discrete sum over all available fates F,

$$\begin{aligned} bV_{g,n}(b,B) = &\sum_{f\in F}\int \mathrm{d}\begin{pmatrix}\text{moduli gluing}\\ \Lambda \text{ to } Y\end{pmatrix} \\ &\times \int_0^b \mathrm{d}p\,\Theta\begin{pmatrix}\gamma_p \text{ has fate}\\ f \text{ within } \Lambda\end{pmatrix}\int \frac{\mathrm{d}(\text{moduli of } Y)}{\mathrm{MCG}(Y)}. \end{aligned} \tag{4.44}$$

The indicator function Θ ensures that the point p actually admits a geodesic with fate f. Denoting the three boundary lengths of Λ by b_1, b_2, b_3, we can express the Weil-Petersson volume in terms of

$$\mathrm{T}(b_1 \to b_2; b_3) = \log\left(\frac{\cosh\frac{b_3}{2} + \cosh\frac{b_1+b_2}{2}}{\cosh\frac{b_3}{2} + \cosh\frac{b_1-b_2}{2}}\right), \tag{4.45}$$

which is the length of the portion of the boundary 1 such that γ_p goes to boundary 2 without leaving Λ. A similar object is

$$\mathrm{D}(b_1; b_2, b_3) = b_1 - \mathrm{T}(b_1 \to b_2; b_3) - \mathrm{T}(b_1 \to b_3; b_2), \tag{4.46}$$

i.e., the length of the portion of boundary 1 such that γ_p either self-intersects or returns to boundary 1 without leaving Λ[7]. The crucial point here is that eq. (4.44) expresses the WP volume for a given genus g in terms of manifolds with lower genus, albeit higher number of boundaries. This makes the equation

[7]It is a straightforward mental exercise to convince oneself that these possibilities are exhaustive and hence to make sense of the sum rule (4.46).

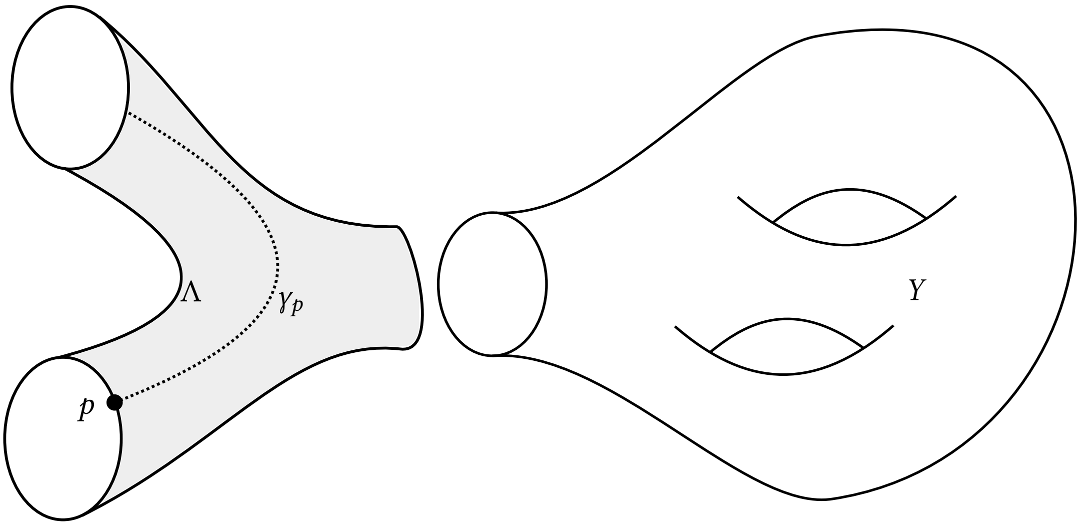


FIGURE 4.4: Decomposition of a hyperbolic surface X of genus 2 with two boundaries into a three-holed sphere Λ and a manifold Y with fewer boundaries. A point p on one of the external boundaries of X (i.e., the ones not glued to Y) serves as the base point of a geodesic γ_p that terminates at the other external boundary without leaving Λ. Adapted from [119].

recursive; indeed, we can use eqs. (4.45) and (4.46) to express $V_{g,n}$ in terms of lower-genus WP volumes by Mirzakhani's celebrated recursion relation [62],

$$\begin{aligned} bV_{g,n}(b,B) =& \sum_{k=2}^{|B|} \int_0^\infty b'\,\mathrm{d}b'\,[b - \mathrm{T}(b \to b'; b_k)]V_{g,n-1}(b', B\backslash b_k) \\ &+ \frac{1}{2}\int_0^\infty b'\,\mathrm{d}b'\,b''\,\mathrm{d}b''\,\mathrm{D}(b;b',b'') \\ &\times \left[V_{g-1,n+1}(b',b'',B) + \sum_{\text{stable}} V_{h_1,|B_1|+1}(b',B_1)V_{h_2,|B_2|+1}(b'',B_2) \right]. \end{aligned} \tag{4.47}$$

The sum indexed "stable" ensures that $h_1 + h_2 = g$ and $B_1 \cup B_2 = B$, excluding the appearance of volumes formally corresponding to the disk and double trumpet geometries. The only thing missing for the evaluation of this recursion are two initial conditions, which are usually chosen

$$V_{0,3}(b_1,b_2,b_3) = 1, \quad V_{0,2}(b_1,b_2) = \frac{1}{b_1}\delta(b_1 - b_2). \tag{4.48}$$

It should be noted at this point that we may choose a different normalisation for the lengths of the gluing geodesics such that

$$2b_{\text{a(lternative)}} = b_{\text{s(tandard)}}. \tag{4.49}$$

This changes the Weil-Petersson measure, upon integrating out the (now $2b_a$-periodic) twist τ, to

$$4b_a\,\mathrm{d}b_a = b_s\,\mathrm{d}b_s\,, \tag{4.50}$$

and the trumpet partition function to

$$Z_{\mathrm{t}}(\beta, b_a) = \frac{1}{\sqrt{\pi}\beta^{1/2}} e^{-b_a^2/\beta}, \tag{4.51}$$

with the remaining factors from the trumpet normalisation being absorbed into the WP volumes as a global rescaling, leaving the overall physics invariant. This convention, while somewhat unnatural in JT gravity, will allow me to more easily make contact with the system introduced in chapter 3 using the semiclassical methods discussed in chapter 2, and it is also the convention we used in [3]. For this reason, I will be using this convention in chapter 9, while adopting the standard JT convention in part II, where I only work in the more established setting of JT/RMT duality, to be introduced in chapter 5.

From the drastically shortened and simplified presentation above, the reader may appreciate that Mirzakhani's recursion, while enormously alleviating the difficulty of computing moduli space volumes appearing in the JT gravity path integral, is still a bit unwieldy. For this reason, one often uses the equivalent topological recursion of the dual matrix model that I will discuss in chapter 5. However, Mirzakhani's recursion still builds invaluable geometric intuition that is useful in the generalisation to the unorientable case that I will turn to in the following section.

4.1.3 (Un)orientability

What we discussed so far suffices to study JT gravity with a specific set of boundary conditions, as long as we allow only orientable manifolds in the path integral. Nothing precludes us from admitting such manifolds however, and there are versions of JT gravity that allow precisely such manifolds, as well as additional spin or pin structures on them, corresponding to different symmetries in a putative boundary dual [56]. In particular, unorientable JT gravity corresponds to a boundary dual with intact time-reversal invariance. This is especially interesting from the perspective of trying to find a quantum chaotic boundary dual and treating it semiclassically. As we have seen in chapter 2, the leading corrections from periodic orbit theory are simpler in this case, making the comparison of computations on both sides of such a putative duality more feasible; I will exploit this fact in section 9.3.

Unorientability is introduced by "simply" gluing one or multiple crosscaps to an otherwise orientable manifold, as depicted in fig. 4.5. A crosscap can be imagined as a geodesic boundary of the manifold, together with an antipodal identification of points. A geodesic of length $a/2$ winding halfway around the boundary is closed by this identification; the neighbourhood of this geodesic can then be visualised as a Möbius strip. In the moduli space integral, crosscaps can be included using Norbury's measure [192, 193],

$$\frac{\mathrm{d}a}{2\tanh\frac{a}{4}}, \tag{4.52}$$

which notably diverges as $a \to 0$. This is precisely why unorientable JT gravity is naively ill-defined. As shown in [119] however, Mirzakhani's recursion can be formally generalised to include crosscaps, and subsequently regularised. To do so, we need to include two new fates for the geodesic γ_p: it can pass through a crosscap and then either return to the original boundary (cf. fig. 4.5) or self-intersect. In the recursion, these fates are accounted for by defining $\mathrm{C}(b_1, a; b_2)$ as the length of the segment on boundary 1 such that either of the two new fates are realised, and then integrating this function against the crosscap measure (4.52),

$$\begin{aligned} \mathrm{c}(b_1; b_2) &:= \int_0^\infty \frac{\mathrm{d}a}{2\tanh\frac{a}{4}} \mathrm{C}(b_1, a; b_2) \\ &= b_1 b_2 - 2b_2 \log \frac{\cosh\frac{b_1+b_2}{4}}{\cosh\frac{b_1-b_2}{4}} + 4\left[\mathrm{Li}_2\left(-e^{-\frac{b_1+b_2}{2}}\right) - \mathrm{Li}_2\left(-e^{-\frac{b_1-b_2}{2}}\right)\right], \end{aligned} \tag{4.53}$$

where

$$\mathrm{Li}_2(z) = \sum_{k=1}^\infty \frac{z^k}{k^2} \tag{4.54}$$

is the dilogarithm. In terms of $c(b_1; b_2)$, the recursion reads [119]

$$\begin{aligned} bV_{g,n}(b, B) =& 2\sum_{k=2}^{|B|} \int_0^\infty b'\,\mathrm{d}b'\,[b - \mathrm{T}(b \to b'; b_k)] V_{g,n-1}(b', B\backslash b_k) \\ &+ \frac{1}{2}\int_0^\infty b'\,\mathrm{d}b'\,b''\,\mathrm{d}b''\,\mathrm{D}(b; b', b'') \\ &\times \left[V_{g-1,n+1}(b', b'', B) + \sum_{\text{stable}} V_{h_1,|B_1|+1}(b', B_1) V_{h_2,|B_2|+1}(b'', B_2)\right] \\ &+ \frac{1}{2}\int_0^\infty b'\,\mathrm{d}b'\,\mathrm{c}(b; b') V_{g-\frac{1}{2},n}(b', B), \end{aligned} \tag{4.55}$$

where the factor 2 in the first line accounts for possible orientation reversal introduced in the gluing. A crosscap contributes a genus of 1/2 to a manifold[8], explaining the appearance of half-integer genus WP volumes in eq. (4.55)[9]. In order for this recursion to close, we now need three initial conditions, which read

$$V_{0,3}(b_1, b_2, b_3) = 4, \quad V_{0,2}(b_1, b_2) = \frac{2}{b_1}\delta(b_1 - b_2), \quad V_{\frac{1}{2},1}(b) = \frac{1}{2b\tanh\frac{b}{4}}, \tag{4.56}$$

with the factors 2 and 4 compared to eq. (4.48) accounting for possible orientation reversals at the gluing boundaries. For the purposes of this thesis, the most relevant unorientable WP volume – for manifolds with two boundaries and one

[8]The Euler characteristic of a bare crosscap can be computed, e.g., by a cell decomposition to be $\chi = 1 \equiv 2 - 2g$, which defines the unorientable genus of the crosscap to be 1/2 [56].

[9]Note that even numbers of crosscaps contribute an overall integer genus. This means that in the unorientable theory, integer WP volumes change nontrivially as well; the moduli space of, e.g., genus 1 manifolds now also contains manifolds with two crosscaps and no handles.

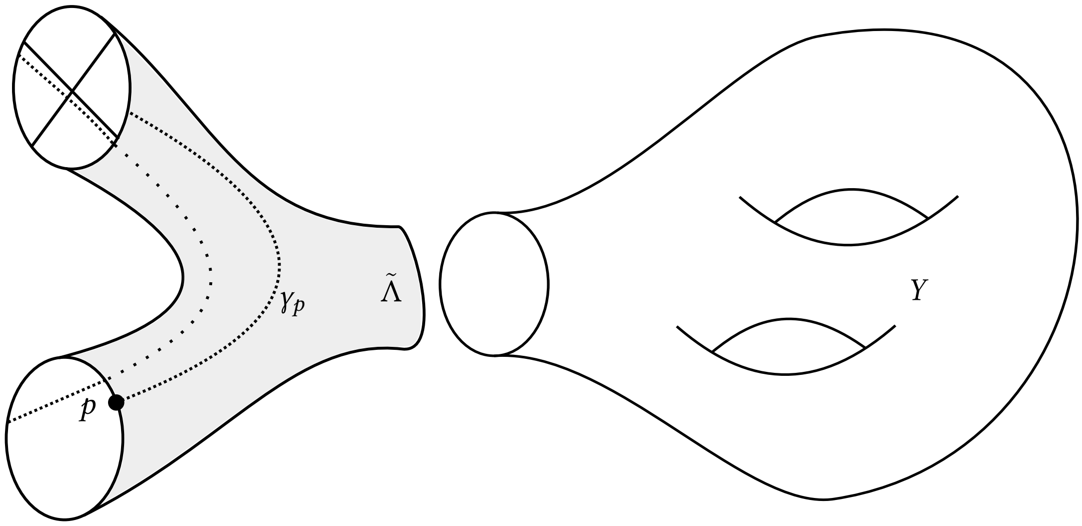


FIGURE 4.5: An unorientable manifold X with one boundary and genus 5/2, decomposed into a three-holed sphere $\tilde{\Lambda}$ with one of the external boundaries closed off by a crosscap $\otimes$, and a rest Y, as in fig. 4.4. The geodesic γ_p originates at the point p of the external boundary, moves through the crosscap and returns to the original boundary without leaving $\tilde{\Lambda}$. Adapted from [119].

crosscap – is exceptional, but has been computed in [117] as

$$V^{(\epsilon)}_{\frac{1}{2},2} \approx 2\log\frac{\cosh\frac{b_1}{2}+\cosh\frac{b_2}{2}}{2} - 4\log\frac{\epsilon}{4}, \tag{4.57}$$

where ϵ is a cutoff for small geodesic lengths in the crosscap integral. In particular, we will be interested in the limit of eq. (4.57) as the geodesic lengths b_1, b_2 tend to infinity; this corresponds to the low-energy limit of JT gravity, which isolates the leading power of the regularisation-independent parts of the unorientable Weil-Petersson volumes. I will therefore briefly introduce this limit in the next section.

4.1.4 The Airy model

The Airy model [61], also known as topological gravity or the Kontsevich-Witten model, is obtained as the low-energy limit of JT gravity, i.e., by taking the limit

$$\rho_0^{\mathrm{JT}}(E) = \frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right) \xrightarrow{E\ll 1} \frac{\sqrt{E}}{\pi} =: \rho_0^{\mathrm{Airy}}(E), \tag{4.58}$$

in the leading order density of states. The relation between eq. (4.58) and the fixed-boundary length one-point function eq. (4.34) suggests a geometric interpretation of this limit: at low energies, the path integral ought to be dominated by long boundaries. Indeed, an equivalent, operationally convenient way of recovering the Airy model is to rescale the boundary lengths according to [117]

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle_{\mathrm{Airy}} = \lim_{N\to\infty} N^{\frac{3}{2}\chi}\,\langle Z(N\beta_1)\dots Z(N\beta_n)\rangle_{\mathrm{JT}}, \tag{4.59}$$

and shifting

$$S_0 \to S_0 + \frac{3}{2}\log N. \tag{4.60}$$

If we want to write eq. (4.59) in terms of integrals over trumpets times a moduli space “Airy volume”, the functional form of the trumpet (4.39) suggests rescaling the geodesic gluing boundaries by $\sqrt{N}$ and defining the Airy volumes by

$$V_{g,n}^{\text{Airy}}(b_1,\dots,b_n) = \lim_{N\to\infty} N^{3-3g-n} V_{g,n}(\sqrt{N}b_1,\dots,\sqrt{N}b_n). \tag{4.61}$$

In the orientable case, the WP volumes are simple polynomials, and hence, the Airy limit can be seen as isolating only the highest powers of the gluing boundary lengths. Elaborating on the geometric interpretation, eq. (4.61) means that the manifolds relevant in this limit must admit very long geodesics in the bulk that serve as suitable gluing boundaries. Combined with the restriction of the area of the manifold due to the Gauß-Bonnet theorem, this means that the manifolds have to compensate for these long geodesics by being very thin.

Indeed, the moduli space of the Airy model can be decomposed into so-called Kontsevich graphs, i.e., trivalent ribbon graphs that respect precisely this thin-strip form of the bulk manifolds, see fig. 4.6 for the possible graphs at $g = 1/2$. In this formalism, the Airy volumes can be directly computed via their Laplace transforms by [117]

$$\tilde{V}_{g,n}^{\text{Airy}}(z_1,\dots,z_n) = \sum_{\Gamma\in\Gamma_{g,n}} \frac{2^{2g-2+n}}{|\text{Aut}(\Gamma)|} \prod_{k=1}^{6g-6+3n} \frac{1}{z_{l(k)} + z_{r(k)}}, \tag{4.62}$$

where $\Gamma_{g,n}$ is the set of possible Kontsevich graphs with genus g made up of n lines, $|\text{Aut}(\Gamma)|$ is the order of the automorphism group of a graph Γ, and $l(k)$ and $r(k)$ label which boundary of the manifold the left/right side of the ribbon in question belongs to. Evaluating eq. (4.62) for the graphs in fig. 4.6, labelled (n_{11}, n_{22}) by their number of 11- and 22-propagators[10], and Laplace transforming back to the domain of geodesic lengths, we find

$$(1,0) + (0,1) = \min(b_1, b_2) \tag{4.63}$$

$$(2,0) + (0,2) = |b_1 - b_2| \tag{4.64}$$

$$V_{\frac{1}{2},2}(b_1, b_2) = \max(b_1, b_2). \tag{4.65}$$

The same answer could have been obtained from the full JT result eq. (4.57) by dropping the regulator-dependent term and approximating $\log\cosh x \approx x$ for large x. Interestingly, in [117], the (1,0) and (0,1) graphs were connected to the encounter formalism, and in particular, to Sieber-Richter pairs, while there is no such obvious connection for the (2,0) and (0,2) graphs. Nevertheless, both contributions combine to give something that is compatible with the universal result for the spectral form factor of the Airy model that can be computed from the two-point function. In section 9.3, I will perform a computation of $V_{\frac{1}{2},2}^{\text{Airy}}$

[10]This means the number of double lines where both sides of the ribbon belong to boundary 1 or 2, respectively.

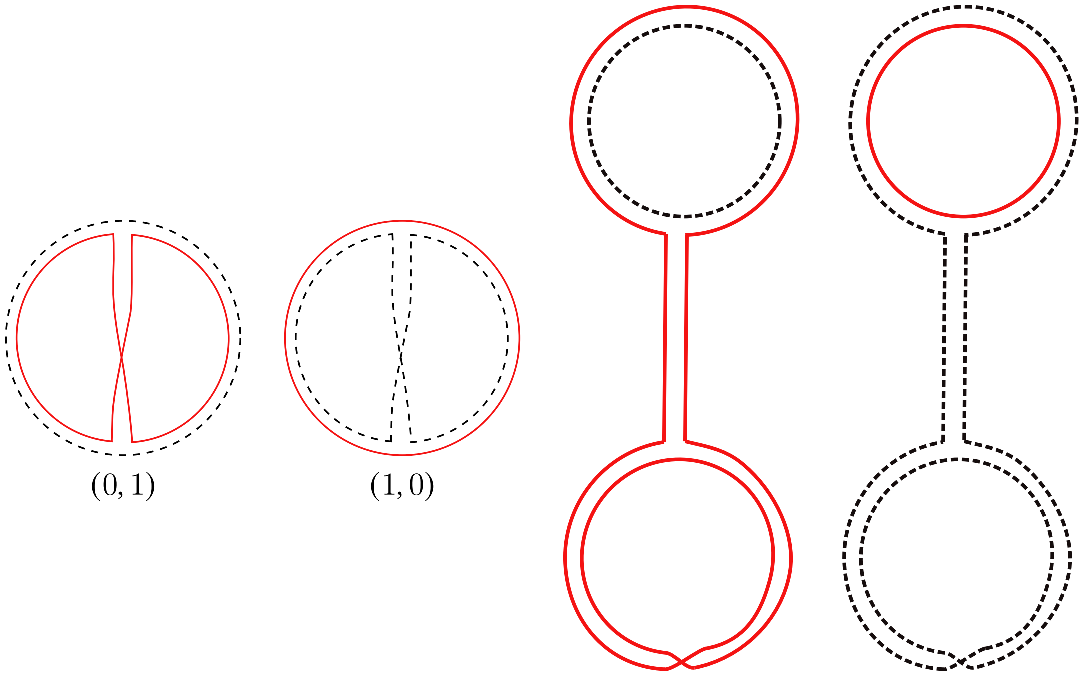


Figure 4.6: Kontsevich ribbon graphs contributing to the Airy volume $V_{\frac{1}{2},2}(b_1, b_2)$, labelled by their numbers (n_{11}, n_{22}) of 11-propagators (black dashed double lines) and 22-propagators (red solid double lines). We can identify two closed lines corresponding to the two boundaries of length b_1 and b_2 respectively, and we find exactly one twisted ribbon corresponding to the crosscap. The reader can appreciate from this depiction that the geometry is almost entirely determined by the longer of the two boundaries, with the shorter boundary simply following the geometry, as dictated by the thin-strip limit. This property is particularly clear from the form (4.65), which depends only on one of the boundary lengths at fixed b_1, b_2. It should also be noted that while the ribbon graphs are labelled by the propagators, i.e., a property of the Laplace transforms of the Airy volumes, it is indeed the Riemann surfaces appearing in the gravitational path integral themselves that turn into ribbon graphs in the Airy limit [118]. Adapted from [117].

in the model introduced in chapter 3 where something similar, but in a sense orthogonal happens: as we will see, it is going to be the *encounter* contribution that will produce both a universal and a non-universal term; these two will have to combine to give the full Airy moduli space volume.

4.2 The factorisation problem

Paradigmatic instances of holographic theories, such as the correspondence between string theory on $AdS_5 \times S_5$ and $\mathcal{N} = 4$ super Yang-Mills theory [47, 49, 50], or the AdS_3/CFT_2 duality that has garnered much attention in recent times (e.g. [51]), all exhibit (conformal) quantum field theories as the boundaries dual to the quantum gravitational bulk theory. Being quantum field theories, these boundary duals are expected to have the cluster decomposition property [9, 75],

$$\lim_{|x|\to\infty} \langle O_1(x)O_2(0)\rangle = \langle O_1\rangle \langle O_2\rangle\,, \tag{4.66}$$

where $\langle\cdot\rangle$ is the vacuum expectation value and $O_{1,2}$ are arbitrary local, translation invariant operators. In simple language, eq. (4.66) means that two local, far separated observables should not influence one another, and corresponding correlation functions should factorise. In holographic theories, this should in particular hold true if the boundary consists of multiple disconnected regions and the operators are localised on different disconnected pieces of said boundary. By the holographic dictionary however, such correlation functions in the boundary theory may just as easily be computed as gravitational partition functions with appropriate boundary conditions, and as we have seen above, these partition functions can include wormhole geometries that connect different boundary pieces and explicitly prevent factorisation. This apparent contradiction is referred to as the factorisation problem. Initial attempts to deal with this problem yielded a proof that Euclidean AdS bulk spacetimes cannot have a disconnected boundary with positive curvature [194], as such a manifold would not be a classical solution of Einstein's equations. If the bulk is Lorentzian on the other hand, disconnected boundaries can exist, but the pieces must be separated by event horizons [195]. In more elaborate theories with, e.g., Yang-Mills fields turned on, such geometries can appear however [74].

A naive solution to this problem might be to simply exclude them from the theory, e.g., by not allowing them in the path integral (4.20). However, so-called replica wormholes have been shown to be crucial ingredients in the recovery of the Page curve for the black hole entropy [66–69]. Replica wormholes appear when trying to compute the entanglement entropy S_{BH} of the black hole by analytic continuation of the Rényi entropies [83, 84],

$$S_{\mathrm{BH}} = -\operatorname{tr}(\rho_{\mathrm{TMD}} \log \rho_{\mathrm{TMD}}) = -\lim_{n\to 1} \operatorname{tr}\big(\rho^n_{\mathrm{TMD}}\big), \tag{4.67}$$

where

$$\rho_{\mathrm{TMD}} = \overline{\rho_{\mathrm{TFD}}(t)} = \frac{1}{Z(\beta)} = \sum_i e^{-\beta E_i} \,|i\rangle_L \langle i|_L \otimes \big|\bar{i}\big\rangle_R \big\langle\bar{i}\big|_R \tag{4.68}$$

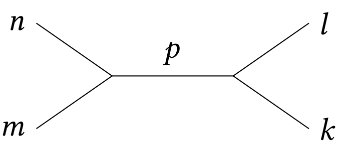
is the so-called *thermo-mixed double state*, the time average of the more familiar thermofield double state. These replica wormholes are not a priori the same as, e.g., the double trumpet geometry that is responsible for the ramp region in the spectral form factor of JT gravity; however, they have been shown to be closely related in [83, 84], and indeed, one can make an ansatz for the SFF in terms of the TMD state,

$$\langle Z(\beta)Z(\beta)\rangle = \langle Z(\beta)\rangle^2 \operatorname{tr}\left(\rho^2_{\mathrm{TMD}}\right). \tag{4.69}$$

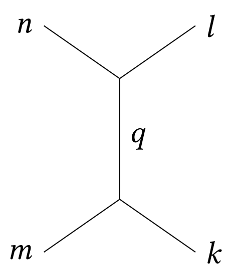
Consequently, simply excluding wormholes from the theory spoils the solution to other important problems in the study of quantum gravity.

A possible way out of the factorisation problem – indeed the one realised in JT gravity – is to relieve the boundary theory from the requirement of cluster decomposition by considering an ensemble of boundary duals, rather than a single, fixed Hamiltonian. The matrix model dual to JT gravity that I will discuss in chapter 5 is precisely such an ensemble. This programme of ensemble holography, first introduced in [196, 197] and more recently developed in [76], relies on the fact that an ensemble average can introduce correlations between observables on otherwise disconnected boundary segments, hence making the ensemble correlation functions not factorise, in agreement with the non-factorising gravitational bulk. In higher-dimensional holographic theories, where the boundary theory is a conformal field theory (CFT), one might now try to define an ensemble of CFTs and average over the CFT data, i.e., the conformal dimensions Δ_i of the primary fields and the three-point structure constants C^k_{ij}. However, CFTs are highly constrained; in particular, their four-point functions must obey crossing symmetry [198],

$$\sum_p C^p_{nm} C^p_{lk}$$



$$= \sum_q C^q_{nl} C^q_{mk}$$



. (4.70)

The requirement (4.70) is very stringent and generic structure constants do not yield crossing-symmetric four-point functions. Meanwhile, there is no known ensemble of chaotic CFTs capturing the non-factorisation properties expected in quantum gravity[11], although attempts to define such an ensemble persist [201], particularly in the context of the AdS_3/CFT_2 duality[12].

Alternative approaches exist that try to understand and cure the failure of bulk correlators to factorise. In the SYK model for instance, an explicit disorder average introduces ensemble correlations. Single disorder realisations on the other hand exhibit so-called half-wormhole saddles [202–204] that disappear

[11]The Narain ensemble [199, 200] is one of the few known examples of CFT ensembles, but it consists of integrable 2d CFTs and does not produce non-factorising correlation functions.

[12]This setting seems particularly promising since AdS_3 gravity, like JT gravity, does not have local bulk degrees of freedom, which might make cluster decomposition on the boundaries easier to satisfy.

under the disorder average, but restore factorisation in individual SYK systems. For an interpolation between this setting and ordinary SYK, see [205]. A solution proposed in [78] shows that integrating out UV degrees of freedom in a more complex, factorising theory leads to non-factorising correlators in the coarse-grained theory. This proposal is further supported by [206], who find that nonperturbative effects due to wormholes can actually restore bulk factorisation. The failure of bulk factorisation can also be traced to the existence of a non-exact symplectic form in the phase space of the theory, leading to a non-vanishing Berry phase that corresponds to the appearance of wormholes in the bulk [207].

A somewhat orthogonal idea attempts not to understand factorisation in the bulk, but failure to do so in the boundary, in line with [80]: the goal is to take a single boundary dual system, e.g., one member of the JT ensemble, and identify mechanisms that can make correlators computed in this system appear as though they originate from an ensemble. In the context of AdS_3/CFT_2 for instance, [51] proposes an average over microstates within a single CFT, evaluated using a trace formula very similar to the programme pursued in this thesis. Indeed, in chapter 9, I will discuss the mechanism we introduced in [3], that aims to understand precisely this question: how the correlators of an individual quantum mechanical system can appear not to factorise, as one would expect in a bulk theory dual to an ensemble.

Chapter 5

Matrix models

I introduce matrix models by first deriving the measure in the two Dyson symmetry classes relevant for the thesis, i.e., ensembles invariant under unitary and orthogonal transformations respectively. I then illustrate the loop equations and topological recursion as methods of computing perturbative expansions of correlation functions in matrix models relying only on the spectral density, or more precisely, the spectral curve of the matrix model. Finally, I introduce the matrix model whose perturbative expansion is equivalent to the topological expansion in JT gravity as determined by Mirzakhani's recursion and Stanford's generalisation thereof.

A matrix model is a theory of $N \times N$ matrices H with certain symmetries, defined by a *matrix integral*

$$\mathcal{Z} = \int \mathrm{d}H e^{-N \operatorname{tr} V(H)}, \tag{5.1}$$

where $V(H)$ is some potential. Observables in such a model are functions of the matrix H and their expectation values can be computed by inserting them in the matrix integral, e.g.,

$$\langle O(H) \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{d}H e^{-N \operatorname{tr} V(H)} O(H). \tag{5.2}$$

For the cases of relevance for this thesis, we can interpret H as the Hamiltonian of some physical system, and hence, eq. (5.1) defines an ensemble of inequivalent physical systems that could, e.g., be realised as a disorder average[1]. In this chapter, I will provide a concise introduction to matrix models, in particular the matrix model dual to JT gravity, compiled mostly from [55, 56] unless otherwise indicated.

In order to specify a matrix model, we need to provide the potential

$$V(H) = \frac{1}{2} H^2 - \sum_{k=3}^{\infty} \frac{t_k}{k} H^k, \tag{5.3}$$

as well as the measure $\mathrm{d}H$. If all parameters $t_k = 0$, eq. (5.1) is simply a Gaussian integral; the resulting ensembles are therefore called Gaussian ensembles [208]. These ensembles are the subject of what is usually called *Random Matrix Theory* (RMT). Their behaviour is in a sense universal: many observables in more

[1] In other cases, it is more sensible to interpret the matrix degree of freedom as a supercharge that squares to the physical Hamiltonian instead [56].

complicated ensembles with some $t_k \neq 0$ agree with the corresponding Gaussian ensemble, cf. chapter 6. I will return to the matrix potential below, but first, I want to discuss the classification of matrices potentially appearing in the matrix integral by their symmetries.

The simplest choice would be to integrate over all Hermitean $N \times N$ matrices and preserve no further antiunitary symmetries, in which case $\mathrm{d}H$ would simply be the flat Lebesgue measure on the space of such matrices. Often, we want to consider observables that depend only on the eigenvalues of H, in which case it is useful to diagonalise the matrix and integrate out the transformations needed to do so. Hermitean $N \times N$ matrices can be diagonalised by a unitary $N \times N$ matrix, so if we consider a diagonal matrix

$$S = \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & \ddots \end{pmatrix}, \tag{5.4}$$

as well as a unitary matrix

$$M = e^{\epsilon X}, \quad X = -X^\dagger, \tag{5.5}$$

where X is purely off-diagonal, then X generates infinitesimal shifts on the space of Hermitean matrices that leave the eigenvalues intact, according to

$$\delta H_{ij} = ([X, H])_{ij} = (s_i - s_j) X_{ij}, \tag{5.6}$$

where summation over repeated indices is not implied. Integrating out the unitary transformations therefore contributes a factor $(s_i - s_j)$ for every independent generator shifting the eigenvectors associated to the eigenvalue pair s_i, s_j. For Hermitean matrices, the eigenvectors are complex, so can be rotated and given a relative phase. The associated two generators, upon integrating out, give an overall measure factor

$$\mathrm{d}H = \prod_{i<j} (s_i - s_j)^2 \prod_k \mathrm{d}s_k. \tag{5.7}$$

The factor $\prod_{i<j}(s_i - s_j)^2$ is often called the *Vandermonde determinant*. The matrix ensemble with the measure (5.7) and the Gaussian potential is called the *Gaussian Unitary Ensemble* (GUE), and ensembles with the measure (5.7) and a more general potential (5.3) are said to belong to the GUE symmetry class. The matrix model dual to orientable JT gravity that I will introduce below is one such model.

The most important antiunitary symmetry we can include, that leads to the classification in terms of Dyson's threefold way [209], is time-reversal symmetry T. Depending on whether this symmetry squares to +1 or −1, the group of transformations leaving the eigenvalues of H invariant reduces to the group of orthogonal ($\mathrm{T}^2 = +1$) or symplectic ($\mathrm{T}^2 = -1$) $N \times N$ matrices. In this thesis, I will restrict to the orthogonal case[2].

[2]It should be noted that we can allow further unitary symmetries that anticommute with the Hamiltonian, namely particle-hole symmetry and chiral symmetry. This imposes

The matrices commuting with T in this case are not all Hermitean matrices, but only the real symmetric ones, and an infinitesimal orthogonal transformation obeys

$$X = -X^t. \tag{5.8}$$

There is now only one generator inducing a rotation of the (real) eigenvectors corresponding to an eigenvalue pair s_i, s_j, so upon integrating out the orthogonal transformations, the measure reads

$$\mathrm{d}H = \prod_{i<j}(s_i - s_j)\prod_k \mathrm{d}s_k. \tag{5.9}$$

The corresponding Gaussian ensemble is called the *Gaussian Orthogonal Ensemble* (GOE), defining the GOE symmetry class as above. Unorientable JT gravity is dual to a matrix model in the GOE symmetry class.

Returning to the matrix potential (5.3), we may interpret terms in the sum over k as interaction terms for the matrix degree of freedom H, with coupling constants t_k, while the quadratic term provides the propagator[3]. For suitable potentials that yield a convergent matrix integral, this allows a Taylor expansion around the Gaussian integral that can be organised in terms of double-line Feynman diagrams, called ribbon graphs [211]. Organising this expansion in terms of the topology of the ribbon graphs gives a *genus expansion* for correlation functions of observables, and it is often possible to determine these correlation functions order by order using a recursive procedure that I will describe below. Formally, this procedure can also be carried out for potentials that do not yield a convergent matrix integral[4], even though in this case, the matrix integral and Taylor expansion do not commute. As we will see below however, the recursion allows for the elimination of the matrix potential in favour of an object called the spectral curve that is also well-defined for these *formal* matrix models [212].

Expressed in terms of the eigenvalues of H, the full matrix integral therefore takes the form

$$\mathcal{Z} = \int \prod_{i<j}(s_i - s_j)^\beta \prod_k e^{-N\frac{\beta}{2}V(s_k)}\mathrm{d}s_k, \tag{5.10}$$

where $\beta = 1(2)$ in the orthogonal (unitary) symmetry class is called the *Dyson index*, cf. also chapter 6. The most important observable that I will consider is the so-called *resolvent*,

$$R(E) = \mathrm{tr}\,\frac{1}{E-H} = \sum_i^N \frac{1}{E - s_i}. \tag{5.11}$$

constraints on the matrices integrated over and leads to the seven Altland-Zirnbauer ensembles [210].

[3]It is the "kinetic term" of the matrix model understood as a 0-dimensional quantum field theory.

[4]Such as the one of JT gravity, which has a divergent potential [55].

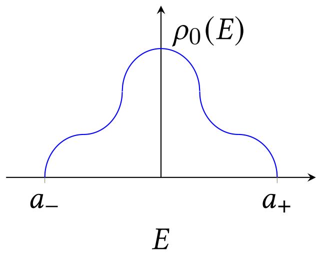


FIGURE 5.1: Example of a possible smooth spectral density of a matrix model. It is compactly supported between the end points $a_\pm$ on the real energy axis and normalisable. Adapted from [55].

It is connected to the spectral density via

$$\rho(E) = \sum_i \delta(E - s_i) = -\frac{1}{2\pi i}(R(E + i\epsilon) - R(E - i\epsilon)), \tag{5.12}$$

i.e., the spectral density is given by the discontinuity of the resolvent when moving across the real axis in the complex energy plane. From eq. (5.12), it is clear that the resolvent readily determines all other single trace observables in the matrix model via

$$\operatorname{tr} O(H) = \sum_i O(s_i) = \int \mathrm{d}E \rho(E) O(E). \tag{5.13}$$

Defining the n-point resolvents

$$R_n(\underbrace{E_1, E_2, \ldots, E_n}_{I}) = R(E_1)R(E_2)\ldots R(E_n), \tag{5.14}$$

we can express their expectation values using the (perturbative) genus expansion alluded to above,

$$\langle R_n(I) \rangle = \sum_{g=0}^{\infty} \frac{R_{g,n}(I)}{N^{2g+n-2}}. \tag{5.15}$$

In particular, we can use this to define the smoothed spectral density via

$$\rho_0(E) = \lim_{N\to\infty} \frac{1}{N} \langle \rho(E) \rangle = -\frac{1}{2\pi i}(R_{0,1}(E + i\epsilon) - R_{0,1}(E - i\epsilon)). \tag{5.16}$$

This density is a smooth, normalised function of compact support[5] between some points $a_\pm$ on the energy axis and might look for example as in fig. 5.1.

The smoothed density also determines a rich geometric structure called the *spectral curve*

$$(x(z), y(z)) \subset \mathbb{C}^2, \tag{5.17}$$

[5]Essentially, the exact spectral density obtains compact support in the limit $N \to \infty$, since the eigenvalues have to become very finely spaced. This allows the density to be approximated by a smooth function [55].

$$x(z) = z^2 = -E, \quad y(x) = R_{0,1}(x) - \frac{V'(x)}{2}. \tag{5.18}$$

Typically, the smoothed density itself has a branch cut along the real axis[6], meaning $\rho_0(E)$ is a multivalued function of the complex energy E. The coordinate z covers the complex energy plane twice and thus resolves this multivaluedness.

The genus-zero contribution to the two-point resolvent can be determined quite simply from the spectral curve (or rather, the genus-zero one-point resolvent) by an appropriate derivative [55], and reads

$$R_{0,2}(E_1, E_2) = \frac{1}{2(E_1 - E_2)^2}\left(\frac{E_1 E_2 - a^2}{\sqrt{\sigma(E_1)}\sqrt{\sigma(E_2)}} - 1\right), \tag{5.19}$$

where we defined $\sigma(x) = (x - a_+)(x - a_-)$ and the smoothed spectral density to be supported in a symmetric interval, $a_\pm = \pm a$. Remarkably, this function depends on the matrix potential only through the end points of the support of the spectral density, meaning that the form (5.19) is universal for all one-cut matrix models[7].

Unfortunately, computing the rest of the genus expansions, or higher-point resolvents, is not quite so easily achieved. It can be done by the method of *loop equations* however; they can be derived from the observation that [56]

$$0 = \int_{-\infty}^{\infty} \mathrm{d}^N s \, \frac{\partial}{\partial s_a}\left[\frac{1}{x - s_a} R_n(I) \prod_{i<j} \left|s_i - s_j\right|^\beta \prod_k e^{-N\frac{\beta}{2}V(s_k)}\right], \tag{5.20}$$

or equivalently, performing the derivative,

$$0 = \left\langle \left[\frac{1}{(x - s_a)^2} + \frac{\beta}{x - s_a}\sum_{j\neq a}\frac{1}{s_a - s_k} - \frac{N\beta}{2}\frac{V'(s_a)}{x - s_a}\right] R_n(I) + \frac{1}{x - s_a}\partial_{s_a} R_n(I)\right\rangle, \tag{5.21}$$

where $\langle\cdot\rangle$ denotes the ensemble average defined by eq. (5.10). Using the relation (5.18), we may eliminate the potential V in favour of the spectral curve. In practice, it is therefore often more convenient to specify a matrix model by giving its spectral curve, rather than the potential. Inserting the spectral curve and the genus expansion for the resolvents, we find following recursive equation:

$$2y(x) R_{g,n}(x, I) + F_g(x, I) \sim 0, \tag{5.22}$$

[6]For instance, the Gaussian ensembles for large N exhibit the Wigner semicircle $\rho_0(E) = 2\sqrt{1 - E^2}/\pi$ as their smoothed spectral density [208, 213]. The square root possesses a branch cut along the real axis.

[7]As long as the spectral density is supported in a compact interval, we may always shift the energy such that the interval is symmetric around 0.

where ~ means up to terms that are analytic near the branch cut and

$$F_g(x,I) = \left(1-\frac{2}{\beta}\right)\partial_x R_{g-\frac{1}{2},n}(x,I) + R_{g-1,n+1}(x,x,I) + \sum_{\text{stable}} R_{h,|J|}(x,J)R_{g-h,n-|J|}(x,I\setminus J)$$
$$+ 2\sum_{k=1}^{n}\left(R_{0,2}(x,x_k) + \frac{1}{\beta}\frac{1}{(x-x_k)^2}\right)R_{g,n-1}(x,I\setminus x_k). \tag{5.23}$$

The sum indexed "stable" is defined in full analogy to eq. (4.47), i.e., it excludes the appearance of $R_{0,1}(x)$ and $R_{0,2}(x,x_k)$.

Equation (5.22) can be solved by contour integration [56],

$$R_{g,n}(x,I)\sqrt{\sigma(x)} = \frac{1}{2\pi i}\int_C \frac{\mathrm{d}x'}{x-x'}F_g(x',I)\frac{\sqrt{\sigma(x')}}{2y(x')}, \tag{5.24}$$

where the contour C encloses the branch cut. Together with the starting points given by the spectral curve and eq. (5.19), this allows the recursive determination of the genus expansions of arbitrary n-point resolvents, requiring however the solving of a fairly involved contour integral. A stark simplification occurs for matrix models in the GUE symmetry class: in this case, $\beta = 2$ and the first term in eq. (5.23) vanishes. This means not only that there are no half-integer genus contributions to the resolvents, but also that the contour integral can be replaced by a sum over residues at the end points $a_\pm$.

Reformulating the loop equations in this way leads to the celebrated *topological recursion* due to Eynard and Orantin [214, 215]. I will report the form of this recursion below, but before doing so, I will specialise to the JT gravity matrix model alluded to above. To do so, I specify the *total* smoothed spectral density $\rho_0^T(E)$, which means simply that I do not divide by N in eq. (5.16), such that $\rho_0^T(E)$ is normalised to N, rather than to unity:

$$\rho_0^T(E) = \frac{e^{S_0}}{(2\pi)^2}\sinh\left(2\pi\sqrt{\frac{a^2-E^2}{2a}}\right), \tag{5.25}$$

which is easily seen to be supported in in $E \in [-a,a]$. The free parameter e^{S_0} should be identified with the topological weighting parameter of JT gravity. The density (5.25) does not agree with the spectral density of JT gravity given by eq. (4.34), but it can be brought to this form by shifting the left endpoint to 0 and pushing the right one to $2a \to \infty$. If we aim to keep the density normalised to N, we also need to let the ratio N/e^{S_0}, and thereby the total number of eigenvalues of the matrix integral, tend to infinity; the *local* eigenvalue density however is controlled by e^{S_0} and remains finite. This leaves a formal matrix model with infinitely many eigenvalues, but that is still nontrivial, i.e., the correlation functions of resolvents are not simply given by products of the genus-zero one-point function. The procedure just described is usually referred to as taking the double scaling limit of a matrix model, and means effectively zooming in on the left spectral edge of the smoothed density. The resulting density is simply given by the spectral density of JT gravity, and one can recover the spectral curve (and therefore the input to the loop equations and the topological recursion) via

$$\rho_0(E) = e^{-S_0}\rho_0^T(E) = \frac{1}{4\pi^2}\sinh\left(2\pi\sqrt{E}\right). \tag{5.26}$$

The result of the recursion now gives a genus expansion not in $1/N$, but in e^{-S_0}:

$$\langle R_n(E_1,\dots,E_n)\rangle = \sum_{g=0}^{\infty}\frac{R_{g,n}(E_1,\dots,E_n)}{e^{(2g+n-2)S_0}}, \tag{5.27}$$

which is exactly the form of a genus expansion in JT gravity.

A similar process may be carried out for the low-energy limit of this matrix model, dual to Witten-Kontsevich topological gravity, also known as the Airy model [61]. The pre-double scaled smoothed spectral density in this case is

$$\rho_0^T(E) = \frac{e^{S_0}}{\pi}\sqrt{\frac{a^2-E^2}{2a}}, \tag{5.28}$$

which gives the spectral density of the Airy model in the double scaling limit,

$$\rho_0^T(E) = \frac{e^{S_0}}{\pi}\sqrt{E}. \tag{5.29}$$

To finally write down the topological recursion for completeness' sake, it is helpful to define functions of the double cover coordinates z_i,

$$W_{g,n}(z_1,\dots,z_n) = (-1)^n 2^n z_1\dots z_n R_{g,n}(-z_1^2,\dots,-z_n^2), \tag{5.30}$$

which obey the topological recursion

$$\begin{aligned} W_{g,n}(z_1,\underbrace{z_2,\dots,z_n}_{I}) =& \operatorname*{Res}_{z\to 0}\left\{\frac{1}{z_1^2-z^2}\frac{4}{y(z)}\left[W_{g-1,n+1}(z,-z,I)\right.\right.\\ &\left.\left.+\sum_{\text{stable}} W_{h,1+|J|}(z,J)W_{g-h,n-|J|}(-z,I\setminus J)\right]\right\}. \end{aligned} \tag{5.31}$$

Importantly, it was shown in [211] that the topological recursion with the JT gravity spectral curve is satisfied by

$$W_{g,n}(z_1,\dots,z_n) = \int_0^\infty b_1\mathrm{d}b_1 e^{-b_1z_1}\dots\int_0^\infty b_n\mathrm{d}b_n e^{-b_nz_n}V_{g,n}(b_1,\dots,b_n), \tag{5.32}$$

where $V_{g,n}(b_1,\dots,b_n)$ are the Weil-Petersson volumes introduced in chapter 4. In other words, Mirzakhani's recursion for the moduli space of bordered Riemann surfaces is simply a special case of the Eynard-Orantin topological recursion for matrix models! This observation is what finally allowed the proof in [55] of the duality between orientable JT gravity and the GUE class matrix model specified by the spectral curve (5.26).

In part II of the thesis, I will report our work [1, 2] on studying JT and topological gravity through their dual matrix models, and in particular, on relations between the Weil-Petersson volumes. These were computed using topological recursion in the orientable case and directly from the loop equations in the unorientable case, and the relations conjectured on the basis of random matrix universality of the matrix models, cf. chapter 6, underlining the power of matrix models as a tool in quantum gravity.

Part II

Connecting Jackiw-Teitelboim Gravity and Random Matrix Theory in the τ-scaling Limit

Chapter 6

Universality in the microcanonical spectral form factor

I review the emergence of the universal form of the microcanonical spectral form factor in the circular unitary and orthogonal ensembles. They exhibit the same functional form as in the analogous Gaussian ensembles, but avoid the added complication of spectral unfolding.

One of the most important observables for the study of chaotic quantum systems is the so-called spectral form factor (SFF). The SFF can be obtained by taking the double Laplace transform of the two-point correlation function of (total[1]) level densities,

$$\langle Z(\beta_1)Z(\beta_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\langle \rho^T(E_1)\rho^T(E_2)\right\rangle e^{-\beta_1 E_1-\beta_2 E_2}\,\mathrm{d}E_1\,\mathrm{d}E_2\,, \tag{6.1}$$

and evaluating at $\beta_1 = \beta_2^* = \beta + it$:

$$\kappa_\beta(t) \coloneqq \langle Z(\beta+it)Z(\beta-it)\rangle\,. \tag{6.2}$$

I will sometimes refer to this quantity as the canonical SFF, so as to distinguish it from the closely related *microcanonical* SFF, obtained by an inverse Laplace transform of (6.2):

$$\kappa_E(t) = \int_{c-i\infty}^{c+i\infty} \kappa_\beta(t)e^{2\beta E}\mathrm{d}\beta, \tag{6.3}$$

with a suitably chosen constant c such that all singularities of the integrand are to the left of the contour of integration. The microcanonical SFF importantly exhibits a *universal* form depending only on the symmetry class and the spectral density of the physical system under study[2].

Given that the microcanonical spectral form factor is a measure of spectral fluctuations, which are expected to be universally described by random matrix theory [105, 216], it is perhaps not surprising that it too follows the behaviour found in the analogous RMT computation using the appropriate Gaussian

[1]In a sense compatible with [55], i.e., the density is normalised to N, not to 1.

[2]It should be noted that naively computing the microcanonical SFF for a single system will not reveal this universal form, since (6.2) is not a self-averaging quantity, i.e., if the microcanonical SFF is $\mathcal{O}(1)$, it will display $\mathcal{O}(1)$ system-specific fluctuations as well. The universal shape only emerges after averaging over multiple systems in some way (e.g. averaging over disorder).

ensemble in a certain "universal" limit. Non-Gaussian matrix models have been shown to reproduce the microcanonical SFF of the corresponding Gaussian model in that same limit [111]. The double-scaled JT gravity matrix model is only a formal matrix model[3] however, requiring a modification of the limiting procedure, which I will discuss in due time.

In this chapter, I will briefly sketch the derivation in [94] of the so-called cluster function, which will depend only on the symmetry class, and which the density-density correlator in eq. (6.1) tends towards in the universal limit. The canonical SFF, on the other hand, is system-dependent, and its computation for both orientable and unorientable JT gravity will be postponed to chapters 7 and 8, where I will discuss the results we obtained in [1, 2].

For the three Gaussian ensembles, taking the universal limit brings with it the difficulty of "unfolding" the energy spectrum, which is to say, to rescale the spectrum such that the mean level spacing is unity. This ensures a clean separation of the scale on which the smoothed spectral density fluctuates locally from the scale where it varies systematically, and allows for the comparison of quantities in different regions of the spectrum, making it "homogeneous". Since unfolding, albeit a standard procedure, only adds a layer of complexity without providing much insight for our purposes, I will forgo it in favour of deriving the universal limit for Dyson's circular ensembles [217, 218]. They consist of random unitary $N \times N$ matrices H invariant under either orthogonal, unitary or symplectic transformations, and are correspondingly named circular orthogonal (COE), unitary (CUE) or symplectic (CSE) ensemble. Since the matrices are unitary, their eigenvalues are unimodular,

$$H \left|\phi_j\right\rangle = e^{-i\phi_j} \left|\phi_j\right\rangle . \tag{6.4}$$

A special property of these ensembles is that the joint eigenvalue distribution,

$$P(\{\phi\}) = \frac{1}{\mathcal{N}_\beta} \prod_{i<j} \left| 2 \sin \frac{\phi_i - \phi_j}{2} \right|^\beta , \tag{6.5}$$

with a normalisation constant $\mathcal{N}_\beta$ and the Dyson index

$$\beta = \begin{cases} 1 & \text{COE} \\ 2 & \text{CUE}\,, \\ 4 & \text{CSE} \end{cases} \tag{6.6}$$

depends only on the distance between two eigenvalues, not their location. Recall that one of the purposes of unfolding the spectrum is to ensure that the smoothed

[3] In the sense discussed in chapter 5, i.e., it has a divergent potential.

spectral density is not sensitive to local energy fluctuations. Considering[4]

$$\left\langle \rho^T(\phi) \right\rangle = \int \mathrm{d}^N\phi\, P(\{\phi\})\delta(\phi - \phi_1) = \frac{1}{2\pi}, \tag{6.7}$$

this is evidently the case for the circular ensembles[5]. Using the circular ensembles therefore avoids the issue of dealing with unfolding explicitly, but the universal form of the SFF turns out to be the same as for the Wigner ensembles [94]. In the following, I will also restrict to the COE and CUE cases as the only ones relevant for the work presented in this thesis. However, fairly similar steps would lead to the CSE result as well, after correctly accounting for Kramers' degeneracy[6].

We begin by Fourier-decomposing the exact spectral density of a given matrix H with eigenvalues $\{\phi_i\}$,

$$\rho^T(\phi) = \sum_{i=1}^{N} \delta(\phi - \phi_i) = \frac{1}{2\pi N} \sum_{i=1}^{N} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi_i)} =: \frac{1}{2\pi N} \sum_{n=-\infty}^{\infty} t_n e^{in\phi}, \tag{6.8}$$

where

$$t_n = \sum_{i=1}^{N} e^{-in\phi_i} = \operatorname{tr} H^n. \tag{6.9}$$

To compute the cluster function, we need the correlation function of spectral densities,

$$\left\langle \rho^T(\phi)\rho^T(\phi') \right\rangle = \left(\frac{1}{2\pi N}\right)^2 \sum_{n,n'} t_n^* t_{n'} e^{in\phi - in'\phi'}. \tag{6.10}$$

The Fourier coefficients t_n are related to the density-density correlator by a pair of inverse Fourier transforms,

$$\left\langle t_n^* t_{n'} \right\rangle = \int_0^{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{d}\phi' \left\langle \rho^T(\phi)\rho^T(\phi') \right\rangle e^{in\phi - in'\phi'}. \tag{6.11}$$

Now since the spectra of the circular ensembles are homogeneous, $\left\langle \rho^T(\phi)\rho^T(\phi') \right\rangle \equiv \left\langle \rho^T\rho^T(\phi - \phi') \right\rangle$. Substituting $E = \frac{\phi+\phi'}{2}, \Omega = \phi - \phi'$ then, we find

$$\left\langle t_n^* t_{n'} \right\rangle = \underbrace{\int_0^{2\pi} \mathrm{d}E\, e^{i(n-n')E}}_{2\pi\delta_{n,n'}} \int_{-2E}^{2E} \mathrm{d}\Omega \left\langle \rho^T\rho^T(\Omega) \right\rangle e^{i(n+n')\Omega/2}, \tag{6.12}$$

[4] For matrix model quantities, I will be a bit careless in distinguishing ensemble averages and spectral averages by notation. In the limit we are working in in JT gravity, they turn out to be the same, and for the calculations presented in this chapter, I will only need one type of average, the ensemble one.

[5] Equation (6.7) is easily checked for $N = 2$, $\mathcal{N}_1 = 16\pi$, $\mathcal{N}_2 = 8\pi^2$, $\mathcal{N}_4 = 24\pi^2$.

[6] For every energy eigenstate of a time-reversal invariant system with half-integer spin, there is a time-reversed state with the same energy [219, 220]. To account for this, factors of 1/2 are often necessary, e.g., in front of traces, to only consider one eigenvalue of a degenerate pair.

with the Ω integration range understood $\mod 2\pi$. Given that the correlation function of Fourier coefficients is diagonal in n, (6.10) simplifies to

$$\left\langle \rho^T(e)\rho^T(e')\right\rangle = \left(\frac{1}{2\pi N}\right)^2 \sum_n \left\langle \left|t_n\right|^2 \right\rangle e^{i\frac{2\pi}{N}n\omega} \tag{6.13}$$

upon trivially rescaling the phases with the mean spacing,

$$e = N\left\langle \rho^T(\phi)\right\rangle \phi = \frac{N}{2\pi}\phi, \quad \omega = \frac{N}{2\pi}\Omega. \tag{6.14}$$

The cluster function we need for (6.1) can be obtained from (6.13) by subtracting the disconnected part $\left\langle \rho^T(e)\right\rangle \left\langle \rho^T(e')\right\rangle = (1/2\pi)^2$, normalising by the squared mean density, and subtracting a contact term $\delta(\omega)$. This yields

$$Y(\omega) = \delta(\omega) - \frac{2}{N^2}\sum_{n=1}^{\infty} \left\langle \left|t_n\right|^2 \right\rangle \cos\left(\frac{2\pi n}{N}\omega\right). \tag{6.15}$$

What remains is to compute $\left\langle \left|t_n\right|^2 \right\rangle$ for the CUE and COE respectively, as well as to evaluate the universal limit. The latter is the limit in which the spectral average and the ensemble average coincide, and the variance of the spectral average vanishes. In the ensembles at hand, this limit is simply $N \to \infty$ with $\tau = n/N$ held constant [94], though in general, with a not-yet-unfolded spectrum, a more subtle scaling is required (such as $N \to \infty$, the energy window to be averaged over $\Delta E \to 0$, while the number of levels contained in this window $\Delta N \to \infty$ [221]).

For the computation of the Fourier coefficients $\left\langle \left|t_n\right|^2 \right\rangle$, I refer the reader to [94] as well, noting only that the economical way to proceed would be to compute the spectral average of the joint eigenvalue distribution (6.5) and extract the Fourier coefficients as the second derivative. Doing so yields for the CUE,

$$\left\langle \left|t_n\right|^2 \right\rangle = \begin{cases} N^2\delta_{n,0} + n & 0 \le n \le N \\ N & N \le n \end{cases}. \tag{6.16}$$

To obtain the final result for the CUE cluster function we still need to work a little bit: first, we plug (6.16) into eq. (6.15),

$$Y_{\text{CUE}}(\omega) = \delta(\omega) - \frac{2}{N^2}\sum_{n=1}^{N} n\cos\left(\frac{2\pi n}{N}\omega\right) - \frac{2}{N^2}\sum_{n=N+1}^{\infty} N\cos\left(\frac{2\pi n}{N}\omega\right). \tag{6.17}$$

We can complete the second sum by adding and subtracting $N\cos\left(\frac{2\pi n}{N}e\right)$ for the first N values of n,

$$Y_{\text{CUE}}(\omega) = \delta(\omega) + \frac{2}{N^2}\sum_{n=1}^{N}(N-n)\cos\left(\frac{2\pi n}{N}\omega\right) - \frac{2}{N^2}\sum_{n=1}^{\infty} N\cos\left(\frac{2\pi n}{N}\omega\right). \tag{6.18}$$

The term $\delta(\omega)$ is to be understood, distributionally, as a Dirac comb of period N,

$$\delta(\omega) = \sum_{n\in\mathbb{Z}} \delta(\omega - nN), \tag{6.19}$$

since in the circular ensembles, contact terms arise when $e = e' \mod N$. Fourier-decomposing the Dirac comb[7],

$$\delta(\omega) = \frac{1}{N}\sum_{n\in\mathbb{Z}} e^{i\frac{2\pi n}{N}\omega} = \frac{1}{N} + \frac{2}{N}\sum_{n=1}^{\infty}\cos\left(\frac{2\pi n}{N}\omega\right), \tag{6.20}$$

we identify precisely the divergent sum in eq. (6.18). In other words, the contact term regularises the divergent part of the density-density correlator. Putting everything together, we arrive at

$$Y_{\mathrm{CUE}}(\omega) = \frac{1}{N} + \frac{2}{N^2}\sum_{n=1}^{N}(N-n)\cos\left(\frac{2\pi n}{N}\omega\right). \tag{6.21}$$

In the universal limit $N \to \infty, \tau = n/N = \text{const.}$, this reads

$$\kappa_{\mathrm{CUE}}(\tau) = \begin{cases} \delta(\tau) + |\tau| & |\tau| \le 1 \\ 1 & |\tau| > 1 \end{cases} \tag{6.22}$$

$$Y_{\mathrm{CUE}}(\omega) = \frac{1-\cos(2\pi\omega)}{2\pi^2\omega^2} = \left(\frac{\sin\pi\omega}{\pi\omega}\right)^2, \tag{6.23}$$

i.e., we find the well-known *sine kernel.* Note that $\frac{1}{N}\left\langle |t_n|^2 \right\rangle \to \kappa_{\mathrm{CUE}}(\tau)$ is precisely the microcanonical SFF, exhibiting the characteristic ramp-plateau structure expected for the unitary symmetry class.

Repeating the same calculation for the COE gives the Fourier coefficients

$$\left\langle |t_n|^2 \right\rangle = \begin{cases} N^2 & n = 0 \\ 2n - n\sum_{m=1}^{n}\frac{1}{m+(N-1)/2} & 1 \le n \le N \\ 2N - n\sum_{m=1}^{N}\frac{1}{m+n-(N+1)/2} & N \le n. \end{cases} \tag{6.24}$$

In the universal limit, the sums can be approximated as integrals, and these integrals carried out, giving

$$\kappa_{\mathrm{COE}}(\tau) = \begin{cases} \delta(\tau) - 2|\tau| - |\tau|\log(2|\tau|+1) & |\tau| \le 1 \\ 2 - |\tau|\log\frac{2|\tau|+1}{2|\tau|-1} & |\tau| > 1 \end{cases} \tag{6.25}$$

$$Y_{\mathrm{COE}}(\omega) = \left(\frac{\sin\pi\omega}{\pi\omega}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{\sin\pi\omega}{\pi\omega}\right)\int_{\omega}^{\infty}\frac{\sin\pi s}{\pi s}\,\mathrm{d}s\,. \tag{6.26}$$

As mentioned above, these results hold for the Gaussian unitary and orthogonal ensembles as well, and in particular, also for any matrix model in the corresponding symmetry class. The only adjustment necessary to use the cluster

[7] A back-of-the-envelope calculation easily finds all Fourier coefficients to be $1/N$.

functions in eq. (6.1) is to perform the spectral unfolding in the way indicated by eq. (6.14). The constant mean level spacing of the circular ensembles has to be replaced by the energy-dependent one of the matrix model, i.e., we need to evaluate the cluster function at

$$Y(\omega) = Y(N \left\langle \rho^T(E) \right\rangle \Omega). \tag{6.27}$$

Chapter 7

The spectral form factor in orientable JT gravity

I report the results we obtained in [1] about cancellations in the Weil-Petersson volumes by way of a computation of the spectral form factor in orientable JT gravity. In section 7.1, the spectral form factor is evaluated in the late-time limit of the dual matrix model, using the (nonperturbative) sine kernel. This computation is assumed to give the same result as the gravitational (or matrix model perturbative) computation in the so-called τ-scaling limit, based on topological recursion, carried out in section 7.2. An order-by-order comparison allowed us to conjecture that nontrivial cancellations must occur in the coefficients of Weil-Petersson volumes; all predicted cancellations that have been checked so far are satisfied.

The computation in chapter 6 gives a universal form for the cluster function $Y(\omega)$ for a wide range of matrix models, and this form can be used to compute the canonical spectral form factor in the late-time limit according to eq. (6.1). For ordinary matrix models, with compactly supported, finite spectral densities, it is clear that this integral converges, and also that the late-time regime is accessible, at least in the middle of the spectrum, where the spectral density can be assumed to vary only slowly with the energy (so that variances of spectral averages can vanish in a meaningful sense).

For the JT gravity matrix model, the situation is seemingly different: the spectral density is obtained by taking such a compactly supported, finite density [55],

$$\rho_0^{\text{not double scaled}}(E) = \frac{e^{S_0}}{4\pi^2}\sinh\left(2\pi\sqrt{\frac{1}{2a}(a^2-E^2)}\right), \quad -a < E < a, \tag{7.1}$$

shifting the energy such that the density is supported from 0 to $2a$, and then taking $a \to \infty$, $N \to \infty$ in such a way that the integral over the density remains "normalised" in a controlled way, i.e.,

$$C(a) = \int_0^{2a} \mathrm{d}E\rho_0^{\text{not double scaled}}(E) \xrightarrow{a\to\infty} \infty, \quad e^{S_0} = \frac{N}{C(a)} = \text{const.} \tag{7.2}$$

This double-scaling limit introduces multiple problems for the computation I want to present in this chapter: for one, the $N \to \infty$ limit, usually part of the universal limit, is already performed. However, as I discussed in chapter 6, the important parameter that has to be adjusted to access the late-time limit is the mean level spacing, which is set by e^{S_0} in the double scaled JT gravity matrix model. This leaves enough control to still take a limit akin to the late-time one, as we argued in [1].

Correlation functions in matrix models usually admit a $1/N$ expansion, and this expansion is replaced by one in e^{-S_0} in the double-scaling limit. Taking the limit of the (inverse) mean level spacing scale $e^{S_0} \to \infty$, in the sense of isolating the most divergent terms, plays the role of the late-time limit in JT gravity. In the SFF, this limit is equivalent to considering times scaling with the Heisenberg time[1] $t = e^{S_0}\tau$ with $\tau \sim \mathcal{O}(1)$. This limit was first discussed in [116] and the SFF evaluated in this limit in [117]. In the literature [222], it is often referred to as the τ-scaling limit. When I mention the "late-time limit" of the matrix model in the following, the τ-scaling limit is what I will be referring to.

A somewhat similar limit has been considered in [223] for the one-point function evaluated at temperatures $\beta \sim \mathcal{O}\left(e^{2/3S_0}\right)$. This scaling is fine-tuned to cancel the suppression originating from the e^{-S_0} expansion

$$\langle Z(\beta)\rangle = \sum_{g=0}^{\infty} e^{-(2g-1)S_0} Z_g(\beta), \tag{7.3}$$

meaning that the expansion naively doesn't make sense any more, and all topologies contribute to the same order. A more careful evaluation shows that this limit selects the top-degree term of the polynomial Weil-Petersson volumes $V_{g,1}$. Due to the (to my knowledge coincidental) agreement between moduli space dimensions and top-degree monomial coefficients of the one-boundary, genus-g, and the $3g+1$-boundary, genus-zero quantities respectively, the sum in (7.3) can be reinterpreted as a grand canonical partition function of cusp triplets on a fixed single-boundary topology[2].

It would be interesting to study whether the τ-scaling limit can be given a similar geometric interpretation, but as we will see later, this limit is a bit more subtle in its effects on the genus expansion of the JT gravity two-point function; in particular, it does not select the top-degree terms of the Weil-Petersson volumes, making it inequivalent to the well-studied Airy limit, unlike in [223].

The other problems introduced by the double-scaling limit originate from the shifting, and the diverging normalisation, of the spectral density, cf. eq. (7.2). This can equivalently be viewed as zooming in on the left spectral edge. Doing

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[1] To be a bit more precise, the Heisenberg time is $T_H = 2\pi\rho_0^T(E)$ [100], with $\rho_0^T(E)$ given in eq. (7.4). As such, the Heisenberg time is energy-dependent, and thus ill-defined in the canonical spectral form factor. It scales with e^{S_0} however, and so in the canonical SFF, we consider times scaling in the same way.

[2] The cusps are simply boundaries of length 0.

so results in a non-normalisable spectral density[3],

$$\rho_0^T(E) = \frac{e^{S_0}}{4\pi^2}\sinh\Big(2\pi\sqrt{E}\Big), \tag{7.4}$$

meaning that it is no longer clear a priori that eq. (6.1) converges at all. We will see by direct computation that, in the orientable JT case, $e^{-S_0}\kappa_\beta(e^{S_0}t)$ *is* a convergent quantity, however. Perhaps more seriously, the derivation of the universal form of the cluster function in chapter 6 relied on the spectral density varying only slowly in the spectral region of interest. This is anything but the case on the spectral edge, where $\rho_0^T(E)$ changes quite rapidly (indeed, as $e^{\sqrt{E}}$). For this reason, we should not replace the density-density correlator in eq. (6.1) with the sine kernel (in the orientable JT gravity case), but use the fully general Christoffel-Darboux kernel instead. This kernel is sensitive to the position in the spectrum, as well as to higher-order corrections in e^{-S_0} to the spectral density, making it the appropriate choice for computing the two-point function at the spectral edge. However, as shown in [222], the Christoffel-Darboux kernel admits an expansion in e^{-S_0} itself, and the leading order term is precisely the sine kernel, evaluated at $\rho_0^T(E)\Omega$. All corrections are suppressed by additional powers of e^{-S_0}. Now since in [1], we were only interested in isolating the leading order in e^{S_0}, it is perfectly sufficient to use the sine kernel in the following, and ignore subleading corrections. This property is essentially a realisation of JT gravity being a "dense" system in the sense of [224], i.e., its spectrum is governed by universal RMT correlations immediately after the spectral edge.

In the rest of this chapter, I will reproduce our calculation in [1] of the SFF in the τ-scaling limit of the Hermitean JT gravity matrix model (i.e., in the GUE symmetry class), the result of which was first reported in [117], and first conjectured as an ansatz in [54], in section 7.1. Then, in section 7.2, I will repeat the calculation starting from the genus expansion of orientable JT gravity, using the general polynomial structures of the Weil-Petersson volumes. Demanding that both calculations agree will allow me to identify a set of relations between the coefficients of the volumes, which we termed constraints in [1], that have to be satisfied for the expected agreement to be possible. These constraints were independently found in [118] using an approach based on intersection numbers, and [1] and [118] were originally submitted to the arXiv in a coordinated manner.

7.1 The SFF of JT gravity in the τ-scaling limit

Our starting point for the computation of the SFF in the late-time RMT limit is eq. (6.2),

$$\kappa_\beta(t) = \int_0^\infty \mathrm{d}E e^{-2\beta E}\int_{-\infty}^\infty \mathrm{d}\Omega\, e^{-it\Omega}\left\langle\rho^T\left(E+\frac{\Omega}{2}\right)\rho^T\left(E-\frac{\Omega}{2}\right)\right\rangle, \tag{7.5}$$

[3]The superscript T denotes the *total* spectral density in the convention of [55], meaning that the leading term in the topological expansion is $O\big(e^{S_0}\big)$. I will sometimes work with a spectral density of $O(1)$, for which I will omit this superscript. The difference is essentially that before double scaling, the total density is normalised to N, while the density without the superscript is normalised to 1.

where I substituted $E = (E_1 + E_2)/2$, $\Omega = E_1 - E_2$. Note that the integration boundaries in the Ω integral have been extended to $\pm\infty$, so this integral does not cover the same area in the (E_1, E_2) plane as (6.2). However, in the GUE symmetry class, the difference between the two integrals can be shown to be subleading in e^{S_0}, so the (more convenient) choice of integration boundaries in eq. (7.5) is justified.

We rescale the time in the manner anticipated in the introduction to this chapter, $t = e^{S_0}\tau$, and make the scaling with e^{S_0} explicit in the level densities, $\rho^T = e^{S_0}\rho$:

$$\kappa_\beta(\tau) = e^{2S_0}\int_0^\infty \mathrm{d}E\, e^{-2\beta E}\int_{-\infty}^{\infty}\mathrm{d}\Omega\, e^{-ie^{S_0}\tau\Omega}\left\langle \rho\left(E+\frac{\Omega}{2}\right)\rho\left(E-\frac{\Omega}{2}\right)\right\rangle. \tag{7.6}$$

If we take e^{S_0} large, and $\tau \sim \mathcal{O}(1)$, the factor $e^{-ie^{S_0}\tau\Omega}$ oscillates rapidly, and therefore localises the integral near small energy differences $\Omega \sim \mathcal{O}(e^{-S_0})$. This is precisely the late-time regime, as anticipated. In this regime then, we are allowed to express the density-density correlator in the SFF using the cluster function as (cf. the deliberations in chapter 6)

$$\left\langle \rho\left(E+\frac{\Omega}{2}\right)\rho\left(E-\frac{\Omega}{2}\right)\right\rangle = \langle\rho(E_1)\rangle\langle\rho(E_2)\rangle + \delta(\omega)\langle\rho(E)\rangle - \frac{\sin^2(\pi\langle\rho(E)\rangle\omega)}{\pi^2\omega^2}, \tag{7.7}$$

where we rescaled $\omega = e^{S_0}\Omega$. Plugging this expression back into eq. (7.6), we find

$$\begin{aligned}\kappa_\beta(\tau) =& \left\langle Z\left(\beta + ie^{S_0}\tau\right)\right\rangle\left\langle Z\left(\beta - ie^{S_0}\tau\right)\right\rangle \\ &+ e^{S_0}\int_{-\infty}^{\infty}\mathrm{d}E e^{-2\beta E}\int_{-\infty}^{\infty}\mathrm{d}\omega\, e^{-i\tau\omega}\left[\delta(\omega)\langle\rho(E)\rangle - \frac{\sin^2(\pi\langle\rho(E)\rangle\omega)}{\pi^2\omega^2}\right].\end{aligned} \tag{7.8}$$

Note that the integral in eq. (7.8) does not depend on e^{S_0} any more, so the entire term is $\mathcal{O}(e^{S_0})$. We can compare this to the first term, using the explicit form of the disk partition function (4.33), since higher topology contributions are of higher order in e^{-S_0}:

$$\begin{aligned}&e^{2S_0}Z_0(\beta+ie^{S_0}\tau)Z_0(\beta-ie^{S_0}\tau)\\ &= e^{2S_0}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{(2\beta+ie^{S_0}\tau)}\right)^{\frac{3}{2}} e^{\frac{\pi^2}{\beta+ie^{S_0}\tau}}\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2(\beta-ie^{S_0}\tau)}\right)^{\frac{3}{2}} e^{\frac{\pi^2}{\beta-ie^{S_0}\tau}}\\ &= e^{-S_0}\frac{1}{16\pi}\left(\frac{1}{e^{-2S_0}\beta^2+\tau^2}\right)^{\frac{3}{2}} e^{\pi^2\frac{2\beta}{\beta^2+e^{2S_0}\tau^2}},\end{aligned} \tag{7.9}$$

which is obviously suppressed compared to the integral in (7.8). Had I defined the SFF using connected correlators to begin with, as is more common, this term would have been absent from the start. What we are left with then, is

$$\kappa_\beta(\tau) = e^{S_0}\int_0^\infty \mathrm{d}E e^{-2\beta E}\left[\langle\rho(E)\rangle - \int_{-\infty}^{\infty}\mathrm{d}\omega e^{-i\tau\omega}\frac{\sin^2(\pi\langle\rho(E)\rangle\omega)}{\pi^2\omega^2}\right]. \tag{7.10}$$

The remaining ω integral can be solved using the Fourier convolution theorem. Define

$$h(\omega) = \frac{\sin(\pi \langle\rho(E)\rangle \omega)}{\pi\omega}, \tag{7.11}$$

whose Fourier transform is the rectangle function,

$$\mathcal{F}[h(\omega)](\tau) := \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\tau\omega} h(\omega) = \frac{1}{2}\big[\operatorname{sign}(\pi \langle\rho(E)\rangle - \tau) + \operatorname{sign}(\pi \langle\rho(E)\rangle + \tau)\big]. \tag{7.12}$$

Then, the Fourier transform of the square of $h(\omega)$ is

$$\mathcal{F}[h^2(\omega)](\tau) = \frac{1}{2\pi}\mathcal{F}[h(\omega)] * \mathcal{F}[h(\omega)](\tau), \tag{7.13}$$

where $*$ denotes the convolution,

$$\mathcal{F}[h(\omega)] * \mathcal{F}[h(\omega)](\tau) = \int_{-\infty}^{\infty} \mathrm{d}s h(s) h(\tau - s). \tag{7.14}$$

The integrand is equal to 1 when

$$s \in (-\pi \langle\rho(E)\rangle, \pi \langle\rho(E)\rangle) \cap (\tau - \pi \langle\rho(E)\rangle, \tau + \pi \langle\rho(E)\rangle), \tag{7.15}$$

so the value of the integral is simply the length of this interval. Clearly, if $\tau > 2\pi \langle\rho(E)\rangle$, the intersection in eq. (7.15) is empty, while otherwise, it has length $2\pi \langle\rho(E)\rangle - \tau$, so

$$\mathcal{F}[h^2(\omega)](\tau) = \max(2\pi \langle\rho(E)\rangle - \tau, 0). \tag{7.16}$$

Plugging this back into eq. (7.10), we find

$$\kappa_\beta(\tau) = e^{S_0} \int_0^\infty \mathrm{d}E e^{-2\beta E} \min\left\{\frac{\tau}{2\pi}, \rho_0(E)\right\}, \tag{7.17}$$

after neglecting subleading terms in

$$\langle\rho(E)\rangle = \rho_0(E) + \mathcal{O}\left(e^{-2S_0}\right). \tag{7.18}$$

The result (7.17) is the Laplace transform of the microcanonical SFF to leading order in e^{S_0}, and has been shown in [117] to be a simple power series in τ,

$$\kappa_\beta(\tau) = e^{S_0} \sum_{n=0}^{\infty} a_n(\beta) \tau^{2n+1}, \tag{7.19}$$

with a finite radius of convergence in τ. Moreover, this expansion saturates as $\tau \to \infty$. This saturation is produced by the $\delta(\omega)$ self-correlation term in the two-point function – a key diagnostic of the discreteness of the energy spectrum of the system[4]. Recovering this discreteness is a good sign for a theory modelling the behaviour of a black hole, since the spectrum of quantum black holes has

[4] Recall chapter 6.

been conjectured to be discrete [225]. Needless to say, the appearance of a late-time plateau in the SFF of a system is not surprising when the system is an (albeit formal) matrix model and the calculation being done explicitly relies on the universal limit of said matrix model. It is also a point of discussion how much the above calculation can be trusted *as a statement about JT gravity*, given that the duality between JT gravity and the matrix model of [55] was initially proven only at the level of the perturbative expansion of the matrix model, respectively the topological expansion of JT gravity. The late-time behaviour of the SFF meanwhile is nonperturbative from the matrix model standpoint. Plenty of work (e.g. [182, 183, 185–187, 226–229]), including by the author and collaborators [1, 2, 112, 113], has since been aimed at establishing the duality between the nonperturbative sectors of the theories as well. One such aspect is the computation of the late-time (i.e., τ-scaled) SFF from the genus expansion of JT gravity, which I will present in the next section.

7.2 The SFF computed from the genus expansion

In orientable JT gravity, the genus expansion of the SFF, and consequently, its τ-scaling limit, are fairly straightforward to compute. Recall that n-point correlation functions are given by

$$\langle Z(\beta_1)\dots Z(\beta_n)\rangle^{\mathrm{c}} = \sum_{g=0}^{\infty} e^{(2-2g-n)S_0} Z_{g,n}(\beta_1,\dots,\beta_n), \tag{7.20}$$

where

$$Z_{g,n}(\beta_1,\dots,\beta_n) = \int_0^\infty b_1\,\mathrm{d}b_1\dots b_n\,\mathrm{d}b_n\, Z_{\mathrm{t}}(\beta_1,b_1)\dots Z_{\mathrm{t}}(\beta_n,b_n) V_{g,n}(b_1,\dots,b_n), \tag{7.21}$$

and

$$Z_{\mathrm{t}}(\beta,b) = \sqrt{\frac{1}{4\pi\beta}}\, e^{-\frac{b^2}{4\beta}} \tag{7.22}$$

is the trumpet partition function. The Weil-Petersson volumes $V_{g,n}$ are symmetric polynomials in the squared geodesic boundary lengths b_i, and in particular,

$$V_{g,2}(b_1,b_2) = \sum_{n,m}^{n+m\leq 3g-1} C^{(g)}_{n,m} b_1^{2n} b_2^{2m}, \tag{7.23}$$

with coefficients $C^{(g)}_{n,m} \in \pi^{6g-2-2(n+m)}\cdot\mathbb{Q}$. This simple form means that the genus expansion of the two-point function is determined by simple double Gaussian integrals, yielding

$$\begin{aligned}\langle Z(\beta_1)Z(\beta_2)\rangle^{\rm c} &= \sum_{g=0}^{\infty} e^{-2gS_0} \int_0^{\infty} b_1\, {\rm d}b_1\, b_2\, {\rm d}b_2\, Z_{\rm t}(\beta_1, b_1) Z_{\rm t}(\beta_2, b_2) \sum_{n,m}^{n+m\leq 3g-1} C^{(g)}_{n,m} b_1^{2n} b_2^{2m} \\ &= \sum_{g=0}^{\infty} e^{-2gS_0} \sum_{n,m}^{n+m\leq 3g-1} C^{(g)}_{n,m} \frac{\sqrt{\beta_1\beta_2}}{\pi} n! m! 4^{n+m} \beta_1^n \beta_2^m,\end{aligned} \tag{7.24}$$

and specialising to the spectral form factor,

$$\begin{aligned}\kappa_\beta(t) &= \sum_{g=0}^{\infty} e^{-2gS_0} \sum_{n,m}^{n+m\leq 3g-1} C^{(g)}_{n,m} \frac{\sqrt{\beta^2+t^2}}{\pi} n! m! 4^{n+m} (\beta+it)^n (\beta-it)^m \\ &=: \sum_{g=0}^{\infty} e^{-2gS_0} \sum_{n,m}^{n+m\leq 3g-1} C^{(g)}_{n,m} \left(\kappa^g_\beta(t)\right)^{n,m},\end{aligned} \tag{7.25}$$

where

$$\left(\kappa^g_\beta(t)\right)^{n,m} = \frac{\sqrt{\beta^2+t^2}}{\pi} n! m! 4^{n+m} \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{m}{j} \binom{n}{k} i^{j+k} (-1)^j \beta^{m+n-k-j} t^{j+k}, \tag{7.26}$$

which is found by applying the binomial theorem. For JT gravity, many of the coefficients $C^{(g)}_{n,m}$ are reported in the literature, and more are easily computable for anyone with enough patience. Evaluating (7.25) in the τ-scaling limit and comparing to the τ expansion of (7.17) is therefore a straightforward exercise, at least for the first few orders.

Doing so explicitly, as, e.g., in [117], one does indeed find agreement, but this is not the approach we followed in [1]. With an eye towards theories where the genus expansion might be more troublesome to compute, or obtaining the late-time result more involved (particularly in matrix models with a different symmetry class, as I will discuss in chapter 8), we asked the question what can be predicted about the genus expansion by assuming that it is compatible with the late-time result in the appropriate limit – in particular, can we draw any conclusions if all we know is that the SFF must be a function that diverges at most like $\mathcal{O}(e^{S_0})$ in the τ-scaling limit? The answer to this question is yes, as the reader may appreciate from the following simple power-counting argument: In the τ-scaling limit, we replace $t = e^{S_0}\tau$ with $\tau \sim \mathcal{O}(1)$, so every power of t in (7.26) contributes a factor of e^{S_0}. Take for instance the highest power t^{n+m+1} (including the square-root factor in front). Now in the second sum in eq. (7.25), the sum $n+m$ is limited by the genus, $n+m \leq 3g-1$. Meanwhile, each contribution at genus g is suppressed by a factor e^{-2gS_0} from the topological expansion. Combining all powers of e^{S_0}, we find, e.g., the most strongly divergent contribution to scale as

$$e^{(3g-2g)S_0} = e^{gS_0}. \tag{7.27}$$

The expectation from the late-time limit, however, is that the SFF should only diverge linearly in with e^{S_0}. The only way for the two calculations to be reconcilable, we conjectured, is that all contributions to orders e^{2S_0} or higher vanish identically. A different resolution with a description, e.g., as nonperturbative effects on the matrix model side seemed unlikely, since known nonperturbative effects in JT gravity do not contribute as simple powers of e^{S_0}, but rather as exponentials of the small parameter, e.g., $e^{-e^{S_0}}$ or $e^{ie^{S_0}}$.

Rather than simply summing up all contributions to a given order in e^{S_0} and checking if they vanish, we can refine the procedure somewhat by realising that terms can exactly cancel one another only if they agree not just in their order in e^{S_0}, but also in β and τ. Comparing with eq. (7.26), we see that this can only happen for terms agreeing in g, $n+m$ and $j+k$. Hence, we need to sum, in general, over partitions (n,m) of the fixed sum $n+m$, and should find that all contributing terms cancel exactly[5]. Since $n+m$ needs to be fixed,

$$\frac{\sqrt{\beta^2+t^2}4^{n+m}}{\pi}$$

is simply a global prefactor that we may ignore. I will denote

$$\sum_{n+m} := \sum_{n=0}^{\alpha}\sum_{m=0}^{\alpha-n}, \tag{7.28}$$

leaving the fixed value of the sum $n+m$ implicit, and by $\sum_{n+m}^{n\geq m}$, I mean the obvious restriction of the above double sum. With this notation, we can write down equations that the genus expansion of the SFF needs to satisfy in order to be compatible with the late-time RMT result. The simplest such equations, stemming from terms in eq. (7.26) with $n+m=j+k$, take the form

$$\sum_{n+m} C_{n,m}^{(g)} n!m!(-1)^m = \sum_{n+m}^{n\geq m} \frac{1}{1+\delta_{n,m}} C_{n,m}^{(g)} n!m!\left[(-1)^n+(-1)^m\right] = 0, \tag{7.29}$$

where $\delta_{n,m}$ is the Kronecker delta. From the second form of (7.29), it is obvious that some of these equations are trivially satisfied. Namely, if $n+m$ is odd, a term with some (n_0,m_0) will exactly cancel the one with (m_0,n_0), given the symmetry of the WP coefficients, $C_{n,m}^{(g)} = C_{m,n}^{(g)}$. However, if $n+m$ is even, this trivial cancellation does not occur, and eq. (7.29) can be understood as a *constraint* on the coefficients of the Weil-Petersson volumes, meaning that even if the set $\{C_{n,m}^{(g)} | n,m : n+m \text{ fixed}\}$ were not known, it still has to obey (7.29).

Indeed, if we decrease $j+k$ in eq. (7.26), but leave $n+m$ fixed, we can find a whole hierarchy of constraints for the same set of coefficients (in the sense that "higher" constraints can be vastly simplified by applying the "lower" constraints). These constraint hierarchies take the form

$$\sum_{n+m} C_{n,m}^{(g)} n!m!(-1)^m(n-m) = 0 \tag{7.30}$$

[5]The sum over partitions (j,k) of fixed $j+k$ merely provides a combinatorial factor.

$$\sum_{n+m} C^{(g)}_{n,m} n!m!(-1)^m (n-m)^2 = 0 \tag{7.31}$$

$$\vdots$$

$$\sum_{n+m} C^{(g)}_{n,m} n!m!(-1)^m (n-m)^l = 0, \tag{7.32}$$

$$\vdots$$

i.e., for each new constraint, we decrease $j+k$ by one to obtain a different combinatorial factor, which can be brought to the simple form $(n-m)^l$ by cleverly subtracting "lower" constraints. The concise form (7.32) can be found using an elementary combinatorial calculation, which I will not report here, referring the reader instead to appendix A of [1]. Since we move from one constraint to the next by essentially multiplying with an odd function of $n-m$, the constraints will be alternately trivial and nontrivial, similar to the discussion after eq. (7.29).

All possible constraints we have checked so far have been satisfied, reported, e.g., up to $g=5$ in [1]. The simplest example occurs at $g=3$, and reads

$$280C^{(3)}_{8,0} - 35C^{(3)}_{7,1} + 10C^{(3)}_{6,2} - 5C^{(3)}_{5,3} + 2C^{(3)}_{4,4} = 0, \tag{7.33}$$

and is indeed satisfied by the coefficients [230]

$$\begin{aligned} C^{(3)}_{8,0} &= \frac{1}{856141332480}, \; C^{(3)}_{7,1} = \frac{1}{21403533312}, \; C^{(3)}_{6,2} = \frac{77}{152882380800} \\ C^{(3)}_{5,3} &= \frac{503}{267544166400}, \; C^{(3)}_{4,4} = \frac{607}{214035333120}. \end{aligned} \tag{7.34}$$

Given that the above calculation can be translated into a perturbative matrix model calculation, while we performed a nonperturbative calculation in the same matrix model in section 7.1, this may not seem too surprising: a matrix model is compatible with its own late-time limit. However, from the standpoint of quantum gravity, in the sense of computing correlation functions using Mirazkhani's recursion, and keeping in mind that the JT gravity matrix model is merely a formal one, with a non-normalisable spectral density, the equality is far from trivial. Indeed, showing the agreement of the analogous computations in the orthogonal symmetry class will prove much more difficult.

Sequentially finding all the constraints in the way described above will at some point yield a term of $\mathcal{O}(e^{S_0})$: this is precisely the one contributing the late-time result[6], marking the end of the constraint hierarchy we can find simply by demanding compatibility between the two calculations. A sensible question to ask then is whether these constraints are sufficient to fully determine the WP volumes. For a given value of $n+m$ to admit nontrivial constraints, it must provide enough powers of e^{S_0} to exceed the suppression of e^{-2gS_0} due to the topological expansion. If $n+m=2g$, the additional square root factor will give in

[6] Interestingly, the late-time result isolates neither the highest, nor lowest order binomials in the SFF, but a subleading power.

total $O(e^{S_0})$, i.e., the term contributing to the late-time result. For $n+m=2g+1$, the resulting constraint is trivially satisfied due to $n+m$ being odd. Therefore,

$$2g+1 < n+m \leq 3g-1, \tag{7.35}$$

since $n+m$ is also bounded from above due to the structure of the Weil-Petersson volumes. We can therefore obtain $g-1$ different hierarchies with $n+m-2g$ independent constraints, half of which are going to be trivially satisfied. The number of nontrivial equations in each hierarchy, which can be used to actually constrain the WP coefficients, is

$$\begin{aligned} &\frac{n+m}{2} - g && : n+m \text{ even} \\ &\frac{n+m-1}{2} - g && : n+m \text{ odd.} \end{aligned} \tag{7.36}$$

On the other hand, the number of independent coefficients within a given hierarchy is simply the number of unordered pairwise partitions of $n+m$, i.e., $\frac{n+m}{2}+1$ if $n+m$ is even and $\frac{n+m+1}{2}$ if $n+m$ is odd. This leaves a total of

$$\#\,\text{coefficients} - \#\text{constraints} = 1+g \tag{7.37}$$

coefficients more than there are constraints. The constraint hierarchies are evidently not enough to completely determine the WP volumes, but they do serve as a powerful consistency check for any theory with a similar structure to JT gravity.

One such theory where this check has been performed is the so-called Virasoro minimal string [231]. This model can be viewed as a deformation of JT gravity with the bulk dilaton action

$$S_{\text{Virasoro}}[g,\phi] = -\frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^2x\,\sqrt{g}(\phi R + W(\phi)), \quad W(\phi) = \frac{\sinh\left(2\pi b^2\phi\right)}{\sin\left(\pi b^2\right)}, \tag{7.38}$$

where b is a continuous parameter related to the central charge of the model when understood as two coupled Liouville CFTs. For $b \to 0$, $W(\phi) \to 2\phi$, and the model reduces to (orientable) JT gravity. Alternatively, the Virasoro minimal string can be described as a double-scaled matrix model in the unitary symmetry class, defined by the spectral density

$$\rho_0(E) = 2\sqrt{2}\frac{\sinh\left(2\pi b\sqrt{E}\right)\sinh\left(2\pi b^{-1}\sqrt{E}\right)}{\sqrt{E}}. \tag{7.39}$$

Correlation functions can be computed from topological recursion, or alternatively, a deformed Mirzakhani recursion, yielding polynomials $V^{(b)}_{g,n}(P_1,\ldots,P_n)$, termed *quantum volumes* by the authors in [231][7]. The Virasoro trumpet partition function, importantly, is still only a Gaussian in P, so the same computation as we did in JT gravity can be repeated for this model, and consequently, the

[7] The Virasoro momenta P_i are analogues of the geodesic boundary lengths in ordinary JT gravity.

quantum volumes must satisfy the exact same constraints as the ones we found in [1]. The cancellations of coefficients of the quantum volumes were checked[8] by the authors of [231] up to $g = 10$.

As a final note, the constraints we found in [1] were discovered simultaneously in [118] using an approach that allows to link these constraints to certain properties of intersection numbers, potentially providing an unexplored perspective on generalisations of intersection theory.

[8]Notably, this also serves as a check for the cancellations in JT gravity, since the Weil-Petersson volumes are simply the $b = 0$ part of the quantum volumes, up to a rescaling.

Chapter 8

The spectral form factor in unorientable topological gravity

I report the results we obtained in [2] on computing the spectral form factor in the unorientable Airy model. We showed that the (nonperturbative) computation in the late-time τ-scaling limit of the matrix model, conducted in section 8.1, and the analogous perturbative computation in the Airy model, based on loop equations and carried out in section 8.2, agree, proving quantum chaoticity of the unorientable Airy model. We showed the equivalence with a bootstrapping prescription using what I call exchange functions in section 8.3, where I aimed to give a cleaned-up presentation of the procedure that is hopefully a bit more transparent than the one in [2]. Finally, in section 8.4, I report the novel cancellations in the moduli space volumes of the Airy model that we found in [2], and that are in the spirit of, but qualitatively different to the ones discussed in chapter 7.

The calculations I presented in the preceding chapter are based on the one hand on the availability of a matrix model whose spectral form factor can be computed in the τ-scaling limit, and on the other, on the genus expansion of orientable JT gravity, which is reasonably easy to determine using a well-defined recursive method, either the topological or Mirzakhani recursion. In particular, recalling eq. (7.19), the τ-scaled SFF can be written as e^{S_0} times a simple power series in τ with coefficients independent of e^{S_0}. The polynomial structure of the Weil-Petersson volumes then allowed the analogous computation on the JT side via the simple evaluation of Gaussian integrals, giving a power series in τ as well. A priori, using only the general polynomial structure of the WP volumes and not their explicit coefficients, this series could diverge more strongly in the τ-scaling limit than the SFF computed from RMT. This discrepancy led us to conjecture the existence of certain linear combinations of the coefficients that have to sum to 0, serving as a powerful check on the validity of similar proposed gravity/matrix model dual pairs in the unitary symmetry class.

A reasonable question then is whether our calculation can be generalised to other settings, in particular ones where the gravity side is not as well understood as in orientable JT gravity. On the matrix model side, the most straightforward generalisation is the one to other symmetry classes, i.e. either the other Wigner-Dyson classes, defined by the matrix integral measure

$$\mathrm{d}H = \prod_{i<j} \left|\lambda_i - \lambda_j\right|^\beta \prod_k e^{-N\,\mathrm{tr}(V(\lambda_k))}\,\mathrm{d}\lambda_k\,, \tag{8.1}$$

with the Dyson index $\beta = 1, 2, 4$ corresponding to GOE, GUE or GSE symmetry, or even the more general Altland-Zirnbauer classes [210], defined by the measure

$$\mathrm{d}H = \prod_{i<j} \left|\lambda_i^2 - \lambda_j^2\right|^\beta \prod_k |\lambda_k|^\alpha e^{-N\,\mathrm{tr}(V(\lambda_k))}\,\mathrm{d}\lambda_k\,, \tag{8.2}$$

with an additional index $\alpha = 0, 1, 2$. All of these cases are interesting as they have been conjectured to be dual to dilaton gravities as well [56], but concrete calculations are much harder in some of these settings. The generalisation I will concern myself with in this thesis is the one to the GOE symmetry class, which corresponds to allowing unorientable manifolds, but no additional spin or pin structures[1] in the JT gravity path integral.

We made a first foray into the GOE case in [2], where we repeated our calculation in [1] in the simplified setting of unorientable topological (or Airy) gravity. This theory, as the reader will recall from section 4.1.4, is the low-energy limit of JT gravity and has a monomial density of states,

$$\rho_0^{\mathrm{Airy}}(E) = \frac{e^{S_0}}{\pi}\sqrt{E}, \tag{8.3}$$

which greatly simplifies the RMT computation. Even so, it will be considerably more difficult to perform than in the GUE case, since the GOE cluster function (6.26) is not simply the sine kernel.

Another complication arising in the unorientable JT context is the fact that the moduli space volumes of manifolds with crosscaps are not well-defined. Although they can be computed using a modified Mirzakhani-type recursion relation [119] as discussed in section 4.1.3, the result is in general divergent and needs to be regularised to recover finite correlation functions (see also [117]). Airy gravity extracts precisely the leading order of the regularisation-independent piece of these volumes. In the orientable setting, this would correspond to the highest powers of the polynomial Weil-Petersson volumes, justifying the interpretation of Airy gravity as the "thin-strip limit" of JT gravity. As we will see below however, the structure of the volumes in the unorientable case is more complicated and in general not analytic, which will lead to drastic changes in the outcome of the computation on the gravity side.

In the rest of this chapter, I will relay the contents of [2], where we managed to perform both the matrix model and the gravity calculation, and show (nontrivially) that they agree. I will proceed as follows: in section 8.1, I will report the calculation of the SFF from the τ-scaling limit of the Airy matrix model, analogous to section 7.1. Unlike in the unitary case, we will see that the SFF is not a simple power series, but contains terms logarithmic in τ as well. In section 8.2,

[1] "Pin is to $O(N)$ as spin is to $SO(N)$" [232].

I will briefly sketch the computation of the unorientable Airy equivalent of the WP volumes using the loop equations of the corresponding matrix model. This calculation, entirely due to Torsten Weber, may not seem gravitational at first, but it can be seen in analogy to the option of using the topological recursion on the matrix model side as opposed to the Mirzakhani recursion on the gravity side in the orientable case. The result of this computation will be different from the one in section 8.1, and indeed will contain terms that, even after τ-scaling, come with additional powers of $\sqrt{t}$ or with terms logarithmic in t. Using a procedure devised by Jarod Tall and refined by our Master's student Marco Lents [233], I will show in section 8.3 that at each considered order in τ, the difference between these terms and the (different) logarithmic terms in the RMT result are terms arising from either the convergent series expansion, or the asymptotic expansion, of certain combinations of generalised hypergeometric functions. The corrections coming from the asymptotic expansion are suppressed by (fractional) powers of e^{S_0} compared to the leading behaviour of the SFF, while the higher orders in τ coming from the convergent expansion are irrelevant when comparing the coefficients of a fixed power of τ in both calculations. In this way, one can show order by order, using different combinations of hypergeometric functions, that the two computations agree in the τ-scaling limit in an asymptotic sense.

The reader may be sceptical of the validity of this procedure, as were we when we first implemented it, and we make no claim to it being the unique way of getting the RMT and gravity calculations to agree. However, conjecturing agreement in the sense we proposed in [2], it is possible to once again derive constraints on the Airy WP volumes, i.e., relations between the coefficients that must be fulfilled for the two computations to be able to agree. I will describe how to derive these constraints in section 8.4, referring the reader to [2] for explicit expressions for both constraints and coefficients. As in the unitary case, all such constraints that we have checked have been verified, providing strong evidence not only for the validity of our procedure, but also for the quantum chaoticity of Airy gravity in the sense of reproducing the late-time RMT result in the τ-scaled SFF.

I want to note at this point that my involvement in this part of the project was much more limited than in what I reported in chapter 7, which is why I will gloss over some details that are included in [2]. However, as the computation is considerably more involved, I will still deal with our calculation at the length I believe necessary to appreciate it to the same extent as the orientable/GUE one.

The approach of [2] was generalised by Tall and Weber to full JT gravity up to genus 1 in [113] using a regularisation derived from the $(2, 2p + 1)$ minimal string, which limits to JT gravity as $p \to \infty$. The authors of [113] were furthermore able to show that the SFF itself is regularisation-independent. A further generalisation to different symmetry classes was performed by Weber and Lents [112]: solving the loop equations and thereby determining the analogues of Weil-Petersson volumes turns out to be just as hard for arbitrary Dyson index β as it is for any $\beta \neq 2$. This allows for the formal generalisation of the SFF computation to arbitrary β-ensembles, which were also considered in the context of matrix models in [234].

8.1 The SFF of the orthogonal Airy model in the τ-scaling limit

I will begin, as in the unitary case, by presenting the derivation of the SFF in the τ-scaling limit. To this end, recall eq. (7.10) for an expression of the canonical SFF in terms of the cluster function $Y(\omega)$. We will directly replace $\langle\rho(E)\rangle \approx \rho_0(E)$ and write the ω integral in a more convenient form exploiting the symmetry of the integrand:

$$e^{-S_0}\kappa_\beta(\tau) = \int_0^\infty \mathrm{d}E e^{-2\beta E}\rho_0(E) - 2\int_0^\infty \mathrm{d}E e^{-2\beta E}\rho_0(E)\int_0^\infty \mathrm{d}\omega \cos\left(\frac{\tau}{\rho_0(E)}\omega\right)Y(\omega). \tag{8.4}$$

The integral over ω is solved, e.g., in [208], and yields

$$2\int_0^\infty \mathrm{d}\omega \cos\left(\frac{\tau}{\rho_0(E)}\omega\right)Y(\omega) = \begin{cases} 1 - \frac{\tau}{\pi\rho_0(E)} + \frac{\tau}{2\pi\rho_0(E)}\log\left(1 + \frac{\tau}{\pi\rho_0(E)}\right) & : \frac{\tau}{2\pi} \leq \rho_0(E), \\ -1 + \frac{\tau}{2\pi\rho_0(E)}\log\left(\frac{\frac{\tau}{\pi}+\rho_0(E)}{\frac{\tau}{\pi}-\rho_0(E)}\right) & : \frac{\tau}{2\pi} \geq \rho_0(E). \end{cases} \tag{8.5}$$

Plugging this into the SFF and using the explicit form of the Airy spectral density, we are left with the following integrals to compute:

$$\begin{aligned} \kappa_\beta(\tau) =& e^{S_0}\int_0^{\tau^2} \mathrm{d}E\, e^{-2\beta E}\left[\frac{\sqrt{E}}{\pi} - \frac{\tau}{2\pi}\log\left(\frac{\tau}{\pi} + \frac{\sqrt{E}}{2\pi}\right) + \frac{\tau}{2\pi}\log\left(\frac{\tau}{\pi} - \frac{\sqrt{E}}{2\pi}\right)\right] \\ &+ e^{S_0}\int_{\tau^2}^\infty \mathrm{d}E\, e^{-2\beta E}\left[\frac{\tau}{\pi} - \frac{\tau}{2\pi}\log\left(1 + \frac{2\tau}{\sqrt{E}}\right)\right]. \end{aligned} \tag{8.6}$$

The first term in either integral can be identified as (twice) the GUE spectral form factor for the Airy model,

$$\kappa_\beta(\tau) = 2\kappa_\beta^{\mathrm{GUE}}(\tau) \underbrace{- e^{S_0}\frac{\tau}{2\pi}\left[\int_0^\infty \mathrm{d}E\, e^{-2\beta E}\log\left(1 + \frac{2\tau}{\sqrt{E}}\right) - \int_0^{\tau^2} \mathrm{d}E\, e^{-2\beta E}\log\left(-1 + \frac{2\tau}{\sqrt{E}}\right)\right]}_{=:\chi_\beta(\tau)}, \tag{8.7}$$

$$2\kappa_\beta^{\mathrm{GUE}}(\tau) = 2e^{S_0}\int_0^\infty \mathrm{d}E e^{-2\beta E}\min\left\{\frac{\tau}{2\pi}, \rho_0(E)\right\} = \frac{e^{S_0}}{2\,(2\beta)^{3/2}\sqrt{\pi}}\operatorname{erf}\left(\sqrt{2\beta\tau^2}\right), \tag{8.8}$$

where $\kappa_\beta^{\mathrm{GUE}}$ was first computed in [117] and $\operatorname{erf}(z)$ is the error function. We evaluated $\chi_\beta(\tau)$ in appendix E of [2]. The computation is quite lengthy and technical, so I will not reproduce it in full here, focusing instead on a few key steps. Adding and subtracting a term

$$\int_0^\infty \mathrm{d}E \log\left(\sqrt{E}\right)$$

and integrating by parts, we can rewrite the integrals (omitting the $-e^{S_0}\tau/2\pi$ prefactor) as

$$\frac{1}{2\beta}\left[\int_0^\infty \mathrm{d}x \left(e^{-2\beta(x-\tau)^2}\frac{1}{(x+\tau)} - e^{-2\beta(x+\tau)^2}\frac{1}{(x+\tau)}\right)\right], \tag{8.9}$$

and make use of a translation operator $e^{-2\tau\frac{\mathrm{d}}{\mathrm{d}x}}$ to obtain an expression in terms of Hermite polynomials,

$$\frac{1}{2\beta}\sum_{n=1}^{\infty}\frac{(2\sqrt{2\beta}\tau)^n}{n!}\int_{\sqrt{2\beta}\tau}^{\infty} dx\frac{e^{-x^2}\overbrace{(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-(x)^2}}^{H_n(x)}}{x}. \tag{8.10}$$

Plugging in the explicit form of the Hermite polynomials allows for a solution in terms of incomplete Γ functions, yielding

$$\chi_\beta(\tau) = -e^{S_0}\frac{\tau}{2\pi}\left[\frac{1}{4\beta}\sum_{n=1}^{\infty}\sum_{l=0}^{\mathrm{floor}(n/2)}\frac{(-1)^l(2)^{n-2l}}{(l)!(n-2l)!}(2\sqrt{2\beta}\tau)^n\Gamma\left(\frac{n}{2}-l,2\beta\tau^2\right)\right]. \tag{8.11}$$

This expression can be massaged a bit to find

$$\begin{aligned}\chi_\beta(\tau) =& -e^{S_0}\frac{1}{4\beta}\frac{\tau}{2\pi}\Bigg[\Gamma(0,2\beta\tau^2)(e^{-8\beta\tau^2}-1)\\ &+e^{-8\beta\tau^2}\sum_{n=1}^{\infty}\left(\frac{(2)^n(2\sqrt{2\beta}\tau)^n}{(n)!}\Gamma\Big(\frac{n}{2}\Big)-\sum_{m=1}^{n}\frac{(-1)^{n+m}(2)^{2m}(\sqrt{2\beta}\tau)^{2n}}{(m)!(n-m)!(n-\frac{m}{2})}\right)\Bigg],\end{aligned} \tag{8.12}$$

and the first sum over n can be performed exactly,

$$\sum_{n=1}^{\infty}\frac{2^{\frac{5n}{2}}\Gamma\left(\frac{n}{2}\right)\left(\sqrt{\beta}\tau\right)^n}{n!} = 16\beta\tau^2\,{}_2F_2\left(1,1;\frac{3}{2},2;8\beta\tau^2\right)+\pi\,\mathrm{erfi}\left(2\sqrt{2}\sqrt{\beta}\tau\right), \tag{8.13}$$

where $\mathrm{erfi}(z) = -i\,\mathrm{erf}(iz)$ is the imaginary error function and ${}_2F_2(a_1,a_2;b_1,b_2;z)$ a generalised hypergeometric function. Together with the GUE result of [117], we find the exact canonical SFF of the orthogonal Airy matrix model in the τ-scaling limit as

$$\begin{aligned}e^{-S_0}\kappa_\beta(\tau) =& \frac{1}{2\,(2\beta)^{3/2}\sqrt{\pi}}\,\mathrm{erf}\left(\sqrt{2\beta\tau^2}\right) - \frac{\tau e^{-8\beta\tau^2}}{8\pi\beta}\Bigg[\Gamma(0,2\beta\tau^2)(1-e^{8\beta\tau^2})\\ &+16\beta\tau^2\,{}_2F_2\left(1,1;\frac{3}{2},2;8\beta\tau^2\right)+\pi\,\mathrm{erfi}\left(\sqrt{8\beta\tau^2}\right)\\ &-\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{(-1)^{n+m}(2)^{2m}}{(m)!(n-m)!(n-\frac{m}{2})}\right)\left(2\beta\tau^2\right)^n\Bigg].\end{aligned} \tag{8.14}$$

While it is remarkable that an exact solution for the SFF can be found at all, it is not terribly helpful in this form if our aim is to compare it to the gravitational calculation, which will involve the genus expansion, i.e., an order-by-order expansion[2] in τ. To facilitate this comparison, I report the first few orders of the small τ expansion of (8.14):

$$\begin{aligned} e^{-S_0}\kappa_\beta(\tau) =& \frac{\tau}{2\pi\beta} - \frac{\tau^2}{\sqrt{2\pi\beta}} - \frac{\gamma + \log\left(2\beta\tau^2\right) + \frac{1}{3}}{\pi}\tau^3 + \frac{8\sqrt{2\pi\beta}}{3\pi}\tau^4 \\ &+ \frac{\beta\left(4\gamma + 4\log\left(2\beta\tau^2\right) - \frac{7}{15}\right)}{\pi}\tau^5 - \frac{64\,(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}\tau^6 + O\left(\tau^7\right), \end{aligned} \tag{8.15}$$

with γ denoting the Euler-Mascheroni constant. For even powers of τ, we can observe that the SFF behaves as expected from the GUE case, i.e., there is simply a (β-dependent) coefficient that is not too complicated. For odd orders however, besides the appearance of the Euler-Mascheroni constant, we immediately note the presence of logarithmic terms $\log\left(2\beta\tau^2\right)$. These do not bode well for a comparison with the gravitational result: if logarithmic terms appear in the genus expansion, they would produce factors of S_0 in the τ-scaling limit, while the only known mechanism that suppresses certain terms in the expansion is the topological weighting by factors e^{-2gS_0}. It seems impossible from the outset that such terms could appear in the τ-scaling limit then. As we will see in the next section, the genus expansion will produce logarithmic terms (albeit different ones) – and getting the result to match with eq. (8.15) will require a fairly unintuitive method using generalised hypergeometric functions, that I will detail in section 8.3.

8.2 The SFF of unorientable topological gravity computed from the genus expansion

As I already alluded to in the introduction to this chapter, the gravitational calculation we performed in [2] is based on the perturbative loop equations of the matrix model dual to topological gravity. The loop equations, as the reader will recall from chapter 5, are a set of recursive equations to determine the resolvents of the matrix model,

$$R(E_1,\ldots,E_n) = \operatorname{tr}\frac{1}{E_1 - H}\ldots\operatorname{tr}\frac{1}{E_n - H}, \tag{8.16}$$

or more precisely, for their genus expansion,

$$\langle R(E_1,\ldots,E_n)\rangle^{\mathrm{c}} = \sum_{g=0,\frac{1}{2},1,\ldots} \frac{R_g(E_1,\ldots,E_n)}{e^{(2g+n-2)S_0}}, \tag{8.17}$$

by solving certain contour integrals. The way this is done is described in [2], and will not be the focus of the presentation here. The genus-g contribution to

[2]With some caveats that I will attend to in the next section.

the two-level resolvent has the general form

$$R_g(z_1, z_2) = \frac{P_g(z_1, z_2)}{(z_1 z_2)^{6g+1}(z_1 + z_2)^{2g+2}}, \tag{8.18}$$

with some symmetric polynomial $P_g(z_1, z_2)$ of degree $8g$ in the double cover coordinates z_i, $E_i = -z_i^2$. These coordinates live on the Riemann sphere and are typically introduced to make the contour integrals appearing in the course of solving the loop equations easier to perform. The resolvents are connected to the Airy WP volumes via a double inverse Laplace transform,

$$V_{g,2}(b_1, b_2) = \mathcal{L}^{-1}\left[\frac{4z_1 z_2}{b_1 b_2} R_g(z_1, z_2), (b_1, b_2)\right] \tag{8.19}$$

$$= -\int_{c+i\mathbb{R}} \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{\pi^2} \frac{z_1 z_2}{b_1 b_2} R_g\left(z_1, z_2\right) e^{b_1 z_1} e^{b_2 z_2}. \tag{8.20}$$

In the orientable case, it turned out to be very useful that the Weil-Petersson volumes (and consequently, their large-b_i limit) had the very simple general form of symmetric polynomials. This allowed for the computation of generic contributions to the canonical SFF without knowing the coefficients of the volumes exactly, and consequently, for the formulation of constraints that the coefficients needed to satisfy. We would like to have something similar in the unorientable case, but it turns out that here, the Airy WP volumes are slightly more complicated in general. At fixed lengths, they will still be polynomials in b_1, b_2, as can be argued from eq. (8.18), and the full volumes are still symmetric in the lengths. However, in the inverse Laplace transform, Heaviside functions and δ functions could appear[3]. The form we conjectured in [2] therefore is

$$V_{g,2}(b_1, b_2) = V_g^{>}(b_1; b_2)\Theta(b_1 - b_2) + V_g^{>}(b_2; b_1)\Theta(b_2 - b_1), \tag{8.21}$$

where

$$V_g^{>}(b_1; b_2) = \sum_{\alpha_1, \alpha_2 \in \mathbb{N}_0}^{\alpha_1 + \alpha_2 = 6g-2} D_{\alpha_1, \alpha_2} b_1^{\alpha_1} b_2^{\alpha_2}, \tag{8.22}$$

and the coefficients[4] D_{α_1,α_2} are in general *not* symmetric in α_1, α_2. To make this easier to keep in mind, I chose a different symbol than for the coefficients in the orientable WP volumes[5] and denote the non-symmetric dependence of $V_g^{>}(b_1; b_2)$ on its arguments with a semicolon. The entire Airy volume however is still symmetric, as easily verified from eq. (8.21).

This general form can also be argued for using the formalism of Kontsevich diagrams (see appendix C of [2]) and has been confirmed by all volumes we have computed so far. The simplest example, illustrating the difference to the

[3]Note for example that $\mathcal{L}^{-1}[(z_1 + z_2)^{-n}, (b_1, b_2)] = \delta(b_1 - b_2) b_1^n/(n-1)!$.

[4]I suppress the dependence of the coefficients on the genus for notational simplicity.

[5]A further difference is that the indices of D_{α_1,α_2} are the actual powers of b_1, b_2 in the corresponding monomial, while the indices of $C_{m,n}$ in the orientable case were only half the power of b_1, b_2.

orientable case, is

$$V^{>}_{\frac{1}{2}}(b_1;b_2) = b_1, \tag{8.23}$$

$$V_{\frac{1}{2},2}(b_1,b_2) = b_1\Theta(b_1-b_2) + b_2\Theta(b_2-b_1) = \max(b_1,b_2), \tag{8.24}$$

in agreement with [117]. The Airy WP volumes evidently exhibit non-analytic behaviour along the diagonal $b_1 = b_2$, which will produce non-polynomial behaviour in the canonical SFF. This behaviour is needed to recover the logarithmic terms we found in the RMT computation, but it will also introduce some apparent problems for the convergence of the SFF in the τ-scaling limit.

For the moment however, we can proceed with the gravitational computation much like in the orientable case, using these volumes as input. The critical reader may ask why we call this calculation gravitational, given that we use a recursive method on the matrix model and then simply compute a Laplace transform. As shown in [119] however, the loop equations for the orthogonal JT matrix model map exactly onto the Mirzakhani-like recursion for unorientable manifolds, and by taking the low-energy/thin-strip (Airy) limit, the same exact mapping holds between topological gravity and the Airy model as well. Hence, due to this exact duality, the (in our view simpler) loop equations approach can be viewed to be properly gravitational as well.

The computation we have to perform to arrive at the canonical SFF, as in the orientable case, is to integrate the (Airy) WP volumes against two trumpet partition functions,

$$Z_{g,2}(\beta_1,\beta_2) = \int_0^\infty \mathrm{d}b_1\, b_1 \int_0^\infty \mathrm{d}b_2\, b_2 Z_{\mathrm{t}}(\beta_1,b_1)Z_{\mathrm{t}}(\beta_2,b_2)V_{g,2}(b_1,b_2), \tag{8.25}$$

to obtain the genus-g contribution to the two-point function, and then to set $\beta_1 = \beta + it = \beta_2^*$. All terms appearing at genus g will be suppressed by the topological factor e^{-2gS_0}, so only terms of at least power t^{2g} can survive the τ-scaling limit. Neglecting lower-order terms, the first few contributions to the spectral form factor, which will suffice to discuss any qualitative differences to the orientable case, read

$$\kappa^0_\beta(t) = \frac{\sqrt{t^2+\beta^2}}{2\pi\beta}, \tag{8.26}$$

$$\kappa^{\frac{1}{2}}_\beta(t) = -\frac{t^2+\beta^2}{\sqrt{2\pi\beta}}, \tag{8.27}$$

$$\kappa^1_\beta(t) = \left[\frac{-10}{3} + i\left(\arctan\left(\sqrt{\frac{\beta-it}{\beta+it}}\right) - \arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right)\right)\right]\frac{t^3}{\pi}, \tag{8.28}$$

$$\kappa^{\frac{3}{2}}_\beta(t) = \frac{8\sqrt{2\pi\beta}}{3\pi}t^4 - \frac{it^4}{3\sqrt{\pi}}\left(\sqrt{\beta-it} - \sqrt{\beta+it}\right), \tag{8.29}$$

$$\kappa^2_\beta(t) = \frac{\beta t^5}{\pi}\left[\frac{163}{15} - 2\pi i + 8i\arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right)\right], \tag{8.30}$$

$$\kappa_\beta^{\frac{5}{2}}(t) = -\frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}t^6 + \frac{t^6\sqrt{t^2+\beta^2}}{30\sqrt{\pi}}\left(\sqrt{\beta-it}+\sqrt{\beta+it}\right)$$
$$+\frac{21it^5\sqrt{t^2+\beta^2}\beta}{5\sqrt{\pi}}\left(\sqrt{\beta-it}-\sqrt{\beta+it}\right), \tag{8.31}$$

$$\kappa_\beta^3(t) = \frac{-2t^8\sqrt{\beta^2+t^2}}{45\pi} - \frac{1658\beta^2t^6\sqrt{\beta^2+t^2}}{63\pi} + \frac{16i\beta^2t^7\left(\pi-4\arctan\left(\frac{\sqrt{\beta+it}}{\sqrt{\beta-it}}\right)\right)}{3\pi}, \tag{8.32}$$

$$\kappa_\beta^{\frac{7}{2}}(t) = \frac{it^7\sqrt{\beta^2+t^2}}{210\sqrt{\pi}}\left(\left(756\beta^2-t^2\right)\left(\sqrt{\beta+it}-\sqrt{\beta-it}\right)+31i\beta t\left(\sqrt{\beta-it}+\sqrt{\beta+it}\right)\right)$$
$$+\frac{2048}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}t^8. \tag{8.33}$$

There are many striking features in these results that were not present in the orientable case; perhaps most crucially, the orientable SFF consisted of simple polynomials in t of order no greater than $2g+1$, such that $e^{-S_0}\kappa_\beta^{\mathrm{GUE}}(\tau)$ was finite. This is evidently no longer the case: the $g = 3/2$ contribution contains a term proportional to

$$e^{-4S_0}it^4\left(\sqrt{\beta-it}-\sqrt{\beta+it}\right) = e^{-4S_0}\sqrt{2}t^{9/2}+O\left(\frac{\beta}{t}\right) \to e^{S_0/2}\sqrt{2}\tau^{9/2},$$

meaning that $e^{-S_0}\kappa_\beta(\tau)$ seems to be divergent. Furthermore, we have arctan terms that were not present in the orientable case, although these terms at least do not manifestly spoil the finiteness in the τ-scaling limit. Finally, the SFF appears to no longer be manifestly real starting from $g = 2$. With a view toward comparison with eq. (8.15), we also do not see any logarithmic terms in the above results. As I will discuss presently however, at least some of these problems can be solved by a more careful treatment of the arctan terms as functions on the complex plane $\mathbb{C}$.

I will elaborate on the general structure of terms appearing in the SFF in section 8.4 in the course of deriving constraints analogous to the ones in [1], but for now I will simply state without proof the form of a generic arctan contribution to the SFF,

$$I_{\alpha,\gamma}^{\arctan}(\beta,t) := \frac{2^{6g-1}\Gamma(1+\frac{\alpha}{2})\Gamma(1+\frac{\gamma}{2})}{\pi^2}(\beta+it)^{\frac{\alpha+1}{2}}(\beta-it)^{\frac{\gamma+1}{2}}\arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right), \tag{8.34}$$

originating from a term $b_1^\alpha b_2^\gamma\Theta(b_1-b_2)$ in the corresponding WP volume, where α,γ are odd integers. Terms originating from $\Theta(b_2-b_1)$ can be treated fully analogously by swapping $\beta+it \leftrightarrow \beta-it$. Using $z_\pm = \sqrt{\frac{\beta\pm it}{\beta\mp it}}$, as well as the identity

$$\arctan z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right), \tag{8.35}$$

we can rewrite the arctan terms as

$$\arctan(z_\pm) = \frac{1}{2i}\log\left(i\frac{\mp t + \sqrt{t^2+\beta^2}}{\beta}\right) = \frac{1}{2i}\left[\log\left(\frac{\mp t + \sqrt{t^2+\beta^2}}{\beta}\right) + i\frac{\pi}{2}\right]. \quad (8.36)$$

As a first benefit to this rewriting, note the appearance of constants $i\pi/2$, which exactly cancel the imaginary constants, e.g., in eq. (8.30), in all cases we have computed so far, making the SFF manifestly real again. Note, however, that while the arctan terms were manifestly finite in the τ-scaling limit, the logarithms naively scale with S_0, so we seem to be introducing a different type of divergence in the result. This does not appear to be a promising candidate to compensate the divergences of extra fractional powers of t in, e.g., eq. (8.29). However, as we will see in the next section, this is precisely what happens.

To access the τ-scaling limit, we can easily expand the logarithms in eq. (8.36) around $\beta/t = 0$ and find

$$\log\left(\frac{\mp t + \sqrt{t^2+\beta^2}}{\beta}\right) = \pm\log\left(\frac{\beta}{2t}\right) \pm \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}2^n\left(\sum_{k=2}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{\beta}{t}\right)^{2k-2}\right)^n, \quad (8.37)$$

i.e., the logarithmic terms contribute as $\pm\log(\beta/2t) + O(e^{-2S_0})$ in the τ-scaling limit. Plugging this result back into the genus expansion of the late-time SFF, we find the first few contributions to be[6]

$$\kappa_\beta^0(t) \to \frac{t}{2\pi\beta}, \quad (8.38)$$

$$\kappa_\beta^{\frac{1}{2}}(t) \to -\frac{t^2}{\sqrt{2\pi\beta}}, \quad (8.39)$$

$$\kappa_\beta^1(t) \to -\left[\frac{10}{3} + \log\left(\frac{\beta}{2t}\right)\right]\frac{t^3}{\pi}, \quad (8.40)$$

$$\kappa_\beta^{\frac{3}{2}}(t) \to -\frac{\sqrt{2\pi}t^{\frac{9}{2}}}{3\pi} + \frac{8\sqrt{2\pi\beta}}{3\pi}t^4, \quad (8.41)$$

$$\kappa_\beta^2(t) \to \frac{\beta t^5}{\pi}\left[\frac{163}{15} + 4\log\left(\frac{\beta}{2t}\right)\right], \quad (8.42)$$

$$\kappa_\beta^{\frac{5}{2}}(t) \to -\frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}t^6 + \frac{t^7\sqrt{2t}}{15\sqrt{\pi}} + \frac{17\beta t^6\sqrt{t}}{6\sqrt{2\pi}}, \quad (8.43)$$

$$\kappa_\beta^3(t) \to \frac{-2t^9}{45\pi} - \frac{8297\beta^2t^7}{315\pi} - \frac{32\beta^2t^7\log\left(\frac{\beta}{2t}\right)}{3\pi}, \quad (8.44)$$

$$\kappa_\beta^{\frac{7}{2}}(t) \to \frac{t^{10}\sqrt{t}}{105\sqrt{2\pi}} - \frac{3\beta t^9\sqrt{t}}{10\sqrt{2\pi}} - \frac{881\beta^2t^8\sqrt{t}}{120\sqrt{2\pi}} + \frac{2048}{105}\sqrt{\frac{2}{\pi}}\beta^{5/2}t^8, \quad (8.45)$$

where $\to$ means "contributes to the SFF as". Comparing these results with eq. (8.15), we observe that the linear order in t (respectively τ), as well as all

[6]Since the contributions do not cleanly factorise into e^{S_0} times a polynomial of $O(e^{0\cdot S_0})$, I will report only those terms surviving the τ-scaling limit, but leave the dependence on t instead of τ.

the even orders agree between both approaches, but the constants in front of odd powers disagree; similarly, the logarithmic terms are not quite the same (the coefficients of $\log(\beta)$ agree, but not even the sign of the $\log(t)$ coefficient is correct), and the fractional powers in the gravitational calculation do not appear on the RMT side. At this point, one might conclude that the two approaches simply do not give the same result. This would be unexpected, meaning that a perturbative (formal) matrix model calculation is not compatible with the late-time limit of that same matrix model. However, the remarkable agreement between the two approaches for some of the terms hints at there being a way to reconcile the two computations after all. I will elaborate how we achieved this in the following section.

8.3 Nontrivial equivalence via asymptotic series

Let's begin by collecting eqs. (8.38) to (8.45) to write down the τ-scaled SFF. I will do so in a way that in each term, the amount of factors of t replaced by $e^{S_0}\tau$ will be exactly what is needed to cancel the suppression due to the topological expansion; the remaining factors of t, as well as occurrences inside logarithms, will be kept for now. In this section, I will also distinguish the SFF computed from the Airy Weil-Petersson volumes and the one computed from RMT by the superscripts WP and RMT, respectively. The τ-scaled gravitational SFF then reads, after slightly reordering and rewriting,

$$\begin{aligned} e^{-S_0}\kappa_\beta^{\mathrm{WP}}(\tau) &= \frac{\tau}{2\pi\beta} - \frac{\tau^2}{\sqrt{2\pi\beta}} + \frac{\tau^3}{\pi}\left[\frac{-10}{3} + \log\left(\frac{2t}{\beta}\right)\right] - \frac{\sqrt{2\pi\beta}\tau^4}{3\pi}\left(\frac{t}{\beta}\right)^{1/2} \\ &+ \frac{8\sqrt{2\pi\beta}}{3\pi}\tau^4 + \frac{\beta\tau^5}{\pi}\left[\frac{163}{15} - 4\log\left(\frac{2t}{\beta}\right)\right] - \frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}\tau^6 + \frac{17\tau^6\beta\sqrt{2\pi\beta}}{12\pi}\left(\frac{t}{\beta}\right)^{1/2} \\ &+ \frac{\tau^6\,(2\pi\beta)^{3/2}}{60\pi^2}\left(\frac{t}{\beta}\right)^{3/2} + \frac{\beta^2\tau^7}{\pi}\left[-\frac{8297}{315} + \frac{32}{3}\log\left(\frac{2t}{\beta}\right)\right] - \frac{2\tau^7\beta^2}{45\pi}\left(\frac{t}{\beta}\right)^2 \\ &+ \frac{\tau^8(2\pi\beta)^{\frac{5}{2}}}{840}\left(\frac{t}{\beta}\right)^{\frac{5}{2}} - \frac{3\tau^8\beta(2\pi\beta)^{\frac{3}{2}}}{40\pi^2}\left(\frac{t}{\beta}\right)^{\frac{3}{2}} - \frac{881\tau^8\beta^2\sqrt{2\pi\beta}}{240}\sqrt{\frac{t}{\beta}} + \frac{512(2\pi\beta)^{\frac{5}{2}}}{105\pi^3}\tau^8 \\ &+ \mathcal{O}\left(\tau^9\right) \end{aligned} \tag{8.46}$$

We will reorganise this expansion a bit, using the following strategy: the linear and even orders in τ already agree with $\kappa_\beta^{\mathrm{RMT}}(\tau)$, so we will leave those terms alone. For the remainder, we have terms like $-\log(t)$ in $\kappa_\beta^{\mathrm{WP}}$, and $+\log(\tau^2)$ in $\kappa_\beta^{\mathrm{RMT}}(\tau)$ (compared to the sign of $\log(\beta)$). If we somehow manage to add $\log(t\tau^2)$ to the gravitational result, these terms could be reconciled, so it stands to reason that we should isolate the functional dependence of $\kappa_\beta^{\mathrm{WP}}(\tau)$ on $t\tau^2$. Hence, we will group every (fractional) power of t with a factor of τ^2, and organise the expansion in terms of the remaining powers of τ. Equivalently, we could group terms according to their dependence on powers β (but not $\log(\beta)$), as suggested by the RMT result (8.15). This yields

$$\begin{aligned}
e^{-S_0}\kappa_\beta^{\mathrm{WP}}(\tau) &= \frac{\tau}{2\pi\beta} - \frac{\tau^2}{\sqrt{2\pi\beta}} + \frac{8\sqrt{2\pi\beta}}{3\pi}\tau^4 - \frac{64(2\pi\beta)^{\frac{3}{2}}}{15\pi^2}\tau^6 + \frac{512(2\pi\beta)^{5/2}}{105\pi^3}\tau^8 + \dots \\
&+ \frac{\tau^3}{\pi}\left[\frac{-10}{3} + \log\left(\frac{2t}{\beta}\right) - \frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2} + \frac{\sqrt{2\pi}}{30}\left(t\tau^2\right)^{3/2} - \frac{2\left(t\tau^2\right)^2}{45} + \frac{\sqrt{2\pi}}{210}(t\tau^2)^{5/2} + \dots\right] \\
&+ \frac{\tau^5\beta}{\pi}\left[\frac{163}{15} - 4\log\left(\frac{2t}{\beta}\right) + \frac{17\sqrt{2\pi}}{12}\left(t\tau^2\right)^{1/2} - \frac{3\sqrt{2\pi}}{20}\left(t\tau^2\right)^{3/2} + \dots\right] \\
&+ \frac{\tau^7\beta^2}{\pi}\left[-\frac{8297}{315} + \frac{32}{3}\log\left(\frac{2t}{\beta}\right) - \frac{881\sqrt{2\pi}}{240}(t\tau^2)^{1/2} + \dots\right] \\
&+ \dots .
\end{aligned} \tag{8.47}$$

The dots stand for contributions from higher genus, which are also higher order in $t\tau^2$. I will neglect them from now on, and it will become clear later that they do not interfere with the statements we made in [2]. Focussing on the coefficient of τ^3, i.e., the second line above, the task is to show that it asymptotically agrees with the RMT result for large t, but as we argued in [2], the method I discuss below can be reiterated for higher orders of τ. This method consists, essentially, of cleverly guessing a function whose convergent Taylor expansion around $\tau = 0$ starts with a term we want to get rid of in $\kappa_\beta^{\mathrm{WP}}(\tau)$, and which simultaneously has an asymptotic expansion for large t that produces the terms we expect to find in $\kappa_\beta^{\mathrm{RMT}}(\tau)$. If we can find such an "exchange" function, we can argue that the two computations agree up to higher order corrections in $\tau \sim \mathcal{O}(1)$ and $1/t \ll 1$. Repeating this procedure for the next coefficient would allow us to bootstrap the agreement between the two calculations to higher and higher orders in τ. In particular, we want to correctly fix the constants, and to add a term $\log\left(4t\tau^2\right)$ in order to recover the logarithms in the RMT result.

This procedure may seem fairly arbitrary, but there are still important consistency checks to be performed. In order for the method to work, we have to be able to eliminate the $t\tau^2$-dependence in the τ^3 coefficient. We are in a sense "pushing the problem to higher orders", but we know that the even orders in τ already agree. This agreement should not be spoiled by the bootstrapping procedure. With all these constraints in mind, one exchange function that does the job is

$$\begin{aligned}
&\frac{\tau^3\left(t\tau^2\right)^2}{45\pi}\left(6\,{}_2F_2\left(2,2;3,\frac{7}{2};-t\tau^2\right) - 4\,{}_1F_1\left(\frac{3}{2};\frac{7}{2};\frac{-t\tau^2}{2}\right)\right) \\
&= \frac{2\tau^7 t^2}{45\pi} + \underbrace{\frac{\tau^3\left(t\tau^2\right)^2}{45\pi}\sum_{k=1}^{\infty} a_k (t\tau^2)^k}_{\mathcal{O}(\tau^9)},
\end{aligned} \tag{8.48}$$

where a_k are known, but for the claim we want to make, irrelevant coefficients. They can, if necessary, be read off from the definition of the generalised hypergeometric functions ${}_2F_2(a_1, a_2; b_1, b_2; z)$ and ${}_1F_1(a; b; z)$, respectively. The leading $O(\tau^7)$ term in eq. (8.48) is precisely the fifth term in the τ^3 coefficient of (8.47). Now the asymptotic expansion of the exchange function reads [235]

$$\begin{aligned}
&\frac{\tau^3\left(t\tau^2\right)^2}{45\pi}\left(6\,{}_2F_2\left(2,2;3,\frac{7}{2};-t\tau^2\right)-4\,{}_1F_1\left(\frac{3}{2};\frac{7}{2};\frac{-t\tau^2}{2}\right)\right)\\
&\overset{t\to\infty}{=}\frac{\tau^3}{\pi}\left(-\frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2}+\log\left(4t\tau^2\right)+\gamma-3\right)+O\left(t^{-1/2}\right),
\end{aligned} \tag{8.49}$$

where $O\left(t^{-1/2}\right)$ consists of terms that do not survive the τ-scaling limit. After recognising that (8.48) equals (8.49) up to $O(\tau^9)$ corrections in the τ-scaling limit, we can plug the asymptotic expansion into eq. (8.47), leaving the τ^3 coefficient

$$\begin{aligned}
&\left[\frac{-10}{3}+\log\left(\frac{2t}{\beta}\right)-\frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2}+\frac{\sqrt{2\pi}}{30}\left(t\tau^2\right)^{3/2}-\frac{2\left(t\tau^2\right)^2}{45}+\frac{\sqrt{2\pi}}{210}(\tau^2 t)^{5/2}\right]\\
&\overset{t\to\infty}{=}\left[\frac{-10}{3}+\log\left(\frac{2t}{\beta}\right)-\frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2}+\frac{\sqrt{2\pi}}{3}\left(t\tau^2\right)^{1/2}-\log\left(4t\tau^2\right)-\gamma+3\right.\\
&\qquad\left.+\frac{\sqrt{2\pi}}{30}\left(t\tau^2\right)^{3/2}+\frac{\sqrt{2\pi}}{210}(\tau^2 t)^{5/2}+O\left(\tau^9\right)\right]\\
&=-\left[\log\left(2\beta\tau^2\right)+\gamma+\frac{1}{3}\right]+\underbrace{\left(\frac{\sqrt{2\pi}}{30}\left(t\tau^2\right)^{3/2}+\frac{\sqrt{2\pi}}{210}(\tau^2 t)^{5/2}+O\left(\tau^9\right)\right)}_{O(\tau^3)}.
\end{aligned} \tag{8.50}$$

The first bracket is now simply the correct coefficient of τ^3/π in the RMT result (8.15), while the second bracket only contributes to the SFF at $O(\tau^6)$. This procedure proves the equivalence, in an asymptotic sense, of the gravitational and the non-perturbative matrix model calculation to order τ^3. A nice side effect of our choice (8.49) is that the t-dependent τ^4-term is cancelled as well, pushing the agreement between the two results one order higher. In summary, we have shown that

$$\kappa_\beta^{\mathrm{WP}}(\tau)\xrightarrow{t\to\infty}\kappa_\beta^{\mathrm{RMT}}(\tau)+O\left(\tau^5\right). \tag{8.51}$$

In a similar manner, the agreement could be bootstrapped to higher and higher orders in τ by finding suitable exchange functions analogous to (8.49), i.e., functions whose lowest-order truncated Taylor series appears in the gravitational SFF, but which also have an asymptotic expansion capable of swapping the logarithms and constants of the gravitational result for the ones on the RMT side. We argued in [2] that such a choice should always be possible, but it may seem to be (and is) quite arbitrary, and begs the question whether one could not get any given pair of series to agree in this sense. However, I want to re-emphasise the remarkable agreement between the two calculations even before we apply our bootstrapping procedure: *all* even orders in τ that we have computed agree (neglecting t-dependent terms), as well as all coefficients of $\log(\beta)$. Furthermore, after transforming the gravitational result into the RMT one using the bootstrapping procedure, we are left with corrections that do not depend on β. Such corrections do not appear at higher orders in eq. (8.15), i.e., they must be treated on their own and shown to vanish asymptotically, in a spirit similar to the original bootstrapping step. Meanwhile, the τ^5 coefficient would have to be fixed independently by yet another choice of exchange function.

In principle, if the entire genus expansion of the SFF could be computed and summed up, we would expect the result to agree with the RMT calculation without any need for bootstrapping, but this is obviously not possible to carry out in practice. We assume that the reason for the stark difference between the GOE/unorientable case and the GUE/orientable one is that the limit of $t \to \infty$ and the summation of the topological series simply happen to commute in the orientable case, but not in the unorientable one. Why this happens is not clear to us however.

To support this conjecture, as well as the calculation reported in this section, it is possible to use the general structure of the genus expansion, and the assumption that our conjecture holds, to predict cancellations between certain terms in the Airy WP volumes, in a manner similar to the constraint hierarchies in the orientable setting. These cancellations are the subject of the next section.

8.4 Cancellations in the unorientable Airy Weil-Petersson volumes

In order to infer relations between the Airy Weil-Petersson coefficients from the late-time spectral form factor, we first need to compute a generic contribution to the SFF arising from the Airy WP volumes. We can read off from eqs. (8.21) and (8.22) that such contributions are either going to arise from integrating $b_1^\alpha b_2^\gamma$, fairly similarly to the orientable case[7], or from terms containing Heaviside functions, i.e., $b_1^\alpha b_2^\gamma \Theta(b_1 - b_2)$ (and analogous terms with $\Theta(b_2 - b_1)$). The more interesting type of cancellations, not present in the orientable case, arise from this latter type of term in the volumes. A generic contribution of this type to the two-point function reads

$$\begin{aligned} I(\alpha,\gamma) &:= \int_0^\infty \mathrm{d}b_1\, b_1 \int_0^\infty \mathrm{d}b_2\, b_2 \frac{e^{-\frac{b_1^2}{4\beta_1}}}{\sqrt{4\pi\beta_1}} \frac{e^{-\frac{b_2^2}{4\beta_2}}}{\sqrt{4\pi\beta_2}} b_1^\alpha b_2^\gamma \Theta(b_1 - b_2) \\ &= \frac{1}{4\pi\sqrt{\beta_1\beta_2}} \int_0^\infty \mathrm{d}b_1\, e^{-\frac{b_1^2}{4\beta_2}} b_1^{\alpha+1} \int_0^{b_1} \mathrm{d}b_2\, e^{-\frac{b_2^2}{4\beta_2}} b_2^{\gamma+1}. \end{aligned} \tag{8.52}$$

[7]Such terms will arise after adding contributions from either Θ function in eq. (8.21) when the corresponding coefficients in eq. (8.22) are symmetric.

The integrals can be solved using elementary methods. Abbreviating $a = 1/4\beta_1$ and $b = 1/4\beta_2$, we find

$$\gamma \text{ even}, \alpha \text{ even} \quad I(\alpha,\gamma) = \frac{1}{16\pi\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha}{2}}\frac{1}{b(a+b)}, \tag{8.53}$$

$$\gamma \text{ even}, \alpha \text{ odd} \quad I(\alpha,\gamma) = \frac{1}{16\sqrt{\pi}\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha+1}{2}}\frac{1}{a}\left[\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a+b}}\right], \tag{8.54}$$

$$\gamma \text{ odd}, \alpha \text{ even} \quad I(\alpha,\gamma) = \frac{1}{16\sqrt{\pi}\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma+1}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha}{2}}\frac{1}{b\sqrt{a+b}}, \tag{8.55}$$

$$\gamma \text{ odd}, \alpha \text{ odd} \quad I(\alpha,\gamma) = \frac{1}{8\pi\sqrt{\beta_1\beta_2}}\left(-\frac{\partial}{\partial a}\right)^{\frac{\gamma+1}{2}}\left(-\frac{\partial}{\partial b}\right)^{\frac{\alpha+1}{2}}\frac{\arctan\left(\sqrt{\frac{a}{b}}\right)}{\sqrt{ab}}. \tag{8.56}$$

Much like terms originating from integrals over simple binomials $b_1^\alpha b_2^\gamma$, most of these contributions yield simple polynomials in β_1, β_2 and could be analysed in a manner completely analogous to the orientable case, with two caveats: first, as we have seen in section 8.2, the SFF computed from the Airy volumes explicitly contains terms diverging in the τ-scaling limit that are later transformed away using the exchange function approach introduced in section 8.3. Hence, we can not conclude from the mere scaling with e^{S_0} that certain combinations of coefficients must vanish. However, the more careful analysis of coefficients we performed in [2] shows that by tracking the powers of π in both computations, their compatibility still demands the cancellation of higher-order terms in e^{S_0}, as in the orientable case. Second, since the coefficients in the unorientable Airy WP volumes are not symmetric under index exchange, *none* of the constraints we can predict are trivially satisfied, unlike in the orientable case.

Other than this, the purely polynomial contributions to the SFF do not bear any differences to the orientable case[8]. Here, I will focus on constraints arising from logarithmic terms. Logarithms can only come from eq. (8.56), i.e., from contributions to the SFF where we integrate distinct, odd powers of b_1 and b_2. In particular, acting with a derivative on the arctan also yields a simple polynomial term in the SFF, so we only need to be concerned with the piece of eq. (8.56) where the derivatives do not act on the arctan. The constraints arising from the same type of term *with* derivatives acting on the arctan are again analogous to the orientable case. We computed the first two such constraints in [2] and verified that they are satisfied.

The interesting piece for qualitatively different constraints, as mentioned above, is eq. (8.34). At fixed genus g, we can add all these terms up and find

[8]The interested reader is referred to [112], where explicit examples are reported.

the following contribution to the SFF:

$$e^{-2gS_0}\arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right)\frac{t^{3g}}{\pi}\sum_{l=0}^{3g}G(g,l)\beta^l t^{-l}$$
$$\underbrace{\sum_{\substack{\alpha+\gamma=6g-2\\ \alpha,\gamma\text{ odd}}} D_{\alpha,\gamma}\frac{\Gamma(1+\frac{\alpha}{2})\Gamma(1+\frac{\gamma}{2})}{\pi}(-1)^{\frac{\gamma+1}{2}}\sum_{n+m=l}\binom{\frac{\alpha+1}{2}}{n}\binom{\frac{\gamma+1}{2}}{m}(-1)^m}_{=:K_g(l)}, \tag{8.57}$$

where $G(g,l)$ is some non-vanishing function irrelevant for the cancellations. As I discussed in section 8.2, the τ-scaling limit extracts the logarithmic part of the arctan, and indeed,

$$t^{\text{even power}}\arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right)\to\frac{\pi}{2}, \tag{8.58}$$

$$t^{\text{odd power}}\arctan\left(\sqrt{\frac{\beta+it}{\beta-it}}\right)\to\log\left(\frac{\beta}{2t}\right). \tag{8.59}$$

Hence, after adding eq. (8.57) and the converse expression obtained from swapping $b_1\leftrightarrow b_2$, the τ-scaled SFF contains

$$e^{(g-l)S_0}K_g(l)\frac{\tau^{3g-l}}{\pi}\begin{cases}\log\left(\frac{\beta}{2t}\right) & 3g-l\text{ odd},\\ \frac{\pi}{2} & 3g-l\text{ even}.\end{cases} \tag{8.60}$$

We see then that for $g-l>1$, there are logarithmic terms in the SFF that diverge in the τ-scaling limit. Such terms can not be transformed away by the method discussed in section 8.3, since we only exchange $\log(\beta/2t)$ for $\log(2\beta\tau^2)$. Hence, we can once again predict that

$$K_g(l)=0\quad\text{for }3g-l\text{ odd},\ g-l>1 \tag{8.61}$$

purely from the finiteness of the logarithmic piece of $e^{-S_0}\kappa_\beta^{\text{RMT}}(\tau)$. The only such constraint we are actually able to check with the volumes computed up to $g=3$ is

$$K_3(0)\propto 715D_{1,15}-143D_{3,13}+55D_{5,11}-35D_{7,9}+35D_{9,7}-55D_{11,5}+143D_{13,3}-715D_{15,1}, \tag{8.62}$$

whose vanishing can be verified by explicitly plugging in the $g = 3$ Airy WP volume,

$$\begin{aligned} V_3^>(b_1;b_2) \propto & 887887 b_1^{16} + 35515480 b_2^2 b_1^{14} + 18350080 b_2^3 b_1^{13} + 348079732 b_2^4 b_1^{12} \\ & + 143130624 b_2^5 b_1^{11} + 1242409168 b_2^6 b_1^{10} + 449839104 b_2^7 b_1^9 + \\ & + 1735393660 b_2^8 b_1^8 + 599785472 b_2^9 b_1^7 + 1049704656 b_2^{10} b_1^6 \\ & + 286261248 b_2^{11} b_1^5 + 269432436 b_2^{12} b_1^4 + 55050240 b_2^{13} b_1^3 \\ & + 21347416 b_2^{14} b_1^2 + 3670016 b_2^{15} b_1 + 447567 b_2^{16}. \end{aligned} \tag{8.63}$$

Part III

Connecting Jackiw-Teitelboim Gravity and Motion on High-Dimensional Hyperbolic Manifolds

Chapter 9

JT gravity as a sum over periodic orbits

I report the results we obtained in [3], where we showed that the system I introduced in chapter 3 exhibits correlation functions that are identical to their analogues in JT gravity. In particular, I show that the disk partition function emerges from the Weyl term within the Selberg trace formula in the limit of configuration space dimension $f \to \infty$ in section 9.1.2; I also review the computation of the disk partition function in BF theory to argue that the Weyl term in the STF plays the same role in section 9.1.1. I further show our computation of the double trumpet partition function in Berry's diagonal approximation in section 9.2.1, as well as our extension to arbitrary times in section 9.2.2. Finally, I show ongoing research on the obtaining of a genus expansion for the two-point function in the time-reversal invariant $f = 3$ version of our system, corresponding to the unorientable Airy model in section 9.3.

After exploring JT gravity through random matrix theory, the most generic, universal means available, in part II, I turn to a more specialised tool in this part: my aim is to convince the reader that a system of the type introduced in chapter 3 is perturbatively "dual" to JT gravity in a sense that is more conventional than the matrix model duality discussed above, but not quite as strong as the notion of duality usually employed when talking about such prototypical examples as $AdS_5 \times S^5/\mathcal{N} = 4$ SYM duality [47, 49, 50]. This strong notion of duality requires the identification of two systems' Hilbert spaces, as well as the spectra of their Hamiltonians. As we discussed in [3], this was not a realistic goal for us for the following reasons: JT gravity, being dual to a matrix model [55], does not have a fixed energy spectrum with discrete levels, as one would expect from a single realisation of a quantum system, but is rather described by a spectrum averaged over the matrix ensemble, with only a smoothed spectral density remaining.

A concrete spectrum could be obtained by picking a specific member of the matrix ensemble, or a specific α state in the nomenclature of [76], by inserting a large number of fixed-energy branes in the path integral [78, 226]. This insertion would fix a large number of eigenvalues in the matrix integral, but lead in turn to a fairly nontrivial gravitational interpretation [78]. One could then go on trying to find a concrete perturbative dual to this member of the JT gravity ensemble, but this does not seem to promise any particular insight on quantum gravity in general, in addition to being fairly difficult to achieve.

A related notion of finding a dual of JT gravity is the universe field theory approach of [182], who showed that the third-quantised Kodaira-Spencer (KS) theory of [236] is dual to JT gravity by proving that the Schwinger-Dyson equations of that model imply topological recursion. The KS theory can be understood as a string field theory whose target space is the JT spectral curve and whose world sheets are the JT universes, with splitting and joining of "baby universes" in the terminology of [76], i.e., disconnected boundary segments, being described as cubic vertices in the string field theory. Nonperturbative effects, such as the fixed energy branes often used in JT gravity and interpreted in terms of nonperturbative completions of the dual matrix model (e.g. [55, 226]), can be given a rigorous interpretation as complex fermionic fields within KS theory, and indeed, the matrix model and KS theory are connected through an open/closed string duality.

It was shown in [79] that generically in quantum chaotic systems, but particularly in the gravitational context, systems that experience so-called *causal symmetry* between retarded and advanced propagators upon introduction of suitable auxiliary degrees of freedom can be described by the Efetov-Wegner σ model of quantum chaos [237, 238] as an effective field theory in the late-time limit after integrating out UV degrees of freedom, breaking the causal symmetry. The symmetry however allows access to nonperturbative information that is universal in the RMT sense, such as the plateau in the spectral form factor, even if only the perturbative expansion of the system is known. Subsequently, the authors of [79, 182] showed in [183] that KS theory can be understood as precisely the theory that realises this programme for JT gravity and maps onto the σ model in the late-time limit, thus proving the quantum chaoticity of JT gravity (and indeed, KS theory and the JT gravity matrix model) at the nonperturbative level.

I will return to the question of how strong our notion of duality is by comparison, and the possibility of identifying Hilbert spaces once I have introduced the model in some more specificity, but for now, I want to take the spectral information that is present in standard JT gravity as guidance.

One such piece of information is the characteristic density of states [102] $\sim \sinh\left(2\pi\sqrt{E}\right)$, a hallmark of systems with a black hole dual [120]. As we saw in chapter 4 it arises from the Schwarzian action describing the leading-order (disk) contribution to the 1-point spectral function. Among the known approaches to studying JT gravity, the matrix ensemble of [55] and the extension of [182, 183] that realises JT correlators within a larger universe field theory both require the Schwarzian density as an external input, while the SYK [110, 239, 240] and related models [54, 115, 241] display the Schwarzian action, but only match the gravity side in the universal regime [242]. The origin of the Schwarzian action, or at least the associated density of states, on a microscopic level independent of the parameter regime is therefore an important gap in our understanding of JT gravity in the context of holography.

Another important question is the inclusion of connected geometries with disconnected boundary segments. In particular, wormhole geometries correspond to non-factorising correlation functions in the dual theory. As discussed in section 4.2, the emergence of non-vanishing connected 2-point correlators can be explained by an ensemble interpretation of the theory dual to JT gravity [76–79], but in higher dimensions this interpretation is not available and the so-called factorisation problem remains [74, 80–82].

This problem calls for finding a way to describe apparent ensemble features at the level of single systems with gravitational duals, with ideas being pursued in the fields of three-dimensional gravity [51], constrained matrix models [78], non-perturbative corrections [206] and the effective description as a universe field theory [182, 183].

Finally, a notoriously difficult question in the study of quantum gravity is the inclusion of non-trivial topologies in the path integral [243–245]. JT gravity allows this inclusion in a fairly straightforward manner, while its duals, such as the SYK model [242, 246], often struggle to do the same.

In [3], we address all of these problems, in a manner guided by periodic orbit theory [93, 94], whose successes in explaining both universal [98, 99, 247] and non-universal [248, 249] features of quantum chaotic systems seem particularly well suited to achieve this task, see [97] for a recent review.

In the rest of this chapter, I will report results we obtained in [3], where we construct an individual chaotic quantum system after the blueprint detailed in chapter 3 that exhibits not only the Schwarzian spectral density of JT gravity (section 9.1), but also, exploiting the discreteness of the set of periodic orbits of the system, yields the full JT gravity double trumpet partition function, i.e., the leading-topology 2-boundary partition function, both in and away from the late-time regime (section 9.2). As I detailed in part II, random-matrix universality is powerful enough that no appeal to a specific system is necessary, but also limited in that it only obtains in a very particular parameter regime. Further, agreement with universal RMT behaviour cannot be the basis of dualities between specific theories, such as the one I am arguing for in this part of the thesis.

Finally, in section 9.3, I will introduce a construction that leverages length correlations between periodic orbits already used in [3] to systematically compute corrections to the leading-order results that can be matched with the JT gravity genus expansion, thereby defining a periodic orbit genus expansion for chaotic quantum systems.

The setting for all of the above will be the computation of correlation functions of eq. (3.57) evaluated using the Selberg trace formula (3.1), i.e.,

$$\begin{aligned} Z(T) = \operatorname{tr} e^{-i\hat{H}T} &= \int_0^\infty e^{-ip^2T/4}\Phi_f(p)\,\mathrm{d}p + \sum_{\gamma\in\Gamma^*}\sum_{m=1}^{\infty} \frac{\chi_\gamma^m l_\gamma}{S_f(m,l_\gamma)} \frac{e^{i\frac{m^2 l_\gamma^2}{T}}}{\sqrt{\pi i T}} \\ &=: Z_{\mathrm{Weyl}}(T) + Z_{\mathrm{osc}}(T), \end{aligned} \tag{9.1}$$

where I identified the integral with the Weyl term and the periodic orbit sum with the oscillatory term, following the discussion in chapter 2.

9.1 The disk partition function

In this section, I will discuss how to recover the disk partition function of JT gravity in a specific limit of a Hadamard-Gutzwiller-like model, as discussed in chapter 3; namely, I will take the limit of the number of degrees of freedom, in the form of the configuration space dimension, going to infinity. I will discuss how to realise this limit in section 9.1.2, but before doing so, I want to review the calculation of the disk in a particular formulation of JT gravity, as a *BF* gauge theory[1]. In this formulation, the appearance of the sinh spectral density in particular becomes very transparent, and amenable to a comparison to our results in [3].

9.1.1 Interlude: The JT gravity disk in gauge theory

I will proceed with a review of the computation of the disk partition function in JT gravity understood as an $\mathfrak{sl}(2,\mathbb{R})$ *BF* theory[2]. This is to say, I want to consider the theory with the action

$$S_{BF} = -i \int \mathrm{tr}(BF). \tag{9.2}$$

Here, B is a scalar field and $F = dA + A \wedge A$ is the usual field strength tensor associated to a gauge field A. Everything presented in this subsection is taken from [250] unless otherwise indicated, but an excellent presentation with a slightly different approach can be found in [251, 252].

In the form (9.2), it is easy to see that B acts as a Lagrange multiplier setting $F = 0$, i.e., requiring the gauge connection to be flat. One could now go on and compute the volume of the moduli space of flat $\mathfrak{sl}(2,\mathbb{R})$ connections, which turns out to be equivalent to finding the Weil-Petersson volumes [56]. Further, after setting

$$A = e^a P_a + \omega P_0, \qquad B = B^a P_a + B^0 P_0, \tag{9.3}$$

where e^a is the zweibein, ω the spin connection and $P_{\{0,1,2\}}$ the generators of $\mathfrak{sl}(2,\mathbb{R})$, a standard computation leads to

$$S_{BF} = \frac{i}{4} \int d^2x \sqrt{g} B^0 (R+2). \tag{9.4}$$

To see this, recall that in 2 dimensions, the metric g and Ricci scalar R are connected to the zweibein and spin connection as [176, 253]

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \quad R^{ab} = d\omega^{ab} = \frac{1}{2} R e^a \wedge e^b, \tag{9.5}$$

where η_{ab} is the Minkowski (or Euclidean, depending on the signature) metric. x are coordinates parametrising the spacetime $\mathcal{M} = AdS_2$, which arises because AdS_2 is simply the quotient of $\mathrm{SL}(2,\mathbb{R})$[3] by the two-dimensional Lorentz group $\mathrm{SO}(1,1)$. Equation (9.4) is just the JT gravity bulk action with dilaton $\phi = iB^0/4$. However, let us stick with the *BF* theory for now.

[1] *BF* is *not* an abbreviation for "background field", as sometimes claimed. Indeed, I am not going to be using the background field method familiar from perturbation theory at all.

[2] $\mathfrak{sl}(2,\mathbb{R})$ is the Lie algebra that exponentiates to the group $\mathrm{SL}(2,\mathbb{R})$ of real 2×2 matrices of determinant $+1$. Hence, $\mathfrak{sl}(2,\mathbb{R})$ consists of the real traceless 2×2 matrices.

[3] Or more precisely, $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$.

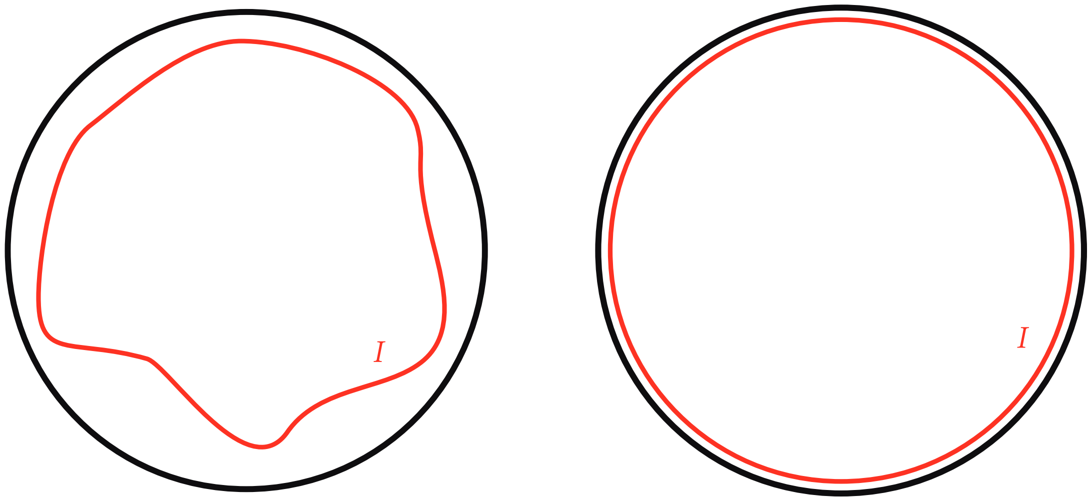

FIGURE 9.1: Left: AdS_2 with the boundary at infinity (black), and a circular defect I (orange). Right: The defect I is identified with the boundary to recover the correct boundary conditions, as per the discussion after eq. (9.8). Adapted from [250].

To recover the boundary action, we need to insert a circular defect I (which can be chosen arbitrarily close to the boundary $\partial\mathcal{M}$) by adding a term

$$S_I = -\frac{e}{4}\int_0^\beta du\, \mathrm{tr}\, B^2(u) \tag{9.6}$$

with some coupling constant e to the BF action, see fig. 9.1 for an illustration. Here, u is the proper length measured along I and β the defect's total length. Choosing a parametrisation $\tau(u)$ of $\partial\mathcal{M}$ with $\tau(\beta) - \tau(0) \equiv L$ and substituting

$$F(u) = \tan\left(\sqrt{\det A_\tau}\tau(u)\right), \tag{9.7}$$

where A_τ is the gauge field component along the boundary, we find the total action of the BF theory,

$$S = -\frac{1}{e}\int_0^\beta du\{F(u), u\}, \tag{9.8}$$

where $\{F(u), u\}$ denotes the Schwarzian derivative. Hence, we recover the well-known Schwarzian action of JT gravity. What is left to do is to match the boundary conditions; in particular, the (renormalised) proper length of $\partial\mathcal{M}$ is usually taken to be $L = \beta$, i.e., the defect I should be identified with the boundary, cf. fig. 9.1. To do so, we need to require that the holonomy of A around I is trivial,

$$e^{i\oint_I A} = 1. \tag{9.9}$$

This however depends on the global structure of the as yet unspecified gauge group, not just the gauge algebra $\mathfrak{sl}(2,\mathbb{R})$.

Consider then the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of Lie groups with the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and note that the disk partition function for any subgroup $\mathcal{G} \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$

can be written as

$$Z_{\mathrm{d}} \propto \int dR\rho(R) e^{-\frac{\beta}{2C}\left[C_2(R)-\frac{1}{4}\right]}, \tag{9.10}$$

with an appropriate constant C. The integral is over irreducible representations (irreps) of $\mathcal{G}$, $C_2(R)$ is the quadratic Casimir of the irrep R, and

$$\rho(R) d\mu ds = \frac{(2\pi)^{-2} s \sinh(2\pi s)}{\cosh(2\pi s) + \cos(2\pi\mu)} d\mu ds \tag{9.11}$$

is the Plancherel measure for $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ irreps of the principal series, labelled by $-1/2 \le \mu \le 1/2$ and $s \in \mathbb{R}_0^+$ [254]. More precisely, $C_2(R) = 1/4 - s^2$ is the value of the Casimir of R, and $\exp(2\pi i\mu)$ is the eigenvalue of the generator of the $\mathbb{Z}$ centre of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$[4].

A particularly natural choice would be $\mu = 0$, which amounts to choosing $\mathcal{G} = \mathrm{SL}(2,\mathbb{R})$. This would yield

$$Z_{\mathrm{d}} \propto \int_0^\infty ds s \tanh(2\pi s) e^{-\frac{\beta}{2C}s^2}, \tag{9.12}$$

and comparing to eq. (3.34), this expression would be a very natural candidate to be described by the Hadamard-Gutzwiller model in 2 dimensions[5]. The correct JT partition function however, is

$$Z_{\mathrm{d}} \propto \int_0^\infty ds s \sinh(2\pi s) e^{-\frac{\beta}{2C}s^2}. \tag{9.13}$$

To obtain the sinh from the Plancherel measure (9.11), we need to analytically continue $\mu \to i\infty$, which can be achieved by centrally extending $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and subsequently projecting onto a sector in which only hyperbolic elements of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ contribute[6]. Having done so, one recovers the JT gravity disk partition function (4.33), and an appropriate computation yields the trumpet (4.39) as well.

A further interesting point of comparison that we noted in [3] concerns the role of the Plancherel measure. Within JT gravity [250], wave functions can be expressed as class functions on the gauge group $\mathcal{G}$. A natural basis for this is the representation basis in terms of characters of the irreps U_R of $\mathcal{G}$.

Then, expressing the Dirac δ function on $\mathcal{G}$ as

$$\delta(g) = \int \rho(R)\,\mathrm{tr}(U_R(g)) dR, \tag{9.14}$$

[4]There are also two discrete series of irreps that one needs to account for in principle. However, in the limit where one recovers the JT gravity disk partition function, the discrete series irreps are suppressed [250].

[5]Note however that the aforementioned discrete irreps are not negligible in this case, and would in fact be the dominant contribution.

[6]An alternative choice is to restrict directly to a positive sub-semigroup of $\mathrm{SL}(2,\mathbb{R})$ without performing the central extension first [251, 252].

we can write functions on the group as

$$x(g_0) = \int \rho(R)\,\mathrm{tr}\Big(U_R(x)U_R(g_0^{-1})\Big)dR, \tag{9.15}$$

where by abuse of notation, $U_R(x)$ is the Fourier transform of x associated to the irrep U_R. Finally, by the Plancherel inversion formula [255, 256], we can use the irreps themselves as states and recall their orthonormality relation,

$$(U_R, U_{R'}) = \frac{1}{\rho(R)}\delta(R - R'), \tag{9.16}$$

where $\delta(R - R')$ is to be understood as a product of a Dirac delta for each continuous representation label and a Kronecker delta for each discrete one. Equation (9.16) is the analogue of the orthonormality relation (3.33) of eigenfunctions of the Laplacian in the Hadamard-Gutzwiller-like model [145, 154], and indeed, upon restricting the representation labels as detailed above, the two equations are simply identical, provided that the Plancherel measures agree between the two settings (which they do, as I will show in section 9.1.2).

Combined with the realisation that the quadratic Casimir of $\mathfrak{sl}(2,\mathbb{R})$ and the spectral representation of the Laplacian on $\mathbb{H}^f/\Gamma$ (cf. chapter 3) are identical functions, we conclude that at least at the level of the Weyl term, respectively the disk partition function, the Selberg trace formula and JT gravity compute the same mathematical object, albeit for a different (isometry/gauge) group. This close analogy between the two computations suggests an interesting direction for studying quotients of the isometry groups of high-dimensional hyperbolic spaces by their discrete subgroups, perhaps with a view similar to what is described in [257][7].

9.1.2 Hadamard-Gutzwiller-like models in the limit of infinite dimension

As we saw in chapter 3, the construction of the Selberg trace formula generalises completely straightforwardly to arbitrarily high configuration space dimension f. In this subsection, I will use this fact to study a high-dimensional Hadamard-Gutzwiller-like model, similar to the one I introduced in section 3.3, although absent further discrete symmetries. My only aim at the moment is to recover the disk partition function of JT gravity, which we showed in [3] to emerge from the Weyl term of eq. (3.1). This term, as I discussed in section 3.1, depends only on point orbits, and hence is insensitive to the presence or absence of time-reversal invariance. Likewise, the JT gravity disk partition function corresponds to a purely orientable geometry, and as such, does not depend on the admission of other unorientable manifolds to the path integral. For this reason, I will postpone the discussion of magnetic fluxes à la section 3.2 and the resulting symmetry class of the system until sections 9.2 and 9.3, where it will become relevant.

[7] I would like to thank Thomas Mertens for bringing this to my attention.

The first step I want to take is the recovery of the JT gravity spectral density (4.34). To this end, recall the form of the Plancherel measure that serves as the smoothed spectral density in the STF, eq. (3.34). We showed in [3] that this measure yields the JT gravity spectral density upon taking the limit $f \to \infty$, i.e., for infinitely many configuration space degrees of freedom. To see this, it is instructive to treat the odd- and even-dimensional case separately.

Starting with the odd-dimensional case, we factor out the $k = 0$ term from the product, and compute the product over the k^2 factor,

$$\prod_{k=1}^{\frac{f-3}{2}} k^2 = \left(\frac{f-3}{2}!\right)^2 , \tag{9.17}$$

to find

$$\Phi_f^{\text{odd}}(p) = \frac{\frac{f-1}{2}!\left(\frac{f-3}{2}!\right)^2}{\pi^{(f+3)/2}(f-1)!} p \left(\pi p \prod_{k=1}^{\frac{f-3}{2}} \left(1 + \frac{p^2}{k^2}\right)\right). \tag{9.18}$$

Upon comparison to the infinite product representation of $\sinh x$,

$$\sinh x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2 \pi^2}\right), \tag{9.19}$$

we identify

$$\Phi_{f\to\infty}^{\text{odd}}(p) = \frac{p}{2\pi^2} \mathcal{N}_\infty^{\text{odd}} \sinh(\pi p), \tag{9.20}$$

with the normalization constant $\mathcal{N}_\infty^{\text{odd}} = \lim_{f\to\infty} \mathcal{N}_f^{\text{odd}}$,

$$\mathcal{N}_f^{\text{odd}} = \frac{2\frac{f-1}{2}!\left(\frac{f-3}{2}!\right)^2}{\pi^{(f-1)/2}(f-1)!}. \tag{9.21}$$

An important aspect of the above for chapter 10 is the fact that at every finite f, the spectral density grows only polynomially with p, whereas in the limit $f \to \infty$, it grows exponentially.

The even-dimensional case requires essentially the same calculation. We again compute the product over $(k + 1/2)^2$ to find

$$\Phi_f^{\text{even}}(p) = \frac{\Gamma\left(\frac{f-1}{2}\right)^2 p \tanh(\pi p)}{\pi (2\pi)^{f/2} (f-2)!!} \prod_{k=0}^{\frac{f-4}{2}} \left(1 + \frac{p^2}{(k+\frac{1}{2})^2}\right), \tag{9.22}$$

and compare once more to the infinite product representation

$$\cosh x = \prod_{k=0}^{\infty} \left(1 + \frac{x^2}{(k+1/2)^2 \pi^2}\right), \tag{9.23}$$

yielding

$$\Phi^{\text{even}}_{f\to\infty}(p) = \frac{p}{2\pi^2}\mathcal{N}^{\text{even}}_{\infty}\tanh(\pi p)\cosh(\pi p) = \frac{p}{2\pi^2}\mathcal{N}^{\text{even}}_{\infty}\sinh(\pi p), \tag{9.24}$$

with the normalization constant $\mathcal{N}^{\text{even}}_{\infty} = \lim_{f\to\infty}\mathcal{N}^{\text{even}}_{f}$,

$$\mathcal{N}^{\text{even}}_{f} = \frac{\Gamma\left(\frac{f-1}{2}\right)^2}{(2\pi)^{\frac{f-2}{2}}(f-2)!!}. \tag{9.25}$$

Hence, the two cases agree in the infinite-dimensional limit up to the normalisation constants, for which

$$\frac{\mathcal{N}^{\text{even}}_{f}}{\mathcal{N}^{\text{odd}}_{f}} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}\sin\left(\frac{f\pi}{2}\right)^2} = \begin{cases} 1 & f \text{ even}, \\ \sqrt{\frac{\pi}{2}} & f \text{ odd}. \end{cases} \tag{9.26}$$

With a view towards chapter 10, eq. (9.22), unlike eq. (9.18), is not a polynomial at finite f, but its growth with p is still bounded by a polynomial, since $\tanh(\pi p) < 1$. In the infinite-dimensional limit, we once again recover exponential growth.

The exactly known Plancherel measure carries a distinct advantage over other systems that are typically studied using semiclassical methods: usually, the spectral density is only known at highest order in the energy through the Thomas-Fermi approximation [163, 258].

At large energies, this method agrees with the Plancherel measure,

$$\mathcal{V}'\Phi_f(p) = \rho^{\text{TF}}_f(p^2)\left(1 + \mathcal{O}(1/p)\right), \tag{9.27}$$

where $\mathcal{V}'$ is the configuration space volume of the system, and in this sense, the Plancherel measure can be understood to contain *all quantum corrections* to the Weyl volume law [259]. Absorbing the normalisation constant into the rescaled volume $\mathcal{V}$, we finally recover

$$\mathcal{V}'\Phi_\infty(p) = \mathcal{V}\sinh(\pi p), \tag{9.28}$$

which is simply the momentum representation of the JT gravity spectral density (4.34), up to the volume prefactor. If we now choose the spectral function that we plug into the STF as the time evolution operator eq. (3.57)[8], the Weyl term takes the form

$$Z_{\text{Weyl}}(T) = \mathcal{V}\int_0^\infty \frac{p\,\mathrm{d}p}{2\pi^2}e^{-iTp^2/4}\sinh\pi p = \mathcal{V}\int_0^\infty e^{-iTE}\frac{1}{\pi^2}\sinh\left(2\pi\sqrt{E}\right)\mathrm{d}E, \tag{9.29}$$

where we substitute $p^2 = 4E$. This is, again up to the volume scaling, exactly the JT gravity disk partition function (4.34). In order to obtain full agreement

[8]Note that $U_T(p) = e^{-iTp^2/4}$ explicitly satisfies all requirements to apply the STF; it is obviously even and analytic everywhere, and for $\operatorname{Im} T < 0$, it also decays fast enough for large p.

with JT gravity, it would be desirable to set the rescaled volume $\mathcal{V} = e^{S_0}/4$. A potential difficulty in this endeavour is the theorem due to Mostow [260], Prasad [261] and Marden [262], stating that the volumes of hyperbolic manifolds in three or more dimensions are topological invariants. In particular, the spectrum of available volumes is discrete and *not* dense in $\mathbb{R}_0^+$, so it is not possible to pick a manifold approximating a given desired volume arbitrarily closely, and this is not remedied by allowing non-compact finite-volume manifolds either. This does not pose a problem for our desire to identify the rescaled volume with e^{S_0} however; the former may not be arbitrary, but it can be chosen arbitrarily large[9]. The latter, on the other hand, is supposed to be a large, but otherwise arbitrary number, which can hence be tuned to a value compatible with the volume of the chosen manifold in a given dimension.

Going beyond the strict limit $f \to \infty$, I want to emphasize that Hadamard-Gutzwiller-like models are of broader interest for the study of quantum gravity. Consider $f = 3$, for which

$$\Phi_3(p) = \frac{p^2}{2\pi^2}. \tag{9.30}$$

With this measure, the Weyl term for the 3-dimensional model reads

$$\mathcal{V} \int_0^\infty e^{-iTE} \frac{2\sqrt{E}}{\pi^2} \, \mathrm{d}E, \tag{9.31}$$

which is exactly the disk partition function for the Airy model [61] if $\mathcal{V} = \frac{\pi}{2} e^{S_0}$. This model, as we have seen in chapter 8, is particularly useful in the study of unorientable gravity, where full JT gravity is much more difficult to deal with [2, 113, 119]. Indeed, since unorientability on the gravity side corresponds to intact time-reversal invariance in the dual theory, unorientable Airy gravity is *easier* to study in the Hadamard-Gutzwiller-like models that are the subject of this thesis, as we don't have to include magnetic fluxes in the spirit of section 3.2. The 3-dimensional case has another advantage: the geodesic length spectrum is still feasibly computable for many manifolds, e.g., using the Python based software SnapPy [264]. Doing so, one could then run into the problem posed by the Mostow-Prasad rigidity theorem discussed above: the volume spectrum is discrete and not dense in $\mathbb{R}_0^+$, meaning it might be difficult to find a manifold with computable solutions that is large enough to sensibly identify its volume with e^{S_0}. A particularity[10] of 3-dimensional hyperbolic geometry however is the fact that there are infinitely many manifolds with an identical *initial* geodesic spectrum [265]. That means that one could, in principle, identify two such manifolds and compute the geodesic spectrum on the simpler, typically smaller one, and use this spectrum in the Selberg trace formula applied to the larger, more complicated manifold. In most known systems [266, 267], and particularly in 2-dimensional Hadamard-Gutzwiller-like models [173], only fairly short periodic orbits are necessary to achieve relatively well-converged results for typical observables, making this a valuable approach in the 3-dimensional case.

[9] Note however that the *minimal* hyperbolic volume decays super-exponentially with the dimension [263], and in particular, faster than the growth of eqs. (9.21) and (9.25).

[10] At least I am not aware of a comparable result in more than 3 dimensions.

With this in mind, it is possible to address (speculatively) the question about identifying Hilbert spaces that I raised in the introduction to this chapter. The Hilbert space of JT gravity [268] is parametrised by the lengths of certain geodesics in the spacetimes included in the path integral, and contains a large number of null states. The natural Hilbert space of our model on the other hand is $L^2(\mathcal{M})$, where $\mathcal{M}$ is our manifold. These seem very different at first sight, but what we might observe is a Hilbert space with null states already quotiented out, cf. [269], given also that there are no natural candidates for null states in our theory.

In [3], we speculated that there is a physical interpretation of such null states in our system, similar to [202, 269]. In particular, they might be realised as different manifolds with the same geodesic length spectrum. Given the close relation between this spectrum and the spectrum of the Laplacian on hyperbolic manifolds, i.e., the Hamiltonian of our system, this possibility is certainly intriguing, but the fact that no analogy to results of [265] in higher dimensions is known makes this issue difficult to address with any sort of rigour.

The agreement of the Hadamard-Gutzwiller-like models with quantum gravitational models in $f = 3$ dimensions and in the limit $f \to \infty$ begs the question as to whether similar gravitational models can be described by intermediate dimensions as well. There are indeed models such as the $(2, 2k-1)$ minimal string that interpolate between the Airy model (for $k = 1$) and JT gravity ($k \to \infty$), which has the spectral density [270][11]

$$\rho_k(E) = \frac{(-1)^{k+1}}{2\sqrt{2}\pi} \frac{T_k(-E) - T_{k-1}(-E)}{\sqrt{-E-1}}, \tag{9.32}$$

where $T_k(x)$ are the Chebyshev polynomials of the first kind defined by $T_k(\cos\theta) = \cot k\theta$. Much like eq. (3.34) in odd dimensions, this is a polynomial density of states with lower order corrections to the top degree monomial[12] in $p \sim \sqrt{E}$. The $(2, 2k-1)$ minimal string can be understood as a deformation of JT gravity and admits black hole-inspired geometries [270]. For this reason, I will briefly discuss them again in the context of the OTOC calculation I present in chapter 10.

9.2 The double trumpet partition function

In the previous section, we saw that the JT gravity disk partition function emerges from the Weyl term in the STF (3.1), which corresponds to the contributions of zero-length periodic orbits, in the limit of the number of configuration space degrees of freedom $f \to \infty$. I have not made use of the sum over finite-length orbits yet, however, and I will show in this section that this sum is the crucial ingredient for recovering correlation functions that describe connected multi-boundary geometries, as well as higher topologies, the subject of section 9.3. It

[11]In a normalisation slightly different from the one usually used in JT gravity, but the salient points for the comparison I endeavour to make are still present.

[12]The polynomials only agree for $f = 3, k = 1$ and $f \to \infty, k \to \infty$ however.

is convenient for this reason to recall the splitting

$$Z(T) = \operatorname{tr} e^{-i\hat{H}T} = Z_{\text{Weyl}}(T) + Z_{\text{osc}}(T), \tag{9.33}$$

where $T = t - i\beta$. The focus in the following sections will be on the oscillatory term

$$Z_{\text{osc}}(T) = \sum_{\gamma\in\Gamma^*}\sum_{m=1}^{\infty} \frac{\chi_\gamma^m l_\gamma}{S_f(m, l_\gamma)} \frac{e^{i\frac{m^2 l_\gamma^2}{T}}}{\sqrt{\pi i T}}. \tag{9.34}$$

Due to the exponential proliferation of periodic orbits with their length [94], multiple traversals of the same primitive orbit are not as important as including longer primitive orbits[13]. For notational convenience, I will therefore only include terms with $m = 1$ in eq. (9.34) from now on. Later, we will see that multiple traversals are explicitly suppressed with respect to single traversals, justifying this omission.

The term

$$Z_{\text{t}}^{\text{sc}}(T, l_\gamma) = \frac{e^{i\frac{l_\gamma^2}{T}}}{\sqrt{\pi i T}}, \tag{9.35}$$

remarkably, is exactly the trumpet partition function (4.51) of JT gravity with the modified convention (4.49) for geodesic lengths. Recalling section 3.3, this means that we have identified the correct operator (3.57) of the boundary theory to represent the fixed-length, fixed-dilaton boundary conditions that are standard in JT gravity (cf. chapter 4).

Dropping also the superscript from Γ^* for convenience, we can rewrite the periodic orbit sum as

$$Z_{\text{osc}}(T) = \sum_{\gamma\in\Gamma} A_\gamma(T) e^{iS_\gamma(T)}, \tag{9.36}$$

where I identified the classical action of a freely moving particle evaluated along a periodic orbit of length l_γ in time T [94],

$$S_\gamma(T) = \frac{l_\gamma^2}{T}. \tag{9.37}$$

Correlators of (9.33) are typically computed using the semiclassical average $\langle\cdot\rangle_{\text{sc}}$ (2.32). This average has been fruitfully used in, e.g., [100] to understand the emergence of universal spectral statistics in chaotic quantum systems in terms of periodic orbit theory, but it is straightforward to see (most easily in the 1-point function of (9.33)), that it is not suitable for obtaining genus expansions:

$$\langle Z(T)\rangle_{\text{sc}} = Z_{\text{Weyl}}(T), \tag{9.38}$$

i.e., the average kills the rapidly oscillating contribution Z_{osc}. As we will see in section 9.3 however, we can obtain a genus-like expansion from correlators of eq. (9.34) *if* we reconsider what we mean by the average $\langle\cdot\rangle$ in a specific way[14].

[13]See however [271, 272] on quantum graphs.

[14]It should be noted however that the exact vanishing of beyond-Weyl term contributions to correlators is a feature peculiar to the one-point function, while the formalism for obtaining

Before discussing this expansion however, I will focus on the leading-order two-point function in the rest of this section. In section 9.2.1, I will first discuss the (by now well-understood [273]) emergence of the ramp in the spectral form factor from Berry's diagonal approximation [98], highlighting the limitations of this approach as far as parameter regimes are concerned. Then, in section 9.2.2, I will report how we recovered the full double trumpet from the aforementioned modified average in [3].

9.2.1 Berry's diagonal approximation: the ramp

Adopting the notation of chapter 4, the two-point correlation function of eq. (9.33) is computed using standard periodic orbit theory as

$$\mathcal{Z}_2(T_1, T_2) = \langle Z(T_1) Z(T_2) \rangle_{\rm sc} \,. \tag{9.39}$$

From eq. (9.38), we see that terms including one instance of the Weyl term and one of the periodic orbit sum will simply vanish under the average, while the term including both Weyl terms just gives the product of two disk partition functions. The important term for the computation of the connected correlators of interest in JT gravity is therefore the one including both periodic orbit sums. This term reads

$$\mathcal{Z}_2^{\rm c}(T_1, T_2) = \langle Z_{\rm osc}(T_1) Z_{\rm osc}(T_2) \rangle_{\rm sc}^{\rm c} = \left\langle \sum_{\gamma,\gamma'} \frac{l_\gamma}{S_f(l_\gamma)} \frac{l_{\gamma'}}{S_f(l'_\gamma)} \frac{e^{i\frac{l_\gamma^2}{T_1}}}{\sqrt{\pi i T_1}} \frac{e^{i\frac{l_{\gamma'}^2}{T_2}}}{\sqrt{\pi i T_2}} \right\rangle_{\rm sc}^{\rm c} \tag{9.40}$$

To evaluate this expression, we can use the intuition familiar from the 1-point function: $Z_{\rm osc}$ is a very rapidly oscillating function, and so any small average over it is expected to vanish. Therefore, if we let $Z_{\rm osc}(T_1)$ and $Z_{\rm osc}(T_2)$ vary independently, we would expect $\mathcal{Z}_2^{\rm c}(T_1, T_2)$ to be zero. However, if we have $T_1 \approxeq -T_2$, we see that the rapidly oscillating terms nearly cancel each other, producing an enhanced, nonzero contribution to the two-point function [99]. This observation can be developed into the so-called encounter formalism [100] that I reviewed in section 2.2. Before reevaluating eq. (9.40) with this tool, I will discuss only the *most* enhanced contribution, i.e., the one originating from the diagonal terms $\gamma = \gamma'$ in the double sum. This is Berry's diagonal approximation [98]. Under this approximation then, eq. (9.40) can be evaluated as

$$\mathcal{Z}_2^{\rm c}(T_1, T_2) = \begin{cases} 0 & T_1 T_2 > 0, \\ \mathcal{Z}^{\rm dg}(T) & -T_1 \simeq T_2 = T, \end{cases} \tag{9.41}$$

where

$$\mathcal{Z}^{\rm dg}(T) = \sum_\gamma \frac{\kappa l_\gamma^2}{S_f^2(l_\gamma)} \frac{e^{-\frac{2 l_\gamma^2 \beta}{|T|^2}}}{\pi |T|} . \tag{9.42}$$

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a genus expansion that I introduce in section 9.3 is right now only developed for the two-point function. Bridging this gap is a goal of ongoing research.

To arrive at this result, we use the requirement $\gamma = \gamma'$ in the double sum in eq. (9.40) to eliminate the sum over γ'. Then, the oscillating factors exactly cancel each other, yielding eq. (9.42). The factor $\kappa = 1(2)$ for broken (intact) time reversal invariance indicates the symmetry class of the system and appears here because of the degenerate action of γ and its time-reverse in the case of intact TRI. Comparing to the matrix model nomenclature of part II, it is connected to the Dyson index via $\kappa = 2/\beta$. To recover the characteristic ramp form of the canonical spectral form factor (cf. chapter 6), we introduce the length density,

$$\eta(l) = \sum_{\gamma} \delta(l - l_{\gamma}), \tag{9.43}$$

with which we can exactly rewrite eq. (9.42) as an integral over the length of the periodic orbit,

$$\mathcal{Z}^{\mathrm{dg}}(T) = \kappa \int_{l_0}^{\infty} \mathrm{d}l \, \eta(l) \frac{l^2}{S_f^2(l)} \frac{e^{-\frac{2l^2\beta}{|T|^2}}}{\pi|T|}. \tag{9.44}$$

Here, l_0 is the lower cutoff to the geodesic length spectrum set by the shortest periodic orbit, as discussed in chapter 3. Due to the well-known exponential proliferation of periodic orbits [274, 275], we can approximate the length density as

$$\eta(l) \approx \overline{\eta}(l) = \frac{2\sinh[(f-1)l]}{l}. \tag{9.45}$$

Comparing to eq. (3.28), this exactly cancels the stability amplitudes $S_f^2(l_\gamma)$ for large enough lengths. As an aside, we also see from eq. (3.28), that orbits with multiple traversals are suppressed by a stronger exponential decay in the stability than is compensated by the density, justifying my leaving them out of equations. Using this approximation, extending the lower integration boundary to zero which produces only subleading corrections, and slightly relaxing the condition that $T_1 = -T_2$ in the two-point function, i.e., evaluating at $T_1 = t - \epsilon - i\beta, T_2 = -t - \epsilon - i\beta$ where $t \gg \epsilon, \beta$, we obtain

$$\mathcal{Z}^{\mathrm{dg}}(T_1, T_2) = \kappa \int_0^{\infty} l \, \mathrm{d}l \, \frac{2e^{-\frac{2l^2}{t^2}(\beta + i\epsilon)}}{\pi t} + O\left(\frac{\beta}{t}, \frac{\epsilon}{t}\right) \simeq \frac{\kappa t}{2\pi} \frac{1}{\beta + i\epsilon}, \tag{9.46}$$

which is exactly the ramp SFF as expected [276]. In the orthogonal symmetry class, there will be corrections to this result [99, 100], though for the unitary symmetry class, these vanish identically. In JT gravity meanwhile, it is known that the SFF exhibits a non-vanishing genus expansion regardless of the symmetry class [55, 119]. Hence, the standard semiclassical method of computing corrections via correlated actions (cf. chapter 2) is not suitable to capture the entirety of JT gravity correlators. However, eq. (9.46) reveals a further problem with the standard approach even before trying to reproduce the genus expansion: the equation is only valid so long as the two times are sufficiently close[15]. The double trumpet in JT gravity meanwhile is not bound to any such restriction. To

[15] More precisely, we should have $\mathrm{Re}\, T_1 \approx -\mathrm{Re}\, T_2$.

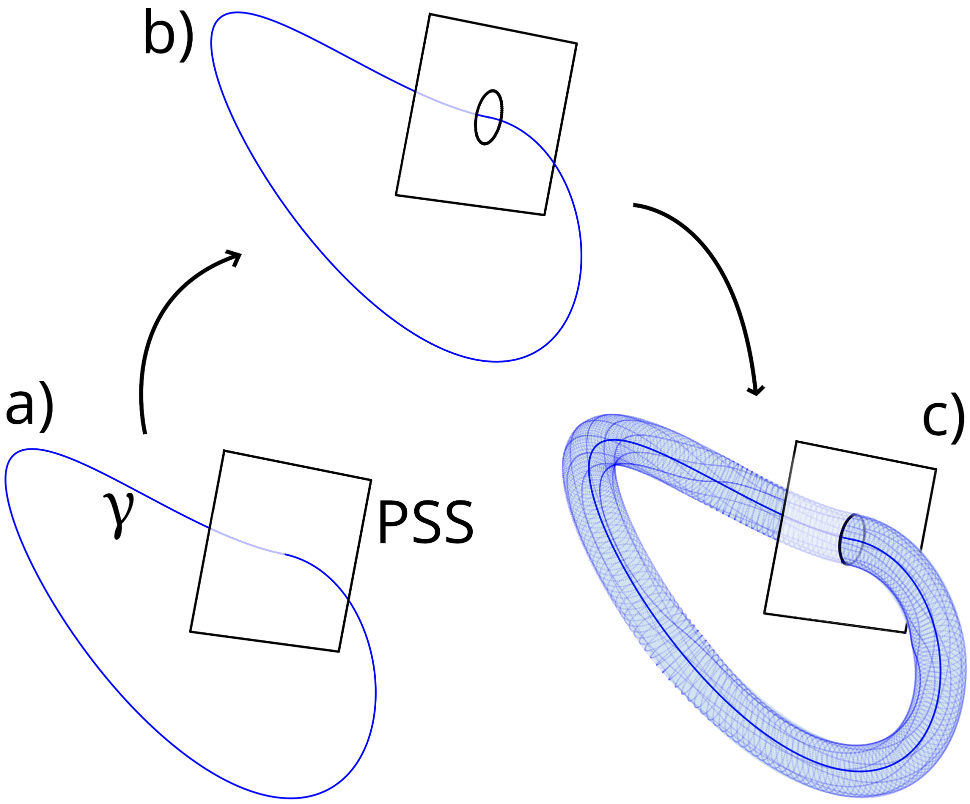

FIGURE 9.2: a) A schematic representation of a periodic orbit γ in phase space, piercing a Poincaré surface of section (PSS) [278]. For a system to be chaotic, it needs to have at least two configuration space degrees of freedom and hence at least a four-dimensional phase space [94], but I suppress all but three phase space dimensions for illustrative purposes. b) A finite resolution (oval) on the PSS around the actual piercing point of γ is introduced. c) This finite resolution blurs the actual periodic orbit in the phase space (blue tube), and thereby the actual length of γ. The existence of orbits describing the tube is guaranteed, since a chaotic system possesses a dense set of periodic orbits [94]. Taken from [3], illustration by Florian Schöppl.

remedy this and extend eq. (9.46) to the entire T_1, T_2 domain, I will use the next subsection to introduce the refined procedure we developed in [3] to compute correlation functions, using correlated *lengths* of periodic orbits, rather than their associated actions.

9.2.2 Length correlations: the full double trumpet

In order to properly address the question of how to include connected multi-boundary geometries, as well as higher topologies in holographic theories, it is crucial to keep the factorisation problem (cf. section 4.2) in mind. Concretely, this means that any modification of the standard semiclassical (microcanonical) average (2.32) that I propose ought to preserve the ability to operate within a single quantum system, or as close to it as possible. We achieved this in [3] by introducing a coarse graining prescription – familiar from standard statistical mechanics [277] – on the phase space of the system.

This coarse graining prescription is illustrated in fig. 9.2 and can be described in the following manner: from the encounter formalism [99, 100] (cf. section 2.2), we recall that enhanced contributions to the two-point function originate from so-called *self-encounters* of a periodic orbit, i.e., regions in the phase space where the orbit comes close to itself (in the sense that the area spanned by the points where it pierces a Poincaré surface of section is smaller than some classical scale c^2 [278]). If we now coarse grain the system's phase space on such a scale, we lose the information about any actual periodic orbit of the system that we would like to consider. Instead, we can only resolve bundles of periodic orbits that are guaranteed to exist by the density of periodic orbits in chaotic systems [94], and that might, e.g., differ only in self-encounter regions, but run very close to one another everywhere else.

This coarse graining consequently induces a loss of information about the actual length spectrum of the system, i.e., the crucial information entering the STF. While the use of such "generic" coarse graining prescriptions is not without precedent in the study of (particularly low-dimensional) holography [279], it would be very interesting to identify a concrete microscopic origin of the loss of information, potentially in the spirit of embedding the theory in a larger universe field theory [182, 183], and integrating out certain degrees of freedom.

We encode the loss of information by considering the exact microscopic distribution of lengths in the system,

$$P^{\mathrm{mc}}_{\vec{l}_\Gamma}(\vec{l}) = \prod_{\gamma\in\Gamma} \delta(l^{(\gamma)} - l_\gamma). \tag{9.47}$$

The notation unfortunately carries the potential to be somewhat confusing, but it means quite simply that the lengths $\vec{l} = \{l^{(\gamma_1)}, l^{(\gamma_2)}, \ldots\}$ of the periodic orbits $\gamma_1, \gamma_2, \ldots$ of the system are drawn from a distribution that, in this case, fixes their length to exactly $\vec{l}_\Gamma = \{l_{\gamma_1}, l_{\gamma_2}, \ldots\}$. The coarse graining now proceeds by replacing (9.47) by a coarse-grained distribution that does not possess the information about the exact length spectrum of the system, but does encode the correlations between lengths of periodic orbits that have been rigorously shown to exist, e.g., in [280, 281]. These correlations, even in a standard semiclassical treatment, are not a mere artifact of the mathematical peculiarity of the system, but are indeed responsible for the emergence of universal spectral fluctuations in the sense of RMT [247] (cf. chapter 6). What may be seen as a peculiarity of our system however is the fact that these correlations are of classical origin and independent of $\hbar$.

While the full coarse-grained distribution is a highly complicated object that is difficult to determine exactly, it still defines an average $\langle\cdot\rangle$ over the length spectrum, and we can formally give, e.g., the first few moments of the length density,

$$\langle \eta(l) \rangle = \sum_{\gamma} \int \mathrm{d}\vec{l}\, P^{\mathrm{cg}}(\vec{l})\delta(l - l^{(\gamma)}), \tag{9.48}$$

$$\langle \eta(l)\eta(l') \rangle = \sum_{\gamma,\gamma'} \int \mathrm{d}\vec{l}\, P^{\mathrm{cg}}(\vec{l})\delta(l - l^{(\gamma)})\delta(l' - l^{(\gamma')}). \tag{9.49}$$

These moments can now be used to compute correlation functions of eq. (3.57) as in the rest of this chapter up to now. In particular, the connected two-point function now reads[16]

$$\langle Z_{\mathrm{osc}}(T_1) Z_{\mathrm{osc}}(T_2)\rangle^{\mathrm{c}} = \int_0^\infty \frac{l\,\mathrm{d}l}{S_f(l)} \frac{l'\,\mathrm{d}l'}{S_f(l')} \langle \eta(l)\eta(l')\rangle^{\mathrm{c}} \frac{e^{i\frac{l^2}{T_1}}}{\sqrt{\pi i T_1}} \frac{e^{i\frac{(l')^2}{T_2}}}{\sqrt{\pi i T_2}}, \tag{9.50}$$

where I emphasise that the average $\langle \cdot \rangle$ is the *only* average used here – we do not apply the semiclassical average $\langle \cdot \rangle_{\mathrm{sc}}$.

In order to evaluate this further, I have to specify the kind of length correlations I described above. At the level of the double trumpet, it turns out that the most obvious length correlation suffices – the one between an orbit and itself [282]. Obviously, within eq. (9.49), there occur contact terms [208] when $\gamma = \gamma'$ (as well as when γ and γ' describe mutually time-reversed orbits in the GOE case, yielding an extra factor $\kappa = 2$ [283]). Making the reasonable, and later on justified assumption that this type of correlation gives in some sense a *leading* contribution to the two-point function of length densities, and posing as the sole other condition on $\langle \cdot \rangle$ that $\langle \eta(l)\rangle = \overline{\eta}(l)$, we can determine the connected two-point function as

$$\begin{aligned} \langle \eta(l)\eta(l')\rangle^{\mathrm{c}} &= \left\langle \sum_{\gamma,\gamma'} \delta(l - l^{(\gamma)})\delta(l' - l^{(\gamma')}) \right\rangle^{\mathrm{c}} \\ &= \kappa \left\langle \sum_{\gamma} \delta(l - l^{(\gamma)})\delta(l - l') \right\rangle + \left\langle \sum_{\gamma \neq \gamma'} \delta(l - l^{(\gamma)})\delta(l' - l^{(\gamma')}) \right\rangle^{\mathrm{c}} \\ &= \kappa \overline{\eta}(l)\delta(l - l'), \end{aligned} \tag{9.51}$$

up to some as yet unspecified corrections from nontrivial correlations between orbit lengths. In the second line of eq. (9.51), I separated the diagonal ($\gamma = \gamma'$) from the off-diagonal terms and used the identity

$$\delta(a - c)\delta(b - c) = \delta(a - c)\delta(a - b).$$

This allows the identification of $\langle \eta(l)\rangle$ in the first term, while I neglect the second as per the arguments above. The index κ reappears since in systems with TRI, every orbit has two partners with perfectly correlated lengths: itself, and its time-reverse. It is therefore particularly transparent that the appearance of contact terms in the length density two-point correlator is due to the discreteness of the set of periodic orbits [278] – if there were a continuous family of orbits, this two-point function would not exhibit contact terms, unless there is an accumulation point somewhere in the length spectrum[17]. We can hence rewrite

[16] Note that while the one-point function does change, the leading order term corresponding to the JT gravity disk that we computed in section 9.1, being independent of all finite-length periodic orbit contributions, remains the same.

[17] I am not aware of a direct reference for this statement, but it can be easily checked by explicit computation analogous to the one presented above for the discrete case. In essence, by the Lebesgue decomposition theorem, the diagonal where both lengths agree is of measure 0

eq. (9.50), following the discussion after eq. (9.45), as

$$\langle Z_{\rm osc}(T_1) Z_{\rm osc}(T_2)\rangle^{\rm c} = \int_0^\infty l\, {\rm d}l\, l'\, {\rm d}l' \, \frac{e^{i\frac{l^2}{T_1}}}{\sqrt{\pi i T_1}} \frac{e^{i\frac{(l')^2}{T_2}}}{\sqrt{\pi i T_2}} V^{\rm sc}_{0,2}(l,l'), \tag{9.52}$$

where $V^{\rm sc}_{0,2}(l,l')$ is the semiclassical version of the formal Weil-Petersson volume (4.48), respectively (4.56),

$$V^{\rm sc}_{0,2}(l,l') = \kappa \frac{\delta(l-l')}{l}. \tag{9.53}$$

We may now carry out the l' integral to recover

$$\langle Z_{\rm osc}(T_1) Z_{\rm osc}(T_2)\rangle^{\rm c} = \kappa \int_0^\infty l\, {\rm d}l \, \frac{e^{i\frac{l^2}{T_1}}}{\sqrt{\pi i T_1}} \frac{e^{i\frac{l^2}{T_2}}}{\sqrt{\pi i T_2}} = \kappa Z_{0,2}^{\rm JT\ gravity}(T_1,T_2), \tag{9.54}$$

i.e., the *exact* JT gravity double trumpet partition function, in its form as an integral over two trumpet partition functions (9.35) against the Weil-Petersson form $l\,{\rm d}l$, and including the symmetry factor κ reflecting (un)orientability, in *any* regime of the complex times/boundary lengths T_1, T_2! It should be emphasised that T_1, T_2 in this calculation do *not* play the role of periods of specific periodic orbits, but merely enter as the arguments of the correlation functions.

Our formalism is therefore capable of representing the splitting of the gravitational path integral into the computation of bulk moduli space volumes depending only on the lengths of "gluing geodesics" (pairs of periodic orbits), and a separate evaluation of the boundary mode path integral, in the form of the trumpet partition functions. In particular, at leading order in the two-point function, the contact term accounting for pairs with perfectly correlated lengths represent the ill-defined moduli space volume of the gravitational double trumpet that can only be glued if the lengths of the two geodesics exactly coincide.

Together with the results of section 9.1, we were able in [3] to establish a dictionary between JT gravity and our high-dimensional Hadamard-Gutzwiller-like model that includes: the disk partition function of JT gravity (9.29) (or of the Airy model (9.31) in $f = 3$ dimensions), the role of the volume of our manifold as the large parameter of the theory, the trumpet boundary contribution (9.35), the formal Weil-Petersson volume (9.53) and with it, the double trumpet partition function (9.54), all in both orientable and unorientable JT gravity, with the physical mechanisms leading to these results being strikingly reminiscent of their gravitational analogues.

In the following section, I will report our progress on adding another entry to this dictionary: the semiclassical version of the genus-1/2 moduli space volume in the unorientable Airy model. In the course of section 9.3, we will be able to appreciate the continuity of the physical mechanisms identified in the chapter so far, lending credence to the proposition that we have indeed found the correct physics to represent gravitational concepts in periodic orbit theory in [3].

in the integration domain, and hence cannot be isolated, unless the integrand is itself singular there [284].

9.3 Genus expansions?

As discussed previously, we expect the lengths of periodic orbits of our system to exhibit correlations beyond the obvious perfect correlation of an orbit with itself or its time-reverse [280, 281], implied already by the action correlations that are guaranteed to exist (cf. chapter 2). In the standard semiclassical encounter formalism of [100] discussed in section 2.2, such correlations produce corrections to the leading order behaviour of the SFF, with the first correction stemming from Sieber-Richter pairs in time-reversal invariant systems. This contribution is evaluated using the semiclassical average (2.32) (or rather, a slightly modified version where the time-window average is replaced by an energy-window average) in great detail in [285] for arbitrary, but finite dimension. In this section, the contents of which are the subject of currently ongoing research [121], I aim to perform a similar evaluation in our system using the correlated length average introduced in section 9.2. It should be noted that it is not a priori clear that Sieber-Richter pairs are (fully or partially) responsible for the first correction to the exactly correlated contact terms in eq. (9.50); our formalism is based on the *existence* of length correlations, not on the smallness of length differences. Nevertheless, chaoticity in our system guarantees the existence of Sieber-Richter pairs, and the geometrisability of the encounter formalism in our model (to be discussed below) guarantees the existence of orbit pairs with small length differences in turn. Hence, I will simply set out to evaluate these contributions using our length average, and justify post hoc that this was a sensible undertaking by discussing the result of the computation.

While performing said computation for full JT gravity, corresponding to the limit of infinite configuration space dimension on the periodic orbit side, is still work in progress, we may perfectly well mimic the calculation of [285] in the $f = 3$ version of our system, which we claim to be perturbatively dual to the Airy limit of JT gravity. We have seen in eq. (9.31) that our system correctly gives the disk partition function of the Airy model, and as the reader will have noticed in section 9.2, the double trumpet result is independent of the configuration space dimension. The computation I will present in this section therefore serves as an additional test of our proposed perturbative duality. Standard semiclassics teaches us that Sieber-Richter pairs only contribute in time-reversal invariant systems (cf. chapter 2), meaning that we have to compare to the unorientable Airy model. There, the first correction to the two-point function originates from genus $g = 1/2$, with the Airy volume[18] [117]

$$V^{\text{Airy}}_{\frac{1}{2},2}(b_1, b_2) = 8\max(b_1, b_2). \tag{9.55}$$

The goal therefore is to check whether the correction due to the Sieber-Richter pair, i.e., the analogue of the next-to-leading order correction to the spectral form factor in standard semiclassics, gives eq. (9.55) as well.

Our starting point is eq. (9.50) with the contact term contribution (9.51) removed. I will rewrite this expression (still in arbitrary configuration space

[18]Recall the length normalisation we use on the gravity side, cf. chapter 4.

dimension f) as

$$\mathcal{Z}_2^{c,(2)}(T_1,T_2) = \int_0^\infty \frac{l\,\mathrm{d}l}{S_f(l)}\frac{l'\,\mathrm{d}l'}{S_f(l')}\frac{e^{i\frac{l^2}{T_1}}}{\sqrt{\pi i T_1}}\frac{e^{i\frac{l'^2}{T_2}}}{\sqrt{\pi i T_2}}\overline{\eta}\left(\frac{l+l'}{2}\right)P(l-l'), \tag{9.56}$$

where $P(l-l')$ is the density of orbit pairs with length difference $l-l'$. In other words, I use $\overline{\eta}$ to estimate the number of orbits with an average length of $L=(l+l')/2$ and P to account for partner orbits within $\delta l = l - l'$ around the average length[19].

We can use the cancellation of the exponential proliferation of periodic orbits against their stabilities,

$$\frac{\overline{\eta}((l+l')/2)}{S_3(l)S_3(l')} = \frac{2}{l+l'} = \frac{1}{L}, \tag{9.57}$$

and hence our candidate semiclassical Weil-Petersson volume is simply determined by the density of length differences,

$$\mathcal{Z}_2^{c,(2)}(T_1,T_2) = \int_0^\infty l\,\mathrm{d}l\,l'\,\mathrm{d}l'\,\frac{e^{i\frac{l^2}{T_1}}}{\sqrt{\pi i T_1}}\frac{e^{i\frac{l'^2}{T_2}}}{\sqrt{\pi i T_2}}V^{\mathrm{sc}}_{\frac{1}{2},2}(l,l'), \quad V^{\mathrm{sc}}_{\frac{1}{2},2}(l,l') = \frac{P(\delta l)}{L}. \tag{9.58}$$

The task is now to find a way to determine $P(\delta l)\,\mathrm{d}\delta l$. To this end, I will use the fact that in our system, length differences are related to action differences at fixed energy in a very simple way,

$$\Delta S = p(E)\delta l, \tag{9.59}$$

where

$$p(E) = 2\sqrt{E} \tag{9.60}$$

is the momentum. Using this translation rule, we can determine $P(\delta l)$ via the density of action differences at fixed energy, i.e.,

$$P(\delta l)\,\mathrm{d}\delta l = \frac{P(\Delta S = 2\sqrt{E}\delta l)}{2\sqrt{E}}\,\mathrm{d}\Delta S\,. \tag{9.61}$$

The reader may pause at the prospect of introducing an energy dependence in what is otherwise a completely geometrical expression. Indeed, this programme only gives sensible answers because, as we will see in the following, the expression found in [285] for $P(\Delta S)\,\mathrm{d}\Delta S$ can be completely geometrised in our system, i.e., it can be written purely in terms of geometric variables, with all energy dependences cancelling.

[19] A comment on notation is in order here: the density $P(l-l')$ is *not only* a function of the length difference, but also of the average length. I merely suppress the dependence on the average length for readability.

Assuming that the dominant contribution will originate from Sieber-Richter pairs, we can compute this expression via [94, 285]

$$P(\Delta S) = \Theta(c^2 - \Delta S) \int_{-c^2}^{c^2} \mathrm{d}S_1 \ldots \mathrm{d}S_{f-1} \frac{\mathrm{d}^{f-1} N_\gamma(\{S_j\})}{\mathrm{d}S_1 \ldots \mathrm{d}S_{f-1}} \delta\left(\Delta S - \sum_i^{f-1} S_i\right), \tag{9.62}$$

where c^2 is the encounter linearisation scale (cf. section 2.2) and S_i parametrise the action differences in the available degrees of freedom. The step function ensures that the total action difference is compatible with linearisation. The quantity

$$\frac{\mathrm{d}^{f-1} N_\gamma(\{S_j\})}{\mathrm{d}S_1 \ldots \mathrm{d}S_{f-1}} \tag{9.63}$$

is the density of partner orbits for a given orbit γ. Determining this density is a task of considerable difficulty, though for the correction at hand, it can be approximated by

$$\frac{\mathrm{d}^{f-1} N_\gamma(\{S_j\})}{\mathrm{d}S_1 \ldots \mathrm{d}S_{f-1}} \approx 2^{f-1} \frac{T^2}{\Omega} \lambda \sum_{j=1}^{f-1} \prod_{i \neq j}^{f-1} \log\left(\frac{c^2}{|S_i|}\right) - 2^f \frac{T}{\Omega} \prod_{i=1}^{f-1} \log\left(\frac{c^2}{|S_i|}\right), \tag{9.64}$$

where Ω is the volume of the energy shell under consideration, λ is the Lyapunov exponent of the system[20], and T is the period of the orbits considered. Importantly, T is *not* to be identified with either of the time arguments T_1, T_2 of the correlation function: the semiclassical Airy volume, as can be read off from eq. (9.58), depends *only* on the lengths of the periodic orbit pairs involved, and not on the arguments of the correlation functions, which only enter via the semiclassical trumpets (9.35). As in the genus-zero case (cf. section 9.2.2), this is conceptually compatible with the splitting of the gravitational path integral into the computation of bulk moduli space volumes and a separate evaluation of the boundary mode path integral.

It should be noted that arriving at eq. (9.64) in [285] requires performing the average over the periodic orbit ensemble, as well as a phase space average. However, in our model, the quantities being averaged over, i.e., the local stretching factors of the linearised dynamics (cf. chapter 2), are all constant and equal to the Lyapunov exponent of the system, irrespective of the periodic orbit and the location within the energy shell, rendering all the ensemble averages trivial and indeed unnecessary.

Returning to the evaluation of eq. (9.64), the first term is much larger than the second, but averages to zero if the usual semiclassical smoothing is employed. Hence, remarkably, the leading universal correction to the spectral form factor is entirely given by the *subleading* term in the density (9.64), while the first term is system-specific and encodes information about the Lyapunov spectrum.

To map this back to our model, specifically in $f = 3$ dimensions, we can introduce geometrized variables in the following way: taking inspiration from

[20] In generic systems with multiple different Lyapunov exponents, we would have the exponent λ_j associated to the jth direction appearing inside the sum instead.

eq. (9.59), we associate a length variable to each action difference,

$$S_i = p(E)l_i = 2\sqrt{E}l_i, \tag{9.65}$$

and we do the same for the linearisation scale,

$$c^2 = p(E)l_c = 2\sqrt{E}l_c. \tag{9.66}$$

The encounter formalism, being defined within a microcanonical energy window, usually treats c^2 as a constant [100], but it sets the scale of action differences that allow for a linearisation of the dynamics, and as such, may very well depend on the energy. This dependence is irrelevant in semiclassics because physical quantities are usually independent of c^2 at leading order[21]. In the context of the length correlation average I intend to compute however, it is important to take this energy dependence seriously, and being an action scale, it is sensible to assume that the energy dependence follows eq. (9.66), much like the other actions appearing in the formalism. The Lyapunov exponent is given by eq. (3.42), and the periods are connected to the lengths via the velocities,

$$v(E) = \{x, H\}(E) = \frac{p(E)}{2}, \quad T_i = \frac{l_i}{v(E)} = \frac{2l_i}{p(E)} = \frac{l_i}{\sqrt{E}} \approx \frac{L}{\sqrt{E}}. \tag{9.67}$$

Finally, the classical energy shell volume is computed in appendix A to be

$$\Omega = 16\pi\mathcal{V}\sqrt{E}. \tag{9.68}$$

Putting everything together, and accounting for the extra factor $1/2\sqrt{E}$ from the Jacobian (9.61), we recover the density of length differences,

$$\begin{aligned} P(\delta l) = {} & \frac{2}{\pi\mathcal{V}}\Theta(l_c - |\delta l|)\Theta(|\delta l| - l_c e^{-L}) \\ & \times \int_{-l_c}^{l_c} \mathrm{d}l_1\,\mathrm{d}l_2\,\delta(\delta l - l_1 - l_2)\left[L^2 \log\left(\frac{l_c^2}{|l_1 l_2|}\right) - L\log\left(\frac{l_c}{|l_1|}\right)\log\left(\frac{l_c}{|l_2|}\right)\right]. \end{aligned} \tag{9.69}$$

The first Heaviside step function in eq. (9.69) enforces that the length difference is small enough that the involved orbits actually form a Sieber-Richter pair, i.e., that they differ only within an encounter. The second step function on the other hand ensures that the orbits are long enough to actually form an encounter; the condition inside the Θ function is simply the geometrised version of the relation (2.40) between the encounter time and the ratio of action differences to the linearisation scale. Since l_c is the length scale associated to an encounter, $\delta l/l_c < 1$ and we can evaluate eq. (9.69) by expanding in $\delta l/l_c$, taking care to pick up a contribution from the non-oscillatory tail in the asymptotic expansion of the second term. I perform this computation in appendix B, and find

[21] For instance, the $\log c^2$ contributions of encounter integrals [100] cancel against Ehrenfest time factors. The only impact of an energy-dependent linearisation scale is then an additive shift of the Ehrenfest time.

$$V^{\rm sc}_{\frac{1}{2},2}(l,l') = \frac{2}{\pi\mathcal{V}} l_c \left(l + l' - 4 + \frac{\pi^2}{2}|\delta l|/l_c\right) \Theta(l_c - |\delta l|)\Theta(|\delta l| - l_c e^{-L}) \quad (9.70)$$
$$= 2e^{-S_0}\max(l-2, l'-2)\Theta\left(\frac{\pi^2}{2} - |l-l'|\right)\Theta\left(|l-l'| - \frac{\pi^2}{2}e^{-\frac{l+l'}{2}}\right). \quad (9.71)$$

Up to a multiplicative constant[22], eq. (9.71) at fixed lengths is simply the Airy volume (9.55), modified by a lower cutoff on the periodic orbit lengths. This cutoff needs to be respected, i.e., all orbits must have at least length 2, if the volume is to be positive semi-definite[23]. I needed to set $l_c = \pi^2/2$ to obtain the max form in eq. (9.71) that most closely resembles the actual Airy volume. Finding a physical interpretation for l_c is an important step in the ongoing development of the techniques introduced herein and eventually computing higher-order contributions to the genus expansion. However, the fundamental content in eq. (4.65) – symmetry in the arguments, semi-positive definiteness, and nonanalyticity along the diagonal $b_1 = b_2$ – is already present in eq. (9.70).

Note that it is expected, but nontrivial, that the suppression by e^{-S_0} appears in the volume, since our formalism of length correlations (computed via action correlations in this case) directly produces both the functional form of the volume *and* accounts for its smallness. This is in contrast to Mirzakhani's recursion (cf. section 4.1.2), which is purely geometrical and does not in itself order different contributions to the gravitational path integral.

A crucial realisation, to close this chapter, is that eqs. (9.70) and (9.71) contain both a universal contribution $\supset |l - l'|$ *and* a system-specific contribution $\supset l + l'$ that cannot be captured by standard semiclassics. Both of these contributions combine to give the full gravitational result, underlining the importance of our approach and confirming the intuition from part II that gravitational correlators are much richer in structure than their universal RMT sector would suggest.

This conspiracy of universal and system-specific contributions from the encounter to produce the full Airy volume is in a sense the inverted logic of the computation performed by the authors of [117], who associate some of the Kontsevich graphs contributing to the full Airy volume to encounters, but not others; both types of Kontsevich graphs are required to recover the *universal* spectral form factor however. Interestingly, our result only gives the Airy volume for lengths that are not too different due to the step function in eq. (9.71), as already discussed, while the computation in [117] suffers from no such restriction. This suggests the intriguing possibility of extending our method, and indeed the encounter formalism, outside the regime of linearisability.

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[22]Finding the origin of this difference is a subject of ongoing research. Hints may be available by computing the short-time expansion of the spectral form factor.

[23]Recall however that the integrals in eq. (9.58) should have a lower cutoff set by the shortest periodic orbit l_0. Depending on its value, the semiclassical Airy volume may automatically be positive semi-definite.

Chapter 10

The Maldacena-Shenker-Stanford bound on hyperbolic manifolds

I report the results we obtained in [4], where we developed a semiclassical method to compute the quantum Lyapunov exponent based on Wigner-Moyal phase space quantisation of the out-of-time ordered commutator. We also showed that in the infinite-dimensional limit, the system introduced in chapter 3 saturates the Maldacena-Shenker-Stanford bound, which I review in section 10.1. I introduce the necessary basics of Wigner-Moyal quantisation, including a formalism to compute Weyl symbols of time-evolved operators, in section 10.2, and restore physical units to the system of chapter 3 in section 10.3. In section 10.4.1, I reproduce the computation of the quantum Lyapunov exponent in general chaotic systems, before specialising to the system of chapter 3 in section 10.4.2, discussing both the classical and deep quantum regimes of the Lyapunov exponent, as well as the infinite-dimensional limit, where the MSS bound is saturated. I also show our results that indicate the existence of a universal curve for the quantum Lyapunov exponent in different dimensions. Finally, in section 10.5, I show our progress on evaluating subleading corrections to the leading Lyapunov growth, which are expected to respect bounds of their own by the arguments of [122].

The out-of-time ordered commutator (OTOC) is a quantity of great interest in the study of quantum chaos, going back to the work of Larkin and Ovchinnikov [89]. For a given quantum system with a Hamiltonian $\hat{H}$, consider self-adjoint operators $\hat{V}, \hat{W}$, and denote time-evolved operators[1] by $\hat{W}(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{W}e^{-\frac{i}{\hbar}\hat{H}t}$. Then, the OTOC reads

$$C(t) = -\left\langle [\hat{W}(t), \hat{V}]^2 \right\rangle, \tag{10.1}$$

where $\langle\cdot\rangle = \mathrm{tr}\left(\frac{e^{-\beta\hat{H}}}{\mathrm{tr}\left(e^{-\beta\hat{H}}\right)}\cdot\right)$ is the expectation value in the thermal state at temperature $T = 1/k_B\beta$. In some of the literature, OTOC refers to the closely related out-of-time ordered *correlator* $F(t) = \left\langle \hat{W}(t)\hat{V}\hat{W}(t)\hat{V} \right\rangle$, but I will mean only the commutator in the following.

In many quantum chaotic systems the OTOC is characterised by an initial exponential growth for times shorter than the so-called *scrambling time* t^* of

[1]Note that in this chapter, I restore physical units everywhere, hence the appearance of $\hbar$ in the following formulae.

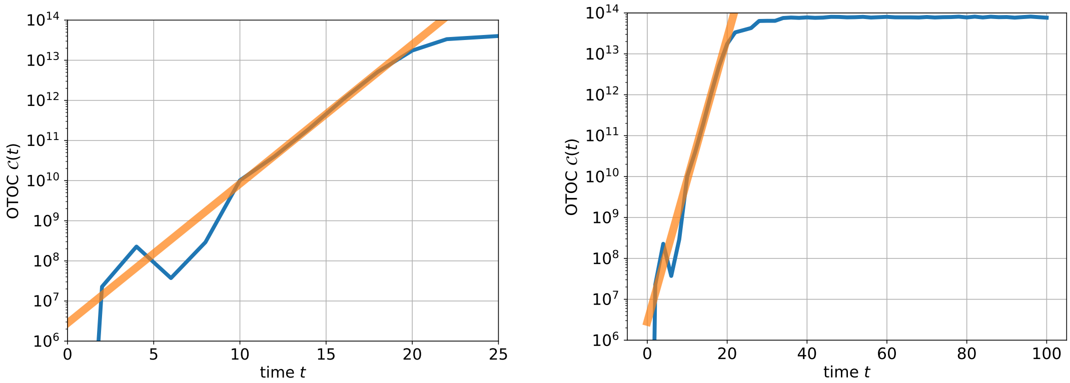

FIGURE 10.1: The OTOC $C(t) = \langle |[\hat{n}_1(t), \hat{n}]|^2 \rangle_\psi$ in a strongly kicked Bose-Hubbard system with $N = 10000$ particles and $L = 2$ sites, where $\hat{n}_1$ is the particle number operator on the first site and ψ is a suitably localised coherent state at energy E. Left: early times. The nearly linear slope indicates the initial exponential growth with a constant rate corresponding to twice the classical Lyapunov exponent. Right: The OTOC saturates at late times. Blue solid line: numerical data. Orange opaque line: linear fit. Taken from [4], data and plot by Mathias Steinhuber [288].

the system, i.e., the time necessary for initial state information to propagate through the entire system (although there are exceptions, see [90] and references therein). The rate of this exponential growth, Λ, defines the system's temperature-dependent quantum Lyapunov exponent,

$$C(t) \sim e^{\Lambda t}. \tag{10.2}$$

Note that there are differing conventions in the literature on whether to call the growth exponent of $C(t)$ Λ or 2Λ. The latter seems to be more common in the quantum chaos community [90], while the former is more prevalent in quantum gravity contexts [87, 286, 287], and it is also the convention I will be adopting in the rest of this chapter.

The reason for the nomenclature becomes obvious by picking for the operators, e.g., a canonical pair $\hat{x}, \hat{p}$ and corresponding classical phase-space degrees of freedom x, p. Provided the system has a classical limit, heuristic application of canonical quantisation demands that

$$\langle [\hat{x}(t), \hat{p}] \rangle_\psi \sim i\hbar \{x(t), p\}, \tag{10.3}$$

where the expectation value is defined with respect to a state ψ, suitably localised near a fixed energy E, and $x(t)$ the initial coordinate x evolved for a time t. Since for systems displaying chaotic dynamics in their classical limit,

$$\{x(t), p\} = \frac{\partial x(t)}{\partial x} \sim e^{\lambda t}, \tag{10.4}$$

where $\lambda(E)$ is the Lyapunov exponent of the classical counterpart, the corresponding OTOC, in this *microcanonical* framework, initially increases exponentially with a rate 2λ, as illustrated in fig. 10.1. More generally, in this microcanonical situation, the exponential OTOC growth rate has been shown to agree with twice the classical $\lambda(E)$ [90, 289, 290][2]. Also, eqs. (10.2) and (10.4) heuristically suggest

$$\Lambda = 2\lambda. \tag{10.5}$$

However, since the quantum Lyapunov exponent in eq. (10.2) is defined by a canonical initial state (and cannot be directly related to $\lambda(E)$), this naive equivalence must not be taken as correct without further caveats.

In this chapter, I will report results we obtained in [4], where we proposed a consistent quasiclassical theory of the canonical quantum Lyapunov exponent in quantum systems with a large number of degrees of freedom and a classical limit. Our work provides a link between quantum canonical and classical microcanonical Lyapunov exponents that has so far been elusive in general, in part due to the erroneous identification of $\lambda(E)$ and $\Lambda(\beta)$. Two notable exceptions to the latter are the work of Jalabert *et al.* for systems with few degrees of freedom [289], and Hashimoto *et al.* for general scaling systems [291]. In the former, the temperature-dependent exponent is calculated by applying the statistical definition of mean energy in the canonical ensemble to the microcanonical classical exponent $\lambda(E)$. As we showed in [4] however, the use of the *classical* Thomas-Fermi approximation for the microcanonical level density automatically renders the result classical, and the quantum features of $\Lambda(\beta)$ totally absent. In fact, our analysis showed that the very definition of $\Lambda(\beta)$ in systems with few degrees of freedom (such as the ones considered in [289] and [292, 293]) is ambiguous because the emergence of the exponential form of the OTOC will strongly rely on the standard assumptions behind the ensemble equivalence in statistical mechanics.

The analysis presented in [291] is closer in spirit to our approach, as it applies to systems with many degrees of freedom, but suffers from the lack of precise analytical results for $\lambda(E)$ and absence of quantum effects in the many-body level density. For an expanded discussion of the relation of the work of Hashimoto et al. [291, 294], I refer the reader to [3].

Our approach to the computation of the quantum Lyapunov exponent critically depends on two ingredients, namely the exact energy dependence of the *classical microcanonical* Lyapunov exponent, and the full quantum mechanical mean level density. As the reader will recall from chapter 3, both of these ingredients are indeed available in the chaotic system I studied throughout this thesis. Since the system exhibits the same correlation functions as JT gravity, which has been shown to saturate the Maldacena-Shenker-Stanford (MSS) bound on chaos [87, 295]

$$\Lambda \leq \frac{2\pi}{\hbar\beta}, \tag{10.6}$$

it is natural to ask whether our model saturates the bound as well. With it possessing a well-defined classical limit, we followed a natural approach for the calculation of the quantum canonical Lyapunov exponent from a quasiclassical perspective, where quantum properties of the system are appropriately expressed

[2] For times longer than t^*, the OTOC saturates, as evident, e.g., in fig. 10.1 and predicted in [290].

through its classical phase space structures. Due to the great degree of analytical control, afforded, e.g., by the semiclassical description via the Selberg trace formula (cf. chapter 3), as well as the exactly known density of states and microcanonical Lyapunov exponent, we were not only able to rigorously show the saturation of the MSS bound, but also that this saturation is a quantum effect. It stems from *quantum corrections to the leading power law behaviour of the density of states*.

Moreover, we were able to provide explicit results for the quantum Lyapunov exponent as a function of temperature and number of degrees of freedom, beyond the limiting cases, and took initial steps towards evaluating the first subleading $\hbar^2$ correction to the OTOC, which can be systematically determined by the mechanism of Wigner-Moyal phase space quantisation, and which has to exhibit exponential behaviour with a different, likewise bounded growth exponent by the arguments of [122].

In the remainder of the chapter, I will report the results of [4], which has been accepted and is being prepared for publication in the *Journal of High Energy Physics* (JHEP) at the time of writing. Having been submitted very recently, the contents of [4] represent the state of the art of our research on this topic, and my presentation in the following will not differ substantially from the one in [4]. I have merely made an effort to embed said contents into the context of the thesis, harmonising the notation and making minor adjustments wherever sensible, such as omitting introductory material already presented elsewhere in the thesis.

The rest of the chapter is organised as follows: in section 10.1, I quickly recapitulate the MSS bound. In section 10.2, I introduce the formalism of Wigner-Moyal quantisation, and particularly a way to obtain $\hbar$ expansions for Heisenberg operators developed in [296]. In section 10.3, I briefly discuss the reinsertion of physical units in the Selberg trace formula to allow for a sensible expansion in small $\hbar$. Finally, I will report our computation of the quantum Lyapunov exponent at leading and subleading order in sections 10.4 and 10.5.

10.1 The Maldacena-Shenker-Stanford bound

A particularly interesting facet of the study of OTOCs is the appearance of an analytical bound on its decay rate [87]. Namely, it can be shown that $f(t) = F(t)/F_d$, with $F_d = \langle \hat{V}\hat{V} \rangle \langle \hat{W}\hat{W} \rangle$ the (constant) factorised value of the out-of-time ordered correlator, satisfies

1. $f(t + i\tau)$ is analytic in the half strip $0 < t$, $-\frac{\beta}{4} \le \tau \le \frac{\beta}{4}$ and real for $\tau = 0$.

2. $|f(t + i\tau)| \le 1$ in the entire half-strip.

For such a function, then,

$$\frac{1}{1-f}\left|\frac{df}{dt}\right| \le \frac{2\pi}{\hbar\beta}, \tag{10.7}$$

which implies, upon *assuming* the form

$$F(t) = F_d - \epsilon e^{\Lambda t}, \tag{10.8}$$

that the decay rate Λ must obey

$$\Lambda \leq \frac{2\pi}{\hbar\beta}, \tag{10.9}$$

which is the famous Maldacena-Shenker-Stanford (MSS) bound[3]. The proof of (10.9) applies under fairly general assumptions, namely that the correlation functions factorise in a long enough time regime, and that there is a strong hierarchy between the scrambling time t^* and the (shorter) *dissipation time* t_d, which is roughly the decay time of two-point correlators like $\langle \hat{V}\hat{V}(t) \rangle$. This hierarchy is expected particularly in systems with many degrees of freedom and Hamiltonians built from finite products of simple operators. This is the case in many typical many-body systems, and also in large-dimensional few- or one-body systems such as large-dimensional Hadamard-Gutzwiller-like models.

It is important to realise that the bound applies under the assumption that there is a physical mechanism leading to the exponential growth of the second term in eq. (10.8). Therefore, verifying its validity starting from a full-fledged microscopic description of a given system automatically poses the problem of understanding the emergence of such exponential behaviour. This is in general a formidable task, as Lyapunov exponents are emergent, non-perturbative features of the dynamics and there are few physical systems where their existence and dependence on the energy or temperature are known explicitly. For a type of system satisfying certain homogeneity conditions [297], the dependence of the classical Lyapunov exponent on the energy can be inferred, and a theory of the quantum Lyapunov exponent, i.e., growth rate, starting from a given dependence on the energy, has been proposed in [291]. In all these cases, however, the exact form of the results beyond the scaling with the energy is lacking and therefore the closer study of the bound remains out of reach.

So far, no system with a unitary time evolution satisfying these assumptions has been found to violate eq. (10.9) to the best of my knowledge, and the cases where the equality is saturated are particularly interesting[4]. Such systems are usually gravitational, or dual to a gravitational system, and often involve black holes as well[5]. While it is expected that systems with black holes saturate the bound, there is no consensus on whether a black hole is necessary, or on "how gravitational" a system has to be to saturate the bound. An example of such a system is Jackiw-Teitelboim gravity [295] and its dual, the Sachdev-Ye-Kitaev model [104]. Independent calculations on both sides of the duality show the saturation of the MSS bound in this case. Interestingly, there is a class of gravitational models, the $(2, 2p-1)$ minimal string theories, that limit to JT gravity as $p \to \infty$, and for which the question of saturation of (10.9) or not is as yet unanswered [270]. We will briefly comment on these models in section 10.4.

A more careful analysis of the analytic structure of $F(t)$ that I will touch on in section 10.5 reveals that (10.8) are only the first two terms in a short-time

[3] I emphasise for clarity that the decay rate of $F(t)$, and thereby also the decay rate of the OTOC $C(t)$, is bounded by $\frac{2\pi}{\hbar\beta}$, irrespective of whether we call this decay rate Λ or 2Λ.

[4] Note however that in non-unitary theories, the bound can be violated [298].

[5] Beyond the examples already mentioned in the introduction to the chapter [52, 53, 108, 109, 178], see, e.g., [299].

expansion of the out-of-time ordered correlator [122],

$$F_d - F(t) = \epsilon f_1 e^{\Lambda t} + \epsilon^2 f_2 e^{\Lambda_2 t} + O\left(\epsilon^3\right). \tag{10.10}$$

Here, ϵ is a small parameter ensuring that the corrections are subleading compared to the leading Lyapunov growth. Note that, consequently, what is referred to in the literature as quantum Lyapunov exponent is *defined* as the rate of growth of the leading contribution to the correlator. In systems such as the Hadamard-Gutzwiller-like models introduced in chapter 3, where the scrambling time can be identified with the Ehrenfest time[6] $t_E = \lambda^{-1}\log(\text{const.}/\hbar)$ [290, 300, 301], a natural candidate for these subleading terms are quantum corrections, and it is sensible to set $\epsilon = \hbar^2$.

10.2 Wigner-Moyal quantisation

10.2.1 Basics and definitions

Given the close, but not yet well-understood connection between classical and quantum Lyapunov exponents, it is helpful to make use of the Wigner-Moyal formalism, which makes the connection between classical (Hamiltonian) and quantum mechanics particularly transparent when post-Ehrenfest time interference effects can be neglected. Being essentially a thermodynamic object, the temperature-dependent quantum Lyapunov exponent is expected to satisfy this condition.

A naive (and for many purposes sufficient) understanding of quantisation supposes that one can define a quantum theory by finding operators $\hat{x}^i, \hat{p}_i$ for any phase-space degrees of freedom with coordinates x^i, p_i and identifying commutators with the classical Poisson algebra,

$$\{x^i, p_j\} = \delta^i_j \xrightarrow{\text{Quantisation}} \frac{1}{i\hbar}[\hat{x}^i, \hat{p}_j] = \delta^i_j, \tag{10.11}$$

i.e., by taking a "reverse classical limit". Trying to generalise this prescription to general phase-space functions, one quickly runs into problems however: Groenewold's theorem states that no quantisation map can be found that preserves the classical Poisson structure for all polynomials in $x = (x^1, x^2, \ldots, x^f)$, $p = (p_1, p_2, \ldots, p_f)$ of degree 3 or less [302].

Moyal subsequently showed that the correct phase-space algebra to represent the quantum operator algebra is not given by Poisson brackets, but by an $\hbar$ deformation of theirs, usually referred to as the Moyal bracket [303].

To make use of this representation, we need to define the *Weyl symbol* $W_{\hat{A}}$ of an operator $\hat{A}$,

$$W_{\hat{A}}(x,p) = \int dx' \sqrt[4]{g(x+x'/2)g(x-x'/2)}\, e^{\frac{i}{\hbar}p_i (x')^i} \langle x - x'/2|\hat{A}|x + x'/2\rangle, \tag{10.12}$$

[6] The Ehrenfest time is the characteristic time scale that signals the dominance of interference effects in chaotic systems with classical Lyapunov exponent λ.

with $g(x)$ the determinant of the configuration space metric at the point x. Depending on the ordering of the operator $\hat{A}$, Weyl symbols of different operators with the same classical limit may differ by quantum corrections.

Some particularly important Weyl symbols are those of phase-space polynomials in the so-called Weyl ordering,

$$(ax + bp)^n \xrightarrow{\text{Weyl quantisation}} (a\hat{x} + b\hat{p})^n. \tag{10.13}$$

In this case, one can just reverse the arrow to find the Weyl symbol,

$$W_{(a\hat{x}+b\hat{p})^n} = (ax + bp)^n. \tag{10.14}$$

Additionally, we need the Weyl symbol of a density matrix $\hat{\rho}$, called the Wigner function of the state $\hat{\rho}$,

$$W(x,p) \equiv W_{\hat{\rho}}(x,p) = \int dx' \sqrt[4]{g(x + x'/2)g(x - x'/2)} e^{\frac{i}{\hbar} p_i (x')^i} \langle x - x'/2|\hat{\rho}|x + x'/2\rangle . \tag{10.15}$$

The Wigner function is a quasiprobability distribution on the phase space, and allows for the computation of expectation values in the corresponding state,

$$\langle \hat{A} \rangle_{\hat{\rho}} = \int dxdp\, W(x,p) W_{\hat{A}}(x,p). \tag{10.16}$$

We further introduce the (Moyal) $\star$ product,

$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n \Pi^n(f,g), \tag{10.17}$$

with the Poisson bivector $\Pi = \nabla J \nabla$, where $J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$ is the standard symplectic form[7] (on a two-dimensional phase space[8]):

$$\begin{aligned} \Pi^0(f,g) = fg, \qquad \Pi^1(f,g) = \{f,g\}, \\ \Pi^n(f,g) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{\partial^k}{\partial p^k} \frac{\partial^{n-k} f}{\partial x^{n-k}} \right) \left(\frac{\partial^{n-k}}{\partial p^{n-k}} \frac{\partial^k g}{\partial x^k} \right), \end{aligned} \tag{10.18}$$

and suitably generalised in more dimensions, as well as the Moyal bracket,

$$\{f,g\}_M = f \star g - g \star f. \tag{10.19}$$

[7]One can easily see from this definition that $\Pi^{2m+1}(f,g) = -\Pi^{2m+1}(g,f)$, and $\Pi^{2m}(f,g) = \Pi^{2m}(g,f)$.

[8]This holds for a phase space with a flat symplectic form $J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$. One can always choose coordinates in which this is locally the case, i.e., for a system with configuration space $\mathcal{X}$, one picks the local trivialisation $\mathcal{X} \times \mathbb{R}^f$ of the phase space, and then chooses Riemannian normal coordinates on $\mathcal{X}$. If this is not desired, one has to employ the more general Kontsevich quantisation formula [304].

Equation (10.17) also induces an $\hbar$ expansion in the Moyal bracket,

$$\{f,g\}_M = \sum_{n=0}^{\infty} \frac{(-1)^n(\hbar/2)^{2n}}{(2n+1)!}\Pi^{2n+1}(f,g). \tag{10.20}$$

With these definitions, we can use the following properties for Weyl symbols:

$$W_{\hat{A}\hat{B}} = W_{\hat{A}} \star W_{\hat{B}}, \tag{10.21}$$
$$W_{[\hat{A},\hat{B}]} = i\hbar\{W_{\hat{A}}, W_{\hat{B}}\}_M. \tag{10.22}$$

10.2.2 Heisenberg operators

In order to compute the OTOC, we need a way to find the Weyl symbols of time-evolved (Heisenberg picture) operators. Naively, one could simply cast a Heisenberg operator,

$$\hat{A}(t) = e^{i\hat{H}t/\hbar}\hat{A}e^{-it\hat{H}/\hbar}, \tag{10.23}$$

in terms of the Weyl symbols of $\hat{A}$ and the time evolution operator $e^{-i\hat{H}t/\hbar}$ using eq. (10.21). However, the latter exhibits an essential singularity at $\hbar = 0$ and therefore does not admit a regular Taylor expansion around that point. Since this is exactly what we want to compute, an alternative way of determining the Weyl symbols of operators like (10.23) is needed.

Osborn and Molzahn [296] provide such a way, which I will closely follow for the remainder of this section. Consider Hamiltonians with a Weyl symbol of the form

$$W_{\hat{H}}(t,\hbar;\zeta) = H_c(t;\zeta) + \sum_{r=1}^{\infty} \frac{\hbar^r}{r!} H_r(t;\zeta), \tag{10.24}$$

where $\zeta = (x,p)$ is an initial-time phase-space point, and the argument t refers to a possible explicit time dependence of the Hamiltonian. We can time evolve operators by acting with some $\hat{\Gamma}(s,t)$,

$$\hat{\zeta}(t,s) = \hat{\Gamma}(s,t)\hat{\zeta}, \quad \hat{H}(t,s) = \hat{\Gamma}(s,t)\hat{H}(t). \tag{10.25}$$

such that the time evolution is determined by the Heisenberg equation,

$$i\hbar\frac{d}{dt}\hat{\zeta}(t,s) = [\hat{\zeta}(t,s),\hat{H}(t,s)] = \hat{\Gamma}(s,t)[\hat{\zeta},\hat{H}(t)]. \tag{10.26}$$

The second equation usefully reveals the dependence of the so-called quantum trajectory $\hat{\zeta}(t,s)$ on the commutator between the initial condition $\hat{\zeta}$ and the Hamiltonian. Equation (10.26) is solved by the ansatz

$$W_{\hat{\zeta}}(t,s,\hbar;\zeta) = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!}\zeta_r(t,s;\zeta), \tag{10.27}$$

where the expansion coefficients are determined as follows:

The classical limit of eq. (10.26) is

$$\frac{d}{dt}\zeta_0(t,s;\zeta) = J\nabla H_c(t,\zeta_0(t,s;\zeta)), \tag{10.28}$$

which is precisely Hamilton's equation [103] and, therefore, is solved by the classical flow generated by the Hamiltonian H_c,

$$\zeta_0(t,s;\cdot) = \gamma(t,s|\cdot), \tag{10.29}$$

i.e., $\gamma(t,s|(x_s,p_s)) = (x(t),p(t))$, or simply, the classical solution of the equations of motion. For higher order coefficients, we define the Jacobi operator

$$\mathcal{J}(t;s,\zeta) = \frac{d}{dt} - J\nabla\nabla H_c(t,\gamma(t,s|\zeta)), \tag{10.30}$$

which gives equations of the form

$$\mathcal{J}(t;s,\zeta)\zeta_r = f_r(t,s;\zeta). \tag{10.31}$$

In the following, we will only consider the first

$$\mathcal{J}(t;s,\zeta)\zeta_1(t,s;\zeta) = J\nabla H_1(t,\gamma(t,s|\zeta)), \tag{10.32}$$

and second equation[9],

$$\mathcal{J}(t;s,\zeta)\zeta_2(t,s;\zeta) = \left[(\zeta_1\cdot\nabla)^2 - \frac{1}{8}\Pi^2(\gamma\cdot\nabla)^2 + \frac{1}{12}\Pi_{12}\Pi_{23}(\gamma\cdot\nabla)^3\right] J\nabla H_c(t,\gamma(t,s|\zeta)) \\ + 2(\zeta_1\cdot\nabla)J\nabla H_1(t,\gamma(t,s|\zeta)) + J\nabla H_2(t,\gamma(t,s|\zeta)). \tag{10.33}$$

These equations can then be integrated using a Green's function of the Jacobi operator $\mathcal{J}$, which can be chosen as a function of derivatives of the classical flow γ [296],

$$\zeta_r(t;\zeta) = \int_0^t ds\nabla\gamma(t,0|\zeta)J\nabla\gamma(s,0|\zeta)^T J^{-1} f_r(s,0;\zeta). \tag{10.34}$$

A similar treatment allows for finding the Weyl symbols of more complicated operators, but in this work, I am only concerned with quantum trajectories $\hat{\zeta}(t,s)$. A nice side effect of this method is that it does not make explicit reference to the potentially curved configuration space: any difficulties arising in that context are encoded in the classical flow γ, and in the Weyl symbol of the Hamiltonian (10.24).

[9] Expressions such as $\Pi_{ij}f_1\ldots f_n$ are to be understood as taking derivatives w.r.t. the arguments i and j, then evaluating at $\zeta_k = \zeta$ for all $k = 1,\ldots,n$.

10.3 The Hadamard-Gutzwiller-like model in physical units

In this section, I will restore units to the Hadamard-Gutzwiller-like model introduced in chapter 3. Concretely, I want to consider a particle of mass m moving freely on a manifold $\mathcal{M}$ of dimension f and constant curvature $R = -2/L^2$.

An example of such a system is sketched in fig. 3.4, together with a periodic orbit of the classical dynamics, generated by the Hamilton function

$$h = \frac{1}{2m} p_i g^{ij} p_j, \tag{10.35}$$

and the metric g_{ij} reads

$$g_{ij} = \frac{4L^2}{(1-(x^1)^2 - \ldots - (x^f)^2)^2}\delta_{ij}. \tag{10.36}$$

The associated quantum Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2mL^2}\Delta, \tag{10.37}$$

such that $\hbar^2/2mL^2$ sets the energy scale of the system. Inserting this energy scale in the Selberg trace formula (3.1), we obtain

$$\sum_n u\left(\sqrt{\frac{2mL^2}{\hbar^2}E_n}\right) = \frac{L^f \mathcal{V}}{(4\pi)^{f/2}} \int_0^\infty dr\, u(r)\Phi_f(r) + \sum_{\gamma\in\Gamma^*}\sum_{k=1}^{\infty} A_\gamma\, \tilde{u}\left(\frac{kl_{\text{PO}}}{L}\right). \tag{10.38}$$

The STF is not particularly useful for the computation of the OTOC we performed in [4] (although there are examples of OTOC calculations using semiclassical theory, e.g. [289, 290]), but it reveals one of the ingredients needed for our computation: the Plancherel measure $\Phi_f(r)$[10]. After substituting $r = \sqrt{\frac{2mL^2E}{\hbar^2}}$,

$$\Phi_f(r)dr \equiv \varrho_f(E)\sqrt{\frac{2mL^2}{\hbar^2 E}}dE, \tag{10.39}$$

where $\varrho_f(E)$ denotes the density of states, simply counts the number of states in an energy interval. It therefore doubles as the microcanonical partition function $Z(E)$ of the theory, which will be important in section 10.4. After absorbing the

[10] It should be noted that even though we do not explicitly use the STF to compute the OTOC, interfering classical paths are still expected to contribute and, eventually, rival the Plancherel measure contribution, particularly for states that are extended over the entire system. If we start with a classical state that is localised in phase space, the time it takes to spread over the entire system is precisely the Ehrenfest time, i.e., when the OTOC is expected to deviate significantly from the simple exponential growth (10.2). The saturation of the OTOC around the Ehrenfest time has been shown [290] to result from interfering paths. It is therefore justified to leave out such contributions in the computation of the early-time exponential growth rate.

Jacobi determinant factor, the density of states in physical units reads

$$\varrho_f(E) = \sqrt{\frac{2mL^2}{\hbar^2 E}} \frac{f}{(4\pi)^{f/2}\Gamma\left(\frac{f+2}{2}\right)} \frac{\left|\Gamma\left(i\sqrt{\frac{2mL^2}{\hbar^2}E} + (f-1)/2\right)\right|^2}{\left|\Gamma\left(i\sqrt{\frac{2mL^2}{\hbar^2}E}\right)\right|^2}$$
$$= \begin{cases} \frac{2mL^2}{\hbar^2} \frac{\tanh\left(\pi\sqrt{\frac{2mL^2}{\hbar^2}E}\right)}{(2\pi)^{f/2}(f-2)!!} \prod_{k=0}^{\frac{f-4}{2}} \left(\frac{2mL^2}{\hbar}E + \left(k + \frac{1}{2}\right)^2\right) & f \text{ even,} \\ \frac{1}{2^{(f-1)/2}\pi^{(f+1)/2}(f-2)!!} \sqrt{\frac{2mL^2}{\hbar^2 E}} \prod_{k=0}^{\frac{f-3}{2}} \left(\frac{2mL^2}{\hbar^2}E + k^2\right) & f \text{ odd,} \end{cases} \tag{10.40}$$

and in the $f \to \infty$ limit, we recover the sinh density of states of JT gravity,

$$\varrho_\infty(E) = \frac{2mL^2}{\hbar^2} \sinh\left(\pi\sqrt{\frac{2mL^2}{\hbar^2}E}\right). \tag{10.41}$$

As I will show in the next section, the quasiclassical calculation of the quantum Lyapunov exponent critically depends not only on the precise knowledge of the level density, eq. (10.40), but also on the specific form of the classical, microcanonical Lyapunov exponent. Remarkably, in our case, the latter is not only fully independent of the dimension[11], but its rigorously exact dependence on the microscopic parameters of the theory and the energy is known and given by [167]

$$\lambda(E) = \sqrt{\frac{2E}{mL^2}}, \tag{10.42}$$

where $E = p^2/2m$.

Equipped with these two ingredients, the exact mean level density and the classical microcanonical Lyapunov exponent, we can proceed to the leading order computation of the quantum Lyapunov exponent.

10.4 Leading-order quantum Lyapunov exponent

10.4.1 General strategy

We start with the computation of the OTOC for systems with classical chaotic limit, up to leading order in $\hbar$ with the operator choice $\hat{V} = \hat{x}, \hat{W} = \hat{p}$:

$$C(t) = -\operatorname{tr}\left(\hat{\rho}(\beta)[\hat{p}(t), \hat{x}]^2\right), \tag{10.43}$$

where $\hat{\rho}(\beta) = \frac{e^{-\beta\hat{H}}}{Z(\beta)}$ is the thermal state, as specified in section 10.1. Following standard methods of statistical physics [305], we can rewrite this expression in

[11]Recall chapter 3.

terms of an energy integral in exchange for introducing a factor $\delta(E-\hat{H})$. The result can then be related to the microcanonical average of the commutator under consideration:

$$\begin{aligned} C(t) &= -\int_0^\infty dE \, \mathrm{tr}\left(\frac{e^{-\beta\hat{H}}}{Z(\beta)}[\hat{p}(t),\hat{x}]^2\delta(E-\hat{H})\right) \\ &= -\int_0^\infty dE \frac{e^{-\beta E}}{Z(\beta)} Z(E) \frac{1}{Z(E)} \, \mathrm{tr}\left(\delta(E-\hat{H})[\hat{p}(t),\hat{x}]^2\right) \\ &= -\int_0^\infty dE \frac{Z(E)}{Z(\beta)} e^{-\beta E} \left\langle [\hat{p}(t),\hat{x}]^2 \right\rangle_{\mathrm{mc}}, \end{aligned} \tag{10.44}$$

where we used the fact that $\delta(E-\hat{H})/Z(E)$ is precisely the microcanonical density matrix, and $Z(E) = \mathrm{tr}\,\delta(E-\hat{H})$. Canonical OTOCs were previously computed via the microcanonical OTOC in [294] by explicitly summing over a system's energy eigenvalues and eigenstates. This approach allows for a fairly precise (numerical) evaluation of the OTOC when such spectral information is available, but analytical control is difficult to obtain even in integrable systems. The examples studied in [294] also do not exhibit exponential growth in the OTOC. As we will see in the following, our approach allows us to robustly and analytically identify the leading quantum Lyapunov exponent of classically chaotic systems (or even just systems with a classical instability, e.g., [91, 92, 306]) *and* the time regime for which this exponential growth obtains, based only on analytic control over the *smoothed* spectral density and the classical Lyapunov (or instability) exponent. To do so, we express the microcanonical OTOC on the classical phase space through its Wigner-Moyal quantisation,

$$C(t) = -\int_0^\infty dE \frac{Z(E)}{Z(\beta)} e^{-\beta E} \int dx dp \, W_{[\hat{p}(t),\hat{x}]^2} W(x,p), \tag{10.45}$$

where $(x,p) \in \mathbb{R}^{2f}$ are coordinates parametrizing the phase space of the system, $W_{\hat{A}}$ is the Weyl symbol of the operator $\hat{A}$ and $W(x,p) = W_{\delta(E-\hat{H}(\hat{x},\hat{p}))}$ is the Wigner function. For simplicity of notation, we will assume that the phase space is covered by a single coordinate patch, so that there is only one region that contributes to the phase-space integral in eq. (10.45). A further subtlety arises from the fact that on compact manifolds, momenta conjugate to coordinates with finite range (e.g. angles) are quantised, leading to a discrete phase space. However, after replacing the corresponding momentum integrals by sums, our arguments are still applicable. In the quasiclassical limit, the continuous phase space can be recovered. These subtleties aside, eq. (10.45) is exact.

Directly computing the Weyl symbol of such a complicated operator is quite hard, but we can simplify the task somewhat:

$$\begin{aligned} W_{[\hat{p}(t),\hat{x}]^2} &= W_{[\hat{p}(t),\hat{x}]} \star W_{[\hat{p}(t),\hat{x}]} \\ &= -\hbar^2 \{W_{\hat{p}(t)}, W_{\hat{x}}\}_M \star \{W_{\hat{p}(t)}, W_{\hat{x}}\}_M \end{aligned} \tag{10.46}$$

The Weyl symbol of Weyl ordered initial time operators is simply given by the direct replacement of all factors $\hat{x}, \hat{p} \to x, p$, i.e., by their corresponding phase-space functions. For time-evolved or not Weyl-ordered operators, finding the Weyl symbol is much more involved (as we will see in section 10.5), but the beauty of the Wigner-Moyal approach lies in the fact that at leading order in $\hbar$, we can simply replace

1. the Weyl symbol $W_{\hat{p}(t)}$ by the classical solution $p(t)$,
2. the Moyal bracket by the Poisson bracket and
3. the $\star$ product by the usual phase-space product.

Likewise, at leading order, the Wigner function is given by

$$W_E(x,p) = \frac{\delta(E - h(x,p))}{Z(h(x,p))} + \mathcal{O}(\hbar). \tag{10.47}$$

A crucial observation is that, for the OTOC understood as an expansion in a small parameter [87] with corrections becoming important at later times [122, 123], our calculation up to this point shows that the leading order Lyapunov exponent is *independent* of the regularisation (i.e., choice of distribution of the density matrix factors) – a property so far only assumed and confirmed numerically, but never rigorously shown [307].

After the above replacements, the OTOC reads, to leading order,

$$C(t) \approx \hbar^2 \int_0^\infty dE\, \frac{Z(E)}{Z(\beta)} e^{-\beta E} \int dx dp\, W_E(x,p) \left| \frac{\partial p^i(t;x,p)}{\partial p^j} \right|^2, \tag{10.48}$$

where we made the dependence of $p(t)$ on the initial conditions explicit. Since we are only interested in the growth rate of this integral, evaluating it exactly is not necessary. Now the classical microcanonical Lyapunov exponent λ enters, capturing the exponential growth of the off-diagonal blocks $\{p^i(t), q_j\}$,

$$\{p^i(t), q_j\} = -\frac{\partial p^i(t;x,p)}{\partial p^j} = F^i_j(x,p) e^{\lambda(x,p)t}, \tag{10.49}$$

of the stability matrix [308].

Up to now, our only assumption is that the system is chaotic and admits a well-defined classical limit, with a leading-order approximation in $\hbar$ to the OTOC given by inserting eq. (10.49) into eq. (10.48). The canonical quantum Lyapunov exponent is then obtained by identifying the corresponding leading order-contribution in $\hbar$ to the exponential growth (if any) of the OTOC. Progress in this direction is only possible if the specific dependences of both the thermodynamic $Z(E), Z(\beta)$ and dynamical $\lambda(x,p)$ functions entering eq. (10.48) are known.

10.4.2 The canonical quantum Lyapunov exponent for high-dimensional hyperbolic motion

To apply the method above to high-dimensional hyperbolic motion, we note that in this case, the Lyapunov exponent is constant on the classical energy shell, namely where $\lambda(x,p)$ depends on the initial conditions (x,p) only through the energy $E=h(x,p)$, such as the ones considered in [291] and [297, 308]. This is a special and convenient feature, as the dependence of $\lambda(x,p)$ on the initial phase space region in general systems is expected to be complicated and, in most cases, simply not known. $F^i_j(x,p)$ is a slowly varying phase-space function that will not be important for our purposes[12]. Using eq. (10.47), we have

$$W_E(x,p)e^{2\lambda(h(x,p))t} = W_E(x,p)e^{2\lambda(E)t} + O(\hbar), \tag{10.50}$$

so we can pull the exponential term out of the phase-space integral and are left with

$$\begin{aligned} C(t) &= \hbar^2 \int_0^\infty dE \frac{Z(E)}{Z(\beta)} e^{2\lambda(E)t-\beta E} \overline{\left|F^i_j\right|^2}(E) \\ &= \frac{\hbar^2}{Z(\beta)} \int_0^\infty dE\, e^{2\lambda(E)t-\beta E+\log Z(E)} \overline{\left|F^i_j\right|^2}(E) \end{aligned} \tag{10.51}$$

where $\overline{\left|F^i_j\right|^2}(E)$ is the phase-space average of the slowly varying part of the stability matrix, and hence is expected to only weakly depend on the energy as well.

When $f=2$, this result for the leading $O(\hbar^2)$ contribution to the OTOC coincides with the result of [289] for the appropriate choice of operators. However, it does not allow for an unambiguous identification of a quantum Lyapunov exponent as there is no clear region of exponential growth. As we show below, this is because the key ingredient for such an identification is a saddle point analysis only justified in the regime of ensemble equivalence, $f\to\infty$, where the bound on chaos was originally derived [87].

We will invoke the standard tools of ensemble equivalence well known in statistical physics [309], that are asymptotically exact in the limit $f\to\infty$. From eq. (10.51), the growth rate of the integral, and thereby the Lyapunov exponent, can be estimated (up to loop corrections) by evaluation at the stationary point E^*_β of the integrand,

$$C(t) \approx \hbar^2 \overline{\left|F^i_j\right|^2}(E^*_\beta) e^{2\lambda(E^*_\beta)t}, \tag{10.52}$$

where we used that $Z(\beta) = Z(E^*_\beta)e^{-\beta E^*_\beta}$ [309, 310] as dictated by the standard thermodynamic relation between the entropy and the free energy. This result, derived using only the Plancherel measure corresponding to zero-length classical orbits as input, is in particular consistent with [290], which also finds the OTOC to be growing with a rate given by twice the classical Lyapunov exponent. The computation in [290] uses extended classical paths and finds the exponential growth regime to be dominated by multiple traversals of short reference paths. After the Ehrenfest time, encounters from longer classical paths produce interference effects that destroy this exponential growth. Given that we are only interested in the pre-Ehrenfest time regime and find the same growth rate as

[12] Roughly, $F^i_j(x,p)$ is a function of the basis vectors of the tangent space at the phase-space point (x,p), while the exponential behaviour is captured by the stretching factor $e^{\lambda t}$ [308].

[290], we are justified in ignoring contributions from extended classical paths altogether.

Plugging in the microcanonical Lyapunov exponent (10.42) of the system, the stationarity condition reads

$$0 = 2\lambda'(E^*_\beta)t - \beta + \frac{Z'(E^*_\beta)}{Z(E^*_\beta)} \equiv \frac{\sqrt{2}t}{\sqrt{mL^2E^*_\beta}} - \beta + G_f(\beta), \tag{10.53}$$

where $G_f(\beta) = \varrho'_f(E)/\varrho_f(E)$ is determined by approximating the microcanonical partition function $Z(E)$ by its smooth part, $Z(E) \approx \varrho_f(E)$.

We can analytically evaluate eq. (10.53) in two interesting regimes. The density of states (10.40) of our model (we may restrict to odd f for simplicity) takes the form of a curvature expansion [162–164]. This expansion comprises the highest-degree monomial given by the Weyl volume law [94], as well as all quantum corrections in the form of a lower-degree polynomial [259] that become increasingly important for low energies. Therefore, if we want to access the classical regime of the system, the highest-degree term in eq. (10.40) should dominate the others [305]. We can translate this to the simple condition (see appendix C for a short derivation),

$$\frac{f^2}{24} \ll \frac{2mL^2}{\hbar^2\beta} \equiv \frac{4\pi L^2}{\lambda^2_{\text{th}}}, \tag{10.54}$$

where we introduced the particle's thermal de Broglie wavelength,

$$\lambda_{\text{th}} = \sqrt{\frac{2\pi\hbar^2\beta}{m}}. \tag{10.55}$$

If the number of degrees of freedom f is large, which is necessary for the applicability of the saddle point approximation (10.52), the condition (10.54) simply means that their thermal wavelength has to be small enough not to experience curvature effects. Since curvature corrections to the density of states play the role of quantum corrections in our system, this is consistent with the aim of studying the system in the classical regime. Neglecting the quantum corrections then, we can solve the stationarity condition (10.53) with

$$G_f(\beta) = \frac{f}{2E^*_\beta} \tag{10.56}$$

and find the growth rate of the OTOC to be

$$2\lambda(E^*_\beta) = \frac{\sqrt{\frac{4t^2}{mL^2}} \pm \sqrt{\frac{4t^2}{mL^2} + 4\beta f}}{\sqrt{mL^2\beta^2}}, \tag{10.57}$$

which is independent of $\hbar$, as expected. We may neglect the t-dependent terms in eq. (10.57) if

$$\frac{1}{f\beta} \ll \frac{mL^2}{t^2} \Leftrightarrow \frac{L}{\lambda_{\text{th}}} \ll \sqrt{\frac{f}{2\pi}\frac{mL^2}{\hbar t}}. \tag{10.58}$$

In this case, we can solve for the quantum Lyapunov exponent,

$$2\lambda(E^*_\beta) = \sqrt{\frac{4f}{mL^2\beta}}. \tag{10.59}$$

Recalling eq. (10.42), this is simply the statement that the thermal Lyapunov exponent is determined by a temperature that is dictated by the classical equipartition theorem. Usually, one would associate larger, rather than smaller temperatures to more classical behaviour, but in the system at hand, it is known that quantum corrections to the density of states take the form of a curvature expansion [162–164], and hence, it is sensible to stay far away from the regime in which such corrections start mattering to observe the system at its "most classical".

This result should be consistent with eqs. (10.54) and (10.58) if the time is at least of the order of the dissipation time $t_d = 1/\lambda(E^*_\beta)$. This condition yields the following chain of inequalities,

$$\frac{f}{\sqrt{96\pi}} \ll \frac{L}{\lambda_{\text{th}}} \ll f\frac{L}{\lambda_{\text{th}}}, \tag{10.60}$$

which are obviously satisfied for large f and if the temperature is high enough.

As the thermal wavelength of the particle grows to comparable size to the curvature radius, i.e., as the temperature decreases, quantum corrections become very important, qualitatively changing the scaling of $G_f(E)$. Eventually, the entire polynomial (10.40) will contribute, approximating the infinite-dimensional sinh-behaviour. It is beneficial to exploit the product structure of eq. (10.40) and write the density of states (again in the f odd case for simplicity) as the $f \to \infty$ result up to multiplicative corrections,

$$\varrho_f(E) \propto \frac{\sinh\left(\pi\sqrt{\frac{2mL^2}{\hbar^2}E}\right)}{\prod_{k=\frac{f-1}{2}}^{\infty} k^2 + \frac{2mL^2}{\hbar^2}E}, \tag{10.61}$$

which yields

$$G_f(\beta) = \sqrt{\frac{mL^2}{2\hbar^2}}\left(\frac{\pi}{\sqrt{E^*_\beta}}\coth\left(\pi\sqrt{\frac{2mL^2}{\hbar^2}E^*_\beta}\right) - \sum_{k=1}^{\infty}\frac{1}{(k+f)^2/4 + 2mL^2E^*_\beta/\hbar^2}\right). \tag{10.62}$$

Using the fact that

$$\coth x = \frac{1+e^{-2x}}{1-e^{-2x}} \approx 1 \quad \text{for } x \gtrsim 1, \tag{10.63}$$

we can neglect the coth factor whenever

$$E_\beta^* \gtrsim \frac{\hbar^2}{2mL^2}\frac{1}{\pi^2}. \tag{10.64}$$

With this simplification, it remains to solve the stationarity condition (10.53) for

$$G_f(\beta) \approx \sqrt{\frac{mL^2}{2\hbar^2}}\left(\frac{\pi}{\sqrt{E_\beta^*}} - \sum_{k=1}^{\infty}\frac{1}{(k+f)^2/4 + 2mL^2E_\beta^*/\hbar^2}\right). \tag{10.65}$$

If the dimension is sufficiently large, the sum in eq. (10.65) should only result in a small correction to the stationarity condition (10.53),

$$\frac{f^2}{4} \gg \frac{2mL^2E_\beta^*}{\hbar^2} \approx \frac{4L^4}{\lambda_{\text{th}}^4}, \tag{10.66}$$

where the "≈" comes from plugging in the solution

$$E_\beta^* = \frac{mL^2\pi^2}{\hbar^2\beta^2}, \tag{10.67}$$

obtained from solving the saddle point equation neglecting the sum in eq. (10.65)[13]. Note that this describes (roughly) the opposite extreme to the classicality condition (10.54), i.e., at fixed dimension, the thermal wavelength needs to increase until curvature (and thus quantum) effects become relevant to access it. This justifies solving eq. (10.53) iteratively. Neglecting the last term, solving for E_β^*, see eq. (10.67), inserting this again into the full equation, and solving a second time, results in

$$\begin{aligned} 2\lambda(E_\beta^*) &= \frac{2(\hbar t + \pi mL^2)}{mL^2\hbar\beta}\frac{1}{1 + \frac{2}{\pi}\,\mathrm{Im}(\Psi(\frac{f-1}{2} + i\frac{\hbar t + \pi mL^2}{\hbar^2\beta}))} \\ &\xrightarrow{\hbar \ll \frac{mL^2}{t}} \frac{2\pi}{\hbar\beta}\left(1 - \frac{8\pi L^2}{\lambda_{\text{th}}^2}\frac{1}{f} + \ldots\right), \end{aligned} \tag{10.68}$$

with the digamma function $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. The regime indicated by the arrow is accessed if the typical action of a trajectory of the particle $\frac{mL^2}{t}$ is large against $\hbar$. This regime can be made sense of by recalling that for the applicability of the arguments of [87] used to derive the MSS bound, one needs a strong separation of the dissipation and scrambling time scales. That is to say,

$$t_d = \frac{1}{\Lambda} \ll \frac{1}{\Lambda}\log(S_{\text{cl}}/\hbar) = t_E, \tag{10.69}$$

where $S_{\text{cl}} = \frac{mL^2}{t}$ is a typical classical action relevant for the definition of the Ehrenfest time [290, 300, 301]. In order for these time scales to obey the hierarchy

[13]This energy has to be in particular consistent with eq. (10.64).

(10.69), we need

$$1 \ll \frac{S_{\text{cl}}}{\hbar} = \frac{mL^2}{\hbar t}, \tag{10.70}$$

which is exactly the short-time condition we used in eq. (10.68). This short-time condition can hence be understood in the sense of a semiclassical limit[14].

In particular, we may chain the short-time condition (10.69) and the condition (10.66) to access the quantum regime to find

$$\frac{f}{2} \gg \frac{2L^2}{\lambda_{\text{th}}^2} \gg \frac{2\hbar t}{\lambda_{\text{th}}^2 m}. \tag{10.71}$$

Naively, this chain of inequalities can be satisfied by choosing the dimension sufficiently large and the time sufficiently short. However, in order for the OTOC to exhibit Lyapunov growth, we need to wait for at least a dissipation time $t_d = 1/\Lambda$. Plugging in the short-time quantum Lyapunov exponent (10.68) while neglecting terms of $\mathcal{O}\left(\frac{L^2}{\lambda_{\text{th}}^2 f}\right) \ll 1$, this chain of inequalities reads

$$\frac{f}{2} \gg \frac{mL^2}{\pi\hbar^2\beta} \gg \frac{1}{2\pi^2}, \tag{10.72}$$

which can be satisfied for temperatures that are not too low. This is consistent with the condition (10.64), which can be restated as

$$\frac{mL^2}{\pi\hbar^2\beta} \gtrsim \frac{1}{\sqrt{2}\pi^3}, \tag{10.73}$$

and hence provides a (smaller) lower bound on the temperature that is required for the computation to be consistent to begin with. Equation (10.64) may be interpreted as demanding that the saddle point energy is in some sense sufficiently far away from the spectral edge; the chain (10.71) ensures that this is satisfied.

From eq. (10.68), we can see the correction coming from the sum in eq. (10.65) dying away if the thermal wavelength increases, or alternatively, if the dimension becomes very large. Indeed, in the infinite-dimensional limit, the sum in

[14]It should be noted at this point that particularly in many-body semiclassics, there are typically two complementary notions of the classical limit [97]: the usual classical limit $\hbar \to 0$, as well as the limit of a vanishing "effective" $\hbar$, i.e., the number of degrees of freedom of the system $f \to \infty$. It has been argued for bosonic systems in [290] that the limit $f \to \infty$ produces an expansion of the OTOC akin to the one of [87], in apparent tension with our results. A key difference between the case of [290] and ours, however, is that the degrees of freedom described by eq. (10.37) are distinguishable, calling into question the simple applicability of results for bosonic systems. Indeed, the limit $f \to \infty$ is not a natural classical limit in our case; it is rather the limit that emphasises the quantum regime *the most*, in the sense of making the system maximally chaotic at every temperature, see fig. 10.3.

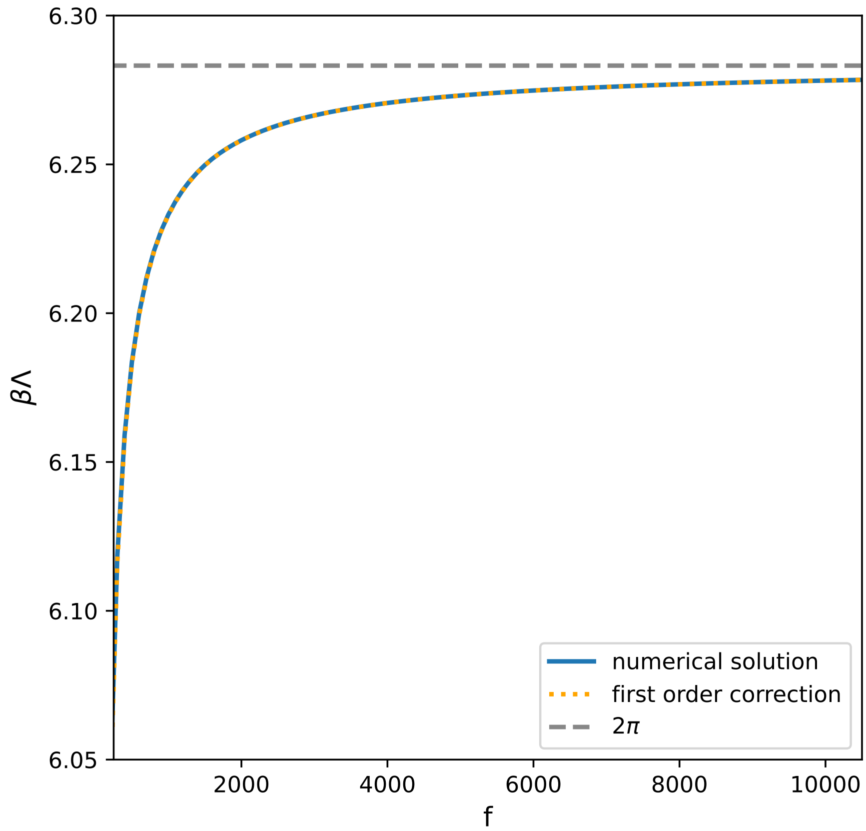

FIGURE 10.2: Quantum Lyapunov exponent $\Lambda = 2\lambda(E_\beta^*)$ for inverse temperature $\beta = 1$, evaluated for different dimensions f, plotted in units where $\hbar = 1, m = 2, L = 1, \lambda_{\text{th}} = 1/2$. Solid blue line: full numerical solution of the stationarity condition (10.53). Dotted orange line: first correction to the infinite-dimensional solution for large, finite dimension (10.68). Grey dashed line: 2π, i.e., the MSS bound, eq. (10.9). One can see that both finite-dimensional results approach the MSS bound for (very) large dimensionality. Taken from [4], plot by Gerrit Caspari.

eq. (10.65) vanishes entirely, and we are only left with

$$G_f(\beta) \approx \sqrt{\frac{2mL^2}{\hbar^2}} \frac{\pi}{\sqrt{E_\beta^*}}, \quad f = \infty, \tag{10.74}$$

which yields for the Lyapunov exponent the central result

$$2\lambda(E_\beta^*) = \frac{2}{\hbar}\left(\frac{\pi}{\beta} + \frac{\hbar}{mL^2}\frac{t}{\beta}\right) \xrightarrow{\hbar \ll \frac{mL^2}{t}} \frac{2\pi}{\hbar\beta}. \tag{10.75}$$

Remarkably, our chaotic quantum system saturates the MSS bound, corresponding to maximally fast scrambling, in the limit of infinite configuration space dimension[15]. Since this is precisely the limit in which it starts to exhibit correlation functions akin to the ones found in JT gravity (cf. chapter 9), this result serves to further support the status of this model as dual to JT gravity, where saturation of the MSS bound has been confirmed independently [295]. The approach of the system's quantum Lyapunov exponent to the bound by increasing the dimension can be seen in fig. 10.2, where we plot the full solution of the stationarity condition (10.53), as well as the first correction to the infinite-dimensional result (10.68). In both cases, we can see that the MSS bound is approached when increasing the dimension f.

Figure 10.3 shows the quantum Lyapunov exponent as a function of the rescaled inverse temperature $f\beta$ for fixed f. From the discussion above, as well as from fig. 10.3, it is also clear that our system approaches the MSS bound for fixed f, if the temperature becomes sufficiently small. This means that we have a class of f-dimensional quantum systems with chaotic classical limits that become maximally fast scramblers in the low-temperature limit.

Moreover, the simple form of eq. (10.59) suggests that upon rescaling by $1/f$, the quantum Lyapunov exponent should be described by a universal function of $f\beta$ with no (or only very weak) additional dependence on the dimension. The slightly opaque solid lines show $(1/f)\Lambda(f\beta)$, obtained from the full numerical solution of the stationarity condition (10.53) for two representative dimensions, f=301 and f=4301, in units where $\hbar = 1, m = 2, L = 1$. The two numerical curves are so close to each other that their difference cannot be resolved in the plot. This, together with further analysis for other f values, indeed points towards a unique curve $(1/f)\Lambda(f\beta)$ for describing the quantum Lyapunov exponent for large f.

Furthermore the convergence towards the chaos bound at fixed f suggests that even without taking the JT gravity limit, there might be relatively simple gravitational duals. This notion is supported by the Airy model (cf. section 4.1.4). As shown in chapter 9, the density of states of the Airy model is identical to the one of our model in $f = 3$ dimensions. At intermediate finite dimensions, our model can be seen as interpolating between topological and JT gravity by changing the dimension, in a manner similar, but not identical, to the $(2, 2p - 1)$ minimal string [55, 311], which has recently been found to admit a black-hole-like geometry [270].

A somewhat similar behaviour of the quantum Lyapunov exponent has been found in the SYK model in [104]. While for simple gravitational theories such as Einstein gravity, one expects Λ to saturate the MSS bound, stringy corrections [312] can hinder the development of chaos and decrease the quantum Lyapunov exponent. This effect has also been discussed explicitly for the Schwarzian theory in [313], and a description in terms of so-called scramblon modes has been shown to be applicable in SYK-like models [314]. The submaximal chaos apparent in the quantum Lyapunov exponent in fig. 10.3, viewed in this light, therefore hints at an interesting, as yet unexplored interpretation of the high-temperature regime of our model (at finite dimension) in terms of a more

[15]In this context, the conjecture proposed in [291] can be interpreted as a *classical* bound, that is corrected by increasingly strong quantum corrections, eventually producing the *quantum* bound, with a transition that happens around the f-dependent crossing point seen in fig. 10.3.

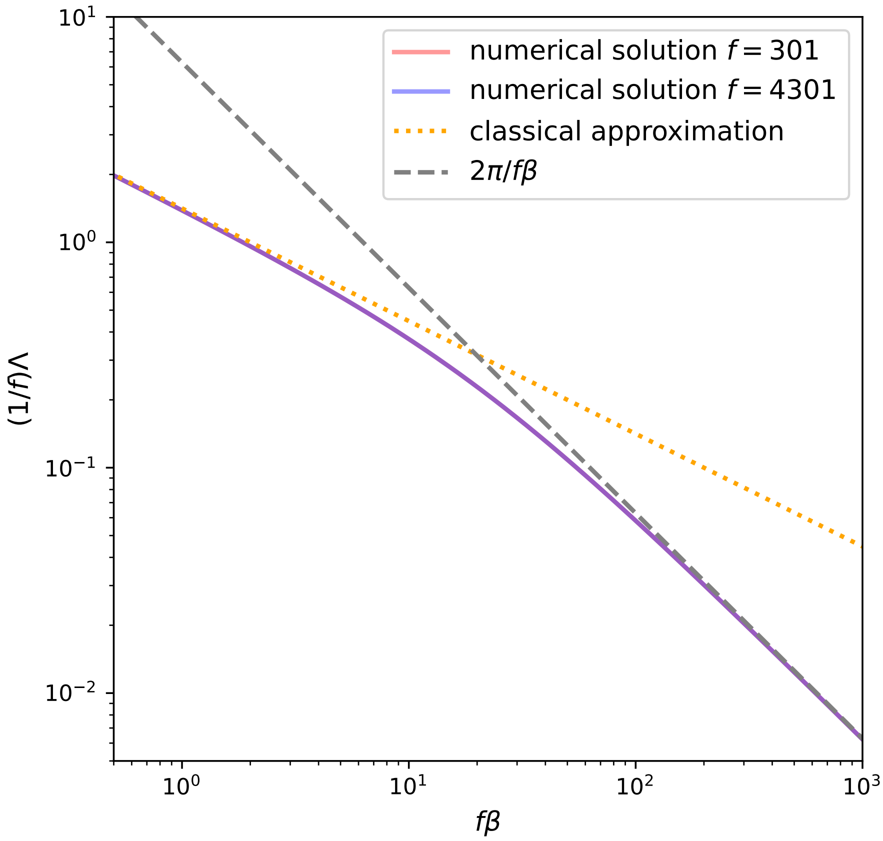

FIGURE 10.3: Double-log plot of the system's rescaled leading-order quantum Lyapunov exponent $(1/f)\Lambda$ (where $\Lambda = 2\lambda(E^*_\beta)$) as a function of the rescaled inverse temperature $f\beta$. The slightly opaque solid lines show $(1/f)\Lambda(f\beta)$, obtained from the full numerical solution of the stationarity condition (10.53) for two representative dimensions, $f = 301$ and $f = 4301$, in units where $\hbar = 1, m = 2, L = 1$. The two numerical curves are so close to each other that their difference cannot be resolved in the plot. This indicates $(f\beta)$-scale invariance in the large-f limit. Dotted orange line: classical approximation to the Lyapunov exponent (10.59), given by the equipartition theorem. Dashed grey line: $2\pi/\beta$, i.e., the MSS bound eq. (10.75). The system saturates the MSS bound at low (scaled) temperatures, while it is more appropriately described by classical equipartition (eq. (10.59)) at high (scaled) temperatures. Taken from [4], plot by Gerrit Caspari.

complicated gravitational theory with stringy (or similar) effects that disappear at low temperature, and that might even be explicable in terms of the corrections (10.61) to the infinite-dimensional pure JT gravity density of states (10.41).

10.5 Subleading corrections

As already mentioned above, the term whose growth rate is bounded by $2\pi/\hbar\beta$ in the out-of-time ordered correlator $F(t)$ is predicted [87] to only be the first in an expansion,

$$F_d - F(t) = \hbar^2 f_1 e^{\frac{2\pi}{\hbar\beta}t} + O\left(\hbar^4\right), \tag{10.76}$$

where F_d is the factorised value of $F(t)$, as introduced in section 10.1. Indeed, defining moments

$$\mu_J(t) = e^{\frac{4\pi J}{\hbar\beta}t} \int_{t-i\hbar\beta/4}^{t+i\hbar\beta/4} dt' e^{-\frac{2\pi}{\hbar\beta}(t'-i\hbar\beta/4)(2J+1)} (F(t') - F_d), \tag{10.77}$$

the analytic structure of $F(t)$ imposes a set of conditions on the moments [122]:

$$0 < \mu_J(t) < \frac{2\hbar\beta F_d}{\pi(2J+1)} e^{-\frac{2\pi}{\hbar\beta}t}, \tag{10.78}$$

$$\mu_{J+1}(t) < \mu_J(t), \tag{10.79}$$

$$\mu_{J+1}(t)^2 \leq \mu_J(t)\mu_{J+2}(t). \tag{10.80}$$

These conditions must be satisfied by the out-of-time ordered correlator, and eqs. (10.78) and (10.79) imply that at late enough times[16], corrections to the Lyapunov growth of, e.g., the form

$$F_d - F(t) = \hbar^2 f_1 e^{\frac{2\pi}{\hbar\beta}t} + \hbar^4 f_2 e^{\Lambda_2 t} + O\left(\hbar^6\right) \tag{10.81}$$

must appear in systems with maximal Lyapunov growth $\Lambda = \frac{2\pi}{\hbar\beta}$. Further accounting for eq. (10.80) produces a bound on this new, subleading exponential growth as well,

$$\Lambda_2 \leq \frac{6\pi}{\hbar\beta}. \tag{10.82}$$

The saturation of this equality again forces similar late-time corrections $\Lambda_3, \Lambda_4, \ldots$, which are then again bounded by $\frac{10\pi}{\hbar\beta}, \frac{14\pi}{\hbar\beta}, \ldots$ and so on, by repeatedly applying eqs. (10.78) to (10.80) [123]. Since our system, as we have shown in section 10.4, saturates the MSS bound at leading order in the OTOC, and the Wigner-Moyal quantisation gives a systematic way to compute $\hbar$ corrections to the leading result, it is natural to examine those corrections and determine whether subleading bounds are saturated as well. In this section, I will attempt to characterise the

[16] A bit more precisely: the conditions eqs. (10.78) and (10.79) imply that there have to be corrections to the maximal Lyapunov growth (10.76) of the form (10.81) with $\Lambda_2 > \frac{2\pi}{\hbar\beta}$. These corrections then imply the existence of a timescale $t_1 \ll t^*$ where the approximation (10.76) to the OTOC breaks down and the second term in eq. (10.81) starts to dominate [122].

first nonzero correction to the leading order OTOC in our system and estimate its growth rate. However, as we will see, the complexity of the computation is drastically higher than at leading order.

To recap some of the observations in section 10.2, we expect corrections to the leading order in $\hbar$ from

1. the Weyl symbol for the time-evolved operator $\hat{p}(t)$ (though not for $\hat{x}$),
2. the Moyal bracket and the $\star$ product,
3. and the Wigner function $W(x, p)$.

Naively, one might also expect these corrections to be sensitive to the non-flat metric of the configuration space, since Weyl symbols in configuration spaces must be modified according to eq. (10.12), i.e., by including the determinant $g(x)$ of the configuration space metric at x. Given the essential singularity at $\hbar = 0$ in the prescription (10.12), tracing the influence of the metric on the small $\hbar$ expansion of a Weyl symbol is a highly nontrivial endeavor. Fortunately, the formalism developed by Osborn and Molzahn [296] to determine Weyl symbols of Heisenberg operators takes care of the metric dependence automatically via the classical flow that enters the computation. Since the Weyl symbols of initial time Weyl-ordered polynomials in $\hat{x}, \hat{p}$ remain unchanged by the curved configuration space, the only explicit modification of the Weyl symbol computation we have to account for come from $\hbar$ corrections to the Hamiltonian, which I discuss in appendix D. In the notation of section 10.2, our Hamiltonian's Weyl symbol has the form [315]

$$H_c = \frac{1}{2m} p_i g^{ij}(x) p_j, \tag{10.83}$$

$$H_2 = \frac{R}{12m}, \tag{10.84}$$

$$H_{r\neq 2} = 0, \tag{10.85}$$

where R is the Ricci scalar of the manifold. Due to the vanishing of H_1, we can conclude from eq. (10.32) that

$$\zeta_1 = 0, \tag{10.86}$$

and crucially, H_2 is a constant in the system we consider, leading to a simplification of eq. (10.33). We can therefore determine the first nonzero correction to the quantum trajectory $\hat{\zeta}$ by integrating eq. (10.33) according to eq. (10.34), and find

$$\zeta_2(t;\zeta) = \int_0^t ds \nabla\gamma(t,0|\zeta) J \nabla\gamma(s,0|\zeta)^T J^{-1} \times \left[-\frac{1}{8}\Pi^2(\gamma\cdot\nabla)^2 + \frac{1}{12}\Pi_{12}\Pi_{23}(\gamma\cdot\nabla)^3 \right] J\nabla H_c(t,\gamma(t,s|\zeta)). \tag{10.87}$$

We can simplify this a bit further in order to facilitate the discussion that follows, by realizing that

$$\nabla\gamma(t,0|\zeta) \equiv M(t) \tag{10.88}$$

is simply the monodromy of the classical flow. For a Hamiltonian flow, the monodromy preserves the symplectic form [103],

$$M(t)^T J M(t) = J, \tag{10.89}$$

and hence, we can combine

$$M(t) J M(s)^T J^{-1} = M(t) M(s)^{-1} = \nabla\gamma(t,s|\zeta), \tag{10.90}$$

since $J^2 = -1$ and multiplying the above equation with M^{-1} from the right and J^{-1} from the left. With this rewriting, we can express the correction to the quantum trajectory as

$$\zeta_2(t;\zeta) = \int_0^t ds \nabla\gamma(t,s|\zeta) \left[-\frac{1}{8}\Pi^2(\gamma\cdot\nabla)^2 + \frac{1}{12}\Pi_{12}\Pi_{23}(\gamma\cdot\nabla)^3 \right] J\nabla H_c(t,\gamma(t,s|\zeta)). \tag{10.91}$$

To the desired order in $\hbar$, this accounts for all contributions to the OTOC coming from the Weyl symbols themselves. The remaining corrections stem from the Moyal bracket,

$$\{\cdot,\cdot\}_M = \{\cdot,\cdot\} - \frac{\hbar^2}{12}\Pi^3(\cdot,\cdot) + \mathcal{O}\left(\hbar^4\right), \tag{10.92}$$

and the $\star$ product,

$$\star = \cdot + \frac{i\hbar}{2}\Pi - \frac{\hbar^2}{8}\Pi^2 + \mathcal{O}\left(\hbar^3\right). \tag{10.93}$$

Note however, that in eq. (10.46), the $\star$ product is taken between two copies of the same object. Plugging in the $\hbar$ expansion for the Moyal brackets, the only terms appearing at $\mathcal{O}(\hbar)$ are the two "fully classical" ones, and the $\star$ product of these vanishes, cf. footnote 7.

There may be further corrections from the Wigner function which we have not considered. However, as we are interested only in the (exponential) growth of the OTOC, we argue for neglecting them, since they are generic for any correlation function of any set of operators, depending only on the state. For this reason, we do not expect them to contribute in an interesting manner to the growth rate.

All assembled then, the first correction to eq. (10.46) is

$$\hbar^4\left(- \{p_2(t),x\}\{p(t),x\} + \frac{1}{6}\{p(t),x\}\Pi^3(p(t),x) + \frac{1}{8}\Pi^2\left(\{p(t),x\},\{p(t),x\}\right)\right), \tag{10.94}$$

where we generalised the notation in eq. (10.27) to $W_{\hat{p}(t)}$ and suppressed the dependence on the initial condition ζ. At present, we do not know how to evaluate this expression or compute Λ_2 from it, although numerical estimates for some simple manifolds might be possible.

As a very rough estimate, we could use as a guideline that

$$\{p(t),x\} \sim e^{\lambda t}. \tag{10.95}$$

If we take this to mean that phase space derivatives acting on time-evolved quantities produce Lyapunov growth $e^{\lambda t}$, then we can give a broad-strokes prediction for the time dependence of each term in eq. (10.94). Most obviously, for the middle term, we have the following:

$$\Pi^3(p(t), x) = \sum_{k=0}^{3}(-1)^k \binom{3}{k} \left(\frac{\partial^k}{\partial p^k} \frac{\partial^{3-k}}{\partial x^{3-k}} p(t) \right) \left(\frac{\partial^{3-k}}{\partial p^{3-k}} \frac{\partial^k}{\partial x^k} x \right). \tag{10.96}$$

Now in this sum, if $k \neq 3$, the second term produces derivatives of initial x with respect to initial p, which vanish. But if $k = 3$, we have $\partial^3 x / \partial x^3 = 0$. Hence, this contribution vanishes identically,

$$\{p(t), x\} \Pi^3(p(t), x) = 0. \tag{10.97}$$

For the third term, a superficial examination yields that each of the Poisson brackets acted upon by Π^2 already grows exponentially, so each application of the Poisson bivector should produce a factor $e^{2\lambda t}$, giving a total growth of $e^{6\lambda t}$. Indeed, we can evaluate this expression:

$$\Pi^2(\{p(t), x\}, \{p(t), x\}) \tag{10.98}$$

$$= \sum_{k=0}^{2}(-1)^k \binom{2}{k} \left(\frac{\partial^k}{\partial p^k} \frac{\partial^{2-k}}{\partial x^{2-k}} \frac{\partial p(t)}{\partial p} \right) \left(\frac{\partial^{2-k}}{\partial p^{2-k}} \frac{\partial^k}{\partial x^k} \frac{\partial p(t)}{\partial p} \right) \tag{10.99}$$

$$= 2 \frac{\partial^3 p(t)}{\partial x^2 \partial p} \frac{\partial^3 p(t)}{\partial p^3} - 2 \left(\frac{\partial^3 p(t)}{\partial x \partial^2 p} \right)^2. \tag{10.100}$$

Evidently, this includes 6 total derivatives of the time-evolved momentum. A sensible expectation is then that the term grows no more strongly than

$$\Pi^2(\{p(t), x\}, \{p(t), x\}) \sim e^{6\lambda t}, \tag{10.101}$$

as anticipated, and still in agreement with the bound (10.82).

Finally, eq. (10.91) is the hardest term to evaluate explicitly, since it depends in a nontrivial manner on the classical solutions on the hyperbolic manifold $\mathcal{M}$, which are not generally known analytically. The only way to provide any estimate at the moment is to be even more speculative.

In the simpler free particle case on $\mathbb{H}^f$, the $\gamma \cdot \nabla$ derivatives act in more or less the same way as the $J\nabla$ on the Hamiltonian, i.e., they are derivatives along the flow direction. The Π derivatives are not directional and thus act differently. Naively counting all derivatives, the integrand might grow as fast as 8λ. If we use the refined assumption that only the Π derivatives produce exponential growth however, we can bound the growth of the integrand to at most 5λ. Finally, when looking at the free particle solutions [316], the only function of the initial coordinates that we take Π type derivatives of that seems to be sufficiently complicated to produce exponential growth are derivatives of the conformal factor of the metric:

$$g_{ij}(x) = \Omega(x)\delta_{ij}, \quad \Omega(x) = \frac{2}{1-|x|^2}. \tag{10.102}$$

Since this only depends on x, it is conceivable that only the x derivatives in Π produce exponential growth, bounding the total exponent of the integral to 3λ, although this is not the expected behaviour in typical chaotic quantum systems.

If we then plug these estimates into the actual contribution to the OTOC, we estimate the growth to be bounded by

$$-\{p_2(t), x\}\{p(t), x\} \sim \begin{cases} e^{10\lambda t} & \text{if all derivatives contribute,} \\ e^{7\lambda t} & \text{if only } \Pi \text{ derivatives contribute,} \\ e^{5\lambda t} & \text{if only } x \text{ derivatives in } \Pi \text{ contribute.} \end{cases} \tag{10.103}$$

In our particular system, we were able to find in section 10.4 that in the limit $f \to \infty$, the classical Lyapunov exponent is evaluated at the saddle point as $\frac{\pi}{\hbar\beta}$, which would mean that the total growth of the $\hbar^4$ correction to the OTOC grows with

$$\Lambda_2 = \begin{cases} \frac{10\pi}{\hbar\beta} & \text{if all derivatives contribute,} \\ \frac{7\pi}{\hbar\beta} & \text{if only } \Pi \text{ derivatives contribute,} \\ \frac{6\pi}{\hbar\beta} & \text{if only } x \text{ derivatives in } \Pi \text{ contribute,} \end{cases}, \tag{10.104}$$

since in the third case, the contribution from eq. (10.101) dominates the one from eq. (10.103). This means that only in the third case, we could guarantee the bound (10.82) being respected at all, and our estimate would leave room for the bound to be saturated as well. Interestingly, this saturation would, if at all, arise from the correction to the $\star$ product, not the quantum trajectory. In the other cases meanwhile, we can not say anything definite, but it is at least encouraging that our method gives an estimate for the growth bound that is reasonably close, and structurally identical to eq. (10.82), and not orders of magnitude off.

We can compare these estimates to the Schwarzian theory, which describes both JT gravity and the SYK model. In this theory, the out-of-time ordered correlator can be computed exactly in the limit $\beta \ll 1$, and is found to be (in the usual natural units for JT gravity) [123, 317]

$$\frac{F(t)}{F_d} = \frac{1}{z^{2\Delta}} U(2\Delta, 1, 1/z), \qquad z = \frac{\beta}{8\pi} e^{\frac{2\pi t}{\beta}}, \tag{10.105}$$

for a pair of operators $\hat{V}, \hat{W}$ with scaling dimension Δ. Here, $U(a, b, y)$ is the confluent hypergeometric function. At early enough times, it has an asymptotic expansion in terms of the generalised hypergeometric function [318]

$$U(a, b, y) \sim y^{-a} \; {}_2F_0(a, a-b+1;;-1/y), \tag{10.106}$$

where

$$ {}_2F_0(a_1, a_2; ; z) = \frac{1}{\Gamma(a_1)\Gamma(a_2)} \sum_{n=0}^{\infty} \Gamma(n + a_1)\Gamma(n + a_2) \frac{z^n}{n!}. \tag{10.107} $$

Expanding eq. (10.105) up to second order in small z, we find

$$ \frac{F(t)}{F_d} = 1 - \frac{\beta\Delta^2}{2\pi} e^{\frac{2\pi t}{\beta}} + \frac{\beta^2\Delta^2(2\Delta + 1)^2}{32\pi^2} e^{\frac{4\pi t}{\beta}} + \ldots \tag{10.108} $$

Clearly, the subleading correction grows with $\Lambda_2 = \frac{4\pi}{\beta}$, i.e., far away from the bound (10.82). This tension with our rough estimate (10.104) suggests that a more careful examination of the OTOC in our system is needed. It should be noted however, that eq. (10.108) is an expansion for small β, whereas our system's OTOC at fixed dimension approaches the MSS bound only for relatively large β, cf. fig. 10.3, meaning that the two behaviours need not necessarily agree.

Chapter 11

Discussion

In this thesis, I have used two very different approaches to quantum chaos in my study of quantum gravity: universality in random matrix theory facilitated the extraction of nonperturbative information from the perturbative sector of JT (and topological) gravity in part II, while in part III, semiclassical methods – in particular, periodic orbit theory and the Wigner-Moyal phase space quantisation – together with the use of system-specific information provided strong arguments for a duality between JT gravity and a high-dimensional Hadamard-Gutzwiller-like model at the perturbative level, and made it possible to address questions regarding factorisation and maximal scrambling on the boundary side of the duality. At first glance, these approaches might seem fairly disparate, but we already know from part I that there are intricate connections between the two. I want to use this final chapter of the thesis to review some key results of the foregoing and highlight some unexplored or half-explored connections between them that may otherwise go unappreciated.

The spectral form factor in orientable JT gravity

Beginning with chapter 7, we have seen the computation of the canonical spectral form factor of orientable JT gravity in the late-time, τ-scaling limit [1]. This limit was first correctly identified in [117] as the one in which the form of the SFF dictated by the universal RMT result for the microcanonical SFF emerges, characterised by a saturation at late times that is not obvious from the perturbative contribution to the microcanonical SFF. Demanding that this saturation can occur is what allowed us to predict nontrivial relations between the coefficients of Weil-Petersson volumes that must be satisfied, and while these relations are not enough to completely determine the $V_{g,2}(b_1, b_2)$, they generalise to a large class of (matrix) models and serve as a powerful consistency check for models of quantum gravity with matrix model duals [231].

Two open questions come to mind regarding this chapter: first, is there a useful geometric interpretation of the τ-scaling limit? For the one-point function, [223] (but see also [224]) shows that in a different late-time limit, the genus expansion becomes meaningless because all topological sectors give $\mathcal{O}(1)$ contributions. Nevertheless, due to an accidental degeneracy of moduli space dimensions, the one-point function can be shown to have a geometric interpretation in terms of a gas of cusp triplets on a single geometry. A similar interpretation of the two-point function in the τ-scaling limit might prove interesting; finding such an interpretation however is complicated by the fact that the τ-scaling limit does not isolate the top-degree term in the Weil-Petersson volumes, corresponding to the Airy limit, but a subleading one. The τ-scaling limit does hold the advantage however that it doesn't invalidate the entire genus expansion, producing instead a series with a finite radius of convergence, but one that nevertheless obtains contributions to the same order in e^{-S_0} from every topological sector.

Secondly, due to the cancellations predicted in [1], the universal result emerges from a subleading power of the gluing boundary lengths in the Weil-Petersson volumes, as mentioned already. Interestingly, as we have seen in section 9.3, something similar happens in the encounter formalism in periodic orbit theory, particularly for the Sieber-Richter pair: the leading, system-specific term dies away under the semiclassical average, while the subleading term provides the universal result. In order to better determine if this similarity goes any deeper, extending our formalism from section 9.3 to full unorientable JT gravity would be desirable.

A final observation on complementary regions regards the role of discreteness: the saturation of the canonical SFF stems from the contact term in the cluster function (cf. chapter 6) and is hence a diagnostic of the discreteness of the energy spectrum. For the ramp regime on the other hand, we have seen in section 9.2 that it can be explained by the discreteness in the spectrum of periodic orbits instead. It would be interesting to see whether, using nonperturbative techniques from periodic orbit theory (such as Riemann-Siegel lookalikes [94]), this property could be transferred to the saturation of the SFF as well.

The spectral form factor in unorientable topological gravity

In chapter 8, I reported our extension of the late-time SFF computation to the unorientable Airy model in [2]. The computation naively yields a disagreement between the nonperturbative calculation using the universal RMT limit and the perturbative calculation based on loop equations, with the latter approach producing terms that diverge in the τ-scaling limit. We were however able to devise a procedure based on "exchange functions" that produces agreement between the two sides by subtracting the convergent short-time expansion of an exchange function and adding the asymptotic late-time expansion, in this way bootstrapping the agreement to higher and higher orders in the expansion in τ. Remarkably, this procedure is only necessary for odd orders, whereas even orders already agree.

The agreement partially persists for the newly appearing logarithmic terms that are present at odd orders – there are no such terms in the orientable case – for the coefficient of $\log \beta$, but not for the $\log \tau$ coefficient. Importantly, the analogues of the Weil-Petersson volumes in the unorientable Airy model are nonanalytic along the diagonal $b_1 = b_2$, and the correct combination of terms on both sides of the diagonal is crucial for recovering the correct coefficient of the $\log \beta$ terms.

A potentially interesting observation from this chapter regards the direction of the exchange function approach: what we see from a naive computation of the SFF using the perturbative loop equations has been identified as terms arising from the short-time convergent expansion of some exchange function. In order to recover the nonperturbative universal/late-time result from the RMT computation, we needed to replace this convergent expansion by the asymptotic late-time expansion of the exchange function. Can this be seen as choosing a nonperturbative completion of the genus expansion of topological gravity in some sense? While the choice of exchange function is not fully arbitrary, it cannot be excluded that there is at least some freedom to choose different functions from the ones we used in [2]. Re-examining this mechanism in light of potentially inequivalent nonperturbative completions might prove an interesting direction for future research.

JT gravity as a sum over periodic orbits

There are two central results in chapter 9. First, we were able to provide strong arguments for the duality of a high-dimensional Hadamard-Gutzwiller-like model to JT gravity in [3]. In the process, we established a dictionary between gravity and periodic orbit theory/the Selberg trace formula that contains the following entries:

- The disk partition function of JT gravity is equivalently computed by the Weyl term in the infinite-dimensional version of the STF, which performs the same computation as the *BF* theory formulation of JT gravity.

- In non-disk correlation functions, JT gravity exhibits a splitting between the boundary mode path integral and the bulk moduli space integral. On the periodic orbit side, this is realised as the independence of the choice of operator or spectral function to insert in the STF and the length spectrum, and in particular, the length correlations that we used to evaluate correlation functions.

- The trumpet partition function of JT gravity is reproduced by choosing the spectral function eq. (3.57).

- The formal Weil-Petersson volume $V_{0,2}$ (cf. eq. (4.48)) in JT gravity enforces that gluing two trumpets together without inserting a genus-carrying convex core is only possible along geodesic boundaries that are exactly degenerate in length. On the periodic orbit side, this manifests as a contact term from orbits with exactly correlated lengths (i.e. the orbit with itself or its time-reverse). Interestingly, this contact term crucially depends on the discreteness of the length spectrum of periodic orbits, whereas there is a continuous family of gluing geodesics that the JT trumpet partition function can terminate in.

- The volume of the manifold on which we apply the Selberg trace formula plays the role of the large parameter e^{S_0} in JT gravity. This association gives the correct scaling for the correlators we have computed so far, i.e.

$$\mathcal{Z}_{0,1} \sim e^{S_0},\ \mathcal{Z}_{0,2} \sim e^{0S_0},\ \mathcal{Z}_{\frac{1}{2},2} \sim e^{-S_0}. \tag{11.1}$$

- The Weil-Petersson measure (4.42) is provided by the intricate cancellation of the exponential proliferation of periodic orbits against their stabilities.

Meanwhile, we have not been able to identify Norbury's measure (4.52) for the crosscap, noting however that it would only be expected to appear in full unorientable JT gravity, not in the Airy model.

- We were able to compute the double trumpet partition function in any regime of parameters using the results mentioned above. Remarkably, the double trumpet is fairly universal: it depends on the smoothed spectral density of the theory only through the end points of its support, meaning that any one-cut matrix model exhibits the same double trumpet. On the periodic orbit side, we have seen that the double trumpet is recovered independently of the dimensionality of the hyperbolic system. Since the main impact of changing the dimensionality in our system is to change the Plancherel measure, and thereby, the spectral density, the robustness of the double trumpet, depending as it does on the discreteness of periodic orbits, may be seen as a manifestation of this universality.

In section 9.3 then, I have attempted to extend our computation based on length correlations from the fully correlated lengths yielding the double trumpet to orbits with small length differences. While I used the encounter formalism as a guide, the calculation in section 9.3 goes beyond standard encounters, which provide a powerful link between periodic orbit theory and random matrix theory, but achieve this by washing out system-specific information to a large extent. My calculation, which is currently in the process of being developed into a (more) complete formalism [121], is able to preserve some system-specific information in the form of the system's Lyapunov spectrum, and thereby is able to reproduce the moduli space volume $V_{\frac{1}{2},2}$ of the unorientable Airy model, up to a multiplicative constant and some modifications.

These are: restrictions on the lengths of periodic orbits involved, as well as their difference, that stem from using an encounter computation to select the contributions to the two-point function we evaluate, as well as a length cutoff in the functional form of the candidate Weil-Petersson volume, and a remnant free parameter l_c. Lifting the restrictions on lengths may require a more complete understanding of periodic orbit theory, or at least of the length correlations of geodesics on hyperbolic manifolds, perhaps in line with [117]. The role of the length cutoff is mainly to shift the point in the length spectrum from which the WP volume is positive semi-definite, and may even be trivially respected depending on the systole, i.e., shortest geodesic on the manifold under study. The encounter length scale l_c meanwhile is interesting in that the term in the WP volume yielding the universal part of the spectral form factor happens to be independent of l_c, suggesting that this scale may be viewed as system-specific, or at least, that the recovery of the universal result is robust against different choices of l_c. As a final observation regarding $V_{\frac{1}{2},2}(b_1, b_2)$, note that the Airy volume (4.65) exhibits a nonanalyticity along the diagonal while the full JT volume (4.57) does not. This should be reflected in our formalism as well, suggesting that the nonanalyticity should be resolved when pushing the computation to higher (or at least to infinite) dimension.

The second key result of chapter 9 is an explanation of nonfactorisation in the two-point function of our proposed boundary dual. As argued in section 4.2, the prevalent explanation in JT gravity of the boundary dual being an ensemble of physical systems, rather than a single one, is expected to fail in higher-dimensional gravity, necessitating a mechanism that prevents factorisation at the single-system level. We provide such a mechanism by partially coarse graining out system-specific information and then evaluating correlation functions using an ensemble of lengths that nevertheless exhibit certain known, purely classical correlations. In particular, the double trumpet relies only on the contact term originating from orbit pairs with exactly correlated lengths: namely, we can pair an orbit with itself, or with its time-reverse in time-reversal invariant systems. The double trumpet can therefore be manifestly computed at the level of a single system, i.e., a single realisation of the geodesic length spectrum. Factorisation would be restored in our formalism by summing up the entire set of periodic orbits.

An interesting open question about this coarse graining formalism is whether it can be understood as akin to integrating out UV degrees of freedom in a more complicated theory, along the lines of [182, 183] or [78], where factorisation is restored in, in some sense, "more realistic" models than JT gravity.

The Maldacena-Shenker-Stanford bound on hyperbolic manifolds

In chapter 10, I report the results of [4], where we developed a quasiclassical formalism for computing the out-of-time ordered commutator and applied this formalism to the Hadamard-Gutzwiller-like model of [3]. We were able to show that in the infinite-dimensional limit where the system is expected to be dual to JT gravity, the OTOC saturates the Maldacena-Shenker-Stanford bound on chaos, as well as that even at finite dimension, the bound is approached at low temperatures. Moreover, we found a universal curve that the rescaled quantum Lyapunov exponent $\Lambda(f\beta)/f$ follows for arbitrary (large) dimension.

This computation is based on the Wigner-Moyal phase space quantisation, which naturally yields an expansion in small $\hbar$. In the context of field theories with N degrees of freedom, the case of interest for [87], the OTOC is expanded in $1/N$ however. This expansion breaks down around the scrambling time

$$t^* = \frac{1}{\Lambda}\log N. \tag{11.2}$$

In single-particle systems like the Hadamard-Gutzwiller-like model we used in [3, 4], the scrambling time is identified with the Ehrenfest time

$$t_E = \frac{1}{\lambda}\log(S_{\mathrm{cl}}/\hbar), \tag{11.3}$$

which justifies the $\hbar$ expansion. Indeed, the leading-order Wigner-Moyal approximation generically breaks down at t_E as well. In other words, in our system, we shouldn't expect to be able to distinguish scrambling time effects from Ehrenfest time effects, i.e., it stands to reason that around t_E, we need to account for effects of extended classical paths in the OTOC, following the discussion in chapter 10.

For a final observation, let me return to semiclassics and the emergence of periodic orbit theory from the quantum mechanical path integral, cf. chapter 2.

This emergence is based on a saddle point approximation to the path integral that enforces not only the classicality of paths, but (in the case of traces), also their smooth closure, identifying periodic orbits of the classical dynamics as the chief contributions to the trace of the semiclassical Green's function. In other words, the physical content of semiclassics originates from the saddle point condition.

Something similar happens in our computation of the OTOC, which relies on a saddle point approximation as well. In this case, the saddle point condition enforces that the quantum Lyapunov exponent $\Lambda(\beta)$ is given by the classical Lyapunov exponent $\lambda(E^*_\beta)$ evaluated at the saddle point energy E^*_β. In the classical regime, this energy is simply the one expected from the classical equipartition theorem, though in the quantum regime, no such simple interpretation is available. The computation does however require the saddle point energy to be larger than a certain energy gap,

$$E^*_\beta \gtrsim \frac{\hbar^2}{2mL^2}\frac{1}{\pi^2}. \tag{11.4}$$

This condition is precisely one of the necessary conditions for the scale separation required to derive the MSS bound – the other condition is simply that the time at which we evaluate the OTOC is shorter than the scrambling time.

Appendix A

Computation of the energy shell volume on a hyperbolic 3-manifold

The classical energy shell of a system with Hamiltonian $H(q,p)$ defined on the cotangent bundle $\mathcal{T}^*\mathcal{M}$ to some f-dimensional configuration space $\mathcal{M}$ of volume $\mathcal{V}$ is

$$\Sigma_E = \{(q,p) \in \mathcal{T}^*\mathcal{M} | H(q,p) = E\}, \tag{A.1}$$

and its volume can be computed as

$$\Omega = \int \mathrm{d}^f q \int \mathrm{d}^f p \, \delta^{(f)}(E - H(q,p)). \tag{A.2}$$

If the Hamiltonian of the system is eq. (3.40), we get

$$\begin{aligned} \Omega &= \int \mathrm{d}^f q \int \mathrm{d}^f p \, \delta^{(f)}(E - |p|^2/4) \\ &= \left(\int \mathrm{d}^f q \right) \mathrm{Vol}(S_{f-1}) \int_0^\infty \mathrm{d}|p| \, |p|^{f-1} \delta(E - |p|^2/4) \\ &= \mathcal{V} \frac{2\pi^{f/2}}{\Gamma(f/2)} \int_0^\infty \mathrm{d}|p| \, \frac{|p|^{f-1}}{\sqrt{E}} \delta(|p| - 2\sqrt{E}) \\ &= \frac{2^f \pi^{f/2}}{\Gamma(f/2)} \mathcal{V} \sqrt{E}^{f-2}. \end{aligned} \tag{A.3}$$

In particular, for $f = 3$, we have

$$\Omega = 16\pi \mathcal{V} \sqrt{E}. \tag{A.4}$$

Appendix B

Computation of the semiclassical Airy volume $V^{\text{sc}}_{\frac{1}{2},2}$

The semiclassical Airy volume $V^{\text{sc}}_{\frac{1}{2},2}$ contains two terms, the first of which reads

$$\frac{2}{\pi\mathcal{V}}\int_{-l_c}^{l_c}\mathrm{d}l_1\,\mathrm{d}l_2\,\delta(\delta l-l_1-l_2)L^2\log\left(\frac{l_c^2}{|l_1 l_2|}\right). \tag{B.1}$$

Using the integral representation of $\delta(\delta l-l_1-l_2)$, this can be rewritten as

$$\begin{aligned}&\frac{L^2}{\pi^2\mathcal{V}}\int_{-\infty}^{\infty}\mathrm{d}p\,e^{ip\delta l}\int_{-l_c}^{l_c}\mathrm{d}l_1\,\mathrm{d}l_2\log\left(\frac{l_c^2}{|l_1 l_2|}\right)e^{-ipl_1}e^{-ipl_2}\\ =&\frac{L^2}{\pi^2\mathcal{V}}\int_{-\infty}^{\infty}\mathrm{d}p\,e^{ip\delta l}\left[\int_{-l_c}^{l_c}\mathrm{d}s\,e^{-ips}\log\left(\frac{l_c}{|s|}\right)\right]\frac{2\sin(pl_c)}{p}\\ =&\frac{L^2}{\pi^2\mathcal{V}}\int_{-\infty}^{\infty}\mathrm{d}p\,e^{ip\delta l}\left[2l_c\int_0^1\mathrm{d}x\cos(pl_c x)\log\left(\frac{1}{x}\right)\right]\frac{2\sin(pl_c)}{p}.\end{aligned} \tag{B.2}$$

The x integral can be solved,

$$\int_0^1\mathrm{d}x\cos(ax)\log\left(\frac{1}{x}\right)=\left[-\frac{1}{a}(\sin(ax)\log(x)-\mathrm{Si}(ax))\right]_0^1=\frac{\mathrm{Si}(a)}{a}, \tag{B.3}$$

where $\mathrm{Si}(a)$ is the sine integral. The first contribution to the Airy volume therefore reads

$$\frac{4L^2l_c^2}{\pi^2\mathcal{V}}\int_{-\infty}^{\infty}\mathrm{d}p\,e^{ip\delta l}\frac{\mathrm{Si}(pl_c)}{pl_c}\frac{\sin(pl_c)}{pl_c}. \tag{B.4}$$

By a similar calculation, the second contribution is found to be

$$-\frac{2}{\pi\mathcal{V}}L\int_{-l_c}^{l_c}\mathrm{d}l_1\,\mathrm{d}l_2\,\delta(\delta l-l_1-l_2)\log\left(\frac{l_c}{|l_1|}\right)\log\left(\frac{l_c}{|l_2|}\right)=-\frac{4Ll_c^2}{\pi^2\mathcal{V}}\int_{-\infty}^{\infty}\mathrm{d}p\,e^{ip\delta l}\left(\frac{\mathrm{Si}(pl_c)}{pl_c}\right)^2. \tag{B.5}$$

Writing $z = \delta l / l_c < 1$, we can write the full semiclassical Airy volume as

$$V^{\rm sc}_{\frac{1}{2},2}(l_1,l_2) = \frac{8 l_c}{\pi^2 \mathcal{V}} \left[L \underbrace{\int_0^\infty \mathrm{d}x \cos(xz) \frac{\mathrm{Si}(x)}{x}\frac{\sin(x)}{x}}_{=:I(z)} - \underbrace{\int_0^\infty \mathrm{d}x \cos(xz) \left(\frac{\mathrm{Si}(x)}{x}\right)^2}_{=:J(z)} \right]. \tag{B.6}$$

We can solve these integrals, being careful to account for the asymptotic behaviour of the sine integral at large argument [319],

$$\mathrm{Si}(x) \sim \frac{\pi}{2} - \frac{\cos x}{x} + \ldots, \tag{B.7}$$

where ... encompasses further oscillatory terms. For the first integral $I(z)$, the large x asymptotics of the integrand is given by

$$\frac{\mathrm{Si}(x)}{x}\frac{\sin(x)}{x} \sim \frac{\pi \sin(x)}{2x^2} + \ldots, \tag{B.8}$$

which is oscillatory and hence does not produce any non-analytic behaviour in the integral. We can therefore simply Taylor expand the cosine to find

$$I(z) = I(0) + \mathcal{O}\left(z^2\right) = \int_0^\infty \mathrm{d}x \, \frac{\mathrm{Si}(x)}{x}\frac{\sin(x)}{x} + \mathcal{O}\left(z^2\right) = \frac{\pi}{2} + \mathcal{O}\left(z^2\right). \tag{B.9}$$

For the second integral, we have

$$\left(\frac{\mathrm{Si}(x)}{x}\right)^2 \sim \frac{\pi^2}{4x^2} + \ldots, \tag{B.10}$$

which is non-oscillatory and hence *does* produce a non-analytic contribution [320]. We can capture this contribution by adding and subtracting the integral at $z = 0$, which isolates precisely the contribution stemming from the non-oscillatory tail. We rewrite

$$J(z) = J(0) - \int_0^\infty \mathrm{d}x \, \underbrace{(1 - \cos(xz)) \left(\frac{\mathrm{Si}(x)}{x}\right)^2}_{=:K(x,z)}, \tag{B.11}$$

and consider the integrand $K(x,z)$ in different regimes:

$$K(x,z) = \begin{cases} \left(\frac{(xz)^2}{2} + \mathcal{O}\left((xz)^4\right)\right) \left(\frac{\mathrm{Si}(x)}{x}\right)^2 & : xz \ll 1 \\ (1 - \cos(xz)) \left(\frac{\pi^2}{4x^2} + \ldots\right) & : x \gg \frac{1}{z} \xrightarrow{z \to 0} \infty. \end{cases} \tag{B.12}$$

In the former case, the integrand is far away from the regime of validity of the asymptotic expansion of the sine integral part; it is simply analytic and can be replaced by its Taylor expansion in small z. The contribution of this part of the integral is $\mathcal{O}(z^2)$ and hence smaller than the order that we wish to consider. In the complementary regime, the sine integral part may be replaced

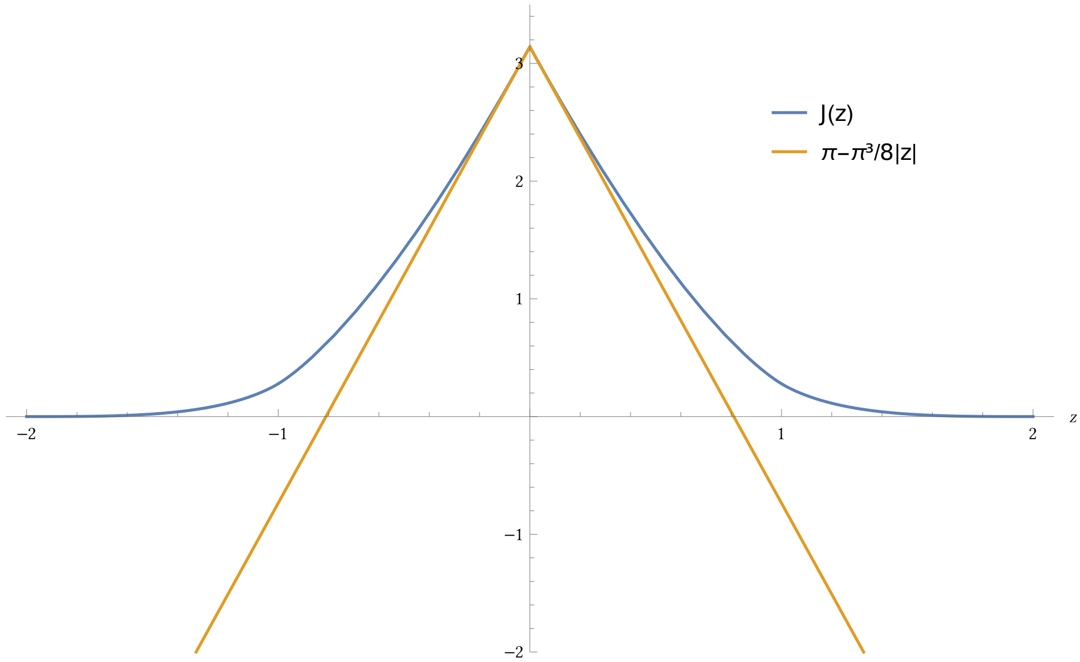

FIGURE B.1: Blue: Numerical evaluation of $J(z)$. Orange: Asymptotics (B.13). The Taylor expansion of $J(z)$ requires $|z| < 1$ to be well-defined, though the asymptotics deviates from the exact result earlier already. In particular, eq. (B.13) becomes negative for $z \approx 0.8$, whereas $J(z)$ remains positive everywhere.

by its asymptotic expansion, justified by the formal condition $x \gg \infty$ in the limit of small z. In this case, we pick up a contribution from the non-oscillatory tail; the oscillatory terms collected in $\ldots$ are terms of the form $\cos(x)/x^3$ or $\sin(x)/x^4$; the corresponding integrals are again absolutely convergent, allowing once more the replacement of $\cos(xz)$ and thus yielding an $\mathcal{O}(z^2)$ contribution. The only relevant contribution that is of lower order therefore stems from the non-oscillatory tail; accounting for it, the integral reads

$$J(z) = J(0) - \frac{\pi^2}{4}\int_0^\infty \mathrm{d}x\,\frac{1-\cos(xz)}{x^2} + \mathcal{O}\left(z^2\right) = \pi - \frac{\pi^3}{8}|z|, \tag{B.13}$$

which can be evaluated using a suitable Tauberian theorem [320]. The exact integral $J(z)$ is compared to the asymptotics (B.13) in fig. B.1.

Putting everything together, we recover the semiclassical Airy volume,

$$\begin{aligned} V^{\mathrm{sc}}_{\frac{1}{2},2}(l,l') &= \frac{8l_c}{\pi^2\mathcal{V}}\left(\frac{L\pi}{2} - \pi + \frac{\pi^3}{8}|z|\right)\Theta(1-|z|)\Theta(|z|-e^{-L}) \\ &= \frac{2}{\pi\mathcal{V}}l_c\left(l + l' - 4 + \frac{\pi^2}{2}|\delta l|/l_c\right)\Theta(l_c - |\delta l|)\Theta(|\delta l| - l_c e^{-L}) \\ &= 2e^{-S_0}\max(l-2, l'-2)\Theta(\pi^2/2 - |l-l'|)\Theta\left(|l-l'| - \frac{\pi^2}{2}e^{-\frac{l+l'}{2}}\right), \end{aligned} \tag{B.14}$$

where the last equality demands setting $l_c = \pi^2/2$, and I plugged in $\mathcal{V} = \frac{\pi}{2}e^{S_0}$, as discussed after eq. (9.31).

Appendix C

Accessing the classical regime of the OTOC

We consider the product

$$\prod_{k=0}^{\frac{f-3}{2}} \left(\frac{2mL^2}{\hbar^2} E + k^2 \right) \equiv \prod_{k=0}^{M} (x + k^2) = x^M \prod_{k=0}^{M} \left(1 + \frac{k^2}{x} \right). \tag{C.1}$$

For this product to be dominated by the highest-degree monomial, we need

$$\prod_{k=0}^{M} \left(1 + \frac{k^2}{x} \right) \approx 1, \tag{C.2}$$

which is true when

$$\left| \prod_{k=0}^{M} \left(1 + \frac{k^2}{x} \right) - 1 \right| \leq e^{\sum_{k=0}^{M} \frac{k^2}{x}} - 1 \approx 0, \tag{C.3}$$

i.e.,

$$\sum_{k=0}^{M} \frac{k^2}{x} \ll 1. \tag{C.4}$$

This sum can be evaluated exactly as

$$\sum_{k=0}^{M} k^2 = \frac{M(M+1)(2M+1)}{6} = \frac{(f-3)(f-2)(f-1)}{24} \approx \frac{f^3}{24} \tag{C.5}$$

for large f. Hence, the condition to access the classical regime in our system is

$$\frac{f^3}{24} \ll \frac{2mL^2}{\hbar^2} E, \tag{C.6}$$

and if we assume that in the classical regime, the energy is given roughly by the thermal energy,

$$E \approx \frac{f}{\beta}, \tag{C.7}$$

we obtain the classicality condition reported in the main text,

$$\frac{f^2}{24} \ll \frac{2mL^2}{\hbar^2\beta} = \frac{4\pi L^2}{\lambda_{\mathrm{th}}^2}. \tag{C.8}$$

Appendix D

Weyl symbol of the Hamiltonian

To find the Weyl symbol for our Hamiltonian (10.37), consider

$$\begin{aligned}
\frac{2m}{\hbar^2}\hat{H} &= -\Delta = -g^{-1/2}\partial_i g^{1/2} g^{ij}\partial_j \\
&= -g^{-1/2}(\partial_i g^{1/2} g^{ij})\partial_j - g^{ij}\partial_i\partial_j \\
&= -\partial^2 - g^{-1/2}(\partial_i g^{1/2})g^{ij}\partial_j - (\partial_i g^{ij})\partial_j \\
&= -\partial^2 - (\partial_i g^{ij})\partial_j - (\partial_i \log\sqrt{g})g^{ij}\partial_j \\
&= -\partial^2 - (\partial_i g^{ij})\partial_j - \frac{1}{2}g^{ij}\left[(\partial_i \log\sqrt{g})\partial_j + (\partial_j \log\sqrt{g})\partial_i\right].
\end{aligned} \tag{D.1}$$

Compare this to DeWitt's ordering [321–323],

$$\hat{H}_{\text{DeWitt}} = \frac{1}{2m}\hat{p}_i g^{ij}(\hat{x})\hat{p}_j + \hbar^2 Q(\hat{x}), \tag{D.2}$$

with the momentum operator

$$\hat{p}_i = \frac{\hbar}{i}\left(\partial_i + \frac{1}{2}\Gamma^j_{ji}(x)\right) \tag{D.3}$$

and the so-called quantum potential

$$Q(x) = \frac{1}{4m}g^{ij}\left[\partial_j\Gamma^k_{ki} - \Gamma^k_{ij}\Gamma^l_{lk} - \frac{1}{2}\Gamma^k_{ki}\Gamma^l_{lj}\right]. \tag{D.4}$$

Γ^k_{ij} are the standard Christoffel symbols. First, note that using

$$\Gamma_i = \Gamma^j_{ji} = \partial_i \log\sqrt{g}, \tag{D.5}$$

the quantum potential can be rewritten as [315]

$$Q(x) = \frac{1}{4m}\partial_i(g^{ij}\Gamma_j) + \frac{1}{8m}g^{ij}\Gamma_i\Gamma_j. \tag{D.6}$$

Furthermore, the Hamiltonian

$$\frac{2m}{\hbar^2}\hat{H}_{\text{DeWitt}} = 2mQ(x) - \left(\partial_i + \frac{1}{2}\Gamma_i(x)\right)g^{ij}(x)\left(\partial_j + \frac{1}{2}\Gamma_j(x)\right)$$

$$
\begin{aligned}
&= 2mQ - \frac{1}{2}(\partial_i g^{ij})\Gamma_j - \frac{1}{4}\Gamma_i g^{ij}\Gamma_j - \frac{1}{2}(\partial_i \Gamma_j) g^{ij} \\
&\quad - (\partial_i g^{ij})\partial_j - \partial^2 - \frac{1}{2} g^{ij}\Gamma_j \partial_i - \frac{1}{2}\Gamma_i g^{ij}\partial_j \\
&= -(\partial_i g^{ij})\partial_j - \partial^2 - \frac{1}{2} g^{ij}\left(\Gamma_j \partial_i + \Gamma_i \partial_j\right)
\end{aligned}
\tag{D.7}
$$

agrees with eq. (D.1). Fortuitously, the Weyl symbol for the DeWitt Hamiltonian is reported in [323] as

$$
W_{\hat{H}_{\mathrm{DeWitt}}}(x,p) = W_{\hat{H}}(x,p) = \frac{1}{2m} p_i g^{ij}(x) p_j + \hbar^2 Q(x) + \frac{\hbar^2}{8m}\partial_i \partial_j g^{ij}(x). \tag{D.8}
$$

After changing into Riemannian normal coordinates, i.e., coordinates such that at a point q, $g_{ij}(q) = \delta_{ij}$, $\partial_k g_{ij}(q) = 0$, $\Gamma^k_{ij}(q) = 0$, but $\partial_l \Gamma^k_{ij}(q) \neq 0$, a somewhat tedious calculation shows that the $\hbar^2$ correction to the Weyl symbol of the Hamiltonian is simply given by the Ricci scalar of the manifold,

$$
h_2 = \frac{R}{12m}. \tag{D.9}
$$

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