Shintani cocycle decomposition of topological polylogarithms

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Chapter One

Introduction

1.1 Zeta functions at nonpositive integers

Riemann zeta function. We begin this section by briefly discussing some properties of the *Riemann zeta function* ζ and the *Gamma function* Γ, defined respectively as

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad \Gamma(s) := \int_0^{\infty} e^{-u} u^{s-1} du,$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and respectively for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Both of these functions can be continued to a meromorphic function in $\mathbb{C}$ (see [Col09, Théorème VII.2.1] and Lemma 1.1.1 below).

A simple way to relate these two functions is by noting that, for $\lambda \in \mathbb{R}_+$,

$$\lambda^{-s}\Gamma(s) = \int_0^{\infty} e^{-t\lambda} t^{s-1} dt \tag{1.1}$$

after using the change of variables $u = \lambda t$. Through a simple contour integration argument, this can be extended to $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$. Then one can set $\lambda = n$ and sum over all natural numbers, and after justifying the interchange of sum and integral, we get

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nt} t^{s-1} dt.$$

Thus, by setting $G(t) := \frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^{\infty} e^{-nt}$, we have:

$$\zeta(s)\Gamma(s) = \int_0^\infty G(t)t^{s-1}dt. \tag{1.2}$$

More conceptually, the integral on the right hand side is defined as the *Mellin transform* of the function $G(t)$. The following Lemma can be used to give an analytic continuation of ζ and to evaluate it at negative integers:

Lemma 1.1.1. *Suppose $f \in C^\infty(\mathbb{R}_{\geq 0})$ and all of its derivatives rapidly decrease (that is, they are $O(|x|^{-N})$ as $x \to \infty$ for all $N \in \mathbb{N}$). Then $M(f,s) := \frac{1}{\Gamma(s)}\int_0^\infty f(t)t^{s-1}dt$, defined for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, admits an analytic continuation to all of $\mathbb{C}$, and for $k \in \mathbb{N}$, $M(f,-k) = (-1)^k(\frac{d}{dt})^k f(t)\big|_{t=0}$.*

Proof. We follow [Col09, Proposition VII.2.6]: first, it is clear that $\int_0^\infty |f(t)t^{s-1}dt|$ converges on compacts of $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$ due to the hypotheses on f decreasing rapidly, and we have that $\Gamma(s)$ is holomorphic and never equal to zero, so $M(f,s)$ is holomorphic for $\mathrm{Re}(s) > 0$.
Moreover, integration by parts shows that $M(f', s+1)+M(f,s) = M(f', s+1)+\frac{s}{\Gamma(s+1)}\int_0^\infty f(t)t^{s-1}dt = \frac{1}{\Gamma(s+1)}\int_0^\infty f'(t)\cdot(t^s)dt+\frac{1}{\Gamma(s+1)}\int_0^\infty f(t)\cdot(t^s)'dt = \frac{1}{\Gamma(s+1)}[f(t)t^s]_0^\infty = 0$ for all $\mathrm{Re}(s) > 0$. Repeating the argument, we have $M(f,s) = (-1)^k M(f^{(k)}, s+k)$ for $\mathrm{Re}(s) > 0$ and $k \in \mathbb{N}$. But this shows that $M(f,s) = (-1)^k M(f^{(k)}, s+k)$ is holomorphic for $\mathrm{Re}(s) > -k$ by applying the first paragraph to $f^{(k)}$ instead of f. Since this is true for all $k \in \mathbb{N}$, we have that $M(f,s)$ is entire. Lastly, we have that $M(f,-k) = (-1)^{k+1}M(f^{(k+1)},1) = (-1)^{k+1}\int_0^\infty f^{(k+1)}(t)dt = (-1)^{k+1}[f^{(k)}]_0^\infty = (-1)^k f^{(k)}(0)$. ◻

Remark 1.1.2. The Lemma above can be reinterpreted in the language of distributions: for all $s \in \mathbb{C}$, we define $\mathcal{M}_s$ as the tempered distribution $f \mapsto M(f,s)$ for $f \in \mathcal{S}(\mathbb{R}_{\geq 0})$ the Schwartz space (that is, space of smooth functions whose derivatives rapidly decrease, as in the Lemma 1.1.1). Note that for $k \in \mathbb{N}$, $\mathcal{M}_{-k} = \delta_0^{(k)}$ the k-th derivative of the delta function at zero.

Note that we cannot apply this Lemma directly, since $G(t)$ is not well defined at $t = 0$. Instead, we use $F(t) := tG(t)$ and the identity

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty G(t) t^{s-1} dt = \frac{1}{(s-1)\Gamma(s-1)} \int_0^\infty F(t) t^{s-2} dt = \frac{1}{s-1} M(F, s-1). \quad (1.3)$$

Now we may use Lemma 1.1.1 to deduce that $\zeta(-k) = \frac{(-1)^k B_{k+1}}{k+1}$, where $B_k \in \mathbb{Q}$ is the *k-th Bernoulli number*, defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} = \frac{te^{-t}}{1 - e^{-t}} = F(t).$$

Hurwitz zeta function and Dirichlet L-series. The above discussion can be generalized to *Dirichlet L-functions*, defined for a multiplicative character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ as

$$L(s, \chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s},$$

where $\chi(n) := \chi(n \bmod N)$ if $\gcd(n, N) = 1$ and $\chi(n) = 0$ otherwise. This series is absolutely convergent for $\mathrm{Re}(s) > 1$.

In order to proceed with the generalization it is convenient to introduce another zeta function, the *Hurwitz zeta function*, defined as

$$\zeta(s, x) := \sum_{n=0}^{\infty} (n + x)^{-s}$$

for $\mathrm{Re}(s) > 1$ and $0 < x \leq 1$. Now we observe that we can obtain the Dirichlet L-function as a linear combination of Hurwitz zeta functions in the following manner:

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} = \sum_{a=1}^{N} \chi(a) \sum_{n=1}^{\infty} (Nn+a)^{-s} = \sum_{a=1}^{N} \chi(a) N^{-s} \sum_{n=1}^{\infty} \left(n + \frac{a}{N}\right)^{-s} = \sum_{a=1}^{N} \chi(a) N^{-s} \zeta\left(s, \frac{a}{N}\right).$$

We have thus reduced the problem of analytically continuing $L(s, \chi)$ and determining the values of $L(-k, \chi)$ to doing so for $\zeta(s, x)$.

Now we may proceed as in the case of the Riemann zeta, obtaining

$$\Gamma(s)\zeta(s, x) = \int_0^\infty G(t, x) t^{s-1} dt$$

for $G(t,x) := \frac{e^{-xt}}{1-e^{-t}} = \sum_{n=0}^{\infty} e^{-(n+x)t}$. Applying the same reasoning as in (1.3) and Lemma 1.1.1 we get the desired analytic continuation, as well as

$$\zeta(-k,x) = -\frac{B_{k+1}(x)}{k+1} \quad \text{and} \quad L(-k,\chi) = -\sum_{a=1}^{N} \chi(a) N^k \frac{B_{k+1}(a/N)}{k+1},$$

where $B_n(x) \in \mathbb{Q}[x]$ is the *n-th Bernoulli polynomial*, defined by

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = t\frac{e^{tx}}{e^t - 1} = tG(t, 1-x) = -tG(-t, x)$$

and satisfying

$$B_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} B_j x^{n-j}$$

with B_j the previously defined j-th Bernoulli number (see [AIK14, Proposition 4.9]).

We also remark that both in the case of the Riemann zeta as well as in the case of the Hurwitz zeta, the meromorphic continuation may be achieved after converting the corresponding integral to a contour integral which avoids the unique singularity of G at zero (see [Hid93, Chapter 2]).

Lattices and partial zeta functions. The next step would be to generalize this method to Hecke L-functions for finite Hecke characters (that is, characters of the ray class group). Let F be a totally real number field and L a lattice in $F \otimes \mathbb{R} \cong \mathbb{R}^n$ for $n = \dim F$ defined by a fractional ideal of F, and denote by $(-)^+$ the totally positive elements, that is, those in the totally positive orthant of $\mathbb{R}^n$. For a non-zero torsion point $\boldsymbol{t} \in F \otimes \mathbb{R}/L$, denote by $\Gamma = \mathcal{O}_{\boldsymbol{t}}^{\times,+}$ the group of totally positive units which stabilize $\boldsymbol{t}$ modulo L.

Definition 1.1.3. The zeta function associated to L and $\boldsymbol{t}$ is

$$\zeta(\boldsymbol{t}, L, s) := \sum_{\alpha \in (\boldsymbol{t}+L)^+/\Gamma} N(\alpha)^{-s} \tag{1.4}$$

which as usual converges for $\mathrm{Re}(s) > 1$.

Definition 1.1.4. Let now $\mathfrak{f}$ and $\mathfrak{b}$ integral ideals of $\mathcal{O}_F$. The *partial zeta function* of the ray class of $\mathfrak{b}$ modulo $\mathfrak{f}$ is

$$\zeta(\mathfrak{b},\mathfrak{f},s) := \sum_{\mathfrak{g}\sim_{\mathfrak{f}}\mathfrak{b}} N(\mathfrak{g})^{-s} \tag{1.5}$$

where the sum is over all integral ideals $\mathfrak{g}$ in the ray class of $\mathfrak{b}$ modulo $\mathfrak{f}$.

Since all of these ideals $\mathfrak{g}$ are of the form $\mu\mathfrak{b}$ with $\mu \in (1+\mathfrak{f}\mathfrak{b}^{-1})^+ = (1+L)^+$ for $L = \mathfrak{f}\mathfrak{b}^{-1}$, we have that

$$\zeta(\mathfrak{b},\mathfrak{f},s) = N(\mathfrak{b})\zeta(1,\mathfrak{f}\mathfrak{b}^{-1},s). \tag{1.6}$$

Note that in the 1 dimensional case we recover the Hurwitz zeta function and the decomposition of the Dirichlet L-series. These zeta functions were the ones studied by Siegel (see e.g. [Sie70]) in order to prove the Klingen-Siegel theorem on the rationality of negative integer zeta values (see Corollary 4.2.2).

Another such generalization, also capable of obtaining the rationality of negative integer zeta values, was given by Shintani in [Shi76]. Following Shintani's work, it is possible to decompose the partial zeta functions as a linear combination of now-called Shintani zeta functions, similarly to the case above. However, this decomposition does not follow from a simple algebraic manipulation, but from a remarkable result of geometric nature called Shintani's Unit Theorem. This Theorem, as well as the decomposition of the L-function, will be presented in the next sections. For now, we finish this section by introducing the Shintani zeta function within our familiar framework, and therefore obtaining its values at negative integers in a particularly nice case.

Shintani zeta function. Let $A = (a_{ij})$ be a complex $r \times m$ matrix with $Re(a_{ij}) > 0$ for all i and j; $\chi \in \mathbb{C}^r$ with $|\chi_i| \le 1$ for all i; and $\boldsymbol{x} \in \mathbb{R}^r$ such that $0 \le x_i \le 1$ for all i, but not all x_i are 0. We define linear forms L_i on $\mathbb{C}^m$ and L_j^* on $\mathbb{C}^r$ by

$$L_i(\boldsymbol{z}) = \sum_{k=1}^{m} a_{ik}z_k, \quad L_j^*(\boldsymbol{w}) = \sum_{k=1}^{r} a_{kj}w_k \quad \text{for} \quad \boldsymbol{z} = (z_1,\dots,z_m),\ \boldsymbol{w} = (w_1,\dots,w_r).$$

$L_i(\boldsymbol{z})$ can be seen as the i-th row of $A\boldsymbol{z}$ and $L_j^*(\boldsymbol{w})$ as the j-th column of $\boldsymbol{w}^T A$.

Definition 1.1.5. The *Shintani zeta function* of $\boldsymbol{s} \in \mathbb{C}^m$ is

$$\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \sum_{\boldsymbol{n}\in\mathbb{N}^r} \chi^{\boldsymbol{n}} \prod_{j=1}^{m} L_j^*(\boldsymbol{n}+\boldsymbol{x})^{-s_j}.$$

Note that when $A = 1$ and $\chi = 1$, $\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \zeta(s, x)$ the Hurwitz zeta function. We also note that the Shintani zeta function converges absolutely when $\mathrm{Re}(s_i) > \frac{r}{m}$ for all i.

We proceed as before, introducing the function

$$G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) := \sum_{\boldsymbol{n}\in\mathbb{N}^r} \chi^{\boldsymbol{n}} e^{-\sum_{j=1}^m L_j^*(\boldsymbol{n}+\boldsymbol{x})t_j}$$

for $\boldsymbol{t} \in \mathbb{R}_+^m$ and noting that

$$G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) = \prod_{i=1}^{r} \frac{e^{-x_i L_i(\boldsymbol{t})}}{1 - \chi_i e^{-L_i(\boldsymbol{t})}}.$$

It then follows as before that

$$\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \int_0^\infty \cdots \int_0^\infty G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) \prod_{j=1}^{m} \frac{t_j^{s_j-1}}{\Gamma(s_j)} dt_j. \tag{1.7}$$

Converting this integral into a contour integral convergent for all $\boldsymbol{s} \in \mathbb{C}^m$ cannot be done in a straightforward manner, since there are poles along the hyperplanes defined by the forms L_i, but first requires a partition of $\mathbb{R}_+^m$ into m special regions together with a change of variables in each region, a method ingeniously devised by Shintani in [Shi76] (an algebraic interpretation of this trick is discussed in section 3.3). In this manner, it is possible to obtain a meromorphic continuation of $\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi)$ to all of $\mathbb{C}^m$.

In the case of $\chi_i \neq 1$ for all $i \leq m$, we are able to use the following generalization of 1.1.1 to obtain the analytic continuation and the values of the Shintani zeta function at negative integers:

Lemma 1.1.6. *Suppose $f \in C^\infty(\mathbb{R}_{\geq 0}^m)$ rapidly decreases, as well as all of its partial derivatives. Then $M(f, \boldsymbol{s}) := \int_0^\infty \cdots \int_0^\infty f(\boldsymbol{t}) \prod_{j=1}^m \frac{t_j^{s_j-1}}{\Gamma(s_j)} dt_j$, defined for all $\boldsymbol{s} \in \mathbb{C}^m$ with $Re(s_j) > 0$ for all j, admits an analytic continuation to all of $\mathbb{C}^m$, and for $\boldsymbol{k} \in \mathbb{N}^m$, $M(f, -\boldsymbol{k}) = \prod_{i=1}^m (-\frac{\partial}{\partial t_i})^{k_i} f(t)\big|_{t=0}$.*

Proof. This follows easily from 1.1.1. Also appears in [Col88, Lemme 3.2]. □

Indeed, the function $G(\boldsymbol{t}, A, \boldsymbol{x}, \chi)$ satisfies the hypotheses of the Lemma for $\chi_i \neq 1 \ \forall i \leq m$. In fact, it has no singularities in a neighbourhood of zero, so we may obtain its power series expansion around 0 as

$$G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) = \sum_{\boldsymbol{n} \in \mathbb{N}^m} B_{\boldsymbol{n}+1}(\boldsymbol{x}) \prod_{i=1}^{m} \frac{t_i^{n_i}}{(n_i+1)!}.$$

We may call the $B_{\boldsymbol{n}}(\boldsymbol{x})$ *generalized Bernoulli polynomials*, since they reduce to Bernoulli polynomials when $m = 1$, and remark that they have rational coefficients. Just like before, we conclude that

$$\zeta((-k, \ldots, -k), A, \boldsymbol{x}, \chi) = (-1)^{mk} \frac{B_{\boldsymbol{k}+1}(\boldsymbol{x})}{(k+1)^m} \quad \text{for } k \in \mathbb{N}.$$

1.2 Outline

The aim of this thesis is to compare two cohomological constructions encoding special values of zeta functions associated to totally real number fields: the Shintani cocycles arising from geometric cone decompositions, and the Eisenstein classes defined via the topological polylogarithm. In order to make this comparison precise, it is necessary to fix compatible analytic, geometric, and cohomological frameworks, which is the purpose of the first three chapters.

In the next chapter, we will see how these Shintani zeta functions are related to the partial zeta functions through the Shintani unit theorem, how we may obtain their values at negative integers when $\chi = 1$, and how to interpret these results in terms of cohomology, giving rise to the Shintani cocycles. We mostly follow the ideas from [CDG15] and [BHYY23].

In the third chapter, we present the theory of topological polylogarithms, developed in [BKL18] and [Gra16], giving a geometric picture through which one can study the generating series of special values of zeta functions associated to lattices as sections of the Logarithm sheaf. We give an outline of the theory, explaining how one can use currents to explicitly represent the topological polylogarithm class in the equivariant cohomology of the Logarithm sheaf. The final section addresses the problem of choosing compatible coefficient rings for both the Shintani and polylogarithm constructions, introducing a suitable field of Laurent series that accommodates both theories and enables a meaningful comparison.

Finally, in the fourth and last chapter we compare both cohomological theories by decomposing the topological polylogarithm into Shintani cocycles via a simplicial cone decomposition of the auxiliary space used to define the topological polylogarithm. This results in the Main Theorem of the thesis, which roughly states that the Eisenstein class given by the topological polylogarithm is equal to the Shintani class defined by the Shintani cocycles, up to an explicit sign and a correction term given by the polar cocycle previously studied in the literature.

Chapter Two

Shintani decompositions

2.1 Setup

The following identifications are useful for comparing different approaches, with notations based on [BKL18] and [CDG15], but not always the same. All throughout we follow the principle "right action on spaces, left action on cohomology".

Orientations, bases and identifications. Let F be an n-dimensional totally real number field, $\mathfrak{f}$ and $\mathfrak{b}$ nonzero integral ideals of $\mathcal{O}_F$ and Γ a subgroup of finite index of the group of totally positive units $\mathcal{O}_F^{\times,+}$.

We will often use the following identification, which is both an isomorphism of $\mathbb{Q}$-algebras as well as of $\mathbb{R}$-vector spaces:

$$\begin{aligned} F \otimes_{\mathbb{Q}} \mathbb{R} &\xrightarrow[\cong]{J} \prod_{J_i : F \hookrightarrow \mathbb{R}} \mathbb{R} = \mathbb{R}^n \\ x \otimes r &\longmapsto r(J_1(x_1), \ldots, J_n(x_n)) \end{aligned}$$

where the product runs over all real embeddings of F, with a fixed order given by $i \in \{0, \ldots, n\}$.

We then have the following isomorphisms of $\mathbb{R}$-vector spaces and of lattices:

Let $\mathbf{w} := (w_1, \ldots, w_n)$ be a $\mathbb{Z}$-basis of $\mathfrak{f}\mathfrak{b}^{-1}$, therefore also a $\mathbb{Q}$-basis of F.

$$\begin{array}{ccccc} \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow[\cong]{\cdot \mathbf{w}} & F \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow[\cong]{J} & \mathbb{R}^n \\ \mathbb{Z}^n \otimes 1 & \longmapsto & \mathfrak{f}\mathfrak{b}^{-1} \otimes 1 & \longmapsto & L \\ v + \mathbb{Z}^n & \longmapsto & 1 + \mathfrak{f}\mathfrak{b}^{-1} & \longmapsto & 1 + L \\ Q & & \longmapsto & & (0, \ldots, 0, 1) \end{array} \tag{2.1}$$

Where $v := (Tr(w_1^*), \ldots, Tr(w_n^*))$ and $Q := (J_n(w_1^*), \ldots, J_n(w_n^*))$ for w_i^* corresponding to the dual of w_i with respect to the perfect trace pairing $Tr(- \cdot -) : F \times F \to \mathbb{Q}$, which corresponds to the usual dot product on the right side since J is a $\mathbb{Q}$-algebra isomorphism. Note that (2.1) gives immediately $\zeta(v, \mathbb{Z}^n, s) = \zeta(1, L, s)$ for Γ the group of totally positive units that stabilize $\boldsymbol{t}$ as in (1.4). Therefore such a choice of J and $\mathbf{w}$ gives $\zeta(\mathfrak{b}, \mathfrak{f}, s) = N(\mathfrak{b})\zeta(v, \mathbb{Z}^n, s)$.

Norm 1 subspace. Denote $V = \mathbb{R}^n$ and $V_{>0} = \mathbb{R}^n_{>0}$. Then $F \otimes \mathbb{R}$ inherits through J the topology of V, and we may define $(F \otimes \mathbb{R})_{>0} := J^{-1}(V_{>0})$ the connected component of $1 \in (F \otimes \mathbb{R})^\times$. Therefore we get $J : (F \otimes \mathbb{R})_{>0} \xrightarrow{\cong} V_{>0}$ isomorphism of multiplicative topological groups, which distribute over the additive semigroup structure.

As multiplicative topological groups, they fit in the following split short exact sequence

$$1 \longrightarrow V^1 \underset{p_1}{\leftrightarrows} V_{>0} \xrightarrow{N} \mathbb{R}_{>0} \longrightarrow 1 \tag{2.2}$$

where V^1 is defined as the norm 1 subspace of $V_{>0}$, forming a connected hypersurface, and p_1 is the projection onto this subspace. We have explicitly

$$\begin{array}{ccc} V_{>0} & \xrightarrow{\cong} & V^1 \times \mathbb{R}_{>0} \\ v & \longmapsto & \left(\frac{v}{|N(v)^{1/n}|}, N(v)\right) \end{array} \qquad \begin{array}{ccc} (F \otimes \mathbb{R})_{>0} & \xrightarrow{\cong} & (F \otimes \mathbb{R})^1 \times \mathbb{R}_{>0} \\ x \otimes r & \longmapsto & \left(x \otimes \frac{1}{|N(x)^{1/n}|}, r^n N(x)\right) \end{array}$$

Group actions. We also have a natural $F^\times$ action on F, which induces a representation $\rho_{\mathbf{w}} : F^\times \to \mathrm{GL}_n(\mathbb{Q})$ defined in such a way as to make the previous identifications equivariant, i.e., $u \cdot \mathbf{w} = \mathbf{w} \cdot \rho_{\mathbf{w}}(u)$ for all $u \in F^\times$. We then have the following equivariant isomorphisms

$$\begin{array}{ccccc} \rho_{\mathbf{w}}(F^\times) \circlearrowright & & F^\times \circlearrowright & & diag(J(F^\times)) \circlearrowright \\ \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow[\cong]{\cdot \mathbf{w}} & F \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow[\cong]{J} & \mathbb{R}^n \end{array} \tag{2.3}$$

Where $\rho_{\mathbf{w}}(F^\times) \subset \mathrm{GL}_n(\mathbb{Q})$ acts on $\mathbb{Q}^n$ on the right via multiplication by $\rho_{\mathbf{w}}(u)^\top$ for all $u \in F^\times$. We also have that $J(A \cdot \mathbf{w}) = A \cdot J(\mathbf{w})$ for all $A \in \mathrm{GL}_n(\mathbb{Q})$.

Note that $\rho_{\mathbf{w}} : \Gamma \hookrightarrow \mathrm{SL_n}(\mathbb{Q})$ since $\det(\rho_{\mathbf{w}}(-)) = N(-)$ the norm, for all bases $\mathbf{w}$. So in particular, the action of Γ descends through p_1 to V^1.

Lastly, also note that the action of Γ is closed on the lattice, that is, each $\rho_{\mathbf{w}}$ induces $\Gamma \hookrightarrow \mathrm{GL}(L)$, so we get as before

$$\begin{array}{ccccc} \rho_{\mathbf{w}}(\Gamma) \circlearrowright & & \Gamma \circlearrowright & & diag(J(\Gamma)) \circlearrowright \\ \mathbb{Z}^n & \xrightarrow[\cong]{\cdot \mathbf{w}} & \mathfrak{f}\mathfrak{b}^{-1} & \xrightarrow[\cong]{J} & L \end{array} \tag{2.4}$$

2.2 Shintani cone decompositions

We follow [BHYY23] for the conventions regarding cone decompositions. Note that this differs from [CDG15], as we will remark throughout the text.

Cones.

Definition 2.2.1. A rational closed polyhedral cone in $V_{>0} \cup \{0\}$, which we simply call a cone, is any set of the form

$$\sigma_{\underline{\alpha}} \coloneqq \{x_1\alpha_1 + \ldots + x_m\alpha_m \mid x_1, \ldots, x_m \in \mathbb{R}_{\geq 0}\}$$

for some $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in L^m_{>0}$. In this case, we say that $\underline{\alpha}$ is a generator of $\sigma_{\underline{\alpha}}$. By considering the case $m = 0$, we see that $\sigma = \{0\}$ is a cone.

Remark 2.2.2. Note that in [CDG15] they define cones as being open. The advantage of working with closed cones is that we can consider only cones of maximal dimension in the Shintani decomposition below.

We define the dimension $\dim\sigma$ of a cone σ to be the dimension of the $\mathbb{R}$-vector space generated by σ. For any subset $R \subset V_{>0}$, we let

$$\breve{R} := \{(u_{\tau_1}, \ldots, u_{\tau_n}) \in V_{>0} \mid \exists\delta > 0,\ 0 < \forall\delta' < \delta, (u_{\tau_1}, \ldots, u_{\tau_{n-1}}, u_{\tau_n} - \delta') \in R\}.$$

Note that by definition, if $\dim\sigma < n$, then we have $\breve{\sigma} = \emptyset$, where $\breve{\sigma}$ is the cone without the $(n-1)$-dimensional lower face, which corresponds to the Q-perturbation $c(\underline{\alpha})_Q$ in the notation of [CDG15] §2.1, under our choice of Q in 2.1.

In [Shi76] Shintani proved what is now known as Shintani's Unit Theorem, which states that the fundamental domain of the action of Γ on $V_{>0}$ is given by a finite set of rational polyhedral cones. This implies that the partial zeta functions may be expressed as a finite sum of geometric Shintani zeta functions $\zeta_\sigma(1, (s, \ldots, s))$ using a Shintani decomposition, as we will show below.

Cone decompositions. Following [BHYY23] Definition 2.10 we say that a cone σ is *simplicial*, if there exists a generator of σ that is linearly independent over $\mathbb{R}$. Any cone generated by a subset of such a generator is called a *face* of σ. A simplicial fan Φ is a set of simplicial cones such that for any $\sigma \in \Phi$, any face of σ is also in Φ, and for any cones $\sigma, \sigma' \in \Phi$, the intersection $\sigma \cap \sigma'$ is a common face of σ and σ'.

Definition 2.2.3. A *Shintani decomposition* is a simplicial fan Φ satisfying the following properties.

1. $V_{>0} \cup \{0\} = \coprod_{\sigma\in\Phi} \sigma^\circ$, where σ° is the relative interior of σ, i.e., the interior of σ in the $\mathbb{R}$-linear span of σ.
2. For any $\sigma \in \Phi$ and $\gamma \in \Gamma$, we have $\sigma\gamma \in \Phi$.
3. The quotient Φ/Γ is a finite set.

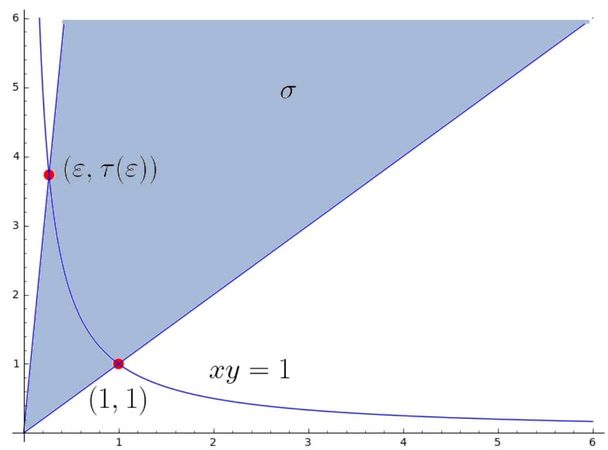
Theorem 2.2.4. *(Shintani's Unit Theorem) A Shintani decomposition as above always exists.*

Proof. Such a decomposition can be obtained by slightly modifying the construction of Shintani [Shi76, Theorem 1] (see also [Hid93, §2.7 Theorem 1], [Yam10, Theorem 4.1]). □

For any integer $q \geq 0$, we denote by Φ_q the subset of Φ consisting of cones of dimension q. From our choice to work with closed cones, it follows as in [Yam10, Proposition 5.6] that Φ_n satisfies

$$V_{>0} = \coprod_{\sigma \in \Phi_n} \breve{\sigma}. \tag{2.5}$$

Example 2.2.5. Let F be a real quadratic extension of $\mathbb{Q}$. Then by Dirichlet's Unit Theorem, Γ is generated by a single fundamental unit ε, and the cone generated by $(1, \varepsilon)$ can be visualized as



where all the units in Γ are obviously contained in the line $xy = 1$ which corresponds to our V^1, and the Γ action on σ clearly covers the whole positive quadrant $V_{>0}$. For this image, $F = \mathbb{Q}(\sqrt{3})$ and $\varepsilon = 2 + \sqrt{3}$.

2.3 Cone zeta functions

In this section we will define zeta functions associated to cones, and relate them with the previous zeta functions from the introduction. In particular, we will use the Shintani decomposition to relate them with the partial zeta functions.

Definition 2.3.1. Let σ be a cone, and let $\boldsymbol{t} \in V/L$ be a torsion point. We define the *cone zeta function* $\zeta_\sigma(\boldsymbol{t}, \boldsymbol{s})$ by the series

$$\zeta_\sigma(\boldsymbol{t}, \boldsymbol{s}) := \sum_{\alpha \in \breve{\sigma} \cap \boldsymbol{t}+L} \prod_{j=1}^{n} (J_j(\alpha))^{-s_j}, \tag{2.6}$$

where $\boldsymbol{s} = (s_j) \in \mathbb{C}^n$. The series converges if $\mathrm{Re}(s_j) > 1$ for all $j \leq n$.

If we let $\boldsymbol{s} = (s, \ldots, s)$ for $s \in \mathbb{C}$, then we have

$$\zeta_\sigma(\boldsymbol{t}, (s, \ldots, s)) = \sum_{\alpha \in \breve{\sigma} \cap \boldsymbol{t} + L} N(\alpha)^{-s}. \tag{2.7}$$

Proposition 2.3.2. *The zeta function associated to a n-dimensional cone σ can be written as a linear combination of Shintani zeta functions from Definition 1.1.5.*

Proof. Let $\sigma = \sigma_{\underline{\alpha}}$ be a n-dimensional cone generated by $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and we let $P_{\underline{\alpha}} \coloneqq \{x_1\alpha_1 + \ldots + x_n\alpha_n \mid \forall i \; 0 \leq x_i \leq 1\}$ be the parallelepiped spanned by $\alpha_1, \ldots, \alpha_n$. First, note that $\breve{P}_{\underline{\alpha}} \cap t + L$ is finite since $P_{\underline{\alpha}}$ is compact and L is a lattice.

Now, let $R^\sigma_{\underline{\alpha}}$ be the set of $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x} \cdot \underline{\boldsymbol{\alpha}} = (x_1\alpha_1, \ldots, x_n\alpha_n) \in \breve{\sigma} \cap \boldsymbol{t} + L$ and $R^P_{\underline{\alpha}}$ the set of $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{x} \cdot \underline{\boldsymbol{\alpha}} \in \breve{P}_{\underline{\alpha}} \cap \boldsymbol{t} + L$. Then we claim that $R^\sigma_{\underline{\alpha}} = R^P_{\underline{\alpha}} \oplus \mathbb{N}^n$. Indeed, given $\boldsymbol{x} \in R^\sigma_{\underline{\alpha}}$, it can be uniquely written as $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor) + (\{x_1\}, \ldots, \{x_n\})$, where $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor) \in \mathbb{N}^n$ and $(\{x_1\}, \ldots, \{x_n\}) \in R^P_{\underline{\alpha}}$.

Furthermore $\boldsymbol{x} \in R^\sigma_{\underline{\alpha}}$ implies that $\boldsymbol{x} \in \mathbb{Q}^n$, thus we may conclude that for every i, $J_i(x_1\alpha_1 + \ldots + x_n\alpha_n) = x_1 J_i(\alpha_1) + \ldots + x_n J_i(\alpha_n)$. We may then rewrite $\zeta_\sigma(t, \boldsymbol{s})$ as

$$\zeta_\sigma(\boldsymbol{t}, \boldsymbol{s}) = \sum_{\beta \in \breve{\sigma} \cap \boldsymbol{t} + L} \phi(\beta) \prod_{j=1}^{n} (J_j(\beta))^{-s_j} = \sum_{\boldsymbol{x} \in R^\sigma_{\underline{\alpha}}} \left[\prod_{j=1}^{n} \left(\sum_{i=1}^{n} x_i J_j(\alpha_i) \right)^{-s_j} \right] =$$

$$= \sum_{\boldsymbol{x} \in R^P_{\underline{\alpha}}} \sum_{\boldsymbol{n} \in \mathbb{N}^n} \prod_{j=1}^{n} \left(\sum_{i=1}^{n} (x_i + n_i) J_j(\alpha_i) \right)^{-s_j} =$$

$$= \sum_{\boldsymbol{x} \in R^P_{\underline{\alpha}}} \zeta(\boldsymbol{s}, A, \boldsymbol{x}, 1),$$

where

$$A \coloneqq A_{\underline{\alpha}} \coloneqq J(\underline{\alpha}) \coloneqq \begin{pmatrix} J_1(\alpha_1) & \cdots & J_n(\alpha_1) \\ \vdots & \ddots & \vdots \\ J_1(\alpha_n) & \cdots & J_n(\alpha_n) \end{pmatrix} \quad \text{and} \quad \chi = 1.$$

The claim now follows from the finiteness of $R^P_{\underline{\alpha}}$.

□

This zeta functions give rise to the following generating functions:

Definition 2.3.3. Let σ be the cone generated by $\underline{\boldsymbol{\alpha}}$ and $\boldsymbol{t} \in V/L$ be a torsion point. Then we define $\mathcal{G}_{\sigma_{\underline{\alpha}}}(\boldsymbol{t}, z)$ to be the meromorphic function on $\mathbb{C}^n$

$$\mathcal{G}_{\sigma_{\underline{\alpha}}}(t, z) \coloneqq \frac{\sum_{\alpha \in \check{P}_{\underline{\alpha}} \cap t+L} e^{\langle \alpha, z\rangle}}{\left(1 - e^{\langle \alpha_1, z\rangle}\right) \cdots \left(1 - e^{\langle \alpha_n, z\rangle}\right)}$$

with $\check{P}_{\underline{\boldsymbol{\alpha}}} \cap L$ the fundamental parallelepiped defined in the proof of Proposition 2.3.2. The definition of $\mathcal{G}_\sigma(\boldsymbol{t})$ depends only on the cone and is independent of the choice of the generator $\underline{\boldsymbol{\alpha}}$.

Remark 2.3.4. This generating function is given by the Solomon-Hu pairing in [CDG15] Proposition 3.1. If $F = \mathbb{Q}$ and $\sigma = \mathbb{R}_{\geq 0}$, then we have $\mathcal{G}_\sigma(t, z) = \frac{e^{tz}}{1-e^z}$, which is exactly the previously defined $G(t)$ from (1.2).

We have the following result.

Proposition 2.3.5. *Let $L = J(\mathfrak{f}\mathfrak{b}^{-1})$, $\boldsymbol{t} \in V/L$ a non-zero torsion point and $\Gamma = O_{\boldsymbol{t}}^{\times,+}$ the group of totally positive units which stabilize $\boldsymbol{t}$ modulo L. If Φ is a Shintani decomposition, then we have*

$$\zeta(\boldsymbol{t}, L, s) = \sum_{\sigma \in \Phi_n/\Gamma} \zeta_\sigma(\boldsymbol{t}, (s, \ldots, s)). \tag{2.8}$$

Proof. By (2.5), if C is a set of representatives of Φ_n/Γ, then $\coprod_{\sigma \in C} \breve{\sigma}$ is a representative of the set $V_{>0}/\Gamma$. Our result follows from the definition (1.4) of the lattice zeta function and (2.7). □

It follows immediately from (1.6) that

$$N(\mathfrak{b})^s \zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{\sigma \in \Phi_n/\Gamma} \zeta_\sigma(1, (s, \ldots, s)). \tag{2.9}$$

2.4 Shintani cocycles

Lattice resolutions. For any integer $q \geq 0$, we let

$$C_q(L) \coloneqq \bigoplus_{\underline{\alpha} \in L^{q+1}}^{\text{alt}} \mathbb{Z}\underline{\boldsymbol{\alpha}}$$

be the quotient of $\bigoplus_{\underline{\alpha}\in L^{q+1}} \mathbb{Z}\underline{\boldsymbol{\alpha}}$ by the submodule generated by

$$\{\rho(\underline{\boldsymbol{\alpha}}) - \operatorname{sgn}(\rho)\underline{\boldsymbol{\alpha}} \mid \underline{\boldsymbol{\alpha}} \in L^{q+1}, \rho \in \mathfrak{S}_{q+1}\} \cup \{\underline{\boldsymbol{\alpha}} = (\alpha_0, \dots, \alpha_q) \mid \alpha_i = \alpha_j \text{ for some } i \neq j\}.$$

We denote by $\langle\underline{\boldsymbol{\alpha}}\rangle$ the class represented by $\underline{\boldsymbol{\alpha}}$ in $C_q(L)$. We see that $C_q(L)$ has a natural action of Γ and is a free $\mathbb{Z}[\Gamma]$-module. Indeed, if we let B be the subset of L^{q+1} consisting of elements $\underline{\boldsymbol{\alpha}} = (\alpha_0, \dots, \alpha_q)$ such that $\alpha_0 < \cdots < \alpha_q$, then $C_q(L)$ is generated by a representative of the quotient B/Γ. Then $C_\bullet(L)$ is a complex of $\mathbb{Z}[\Gamma]$-modules with respect to the standard differential operator $d_q\colon C_q(L) \to C_{q-1}(L)$ given by

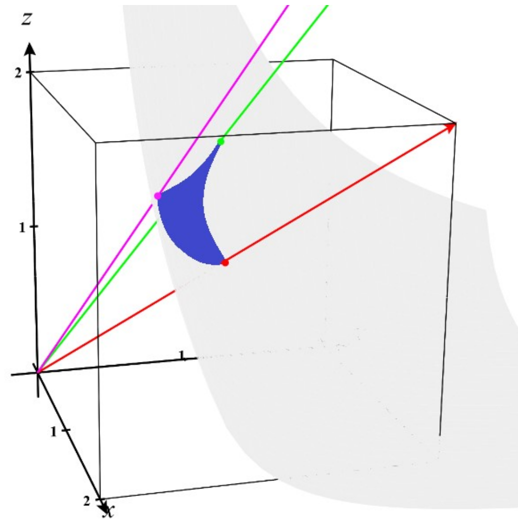
$$d_q(\langle\alpha_0, \dots, \alpha_q\rangle) \coloneqq \sum_{j=0}^{q} (-1)^j \langle\alpha_0, \dots, \check{\alpha}_j, \dots, \alpha_q\rangle$$

for any $\underline{\boldsymbol{\alpha}} = (\alpha_0, \dots, \alpha_q) \in L^{q+1}$. If we view $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module with trivial Γ-action and let $d_0\colon C_0(L) \to \mathbb{Z}$ be the unique $\mathbb{Z}$-linear homomorphism defined by $d_0(\langle\alpha\rangle) = 1$ for any $\alpha \in L$, then $C_\bullet(L)$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module.

We will next use a Shintani decomposition to construct a complex which is quasi-isomorphic to the complex $C_\bullet(L)$. Note that for any Shintani decomposition Φ, we can assume that each $\sigma \in \Phi$ is of the form $\sigma = \sigma_{\underline{\alpha}}$ for some $\underline{\boldsymbol{\alpha}} \in L^{q+1}$, since the generators of our cones are always assumed to be rational.

Simplicial resolutions. Remember that through log, V^1 is a real manifold homeomorphic to $\mathbb{R}^{n-1}$ and for any $\sigma \in \Phi_{q+1}$ the intersection $\sigma \cap V^1$ is a subset of V^1 which is homeomorphic to a simplex of dimension q. Recall that by Definition 2.2.3, Φ is a simplicial fan, therefore the set $\Phi^1 := \{\sigma \cap V^1 \mid \sigma \in \Phi\} = p_1(\Phi)$ gives a simplicial decomposition of the topological space V^1 that is compatible with the action of Γ.

Remark 2.4.1. If $n = 3$, the cone generated by the pink, green and red lines in the figure below intersects the *norm one surface* V^1 for $V = \mathbb{R}^3$, colored in gray, giving rise to the blue simplex:



In what follows, for any $\sigma \in \Phi_{q+1}$, we denote by $\langle\sigma\rangle$ the class $\operatorname{sgn}(\underline{\alpha})\langle\underline{\alpha}\rangle$ in $C_q(L)$, where sgn corresponds to the sign of the determinant of the matrix $A_{\underline{\alpha}}^{\top}$ whose columns are given by the α_i, and $\underline{\alpha} \in L^{q+1}$ is a generator of σ that is minimal in the sense that dividing it by an integer means $\underline{\alpha}$ is no longer in L^{q+1}. This is well-defined since such a generator $\underline{\alpha}$ is uniquely determined up to permutation. We then have the following.

Lemma 2.4.2. *For any integer $q \geq 0$, we let $C_q(\Phi)$ be the $\mathbb{Z}[\Gamma]$-submodule of $C_q(L)$ generated by $\langle\sigma\rangle$ for all $\sigma \in \Phi_{q+1}$. Then $C_\bullet(\Phi)$ is a subcomplex of $C_\bullet(L)$ which also gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module. In particular, the natural inclusion induces a quasi-isomorphism of complexes*

$$C_\bullet(\Phi) \stackrel{\text{qis}}{\hookrightarrow} C_\bullet(L)$$

compatible with the action of Γ.

Proof. Note that $C_q(\Phi)$ for any integer $q \geq 0$ is a free $\mathbb{Z}[\Gamma]$-module generated by representatives of the quotient Φ_{q+1}/Γ. By construction, $C_\bullet(\Phi)$ can be identified with the chain complex associated to the simplicial decomposition Φ^1 of the topological space V^1. Since $V^1 \cong \mathbb{R}^{n-1}$ is contractible, the chain complex associated to the simplicial decomposition is exact, so we see that the complex $C_\bullet(\Phi)$ is exact and gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module. Since any two free resolutions of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module are quasi-isomorphic, the claim follows from the fact that $C_\bullet(L)$ also gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Gamma]$-module. □

Comparison with cohomology of torus. Since F is totally real, the Dirichlet Unit Theorem shows that the discrete subset $log(\Gamma) \subset log(V^1)$ is a free $\mathbb{Z}$-module of rank $n-1$. Hence we have

$$V^1 \big/ \Gamma \cong \mathbb{R}^{n-1} \big/ \mathbb{Z}^{n-1}.$$

We consider the coinvariant

$$C_q(\Phi/\Gamma) := C_q(\Phi)_\Gamma$$

of $C_q(\Phi)$ with respect to the action of Γ, that is, the $\mathbb{Z}$-module obtained by quotienting $C_q(\Phi)$ by the subgroup generated by $\langle\sigma\rangle - \langle\sigma\gamma\rangle$ for $\sigma \in \Phi_{q+1}$ and $\gamma \in \Gamma$. For any $\sigma \in \Phi_{q+1}$, we denote by $\overline{\sigma}$ the image of σ in the quotient Φ_{q+1}/Γ, and we denote by $\langle\overline{\sigma}\rangle$ the image of $\langle\sigma\rangle$ in $C_q(\Phi/\Gamma)$, which depends only on the class $\overline{\sigma} \in \Phi_{q+1}/\Gamma$. Now, for each such $\overline{\sigma} \in \Phi_{q+1}/\Gamma$, we associate the set $(\sigma\Gamma \cap V^1)/\Gamma$, which gives a q-dimensional simplex of the $(n-1)$-dimensional torus $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. Doing so for every $q \geq 0$, or equivalently for each $\overline{\sigma} \in \Phi/\Gamma$, we obtain a simplicial decomposition of $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$ and $C_\bullet(\Phi/\Gamma)$ may be identified with the associated chain complex. Hence we have

$$H_m(C_\bullet(\Phi/\Gamma)) = H_m(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z}), \qquad H^m\Big(\mathrm{Hom}_\mathbb{Z}\big(C_\bullet(\Phi/\Gamma), \mathbb{Z}\big)\Big) = H^m(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z}),$$

where $H_m(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z})$ (respectively $H^m(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z})$) denotes the integral m-th (co)homology group of the $(n-1)$-dimensional torus $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, and therefore is a free abelian group of rank $\binom{n-1}{m}$. Therefore, Kronecker duality implies that the dual pairing

$$H^m(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z}) \times H_m(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z}) \to \mathbb{Z} \tag{2.10}$$

is perfect (see for example [Mun84, Theorem 45.8]). Consequently, the bilinear pairing

$$\begin{aligned} \mathrm{Hom}_\mathbb{Z}\big(C_m(\Phi/\Gamma), \mathbb{Z}\big) \times C_m(\Phi/\Gamma) &\xrightarrow{\ \cong\ } \mathbb{Z} \\ (\varphi, u) &\longmapsto \varphi(u) \end{aligned} \tag{2.11}$$

is also perfect.

We also note that the generator of the homology group

$$H_{n-1}\big(C_\bullet(\Phi/\Gamma)\big) = H_{n-1}(\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$$

is given by the fundamental class

$$\sum_{\overline{\sigma}\in\Phi_g/\Gamma} \langle\overline{\sigma}\rangle \in C_{n-1}(\Phi/\Gamma). \tag{2.12}$$

Now, for a field $\mathbb{K}$, the Universal Coefficient Theorem (see [Mun84, Theorem 53.5]) together with the Kronecker pairing (2.11) gives us the following canonical isomorphism

$$\begin{aligned} H^{n-1}\Big(\mathrm{Hom}_{\mathbb{Z}}(C_\bullet(\Phi/\Gamma),\mathbb{K})\Big) &\xrightarrow{\quad\cong\quad} \mathbb{K} \\ \varphi &\longmapsto \textstyle\sum_{\overline{\sigma}\in\Phi_n/\Gamma}\varphi(\langle\overline{\sigma}\rangle), \end{aligned} \tag{2.13}$$

induced by the fundamental class (2.12).

Cocycles of generating functions and the Shintani class. We would like to consider cocycles given by functions defined through cones, in particular for the cone generating functions of definition 2.3.3. First, note that by [Yam10, Proposition 6.2], we have the following cocycle relation

$$\sum_{j=0}^{n}(-1)^j \operatorname{sgn}(\alpha_0,\dots,\check{\alpha}_j,\dots,\alpha_n)\mathbf{1}_{\breve{\sigma}(\alpha_0,\dots,\check{\alpha}_j,\dots\alpha_n)} \equiv 0 \tag{2.14}$$

as a function on V^1, where $\mathbf{1}_R$ is the characteristic function of $R \subset V_{>0}$ satisfying $\mathbf{1}_R(x) = 1$ if $x \in R$ and $\mathbf{1}_R(x) = 0$ if $x \notin R$.

Remark 2.4.3. In [CDG15], they consider spaces of cones $\mathcal{K}$ which correspond to the submodules of $\mathrm{Maps}(V,\mathbb{Z})$ generated by characteristic functions of cones. The cocycle relation 2.14 is given in §2.2 and can be defined in the dual complex $\mathrm{Hom}(C_\bullet(\Phi),\mathcal{K})$. Note that even though they consider open cones, the characteristic functions include the boundaries (except the ones excluded by the Q deformation process, with Q given as in 2.1), so they correspond to the ones above. By considering the Γ action on the generators of the cone, they define the n-1 Γ-cocycle ϕ_Γ (ϕ_U in their notation) and the $\mathrm{GL}_n(\mathbb{Q})$-cocycle Φ_{Sh}, which are related precisely by 2.3: $J(w)^*[\phi_U] = \rho_{\mathbf{w}}^*[\Phi_{Sh}]$ (see [CDG15] §3.1 and fix Q as above).

As in the introduction, the coefficients of the generating functions should give the negative integer zeta values. However, to obtain these coefficients as well as rationality results, it‘s important to consider suitable coefficient rings in which they are defined. This requires some care and will be done later in section 3.3, where we will define the field $\widetilde{R}$. For now one can think of $\widetilde{R}$ as the field of fractions of the ring of formal power series $\mathbb{R}((z)) := \mathrm{Frac}(\mathbb{R}[[z]])$. This field will have a natural action of Γ, and the coinvariants with respect to this action will be $\widetilde{R}_\Gamma$, which will be identified with a field of Laurent series in 1 single variable. We then have

Definition 2.4.4. For a fixed nonzero torsion point $\boldsymbol{t} \in V/L$ we define the *Shintani class*

$$[\mathcal{G}] \in H^{n-1}(\mathrm{Hom}(C_\bullet(\Phi/\Gamma, \widetilde{R}_\Gamma))$$

as the class defined by the assignment $\underline{\boldsymbol{\alpha}} \mapsto \mathrm{sgn}(\underline{\alpha})\mathcal{G}_{\sigma_{\underline{\alpha}}}(\boldsymbol{t}, z)$ in $\mathrm{Hom}(C_\bullet(\Phi/\Gamma), \widetilde{R}_\Gamma)$.

It is clear that the assignment induces an $n-1$ cocycle since 2.14 implies that

$$\sum_{j=1}^{g}(-1)^j \mathrm{sgn}(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g)\mathcal{G}_{\sigma_{(\alpha_0,\ldots,\check{\alpha}_j,\ldots,\alpha_g)}}(\boldsymbol{t}, \boldsymbol{z}) =$$

$$= \sum_{j=1}^{g}(-1)^j \mathrm{sgn}(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g) \sum_{\alpha \in \check{\sigma}_{(\alpha_0,\ldots,\check{\alpha}_j,\ldots,\alpha_g)} \cap \boldsymbol{t}+L} e^{\langle \alpha, z\rangle} =$$

$$= \sum_{\alpha \in \boldsymbol{t}+L} e^{\langle \alpha, z\rangle} \left(\sum_{j=0}^{g}(-1)^j \mathrm{sgn}(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g)\mathbf{1}_{\check{\sigma}_{(\alpha_0,\ldots,\check{\alpha}_j,\ldots\alpha_g)}}(\alpha) \right) = 0.$$

In section 4.2 we will use the pairing 2.11 above to send the Shintani class to a Laurent series whose coefficients will give the negative integer zeta values.

Chapter Three

Topological Polylogarithm

3.1 The geometric picture

We now introduce the theory of the topological polylogarithm developed in [BKL18] and [Gra16], which is more geometric in nature and is closely related to the notions in the previous chapter, as we will show later.

Torus Family. Let V, L and Γ be as in the setup 2.1. Let $\mathcal{S} := {}^{V_{>0}}\!/_{\Gamma}$. Further, write $\pi : V \to T := {}^{V}\!/_{L}$ for the quotient map. Following [Gra16] §2.1.1, since $\pi_1(\mathcal{S}) = \Gamma$, we have a correspondence between lattices with a Γ action and families of real tori over $\mathcal{S}$

$$\begin{array}{ccc} \{\text{Lattices with } \Gamma \text{ action}\} & \longrightarrow & \{{}^{\text{Tori}}\!/_{\mathcal{S}}\} \\ L & \longmapsto & {}^{V_{>0} \times T}\!/_{\Gamma} \end{array}$$

and if L acts on $V_{>0}$ trivially, we have

$$\begin{array}{ccc} {}^{V_{>0} \times T}\!/_{\Gamma} & = & {}^{V_{>0} \times V}\!/_{L \rtimes \Gamma} \\ \downarrow & & \downarrow \\ \mathcal{S} & = & \mathcal{S}. \end{array}$$

Logarithm sheaf. Let now A be a Noetherian commutative ring. Then we may consider the group ring $A[L]$, where $A[L] \cong A[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ after a choice of $\mathbb{Z}$-basis $(X_1, \ldots, X_n)$ for L, as well as its completion $R := R_A := \varprojlim_k {}^{A[L]}\!/_{I^k} \cong A[\![X_1 - 1, \ldots, X_n - 1]\!]$ with $I := \ker(\mathrm{aug})$ being the augmentation ideal. It is clear that these rings admit naturally an L-action and that the action $\Gamma \curvearrowright L$ extends to them. For a given choice of ring A we will generally denote $L_A := L \otimes A$.

Definition 3.1.1. On $V \times R$ we define the L action given by $l \cdot (v, r) = (v + l, l^{-1} r)$. The *logarithm sheaf* $\mathcal{L}\text{og}$ is defined as the local system on T obtained by considering the sheaf of sections of the quotient map $\tilde{\pi} : {}^{V \times R}\!/_{L} \to T$.

Note that $L = \pi_1(T)$, so its action on $V \times R$ defines a monodromy action and the corresponding locally constant sheaf is $\mathcal{L}\text{og}$.

By looking at the action of Γ on its espace étale, we conclude that $\mathcal{L}\text{og}$ is a Γ-equivariant sheaf, that is, the following diagram commutes (see [BKL18] §3.7):

$$\begin{array}{ccc} \left({}^{V \times R}\!/_{L}\right) \curvearrowleft \Gamma & \longrightarrow & {}^{V \times R}\!/_{L} \\ \Big\downarrow{\scriptstyle (\tilde{\pi},1)} & & \Big\downarrow{\scriptstyle \tilde{\pi}} \\ \left({}^{V}\!/_{L}\right) \curvearrowleft \Gamma & \longrightarrow & {}^{V}\!/_{L} \end{array}$$

Remark 3.1.2. $\mathcal{L}\text{og}$ does not admit global sections in general, but for torsion points of order invertible in A, denoted by $T^{(A)}$, it admits a canonical multiplicative trivialization (in the sense of [BKL18] §3.5).

Since $\mathcal{L}\text{og}$ is a local system on T by definition, locally isomorphic to R, we have the following correspondences (the first one in general, the second one if A is an $\mathbb{R}$-algebra)

$$\{\pi_1(T) - \text{reps}\} \longleftrightarrow \{\text{Local Systems}\} \longleftrightarrow \{\text{(pro) Vector Bundles with flat connection}\}$$

$$R \longmapsto \mathcal{L}\text{og} \longmapsto (\mathcal{L}\text{og}^\infty, \nabla)$$

where R is seen as an $A[L]$-mod and $\mathcal{F}^\infty := \mathcal{F} \otimes_{\underline{\mathbb{R}}} \mathscr{C}^\infty$ for any sheaf of $\underline{\mathbb{R}}$-algebras $\mathcal{F}$.

Remark 3.1.3. In the case $A = \mathbb{C}$ we also note that there exists a canonical continuous trivialization $\varrho_{cont} : \underline{R}^\infty \xrightarrow{\cong} \mathcal{L}\text{og}^\infty$ which agrees with the multiplicative trivialization at torsion points.

In order to relate these constructions with the theory of Shintani, we will later consider a field $\widetilde{R}$, which will correspond to $\widetilde{\mathcal{L}\text{og}}$ and $\widetilde{\mathcal{L}\text{og}}^{\infty}$ in the above equivalences.

Diagram of spaces. To complete our geometric picture, we consider also modifications of our base space and tori family to the norm 1 subspaces, as well as an auxiliary space X, together with smooth maps, which will all be specified later. This will be needed for explicit computations.

Let now $p_1 : V_{>0} \to V^1$ and analogously $p_1 : \mathcal{S} \to \mathcal{S}^1 := V^1/\Gamma$ smooth projections, noting that $\pi_1(\mathcal{S}^1) = \Gamma$ too. We consider also an auxiliary space X equipped with a Γ-action and trivial L-action, together with a smooth map $\psi : V^1 \to X$ (which in [BKL18] §4.1, $X = \mathscr{P} \subset \mathrm{Hom}(V, V^*)$ is the space of positive definite symmetric bilinear forms on V). So we get the following commutative diagram of spaces

$$\begin{array}{ccccc}
V_{>0} \times V/_{L \rtimes \Gamma} & \xrightarrow{p_1} & V^1 \times V/_{L \rtimes \Gamma} & \xrightarrow{\psi} & X \times V/_{L \rtimes \Gamma} \\
\| & & \| & & \| \\
V_{>0} \times T/_{\Gamma} & \xrightarrow{p_1} & V^1 \times T/_{\Gamma} & \xrightarrow{\psi} & X \times T/_{\Gamma} \\
\downarrow & & \downarrow \uparrow t & & \downarrow \uparrow t \\
\mathcal{S} & \xrightarrow{p_1} & \mathcal{S}^1 & \xrightarrow{\psi} & X/_{\Gamma}
\end{array} \tag{3.1}$$

where t is a torsion section, which corresponds to the choice of a point $t \in T^{(A)}$ stabilized by Γ.

The idea is that classes in the Γ-equivariant cohomology of the punctured torus correspond to polylogarithm functions, which specialize to the power series given by the generating functions of zeta values.

We want to compute the equivariant cohomology of $\mathcal{L}\text{og}$ in the relative case of the torus punctured at zero $\mathring{T} := T \setminus \{0\}$, analogously $\mathring{V} := V \setminus \pi^{-1}\{0\} = V \setminus L$. We may choose an isomorphism J as in 2.1 which defines an orientation on our manifolds. Equivalently, it trivializes the orientation sheaf $\lambda_L := H_n(L, \mathbb{Z})$ of [BKL18] (3.11). We choose our auxiliary space X such that we have the following isomorphism (in the middle):

$$H^{n-1}(\overset{\circ}{T},\Gamma,\mathcal{L}og) \xrightarrow[\varrho_{cont}]{\cong} H^{n-1}(\overset{\circ}{T},\Gamma,R) \xrightarrow{\cong} H^{n-1}\left(X\times\overset{\circ}{V}\big/_{L\rtimes\Gamma},R\right) \xrightarrow{\cong} H^{n-1}\left(X\times\overset{\circ}{T}\big/_{\Gamma},R\right).$$

Now we may pullback through ψ and $\boldsymbol{t}$, following the diagram of spaces 3.1, to get classes in the cohomology of $\mathcal{S}^1$:

$$H^{n-1}\left(X\times\overset{\circ}{T}\big/_{\Gamma},R\right) \xrightarrow{\psi^*} H^{n-1}\left(V^1\times\overset{\circ}{T}\big/_{\Gamma},R\right) \xrightarrow{t^*} H^{n-1}(\mathcal{S}^1,R)$$

and finally, since $\pi_1(\mathcal{S}^1)=\Gamma$, we have

$$H^{n-1}(\mathcal{S}^1,R) = H^{n-1}(\Gamma,R) \xrightarrow{ev} (R)_\Gamma \xrightarrow{\cong} A[\![w]\!]$$

where the evaluation map "ev" arises from cap product with the fundamental class induced by the choice of orientation (see 4.2.1).

Remark 3.1.4. These cohomological results can be obtained by considering families of tori as in [Gra16] §2, but through a suitable choice of X it is enough in our case, exactly as in [BKL18], to simply consider the torus T given as a fiber of the family.

The topological polylogarithm pol $\in H^{n-1}(\overset{\circ}{T},\Gamma,\mathcal{L}og)$ will be a special class, defined by a certain residue condition, which when evaluated through ev gives the negative zeta values in A (see Definition 3.3.11 below, where it is defined for a larger local system $\widetilde{\mathcal{L}og}$).

In the next chapter, we show that one can explicitly compute the equivariant cohomology of $\mathcal{L}og$ on the punctured torus by assuming that $A=\mathbb{C}$ and, using the Poincaré lemma for connections, the fact that classes in $H^{n-1}\left(X\times\overset{\circ}{T}\big/_{\Gamma},R_{\mathbb{C}}\right)$ are given by residue conditions of the corresponding connection:

$$H^{n-1}\left(X\times\overset{\circ}{T}\big/_{\Gamma},R_{\mathbb{C}}\right) = H^{n-1}\left(\left(\widehat{\Omega}^\bullet(X\times\overset{\circ}{T})\widehat{\otimes}R_{\mathbb{C}}\right)^\Gamma,\nabla\right).$$

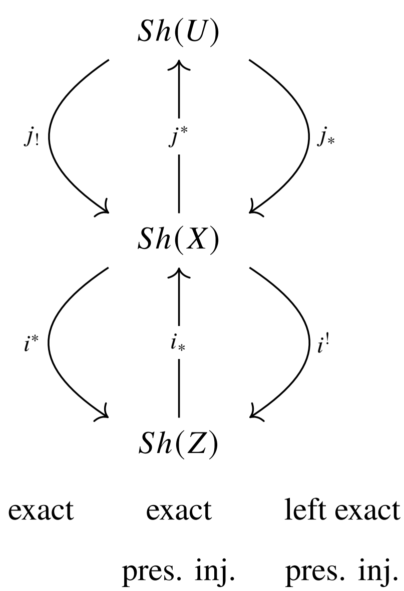
3.2 Cohomology, localization and currents

This section consists of a review of classical facts about cohomology of sheaves on manifolds, mainly following [Ive86] and [KS90]. The only result is a proof that the residue of currents computes the connecting homomorphism in the long exact sequence of cohomology with support, a well known fact for which a proof is hard to find in the literature.

Sheaves on manifolds. Consider the following complementary diagram of real smooth manifolds:

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

where the crossed arrow means that Z is closed and the arrow with a circle means that $U := X \setminus Z$ is open. Let $Sh(M)$ denote the category of sheaves of abelian groups on the manifold M. Then the inclusions above induce the following adjoint functors, which are left/right adjoint to the other on the same horizontal level, based on which side of the diagram they are written:



$$i_* = i_!$$
$$j^* = j^!$$
$$j^* j_* = 1$$
$$j^* j_! = 1$$
$$i^! i_* = 1$$
$$i^* i_* = 1$$

Distinguished triangles. From these we get the following distinguished triangles in the derived category $\mathcal{D}(Sh(X))$

$$j_! j^* \longrightarrow 1 \longrightarrow i_* i^* \longrightarrow [+1] \qquad (\triangle_1)$$

$$i_* Ri^! \longrightarrow 1 \longrightarrow Rj_* j^* \longrightarrow [+1] \qquad (\triangle_2)$$

These triangles are dual to each other in the sense of Verdier duality. For now, we note that they give rise to the following long exact cohomology sequences, for $\mathcal{F} \in Sh(X)$:

$$\ldots \to H^p_Z(X, \mathcal{F}) \to H^p(X, \mathcal{F}) \xrightarrow{j^*} H^p(U, \mathcal{F}|_U) \to \ldots \qquad (\triangle_2)$$

and after applying $Rf_!$ to ($\triangle_1$), for $f : X \to \{*\}$

$$\ldots \to H^p_c(U, \mathcal{F}|U) \xrightarrow{j_!} H^p_c(X, \mathcal{F}) \xrightarrow{i^*} H^p_c(Z, \mathcal{F}|_Z) \to \ldots \qquad (Rf_! \triangle_1)$$

where H_c denotes cohomology with compact support, and H_Z cohomology with support on Z.

From now on we assume that $dim(X) = n$.

Alexander duality. Let's first briefly analyze the situation when $\mathcal{F} = \underline{\mathbb{C}}$ and the manifold X is assumed to be oriented. In this case, the theory of Verdier duality summarized in the next paragraph gives rise to what is classically known as Alexander duality, which guarantees the existence of a cup product pairing

$$H^p_Z(X, \underline{\mathbb{C}}) \times H^q_c(Z, \underline{\mathbb{C}}) \longrightarrow H^{p+q}_c(X, \underline{\mathbb{C}})$$
$$(\alpha, \beta) \longmapsto \alpha \smile \beta$$

with properties as follows. The orientation induces a trace map $\int_X : H^n_c(X, \underline{\mathbb{C}}) \xrightarrow{\cong} \mathbb{C}$ which, together with the cup product, gives

$$H^p_Z(X, \underline{\mathbb{C}}) \xrightarrow{\cong} H^{n-p}_c(Z, \underline{\mathbb{C}})^\vee$$
$$\alpha \longmapsto (\beta \mapsto \int_X \alpha \smile \beta)$$

where the superscript $()^\vee$ denotes duality in the sense of complex vector spaces. Applying Alexander duality to the previous long exact sequences gives rise to the following relations:

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{j^*} & H^{p-1}(U,\underline{\mathbb{C}}) & \xrightarrow{\partial} & H^p_Z(X,\underline{\mathbb{C}}) & \xrightarrow{r} & H^{p-1}(X,\underline{\mathbb{C}}) & \xrightarrow{j^*} & H^p(U,\underline{\mathbb{C}}) & \xrightarrow{\partial} & \dots \\
 & & \Psi & & \Psi & & \Psi & & & & \\
 & & \alpha & & \beta & & \gamma & & & & \\
\dots & \xleftarrow{j_!} & H^{n-p+1}_c(U,\underline{\mathbb{C}}) & \xleftarrow{\partial} & H^{n-p}_c(Z,\underline{\mathbb{C}}) & \xleftarrow{i^*} & H^{n-p}_c(X,\underline{\mathbb{C}}) & \xleftarrow{j_!} & H^{n-p}_c(U,\underline{\mathbb{C}}) & \xleftarrow{\partial} & \dots \\
 & & & & \Psi & & \Psi & & \Psi & & \\
 & & & & \xi & & \eta & & \zeta & &
\end{array}$$

- $(-1)^p \partial\alpha \smile \xi = j_!(\alpha \smile \partial\xi)$
- $r\beta \smile \eta = \beta \smile i^*\eta$
- $j_!(j^*\gamma \smile \zeta) = \gamma \smile j_!\zeta$

Verdier duality. The hypotheses on the orientation and on the sheaf of coefficients can be relaxed, so considering any sheaf of modules of a Noetherian ring $\mathcal{F}$ we get

$$H^p_c(X,\mathcal{F}^\vee) \cong H^{n-p}_Z(X,\mathcal{F}^\vee \otimes or_X)$$

where or_X denotes the orientation sheaf, defined as the sheafification of the presheaf

$$or_X(W) := W \mapsto H^n_c(W,\underline{\mathbb{C}})^\vee.$$

The dualizing complex is the complex in the derived category $\mathscr{D}^\bullet_X \in \mathcal{D}(Sh(X))$ satisfying

$$\mathrm{Hom}(I^\bullet,\mathscr{D}^\bullet_X) \cong \mathrm{Hom}(\Gamma_c(X,I^\bullet),\underline{\mathbb{C}})$$

for every $I^\bullet$ injective or soft resolution. So $\mathscr{D}^\bullet_X \in [-n,0]$ is bounded (that is, the complex is zero outside of this interval), and we have that

$$W \mapsto H^p_c(W,\underline{\mathbb{C}})^\vee$$

is a sheaf and that $or_X[n] \cong \mathscr{D}^\bullet_X$.

Taking $I^\bullet = \mathscr{D}_X^\bullet$, we get

$$\begin{array}{ccc} \mathrm{Hom}(\mathscr{D}_X^\bullet, \mathscr{D}_X^\bullet) & \xrightarrow{\cong} & \mathrm{Hom}(\Gamma_c(X, \mathscr{D}_X^\bullet), \underline{\mathbb{C}}) \\ \mathrm{id} & \longmapsto & \int_X \end{array}$$

and $\Gamma_c(X, \mathscr{D}_X^\bullet) = H_c^n(X, or_X) \xrightarrow{\int_X} \mathbb{C}$. We also have $i^! \mathscr{D}_X^\bullet = \mathscr{D}_Z^\bullet$ and the unit of the adjunction $i_! i^! : \mathscr{D}_X^\bullet \to \mathscr{D}_X^\bullet$ gives

$$\int_Z : \Gamma_c(Z, i^! \mathscr{D}_X^\bullet) \xrightarrow{i_* = i_!} \Gamma_c(X, \mathscr{D}_X^\bullet) \xrightarrow{\int_X} \mathbb{C}$$

and analogously

$$\begin{array}{ccc} \int_U : H_c^n(U, j^* or_X) & \longrightarrow & \mathbb{C} \\ \alpha & \longmapsto & \int_X j_! \alpha \end{array}$$

All of the above are consequences of Verdier duality, and extends to general coefficients in a Noetherian commutative ring. Lastly, another consequence of Verdier duality is the purity isomorphism $Ri^! \underline{\mathbb{Z}}_X = \underline{\mathbb{Z}}_Z[-c]$, for $c := codim(Z)$. In other words, we have that $H_Z^p(X, \underline{\mathbb{Z}}) = H^{p-c}(Z, \underline{\mathbb{Z}})$.

Differential forms and currents. This paragraph follows mostly [dR84] Chapter 3 and [GH94] Chapter 3.

We can resolve the sheaf $\underline{\mathbb{C}}$ by smooth forms

$$0 \to \underline{\mathbb{C}} \to \mathscr{C}^\infty \to \Omega^1 \to \Omega^2 \to \dots$$

which is a complex of fine sheaves and therefore soft. Applying Verdier duality for this resolution presents the dualizing complex as the complex of functionals $\mathscr{D}_X^p(W) = \Gamma_c(W, \Omega^{-p})^\vee$.

After introducing natural topologies on these spaces, we are lead to the definition of the sheaves of currents $\widehat{\Omega}^p(W) \subset \Gamma_c(W, \Omega^{n-p})^\vee$. One notes that distributions $\mathscr{C}_c^\infty(W) \to \mathbb{C}$ are simply n-currents (and also 0-currents, see Remark 3.2.1 below).

The differentials on currents are given by

$$\begin{aligned} \widehat{\Omega}^p &\xrightarrow{\quad d \quad} \widehat{\Omega}^{p+1} \\ \mathcal{T} &\longmapsto (\omega \mapsto (-1)^{p+1}\mathcal{T}(d\omega)) \end{aligned}$$

and given a coordinate chart and $\mathcal{T} \in \widehat{\Omega}^p$, we have $\mathcal{T} = \sum_{|I|=p} \mathcal{T}_I \wedge dx_I$ for $\mathcal{T}_I \in \widehat{\Omega}^n$ distributions. This shows that the complex $\widehat{\Omega}^\bullet$ is fine. Furthermore, for an oriented manifold we have that the inclusion

$$\begin{aligned} \Omega^\bullet &\hookrightarrow \widehat{\Omega}^\bullet \\ \omega &\longmapsto \mathcal{T}_\omega := (\eta \mapsto \textstyle\int_X \omega \wedge \eta) \end{aligned}$$

is a quasi-isomorphism. The currents coming from smooth forms are called smooth, and they are dense among all currents.

Remark 3.2.1. If the manifold is not oriented, one must consider instead $\Omega^\bullet \otimes or_X$ so that the integral giving the trace map $\int_X$ is defined. In general one cannot integrate forms, but rather densities: any non-vanishing element of $\Omega^n \otimes or_X$ gives rise to a volume element, which is a (positive) density. If X is already oriented, then any non-vanishing form $\omega \in \Omega^n$ defines a density by simply taking $|\omega|$, and is called a volume form. In this way, distributions can be identified both with 0-currents as well as n-currents, by multiplication with a volume element.

Note also that the previous cup product of Alexander duality can be given in terms of currents and smooth forms, as follows

$$\begin{aligned} \widehat{\Omega}^k \times \Omega^l &\longrightarrow \widehat{\Omega}^{k+l} \\ (\mathcal{T}, \omega) &\longmapsto \mathcal{T} \wedge \omega := (\eta \mapsto \mathcal{T}(\omega \wedge \eta)) \end{aligned}$$

or anti-symmetrically $(\omega, \mathcal{T}) \mapsto \omega \wedge \mathcal{T} := (-1)^{kl}\mathcal{T} \wedge \omega$.

Residue of currents. We want to compute the boundary map in the long exact cohomology sequence arising from ($\triangle_2$), which is called the *topological residue*

$$H^{p-1}(U, \mathcal{F}|_U) \xrightarrow{res} H^p_Z(X, \mathcal{F}).$$

However, neither the resolution by differential forms $\Omega^\bullet$ nor by currents $\widehat{\Omega}^\bullet$ is flasque, so they cannot be used directly to compute the topological residue. Instead, we must first look at the cocone of j^* in the homotopy category, which gives rise to a shift of the distinguished triangle in the homotopy category giving rise to ($\triangle_2$):

$$i_* i^! \to 1 \to j_* j^* \xrightarrow{res} [+1] \tag{$\triangle_2$}$$

$$j_* j^*[-1] \xrightarrow{-res} C(j^*) \to 1 \to j_* j^* \tag{$\triangle_{C(j^*)}$}$$

which shows that $C(j^*)$, the cocone of the pullback by j, computes the cohomology with support in Z.

The triangle ($\triangle_{C(j^*)}$) applied to the complex $\Omega^\bullet$ gives a split short exact sequence of complexes, which at degrees $(p-1, p)$ is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{p-1}(U) & \longrightarrow & \Omega^{p-1}(U) \oplus \Omega^p(X) & \longrightarrow & \Omega^p(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \Omega^p(U) & \longrightarrow & \Omega^p(U) \oplus \Omega^{p+1}(X) & \longrightarrow & \Omega^{p+1}(X) & \longrightarrow & 0
\end{array}$$

where on the left we have $-d$ as per the convention on the shift $[-1]$ for distinguished triangles, and the differential d in the middle is given by $(\theta, \omega) \mapsto (j^*\omega - d\theta, d\omega)$. The cohomology arising from the complex $C(j^*\Omega^\bullet_X)$ is sometimes called *relative de Rham cohomology*.

We then have the following long exact sequence in cohomology

$$\begin{array}{ccccccc}
\ldots \longrightarrow & H^{p-1}(\Omega^\bullet_U) & \longrightarrow & H^p(C(j^*\Omega^\bullet_X)) & \longrightarrow & H^p(\Omega^\bullet_X) & \longrightarrow \ldots \\
 & [\theta] & \longmapsto & [\theta, 0] & & & \\
 & & & [\theta, \omega] & \longmapsto & [\omega] & \\
 & [j^*\omega - \theta] & \longmapsto & [j^*\omega - \theta, 0] \equiv [\theta, d\omega] & \longmapsto & 0 &
\end{array}$$

We want to explicitly compute these relative de Rham cohomology groups in terms of cohomology with support in Z. It is now that the use of currents becomes important, since smooth forms of U cannot in general be extended to smooth forms of X, but they do define currents of X.

Theorem 3.2.2. *The isomorphism of distinguished triangles gives*

$$H^p(C(j^*\Omega^\bullet_X)) \cong H^p_Z(\widehat{\Omega}^\bullet_X) \cong H^p(\widehat{\Omega}^\bullet_Z)$$

and the isomorphism is given explicitly by

$$\begin{aligned} \Phi : H^p(C(j^*\widehat{\Omega}^\bullet_X)) &\xrightarrow{\cong} H^p(\widehat{\Omega}^\bullet_Z) \\ [(\mathcal{T}_\theta, \mathcal{T}_\omega)] &\longmapsto \mathcal{T}_\omega - d\mathcal{T}_\theta \\ [(0, \mathcal{T}_\omega)] &\longleftarrow\!\shortmid \mathcal{T}_\omega \end{aligned}$$

Proof. First, note that these morphisms are inverse to each other, since $[(0, \mathcal{T}_\omega - d\mathcal{T}_\theta)] \equiv [(\mathcal{T}_\theta, \mathcal{T}_\omega)] - [(\mathcal{T}_\theta, d\mathcal{T}_\theta)] \equiv [(\mathcal{T}_\theta, \mathcal{T}_\omega)] - [d(0, \mathcal{T}_\theta)] \equiv [(\mathcal{T}_\theta, \mathcal{T}_\omega)]$. While we have written $\mathcal{T}_\theta, \mathcal{T}_\omega$, we did not need to assume that these currents are smooth. However note that $\Omega^p(U) \to \widehat{\Omega}^p(X)$, $\theta \mapsto \mathcal{T}_\theta$ makes sense, since $\mathcal{T}_\theta$ can be defined for any locally integrable form $\theta \in (L^1, loc, X)$ and any smooth form $\theta \in \Omega^p(U)$ can be extended by zero to a locally integrable form on X.

Note also that $j^*\omega = d\theta$ since $[(\theta, \omega)]$ is closed, so $j^*\omega - d\theta = 0$ on U. Using the quasi-isomorphism $\Omega^\bullet \hookrightarrow \widehat{\Omega}^\bullet$ and the fact that smooth currents are dense among all currents shows that $\mathcal{T}_\omega - d\mathcal{T}_\theta$ indeed has support in Z.

Lastly, if ω is exact, then $\omega = d\varphi$, so after applying Φ, we get $\mathcal{T}_{d\varphi} - d\mathcal{T}_\theta$ with $d\varphi|_U = d\theta$. This shows that the principle "res$(\mathcal{T}_\theta) = d\mathcal{T}_\theta - \mathcal{T}_{d\theta}$" holds for smooth currents, as in [GH94] page 368. Remember the minus sign for res in ($\triangle_{C(j^*)}$).

□

Remark 3.2.3. Note that if

$$Z \overset{i}{\hookrightarrow} X$$

is of codimension c, then $i^* : \Omega^k_c(X) \twoheadrightarrow \Omega^k_c(Z)$. Therefore we have $\widehat{\Omega}^k(Z) \xrightarrow{(i^*)^\vee} \widehat{\Omega}^{k+c}(X)$, which explicitly is

$$\begin{aligned} \mathrm{Hom}(\widehat{\Omega}^{n-c-k}_c(Z), \mathbb{C}) &\xhookrightarrow{(i^*)^\vee} \mathrm{Hom}(\widehat{\Omega}^{n-c-k}_c(Z), \mathbb{C}) \\ \mathcal{T}_Z &\longmapsto (\omega \mapsto \mathcal{T}_Z(i^*(\omega))) \end{aligned}$$

and in the case where $c = n$ (for example if $Z = \{p\}$ consists of a single point), we have that $k = 0$ and $\widehat{\Omega}^0(\{p\}) \xhookrightarrow{(i^*)^\vee} \widehat{\Omega}^n(X)$, which will be given by $z \mapsto z\delta_{\{p\}}$, where $z \in \mathbb{C} \cong \widehat{\Omega}^0(\{p\})$.

3.3 Field of coefficients

In order to compare the topological polylogarithm with the Shintani class, we would like to find a common ring of coefficients where both of them live. For this, it is very helpful to fix a choice of orientation, as is explained below.

Fixing an orientation. Following the notation for the logarithm sheaf in 3.1 for $A = \mathbb{C}$, we consider the group ring $\mathbb{C}[L]$ as well as its completion $R_{\mathbb{C}}$. There exists an isomorphism $\exp^* : R_{\mathbb{C}} \cong \widehat{\mathrm{Sym}} L_{\mathbb{C}}$ by [BKL18] Corollary 3.10, which together with a basis $\mathbf{w}$ and isomorphism J as in 2.1, gives rise to the following commutative diagram

$$\begin{array}{ccccccc}
\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] & \hookrightarrow & \mathbb{C}[\![X_1 - 1, \ldots, X_n - 1]\!] & \xrightarrow[\exp^*]{\cong} & \widehat{\mathrm{Sym}}\mathbb{Z}^n \otimes \mathbb{C} & = & \mathbb{C}[\![z]\!] \\
\cong \downarrow \mathbf{w} & & \cong \downarrow \mathbf{w} & & \cong \downarrow \mathbf{w} & & \cong \downarrow J(\mathbf{w}) \\
\mathbb{C}[L] & \hookrightarrow & R_{\mathbb{C}} & \xrightarrow[\exp^*]{\cong} & \widehat{\mathrm{Sym}} L_{\mathbb{C}} & \xrightarrow[J]{\cong} & \mathbb{C}[\![z]\!]
\end{array} \tag{3.2}$$

where $X_i^a \mapsto a\mathbf{w}_i$ for any $a \in \mathbb{Z}$, and $\exp^*(X_i) = e^{z_i}$. While the choice of basis $\mathbf{w}$ is not important if one wishes to work only with the lower part of the diagram, a choice of orientation J is needed to consider $R_{\mathbb{C}} \cong \mathbb{C}[\![z]\!]$ as in [BKL18] §5.2. As it is explained there, such a choice of orientation also trivializes the orientation sheaf $\lambda := \lambda(L) := \Lambda^n L = or_T$ and defines a fundamental class as in our definition 2.12.

Remark 3.3.1. The automorphism $J(\mathbf{w})$ is important when comparing the left vs right side of 2.1 as well as the upper vs lower side of the above diagram 3.2, as well as when comparing with [CDG15]. It appears there often written as a matrix M, which is ultimately chosen as $J(\mathbf{w})$ or its transpose, depending on from which side it is multiplying (see also 4.2.3).

Multivariate Laurent series. Since the generating function associated to a cone 2.3.3 has poles along each hyperplane orthogonal to each generator of its cone, we need to work with Laurent series instead of just power series. However, the notion of coefficient is not well defined for Laurent series of multiple variables since $\mathbb{Q}((X,Y)) \subsetneq \mathbb{Q}((X))((Y))$ and $\mathrm{Frac}(\mathbb{Q}((X,Y))) \neq \{\sum_{i\geq n}\sum_{j\geq m} a_{i,j}X^iY^j | a_{i,j} \in \mathbb{Q}, n,m \in \mathbb{Z}\}$. Indeed, just look at $(Y-X)^{-1} = \sum_{i\geq 0} Y^{-(i+1)}X^i$.

To avoid these pathologies, we use cones to construct suitable $\mathbb{Z}^n$-graded rings of coefficients.

Remark 3.3.2. Our ring of coefficients will also be a field, which simplifies certain cohomological proofs and is sufficient for obtaining rationality results on zeta values (but not sufficient if one is also interested in integrality results!).

We now follow [MK13]. Fix a choice of orientation J as in the previous paragraph and in 2.1. The idea is to define an order on our lattice and at first consider only power series with exponents belonging to the intersection of the lattice with a cone that is compatible with the order. It is then possible to invert these power series by enlarging the cone in a suitable way, using minimal elements with respect to the order. By taking the union over all possible cones and inverting all variables, we get a field.

Definition 3.3.3. We define $\leq_{lex}$ to be the *lexicographic order* on L defined by ordering each real embedding J_i at a time. That is, for $l, l' \in L$, $l \leq_{lex} l'$ if $l = l'$ or $J_{i_0}(l'-l) > 0$ for $i_0 := \min\{i | J_i(l'-l) \neq 0\}$.

This defines an additive total order as in [MK13] Definition 5. Let $\mathscr{C}_{lex}$ be the set of all cones compatible with this lexicographic order, that is, all cones σ such that $\alpha \in \sigma \cap L \Rightarrow 0 \leq_{lex} \alpha$. Note that for any Shintani cone decomposition $\Phi \subset \mathscr{C}_{lex}$, that is, every totally positive cone is compatible with the lexicographic order.

Definition 3.3.4. Let $\mathbf{w}$ be a $\mathbb{Z}$-basis of L as in 2.1. Then we define $\leq_{\mathbf{w}}$ to be the additive total order on $\mathbb{Z}^n \subset \mathbb{Q}^n \otimes \mathbb{R}$ such that $a \leq_{\mathbf{w}} b$ if $a = b$ or $a \cdot \mathbf{w} \leq_{lex} b \cdot \mathbf{w}$.

Note that this is well defined since $\mathbf{w}$ is a basis and $\leq_{lex}$ is also an additive total order on L. For $\sigma_{\underline{\alpha}} \in \mathscr{C}_{lex}$, let $\underline{y} \in \mathbb{Q}^n \otimes \mathbb{R}$ be the n-tuple of vectors which are sent through the isomorphism $J(\mathbf{w})$ of 2.1 to $\underline{\alpha}$, that is, $J(\mathbf{w})^\top \boldsymbol{y}_i = \alpha_i$. Then $C_{\underline{y}}$ forms a cone in $\mathbb{Q}^n \otimes \mathbb{R}$ compatible with $\leq_{\mathbf{w}}$. We denote the set of these cones by $\mathscr{C}_{\leq_{\mathbf{w}}}$ and note that $J(\mathbf{w})$ induces a bijection $\mathscr{C}_{\leq_{\mathbf{w}}} \cong \mathscr{C}_{lex}$.

Proposition 3.3.5. *Let $\mathbb{K}$ be a field of characteristic zero and for any $\sigma \in \mathscr{C}_{lex}$, respectively $C \in \mathscr{C}_{\leq_{\mathbf{w}}}$, let $\mathbb{K}_\sigma[\![\boldsymbol{x}]\!] = \{\sum_{\boldsymbol{e}\in L} a_{\boldsymbol{e}} \boldsymbol{x}^{\boldsymbol{e}} | \quad \boldsymbol{e} \in \sigma,\ a_{\boldsymbol{e}} \in \mathbb{K}\}$ and respectively $\mathbb{K}_C[\![\boldsymbol{x}]\!] = \{\sum_{\boldsymbol{e}\in\mathbb{Z}^n} a_{\boldsymbol{e}} \boldsymbol{x}^{\boldsymbol{e}} | \quad \boldsymbol{e} \in C,\ a_{\boldsymbol{e}} \in \mathbb{K}\}$. Then we have that $\mathbb{K}_{\leq_{lex}}[\![\boldsymbol{x}]\!] := \bigcup_{\sigma\in\mathscr{C}_{lex}} \mathbb{K}_\sigma[\![\boldsymbol{x}]\!]$ and $\mathbb{K}_{\leq_{\mathbf{w}}}[\![\boldsymbol{x}]\!] := \bigcup_{C\in\mathscr{C}_{\leq_{\mathbf{w}}}} \mathbb{K}_\sigma[\![\boldsymbol{x}]\!]$ are integral domains isomorphic through $J(\mathbf{w})$, and $\mathbb{K}_{\leq_{lex}}(\!(\boldsymbol{x})\!) := \bigcup_{\boldsymbol{e}\in L} \boldsymbol{x}^{\boldsymbol{e}} \mathbb{K}_{\leq_{lex}}[\![\boldsymbol{x}]\!]$ and $\mathbb{K}_{\leq_{\mathbf{w}}}(\!(\boldsymbol{x})\!) := \bigcup_{\boldsymbol{e}\in\mathbb{Z}^n} \boldsymbol{x}^{\boldsymbol{e}} \mathbb{K}_{\leq_{\mathbf{w}}}[\![\boldsymbol{x}]\!]$ are fields isomorphic through $J(\mathbf{w})$.*

Proof. This is shown in [MK13] Theorem 15. □

Remark 3.3.6. We note that in [Kat78] 1.1 a similar "artifice" is used, where a set S of linearly independent linear forms is considered, which can be taken as the α_i^* for each generator of a totally positive cone $\sigma_{\underline{\alpha}}$. Each such linear form divides the space into two components, one positive and one negative, with the boundary being the orthogonal hyperplane. Katz then considers the series with exponents in the enlarged cone defined by the intersection of all positive components defined for each generator of the cone given by S. This larger cone strictly contains the totally positive orthant. Katz also considers the ring of Laurent series associated to this ring by inverting the totally positive exponents.

Shintani operator and coefficients. We want to compare this field with the algebra considered in [CDG15] Definition 3.3.

Let's rewrite [CDG15]'s algebraic method of extracting coefficients from the generating Laurent series. Define $\mathbb{K}(\!(z)\!)^{hd} \subset \mathbb{K}(\!(z)\!) := \mathrm{Frac}(\mathbb{K}[\![z]\!])$ as the subfield of fractions $G = \frac{g}{p}$ where $g \in \mathbb{K}[\![z]\!]$ is a power series and p is a homogeneous polynomial of degree d whose monomial coefficients are all non-zero (i.e. $a_i \neq 0 \quad \forall \quad a_i z_i^d$). For example, for a cone with associated matrix A, $p_A(z) := N(Az) = \prod_{i=1}^n L_i(z) \in \mathrm{Sym}^n(\mathbb{K}^n)$ is complete homogeneous polynomial of degree n.

Define for every $i \leq n$

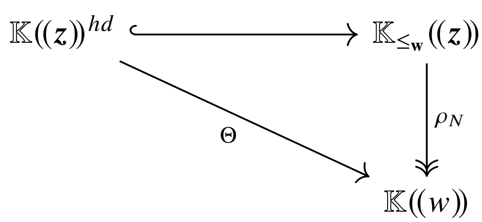
$$\theta_i(z) := (z_i z_1, \dots, z_i z_{i-1}, z_i, z_i z_{i+1}, \dots, z_i z_n)$$

and note that $p(\theta_i(z)) \in z_i^n \mathbb{K}[\![z_1, \dots, \check{z}_i, \dots, z_n]\!]^\times$. Therefore $\theta_i(G(z)) := G(\theta_i(z)) \in z_i^{-d}\mathbb{K}[\![\boldsymbol{z}]\!]$ for all $G \in \mathbb{K}(\!(\boldsymbol{z})\!)^{hd}$. Let $\rho_N : \mathbb{K}[\![\boldsymbol{z}]\!] \to \mathbb{K}[\![w]\!]$ be the map induced by $N(z) = w$, that is, only the terms of the form $(z_1 \cdots z_n)^k \mapsto w^k$ survive. Now, after setting $\theta := (\theta_1, \dots, \theta_n)$, we may define $\Theta : \mathbb{K}(\!(\boldsymbol{z})\!)^{hd} \to \mathbb{K}[\![w]\!]$ as the composition of the following maps:

$$\mathbb{K}(\!(\boldsymbol{z})\!)^{hd} \xrightarrow{\theta} \prod_{i=1}^n z_i^{-d}\mathbb{K}[\![\boldsymbol{z}]\!] \xrightarrow{(\rho_N)_{i\leq n}} \prod_{i=1}^n \mathbb{K}(\!(w)\!) \xrightarrow{\text{Avg}} \mathbb{K}(\!(w)\!).$$

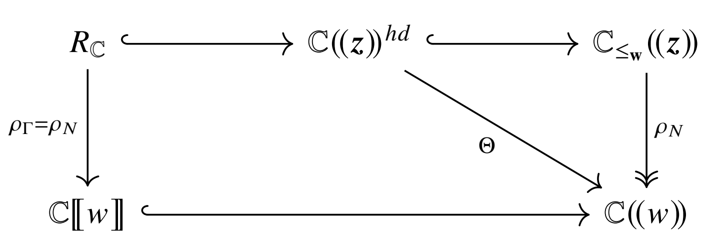
For Δ the Shintani operator of [CDG15] Definition 3.5, we have $\Theta = \sum_{k\in\mathbb{Z}} \frac{\Delta^{(k)}}{k!^n} w^k$.

Lemma 3.3.7. *Fix J and $\mathbf{w}$ as in 2.1. The following diagram is commutative*



Proof. Let $G(\boldsymbol{x}) = \sum_{\boldsymbol{e}\in\mathbb{Z}^n} a_{\boldsymbol{e}} \boldsymbol{x}^{\boldsymbol{e}}$. Then by abuse of notation we write $\theta_i : \mathbb{Z}^n \to \mathbb{Z}^n, \boldsymbol{e} \mapsto (e_1, \dots, Tr(\boldsymbol{e}), \dots, e_n)$ on the i-th position, so that $G(\theta_i(\boldsymbol{x})) = \sum_{\boldsymbol{e}\in\mathbb{Z}^n} a_{\boldsymbol{e}} \boldsymbol{x}^{\theta_i(\boldsymbol{e})} = \sum_{\boldsymbol{e}\in\mathbb{Z}^n} a_{e_1,\dots,e_i - \sum_{j\neq i} e_j, \dots, e_n} \boldsymbol{x}^{\boldsymbol{e}}$. Indeed θ_i is bijective for each i, that is, it is an automorphism of the lattice. The coefficient of $G(\theta_i(\boldsymbol{x}))$ selected by $\Delta^{(k)}/k!^n$ is the one corresponding to the exponent $\boldsymbol{e} = (k, \dots, nk, \dots, k)$, so $a_{(k,\dots,nk-\sum_{j\neq i} k, \dots, k)} = a_{(k,\dots,nk-(n-1)k),\dots,k)} = a_{(k,\dots,k,\dots,k)}$. So every θ_i selects the same coefficient and the average is just that coefficient. $\square$

Corollary 3.3.8. *For $\mathbb{K} = \mathbb{C}$ we have the commutativity of the following diagram*



(3.3)

Proof. This follows from the Lemma above, where $R_{\mathbb{C}} \cong \mathbb{C}[\![z]\!]$ via 3.2 (and applying the automorphism $J(\mathbf{w})^{-1}$ there), and $\rho_\Gamma : R_{\mathbb{C}} \to R_{\mathbb{C},\Gamma} = \mathbb{C}[\![w]\!]$ is the projection onto the Γ-coinvariants by Lemma 5.4 of [BKL18]. □

Topological polylogarithm. From now on we call $\widetilde{R_{\mathbb{K}}} := \mathbb{K}_{\leq_{lex}}((\boldsymbol{x}))$.

Proposition 3.3.9. *Let $A = \mathbb{K}$ be a field extension of $\mathbb{Q}$ as before. Define $\mathcal{L}\mathrm{og}_{\leq} := \underline{A_{\leq_{lex}}[\![\boldsymbol{x}]\!]} \otimes_{\underline{A[L]}} \pi_! \underline{A}$ and $\widetilde{\mathcal{L}\mathrm{og}} := \mathcal{L}\mathrm{og}_{\leq} \otimes_{\underline{A_{\leq_{lex}}[\![\boldsymbol{x}]\!]}} \underline{\widetilde{R}}$. Then we have that $H^k(T, \mathcal{L}\mathrm{og}_{\leq}) = 0$ if $k \neq n$ and $H^n(T, \mathcal{L}\mathrm{og}_{\leq}) = A$; and that $H^k(T, \widetilde{\mathcal{L}\mathrm{og}}) = 0$ for all k.*

Proof. Note that we have fixed an orientation on the torus through J, so λ_L is trivial. The proof is the same as in [BKL18] Theorem 3.25 (see also [BL94] 2.3.4) after noting that $A_{\leq_{lex}}[\![\boldsymbol{x}]\!]$ is a flat $A[L]$-module, since it is a flat A-module for A a field, and it admits a natural L action. This proves that $H^n(T, \mathcal{L}\mathrm{og}_{\leq}) = A$ and zero for other cohomological degrees. Furthermore, the field $\widetilde{R}$ is naturally a free module over $\mathbb{K}_{\leq_{lex}}[\![\boldsymbol{x}]\!]$, so we may take the tensor outside of the cohomology group, and the vanishing for all $k \neq n$ of $\mathcal{L}\mathrm{og}_{\leq}$ implies we may restrict our attention only to the case $k = n$, where the cohomology group is equal to A, but $A \otimes_{A_{\leq_{lex}}[\![\boldsymbol{x}]\!]} \widetilde{R} = 0$ since $\boldsymbol{x}$ acts trivially on the left but not on the right. □

We now apply ($\triangle_2$) to our logarithm sheaves:

Corollary 3.3.10. *Let $D \subset T$ be a finite and non-empty subset. Then for $i \neq n-1$*

$$H^i(T \setminus D, \widetilde{\mathcal{L}\mathrm{og}}) = H^i(T \setminus D, \mathcal{L}\mathrm{og}) = 0$$

which induces a short exact sequence

$$0 \to H^{n-1}(T \setminus D, \mathcal{L}\mathrm{og}) \xrightarrow{\mathrm{res}} \mathcal{L}\mathrm{og}\,|_D \xrightarrow{\sigma_D} A \to 0,$$

where $\mathcal{L}\mathrm{og}\,|_D = \bigoplus_{d \in D} \mathcal{L}\mathrm{og}_d$ (respectively for $\widetilde{\mathcal{L}\mathrm{og}}$) is the restriction of $\mathcal{L}\mathrm{og}$ to D and σ_D is the sum of the maps $aug : \mathcal{L}\mathrm{og}_d \to \mathcal{L}\mathrm{og}_d / I\mathcal{L}\mathrm{og}_d = A$.

In the case of $\widetilde{\mathcal{L}\mathrm{og}}$ the vanishing of the n-th cohomology reduces the short exact sequence above to an isomorphism given by the topological residue

$$H^{n-1}(T \setminus D, \widetilde{\mathcal{L}\mathrm{og}}) \xrightarrow[\cong]{\mathrm{res}} \widetilde{\mathcal{L}\mathrm{og}}\,|_D$$

Proof. This is Corollary 3.26 of [BKL18] together with Proposition 3.3.9 above. It follows easily from the previous section by taking $Z = D$ and the purity isomorphism. □

Definition 3.3.11. The *topological polylogarithm* pol $\in H^{n-1}(T \setminus \{0\}, \Gamma, \widetilde{\mathcal{L}\mathrm{og}})$ is the class whose residue is given by $\mathrm{res}(\mathrm{pol}) = \delta_0$.

Remark 3.3.12. Compare with Definition 3.30 of [BKL18], noting that with our choice of coefficients $\widetilde{R}$, the degree zero function α becomes simply an element of $\widetilde{R}$, which in our case is taken to be $\alpha = 1$. This is also equivalent to Definition 3.40 of [BKL18], which is the second variant of the topological polylogarithm considered there, where with our simplified choice of coefficients $\widetilde{R}$, $\varpi = 1$.

Remark 3.3.13. For the Γ-equivariant cohomology, we can simply consider Γ-invariant currents, as is shown in [Gra16] Proposition 2.4.21 and is done in [BKL18] §4.2.

Remark 3.3.14. In the 1-dimensional case, we get precisely the contents of [BL94] §2.3. In particular, for $\mathring{T} = \mathbb{S}^1 \setminus \{0\}$, to find the section explicitly corresponding to the polylogarithm class we need to find the section that gets sent to δ_0 through res, as in 3.2.3. When using the trivialization e^{yt}, for $y \in \mathbb{S}^1$ and $\widetilde{R} = \mathbb{R}((t))$, we have by 3.2.2 that "$\mathrm{res}(\mathcal{T}_\theta) = d\mathcal{T}_\theta - \mathcal{T}_{d\theta}$", so for f smooth with compact support in $\mathbb{S}^1$, we have that $f(1) = f(0)$ and then

$$\mathrm{res}(e^{yt}) = \int_0^1 e^{yt} f'(y)dy - \int_0^1 te^{yt} f(y)dy = e^{yt} f(y)\Big|_0^1 = (1 - e^t)f(0)$$

so $e^{yt} = (1 - e^t)\delta_0$ and $\mathrm{res}(\frac{e^{yt}}{1-e^t}) = \delta_0$. In other words, we had to find the function $g(y)$ on $\mathbb{R}$, smooth on $\mathbb{S}^1 \setminus \{0\}$, such that $g(0) - g(1) = 1$ (so clearly not smooth on $\mathbb{S}^1$), as claimed in [BKL18] §2. We therefore have that $\frac{e^{yt}}{1-e^t} \in \widetilde{R}$, precisely the generating function for the Hurwitz zeta function, represents the polylogarithm. In particular, if $y \in \mathbb{Q}$ we get series with coefficients in $\mathbb{Q}$, so the zeta values are rational.

Chapter Four

Explicit computations and comparison of classes

4.1 Green current and generating functions

We now define a suitable auxiliary space X which will help with explicit computations.

Definition 4.1.1. Let $\mathbb{S}^{n-1}$ be the $n-1$ dimensional unit sphere centered at zero inside V. Our auxiliary space X will be defined as the intersection $X := \mathbb{S}^{n-1} \cap V_{>0}$ of the sphere with the totally positive orthant.

The canonical projections $p_X : V_{>0} \to X$, $x \mapsto \frac{x}{||x||}$ and $p_1 : V_{>0} \to V^1$ induce the smooth isomorphism $\psi : V^1 \to X$, where $\psi = p_X|_{V^1}$ and $\psi^{-1} = p_1|_X$, through which X inherits the structure of a topological group.

Furthermore, note that the isomorphism ψ can be given as the function mapping each point x in V^1 to the unique point in X belonging to the line passing through x and 0 in V. This shows that a cone $\sigma_{\underline{\alpha}}$ defines a simplex on V^1 (as we have seen in 2.2) as well as on X, denoted by $X_\sigma = X_{\underline{\alpha}}$, and a Shintani decomposition Φ of $V_{>0}$ defines simplicial decompositions Φ^1 and Φ^X of V^1 and X, respectively. Note that while the Γ action is not stable on X, it is on Φ^X, that is, when it is taken together with p_X.

Sczech cocycle. Let $\underline{\alpha}$ and the corresponding matrix $A_{\underline{\alpha}} = J(\underline{\alpha})$ be as in the proof of Proposition 2.3.2. In [Scz93], Sczech considers the following function:

$$f(\underline{\alpha})(z) := \frac{det(A_{\underline{\alpha}})}{\prod_{j=1}^{n} \langle z, \alpha_j \rangle}$$

which is shown to satisfy the following cocycle condition in [Scz93] Lemma 1:

$$\sum_{i=0}^{n} (-1)^i f(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n) = 0.$$

We now state the following proposition due to Hurwitz:

Proposition 4.1.2. ***(Hurwitz)*** *Let $X_{\underline{\alpha}}$ be the simplex of X defined by the generators $\underline{\alpha}$ of a simplicial cone and $z \in V_{\mathbb{C}}$ such that $\langle z, \alpha_j \rangle \neq 0 \quad \forall j$. Consider also the $n-1$ differential form $\omega_y := \sum_{i=1}^{n} (-1)^{i-1} y_i dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n$. Then*

$$(n-1)! \int_{X_{\underline{\alpha}}} \frac{\omega_y}{\langle z, y \rangle^n} = f(\underline{\alpha})(z)$$

Proof. Hurwitz proves this in [Hur22] (see also [Scz93] §2.1) by considering the integral over the unique simplex of $\mathbb{P}^{n-1}(\mathbb{R})$ defined by $\underline{\alpha}$ where $\langle z, y \rangle$ does not vanish. When $\underline{\alpha}$ consists of totally positive generators, as in our case, this simplex corresponds to $X_{\underline{\alpha}}$ under the equivalence between $\mathbb{P}^{n-1}(\mathbb{R})$ and the $n-1$ sphere modulo the antipodal map. □

Corollary 4.1.3.

$$(n-1)! \int_{X} \frac{\omega_y}{\langle z, y \rangle^n} = f(e_1, \ldots, e_n)(z) = \frac{1}{N(z)}.$$

Remark 4.1.4. Note that this Corollary is proved in [BKL18] Theorem 5.6 for V^1 using 2.2 and pulling back through $\psi : V^1 \cong X$.

Remark 4.1.5. This corollary is also equivalent to [Col24] (0.3). There, in (0.2) Colmez decomposes the totally positive orthant into cones as in Definition 2.2.3 and takes the Laplace transform of the identification function, which gives (0.3) equating $\frac{1}{N(z)}$ and the sum of all the functions $f(\underline{\alpha})(z)$ for all cones. This sum is precisely the integral in Proposition 4.1.2 decomposed into the simplexes defined by each cone.

Explicit currents and distributions. Throughout this section, we fix $\widetilde{R} = \widetilde{R_{\mathbb{C}}}$.

Proposition 4.1.6. *Let* $\mathcal{G}_{\sigma_{\underline{\alpha}}}(\boldsymbol{t}, z) := \frac{\sum_{\alpha \in \check{P}_{\underline{\alpha}} \cap t+L} e^{\langle \alpha, z\rangle}}{\left(1-e^{\langle \alpha_1, z\rangle}\right)\cdots\left(1-e^{\langle \alpha_n, z\rangle}\right)}$ *be the generating function defined in 2.3.3. It defines a distribution almost everywhere on T with values on $\widetilde{R}$, which satisfies*

$$\operatorname{vol}(L)^{-1} \sum_{\mu \in L^*} e^{2\pi i \langle t, \mu\rangle} f(\underline{\alpha})(2\pi i \mu - z) = \operatorname{sgn}(\underline{\alpha}) \mathcal{G}_{\sigma_{\underline{\alpha}}}(z).$$

Proof. For the first claim, by definition the power series of the exponentials appearing in $\mathcal{G}_{\sigma_{\underline{\alpha}}}(z)$ have all coefficients and exponents in $\sigma_{\underline{\alpha}}$, and since any such cone is compatible with $\leq_{lex}$, we clearly have that $\mathcal{G}_{\sigma_{\underline{\alpha}}}(z) \in \mathbb{C}_{\leq_{lex}}((z)) = \widetilde{R_{\mathbb{C}}}$.

By definition,

$$\sum_{\mu \in L^*} e^{2\pi i \langle \boldsymbol{t}, \mu\rangle} f(\underline{\alpha})(z) = \sum_{\mu \in L^*} \frac{det(A_{\underline{\alpha}}) e^{2\pi i \langle \boldsymbol{t}, \mu\rangle}}{\prod_{j=1}^n \langle z, \alpha_j\rangle}$$

and the rest can be shown by what is essentially Poisson summation as in ([CDG15] Proposition 4.5), but we will instead follow ([BV00] Lemma 18) for a more straightforward proof.

We first lift the problem to V, the universal cover of T. Let $V_{\sigma_{\underline{\alpha}}} \subset V$ be defined as V minus the points given by all the L-translates of the walls of $\sigma_{\underline{\alpha}}$. Let $L_{\underline{\alpha}} \subset L$ be the sub-lattice generated over $\mathbb{Z}$ by the $\underline{\alpha}$. Then a set of representatives for the finitely many elements in $L/L_{\underline{\alpha}}$ can be given for each $\boldsymbol{t} \in V_{\sigma_{\underline{\alpha}}}$ by $\mathcal{R}(\boldsymbol{t}, \underline{\alpha}) := (\boldsymbol{t} - \check{P}_{\underline{\alpha}}) \cap L$. Define now

$$F_{\underline{\alpha}}(\boldsymbol{t}, z) := |L/L_{\underline{\alpha}}|^{-1} \sum_{x \in \mathcal{R}(\boldsymbol{t}, \underline{\alpha})} \frac{e^{\langle x, z\rangle}}{\prod_{i \leq n}(1 - e^{-\langle \alpha_i, z\rangle})}$$

and note that [BV00] Lemma 18 gives the equality

$$F_{\underline{\alpha}}(\boldsymbol{t}, -z) = \sum_{\mu \in L^*} e^{\langle t, 2\pi i \mu - z\rangle} f(\underline{\alpha})(2\pi i \mu - z)$$

as locally constant distributions of $V_{\sigma_{\underline{\alpha}}}$, since $|L/L_{\underline{\alpha}}| = \frac{|det(A_{\underline{\alpha}})|}{|\operatorname{vol}(L)|}$. Now to conclude simply note that we have a bijection $[(t - \check{P}_{\underline{\alpha}}) \cap L] \to [\check{P}_{\underline{\alpha}} \cap \boldsymbol{t} + L]$ by $x \mapsto \boldsymbol{t} - x$, and multiply both sides of the above equation by $e^{\langle \boldsymbol{t}, z\rangle}$ to get the claim on the torus. □

Remark 4.1.7. In the 1 dimensional case, $V_{\sigma_{\underline{\alpha}}}$ is just $\mathbb{R} \setminus \mathbb{Z} = \mathring{V}$ and we have

$$\sum_{\mu \in \mathbb{Z}} \frac{e^{t(2\pi i \mu - z)}}{2\pi i \mu - z} = \frac{e^{-[t]z}}{1 - e^z} \tag{4.1}$$

as in [BV00] Lemma 16. Multiplying by e^{tz} gives precisely $\frac{e^{\{t\}z}}{1-e^z}$, the distribution on $\mathring{T} = S^1 \setminus \{0\}$ representing the polylogarithm as in remark 3.3.14.

Remark 4.1.8. It is easy to see that for a fixed nontrivial torsion point $\boldsymbol{t}$, any Shintani decomposition Φ can be slightly deformed into a new one Φ' so that $\boldsymbol{t}$ is not in any of the walls of any of the cones of the decomposition. In other words, such that $\mathcal{G}_{\sigma_{\underline{\alpha}}}(\boldsymbol{t}, z) \in \widetilde{R}$ is well defined for all $\sigma_{\underline{\alpha}} \in \Phi'$. Compare with [BHYY23] Lemma 5.3.

We will now see how the above results fit into the topological polylogarithm picture. In Theorem 4.7 of [BKL18], a Green current $\widetilde{\mathscr{G}}$ is defined, which represents the polylogarithm class for the punctured torus. Note that by Remark 3.3.12 this currents satisfies the residue condition of Definition 3.3.11 with $\varpi = 1$. The specialization of this current to a torsion point of the torus is closely related to the objects above, as the next proposition shows.

Proposition 4.1.9. *Let $\widetilde{\mathscr{G}}$ be the Green current of* [BKL18] *Theorem 4.7 representing the polylogarithm. Then, for a torsion section $\boldsymbol{t}$ as in* (3.1)*, we have that*

$$\boldsymbol{t}^*\widetilde{\mathscr{G}} = (-1)^{n-1}(n-1)! \sum_{\mu \in L^* \setminus \{0\}} \frac{e^{2\pi i \langle \boldsymbol{t}, \mu \rangle}}{\langle 2\pi i \mu - z, y \rangle^n} \omega_y$$

through the isomorphism $\psi : V^1 \to X$, where $\omega_y := \sum_{i=1}^n (-1)^{i-1} y_i dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n$ as in Proposition 4.1.2.

Proof. $\widetilde{\mathscr{G}}$ is a Fourier expansion with coefficients E^a_μ in

$$\bigoplus_{a=0}^{n-1} \Omega^a(\mathscr{P}) \otimes \Omega^{n-1-a}(T) = \Omega^{n-1}(\mathscr{P} \times T)$$

where $\mathscr{P} \subset \mathrm{Hom}(V, V^*)$ is the space of positive definite symmetric bilinear forms on V, as in [BKL18] §4.1. Taking the pullback through $\boldsymbol{t}$ means only the coefficient $E^{n-1}_\mu \in \Omega^{n-1}(\mathscr{P}) \otimes \Omega^0(T)$ survives, and using the explicit computation of [BKL18] Lemma 4.12, as well as the inclusion $V^1 \hookrightarrow \mathscr{P}$ given by $q \mapsto B_q(v, w) = \sum_{j=1}^n q^{-1} v_j w_j$, we have the result. □

4.2 Evaluation and comparison of polylogarithm and Shintani classes

We are finally ready to combine all the previous results to compare the Shintani class with the topological polylogarithm and obtain the special values of partial zeta functions.

Let's briefly recall our Definition 2.4.4 of the Shintani class. For Φ^* a Shintani decomposition, with $* \in \{\emptyset, 1, X\}$, and $\boldsymbol{t} \in \overset{\circ}{T}$ a fixed torsion section as in (3.1), the assignment $\underline{\alpha} \mapsto \operatorname{sgn}(\underline{\alpha})\mathcal{G}_{\sigma_{\underline{\alpha}}}(\boldsymbol{t}, z)$ defines an element in $\operatorname{Hom}(C_\bullet(\Phi^*), \widetilde{R})$, and through ρ_Γ as in (3.3), an element in $\operatorname{Hom}(C_\bullet(\Gamma\backslash\Phi^*), \widetilde{R}_\Gamma)$. The cocycle relation (2.14) implies we get a class

$$[\mathcal{G}] \in H^{n-1}(\operatorname{Hom}(C_\bullet(\Gamma\backslash\Phi^*), \widetilde{R}_\Gamma))$$

which we called the Shintani class. Note that through the quasi-isomorphism from Lemma 2.4.2, this class does not depend on the cone decomposition Φ^* and the fundamental class of (2.12) is sent to $[\alpha] \in \lambda$ of [BKL18] §5.2.

As in [BKL18], our choice of orientation gives an evaluation map ev: $H^{n-1}(\mathcal{S}^1, \widetilde{R}) \to \widetilde{R}_\Gamma$, given by $\eta \mapsto \rho_\Gamma(\operatorname{vol}(L)^{-1} \int_{\mathcal{S}^1} \eta)$ for η a differential form. This is precisely the same cap product as in (2.11) via the identification between Betti and de Rham cohomology.

Finally, note that in Proposition 4.1.9 we sum over the non-zero lattice points, whereas in Proposition 4.1.6 we also add the $\mu = 0$ term, which is $(-1)^n f(\underline{\alpha})(z)$. This corresponds precisely to $-\Psi_P$ the polar cocycle of [CDG15] Theorem 4.1, which appears when comparing their cocycle with the one of Sczech in [Scz93] (see also Remark 4.2.3 below).

Combining everything we get our main theorem:

Theorem 4.2.1. *Let $\boldsymbol{t} \in \overset{\circ}{T}$ be a torsion point stabilized by Γ as above. The Eisenstein class* $\operatorname{Eis}(t) := t^* \operatorname{pol} \in H^{n-1}(\Gamma, \widetilde{R})$ *of* [BKL18] *Definition 3.32 satisfies*

$$\operatorname{Eis}(t) = t^* \operatorname{pol} = (-1)^{n-1}[\mathcal{G}] + [\Psi_P] \tag{4.2}$$

and

$$\mathrm{ev}(\mathrm{Eis}(t)) = \mathrm{ev}(t^*\widetilde{\mathscr{G}}) = \mathrm{ev}((-1)^{n-1}([\mathcal{G}]) + [\Psi_P]) = \sum_{k\geq 0} \zeta(\boldsymbol{t}, L, -k)\frac{w^k}{(k!)^n} \in \mathbb{Q}[\![w]\!]. \tag{4.3}$$

Proof. First note that the evaluation map coincides with our pairing (2.11) induced by the cap product. For $\widetilde{R}$ a field, this pairing is perfect, so the first equation (4.2) follows from the second (4.3), which is proved in [BKL18] Theorem 5.6.

However, we would like to give a more direct proof, which shows that the Green current representing the polylogarithm class for the punctured torus can be evaluated at each cone of a Shintani decomposition Φ^*, giving the corresponding generating class. Such a proof will not depend on [BKL18] Theorem 5.6 and in particular on the functional equation for the associated zeta functions, but only on the propositions 4.1.2 and 4.1.6, whereas the latter can be interpreted to give the functional equation for integral values of the zeta functions.

First, note that the isomorphism ψ gives $H^{n-1}(X/\Gamma, \widetilde{R}) \cong H^{n-1}(\mathcal{S}^1, \widetilde{R})$ and these groups are equivalent to $H^{n-1}(\Gamma, \widetilde{R})$ as its classifying space. They are also equivalent to $H^{n-1}(\mathrm{Hom}(C_\bullet(\Gamma\backslash\Phi^*), \widetilde{R}_\Gamma))$ via the corresponding Shintani simplicial decompositions for $* = 1, X$. An $\underline{\alpha} \in C_{n-1}(\Phi)$ defines a simplex $X_{\underline{\alpha}}$ of X, and by integrating the form $t^*\widetilde{\mathscr{G}}$ of Proposition 4.1.9 over this simplex, we get by Proposition 4.1.2 that

$$\int_{X_{\underline{\alpha}}} t^*\widetilde{\mathscr{G}} = (-1)^{n-1} \sum_{\mu\in L^*\backslash\{0\}} e^{2\pi i\langle t,\mu\rangle} f(\underline{\alpha})(2\pi i\mu - z)$$

and further, using proposition 4.1.6, we have that

$$\mathrm{vol}(L)^{-1}\int_{X_{\underline{\alpha}}} t^*\widetilde{\mathscr{G}} = (-1)^{n-1}\mathrm{sgn}(\underline{\alpha})\mathcal{G}_{\sigma_{\underline{\alpha}}}(z).$$

Now since $\Gamma\backslash\Phi^X$ is a finite set, we may sum over the representatives of this quotient and apply ρ_Γ, which on the left hand side gives precisely the evaluation of the Eisenstein class at $\boldsymbol{t}$, $\mathrm{ev}(\mathrm{Eis}(t)) = \rho_\Gamma(\mathrm{vol}^{-1}(L)\int_{\mathcal{S}^1} t^*\widetilde{\mathscr{G}})$, and on the right hand side gives $(-1)^{n-1}$ times the sum of the cone zeta functions over these representatives, which by Proposition 2.3.5 and (3.3) is, up to the factor $\frac{(-1)^{n-1}}{w}$ coming from $\rho_\Gamma(\Psi_P)$, equal to

$$(-1)^{n-1}\sum_{k\geq 0}\sum_{\sigma\in\Gamma\backslash\Phi_n} \zeta_\sigma(\boldsymbol{t}, (-k,\dots,-k)\frac{w^k}{(k!)^n} = (-1)^{n-1}\sum_{k\geq 0}\zeta(\boldsymbol{t}, L, -k)\frac{w^k}{(k!)^n} \in \mathbb{Q}(\!(w)\!).$$

The rationality of the zeta values follows from the fact that the classes can be defined for $\widetilde{R_{\mathbb{Q}}}$, and therefore must have coefficients in $\mathbb{Q}$ under the evaluation map, as in [BKL18] (5.9). Alternatively, one can see that the Shintani class can be defined with $\mathbb{Q}$ coefficients. □

Corollary 4.2.2. *(Klingen-Siegel) The values of partial zeta functions of totally real number fields at negative integers are rational. That is*

$$\zeta(\mathfrak{b}, \mathfrak{f}, -k) \in \mathbb{Q}$$

for $\mathfrak{f} \neq \mathcal{O}_F$ and $\mathfrak{b}$ integral ideals and $k \geq 0$. If $\mathfrak{f} = \mathcal{O}_F$, $\zeta(\mathfrak{b}, \mathfrak{f}, -k) \in \mathbb{Q}$ for $k \geq 1$.

Proof. If $\mathfrak{f} \neq \mathcal{O}_F$, 1 is a non-zero torsion point for the corresponding lattice $\mathfrak{f}\mathfrak{b}^{-1}$, and the Corollary follows directly from Theorem 4.2.1 above and (1.6). If $\mathfrak{f} = \mathcal{O}_F$, use the result for $\mathfrak{f} = \mathfrak{p}$ and deduce by removing the corresponding Euler factor of $\zeta(\mathfrak{b}, \mathfrak{f}, -k) \in \mathbb{Q}$ as in [BKL18] (5.11). □

Comparison with Shintani cocycle of [CDG15].

Remark 4.2.3. In [CDG15] they work on the left hand side of (2.1), where we fix Q and v as in (2.1), as they do in §3.6, as well as $J(\mathbf{w})$, which corresponds to the matrix M in their notation (they normalize it by the norm of $\mathfrak{b}$, which they call $\mathfrak{a}$, to obtain partial zeta values directly through (1.6)). As remarked in Remark 2.4.3, they consider cocycles with respect to both groups $\rho_{\mathbf{w}} : \Gamma \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$, related by pulling back through this inclusion and $J(\mathbf{w})$. They then apply their Solomon-Hu pairing to this cocycle, which by Remark 2.3.4 corresponds to our Shintani class above. In their paper, this class is denoted by $[\Psi_{Sh}]$ and has values in a certain space of functions $\mathcal{F}$ which is described below. These functions have values in $\mathbb{K}((z))^{hd} \subset \widetilde{R_{\mathbb{K}}}$, and when pairing with the same fundamental class as above and applying the Shintani operator Δ_{Sh}, they obtain the same evaluation as above, by section 3.3. Their Theorem 4.1 is essentially equivalent to our main Theorem 4.2.1 above.

The space $\mathcal{F}$ in [CDG15] §3.5 corresponds to the sheaf of sections of $V \times \mathbb{R}((z))^{hd}$ equivariant with respect to the $L \rtimes \Gamma$ action, however equipped with a $\mathrm{GL}_n(\mathbb{Q})$-action, which will correspond to the trace compatibility of the polylogarithm as in [BKL18] §3.11. This comes from the fact that the distribution property satisfied by the elements of $\mathcal{F}$ is precisely the distribution property of Eisenstein series giving rise to the trace compatibility in [BKL18].

To see this, note that for Q, M, v fixed as above, the distribution property for $f \in \mathcal{F}$ becomes $\forall \lambda \in \mathbb{Z} \setminus \{0\}$, $\lambda w - v \in \mathbb{Z}^n$, whence $w = \lambda^{-1} l + \lambda^{-1} v$ for $l \in \mathbb{Z}^n$,

$$f(M, v)(z) = \operatorname{sgn}(\lambda)^n \sum_{l \in \mathbb{Z}^n} f(\lambda^{-1} l + \lambda^{-1} v)(z \lambda M^\top).$$

This guarantees that if $A \in \mathrm{GL}_n(\mathbb{Q}) \cap \mathrm{M}_n(\mathbb{Z})$, then A is an isogeny of L through (2.1) in the sense of [BKL18] Definition 3.4 (injective with finite cokernel). Then

$$A \cdot f(M, v)(z) = \sum_{r \in L/_{AL}} \operatorname{sgn}(\det A) f(A^\top M, A^{-1} r + A^{-1} v)(z)$$

which is the trace compatibility of [BKL18]. If A comes from $\gamma \in \Gamma \to \mathrm{GL}(L)$ as in (2.4), then the cokernel of this isogeny is trivial as $\Gamma \to \mathrm{SL}_n(\mathbb{Q})$, so the action is precisely the action of $L \rtimes \Gamma$ as in [BKL18] §3.7.

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