



Semi-Riemannian spin^c manifolds carrying generalized Killing spinors and the classification of Riemannian spin^c manifolds admitting a type I imaginary generalized Killing spinor

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Abstract

We classify Riemannian spin^c manifolds carrying a type I imaginary generalized Killing spinor, by explicitly constructing a parallel spinor on each leaf of the canonical foliation given by the Dirac current. We also provide a class of Riemannian spin^c manifolds carrying a type II imaginary generalized Killing spinor, by considering spacelike hypersurfaces of Lorentzian spin^c manifolds. We carry out much of the work in the setting of semi-Riemannian spin^c -manifolds carrying generalized Killing spinors, allowing us to draw conclusions in this setting as well. In this context, the Dirac current is not always a closed vector field. We circumvent this in even dimensions, by considering a modified Dirac current, which is closed in the cases when the original Dirac current is not. On the path to these results, we also study semi-Riemannian manifolds carrying closed and conformal vector fields.

Keywords Spin^c geometry · Generalized Killing spinors · Dirac current · Closed & conformal vector fields

Contents

1	Introduction
2	Preliminaries and notation
3	Generalized Killing spinors
4	The geometry of semi-Riemannian manifolds carrying closed and conformal vector fields
4.1	Non-local splittings without completeness assumptions
4.2	Global splittings under completeness assumptions
5	Returning to generalized Killing spinors
5.1	The semi-Riemannian spin^c setting
5.2	The Riemannian classification
5.2.1	Type I imaginary generalized Killing spinors
5.2.2	Type II imaginary generalized Killing spinors
	References

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1 Introduction

We say that a non-trivial complex spinor field φ on a semi-Riemannian spin^c manifold (M, g) is a *generalized Killing spinor* if there is some g -symmetric endomorphism field A of TM such that

$$\nabla_X^S \varphi = \ell A(X) \cdot \varphi \quad \forall X \in \Gamma(TM), \quad (1)$$

where $\ell \in \{1, i\}$. We call A the *Killing endomorphism* and if $\ell = 1$ ($\ell = i$) we say that φ is a *real (imaginary) generalized Killing spinor*. This is not the original definition from [37] of a generalized Killing spinor, but it is the one that is most frequent. The original definition only allowed for the Killing endomorphism A to be of the form $A = \mu \text{Id}$, where μ is some real valued function on M . For this reason, we introduce the term "special generalized Killing spinor" to refer to this case. When the Killing endomorphism A is a constant multiple of the identity, we simply call the spinor field a *Killing spinor*.

Killing spinors were first considered in physics. They were originally introduced in general relativity and later appeared in supergravity theories. See the introduction of [9] for a full description of their origin. Killing spinors are also of purely geometric interest. The existence of a Killing spinor on a semi-Riemannian spin manifold implies that the manifold has constant scalar curvature, and in the Riemannian case, the manifold has to be Einstein [9]. Real Killing spinors on compact Riemannian spin manifolds appear as the limiting case of Friedrich's eigenvalue estimate for the Dirac operator [15]. Killing spinors are also related to supergeometry [1]. Generalized Killing spinors appear on hypersurfaces of semi-Riemannian spin and spin^c manifolds carrying parallel spinors, which is a motivation for their study [3, 6, 12, 13].

Riemannian spin manifolds with parallel spinors are Ricci-flat and have special holonomy group [16, 27, 39]. Baum classifies complete Riemannian spin manifolds carrying imaginary Killing spinors in [8] and [7]. Baum splits the classification into two cases, called type I and type II imaginary Killing spinors. For the type I case, Baum finds a suitable foliation of the manifold consisting of hypersurfaces carrying parallel spinors. From such a hypersurface, it is shown that a flow in the normal direction gives rise to a warped product, isometric with the original manifold. For the type II case, Baum shows that the manifold must be isometric to hyperbolic space. The classification of complete Riemannian spin manifolds with real Killing spinors goes back to the group of Friedrich, and their collaborators [18–21, 24]. The full classification of complete, simply connected Riemannian spin manifolds carrying real Killing spinors is due to Bär [4]. This is done by showing that the cone over the given manifold carries a parallel spinor.

The above results have been extended by many in various directions. Among these, the following classifications are directly related to this article. Moroianu [35] classifies complete, simply connected Riemannian spin^c manifolds carrying parallel spinors. Moroianu shows that the manifold splits as a Riemannian product between a Kähler manifold (equipped with its canonical spin^c -structure) and a spin manifold carrying a parallel spinor [35, Theorem 1.1]. Shortly after Baum, Rademacher extends Baum's results to what we call special imaginary generalized Killing spinors [37]. Große and Nakad have extended Rademacher's classification to the spin^c setting [23, Theorem 4.1]. An extension of Rademacher's result to the setting of Riemannian spin manifolds carrying a type I imaginary generalized Killing spinor have been made by Leistner and Lischewski [33, Theorem 4].

2 Preliminaries and notation

This section is a summary of some preliminaries from spin and spin^c geometry. The reader is directed to [14, 17, 31] for details.

For $n > 1$, let $\theta : \text{Spin}_{\text{GL}}(n) \rightarrow \text{GL}_n^+$, be the connected double covering group of GL_n^+ , where GL_n^+ is the connected component of the identity in $\text{GL}_n(\mathbb{R})$. If $n > 2$, this corresponds to the universal covering group. Let M be an oriented smooth manifold of dimension n and let P_{GL^+} be the bundle of oriented frames on M . A *topological spin structure* on M is a $\text{Spin}_{\text{GL}}(n)$ -principal bundle $P_{\text{Spin}_{\text{GL}}} \rightarrow M$, such that the associated GL_n^+ -principal bundle $P_{\text{Spin}_{\text{GL}}} \times_{\theta} \text{GL}_n^+$ coincides with P_{GL^+} , meaning there is a bundle isomorphism $I : P_{\text{Spin}_{\text{GL}}} \times_{\theta} \text{GL}_n^+ \rightarrow P_{\text{GL}^+}$, covering the identity. We say that two topological spin structures $P_{\text{Spin}_{\text{GL}}}$ and $P'_{\text{Spin}_{\text{GL}}}$ are isomorphic if there is an isomorphism $P_{\text{Spin}_{\text{GL}}} \rightarrow P'_{\text{Spin}_{\text{GL}}}$ of $\text{Spin}_{\text{GL}}(n)$ -principal bundles, compatible with the corresponding isomorphisms $I : P_{\text{Spin}_{\text{GL}}} \times_{\theta} \text{GL}_n^+ \rightarrow P_{\text{GL}^+}$ and $I' : P'_{\text{Spin}_{\text{GL}}} \times_{\theta} \text{GL}_n^+ \rightarrow P_{\text{GL}^+}$. Further, we say that $\text{Spin}_{\text{GL}}^c(n) := \text{Spin}_{\text{GL}}(n) \times_{\mathbb{Z}/2} S^1$ is the *topological spin^c group* and we note that there are two natural maps, $\lambda : \text{Spin}_{\text{GL}}^c(n) \rightarrow \text{GL}_n^+$ and $\rho : \text{Spin}_{\text{GL}}^c(n) \rightarrow S^1$, defined via $\lambda([A, z]) := \theta(A)$ and $\rho([A, z]) = z^2$. We define a *topological spin^c structure* on M to be a $\text{Spin}_{\text{GL}}^c(n)$ -principal bundle $P_{\text{Spin}_{\text{GL}}^c} \rightarrow M$ such that the associated GL_n^+ -principal bundle $P_{\text{Spin}_{\text{GL}}^c} \times_{\lambda} \text{GL}_n^+$ coincides with P_{GL^+} . The isomorphism classes of such structures are defined in analogy with the aforementioned spin case. To any topological spin^c structure, we have a naturally associated S^1 -principal bundle $P_{S^1} := P_{\text{Spin}_{\text{GL}}^c} \times_{\rho} S^1$. A topological spin structure $P_{\text{Spin}_{\text{GL}}}$ gives rise to a double covering $P_{\text{Spin}_{\text{GL}}} \rightarrow P_{\text{GL}^+}$ of principal bundles. Similarly, a topological spin^c structure $P_{\text{Spin}_{\text{GL}}^c}$ gives rise to a double covering $\Lambda : P_{\text{Spin}_{\text{GL}}^c} \rightarrow P_{\text{GL}^+} \oplus P_{S^1}$ of principal bundles. Note that a topological spin structure $P_{\text{Spin}_{\text{GL}}}$ induces a topological spin^c structure via $P_{\text{Spin}_{\text{GL}}^c} := P_{\text{Spin}_{\text{GL}}} \times_{\mathbb{Z}/2\mathbb{Z}} P_{S^1}$, where P_{S^1} denotes the trivialized S^1 -principal bundle on M .

Remark 1 In terms of classifying spaces, a topological spin^c structure on a smooth manifold M of dimension n is a lift $M \rightarrow \text{BSpin}_{\text{GL}}^c(n)$ of the classifying map $M \rightarrow \text{BGL}_n^+$ for P_{GL^+} along the natural map $\text{BSpin}_{\text{GL}}^c(n) \rightarrow \text{BGL}_n^+$ induced by the map $\lambda : \text{Spin}_{\text{GL}}^c(n) \rightarrow \text{GL}_n^+$ from above. An isomorphism class of topological spin^c structures then corresponds to a homotopy class of such lifts. The corresponding remark for topological spin structures is analogous.

Next, let g be a semi-Riemannian metric on M of index r such that (M, g) is space and time oriented. We allow $r = 0$, corresponding to the Riemannian case, but we exclude the case $r = n$. This yields a reduction $\iota : P_{\text{SO}^0} \hookrightarrow P_{\text{GL}^+}$ of principal bundles along the canonical inclusion $\text{SO}^0(r, n - r) \hookrightarrow \text{GL}^+$, where $\text{SO}^0(r, n - r)$ denotes the identity component of $\text{SO}(r, n - r)$. Given a topological spin^c structure on M , we define the corresponding (geometric) *spin^c structure* as the pullback P_{Spin^c} along Λ and $\iota \oplus \text{Id} : P_{\text{SO}^0} \oplus P_{S^1} \rightarrow P_{\text{GL}^+} \oplus P_{S^1}$. In other words, the principal bundle P_{Spin^c} fits into the following pullback diagram

$$\begin{array}{ccc}
 P_{\text{Spin}^c} & \longrightarrow & P_{\text{Spin}_{\text{GL}}^c} \\
 \downarrow & \lrcorner & \downarrow \Lambda \\
 P_{\text{SO}^0} \oplus P_{S^1} & \xrightarrow{\iota \oplus \text{Id}} & P_{\text{GL}^+} \oplus P_{S^1}.
 \end{array}$$

The definition of a spin structure is analogous. A space and time oriented semi-Riemannian manifold (M, g) , equipped with a spin^c (spin) structure is said to be a *time oriented semi-Riemannian spin^c (spin) manifold*.

Let (M, g) be a time oriented semi-Riemannian spin^c manifold of dimension n and index r . The bundle P_{spin^c} is a $\text{spin}^c(r, n-r)$ -principal bundle, where $\text{spin}^c(r, n-r) := \text{spin}(r, n-r) \times_{\mathbb{Z}/2\mathbb{Z}} S^1$ and where we write $\text{spin}(r, n-r)$ to denote the connected component of the identity of the spin group sitting inside the Clifford algebra $\mathcal{C}\ell_{r,n-r}$. Letting $\delta : \text{spin}(r, n-r) \rightarrow \text{GL}(\Sigma)$ be the restriction of an irreducible representation of the complexified Clifford algebra $\mathbb{C}\ell_{r,n-r}$, we define $\Delta : \text{spin}^c(r, n-r) \rightarrow \text{GL}(\Sigma)$ via $\Delta([s, z])v = z\delta(s)v$. The associated vector bundle $S := P_{\text{spin}^c} \times_{\Delta} \Sigma$ is called the *complex spinor bundle* $S \rightarrow M$, which is a vector bundle of complex dimension $2^{\lfloor \frac{n}{2} \rfloor}$. When n is even, S splits as a direct sum $S = S^+ \oplus S^-$.

Structures on the spinor bundle *In any dimension, the spinor bundle S carries the following structures:*

1. *Clifford multiplication $\cdot : TM \otimes S \rightarrow S$, $X \otimes \varphi \mapsto X \cdot \varphi$, which satisfies the Clifford relation*

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2g(X, Y)\varphi.$$

Using the metric, we can also define Clifford multiplication by one-forms. In case n is even, Clifford multiplication maps elements $\varphi \in S^+$ to $X \cdot \varphi \in S^-$.

2. *A hermitian bundle metric $\langle \cdot, \cdot \rangle$, which is positive definite when $r = 0$ and has signature $2^{\lfloor \frac{n}{2} \rfloor - 1}$ when $r > 0$. It obeys the rule*

$$\langle X \cdot \varphi, \psi \rangle = (-1)^{r+1} \langle \varphi, X \cdot \psi \rangle.$$

In case n is even, the bundle metric makes the splitting $S^+ \oplus S^-$ orthogonal.

3. *Given a connection $A \in \Omega^1(P_{S^1}, i\mathbb{R})$, we obtain a metric connection ∇^S on S which satisfies*

$$\nabla_X^S(Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla_X^S \varphi,$$

where ∇ denotes the Levi-Civita connection on (M, g) . In case n is even, the splitting $S^+ \oplus S^-$ is ∇^S -parallel.

The curvature R^S of ∇^S is related to the Ricci curvature, Ric , of g via

$$\sum_{k=1}^n \varepsilon_k e_k \cdot R^S(X, e_k)\varphi = -\frac{1}{2} \text{Ric}(X) \cdot \varphi + \frac{i}{2} (X \lrcorner \omega) \cdot \varphi, \tag{2}$$

where $\{e_1, \dots, e_n\}$ is a pseudo-orthonormal frame with $\varepsilon_k = g(e_k, e_k)$ and $i\omega \in \Omega^2(M, i\mathbb{R})$ denotes the curvature 2-form coming from the connection 1-form $A \in \Omega^1(P_{S^1}, i\mathbb{R})$ [17, Page 65].

Remark 2 The reason why we only consider time oriented manifolds, is to be able to define the above bundle metric. It can not be defined unless this assumption is included, see Remark 6.2.14 of [5].

3 Generalized Killing spinors

Let (M, g) denote a time oriented semi-Riemannian spin^c manifold of dimension n and index r , with all the associated structure from the above section. For clarity, we repeat the definition

from the introduction. A non-trivial spinor field $\varphi \in \Gamma(S)$ is said to be a *generalized Killing spinor*, if there is some g -symmetric endomorphism field A of TM such that

$$\nabla_X^S \varphi = \ell A(X) \cdot \varphi \quad \forall X \in \Gamma(TM), \tag{3}$$

where $\ell \in \{1, i\}$. We call A the *Killing endomorphism* and if $\ell = 1$ ($\ell = i$) we say that φ is a *real (imaginary) generalized Killing spinor*. In case $A = \mu \text{Id}$ for some real valued function $\mu \in C^\infty(M)$, we say in addition that φ is a *special generalized Killing spinor* with *Killing function* $\ell\mu$. Lastly, if μ is a real constant, we say that φ is a *Killing spinor* with *Killing number* $\ell\mu$. If φ is a generalized Killing spinor, we will simply say that it is *special* to mean that it is a special generalized Killing spinor.

Given a spinor field $\varphi \in \Gamma(S)$, we define the *Dirac current* $V_\varphi \in \Gamma(TM)$ as the vector field that is metrically equivalent to the 1-form $\omega_\varphi : X \mapsto i^{r+1} \langle \varphi, X \cdot \varphi \rangle$. When M has even dimension, we write $\varphi = \varphi^+ + \varphi^-$ and use the notation $\bar{\varphi} := \varphi^+ - \varphi^-$ to define the one-form

$$\bar{\omega}_\varphi : X \mapsto \langle \varphi, X \cdot \bar{\varphi} \rangle. \tag{4}$$

Using the symmetry property $\langle X \cdot \psi_1, \psi_2 \rangle = (-1)^{r+1} \langle \psi_1, X \cdot \psi_2 \rangle$ mentioned in 2 of [structures on the spinor bundle](#), we see that when the index r is even, the above one-form is real valued, and when r is odd, this one-form is imaginary valued. When r is even, we define the *modified Dirac current* \bar{V}_φ as the metrically equivalent vector field associated to $\bar{\omega}_\varphi$. When r is odd, we define the *modified Dirac current*, \bar{V}_φ as the metrically equivalent vector field to the 1-form $i\bar{\omega}_\varphi$.

We say that a vector field is closed if its associated 1-form is closed and we say that a vector field V is conformal if it satisfies $L_V g = \psi g$, for some function $\psi \in C^\infty(M)$. Recall that a vector field V is closed if and only if $g(\nabla_X V, Y) = g(\nabla_Y V, X)$, for all $X, Y \in TM$. Note that if V is closed, then we have $(L_V g)(X, Y) = 2g(\nabla_X V, Y)$, for all $X, Y \in TM$.

Lemma 3 *Let $\varphi \in \Gamma(S)$ be a generalized Killing spinor and suppose that either*

1. *the index r is odd, and $\ell = 1$, or*
2. *the index r is even, and $\ell = i$, or*
3. *the index r is odd, the dimension n is even, and $\ell = i$, or*
4. *the index r is even, the dimension n is even, and $\ell = 1$.*

Then in Case 1 and Case 2, V_φ is a closed vector field with

$$\nabla_X V_\varphi = c \langle \varphi, \varphi \rangle A(X), \tag{5}$$

where $c = -2$ if $r \pmod 4 \in \{0, 3\}$ and $c = 2$ otherwise. In Case 3 and Case 4, \bar{V}_φ is a closed vector field with

$$\nabla_X \bar{V}_\varphi = 2 \langle \varphi, \bar{\varphi} \rangle A(X). \tag{6}$$

If φ is furthermore a special generalized Killing spinor with $A = \mu \text{Id}$, then in Case 1 and Case 2, V_φ is conformal with $L_{V_\varphi} g = 2c\mu \langle \varphi, \varphi \rangle g$. In Case 3 and Case 4, \bar{V}_φ is conformal with $L_{\bar{V}_\varphi} g = 4\mu \langle \varphi, \bar{\varphi} \rangle g$

Proof Given a generalized Killing spinor $\varphi \in \Gamma(S)$, we compute for vector fields X and Y that

$$\begin{aligned} g(\nabla_X V_\varphi, Y) &= X(g(V_\varphi, Y)) - g(V_\varphi, \nabla_X Y) \\ &= i^{r+1} (X(\langle \varphi, Y \cdot \varphi \rangle) - \langle \varphi, \nabla_X Y \cdot \varphi \rangle) \end{aligned}$$

$$\begin{aligned}
 &= i^{r+1}(\langle \nabla_X^S \varphi, Y \cdot \varphi \rangle + \langle \varphi, \nabla_X^S(Y \cdot \varphi) \rangle - \langle \varphi, \nabla_X Y \cdot \varphi \rangle) \\
 &= i^{r+1}(\langle \nabla_X^S \varphi, Y \cdot \varphi \rangle + \langle \varphi, Y \cdot \nabla_X^S \varphi \rangle) \\
 &= i^{r+1}(\langle \ell A(X) \cdot \varphi, Y \cdot \varphi \rangle + \langle \varphi, Y \cdot \ell A(X) \cdot \varphi \rangle). \tag{7}
 \end{aligned}$$

First, suppose we are in Case 1, i.e. the index r is odd, and the generalized Killing spinor is real. Then by (7) we get that

$$\begin{aligned}
 g(\nabla_X^S V_\varphi, Y) &= i^{r+1}(\langle A(X) \cdot \varphi, Y \cdot \varphi \rangle + \langle \varphi, Y \cdot A(X) \cdot \varphi \rangle) \\
 &= i^{r+1}(\langle \varphi, A(X) \cdot Y \cdot \varphi \rangle + \langle \varphi, Y \cdot A(X) \cdot \varphi \rangle) \\
 &= -2i^{r+1} \langle \varphi, \varphi \rangle g(A(X), Y) \\
 &= c \langle \varphi, \varphi \rangle g(A(X), Y)
 \end{aligned}$$

which shows (5). Note that since A is g -symmetric, we have also showed that V_φ is closed. Using this, the conformal statement follows straightforwardly. Case 2 is obtained by a similar calculation. Indeed, we use (7) to get that

$$\begin{aligned}
 g(\nabla_X V_\varphi, Y) &= i^{r+1}(\langle iA(X) \cdot \varphi, Y \cdot \varphi \rangle + \langle \varphi, Y \cdot iA(X) \cdot \varphi \rangle) \\
 &= i^{r+2}(\langle A(X) \cdot \varphi, Y \cdot \varphi \rangle - \langle \varphi, Y \cdot A(X) \cdot \varphi \rangle) \\
 &= -i^{r+2}(\langle \varphi, A(X) \cdot Y \cdot \varphi \rangle + \langle \varphi, Y \cdot A(X) \cdot \varphi \rangle) \\
 &= 2i^{r+2} \langle \varphi, \varphi \rangle g(A(X), Y) \\
 &= c \langle \varphi, \varphi \rangle g(A(X), Y),
 \end{aligned}$$

which completes Case 2. Suppose now that we are in Case 3. We note that $\nabla_X^S \bar{\varphi} = -iA(X) \cdot \bar{\varphi}$ and use this to compute that

$$\begin{aligned}
 g(\nabla_X \bar{V}_\varphi, Y) &= X(g(\bar{V}_\varphi, Y)) - g(\bar{V}_\varphi, \nabla_X Y) \\
 &= i(X(\langle \varphi, Y \cdot \bar{\varphi} \rangle) - \langle \varphi, \nabla_X Y \cdot \bar{\varphi} \rangle) \\
 &= i(\langle \nabla_X^S \varphi, Y \cdot \bar{\varphi} \rangle + \langle \varphi, \nabla_X^S(Y \cdot \bar{\varphi}) \rangle - \langle \varphi, \nabla_X Y \cdot \bar{\varphi} \rangle) \\
 &= i(\langle iA(X) \cdot \varphi, Y \cdot \bar{\varphi} \rangle + \langle \varphi, Y \cdot \nabla_X^S \bar{\varphi} \rangle) \\
 &= -\langle A(X) \cdot \varphi, Y \cdot \bar{\varphi} \rangle - i \langle \varphi, Y \cdot iA(X) \cdot \bar{\varphi} \rangle \\
 &= -\langle \varphi, A(X) \cdot Y \cdot \bar{\varphi} \rangle - \langle \varphi, Y \cdot A(X) \cdot \bar{\varphi} \rangle) \\
 &= 2 \langle \varphi, \bar{\varphi} \rangle g(A(X), Y),
 \end{aligned}$$

which completes Case 3. Similarly, for Case 4, we compute that

$$\begin{aligned}
 g(\nabla_X \bar{V}_\varphi, Y) &= X(g(\bar{V}_\varphi, Y)) - g(\bar{V}_\varphi, \nabla_X Y) \\
 &= X(\langle \varphi, Y \cdot \bar{\varphi} \rangle) - \langle \varphi, \nabla_X Y \cdot \bar{\varphi} \rangle \\
 &= \langle \nabla_X^S \varphi, Y \cdot \bar{\varphi} \rangle + \langle \varphi, \nabla_X^S(Y \cdot \bar{\varphi}) \rangle - \langle \varphi, \nabla_X Y \cdot \bar{\varphi} \rangle \\
 &= \langle A(X) \cdot \varphi, Y \cdot \bar{\varphi} \rangle + \langle \varphi, Y \cdot \nabla_X^S \bar{\varphi} \rangle \\
 &= \langle A(X) \cdot \varphi, Y \cdot \bar{\varphi} \rangle - \langle \varphi, Y \cdot A(X) \cdot \bar{\varphi} \rangle \\
 &= -(\langle \varphi, A(X) \cdot Y \cdot \bar{\varphi} \rangle + \langle \varphi, Y \cdot A(X) \cdot \bar{\varphi} \rangle) \\
 &= 2 \langle \varphi, \bar{\varphi} \rangle g(A(X), Y),
 \end{aligned}$$

which completes the proof. □

Our goal is to utilize these closed and sometimes conformal vector fields to draw geometric conclusions of the underlying manifold.

4 The geometry of semi-Riemannian manifolds carrying closed and conformal vector fields

Closed and conformal vector fields on semi-Riemannian manifolds have been studied by Kühnel and Rademacher in [29] and [30]. In [29], they show the following. Note that they are considering gradient vector fields, which are locally the same as closed vector fields.

Lemma 4 (Kühnel & Rademacher, 1994, [29]) *Let (M, g) be a semi-Riemannian manifold admitting a non-trivial conformal gradient field $V = \text{grad } \psi$. Then the following holds.*

1. *In a neighborhood U of any point $p \in M$ with $g(V, V)_p \neq 0$, we have that $(U, g|_U)$ is isometric to a warped product of the form $(I \times F, \epsilon dr^2 + (\psi')^2 g|_F)$, where $\epsilon = \text{sign}(g(V, V))$ and the submanifolds $\{t\} \times F$ corresponds to the level sets of ψ . Furthermore, the integral curves of $\frac{V}{|V|}$ are geodesics and the function $\psi : I \rightarrow \mathbb{R}$ satisfies $\psi'' = \epsilon \frac{\Delta \psi}{n}$, where Δ denotes the Laplacian.*
2. *The zeroes of V are isolated. For each such zero $p \in M$, there exists a normal neighborhood $U \subset T_p M$, for which there are coordinates on $U \setminus C_p$, with C_p denoting the light cone, where the metric takes the form $g_{(r,x)} = \epsilon dr^2 + (\frac{\psi'_\epsilon(r)}{\psi'_\epsilon(0)})^2 \tilde{g}_x$, with $\epsilon = \text{sign } g(V, V)$, and \tilde{g} denotes the standard metric on $\{x \in \mathbb{R}^n \mid \langle x, x \rangle_{r,n-r} = \epsilon\}$. In these coordinates, we have that $\psi_\epsilon(r) = \psi(r, x)$. Furthermore, these coordinates extend to a conformally flat metric on all of U .*

Stronger results in the Riemannian setting have been proved by Rademacher in [37]. Our goal in this section is to strengthen the result 1 of Lemma 4 above. We intend to do so by expanding on the ideas and techniques developed by Olea and Gutiérrez in [25] and [26]. First, we proceed without assuming completeness.

4.1 Non-local splittings without completeness assumptions

Throughout, we assume that (M, g) is a connected semi-Riemannian manifold of dimension at least 2.

Let V be a closed vector field with nowhere vanishing norm $|V| := \sqrt{|g(V, V)|}$ and put $E := \frac{V}{|V|^2}$. Let $\Phi : D \rightarrow M$, with $D \subset \mathbb{R} \times M$, be the unique maximal flow of E . Recall that E is said to be a complete vector field if $D = \mathbb{R} \times M$. For each $t \in \mathbb{R}$, we get a map $\Phi_t : M_t \rightarrow M_{-t}$, where $M_t := \{p \in M \mid (t, p) \in D\}$. The map Φ_t is a diffeomorphism with inverse Φ_{-t} . In case V is also conformal, we will also consider the maximal flow $\tilde{\Phi}$ of the vector field $\tilde{E} := \frac{V}{|V|}$. We let $\tilde{D} \subset \mathbb{R} \times M$ denote the domain of $\tilde{\Phi}$.

One reason why we are interested in closed vector fields V is that the distribution $V^\perp \subset TM$ is integrable and thus induces a foliation $\mathcal{F} = \{L_p\}_{p \in M}$ of codimension 1 submanifolds L_p .

In [25] and [26] it is assumed in addition that \tilde{E} is complete. This yields global results, which we will come back to later. To clarify matters in the non-complete case, we need to introduce the following definition. We say that a connected open subset $L'_p \subset L_p$ is a *uniform subleaf* with respect to the flow Φ ($\tilde{\Phi}$), if there exists some non-empty open interval I , containing 0, such that $I \times L'_p \subset D$ ($I \times L'_p \subset \tilde{D}$). We call the pair (L'_p, I) a *uniform pair* for Φ ($\tilde{\Phi}$). Given a uniform pair (L'_p, I) for Φ ($\tilde{\Phi}$), we write $\Phi^{L'_p}$ ($\tilde{\Phi}^{L'_p}$) to denote the restriction of Φ to $I \times L'_p$ ($\tilde{\Phi}$ to $I \times L_p$). This notation will be unambiguous since the interval I will be clear from context. Note that the point p might not be a point of L'_p . However, for

any $q \in L_p$, we have that $L_q = L_p$, and therefore we will always assume that p is chosen so that $p \in L'_p$.

We will sometimes use the following fact, which follows straightforwardly from the definition of a flow. Given a uniform pair (L'_p, I) for Φ , the map $\Phi^{L'_p}$ is injective if and only if each maximal integral curve γ of E , with $\gamma(0) \in L'_p$, satisfies $\gamma(I) \cap L'_p = \gamma(0)$.

Assume that V is also conformal with $L_V g = \psi g$. Then it follows from the definitions that $\psi = \frac{2 \operatorname{div} V}{n}$ and $\nabla_X V = \frac{\operatorname{div} V}{n} X$, for all $X \in TM$. By the last property, $|V|$ is constant along the leaves L_p . We collect some straightforward formulas which will be used later. We compute for any $X, Y \perp V$ that

$$(L_{\tilde{E}}g)(X, Y) = \frac{(L_V g)(X, Y)}{|V|} = \frac{2 \operatorname{div} V}{n|V|} g(X, Y). \tag{8}$$

Using the formula $\nabla_X V = \frac{\operatorname{div} V}{n} X$ from above, we obtain that

$$\tilde{E}(|V|) = \frac{\operatorname{div} V}{n}. \tag{9}$$

We say that $\Phi(\tilde{\Phi})$ preserves the foliation \mathcal{F} , if for every uniform pair (L'_p, I) for $\Phi(\tilde{\Phi})$, we have that $\Phi_t(L'_p) \subset L_{\Phi_t(p)}$ ($\tilde{\Phi}_t(L'_p) \subset L_{\tilde{\Phi}_t(p)}$).

Proposition 5 *Let V be a closed vector field with nowhere vanishing norm and use the notation from above. Then Φ preserves the foliation \mathcal{F} . If V is also conformal, then $\tilde{\Phi}$ also preserves the foliation \mathcal{F} .*

Proof Fix some uniform pair (L'_p, I) for Φ . First, assume that $\Phi|_{I \times L'_p}$ is injective. Take some point $q \in L'_p$, with $q \neq p$. Since L'_p is connected, there exists a path with embedded image $l \cong (0, 1)$ in L'_p , containing both the point p and the point q . Take a tubular neighborhood \mathcal{U} of l in L'_p . Since V is nowhere tangent to \mathcal{U} and $\Phi|_{I \times \mathcal{U}}$ is injective, we have that $\Phi|_{I \times \mathcal{U}}$ is a diffeomorphism onto an open submanifold N of M [32, Theorem 9.20]. Since N has trivial de Rham cohomology, we can write $V|_N = \operatorname{grad} h$, for some function $h : N \rightarrow \mathbb{R}$ and note that the intersection of each leaf $L_m \in \mathcal{F}$ with N is either empty or a level set of h . Hence, if we show that $h(\Phi_t(p)) = h(\Phi_t(q))$, for all $t \in I$, then $\Phi_t(q) \in L_{\Phi_t(p)}$ and we could conclude the current case. By assumption, it is true for $t = 0$. Therefore, it suffices to show that $\frac{d}{dt} h(\Phi_t(p)) = \frac{d}{dt} h(\Phi_t(q))$ for all $t \in I$. We compute that

$$\frac{d}{dt} h(\Phi_t(p)) = dh(E_{\Phi_t(p)}) = g(V, \frac{V}{|V|^2})_{\Phi_t(p)} = \operatorname{sign} g(V, V). \tag{10}$$

With the same computation, we see that $\frac{d}{dt} h(\Phi_t(q)) = \operatorname{sign} g(V, V)$ and we can conclude the case when $\Phi|_{I \times L'_p}$ is injective. Now, suppose instead that $\Phi|_{I \times L'_p}$ is not injective. Let $T \in \mathbb{R}$ be the largest positive real number such that $\Phi|_{(I \cap (-T, T)) \times L'_p}$ is injective. Hence, there exists an integral curve for E , starting at a point of L'_p , which returns to L'_p either at T or at $-T$, and we assume without loss of generality that it returns at T . There exists some $\varepsilon > 0$ such that $\Phi|_{(-\varepsilon, \varepsilon) \times \Phi_T(L'_p)}$ is injective, and so by the first part of the proof it follows that every integral curve for E starting at a point of L'_p returns to L_p at T . By induction, each integral curve starting at a point of L'_p returns to L_p exactly at $t \in \{kT\}_{k \in \mathbb{Z}} \cap I$. Hence $\Phi|_{((k-1)T, (k+1)T) \cap I \times L'_p}$ is injective for all k . It follows that $\Phi|_{((-T, T) \cap (I - kT)) \times \Phi_{kT}(L'_p)}$ is injective, where we have used the notation $(a, b) - kT := (a - kT, b - kT)$. For all $k \in \mathbb{Z}$, we apply the first part of the proof to $\Phi|_{((-T, T) \cap (I - kT)) \times \Phi_{kT}(L'_p)}$ and conclude that for all $t \in (-T, T) \cap (I - kT)$, we have that $\Phi_{t+kT}(L'_p) = \Phi_t(\Phi_{kT}(L'_p)) \subset L_{\Phi_t(\Phi_{kT}(p))} =$

$L_{\Phi_{t+kT}(p)}$, and it follows that Φ preserves the foliation. Assume that V is also conformal. Then $|V|$ is constant along the leaves of the foliation \mathcal{F} . We can, except for the argument that $\frac{d}{dt}h(\Phi_t(p)) = \frac{d}{dt}h(\Phi_t(q))$, use the same proof as above to show that $\tilde{\Phi}$ preserves the foliation \mathcal{F} . To show that $\frac{d}{dt}h(\tilde{\Phi}_t(p)) = \frac{d}{dt}h(\tilde{\Phi}_t(q))$, we use that V is constant along the leaves to compute that $\frac{d}{dt}h(\Phi_t(p)) = (\text{sign } g(V, V))|V|_p = (\text{sign } g(V, V))|V|_q = \frac{d}{dt}h(\Phi_t(q))$ and thus we may complete the proof. \square

Proposition 6 *Let V be a closed vector field on (M, g) with nowhere vanishing norm, and let Φ and $\tilde{\Phi}$ be given as above. Then, for each uniform pair (L'_p, I) for Φ , the map*

$$\Phi^{L'_p} : (I \times L'_p, \frac{dt^2}{g(V, V)} + h_t) \rightarrow (M, g),$$

where $h_t = \Phi_t^*(g|_{L_{\Phi_t(p)}})$, is a local isometry. Furthermore, if V is also conformal, then for each uniform pair (L'_p, I) for $\tilde{\Phi}$, the map

$$\tilde{\Phi}^{L'_p} : (I \times L'_p, \epsilon dt^2 + f^2 g|_{L'_p}) \rightarrow (M, g),$$

where $\epsilon = g(\tilde{E}, \tilde{E})$ and $f(t) = \exp \int_0^t \frac{\text{div } V}{n|V|}(s) ds$, is a local isometry.

Proof By construction, $\Phi^{L'_p} : I \times L'_p \rightarrow M$ is a local diffeomorphism relating $\frac{\partial}{\partial t}$ with $\frac{V}{|V|^2}$. By Proposition 5, Φ preserves the foliation induced by V^\perp and therefore the pullback metric $(\Phi^{L'_p})^*(g)$ on $I \times L'_p$ can be written as $\frac{dt^2}{g(V, V)} + h_t$, where $h_t = \Phi_t^*(g|_{L_{\Phi_t(p)}})$. This proves the first part. Assume now that V is also conformal. By an argument similar to the above, we get that the pullback metric $(\tilde{\Phi}^{L'_p})^*(g)$ is of the form $\epsilon dt^2 + m_t$, where $m_t = \tilde{\Phi}_t^*(g|_{L_{\Phi_t(p)}})$. It remains to determine m_t . Fix $v, w \in T_x(L'_p)$. We intend to show that $m_t(v, w)$ satisfies the differential equation $\frac{d}{dt}m_t(v, w) = (f^2)'m_t(v, w)$. Using (8), we compute that

$$\begin{aligned} \frac{d}{dt}(m_t(v, w)) &= \frac{d}{dt}((\tilde{\Phi}_t^*g)(v, w)) \\ &= (L_{\tilde{E}}g)((\tilde{\Phi}_t)_*v, (\tilde{\Phi}_t)_*w) \\ &= \frac{2 \text{div } V}{n|V|}(\tilde{\Phi}_t(x))g((\tilde{\Phi}_t)_*v, (\tilde{\Phi}_t)_*w), \\ &= \frac{2 \text{div } V}{n|V|}(\tilde{\Phi}_t(x))(m_t(v, w)), \end{aligned}$$

and hence the abovementioned differential equation is satisfied. Showing that $\frac{2 \text{div } V}{n|V|}(\tilde{\Phi}_t(x))$ is only a function of t , completes the proof. As stated above, $|V|$ is constant along the leaves. Since $\tilde{\Phi}^{L'_p}$ is a local diffeomorphism, we have that around any point $p \in M$ there is a local frame of the form $\{\tilde{E}, \partial_1, \dots, \partial_{n-1}\}$, where $\partial_k \perp V$ for all $k \in \{1, \dots, n - 1\}$. We use (9) and that $|V|$ is constant along the leaves, to compute that

$$\partial_k \text{div } V = n \partial_k \tilde{E}(|V|) = n \tilde{E} \partial_k(|V|) = 0.$$

Thus the function $\text{div } V$ is constant along the leaves and consequently, so is $\frac{2 \text{div } V}{n|V|}(\tilde{\Phi}_t(x))$. Hence we may unambiguously write $\frac{2 \text{div } V}{n|V|}(\tilde{\Phi}_t(x)) = \frac{2 \text{div } V}{n|V|}(t)$ and conclude the proof. \square

When V is both closed and conformal, one can give conditions for when the above local isometry is an isometry onto its image. This is the content of the following proposition, which is a generalization of a result due to Gutierrez and Olea in [25] and [26].

Proposition 7 *Let V be a closed and conformal vector field on (M, g) with nowhere vanishing norm, and let $\tilde{\Phi}$ be given as above. Further, let (L'_p, I) be a uniform pair for $\tilde{\Phi}$ such that $\operatorname{div} V_p \neq 0$, and write $N := \tilde{\Phi}^{L'_p}(I \times L'_p)$. If either*

1. $\operatorname{div}(V)|_N$ is non-positive or non-negative, or
2. $\operatorname{Ric}(V, V)|_N$ is non-positive or non-negative,

then the map $\tilde{\Phi}^{L'_p}$ is injective.

Proof The map $\tilde{\Phi}^{L'_p}$ is injective if and only if each integral curve of \tilde{E} , starting at some point of L'_p , with image contained in N , intersects L'_p only at $t = 0$. Fix an integral curve γ of \tilde{E} with $\gamma(0) = q \in L'_p$ and image contained in N . Since $|V|$ is constant along the leaves of \mathcal{F} , the function $|V|_{\gamma(t)}$ only depends on which leaf $\gamma(t)$ is in. Also, by (9), the condition that $\operatorname{div} V_p \neq 0$ means that $\tilde{E}(|V|)_q \neq 0$. Hence the function $|V|_{\gamma(t)}$ is not constant.

First, suppose that $\operatorname{div}(V)|_N \geq 0$. By (9), we have that $\frac{d}{dt}|V|_{\gamma(t)} \geq 0$ and therefore $|V|_{\gamma(t)}$ is a non-constant monotone function. Hence, $\gamma(t)$ can never return to L'_p for any $t \neq 0$. The proof is completely analogous when $\operatorname{div}(V)|_N \leq 0$.

Secondly, using (9) together with the formula $\nabla_X V = \frac{\operatorname{div} V}{n} X$, we get that $\nabla_X V = \tilde{E}(|V|)X$, so we may compute that

$$R(X, V)V = \nabla_X \nabla_V V - \nabla_V \nabla_X V - \nabla_{[X, V]}V = X(\tilde{E}(|V|))V - V(\tilde{E}(|V|))X,$$

and therefore, $\operatorname{Ric}(V, V) = -(n - 1)V(\tilde{E}(|V|))$. Assume that $\operatorname{Ric}(V, V)|_N$ is non-negative and suppose for a contradiction that there is some point $0 \neq t_0 \in I$ such that $\gamma(t_0) \in L'_p$. By our computation of $\operatorname{Ric}(V, V)$, the function $\frac{\partial^2}{\partial t^2}|V|_{\gamma(t)}$ is non-positive. By Proposition 5, the flow $\tilde{\Phi}$ preserves the foliation, and therefore we have that $|V|_{\gamma(t)} = |V|_{\gamma(t_0+t)}$ for all t for which both sides are well defined. This means that we can extend the non-constant smooth function $|V|_{\gamma(\cdot)}$ to a non-constant smooth and periodic function $\mathbb{R} \rightarrow \mathbb{R}$ with period t_0 , whose second derivative is non-positive. This is a contradiction since smooth non-constant periodic functions must have both positive and negative second derivatives. The same proof applies if we assume that $\operatorname{Ric}(V, V)|_N$ is non-positive. \square

4.2 Global splittings under completeness assumptions

We proceed with the notation from the previous section. Given a flow Ψ and an interval $(t_1, t_2) \subset \mathbb{R}$, we will sometimes write $\Psi_{(t_1, t_2)}(V)$, to denote $\Psi((t_1, t_2) \times V)$.

Proposition 8 *Let Ψ be the flow of a nowhere vanishing complete vector field W on a connected smooth manifold P , preserving a codimension 1 foliation $\mathcal{Q} = \{Q_p\}_{p \in P}$. Then, for any leaf $Q_p \in \mathcal{Q}$, the map $\Psi|_{\mathbb{R} \times Q_p} : \mathbb{R} \times Q_p \rightarrow P$ is a normal covering map.*

Proof Fix some $Q_p \in \mathcal{Q}$. First, we show that $\Psi|_{\mathbb{R} \times Q_p} : \mathbb{R} \times Q_p \rightarrow P$ is surjective. We will show that the image $\Psi(\mathbb{R} \times Q_p) =: N$ is both open and closed. Since the vector field W is nowhere vanishing, we have that $\Psi|_{\mathbb{R} \times Q_p}$ is a local diffeomorphism and thus, N is open. If N^c is empty, then we are done. Suppose N^c is not empty and take some $x \in N^c$. It follows that $\Psi(\mathbb{R} \times Q_x) \subset N^c$. Indeed, if there were points $t_1, t_2 \in \mathbb{R}, y_1 \in Q_x, y_2 \in Q_p$ such that $\Psi(t_1, y_1) = \Psi(t_2, y_2)$, then we can combine the integral curves $\Psi(-, y_1)$ and $\Psi(-, y_2)$ to obtain an integral curve from Q_p to Q_x , which would mean that $x \in N$. Hence, x is

contained in the open subset $\Psi(\mathbb{R} \times Q_x)$, which is a subset of N^c . This holds for all points in N^c and thus N^c is open. Hence $\Psi|_{\mathbb{R} \times Q_p} : \mathbb{R} \times Q_p \rightarrow P$ is surjective. Now, we will show that every $x \in P$ has an evenly covered neighborhood. If $\Psi|_{\mathbb{R} \times Q_p}$ is injective, then we are done because $\Psi|_{\mathbb{R} \times Q_p}$ would be a diffeomorphism. Suppose therefore that $\Psi|_{\mathbb{R} \times Q_p}$ is not injective. Take some integral curve γ of W starting at a point of Q_p . Let $T \in \mathbb{R}$ be the smallest positive real number such that $\gamma(T) \in Q_p$. Since the flow Ψ preserves the foliation \mathcal{Q} , we have that each integral curve of W starting at a point on Q_p will return to the leaf Q_p at exactly the times $\{kT \mid k \in \mathbb{Z}\}$. Fix some point $x \in P$. Since W is nowhere vanishing, there exists some $\varepsilon > 0$ such that $\Psi|_{(-\varepsilon, \varepsilon) \times Q_x} : (-\varepsilon, \varepsilon) \times Q_x \rightarrow \Psi((-\varepsilon, \varepsilon) \times Q_x) =: U$ is a diffeomorphism. Hence U is open. Let $t_x \in \mathbb{R}$ be the smallest positive real number such that $\Psi_{t_x}(Q_p) = Q_x$. We note that

$$\begin{aligned} \Psi((t_x + kT - \varepsilon, t_x + kT + \varepsilon) \times Q_p) &= \Psi_{(t_x + kT - \varepsilon, t_x + kT + \varepsilon)}(Q_p) \\ &= \Psi_{(-\varepsilon, \varepsilon)} \circ \Psi_{t_x} \circ \Psi_{kT}(Q_p) \\ &= \Psi_{(-\varepsilon, \varepsilon)} \circ \Psi_{t_x}(Q_p) \\ &= \Psi_{(-\varepsilon, \varepsilon)}(Q_x) = U, \end{aligned}$$

and by the minimality assumptions on T and t_x , we conclude that

$$(\Psi|_{\mathbb{R} \times Q_p})^{-1}(U) = \bigcup_{k \in \mathbb{Z}} ((t_x + kT - \varepsilon, t_x + kT + \varepsilon) \times Q_p).$$

Hence, $\Psi|_{\mathbb{R} \times Q_p}$ is a covering map. We show that the group of deck transformations of the covering act transitively on the fibers. Take any $(t_0, p_0), (t_1, p_1) \in \mathbb{R} \times Q_p$ in the fiber over some $y \in P$. We claim that the diffeomorphism

$$\begin{aligned} f : \mathbb{R} \times Q_p &\rightarrow \mathbb{R} \times Q_p \\ (t, m) &\mapsto (t + (t_0 - t_1), \Psi((t_1 - t_0), m)) \end{aligned}$$

is a deck transformation mapping (t_1, p_1) to (t_0, p_0) . It is a deck transformation since

$$\begin{aligned} \Psi(f(t, m)) &= \Psi(t + (t_0 - t_1), \Psi((t_1 - t_0), m)) \\ &= \Psi_t \circ \Psi((t_0 - t_1), \Psi((t_1 - t_0), m)) \\ &= \Psi_t \circ \Psi(0, m) = \Psi(t, m), \end{aligned}$$

and it maps (t_1, p_1) to (t_0, p_0) since

$$f(t_1, p_1) = (t_0, \Psi((t_1 - t_0), p_1)) = (t_0, \Psi_{-t_0} \circ \Psi(t_1, p_1)) = (t_0, \Psi_{-t_0}(y)) = (t_0, p_0).$$

Hence we conclude that $\Psi|_{\mathbb{R} \times Q_p}$ is a normal covering map. □

With this, we improve Proposition 6 under the additional assumption of E or \tilde{E} being complete.

Proposition 9 *Let V be a closed vector field on (M, g) with nowhere vanishing norm, and let E, Φ, \tilde{E} , and $\tilde{\Phi}$ be given as in the previous section. If E is complete, then for any leaf L_p of \mathcal{F} , we have that*

$$\Phi^{L_p} : (\mathbb{R} \times L_p, \frac{dt^2}{g(V, V)} + h_t) \rightarrow (M, g),$$

where $h_t = \Phi_t^*(g|_{L_{\Phi_t(p)}})$, is a normal, semi-Riemannian covering. Assume in addition that V is also conformal. If \tilde{E} is complete, then for any leaf L_p of \mathcal{F} , we have that

$$\tilde{\Phi}^{L_p} : (\mathbb{R} \times L_p, \epsilon dt^2 + f^2 g|_{L_p}) \rightarrow (M, g),$$

where $\epsilon = g(\tilde{E}, \tilde{E})$ and $f(t) = \exp \int_0^t \frac{\operatorname{div} V}{n|V|}(s) ds$ is a normal, semi-Riemannian covering.

Proof The statement follows by Proposition 8 and Proposition 6. □

By applying Proposition 7 we obtain the following corollary.

Corollary 10 *Let V be a closed and conformal vector field on (M, g) with nowhere vanishing norm. Suppose \tilde{E} is complete, and $\operatorname{div} V$ is not constantly zero. If either $\operatorname{div} V$ or $\operatorname{Ric}(V, V)$ is non-positive or non-negative, then the covering $\tilde{\Phi}^{L_p}$ from Proposition 9 is an isometry for all leaves L_p .*

Proof Let p be such that $\operatorname{div} V_p \neq 0$. By Proposition 7, we have that $\tilde{\Phi}^{L_p}$ is an isometry. By Proposition 5, $\tilde{\Phi}$ preserves the foliation, and therefore the same is true for any other leaf $L_q \in \mathcal{F}$. □

5 Returning to generalized Killing spinors

We will apply the results of the previous section to semi-Riemannian spin^c manifolds carrying generalized Killing spinors. In the Riemannian case, we have a natural distinction of type I and type II imaginary generalized Killing spinors, and this allows us to further develop the classification. Given a type I imaginary generalized Killing spinor on a Riemannian spin^c manifold, we will find an explicit parallel spinor on each leaf of the foliation given by the orthogonal complement of the Dirac current. One could make ad hoc conditions for when similar results hold in the semi-Riemannian setting, but we do not conform to this. Nevertheless, due to a result by Bohle [11], we can find Killing and sometimes parallel spinors on the leaves when we restrict our attention to Killing spinors on time oriented semi-Riemannian spin manifolds.

5.1 The semi-Riemannian spin^c setting

Let (M, g) be a connected time oriented semi-Riemannian spin^c manifold of index r and dimension n . Let φ be a generalized Killing spinor as in either of the cases of Lemma 3. To simplify notation, we shall throughout the remainder of this section write V to denote either V_φ , corresponding to Case 1 and Case 2 of Lemma 3, or \overline{V}_φ , corresponding to Case 3 and Case 4 of Lemma 3. Hence, by Lemma 3, V is a closed vector field and if φ is a special generalized Killing spinor, then V is also conformal. Let $\mathcal{F} = \{L_p\}_{p \in M}$ be the foliation induced by V^\perp . Further, we write β to denote either $\langle \varphi, \varphi \rangle$ or $\langle \varphi, \overline{\varphi} \rangle$, with the analog case distinction as for V . Since $\nabla_X V = c\beta A(X)$, we get that

$$\operatorname{div} V = c\beta \operatorname{tr}(A), \tag{11}$$

where in Case 1 and Case 2, $c \in \{\pm 2\}$ corresponds to the constant c given in Lemma 3, and in Case 3 and Case 4, $c = 2$. If $|V|$ is nowhere vanishing, then we put $E := \frac{V}{|V|^2}$ and let Φ be the unique maximal flow of E . Similarly, let $\tilde{\Phi}$ be the unique maximal flow of $\tilde{E} := \frac{V}{|V|}$.

The following facts are straightforward computations. In Case 1 and Case 4, the function $Q_+ = \beta^2 + g(V, V)$ is locally constant. In Case 2 and Case 3, the function $Q_- = \beta^2 - g(V, V)$ is locally constant. Hence, in the corresponding cases, we have that $|V| = \sqrt{|Q_{\pm} - \beta^2|}$. If $|V|$ is nowhere vanishing, we shall write $\tilde{\beta} := c \frac{\beta}{|V|} = c \frac{\beta}{\sqrt{|Q_{\pm} - \beta^2|}}$ and $\epsilon = \text{sign}(g(V, V))$, to simplify the formulas below.

Given a hypersurface $N \subset M$ and a unit normal ξ , we write $W^\xi : TN \rightarrow TN$ to denote the Weingarten map $X \mapsto -(\nabla_X \xi)^T$, where $(-)^T$ denotes the orthogonal projection $TM \rightarrow TN$.

Proposition 11 *Let (M, g) be a time oriented semi-Riemannian spin^c manifold of index r and dimension n carrying a generalized Killing spinor φ , with $|V|$ nowhere vanishing, such that any of the cases of Lemma 3 holds. For any leaf L_p , the Weingarten map $W^{\tilde{E}}$ for L_p can be written as*

$$W^{\tilde{E}}(X) = -\tilde{\beta}(A(X) - \epsilon g(A(X), \tilde{E})\tilde{E}). \tag{12}$$

For any uniform pair (L'_p, I) for Φ , the map $\Phi^{L'_p} : (I \times L'_p, \frac{dt^2}{g(V, V)} + h_t) \rightarrow (M, g)$, where $h_t = \Phi_t^*(g|_{L_{\Phi_t(p)}})$, is a local isometry. If φ is special with $A = \mu \text{Id}$, then for any uniform pair (L'_p, I) for $\tilde{\Phi}$ we get the local isometry $\tilde{\Phi}^{L'_p} : (I \times L'_p, \epsilon dt^2 + f^2 g|_{L'_p}) \rightarrow (M, g)$, where

$$f(t) = \exp \int_0^t \mu \tilde{\beta} ds. \tag{13}$$

Furthermore, if the subleaf L'_p is such that $(\mu\beta)_p \neq 0$ and the interval I is such that either the function $\mu\beta$ or the function

- $4(n - 1)(\ell\mu)^2 g(V, V) + 2\bar{\ell}i^{r+1}(nV(\mu)\langle\varphi, \varphi\rangle + \langle\varphi, \text{grad}(\mu) \cdot V \cdot \varphi\rangle)$ in Case 1 and Case 2
- $4(n - 1)(\ell\mu)^2 g(V, V) - 2nV(\mu)\langle\varphi, \bar{\varphi}\rangle - 2\langle\text{grad}(\mu) \cdot V \cdot \varphi, \bar{\varphi}\rangle$ in Case 3 and Case 4

is non-negative or non-positive on $\tilde{\Phi}(I \times L'_p)$, then the map $\tilde{\Phi}^{L'_p}$ is injective.

Proof We compute that

$$\begin{aligned} \nabla_X \tilde{E} &= X(|g(V, V)|^{-\frac{1}{2}})V + |V|^{-1} \nabla_X V \\ &= -\frac{1}{2}|g(V, V)|^{-\frac{3}{2}} 2\epsilon g(\nabla_X V, V)V + |V|^{-1} c\beta A(X) \\ &= -|g(V, V)|^{-\frac{3}{2}} c\beta \epsilon g(A(X), V)V + |V|^{-1} c\beta A(X) \\ &= \frac{c\beta}{|V|} (A(X) - \epsilon g(A(X), \tilde{E})\tilde{E}) \\ &= \tilde{\beta} (A(X) - \epsilon g(A(X), \tilde{E})\tilde{E}), \end{aligned}$$

and hence we obtain the formula for the Weingarten map. The statements about the local isometries follows by Lemma 3 and Proposition 6. The injectivity statement is a translation of the conditions of Proposition 7 into this setting. By (11), we get that $\text{div } V = cn\mu\beta$. To compute $\text{Ric}(V, V)$, we first compute that

$$R^S(X, e_k)\varphi = \nabla_X^S \nabla_{e_k}^S \varphi - \nabla_{e_k}^S \nabla_X^S \varphi - \nabla_{[X, e_k]}^S \varphi$$

$$\begin{aligned}
 &= \ell(\nabla_X^S \mu e_k \cdot \varphi - \nabla_{e_k}^S \mu X \cdot \varphi - \mu[X, e_k] \cdot \varphi) \\
 &= \ell(X(\mu)e_k \cdot \varphi + \mu e_k \cdot \nabla_X^S \varphi - e_k(\mu)X \cdot \varphi - \mu X \cdot \nabla_{e_k}^S \varphi) \\
 &= \ell(X(\mu)e_k \cdot \varphi + \ell\mu^2 e_k \cdot X \cdot \varphi - e_k(\mu)X \cdot \varphi - \ell\mu^2 X \cdot e_k \cdot \varphi) \\
 &= \ell(X(\mu)e_k - e_k(\mu)X) \cdot \varphi + \ell^2\mu^2(e_k \cdot X \cdot \varphi - X \cdot e_k \cdot \varphi),
 \end{aligned}$$

and hence the left hand side of (2) becomes

$$\begin{aligned}
 LHS &= \sum_{k=1}^n \varepsilon_k e_k \cdot (\ell(X(\mu)e_k - e_k(\mu)X) \cdot \varphi + \ell^2\mu^2(e_k \cdot X \cdot \varphi - X \cdot e_k \cdot \varphi)) \\
 &= -\ell n X(\mu)\varphi - \ell \operatorname{grad}(\mu) \cdot X \cdot \varphi + 2\ell^2\mu^2 \sum_{k=1}^n \varepsilon_k e_k \cdot (e_k \cdot X \cdot \varphi + g(X, e_k)\varphi) \\
 &= -\ell n X(\mu)\varphi - \ell \operatorname{grad}(\mu) \cdot X \cdot \varphi - 2(n-1)\ell^2\mu^2 X \cdot \varphi,
 \end{aligned}$$

and thus we can write the identity (2) as

$$\operatorname{Ric}(X) \cdot \varphi = 2\ell n X(\mu)\varphi + 2\ell \operatorname{grad}(\mu) \cdot X \cdot \varphi + 4(n-1)\ell^2\mu^2 X \cdot \varphi + i(X \lrcorner \omega) \cdot \varphi.$$

With this, we compute in Case 1 and Case 2 that

$$\begin{aligned}
 \operatorname{Ric}(V, V) &= g(\operatorname{Ric}(V), V) \\
 &= i^{r+1} \langle \varphi, \operatorname{Ric}(V) \cdot \varphi \rangle \\
 &= i^{r+1} \langle \varphi, 2\ell n V(\mu)\varphi + 2\ell \operatorname{grad}(\mu) \cdot V \cdot \varphi + 4(n-1)\ell^2\mu^2 V \cdot \varphi + i(V \lrcorner \omega) \cdot \varphi \rangle \\
 &= 4(n-1)\ell^2\mu^2 g(V, V) + 2\bar{\ell}i^{r+1} (nV(\mu)\langle \varphi, \varphi \rangle + \langle \varphi, \operatorname{grad}(\mu) \cdot V \cdot \varphi \rangle),
 \end{aligned}$$

where in the last step we used that ω is a 2-form. Similarly, for Case 3 we compute that

$$\begin{aligned}
 \operatorname{Ric}(V, V) &= g(\operatorname{Ric}(V), V) = i \langle \varphi, \operatorname{Ric}(V) \cdot \bar{\varphi} \rangle = i \langle \operatorname{Ric}(V) \cdot \varphi, \bar{\varphi} \rangle \\
 &= i \langle 2inV(\mu)\varphi + 2i \operatorname{grad}(\mu) \cdot V \cdot \varphi - 4(n-1)\mu^2 V \cdot \varphi + i(V \lrcorner \omega) \cdot \varphi, \bar{\varphi} \rangle \\
 &= -4(n-1)\mu^2 g(V, V) - 2\langle nV(\mu)\varphi + \operatorname{grad}(\mu) \cdot V \cdot \varphi, \bar{\varphi} \rangle \\
 &= -4(n-1)\mu^2 g(V, V) - 2\langle nV(\mu)\varphi, \bar{\varphi} \rangle - 2\langle \operatorname{grad}(\mu) \cdot V \cdot \varphi, \bar{\varphi} \rangle,
 \end{aligned}$$

and for case 4, we get that

$$\begin{aligned}
 \operatorname{Ric}(V, V) &= g(\operatorname{Ric}(V), V) = \langle \varphi, \operatorname{Ric}(V) \cdot \bar{\varphi} \rangle = -\langle \operatorname{Ric}(V) \cdot \varphi, \bar{\varphi} \rangle \\
 &= -\langle 2nV(\mu)\varphi + 2 \operatorname{grad}(\mu) \cdot V \cdot \varphi + 4(n-1)\mu^2 V \cdot \varphi + i(V \lrcorner \omega) \cdot \varphi, \bar{\varphi} \rangle \\
 &= 4(n-1)\mu^2 g(V, V) - 2\langle nV(\mu)\varphi + \operatorname{grad}(\mu) \cdot V \cdot \varphi, \bar{\varphi} \rangle \\
 &= 4(n-1)\mu^2 g(V, V) - 2\langle nV(\mu)\varphi, \bar{\varphi} \rangle - \langle \operatorname{grad}(\mu) \cdot V \cdot \varphi, \bar{\varphi} \rangle,
 \end{aligned}$$

which finishes the proof. □

Applying part 2 of Lemma 4, we can also compute the metric around any zero of V , when φ is a special generalized Killing spinor.

Proposition 12 *Let (M, g) be a time oriented semi-Riemannian spin^c manifold of index r and dimension n carrying a special generalized Killing spinor φ with Killing function μ such that any of the cases of Lemma 3 holds. Then the zeroes of V are isolated. For each such zero $p \in M$, there exists a normal neighborhood $U \subset T_p M$, for which there are coordinates on $U \setminus C_p$, with C_p denoting the light cone, where the metric takes the form*

$g_{(r,x)} = \epsilon dr^2 + (\frac{\psi'_\epsilon(r)}{\psi_\epsilon(0)})^2 \tilde{g}_x$, with $\epsilon = \text{sign } g(V, V)$, and \tilde{g} denotes the standard metric on $\{x \in \mathbb{R}^n \mid \langle x, x \rangle_{r,n-r} = \epsilon\}$. In these coordinates, we have that $V = \text{grad } \psi$, where $\psi(r, x) = \psi_\epsilon(r)$. Furthermore, these coordinates extend to a conformally flat metric on all of U .

Proof This follows by Lemma 3 and Part 2 of Lemma 4. □

Remark 13 In case the Killing function is a non-zero constant, i.e. the spinor is a non-parallel Killing spinor, then we have globally that V is a gradient vector field $V = \text{grad } \psi$. A straightforward case by case computation of $\text{grad } \beta$ yields that the function Ψ can, up to addition by a constant, be computed as

Case 1	Case 2	Case 3	Case 4	(14)
$\frac{i^{r+1}\beta}{2\mu}$	$-\frac{i^r\beta}{2\mu}$	$\frac{\beta}{2\mu}$	$-\frac{\beta}{2\mu}$	

In case E or \tilde{E} is complete, we get the following improvement of Proposition 11.

Proposition 14 *Let (M, g) be a connected time oriented semi-Riemannian spin^c manifold of index r and dimension n carrying a generalized Killing spinor φ , with $|V|$ nowhere vanishing, such that any of the cases of Lemma 3 holds. For any leaf L_p , the Weingarten map $W^{\tilde{E}}$ is as in (12). If E is complete, then for any leaf L_p we get that the map $\Phi^{L_p} : (\mathbb{R} \times L_p, \frac{dt^2}{g(V,V)} + h_t) \rightarrow (M, g)$, where $h_t = \Phi_t^*(g|_{L_{\Phi_t(p)}})$, is a normal, semi-Riemannian covering. If instead \tilde{E} is complete and φ is special with $A = \mu \text{Id}$, then for any leaf L_p , the map $\tilde{\Phi}^{L_p} : (\mathbb{R} \times L_p, \epsilon dt^2 + f^2 g|_{L_p}) \rightarrow (M, g)$, where f is as in (13), is a normal, semi-Riemannian covering. If in addition the function $\mu\beta$ is not constantly zero and either the function $\mu\beta$ or the function*

- $4(n - 1)\ell^2\mu^2g(V, V) + 2\bar{\ell}i^{r+1}(nV(\mu)\langle\varphi, \varphi\rangle + \langle\varphi, \text{grad}(\mu) \cdot V \cdot \varphi\rangle)$ in Case 1 and Case 2
- $4(n - 1)\ell^2\mu^2g(V, V) - 2nV(\mu)\langle\varphi, \bar{\varphi}\rangle - 2\langle\text{grad}(\mu) \cdot V \cdot \varphi, \bar{\varphi}\rangle$ in Case 3 and Case 4

is non-negative or non-positive, then the map $\tilde{\Phi}^{L_p}$ is an isometry for any leaf L_p .

Proof The computation of the Weingarten map is done in Proposition 11. The statements about the coverings follow from Lemma 3 and Proposition 9. The injectivity condition is a translation of the conditions of Corollary 10 and is in complete analogue to the proof of Proposition 11. □

The above results contain some geometric information of manifolds carrying generalized Killing spinors, but we are unable to draw conclusions about what type of spinor equation one can obtain on the fiber. However, when restricted to spin manifolds with Killing spinors, we have the following result due to Bohle [11, Theorem 6 & Theorem 7].

Theorem 15 (Bohle, 2003, [11]) *A time oriented semi-Riemannian warped product spin manifold of the form $(I \times F, \epsilon dt^2 + f^2 g_F)$ admits a Killing spinor with Killing number $\lambda \in \{\pm\frac{1}{2}, 0, \pm\frac{i}{2}\}$ if and only if the warping function satisfies $f'' = -4\epsilon\lambda^2 f$ and the fiber F admits a Killing spinor to the Killing number $\pm\lambda_F$, where $\lambda_F^2 = \lambda^2 f^2 + \epsilon\frac{1}{4}(f')^2$.*

Note that by rescaling the metric, one can always assume that Killing numbers is of the above modulus. The function computed in (13) satisfies the equation $f'' = -4\epsilon(\ell\mu)^2 f$ in case the special generalized Killing spinor of Proposition 11 is a Killing spinor with Killing number $\lambda \in \{\pm\frac{1}{2}, 0, \pm\frac{i}{2}\}$. Using Theorem 15, we improve Proposition 11 and Proposition 14 in the case of Killing spinors, by concluding that the fibers carries Killing spinors.

Theorem 16 *Let (M, g) be a connected time oriented semi-Riemannian spin manifold carrying a Killing spinor φ having Killing number $\lambda \in \{\pm\frac{1}{2}, 0, \pm\frac{i}{2}\}$, with $|V|$ nowhere vanishing such that any of the cases of Lemma 3 holds. Then for any uniform pair (L'_p, I) for $\tilde{\Phi}$, we have that the map $\tilde{\Phi}^{L'_p} : (I \times L'_p, \epsilon dt^2 + f^2 g|_{L'_p}) \rightarrow (M, g)$, where f is as in (13), is an isometry onto its image. The fiber $(L'_p, g|_{L'_p})$ carries a Killing spinor with Killing number $\pm\lambda_{L'_p}$, where $\lambda_{L'_p}^2 = \lambda^2 f^2 + \epsilon\frac{1}{4}(f')^2$.*

If \tilde{E} is complete, then for any leaf L_p , the map $\tilde{\Phi}^{L_p} : (\mathbb{R} \times L_p, \epsilon dt^2 + f^2 g|_{L_p}) \rightarrow (M, g)$, where f is as in (13), is an isometry. The fiber $(L_p, g|_{L_p})$ carries a Killing spinor with Killing number $\pm\lambda_{L_p}$ satisfying $\lambda_{L_p}^2 = \lambda^2 f^2 + \epsilon\frac{1}{4}(f')^2$.

Proof By proposition 11 we get that $\tilde{\Phi}^{L'_p}$ is a local isometry. However, since the function $4(n-1)\lambda^2 g(V, V)$, is either positive or negative, we have by the last condition of Proposition 11 that $\tilde{\Phi}^{L'_p}$ is in fact an isometry. The statement regarding the Killing spinor on the fiber follows from Theorem 15. For the case when \tilde{E} is complete, one argues identically, using Proposition 14. □

In the Lorentzian case, we have stronger results due to Bohle [11, Section 5] and Leitner [34, Theorem 5.3]. In the Riemannian case, the global statement is essentially the same as due to Baum in [7]. The author is not aware of similar local statements in the Riemannian case but is sure that they are known to experts. There are also stronger results for $r = 2$, due to Shafiee and Bahrampour [38]. In case of parallel spinors we have stronger results in [10, 16, 27, 39].

5.2 The Riemannian classification

In this section, we specialize to the Riemannian setting, showing how one can generalize previous classifications made by Große and Nakad [23], and Leistner and Lischewski [33].

5.2.1 Type I imaginary generalized Killing spinors

We say that an imaginary generalized Killing spinor φ on a Riemannian spin^c manifold is of *type I* if the locally constant function $Q_- = \langle \varphi, \varphi \rangle^2 - g(V_\varphi, V_\varphi)$ equals 0 everywhere. If Q_- does not equal 0, then we say that φ is of *type II*. By the Cauchy-Schwarz inequality, we always have that Q_- is non-negative. It is a general fact that generalized Killing spinors have no zeroes. Indeed, a generalized Killing spinor is parallel with respect to the modified connection $\tilde{\nabla}_X^S \psi := \nabla_X^S \psi - \ell A(X) \cdot \psi$ and thus if φ had one zero, it would be zero throughout by $\tilde{\nabla}^S$ -parallel transport. With this, we see in the Riemannian case that type I imaginary generalized Killing spinors are such that $|V_\varphi| \neq 0$. This is the first reason why we are able to classify type I imaginary generalized Killing spinors. As in Section 5.1, let Φ be the flow of $E := \frac{V_\varphi}{|V_\varphi|^2}$, let $\tilde{\Phi}$ to be the flow of $\tilde{E} := \frac{V_\varphi}{|V_\varphi|}$, and let $\mathcal{F} = \{L_p\}_{p \in M}$ be the foliation induced by E^\perp .

Recall from Proposition 5 that Φ preserves the foliation \mathcal{F} and if φ is special, then $\tilde{\Phi}$ also preserves the foliation \mathcal{F} . The following lemma shows the second reason why we are able to classify type I imaginary generalized Killing spinors. The proof of the following lemma was pointed out to the author by Jonathan Glöckle and is essentially the same computation as in section 1.3 of [22]. The following lemma does not assume connectedness.

Lemma 17 *Given a spinor field φ on a Riemannian spin^c manifold such that $\langle \varphi, \varphi \rangle^2 - g(V_\varphi, V_\varphi) = 0$ everywhere, we also have that*

$$V_\varphi \cdot \varphi = i|V_\varphi|\varphi. \tag{15}$$

Conversely, if φ is an imaginary generalized Killing spinor which is not parallel on any component of M , and satisfies equation (15), then φ is of type I.

Proof We compute that:

$$\begin{aligned} \langle V_\varphi \cdot \varphi - i|V_\varphi|\varphi, V_\varphi \cdot \varphi - i|V_\varphi|\varphi \rangle &= \langle V_\varphi \cdot \varphi, V_\varphi \cdot \varphi \rangle - 2 \operatorname{Re}\langle V_\varphi \cdot \varphi, i|V_\varphi|\varphi \rangle + \langle i|V_\varphi|\varphi, i|V_\varphi|\varphi \rangle \\ &= g(V_\varphi, V_\varphi)\langle \varphi, \varphi \rangle - 2i|V_\varphi|\langle \varphi, V_\varphi \cdot \varphi \rangle + |V_\varphi|^2 \langle \varphi, \varphi \rangle \\ &= 2g(V_\varphi, V_\varphi)\langle \varphi, \varphi \rangle - 2|V_\varphi|g(V_\varphi, V_\varphi) \\ &= 2|V_\varphi|^2(\langle \varphi, \varphi \rangle - |V_\varphi|). \end{aligned} \tag{16}$$

First, assume that φ is a type I imaginary generalized Killing spinor. Then (15) follows because both $g(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ are positive definite. Suppose now that φ is an imaginary generalized Killing spinor which is not parallel on any component of M such that $V_\varphi \cdot \varphi = i|V_\varphi|\varphi$. Then, for each point $x \in M$, we have by (16), that either $V_\varphi(x) = 0$, or that $\langle \varphi, \varphi \rangle(x) - |V_\varphi|(x) = 0$. Since the function $\langle \varphi, \varphi \rangle^2 - g(V_\varphi, V_\varphi)$ is locally constant, it suffices to show that the set $N := \{x \in M \mid V_\varphi(x) = 0\}$ does not contain any component of M . Suppose for a contradiction that N contains some component M_1 of M . By assumption, there is some point $x \in M_1$ and a vector $Y \in T_x M_1$ such that $A(Y) \neq 0$. Pick any local extension \tilde{Y} of Y to compute that

$$\begin{aligned} 0 &= Y(g(V_\varphi, A(\tilde{Y}))) = Y(i\langle \varphi, A(\tilde{Y}) \cdot \varphi \rangle) \\ &= i(\langle \nabla_Y \varphi, A(\tilde{Y}) \cdot \varphi \rangle + \langle \varphi, \nabla_Y A(\tilde{Y}) \cdot \varphi \rangle + \langle \varphi, A(\tilde{Y}) \cdot \nabla_Y \varphi \rangle) \\ &= i(\langle iA(Y) \cdot \varphi, A(Y) \cdot \varphi \rangle + \langle \varphi, A(Y) \cdot iA(Y) \cdot \varphi \rangle) \\ &= 2\langle \varphi, A(Y) \cdot A(Y) \cdot \varphi \rangle = -2g(A(Y), A(Y))\langle \varphi, \varphi \rangle_x, \end{aligned}$$

and since generalized Killing spinors have no zeroes, we conclude that $A(Y) = 0$ which establishes the contradiction and completes the proof. \square

Remark 18 To see that the assumption that φ is not allowed to be parallel on any component of M in Lemma 17 is necessary, consider a complete and simply connected spin manifold (M, g) that carries a parallel spinor and has irreducible holonomy. In this case, the Dirac current would have to vanish identically, because otherwise the manifold (M, g) would carry a nowhere vanishing parallel vector field and the holonomy of (M, g) would be reducible. Hence the Dirac current vanishes but $\langle \varphi, \varphi \rangle$ does not, and so φ is of type II but satisfies equation (15) trivially.

Remark 19 Baum introduced the terms type I and type II imaginary Killing spinors in [7]. It is shown in [7] and [8] that a type I imaginary Killing spinor on a Riemannian spin manifold satisfies (15). This proof can be adapted to imaginary generalized Killing spinors as well, but is very different from the above proof of Lemma 17. In [33], Leistner and Lischewski uses equation (15) directly.

In what follows, we will apply the spin^c Gauß formula from [36, Section 3]. We follow the setup of that paper. Given a leaf $L_p \in \mathcal{F}$, we equip it with the induced spin^c structure and we write S_{L_p} to denote the induced spinor bundle on the hypersurface L_p . When the dimension n of M is odd, we have that $S_{L_p} \cong S|_{L_p}$ and when n is even, we have that $S_{L_p} \cong S^+|_{L_p}$. Hence, when n is odd, we can simply write $\psi|_{L_p}$ to denote the restriction of a section, but when n is even we write $\psi|_{L_p}$, to mean $\psi^+|_{L_p}$. In this setup, the S^1 -bundle associated with the spin^c structure on L_p is simply the restriction $P_{S^1}|_{L_p}$, where P_{S^1} is the S^1 -bundle associated to the spin^c structure on M . Hence, given a connection 1-form $A \in \Omega^1(P_{S^1}, i\mathbb{R})$, we get induced connections ∇^S on S and $\nabla^{S_{L_p}}$ on S_{L_p} . If we let $\bullet : T(L_p) \otimes S_{L_p} \rightarrow S_{L_p}$ denote the Clifford multiplication on S_{L_p} , we have that $X \bullet \eta = \tilde{E} \cdot X \cdot \eta$, where on the right hand side, η is either viewed as an element of $S|_{L_p}$ or $S^+|_{L_p}$, depending on if n is odd or even.

Remark 20 It is important to point out that the above identifications of spinor bundles and Clifford multiplication on hypersurfaces is not the only one. It boils down to the choice of inclusion $C\ell_{n-1} \rightarrow C\ell_n$ of Clifford algebras. The choice which is made in [36] is the inclusion $e_j \rightarrow e_n \cdot e_j$, where $\{e_1, \dots, e_n\}$ is the standard basis for \mathbb{R}^n and thus also the set of standard generators for $C\ell_n$. Another choice that can be made is to choose the inclusion $e_j \rightarrow e_j$. The second choice would have worked equally well for the main results of this article. However, the below formula (17) would have had to been adjusted accordingly.

Translating Proposition 3.3 of [36], to our notation, we get for any spinor-field $\psi \in \Gamma(S)$ and any vector field $X \in \Gamma(T(L_p))$ that

$$\nabla_X^{S_{L_p}}(\psi|_{S_{L_p}}) = (\nabla_X^S \psi)|_{L_p} + \frac{1}{2} W^{\tilde{E}}(X) \bullet \psi|_{L_p}. \tag{17}$$

With this, we use (12) to compute for any type I imaginary generalized Killing spinor φ on a Riemannian spin^c manifold, that the spinor $\varphi|_{L_p}$, satisfies

$$\begin{aligned} \nabla_X^{S_{L_p}} \varphi|_{L_p} &= (\nabla_X^S \varphi)|_{L_p} + \frac{1}{2} W^{\tilde{E}}(X) \bullet \varphi|_{L_p} \\ &= (iA(X) \cdot \varphi)|_{L_p} + \frac{1}{2} (\tilde{E} \cdot W^{\tilde{E}}(X) \cdot \varphi)|_{L_p} \\ &= (iA(X) \cdot \varphi)|_{L_p} + (\tilde{E} \cdot (A(X) - g(A(X), \tilde{E})\tilde{E}) \cdot \varphi)|_{L_p} \\ &= (iA(X) \cdot \varphi)|_{L_p} - ((A(X) - g(A(X), \tilde{E})\tilde{E}) \cdot \tilde{E} \cdot \varphi)|_{L_p} \\ &= (iA(X) \cdot \varphi)|_{L_p} - \underbrace{(A(X) \cdot \tilde{E} \cdot \varphi + g(A(X), \tilde{E})\varphi)}_{=i\varphi}|_{L_p} \\ &= -g(A(X), \tilde{E})\varphi|_{L_p}. \end{aligned} \tag{18}$$

Now, computing as in Lemma 3, we see that $X(\langle \varphi, \varphi \rangle) = -2|V_\varphi|g(A(X), \tilde{E}) = -2\langle \varphi, \varphi \rangle g(A(X), \tilde{E})$, since φ is of type I. We have obtained the formula:

$$\frac{X(\langle \varphi, \varphi \rangle)}{2\langle \varphi, \varphi \rangle} \varphi|_{L_p} = -g(A(X), \tilde{E})\varphi|_{L_p} = \nabla_X^{S_{L_p}} \varphi|_{L_p}. \tag{19}$$

Lemma 21 *Let φ be a type I imaginary generalized Killing spinor on a Riemannian spin^c manifold. Then the spinor $\langle \varphi, \varphi \rangle^{-\frac{1}{2}} \varphi|_{L_p}$ is $\nabla^{S_{L_p}}$ -parallel, for any leaf $L_p \in \mathcal{F}$.*

Proof We compute that

$$\begin{aligned} \nabla_X^{S_{L_p}} \langle \varphi, \varphi \rangle^{-\frac{1}{2}} \varphi|_{L_p} &= X(\langle \varphi, \varphi \rangle^{-\frac{1}{2}}) \varphi|_{L_p} + \langle \varphi, \varphi \rangle^{-\frac{1}{2}} \nabla_X^{S_{L_p}} \varphi|_{L_p} \\ &\stackrel{(19)}{=} -\frac{X(\langle \varphi, \varphi \rangle)}{2\langle \varphi, \varphi \rangle^{3/2}} \varphi|_{L_p} + \langle \varphi, \varphi \rangle^{-\frac{1}{2}} \frac{X(\langle \varphi, \varphi \rangle)}{2\langle \varphi, \varphi \rangle} \varphi|_{L_p} = 0, \end{aligned}$$

and complete the proof. □

Remark 22 By (18), we see that the condition for a type I imaginary generalized Killing spinor to restrict to a $\nabla^{S_{L_p}}$ -parallel spinor on L_p is that $g(X, A(\tilde{E})) = 0$, for all $X \in T(L_p)$. In case φ in Lemma 21 is special, then this condition is satisfied and so φ restricts to a $\nabla^{S_{L_p}}$ -parallel spinor on L_p . This is the parallel spinor found in the proof of Theorem 4.1 of [23].

We can now state and prove our main theorems.

Theorem 23 *Let (M, g) be a Riemannian spin^c manifold carrying a type I imaginary generalized Killing spinor φ .*

- a) *For any uniform pair (L'_p, I) for Φ , the map $\Phi^{L'_p} : (I \times L'_p, \frac{dt^2}{g(V_\varphi, V_\varphi)} + h_t) \rightarrow (M, g)$ is a local isometry, where $h_t := \Phi_t^*(g|_{L_{\Phi_t(p)}})$ is a family of metrics on L'_p , such that (L'_p, h_t) carries a parallel spinor for each t .*

Suppose in addition that φ is special with Killing function $i\mu$.

- b) *For any uniform pair (L'_p, I) for $\tilde{\Phi}$, the map $\tilde{\Phi}^{L'_p} : (I \times L'_p, dt^2 + e^{-4 \int_0^t \mu ds} g|_{L_p}) \rightarrow (M, g)$ is a local isometry and φ restricts to a parallel spinor on $(L'_p, g|_{L_p})$. If μ is not identically zero on the image of $\tilde{\Phi}^{L'_p}$ and is either non-negative or non-positive on the image of $\tilde{\Phi}^{L'_p}$, then the local isometry $\tilde{\Phi}^{L'_p}$ is an isometry onto its image.*

Proof Everything except the last sentence follows directly from Proposition 11, Lemma 21 and Remark 22. To prove the last sentence, take a leaf $\Lambda \in \mathcal{F}$ that have non-trivial intersection with the image of $\tilde{\Phi}^{L'_p}$ and such that $\mu|_\Lambda \neq 0$. Since the flow of $\tilde{\Phi}$ preserves the foliation, there exists a $t_0 \in I$ such that $\tilde{\Phi}(t_0, L'_p) \subset \Lambda$. The pair $(\tilde{\Phi}(t_0, L'_p), I - t_0)$, where $(a, b) - t_0 := (a - t_0, b - t_0)$, is uniform for $\tilde{\Phi}$ and satisfies the first injectivity condition of Proposition 11. Therefore, the map $\tilde{\Phi}|_{(I-t_0) \times \tilde{\Phi}(t_0, L'_p)}$ is injective and since $\tilde{\Phi}$ preserves the foliation \mathcal{F} , the map $\tilde{\Phi}^{L'_p}$ is also injective. Hence $\tilde{\Phi}^{L'_p}$ is an isometry onto its image. □

Assuming completeness of E or \tilde{E} , we get the following global analogue of the above theorem.

Theorem 24 *Let (M, g) be a connected Riemannian spin^c manifold carrying a type I imaginary generalized Killing spinor.*

a) If E is complete, then for any leaf L_p , we have that $\Phi^{L_p} : (\mathbb{R} \times L_p, \frac{dt^2}{g(V_\varphi, V_\varphi)} + h_t) \rightarrow (M, g)$ is a normal, Riemannian covering, where $h_t = \Phi_t^*(g|_{L_{\Phi_t(p)}})$ is a family of metrics on L_p such that for each t , the manifold (L_p, h_t) carries a parallel spinor.

Suppose in addition that φ is special with Killing function $i\mu$.

b) If \tilde{E} is complete, then for any leaf L_p , we have that $\tilde{\Phi}^{L_p} : (\mathbb{R} \times L_p, dt^2 + e^{-4\int_0^t \mu ds} g|_{L_p}) \rightarrow (M, g)$ is a normal, Riemannian covering and φ restricts to a parallel spinor on $(L_p, g|_{L_p})$. If μ is not identically zero and is either non-negative or non-positive, then the covering $\tilde{\Phi}^{L_p}$ is an isometry.

Proof Everything except the last sentence follows from Proposition 14, Lemma 21, and Remark 22. To prove the last statement, one appeals to the injectivity condition of Proposition 14 and argues as in the proof of Theorem 23. □

Theorem 24 generalizes Theorem 4.1 and Corollary 4.2 of [23]. In particular, it extends Baum’s classification of type I imaginary Killing spinors on spin manifolds [7, Theorem 3] to the spin^c setting. Theorem 24 a) also generalizes a part of Theorem 4 in [33].

5.2.2 Type II imaginary generalized Killing spinors

Recall that an imaginary generalized Killing spinor on a Riemannian spin^c manifold is said to be of type II if the constant $Q_- = \langle \varphi, \varphi \rangle^2 - g(V_\varphi, V_\varphi)$ is positive. In case the type II imaginary generalized Killing spinor is a Killing spinor, then Große and Nakad [23, Proposition 4.3] showed that the manifold is isometric to hyperbolic space, provided it is complete. It was shown by Rademacher [37] that on spin manifolds, all type II special imaginary generalized Killing spinors have constant Killing function, i.e. they are Killing spinors. Grosse and Nakad [23, Section 4.2.2] extends this to the spin^c case for dimension at least 3 and provide a counterexample in dimension 2.

To clarify matters in the case when the generalized Killing spinor need not be special, we want to consider hypersurfaces of Lorentzian manifolds. The following identifications are described in [2] and [5] in the spin case and in [36] for the spin^c setting. We give a quick summary similar to the above discussion on Riemannian hypersurfaces of Riemannian spin^c manifolds. Let (\bar{M}, \bar{g}) be a time oriented Lorentzian spin^c manifold with spinor bundle S and let (M, g) be a spacelike hypersurface with unit normal N . We give (M, g) the induced spin^c structure and write S_M for the corresponding spinor bundle. When $\dim \bar{M}$ is odd, we have that $S_M \cong S|_M$ and when $\dim \bar{M}$ is even, we have that $S_M \cong S^+|_M$. In this setup, the S^1 -bundle associated with the spin^c structure on M is simply the restriction $P_{S^1}|_M$, where P_{S^1} is the S^1 -bundle associated to the spin^c structure on \bar{M} . Hence, given a connection $A \in \Omega^1(P_{S^1}, i\mathbb{R})$, we get induced connections ∇^S on S and ∇^{S_M} on S_M . Clifford multiplication \bullet on S_M is related to Clifford multiplication \cdot on S via $X \bullet \psi = iN \cdot X \cdot \psi$. Correspondingly, the connection ∇^S relates to the connection ∇^{S_M} , via

$$(\nabla_X^S \psi)_M = \nabla_X^{S_M}(\psi|_M) - \frac{i}{2} W^N(X) \bullet (\psi|_M). \tag{20}$$

The positive definite hermitian bundle metric (\cdot, \cdot) in S_M from structures on the spinor bundle is given explicitly by $(\varphi, \psi) := \langle N \cdot \varphi, \psi \rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product in S . With this setup, we show that when generalized Killing spinors are defined in this generality, type II imaginary generalized Killing spinors exist in abundance.

Theorem 25 *Let $(\overline{M}, \overline{g})$ be a time oriented Lorentzian spin^c manifold carrying a parallel spinor φ . Then, for any spacelike hypersurface in $(\overline{M}, \overline{g})$ with unit normal N , φ restricts to an imaginary generalized Killing spinor with Killing endomorphism given by $\frac{1}{2}$ times the Weingarten map of that hypersurface. If the Dirac current V_φ , with respect to φ on $(\overline{M}, \overline{g})$ is timelike, then the restricted spinor is of type II, and if it is lightlike, the restricted spinor is of type I.*

Proof By (20), we have that φ restricted to any hypersurface is an imaginary generalized Killing spinor ϕ with Killing endomorphism $\frac{1}{2}W^N$. Let U_ϕ denote the Dirac current on the hypersurface with respect to this restricted spinor. For any vector field X tangent to the hypersurface, we compute that

$$\begin{aligned}\overline{g}(U_\phi, X) &= i(\phi, X \bullet \phi) = i\langle N \cdot \varphi, iN \cdot X \cdot \varphi \rangle \\ &= \langle N \cdot N \cdot \varphi, X \cdot \varphi \rangle = \langle \varphi, X \cdot \varphi \rangle \\ &= -\overline{g}(V_\varphi, X).\end{aligned}$$

Further, we see that

$$\overline{g}(V_\varphi, N) = -\langle \varphi, N \cdot \varphi \rangle = -\langle N \cdot \varphi, \varphi \rangle = -(\phi, \phi),$$

and thus, along the hypersurface, we have that $V_\varphi = -U_\phi + (\phi, \phi)N$. If V_φ is timelike, we get that

$$0 > \overline{g}(V_\varphi, V_\varphi) = \overline{g}(U_\phi, U_\phi) + (\phi, \phi)^2 \overline{g}(N, N) = \overline{g}(U_\phi, U_\phi) - (\phi, \phi)^2,$$

which precisely means that φ restricts to a type II imaginary generalized Killing spinor on the hypersurface. Similarly, if V_φ is lightlike, the restricted spinor will be of type I. \square

Lorentzian spin^c manifolds carrying a parallel spinor such that the Dirac current is timelike, exists, and were studied in [28]. The second part of the above statement is essentially known and is a motivation for studying type I imaginary generalized Killing spinors; see for example, [33].

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