Synthetic Approaches to Lax Phenomena in Higher Category Theory

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Introduction

Higher Category Theory and Applications

The goal of category theory is to understand mathematical objects by studying the structure preserving morphisms between each other. Higher category theory extends this to examples where morphisms between morphisms are present, called 2-morphisms, and by iterating this concept even allowing for arbitrary n-morphisms. A special case is the theory of $(\infty, 1)$-categories, where all n-morphisms for $n \geq 2$ are invertible and which behave much like higher homotopies between continuous maps. $(\infty, 1)$-categories provide an efficient abstract theory capturing contexts where morphisms can be identified in many different ways and where one can formulate constructions that depend on these higher isomorphisms. Such higher isomorphisms do not just exist for topological spaces, but naturally also for other geometric objects such as schemes as well as for algebraic objects like rings and modules.

We may illustrate the usefulness of $(\infty, 1)$-category theory by a very incomplete list of prominent examples, it has been applied to geometric topology through embedding calculus by Krannich and Kupers [KK24], to number theory through the proof of Weil's conjecture on Tamagawa numbers over function fields by Lurie and Gaitsgory [GL19], to K-theory through the cyclotomic trace by Nikolaus and Scholze [NS18], to string topology by Naef and Safranov [NS24], and to stable homotopy theory through chromatic homotopy theory by Burklund, Hahn, Levy, Schlank, and Yuan among many others [Bur+23], [BSY22].

Generalising to (∞, n)-categories, i. e. allowing non-invertible k-morphisms until level n, the theory has been used in a variety of contexts. $(\infty, 2)$-categories appear in Lurie's treatment of Goodwillie Calculus [Lur09a] and in the construction of six functor formalisms by Gaitsgory and Rozenblyum [GR17], as well as Cnossen, Lenz and Linskens [CLL26], or Heyer and Mann [Man22], [HM24]. General (∞, n)-categories appear as the domain of n-dimensional Topological Quantum Field Theories in the form of bordism categories, whose k-morphisms are k-dimensional cobordisms, or as Morita categories of E_n-algebras, whose 1-morphisms for example are bimodules which are themselves E_{n-1}-algebras.

Lax Phenomena in Higher Category Theory

The crucial new aspect in the theory of (∞, n)-categories, which sets it apart from $(\infty, n-1)$Cat-enriched category theory, is the existence of so-called *lax phenomena*, i. e. situations where commutative diagrams are replaced by analogous diagrams containing possibly non-invertible higher morphisms.

On the practical side they appear in many areas of homotopy coherent mathematics and have been used to solve a variety of problems. To showcase this with another incomplete list of examples, let us start with the observation that they are not even exclusive to (∞, n)-category theory, but already appear in their simplest forms in the study of $(\infty, 1)$-categories. For example, suppose one has a natural transformation α between functors $F, G \colon C \to (\infty, 1)\mathrm{Cat}$ into the $(\infty, 1)$-category of $(\infty, 1)$-categories, such that each component of α is a left adjoint functor. The naturality squares for α will canonically give rise to a not-necessarily invertible natural transformation filling the square where the left adjoints are replaced by their right adjoints.

$$\begin{tikzcd} F(c) \arrow[d, "F(f)"'] & G(c) \arrow[l, "\alpha_c^R"'] \arrow[d, "G(f)"] \\ F(\tilde{c}) & G(\tilde{c}) \arrow[l, "\alpha_{\tilde{c}}^R"] \arrow[from=1-1, to=2-2, phantom, "\searrow"] \end{tikzcd}$$

This situation arises for example when considering morphisms between six-functor-formalisms, see [CLL26]. Such a transformation is an example of a *lax natural transformation*, an $(\infty, 2)$-categorical generalisation of natural transformations whose naturality squares do not commute anymore, but have only a directed 2-morphism like above. The notion of *lax (co)limit* derived from these generalised natural transformations naturally arises in the process of categorifying homological algebra and perverse sheaves, as can be seen for example in the work of Dyckerhoff and collaborators in [CDW24]. Lax natural transformations and their n-dimensional analogues are also the essential building blocks for the so-called *twisted quantum field theories* by Johnson-Freyd and Scheimbauer [JS17]. 2-Functors, lax natural transformations and the canonical transformations between those assemble into an $(\infty, 2)$-category $\mathrm{Fun}_{\mathrm{lax}}(A, B)$. These $(\infty, 2)$-categories are the internal Hom-objects for a monoidal structure on the $(\infty, 1)$-category $(\infty, 2)\mathrm{Cat}$ of $(\infty, 2)$-categories, called the *lax Gray tensor product* $A \otimes_{\mathrm{lax}} B$. In current work in progress, Keidar, Neuhauser and Yanovski observe that the product of cobordisms on the bordism categories naturally assembles into an algebra structure with respect to this lax Gray tensor product. Other examples of lax phenomena arise from the categorical study of homotopy-coherent algebra, through lax monoidal functors between monoidal $(\infty, 1)$-categories, as well as the theory of algebras of ∞-operads and ∞-monads themselves, which has been crucial in Lurie's treatment [Lur17] of Higher Algebra. These three notions, as well as their (∞, n)-categorical generalisations, admit a common framework through the notion of *lax functors* $A \rightsquigarrow B$ between higher categories. These have been used in the proof of a generalised chain rule in Goodwillie Calculus by Blans and Blom in [BB25].

In addition to having lax versions of (∞, n)-categorical definitions, one can also prove lax versions of the higher categorical theorems, such as the Yoneda Lemma, Straightening-Unstraightening and the reconstruction of adjoints via objectwise data.

On the theoretical side as a crude approximation to study (∞, n)-category theory in its entirety, one may factorise it into its *n-strict* part, by which we roughly mean everything attainable through the methods of $(\infty, n-1)\mathrm{Cat}$-enriched category theory, and its genuinely *n-dimensional* lax phenomena. However as we have seen above, there exist a lot of lax concepts that are interrelated to one another. In order to not have to treat each example on a case by case basis one may raise the question how they all relate and whether they adhere to a hierarchical order, where some lax phenomena might be more fundamental, or *primitive*, than others. For example the general concept of lax colimit seems to depend on the notion of lax natural transformation as its classical universal property characterises it as the initial lax cocone under the colimit diagram, and cocones themselves can be defined as natural transformations from constant functors to the diagram. On the other hand, in the theory of Straightening-Unstraightening, fundamental in type-theoretic approaches like Homotopy Type Theory [Pro13] and the synthetic $(\infty, 1)$-category theory of [Cis+25], the Grothendieck constructions turn out to be lax colimits of the corresponding functors into the $(\infty, n+1)$-category of (∞, n)-categories all along. In the hope that the study of the primitive lax phenomena lets one economically reconstruct the whole theory we formulate the following question.

Question 0.0.1 What are the *primitive* lax phenomena?

Answering this question might also help in dealing with the examples of lax phenomena outside of abstract category theory.

From the above list of examples two candidates can be identified, the notion of lax natural transformation and the lax functors. One coarse reason for this choice is that all concepts building on laxly commutative squares seem to be reducible to the former, and all algebraic examples, like lax monoidal functors and monads, seem to be reducible to the latter.

To make our separation of (∞, n)-category theory into $(\infty, n-1)$Cat-enriched category theory and lax phenomena more concrete, we also have to clarify what our primitive lax phenomena are based on. We know that $(\infty, n-1)$Cat-enriched category theory is based on $(\infty, n-1)$-category theory and the theory of enriched categories.

Question 0.0.2 Can we base primitive lax phenomena solely on $(\infty, n-1)$Cat-enriched category theory?

However, some lax constructions, like the lax generalisation of the arrow category for (∞, n)-categories, themselves seem to be so primitive to category theory that one might wonder if we even need the whole theory of enriched categories itself. To this end, let us switch gears to another approach to higher categories.

Synthetic (∞, n)-Category Theory and Formalisation

The theory of $(\infty, 1)$-categories has traditionally been developed using analytic approaches, i. e. through explicit combinatorial models such as quasi-categories [BV73], [Joy02], [Lur09b] or complete Segal spaces [Rez01], studied via model category theory and ordinary category theory. While these analytic foundations have been tremendously successful, there is a growing recognition that the way higher category theory is used in practice often differs dramatically from these set-level implementations. The homotopical behavior of the objects of interest is difficult to encode directly in analytic models, so researchers increasingly employ model-independent language that focuses on intrinsic universal properties rather than model-specific details. This problem is significantly amplified when trying to formalise homotopy-coherent mathematics in proof assistants based on classical set-theory.

This observation motivates a complementary, more *synthetic*, approach to higher category theory. Rather than constructing higher categories from lower-dimensional building blocks, a synthetic approach takes (∞, n)-categories as primitive objects and provides rules to manipulate them through *universal properties*, formulated in terms that are invariant under equivalence. This strategy builds naturally on Homotopy Type Theory [Pro13], which provides a synthetic treatment of ∞-groupoids, i. e. a framework where ∞-groupoids are the basic objects of study rather than being constructed from sets and combinatorics. Recent work, such as [Cis+25] by Cisinski-Cnossen-Nguyen-Walde, or [RS17] and [BW23] by Riehl, Shulman, Buchholtz and Weinberger, extends this program to $(\infty, 1)$-categories, developing type-theoretic foundations that both support easier computer formalisation in proof assistants and admit interpretation in the internal higher category theory of any $(\infty, 1)$-topos [Mar21], [MW21].

In order to arrive at a similarly formalisable theory of (∞, n)-categories one also needs synthetic universal properties capturing and relating the different lax phenomena of the previous section, which up to now have solely been treated via combinatorial descriptions in the literature, like in [Ver08a] and [GH25].

Synthetic Approaches to Lax Phenomena

In this sense, combining this last necessity statement with our Question 0.0.2 leads to the following question.

Question 0.0.3 Can we base primitive lax phenomena solely on primitive concepts of synthetic (∞, n)-category theory?

The hierarchy of our lax phenomena of course depends on our axiomatic foundation of higher category theory. So changing the axiom system to a synthetic approach, for example a type-theoretic one, will change also our viewpoint on how lax phenomena are related. However, as synthetic approaches are typically formulated through universal properties, trying to base primitive lax phenomena on the primitives of a synthetic theory only through universal properties might not just lead to new insights but hopefully also to more conceptual definitions.

The goal of this work is to showcase possible answers of Question 0.0.3 in the cases of the rough candidates we chose above, for dimension $n = 2$. More precisely, the synthetic universal properties we define will only refer to underlying object spaces $A^{\simeq} \to A$, Hom-functors $A(-,-)\colon (A^{\simeq})^{\mathrm{op}} \times A^{\simeq} \to (\infty,1)\mathrm{Cat}$ restricted to object spaces, the fact that 2-functors also induce actions on object spaces and Homs, and the existence of an $(\infty,1)$-category $(\infty,2)\mathrm{Cat}$ of $(\infty,2)$-categories.

In the case of lax functors, we choose their classifying objects, to be the objects endowed with an elementary synthetic universal property. The *lax functor classifier* of an $(\infty,2)$-category A for example, is another $(\infty,2)$-category $\mathrm{Lax}(A)$, such that we have an equivalence

$$\{\text{lax 2-functors} \quad A \rightsquigarrow B\} \qquad \simeq \qquad \{\text{2-functors} \quad \mathrm{Lax}(A) \to B\}$$

In the case of lax natural transformations, we also chose their classifier, namely the left adjoint lax Gray tensor product

$$\{\text{2-functors} \quad A \to \mathrm{Fun}_{\mathrm{lax}}(B, C)\} \qquad \simeq \qquad \{\text{2-functors} \quad A \otimes_{\mathrm{lax}} B \to C\}$$

Our approach now will be to *define* the left hand sides of these above correspondences by *defining* the classifying objects on the right hand side.

The reason we chose both times to formalise their classifying objects, is that we *can* endow both these classifiers with universal properties *not* relating to the left hand sides.

In the classical theory of ordinary 2-categories, the lax functor classifier $\mathrm{Lax}(A)$ as well as the lax Gray tensor product $A \otimes_{\mathrm{lax}} B$, have been constructed first by explicit generators and relations descriptions. However, from their classification equivalences from above these lax classifiers always admit actual 2-functors to their *strict* versions. In the case of lax functors, we have the 2-functor $\mathrm{Lax}(A) \to A$ corresponding to the identity lax functor on A, which is in fact strict. In the case of the lax Gray tensor product, we have the 2-functor $A \otimes_{\mathrm{lax}} B \to A \times B$ corresponding to the 2-functor $A \to \mathrm{Fun}(B, A \times B)$ given by the universal property of the product, postcomposed with the Hom-locally fully-faithful inclusion $\mathrm{Fun}(B, A \times B) \hookrightarrow \mathrm{Fun}_{\mathrm{lax}}(B, A \times B)$ of 2-natural transformations into lax natural transformations. The fundamental observation in this work now is that these comparison 2-functors $\mathrm{Lax}(A) \to A$ and $A \otimes_{\mathrm{lax}} B \to A \times B$ admit on Hom-$(\infty,1)$-categories

$$\mathrm{Lax}(A)(a, \tilde{a}) \to A(a, \tilde{a}) \quad \text{and} \quad (A \otimes_{\mathrm{lax}} B)((a,b),(\tilde{a},\tilde{b})) \to (A \times B)((a,b),(\tilde{a},\tilde{b}))$$

fully-faithful right adjoints. One may now observe furthermore that the data of these fully-faithful Hom-right adjoints generate all 1- and 2-morphisms of $\mathrm{Lax}(A)$, respectively $A \otimes_{\mathrm{lax}} B$. This raises the question whether the data of these *strictifying* 2-functors plus the property of having fully-faithful Hom-right adjoints already characterise both objects as freely generated from this data. In what sense they are *actually* freely generated by this data and properties is formalised in the following two definitions as initiality in certain $(\infty,1)$-categories, giving an affirmative answer to Question 0.0.3 for lax 2-functors and 2-lax Gray tensor products.

Definition (Definition 1.1.1)
Let A be an $(\infty,2)$-category. The lax 2-functor classifier of A is the initial $(\infty,2)$-category

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \end{tikzcd}$$

such that for every pair of objects $a,\widetilde{a}$ of $A^{\simeq}$, the induced functor

$$F_{U(a),U(\widetilde{a})}\colon X(U(a),U(\widetilde{a})) \to A(F(U(a)),F(U(\widetilde{a}))) \simeq A(a,\widetilde{a})$$

on Hom-$(\infty,1)$-categories admits a fully-faithful right adjoint. Morphisms between such triangles hereby have to commute with these Hom-adjoints.

Definition (Definition 3.4.13)
Let A,B be $(\infty,2)$-categories. The 2-lax Gray tensor product $A\otimes_{\mathrm{lax}} B$ of A and B is the initial $(\infty,2)$-category

$$\begin{tikzcd} A\times B^{\simeq} \underset{A^{\simeq}\times B^{\simeq}}{\cup} A^{\simeq}\times B \arrow[r, "U"] \arrow[dr, "\mathrm{can}"'] & X \arrow[d, "F"] \\ & A\times B \end{tikzcd}$$

such that for every pair of objects $a,\widetilde{a}$ of $A^{\simeq}$ and $b,\widetilde{b}$ of $B^{\simeq}$, the induced section of

$$F_{U(a,b),U(\widetilde{a},\widetilde{b})}\colon X(U(a,b),U(\widetilde{a},\widetilde{b})) \to (A\times B)(F(U(a,b)),F(U(\widetilde{a},\widetilde{b}))) \simeq A(a,\widetilde{a})\times B(b,\widetilde{b})$$

given by $U(-,\mathrm{id}_{\widetilde{b}})\circ U(\mathrm{id}_a,-)$ constitutes a right adjoint section.

In order to connect these two synthetic universal properties to the existing models we will first show that the canonical classifier of the particular combinatorial model of lax functors as *inert-cocartesian functors* satisfies our universal property of the lax 2-functor classifier. We will then use this result to bridge the gap between the models of the 2-lax Gray tensor product and our definition, by showing that the canonical classifier of the particular model based on so-called *cubical lax 2-functors* $A\times B\rightsquigarrow C$ satisfies our universal property.

Furthermore we will showcase the applicability of the synthetic universal property of the lax functor classifier by two instances. In the first instance we will redevelop synthetically a portion of higher categorical theory, namely a tiny bit of monad theory. Monad theory for higher categories has up to now only been developed based on ordinary 2-category theory by Riehl and Verity in [RV16], or based on combinatorial models of monoidal $(\infty,1)$-categories and ∞-operads by Lurie in [Lur17]. In the second instance we will introduce a lax upgrade of the usual objectwise existence criterion of adjoints to synthetic lax 2-functors. This allows one to also synthetically construct examples of lax functors in practice. To illustrate this we will give various examples of constructing lax and oplax functors with this objectwise existence criterion.

Organisation of this thesis

In chapter 1 we will start by motivating our synthetic universal property of the lax 2-functor classifier. Afterwards we introduce the model of lax 2-functors called *inert-cocartesian functors* and its corresponding model of $(\infty, 2)$-categories. Then we will recollect on the canonical classifying object for this combinatorial model of lax functors and prove that it satisfies our synthetic definition Definition 1.1.1. In chapter 2 we start by developing a tiny portion of synthetic monad theory. Exploiting the tight bond between monads and lax functors we introduce basic notions and apply our Definition 1.1.1 to synthetically access the archetypal source of monads, extracting them from adjunctions. We proceed to prove a lax 2-dimensional reincarnation of the objectwise existence criterion of adjoints, to synthetically construct examples of lax functors in an easy objectwise manner. To showcase this lax objectwise existence criterion we use it to construct examples of lax and oplax functors between span-2-categories, profunctor 2-categories, discuss how *deriving functors* leads to lax functoriality, as well as how it relates to other notions of lax 2-adjunctions. In chapter 3 we first prove a general result on how the extended universal property of the lax functor classifiers Definition 1.1.1 descends along Hom-adjunction unit localisations and derive a first synthetic universal property of the lax Gray product that relates with a particular model via the results from chapter 1. We then reformulate this rough first universal property twice, and finally arrive at the synthetic and concise universal property Definition 3.4.13.

Foundations

As the heart of this work are synthetic formulations of universal properties of $(\infty, 2)$-categorical notions it is desirable to make precise which model-independent aspects of $(\infty, 2)$-category theory it relies on. Furthermore, this also entails the need to be synthetic in our use of $(\infty, 1)$-category theory. To this end, we kept the interpretation of the aforementioned model of $(\infty, 2)$-categories and lax 2-functors between them, as well as the proofs involved in attaining Theorem 1.4.5 general and model-independent enough so that they will be interpretable, i. e. make sense, in any *synthetic* $(\infty, 1)$-category theory, such as the ones that are being developed by Cisinski, Cnossen, Nguyen and Walde in their book project [Cis+25], or by Riehl and Shulman in [RS17], and by Buchholtz and Weinberger in [BW23]. To this end, for 1-categories in this work we want to avoid specific set-theoretic implementations, such as model structures, and combinatorial descriptions of 1-categories, like complete Segal spaces, and the use of strict 1-categories, except for the n-simplices Δ^n, i. e. the finite non-empty totally ordered sets. Synthetic $(\infty, 1)$-category theory however means even more than this. It entails that our use of $(\infty, 1)$-category theory will be general enough to be also interpretable in the theory of internal higher categories in any $(\infty, 1)$-topos, as for example developed by Martini and Wolf in for example [Mar21], [MW21].

We refer the reader to Remark 1.2.1 for more details on the relevant model-independent aspects of $(\infty, 2)$-category theory we need for our synthetic universal properties specifically.

Let us now give a short overview of what more in-depth aspects of $(\infty, 2)$-category theory we need in our chapters separately. In chapter 1, we discuss at the end the interaction of lax functor classifiers with 2-categorical opposites, so we'll use these there. In chapter 2 we will make use of the 2-categorical theory of straightening-unstraightening, as well as the 2-Yoneda Lemma and its many incarnations, existence of the adjoints of the inclusions $\mathcal{S} \hookrightarrow (\infty, 1)\mathrm{Cat} \hookrightarrow (\infty, 2)\mathrm{Cat}$ on the level of $(\infty, 1)$-categories, opposites and strict $(\infty, 2)$-categorical left Kan extensions. In chapter 3 we only use the 2-Yoneda Lemma, and the fact that 2-morphism localisations exist. In Appendix A we give a more in-depth and thorough development of the necessary synthetic $(\infty, 1)$-categorical statements involved in the proof of Theorem 1.4.5, which will then also be useful in chapter 2, where we use them to interpret a version of the fibrational 1-Yoneda Lemma in $(\infty, 2)$-categorical presheaf $(\infty, 2)$-categories. Note also that we will use the ability of taking full or non-full, 1- and 2-categorical subcategories throughout in the main body, but not in Appendix A.

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Conventions and Notation

We will refer to $(\infty, 1)$-categories as just 1-*categories*, or even just *categories*. Furthermore we will refer to $(\infty, 2)$-categories as just 2-*categories*. Nevertheless, for intuition we will often adhere to the classical theory of strict $(2, 2)$- and $(1, 1)$-categories, we will refer to them as *ordinary* 2- and 1-categories, respectively.

Notation 0.0.4

We write $\mathrm{Fun}(A, B)$ for the category of functors from A to B. We will denote natural transformations α from a functor F to a functor G by $\alpha\colon F \Rightarrow G$ to distinguish them from ordinary morphisms. For a functor $F\colon A \to B$ we write $F_*\colon \mathrm{Fun}(T, A) \to \mathrm{Fun}(T, B)$ for the postcomposition with F and ${}_*F\colon \mathrm{Fun}(B, T) \to \mathrm{Fun}(A, T)$ for the precomposition with F. We write $\Delta_A\colon A \to \mathrm{Fun}(\Delta^1, A)$ for the precomposition functor ${}_*(\Delta^1 \to \Delta^0)$, and $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, A) \to A$ for the domain and the codomain functor, respectively. We denote the (∞-)category of small 1-categories by Cat, its full subcategory of small spaces, also called ∞-groupoids, by $\mathcal{S}$ and its full subcategory of simplices, i. e. non-empty finite totally ordered sets, by $\mathbf{\Delta}$. We write $(-)^{\simeq}\colon \mathrm{Cat} \to \mathcal{S}$ for the right adjoint to the inclusion functor $\mathcal{S} \hookrightarrow \mathrm{Cat}$, so that for a category A we get its underlying space of objects denoted by $A^{\simeq}$. We denote by $\mathrm{Map}(A, B) := \mathrm{Fun}(A, B)^{\simeq}$ the space of functors from A to B. For a cospan of functors $f\colon A \to C \longleftarrow B\colon g$ we sometimes write its pullback as $A \times_C^{(f,g)} B$ to emphasise the dependence on f and g, in contrast to the notation $A \times_C B$. When a functor $F\colon A \to B$ is left adjoint to a functor $U\colon B \to A$ we denote this by $F \dashv U$. Functors X to B that are cocartesian or cartesian fibrations will be denoted by $X \twoheadrightarrow B$. For a category A we write $\mathrm{comp}\colon \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \to \mathrm{Fun}(\Delta^1, A)$ for the composition functor of its morphisms. For a category A and on object a of A we write $A_{/a}$ for the slice category over a and ${}^{a/}A$ for the slice category under a. For two functors $p\colon A \to I$ and $q\colon B \to I$, seen as objects in $\mathrm{Cat}_{/I}$, we define the relative functor category over I from (A, p) to (B, q) via the following pullback square.

$$\begin{tikzcd}
\mathrm{Fun}_{/I}((A,p),(B,q)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(A,B) \arrow[d, "q_*"] \\
\Delta^0 \arrow[r, "p"'] & \mathrm{Fun}(A,I)
\end{tikzcd}$$

Chapter 1

A Synthetic Universal Property for the Lax 2-Functor Classifier

1.1 Introduction

Motivation

One of the strengths of (higher) category theory is the abstract formulation of universal properties as well as the ability to work with and reason about them. For a mathematical object to have a simple universal property often helps to reduce the complexity of problems involving this object. In this chapter we give a model-independent definition of the ubiquitous notion of lax 2-functors from 2-category theory by providing such a simple universal property for their classifying objects, not depending on a particular combinatorial model or the use of ordinary 2-category theory.

From the viewpoint of classical ordinary 2-category theory, lax 2-functors between 2-categories appear in a lot of places. Monad objects inside a 2-category $\mathcal{K}$ can be identified with lax functors $* \rightsquigarrow \mathcal{K}$ out of the one-object 2-category, lax monoidal functors $(\mathcal{V}, \otimes) \rightsquigarrow (\mathcal{W}, \otimes)$ are the same thing as lax functors $B\mathcal{V} \rightsquigarrow B\mathcal{W}$ between the monoidal categories viewed as one-object 2-categories $B\mathcal{V}$, respectively $B\mathcal{W}$ and even $\mathcal{V}$-enriched categories with object set X can be seen equivalent to lax functors $\mathrm{codisc}(X) \rightsquigarrow B\mathcal{V}$, out of the *codiscrete*, sometimes also *chaotic*, 2-category $\mathrm{codisc}\, X$ on the set X. On the other hand, lax 2-functors served as the basis for a vast generalization of the 1-categorical Grothendieck construction, functors $E \to C$ between ordinary categories having small fibers can be equivalently described as lax 2-functors $C \rightsquigarrow \mathrm{Span}(\mathrm{Set})$ into a certain 2-category of spans in the ordinary category Set of small sets. Furthermore, the notion of lax functors extends even deeper into the theory of 2-categories as one can identify 2-functors $A \otimes_{\mathrm{lax}} B \to C$ out of the lax Gray tensor product with special, sometimes called *cubical*, lax functors $A \times B \rightsquigarrow C$ out of the cartesian product of 2-categories.

Homotopy-coherent implementations of these viewpoints using combinatorial models have allowed us to access these structures in the realm of higher category theory. Lax monoidal functors between monoidal 1-categories for example are defined through Lurie's model of ∞-operads and maps between them, so-called inert-cocartesian functors, in [Lur17, Definition 2.1.2.7]. Homotopy-coherent monads on 1-categories were at first also modelled as algebras in monoidal 1-categories of endofunctors in [Lur17, Definition 4.7.0.1]. In [GH15, Definition 2.2.17] Gepner and Haugseng model categories enriched over a monoidal 1-category using a generalization of inert-cocartesian functors between non-symmetric generalized ∞-operads. In [Blo25, Theorem 4.1] Blom proves the generalised straightening-unstraightening equivalence mentioned above in the model of Segal objects in the 1-category of 1-categories and inert-cocartesian functors between those.

Prominently, the viewpoint on the lax Gray tensor product presented above is exploited in Gaitsgory and Rozenblyum's work on Derived Algebraic Geometry [GR17] to serve as a definition of the lax Gray tensor product in a model of 2-categories. Based on the lax Gray tensor product and now proven conjectures about its properties, see [LR25] for an overview of the work involved, they procede to prove many general 2-categorical results about constructing, extending and uniquely characterising homotopy-coherent six functor formalisms from less coherent input data.

A model-independent definition of lax 2-functors could complement these combinatorial descriptions by providing a common ground to compare them and to reason about them and the examples from an abstract point of view.

Intuition

Borrowing intuition from the classical theory of ordinary 2-categories, lax 2-functors are supposed to be a generalization of the usual notion of 2-functors between 2-categories. Usual 2-functors $F\colon A \to B$ are the most immediate realization of the idea of structure-preserving morphism of 2-categories. They have for example for each object a of A an identification $\mathrm{id}_{F(a)} \simeq F(\mathrm{id}_a)$, as well as for each 1-morphism composite $g \circ f$ in A an identification $F(g) \circ F(f) \simeq F(g \circ f)$, i. e. they preserve 1-morphism identities and compositions up to identification. A lax 2-functor F from A to B, denoted by $F\colon A \rightsquigarrow B$, on the other hand, relaxes these preservation properties to the structure of not-necessarily invertible 2-morphisms $\delta\colon \mathrm{id}_{F(a)} \Rightarrow F(\mathrm{id}_a)$ and $\gamma\colon F(g) \circ F(f) \Rightarrow F(g \circ f)$, but keeps their naturality. In order to have these 2-morphisms still behave in a unital and associative way one also needs to retain for example the following identifications of 2-morphisms

$$
\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[rr, bend right, "F(f)"'] & F(\tilde{a}) \arrow[r, "F(\mathrm{id}_{\tilde{a}})"'] \arrow[r, bend left=80, equal] \arrow[r, phantom, bend left=40, "\delta\Downarrow"] \arrow[d, phantom, "\gamma\Downarrow"] & F(\tilde{a}) \\
& {} &
\end{tikzcd}
\simeq
\begin{tikzcd}
F(a) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(f)"'] \arrow[r, phantom, "\Vert"] & F(\tilde{a})
\end{tikzcd}
\simeq
\begin{tikzcd}
F(a) \arrow[r, "F(\mathrm{id}_{\tilde{a}})"'] \arrow[r, bend left=80, equal] \arrow[r, phantom, bend left=40, "\delta\Downarrow"] \arrow[rr, bend right, "F(f)"'] & F(a) \arrow[r, "F(f)"'] \arrow[d, phantom, "\gamma\Downarrow"] & F(\tilde{a}) \\
& {} &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& F(a_1) \arrow[r, "F(g)"] \arrow[d, phantom, "\Downarrow\gamma"] & F(a_2) \arrow[dr, "F(h)"] & \\
F(a_0) \arrow[ur, "F(f)"] \arrow[urr, "F(g\circ f)"'] \arrow[rrr, bend right, "F(h\circ(g\circ f))"'] & {} & {} \arrow[phantom, "\Downarrow\gamma"] & F(a_3)
\end{tikzcd}
\simeq
\begin{tikzcd}
& F(a_1) \arrow[r, "F(g)"] \arrow[drr, "F(h\circ g)"'] & F(a_2) \arrow[dr, "F(h)"] \arrow[d, phantom, "\Downarrow\gamma"] & \\
F(a_0) \arrow[ur, "F(f)"] \arrow[rrr, bend right, "F((h\circ g)\circ f)"'] & {} \arrow[phantom, "\gamma\Downarrow"] & {} & F(a_3)
\end{tikzcd},
$$

which are also implicitely part of the ordinary notion of 2-functor between 2-categories. Let us remark here, that while for ordinary 2-categories these axioms are enough, they are by no means exhaustive in the realm of 2-categories, where e. g. also higher arity associativity needs to be taken into account as structure. Actual 2-functors of 2-categories should still be examples of these more general kinds of 2-functors. In fact, they are then the lax 2-functors that are strict, in the sense that the 2-morphisms δ and γ are invertible.

In ordinary 2-category theory, lax 2-functors can furthermore be classified by certain 2-functors in the following sense. For every 2-category A there exists a 2-category $\mathrm{Lax}(A)$, which we call the lax 2-functor classifier of A, together with a lax 2-functor $\iota\colon A \rightsquigarrow \mathrm{Lax}(A)$ which is supposed to be the initial lax 2-functor out of A, i. e. every lax 2-functor $F\colon A \rightsquigarrow B$ into another 2-category

B can be factored uniquely as

$$A \overset{\iota}{\rightsquigarrow} \mathrm{Lax}(A) \xrightarrow{\overline{F}} B\,,$$

that is a strict 2-functor $\overline{F}\colon \mathrm{Lax}(A) \to B$, precomposed with $A \rightsquigarrow \mathrm{Lax}(A)$. We say that the 2-functor $\overline{F}\colon \mathrm{Lax}(A) \to B$ classifies the lax 2-functor F. Thus, in order to define the notion of lax 2-functors it suffices to define the lax 2-functor classifiers and consider strict 2-functors out of them. As usual 2-functors should also be considered as lax 2-functors, they can be classified too. In the case of the identity 2-functor $\mathrm{id}\colon A \to A$ we obtain the strict 2-functor $\lambda \coloneqq \overline{\mathrm{id}}\colon \mathrm{Lax}(A) \to A$.

Definition

To generalise lax 2-functors into the realm of 2-category theory we now introduce a model-independent universal property for the 2-category $\mathrm{Lax}(A)$ with respect to its strict 2-functor $\lambda\colon \mathrm{Lax}(A) \to A$. More precisely, it will not depend on combinatorial descriptions or the use of ordinary 2-category theory, but solely rely on basic abstract features of 2-categories and 1-category theory.

Definition 1.1.1

Let A be a 2-category. A commutative triangle of 2-functors

$$\begin{tikzcd}
A^{\simeq} \arrow[r, "U"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\
& A\,,
\end{tikzcd}$$

where $\gamma\colon A^{\simeq} \to A$ denotes the inclusion of the underlying space of objects, is said to have local right adjoint sections if it satisfies that for every pair of objects $a, \tilde{a}$ of $A^{\simeq}$, the induced functor

$$F_{U(a),U(\tilde{a})}\colon X(U(a),U(\tilde{a})) \to A(F(U(a)),F(U(\tilde{a}))) \simeq A(a,\tilde{a})$$

on Hom-1-categories admits a fully-faithful right adjoint. For a commutative tetrahedron

$$\begin{tikzcd}
& A^{\simeq} \arrow[dl, "U"'] \arrow[dr, "V"] \arrow[dd, bend left=10, "\gamma" near end] & \\
X \arrow[rr, "H" near start] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\
& A &
\end{tikzcd}$$

between two such 2-functors F and G having local right adjoint sections with respect to U, respectively V, we say that it commutes with the local right adjoint sections if the induced square in the middle of

$$\begin{tikzcd}
A(FU(a),FU(\tilde{a})) \arrow[r, hook, dashed, "U_{a,\tilde{a}}"] \arrow[dr, equal, "\varepsilon\simeq"'] & X(U(a),U(\tilde{a})) \arrow[r, "H_{U(a),U(\tilde{a})}"] \arrow[d, "F_{U(a),U(\tilde{a})}"] & Y(HU(a),HU(\tilde{a})) \arrow[d, "G_{V(a),V(\tilde{a})}"'] \arrow[dr, equal] \arrow[d, phantom, "\Leftarrow\eta" near end, shift left=40] & \\
& A(FU(a),FU(\tilde{a})) \arrow[r, "\simeq"] & A(GHU(a),GHU(\tilde{a})) \arrow[r, hook, dashed, "V_{a,\tilde{a}}"'] & Y(HU(a),HU(\tilde{a}))
\end{tikzcd}$$

is vertically right adjointable, i. e. if the above composite 2-morphism is invertible. These two conditions define a non-full subcategory $\mathrm{LocRARI}(A)$ of the strict slice under 1-category ${}^{\gamma/}(2\text{-}\mathrm{Cat}_{/A})$,

i. e. the 1-categorical slice under γ of the strict slice 1-category $2\text{-Cat}_{/A}$ of 2-categories over A. A lax 2-functor classifier of A is an initial object $\lambda\colon \mathrm{Lax}(A) \to A$ in $\mathrm{LocRARI}(A)$. We will also call this initiality the extended universal property of the lax 2-functor classifier. Asking for fully-faithful Hom-left-adjoints we can entirely similarly define a non-full subcategory $\mathrm{LocLALI}(A)$ and the oplax 2-functor classifier of A, denoted by $\mathrm{OpLax}(A)$.

To illustrate this definition with an example, let us denote the Hom-local right adjoints of $\lambda_{a,\widetilde{a}}$ by $\iota_{a,\widetilde{a}}$. Then the units of these Hom-adjunctions will provide us with the desired identity comparison 2-cells $\mathrm{id}_a \Rightarrow \iota_{a,a}(\lambda_{a,a}(\mathrm{id}_a)) \simeq \iota_{a,a}(\mathrm{id}_a)$ and composition comparison 2-cells

$$\iota_{a_1,a_2}(g) \circ \iota_{a_0,a_1}(f) \Rightarrow \iota_{a_0,a_2}(\lambda_{a_0,a_2}(\iota_{a_1,a_2}(g) \circ \iota_{a_0,a_1}(f))) \simeq \iota_{a_0,a_2}(g \circ f) \ .$$

Comparison to a combinatorial model

The main goal of this chapter is to review an existing model-dependent definition of lax functors, discussed for example in [GR17, Chapter 10.3], as well as the construction of their classifiers, and to show that these classifiers satisfy our universal property. More precisely, the model is obtained by starting with the so-called complete 2-fold Segal space model for 2-categories from [BS21], and then unstraightening these complete Segal objects X in the 1-category of small 1-categories with X_0 a space to cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$. In this model, the cocartesian functors play the role of strict 2-functors. By restricting the class of morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to the so-called inert simplex maps, one obtains a definition of lax functors between globular complete Segal cocartesian fibrations by considering functors over $\mathbf{\Delta}^{\mathrm{op}}$ that are only required to preserve cocartesian lifts along the inert simplex maps. Inert-cocartesian functors out of an 2-category A, modelled this way, also admit an explicitely constructed classifier, the envelope $\mathrm{Env}(A)$ of A, together with a distinguished strict 2-functor $\lambda\colon \mathrm{Env}(A) \to A$ classifying the identity inert-cocartesian functor on A, which happens to be an equivalence on underlying spaces of objects.

This also already appeared in the literature for example as [Aya+24]. Our main goal is then made precise by Theorem 1.4.5, or, to put it shortly, by

Theorem

The 2-functors $A^{\simeq} \xrightarrow{\iota^{\simeq}} \mathrm{Env}(A) \xrightarrow{\lambda} A$ satisfy the extended universal property of the lax functor classifier in the sense of Definition 1.1.1.

Organisation of the chapter

In section 1.2 we start by recalling the aforementioned model of 2-categories then make precise in what sense the above introduced universal property of the lax 2-functor classifier is actually synthetic and on what aspects of 2-category theory it relies on. Afterwards, we introduce the tractable model of lax 2-functors therein, given by the inert-cocartesian functors. Then we proceed to give the explicit classifier construction that exists for these inert-cocartesian functors and discuss some of its relevant properties.

In section 1.3 we start by relating the synthetic notions of Definition 1.1.1, i. e. of 2-functors having local right adjoint sections and 2-functors commuting with them, to more model-dependent counterparts and then finally prove that the envelope satisfies a simpler version of the lax functor classifier universal property, i. e. Definition 1.3.1 in Theorem 1.3.8.

In section section 1.4 we prove that the envelope construction also satisfies the extended universal property of the lax functor classifier Definition 1.1.1.

In the Appendix A we prove the relevant statements from 1-category theory that the other sections rely on by developing the necessary background material on adjunctions and cocartesian fibrations in more detail, starting with the definitions.

1.2 Inert-Cocartesian Functors and the Envelope Construction

In this section we will start with a recollection of a particular model of 2-categories, one closely related to the notion of complete 2-fold Segal space, which is also the one used in [GR17, Appendix on $(\infty, 2)$-categories]. Afterwards, we will introduce a model of lax 2-functors for this model of 2-categories, which is the one used in [GR17, Chapter 10.3] to define and work with the lax Gray tensor product. For this model there exists an explicit construction of its classifying object, for which we will later prove the model-independent universal property. The construction of the classifying objects in this generality already appeared for example in [GH15]. Lastly we will show that this classifier exhibits several properties which are very similar to the ones used in Definition 1.1.1 of a 2-functor having local right adjoint sections. All these statements can also be found in [GR17, Chapter 11.A].

1.2.1 Globular Complete Segal Cocartesian Fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ and Inert-Cocartesian Functors

It is a classical result, for example discussed in [BS21], that one can model 2-categories as complete 2-fold Segal spaces, defined as certain bisimplicial objects in spaces. For our purposes it is actually more convenient to use the equivalent formulation as simplicial objects $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ in the category Cat of (small) 1-categories, which are Segal, complete and globular, i. e. satisfy the extra assumption that X_0 is a space. The corresponding notion of 2-functors in this model is simply given by natural transformations.

Using the Straightening-Unstraightening equivalence

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}) \simeq \mathrm{Cocart}(\mathbf{\Delta}^{\mathrm{op}})$$

between functors to Cat and cocartesian fibrations we can alternatively describe these special simplicial objects in Cat as cocartesian fibrations $X \longrightarrow\!\!\!\!\!\rightarrow \mathbf{\Delta}^{\mathrm{op}}$ over the category $\mathbf{\Delta}^{\mathrm{op}}$ which satisfy equivalently expressed Segal axioms, completeness axioms and satisfy that the fiber X_0 over Δ^0 is a space. Let us denote this category by $\mathrm{CSCocart}^{\mathrm{glob}}(\mathbf{\Delta}^{\mathrm{op}})$ for later reference. Here, the corresponding role of 2-functors is played by the cocartesian functors between such globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$.

To give some intuition, let us take a globular complete Segal object $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$, or equivalently unstraightened as a cocartesian fibration $p\colon X \longrightarrow\!\!\!\!\!\rightarrow \mathbf{\Delta}^{\mathrm{op}}$. We interpret the space X_0 as the space of objects $X^{\simeq}$ of the 2-category X. We think of X_1 as the category of 1-morphisms and 2-cells between those, and the two evaluations functors $X_0 \leftarrow X_1 \rightarrow X_0$ as the source and target functors i. e. the fibers of $(s, t)\colon X_1 \to X_0 \times X_0$ over a pair of objects (a, b) should exactly model the Hom-1-categories $X(a, b)$ varying appropriately in both variables over the space of objects $X^{\simeq} := X_0$. The functor $X_0 \to X_1$ maps objects to their identity 1-morphisms. By the Segal axiom we can identify $X_2 \simeq X_1 \times_{X_0} X_1$. In the fibrational viewpoint the composition of two 1-morphisms is then given by cocartesian pushforward along $(0, 2)\colon \Delta^1 \hookrightarrow \Delta^2$ in $\mathbf{\Delta}^{\mathrm{op}}$.

We now continue by reviewing what we actually need to know about 2-categories and 2-category theory in order to model-independently define our universal property Definition 1.1.1. Essentially, we require the existence of the 1-category 2-Cat of 2-categories, and our universal property will be stated entirely in terms of basic properties of 2-Cat, that do not depend on how it was defined.

Remark 1.2.1

We will only use the following basic ingredients from 2-category theory. Note that this list is by no means exhaustive, i. e. it does not intend to capture all of 2-category theory, but it should be a minimal subset with respect to Definition 1.1.1 that is interpretable in any model of 2-category theory, not just the above introduced complete 2-fold Segal spaces or the globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$.

1. We have a 1-category 2-Cat of all (small) 2-categories, (strict) 2-functors between them, 2-natural isomorphisms between those and so on.
2. Every 2-category A has an underlying space of objects $A^{\simeq}$. More precisely, this assignment should be functorial and $(-)^{\simeq}\colon \text{2-Cat} \to \mathcal{S}$ is supposed to be the right adjoint to the fully-faithful inclusion $\mathcal{S} \hookrightarrow \text{2-Cat}$ of spaces into 2-categories. In particular we get that every (strict) 2-functor $F\colon A \to B$ induces a functor on underlying spaces of objects $F^{\simeq}\colon A^{\simeq} \to B^{\simeq}$.
3. Every 2-category A has Hom-1-categories, parametrized by the product of its space of objects with the dual of itself

$$A(-,-)\colon (A^{\simeq})^{\mathrm{op}} \times A^{\simeq} \to \mathrm{Cat}\ .$$

   For our purposes, we do not need further functoriality in non-invertible 1- or 2-morphisms.
4. Every (strict) 2-functor $F\colon A \to B$ induces 1-functors on the Hom-1-categories, natural in the space of objects of A, $F_{-,-}\colon A(-,-) \Rightarrow B(F^{\simeq}(-), F^{\simeq}(-))$. We will usually abuse notation and just write $B(F(-), F(-))$ for the codomain of these functors.
5. For every triangle of 2-functors

$$\begin{tikzcd} A \arrow[rr, "H"] \arrow[dr, "F"'] & & B \arrow[dl, "G"] \\ & C & \end{tikzcd} \tag{1.2.1.1}$$

   commuting by a 2-natural isomorphism we already know that we get a commutative triangle of functors on object spaces

$$\begin{tikzcd} A^{\simeq} \arrow[rr, "H"] \arrow[dr, "F"'] & & B^{\simeq} \arrow[dl, "G"] \\ & C^{\simeq} & \end{tikzcd} \tag{1.2.1.2}$$

   and we want this to also extend to commutative squares of actions on Hom-categories

$$\begin{tikzcd} A(a,\widetilde{a}) \arrow[r, "H"] \arrow[d, "F"'] & B(Ha, H\widetilde{a}) \arrow[d, "G"] \\ C(Fa, F\widetilde{a}) \arrow[r, "\simeq"'] & C(GHa, GH\widetilde{a})\,, \end{tikzcd} \tag{1.2.1.3}$$

   natural in the space of pairs $(a, \widetilde{a})$ of objects of A, more precisely in space $(A^{\simeq})^{\mathrm{op}} \times A^{\simeq}$.

6. Additionally, the last three points can also be made coherent by asking for a section of the large categorical presheaf $\mathrm{Fun}(((-)^{\simeq})^{\mathrm{op}} \times (-)^{\simeq}, \mathrm{Cat})\colon 2\text{-}\mathrm{Cat}^{\mathrm{op}} \to \mathrm{CAT}$, although we will not need this further coherence.

Note in particular, that these ingredients are agnostic to whether one implements 2-categories as so-called Cat-enriched categories or vertically trivial and complete internal category objects in Cat, which, in a combinatorial way we do here. So the model-independent universal property of the lax 2-functor classifier of Definition 1.1.1 will also be agnostic to these choices.

Since our main theorem is the comparison of the model-independent universal property Definition 1.1.1 and the aforementioned model-dependent definition of lax 2-functors between unstraightened globular complete Segal objects in Cat, we will need to instantiate the above list of 2-categorical ingredients in this model. Nevertheless for the proof of our main theorem we still avoid specific set-theoretic implementations, such as model structures, combinatorial descriptions of 1-categories, like complete Segal spaces, and the use of ordinary 1-categories.

Proof. 1. As outlined in subsection 1.2.1, we obtain a 1-category of our combinatorial model of 2-categories by starting with the functor 1-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat})$ and take its full subcategory of complete Segal objects X such that X_0 is a space. By unstraightening this is equivalent to $\mathrm{CSCocart}^{\mathrm{glob}}(\mathbf{\Delta}^{\mathrm{op}})$.

2. In this model the adjunction for underlying spaces of objects is obtained by restricting the adjunction

$$\begin{tikzcd} \mathrm{Cat} \arrow[rr, bend left, hook, "\mathrm{disc}"] \arrow[rr, phantom, "\perp"] && \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}) \arrow[ll, bend left, "\mathrm{ev}_0"] \end{tikzcd} \tag{1.2.1.4}$$

to the left hand side to the full subcategory on spaces and the right hand side to the full subcategory on globular complete Segal objects. This means we define $X^{\simeq} := X_0$.

3. For the Hom-1-categories and their functoriality and their functoriality as outlined in the last point we start with the cospan $\Delta^0 \xrightarrow{0} \Delta^1 \xleftarrow{1} \Delta^0$ in $\mathbf{\Delta}$ and apply $\mathrm{Fun}((-)^{\mathrm{op}}, \mathrm{Cat})$ to get

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}) \times_{\mathrm{Cat}}^{(\mathrm{ev}_0, \mathrm{incl})} \mathcal{S} \xrightarrow{\mathrm{ev}_0 \leftarrow \mathrm{ev}_1 \rightarrow \mathrm{ev}_0} \mathcal{S} \times_{\mathrm{Cat}} \mathrm{Fun}(\{2 \leftarrow 0 \rightarrow 1\}, \mathrm{Cat}) \times_{\mathrm{Cat}} \mathcal{S} \ ,$$

which takes a simplicial object $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ with X_0 a space and associates to it its source-target span $X^{\simeq} := X_0 \xleftarrow{s} X_1 \xrightarrow{t} X_0 =: X^{\simeq}$. We may think of the fibers of this span at pairs of objects (x, y) of $X^{\simeq}$ as the Hom-categories $X(x, y)$ of X in this model. To turn this into the shape of an actual Hom-functor parametrised by $(X^{\simeq})^{\mathrm{op}} \times X^{\simeq}$ we need to straighten this span appropriately. The codomain of this functor, i. e. the category of spans $Y \leftarrow C \to Z$ with Y and Z spaces, is itself equivalent to the total category of the unstraightening of the functor

$$\mathrm{Cat}_{/(-)_1 \times (-)_2}\colon \mathcal{S} \times \mathcal{S} \xrightarrow{\times} \mathcal{S} \hookrightarrow \mathrm{Cat} \xrightarrow{\mathrm{Cat}_{/(-)}} \mathrm{CAT} \ .$$

To this we first apply the cartesian straightening equivalence $\mathrm{Str}^{\mathrm{cart}}$, in the special case over spaces Y, and then apply the cocartesian straightening, also in the special case over spaces Z, i. e.

$$\mathrm{Str}_Y^{\mathrm{cart}}\colon \mathrm{Cat}_{/Y} \simeq \mathrm{Fun}(Y^{\mathrm{op}}, \mathrm{Cat}) \qquad \text{and} \qquad \mathrm{Str}_Z^{\mathrm{cocart}}\colon \mathrm{Cat}_{/Z} \simeq \mathrm{Fun}(Z, \mathrm{Cat})$$

to get the chain of equivalences

$$\begin{tikzcd}
\mathrm{Cat}_{/(-)_1\times(-)_2} \arrow[d, "\simeq"'] & & \mathrm{Fun}((-)_1^{\mathrm{op}}, \mathrm{Cat}_{/(-)_2}) \arrow[d, "(\mathrm{Str}^{\mathrm{cocart}})_*"] \\
(\mathrm{Cat}_{/(-)_1})_{/(\mathrm{pr}_1 \colon (-)_1\times(-)_2\to(-)_1)} \arrow[d, "(\mathrm{Str}^{\mathrm{cart}})_{/\,\mathrm{pr}_1}"'] & & \mathrm{Fun}((-)_1^{\mathrm{op}}, \mathrm{Fun}((-)_2, \mathrm{Cat})) \arrow[d, "\simeq"] \\
\mathrm{Fun}((-)_1^{\mathrm{op}}, \mathrm{Cat})_{/\,\mathrm{const}_{(-)_2}} \arrow[uurr, "\simeq"] & & \mathrm{Fun}((-)_1^{\mathrm{op}} \times (-)_2, \mathrm{Cat}))
\end{tikzcd}$$

that realize the desired straightening to functors into Cat with the correct variances.

□

Next we review a model of lax functors between 2-categories modelled as globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$. This is done by relaxing the cocartesian lift preservation of functors over $\mathbf{\Delta}^{\mathrm{op}}$ between such fibrations. For this we will now define two special subclasses of morphisms in $\mathbf{\Delta}$.

Definition 1.2.2

Let $\alpha\colon \Delta^n \to \Delta^m$ be a simplex morphism.

1. α is called active if $\alpha(0) = 0$ and $\alpha(n) = m$.
2. α is called inert if $\forall i \in \{0, \dots, n-1\}\colon \alpha(i+1) = \alpha(i) + 1$.

Remark 1.2.3

At least under additional knowledge, which we will exemplify now, on how to characterise functors into simplices Δ^n, these two classes of morphisms constitute a factorisation system on $\mathbf{\Delta}$. For example if we know that $\mathrm{Map}(C, \Delta^n)$ is actually equivalent to the subspace of $\mathrm{Map}(C^{\simeq}, \{0, \dots, n\})$ on those functors $F\colon C^{\simeq} \to \{0, \dots, n\}$ that satisfy for all morphisms $f\colon c \to \tilde{c}$ in C that $F(c) \leq F(\tilde{c})$ with respect to the poset structure $\{0 < \cdots < n\}$ on $\{0, \dots, n\}$, one can deduce the factorisation system. Explicitely, for a simplex morphism $\alpha\colon \Delta^n \to \Delta^m$ we can then factorise it as

$$\Delta^n \xrightarrow{\text{active}} \Delta^{\{\alpha(0),\dots,\alpha(n)\}} \xhookrightarrow{\text{inert}} \Delta^m \ ,$$

where we denoted by $\Delta^{\{\alpha(0),\dots,\alpha(n)\}}$ the convex subsimplex of Δ^m spanned by the vertices of Δ^m between $\alpha(0)$ and $\alpha(n)$.

A model of lax functors in this situation is now given by looking at functors over $\mathbf{\Delta}^{\mathrm{op}}$ that are only required to preserve cocartesian lifts along (opposites of) inert morphisms.

Recollection 1.2.4

As expressed in [GR17, Chapter 10.3], this unstraightened model of 2-categories now supports a very tractable model of lax 2-functors, the inert-cocartesian functors, i. e. the functors between cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, two heads] & & B \arrow[dl, two heads] \\
& \mathbf{\Delta}^{\mathrm{op}} ,
\end{tikzcd} \tag{1.2.1.5}$$

which preserve only the cocartesianness of cocartesian lifts of inert simplex morphisms, or more compactly, which are cocartesian functors after pulling back along the non-full subcategory inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inert}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ on inert simplex morphisms. For this model of lax 2-functors there is also an explicit classifier, generalizing the monoidal envelope construction as for example in [Lur17, Section 2.2.4].

Example 1.2.5

The simplest examples of simplex morphisms which are not inert are $\Delta^1 \to \Delta^0$ and the inclusion $(0,2)\colon \Delta^1 \hookrightarrow \Delta^2$. So for an inert-cocartesian functor

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, two heads] & & B \arrow[dl, two heads] \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd} \tag{1.2.1.6}$$

between 2-categories the fact that F does not necessarily preserve cocartesian lifts along these morphisms gives us in the case of $\Delta^1 \to \Delta^0$ a not necessarily invertible comparison morphisms

$$\mathrm{id}_{F(x)} = (\Delta^1 \to \Delta^0)_! F(x) \to F((\Delta^1 \to \Delta^0)_! x) = F(\mathrm{id}_x)$$

and in the case for $(0,2)\colon \Delta^1 \hookrightarrow \Delta^2$ not necessarily invertible comparison morphisms

$$F(g) \circ F(f) = (0,2)_! F(f,g) = (0,2)_! (F(f), F(g)) \to F((0,2)_!(f,g)) = F(g \circ f)$$

in the category X_1, which we interpret as 2-cells between 1-morphisms.

1.2.2 The Envelope Construction and its Fundamental Properties

We now turn our attention to an explicit construction of classifying objects for this model of lax functors. As not only the definition of inert-cocartesian functors but also their classifier also work in the greater generality of just Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ (which in turn are models for double ∞-categories). We will also adopt this generality for the following construction and statements when applicable.

Definition 1.2.6

Let us denote by $\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ the full subcategory of $\mathrm{Fun}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ on all the functors which correspond to active simplex morphisms. Let $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ be a Segal cocartesian fibration. Then its envelope construction is defined to be the functor

$$\mathrm{ev}_1\colon \mathrm{Env}(A) \coloneqq A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathrm{Fun}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \xrightarrow{*(\Delta^0 \xrightarrow{1} \Delta^1)} \mathbf{\Delta}^{\mathrm{op}} .$$

Proposition 1.2.7

Let $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ be a Segal cocartesian fibration. Then its envelope $\mathrm{ev}_1\colon \mathrm{Env}(A) \to A$ is again a cocartesian fibration and Segal.

Reference. This already appeared as [GH15, Proposition A.1.2]. ■

The following proposition will introduce the model of the, soon to be proven initial, lax functor $\iota\colon A \rightsquigarrow \mathrm{Lax}(A)$ out of A, as well as establish completeness of the envelope construction, even in the generality of starting with a double ∞-category, which is modelled here by a Segal cocartesian fibration.

Proposition 1.2.8

Let $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ be a Segal cocartesian fibration. Then the envelope construction

$$\mathrm{ev}_1\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathbf{\Delta}^{\mathrm{op}}$$

is always also complete. Furthermore, the functor ι defined as

$$A \simeq A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{id}_A \times \mathrm{id}_{\Delta(\mathbf{\Delta}^{\mathrm{op}})}} A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$$

is an inert-cocartesian functor over $\mathbf{\Delta}^{\mathrm{op}}$ and an equivalence on fibers over Δ^0. Hence if A_0 is a space, then so is $(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_0$. In particular, if $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ models a 2-category in the sense that it is Segal, complete and globular, i. e. has as fiber over Δ^0 a space, then so does its envelope $\mathrm{ev}_1\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathbf{\Delta}^{\mathrm{op}}$.

Proof. 1. To check completeness we first need to know how cocartesian lifting in

$$\mathrm{ev}_1\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathbf{\Delta}^{\mathrm{op}}$$

works. We extract the necessary parts for this from the aforementioned [GH15, Proposition A.1.2] to make it explicit now. Another excellent reference for the cocartesian lifting of these kinds of fibrations is [Aya+24, Proposition A.0.1]. Let us start with an arbitrary object in $A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ in the fiber over an arbitrary Δ^n, i. e. a tuple consisting of an active simplex morphism $\alpha\colon \Delta^n \to \Delta^m$ in $\mathbf{\Delta}$, viewed as a morphism $\Delta^m \to \Delta^n$ in $\mathbf{\Delta}^{\mathrm{op}}$, and an object σ in the fiber over Δ^m. Let us furthermore be given another arbitrary simplex morphism $\gamma\colon \Delta^k \to \Delta^n$, viewed as a morphism $\Delta^n \to \Delta^k$ in $\mathbf{\Delta}^{\mathrm{op}}$, along which we want to cocartesian pushforward. From now on we will only talk about simplex morphisms, i. e. morphisms in $\mathbf{\Delta}$, but will implicitely use them as morphisms in $\mathbf{\Delta}^{\mathrm{op}}$. The first step is to factorize the composite $\alpha \circ \gamma$ into an active followed by an inert simplex morphism

$$\begin{tikzcd} \Delta^k \arrow[r, "\beta", "\text{active}"'] \arrow[d, "\gamma"'] & \Delta^l \arrow[d, "\text{inert}"', "\theta"] \\ \Delta^n \arrow[r, "\text{active}", "\alpha"'] & \Delta^m \end{tikzcd} \tag{1.2.2.1}$$

using the corresponding unique factorization system. Then the ev_1-cocartesian pushforward of (α, σ) is exactly $(\beta, \theta_!\sigma)$, where $\theta_!\sigma$ is the p-cocartesian pushforward of the object σ along the morphism θ. For our discussion of completeness let us take an arbitrary 2-simplex (α, σ) in $A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ such that its cocartesian pushforward along $(0, 2)\colon \Delta^1 \to \Delta^2$ is degenerated from a 0-simplex $(\mathrm{id}_{\Delta^0}, a)$. Note that the identity of Δ^0 is the only active map out of Δ^0.

This means explicitely that for the unique active-inert factorization

$$\begin{tikzcd} \Delta^1 \arrow[r, "{(\alpha(0),\alpha(2))}", "\text{active}"'] \arrow[d, "{(0,2)}"'] & \Delta^m \arrow[d, "\text{inert}"', "\text{id}"] \\ \Delta^2 \arrow[r, "\text{active}", "\alpha"'] & \Delta^m \end{tikzcd} \tag{1.2.2.2}$$

and the p-cocartesian pushforward $\mathrm{id}_! \, \sigma = \sigma$, we have that the cocartesian pushforwarded tuple $((\alpha(0), \alpha(2)), \sigma)$ is equivalent to the degeneration of $(\mathrm{id}_{\Delta^0}, a)$, i. e. via the factorization

$$\begin{tikzcd} \Delta^1 \arrow[r, "\text{active}"'] \arrow[d] & \Delta^0 \arrow[d, "\text{inert}"', "\text{id}"] \\ \Delta^0 \arrow[r, "\text{active}", "\text{id}"'] & \Delta^0 \, , \end{tikzcd} \tag{1.2.2.3}$$

which is the tuple $(\Delta^1 \to \Delta^0, a)$. But this means on the one hand side that

$$\begin{tikzcd} \Delta^1 \arrow[r, "{(\alpha(0),\alpha(2))}"] \arrow[dr] & \Delta^m \arrow[d, "\simeq"] \\ & \Delta^0 \, , \end{tikzcd} \tag{1.2.2.4}$$

i. e. that $m = 0$, and that on the other hand side $\sigma \simeq a$. But put together this says that the 2-simplex (α, σ) itself is a degeneration of the 0-simplex $(\mathrm{id}_{\Delta^0}, a)$. Hence we just proved that every 2-simplex with degenerate $0 \to 2$ edge is already degenerate itself. Hence so will be every invertible 1-simplex. In particular the pushforward along $\Delta^1 \to \Delta^0$ functor

$$(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_0 \to (A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_1^{\mathrm{invert}} \, ,$$

whose codomain is restricted to the full subspace on the invertible morphisms, is essentially surjective. We have also

$$(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_0 \simeq A \times_{\mathbf{\Delta}^{\mathrm{op}}} \{\Delta^0 \xrightarrow{\mathrm{id}} \Delta^0\} \, .$$

As seen above, the pushforward along $\Delta^1 \to \Delta^0$ functor factorizes as

$$\begin{tikzcd} A \times_{\mathbf{\Delta}^{\mathrm{op}}} \{\Delta^0 \xrightarrow{\mathrm{id}} \Delta^0\} \arrow[r, "\simeq"'] \arrow[rr, bend left, "(\Delta^1 \to \Delta^0)_!"] & A \times_{\mathbf{\Delta}^{\mathrm{op}}} \{\Delta^1 \to \Delta^0\} \arrow[r, hook] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] & (A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_1 \arrow[d, "(\mathrm{pr}_1)_1"] \\ & \{\Delta^1 \to \Delta^0\} \arrow[r, hook] & (\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_1 \, , \end{tikzcd}$$

where the square is a pullback by definition. The functor including the object

$$\{\Delta^1 \to \Delta^0\} \hookrightarrow (\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}))_1$$

is in fact fully-faithful as Δ^0 has no non-trivial endofunctors, hence so is its pullback in the above diagram. Putting these observations together we proved that the pushforward along $\Delta^1 \to \Delta^0$ functor is actually fully-faithful, hence we proved that

$$(A \times_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \boldsymbol{\Delta}^{\mathrm{op}}))_0 \to (A \times_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \boldsymbol{\Delta}^{\mathrm{op}}))_1^{\mathrm{invert}}$$

is an equivalence.

2. The functor ι is defined as

$$A \simeq A \times_{\boldsymbol{\Delta}^{\mathrm{op}}} \boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{id}_A \times_{\mathrm{id}} \Delta_{(\boldsymbol{\Delta}^{\mathrm{op}})}} A \times_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \boldsymbol{\Delta}^{\mathrm{op}}) \ .$$

By the the explicit formula for cocartesian lifting in the envelope construction and the fact that for any inert simplex morphism $i\colon \Delta^n \hookrightarrow \Delta^m$ the active-inert factorization of the composite $\Delta^n \overset{i}{\hookrightarrow} \Delta^m \xrightarrow{\mathrm{id}} \Delta^m$ is exactly $\Delta^n \xrightarrow{\mathrm{id}} \Delta^n \overset{i}{\hookrightarrow} \Delta^m$, we can deduce that ι is inert-cocartesian. Lastly, the functor $\boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{\Delta_{\boldsymbol{\Delta}^{\mathrm{op}}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \boldsymbol{\Delta}^{\mathrm{op}})$ is in the fiber over Δ^0 just the functor $\{\Delta^0\} \to \{\Delta^0 \xrightarrow{\mathrm{id}} \Delta^0\}$, which is an equivalence. Hence so is ι as a pullback. □

Next we prove that the envelope construction is in fact a classifier for inert-cocartesian functors out of $A \twoheadrightarrow \boldsymbol{\Delta}^{\mathrm{op}}$. This already appeared for example as [GH15, Proposition A.1.3]. Another proof strategy appears in [GR17, Chapter 11, Theorem A.1.5] and builds on the fact that one can relatively (over $\boldsymbol{\Delta}^{\mathrm{op}}$) left Kan extend inert-cocartesian functors from $A \twoheadrightarrow \boldsymbol{\Delta}^{\mathrm{op}}$ to some cocartesian fibration along ι. In this general situation, the relative left Kan extensions in $\mathrm{Cat}_{/\boldsymbol{\Delta}^{\mathrm{op}}}$ are provided by the fact that ι, as a functor between the total categories, is a fully-faithful left adjoint and by the cocartesianness of the target in the extension problem.

One way to make this precise is for example the following general statement, that left adjoints have a directed kind of left lifting property with respect to cocartesian fibrations and its accompanying specialization, which states that for fully-faithful left adjoints this becomes an actual left lifting property. Proofs of these statements can be found in section A.3.

Corollary 1.2.9

Let $F\colon A \to B$ be a left adjoint with right adjoint U and $p\colon X \twoheadrightarrow Y$ a cocartesian fibration. Then the functor

$$\mathrm{Fun}(\Delta^1, \mathrm{Fun}(A,X)) \underset{\mathrm{Fun}(A,X)}{\times} \mathrm{Fun}(B,X) \to \mathrm{Fun}(A,X) \underset{\mathrm{Fun}(A,Y)}{\times} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(A,Y)) \underset{\mathrm{Fun}(A,Y)}{\times} \mathrm{Fun}(B,Y)$$

defined as in Lemma A.3.1 from the adjointable square

$$\begin{tikzcd} \mathrm{Fun}(B,X) \arrow[r, "\mathrm{Fun}(F,X)"] \arrow[d, "\mathrm{Fun}(B,p)"'] & \mathrm{Fun}(A,X) \arrow[d, "\mathrm{Fun}(A,p)"] \\ \mathrm{Fun}(B,Y) \arrow[r, "\mathrm{Fun}(F,Y)"'] & \mathrm{Fun}(A,Y) \end{tikzcd} \tag{1.2.2.5}$$

has a fully-faithful left adjoint.

Corollary 1.2.10

Let $F\colon A \to B$ be a left adjoint and $p\colon X \twoheadrightarrow Y$ a cocartesian fibration. If we furthermore assume the left adjoint F to be fully-faithful, then the adjunction from Corollary 1.2.9 restricts on both sides to an adjunction

$$\begin{tikzcd} \mathrm{Fun}(B,X) \arrow[rr, "({}_*F,\, p_*)"] \arrow[dr, "p_*"'] & & \mathrm{Fun}(A,X) \times^{(p_*,\, {}_*F)}_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y) \arrow[ll, bend right, "\perp"'] \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{Fun}(B,Y) & \end{tikzcd} \tag{1.2.2.6}$$

which even lives over $\mathrm{Fun}(B,Y)$ via the indicated functors in the above diagram.

In this particular situation, by further unpacking the proof of Corollary 1.2.9, we can characterize the essential image of the fully-faithful left adjoint of this adjunction via Lemma A.2.12 as exactly those functors $L\colon B \to X$ whose whiskering $L\varepsilon$, where ε is the counit of the adjunction $F \dashv U$, is a p-cocartesian lift.

The fact that ι is a fully-faithful left adjoint will be deduced from the stability of right adjoints with fully-faithful left adjoints under pullback. This can be found proven in section A.2.

Lemma 1.2.11

Let

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "U"] \\ Y \arrow[r, "K"'] & B \end{tikzcd}$$

be a pullback square such that U has a fully-faithful left adjoint F. Then its pullback V also has a fully-faithful left adjoint G, and the pullback square is furthermore horizontally adjointable. Additionally, one can compute the fully-faithful left adjoint G as the pullback of F along H, and the (co)unit of the adjunction $G \dashv V$ as the pullback of the (co)unit of $F \dashv U$ along K, respectively H.

Proposition 1.2.12

Let $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ be a Segal cocartesian fibration. Then the envelope construction

$$\mathrm{ev}_1\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathbf{\Delta}^{\mathrm{op}}$$

is an inert-cocartesian functor classifier in the sense that for any other Segal cocartesian fibration $q\colon B \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ we have that precomposition with the inert-cocartesian functor

$$\iota\colon A \to A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$$

over $\mathbf{\Delta}^{\mathrm{op}}$ gives an equivalence of restricted relative functor categories over $\mathbf{\Delta}^{\mathrm{op}}$

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), B) \xrightarrow[\simeq]{{}_*\iota} \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A, B)\ ,$$

where on the left hand side we chose the full subcategory on cocartesian functors and on the right hand side the full subcategory on inert-cocartesian functors in $\mathrm{Fun}_{/\mathbf{\Delta}^{\mathrm{op}}}(-,-)$. This tells us that ι is the initial inert-cocartesian functor over $\mathbf{\Delta}^{\mathrm{op}}$ out of $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$.

Proof. We will proceed as layed out in [GR17, Chapter 11, Theorem A.1.5]. First, we observe that the canonical adjunction

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[rr, bend right, "\mathrm{ev}_0"'] \arrow[rr, phantom, "\perp"] & & \mathbf{\Delta}^{\mathrm{op}} \arrow[ll, bend right, "\Delta_{\mathbf{\Delta}^{\mathrm{op}}}"'] \end{tikzcd}$$

restricts from $\mathrm{Fun}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ down to the full subcategory $\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ as identity morphism are in particular active. By applying Lemma 1.2.11 now to the defining pullback square

$$\begin{tikzcd} A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\ \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[r, "\mathrm{ev}_0"'] & \mathbf{\Delta}^{\mathrm{op}} \end{tikzcd}$$

we deduce that the fully-faithful functor ι, which is given by

$$A \simeq A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{id}_A \times_{\mathrm{id}} \Delta_{(\mathbf{\Delta}^{\mathrm{op}})}} A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \ ,$$

actually is left adjoint to $\rho \coloneqq \mathrm{pr}_0\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to A$ on total categories with invertible unit. Note however, that this adjunction does not live over $\mathbf{\Delta}^{\mathrm{op}}$ via ev_1 and p, i. e. its counit will not become invertible after applying ev_1. We know from Corollary 1.2.9 and Corollary 1.2.10 that for the adjunction $\iota \dashv \rho$ and the cocartesian fibration q we have the restricted lifting adjunction

$$\begin{tikzcd} \mathrm{Fun}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), B) \arrow[r, "({}_*\iota, q_*)"] \arrow[dr, "q_*"'] & \mathrm{Fun}(A, B) \times_{\mathrm{Fun}(A, \mathbf{\Delta}^{\mathrm{op}})}^{(q_*, {}_*\iota)} \mathrm{Fun}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), \mathbf{\Delta}^{\mathrm{op}}) \arrow[l, hook, bend left] \arrow[d, "\mathrm{pr}_1"] \\ & \mathrm{Fun}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), \mathbf{\Delta}^{\mathrm{op}}) \ , \end{tikzcd}$$

which even lives over $\mathrm{Fun}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), \mathbf{\Delta}^{\mathrm{op}})$. This means we can pull it back to its fiber over the functor ev_1 and obtain an adjunction

$$\begin{tikzcd} \mathrm{Fun}_{/\mathbf{\Delta}^{\mathrm{op}}}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), B) \arrow[r, "{}_*\iota"] & \mathrm{Fun}_{/\mathbf{\Delta}^{\mathrm{op}}}(A, B) \arrow[l, hook, bend left] \end{tikzcd}$$

between the relative functor categories over $\mathbf{\Delta}^{\mathrm{op}}$. Our goal now is to restrict this adjunction to the desired equivalence. In general, one always can restrict such an adjunction with invertible unit to an equivalence if one restricts on the left hand side to the essential image of the fully-faithful left adjoint. Now as stated in Corollary 1.2.10 we can characterize the essential image of the fully-faithful left adjoint as exactly those functors $L\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to B$ over $\mathbf{\Delta}^{\mathrm{op}}$, whose whiskering $L\varepsilon$, where ε denotes the counit of the adjunction $\iota \dashv \rho$, is a q-cocartesian lift. Next we want to restrict the right hand side of our restricted relative functor category equivalence to inert-cocartesian functors. To prove the desired equivalence it now remains to show that for a functor $L\colon A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to B$ to be cocartesian is actually equivalent to asking $L\varepsilon$ to be a q-cocartesian lift together with asking that $L \circ \iota$ is inert-cocartesian. In order to prove this claim we first need an explicit description of the counit of the adjunction $\iota \dashv \rho$. As the adjunction $\iota \dashv \rho$ is the pullback of the adjunction $\Delta_{(\mathbf{\Delta}^{\mathrm{op}})} \dashv \mathrm{ev}_0$ its component at an object $(\alpha\colon \Delta^n \xrightarrow{\text{active}} \Delta^m, \sigma)$ of $A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ is precisely the square

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha", "\text{active}"'] \arrow[d, "\alpha"'] & \Delta^m \arrow[d, "\mathrm{id}"] \\
\Delta^m \arrow[r, "\text{active}", "\mathrm{id}"'] & \Delta^m
\end{tikzcd} \tag{1.2.2.7}$$

seen as a morphism $\mathrm{id}_{\Delta^m} \to \alpha$ in $\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ and the morphism id_σ over id_{Δ^m} in A. Note that this is in fact a cocartesian lift in $A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$. Also, images of inert-cocartesian lifts in A under ι are cocartesian in $A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ as ι is inert-cocartesian. Now, to prove our remaining claim, let us take an arbitrary cocartesian lift of an arbitrary object $(\alpha\colon \Delta^n \xrightarrow{\text{active}} \Delta^m, \sigma)$ in $A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$, along an arbitrary simplex morphism $\gamma\colon \Delta^k \to \Delta^n$, i. e.

$$\begin{tikzcd}
\Delta^k \arrow[r, "\beta", "\text{active}"'] \arrow[d, "\gamma"'] & \Delta^l \arrow[d, "\theta", "\text{inert}"'] \\
\Delta^n \arrow[r, "\text{active}", "\alpha"'] & \Delta^m
\end{tikzcd} \tag{1.2.2.8}$$

seen as a morphism $\alpha \to \beta$ in $\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$ and the p-cocartesian lift $\sigma \xrightarrow{\text{cocart}} \theta_!\sigma$ of σ along θ. Let us depict this as in the following way, where objects in A are drawn over their fibers in $\mathbf{\Delta}^{\mathrm{op}}$.

$$\begin{tikzcd}
 & & \theta_!\sigma \arrow[d, dash] & \\
\Delta^k \arrow[rr, "\beta", "\text{active}"'] \arrow[dr, "\gamma"'] & & \Delta^l \arrow[dr, hook, "\theta", "\text{inert}"'] & \sigma \arrow[ul, "\text{cocart}"'] \arrow[d, dash] \\
 & \Delta^n \arrow[rr, "\alpha", "\text{active}"'] & & \Delta^m
\end{tikzcd}$$

Let us postcompose this with the component to the counit ε at (α, σ) to get the following morphism.

$$\begin{tikzcd}
 & & \theta_!\sigma \arrow[d, dash] & & \\
\Delta^k \arrow[rr, "\beta", "\text{active}"'] \arrow[dr, "\gamma"'] & & \Delta^l \arrow[dr, hook, "\theta", "\text{inert}"'] & \sigma \arrow[ul, "\text{cocart}"'] \arrow[d, dash] \arrow[dr, equal] & \\
 & \Delta^n \arrow[rr, "\alpha", "\text{active}"'] \arrow[dr, "\alpha"'] & & \Delta^m \arrow[dr, equal] & \sigma \arrow[d, dash] \\
 & & \Delta^m \arrow[rr, equal] & & \Delta^m
\end{tikzcd}$$

This can in fact be factorized also in a different way as

$$
\begin{tikzcd}
 & & & \theta_!\sigma \arrow[d, dash] \arrow[dr, equals] & & \\
 & \Delta^k \arrow[rr, "\beta", "\text{active}"'] \arrow[dr, "\beta"'] & & \Delta^l \arrow[dr, equals] & \theta_!\sigma \arrow[d, dash] & \\
 & & \Delta^l \arrow[rr, equals] \arrow[dr, hook, "\text{inert}", "\theta"'] & & \Delta^l \arrow[dr, hook, "\theta", "\text{inert}"'] & \sigma \arrow[ul, "\text{cocart}"'] \arrow[d, dash] \\
 & & & \Delta^m \arrow[rr, equals] & & \Delta^m .
\end{tikzcd}
$$

Note that we have here another component of the counit ε, but at $(\beta, \theta_!\sigma)$, as well as the image of an inert-cocartesian lift $\sigma \xrightarrow{\text{cocart}} \theta_!\sigma$ under ι. Now if our functor L sends ε to a cocartesian lift as well as images of inert-cocartesian lifts under ι, then it in particular sends the last depicted composite to a cocartesian lift. But as this agrees with the second to last depicted morphism, thus it also gets sent to a cocartesian lift. In this second to last composite, the first morphism is also sent to a cocartesian lift by L, so by the left cancellation property of cocartesian lifts, for example proven as Lemma A.2.18 in the Appendix, we deduce that also our arbitrary chosen cocartesian lift gets sent to something cocartesian. Thus the whole functor L is cocartesian. □

Lastly, we will exhibit the existence of the strict 2-functor $\lambda\colon \mathrm{Env}(A) \to A$ for the envelope construction. But in fact we will prove more: λ will be the left adjoint to the fully-faithful inert-cocartesian functor ι over $\mathbf{\Delta}^{\mathrm{op}}$. The main ingredient for the proof will be the following 1-categorical statement, a proof of which can also be found in the Appendix in section A.4.

Lemma 1.2.13

Let

$$
\begin{tikzcd}
Y \arrow[r, "V"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
A \arrow[r, "U"'] & B
\end{tikzcd}
$$

be a pullback square with p a cocartesian fibration and U has a left adjoint F. Then its pullback V also has a left adjoint, denoted by G, and the square is adjointable. Additionally, V is cocartesian over U and G is also a cocartesian functor over F. Furthermore, if the counit of $F \dashv U$ is invertible, then the counit of $G \dashv V$ will be invertible too.

Lemma 1.2.14

Let $p\colon A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ be a Segal cocartesian fibration. Then the inert-cocartesian functor

$$\iota\colon A \to A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$$

has a left adjoint λ over $\mathbf{\Delta}^{\mathrm{op}}$, such that the counit of this adjunction is invertible, and which is also a cocartesian functor. Furthermore, it corresponds to the identity functor on A under the equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}), A) \xrightarrow{\ast\iota \quad \simeq} \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A, A)$$

of Proposition 1.2.12.

Proof. The left adjoint λ is constructed via Lemma 1.2.13 applied to the defining pullback square for ι, i. e. the left hand pullback square in the pullback factorisation

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, equal] & A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[r, "\mathrm{pr}_0"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads] \\
\mathbf{\Delta}^{\mathrm{op}} \arrow[r, "\Delta_{(\mathbf{\Delta}^{\mathrm{op}})}"'] \arrow[rr, bend right, equal] & \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[r, "\mathrm{ev}_0"'] & \mathbf{\Delta}^{\mathrm{op}},
\end{tikzcd}$$

where we use that the other canonical adjunction $\mathrm{ev}_1 \dashv \Delta_{(\mathbf{\Delta}^{\mathrm{op}})}$ also restricts along the full subcategory inclusion $\mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \hookrightarrow \mathrm{Fun}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}})$, and that its counit is invertible. The adjointability statement of Lemma 1.2.13 automatically gives us that

$$\begin{tikzcd}
A \arrow[d, two heads] & A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[l, "\lambda"'] \arrow[d, two heads, "\mathrm{pr}_1"] \\
\mathbf{\Delta}^{\mathrm{op}} & \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[l, "\mathrm{ev}_1"]
\end{tikzcd}$$

commutes, i. e. λ defines a functor over $\mathbf{\Delta}^{\mathrm{op}}$, which is even cocartesian over ev_1. But we also have that ev_1

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathrm{op}} \arrow[d, two heads, "\mathrm{id}"'] & \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[l, "\mathrm{ev}_1"'] \arrow[d, two heads, "\mathrm{ev}_1"] \\
\mathbf{\Delta}^{\mathrm{op}} & \mathbf{\Delta}^{\mathrm{op}} \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

is itself a cocartesian functor from itself to the identity. Putting these two observations together we obtain, that λ is a cocartesian functor

$$\begin{tikzcd}
A \arrow[d, two heads] & A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[l, "\lambda"'] \arrow[d, two heads, "\mathrm{pr}_1"] \\
\mathbf{\Delta}^{\mathrm{op}} \arrow[d, two heads, "\mathrm{id}"'] & \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[l, "\mathrm{ev}_1"] \arrow[d, two heads, "\mathrm{ev}_1"] \\
\mathbf{\Delta}^{\mathrm{op}} & \mathbf{\Delta}^{\mathrm{op}}, \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

essentially because the needed vertical adjointability boils down to having a whiskered counit

$$(\varepsilon^{\mathrm{cocart}_p \dashv (\mathrm{ev}_0, p_*)}) \circ \lambda_* \circ \mathrm{cocart}_{\mathrm{pr}_1} \circ \mathrm{cocart}_{\mathrm{ev}_1}$$

invertible. Here $(\varepsilon^{\mathrm{cocart}_p \dashv (\mathrm{ev}_0, p_*)})$ denotes the counit of the adjunction $\mathrm{cocart}_p \dashv (\mathrm{ev}_0, p_*)$ which witnesses the cocartesianness of p as in Definition A.1.2. This in turn follows from the cocartesianness of λ over ev_1, i. e. the fact that the whiskered counit $\varepsilon^{\mathrm{cocart}_p \dashv (\mathrm{ev}_0, p_*)} \circ \lambda_* \circ \mathrm{cocart}_{\mathrm{pr}_1}$ is already invertible. The fact that λ is cocartesian and $\lambda \circ \iota \simeq \mathrm{id}$ also gives us the last statement. □

1.3 The Envelope satisfies the Simple Universal Property of the Lax 2-Functor Classifier

In this section we will show that the envelope construction $\mathrm{Env}(A)$ together with its canonical strict 2-functor $\lambda\colon \mathrm{Env}(A) \to A$ satisfies a simpler version of the universal property of the lax 2-functor classifier given in Definition 1.1.1, namely the following.

Definition 1.3.1

Let A be a 2-category. A 2-functor $F\colon X \to A$ is said to have local right adjoint sections if it satisfies the following two properties.

1. The induced functor $F^{\simeq}\colon X^{\simeq} \to A^{\simeq}$ on underlying spaces of objects is an equivalence.
2. For every pair of objects $x, \widetilde{x}$ of X, the induced functor

$$F_{x,\widetilde{x}}\colon X(x, \widetilde{x}) \to A(F(x), F(\widetilde{x}))$$

   on Hom-1-categories admits a fully-faithful right adjoint.

For a 2-functor

$$\begin{tikzcd} X \arrow[rr, "H"] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\ & A & \end{tikzcd} \tag{1.3.0.1}$$

over A between two such 2-functors F and G having local right adjoint sections we say that it commutes with the local right adjoint sections if the induced square

$$\begin{tikzcd} X(x,\widetilde{x}) \arrow[r, "H_{x,\widetilde{x}}"] \arrow[d, "F_{x,\widetilde{x}}"'] & Y(Hx, H\widetilde{x}) \arrow[d, "G_{Hx,H\widetilde{x}}"] \\ A(Fx, F\widetilde{x}) \arrow[r, "\simeq"'] & A(GHx, GH\widetilde{x}) \end{tikzcd} \tag{1.3.0.2}$$

is vertically right adjointable, or in this case equivalently, the unit of the adjunction involving $G_{Hx,H\widetilde{x}}$ becomes invertible after precomposition with $H_{x,\widetilde{x}}$ followed by precomposition with the right adjoint of $F_{x,\widetilde{x}}$. These two conditions define a non-full subcategory $\mathrm{SimpleLocRARI}(A)$ of the strict slice 1-category $2\text{-}\mathrm{Cat}_{/A}$. An object of $\mathrm{SimpleLocRARI}(A)$ is said to satisfy the simple universal property of the lax 2-functor classifier of A if it is initial.

Remark 1.3.2

Actually we will prove in Theorem 1.3.8 a slightly stronger statement than we need for the envelope to satisfy this simple universal property, namely we prove it for *non-complete*, or *non-univalent* 2-categories, which in our model correspond to the non-complete globular Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$. However, we still find it instructive to think about this section as actually working towards a proof of the simple universal property, and realising in the proof that we never used completeness.

As a preliminary step we prove that for a 2-functor

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, two heads] & & A \arrow[dl, two heads] \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd}$$

in our model of 2-categories, having local right adjoint sections is actually equivalent to having a fully-faithful inert-cocartesian right adjoint on total categories, just as observed for $\lambda\colon \mathrm{Env}(A) \to A$ in Lemma 1.2.14. Equipped with this statement and its corresponding statement for 2-functors commuting with local right adjoint sections, we can then reformulate the model-independently defined 1-category $\mathrm{SimpleLocRARI}(A)$, which hosts our universal property, in a way that is closely tailored to the model of globular completewith Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$. After having achieved this goal we then proceed to step-by-step enhance the inert-cocartesian functor classification property of $\mathrm{Env}(A)$ to finally ressemble initiality in our aformentioned model-dependent reformulation of $\mathrm{SimpleLocRARI}(A)$.

The first two main ingredients for the preliminary step are about glueing fiberwise adjoints for a cocartesian functor between cocartesian fibrations together to an actual adjoint on the total categories, which can already be found in [Lur17, Proposition 7.3.2.6], and what we need to know about the fiberwise adjoints in order to deduce further cocartesianness of this global adjoint. These are proven in section A.5.

Proposition 1.3.3

Let $p\colon A \twoheadrightarrow I$, $q\colon B \twoheadrightarrow I$ be two cocartesian fibrations over some category I and $F\colon A \to B$ a cocartesian functor over I with $\alpha\colon p \simeq qF$. Let us denote for every object i of I by $F_i\colon A_i \to B_i$ the restriction of F, A and B to the fiber over i, i. e. the pullbacks along $\Delta^0 \xrightarrow{i} I$. Then the following two statements are equivalent.

1. For all objects i of I the functor F_i has a right adjoint U_i.
2. the functor $F\colon A \to B$ has a right adjoint U, which is also a functor over I, i. e. we have that the canonical mate $q\varepsilon \circ \alpha U\colon pU \Rightarrow qFU \Rightarrow q$ of α is invertible, or equivalently the counit $\varepsilon\colon FU \Rightarrow \mathrm{id}$ of the adjunction is a natural transformation over I, i. e. the whiskering $q\varepsilon$ is invertible.

Corollary 1.3.4

Let F be a cocartesian functor

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, two heads, "p"'] & & B \arrow[dl, two heads, "q"] \\ & I & \end{tikzcd} \tag{1.3.0.3}$$

between cocartesian fibrations p and q. Let us futhermore assume that F has a right adjoint $U\colon B \to A$ over the base I. Then the following statements are equivalent.

1. U is a cocartesian functor.
2. For every morphism $k\colon i \to j$ in I the square

$$
\begin{tikzcd}
A_i \arrow[r, "F_i"] \arrow[d, "k_!"'] & B_i \arrow[d, "k_!"] \\
A_j \arrow[r, "F_j"'] & B_j
\end{tikzcd}
\tag{1.3.0.4}
$$

   is horizontally adjointable.

The second main ingredient for the preliminary step is about pullbacks of left adjoints being left adjoints again, a proof can be found in section A.4.

Lemma 1.3.5

Consider the commutative cube of categories such that the top and bottom faces are pullbacks.

$$
\begin{tikzcd}
A_2 \arrow[dd, "F_2"'] & & A_1 \arrow[ll] \arrow[dd, "F_1"] & \\
& A_0 \arrow[ul] \arrow[dd, "F_0"] & & A \arrow[ll] \arrow[ul] \arrow[dd, "F"] \\
B_2 & & B_1 \arrow[ll] & \\
& B_0 \arrow[ul] & & B \arrow[ll] \arrow[ul]
\end{tikzcd}
\tag{1.3.0.5}
$$

If the F_i are left adjoints and the back face as well as the left face are adjointable, then the pullback induced functor F is also a left adjoint and the right and front faces are also adjointable. Additionally, if all the left adjoints F_i, or all the right adjoints U_i respectively, are fully-faithful, then so is the pulled-back one.

The strategy to relate having local right adjoint sections and having a fully-faithful inert-cocartesian right adjoint over $\boldsymbol{\Delta}^{\mathrm{op}}$ now will be roughly the following.

- First we will pass from the Hom-functor properties of having local right adjoint sections to having right adjoint sections for the functors between the fibers over Δ^1 in $\boldsymbol{\Delta}^{\mathrm{op}}$, as the Hom-categories $X(x, y)$ are exactly the fibers of the source-target functor $(s, t)\colon X_1 \to X_0 \times X_0$ in our model.
- Then we will glue these right adjoint sections using Lemma 1.3.5 along the inverse equivalences of $F^{\simeq}$ to get right adjoint sections on all fibers over arbitrary Δ^n via the Segal axiom.
- At last, we use Proposition 1.3.3 to glue the fiberwise adjoints together to a global adjoint on total categories.

Proposition 1.3.6

Let F be a cocartesian functor

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "p"'] & & A \arrow[dl, "q"] \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd} \tag{1.3.0.6}$$

between Segal cocartesian fibrations p and q. Then the following are equivalent.

1. F has a (fully-faithful) right adjoint U over $\mathbf{\Delta}^{\mathrm{op}}$, i. e. the counit is a natural transformation over $\mathbf{\Delta}^{\mathrm{op}}$, and U is inert-cocartesian.
2. For all $n \in \mathbb{N}_0$ the functor F_n in the fiber over Δ^n has a (fully-faithful) right adjoint U_n and for every inert simplex map $\alpha \colon \Delta^n \hookrightarrow \Delta^m$ the induced square

$$\begin{tikzcd} X_n \arrow[r, "\alpha_!"] \arrow[d, "F_n"'] & X_m \arrow[d, "F_m"] \\ A_n \arrow[r, "\alpha_!"'] & A_m \end{tikzcd} \tag{1.3.0.7}$$

   whitnessing cocartesian pushforward preservation of F, is vertically adjointable.
3. F_0 and F_1 have (fully-faithful) right adjoints U_0, respectively U_1, and for both $s, t \colon \Delta^0 \to \Delta^1$ the vertical adjointablility of Equation 1.3.0.7 holds.

If we furthermore assume that the fibers of p and q over Δ^0 are spaces, then we have even more equivalent statements, extending the list of equivalent statements above.

4. F_0 is an equivalence and F_1 has a (fully-faithful) right adjoint U_1
5. F_0 is an equivalence and for all objects x, y of X the action on hom-categories functors

$$X(x, y) \xrightarrow{F_{x,y}} A(Fx, Fy) \tag{1.3.0.8}$$

   of F have (fully-faithful) right adjoints.

Proof. We will show that each assertion is equivalent to the next one.

1. $\iff$ 2. The first part about the right adjoints follows from Proposition 1.3.3. By definition U being inert-cocartesian means that its pullback along $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inert}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$, let us denote this by $\widetilde{U} \colon \widetilde{A} \to \widetilde{X}$, is a cocartesian functor. Let us also denote the pullback of F along $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inert}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ by $\widetilde{F} \colon \widetilde{X} \to \widetilde{A}$. By stability of relative adjunctions under base change, i. e. pullback, proved for example in Lemma A.5.5, this is still a relative adjunction over $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inert}}$. Now applying Corollary 1.3.4 gives us the second part of the equivalence.
2. $\iff$ 3. For the non-trivial implication, we iteratively apply Lemma 1.3.5 to commutative

cubes like

$$\begin{tikzcd}
X_1 \times_{X_0} X_1 \arrow[rr] \arrow[dr] \arrow[dd, "F_1 \times_{F_0} F_1"'] & & X_1 \arrow[dr, "t"] \arrow[dd, "F_1"] & \\
& X_1 \arrow[rr, "s"] \arrow[dd, "F_1"] & & X_0 \arrow[dd, "F_0"] \\
A_1 \times_{A_0} A_1 \arrow[rr] \arrow[dr] & & A_1 \arrow[dr, "t"] & \\
& A_1 \arrow[rr, "s"'] & & A_0
\end{tikzcd}$$

with top and bottom square pullbacks, and then use the Segal property. To see that we actually get all vertical adjointability of Equation 1.3.0.7 for all inert simplex morphisms from just the ones of s and t, one just needs to observe that every inert simplex morphism can be iteratively decomposed into the simpler inert simplex morphisms

$$\Delta^n \simeq \Delta^n \underset{\Delta^0}{\cup} \Delta^0 \xrightarrow[\mathrm{id}]{\mathrm{id} \cup s} \Delta^n \underset{\Delta^0}{\cup} \Delta^1 \simeq \Delta^{n+1}$$

and

$$\Delta^n \simeq \Delta^0 \underset{\Delta^0}{\cup} \Delta^n \xrightarrow[\mathrm{id}]{t \cup \mathrm{id}} \Delta^1 \underset{\Delta^0}{\cup} \Delta^n \simeq \Delta^{n+1} .$$

The vertical adjointability of Equation 1.3.0.7 for these two types of simplex morphisms follows directly from the ones of s and t and the Segal property.

If we furthermore assume that the fibers of p and q over Δ^0 are spaces, then we can prove also the following equivalences.

3. $\iff$ 4. First note, that any adjunction between spaces automatically also is an adjoint equivalence. Vice versa every equivalence can be upgraded to an adjoint equivalence. To see that we get the vertical adjointability for s and t for free with our assumptions one just needs to observe that the vertical mate

$$\begin{tikzcd}
A_1 \arrow[r, "U_1"] \arrow[dr, equal] \arrow[dr, phantom, "\overset{\varepsilon_1}{\swarrow}"] & X_1 \arrow[r, "s"] \arrow[d, "F_1"] & X_0 \arrow[d, "F_0"'] \arrow[dr, equal] \arrow[dr, phantom, "\underset{\eta_0}{\swarrow}"] & \\
& A_1 \arrow[r, "s"'] & A_0 \arrow[r, "U_0"'] & X_0
\end{tikzcd}$$

will always be invertible as X_0 was by assumption a space.

4. $\iff$ 5. This equivalence follows from Proposition 1.3.3 applied to

$$\begin{tikzcd}
X_1 \arrow[r, "F_1"] \arrow[d, two heads, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & A_1 \arrow[d, two heads, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\
X_0 \times X_0 \arrow[r, "\simeq", "F_0 \times F_0"'] & A_0 \times A_0
\end{tikzcd} \tag{1.3.0.9}$$

and the facts that $(\mathrm{ev}_0, \mathrm{ev}_1)$ are trivially cocartesian fibrations, as their codomains are spaces, and similarly that F_1 is trivially a cocartesian functor.

□

Now that we have seen how to model-dependently reformulate the property of having local right adjoint sections, we also need to know how to reformulate what it means for 2-functors over A to preserve these, i. e. commute with these.

Proposition 1.3.7

Let

$$\begin{tikzcd} X \arrow[rr, "H"] \arrow[dr, "F"'] \arrow[ddr, bend right] & & Y \arrow[dl, "G"] \arrow[ddl, bend left] \\ & A \arrow[d] & \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd} \tag{1.3.0.10}$$

be a commutative triangle of cocartesian functors between cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, such that both F and G satisfy both the assumptions and the equivalent conditions of the first part of Proposition 1.3.6. Then the following are equivalent.

1. The commutative triangle of total categories

$$\begin{tikzcd} X \arrow[rr, "H"] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\ & A & \end{tikzcd} \tag{1.3.0.11}$$

   is vertically adjointable, i. e. as a square after adding in the identity functor on A, with respect to the right adjoint U of F over $\mathbf{\Delta}^{\mathrm{op}}$ and the right adjoint V of G over $\mathbf{\Delta}^{\mathrm{op}}$.
2. For all $n \in \mathbb{N}_0$ the commutative triangle of functors between the fibers over Δ^n

$$\begin{tikzcd} X_n \arrow[rr, "H_n"] \arrow[dr, "F_n"'] & & Y_n \arrow[dl, "G_n"] \\ & A_n & \end{tikzcd} \tag{1.3.0.12}$$

   is vertically adjointable with respect to the fiberwise right adjoints U_n of F_n and V_n of G_n.
3. For $i \in \{0, 1\}$ the commutative triangle of functors between the fibers over Δ^i

$$\begin{tikzcd} X_i \arrow[rr, "H_i"] \arrow[dr, "F_i"'] & & Y_i \arrow[dl, "G_i"] \\ & A_i & \end{tikzcd} \tag{1.3.0.13}$$

   is vertically adjointable with respect to the fiberwise right adjoints U_i of F_i and V_i of G_i.

If we furthermore assume, as in the second part of Proposition 1.3.6, that X_0, Y_0 and A_0 are spaces, then we can extend the above list of equivalent statements by the following.

4. For all objects $x, \widetilde{x}$ of X the commutative triangle of Hom-functors

$$
\begin{tikzcd}
X(x,\widetilde{x}) \arrow[rr, "H"] \arrow[dr, "F"'] & & Y(H(x),H(\widetilde{x})) \arrow[dl, "G"] \\
& A(F(x),F(\widetilde{x})) &
\end{tikzcd}
\tag{1.3.0.14}
$$

is vertically adjointable with respect to the right adjoints on Hom-categories of F and G.

Proof. We will show that each assertion is equivalent to the next one.

1. $\iff$ 2. A natural transformation over a base category is, as every natural transformation, invertible if and only if it is so objectwise. But every object lives in some fiber over the base, hence a natural transformation over a base category is invertible if and only if it is fiberwise invertible.
But by the proof of Proposition 1.3.3 we know that all the data of the global adjunction agrees with the data of the fiber adjunctions, when restricted to the fibers. Hence the formation of the mate for the global commutative triangle and the global adjunction on total categories agrees with the fiberwise formed mates of the fiber commutative triangles and the fiber adjunctions, when restricted to the fibers. Thus our observation about the invertibility of natural transformations over some base tells us that the global mate is invertible if and only if all the fiber mates are invertible.
2. $\iff$ 3. We have the commutative prism

$$
\begin{tikzcd}
X_n \arrow[rr, "H_n"] \arrow[dr, "F_n"'] \arrow[dd, "\simeq"'] & & Y_n \arrow[dl, "G_n"] \arrow[dd, "\simeq"] \\
& A_n \arrow[dd, "\simeq"] & \\
X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[rr, "H_1 \times_{H_0} \cdots \times_{H_0} H_1"] \arrow[dr, "F_1 \times_{F_0} \cdots \times_{F_0} F_1"'] & & Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1 \arrow[dl, "G_1 \times_{G_0} \cdots \times_{G_0} G_1"] \\
& A_1 \times_{A_0} \cdots \times_{A_0} A_1 &
\end{tikzcd}
$$

Furthermore we know from the proof of Proposition 1.3.6, i. e. the application of Lemma 1.2.11, that the two front vertical faces of this prism are in fact horizontally adjointable and that the adjunction data for the two bottom diagonal functors between the iterated fiber products are in fact factorwise. This means in particular that forming the mate upstairs is equivalent to downstairs forming the mate, which in itself is done fiber product factorwise. Now we observe that such a factorwise natural transformation between factorwise given functors

$$
\begin{tikzcd}
A_1 \times_{A_0} \cdots \times_{A_0} A_1 \arrow[dr, "\mathrm{mate}_1 \times_{\mathrm{mate}_0} \cdots \times_{\mathrm{mate}_0} \mathrm{mate}_1"'] & \mathrm{Fun}(\Delta^1, Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1) \\
& \mathrm{Fun}(\Delta^1, Y_1) \times_{\mathrm{Fun}(\Delta^1, Y_0)} \cdots \times_{\mathrm{Fun}(\Delta^1, Y_0)} \mathrm{Fun}(\Delta^1, Y_1) \arrow[u, "\simeq"']
\end{tikzcd}
$$

is invertible if and only if all its factors are invertible, which follows from the fact that a morphism in an iterated fiber product is invertible if and only if its projections to all its factors is invertible.

If we furthermore assume, as in the second part of Proposition 1.3.6, that X_0, Y_0 and A_0 are spaces, then we can also prove the following.

3. $\iff$ 4. First we observe that because F_0 and G_0 are equivalences H_0 also has to be an equivalence. Now our situation can be pictured as

$$\begin{tikzcd}
X_1 \arrow[rr, "H_1"] \arrow[dr, "F_1"] \arrow[dd, "{(s,t)}"'] & & Y_1 \arrow[dl, "G_0"'] \arrow[dd, "{(s,t)}"] \\
& A_1 \arrow[dd, "{(s,t)}"] & \\
X_0 \times X_0 \arrow[rr, "H_0 \times H_0", "\simeq"'] \arrow[dr, "\simeq", "F_0 \times F_0"'] & & Y_0 \times Y_0 \,. \arrow[dl, "\simeq"', "G_0 \times G_0"] \\
& A_0 \times A_0 &
\end{tikzcd}$$

In particular we can now apply the same reasoning as in 1. $\iff$ 2. to get the equivalence of being globally adjointable in fiber over Δ^1 and of being Hom-wise adjointable.

□

Equipped with these two characterisations we are now able to prove the simple universal property Definition 1.3.1 of the lax functor classifier for the envelope construction $\lambda\colon \mathrm{Env}(A) \to A$. The final step in the proof will again rely on the stability of left adjoints with invertible counit under pullbacks, i. e. Lemma 1.2.11.

Theorem 1.3.8

For a globular Segal cocartesian fibration 2-category $A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$, the envelope construction

$$\mathrm{ev}_1\colon \mathrm{Env}(A) \coloneqq A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathbf{\Delta}^{\mathrm{op}}$$

together with its canonical cocartesian functor $\lambda\colon \mathrm{Env}(A) \to A$ from Lemma 1.2.14 is the initial object in the non-full subcategory $\mathrm{LocRARI}^{\mathrm{glob}}_{\mathrm{Segal}}(A)$ of $\mathrm{Cocart}^{\mathrm{glob}}_{\mathrm{Segal}}(\mathbf{\Delta}^{\mathrm{op}})_{/A}$ on those objects $F\colon X \to A$ which

1. induce an equivalence $F_0\colon X_0 \to A_0$ on fibers over Δ^0, and
2. for all objects $x, \widetilde{x}$ of X_0 the induced functors on Hom-1-categories

$$F_{x,\widetilde{x}}\colon X(x,\widetilde{x}) \to A(F(x), F(\widetilde{x}))$$

   have fully-faithful right adjoints,

and morphisms which induce vertically adjointable triangles on Hom-1-categories. Furthermore, as the envelope construction is always complete, it is also initial among objects which are furthermore complete. In particular it satisfies the simple universal property Definition 1.3.1 of the lax functor classifier of $A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ if $A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is also complete, i. e. modeling a 2-category.

Proof. Let us first explain how to deduce the universal property of Definition 1.3.1 from the more general statement. The universal property of the lax functor classifier is stated as initiality in the non-full subcategory $\mathrm{LocRARI}(A)$ of the strict slice-over 1-category $2\text{-}\mathrm{Cat}_{/A}$, defined in Definition 1.3.1. As described in subsection 1.2.1 $2\text{-}\mathrm{Cat}_{/A}$ can be modelled by $\mathrm{CSCocart}^{\mathrm{glob}}(\mathbf{\Delta}^{\mathrm{op}})_{/A}$. Hence $\mathrm{SimpleLocRARI}(A)$ is the full subcategory of $\mathrm{LocRARI}^{\mathrm{glob}}_{\mathrm{Segal}}(A)$ on the complete objects. As the envelope construction is always complete by Proposition 1.2.8, initiality in $\mathrm{LocRARI}^{\mathrm{glob}}_{\mathrm{Segal}}(A)$ will also imply initiality in $\mathrm{SimpleLocRARI}(A)$.

By Proposition 1.3.6 and Proposition 1.3.7 the non-full subcategory $\mathrm{LocRARI}^{\mathrm{glob}}_{\mathrm{Segal}}(A)$ can equivalently be described as the non-full subcategory of $\mathrm{Cocart}^{\mathrm{glob}}_{\mathrm{Segal}}(\boldsymbol{\Delta}^{\mathrm{op}})_{/A}$ on those cocartesian functors $F\colon X \to A$ over $\boldsymbol{\Delta}^{\mathrm{op}}$ such that the functor F on total categories has a fully-faithful right adjoint U over $\boldsymbol{\Delta}^{\mathrm{op}}$ that is also inert-cocartesian, and morphisms

$$\begin{tikzcd} X \arrow[rr, "H"] \arrow[dr, "F"'] \arrow[ddr, bend right] & & Y \arrow[dl, "G"] \arrow[ddl, bend left] \\ & A \arrow[d] & \\ & \boldsymbol{\Delta}^{\mathrm{op}} & \end{tikzcd} \tag{1.3.0.15}$$

between such cocartesian functors over $A \twoheadrightarrow \boldsymbol{\Delta}^{\mathrm{op}}$ such that the commutative triangle on total categories is vertically adjointable.

Now let us start proving the initiality statement. For this we take an arbitrary object of our alternatively described version of $\mathrm{LocRARI}^{\mathrm{glob}}_{\mathrm{Segal}}(A)$, i. e. a cocartesian functor $F\colon X \to A$ between globular Segal cocartesian fibrations over $\boldsymbol{\Delta}^{\mathrm{op}}$ such that the functor F on total categories has a fully-faithful right adjoint U over $\boldsymbol{\Delta}^{\mathrm{op}}$ that is also inert-cocartesian. The first step is to slice the equivalence

$$\mathrm{Fun}^{\mathrm{cocart}}_{/\boldsymbol{\Delta}^{\mathrm{op}}}(A \times_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \boldsymbol{\Delta}^{\mathrm{op}}), X) \xrightarrow[\ ]{{}_*\iota \quad \simeq} \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/\boldsymbol{\Delta}^{\mathrm{op}}}(A, X)$$

from Proposition 1.2.12 over $A \twoheadrightarrow \boldsymbol{\Delta}^{\mathrm{op}}$ itself, i. e. to transform it into a statement in

$$(\mathrm{Cat}_{/\boldsymbol{\Delta}^{\mathrm{op}}})_{/(A \twoheadrightarrow \boldsymbol{\Delta}^{\mathrm{op}})} \simeq \mathrm{Cat}_{/A}$$

by the cube

$$\begin{tikzcd}[column sep=small] & \mathrm{Fun}^{\mathrm{cocart}}_{/\boldsymbol{\Delta}^{\mathrm{op}}}(\mathrm{Env}(A), X) \arrow[rr, "{}_*\iota", "\simeq"'] \arrow[dd, "F_*"] & & \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/\boldsymbol{\Delta}^{\mathrm{op}}}(A, X) \arrow[dd, "F_*"] \\ \mathrm{Fun}^{\mathrm{cocart}}_{/A}((\mathrm{Env}(A), \lambda), (X, F)) \arrow[ur] \arrow[rr, "{}_*\iota", "\simeq"'] \arrow[dd] & & \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/A}((A, \mathrm{id}_A), (X, F)) \arrow[ur] \arrow[dd] & \\ & \mathrm{Fun}^{\mathrm{cocart}}_{/\boldsymbol{\Delta}^{\mathrm{op}}}(\mathrm{Env}(A), A) \arrow[rr, "{}_*\iota", "\simeq"'] & & \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/\boldsymbol{\Delta}^{\mathrm{op}}}(A, A)\,, \\ \Delta^0 \arrow[ur, "\lambda"'] \arrow[rr, equal] & & \Delta^0 \arrow[ur, "\mathrm{id}_A"'] & \end{tikzcd}$$

where both the left and the right face are pullbacks by definition. Note here that the back face of the cube commutes as ι is in particular a functor $(A, \mathrm{id}_A) \to (\mathrm{Env}(A), \mathrm{ev}_1)$ over A as the counit of $\lambda \dashv \iota$ is invertible. We obtain an equivalence

$${}_*\iota\colon\ \mathrm{Fun}^{\mathrm{cocart}}_{/A}((\mathrm{Env}(A), \lambda), (X, F)) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{inert\text{-}cocart}}_{/A}((A, \mathrm{id}_A), (X, F))\ , \tag{1.3.0.16}$$

which is also given by precomposition of ι as a functor $(A, \mathrm{id}_A) \to (\mathrm{Env}(A), \mathrm{ev}_1)$ over A. Now we restrict the left hand side of this equivalence to exactly those functors

$$\begin{tikzcd}
A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[rr, "H"] \arrow[dr, "\lambda"'] \arrow[ddr, bend right] & & X \arrow[dl, "F"] \arrow[ddl, bend left] \\
& A \arrow[d] & \\
& \mathbf{\Delta}^{\mathrm{op}} &
\end{tikzcd} \tag{1.3.0.17}$$

such that the commutative triangle of total categories is vertically adjointable. Let us denote this restriction by $\mathrm{Fun}^{\mathrm{cocart,vadj}}_{/A}((\mathrm{Env}(A), \lambda), (X, F))$. Now we observe that

$$\begin{tikzcd}
A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[rr, "H"] \arrow[dr, "\lambda"'] & & X \arrow[dl, "F"] \\
& A &
\end{tikzcd} \tag{1.3.0.18}$$

is vertically adjointable if and only if its precomposition with ι and the invertible counit $\lambda \circ \iota \simeq \mathrm{id}$

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, equal] & A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \arrow[r, "H"] \arrow[d, "\lambda"'] & X \arrow[dl, "F"] \\
& A &
\end{tikzcd} \tag{1.3.0.19}$$

is as a composed triangle vertically adjointable. Hence we may also restrict the right hand side of the equivalence Equation 1.3.0.16 to the vertically adjointable triangles

$$\begin{tikzcd}
A \arrow[rr, "K"] \arrow[dr, equal] & & X \arrow[dl, "F"] \\
& A\,. &
\end{tikzcd} \tag{1.3.0.20}$$

We will denote this restriction by $\mathrm{Fun}^{\mathrm{inert\text{-}cocart,vadj}}_{/A}((A, \mathrm{id}_A), (X, F))$ and get the restricted equivalence

$$\mathrm{Fun}^{\mathrm{cocart,vadj}}_{/A}((\mathrm{Env}(A), \lambda), (X, F)) \xrightarrow{\ {}_*\iota \quad \simeq\ } \mathrm{Fun}^{\mathrm{inert\text{-}cocart,vadj}}_{/A}((A, \mathrm{id}_A), (X, F))\ . \tag{1.3.0.21}$$

But as F has the fully-faithful right adjoint U over $\mathbf{\Delta}^{\mathrm{op}}$ we also have a postcomposition adjunction

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathbf{\Delta}^{\mathrm{op}}}(A, X) \arrow[rr, bend left, "F_*"] \arrow[rr, phantom, "\perp"] & & \mathrm{Fun}_{/\mathbf{\Delta}^{\mathrm{op}}}(A, A) \arrow[ll, bend left, "U_*"]
\end{tikzcd}$$

with invertible counit. And because F and U are both inert-cocartesian we can further restrict this to full subcategories on inert-cocartesian functors

$$\begin{tikzcd} \mathrm{Fun}^{\text{inert-cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A,X) \arrow[rr, bend left, "F_*"] \arrow[rr, phantom, "\perp"] & & \mathrm{Fun}^{\text{inert-cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A,A) \arrow[ll, bend left, "U_*"] \end{tikzcd}$$

and get again an adjunction with invertible counit. In particular, in the pullback

$$\begin{tikzcd} \mathrm{Fun}^{\text{inert-cocart}}_{/A}((A,\mathrm{id}_A),(X,F)) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}^{\text{inert-cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A,X) \arrow[d, "F_*"] \\ \Delta^0 \arrow[r, "\mathrm{id}_A"'] & \mathrm{Fun}^{\text{inert-cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(A,A) \end{tikzcd} \tag{1.3.0.22}$$

from before we now know that the right hand vertical functor has a fully-faithful right adjoint. So we can deduce by Lemma 1.2.11 that also the left hand vertical functor to the point has a fully-faithful right adjoint. This means explicitely that the category $\mathrm{Fun}^{\text{inert-cocart}}_{/A}((A,\mathrm{id}_A),(X,F))$ has a terminal object, which is exactly

$$\begin{tikzcd} A \arrow[rr, "U"] \arrow[dr, equal] \arrow[rr, phantom, shift right=5ex, "\cong\,\varepsilon"] & & X \arrow[dl, "F"] \\ & A & \end{tikzcd} \tag{1.3.0.23}$$

for ε the invertible counit of $F \dashv U$. On the other hand the essential image of the fully-faithful right adjoint of the left hand functor of Equation 1.3.0.22 can be described as all objects for which the unit of the adjunction becomes invertible, but unpacking Lemma 1.2.11 in this situation tells us that the component of this pulled back unit for an object

$$\begin{tikzcd} A \arrow[rr, "K"] \arrow[dr, equal] & & X \arrow[dl, "F"] \\ & A & \end{tikzcd} \tag{1.3.0.24}$$

of $\mathrm{Fun}^{\text{inert-cocart}}_{/A}((A,\mathrm{id}_A),(X,F))$ is exactly its composition with the unit η of the adjunction $F \dashv U$, i. e.

$$\begin{tikzcd} A \arrow[r, "K"] \arrow[dr, equal] & X \arrow[d, "F"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=2ex, "\Swarrow\eta"] & \\ & A \arrow[r, "U"'] & X\,. \end{tikzcd} \tag{1.3.0.25}$$

Note that this composition is in fact exactly the mate transformation of the triangle Equation 1.3.0.24. Hence the essential image of the fully-faithful right adjoint to the left hand vertical functor in our pullback square Equation 1.3.0.22 consists exactly of the vertically adjointable triangles. Restricting this pulled back adjunction now to its essential image we obtain the second equivalence in

$$\mathrm{Fun}^{\mathrm{cocart,vadj}}_{/A}((\mathrm{Env}(A),\lambda),(X,F)) \xrightarrow{*\iota \quad \simeq} \mathrm{Fun}^{\text{inert-cocart,vadj}}_{/A}((A,\mathrm{id}_A),(X,F)) \xrightarrow{\simeq} \Delta^0 \,. \tag{1.3.0.26}$$

In particular we deduce from this the equivalence on underlying mapping spaces

$$\mathrm{Map}^{\mathrm{cocart,vadj}}_{/A}((\mathrm{Env}(A),\lambda),(X,F)) \xrightarrow{*\iota \quad \simeq} \mathrm{Map}^{\text{inert-cocart,vadj}}_{/A}((A,\mathrm{id}_A),(X,F)) \xrightarrow{\simeq} \Delta^0 \,, \tag{1.3.0.27}$$

which now proves initiality of $(\mathrm{Env}(A),\lambda)$. □

1.4 The Extended Universal Property of the Lax 2-Functor Classifier

In this section we will generalise the universal property of the lax fun classifier from [Glo25] to make it more useful and applicable. More precisely, the following new universal property will exhibit the lax functor classifier $\mathrm{Lax}(A)$ as initial among even more general objects, allowing us to actually construct examples. The limiting factor in [Glo25, Definition 1.1] is the strong assumption that our 2-functors to A must induce equivalences on object spaces. This is not just hard to satisfy in examples, but it also strays from our original motivating ordinary 2-categorical intuition, that the lax functor classifier should be *freely generated* from the objects, 1-morphisms and 2-morphisms of A as well as the lax functoriality data any lax functor out of A should enjoy. To this end, we replace the requirement of inducing equivalences on object spaces by merely asking for object space *sections*. In order to still adhere to our free generation idea we will also restrict the need for fully-faithful Hom-right adjoints to those objects coming from $A^{\simeq}$ through this object space section.

Definition 1.1.1

Let A be a 2-category. A commutative triangle of 2-functors

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \,, \end{tikzcd}$$

where $\gamma\colon A^{\simeq} \to A$ denotes the inclusion of the underlying space of objects, is said to have local right adjoint sections if it satisfies that for every pair of objects $a, \tilde{a}$ of $A^{\simeq}$, the induced functor

$$F_{U(a),U(\tilde{a})}\colon X(U(a),U(\tilde{a})) \to A(F(U(a)),F(U(\tilde{a}))) \simeq A(a,\tilde{a})$$

on Hom-1-categories admits a fully-faithful right adjoint. For a commutative tetrahedron

$$\begin{tikzcd} & A^{\simeq} \arrow[dl, "U"'] \arrow[dr, "V"] \arrow[dd, bend left, "\gamma"] & \\ X \arrow[rr, "H"] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\ & A & \end{tikzcd}$$

between two such 2-functors F and G having local right adjoint sections with respect to U, respectively V, we say that it commutes with the local right adjoint sections if the induced square in the middle of

$$\begin{tikzcd} A(FU(a),FU(\widetilde{a})) \arrow[r, hook, dashed, "U_{a,\widetilde{a}}"] \arrow[dr, equal] \arrow[dr, phantom, "\varepsilon\simeq"] & X(U(a),U(\widetilde{a})) \arrow[r, "H_{U(a),U(\widetilde{a})}"] \arrow[d, "F_{U(a),U(\widetilde{a})}"] & Y(HU(a),HU(\widetilde{a})) \arrow[d, "G_{V(a),V(\widetilde{a})}"'] \arrow[dr, equal] \arrow[dr, phantom, "\eta"] & \\ & A(FU(a),FU(\widetilde{a})) \arrow[r, "\simeq"] & A(GHU(a),GHU(\widetilde{a})) \arrow[r, hook, dashed, "V_{a,\widetilde{a}}"'] & Y(HU(a),HU(\widetilde{a})) \end{tikzcd}$$

is vertically right adjointable, i. e. if the above composite 2-morphism is invertible. These two conditions define a non-full subcategory $\mathrm{LocRARI}(A)$ of the strict slice under 1-category ${}^{\gamma/}(2\text{-Cat}_{/A})$, i. e. the 1-categorical slice under γ of the strict slice 1-category $2\text{-Cat}_{/A}$ of 2-categories over A. A lax 2-functor classifier of A is an initial object $\lambda\colon \mathrm{Lax}(A) \to A$ in $\mathrm{LocRARI}(A)$. We will also call this initiality the extended universal property of the lax 2-functor classifier. Asking for fully-faithful Hom-left-adjoints we can entirely similarly define a non-full subcategory $\mathrm{LocLALI}(A)$ and the oplax 2-functor classifier of A, denoted by $\mathrm{OpLax}(A)$.

Next we record pullback properties of the 1-categories $\mathrm{LocRARI}(A)$ that will be useful to assemble $\mathrm{Lax}(-)$ into a 1-functor.

Remark 1.4.1

As having a fully-faithful right adjoint is a property of a 1-functor which is stable under pullback and such a pullback is automatically adjointable by [Glo25, Lemma 2.11], the canonically pullback induced 1-functor $K^*\colon {}^{\gamma_B/}(2\text{-Cat}_{/B}) \to {}^{\gamma_A/}(2\text{-Cat}_{/A})$ restricts to $K^*\colon \mathrm{LocRARI}(B) \to \mathrm{LocRARI}(A)$ for every 2-functor K.

In the following remark we turn the above observation into a coherent form to make all of those initial objects assemble into a functor.

Remark 1.4.2

The functor evaluating at the $0 \to 2$ edge of Δ^2, $\mathrm{Fun}(\Delta^2, 2\text{-Cat}) \to \mathrm{Fun}(\Delta^1, 2\text{-Cat})$ is in fact a cartesian fibration. Hence we can pull it back along the 1-functor $2\text{-Cat} \to \mathrm{Fun}(\Delta^1, 2\text{-Cat})$ classifying the natural transformation $(-)^{\simeq} \Rightarrow \mathrm{id}$ to get a cartesian fibration

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^2, 2\text{-Cat}) \arrow[d, two heads, "\mathrm{ev}_{02}"] \\ 2\text{-Cat} \arrow[r, "(-)^{\simeq}\Rightarrow \mathrm{id}"'] & \mathrm{Fun}(\Delta^1, 2\text{-Cat})\,, \end{tikzcd}$$

whose cartesian lifts are computed as for ev_{02}. Using 1-categorical straightening-unstraightening, this makes the assigment $A \mapsto {}^{\gamma_A/}(2\text{-Cat}_{/A})$ contravariantly functorial with respect to the above mentioned pullback action. The previous remark now tells us that we may restrict to the non-full subcategories $\mathrm{LocRARI}(A)$ and still get a contravariant 1-functor $\mathrm{LocRARI}(-)$ with respect to the pullback action. After we know that all $\mathrm{LocRARI}(A)$ have such initials these initials will automatically assemble into a fully-faithful left adjoint section of the unstraightening $\int \mathrm{LocRARI} \twoheadrightarrow 2\text{-Cat}$. We may then compose this left adjoint section with the composite $\int \mathrm{LocRARI} \to \mathcal{L} \to \mathrm{Fun}(\Delta^2, 2\text{-Cat}) \xrightarrow{\mathrm{ev}_1} 2\text{-Cat}$ to obtain a 1-functor $\mathrm{Lax}(-)$.

Definition 1.4.3

We define the composition of lax 2-functors $F\colon A \rightsquigarrow B$ and $G\colon B \rightsquigarrow C$ as the composite

$$\mathrm{Lax}(A) \xrightarrow{\nabla} \mathrm{Lax}(\mathrm{Lax}(A)) \xrightarrow{\mathrm{Lax}(F)} \mathrm{Lax}(B) \xrightarrow{G} B \ ,$$

where the first 2-functor is the unique one induced by the universal property Definition 1.1.1 for the object

$$\begin{tikzcd} A^{\simeq} \arrow[r] \arrow[ddr] & \mathrm{Lax}(\mathrm{Lax}(A)) \arrow[d, "\lambda_{\mathrm{Lax}(A)}"] \\ & \mathrm{Lax}(A) \arrow[d, "\lambda_A"] \\ & A \end{tikzcd}$$

of $\mathrm{LocRARI}(A)$.

1.4.1 The Envelope Construction satisfies the Extended Universal Property of the Lax 2-Functor Classifier

In this subsection we show that in the model of 2-categories as globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, the envelope construction also satisfies the extended universal property. This will basically be done by reducing it back down to the simpler universal property, but for this we need to step out of the realm of complete, sometimes also called *univalent* 2-categories. The essential step here will be to restrict the object space of the 2-category X of an object

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \end{tikzcd}$$

of $\mathrm{LocRARI}(A)$ which has local right adjoint sections along the object space section $U^{\simeq}$, so that the object space becomes $A^{\simeq}$ *itself.* Whereas for actual complete 2-categories such an operation does not exist in general, it does so in *non-complete* 2-categories, which can be modeled similarly as above by, not-necessarily complete, globular Segal cocartesian fibrations. For these the underlying space functor, taking fibers over Δ^0, has a further right adjoint codisc, which enables us to built the pullbacks

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta"] \\ A_0 \simeq \mathrm{codisc}((A^{\simeq})_0) \arrow[r, "\mathrm{codisc}((U^{\simeq})_0)"] & \mathrm{codisc}(X_0) \end{tikzcd} \ ,$$

which in turn produce non-complete Segal fibrations with the object space A_0 of A.

Remark 1.4.4

The adjunctions

$$\begin{tikzcd} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}) \arrow[r, "\mathrm{ev}_0"] & \mathrm{Cat} \arrow[l, bend right, "\mathrm{codisc}"'] \arrow[l, bend left, "\mathrm{disc}"] \end{tikzcd}$$

restrict to adjunctions

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{glob}}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}) \arrow[r, "\mathrm{ev}_0"] & \mathcal{S} \arrow[l, bend right=60, "\mathrm{codisc}"'] \arrow[l, bend left=60, "\mathrm{disc}"] \end{tikzcd}$$

between globular simplicial objects categories and spaces, which furthermore restrict on the left hand side to Segal objects, the left adjoint $(-)^{\simeq}$ even further to complete ones. Here, ev_0 takes the role of the underlying object space functor $(-)^{\simeq}\colon \text{2-Cat} \to \mathcal{S}$ and disc its left adjoint, the fully-faithful inclusion $\mathcal{S} \hookrightarrow \text{2-Cat}$.

Theorem 1.4.5

In the model of 2-categories as globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, the envelope construction

$$\mathrm{ev}_1\colon \mathrm{Env}(A) \coloneqq A \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Fun}^{\mathrm{active}}(\Delta^1, \mathbf{\Delta}^{\mathrm{op}}) \to \mathbf{\Delta}^{\mathrm{op}}$$

for a 2-category $A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ together with its canonical 2-functors $A^{\simeq} \xrightarrow{\iota^{\simeq}} \mathrm{Env}(A) \xrightarrow{\lambda} A$ satisfies the extended universal property Definition 1.1.1 of the lax functor classifier of $A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$.

Proof. By Remark 1.4.4, for every object

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr] \arrow[ddr, two heads] & X \arrow[d, "F"] \arrow[dd, bend left=60, two heads] \\ & A \arrow[d, two heads] \\ & \mathbf{\Delta}^{\mathrm{op}} \end{tikzcd}$$

of $\mathrm{LocRARI}(A)$ we write $U_0\colon A_0 \to X_0$ for the, by $\mathrm{disc} \dashv \mathrm{ev}_0$, corresponding functor to $U^{\simeq}\colon A^{\simeq} = \mathrm{disc}(\mathrm{ev}_0(A)) \to X$. Now utilize the adjunction with codisc to perform the following pullback

$$\begin{tikzcd} A_0 \arrow[r, "U_0"] \arrow[d] & X_0 \arrow[r, "F_0"] \arrow[d] & A_0 \arrow[d] \arrow[dr] & \\ X_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r, "F"'] \arrow[d] & A \arrow[d] \arrow[dr, two heads] & \Delta^0 \arrow[d, "0"] \\ \mathrm{codisc}(A_0) \arrow[r, "\mathrm{codisc}(U_0)"] \arrow[rr, bend right, equal] \arrow[rrr, bend right=30, two heads] & \mathrm{codisc}(X_0) \arrow[r, "\mathrm{codisc}(F_0)"] & \mathrm{codisc}(A_0) \arrow[r, two heads] & \mathbf{\Delta}^{\mathrm{op}} \end{tikzcd}$$

and get a, in general no longer complete, Segal cocartesian fibration $X_U \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ whose fiber over Δ^0 is now equivalent to A_0, as the top left square above is also a pullback by pullback factorisation. This procedure also lets us factorise morphisms between such data

$$\begin{tikzcd}
 & A_0 \arrow[dl] \arrow[drr] \arrow[ddl, "U"'] \arrow[ddrr, "V"] \arrow[ddd] & & \\
X_U \arrow[rrr, dashed, "\exists!"] \arrow[d] & & & Y_V \arrow[d] \arrow[ddr, two heads] \\
X \arrow[rrr, "H"] \arrow[dr, "F"'] & & & Y \arrow[dll, "G"] \arrow[dr, two heads] & \\
 & A \arrow[rrrr, two heads] & & & \mathbf{\Delta}^{\mathrm{op}}
\end{tikzcd}$$

via

$$\begin{tikzcd}
X_U \arrow[rrrr] \arrow[drr] \arrow[dd] \arrow[dd, phantom, "\lrcorner", very near start] & & & & Y_V \arrow[dll] \arrow[dd] \\
 & & \mathrm{codisc}(A_0) \arrow[ddll, "\mathrm{codisc}(U_0)"'] \arrow[ddrr, "\mathrm{codisc}(V_0)"] & & \\
X \arrow[rrrr, "H"] \arrow[d] \arrow[ddrr, "F"] & & & & Y \arrow[d] \arrow[ddll, "G"'] \\
\mathrm{codisc}(X_0) \arrow[rrrr, "\mathrm{codisc}(H_0)"] \arrow[ddrr, "\mathrm{codisc}(F_0)"'] & & & & \mathrm{codisc}(Y_0)\,. \arrow[ddll, "\mathrm{codisc}(G_0)"] \\
 & & A \arrow[d] & & \\
 & & \mathrm{codisc}(A_0) & &
\end{tikzcd}$$

Note in particular, that we have $A_{\mathrm{id}_{A_0}} = A$. *In the case* that the map $U_0\colon A_0 \to X_0$ is an *equivalence*, we also have $X \simeq X_U$ and that cocartesian functors $X \to Y$ over $\mathbf{\Delta}^{\mathrm{op}}$ correspond by pullback to cocartesian functors $X \simeq X_U \to Y_V$ over $\mathbf{\Delta}^{\mathrm{op}}$, as we will make precise now. Consider the diagram

$$\begin{tikzcd}
\mathrm{Map}_{/A}^{A^{\simeq}\to A/}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{/A}(X,Y) \arrow[rr, equal] \arrow[d, "{}_*U^{\simeq}"'] & & \mathrm{Map}_{/A}(X,Y) \arrow[d, "\mathrm{ev}_0"'] \\
\Delta^0 \arrow[r, "V^{\simeq}"] \arrow[rrr, bend right, "V_0\circ(U_0)^{-1}"'] & \mathrm{Map}_{/A}(A^{\simeq},Y) \arrow[r, "\mathrm{ev}_0", "\simeq"'] & \mathrm{Map}_{/A_0}(A_0,Y_0) & \mathrm{Map}_{/A_0}(A_0,Y_0)\,, \arrow[l, "\simeq"', "{}_*U_0"]
\end{tikzcd} \tag{1.4.1.1}$$

where the left hand square is a pullback by definition, and the right hand square commutes by functoriality of ev_0. The bottom middle horizontal application of ev_0 is an equivalence by the adjunction disc $\dashv$ ev_0, In particular the composite rectangle is also a pullback. The defining pullback

$$\begin{tikzcd}
Y_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\eta_Y"] \\
\mathrm{codisc}(A_0) \arrow[r, "\mathrm{codisc}(V_0)"] & \mathrm{codisc}(Y_0)
\end{tikzcd}$$

also induces the cube

$$
\begin{tikzcd}
\Delta^0 \arrow[r, "F"] & \mathrm{Fun}(X,A) \arrow[rr, equal] \arrow[dd, "(\eta_A)_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(X,A) \arrow[dd, "(\eta_A)_*"] \\
\mathrm{Fun}(X,Y_V) \arrow[ur] \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(X,Y) \arrow[ur, "G_*"] \arrow[dd, "(\eta_Y)_*"', near end] & \\
& \mathrm{Fun}(X,\mathrm{codisc}(A_0)) \arrow[rr, equal] & & \mathrm{Fun}(X,\mathrm{codisc}(A_0))\, , \\
\mathrm{Fun}(X,\mathrm{codisc}(A_0)) \arrow[ur, equal] \arrow[rr, "\mathrm{codisc}(V_0)_*"] & & \mathrm{Fun}(X,\mathrm{codisc}(Y_0)) \arrow[ur, "\mathrm{codisc}(G_0)_*"'] &
\end{tikzcd}
$$

whose front and back squares are pullbacks, hence we get on fibers over $F\colon \Delta^0 \to \mathrm{Map}(X,A)$ the top pullback in

$$
\begin{tikzcd}
\mathrm{Map}_{/A}(X,Y_V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "\mathrm{ev}_0"'] & \mathrm{Map}_{/A}(X,Y) \arrow[d, "(\eta_Y)_*"] \arrow[dd, bend left=60, "\mathrm{ev}_0"] \\
\mathrm{Map}_{/\,\mathrm{codisc}(A_0)}(X,\mathrm{codisc}(A_0)) \arrow[r, "\mathrm{codisc}(V_0)_*"'] \arrow[d, "\mathrm{ev}_0", "\simeq"'] & \mathrm{Map}_{/\,\mathrm{codisc}(A_0)}(X,\mathrm{codisc}(Y_0)) \arrow[d, "\mathrm{ev}_0"', "\simeq"] \\
\mathrm{Map}_{/A_0}(X_0,A_0) \arrow[r, "(V_0)_*"'] & \mathrm{Map}_{/A_0}(X_0,Y_0)\, . \\
\Delta^0 \arrow[u, "\simeq"'] \arrow[ur, "V_0\circ(U_0)^{-1}"'] &
\end{tikzcd}
\tag{1.4.1.2}
$$

Its bottom square is again as above induced by the adjunction $\mathrm{ev}_0 \dashv \mathrm{codisc}$ and the composite outer rectangle agrees with the one induced by functoriality of ev_0. But by our assumption that U_0 was invertible, we also have that $F_0\colon X_0 \to A_0$ is an equivalence, hence we get the contractibility $\mathrm{Map}_{/A_0}(X_0,A_0) \simeq \Delta^0$ and the lower triangle of the above diagram. The right hand side triangle, the left side is similar, of this diagram commutes as it is the pullback of the diagram

$$
\begin{tikzcd}
& \mathrm{Map}(X,Y) \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "(\eta_Y)_*"] \arrow[dd, "G_*"] & \\
\mathrm{Map}(X_0,Y_0) \arrow[dd, "(G_0)_*"'] & & \mathrm{Map}(X,\mathrm{codisc}(Y_0)) \arrow[ll, "\sim", "\mathrm{ev}_0"'] \arrow[dd, "\mathrm{codisc}(G_0)_*"] \\
& \mathrm{Map}(X,A) \arrow[dl, "\mathrm{ev}_0"'] \arrow[dr, "(\eta_A)_*"] & \\
\mathrm{Map}(X_0,A_0) & & \mathrm{Map}(X,\mathrm{codisc}(A_0)) \arrow[ll, "\sim", "\mathrm{ev}_0"']
\end{tikzcd}
$$

along $F\colon \Delta^0 \to \mathrm{Map}(X,A)$, whose horizontal functors are equivalences by $\mathrm{ev}_0 \dashv \mathrm{codisc}$. Hence our two composite pullbacks Equation 1.4.1.1 and Equation 1.4.1.2 have to agree and we thus get a canonical equivalence

$$
\mathrm{Map}_{/A}^{A^{\simeq}\to A/}(X,Y) \xrightarrow{\simeq} \mathrm{Map}_{/A}(X,Y_V)\ ,
$$

which is furthermore natural in the first variable with respect to objects whose underlying 2-functor $F\colon X \to A$ induces an equivalence after $(-)^{\simeq}$, and where the left hand side is the mapping space of ${}^{(A^{\simeq}\to A)/}(\mathrm{CSCocart}^{\mathrm{glob}}(\mathbf{\Delta}^{\mathrm{op}})_{/(A\longrightarrow \mathbf{\Delta}^{\mathrm{op}})})$ and the right hand side the one of

$\mathrm{Cocart}^{\mathrm{glob}}_{\mathrm{Segal}}(\mathbf{\Delta}^{\mathrm{op}})_{/(A \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}})}$. Here for both 1-categories we can choose our morphisms not just to be fully cocartesian functors, but actually more general and only partially cocartesian functors, like inert-cocartesian ones, as the constructed mapping space equivalence above is agnostic to those choices. For the envelope construction we know that

$$\begin{tikzcd} & A^{\simeq} \arrow[dl] \arrow[dr, "\iota^{\simeq}"] \arrow[ddd, bend right, two heads] & \\ A \arrow[rr, "\iota"] \arrow[dr, equal] \arrow[ddr, bend right, two heads] & & \mathrm{Env}(A) \arrow[dl, "\lambda"'] \arrow[ddl, bend left, two heads] \\ & A \arrow[d, two heads] & \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{tikzcd}$$

is a diagram of equivalences in the fiber over Δ^0, by [Glo25, Proposition 2.8 and Lemma 2.14]. Hence precomposition with ι gives us the following commutative square.

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{cocart},(A^{\simeq}\to A)/}_{/A}((\mathrm{Env}(A),\lambda,\iota^{\simeq}),(Y,G,V^{\simeq})) \arrow[r, "\iota^*"] \arrow[d, "\simeq"'] & \mathrm{Map}^{\text{inert-cocart},(A^{\simeq}\to A)/}_{/A}((A,\mathrm{id}_A,\gamma),(Y,G,V^{\simeq})) \arrow[d, "\simeq"] \\ \mathrm{Map}^{\mathrm{cocart}}_{/A}((\mathrm{Env}(A),\lambda),(Y_V,G_V)) \arrow[r, "\iota^*"'] & \mathrm{Map}^{\text{inert-cocart}}_{/A}((A,\mathrm{id}_A),(Y_V,G_V)) \end{tikzcd} \tag{1.4.1.3}$$

As pullbacks as on the left below are computed on fibers over a Δ^n as on the right below

$$\begin{tikzcd} X_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \mathrm{codisc}(A_0) \arrow[r, "\mathrm{codisc}(U_0)"'] & \mathrm{codisc}(X_0) \end{tikzcd} \qquad \begin{tikzcd} (X_U)_n \arrow[r] \arrow[d, "{(\mathrm{ev}_0,\dots,\mathrm{ev}_n)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d, "{(\mathrm{ev}_0,\dots,\mathrm{ev}_n)}"] \\ A_0 \times \cdots \times A_0 \arrow[r, "U_0\times\cdots\times U_0"'] & X_0 \times \cdots \times X_0 \end{tikzcd}$$

we can in particular conclude that the functors on fibers over a pair $a, \widetilde{a}$ of objects of A_0

$$\begin{tikzcd} X_U(a,\widetilde{a}) \arrow[r, "\simeq"] \arrow[dr, "(F_U)_{a,\widetilde{a}}"'] & X(U_0(a),U_0(\widetilde{a})) \arrow[d, "F_{a,\widetilde{a}}"] \\ & A(a,\widetilde{a}) \end{tikzcd}$$

is an equivalence. This means in particular that an object of ${}^{(A^{\simeq}\to A)/}(2\text{-}\mathrm{Cat}_{/A})$ as on the left below has local right adjoint sections if and only if its induced cocartesian functor as of the right below also has them, but for all its objects in $(X_U)_0 \simeq A_0$.

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U"] \arrow[dr] & X \arrow[d, "F"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} X_U \arrow[r, "F_U"] \arrow[dr, two heads] & A \arrow[d, two heads] \\ & \mathbf{\Delta}^{\mathrm{op}} \end{tikzcd}$$

Similarly, a morphism between such objects as depicted on the left below is vertically adjointable on the objects coming from A_0 if and only if the morphism between their induced cocartesian functors to A as below on the right is vertically adjointable but for all objects $(X_U)_0 \simeq A_0 \simeq (Y_V)_0$.

$$\begin{tikzcd} & A^{\simeq} \arrow[dl, "U"'] \arrow[dr, "V"] \arrow[dd, bend left] & \\ X \arrow[rr, "H"] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} X_U \arrow[rr, "H"] \arrow[dr, "F_U"'] & & Y_V \arrow[dl, "G_V"] \\ & A & \end{tikzcd}$$

Hence we may restrict our commutative square Equation 1.4.1.3 everywhere to vertically adjointable morphisms as in

$$\begin{tikzcd} \mathrm{Map}_{/A}^{\mathrm{cocart,vadj},(A^{\simeq}\to A)/}((\mathrm{Env}(A),\lambda,\iota^{\simeq}),(Y,G,V^{\simeq})) \arrow[r, "\simeq"] \arrow[d, "{}_*\iota"'] & \mathrm{Map}_{/A}^{\mathrm{cocart,vadj}}((\mathrm{Env}(A),\lambda),(Y_V,G_V)) \arrow[d, "{}_*\iota"] \\ \mathrm{Map}_{/A}^{\mathrm{inert\text{-}cocart,vadj},(A^{\simeq}\to A)/}((A,\mathrm{id}_A,\gamma),(Y,G,V^{\simeq})) \arrow[r, "\simeq"'] & \mathrm{Map}_{/A}^{\mathrm{inert\text{-}cocart,vadj}}((A,\mathrm{id}_A),(Y_V,G_V))\,. \end{tikzcd}$$

Thus we may employ the fact that the right hand precomposition with ι functor is an equivalence even in the situation that $Y_V \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is only globular and Segal, as proved in [Glo25, Theorem 3.6] to conclude the desired initiality statement. □

Now that we know that lax functor classifiers satisfying the extended universal property Definition 1.1.1 exist at least in one model of 2-categories, we record some compatibilities of $\mathrm{Lax}(-)$ with respect to the different ways of forming opposite 2-categories, i. e. dualising. For this, we take for granted the 1-morphism opposite $(-)^{1\text{-op}}$, and the 2-morphism opposite $(-)^{2\text{-op}}$ of 2-categories, which in the globular complete Segal objects $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ model for example are given by the opposite on $\mathbf{\Delta}$ and Cat, respectively. These will be useful to dualise the later Theorem 2.4.3 in an efficient and formal way.

Lemma 1.4.6

For every 2-category A, we have a canonical 2-equivalence $\mathrm{Lax}(A^{1\text{-op}}) \simeq \mathrm{Lax}(A)^{1\text{-op}}$.

Proof. The 1-autoequivalence $(-)^{1\text{-op}}\colon 2\text{-Cat} \simeq 2\text{-Cat}$ induces $(-)^{1\text{-op}}\colon 2\text{-Cat}_{/A} \simeq 2\text{-Cat}_{/A^{1\text{-op}}}$ and furthermore $(-)^{1\text{-op}}\colon {}^{(A^{\simeq}\to A)/}(2\text{-Cat}_{/A}) \simeq {}^{((A^{1\text{-op}})^{\simeq}\to A^{1\text{-op}})/}(2\text{-Cat}_{/A^{1\text{-op}}})$ which also sends the non-full subcategory $\mathrm{LocRARI}(A)$ to $\mathrm{LocRARI}(A^{1\text{-op}})$, hence in particular induces a 1-equivalence $(-)^{1\text{-op}}\colon \mathrm{LocRARI}(A) \simeq \mathrm{LocRARI}(A^{1\text{-op}})$. □

Lemma 1.4.7

For every 2-category A, the object

$$\begin{tikzcd} A^{\simeq} \arrow[r, "\simeq"] \arrow[drr] & ((A^{\simeq})^{2\text{-op}})^{2\text{-op}} \arrow[r, "(\iota^{\simeq}_{A^{2\text{-op}}})^{2\text{-op}}"] \arrow[drr] & (\mathrm{Lax}(A^{2\text{-op}}))^{2\text{-op}} \arrow[d, "(\lambda_{A^{2\text{-op}}})^{2\text{-op}}"] \\ & & A \arrow[r, "\simeq"] & (A^{2\text{-op}})^{2\text{-op}} \end{tikzcd}$$

satisfies the extended universal property of the oplax 2-functor classifier of A, i. e. we have a canonical 2-equivalence $\mathrm{OpLax}(A) \simeq \mathrm{Lax}(A^{2\text{-op}})^{2\text{-op}}$.

Proof. The 1-autoequivalence $(-)^{2\text{-op}}\colon 2\text{-Cat} \simeq 2\text{-Cat}$ induces $(-)^{2\text{-op}}\colon 2\text{-Cat}_{/A} \simeq 2\text{-Cat}_{/A^{2\text{-op}}}$ and furthermore $(-)^{2\text{-op}}\colon {}^{(A^{\simeq}\to A)/}(2\text{-Cat}_{/A}) \simeq {}^{((A^{2\text{-op}})^{\simeq}\to A^{2\text{-op}})/}(2\text{-Cat}_{/A^{2\text{-op}}})$ which also sends the non-full subcategory $\mathrm{LocLALI}(A)$ to $\mathrm{LocRARI}(A^{2\text{-op}})$, as applying $(-)^{2\text{-op}}$ to an object which has local left adjoint sections will turn Hom-functors which have fully-faithful left adjoints to Hom-functors which have fully-faithful right adjoints, as $B^{2\text{-op}}(b, \tilde{b}) = B(b, \tilde{b})^{\mathrm{op}}$. Hence it in particular induces a 1-equivalence $(-)^{2\text{-op}}\colon \mathrm{LocLALI}(A) \simeq \mathrm{LocRARI}(A^{2\text{-op}})$. $\square$

Chapter 2

Constructing Lax Adjoints from Objectwise Data

2.1 Introduction

Motivation

In this chapter we want to showcase first of all, that one *can* work synthetically with the definition of lax 2-functors given in Definition 1.1.1. Second of all, we want to showcase *how* one can work synthetically in higher category theory building upon this synthetic definition of lax functors.

More precisely we want to demonstrate this both on the side of theory and on the side of practice.

On the side of theory we exemplify the applicability of Definition 1.1.1 by using it to reformulate the basic building blocks of *monad theory* in a synthetic manner, provide access to its primordial source of examples, i. e. extracting monads from adjunctions, as well as establish some of the foundational ingredients for the theory of monadicity characterisations.

Monad theory and especially the monadicity theorems play the leading role for example in Lurie's setup of higher categorical descent theory in [Lur17]. Explicitly, a morphism f satisfies descent if and only if its appropriate restriction of scalars functor f^* is comonadic. This recovers faithfully-flat descent for animated rings and E_∞-ring spectra in the realm of derived and spectral algebraic geometry, see for example [Lur18].

On the practical side we want ways to construct examples of lax 2-functors in a synthetic way. In higher category theory, the construction of functors is in principle difficult as it entails providing an infinite amount of coherence data, and even more so for their lax analogues. In practice, functors are often constructed by breaking them down into functors themselves induced through universal properties. Among the few ways where it is actually sufficient to provide the assignment on objects is the sometimes called *objectwise existence criterion for adjoints*, i. e. the (re)construction of adjoint functors from objectwise given (co)unit morphisms and their universal properties. We provide a direct lax generalisation of such an objectwise (re)construction principle to the context of lax 2-functors and demonstrate its usefulness by constructing several different kinds of examples.

On the technical side we also try to adhere to synthetic principles and let us guide ourselves by type-theoretic methods to facilitate the above applications. Concretely, our main technical ingredient is a Hom-category characterisation of $(\infty, 2)$-categorical Grothendieck constructions. Such Hom-characterisations are common practice in Homotopy Type Theory and follow a concrete recipe. In HoTT, the Hom-characterisation of $(\infty, 0)$-categorical Grothendieck constructions, there called the *identity-type characterisation of* Σ*-types*, has a simple and conceptual proof we try to replicate and generalise to $(\infty, 2)$-category theory.

Intuition

Monads as Lax Functors Ordinary 2-category theoretical intuition tells us that monad theory is a prime candidate to be developed synthetically based on our synthetic universal properties. More precisely, most of the basic building blocks of monad theory can be rephrased in the language of lax 2-functors. We start with the observation we already made in the introduction of chapter 1, that monads in an ordinary 2-category A can be identified with ordinary lax functors $\Delta^0 \rightsquigarrow A$. In this reformulation, monad morphisms correspond to lax natural transformations between such lax functors. The algebra category of a monad may then be recovered as the lax limit of the lax functors. Even the starting point of all monadicity considerations, the canonical algebra resolution of a T-algebra, can be expressed in terms of the lax functor classifier.

Objectwise (Re)Construction of Lax Functors As stated in the motivation above, objectwise reconstruction principles for functors and natural transformations are indispensable tools to construct examples by synthetically generating the infinite coherence underlying higher categorical concepts without the explicit need to parametrise it combinatorially. The classical literature on ordinary 2-categories already holds such an objectwise reconstruction principle that naturally produces lax functors, in the article [BP88]. There, the authors Betti and Power realised, that lax 2-functors arise canonically through the generalisation of the objectwise existence criterion for 2-adjunctions to the theory of *lax*, or *local* 2-adjunctions. More precisely, for a strict 2-functor $F\colon A \to B$ and for every object b of B an object Ub of A together with 1-morphisms $\varepsilon_b\colon F(Ub) \to b$, we may ask the induced 1-functors between Hom-1-categories

$$B(F(a), b) \xleftarrow{(\varepsilon_b)_*} B(F(a), F(Ub)) \xleftarrow{F_{a,Ub}} A(a, Ub)$$

to be left adjoints, in contrast to just equivalences. Then the objects Ub and the Hom-composites

$$U_{b,\tilde{b}}\colon B(b, \tilde{b}) \xrightarrow{(\varepsilon_b)_*} B(F(Ub), \tilde{b}) \xrightarrow{R_{Ub,\tilde{b}}} A(Ub, U\tilde{b})$$

with the right adjoints to the above, will in general *fail* to constitute a strict 2-functor. However, the universal properties contained in these Hom-adjunctions are enough to turn U into a *lax* 2-functor, as well as the components ε_b into a *lax* natural transformation.

Goal

The overarching goal of this chapter is to showcase that building upon the extended universal property we can both synthetically (re)develop a bit of category theory and synthetically construct examples.

To this end, first actual goal is to develop a bit of synthetic monad theory. Here we will apply the extended universal property of the lax functor classifier to access the most notorious source of monads, i. e. to extract them from adjunctions, as well as the canonical comparison functors to the induced algebra categories.

The second goal is to provide an efficient tool to synthetically construct examples and for this we prove a higher categorical generalisation of Betti and Power's objectwise lax adjoint existence criterion. To showcase its applicability, we proceed to construct various examples between span 2-categories and profunctor 2-categories, and furthermore discuss how *deriving* functors naturally gives rise to lax functors. Finally we also demonstrate its usefulness to (re)construct lax 2-adjunction data, even when both 2-functors turn out to be strict.

Organisation of the chapter

In section 2.2 we prove the main technical ingredient of this chapter, the Hom-category characterisation of 2-categorical Grothendieck constructions, giving us enough control over these Hom-categories to apply the extended universal property. In section 2.3 we tease a bit of synthetic monad theory building upon the extended universal property. In section 2.4 we prove the main result of this chapter, the objectwise (re)construction principle for lax adjoints from [BP88] as an immediate application of the extended universal property and our Hom-characterisation. In section 2.5 we discuss various sources of examples where the objectwise existence criterion can be applied to give synthetic constructions of the desired lax functors.

Additional Conventions

In contrast to the former chapter 1, all objects of interest will be 2-categories in this chapter, unless specified differently. In particular we will also use Cat or 2-Cat to denote the 2-categorical versions of these.

2.2 Characterising the Hom-categories of Unstraightenings

In this section we prove the main technical ingredient, the characterisation of the Hom-1-categories of 2-categorical Grothendieck constructions Theorem 2.2.28, of our two direct applications of the extended universal property Definition 1.1.1, the extraction of monads from adjunctions Corollary 2.3.3 and the objectwise (re)construction principle for lax adjoints Theorem 2.4.3. Let us now motivate Theorem 2.2.28 from this practical point of view.

In both these examples we start with a collection of 1-morphisms in a 2-category, enjoying certain universal properties. The collection of 1-morphisms is parametrised over the object space $A^{\simeq}$ of another 2-category A, out of which we would like to construct our lax 2-functor. To be able to apply Definition 1.1.1, we need to find a commutative triangle of 2-functors

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A\,. \end{tikzcd}$$

With our $A^{\simeq}$-parametrised collection of 1-morphisms we already have a good candidate for what $U^{\simeq}$ could do, i. e. simply realise this assignment. To this end, X needs to be an appropriate 2-category with objects the 1-morphisms we consider, also with respect to their domains and codomains. In category theory the canonical moduli of 1-morphisms is the arrow category, and for 2-categories the *lax arrow 2-category*. The idea behind our two applications Corollary 2.3.3 and Theorem 2.4.3 is that we can look at our $A^{\simeq}$-parametrised collection of 1-morphisms we started with but inside an appropriately chosen moduli 2-category X and transform the individual universal properties the 1-morphisms enjoy into universal properties inside $X(U^{\simeq}(a), U^{\simeq}(\tilde{a}))$ with respect to the Hom-actions $X(U^{\simeq}(a), U^{\simeq}(\tilde{a})) \to A(a, \tilde{a})$ of an appropriately chosen forgetful

2-functor $X \to A$. To be able to achieve this we need enough control over the Hom-1-categories of our 1-morphism moduli X, which in our two applications turn out to be variations of the canonical candidate, the lax arrow 2-category of A, which itself is the Grothendieck construction of the Hom-functors $A(-,-)$.

From a type-theoretic point of view on synthetic category theory, such a Hom-characterisation of a given construction is a very standard procedure and often follows a generic blueprint. The idea is that the Hom-type of a given type former should correspond to the type former applied to appropriate Hom-types of the type former's input. Theorem 2.2.28 follows this blueprint, it characterises the Hom-1-categories of a 2-categorical Grothendieck construction of a $F\colon A \to \mathrm{Cat}$ as the 1-categorical Grothendieck construction for the appropriate Hom-spaces of the individual 1-categories $F(a)$ for objects a of A. However, the type-theoretic point of view does not just provide us with a guess on the general result of such a Hom-characterisation, it also gives a generic recipe on how to prove it, i. e. construct an equivalence between Hom-type of type former and type former applied to appropriate Hom-types. One direction of the equivalence, from the former to the latter, is constructed using the induction principle for the Hom-types, for our Hom-1-categories it is the 2-Yoneda Lemma. The other direction is constructed by applying the induction principle of the type former in question to reduce to the task of constructing a map out of the appropriate Hom-types of the input, which brings us into the situation where we may apply the Hom-induction again, i. e. the Yoneda Lemma. However for dependent type formers, like our 2-categorical Grothendieck construction over some base 2-category A, these appropriate Hom-types, in our case the ones of the $F(a)$, will themselves depend again on the terms of the base, here the a in A, i. e. they will live in context over A, which in our case means that they live in $\mathrm{Fun}(A, \mathrm{Cat})$. Hence also the Yoneda Lemma we would like to employ will be the one *internal* to $\mathrm{Fun}(A, \mathrm{Cat})$.

To this end this section is structured into three parts. In the first part we will introduce the necessary background on 2-categorical fibrations and their Grothendieck constructions as well as the 2-categorical Yoneda Lemma. After this we will venture inside $\mathrm{Fun}(A, \mathrm{Cat})$ to find an appropriate version of its internal Yoneda Lemma. Equipped with this knowledge we will then proceed to copy the type-theoretic recipe to prove the Hom-characterisation of 2-categorical Grothendieck constructions as Grothendieck constructions of the Homs.

2.2.1 Basics on 2-Categorical Fibrations and Straightening-Unstraightening

In this subsection we start by revising a central part of 2-category theory, namely 2-categorical notions of (co)cartesian fibrations and their corresponding Grothedieck constructions, i. e. the 2-categorical straightening-unstraightening equivalence.

As this subsection is just a recollection, all statements here are already present in the literature. All the basic properties of 2-cocartesian fibrations can be found for example in [GR17, Chapter 11].

We start with the very definition of 2-categorical fibrations.

Definition 2.2.1

Let $p\colon X \to A$ be 2-functor. A 1-morphism $\phi\colon x_0 \to x_1$ is called 2-cocartesian if for all objects $\widetilde{x}$ of X the canonical commutative square

$$\begin{tikzcd} X(x_1, \widetilde{x}) \arrow[r, "{}_*\phi"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X(x_0, \widetilde{x}) \arrow[d, "p"] \\ A(p(x_1), p(\widetilde{x})) \arrow[r, "{}_*p(\phi)"'] & A(p(x_0), p(\widetilde{x})) \end{tikzcd}$$

induced by precomposition is a pullback. Dually we call it 2-cartesian if for all objects $\widetilde{x}$ of X the canonical commutative square

$$\begin{tikzcd} X(\widetilde{x},x_0) \arrow[r, "\phi_*"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X(\widetilde{x},x_1) \arrow[d, two heads, "p"] \\ A(p(\widetilde{x}),p(x_0)) \arrow[r, "p(\phi)_*"'] & A(p(\widetilde{x}),p(x_1)) \end{tikzcd}$$

induced by postcomposition is a pullback. We call the 2-functor p a 2-cocartesian fibration, respectively 2-cartesian, if it satisfies

1. for every object x of X and every 1-morphism $f\colon p(x) \to a$ there exists a 2-cocartesian 1-morphism $x \to f_! x$ lifting f, i. e. $p(x \to f_!) \simeq f$, respectively for every object x of X and every 1-morphism $f\colon a \to p(x)$ there exists a 2-cartesian 1-morphism $f^*x \to x$ lifting f,
2. for every pair $x, \widetilde{x}$ of X the action of p on Hom-1-categories
$$p\colon X(x,\widetilde{x}) \to A(p(x),p(\widetilde{x}))$$
is a cartesian, respectively cocartesian, fibration of 1-categories,
3. for every 1-morphism $\phi\colon x_0 \to x_1$ and another arbitrary object $\widetilde{x}$ of X we have that the functors ${}_*\phi$ and ϕ_* induced by pre- and postcomposition with ϕ define cartesian, respectively cocartesian, functors over the respective pre- and postcompostion functors ${}_*p(\phi)$ and $p(\phi)_*$ between the actions of p on Hom-1-categories as in the two squares below.

$$\begin{tikzcd} X(x_1,\widetilde{x}) \arrow[r, "{}_*\phi"] \arrow[d, two heads, "p"'] & X(x_0,\widetilde{x}) \arrow[d, two heads, "p"] \\ A(p(x_1),p(\widetilde{x})) \arrow[r, "{}_*p(\phi)"'] & A(p(x_0),p(\widetilde{x})) \end{tikzcd} \qquad \begin{tikzcd} X(\widetilde{x},x_0) \arrow[r, "\phi_*"] \arrow[d, two heads, "p"'] & X(\widetilde{x},x_1) \arrow[d, two heads, "p"] \\ A(p(\widetilde{x}),p(x_0)) \arrow[r, "p(\phi)_*"'] & A(p(\widetilde{x}),p(x_1)) \end{tikzcd}$$

2-Categorical fibrations enjoy the same stability properties as their 1-categorical analogues.

Remark 2.2.2

2-Cocartesian fibrations are stable under pullback and composition.

Next we define the corresponding notion of morphisms between 2-categorical fibrations.

Definition 2.2.3

A 2-functor F between 2-cocartesian, respectively 2-cartesian, fibrations p and q

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd}$$

over A is called a 2-cocartesian functor, respectively 2-cartesian functor, if

1. F sends p-2-cocartesian, respectively p-2-cartesian, 1-morphisms to q-2-cocartesian 1-morphisms, respectively q-2-cartesian, and
2. on all Hom-1-categories

$$\begin{tikzcd} X(x,\widetilde{x}) \arrow[rr, "F"] \arrow[dr, two heads, "p"'] & & Y(F(x),F(\widetilde{x})) \arrow[dl, two heads, "q"] \\ & A(p(x),p(\widetilde{x})) & \end{tikzcd}$$

F induces a cartesian, respectively cocartesian, functor between cartesian, respectively cocartesian, fibrations between 1-categories.

Also these morphisms satisfy the respective 2-categorical generalisations of stability properties their 1-categorical analogues enjoy.

Remark 2.2.4

2-Cocartesian functors over some 2-category A compose and are are stable under pullback along arbitrary 2-functors $B \to A$.

As we would like to apply our Theorem 2.4.3 in all possible dualised ways we need to have all possible dualised versions of 2-cocartesian fibrations.

Remark 2.2.5

We also have the two other notions of 2-fibrations with the combinations $(\mathrm{cocartesian}, \mathrm{cocartesian})$ and $(\mathrm{cartesian}, \mathrm{cartesian})$ for 1- and 2-morphisms.

One of the most essential core pieces of 2-category theory we will employ throughout is the induction principle for Hom-categories, the 2-categorical Yoneda Lemma.

Theorem 2.2.6 (2-Yoneda Lemma)

For every 2-category A, we have that

1. the 2-Yoneda functor $y\colon A \hookrightarrow \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})$, given by currying the Hom-functor, is 2-fully-faithful, i. e. induces an equivalence on all Hom-1-categories, and
2. We have a 2-natural equivalence

$$\mathrm{Fun}(A, \mathrm{Cat})(y_{A^{1\text{-op}}}, -) \simeq \mathrm{ev}$$

of 2-functors $A \times \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Cat}$ such that for every object a of A and every 2-functor $F\colon A \to \mathrm{Cat}$, the component of this 2-natural equivalence is given by the canonical evaluation at id_a 1-functor

$$\mathrm{Fun}(A, \mathrm{Cat})(A(a, -), F) \xrightarrow{\mathrm{ev}_a} \mathrm{Fun}(A(a, a), F(a)) \xrightarrow{\mathrm{ev}_{\mathrm{id}_a}} F(a)\ .$$

Reference. See [Abe23a, Theorem 6.7] with the identification of the 2-Yoneda equivalence components in its proof. ■

We now formulate the fundamental principle to work with the universe 2-Cat of all small 2-categories, the 2-categorical straightening-unstraightening equivalence.

Theorem 2.2.7 (2-Categorical Straightening-Unstraightening Equivalence)

For small 2-categories A the canonical 2-functor

$$\overrightarrow{\int_{a\colon A}} (-)\colon\ \mathrm{Fun}(A, 2\text{-Cat}) \to 2\text{-Cocart}(A)$$

$$F \mapsto [A \times^{(F, p_{\mathrm{univ}})}_{2\text{-Cat}} 2\text{-Cat}_\bullet \xrightarrow{\mathrm{pr}_A} A]\ ,$$

induced by 2-Yoneda on the universal 2-cocartesian fibration with small fibers $p_{\mathrm{univ}}\colon 2\text{-Cat}_\bullet \twoheadrightarrow 2\text{-Cat}$ of the 2-presheaf 2-Cocart$(-)$ with pullback action is an equivalence of 2-categories 2-natural in A, called the 2-cocartesian straigthening-unstraightening equivalence. We will call its inverse equivalence Str_A. Similary we have the 2-cartesian straightening-unstraightening equivalence

$$\overleftarrow{\int_{a\colon A}}(-)\colon\ \mathrm{Fun}(A^{1\text{-op}}, 2\text{-Cat}) \simeq 2\text{-Cart}(A)$$

and also a version of the equivalence relating 2-functors $A^{2\text{-op}} \to \mathrm{Cat}$ to (cocartesian, cocartesian)-fibrations, as well as an equivalence relating 2-functors $A^{1\&2\text{-op}} \to \mathrm{Cat}$ to (cartesian, cartesian)-fibrations. By restricting to the full sub-2-category $\mathrm{Cat} \subseteq 2\text{-Cat}$ of small 1-categories we can deduce restricted straigthening-unstraightening equivalences

$$\mathrm{Fun}(A, \mathrm{Cat}) \simeq 2\text{-Cocart}_{1\text{-catfib}}(A) \quad \text{and} \quad \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat}) \simeq 2\text{-Cart}_{1\text{-catfib}}(A)\ .$$

Reference. See [Nui24, Theorem 6.20] for $d = 2$. ■

We will later also use the following version of straightening-unstraightening which deals with mixed variance functors, like the Hom-functors $A(-,-)$ to 2-Cat.

Corollary 2.2.8

For mixed variance 2-functors $A^{1\text{-op}} \times B \to 2\text{-Cat}$ we can first apply 2-cartesian unstraightening and then 2-cocartesian unstraightening

$$\begin{aligned}\mathrm{Fun}(A^{1\text{-op}} \times B, 2\text{-Cat}) &\simeq \mathrm{Fun}(B, \mathrm{Fun}(A^{1\text{-op}}, 2\text{-Cat}) \simeq \mathrm{Fun}(B, 2\text{-Cart}(A)) \\ &\hookrightarrow \mathrm{Fun}(B, 2\text{-Cat}_{/A}) \simeq \mathrm{Fun}(B, 2\text{-Cat})_{/\,\mathrm{const}(A)} \simeq 2\text{-Cocart}(B)_{/(A\times B \xrightarrow{\mathrm{pr}_B} B)}\end{aligned}$$

to arrive at the so-called two-sided unstraightening 2-functor.

Now that we introduced straightening-unstraightening, let us give some names to the most frequently used unstraightenings.

Notation 2.2.9

For a 2-category A and an object a of A we have the representable 2-functor $A(a,-)\colon A \to \mathrm{Cat}$ at a and we are going to denote its 2-cocartesian, that is here its 2-cocartesian unstraightening by

$${}^{a/^{\mathrm{lax}}}A \coloneqq \overrightarrow{\int_{\tilde{a}\colon A}} A(a,\tilde{a})\ .$$

We can also 2-cocartesian unstraighten the full Hom-functor $A(-,-)\colon A^{1\text{-op}} \times A \to \mathrm{Cat}$ to get

$$\mathrm{Tw}^{\mathrm{lax}}(A) \coloneqq \overrightarrow{\int_{(a,\tilde{a})\colon A^{1\text{-op}}\times A}} A(a,\tilde{a})\ .$$

Furthermore we denote the two-sided 2-unstraightening of $A(-,-)\colon A^{1\text{-op}} \times A \to \mathrm{Cat}$ of A by

$$\mathrm{Arr}^{\mathrm{lax}}(A) \coloneqq \overrightarrow{\int_{\tilde{a}\colon A}} \overleftarrow{\int_{a\colon A}} A(a,\tilde{a})$$

and we will call its projection onto the first and the second variable of the Hom-categories

$$A \xleftarrow{\mathrm{dom}} \mathrm{Arr}^{\mathrm{lax}}(A) \xrightarrow{\mathrm{cod}} A\ ,$$

respectively. Dually, we can postcompose $A(-,-)$ with the the 2-equivalence $(-)^{\mathrm{op}}\colon \mathrm{Cat} \simeq \mathrm{Cat}^{2\text{-op}}$ use

$$\begin{aligned}\mathrm{Fun}(A^{1\text{-op}} \times A, \mathrm{Cat}) \xrightarrow{((-)^{\mathrm{op}})_*} \mathrm{Fun}(A^{1\text{-op}} \times A, \mathrm{Cat}^{2\text{-op}}) &\simeq \mathrm{Fun}(A^{1\&2\text{-op}} \times A^{2\text{-op}}, \mathrm{Cat}) \\ &\simeq \mathrm{Fun}(A^{2\text{-op}}, \mathrm{Fun}(A^{1\&2\text{-op}}, \mathrm{Cat}))\ ,\end{aligned}$$

then apply first (cartesian, cartesian)-unstraightening inside and lastly (cocartesian, cocartesian)-unstraightening outside to get

$$A \xleftarrow{\mathrm{dom}} \mathrm{Arr}^{\mathrm{oplax}}(A) := \int^{\rightarrow}_{\tilde{a}\colon A} \int^{\leftarrow}_{a\colon A} A(a, \tilde{a}) \xrightarrow{\mathrm{cod}} A\ .$$

Remark 2.2.10

In the rest of this chapter, we will *define* a lax natural transformation $\alpha\colon F \Rightarrow G$ between two 2-functors $F, G\colon A \rightsquigarrow B$, which may even be lax, as a commutative triangle

$$\begin{tikzcd} & & B \\ A \arrow[urr, squiggly, "F"] \arrow[r, squiggly, "\alpha"] \arrow[drr, squiggly, "G"'] & \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[ur, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\ & & B \end{tikzcd}$$

of possibly lax 2-functors. Note here, that this definition should be constrasted with the more standard definition of a lax natural transformation as a 2-functor $\Delta^1 \otimes_{\mathrm{lax}} A \to B$ out of the 2-lax Gray tensor product. A synthetic treatment of the lax Gray tensor product is the purpose of the next chapter 3, however we will not discuss a comparison of these two possible definitions, hence we will stick to the definition using $\mathrm{Arr}^{\mathrm{lax}}(B)$ and *not* use the corresponding notion defined using $\Delta^1 \otimes_{\mathrm{lax}} A$ in any of the following and as well as the next chapter 3.

As we will only deal with 2-categorical fibrations fibered in 1-categories, it is useful to have a simpler characterisation of those when we need to prove 2-functors to be fibrations.

Lemma 2.2.11

Let $p\colon X \to A$ be a 2-functor, then the following are equivalent.

1. p is a 2-cocartesian fibration whose fibers are 1-categories,
2. p satisfies the first part of Definition 2.2.1 and on Hom-1-categories it induces right fibrations.

Proof. The second condition immediately implies the first one. For the other implication consider an arbitrary 2-morphism in X living over some 2-morphism in A, which we will depict as in the following diagram.

$$\begin{tikzcd} x \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[d, dash] & y \arrow[d, dash] \\ a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$$

We will factor it into a whiskering of a 2-morphism in a fiber followed by a Hom-cartesian 2-morphism, hence showing that it must have been Hom-cartesian to begin with. We start by factoring ψ into a 2-cocartesian 1-morphism followed by one in the fiber over b. Along this 2-cocartesian 1-morphism we can then Hom-cartesian pullback along the 2-morphism α in A.

$$
\begin{tikzcd}
x \arrow[r, bend left, "\alpha^* g_! x"] \arrow[r, bend right, "g_! x"'] \arrow[r, phantom, "\Downarrow\alpha^* g_! x"] \arrow[rr, bend right=40, "\psi"] \arrow[d, no head] & g_! x \arrow[r, dashed, "\exists!"] \arrow[d, no head] & y \arrow[dl, no head] \\
a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b &
\end{tikzcd}
$$

By the definition of 2-cocartesian fibrations, this above whiskering is then again Hom-cartesian. By its Hom-cartesianness we may now factor the 2-morphism β uniquely over it.

$$
\begin{tikzcd}
x \arrow[rr, bend left=50, "\phi"] \arrow[r, bend left, "\alpha^* g_! x"] \arrow[r, bend right, "g_! x"'] \arrow[r, phantom, "\Downarrow\alpha^* g_! x"] \arrow[d, no head] & g_! x \arrow[u, phantom, "\exists!\widetilde{\beta}"] \arrow[r, dashed, "\exists!"] \arrow[d, no head] & y \arrow[dl, no head] \\
a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b &
\end{tikzcd}
$$

This $\widetilde{\beta}$ now itself lives over id_f. By applying 2-cocartesian pushforward of $\widetilde{\beta}$ along f we can factor it uniquely into the following whiskering of the dashed 2-morphism in the fiber over b with the 2-cocartesian lift of x along f.

$$
\begin{tikzcd}
 & f_! x \arrow[r, dashed, "\exists!"] \arrow[rr, dashed, bend left, "\exists!"] \arrow[r, phantom, "\exists!"] \arrow[ddr, no head] & g_! x \arrow[r, dashed, "\exists!"] \arrow[dd, no head] & y \arrow[ddl, no head] \\
x \arrow[ur, "f_! x"] \arrow[urr, bend left, "\alpha^* g_! x"] \arrow[urr, bend right, "g_! x"'] \arrow[urr, phantom, "\alpha^* g_! x \Downarrow"] \arrow[urrr, bend right=40, "\psi"] \arrow[d, no head] & & & \\
a \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & b &
\end{tikzcd}
$$

□

The following lemma will be a first step towards internalising the theory of fibrations into the 2-categories $\mathrm{Fun}(A, \mathrm{Cat})$, i. e. every 2-cocartesian fibration between 2-cocartesian fibrations fibered in 1-categories automatically becomes a 1-morphism in $\mathrm{Fun}(A, \mathrm{Cat})$.

Lemma 2.2.12

For any commutative triangle of 2-cocartesian fibrations, whose fibers are 1-categories,

$$
\begin{tikzcd}
X \arrow[rr, two heads, "F"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\
 & A &
\end{tikzcd}
$$

the 2-functor F is automatically a 2-cocartesian functor over A.

Proof. By Lemma 2.2.11 we already know that p and q are on Hom-categories right fibrations, hence we automatically have that F preserves Hom-cartesian 2-morphisms for free. It suffices to show that F preserves 2-cocartesian 1-morphisms. To prove this we adhere to a more general phenomenon, namely that every F-2-cocartesian lift of a q-2-cocartesian 1-morphism is automatically also p-2-cocartesian, which can be deduced directly from pasting the following definitional pullback squares.

$$\begin{tikzcd}
X((f_!F(x))_!x, \widetilde{x}) \arrow[r, "{}_*(f_!F(x))_!x"] \arrow[d, "F"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, "p"'] & X(x, \widetilde{x}) \arrow[d, "F"] \arrow[dd, bend left=60, "p"] \\
Y(f_!F(x), F(\widetilde{x})) \arrow[r, "{}_*f_!F(x)"'] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y(F(x), F(\widetilde{x})) \arrow[d, "q"] \\
A(a, p(\widetilde{x})) \arrow[r, "{}_*f"'] & A(p(x), p(\widetilde{x}))
\end{tikzcd}$$

□

2.2.2 Interpreting Basic 1-Category Theory in 2-Categorical Presheaf Categories

In this subsection we realise the main part of the already hinted at strategy for our type-theoretic proof idea of Theorem 2.2.28, the Yoneda Lemma inside $\mathrm{Fun}(A, \mathrm{Cat})$. More precisely, we will not reprove this from scratch but rather make use of synthetically developed basic 1-category theory in the first three sections of Appendix A, in particular the fibrational version of the 1-categorical Yoneda Lemma in Corollary A.3.5. To this end the purpose of this subsection is to gather all the building blocks for the basic 1-category theory laid out in section A.1, show that they exist inside $\mathrm{Fun}(A, \mathrm{Cat})$, as well as prove that these internal notions agree with their external counterparts. To illustrate this, we would like to know for example that inside $\mathrm{Fun}(A, \mathrm{Cat})$ we have finite limits, a cartesian closure like $\mathrm{Fun}(-, -)$, an object Δ^1 that should parametrise morphisms internally, but this should not just be arbitrary. For 2-functors $F, G \colon A \to \mathrm{Cat}$, we also want to have internally defined natural transformations via $F \Rightarrow \mathrm{Fun}(\Delta^1, G)$ to correspond precisely to 2-morphisms

$$\begin{tikzcd}
F \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & G
\end{tikzcd}$$

in $\mathrm{Fun}(A, \mathrm{Cat})$, i. e. the morphisms in $\mathrm{Fun}(A, \mathrm{Cat})(F, G)$.

After we collect all the necessary properties for the interpretation of the first three sections of Appendix A inside $\mathrm{Fun}(A, \mathrm{Cat})$ we give proofs of these properties which are essentially just 2-categorical generalisations of the corresponding 1-categorical counterparts, which are already well established in the higher categorical literature. To close off, we make the actual interpretation of Corollary A.3.5 inside $\mathrm{Fun}(A, \mathrm{Cat})$ precise.

All statements in this subsection already exist in the literature except for maybe Lemma 2.2.24 and Lemma 2.2.25 for which we could not find any references. The basic properties of Cat and $\mathrm{Fun}(A, \mathrm{Cat})$ can already be deduced from their corresponding more general versions proven in the framework of enriched category theory, see for example [Hin20], [Hin23] and [Hei25], [Hei24]. All the basic properties of 2-cocartesian fibrations, as already in the previous subsection, can be found for example in [GR17, Chapter 11].

Proposition 2.2.13

For any small 2-category A we can interpret the basic 1-category theory developed in Appendix A in the 2-category $\mathrm{Fun}(A, \mathrm{Cat})$, except for everything concerning or using the underlying space operation $(-)^{\simeq}$. More precisely, we have counterparts for the following necessary operations.

1. We have finite limits, i. e. the terminal object $*$, binary products $(-) \times (-)$, as well as pullbacks, because $\mathrm{Fun}(A, \mathrm{Cat})$ is in particular 1-categorically small complete by Lemma 2.2.14.
2. We have finite colimts, i. e. the initial object $\varnothing$, binary coproducts $(-) + (-)$, as well as pushouts, because $\mathrm{Fun}(A, \mathrm{Cat})$ is in particular 1-categorically small cocomplete, also by Lemma 2.2.14.
3. We have the cartesian closure $\underline{\mathrm{Hom}}(-, -)$ of $\mathrm{Fun}(A, \mathrm{Cat})$, proven in Lemma 2.2.19, i. e. a way of formulating an internal version of the functor category operation $\mathrm{Fun}(-, -)$.
4. We have an object representing the walking arrow Δ^1, as well as all the higher Δ^n in $\mathrm{Fun}(A, \mathrm{Cat})$ through the finite product preserving 2-left adjoint $\mathrm{const}\colon \mathrm{Cat} \to \mathrm{Fun}(A, \mathrm{Cat})$ to the 2-limit 2-functor $2\text{-}\lim_A\colon \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Cat}$ by Lemma 2.2.14. In particular const preserves on the one hand side all the adjunctions present in 2-category Cat, but also relations like $\Delta^1 \times \Delta^1 \simeq \Delta^2 \underset{\Delta^1}{\cup}{}^{(d_1, d_1)} \Delta^2$, and $\Delta^n \simeq \Delta^1 \underset{\Delta^0}{\cup} \dots \underset{\Delta^0}{\cup} \Delta^1$.
5. We have a canonical identification $\underline{\mathrm{Hom}}(\mathrm{const}(T), F) \simeq \mathrm{Fun}(T, F(-))$ proven in Corollary 2.2.20.
6. We can identify 2-morphisms of $\mathrm{Fun}(A, \mathrm{Cat})$ with 1-morphisms into $\mathrm{Fun}(\Delta^1, -)$ via taking $T = \Delta^1$ in the canonical natural identification
$$\mathrm{Fun}(T, \mathrm{Fun}(A, \mathrm{Cat})(F, G)) \simeq \mathrm{Fun}(A, \mathrm{Cat})(F, \mathrm{Fun}(T, G))$$
proven in Corollary 2.2.21. Also vertical composition and identities of 2-morphisms agree with the internally defined notions via $\mathrm{Fun}(\Delta^1, G)$.

We first prove the completeness and cocompleteness statements for $\mathrm{Fun}(A, \mathrm{Cat})$.

Lemma 2.2.14

The 2-category Cat of small 1-categories is small conically 2-complete and small conically 2-cocomplete. Furthermore, for all 2-categories A the 2-category $\mathrm{Fun}(A, \mathrm{Cat})$ is so too and the small conical 2-(co)limits are computed pointwise.

Proof. For conical 2-limits we prove that the constant 2-functor
$$\mathrm{Cat} \xrightarrow{\mathrm{const}} \mathrm{Fun}(A, \mathrm{Cat}) \simeq 2\text{-}\mathrm{Cocart}_{1\text{-catfib}}(A)\ ,$$
which sends T to $A \times T \xrightarrow{\mathrm{pr}_A} A$, has a 2-right adjoint. For a 1-category T, the 2-cocartesian 1-morphism of pr_A are exactly the ones whose projection to T becomes invertible. As T is a 1-category, pr_A is on Hom-categories also automatically a left and a right fibration. For a 2-cocartesian fibration $p\colon X \twoheadrightarrow A$ fibered in small 1-categories, our 2-limit candidate will be its 1-category of 2-cocartesian sections $\mathrm{Fun}^{2\text{-cocart}}_{/A}(\mathrm{id}_A, p)$. We then get the desired equivalence of Hom-1-categories by restricting the canonical equivalence $\mathrm{Fun}_{/A}((A \times T, \mathrm{pr}_A), p) \simeq \mathrm{Fun}(T, \mathrm{Fun}_{/A}(\mathrm{id}_A, p))$ to $\mathrm{Fun}^{2\text{-cocart}}_{/A}((A \times T, \mathrm{pr}_A), p) \simeq \mathrm{Fun}(T, \mathrm{Fun}^{2\text{-cocart}}_{/A}(\mathrm{id}_A, p))$. Our conical 2-colimit candidate will be the result Q of first localising X at the 2-cocartesian 1-morphisms and then applying the left adjoint $2\text{-}\mathrm{Cat} \to \mathrm{Cat}$ to the inclusion to get a 1-category. In the globular complete Segal objects $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ model for example, this left adjoint is given by postcomposing with the left adjoint $\mathrm{Cat} \to \mathcal{S}$ of the inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}$. Hence we can restrict the canonical equivalence $\mathrm{Fun}_{/A}(p, (A \times T, \mathrm{pr}_A)) \simeq \mathrm{Fun}(X, T)$ to an equivalence
$$\mathrm{Fun}^{2\text{-cocart}}_{/A}(p, (A \times T, \mathrm{pr}_A)) \simeq \mathrm{Fun}^{\{2\text{-cocart } 1\text{-mor}\}\text{-inv}}(X, T)$$
and then further identify the right hand side canonically with $\mathrm{Fun}(Q, T)$. The last claim follows from the fact that $\mathrm{Fun}(A, -)$ preserves 2-adjunctions and the identification of
$$\mathrm{Fun}(A, \mathrm{Cat}) \xrightarrow{\mathrm{const}} \mathrm{Fun}(B, \mathrm{Fun}(A, \mathrm{Cat})) \simeq \mathrm{Fun}(A, \mathrm{Fun}(B, \mathrm{Cat}))$$

with the 2-functor const_*. □

Next we introduce the main tool to prove properties of $\mathrm{Fun}(A, \mathrm{Cat})$, the so-called *end-formula* for Hom-categories of the 2-categories $\mathrm{Fun}(A, B)$. To prove this we employ the theory of strict 2-left Kan extensions and to this end we cite another aspect of the strict 2-Yoneda Lemma.

Theorem 2.2.15

For a small 2-category A, the 2-Yoneda embedding $y_A\colon A \hookrightarrow \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})$ exhibits the 2-categorical presheaf category as the free small 2-colimit cocompletion of A. This means that every 2-presheaf $F\colon A^{1\text{-op}} \to \mathrm{Cat}$ is the 2-colimit of its *canonical Yoneda density resolution*, i. e.

$$\text{2-colim}((y_A)_{/F} \xrightarrow{\mathrm{pr}} A \xrightarrow{y_A} \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})) \xrightarrow{\mathrm{can}} F$$

is an equivalence and in particular an equivalence of 2-categories

$$_*(y_A)\colon\ \mathrm{Fun}^{\text{2-cocts}}(\mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat}), D) \xrightarrow{\simeq} \mathrm{Fun}(A, D)$$

for all small 2-cocomplete 2-categories D, where the left hand side is the full sub-2-category on 2-colimit preserving 2-functors. From this we can deduce for every 2-functor $F\colon A \to B$ between small 2-categories the existence of the strict 2-left Kan extension 2-adjunction

$$\begin{tikzcd} \mathrm{Fun}(B^{1\text{-op}}, \mathrm{Cat}) \arrow[r, "F^*"', ""{name=D}] & \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat}) \arrow[l, bend right, "F_!"', ""{name=U}] \arrow[from=U, to=D, phantom, "\perp"] \end{tikzcd}.$$

Reference. For convenient reference for the free 2-cocompletion statement we cite the literature on enriched categories [Hin23] and specialise to $(\mathcal{V}, \otimes) = (\mathrm{Cat}, \times)$. Using this we may then compute for every $G\colon A \to D$ with D small 2-cocomplete and Hom-locally small,

$$\begin{aligned} D(\text{2-colim}_{(a,x)\colon (y_A)_{/P}} G(a), d) &\simeq \text{2-lim}_{(a,x)\colon ((y_A)_{/P})^{\mathrm{op}}} D(G(a), d) \\ &\simeq \text{2-lim}_{(a,x)\colon ((y_A)_{/P})^{\mathrm{op}}} \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})(A(-,a), D(G(-), d)) \\ &\simeq \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})(\text{2-colim}_{(a,x)\colon (y_A)_{/P}} A(-,a), D(G(-), d)) \\ &\simeq \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})(P, D(G(-), d)) \end{aligned}$$

to show that the restricted Yoneda 2-functor $D(F, -)\colon D \to \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})$ has a left adjoint, in particular the F^* above for $D = \mathrm{Fun}(B^{1\text{-op}}, \mathrm{Cat})$ and $G = y_B \circ F$. ■

Next we introduce the main tool to prove properties of $\mathrm{Fun}(A, \mathrm{Cat})$ that rely on working with its Hom-1-categories $\mathrm{Fun}(A, \mathrm{Cat})(F, G)$, the so-called *end formula*, a very robust Hom-category characterisation for functor 2-categories. The following proof is adapted from [GHN15, Proposition 5.1] to 2-categories.

Lemma 2.2.16 (2-categorical version of [GHN15])

For two 2-categories A,B and any two 2-functors $F, G\colon A \to B$ we have a canonical equivalence of 1-categories

$$\mathrm{Fun}(A, B)(F, G) \simeq \text{2-lim}(\mathrm{Tw}^{\mathrm{lax}}(A) \to A^{1\text{-op}} \times A \xrightarrow{F^{1\text{-op}} \times G} B^{1\text{-op}} \times B \xrightarrow{B(-,-)} \mathrm{Cat}) \qquad (2.2.2.1)$$

natural in F and G.

Proof. In general, we can compute 2-limits of 2-functors $P\colon K \to \mathrm{Cat}$ as the 2-cocartesian sections of its 2-cocartesian unstraightening via

$$2\text{-}\lim(P) \simeq \mathrm{Fun}(\Delta^0, 2\text{-}\lim(P)) \simeq \mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), P) \simeq \mathrm{Fun}^{2\text{-cocart}}_{/K}(K, \overrightarrow{\int\limits_{k\colon K}} P) \ ,$$

where we used the universal property of 2-limits in the second equivalence and our 2-categorical straightening-unstraightening equivalence for the third equivalence. Using this we may identify the 2-limit on the right hand side of Equation 2.2.2.1 with the category of 2-cocartesian sections of the corresponding 2-cocartesian fibration, i. e. the category

$$\mathrm{Fun}^{2\text{-cocart}}_{/\,\mathrm{Tw}^{\mathrm{lax}}(A)}((\mathrm{Tw}^{\mathrm{lax}}(A), \mathrm{id}), \overrightarrow{\int\limits_{i\to j\colon\ \mathrm{Tw}^{\mathrm{lax}}(A)}} B(F(i), G(j)))\ .$$

By pullback this is equivalent to

$$\mathrm{Fun}^{2\text{-cocart}}_{/A^{1\text{-op}}\times A}(\mathrm{Tw}^{\mathrm{lax}}(A), \overrightarrow{\int\limits_{(a,\tilde{a})\colon\ A^{1\text{-op}}\times A}} B(F(a), G(\tilde{a})))\ .$$

By the straightening-unstraightening equivalence $\mathrm{Fun}(A^{1\text{-op}}\times A, \mathrm{Cat}) \simeq 2\text{-}\mathrm{Cocart}_{1\text{-catfib}}(A^{1\text{-op}}\times A)$ this in turn is equivalent to

$$\mathrm{Fun}(A^{1\text{-op}} \times A, \mathrm{Cat})(A(-,-), B(F(-), G(-)))\ .$$

By the currying equivalence $\mathrm{Fun}(A, \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})) \simeq \mathrm{Fun}(A^{1\text{-op}} \times A, \mathrm{Cat})$ this is equivalent to

$$\mathrm{Fun}(A, \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat}))(y_A, F^* y_B \circ G)\ .$$

By the strict 2-left Kan extension adjunction $F_! \dashv F^*$ along F from Theorem 2.2.15, this is furthermore equivalent to

$$\mathrm{Fun}(A, \mathrm{Fun}(B^{1\text{-op}}, \mathrm{Cat}))(F_! y_A, y_B \circ G)\ .$$

Now we may precompose with the canonical 2-left Kan extension equivalence $F_! y_A \simeq y_B \circ F$ to get that this is equivalent to

$$\mathrm{Fun}(A, \mathrm{Fun}(B^{1\text{-op}}, \mathrm{Cat}))(y_B \circ F, y_B \circ G)\ .$$

Finally, by the 2-fully-faithfulness of the 2-Yoneda embedding $y_B\colon B \hookrightarrow \mathrm{Fun}(B^{1\text{-op}}, \mathrm{Cat})$ and the infered 2-fully-faithfulness of $(y_B)_*\colon \mathrm{Fun}(A, B) \hookrightarrow \mathrm{Fun}(A, \mathrm{Fun}(B^{1\text{-op}}, \mathrm{Cat}))$ this is equivalent to the desired Hom-category $\mathrm{Fun}(A, B)(F, G)$. □

Definition 2.2.17

For 2-categories A, D and a 2-functor $S\colon A^{1\text{-op}} \times A \to D$ we call the 2-limit of the composite $\mathrm{Tw}^{\mathrm{lax}}(A) \to A^{1\text{-op}} \times A \xrightarrow{S} D$, whenever it exists, the 2-end of S and denote it by $\int\limits_{a\colon A} S(a, a)$.

Our first corollary out of the Hom-characterisation Lemma 2.2.16 will the following 2-end version of the 2-Yoneda Lemma.

Corollary 2.2.18 (2-End-Yoneda)
For any 2-category A the 2-Yoneda Lemma gets the following 2-end formula using Lemma 2.2.16.

$$\int_{\tilde{a}\colon A} \mathrm{Fun}(A(a,\tilde{a}), F(\tilde{a})) \simeq \mathrm{Fun}(A,\mathrm{Cat})(A(a,-),F) \simeq F(a)$$

With this under our belt we are fully equipped to prove the cartesian closure of $\mathrm{Fun}(A,\mathrm{Cat})$ with the Yoneda-inferred candidate $\underline{\mathrm{Hom}}(G,H)\colon \mapsto \mathrm{Fun}(A,\mathrm{Cat})(A(a,-)\times G, H)$.

Lemma 2.2.19
For any 2-category A the 2-category $\mathrm{Fun}(A,\mathrm{Cat})$ is again 2-cartesian closed and the internal Hom-object from G to H is given by

$$\underline{\mathrm{Hom}}(G,H)\colon a \mapsto \mathrm{Fun}(A,\mathrm{Cat})(y(a)\times G, H)\ .$$

Proof. We can compute this using Lemma 2.2.16 and Corollary 2.2.18 via the following composition of canonical equivalences natural in F and H.

$$\begin{aligned}
\mathrm{Fun}(A,\mathrm{Cat})(F,\underline{\mathrm{Hom}}(G,H)) &\simeq \int_{a\colon A} \mathrm{Fun}(F(a),\mathrm{Fun}(A,\mathrm{Cat})(y(a)\times G,H))\\
&\simeq \int_{a\colon A} \mathrm{Fun}(F(a), \int_{\tilde{a}\colon A}\mathrm{Fun}(A(a,\tilde{a})\times G(\tilde{a}),H(\tilde{a})))\\
\text{Fun}(F(a),-)\text{ preserves 2-limits } &\simeq \int_{a\colon A}\int_{\tilde{a}\colon A} \mathrm{Fun}(F(a),\mathrm{Fun}(A(a,\tilde{a})\times G(\tilde{a}),H(\tilde{a})))\\
&\simeq \int_{a\colon A}\int_{\tilde{a}\colon A} \mathrm{Fun}(A(a,\tilde{a}),\mathrm{Fun}(F(a)\times G(\tilde{a}),H(\tilde{a})))\\
\text{2-End-Yoneda } &\simeq \int_{a\colon A} \mathrm{Fun}(F(a)\times G(a),H(a))\\
&\simeq \mathrm{Fun}(A,\mathrm{Cat})(F\times G,H)
\end{aligned}$$

□

With our 2-end formulae we can also directly deduce that the internal Hom-object from a constant 2-functor $\mathrm{const}(T)$ to some other 2-functor F can be given by $A \xrightarrow{F} \mathrm{Cat} \xrightarrow{\mathrm{Fun}(T,-)} \mathrm{Cat}$.

Corollary 2.2.20
For any 2-category A, 2-functor $F\colon A\to\mathrm{Cat}$ and 1-category T we have following 2-natural equivalence of 2-functors.

$$\begin{aligned}
\underline{\mathrm{Hom}}(\mathrm{const}(T),F) \coloneqq \mathrm{Fun}(A,\mathrm{Cat})(y(-)\times\mathrm{const}(T),F) &\simeq \int_{\tilde{a}\colon A}\mathrm{Fun}(A(a,\tilde{a})\times T, F(\tilde{a}))\\
&\simeq \int_{\tilde{a}\colon A}\mathrm{Fun}(A(a,\tilde{a}),\mathrm{Fun}(T,F(\tilde{a})))\\
\text{2-End-Yoneda} \quad &\simeq \mathrm{Fun}(T,F)
\end{aligned}$$

Finally we identify internal natural transformations with 2-morphisms in $\mathrm{Fun}(A, \mathrm{Cat})$.

Corollary 2.2.21

For any 2-category A, 2-functors $F, G\colon A \to \mathrm{Cat}$ and 1-category T we have following equivalence of 2-categories

$$\begin{aligned}\mathrm{Fun}(T, \mathrm{Fun}(A, \mathrm{Cat})(F, G)) &\simeq \mathrm{Fun}(T, \int_{\widetilde{a}\colon A} \mathrm{Fun}(F(\widetilde{a}), G(\widetilde{a}))) \\ &\simeq \int_{\widetilde{a}\colon A} \mathrm{Fun}(T, \mathrm{Fun}(F(\widetilde{a}), G(\widetilde{a}))) \\ &\simeq \int_{\widetilde{a}\colon A} \mathrm{Fun}(F(\widetilde{a}), \mathrm{Fun}(T, G(\widetilde{a}))) \\ &\simeq \mathrm{Fun}(A, \mathrm{Cat})(F, \mathrm{Fun}(T, G))\end{aligned}$$

natural in 1-categories T.

Now we have gathered the basic building blocks of the synthetic 1-category theory from section A.1 inside $\mathrm{Fun}(A, \mathrm{Cat})$, i. e. we have finite limits and colimits, a free walking arrow $\mathrm{const}(\Delta^1)$, an internal functor category $\underline{\mathrm{Hom}}(-,-)$. What we now have to do is to relate the internally defined notions from section A.1 with their external counterparts that we will use proof of the Hom-characterisation Theorem 2.2.28. Precisely speaking, by *externalising* we mean the application of the limit preserving 2-functor $\mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), -)$, i. e. taking *global sections*, to internal concepts. To gain an idea how this works let us produce a rough dictionary relating internal concepts defined, through the building blocks from Proposition 2.2.13, using section A.1.

1. The internal one-object category $\mathrm{const}(\Delta^0)$ will externalise to Δ^0
$$\mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), \mathrm{const}(\Delta^0)) \simeq \Delta^0$$
as we preserve limits.
2. Internal pullbacks will externalise to external pullbacks of Hom-categories
$$\begin{tikzcd} \mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), F \times_H G) \arrow[d, "\simeq"] \\ \mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), F) \times_{\mathrm{Fun}(A,\mathrm{Cat})(\mathrm{const}(\Delta^0), G)} \mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), G) \end{tikzcd}$$
as we preserve limits.
3. Internal functor categories will externalise to Hom-categories of $\mathrm{Fun}(A, \mathrm{Cat})$
$$\mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0), \underline{\mathrm{Hom}}(F, G)) \simeq \mathrm{Fun}(A, \mathrm{Cat})(\mathrm{const}(\Delta^0) \times F, G) \simeq \mathrm{Fun}(A, \mathrm{Cat})(F, G)$$
and so does their identities, compositions, whiskerings and in particular their invertibility.
4. Internal natural transformations will externalise to 2-morphisms in $\mathrm{Fun}(A, \mathrm{Cat})$ via
$$\mathrm{Fun}(A, \mathrm{Cat})(F, \mathrm{Fun}(\Delta^1, G)) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}(A, \mathrm{Cat})(F, G))\ .$$
5. Internally defined adjunctions will externalise to adjunctions in the 2-category $\mathrm{Fun}(A, \mathrm{Cat})$.

Next we would like to extend our dictionary by considering internally defined cocartesian fibrations via asking for left adjoints for the internal functors $\mathrm{Fun}(\Delta^1, F) \to F \times_G^{(\alpha,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, G)$ with invertible units. The goal here is to actually relate this via unstraightening to exactly those 2-cocartesian 2-functors between 1-category fibered 2-cocartesian fibrations which themselves are 2-cocartesian fibrations. In this sense internal cocartesian fibrations will correspond to external 2-cocartesian ones.

Rather than proving such an equivalence directly we will pass through a middle ground definition, i. e. 2-cocartesian functors over A which are fiberwise cocartesian fibrations of 1-categories and the pushforward functors $f_!$ along 1-morphisms f in the base A preserve this fiberwise cocartesianness. To this end however, we first need a tool to relate fiberwise adjoints for 2-cocartesian functors over A to the global existence of 2-adjoints on total 2-categories.

The following is just a copy of [Lur17, Proposition 7.3.2.11] for 2-categories. Here we employ the objectwise existence criterion for *strict* 2-adjoints, i. e. the fact that a 2-functor $U\colon A \to B$ has a 2-left adjoint if and only if for each object b of B, the 2-presheaf $B(b, U(-))$ is 2-representable, which can be deduced from the 2-fully-faithfullness of the 2-Yoneda embedding.

Lemma 2.2.22

Let us assume a 2-cocartesian functor

$$\begin{tikzcd} X \arrow[rr, "U"] \arrow[dr, "p"', two heads] & & Y \arrow[dl, "q", two heads] \\ & A & \end{tikzcd}$$

between 2-cocartesian fibrations p and q whose fibers are 1-categories. Then the following assertions are equivalent.

1. The 2-functor U has a 2-left adjoint F relative to A.
2. For every object a of A the restriction $U_a\colon X_a \to Y_a$ to fibers over a has a left adjoint F_a and for all 1-morphisms $f\colon a \to \widetilde{a}$ the commutative square

$$\begin{tikzcd} X_a \arrow[r, "f_!"] \arrow[d, "U_a"'] & X_{\widetilde{a}} \arrow[d, "U_{\widetilde{a}}"] \\ Y_a \arrow[r, "f_!"'] & Y_{\widetilde{a}} \end{tikzcd}$$

is vertically adjointable.

In both situations, the 2-left adjoint F will automatically become a 2-cocartesian functor over A.

Proof. Let us first prove that condition 1 already implies 2-cocartesianness of F. To this end let us be given an object y in Y and a 1-morphism $f\colon q(y) \to a$. For this we can consider the following diagram

$$\begin{tikzcd} X(F(f_!y), x) \arrow[r, "{}_*F(f_!y)"] \arrow[d, "\simeq"] \arrow[dd, bend right=60, "p"'] & X(F(y), x) \arrow[d, "\simeq"] \arrow[dd, bend left=60, "p"] \\ Y(f_!y, U(x)) \arrow[r, "{}_*(f_!y)"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y(y, U(x)) \arrow[d, "q"] \\ A(a, q(x)) \arrow[r, "{}_*f"'] & A(p(y), q(x)) \end{tikzcd},$$

whose upper square commutes by naturality of the 2-adjunction Hom-equivalences, whose lower square is a pullback by 2-cocartesian lifting along f, and whose outer triangles commute by the fact that the 2-adjunction is relative to A. Hence we may deduce that the outer rectangle is also a pullback, i. e. F sends 2-cocartesian 1-morphisms to 2-cocartesian ones. As we are fibered in 1-categories, preservation of Hom-cocartesian 2-morphisms is automatic as on Hom-categories we have right fibrations.

1. $\Longrightarrow$ 2. The objectwise adjunctions can be deduced from the fact that relative 2-adjunctions are stable under pullback. To check that the mate

$$\begin{tikzcd}
Y_a \arrow[r, "F_a"] \arrow[dr, equal] & X_a \arrow[r, "f_!"] \arrow[d, "U_a"] \arrow[dl, phantom, "\Nearrow \eta^a"] & X_{\widetilde{a}} \arrow[d, "U_{\widetilde{a}}"'] \arrow[dr, equal] & \\
& Y_a \arrow[r, "f_!"'] & Y_{\widetilde{a}} \arrow[r, "F_{\widetilde{a}}"'] \arrow[u, phantom, "\Nearrow \varepsilon^{\widetilde{a}}"] & X_{\widetilde{a}}
\end{tikzcd}$$

is an invertible natural transformation, it suffices to show it objectwise. To deduce this, let us denote by $\alpha\colon f_!U(F(y)) \simeq U(f_!F(y))$ the unique equivalence witnessing 2-cocartesian 1-morphism preservation of U

$$\begin{tikzcd}
U(F(y)) \arrow[r, "U(f_!F(y))"] \arrow[dr] & U(f_!F(y)) \\
& f_!U(F(y)) \arrow[u, "\exists! \alpha \simeq"']
\end{tikzcd}$$

so we can contemplate the, by naturality, commutative squares

$$\begin{tikzcd}
X(F(f_!y), f_!F(y))_{\mathrm{id}} \arrow[r, "{}_*F(f_!y)"] \arrow[d, "\simeq"] & X(F(y), f_!F(y))_f \arrow[d, "\simeq"] \\
Y(f_!y, U(f_!F(y)))_{\mathrm{id}} \arrow[r, "{}_*(f_!y)"] & Y(y, U(f_!F(y)))_f \\
Y(f_!y, f_!U(F(y)))_{\mathrm{id}} \arrow[r, "{}_*(f_!y)"'] \arrow[u, "\simeq \alpha_*"] & Y(y, f_!U(F(y)))_f \arrow[u, "\simeq \alpha_*"']
\end{tikzcd}$$

To infer the invertibility of the desired component of the mate transformation one starts with a 2-cocartesian lift $F(y) \to f_!F(y)$ of $F(y)$ along a 1-morphism f in the upper right hand corner of the diagram, goes all the way to the bottom, to the left, and then all the way up and gets exactly the mate component up to equivalence. But as F is also 2-cocartesian, the map $F(y) \to f_!F(y)$ in the upper right hand corner gets send to an equivalence in the top left hand corner. As the diagram is commutative we have that the two ways of going around the diagram are equivalent, hence our mate component is also invertible.

2. $\Longrightarrow$ 1. To construct the 2-left adjoint F we employ the objectwise Hom-category equivalence criterion for 2-adjunctions, i. e. we will show that for all objects y of Y the 2-copresheaf $Y(y, U(-))$ is 2-corepresentable. To this end let us be given an object y of Y living over a and a 1-morphism $f\colon a \to \widetilde{a}$. For a 2-cocartesian lift $F_{q(y)}(y) \to f_!F_{q(y)}(y)$ of $F_{q(y)}(y)$ along f and a 2-cocartesian lift $y \to f_!y$ of y along f let us denote the invertible mate component of your given square at object y by

$$F_{\widetilde{a}}(f_!y) \xrightarrow{F_{\widetilde{a}}(f_!\eta_y)} F_{\widetilde{a}}(f_!U(F_a(y))) \xrightarrow{F_{\widetilde{a}}(\alpha)\simeq} F_{\widetilde{a}}(f_!U(F_a(y))) \xrightarrow{\varepsilon_{f_!F_{\widetilde{a}}(y)}} f_!F_a(y)$$

by m_y. We can construct by 2-Yoneda the following 2-natural transformations

$$\begin{tikzcd}
 & Y(y, U(-)) \\
X(F_a(y), -) \arrow[ur, Rightarrow, "U(-)\circ\eta_y"] & Y(f_! y, U(-)) \arrow[u, Rightarrow, "_*(f_! y)"'] \\
X(f_! F_a(y), -) \arrow[u, Rightarrow, "_*(f_! F_a(y))"] \arrow[r, Rightarrow, "_*(m_y)"'] & X(F_{\widetilde{a}}(f_! y), -) \arrow[u, Rightarrow, "U(-)\circ\eta_{f_! y}"']
\end{tikzcd}$$

so we may also, by 2-Yoneda, check the commutativity of the diagram. So if we start in the lower left corner with the identity and then go up and diagonally right up we get

$$y \xrightarrow{\eta_y} U(F_a(y)) \xrightarrow{U(f_! F_a(y))} U(f_! F_a(y))$$

and if we go the other way around we get

$$y \xrightarrow{f_! y} f_! y \xrightarrow{\eta_{f_! y}} U(F_{\widetilde{a}}(f_! y)) \xrightarrow{U(m_y)} U(f_! F_a(y))$$

which is by the middle four interchange law of natural transformations equivalent to

$$y \to f_! y \xrightarrow{f_! \eta_y} f_! U(F_a(y))) \xrightarrow{\alpha} U(f_! F_a(y)) \xrightarrow{\eta_{U(f_! F_a(y))}} U(F_{\widetilde{a}}(U(f_! F_a(y)))) \xrightarrow{U(\varepsilon_{f_! F_{\widetilde{a}}(y)})} U(f_! F_a(y)) \; ,$$

which is by the triangle identity of $F_{\widetilde{a}} \dashv U_{\widetilde{a}}$ equivalent to

$$y \xrightarrow{f_! y} f_! y \xrightarrow{f_! \eta_y} f_! U(F_a(y))) \xrightarrow{\alpha} U(f_! F_a(y)) \; .$$

Now by the defining square and triangle of $f_! \eta_y$ and α

$$\begin{tikzcd}
y \arrow[r, "\eta_y"] \arrow[d, "f_! y"'] & U(F_a(y)) \arrow[d, "f_! U(F_a(y))"'] \arrow[dr, "U(f_! F_a(y))"] & \\
f_! y \arrow[r, dashed, "\exists ! f_! \eta_y"'] & f_! U(F_a(y)) \arrow[r, dashed, "\exists ! \alpha \simeq"'] & U(f_! F_a(y))
\end{tikzcd}$$

we deduce the equivalence to the other produced map. Now we need to show that the top map in

$$\begin{tikzcd}
X(F_a(y), x) \arrow[rr, "U(-)\circ\eta_y"] \arrow[dr, two heads, "p"'] & & Y(y, U(x)) \arrow[dl, two heads, "q"] \\
 & A(q(y), p(x)) &
\end{tikzcd}$$

is an equivalence for all y and x. By 2-cocartesianness of p and q we know that both functors to $A(q(y), p(x))$ are cartesian fibrations, by fiberedness in 1-categories even right fibrations. Hence it suffices to show invertibility fiberwise over $A(q(y), p(x))$. To this end let us fix a 1-morphism $f \colon q(y) \to p(x)$. Now that we know that the diagram of natural transformations above is commutative, we may restrict it along $Y(y, U(x)) \to A(q(y), p(x)) \leftarrow \{f\}$ and get

$$\begin{tikzcd}
 & Y(y, U(x))_f \\
X(F_a(y), x)_f \arrow[ur, "U(-)\circ\eta_y"] & Y_{p(x)}(f_! y, U(x)) \arrow[u, "_*(f_! y)"'] \\
X_{p(x)}(f_! F_a(y), x) \arrow[u, "_*(f_! F_a(y))"] \arrow[r, "_*(m_y)"'] & X_{p(x)}(F_{\widetilde{a}}(f_! y), x) \, , \arrow[u, "U(-)\circ\eta_{f_! y}"']
\end{tikzcd}$$

where all functors except the upper diagonal one are equivalences by our assumptions, hence we deduce that the upper diagonal one is an equivalence too. The relativeness to A of the produced 2-adjunction comes from the fact that all the unit and counit components of the global 2-adjunctions agree with the fiber ones, hence become invertible when sent to A.

□

Remark 2.2.23

The above statement can be dualized in two ways.

1. In the statement, we can replace 2-cocartesianness by (cocartesian, cocartesian)-ness as in the proof we never needed the exact cartesianness of the Hom-actions, but rather only the fact that functors between right fibrations may be checked to be invertible fiberwise, a claim which is also true about left fibrations.
2. We may replace 2-cocartesianness by 2-cartesianness or by (cartesian, cartesian)-ness *if* we ask for *right adjoints* everywhere, instead of left adjoints.

We are now fully equipped to prove that internal cocartesian fibrations correspond exactly to external 2-cocartesian fibrations.

Lemma 2.2.24

Let us assume a 2-cocartesian functor

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, two heads, "p"'] & & Y \arrow[dl, two heads, "q"] \\ & A & \end{tikzcd}$$

between 2-cocartesian fibrations p and q whose fibers are 1-categories. Then the following assertions are equivalent.

1. The 2-functor F is also a 2-cocartesian fibration whose fibers are 1-categories.
2. For all objects a of A the restriction of F to fibers over a, $F_a\colon X_a \to Y_a$, is a cocartesian fibration between 1-categories and for every 1-morphism $f\colon a \to \widetilde{a}$ in A in the natural square induced via cocartesian pushforward

$$\begin{tikzcd} X_a \arrow[r, "f_!"] \arrow[d, two heads, "F_a"'] & X_{\widetilde{a}} \arrow[d, two heads, "F_{\widetilde{a}}"] \\ Y_a \arrow[r, "f_!"'] & Y_{\widetilde{a}} \end{tikzcd}$$

   the functor $f_!\colon X_a \to X_{\widetilde{a}}$ is a cocartesian functor.
3. The straightened map $F\colon p \Rightarrow q$ is an internal cocartesian fibration inside $\mathrm{Fun}(A, \mathrm{Cat})$.

Proof. Let us start with the easiest implication.

1. $\Longrightarrow$ 2. One obtains the fiberwise cocartesian fibration statements by pullback. So let us write down for an arbitrary F_a-cocartesian 1-morphism in X_a what it means to be pushed forward

along a 1-arrow f in A.

$$
\begin{tikzcd}
x \arrow[rr, "p\text{-2-cocart}"] \arrow[dr, "F\text{-2-cocart}"'] \arrow[dd, no head] & & f_!x \arrow[dr, dashed, "\exists! f_!(\phi_!x)"] \arrow[dd, no head] & \\
& \phi_!x \arrow[rr, "p\text{-2-cocart}"'] \arrow[dd, no head] & & f_!\phi_!x \arrow[dd, no head] \\
y \arrow[rr, "q\text{-2-cocart}"] \arrow[dr, "\phi"] \arrow[ddr, no head] & & f_!y \arrow[dr, dashed, "\exists! f_!\phi"] \arrow[ddr, no head] & \\
& \widetilde{y} \arrow[rr, "q\text{-2-cocart}"] \arrow[d, no head] & & f_!\widetilde{y} \arrow[d, no head] \\
& a \arrow[rr, "f"] & & \widetilde{a}
\end{tikzcd}
$$

As the horizontal p-2-cocartesian 1-morphism here are also F-2-cocartesian we may conclude that the pushforward $f_!(\phi_!x)$ is so too by the right cancellation property of F-2-cocartesianness.

2. $\implies$ 1. As p and q are on Hom-categories right fibrations, so is by basic 1-category theory F. So it suffices to show that we have F-2-cocartesian lifts along all 1-morphisms in Y. We also know that all 1-morphisms in Y may be factored as a q-2-cocartesian one followed by one in a fiber over some a in A. In the first case the diagram

$$
\begin{tikzcd}
X(f_!x, \widetilde{x}) \arrow[r, "{}_*f_!x"] \arrow[d, "F"'] \arrow[dd, bend right=60, "p"'] & X(x, \widetilde{x}) \arrow[d, "F"] \arrow[dd, bend left=60, "p"] \\
Y(f_!y, \widetilde{y}) \arrow[r, "{}_*f_!y"'] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y(y, \widetilde{y}) \arrow[d, "q"] \\
A(a', \widetilde{a}) \arrow[r, "{}_*f"'] & A(a, \widetilde{a}) \,,
\end{tikzcd}
$$

whose outer rectangle is a pullback shows that the corresponding p-2-cocartesian lifts already do the job. We are left with finding F-2-cocartesian lifts for 1-morphisms in the fibers over some a in A. By assuming the second assertion, we already know this for 1-morphisms in the fibers over a in A with respect to all other 1-morphisms in the same fiber. But we want more, we want that for a F_a-cocartesian pushforward $x \to \phi_!x$ over a and any 1-morphism $f \colon a \to \widetilde{a}$ as well as any object $\widetilde{x}$ over $\widetilde{a}$ that the square

$$
\begin{tikzcd}
X(\phi_!x, \widetilde{x}) \arrow[rr, "{}_*\phi_!x"] \arrow[d, "F"'] \arrow[ddr, two heads, "p"] & & X(x, \widetilde{x}) \arrow[d, "F"] \arrow[ddl, two heads, "p"'] \\
Y(y', \widetilde{y}) \arrow[rr, "{}_*\phi"'] \arrow[dr, two heads, "q"'] & & Y(y, \widetilde{y}) \arrow[dl, two heads, "q"] \\
& A(a, \widetilde{a}) &
\end{tikzcd}
$$

over $A(a, \widetilde{a})$ is a pullback. As all functors to $A(a, \widetilde{a})$ are right fibrations it suffices to show the pullback property fiberwise for every f in $A(a, \widetilde{a})$. To this end we contemplate the following commutative cube

$$\begin{tikzcd}
 & X(\phi_! x, \widetilde{x})_f \arrow[rr, "{}_*\phi_! x"] \arrow[dd] & & X(x, \widetilde{x})_f \arrow[dd, "F"] \\
X(f_!\phi_! x, \widetilde{x})_{\mathrm{id}} \arrow[ur, "{}_*f_! x' \simeq"] \arrow[rr, "{}_*f_!\phi_! x"] \arrow[dd, "F"] \arrow[dr, phantom, "\lrcorner", very near start] & & X(f_! x, \widetilde{x})_{\mathrm{id}} \arrow[ur, "{}_*f_! x \simeq"'] \arrow[dd, "F"] & \\
 & Y(y', \widetilde{y})_f \arrow[rr, "{}_*\phi"'] & & Y(y, \widetilde{y})_f \ , \\
Y(f_! y', \widetilde{y})_{\mathrm{id}} \arrow[ur, "{}_*f_! y' \simeq"] \arrow[rr, "{}_*f_!\phi"] & & Y(f_! y, \widetilde{y})_{\mathrm{id}} \arrow[ur, "{}_*f_! y \simeq"'] &
\end{tikzcd}$$

whose diagonal functor are all equivalences and front face is a pullback by F_a-cocartesianness, hence the back face too.

2. $\iff$ 3. The third condition explicitly means that in the 2-category $\mathrm{Fun}(A, \mathrm{Cat})$ the 1-morphism

$$(\mathrm{ev}_0, F_*)\colon\ \mathrm{Fun}(\Delta^1, p) \Rightarrow p \times_q^{(F, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, q)$$

has a left adjoint with invertible unit. By our straightening-unstraightening equivalence of 2-categories $\mathrm{Fun}(A, \mathrm{Cat}) \simeq 2\text{-}\mathrm{Cocart}_{1\text{-catfib}}(A)$ this is equivalent to ask for a relative over A 2-cocartesian 2-left adjoint of its unstraightening, which by Lemma 2.2.22 is equivalent to the second condition above.

□

Furthermore, we would like to extend our dictionary to know that also internal cocartesian functors between internal cocartesian fibrations correspond unstraightened to 2-cocartesian functors between 2-cocartesian fibrations.

Lemma 2.2.25

Let us assume a commutative triangle of 2-cocartesian functors

$$\begin{tikzcd}
X \arrow[rr, "H"] \arrow[dr, "F"'] \arrow[ddr, bend right, two heads, "p"'] & & Y \arrow[dl, "G"] \arrow[ddl, bend left, two heads, "r"] \\
 & Z \arrow[d, two heads, "q"] & \\
 & A &
\end{tikzcd}$$

between 2-cocartesian fibrations p,q and r over A fibered in 1-categories such that F and G satisfy the equivalent statements of Lemma 2.2.24. Then the following assertions are equivalent.

1. H is a 2-cocartesian functor between the 2-cocartesian fibrations F and G over Z.
2. For all objects a in A restricting the commutative triangle to fibers over a we have that the functor H_a

$$\begin{tikzcd}
X_a \arrow[rr, "H_a"] \arrow[dr, two heads, "F_a"'] & & Y_a \arrow[dl, two heads, "G_a"] \\
 & Z_a &
\end{tikzcd}$$

is a cocartesian functor over Z_a.

3. In $\mathrm{Fun}(A, \mathrm{Cat})$, in the straightened triangle

$$\begin{tikzcd} p \arrow[rr, Rightarrow, "H"] \arrow[dr, Rightarrow, "F"'] & & r \arrow[dl, Rightarrow, "G"] \\ & q & \end{tikzcd}$$

H is an internal cocartesian functor between the internal cocartesian fibrations F and G over q.

Proof. We again start with the easy implication.

1. $\implies$ 2. is immediate by pullback.

2. $\implies$ 1. Again, as F and G are by assumption on Hom-categories already right fibrations, H being Hom-cartesian is for free. It remains to show that H sends F-2-cocartesian 1-morphisms to G-2-cocartesian ones. From Lemma 2.2.25 we already know that F_a-cocartesian 1-morphisms are in fact automatically F-2-cocartesian, and H preserves those by assumption. It remains to show that F-2-cocartesian lifts of q-2-cocartesian 1-morphisms get send to G-2-cocartesian ones. But as was also shown in Lemma 2.2.25, F-2-cocartesian lifts of q-2-cocartesian 1-morphisms are exactly the p-2-cocartesian lifts, and the same for G, and those are preserved by the fact that H was assumed to be a 2-cocartesian functor over A.

2. $\iff$ 3. As the third condition is just the invertibility of a mate 2-morphism of the square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, p) \arrow[r, Rightarrow, "H_*"] \arrow[d, Rightarrow, "(\mathrm{ev}_0, F_*)"'] & \mathrm{Fun}(\Delta^1, r) \arrow[d, Rightarrow, "(\mathrm{ev}_0, G_*)"] \\ p \times_q^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, q) \arrow[r, Rightarrow, "H \times_{\mathrm{id}} \mathrm{id}"'] & r \times_q^{(G,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, q) \end{tikzcd}$$

in $\mathrm{Fun}(A, \mathrm{Cat})$ or by unstraightening of a certain 2-natural transformation between 2-cocartesian functors over A, testing the invertibility can be done objectwise, so in particular fiberwise in A.

□

Remark 2.2.26

The above two statements may be dualised in the following two directions.

1. We may change 2-cocartesianness to (cocartesian, cocartesian)-ness as we know by Remark 2.2.23 that the same is true for the crucial ingredient Lemma 2.2.22.
2. We may change 2-cocartesianness to 2-cartesianness or to (cartesian, cartesian)-ness, *if* we change from internal cocartesianness to *internal cartesianness*, as we know by Remark 2.2.23 that the same is true for the crucial ingredient Lemma 2.2.22.

Let us finally make precise how we will interpret the first three sections of Appendix A inside $\mathrm{Fun}(A, \mathrm{Cat})$ to deduce our desired fibrational version of the 1-categorical Yoneda Lemma, i. e. for every 1-morphism $G\colon C \to Y$ in $\mathrm{Fun}(A, \mathrm{Cat})$, the 1-morphism $\mathrm{ev}_1\colon C \times_Y^{(G,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, Y) \to Y$ is the *free internal cocartesian fibration on the internal functor* G.

Remark 2.2.27

In the next proof we will make use of the fact that we can interpret basic 1-category theory inside $\mathrm{Fun}(A, \mathrm{Cat})$ for any small 2-category A, so let us now make precise what we mean by this statement. Precisely speaking we want to use Lemma A.2.8, Corollary A.2.10, as well as Corollary A.3.5.

One of the main problems with interpreting basic 1-category theory in $\mathrm{Fun}(A, \mathrm{Cat})$, instead of Cat, is the availability of certain operations. Whereas Cat, more precisely the underlying 1-category $\mathrm{Cat}^{1\text{-cat}}$ admits an *underlying space of objects functor* $(-)^{\simeq} \colon \mathrm{Cat}^{1\text{-cat}} \to \mathcal{S}$, i. e. a right adjoint to the canonical inclusion, this however *cannot* be upgraded to a functor of 2-categories $\mathrm{Cat} \to \mathcal{S}$. Hence we cannot just postcompose with such a 2-functor to obtain a similar right adjoint to the inclusion $\mathrm{Fun}(A, \mathcal{S}) \hookrightarrow \mathrm{Fun}(A, \mathrm{Cat})$. Carefully looking at the first three sections of Appendix A however we observe that we do not make use of a notion of *underlying space of objects* or *full subcategory on a subspace of the underlying space of objects*, except for the essential image characterisations in Corollary 1.2.10 as well as Corollary A.3.5. Hence our proofs of everything in the first three sections of Appendix A except these two statements can very well be interpreted inside $\mathrm{Fun}(A, \mathrm{Cat})$ through the building blocks provided in Proposition 2.2.13. However, for our use of Corollary A.3.5 in the following Theorem 2.2.28, we do not need it as an equivalence of categories, but rather its externalised version of the statement suffices, which we will infer now. More precisely, for every 1-morphism $G \colon C \to Y$ and every internal cocartesian fibration $p \colon X \twoheadrightarrow Y$ in $\mathrm{Fun}(A, \mathrm{Cat})$ we take in the lifting adjunction Corollary A.3.2

$$\begin{tikzcd} \mathrm{Fun}_{/Y}(\mathrm{ev}_1, p) \arrow[r, "{}_*(\Delta_G)"'] \arrow[r, phantom, shift left=3ex, "\perp"] & \mathrm{Fun}_{/Y}(G, p) \, . \arrow[l, bend right=40] \end{tikzcd}$$

To this we can now apply $\mathrm{Fun}(A, \mathrm{Cat})(\Delta^0, -)$, which is a 2-functor and hence preserves adjunctions, i. e. we externalize our internal adjunction and get

$$\begin{tikzcd} \mathrm{Fun}(A, \mathrm{Cat})_{/Y}(\mathrm{ev}_1, p) \arrow[r, "{}_*(\Delta_G)"'] \arrow[r, phantom, shift left=3ex, "\perp"] & \mathrm{Fun}(A, \mathrm{Cat})_{/Y}(G, p) \arrow[l, bend right=30] \end{tikzcd}$$

as $\mathrm{Fun}(A, \mathrm{Cat})(\Delta^0, -)$ in particular sends internal Hom-objects to Hom-categories and preserves limits. The complementary strengthening Corollary A.3.3 tells us that the essential image of the left adjoint is exactly the internal cocartesian functors $\mathrm{ev}_1 \to p$ over Y. Hence we can restrict to its underlying equivalence on the full subcategory spanned by the essential image of the left adjoint to get

$${}_*(\Delta_G) \colon \, \mathrm{Fun}(A, \mathrm{Cat})_{/Y}^{\text{internally 2-cocart}}(\mathrm{ev}_1, p) \xrightarrow{\simeq} \mathrm{Fun}(A, \mathrm{Cat})_{/Y}(G, p) \; .$$

2.2.3 The Hom-Category Characterisation of 2-Categorical Grothendieck Constructions

Now that we have both an internal version of the 1-Yoneda Lemma inside $\mathrm{Fun}(A, \mathrm{Cat})$ as well as ways that allow us to back and forth between the internal and external perspective we can finally emulate the type-theoretic strategy to characterise Hom-1-categories of 2-categorical Grothendieck constructions.

Theorem 2.2.28

Let $F \colon A \to \mathrm{Cat}$ be a 2-functor, together with an object a of A and an object x of $F(a)$. Then the 2-natural transformation

$$\Big(\overrightarrow{\int_{\tilde{a} \colon A}} F\Big)((a, x), -) \Rightarrow \overleftarrow{\int_{f \colon A(a,-)}} F(-)(F(f)(x), -) \; , \tag{2.2.3.1}$$

whose right hand side will be formally constructed in the proof, induced by the 2-Yoneda Lemma applied to the object $(\mathrm{id}_a, \mathrm{id}_x)$ of $\overleftarrow{\int}_{f\colon A(a,a)} F(a)(F(f)(x), x)$, is an equivalence.

Proof. Our first goal here is to reconstruct the right hand side 2-functor $\overrightarrow{\int} F \to \mathrm{Cat}$ in fibrational terms. Afterwards we will construct the natural maps in both directions and prove that they are inverse to each other. By the 2-Yoneda Lemma the object x of $F(a)$ induces a 2-natural transformation $y(x)\colon A(a,-) \Rightarrow F$. We will abuse notation and also write $y(x)$ for the 2-cocartesian 2-functor induced by unstraightening

$$\begin{tikzcd}
{}^{a/^{\mathrm{lax}}}A \arrow[rr, "y(x)"] \arrow[dr, two heads] & & \overrightarrow{\int} F \arrow[dl, two heads] \\
& A\,. &
\end{tikzcd}$$

In $\mathrm{Fun}(A, \mathrm{Cat}) \simeq 2\text{-}\mathrm{Cocart}_{1\text{-catfib}}(A)$ we have F as an object and we may look at its arrow category two-sided fibration

$$\begin{tikzcd}
& \mathrm{Fun}(\Delta^1, F) \arrow[dl, Rightarrow, "\mathrm{ev}_0"'] \arrow[dr, Rightarrow, "\mathrm{ev}_1"] & \\
F & & F\,.
\end{tikzcd}$$

In order to arrive at a candidate for the right hand side of Equation 2.2.3.1 we now consider $\mathrm{ev}_1\colon A(a,-) \times_F^{(y(x),\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, F) \Rightarrow F$. By Proposition 2.2.13 we may interpret Lemma A.2.8 in $\mathrm{Fun}(A, \mathrm{Cat})$ to deduce that it is internally a cocartesian fibration. Hence by Lemma 2.2.24 its unstraightening

$$\begin{tikzcd}
{}^{a/^{\mathrm{lax}}}A \times_{\overrightarrow{\int} F}^{(y(x),\mathrm{ev}_0)} \overrightarrow{\int} \mathrm{Fun}(\Delta^1, F) \arrow[r, "\mathrm{ev}_1"] \arrow[d, two heads] & \overrightarrow{\int} F \arrow[dl, two heads] \\
A &
\end{tikzcd}$$

over A is a 2-cocartesian fibration with 1-categorical fibers. Living over (a,x) in $\overrightarrow{\int} F$, ev_1 has the fiber

$$\Big({}^{a/^{\mathrm{lax}}}A \times_{\overrightarrow{\int} F}^{(y(x),\mathrm{ev}_0)} \overrightarrow{\int} \mathrm{Fun}(\Delta^1, F)\Big)_{(a,x)} \simeq A(a,a) \times_{F(a)}^{(y(x)_a,\mathrm{pr})} F(a)_{/x}\ ,$$

which contains the object $(\mathrm{id}_a, \mathrm{id}_x)$. So we may apply the 2-Yoneda Lemma to this object to obtain a unique 2-cocartesian functor over $\overrightarrow{\int} F$

$$
\begin{tikzcd}
 & \Delta^0 \arrow[dl, "{((a,x),\mathrm{id}_{(a,x)})}"'] \arrow[dr, "{(\mathrm{id}_a,\mathrm{id}_x)}"] \arrow[dd, bend left, "{(a,x)}"] & \\
{(a,x)}/^{\mathrm{lax}} \overrightarrow{\int} F \arrow[rr, dashed, "{y((\mathrm{id}_a,\mathrm{id}_x))}"] \arrow[dr, two heads, "\mathrm{pr}"'] & & a/^{\mathrm{lax}} A \times^{(y(x),\mathrm{ev}_0)}_{\overrightarrow{\int} F} \overrightarrow{\int} \mathrm{Fun}(\Delta^1, F) \arrow[dl, two heads, "\mathrm{ev}_1"] \\
 & \overrightarrow{\int} F \arrow[d, two heads] & \\
 & A &
\end{tikzcd}
$$

sending $\mathrm{id}_{(a,x)}$ to $(\mathrm{id}_a, \mathrm{id}_x)$. Via a further application of the 2-Yoneda Lemma we can also deduce that the middle triangle in

$$
\begin{tikzcd}
 & \Delta^0 \arrow[dl, "{((a,x),\mathrm{id}_{(a,x)})}"'] \arrow[dr, "{(\mathrm{id}_a,\mathrm{id}_x)}"] \arrow[dd, bend left, "{((a,\mathrm{id}_a),(a,x))}"'] & & \\
{(a,x)}/^{\mathrm{lax}} \overrightarrow{\int} F \arrow[rr, dashed, "{y((\mathrm{id}_a,\mathrm{id}_x))}"] \arrow[dr, "{({(a,x)}/^{\mathrm{lax}}\mathrm{pr},\mathrm{pr})}"'] \arrow[ddr, bend right, "\mathrm{pr}"'] & & a/^{\mathrm{lax}} A \times^{(y(x),\mathrm{ev}_0)}_{\overrightarrow{\int} F} \overrightarrow{\int} \mathrm{Fun}(\Delta^1, F) \arrow[dl, "{(\mathrm{pr}_0,\mathrm{ev}_1)}"] \arrow[ddl, bend left, two heads, "\mathrm{ev}_1"] & \\
 & a/^{\mathrm{lax}} A \times_A \overrightarrow{\int} F \arrow[d, two heads] & & \\
 & \overrightarrow{\int} F \arrow[rr, two heads] & & A
\end{tikzcd}
$$

commutes. On fibers over an object $(\widetilde{a}, \widetilde{x})$ this gives us the triangle

$$
\begin{tikzcd}
(\overrightarrow{\int} F)((a,x),(\widetilde{a},\widetilde{x})) \arrow[rr, dashed, "{y((\mathrm{id}_a,\mathrm{id}_x))_{(\widetilde{a},\widetilde{x})}}"] \arrow[dr, two heads, "\mathrm{pr}"'] & & A(a,\widetilde{a}) \times^{(y(x)_{\widetilde{a}},\mathrm{pr})}_{F(\widetilde{a})} F(\widetilde{a})_{/\widetilde{x}} \arrow[dl, two heads, "\mathrm{pr}_0"] \\
 & A(a,\widetilde{a})\,, &
\end{tikzcd}
$$

where both diagonal functors are cocartesian fibrations of 1-categories, and by the assumption of F taking values in 1-categories even right fibrations. Now 1-category theory also allows us to recognise the right hand fibration as the correct interpretation of the formula on the right hand side in Equation 2.2.3.1

$$
\begin{tikzcd}
A(a,\widetilde{a}) \times^{(y(x)_{\widetilde{a}},\mathrm{pr})}_{F(\widetilde{a})} F(\widetilde{a})_{/\widetilde{x}} \arrow[rr, "\simeq"] \arrow[dr, two heads, "\mathrm{pr}_0"'] & & \overleftarrow{\int}_{f\colon A(a,\widetilde{a})} F(\widetilde{a})(F(f)(x),\widetilde{x}) \arrow[dl, two heads, "\mathrm{pr}"] \\
 & A(a,\widetilde{a})\,. &
\end{tikzcd}
$$

To construct the 2-functor in the other direction we will utilise Corollary A.3.5 internal to $\mathrm{Fun}(A, \mathrm{Cat})$, i. e. through Remark 2.2.27. More precisely, by straightening the composite 2-cocartesian fibration ${}^{(a,x)/^{\mathrm{lax}}} \overrightarrow{\int} F \longrightarrow\!\!\!\!\!\rightarrow \overrightarrow{\int} F \longrightarrow\!\!\!\!\!\rightarrow A$ Lemma 2.2.24 tells us that the resulting map $\mathrm{Str}_A({}^{(a,x)/^{\mathrm{lax}}} \overrightarrow{\int} F) \Rightarrow F$ is an internal cocartesian fibration $\mathrm{Fun}(A, \mathrm{Cat})$. By Remark 2.2.27 we know that

$$\mathrm{ev}_1 \colon A(a,-) \times_F^{(y(x),\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, F) \Rightarrow F$$

is, at least externally, the free internal cocartesian fibration on $y(x) \colon A(a,-) \Rightarrow F$ so it suffices to find a morphism

$$\begin{tikzcd}
A(a,-) \arrow[rr, Rightarrow, dashed] \arrow[dr, Rightarrow, "y(x)"'] & & \mathrm{Str}_A({}^{(a,x)/^{\mathrm{lax}}} \overrightarrow{\int} F) \arrow[dl, Rightarrow, "\mathrm{pr}"] \\
& F &
\end{tikzcd}$$

making the triangle commute. For this we once again employ the 2-Yoneda Lemma to reduce ourselves to the search for an object lifting x in $F(a)$ along $\mathrm{pr}_a \colon \mathrm{Str}_A({}^{(a,x)/^{\mathrm{lax}}} \overrightarrow{\int} F)_a \longrightarrow\!\!\!\!\!\rightarrow F(a)$, i. e. an object of $(\overrightarrow{\int} F)((a,x),(a,x))$, for which we can choose $\mathrm{id}_{(a,x)}$. All in all we get a unique internal cocartesian functor over F

$$\begin{tikzcd}
& A(a,-) \arrow[dl, Rightarrow, "\Delta_{y(x)}"'] \arrow[dr, Rightarrow, "y(\mathrm{id}_{(a,x)})"] \arrow[dd, Rightarrow, bend left, "y(x)"' near end] & \\
A(a,-) \times_F^{(y(x),\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, F) \arrow[rr, Rightarrow, dashed, "\exists! \phi"] \arrow[dr, Rightarrow, "\mathrm{ev}_1"'] & & \mathrm{Str}_A({}^{(a,x)/^{\mathrm{lax}}} \overrightarrow{\int} F) \arrow[dl, Rightarrow, "\mathrm{pr}"] \\
& F &
\end{tikzcd}$$

sending $\Delta_{y(x)}$ to $y(\mathrm{id}_{(a,x)})$. To now prove that ϕ is a right inverse to $y((\mathrm{id}_a, \mathrm{id}_x))$ we just need to observe, by Remark 2.2.27, that in

$$\begin{tikzcd}
& A(a,-) \arrow[dl, Rightarrow, "\Delta_{y(x)}"'] \arrow[d, Rightarrow, "y(\mathrm{id}_{(a,x)})"] \arrow[dr, Rightarrow, bend left, "\Delta_{y(x)}"] & \\
A(a,-) \times_F^{(y(x),\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, F) \arrow[r, Rightarrow, dashed, "\exists! \phi"] \arrow[dr, Rightarrow, "\mathrm{ev}_1"'] & \mathrm{Str}_A({}^{(a,x)/^{\mathrm{lax}}} \int F) \arrow[r, Rightarrow, "y((\mathrm{id}_a, \mathrm{id}_x))"'] \arrow[d, Rightarrow, "\mathrm{pr}"] & A(a,-) \times_F^{(y(x),\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, F) \arrow[dl, Rightarrow, "\mathrm{ev}_1"] \\
& F &
\end{tikzcd}$$

the right hand side upper triangle commutes. From Lemma 2.2.25 we know that the straightening of $y((\mathrm{id}_a, \mathrm{id}_x))$ is an internal cocartesian functor. Thus again by the 2-Yoneda Lemma we may reduce to checking that $y((\mathrm{id}_a, \mathrm{id}_x))_a(\mathrm{id}_{(a,x)}) = (\mathrm{id}_a, \mathrm{id}_x)$ is equivalent to $(\Delta_{y(x)})_a(\mathrm{id}_a)$. But by construction we already have $(\Delta_{y(x)})_a(\mathrm{id}_a) \simeq (\mathrm{id}_a, \mathrm{id}_x)$. To prove that ϕ is a left inverse to $y((\mathrm{id}_a, \mathrm{id}_x))$ we need to know that the unstraightening of ϕ is a 2-cocartesian functor over $\overrightarrow{\int} F$, which we know from Lemma 2.2.25, and that in

$$
\begin{tikzcd}
& \Delta^0 \arrow[dl, "{((a,x),\mathrm{id}_{(a,x)})}"'] \arrow[d, "{(\mathrm{id}_a,\mathrm{id}_x)}"] \arrow[dr, bend left, "{((a,x),\mathrm{id}_{(a,x)})}"] \arrow[dd, bend right, "{(a,x)}"'] & \\
(a,x)/^{\mathrm{lax}} \overrightarrow{\int} F \arrow[r, dashed, "{y((\mathrm{id}_a,\mathrm{id}_x))}"] \arrow[dr, two heads, "\mathrm{pr}"'] & a/^{\mathrm{lax}} A \times^{(y(x),\mathrm{ev}_0)}_{\overrightarrow{\int} F} \overrightarrow{\int} \mathrm{Fun}(\Delta^1, F) \arrow[r, "\phi"] \arrow[d, two heads, "\mathrm{ev}_1"] & (a,x)/^{\mathrm{lax}} \overrightarrow{\int} F \arrow[dl, two heads, "\mathrm{pr}"] \\
& \overrightarrow{\int} F &
\end{tikzcd}
$$

the right hand side upper triangle commutes. But by construction, we already know that ϕ sends $(\mathrm{id}_a, \mathrm{id}_x)$ to $\mathrm{id}_{(a,x)}$. So we may conclude, by the 2-Yoneda Lemma. $\square$

Remark 2.2.29

We may dualise the above Theorem 2.2.28 to the three other variances of unstraightening by Remark 2.2.26 to get similar Hom-category characterisations.

2.3 Application: A Tiny Bit of Synthetic Monad Theory

Let us give a little application of the usefulness of the extended universal property of the lax functor classifier, taking the later proven Theorem 2.2.28 for granted for now. In this subsection we will start to exploit the observation, that lax functors out of the point should correspond to monads in the target 2-category, to define in a synthetic way the most basic players of monad theory. Using just the universal property Definition 1.1.1, we will for general 2-categories synthetically define monads, as well as to produce monads from their most prominent source, i. e. to extract monads from adjunctions. For monads in the universe of 1-categories Cat, we will furthermore use the universal property, straightening-unstraightening and the 2-Yoneda Lemma to define their category of algebras, construct their free algebra and forgetful functors, as well as construct the canonical comparison functors from right adjoints to the algebra category of its induced monad, the starting point of all monadicity statements.

Definition 2.3.1

Let A be a 2-category. We define a monad in A to be a lax 2-functor $\Delta^0 \rightsquigarrow A$.

Before we move on, let us comment on how to compare this definition of monads with, for example, Lurie's definition in [Lur17].

Remark 2.3.2

To be able to compare this definition of monad to Lurie's definition in [Lur17], let us consider a monad on a 1-category C as in the top horizontal lax functor below.

$$
\begin{tikzcd}
\Delta^0 \arrow[rr, rightsquigarrow, "T"] \arrow[dr, rightsquigarrow] & & \mathrm{Cat} \\
& B(\mathrm{End}(C), \circ) \arrow[ur, hook, "\text{full on object C}"'] &
\end{tikzcd}
$$

As the object space of $\mathrm{Lax}(\Delta^0)$ is connected by Lemma 3.2.8, it factors over the full sub-2-category of Cat on the object C, which we denote by $B(\mathrm{End}(C), \circ)$. In the model of 2-categories as globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, we can compare this by Theorem 1.4.5 to inert-cocartesian functors as in

$$\begin{tikzcd}
 & B(\mathrm{End}(C),\circ)^{\int} \arrow[d, hook] & \\
\mathbf{\Delta}^{\mathrm{op}} \arrow[ur, "\text{inert-cocartesian}"] \arrow[r, "\text{inert-cocartesian}"] \arrow[dr, equal] & \mathrm{Cat}^{\int} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{CAT}_{\bullet} \arrow[d, two heads, "p_{\mathrm{univ}}"'] \\
 & \mathbf{\Delta}^{\mathrm{op}} \arrow[r, "\mathrm{Cat}"] & \mathrm{CAT}\,,
\end{tikzcd}$$

where we denote by $(-)^{\int}$ the unstraightenings of the globular complete Segal objects modelling the corresponding 2-categories and where we have as the right hand side vertical functor the universal cocartesian fibration with large fibers. Lurie defines monads in [Lur17] as monoids in the monoidal 1-category $(\mathrm{End}(C),\circ)$ of endofunctors on the 1-category C. Monoidal 1-categories themselves he defines in [Lur09b] as monoids in Cat with respect to the cartesian monoidal structure, which are can be modeled by Segal objects $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ with $X(\Delta^0) \simeq \Delta^0$, or equivalently Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, in our case $\mathrm{End}(C)^{\circ} \twoheadrightarrow \mathbf{\Delta}^{\mathrm{op}}$. Monoids in monoidal 1-categories in turn are modelled by lax monoidal functors out of the point, which can be modelled by inert-cocartesian functors out of the identity fibration. Our claim here now is that in the diagram

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathrm{op}} \arrow[r, dashed, "\exists!\text{inert-cocartesian}"] \arrow[rr, bend left, "\text{inert-cocartesian}"] \arrow[ddr, equal] & \mathrm{End}(C)^{\circ} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B(\mathrm{End}(C),\circ)^{\int} \arrow[r, hook] \arrow[d] & \mathrm{Cat}^{\int} \arrow[d] \\
 & \mathrm{codisc}(\Delta^0) \arrow[r, "\mathrm{codisc}(*)"] \arrow[d, two heads, "\simeq"'] & \mathrm{codisc}(B\,\mathrm{Aut}(C)) \arrow[r, hook, "\mathrm{codisc}(\mathrm{incl})"] \arrow[dl, two heads] & \mathrm{codisc}(\mathrm{Cat}^{\simeq}) \arrow[dll, two heads] \\
 & \mathbf{\Delta}^{\mathrm{op}} & &
\end{tikzcd}$$

the middle indicated square is actually a pullback. Hence inert-cocartesian functors to $\mathrm{End}(C)^{\circ}$ would correspond uniquely to to inert-cocartesian functors to $B(\mathrm{End}(C),\circ)$ and the two monad definitions would agree. We will not pursue a proof of the above claim in this work.

The first real application of the extended universal property enables us to reconstruct the data of a monad, in general consisting of an infinite amount of coherence data, from adjunctions, which can be specified by a finite amount of coherence data.

Corollary 2.3.3

Let A be a 2-category and $f\colon a \to b$ a 1-morphism which has a right adjoint u, with unit $\eta\colon \mathrm{id} \Rightarrow uf$ and counit $\varepsilon\colon fu \Rightarrow \mathrm{id}$. Out of this data we can construct a monad in A on a whose underlying 1-morphism is uf, whose unit is η and whose multiplication is given by $u\varepsilon f\colon ufuf \Rightarrow uf$. This is done by applying the extended universal property of the lax 2-functor classifier of Δ^0 to

$$\begin{tikzcd}
\Delta^0 \arrow[r, "(a,u)"] \arrow[dr, equal] & {}_{b/}^{\mathrm{lax}}A \arrow[d] \\
 & \Delta^0\,.
\end{tikzcd}$$

To this end we need to show that the induced functor on Hom-categories

$$ {}^{b/^{\mathrm{lax}}}A((a,u),(a,u)) \to \Delta^0(*,*) \simeq \Delta^0 $$

has a fully-faithful right adjoint. As we saw in Theorem 2.2.28, we may compute this to be equivalent to

$$ \overleftarrow{\int_{g\colon A(a,a)}} A(b,a)(g\circ u, u) \to \Delta^0 \ . $$

Now, as precomposition with the right adjoint u is actually left adjoint to precomposition with f we actually have

$$ \overleftarrow{\int_{g\colon A(a,a)}} A(b,a)(g\circ u, u) \simeq \overleftarrow{\int_{g\colon A(a,a)}} A(a,a)(g, u\circ f) \to \Delta^0 $$

and the last functor has as fully-faithful right adjoint the terminal object of this slice-over-uf category, which on the left hand side corresponds to the object $(uf, u\varepsilon)$. All that is left to do is to postcompose the thus uniquely induced lax 2-functor

$$ \begin{tikzcd}
 & \Delta^0 \arrow[dl] \arrow[dr, "{(a,u)}"] \arrow[dd, equal] & & \\
\mathrm{Lax}(\Delta^0) \arrow[rr, dashed, "\exists!"] \arrow[dr] & & {}^{b/^{\mathrm{lax}}}A \arrow[r, "\mathrm{pr}"] \arrow[dl] & A \\
 & \Delta^0 & &
\end{tikzcd} $$

by the projection to A.

Now turning to monads in Cat, the main observation we will exploit to give synthetic definitions to classical constructions, is that the algebra category should be the lax limit of the lax functor encoding the monad. Now the general philosophy is that for Cat, we have control over lax limits, as they should correspond to categories of sections for the Grothendieck constructions, i. e. unstraightenings, of the functors to Cat.

Definition 2.3.4

For a monad $T\colon \mathrm{Lax}(\Delta^0) \to \mathrm{Cat}$ on a 1-category we can define its category T-Alg of T-algebras as the 1-category of sections of the associated 2-cocartesian unstraightening $\overrightarrow{\int} T \twoheadrightarrow \mathrm{Lax}(\Delta^0)$. Furthermore we call $T(*)$ its underlying category. We call the canonical 1-functor

$$ T\text{-}\mathsf{Alg} = \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T) \xrightarrow{\mathrm{ev}_*} \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\Delta^0, \overrightarrow{\int} T) \simeq T(*) $$

its forgetful functor.

The following lemma will be the essential ingredient to construct the free algebra functor.

Lemma 2.3.5

The projection 2-functor $\mathrm{pr}\colon {}^{*/^{\mathrm{lax}}}\mathrm{Lax}(\Delta^0) \to \mathrm{Lax}(\Delta^0)$ admits a 2-right adjoint U with invertible counit, such that on objects we have $U^{\simeq} = \mathrm{const}(\iota(\mathrm{id}_*))$ and $\varepsilon^{\simeq}$ constant on the identity $\mathrm{pr}(\iota(\mathrm{id}_*)) = *$.

Proof. We check the objectwise criterion for 2-adjunctions. That means we want to check for all objects x of $\mathrm{Lax}(\Delta^0)$ the induced natural transformation ${*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0)(-,U(x)) \xrightarrow{\mathrm{pr}} \mathrm{Lax}(\Delta^0)(\mathrm{pr}(-),*)$ is invertible. To this end we may assume $x = *$ as $\Delta^0 \to \mathrm{Lax}(\Delta^0)$ is essentially surjective Lemma 3.2.8 and it suffices to show for all objects (y,k) of ${*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0)$ that

$$ {*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0)((y,k),U(*)) \xrightarrow{\mathrm{pr}} \mathrm{Lax}(\Delta^0)(y,*) $$

is an equivalence. By Theorem 2.2.28 we can compute this as

$$ \begin{aligned} {*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0)((y,k),U(*)) &\simeq \overleftarrow{\int_{f\colon \mathrm{Lax}(\Delta^0)(y,*)}} \mathrm{Lax}(\Delta^0)(*,*)(f\circ k, \iota(\mathrm{id}_*)) \\ &\simeq \overleftarrow{\int_{f\colon \mathrm{Lax}(\Delta^0)(y,*)}} \Delta^0 \simeq \mathrm{Lax}(\Delta^0)(y,*) \end{aligned} $$

and we know that in those Hom-categories $\mathrm{Lax}(\Delta^0)(y,*) \simeq \mathrm{Lax}(\Delta^0)(*,*)$ the object $\iota(\mathrm{id}_*)$ is a terminal object. □

Definition 2.3.6

For a monad $T\colon \mathrm{Lax}(\Delta^0) \to \mathrm{Cat}$ in Cat we can define its free algebra functor as the following composite.

$$ \begin{aligned} T(*) \simeq \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\Delta^0, \overrightarrow{\int} T) &\xrightarrow{\text{Yoneda}\simeq} \mathrm{Fun}^{\text{2-cocart}}_{/\,\mathrm{Lax}(\Delta^0)}({*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T) \\ &\subseteq \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}({*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T) \xrightarrow{*U} \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T) \\ &= T\text{-}\mathsf{Alg} \end{aligned} $$

Now that we constructed both the free algebra functor as well as the forgetful functor let us comment on the desired adjunction between those.

Remark 2.3.7

Of course we would like to have an adjunction between the free algebra and the forgetful functor. In the above Definition 2.3.6 we composed the free algebra functor out of two parts. The second part is precomposition with the relative 2-right adjoint U over $\mathrm{Lax}(\Delta^0)$ from Lemma 2.3.5

$$ \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}({*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T) \xrightarrow{*U} \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T) $$

so itself a *left adjoint*. The first part is

$$ \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\Delta^0, \overrightarrow{\int} T) \simeq \mathrm{Fun}^{\text{2-cocart}}_{/\,\mathrm{Lax}(\Delta^0)}({*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T)\ , $$

whose first equivalence is the fibrational, i. e. unstraightened, 2-Yoneda Lemma, followed by the inclusion into all 2-functors over $\mathrm{Lax}(\Delta^0)$. As some models of $(\infty,2)$-categories and the lax Gray tensor products show, not-necessarily 2-cocartesian 2-functors over $\mathrm{Lax}(\Delta^0)$ should correspond to *lax natural transformations* between the straightened 2-functors to Cat. Under this identification, the wish for a right adjoint for the functor

$$\mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\Delta^0, \overrightarrow{\int} T) \simeq \mathrm{Fun}^{\text{2-cocart}}_{/\,\mathrm{Lax}(\Delta^0)}({*}/^{\mathrm{lax}}\,\mathrm{Lax}(\Delta^0), \overrightarrow{\int} T)$$

can be seen to be equivalent to wishing the composite

$$T(*) \simeq \mathrm{Fun}(\mathrm{Lax}(\Delta^0), \mathrm{Cat})(\mathrm{Lax}(\Delta^0)(*, -), T) \subseteq \mathrm{Fun}^{\mathrm{lax}}(\mathrm{Lax}(\Delta^0), \mathrm{Cat})(\mathrm{Lax}(\Delta^0)(*, -), T)$$

has a right adjoint, where $\mathrm{Fun}^{\mathrm{lax}}(-,-)$ denotes a 2-category whose objects are strict 2-functors and whose 1-morphisms are the *lax* natural transformations. This is then an instantiation of the *lax* 2-*Yoneda Lemma* which states in its general form the adjointness of the two 1-functors

$$\begin{tikzcd}
F(a) \arrow[r, "\simeq"] & \mathrm{Fun}(A, \mathrm{Cat})(A(a,-), F) \arrow[r, hook, "\mathrm{incl}"] & \mathrm{Fun}^{\mathrm{lax}}(A, \mathrm{Cat})(A(a,-), F) \arrow[ll, bend left, "(-)_a(\mathrm{id}_a)"]
\end{tikzcd}$$

upgrading the strict 2-Yoneda Lemma equivalence. So finding a proof of the lax 2-Yoneda Lemma, in its fibrational, i. e. unstraightened form, would prove our desired adjointness of free algebra and forgetful functors.

Actually, the use of the extended universal property in Corollary 2.3.3 was throwing away a lot of the coherence data provided by the constructed lax functor. As we will see in the following corollary, this extra coherence is, in a precise sense, encoding the comparison functor from the rightadjoint to the algebra category.

Corollary 2.3.8

Let $F\colon C \to D$ be a functor which has a right adjoint U, then we can construct the canonical comparison functor from D to the category of algebras for the monad $T := UF$ induced by the adjunction. As in Corollary 2.3.3 we get the induced dashed lax 2-functor

$$\begin{tikzcd}
& \Delta^0 \arrow[dl] \arrow[dr, "{(C,U)}"] \arrow[dd, equal] & & & \\
\mathrm{Lax}(\Delta^0) \arrow[rr, dashed, "\exists!"] \arrow[dr] & & {}^{D/^{\mathrm{lax}}}\mathrm{Cat} \arrow[r, equal] \arrow[dl] \arrow[drr, two heads] & \overrightarrow{\int} \mathrm{Fun}(D,-) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(D, \mathrm{Cat}_\bullet) \arrow[d, "(p_{\mathrm{univ}})_*"] \\
& \Delta^0 & & \mathrm{Cat} \arrow[r, "\mathrm{const}"] & \mathrm{Fun}(D, \mathrm{Cat})
\end{tikzcd}$$

and by the definition of ${}^{D/^{\mathrm{lax}}}\mathrm{Cat}$ as $\overrightarrow{\int}\mathrm{Fun}(D,-)$ we also have the right hand side pullback, which can also be taken to be the definition of the 2-functor $\mathrm{Fun}(D,-)\colon \mathrm{Cat} \to \mathrm{Cat}$ by straightening-unstraightening. Thus we can transform this 2-functor into a 2-functor

$$\begin{tikzcd}
\mathrm{Lax}(\Delta^0) \times D \arrow[rr] \arrow[dr, two heads, "\mathrm{pr}_0"'] & & \overrightarrow{\int} T \arrow[dl, two heads] \\
& \mathrm{Lax}(\Delta^0) &
\end{tikzcd}$$

over $\mathrm{Lax}(\Delta^0)$ which itself is equivalent to a 1-functor $D \to \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\mathrm{Lax}(\Delta^0), \vec{\int} T) = T\text{-}\mathsf{Alg}$ from D into the category of T-algebras. Furthermore, by definition of the lax 2-functor represented by $\mathrm{Lax}(\Delta^0) \to {}^{D/^{\mathrm{lax}}}\mathrm{Cat}$, precomposing with $\Delta^0 \to \mathrm{Lax}(\Delta^0)$ gives us a commutative square

$$\begin{tikzcd} D \arrow[r] \arrow[d, "U"'] & \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\mathrm{Lax}(\Delta^0), \vec{\int} T) \arrow[r, equal] \arrow[d] & T\text{-}\mathsf{Alg} \\ C \arrow[r, "\simeq"'] & \mathrm{Fun}_{/\,\mathrm{Lax}(\Delta^0)}(\Delta^0, \vec{\int} T)\,, & \end{tikzcd}$$

i. e. the canonical functor to the category of T-algebras composed with the forgetful functor to C recovers the original right adjoint U.

Now that we have constructed the comparison functor from a right adjoint to the algebra category of its induced monad, we have overcome the first obstacle towards a synthetic monadicity theory, we can now define a right adjoint to be *monadic* if this comparison functor is an equivalence. In general, Beck's monadicity theorem and its higher categorical generalisations, see for example [RV16] and [Lur17], gives concrete conditions to check for a right adjoint to be monadic in terms of, for example, conservativity, as well as the existence and preservation of some limits. The higher categorical Barr-Beck monadicity theorem in [Lur17, Theorem 4.7.3.5] was used crucially in Lurie's setup of descent theory in the realm of higher categories and has proven to be an essential tool that produces non-trivial applications in Derived Geometry. A core aspect of monadicity characterisations is the observation that every algebra α for a monad T as a canonical resolution

$$\begin{tikzcd} \dots & T^2 a \arrow[r, bend left, "T\alpha"] \arrow[r, bend right, "\mu_a"'] & Ta \arrow[l, "T\eta_a"'] \arrow[r, "\alpha"] & a \end{tikzcd}$$

by free T-algebras within the category of algebras itself, and after forgetting down to the base category, this resolution diagram enhances in the most optimal way to an absolute colimit cocone, that is a cocone which will be a colimit cocone whereever one sends it with other functors. It was observed for example in [RV16] that the canonical resolution of T-algebras by free ones in the above way is exactly the *comonad resolution* of the free forgetful adjunction that the algebra category should carry. Here the comonad resolution is just the dual for comonads of the canonical *monad resolution*

$$\begin{tikzcd} \mathrm{id} \arrow[r, Rightarrow, "\eta"] & T \arrow[r, Rightarrow, bend left, "\eta_T"] \arrow[r, Rightarrow, bend right, "T\eta"'] & T^2 \arrow[l, Rightarrow, "\mu"'] & \dots \end{tikzcd}$$

every monad admits in the category $\mathrm{Fun}(C, C)$. As a last comment towards this direction, let us observe that for a monad $\mathrm{Lax}(\Delta^0) \to \mathrm{Cat}$ on a category C its action on Hom-categories $\mathrm{Lax}(\Delta^0)(*, *) \to \mathrm{Fun}(C, C)$ exactly provides this monad resolution diagram, as can be deduced from the comparison to the model of the lax functor classifier given by the envelope construction in Theorem 1.4.5 and Theorem 1.4.5.

2.4 Constructing Lax Adjoints from Objectwise Data

We finally come to the main result of this chapter, an objectwise existence criterion for lax 2-adjoints.

Among the few ways where it is actually sufficient to define a functor of higher categories by providing its assigment on objects, is the, already above in Lemma 2.2.22 used, construction of adjoint functors from their (co)unit morphisms and their objectwise universal properties.

To circle back to our goal in this chapter, i. e. to provide means by which we can produce lax 2-functors, a lax generalisation of such an objectwise existence criterion for lax 2-adjoints has similar potential to become a viable tool to construct examples.

For ordinary 2-categories, it has already been observed by Betti and Power in [BP88], that if we start with a strict 2-functor $F\colon A \to B$ and for every object b of B an object Ub of A together with 1-morphisms $\varepsilon_b\colon F(Ub) \to b$, we may ask the induced 1-functors between Hom-1-categories

$$B(F(a), b) \xleftarrow{(\varepsilon_b)_*} B(F(a), F(Ub)) \xleftarrow{F_{a,Ub}} A(a, Ub)$$

to be something more general than equivalences, for example left adjoints. Whereas the case for equivalences results in a 2-adjunction, i. e. U assembles into a strict 2-right adjoint, asking the induced functors on Hom-1-categories to only have right adjoints $R_{a,b}$, Betti and Power observe in [BP88, Proposition 4.2] that the object assigment $b \mapsto Ub$ and the 1-functors

$$B(b, \widetilde{b}) \xrightarrow{(\varepsilon_b)_*} B(F(Ub), \widetilde{b}) \xrightarrow{R_{Ub,\widetilde{b}}} A(Ub, U\widetilde{b})$$

in general assemble into a lax 2-functor $U\colon B \rightsquigarrow A$. Additionally, instead of having the ε_b assemble into a strict 2-natural transformation $FU \Rightarrow \mathrm{id}$ as in the equivalence case, the ε_b will give rise to a generally lax natural transformation ε. The above can be viewed as a generalisation of the fact that right adjoints of strong monoidal functors automatically acquire a lax monoidal structure.

In [BP88, Definition 4.1], Betti and Power give a name to the above situation and say that F, which they also allow to be oplax, *induces a local right adjoint* if one can find Ub and ε_b giving the adjointability. In fact, they define in [BP88, Definition 3.1] the more general notion of a *local adjunction* between an oplax 2-functor F and a lax 2-functor U to be an oplax natural transformation ϕ from the oplax 2-functor $B(F(-), -)$ to the oplax 2-functor $A(-, U(-))$, whose components $\phi_{a,b}$ have right adjoints.

Local adjunctions by themselves are in a certain global sense better behaved, they compose for example. In stark contrast to that, already for ordinary 2-categories it is not true that the composite $\widetilde{F} \circ F$ of oplax functors that *induce local adjunctions* as in the described setting above again give rise to the data for an *induced* local adjunction, i. e. they do *not* compose. If $\widetilde{F}$ was a strict 2-functor however, this problem vanishes, so the above described generality above is indeed a robust one. Let us remark here however, that having a local adjoint is *not* a property, i. e. local adjoints are *not* unique, neither in their general definition, nor in the induced sense, contrarily to the case for strict adjunctions. A single 2-functor might admit, or induce, several inequivalent local adjoints.

Citing from [BP88], adjunctions usually arise by means of a one-sided universal property. In this spirit, induced local adjunctions can be seen to house a good notion of 2-dimensional universal property, much like 1- and strict 2-adjunctions do. More precisely, one can view the tuple (Ub, ε_b) for a single object b to be a lax version of a 2-presheaf representation, called a *local representation* of the 2-presheaf $B(F(-), b)$. In general, a local representation of a 2-presheaf $G\colon K^{1\text{-op}} \to \mathrm{Cat}$ is a tuple (k, x) with k an object of K, x an object of $G(k)$, such that for all objects l of K the induced 1-functors

$$K(l, k) \xrightarrow{K} \mathrm{Fun}(G(k), G(l)) \xrightarrow{\mathrm{ev}_x} G(l)$$

have right adjoints and we can interpret this situation as exhibiting the following universal property if we specialize to $G = B(F(-), b)$ and $(k, x) = (Ub, \varepsilon_b)$, as described in [BP88]. For every 1-morphism $f\colon F(a) \to b$ there exists a lax triangle as on the left in

$$
\begin{tikzcd}
F(a) \arrow[r, "F(g)"] \arrow[dr, "f"'] & F(Ub) \arrow[d, "\varepsilon_b"] \arrow[dl, phantom, "\Swarrow \theta_g"] \\
& b
\end{tikzcd}
\qquad
\begin{tikzcd}
F(a) \arrow[r, "F(\widetilde{g})"] \arrow[dr, "f"'] & F(Ub) \arrow[d, "\varepsilon_b"] \arrow[dl, phantom, "\Swarrow \widetilde{\theta}"] \\
& b
\end{tikzcd}
\qquad
\begin{tikzcd}
\varepsilon_b F(\widetilde{g}) \arrow[r, Rightarrow, "\varepsilon_b F(\alpha)"] \arrow[dr, Rightarrow, "\widetilde{\theta}"'] & \varepsilon_b F(g) \arrow[d, Rightarrow, "\theta_g"] \\
& f\,,
\end{tikzcd}
\tag{2.4.0.1}
$$

such that for every similar such lax triangle as in the middle above there exists a unique $\alpha\colon \widetilde{g} \to g$ making the right hand triangle of 2-morphisms commute, i. e. exhibiting the triangle (g, θ) as the terminal such one in a precise sense. This is a lax generalisation of the usual universal property the counit object components ε_b enjoy for 1- and strict 2-adjunctions. Furthermore it is exactly these objectwise universal properties that let us reconstruct 1- and 2-adjoints, from their objectwise criteria, which in particular becomes immensely more important once we are in the realm of higher category theory, where the description of 1-functors, 2-functors and even more so lax 2-functor in general entails an infinite amount of coherence data. For example the lax composition comparison 2-morphisms $\gamma_{f,g}\colon Ug \circ Uf \Rightarrow U(gf)$ can be extracted from the above universal property by taking in Equation 2.4.0.1

$$
\begin{tikzcd}
\varepsilon_{\hat{b}} F(Ug \circ Uf) \arrow[rr, Rightarrow, dashed, "\exists! \varepsilon_b F(\gamma_{f,g})"] \arrow[d, Rightarrow, "\simeq"'] & & \varepsilon_{\hat{b}} F(U(g \circ f)) \arrow[dd, Rightarrow, "\theta_{gf\varepsilon_b}"] \\
\varepsilon_{\hat{b}} F(Ug) F(Uf) \arrow[dr, Rightarrow, "\theta_g F(Uf)"'] & & \\
& g \circ \varepsilon_{\tilde{b}} F(Uf) \arrow[r, Rightarrow, "g\theta_f"'] & g \circ f \circ \varepsilon_b\,.
\end{tikzcd}
$$

Hence we have found a prime candidate for a robust, objectwise lax (re)construction principle to build lax 2-functors as induced local adjoints in higher category theory.

Remark 2.4.1

Before we start to prove this lax (re)construction principle with our synthetic definition of lax 2-functors, let us comment on the similarities of the above situation and the simple, as well as the extended, universal properties of the lax 2-functor classifier given in Definition 1.3.1 and Definition 1.1.1. The similarities are no coincidence. Let us take ε_b to be invertible in this discussion, i. e. making $b \mapsto Ub$ into a section of F on object spaces. Then for example the simple universal property Definition 1.3.1 fits exactly into the above framework, when we further take this object section to be an equivalence. In this sense the simple universal property could be read as $\lambda\colon \operatorname{Lax}(A) \to A$ being the initial 2-functor into A inducing a local right adjoint *section*, which induces an equivalence on underlying object spaces. Note here that this in fact is a *property* for a 2-functor. The extended universal property does *not* quite fit the above framework. Whereas one can interpret the object space sections $U^{\simeq}$ as invertible counit data as above, the difference lies in asking for what pairs of objects we ask for, in this case fully-faithful, right adjoints. In Definition 1.1.1 we ask it only for object pairs of the form $(Ub, U\widetilde{b})$, in the above situation we ask it for more, i. e. all object pairs of the from (a, Ub). If all the a would be of the form Ub, ie if the section $b \mapsto Ub$ was essentially surjective, then they would be the same. Note here that this is actually the case for $\iota^{\simeq}\colon A^{\simeq} \to \operatorname{Lax}(A)$, because $\operatorname{Lax}(A)$ ought to be the *initial* object with the Hom-local right adjoint sections with respect to this object space section, i. e. it should be the object *freely generated* by this data and properties, hence morally speaking $\operatorname{Lax}(A)$ only needs to have the objects coming from $A^{\simeq}$, and no more. What is however true, is that for every object of

LocRARI(A) as on the left in

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr, "\gamma"'] & \mathrm{EssIm}(U^{\simeq}) \arrow[d, "F"] \\ & A \end{tikzcd}$$

restricting X to the essential image of $U^{\simeq}$ as on the right above again gives an object of LocRARI(A) whose object space section is now essentially surjective, among which Lax(A) is still initial. We still chose to state Definition 1.1.1 in this more general gown, for sake of readability, cleanness and to avoid the implicit essential image step.

In fact, we will prove the lax adjoint functoriality principle by reducing it down to our extended universal property Definition 1.1.1, by literally replacing the original F to something where the counit data, which could be called a *lax* section on object spaces by a strict section on object spaces, as well as simultanously replacing the Hom-right adjoints by fully-faithful ones. This idea can be seen as a 2-categorical generalisation of the following 1-categorical trick. Replacing a functor by the free cocartesian fibrations on it, we can observe that right adjoints for the original functor correspond to right adjoint sections of the free cocartesian fibration, see Lemma A.2.14 for a precise statement.

To replicate this idea we want access to the 2-categorical analogue of the identity 1-functor $\Delta_C : C \to \mathrm{Fun}(\Delta^1, C)$, i. e. the 2-functor $A \to \mathrm{Arr}^{\mathrm{lax}}(A)$ that sends everything to their respective identities.

Corollary 2.4.2

From ordinary 2-category theory, as well as several models of 2-categories based on simplicial combinatorics, see for example [GHL24] in the scaled simplicial setting, we know we should have for each 2-category A an identity 2-functor $\Delta_A : A \to \mathrm{Arr}^{\mathrm{oplax}}(A)$ which should operate on

1. objects by $a \mapsto (a, a, \mathrm{id}_a)$,
2. 1-morphisms by $f \mapsto (f, f, \mathrm{id}_f)$ and also on
3. 2-morphisms by $\alpha \mapsto (\alpha, \alpha, \mathrm{id}_\alpha)$.

Whereas in a type-theoretic approach such functors could be given as axiomatically supplied as constructors for a potential arrow or Hom-category type, we can also infer its existence by applying the extended universal property of the lax 2-functor classifier of Lax(A) to the commutative triangle

$$\begin{tikzcd} A^{\simeq} \arrow[r, "a \mapsto (a,a,\mathrm{id}_a)"] \arrow[dr] & \mathrm{Arr}^{\mathrm{oplax}}(A) := \overrightarrow{\int}_{\widetilde{a} \colon A} \overleftarrow{\int}_{a \colon A} A(a, \widetilde{a}) \arrow[d, "\mathrm{cod}"] \\ & A \end{tikzcd}$$

and proving the existence of the needed fully-faithful Hom-right-adjoints of the induced functor $\mathrm{cod} \colon \mathrm{Arr}^{\mathrm{oplax}}((a, \mathrm{id}_a), (\widetilde{a}, \mathrm{id}_{\widetilde{a}})) \to A(a, \widetilde{a})$ by computing the domain of these 1-functors using Theorem 2.2.28 be equivalent to

$$\begin{tikzcd} \mathrm{Arr}^{\mathrm{oplax}}(A))((a, \mathrm{id}_a), (\widetilde{a}, \mathrm{id}_{\widetilde{a}})) \arrow[r, "\simeq"] \arrow[d, two heads, "\mathrm{pr}"'] & \overrightarrow{\int}_{g \colon A(a,\widetilde{a})} \left(\overleftarrow{\int}_{a \colon A} A(a, \widetilde{a})^{\mathrm{op}} \right)(g_!(a, \mathrm{id}_a), (\widetilde{a}, \mathrm{id}_{\widetilde{a}})) \arrow[dl, two heads, "\mathrm{pr}"] \\ A(a, \widetilde{a}) & \end{tikzcd}$$

and we may also compute the fibers by Theorem 2.2.28 as

$$
(\overleftarrow{\int_{a\colon A}} A(a,\widetilde{a})^{\mathrm{op}})(g_!(a,\mathrm{id}_a),(\widetilde{a},\mathrm{id}_{\widetilde{a}})) \simeq \overrightarrow{\int_{f\colon A(a,\widetilde{a})}} A(a,\widetilde{a})(g,f)\ ,
$$

which has slice-over terminal object adjunctions

$$
\begin{tikzcd}
\overrightarrow{\int_{f\colon A(a,\widetilde{a})}} A(a,\widetilde{a})(g,f) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \Delta^0 \arrow[l, bend left, "(f,\mathrm{id}_f)"]
\end{tikzcd}\ ,
$$

hence a global fully-faithful right adjoint via *functoriality of universals*, i. e. the 1-categorical fact that a cocartesian fibration has a fully-faithful right adjoint *if and only if* all its fibers have terminal objects. Now we get an *a priori lax* 2-functor $\Delta_A\colon A \rightsquigarrow \mathrm{Arr}^{\mathrm{oplax}}(A)$ for which it is straighforward to show by direct inspection of the Hom-adjunctions that its lax comparison 2-morphisms are actually invertible, hence by employing Lemma 3.2.10 it was a *strict* 2-functor $\Delta_A\colon A \to \mathrm{Arr}^{\mathrm{oplax}}(A)$ to begin with. One can obtain similarly by Remark 2.2.29 a *strict* 2-functor $\Delta_A\colon A \to \mathrm{Arr}^{\mathrm{lax}}(A)$.

Theorem 2.4.3

Let $F\colon A \to B$ be a 2-functor and let us be given for each object b of B an object Ub of A together with a 1-morphism $\varepsilon_b\colon F(Ub) \to b$ in B, such that the canonically induced functors on Hom-categories

$$
B(F(a),b) \xleftarrow{(\varepsilon_b)_*} B(F(a),F(Ub)) \xleftarrow{F_{a,Ub}} A(a,Ub)
$$

admit right adjoints $R_{a,b}$ for all objects a of A and b of B. Then the objects Ub together with the functors $U_{b,\widetilde{b}}$ defined as the composite

$$
B(b,\widetilde{b}) \xrightarrow{{}_*(\varepsilon_b)} B(F(Ub),\widetilde{b}) \xrightarrow{R_{Ub,\widetilde{b}}} A(Ub,U\widetilde{b})
$$

on Hom-categories assemble canonically into a lax 2-functor $U\colon B \rightsquigarrow A$, and the 1-morphisms ε_b together with the following counit 2-morphisms of the Hom-adjunctions

$$
\begin{tikzcd}
F(Ub) \arrow[r, "\varepsilon_b"] \arrow[d, "F(U(f))=F(R_{Ub,\widetilde{b}}(f\circ\varepsilon_b))"'] \arrow[dr, phantom, "\Nearrow\varepsilon_f"] & b \arrow[d, "f"] \\
F(U\widetilde{b}) \arrow[r, "\varepsilon_{\widetilde{b}}"'] & \widetilde{b}
\end{tikzcd}
\qquad
\begin{tikzcd}
F(Ub) \arrow[r, "\varepsilon_b"] \arrow[d, "F(U(f))=F(L_{Ub,\widetilde{b}}(f\circ\varepsilon_b))"'] \arrow[dr, phantom, "\Swarrow\varepsilon_f"] & b \arrow[d, "f"] \\
F(Ub) \arrow[r, "\varepsilon_{\widetilde{b}}"'] & \widetilde{b}
\end{tikzcd}
$$

assemble into a lax 2-natural transformation $\varepsilon\colon FU \Rrightarrow \mathrm{id}_B$.

Proof. Our goal is to directly apply the universal property Definition 1.1.1 the lax functor classifier of B, and to this end we first construct the necessary data. More precisely, we will apply it to the 2-functor

$$
\mathrm{cod}\colon A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{oplax}}(B) \to B\ .
$$

The additional section on objects is supplied by the counit 1-morphism through cocartesian lifting

$$
\begin{tikzcd}
B^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[drr, bend right, "\mathrm{incl}"'] & A \arrow[r, "(\mathrm{id}_A,\Delta_B\circ F)"] \arrow[dr, "F"] \arrow[d, phantom, "\Swarrow\varepsilon^{\simeq}"] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[d, "\mathrm{cod}"] \\
& \, & B
\end{tikzcd}
\simeq
\begin{tikzcd}
B^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[rr, dashed, bend right=15, "(U-,\varepsilon_-)"'] \arrow[drr, bend right, "\mathrm{incl}"'] & A \arrow[r, "(\mathrm{id}_A,\Delta_B\circ F)"] \arrow[d, phantom, "\Swarrow"] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[d, "\mathrm{cod}"] \\
& \, & B
\end{tikzcd}
$$

To apply the universal property of the lax 2-functor classifier of B to

$$\begin{tikzcd} B^{\simeq} \arrow[r, "{(U-,\varepsilon_-)}"] \arrow[dr] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[d, "\mathrm{cod}"] \\ & B \end{tikzcd}$$

we need to show that on pairs of objects coming from $B^{\simeq}$ we have the fully-faithful Hom-right-adjoints. By Theorem 2.2.28 we have the following equivalence of cocartesian fibrations

$$\begin{tikzcd} (A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{oplax}}(B))((Ub,\varepsilon_b),(U\tilde{b},\varepsilon_{\tilde{b}})) \arrow[r, "\simeq"] \arrow[d, two heads, "\mathrm{pr}"'] & \overrightarrow{\int}_{g\colon B(b,\tilde{b})} (\overleftarrow{\int}_{a\colon A} B(F(a),\tilde{b})^{\mathrm{op}})(g_!(Ub,\varepsilon_b),(U\tilde{b},\varepsilon_{\tilde{b}})) \arrow[dl, two heads, "\mathrm{pr}"] \\ B(b,\tilde{b}) & \end{tikzcd}$$

over $B(b,\tilde{b})$. In order to find a fully-faithful right adjoint it suffices to show by functoriality of universals that all its fibers have terminal objects. In order to see this we once again apply Theorem 2.2.28 to its fibers and can compute

$$\begin{aligned} (\overleftarrow{\int}_{a\colon A} B(F(a),\tilde{b})^{\mathrm{op}})(g_!(Ub,\varepsilon_b),(U\tilde{b},\varepsilon_{\tilde{b}}) &\simeq \overleftarrow{\int}_{f\colon A(Ub,U\tilde{b})} B(F(Ub),\tilde{b})(f^*\varepsilon_{\tilde{b}}, g_!\varepsilon_b) \\ &\simeq \overleftarrow{\int}_{f\colon A(Ub,U\tilde{b})} B(F(Ub),\tilde{b})(\varepsilon_{\tilde{b}}\circ F(f), g\circ\varepsilon_b) \\ &\simeq \overleftarrow{\int}_{f\colon A(Ub,U\tilde{b})} A(Ub,U\tilde{b})(f, R_{Ub,\tilde{b}}(g\circ\varepsilon_b))\ . \end{aligned}$$

Now we may conclude as the last 1-category is a slice-over category which always has as a terminal object the identity morphism. □

Remark 2.4.4

Theorem 2.4.3 and its proof can also be given the following stronger and more coherent conclusion. The functor

$$\mathrm{Map}_{/B}^{B^{\simeq}\text{-vadj}}(\lambda,\mathrm{cod}) \xrightarrow{\ {}_*\iota^{\simeq}\ } \mathrm{Map}_{/B}^{B^{\simeq}\text{-RARI}}(\gamma_B,\mathrm{cod})$$

is an equivalence, where we denote on the left hand side the full subspace of 2-functors

$$\begin{tikzcd} \mathrm{Lax}(B) \arrow[r, dashed] \arrow[dr, "\lambda"'] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[d, "\mathrm{cod}"] \\ & B\,, \end{tikzcd}$$

which induce, for all objects coming from $\iota^{\simeq}\colon B^{\simeq} \to \mathrm{Lax}(B)$, on Hom-categories vertically adjointable triangles, and on the right hand side the full subspace of functors

$$\begin{tikzcd} B^{\simeq} \arrow[r, dashed] \arrow[dr, "\gamma_B"'] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[d, "\mathrm{cod}"] \\ & B\,, \end{tikzcd}$$

such that on all objects coming from $B^{\simeq}$ the action of cod on Hom-categories has fully-faithful right adjoints.

Remark 2.4.5

We may dualise Theorem 2.4.3 in the following three ways.

1. By applying $(-)^{2\text{-op}}$ to the input data and using $\mathrm{OpLax}(-) \simeq (\mathrm{Lax}((-)^{2\text{-op}}))^{2\text{-op}}$ from Lemma 1.4.7 we get the following version.
   Starting with a strict 2-functor $F\colon A \to B$, with for every object b of B a counit 1-morphism candidate $\varepsilon_b\colon F(Ub) \to b$ we can again ask for left adjoints for the induced 1-functors
   $$B(Fa, b) \xrightarrow{U} A(U(Fa), U(b)) \xrightarrow{*\eta_a} A(a, U(b))$$
   to get an induced oplax 2-functor U and oplax natural transformation $\varepsilon\colon FU \Rrightarrow \mathrm{id}$.
2. Furthermore we can dualise the above Theorem 2.4.3 by applying $(-)^{1\text{-op}}$ to the input data and using $\mathrm{Lax}(-) \simeq (\mathrm{Lax}((-)^{1\text{-op}}))^{1\text{-op}}$ from Lemma 1.4.6, to get the following version.
   Starting with a strict 2-functor $U\colon B \to A$, the potential right adjoint, and for every object a of A a *unit* 1-morphism candidate $\eta_a\colon a \to U(Fa)$ and then we can again ask for right or left adjoints for the induced 1-functors
   $$B(Fa, b) \xrightarrow{U} A(U(Fa), U(b)) \xrightarrow{*\eta_a} A(a, U(b))$$
   to get induced lax or oplax 2-functors F and transformations $\eta\colon \mathrm{id} \Rrightarrow UF$.

Remark 2.4.6

Note that, as stated in the introduction of this section, [BP88, Proposition 4.2] not just reconstructs the lax right adjoint from the objectwise given counit 1-morphisms ε_b and their local universal properties, it also (re)constructs the entire structure of a lax natural transformation $\varepsilon\colon FU \Rrightarrow \mathrm{id}$ from the objectwise input. As we have seen, the above Theorem 2.4.3, provides this too, hence it is not just a lax functoriality principle for the lax adjoints, but also a lax naturality principle for the accompanying counit transformations, i. e. it may also be used to (re)construct these from the objectwise data. As for lax 2-functors in higher category theory, the description of lax natural transformations in general entails the an infinite amount of coherence data on top of the laxly commuting naturality squares for every 1-morphism.

To close this section let us comment on our naming convention with respect to our examples and the original terminology from [BP88]. We chose not to use the terminology of *inducing local adjoints*, as in the below examples and applications we do not yet make use of the actually induced local adjunction in the sense of Betti and Power's [BP88, Definition 4.1], nor do we construct the actual analogues of local adjunctions in the realm of higher category theory. As the below examples and applications solely revolve around the use of Theorem 2.4.3 to showcase the constructions of (op)lax 2-functors and the lax (co)unit transformations, we chose to make this difference clear by calling the output of Theorem 2.4.3 *induced (op)lax adjoints* and *induced (op)lax natural transformations.*

2.5 Application: Examples of (Op-)Lax Adjoints Constructed from Objectwise Data

In this section we apply Theorem 2.4.3 to synthetically construct several examples of (op)lax adjoint 2-functors. The examples from section 2.5.1 and subsection 2.5.2 have already been considered for ordinary 2-categories, the Span-examples also for $(\infty, 2)$-categories through models, like the inert-cocartesian functors between globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, see for example [Rui25]. For the other examples we could not find a reference for, neither in the ordinary, nor the higher categorical literature.

The examples will be split up into two parts. The first part will address the examples that produce *actually (op)lax* 2-adjoints, i. e. the cases where the by Theorem 2.4.3 produced lax 2-functor are *not strict* in general. After this we also discuss what we can say about the special case where both adjoints turn out to be strict 2-functors, by relating it to another formal definition of *lax* 2*-adjunction* given by Gray in [Gra74] which has this property.

2.5.1 Examples of *actually* (Op-)Lax 2-Adjoints

In this subsection we synthetically construct examples of (op)lax 2-adjoints which are actually not strict.

This set of examples can be further divided into three different sources of lax 2-functors, or more specifically, adjoints.

The first subsection will showcase examples arising from so-called *span* 2*-categories* of finitely complete 1-categories. Span-2-categories are particularly well-fit to parametrise bivariant functoriality, and are prominently used in higher category theory to make coherent the theory of Grothendieck's *six-functor formalisms*, powerful categorifications of cohomology theories providing access for example to much more general duality phenomena. The second subsection uses Theorem 2.4.3 to give a solution to the, above in Remark 2.2.27 already encountered, problem of endowing the underlying object space functor $(-)^{\simeq}\colon \mathrm{Cat}^{1\text{-cat}} \to \mathcal{S}$ with 2-categorical functoriality. The third subsection introduces an entirely different source of lax 2-functoriality, given by two, somewhat related, notions of *deriving* functors between certain kinds of free cocompletions.

Span-2-Categories

This set of examples revolves around span 2-categories, whose 1-morphisms are spans $c \leftarrow d \rightarrow e$ in a 1-category with pullbacks. Span composition is given by pullbacks, i. e. a limit construction.

The general idea here is that 2-functorial operations here can become oplax when these defining limits for 1-morphism composition are not preserved on the nose by the intended assigment, but still manage, by 1-functoriality on Hom-categories for example, to send the limit cone in the domain to a cone over the respective target diagram. Hence the limit universal property produces a canonical comparison, which may be represented by a 2-morphism between the image of a composition and the compostion of the respective images of the 1-morphisms.

Before we give the two examples, let us introduce what we need to know about span 2-categories.

Remark 2.5.1

For every 1-category C which admits pullbacks there is a 2-category $\mathrm{Span}(C)$, which roughly consists of

- objects: objects of C,
- 1-morphisms: spans $c \leftarrow d \rightarrow e$ of morphisms in C,

- 2-morphisms: commutative diagrams

$$\begin{tikzcd} & d_0 \arrow[dl, "f_0"'] \arrow[dd, "h"] \arrow[dr, "g_0"] & \\ c & & e \\ & d_1 \arrow[ul, "f_1"] \arrow[ur, "g_1"'] & \end{tikzcd}$$

  between spans of morphisms in C,
- put together, Hom-1-categories are two-fold slice-over categories

$$\mathrm{Span}(C)(c,d) \coloneqq {}_{c\backslash}C_{/d} \coloneqq (C_{/c}) \times_C (C_{/d}),$$

- composition of 1-morphisms: pullback induced spans via pullback

$$\begin{tikzcd} & & b \times_c d \arrow[dl] \arrow[dr] \arrow[dll, dashed, bend right] \arrow[drr, dashed, bend left] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\ & b \arrow[dl] \arrow[dr] & & d \arrow[dl] \arrow[dr] & \\ a & & c & & e\,. \end{tikzcd}$$

For a synthetic account on how to build this 2-category coherently we refer the reader to [Mac22]. We will only need the Hom-1-categories and the composition functors between those for our purposes here, as well as the fact that pullback-preserving functors $U\colon D \to C$ induce strict 2-functors $\mathrm{Span}(U)\colon \mathrm{Span}(D) \to \mathrm{Span}(C)$ between such span 2-categories, such that

- on objects $\mathrm{Span}(U)(d) = U(d)$,
- on Hom-1-categories $\mathrm{Span}(U)_{d_0,d_1} = U\colon {}_{d_0\backslash}D_{/d_1} \to {}_{U(d_0)\backslash}C_{/U(d_1)}$, i. e. it is just the action of U on spans.

In the first example, we lift adjunctions between 1-categories with pullbacks to the level of their span-2-categories. As left adjoints in general do not preserve pullbacks, the idea is that it will still induce an oplax 2-functor, as we always have compatible comparison maps between the images of two, via pullback, composed spans to the pullback of the images of the spans.

For ordinary 2-categories, this example is recorded in [BP88] as an instance of an induced local adjunction for ordinary 2-categories, but in fact goes back to Jay in [Jay88], who uses a different notion of *local adjunction*, where in general F *and* U are supposed to be both lax, or oplax, but his notion of local adjunction is a special case of Betti and Power's local adjunction in the case where one of the local adjoints is a strict 2-functor.

Example 2.5.2

Let $F\colon C \to D$ be a left adjoint functor between 1-categories which admit pullbacks, with right adjoint U. As right adjoints perserve limits, U induces a strict 2-functor $\mathrm{Span}(U)\colon \mathrm{Span}(D) \to \mathrm{Span}(C)$ between span 2-categories. Then F induces an oplax left adjoint $\mathrm{Span}(F)\colon \mathrm{Span}(C) \rightsquigarrow \mathrm{Span}(D)$. More precisely, let us denote the right adjoint by U and the unit of the adjunction by $\eta\colon \mathrm{id} \Rightarrow UF$. we will apply our Theorem 2.4.3 to $\mathrm{Span}(U)$ with the canonical candidates $\mathrm{Span}(F)(c) \coloneqq F(c)$, as well as the span

$$\begin{tikzcd} & c \arrow[dl, "\mathrm{id}_c"'] \arrow[dr, "\eta_c"] & \\ c & & UF(c) \end{tikzcd}$$

to serve as a candidate of a unit-1-morphism from c to $\mathrm{Span}(U)(\mathrm{Span}(F)(c))$. To this end we must verify that the on Hom-1-categories induced functor

$$\mathrm{Span}(D)(F(c),d) \xrightarrow{\mathrm{Span}(U)} \mathrm{Span}(C)(U(F(c)),U(d)) \xrightarrow{{}_*(\mathrm{id}_c,\eta_c)} \mathrm{Span}(C)(c,U(d))\ ,$$

i. e.

$$U_\eta\colon {}_{F(c)\backslash}D_{/d} \xrightarrow{U} {}_{U(F(c))\backslash}C_{/U(d)} \xrightarrow{{}_*(\mathrm{id}_c,\eta_c)} {}_{c\backslash}C_{/U(d)}$$

has a left adjoint, namely

$${}_{F(c)\backslash}D_{/d} \xleftarrow{(\mathrm{id}_{F(U(d))},\varepsilon_d)_*} {}_{F(c)\backslash}D_{/F(U(d))} \xleftarrow{F} {}_{c\backslash}C_{/U(d)} : F_\varepsilon\ .$$

Proof. To prove the claimed Hom-adjunction, let us fix a span $c \xleftarrow{f} x \xrightarrow{g} U(d)$ in C as well as a span $F(c) \xleftarrow{h} y \xrightarrow{k} d$ in D. By taking apart the respective Hom-spaces we get natural equivalences

$$\begin{array}{c}
{}_{F(c)\backslash}D_{/d}(F_\varepsilon(f,g),(h,k)) \\
\downarrow\simeq \\
\{F(f)\} \times^{h_*}_{D(F(x),F(c))} D(F(x),y)^{k_*} \times_{D(F(x),d)} \{\varepsilon\circ F(g)\} \\
\simeq\downarrow \\
\{\eta_c\circ f\} \times^{U(h)_*}_{C(x,U(F(c)))} C(x,U(y))^{U(k)_*} \times_{C(x,U(d))} \{g\} \\
\simeq\uparrow \\
\{f\} \times_{C(x,c)} C(x, c\times_{UF(c)} U(y)) \times_{C(x,U(d))} \{g\} \\
\simeq\uparrow \\
{}_{c\backslash}C_{/U(d)}((f,g),U_\eta(h,k))\ ,
\end{array}$$

where the dashed second equivalence is induced from the equivalence of spans of spaces

$$\begin{tikzcd}
\{F(f)\} \arrow[r] \arrow[d, "\simeq"] & D(F(x),F(c)) \arrow[d, "\simeq"] & D(F(x),y) \arrow[l, "h_*"'] \arrow[r, "k_*"] \arrow[d, "\simeq"] & D(F(x),d) \arrow[d, "\simeq"] & \{\varepsilon_d\circ F(g)\} \arrow[l] \arrow[d, "\simeq"'] \\
\{\eta_c\circ f\} \arrow[r] & C(x,U(F(c))) & C(x,U(y)) \arrow[l, "U(h)_*"'] \arrow[r, "U(k)_*"] & C(x,U(d)) & \{g\} \arrow[l]
\end{tikzcd}$$

coming in the middle from the naturality of the adjunction Hom-equivalences of $F \dashv U$, and the right by the trivial fact that $\varepsilon\circ F(g)$ gets sent to g. The only thing left to check is that on the left the adjunction Hom-equivalence sends $F(f)$ to $\eta_c\circ f$. But by naturality of the unit we have $\eta_c\circ f \simeq UF(f)\circ\eta_x$, and this is exactly the image of $F(f)$ under the Hom-equivalence $D(F(x),F(c)) \xrightarrow{\simeq} C(x,UF(c))$. □

For the next example, we introduce an oplax forgetful functor that sends spans to their span tips. Let C be a 1-category with finite limits. Let us write $B(C,\times)$ for the full 2-category of $\mathrm{Span}(C)$ on the terminal object $*$ of C. The span tip functor will be an oplax 2-functor $\mathrm{Span}(C) \rightsquigarrow B(C,\times)$ and could be used for example whenever one would like to equip the span tips with extra structure.

Example 2.5.3

Then the full sub-2-category inclusion $i\colon B(C,\times) \hookrightarrow \mathrm{Span}(C)$ on the terminal object $*$ has an oplax right adjoint given on objects by $U\colon c \mapsto *$ and with the counit-data given for every object

x of Span(C), i.e. of C, by the span $* \leftarrow x \xrightarrow{\mathrm{id}_x} x$. This lax functor $\mathrm{Span}(C) \rightsquigarrow B(C, \times)$ may be called the span tip 2-functor as on 1-morphism it sends spans to their span tips. More precisely, on Hom-1-categories it is given by the projection ${}_{x\backslash}C_{/y} \to C$.

Proof. By Theorem 2.4.3 we need to prove that

$$\mathrm{Span}(C)(i(*), x) \xleftarrow{\varepsilon_*} \mathrm{Span}(C)(i(*), i(U(x))) \xleftarrow{i} B(C, \times)(*, U(x)) ,$$

which is just

$$ {}_{*\backslash}C_{/x} \xleftarrow{(x \to *)^*} {}_{*\backslash}C_{/*} \xleftarrow{\simeq} C$$

has a left adjoint, and this is the projection functor $C_{/x} \to C$.

$$C_{/x} \xleftarrow{(x \to *)_*} C_{/*} \xleftarrow{\simeq} C ,$$

which has the right adjoint $(-) \times x \xrightarrow{\mathrm{pr}_x} x$. □

Remark 2.5.4

In fact, in the above example we could have also started with the object assignment $F \colon x \mapsto *$ and could have taken the unit 1-morphisms $x \xleftarrow{\mathrm{id}_x} x \to *$ and this data would produce an oplax left adjoint with the same object assignment and the same underlying action on Hom-1-categories.

The 2-Functoriality of the Underlying Space of Objects Functor

As we already observed above in Remark 2.2.27, the underlying space functor $(-)^{\simeq} \colon \mathrm{Cat} \to \mathcal{S}$ is not compatible with natural transformations, i.e. is not a 2-functor of actual 2-categories. The following example gives a way to turn it into a kind of 2-functor by enlarging the the 1-morphisms, i.e. functors, to so-called *profunctors*.

Remark 2.5.5

There is a 2-category Prof of small 1-categories and profunctors between them, i.e. a profunctor $P \colon C \nrightarrow D$ is a left adjoint functor $\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(D^{\mathrm{op}}, \mathcal{S})$, or equivalently a functor $C \times D^{\mathrm{op}} \to \mathcal{S}$. The 1-morphism composition and the 2-morphisms are exactly organised in $\mathrm{Fun}^L(\mathrm{Fun}(C^{\mathrm{op}}, S), \mathrm{Fun}(D^{\mathrm{op}}, S))$ as in the very large 2-category of large 1-categories and left adjoint functors between them, denoted by $\mathrm{CAT}^{\mathrm{L}}$. This can for example be built by starting with the functor $\mathrm{Fun}((-)^{\mathrm{op}}, S) \colon \mathrm{Cat} \to (\mathrm{CAT}^{\mathrm{R}})^{1\&2\text{-op}}$, realising that it lands in the full sub-2-category Pr^{R} on presentable 1-categories, and then postcomposing with the 2-equivalence $(\mathrm{Pr}^{\mathrm{R}})^{1\&2\text{-op}} \simeq \mathrm{Pr}^{\mathrm{L}}$, see for example Lurie's [Lur09b], and then taking the essential image on objects of this 2-functor.

Example 2.5.6

Let us denote by $\mathrm{Prof}_{\mathcal{S}}$ the full sub-2-category of Prof on 1-categories which are spaces. As profunctors between spaces X and Y are equivalently described by functors $X \times Y^{\mathrm{op}} \to \mathcal{S}$, which in turn are equivalent to spans of functors of spaces $X \leftarrow Z \to Y$, this morally suggests that we should have a 2-equivalence $\mathrm{Span}(\mathcal{S}) \simeq \mathrm{Prof}_{\mathcal{S}}$. We claim that the inclusion $\mathrm{Prof}_{\mathcal{S}} \hookrightarrow \mathrm{Prof}$ has a lax right adjoint $(-)^{\simeq} \colon \mathrm{Prof} \rightsquigarrow \mathrm{Prof}_{\mathcal{S}}$ which on objects is given by taking the underlying space $C^{\simeq}$ of a 1-category C, with counit 1-morphisms the induced left Kan extension functors $(\gamma_C)_!$ along the underlying space inclusion functors $\gamma_C \colon C^{\simeq} \to C$.

Proof. We now have to check that the composite

$$\mathrm{Prof}(\mathrm{incl}(X), C) \xleftarrow{(\gamma_!)_*} \mathrm{Prof}(\mathrm{incl}(X), \mathrm{incl}(C^{\simeq})) \xleftarrow{\mathrm{incl}\,\simeq} \mathrm{Prof}_{\mathcal{S}}(X, C^{\simeq})$$

simplified just

$$\mathrm{Fun}^L(\mathrm{Psh}(X), \mathrm{Psh}(C)) \xleftarrow{(\gamma_!)_*} \mathrm{Fun}^L(\mathrm{Psh}(X), \mathrm{Psh}(C^{\simeq}))$$

has a right adjoint for all 1-categories C and spaces X. But in Prof, for every 1-functor f between small 1-categories, the left Kan extension profunctor $f_!$ has the restriction along f functor f^* as a right adjoint in Prof, by the adjoint functor theorem, see for example [Lur09b], as it itself preserves colimits. So we get a right adjoint to $(\gamma_!)_*$ by postcomposing with γ^*, i. e. $(\gamma^*)_*$. $\square$

Deriving Functors produces Lax 2-Functors

Now we introduce two notions of *deriving* functors, and show that these derivation processes assemble into lax 2-functors. The idea here is that sometimes, the derived functor for composite functor might not agree with the individually derived functors, but usually the universal properties involved in this derivation process supply us with canonical comparison natural transformations. So deriving functors might not be strictly functorial, but it could still be laxly functorial, i. e. assemble into a lax 2-functor. We will illustrate this idea by the following two examples.

The first example will be about deriving as in computing cocontinuous approximations of arbitrary functors between free cocompletions. The second example is about deriving as in *animating* compact-projectively generated $(1,1)$-categories and 1-sifted cocontinuous functors between them.

We want to thank Benjamin Düenzinger for his immense help in devising the arguments for these examples, without him this wouldn't have been possible.

Cocontinuous Approximation This example is based on a generalisation of the 1-categorical fact that any functor $F\colon \mathrm{Psh}(C) \to \mathrm{Psh}(D)$ between 1-categorical presheaf categories can be approximated by a cocontinuous functor $\widetilde{F} := \mathrm{Lan}_y(F \circ y)$ from the left by taking the left Kan extension

$$\begin{tikzcd} C \arrow[d, hook, "y"'] \arrow[dr, "F\circ y"] \arrow[d, phantom, shift left=4ex, "\Leftarrow"] & \\ \mathrm{Psh}(C) \arrow[r, dashed, "\exists!\,\mathrm{Lan}_y\,Fy"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, shift right=3ex, "\exists!\Downarrow"] & \mathrm{Psh}(D)\,. \end{tikzcd}$$

As general functors need not preserve representables, the cocontinuous approximation of a composite $G \circ F$ of two arbitrary functors might in general *not* agree with the composite $t\widetilde{G} \circ \widetilde{F}$ of the individual cocontinuous approximations. However, by the left Kan extension universal property of we always get a canonically induced comparison natural transformation

$$\begin{tikzcd} C \arrow[d, hook, "y"'] & & & & & & \\ \mathrm{Psh}(C) \arrow[r, "\mathrm{Lan}_y\,Fy"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, shift right=3ex, "\exists!\Downarrow"] & \mathrm{Psh}(D) \arrow[r, "\mathrm{Lan}_y\,Gy"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, shift right=3ex, "\Downarrow\exists!"] & \mathrm{Psh}(E) & \arrow[r, mapsto, "\mathrm{Lan}_y"] & {} & \mathrm{Psh}(C) \arrow[r, "\mathrm{Lan}_y\,Fy"] \arrow[rr, bend right=30, "\mathrm{Lan}_y(GFy)"'] \arrow[rr, phantom, shift right=3ex, "\exists!\Downarrow"] & \mathrm{Psh}(D) \arrow[r, "\mathrm{Lan}_y\,Gy"] & \mathrm{Psh}(E)\,. \end{tikzcd}$$

In this light, the process of sending functors between free cocompletions is not strictly functorial, but as we will see below, it still assembles into as lax 2-functor.

As free cocompletions exist for all $(n,1)$-categories and all possible small colimit diagram shapes, we will also adapt this generality and comment later on possible instantiations.

Before we construct the example using Theorem 2.4.3, let us introduce the involved concepts more formally.

Definition 2.5.7

Let $\mathcal{J}$ denote a class of small 1-categories, i. e. a choice of small colimit diagram shapes. We also fix an $n \in \mathbb{N}_0 \cup \{\infty\}$ and call a 1-category n-Hom-truncated if all its Hom-spaces are n-truncated. We define $\mathrm{Psh}^n_{\mathcal{J}}(C)$ to be the free n-Hom-truncated $\mathcal{J}$-cocompletion of C, which can be computed by taking the $\mathcal{J}$-colimit closure of the representables inside $\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}_{\leq n})$ via $C \xhookrightarrow{y} \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}) \xrightarrow{(\tau_{\leq n})_*} \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}_{\leq n})$, where $\mathcal{S}_{\leq n} \longhookrightarrow \mathcal{S}$ is the full subcategory of n-Hom-truncated spaces and $\tau_{\leq n}$ the left adjoint to its inclusion. In particular we have the canonical Yoneda functor $y\colon C \to \mathrm{Psh}^n_{\mathcal{J}}(C)$, in general only fully-faithful when C is n-Hom-truncated, and we also have that $\mathrm{Psh}^n_{\mathcal{J}}(C)$ is n-Hom-truncated. More precisely, the universal property states that for every $\mathcal{J}$-cocomplete n-Hom-truncated 1-category D, precomposing with the Yoneda functor induces an equivalence

$$\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^n_{\mathcal{J}}(A), D) \xrightarrow{{}_*y \simeq} \mathrm{Fun}(A, D)$$

Let us denote the essential image of the 2-functor $\mathrm{Psh}^n_{\mathcal{J}}(-)\colon \mathrm{Cat}^{\simeq} \to \mathrm{CAT}$ by $\mathrm{CAT}^{\mathrm{free}}_{\mathcal{J},n}$.

Example 2.5.8

The inclusion $\mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n} \to \mathrm{CAT}^{\mathrm{free}}_{\mathcal{J},n}$ of the sub-2-category of $\mathcal{J}$-colimit preserving functors has a lax right adjoint, with action on objects as well as counit-1-morphisms the identity, and which acts on 1-morphisms by computing $\mathcal{J}$-colimit preserving approximations of functors.

$$\begin{tikzcd} \mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n} \arrow[r, "\mathrm{incl}"', bend left] \arrow[r, phantom, "\perp"] & \mathrm{CAT}^{\mathrm{free}}_{\mathcal{J},n} \arrow[l, squiggly, bend left, "\mathcal{J}\text{-cocontinuous approximation}"] \end{tikzcd}$$

This can be seen as we have the following canonical factorisation into an equivalence by the universal property of free n-Hom-truncated $\mathcal{J}$-cocompletions followed by the left Kan extension adjunction, as in the following diagram.

$$\begin{tikzcd} \mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^n_{\mathcal{J}}(A), \mathrm{Psh}^n_{\mathcal{J}}(B)) \arrow[rr, hook, "\mathrm{incl}"] \arrow[dr, "{}_*y\simeq"'] & & \mathrm{Fun}(\mathrm{Psh}^n_{\mathcal{J}}(A), \mathrm{Psh}^n_{\mathcal{J}}(B)) \arrow[dl, bend left, "{}_*y"] \\ & \mathrm{Fun}(A, \mathrm{Psh}^n_{\mathcal{J}}(B)) \arrow[ur, "\mathrm{Lan}_y"] \arrow[ur, phantom, "\perp", shift left=3] & \end{tikzcd}$$

This generality may now be instantiated for example to the following cases.

1. In the case where $\mathcal{J}$ is taken to be the class of *all* small 1-categories, we deduce that *deriving functors* between free small cocompletions in the sense of replacing them by their small colimit preserving approximations constitutes a lax 2-functor, as colimit approximating the composite of two arbitrary functors may not coincide with the composite of the separately taken colimit preserving approximations of the two factors. By the left Kan extension universal property we however still have a comparison natural transformation between them.

2. In the case where $\mathcal{J}$ is the class of κ-*filtered* small 1-categories, the free κ-filtered cocompletions are also called κ-accessible 1-categories, κ-filtered colimit preserving functors also κ-accessible functors. If we furthermore restrict the specialised lax 2-adjunction to objects which are κ-filtered cocompletions of small 1-categories which admit all κ-small colimits, i. e. the so-called κ-presentable 1-categories we get an induced lax 2-adjunction

$$\begin{tikzcd} \mathrm{CAT}^{\kappa\text{-acc}}_{\kappa\text{-pres}} \arrow[r, "\mathrm{incl}"] \arrow[r, phantom, shift right=4, "\perp"] & \mathrm{CAT}_{\kappa\text{-pres}} \arrow[l, bend left=60, squiggly, "\kappa\text{-accessible approximation}"] \end{tikzcd}.$$

3. In the case of $\mathcal{J}$ being the class of *sifted* small 1-categories, and n arbitrary, we can further restrict our specialised lax 2-adjunction to free sifted cocompletions of small categories that admit finite coproducts, agreeing for $n = 0$ the compact-projectively generated $(1, 1)$-categories, respectively for $n = \infty$ the compact-projectively generated $(\infty, 1)$-categories, by [Lur09b, Section 5.5.8 and 5.5.9].

Animation The animation of a compact-projectively generated $(1, 1)$-category C, the term coined in [ČS24, Section 5.1.4], like the category of abelian groups Ab, takes the full subcategory $C^{\text{sif-cpt}}$ of sifted compact objects, i. e. those c such that $C(c, -)$ preserves 1-sifted colimits, and freely cocompletes it under ∞-sifted colimits, computable as $\mathrm{Ani}(C) := \mathrm{Fun}^{\times}((C^{\text{sif-cpt}})^{\mathrm{op}}, \mathcal{S})$. The original $(1, 1)$-category C then embeds into its animation $\mathrm{Ani}(C)$ as the subcategory of its 0-truncated objects $C \hookrightarrow \mathrm{Ani}(C)$. For a 1-sifted colimit preserving functor $F : C \to D$ between compact-projectively generated $(1, 1)$-categories, we may now compute its ∞-sifted colimit preserving approximation, in Quillen's terminology from [Qui67] the *nonabelian* ∞-*derived functor* as the left Kan extension

$$\begin{tikzcd} C^{\text{sif-cpt}} \arrow[r, hook] \arrow[dr, "y"'] & C \arrow[r, "F"] & D \arrow[d, hook] \arrow[dl, dashed, Rightarrow, "\mathrm{Lan}\text{-}\mathrm{ext}"'] \\ & \mathrm{Ani}(C) \arrow[r, dashed, "\mathrm{Ani}(F)"'] & \mathrm{Ani}(D) \end{tikzcd}.$$

Empirically, deriving functors often yields valuable geometric information, like the derived cotangent complex.

Now as for cocontinuous approximation, taking ∞-sifted colimit preserving approximation is *not* strictly functorial, as for example remarked by Scholze in [ČS24, Proposition 5.1.5], as 1-sifted colimit preserving functors need not preserve sifted compact objects. However, the involved ∞-sifted cocontinuous approximation universal properties still provide us with canonical composition comparison transformations.

The goal of this example is to lift the animation process from an assignment $C \mapsto \mathrm{Ani}(C)$ on objects to an actual functorial operation, i. e. a lax 2-functor, more precisely even the lax right adjoint to the 2-functor sending a compact-projectively generated $(\infty, 1)$-category to its 0-truncated objects.

Remark 2.5.9

If one regards $C^{\text{sif-cpt}}$ as describing a *multi-sorted algebraic theory*, then we may interpret C as the category of models of this multi-sorted algbraic theory in the $(1, 1)$-category of *sets*. The animation of C in turn may be interpreted as the category of models of this multi-sorted algebraic theory in the $(\infty, 1)$-category of spaces $\mathcal{S}$, i. e. a *homotopical* interpretation of the algebraic theory encoded by $C^{\text{sif-cpt}}$.

In constructing the necessary data for the application of Theorem 2.4.3 we adhere to the generality of the previous example as long as possible to make the actual impact and use of siftedness in this discussion as clear as possible.

First we construct the truncation functors from $(\infty,1)$-cocompletions to $(n,1)$-cocompletions.

Construction 2.5.10

For a general $\mathcal{J}$ and n we construct the n-truncation functor $\gamma_n\colon \mathrm{Psh}^{\infty}_{\mathcal{J}}(A) \to \mathrm{Psh}^{n}_{\mathcal{J}}(A)$ via the universal property of being the free $\mathcal{J}$-cocompletion of $\mathrm{Psh}^{\infty}_{\mathcal{J}}(A)$ applied to the Yoneda functors $y\colon A \to \mathrm{Psh}^{n}_{\mathcal{J}}(A)$, where we use the equivalence $\mathrm{Fun}(\mathrm{ho}_n(A), \mathcal{S}_{\leq n}) \simeq \mathrm{Fun}(A, \mathcal{S}_{\leq n})$ for the n-truncation functor $\tau_n\colon A \to \mathrm{ho}_n(A)$.

$$\begin{tikzcd}
A \arrow[r, hook, "y"] \arrow[d] & \mathrm{Psh}^{\infty}_{\mathcal{J}}(A) \arrow[d, dashed, "\exists!"'] \arrow[dr, dashed, "\exists!\gamma_n"] & \\
\mathrm{ho}_n(A) \arrow[r, hook, "y"'] & \mathrm{Psh}^{n}_{\mathcal{J}}(\mathrm{ho}_n(A)) \arrow[r, "\simeq"'] & \mathrm{Psh}^{n}_{\mathcal{J}}(A)
\end{tikzcd}$$

With these functors we can now construct in all generality the 2-functor sending $(\infty,1)$-cocompletions to $(n,1)$-cocompletions, i. e. the one that will be our starting strict 2-functor for Theorem 2.4.3.

Construction 2.5.11

We can construct a n-truncating 2-functor $\mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},\infty} \to \mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n}$ that on objects sends $\mathrm{Psh}^{\infty}_{\mathcal{J}}(A)$ to $\mathrm{Psh}^{n}_{\mathcal{J}}(A)$, and on 1-morphisms computes the left Kan extensions

$$\begin{tikzcd}
A \arrow[r, hook, "y"] \arrow[dr, "y"'] & \mathrm{Psh}^{\infty}_{\mathcal{J}}(A) \arrow[r, "F"] \arrow[d, "\gamma_n"'] & \mathrm{Psh}^{\infty}_{\mathcal{J}}(B) \arrow[d, "\gamma_n"'] \\
& \mathrm{Psh}^{n}_{\mathcal{J}}(A) \arrow[r, dashed, "\exists!\,\mathrm{Lan}_y(\gamma_n F y)"'] & \mathrm{Psh}^{n}_{\mathcal{J}}(B)\,.
\end{tikzcd}$$

To this end consider the essential image $\mathcal{K}$ of the 2-functor

$$\begin{aligned}
\mathrm{Cat}^{\simeq} &\to \mathrm{Fun}(\Delta^1, \mathrm{CAT}^{\mathcal{J}\text{-cocts}})\,, \\
A &\mapsto [\gamma_n\colon \mathrm{Psh}^{\infty}_{\mathcal{J}}(A) \to \mathrm{Psh}^{n}_{\mathcal{J}}(A)]
\end{aligned}$$

and restrict the projections $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}(\Delta^1, \mathrm{CAT}^{\mathcal{J}\text{-cocts}}) \to \mathrm{CAT}^{\mathcal{J}\text{-cocts}} \times \mathrm{CAT}^{\mathcal{J}\text{-cocts}}$ to the span of 2-functors

$$\mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},\infty} \xleftarrow{\mathrm{ev}_0} \mathcal{K} \xrightarrow{\mathrm{ev}_1} \mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n}$$

Claim: The restricted domain projection ev_0 is in fact a 2-equivalence.

Using this claim we finish the construction of our desired 2-functor by using the inverse of ev_0

$$\mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},\infty} \xrightarrow{(\mathrm{ev}_0)^{-1}} \mathcal{K} \xrightarrow{\mathrm{ev}_1} \mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n}\,.$$

Proof of Claim. By construction it is essentially surjective. It suffices to show 2-fully-faithfulness. To see this let us deconstruct the Hom-categories in the following way.

$$\begin{tikzcd}
\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{\infty}_{\mathcal{J}}(A), \mathrm{Psh}^{\infty}_{\mathcal{J}}(B)) \times^{((\gamma_n)_{*}, {}_{*}(\gamma_n))}_{\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{\infty}_{\mathcal{J}}(A), \mathrm{Psh}^{n}_{\mathcal{J}}(B))} \mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{n}_{\mathcal{J}}(A), \mathrm{Psh}^{n}_{\mathcal{J}}(B)) \arrow[d, "\mathrm{pr}_0"] \\
\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{\infty}_{\mathcal{J}}(A), \mathrm{Psh}^{\infty}_{\mathcal{J}}(B))
\end{tikzcd}$$

We can now factorise this functor into the following equivalences

$$\begin{gathered}
\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{\infty}_{\mathcal{J}}(A), \mathrm{Psh}^{\infty}_{\mathcal{J}}(B)) \times^{((\gamma_n)_{*,*}(\gamma_n))}_{\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{\infty}_{\mathcal{J}}(A), \mathrm{Psh}^{n}_{\mathcal{J}}(B))} \mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{n}_{\mathcal{J}}(A), \mathrm{Psh}^{n}_{\mathcal{J}}(B)) \\
\downarrow \simeq \\
\mathrm{Fun}(A, \mathrm{Psh}^{\infty}_{\mathcal{J}}(B)) \times^{((\gamma_n)_{*,*}\,\mathrm{id})}_{\mathrm{Fun}(A, \mathrm{Psh}^{n}_{\mathcal{J}}(B))} \mathrm{Fun}(A, \mathrm{Psh}^{n}_{\mathcal{J}}(B)) \\
\downarrow \simeq \\
\mathrm{Fun}(A, \mathrm{Psh}^{\infty}_{\mathcal{J}}(B)) \\
\downarrow \simeq \\
\mathrm{Fun}^{\mathcal{J}\text{-cocts}}(\mathrm{Psh}^{\infty}_{\mathcal{J}}(A), \mathrm{Psh}^{\infty}_{\mathcal{J}}(B))
\end{gathered}$$

essentially only using the universal properties, to see that it itself is an equivalence. □

Next we construct the necessary counit data for our application of Theorem 2.4.3, which will turn out to be invertible, i. e. give a section on objects. The proofs of the claims of the following construction are unfortunately not synthetic and reference the theory of idempotent completions developed in [Lur09b].

Construction 2.5.12

We would like to construct a section of the n-truncating 2-functor $\mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},\infty} \to \mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n}$ by assigning $\mathrm{Psh}^{n}_{\mathcal{J}}(A) \mapsto \mathrm{Psh}^{\infty}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})$ where $(-)^{\mathcal{J}\text{-cpt}}$ denotes the full subcategory on $\mathcal{J}$-compact objects, i. e. objects c such that $\mathrm{Map}(c, -)$ preserves $\mathcal{J}$-colimits. Note that for this to be well-defined we also still need to show that $(\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}$ is essentially small again.

Claim: The assignment $\mathrm{Psh}^{n}_{\mathcal{J}}(A) \mapsto \mathrm{Psh}^{\infty}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})$ is a section of the action of the n-truncating 2-functor $\mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},\infty} \to \mathrm{CAT}^{\mathrm{free},\mathcal{J}\text{-cocts}}_{\mathcal{J},n}$ on objects, i. e. we have canonical equivalences

$$\varepsilon_A \colon \mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}) \xrightarrow{\simeq} \mathrm{Psh}^{n}_{\mathcal{J}}(A) \ .$$

Proof. Let us construct these equivalences. For a small 1-category A, $y\colon A \to \mathrm{Psh}^{n}_{\mathcal{J}}(A)$, the Yoneda functors, always land in the full subcategory of $\mathcal{J}$-compact objects, i. e. we get a factorisation $y\colon A \to (\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}} \hookrightarrow \mathrm{Psh}^{n}_{\mathcal{J}}(A)$. This in turn we can left Kan extend to a functor $\mathrm{Psh}^{n}_{\mathcal{J}}(y)\colon \mathrm{Psh}^{n}_{\mathcal{J}}(A) \to \mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})$. On the other hand we can $\mathcal{J}$-colimit extend the inclusion $(\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}} \hookrightarrow \mathrm{Psh}^{n}_{\mathcal{J}}(A)$ to a functor $\varepsilon_A\colon \mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}) \to \mathrm{Psh}^{n}_{\mathcal{J}}(A)$ and by the universal properties the composition

$$\mathrm{Psh}^{n}_{\mathcal{J}}(A) \xrightarrow{\mathrm{Psh}^{n}_{\mathcal{J}}(y)} \mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}) \xrightarrow{\varepsilon_A} \mathrm{Psh}^{n}_{\mathcal{J}}(A)$$

has to be the identity. It suffices to show now that the second functor is fully-faithful. We do this by a two-stage closure argument. Let us call F in $\mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})$ good if for all G in $\mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})$ we have that $\mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})(F, G) \xrightarrow{\varepsilon_A} \mathrm{Psh}^{n}_{\mathcal{J}}(A)(\varepsilon_A(F), \varepsilon_A(G))$ is an equivalence. We now argue that all representables are good. To this end let us first assume G is also a representable. By n-Hom-truncatedness of $(\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}$ we have fully-faithfulness of the Yoneda functor $y\colon (\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}} \hookrightarrow \mathrm{Psh}^{n}_{\mathcal{J}}((\mathrm{Psh}^{n}_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})$ and by definition of ε_A we have that

$$\begin{tikzcd}
(\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}(H,K) \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & \mathrm{Psh}^n_{\mathcal{J}}((\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})(y(H),y(K)) \arrow[d, "\varepsilon_A"] \\
& \mathrm{Psh}^n_{\mathcal{J}}(A)(H,K)
\end{tikzcd}$$

commutes. Now we show that if F is a representable, the class of G with the desired property is closed under $\mathcal{J}$-colimits. This follows from the fact that in

$$\mathrm{Psh}^n_{\mathcal{J}}((\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}})(y(H),-) \xrightarrow{\varepsilon_A} \mathrm{Psh}^n_{\mathcal{J}}(A)(H,\varepsilon_A(-))$$

on the left hand side, mapping out of a representable is equivalent to evalutaion, hence colimit-preserving, and on the right hand side, ε_A preserves by definition $\mathcal{J}$-colimits and mapping out of H, which was assumed to be $\mathcal{J}$-compact, does so too by definition. By universal property of the free n-Hom-truncated $\mathcal{J}$-cocompletion we now conclude that every representable is good. What is now left to show is that the class of good objects is itself closed under $\mathcal{J}$-colimits. This follows from the $\mathcal{J}$-colimit preservation of ε_A and the fact that the first variable of a Hom-space always sends colimits to limits. ☐

Claim: The category $(\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}$ is always small if A is small.

Proof. We are going to prove that in the chain of inclusions

$$\mathrm{ho}_n(A) \subseteq (\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}} \subseteq \mathrm{Psh}^n_{\mathcal{J}}(A) \subseteq \mathrm{Fun}(\mathrm{ho}_n(A)^{\mathrm{op}}, \mathcal{S}_{\leq n}) = \mathrm{Psh}^n(\mathrm{ho}_n(A))$$

it consists of retracts of representables coming form $\mathrm{ho}_n(A)$. In general let us take a $\mathcal{J}$-colimit preserving functor $G\colon \mathrm{Psh}^n_{\mathcal{J}}(A) \to \mathcal{S}_{\leq n}$. We may restrict it along Yoneda to $G \circ y$ and use the free n-Hom-truncated small cocompletion universal property of $\mathrm{Psh}^n(\mathrm{ho}_n(A))$ to extend it to a small colimit preserving functor $\widetilde{G}\colon \mathrm{Psh}^n(\mathrm{ho}_n(A)) \to \mathcal{S}_{\leq n}$. But as the inclusion $\mathrm{Psh}^n_{\mathcal{J}}(A) \subseteq \mathrm{Fun}(\mathrm{ho}_n(A)^{\mathrm{op}}, \mathcal{S}_{\leq n})$ preserves $\mathcal{J}$-colimits by definition, the composition $\mathrm{Psh}^n_{\mathcal{J}}(A) \subseteq \mathrm{Psh}^n(\mathrm{ho}_n(A)) \xrightarrow{\widetilde{G}} \mathcal{S}_{\leq n}$ does so too, hence by universal property of $\mathrm{Psh}^n_{\mathcal{J}}(A)$ it must agree with G. For the extension $\widetilde{G}$ we in fact now have our pointwise left Kan extension formula

$$\widetilde{G}(F) \xleftarrow{\simeq} \mathrm{colim}((y_{\mathrm{ho}_n(A)})_{/F} \to \mathrm{ho}_n(A) \xrightarrow{G\circ y} \mathcal{S}_{\leq n})\ .$$

Applied to $G = \mathrm{Psh}^n(X,-)$ for X an object of $(\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}$ and $F = X$ we get the identification $\mathrm{Psh}^n(X,X) \xleftarrow{\simeq} \mathrm{colim}((y_{\mathrm{ho}_n(A)})_{/X} \to \mathrm{ho}_n(A) \xrightarrow{\mathrm{Psh}^n(X,-)\circ y} \mathcal{S}_{\leq n})$ to which we can furthermore apply the colimit preserving functor π_0 to get

$$\pi_0 \mathrm{Psh}^n(X,X) \xleftarrow{\simeq} \mathrm{colim}((y_{\mathrm{ho}_n(A)})_{/X} \to \mathrm{ho}_n(A) \xrightarrow{\mathrm{Psh}^n(X,-)\circ y} \mathcal{S}_{\leq n} \xrightarrow{\pi_0} \mathcal{S}_{\leq 0} \simeq \mathrm{Set})\ .$$

Exploiting the explicit colimit description in Set in terms of coproducts and coequalizers, and the axiom of choice on the coequalizer surjection, we may now trace through this colimit induced isomorphism of sets by taking id_X on the left hand side and actually get that X is a retract of some representable. As all objects of $(\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}$ will be retracts of representables coming from $\mathrm{ho}_n(A)$, $(\mathrm{Psh}^n_{\mathcal{J}}(A))^{\mathcal{J}\text{-cpt}}$ will actually be contained in the idempotent completion of $\mathrm{ho}_n(A)$ inside $\mathrm{Fun}(\mathrm{ho}_n(A)^{\mathrm{op}}, \mathcal{S}_{\leq n})$ which, by the proof of [Lur09b, Proposition 5.1.4.2], is just the full subcategory spanned by retracts. But if A, and then also $\mathrm{ho}_n(A)$, was small, then its idempotent completion is so as well as it is by [Lur09b, Corollary 4.4.5.15] the essential image of the restricted colimit functor over Idem, the free-living idempotent,

$$\mathrm{Fun}(\mathrm{Idem}, \mathrm{ho}_n(A)) \xhookrightarrow{y_*} \mathrm{Fun}(\mathrm{Idem}, \mathrm{Fun}(\mathrm{ho}_n(A)^{\mathrm{op}}, \mathcal{S}_{\leq n})) \xrightarrow{\mathrm{colim}} \mathrm{Fun}(\mathrm{ho}_n(A)^{\mathrm{op}}, \mathcal{S}_{\leq n})$$

and hence small as $\mathrm{Fun}(\mathrm{Idem}, \mathrm{ho}_n(A))$ is small. □

We are now fully equipped to construct the animation, or nonabelian ∞-deriving lax 2-functor as the lax right adjoint to the 0-truncation 2-functor. In particular, we now specialize to the class $\mathcal{J} = \{\mathrm{sif}\}$ of all sifted 1-categories.

Example 2.5.13

We further restrict the 0-truncating 2-functor $\mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},\infty} \to \mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},0}$ from Construction 2.5.11 on both sides to full sub-2-categories on free sifted-cocompletions of small 1-categories that admit finite coproducts, i. e. we precompose our essential image operation above with the forgetful functor $(\mathrm{Cat}_{\coprod})^{\simeq} \to \mathrm{Cat}^{\simeq}$, where $\mathrm{Cat}_{\coprod}$ is the full sub-2-category on small 1-categories with finite coproducts, This 0-truncating 2-functor

$$\mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},\infty,\coprod} \to \mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},0,\coprod}$$

has a lax right adjoint with counit data exactly the section $\mathrm{Psh}^0_{\mathrm{sif}}(A) \mapsto \mathrm{Psh}^{\infty}_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(A))^{\text{sif-cpt}})$ from Construction 2.5.12, i. e. it sends $\mathrm{Psh}^0_{\mathrm{sif}}(A)$ to its *animation*. Note here this section actually restricts to our more special situation, as $(\mathrm{Psh}^0_{\mathrm{sif}}(A))^{\text{sif-cpt}}$ has finite coproducts if A has them, because sifted colimits commute with finite products in the 1-category of spaces $\mathcal{S}$. On Hom-categories this lax right adjoint precisely computes the following sifted colimit preserving extension of the colored composite

$$\begin{tikzcd}
A \arrow[r, hook, "y"] \arrow[d, "y"'] & \mathrm{Psh}^0_{\mathrm{sif}}(A) \arrow[rr, "F"] \arrow[d, "\varepsilon_A^{-1}"', "\simeq"] & & \mathrm{Psh}^0_{\mathrm{sif}}(B) \arrow[d, "\simeq"', "\varepsilon_B^{-1}"] \\
(\mathrm{Psh}^0_{\mathrm{sif}}(A))^{\text{sif-cpt}} \arrow[r, "y"] \arrow[dr, "y"'] & \mathrm{Psh}^0_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(A))^{\text{sif-cpt}}) \arrow[d, "(\iota_0)_*"'] & & \mathrm{Psh}^0_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}}) \arrow[d, "(\iota_0)_*"] \\
& \mathrm{Psh}^{\infty}_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(A))^{\text{sif-cpt}}) \arrow[rr, dashed] & & \mathrm{Psh}^{\infty}_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}}),
\end{tikzcd}$$

i. e. the *non-abelian left ∞-derived functor* and we get

$$\begin{tikzcd}
\mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},\infty,\coprod} \arrow[r, shift left, "\text{0-truncation}"] \arrow[r, phantom, "\perp"] & \mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},0,\coprod} \arrow[l, squiggly, bend left, "\mathrm{Ani}(-)"] .
\end{tikzcd}$$

Proof. We will now show that the Hom-functors of $\mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},\infty,\coprod} \to \mathrm{CAT}^{\text{free,sif-cocts}}_{\text{sif},0,\coprod}$ have the desired right adjoints, i. e. we have to show that

$$\begin{tikzcd}
\mathrm{Fun}^{\text{sif-cocts}}(\mathrm{Psh}^{\infty}_{\mathrm{sif}}(A), \mathrm{Psh}^{\infty}_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}})) \arrow[d] \\
\mathrm{Fun}^{\text{sif-cocts}}(\mathrm{Psh}^{0}_{\mathrm{sif}}(A), \mathrm{Psh}^{0}_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}})) \arrow[d, "\varepsilon_* \simeq"] \\
\mathrm{Fun}^{\text{sif-cocts}}(\mathrm{Psh}^{0}_{\mathrm{sif}}(A), \mathrm{Psh}^{0}_{\mathrm{sif}}(B))
\end{tikzcd}$$

has a right adjoint. The key ingredient here is the explicit formula for $\mathrm{Psh}^n_{\mathrm{sif}}(C)$ whenever C is small and has finite coproducts. Namely we can then identify $\mathrm{Psh}^n_{\mathrm{sif}}(C) \simeq \mathrm{Fun}^\times(C^{\mathrm{op}}, \mathcal{S}_{\leq n})$, i. e. with product-preserving functors n-truncated presheaves on C. Now we can utilise the adjunction $\pi_0 \colon \mathcal{S} \rightleftharpoons \mathcal{S}_{\leq 0} : \iota_0$ as well as the fact that π_0 preserves products, to get an adjunction via postcomposition $(\pi_0)_* \colon \mathrm{Fun}^\times(C^{\mathrm{op}}, \mathcal{S}) \rightleftharpoons \mathrm{Fun}^\times(C^{\mathrm{op}}, \mathcal{S}_{\leq 0}) : (\iota_0)_*$. Notice here that $(\pi_0)_*$ agrees with our formerly defined functor $\gamma_0 \colon \mathrm{Psh}^\infty_{\mathrm{sif}}(C) \to \mathrm{Psh}^0_{\mathrm{sif}}(C)$. So we can conclude by

$$
\begin{tikzcd}
\mathrm{Fun}^{\text{sif-cocts}}(\mathrm{Psh}^\infty_{\mathrm{sif}}(A), \mathrm{Psh}^\infty_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}})) \arrow[r, "\simeq"'] \arrow[d] & \mathrm{Fun}(A, \mathrm{Psh}^\infty_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}})) \arrow[d, "\gamma_0=(\pi_0)_*"', shift right] \arrow[d, phantom, "\dashv"] \\
\mathrm{Fun}^{\text{sif-cocts}}(\mathrm{Psh}^0_{\mathrm{sif}}(A), \mathrm{Psh}^0_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}})) \arrow[r, "\simeq"] \arrow[d, "\varepsilon_* \simeq"] & \mathrm{Fun}(A, \mathrm{Psh}^0_{\mathrm{sif}}((\mathrm{Psh}^0_{\mathrm{sif}}(B))^{\text{sif-cpt}})) \arrow[u, bend right, "(\iota_0)_*"'] . \\
\mathrm{Fun}^{\text{sif-cocts}}(\mathrm{Psh}^0_{\mathrm{sif}}(A), \mathrm{Psh}^0_{\mathrm{sif}}(B)) &
\end{tikzcd}
$$

□

As an immediate application we can now derive sifted monads on compact-projectively genenerated $(1,1)$-categories.

Corollary 2.5.14

As lax 2-functors in particular send monads to monads, the animation, or nonabelian ∞-deriving lax 2-functor

$$\mathrm{Ani}(-) \colon \mathrm{CAT}^{\text{free,sif-cocts}}_{\mathrm{sif},0,\coprod} \rightsquigarrow \mathrm{CAT}^{\text{free,sif-cocts}}_{\mathrm{sif},\infty,\coprod}$$

sends sifted colimit preserving monads on compact-projectively generated $(1,1)$-categories to sifted colimit preserving monads on compact-projectively generated $(\infty,1)$-categories.

Let us illustrate this with a very concrete example, the monad on abelian groups whose algebras are rings.

Example 2.5.15

Let us write Ab for the 0-Hom-truncated 1-category of ordinary abelian groups. This is a compact-projectively generated $(1,1)$-category, i. e. a free 0-Hom-truncated sifted cocompletions of its sifted-compact objects $\mathrm{Free}_{\mathrm{fin}}(\mathrm{Ab})$, the finite free abelian groups. On Ab we have the ordinary sifted colimit preserving monad Sym whose algebras are monoids for the tensor product of abelian groups, and we can view it, as in section 2.3, as a lax 2-functor

$$\mathrm{Sym} \colon \Delta^0 \rightsquigarrow \mathrm{CAT}^{\text{free,sif-cocts}}_{\mathrm{sif},0,\coprod}$$

we can now send this monad through the lax functor constructed in Example 2.5.13 to obtain a monad

$$\mathrm{Ani}\,\mathrm{Sym} \colon \Delta^0 \overset{\mathrm{Sym}}{\rightsquigarrow} \mathrm{CAT}^{\text{free,sif-cocts}}_{\mathrm{sif},0,\coprod} \overset{\mathrm{Ani}(-)}{\rightsquigarrow} \mathrm{CAT}^{\text{free,sif-cocts}}_{\mathrm{sif},\infty,\coprod}$$

on the 1-category $\mathrm{Fun}^\times(\mathrm{Free}_{\mathrm{fin}}(\mathrm{Ab})^{\mathrm{op}}, \mathcal{S})$, i. e. the *animation of abelian groups*, which by the Dold-Kan equivalence [Lur17, Theorem 1.2.3.7], is equivalent to the ∞-derived category of abelian groups $\mathrm{D}_{\geq 0}(\mathrm{Ab})$ in non-negative homological degrees, which in turn is equivalent to the category of connective $\mathbb{Z}$-modules in spectra, by [Lur17, Theorem 7.1.2.13]. The monad Ani Sym should correspond to the one constructed in [Lur18, Construction 25.2.2.6] and hence its algebras should be the so-called *animated* or *derived rings*, which form the basis for the subject of Derived Algebraic Geometry.

2.5.2 Lax 2-Adjunctions between *Strict* 2-Functors

One well-behaved formal notion of *lax 2-adjunction* was given by Gray in [Gra74, Definition I.7.1] in the realm of ordinary 2-categories and can be stated as follows. For strict 2-functors $F\colon A \to B$, $U\colon B \to A$ as well as, possibly (op-)lax, 2-natural transformations $\eta\colon \mathrm{id} \Rightarrow UF$, $\varepsilon\colon FU \Rightarrow \mathrm{id}$ as well as triangulators

$$\begin{tikzcd}
U \arrow[r, "\eta U"] \arrow[dr, equal] & UFU \arrow[d, "U\varepsilon"] \arrow[dl, phantom, "s \Nearrow"] \\
& U
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, equal] & FUF \arrow[d, "\varepsilon F"] \arrow[dl, phantom, "t \Swarrow"] \\
& F
\end{tikzcd}$$

satisfying the swallow-tail identities

$$\begin{tikzcd}
FU \arrow[dr, "F\eta U"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
& FUFU \arrow[r, "\varepsilon FU"] \arrow[d, "FU\varepsilon"'] \arrow[u, phantom, "\Uparrow tU"] \arrow[l, phantom, "Fs \Rightarrow"] & FU \arrow[d, "\varepsilon"] \\
& FU \arrow[r, "\varepsilon"'] \arrow[ur, phantom, "\varepsilon_\varepsilon \Nearrow"] & \mathrm{id}
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
FU \arrow[dr, "\varepsilon"] & \\
& \mathrm{id}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta"] \arrow[d, "\eta"'] & UF \arrow[d, "\eta UF"] \arrow[dl, phantom, "\eta_\eta \Swarrow"] \arrow[ddr, bend left, equal] & \\
UF \arrow[r, "UF\eta"'] \arrow[drr, bend right, equal] & UFUF \arrow[dr, "U\varepsilon F"] \arrow[d, phantom, "Ut \Downarrow"] \arrow[r, phantom, "\Leftarrow sF"] & \\
& & UF
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
\mathrm{id} \arrow[dr, "\eta"] & \\
& UF\,.
\end{tikzcd}$$

This unit-counit description of a lax 2-adjunction also has a corresponding Hom-category description. More precisely, this data will induce the Hom-adjunctions

$$\begin{tikzcd}
& B(Fa, FUb) \arrow[dl, "(\varepsilon_b)_*"'] & \\
B(Fa, b) \arrow[dr, "U"'] \arrow[rr, phantom, "\perp"] & & A(a, Ub)\,, \arrow[ul, "F"'] \\
& A(UFa, Ub) \arrow[ur, "{}_*\eta_a"'] &
\end{tikzcd}$$

whose units and counits in turn are basically built from the triangulators, and the lax natural structure of η and ε. Furthermore, the swallow-tail identities play the role of the triangle identities here for the Hom-adjunctions. Now applying Theorem 2.4.3 to F and the $\varepsilon_b\colon FUb \to b$ the lax functor we construct is not quite U, on Hom-categories it acts via the solid part in

$$\begin{tikzcd}
B(b, \widetilde{b}) \arrow[r, "{}_*\varepsilon_b"] \arrow[dr, "U_{b,\widetilde{b}}"'] & B(FUb, \widetilde{b}) \arrow[r, "U"] & A(UFUb, U\widetilde{b}) \arrow[r, "{}_*\eta_{Ub}"] \arrow[d, phantom, "\Uparrow {}_*s_b"] & A(Ub, U\widetilde{b})\,, \\
& A(Ub, U\widetilde{b}) \arrow[ur, "{}_*U(\varepsilon_b)"'] \arrow[urr, bend right, dashed, equal] & &
\end{tikzcd}$$

which has a canonical comparison to $U_{b,\widetilde{b}}$ via the dashed part, i. e. $U_{b,\widetilde{b}} \circ s_b$ where $s_b\colon \mathrm{id}_{Ub} \Rightarrow U(\varepsilon_b)\eta_{Ub}$ is the component of the triangulator s. If these 2-morphisms $U_{b,\widetilde{b}}(f) \circ s_b$ however are all invertible, for example if s is invertible, we get that the commutative triangle

$$\begin{tikzcd} B \arrow[r, "(U,\varepsilon)"] \arrow[dr, equal] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[d, "\mathrm{cod}"] \\ & B \end{tikzcd}$$

is on Hom-categories vertically adjointable, hence a morphism in $\mathrm{LocRARI}(B)$ so the lax functor constructed via Theorem 2.4.3 must agree with the composite

$$\begin{tikzcd} \mathrm{Lax}(B) \arrow[r, "\lambda_B"] \arrow[dr, "\lambda_B"'] & B \arrow[r, "(U,\varepsilon)"] \arrow[d, equal] & A \times_B^{(F,\mathrm{dom})} \mathrm{Arr}^{\mathrm{lax}}(B) \arrow[dl, "\mathrm{cod}"] \\ & B & \end{tikzcd}$$

meaning concretely that we can actually reconstruct the coherent U and ε from the objectwise data, or more coherently speaking via Remark 2.4.4.

Note here that this procedure can also be *reversed*, in the case where one only has one adjoint given coherently as a strict 2-functor and *does not a priori know* how to construct coherently the other adjoint, unit, counit and so on.

Let us demonstrate this in a very simple toy example of such a lax 2-adjunction.

Example 2.5.16

Another example of lax 2-adjunctions whose unit and counit are allowed to be lax natural transformations are the 2-dimensional generalisations of the slice category terminal object adjunctions

$$\begin{tikzcd} C_{/c} \arrow[r] \arrow[r, phantom, shift right=3, "\perp"] & \Delta^0 \arrow[l, bend left, "{(c,\mathrm{id}_c)}"] \end{tikzcd}$$

for 1-categories C and objects c of C. Generalising to 2-categories the unstraightenings of the Hom-2-functors $A(-,a)$ are the lax slice 2-categories $A_{/^{\mathrm{lax}}a} \twoheadrightarrow A$. Whereas the strict slice 2-categories $A_{/a} \to A$, i. e. the non-1-morphism-full sub-2-categories of $A_{/^{\mathrm{lax}}a}$ on the 2-cartesian 1-morphisms still admit a strict 2-terminal object 2-adjunction

$$\begin{tikzcd} A_{/a} \arrow[r] \arrow[r, phantom, shift right=3, "\perp"] & \Delta^0 \arrow[l, bend left, "{(a,\mathrm{id}_a)}"] \end{tikzcd}$$

the 2-functor $A_{/^{\mathrm{lax}}a} \to \Delta^0$ however will not in general. More concretely, the strict 2-functor $(a,\mathrm{id})\colon \Delta^0 \to A_{/^{\mathrm{lax}}a}$ is still a section

$$\begin{tikzcd} \Delta^0 \arrow[r, "{(a,\mathrm{id}_a)}"] \arrow[dr, no head] \arrow[dr, phantom, shift left=3, "\simeq\varepsilon"] & A_{/a} \arrow[d] \\ & \Delta^0 \end{tikzcd}$$

so it is still plausible to see this ε as an invertible counit transformation, but on Hom-1-categories we will no longer have an equivalence, but rather the initial object adjunctions

$$\begin{tikzcd} A_{/^{\mathrm{lax}}a}((b,f),(a,\mathrm{id}_a)) \arrow[r, "\simeq"] & \overrightarrow{\int}_{g\colon A(b,a)} A(a,a)(f, \mathrm{id}_a \circ g) \arrow[r] \arrow[r, phantom, shift left=3, "\perp"] & \Delta^0(*,*) \arrow[l, bend right, "{(f,\mathrm{id}_f)}"'] \arrow[r, "\simeq"] & \Delta^0 . \end{tikzcd}$$

In the realm of ordinary 2-categories we can inspect these Hom-adjunctions and directly deduce that one can also reconstruct from it a unit transformation, with object components being the 2-cartesian 1-morphisms

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[d, "f"'] & a \arrow[dl, equal, "\mathrm{id}_a"] \\ a & \end{tikzcd}$$

for every object (b, f) but whose naturality squares will only laxly commute via

$$\begin{tikzcd} (b,f) \arrow[r, "{(f,\mathrm{id}_f)}"] \arrow[d, "{(h,\alpha)}"'] & (a,\mathrm{id}) \arrow[d, equal] \arrow[dl, phantom, "\Swarrow (\alpha,\mathrm{id}_\alpha)"] \\ (c,g) \arrow[r, "{(g,\mathrm{id}_g)}"'] & (a,\mathrm{id}) \end{tikzcd}$$

for 1-morphisms $(h, \alpha): (b, f) \to (c, g)$. Guided by the ordinary 2-category theory initiution, we are now in the situation where we could now

1. apply Theorem 2.4.3 at first to $A_{/^{\mathrm{lax}}a} \to \Delta^0$ with our guessed object section $(a, \mathrm{id}): \Delta^0 \to A_{/^{\mathrm{lax}}a}$, to *construct* an *a priori oplax* 2-functor $(a, \mathrm{id}): \Delta^0 \rightsquigarrow A_{/^{\mathrm{lax}}a}$ and
2. then, by direct inspection of its construction and the Hom-adjunctions, *prove* that its oplax comparison 2-morphisms are actually invertible to conclude that the oplax 2-functor was *strict* to begin with, i. e. for it to factor as

$$\begin{tikzcd} \mathrm{Lax}(\Delta^0) \arrow[r] \arrow[d, "\lambda"'] & A_{/^{\mathrm{lax}}a} \\ \Delta^0 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

via Lemma 3.2.10.
3. Then we might take a second look at our Hom-adjunctions, and *guess* a good candidate for unit 1-morphisms to in turn *reapply* Theorem 2.4.3 to the now *a posteriori strict* 2-functor $(a, \mathrm{id}): \Delta^0 \to A_{/^{\mathrm{lax}}a}$ with respect to the object action of our starting 2-functor $A_{/^{\mathrm{lax}}a} \to \Delta^0$ and the unit 1-morphism guesses $(f, \mathrm{id}_f): (b, f) \to (a, \mathrm{id})$ to construct an *a priori lax* version of $A_{/^{\mathrm{lax}}a} \rightsquigarrow \Delta^0$,
4. which we can in our degenerate situation immediately again identify with our starting 2-functor $A_{/^{\mathrm{lax}}a} \to \Delta^0$, but now Theorem 2.4.3 assembles the unit 1-morphisms for us into a lax functor $\eta: A_{/^{\mathrm{lax}}a} \rightsquigarrow \mathrm{Arr}^{\mathrm{lax}}(A_{/^{\mathrm{lax}}a})$ which we may consider as a lax natural transformation from the identity to the *a priori lax* composite 2-functor $A_{/^{\mathrm{lax}}a} \to \Delta^0 \xrightarrow{(a,\mathrm{id}_a)} A_{/^{\mathrm{lax}}a}$
5. and in our case we can by direct inspection of its construction and the Hom-adjunctions again a posteriori prove to be *strict*. Hence we get out a strict 2-functor $\eta: A_{/^{\mathrm{lax}}a} \to \mathrm{Arr}^{\mathrm{lax}}(A_{/^{\mathrm{lax}}a})$ which we may now interpret as a lax natural transformation between *strict* 2-functors.

Chapter 3

Synthetic Universal Properties for the 2-Lax Gray Tensor Product

3.1 Introduction

Motivation

As we have seen in the introduction, the lax generalisations of strict natural transformations, which have naturality squares

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \arrow[dl, phantom, "\swArrow \alpha_f"] \\ F(\tilde{c}) \arrow[r, "\alpha_{\tilde{c}}"'] & G(\tilde{c}) \end{tikzcd}$$

that commute only laxly up to 2-morphism, are ubiquitous, especially in areas of mathematics where one wants to categorify existing algebraic structures or invariants.

The overarching coherence, naturality and compatibility of lax natural transformations is in first approximation packaged in the 2-categories $\mathrm{Fun}_{\mathrm{lax}}(-,-)$ but fully realised by the fact that it is part of a monoidal biclosed structure on 2-Cat. The corresponding tensor product $\otimes_{\mathrm{lax}}$ is the so-called *lax Gray tensor product*. A lot of operations on lax natural transformations, like the passage to mates mentioned above, produce non-trivial coherence problems in the realm of higher category theory. These problems arise in particular when trying to construct homotopy-coherent six functor formalisms and have been successfully applied there based on intricate knowledge of, even different models of, the lax Gray monoidal biclosed structure by Gaitsgory and Rozenblyum in [GR17], and Cnossen, Lenz, Linskens in [CLL26].

If one assumes knowledge about the abstract existence of this lax Gray monoidal biclosed structure one can solve questions like the passage to mates, as for example Haugseng does in [Hau21]. Abstract existence on the other hand of such a model of the monoidal biclosed structure was for example pioneered in the simplicial setting by Verity in [Ver08b] and [Ver08a], but also proven in the cubical setting by [CKM25], as well as in the globular setting by [Mae21] and [CM23]. These different models of the lax Gray product have by now also been shown equivalent via [DKM23], [Lou24] and [GHL21].

However, if one comes up with other guesses for models of the lax Gray product in different settings and starts working with their explicit description, as in [GR17], then one needs to either reprove the existence of the monoidal biclosed structure, or to compare it to the existing models in the literature. In this situation, considerable effort went into connecting the preferred model of the lax Gray product in [GR17], as a certain 2-morphism localisation of a particular combinatorial model of the lax 2-functor classifier, with other models already proven to admit all the needed properties, but also with other, not formerly considered, model-dependent reformulations of the Gray product.

To circumvent such model-comparison problems, works like [CM23] and [Cam23] tried to reduce the existence and need for comparison via universal properties to known classical instances of the lax Gray tensor product, for example as left Kan extended from the ordinary lax Gray product on globular shapes Θ, or even more fundamental, the *Gray cubes*, i. e. the subcategory of iterated lax Gray products of Δ^1. However these universal properties are still founded on combinatorial descriptions of instances of the ordinary lax Gray product and thus not synthetic and elementary as described in Remark 1.2.1.

In this chapter we try to develop synthetic universal properties of the lax Gray product that are only referring to basic building blocks of $(\infty, 2)$-category theory, based on observations about the ordinary lax Gray product and our synthetic extended universal property of the lax functor classifier Definition 1.1.1.

Intuition

As in the other two chapters/articles, we can gain a lot of intuition from the classical literature on ordinary 2-categories. In [Gra74], Gray constructs his lax tensor product for ordinary 2-categories by giving explicit generators and relations. Let us now dive deeper into this original description, to motivate the higher categorical generalisations we plan to give later. For ordinary 2-categories A and B, Gray starts his construction of $A \otimes_{\mathrm{lax}} B$ with generators on objects and 1-morphisms. He takes as object set $\mathrm{Ob}(A \otimes_{\mathrm{lax}} B) := \mathrm{Ob}(A) \times \mathrm{Ob}(B)$. For 1-morphisms f in A, or g in B, as well as objects $b \in \mathrm{Ob}(B)$ or $a \in \mathrm{Ob}(A)$ the generators of 1-morphisms of $A \otimes_{\mathrm{lax}} B$ are given by tuples (f, id_b) and (id_a, g) subject to the relations

$$(\widetilde{f}, \mathrm{id}_b) \circ (f, \mathrm{id}_b) \sim (\widetilde{f} \circ f, \mathrm{id}_b) \quad \text{and} \quad (\mathrm{id}_a, \widetilde{g}) \circ (\mathrm{id}_a, g) \sim (\mathrm{id}_a, \widetilde{g} \circ g) \ .$$

For 2-morphism generators we now have two different kinds. The first source is 2-morphisms $\alpha \colon f \Rightarrow \widetilde{f}$ in A or $\beta \colon g \Rightarrow \widetilde{g}$ in B, which give rise to 2-morphism generators $(\alpha, \mathrm{id}_b) \colon (f, \mathrm{id}_b) \Rightarrow (\widetilde{f}, \mathrm{id}_b)$ and $(\mathrm{id}_a, \beta) \colon (\mathrm{id}_a, g) \Rightarrow (\mathrm{id}_a, \widetilde{g})$, already subject to the relations

$$(\widetilde{\alpha}, \mathrm{id}_b) \circ (\alpha, \mathrm{id}_b) \sim (\widetilde{\alpha} \circ \alpha, \mathrm{id}_b) \quad \text{and} \quad (\mathrm{id}_a, \widetilde{\beta}) \circ (\mathrm{id}_a, \beta) \sim (\mathrm{id}_a, \widetilde{\beta} \circ \beta) \ .$$

Note here that everything up to this point tells us that we get for all objects induced 2-functors $A \times \{b\} \to A \otimes_{\mathrm{lax}} B \leftarrow \{a\} \times B$. The second source, which now makes this construction *lax* are 2-morphism generators of the form

$$\begin{tikzcd}
(a, b) \arrow[r, "{(\mathrm{id}_a, g)}"] \arrow[d, "{(f, \mathrm{id}_b)}"'] \arrow[dr, phantom, "\overset{\gamma_{f,g}}{\Nearrow}"] & (a, \widetilde{b}) \arrow[d, "{(f, \mathrm{id}_{\widetilde{b}})}"] \\
(\widetilde{a}, b) \arrow[r, "{(\mathrm{id}_{\widetilde{a}}, g)}"'] & (\widetilde{a}, \widetilde{b})
\end{tikzcd}$$

which interacts with the other 2-morphism generators for examples to make the left hand cylinder in

$$
\begin{tikzcd}[column sep=large, row sep=huge]
(a,b) \arrow[r, "{(\mathrm{id}_a,g)}"] \arrow[d, bend right=60, "{(f,\mathrm{id}_b)}"'] \arrow[d, bend left=60, "{(\widetilde{f},\mathrm{id}_b)}"] \arrow[d, phantom, "{\Rightarrow (\alpha,\mathrm{id}_b)}"] \arrow[dr, phantom, "{\Rightarrow \gamma_{f,g}}"] & (a,\widetilde{b}) \arrow[d, bend right=60, "{(f,\mathrm{id}_{\widetilde{b}})}"'] \arrow[d, bend left=60, "{(\widetilde{f},\mathrm{id}_b)}"] \arrow[d, phantom, "{\Rightarrow (\alpha,\mathrm{id}_b)}"] \\
(\widetilde{a},b) \arrow[r, "{(\mathrm{id}_{\widetilde{a}},g)}"'] \arrow[ur, phantom, "{\Rightarrow \gamma_{\widetilde{f},\widetilde{g}}}"] & (\widetilde{a},\widetilde{b})
\end{tikzcd}
\qquad
\begin{tikzcd}
(a,b) \arrow[dr, "{(\mathrm{id}_a,g)}"] \arrow[drr, bend left, "{(\mathrm{id}_a,\widetilde{g}\circ g)}"] \arrow[dd, "{(f,\mathrm{id}_b)}"'] & & \\
& (a,\widetilde{b}) \arrow[r, "{(\mathrm{id}_a,\widetilde{g})}"] \arrow[dd, "{(f,\mathrm{id}_b)}"] \arrow[r, phantom, shift right=3, "{\gamma_{f,\widetilde{g}g} \Rightarrow}"] & (a,\hat{b}) \arrow[dd, "{(f,\mathrm{id}_b)}"] \\
(\widetilde{a},b) \arrow[dr, "{(\mathrm{id}_{\widetilde{a}},g)}"'] \arrow[drr, "{(\mathrm{id}_{\widetilde{a}},\widetilde{g}\circ g)}"] \arrow[ur, phantom, "{\gamma_{f,g} \Rightarrow}"] & & \\
& (\widetilde{a},\widetilde{b}) \arrow[r, "{(\mathrm{id}_a,\widetilde{g})}"'] \arrow[uur, phantom, "{\gamma_{f,\widetilde{g}} \Rightarrow}"] & (\widetilde{a},\hat{b})
\end{tikzcd}
$$

commute, with similar relations for (id_a,β). On top of these we have relations of the lax square 2-morphisms $\gamma_{f,g}$ among themselves, like commutativity of the diagram depicted on the right hand side above, with a similar version for vertical pasting of squares, as well as the relations

$$\gamma_{f,\mathrm{id}_b} \sim (\mathrm{id}_f,\mathrm{id}_b) \quad \text{and} \quad \gamma_{\mathrm{id}_a,g} \sim (\mathrm{id}_a,\mathrm{id}_g)$$

and the one stating middle four interchange for a horizontal composition $\gamma_{\widetilde{f},\widetilde{g}} \cdot \gamma_{f,g}$.

The basic idea for our synthetic universal properties of the lax Gray tensor product is founded on the observation that all the diagrams, like the two above, that one needs to force commutative with the above relations, have as domain a composite of the form $(\mathrm{id},g)\circ(f,\mathrm{id})$ and as codomain a composite of the form $(f,\mathrm{id})\circ(\mathrm{id},g)$. But even more is true, the 1-morphisms of these two forms turn out to be the initial, respectively terminal lifts of (f,g) in $A(a,\widetilde{a})\times B(b,\widetilde{b})$ along the canonical Hom-functors

$$(A\otimes_{\mathrm{lax}} B)((a,b),(\widetilde{a},\widetilde{b})) \to A(a,\widetilde{a})\times B(b,\widetilde{b})$$

In fact, the 1-morphisms of these two forms constitute fully-faithful adjoints

$$
\begin{tikzcd}
(A\otimes_{\mathrm{lax}} B)((a,b),(\widetilde{a},\widetilde{b})) \arrow[d, "\lambda" description] \arrow[d, phantom, shift right=5, "\dashv"] \arrow[d, phantom, shift left=5, "\dashv"] \\
A(a,\widetilde{a})\times B(b,\widetilde{b})\,. \arrow[u, bend left=60, "{(-,\mathrm{id}_{\widetilde{b}})\circ(\mathrm{id}_a,-)}"] \arrow[u, bend right=60, "{(\mathrm{id}_{\widetilde{a}},-)\circ(-,\mathrm{id}_b)}"']
\end{tikzcd}
$$

This property bears resemblance to our synthetic universal properties of the lax functor classifier.

In the classical literature on ordinary 2-categories, there already exists a characterisation of a Gray-type tensor product in terms of all its compatible 2-functors $A\times\{b\}\to A\otimes_{\mathrm{lax}} B\leftarrow\{a\}\times B$, and the properties of the Hom-functors

$$(A\otimes B)((a,b),(\widetilde{a},\widetilde{b})) \to A(a,\widetilde{a})\times B(b,\widetilde{b})\ ,$$

namely in the article [BG17]. There, the authors Bourke and Gurski prove for the *pseudo* version of the Gray tensor product, where one additionally forces the $\gamma_{f,g}$ to be 2-isomorphisms, that it can be uniquely characterised as the initial factorisation of the canonical cogap 2-functor

$$
\begin{tikzcd}
(A\times \mathrm{Ob}(B)) \underset{(\mathrm{Ob}(A)\times\mathrm{Ob}(B))}{\cup} (\mathrm{Ob}(A)\times B) \arrow[r, dashed] \arrow[rr, bend right=15, "\mathrm{can}"'] & A\otimes_{\mathrm{lax}} B \arrow[r, dashed, "\lambda"] & A\times B
\end{tikzcd}
$$

and the fact that λ is fully-faithful on Hom-categories.

Goal

The goal of this chapter is to provide synthetic and elementary universal properties for the lax Gray tensor product in the realm of $(\infty, 2)$-categories. To this end we start by showing that the extended universal property of the lax functor classifiers Definition 1.1.1, descends along appropriate 2-morphism localisations. As a first realisation of our goal we then apply this to the preferred description of the lax Gray tensor product in [GR17], namely as a certain 2-morphism localisation of the lax functor classifier $\mathrm{Lax}(A \times B)$. We then proceed to reformulate this localised universal property twice, where in the last reformulation we generalise the above teased universal property of Bourke and Gurski from [BG17] to the lax Gray tensor product, using our observations. The main technical ingredient in this chapter is hereby another synthetic Hom-characterisation, more precisely the characterisation of the Hom-1-categories of 2-morphism localisations.

Organisation of the chapter

In section 3.2 we prove the aforementioned Hom-category characterisation of 2-morphism localisations, which will be the main technical ingredient for all our following observations on the lax functor classifier and its extended universal property. As a tiny application we finally prove that lax functors whose lax comparison 2-morphisms are invertible actually correspond to strict 2-functors. In section 3.3 we prove the first main result of this chapter, that the extended universal property of the lax functor classifier descends along Hom-unit localisations. This lets us characterise the identity 2-functor on a 2-category, roughly speaking, as the initial strict Hom-locally fully-faithful local adjunction, but also derive a synthetic universal property for *unitary* lax functor classifiers. Finally, in section 3.4 we localise the extended universal property of the lax functor classifier to the lax Gray tensor product to get a prototype for a synthetic standalone universal property. Afterwards, we offer two reformulations of this universal property, where the latter can be seen as a lax generalisation of the pseudo Gray product universal property of [BG17], resulting in a short and concise synthetic universal property.

Additional Conventions

In contrast to the first chapter 1, but in accordance to the former chapter 2, all objects of interest will be 2-categories in this chapter, unless specified differently. In particular we will also use Cat or 2-Cat to denote the 2-categorical versions of these.

3.2 Characterising the Hom-Categories of 2-Morphism Localisations

In this section we provide the main technical ingredient of the first main result of this chapter, the Hom-characterisation of 2-morphism localisations. More precisely, this characterisation will follow the same blueprint as mentioned in section 2.2, it characterises the Hom-1-categories of 2-morphism localisations as localisations of the original Hom-1-categories. As a first quick application of this result we are able to deduce that *strict* lax 2-functors are actually strict 2-functors.

3.2.1 2-Morphism Localisations and their Hom-Characterisation

Before we begin, we will recollect what we need to know about 2-morphism localisations. After that we will quickly revise the most important part of every Hom-characterisation, the induction principle for Hom-1-categories, i. e. the 2-categorical Yoneda Lemma, as well as one of its immediate corollaries we will use.

We start with the universal property of 2-morphism localisations.

Definition 3.2.1

Let A be a 2-category and W a class of 2-morphisms of A. A 2-functor $F\colon A \to B$ is called localisation of A at the 2-morphisms W if

1. F sends the 2-morphisms in W to invertible ones, and
2. for all 2-categories D, precomposing with F gives an equivalence of 2-categories

$$_*F\colon \operatorname{Fun}(B, D) \xrightarrow{\simeq} \operatorname{Fun}^W(A, D) ,$$

   where we denote by $\operatorname{Fun}^W(A, D)$ the full sub-2-category of $\operatorname{Fun}(A, D)$ on those 2-functors that invert the 2-morphisms of W.

We denote such 2-morphism localisations by $\gamma\colon A \to A[W^{-1}]$.

Lemma 3.2.2

All 2-morphism localisations exist.

Proof. Combining the results [GH15, Definition 4.3.21] and [GH15, Theorem 5.6.6] from enriched category theory and specialising them to the cartesian closed category $(\mathcal{V}, \otimes) = (\mathrm{Cat}, \times)$ we have an adjunction of 1-categories

$$\begin{tikzcd} {}^{\Delta^0+\Delta^0}/\text{2-Cat} \arrow[r, shift right, "\mathrm{Hom}"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend right, "\Sigma"'] \end{tikzcd} ,$$

where the right adjoint $\mathrm{Hom}\colon {}^{\Delta^0+\Delta^0}/\text{2-Cat} \to \mathrm{Cat}$ sends a bipointed 2-category (A, a, b) to its Hom-1-category on these prescribed objects $A(a, b)$. The left adjoint Σ applied to a 1-category C may be thought of the free 2-category on the object space $\Delta^0 + \Delta^0$ with only non-trivial Hom-1-category $(\Sigma C)(0, 1) = C$. In this sense we get for $C = \Delta^0$ just $\Sigma(\Delta^0) \simeq \Delta^1$ the free walking arrow, and for $C = \Delta^1$, $\Sigma(\Delta^1)$ becomes the free walking 2-morphism. By 1-category theory we know that Δ^0 is the localisation of the unique non-trivial morphism in Δ^1, i. e. we have $(\Delta^1 \to \Delta^0)\colon \operatorname{Map}(\Delta^0, D) \simeq \operatorname{Map}^{\mathrm{inv}}(\Delta^1, D)$ for every 1-category D. In particular, precomposition with $\Delta^1 \to \Delta^0$ induces a fully-faithful functor on functor spaces. This means $\Delta^1 \to \Delta^0$ is an epi, i. e. a functor $C \to D$ such that its codiagonal $\nabla\colon D \cup_C D \to D$ is an equivalence. These epis get preserved by any left adjoint such as Σ. In particular, precomposition with $\Sigma\Delta^1 \to \Sigma\Delta^0 \simeq \Delta^1$ induces a fully-faithful functor on 2-functor spaces $\operatorname{Map}(\Delta^1, D) \hookrightarrow \operatorname{Map}(\Sigma\Delta^1, D)$. By naturality of the Hom-equivalences of the above adjunction we also have

$$\begin{tikzcd} \operatorname{Map}((\Delta^1, 0, 1), (A, a, b)) \arrow[d, hook] & \operatorname{Map}(\Delta^0, A(a, b)) \arrow[l, "\simeq"'] \arrow[d, hook] \\ \operatorname{Map}((\Sigma\Delta^1, 0, 1), (A, a, b)) & \operatorname{Map}(\Delta^1, A(a, b)) \arrow[l, "\simeq"] \end{tikzcd} .$$

By the fact that Δ^0 is the localisation of Δ^1 at its morphism, we can then further restrict the bottom horizontal equivalence to an equivalence

$$\operatorname{Map}^{\Delta^1\text{-inv}}((\Sigma\Delta^1, 0, 1), (A, a, b)) \xleftarrow{\simeq} \operatorname{Map}^{\mathrm{inv}}(\Delta^1, A(a, b)) ,$$

where the 2-functor space on the left is the full subspace on 2-functors $\Sigma\Delta^1 \to A$ preserving the objects and inverting the unique non-trivial 2-morphism in $\Sigma\Delta^1$. As we already have the fully-faithful functor of spaces

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, A) \arrow[rr, hook] \arrow[dr] & & \mathrm{Map}(\Sigma\Delta^1, A) \arrow[dl] \\ & A^{\simeq} \times A^{\simeq} & \end{tikzcd}$$

fibered over all pairs of objects of $A^{\simeq}$, the just proved equivalence in particular implies that the functor $\mathrm{Map}(\Delta^1, D) \hookrightarrow \mathrm{Map}(\Sigma\Delta^1, D)$ restricts to an equivalence $\mathrm{Map}(\Delta^1, D) \simeq \mathrm{Map}^{\Delta^1\text{-inv}}(\Sigma\Delta^1, D)$ onto the the full subspace of 2-functors inverting the unique non-trivial 2-morphism. We can define our candidate for an arbitrary 2-morphism localisation of a 2-category A at a 2-morphism class W by taking the pushout

$$\begin{tikzcd} \coprod_W \Sigma\Delta^1 \arrow[r] \arrow[d] & A \arrow[d] \\ \coprod_W \Delta^1 \arrow[r] & A[W^{-1}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}.$$

Applying $\mathrm{Map}(-, T)$ we can compute

$$\begin{aligned} \mathrm{Map}(A[W^{-1}], T) &\simeq \mathrm{Map}(A, T) \times_{\prod_W \mathrm{Map}(\Sigma\Delta^1, T)} \prod_W \mathrm{Map}(\Delta^1, T) \\ &\simeq \mathrm{Map}(A, T) \times_{\prod_W \mathrm{Map}(\Sigma\Delta^1, T)} \prod_W \mathrm{Map}^{\Delta^1\text{-inv}}(\Sigma\Delta^1, T) \\ &\simeq \mathrm{Map}^W(A, T) \ . \end{aligned}$$

To bump this up to an equivalence of 2-functor 1-categories we compute via the standard trick

$$\begin{aligned} \mathrm{Map}(\Delta^1, \mathrm{Fun}(A[W^{-1}], T)) &\simeq \mathrm{Map}(A[W^{-1}], \mathrm{Fun}(\Delta^1, T)) && \simeq \mathrm{Map}^W(A, \mathrm{Fun}(\Delta^1, T)) \\ &\simeq \mathrm{Map}^{W \times (\Delta^0 + \Delta^0)}(A \times \Delta^1, T)) && \simeq \mathrm{Map}(\Delta^1, \mathrm{Fun}^W(A, T)) \ . \end{aligned}$$

that the induced functor by precomposition also induces an equivalence on the space of 1-morphisms. With the same trick we get an equivalence on spaces of 2-morphisms. Hence we know that the precomposition 2-functor $\mathrm{Fun}(A[W^{-1}], T)) \to \mathrm{Fun}^W(A, T))$ induces an equivalence on spaces of objects, 1-morphisms, 2-morphisms compatibly, so we may infer that it is first of all an equivalence on all Hom-1-categories, and second of all then an equivalence of 2-categories. □

For convenience and to emphasise its use in this chapter, let us restate the induction principle for Hom-categories, i. e. the 2-categorical Yoneda Lemma.

Theorem 3.2.3 (2-Yoneda Lemma)

For every 2-category A, we have that

1. the 2-Yoneda functor $y\colon A \hookrightarrow \mathrm{Fun}(A^{1\text{-op}}, \mathrm{Cat})$, given by currying the Hom-functor, is 2-fully-faithful, i. e. induces an equivalence on all Hom-1-categories, and
2. We have a 2-natural equivalence

$$\mathrm{Fun}(A, \mathrm{Cat})(y_{A^{1\text{-op}}}, -) \simeq \mathrm{ev}$$

of 2-functors $A \times \mathrm{Fun}(A, \mathrm{Cat}) \to \mathrm{Cat}$ such that for every object a of A and every 2-functor $F\colon A \to \mathrm{Cat}$, the component of this 2-natural equivalence is given by the canonical evaluation at id_a 1-functor

$$\mathrm{Fun}(A, \mathrm{Cat})(A(a,-), F) \xrightarrow{\mathrm{ev}_a} \mathrm{Fun}(A(a,a), F(a)) \xrightarrow{\mathrm{ev}_{\mathrm{id}_a}} F(a)\ .$$

Reference. See [Abe23a, Theorem 6.7] with the identification of the 2-Yoneda equivalence components in its proof. ■

For the proof of Theorem 3.2.5 we will need the following tiny fraction of the theory of 2-categorical Kan extensions. More precisely, the 2-left Kan extension of a corepresented 2-functor $A(a,-)\colon A \to \mathrm{Cat}$ along a 2-functor $F\colon A \to B$ is given by

$$\begin{tikzcd} A \arrow[rr, "A(a{,}-)"] \arrow[dr, "F"'] & \arrow[d, phantom, "\Downarrow F_{a,-}"] & \mathrm{Cat}\,. \\ & B \arrow[ur, "B(F(a){,}-)"'] & \end{tikzcd}$$

Lemma 3.2.4

For a 2-functor $F\colon A \to B$, the tuple consisting of $B(F(a),-)$ and the canonical 2-natural transformation

$$\mathrm{ap}_F\colon A(a,-) \Rightarrow B(F(a), F(-)) = F^*(B(F(a),-))$$

defines a corepresenting object for the copresheaf $\mathrm{Fun}(A, \mathrm{Cat})(A(a,-), F^*(-))\colon \mathrm{Fun}(B, \mathrm{Cat}) \to \mathrm{Cat}$.

Proof. We can consider the following commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(B, \mathrm{Cat})(B(F(a),-), T) \arrow[r, "\mathrm{ap}_{F^*}"] \arrow[d, "(-)_{F(a)}"] \arrow[dd, bend right, "\simeq"'] & \mathrm{Fun}(A, \mathrm{Cat})(F^*(B(F(a),-)), F^*(T)) \arrow[d, "_*(\mathrm{ap}_F)"] \arrow[dl, "(-)_a"'] \\ \mathrm{Fun}(B(F(a), F(a)), T(F(a))) \arrow[d, "\mathrm{ev}_{\mathrm{id}_{F(a)}}"] \arrow[dr, "_*(\mathrm{ap}_F)_{a,a}"] & \mathrm{Fun}(A, \mathrm{Cat})(A(a,-), F^*(T)) \arrow[d, "(-)_a"] \\ T(F(a)) & \mathrm{Fun}(A(a,a), T(F(a))) \arrow[l, "\mathrm{ev}_{\mathrm{id}_a}"] \end{tikzcd}$$

in which both the left hand vertical composite as well as the composite of right hand lower vertical with bottom horizontal are equivalences by the 2-Yoneda Lemma. This allows us to deduce that the top horizontal functor composed with the right hand side upper vertical functor is also an equivalence. □

Equipped with this we are ready to prove the characterisation of the Hom-1-categories of 2-morphism localisations as the localisations of the respective original Hom-1-categories.

We thank Benjamin Dünzinger for his immense help for devising the Hom-characterisation argument. Without him this wouldn't have been possible.

Theorem 3.2.5

Let A be a 2-category and W a class of 2-morphisms stable under equivalences, vertical and horizontal composition. Let us denote by $\gamma\colon A \to A[W^{-1}]$ the localisation of A at this class of 2-morphisms. Let us furthermore denote by $W_{a,\widetilde{a}}$ the subclass of W of exactly those 2-morphisms which lie between 1-morphisms $a \to \widetilde{a}$. For these we will denote the localisations of the Hom-categories $A(a,\widetilde{a})$ by $\phi\colon A(a,\widetilde{a}) \to A(a,\widetilde{a})[W^{-1}_{a,\widetilde{a}}]$. Then for all objects $a, \widetilde{a}$ of A the induced functors on Hom-categories

$$\gamma_{a,\widetilde{a}}\colon A(a,\widetilde{a}) \to A[W^{-1}](\gamma(a),\gamma(\widetilde{a}))$$

invert the 2-morphisms from $W_{a,\widetilde{a}}$ and the by localisation induced functors

$$\begin{tikzcd} & A(a,\widetilde{a}) \arrow[dl, "\phi"'] \arrow[dr, "\mathrm{ap}_\gamma"] & \\ A(a,\widetilde{a})[W^{-1}_{a,\widetilde{a}}] \arrow[rr, dashed, "\exists!", "\simeq"'] & & A[W^{-1}](\gamma(a),\gamma(\widetilde{a})) \end{tikzcd}$$

are in fact equivalences.

Proof. The strategy here is to prove that $A(a,-)[W^{-1}_{a,-}]$ satisfies the universal property of the 2-left Kan extension of $A(a,-)$ along the 2-morphism localisation $\gamma\colon A \to A[W^{-1}]$, which by Lemma 3.2.4, we already know to be $A[W^{-1}](\gamma(a),-)$.

Let us first observe that our assumption that W is stable under equivalences, vertical and horizontal composition implies that the full subcategories $W_{a,\widetilde{a}} \subseteq \mathrm{Fun}(\Delta^1, A(a,\widetilde{a}))$ are preserved under all the functors $f_*\colon A(a,a_0) \to A(a,a_1)$ for all 1-morphisms $f\colon a_0 \to a_1$ in A. Hence they assemble into a full subfunctor $W_{a,-} \subseteq \mathrm{Fun}(\Delta^1, A(a,-))$. The localisations of $\phi\colon A(a,\widetilde{a}) \to A(a,\widetilde{a})[W^{-1}_{a,\widetilde{a}}]$ in Cat then assemble into the localisation of $A(a,-)$ at $W_{a,-}$ in $\mathrm{Fun}(A,\mathrm{Cat})$, i. e. equivalently the pushout

$$\begin{tikzcd} \Delta^1 \times W_{a,-} \arrow[r] \arrow[d] & A(a,-) \arrow[d] \\ \Delta^0 \times W_{a,-} \arrow[r] & A(a,-)[W^{-1}_{a,-}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathrm{Fun}(A,\mathrm{Cat})$, which we will also denote by $\phi\colon A(a,-) \Rightarrow A(a,-)[W^{-1}_{a,-}]$, as conical 2-colimits in functor 2-categories are computed objectwise, for example by Lemma 2.2.14. Hence we obtain an equivalence between natural transformations out of $A(a,-)[W^{-1}_{a,-}]$ and natural transformations out of $A(a,-)$ that invert $W_{a,-}$, i. e.

$$\mathrm{Fun}(A,\mathrm{Cat})(A(a,-)[W^{-1}_{a,-}],G) \xrightarrow[\simeq]{\ \phi^*\ } \mathrm{Fun}(A,\mathrm{Cat})^{W_{a,-}\text{-inv}}(A(a,-),G).$$

by applying the 2-functor $\mathrm{Fun}(A,\mathrm{Cat})(-,G)$ to the above pushout. We also know that γ^* is fully faithful, i. e. that

$$(\gamma^*)_{F,G}\colon \mathrm{Fun}(A[W^{-1}],\mathrm{Cat})(F,G) \xrightarrow{\simeq} \mathrm{Fun}(A,\mathrm{Cat})(\gamma^*(F),\gamma^*(G)).$$

Now we observe that, basically by definition, the functor $A(a,-)[W^{-1}_{a,-}]$ inverts all 2-morphisms in W. Thus by universal property of the 2-morphism localisation we have a unique $P\colon A[W^{-1}] \to \mathrm{Cat}$ with $\alpha\colon A(a,-)[W^{-1}_{a,-}] \simeq \gamma^*(P)$. We can compose all our constructed data in the following diagram, starting with the 2-fully-faithfulness of γ^* from its 2-morphism localisation universal property.

$$\begin{tikzcd} & \mathrm{Fun}(A[W^{-1}],\mathrm{Cat})(P,G) \arrow[r, "\simeq", "\gamma^*"'] & \mathrm{Fun}(A,\mathrm{Cat})(\gamma^*(P),\gamma^*(G)) \arrow[d, "\alpha^*", "\simeq"'] \\ \mathrm{Fun}(A,\mathrm{Cat})^{W_{a,-}\text{-inv}}(A(a,-),\gamma^*(G)) \arrow[d, hook, "\mathrm{incl}"'] & & \mathrm{Fun}(A,\mathrm{Cat})(A(a,-)[W^{-1}_{a,-}],\gamma^*(G)) \arrow[ll, "\simeq", "\phi^*"'] \\ \mathrm{Fun}(A,\mathrm{Cat})(A(a,-),\gamma^*(G))\,. & & \end{tikzcd} \tag{3.2.1.1}$$

To finish the proof it suffices to show that the last inclusion in this composition is not just an inclusion but an equivalence. This means to show that any 2-natural transformation $A(a,-) \Rightarrow \gamma^*(G)$ for any G inverts $W_{a,-}$. There are now two ways to see this. On one side, we know that any such $A(a,-) \Rightarrow \gamma^*(G)$ factors over the universal, here initial, such G, namely $A(a,-) \Rightarrow \gamma^*(\gamma_!(A(a,-)))$. But by Lemma 3.2.4 this is just the action of γ on Hom-categories $A(a,-) \Rightarrow A[W^{-1}](\gamma(a),\gamma(-))$ which indeed inverts all the 2-morphisms from $W_{a,-}$. The other way to see this is to directly inspect the 2-naturalitiy of $A(a,-) \Rightarrow \gamma^*(G)$ for any 2-morphism $\alpha\colon f \Rightarrow g$ in $W_{a,\widetilde{a}}$

$$\begin{tikzcd}
A(a,a) \arrow[rr, "\theta_a"] \arrow[d, bend right=50, "f_*"'] \arrow[d, bend left=50, "g_*"] \arrow[d, phantom, "\overset{\Longrightarrow}{\alpha_*}"] & & \gamma^*(G)(a) \arrow[d, bend right=50, "\gamma^*(G)(f)"'] \arrow[d, bend left=50, "\gamma^*(G)(g)"] \arrow[d, phantom, "\overset{\Longrightarrow}{\gamma^*(G)(\alpha)}"] \\
A(a,\widetilde{a}) \arrow[rr, "\theta_{\widetilde{a}}"] & & \gamma^*(G)(\widetilde{a})\,,
\end{tikzcd}$$

evaluate at the identity id_a and add the observation that $\gamma^*(G)(\alpha) = G(\gamma(\alpha))$ is always invertible. Hence we can conclude from knowing that all functors in Equation 3.2.1.1 are equivalences, that $A(a,-) \Rightarrow \gamma^*(P) \simeq A(a,-)[W_{a,-}^{-1}]$ is a corepresentation of $\mathrm{Fun}(A,\mathrm{Cat})(A(a,-),\gamma^*(-))$. But we already know that is has as corepresentation $A(a,-) \Rightarrow A[W^{-1}](\gamma(a),\gamma(-))))$ by Lemma 3.2.4. This means they must be equivalent, through the canonical map given through the localisation universal properties. □

3.2.2 Application: *Strict* Lax 2-Functors are Strict 2-Functors

As a first application of our Hom-characterisation Theorem 3.2.5 we will show that *strict* lax 2-functors, i. e. those $\mathrm{Lax}(A) \to B$ for which the identity comparison 2-morphism $\mathrm{id}_a \Rightarrow \iota(\mathrm{id}_a)$ for all objects a of $A^{\simeq}$, as well as all the composition comparison 2-morphisms $\iota(g) \circ \iota(f) \Rightarrow \iota(g \circ f)$ for all composable pairs of 1-morphisms in A are invertible.

The first step here is to show that inverting all the lax comparison data, i. e. the unit 2-morphisms of the Hom-adjunctions of $\lambda\colon \mathrm{Lax}(A) \to A$, turns lax 2-functors into strict 2-functors.

Example 3.2.6

For a 2-category A, we can consider for the lax functor classifier $\mathrm{Lax}(A)$ the closure W_{units} of all unit 2-morphisms of all the Hom-adjunctions of $\lambda\colon \mathrm{Lax}(A) \to A$ under equivalences, vertical and horizontal compositions. Then the canonical 2-functor induced by the 2-morphism localisation

$$\mathrm{Lax}(A)[W_{\mathrm{units}}^{-1}] \xrightarrow{\exists! L} A$$

is an equivalence of 2-categories.

Proof. Let us first note here that as $A^{\simeq} \to A$ is essentially surjective, the functor $\lambda\colon \mathrm{Lax}(A) \to A$ is essentially surjective as well. Hence it suffices to show that this 2-functor induces equivalences on all Hom-categories. To this end we employ our characterisation of Hom-categories of 2-morphism localisations Theorem 3.2.5 that tells us that we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Lax}(A)(a,\widetilde{a}) \arrow[d] \arrow[ddr, bend left, "\lambda_{a,\widetilde{a}}"] & \\
\mathrm{Lax}(A)(a,\widetilde{a})[(W_{\mathrm{units}})^{-1}_{a,\widetilde{a}}] \arrow[d, "\simeq"'] \arrow[dr, "\exists!"] & \\
\mathrm{Lax}(A)[W^{-1}_{\mathrm{units}}](a,\widetilde{a}) \arrow[r, "L_{a,\widetilde{a}}"'] & A(a,\widetilde{a})\,.
\end{tikzcd}$$

Basic 1-category theory tells us for Hom-adjunctions $\lambda_{a,\widetilde{a}} \dashv \iota_{a,\widetilde{a}}$ that the left adjoint $\lambda_{a,\widetilde{a}}$ is the localisation of both $\lambda^{-1}_{a,\widetilde{a}}(\{\text{equivalences}\})$ and the subset of all units of the adjunction $\lambda_{a,\widetilde{a}} \dashv \iota_{a,\widetilde{a}}$. In order to show that the localisation at $(W_{\mathrm{units}})_{a,\widetilde{a}}$ also agrees with these localisations, we need to show that all 2-morphisms of $(W_{\mathrm{units}})_{a,\widetilde{a}}$ get inverted by $\lambda_{a,\widetilde{a}}$, as $(W_{\mathrm{units}})_{a,\widetilde{a}}$ already by definition contains all the units of the adjunction $\lambda_{a,\widetilde{a}} \dashv \iota_{a,\widetilde{a}}$. The closure properties of W_{units} under equivalences and vertical composition of 2-morphisms are hereby no problem. As horizontal compositions can be built from whiskerings and vertical compositions we are reduced to considering whiskerings of unit 2-morphisms from the Hom-adjunctions of arbitrary Hom-categories not just the one from a to $\widetilde{a}$. To this end take for arbitrary objects a_0 and a_1 an arbitrary unit 2-morphism $\eta_g \colon g \Rightarrow \iota_{a_0,a_1}(\lambda_{a_0,a_1}(g))$ and whisker it by arbitrary 1-morphisms $f \colon a \to a_0$ and $h \colon a_1 \to \widetilde{a}$. Then we may consider the naturality square of the unit transformation $\eta\colon \mathrm{id} \Rightarrow \iota_{a,\widetilde{a}} \circ \lambda_{a,\widetilde{a}}$ of the adjunction $\lambda_{a,\widetilde{a}} \dashv \iota_{a,\widetilde{a}}$ at the whiskered 2-morphism $h\eta_g f$

$$\begin{tikzcd}
h \circ g \circ f \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "h\eta_g f"] & \iota_{a,\widetilde{a}}(\lambda(h \circ g \circ f)) \arrow[d, Rightarrow, "\iota_{a,\widetilde{a}}(\lambda(h\eta_g f))", "\simeq"'] \\
h \circ \iota_{a_0,a_1}(\lambda(g)) \circ f \arrow[r, Rightarrow, "\eta"] & \iota_{a,\widetilde{a}}(\lambda(h \circ \iota_{a_0,a_1}(\lambda(g)) \circ f))\,.
\end{tikzcd}$$

As we know by 2-functoriality of λ that $h\eta_g f$ is also sent to an invertible 2-morphism in A we may deduce that if the components of $\eta\colon \mathrm{id} \Rightarrow \iota_{a,\widetilde{a}} \circ \lambda_{a,\widetilde{a}}$ are sent to invertible 2-morphisms, so is every whiskering of unit 2-morphisms from other Hom-categories by the above naturality square. We conclude that all these localisations agree, i. e. that $\mathrm{Lax}(A)(a,\widetilde{a})[(W_{\mathrm{units}})^{-1}_{a,\widetilde{a}}] \xrightarrow{\simeq} A(a,\widetilde{a})$. □

However, the intuition from ordinary 2-category theory implies that we should actually have a much stronger version of this consistency result, namely that we should only need to invert the identity comparison 2-morphisms $\mathrm{id}_a \Rightarrow \iota(\mathrm{id}_a)$ as well as the compostion comparison 2-morphisms $\iota(g) \circ \iota(f) \Rightarrow \iota(g \circ f)$ for composable 1-morphisms f and g of A.

To this end we need to know more about the objects and 1-morphisms of $\mathrm{Lax}(A)$ in general, so we will prove a precise description in Proposition 3.4.19.

The main ingredient in this description in turn is the following 1-categorical statement for factorising adjunctions over full subcategory inclusions.

Lemma 3.2.7

Let $F\colon C \to D$ be a left adjoint between 1-categories with right adjoint U. Let furthermore $I\colon E \hookrightarrow C$ be a full subcategory of C such that U factors over the inclusion I

$$\begin{tikzcd}
E \arrow[rr, hook, "I"] & & C\,. \\
& D \arrow[ul, dashed, "\exists\widetilde{U}"] \arrow[ur, "U"'] &
\end{tikzcd}$$

Then FI and $\widetilde{U}$ together with the natural transformation $\widetilde{\varepsilon} := FI\widetilde{U} \simeq FU \xrightarrow{\varepsilon} \mathrm{id}$ can be upgraded to an adjunction $FI \dashv \widetilde{U}$ and the triangle

$$\begin{tikzcd} E \arrow[rr, hook, "I"] \arrow[dr, "FI"'] & & C \arrow[dl, "F"] \\ & D & \end{tikzcd}$$

is vertically adjointable.

Proof. From the fact that I is fully-faithful we get that the canonical square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, E) \arrow[r, hook, "I_*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\ E \times E \arrow[r, hook, "I \times I"'] & C \times C \end{tikzcd}$$

is a pullback. Utilising this pullback we can construct a candidate for the unit of $FI \dashv \widetilde{U}$ as follows.

$$\begin{tikzcd} E \arrow[rr, hook, "I"] \arrow[d, dashed, "\exists!\widetilde{\eta}"'] \arrow[dd, bend right=60, "(\mathrm{id}, FI)"'] & & C \arrow[d, "\eta"] \arrow[dd, bend left=60, "(\mathrm{id}, F)"] \\ \mathrm{Fun}(\Delta^1, E) \times_E^{(\mathrm{ev}_1, \widetilde{U})} D \arrow[rr, hook, "I_*"] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1, U)} D \arrow[d] \\ E \times D \arrow[rr, hook, "I \times \mathrm{id}"'] & & C \times D \end{tikzcd}$$

By Lemma A.4.2 we may factorise this into

$$\begin{tikzcd} E \arrow[rr, hook, "I"] \arrow[d, "\Delta_{FI}"] \arrow[dd, dashed, bend right=60, "\exists!\widetilde{\eta}"'] & & C \arrow[d, "\Delta_F"] \arrow[dd, bend left=60, "\eta"] \\ E \times_D^{(FI, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, D) \arrow[rr, hook, "I \times_{\mathrm{id}} \mathrm{id}"] \arrow[d] & & C \times_D^{(F, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, D) \arrow[d, "\simeq"] \\ \mathrm{Fun}(\Delta^1, E) \times_E^{(\mathrm{ev}_1, \widetilde{U})} D \arrow[rr, hook, "I_* \times_{\mathrm{id}} \mathrm{id}"] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1, U)} D \arrow[d] \\ E \times D \arrow[rr, hook, "I \times \mathrm{id}"'] & & C \times D \end{tikzcd}$$

and conclude that the by $\widetilde{\eta}$ induced vertical functor on the middle left is an equivalence as it is the pullback of the equivalence on the middle right as not just the lowest square is a pullback but also the composition of lowest and middle square is, hence by pullback cancellation also the middle one. From this commutative diagram we can also infer the desired adjointability, by starting with the functor $\Delta_{\widetilde{U}} \colon D \to \mathrm{Fun}(\Delta^1, E) \times_E^{(\mathrm{ev}_1, \widetilde{U})} D$ and tracing through the middle square with its vertical equivalences. $\square$

We are now able to show that all objects of $\mathrm{Lax}(A)$ already come from $\iota^{\simeq} \colon A^{\simeq} \to \mathrm{Lax}(A)$ and describe the 1-morphisms of $\mathrm{Lax}(A)$ as arbitray finite composites of 1-morphisms $\iota^{\simeq}(e)$ for e a morphism in $A^{\simeq}$ and 1-morphisms $\iota_{a,\widetilde{a}}(f)$ for 1-morphisms $f \colon a \to \widetilde{a}$ in A.

Lemma 3.2.8

For a 2-category A the canonical 2-functor $\iota^{\simeq}\colon A^{\simeq} \to \mathrm{Lax}(A)$ is essentially surjective and furthermore every 1-morphism of $\mathrm{Lax}(A)$ is either the identity or an arbitrary length composition of $\iota_{a,\widetilde{a}}(f)$, for $f\colon a \to \widetilde{a}$ a 1-morphism in A, and invertible 1-morphisms coming from $\iota^{\simeq}\colon A^{\simeq} \to \mathrm{Lax}(A)$.

Proof. To prove both assertions at once, let us restrict $\mathrm{Lax}(A)$ to first the full sub-2-category on all objects coming from $\iota^{\simeq}$, and then further restrict this to the not-1-morphism-full but 2-morphism-full sub-2-category K on those 1-morphisms which are composites of 1-morphisms of the form $\iota(f)$, i. e. the images of 1-morphisms f of A under the supplied fully-faithful Hom-right-adjoints, and images of the invertible 1-morphisms coming from $\iota^{\simeq}\colon A^{\simeq} \to \mathrm{Lax}(A)$. This tells us that the $\iota^{\simeq}\colon A^{\simeq} \to \mathrm{Lax}(A)$ factors over K. We in particular also have the Hom-locally fully-faithful inclusion 2-functor I

$$\begin{tikzcd}
& C^{\simeq} \arrow[dl, "\widetilde{\iota^{\simeq}}"'] \arrow[dr, "\iota^{\simeq}"] \arrow[dd, bend right] & \\
K \arrow[rr, "I"'] \arrow[dr, "\widetilde{\lambda}"'] & & \mathrm{Lax}(C)\,, \arrow[dl, "\lambda"] \\
& C &
\end{tikzcd}$$

which on Hom-categories is

$$\begin{tikzcd}
K(c,\widetilde{c}) \arrow[rr, hook, "I_{c,\widetilde{c}}"] \arrow[dr, "\widetilde{\lambda}_{c,\widetilde{c}}"'] & & \mathrm{Lax}(C)(c,\widetilde{c})\,, \arrow[dl, "\lambda_{c,\widetilde{c}}"] \\
& C(c,\widetilde{c}) &
\end{tikzcd}$$

By construction, the fully-faithful Hom-right-adjoints $\iota_{c,\widetilde{c}}$ of $\lambda_{c,\widetilde{c}}$ factor through the fully-faithful Hom-inclusions $I_{c,\widetilde{c}}$. Hence by Lemma 3.2.7 we get that $\widetilde{\lambda}_{c,\widetilde{c}}$ inherits the fully-faithful right adjoint from $\lambda_{c,\widetilde{c}}$ and that the above Hom-triangle is automatically adjointable. Hence by the extended universal property, i. e. initiality, of the lax functor classifier, we obtain a unique 2-functor the other way around

$$\begin{tikzcd}
& C^{\simeq} \arrow[dl, "\iota^{\simeq}"'] \arrow[dr, "\widetilde{\iota^{\simeq}}"] \arrow[dd, bend right] \arrow[drr, bend left, "\iota^{\simeq}"] & & \\
\mathrm{Lax}(C) \arrow[rr, dashed, "\exists!"] \arrow[dr, "\lambda"'] & & K \arrow[r, "I"] \arrow[dl, "\widetilde{\lambda}"] & \mathrm{Lax}(C) \arrow[dll, "\lambda"] \\
& C & &
\end{tikzcd}$$

and additionally that this composite must be the identity. On Hom-categories this tells us that the composite

$$\begin{tikzcd}
\mathrm{Lax}(C)(c,\widetilde{c}) \arrow[r] \arrow[drr, "\lambda_{c,\widetilde{c}}"'] & K(c,\widetilde{c}) \arrow[rr, hook, "I_{c,\widetilde{c}}"] \arrow[dr, "\widetilde{\lambda}_{c,\widetilde{c}}"] & & \mathrm{Lax}(C)(c,\widetilde{c}) \arrow[dl, "\lambda_{c,\widetilde{c}}"] \\
& & C(c,\widetilde{c}) &
\end{tikzcd}$$

is the identity as well, hence the first functor on Hom-categories must be fully-faithful as $I_{c,\breve{c}}$ is. The other way around

$$\begin{tikzcd}
 & C^{\simeq} \arrow[dl, "\widetilde{\iota^{\simeq}}"'] \arrow[dr, "\iota^{\simeq}"] \arrow[drr, "\widetilde{\iota^{\simeq}}"] \arrow[dd, bend right] & & \\
K \arrow[rr, "I"] \arrow[dr, "\widetilde{\lambda}"'] & & \mathrm{Lax}(C) \arrow[r, dashed, "\exists!"] \arrow[dl, "\lambda"'] & K \arrow[dll, "\widetilde{\lambda}"] \\
 & C & &
\end{tikzcd}$$

we have essential surjectivity of the composite by construction of K and we get on Hom-categories

$$\begin{tikzcd}
K(c,\breve{c}) \arrow[rr, hook, "I_{c,\breve{c}}"] \arrow[dr, "\widetilde{\lambda}_{c,\breve{c}}"] & & \mathrm{Lax}(C)(c,\breve{c}) \arrow[r, hook] \arrow[dl, "\lambda_{c,\breve{c}}"'] & K(c,\breve{c}) \arrow[dll, "\widetilde{\lambda}_{c,\breve{c}}"] \\
 & C(c,\breve{c}) & &
\end{tikzcd}$$

that the composite functor on Hom-categories is fully-faithful. But by construction, our Hom-adjointabilites and the commutative triangles under $C^{\simeq}$ we also know that it is essentially surjective, hence an equivalence. In particular we have shown that one way to compose our two 2-functors is the identity and the other way composite is an equivalence of 2-categories, in particular both are 2-equivalences. $\square$

The last ingredient we need to show that it suffices to invert the identity comparison 2-morphisms $\mathrm{id}_a \Rightarrow \iota(\mathrm{id}_a)$ as well as the composition comparison 2-morphisms $\iota(g) \circ \iota(f) \Rightarrow \iota(g \circ f)$, is the following 2-categorical statement about horizontal invertibility of 2-morphisms.

Lemma 3.2.9

In a 2-category A, let

$$\begin{tikzcd}
a \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
b \arrow[r, bend left, "g"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & a
\end{tikzcd}$$

be two 2-morphisms such that both $\beta \circ \alpha$ and $\alpha \circ \beta$ become invertible 2-morphisms between invertible 1-morphisms. Then both α and β are invertible 2-morphisms between invertible 1-morphisms.

Proof. The assertion on 1-morphisms follows from the two out of six property of invertible 1-morphisms. The same applies to the 2-morphism claim by considering the horizontal composition

$$\begin{tikzcd}
a \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left, "g"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\beta"] & a \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & b
\end{tikzcd}$$

and by middle four interchange applying the two out of six property of invertible 2-morphisms to

$$\begin{tikzcd}
 & fgh \arrow[dr, Rightarrow, "\alpha\beta h\simeq"] \arrow[dd, Rightarrow, "f\beta h"] & \\
fgf \arrow[ur, Rightarrow, "fg\alpha"] \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "f\beta\alpha\simeq"'] & & hkh\,. \\
 & fkh \arrow[ur, Rightarrow, "\alpha kh"'] &
\end{tikzcd}$$

□

The following consistency result tells us that strict 2-functors can be identified with lax 2-functors whose identity and composition comparison 2-cells are invertible.

Lemma 3.2.10

For a 2-category A, the 2-morphism localisation

$$\begin{tikzcd} \mathrm{Lax}(A) \arrow[r, "\mathrm{loc}"] \arrow[dr, "\lambda"'] & \mathrm{Lax}(A)[\{\delta,\gamma\}^{-1}] \arrow[d, dashed, "\exists! \,\simeq"] \\ & A \end{tikzcd}$$

at the identity and composition comparison 2-morphisms δ and γ already induces an equivalence of 2-categories to A.

Proof. By Example 3.2.6 we already know that the localisations on the top of

$$\begin{tikzcd} \mathrm{Lax}(A) \arrow[r] \arrow[dr] & \mathrm{Lax}(A)[\{\text{Hom-units}\}^{-1}] \arrow[r, "\simeq"] & \mathrm{Lax}(A)[W_{\text{units}}^{-1}] \arrow[r, "\simeq"] & A \\ & \mathrm{Lax}(A)[\{\delta,\gamma\}^{-1}] \arrow[u, "\exists!"'] \arrow[urr] & & \end{tikzcd}$$

are equivalences. Hence it suffices to show that the localisation of $\mathrm{Lax}(A)$ at the unit 2-morphisms δ and γ is the same as the localisation at the class of all unit 2-morphisms for all Hom-adjunctions. To this end we employ Lemma 3.2.8 to conclude that inverting δ and γ suffice as every 1-morphism of $\mathrm{Lax}(A)$ is either the identity or an arbitrary length composition of $\iota(f)$, for f a 1-morphism in A, and invertible 1-morphisms coming from $A \simeq \iota^{\simeq}\,\mathrm{Lax}(A)$. As our Hom-adjunctions with invertible counits tell us that for example for ternary composites of $\iota(f)$ that the following triangle of 2-morphisms commutes.

$$\begin{tikzcd} \iota(h)\circ\iota(g)\circ\iota(f) \arrow[r, Rightarrow, "\gamma_{g,h}\circ\iota(f)"] \arrow[d, Rightarrow, "\eta"] & \iota(h\circ g)\circ\iota(f) \arrow[dl, Rightarrow, "\gamma_{f,h\circ g}"] \\ \iota(h\circ g\circ f) & \end{tikzcd}$$

For invertible 1-morphisms e coming through $\iota^{\simeq}\colon A^{\simeq} \to \mathrm{Lax}(A)$ we can consider the following commutative diagram

$$\begin{tikzcd} \mathrm{id} \arrow[r, Rightarrow, "\alpha\simeq"] \arrow[d, Rightarrow, "\delta"] & e\circ e^{-1} \arrow[r, Rightarrow, "\eta_e\circ\eta_{e^{-1}}"] \arrow[d, Rightarrow, "\eta_{e\circ e^{-1}}"] & \iota(e)\circ\iota(e^{-1}) \arrow[dl, Rightarrow, "\gamma_{e^{-1},e}"] \\ \iota(\mathrm{id}) \arrow[r, Rightarrow, "\iota(\alpha)\simeq"] & \iota(e\circ e^{-1}) & \end{tikzcd},$$

where $\alpha\colon \mathrm{id} \simeq e\circ e^{-1}$ witnesses the fact that e^{-1} is left inverse to e, as well as a similar diagram with roles of e and e^{-1} exchanged, to conclude that both $\eta_e \circ \eta_{e^{-1}}$ and $\eta_{e^{-1}} \circ \eta_e$ get inverted. This in particular implies that both $\iota(e)$ and $\iota(e^{-1})$ are sent to inverses of each other. From this we can conclude that both η_e and $\eta_{e^{-1}}$ are sent to invertible 2-morphisms by Lemma 3.2.9. Using the commutative diagram

$$\begin{tikzcd} \iota(f)\circ e \arrow[r, Rightarrow, "\iota(f)\circ\eta_e"] \arrow[d, Rightarrow, "\eta"'] & \iota(f)\circ\iota(e) \arrow[dl, Rightarrow, "\gamma_{e,f}"] \\ \iota(f\circ e) & \end{tikzcd}$$

we may also infer that unit 2-morphisms for 1-morphisms of the form $f \circ e$ for f a 1-morphism in A and e an invertible 1-morphism coming from $\iota^{\simeq}\colon A^{\simeq} \to \mathrm{Lax}(A)$, for example.

□

3.3 Localising the Extended Universal Property of the Lax Functor Classifier

In this section we will prove the first main result of this chapter, namely that the extended universal property of the lax functor classifier $\mathrm{Lax}(A)$ Definition 1.1.1 descends along localisations of $\mathrm{Lax}(A)$ along subcollections of unit 2-morphisms for the Hom-adjunctions of $\lambda\colon \mathrm{Lax}(A) \to A$. More precisely we will show that the localised $\mathrm{Lax}(A)[W^{-1}]$ initial among all those objects of $\mathrm{LocRARI}(A)$ for whom these unit 2-morphisms are invertible.

3.3.1 Descending the Extended Universal Property along Unit 2-Morphism Localisations

The first step in the proof will be to show that the fully-faithful Hom-adjoints descend along such localisations, and to this end we first prove the corresponding statement for 1-categories, to which we will reduce to via Theorem 3.2.5.

Lemma 3.3.1

Let $F\colon C \to D$ be a left adjoint between 1-categories with fully faithful right adjoint U. Let furthermore $W \subseteq F^{-1}(\{\text{equivalences}\})$ be a subclass of the class of morphisms in C that get sent to equivalences in D. Then by the universal property of localisation of C at W we obtain a functor G factorising F as in the following diagram.

$$\begin{tikzcd}
C \arrow[rr, "\gamma"] \arrow[dr, "F"'] & \arrow[d, phantom, "\alpha\simeq"] & C[W^{-1}] \arrow[dl, dashed, "\exists! G"] \\
& D &
\end{tikzcd}$$

Then we get that

1. the functor G also has a fully faithful right adjoint $V \coloneqq \gamma \circ U$, and
2. the triangle above is vertically adjointable with respect to these right adjoints.

Proof. 1. We define $V \coloneqq \gamma \circ U$. As F inverts W, so do UF and γUF. Hence $\gamma\eta$ is a morphism in $\mathrm{Fun}^W(C, C[W^{-1}])$ and by the localisation equivalence

$$\gamma^*\colon \mathrm{Fun}(C[W^{-1}], C[W^{-1}]) \simeq \mathrm{Fun}^W(C, C[W^{-1}])$$

we have a unique $\widetilde{\eta}\colon \Delta^1 \to \mathrm{Fun}(C[W^{-1}], C[W^{-1}])$ with $\widetilde{\eta}\gamma \simeq (\gamma U\alpha) \circ (\gamma\eta)$, hence also $\widetilde{\eta}\colon \mathrm{id} \Rightarrow VG$. We define $\widetilde{\varepsilon} \coloneqq \varepsilon \circ \alpha^{-1}U\colon G\gamma U \simeq FU \simeq \mathrm{id}$. From the commutativity of the diagram

$$\begin{tikzcd}
& D \arrow[ddr, "\varepsilon^{-1}"] & \\
C \arrow[ur, "F"] \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "\gamma_*"] \arrow[dr, "F_*"] & \\
C[W^{-1}] \arrow[r, dashed, "\exists!\widetilde{\eta}"'] & \mathrm{Fun}(\Delta^1, C[W^{-1}]) \arrow[r, "G_*"'] & \mathrm{Fun}(\Delta^1, D)
\end{tikzcd}$$

we may deduce that $G\widetilde{\eta} \simeq \widetilde{\varepsilon}^{-1}G$. As for the other triangle identity we compute

$$\widetilde{\eta}V = \widetilde{\eta}\gamma U \simeq ((\gamma U\alpha) \circ \gamma\eta)U \simeq \gamma U(\alpha U \circ \varepsilon^{-1}) \simeq V\widetilde{\varepsilon}^{-1}$$

using the triangle identity $\eta U \simeq U\varepsilon^{-1}$.

2. To prove adjointability of the new triangle it suffices to check by invertibility of ε that $\widetilde{\eta}\gamma U$ is invertible, but as we identified this with $V\widetilde{\varepsilon}^{-1}$ in the last step this is immediate.

□

The next step in the proof of Theorem 3.3.3 will be to observe that passing to such localisations will not destroy the existing adjointability on the respective Hom-triangles. Again we prove the 1-categorical counterpart of this statement, as Theorem 3.2.5 allows us to reduce to it.

Lemma 3.3.2

Let

$$\begin{tikzcd} C \arrow[rr, "J"] \arrow[dr, "F"'] & & E \arrow[dl, "H"] \\ & D & \end{tikzcd}$$

be a commutative triangle of functors between 1-categories such that F and H have fully faithful right adjoints U, respectively W, and such that the triangle is vertically adjointable. Let furthermore $W \subseteq F^{-1}(\{\text{equivalences}\})$ be a subclass of the class of morphisms in C that get sent to equivalences in D. If we now assume that the functor J inverts the morphisms in W, then we can factorise the triangle into two triangles

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[rr, bend left, "J"] \arrow[dr, "F"'] & C[W^{-1}] \arrow[r, dashed, "\exists! I"] \arrow[d, dashed, "\exists! G"] & E \arrow[dl, "H"] \\ & D & \end{tikzcd}$$

and obtain by Lemma 3.3.1 that G has a fully faithful right adjoint V and the left hand triangle is adjointable. However, in this situation we can say even more, namely that the right hand side triangle automatically also becomes vertically adjointable.

Proof. As we know from Lemma 3.3.1 that the counit $\widetilde{\varepsilon}$ of $G \dashv V$ is also invertible, adjointability of the right hand side triangle boils down to invertibility of μIV where μ denotes the unit of the adjunction $H \dashv W$. By adjointability of the original outer triangle and invertibility of ε we already have invertibility of μJU, so by $\mu IV = \mu I\gamma U \simeq \mu JU$ the desired invertibility follows directly. □

With the 1-categorical knowledge on how fully-faithful adjointability descends along intermediate localisations we may know localise the extended universal property Definition 1.1.1 of the lax functor classifier.

Theorem 3.3.3

Let A be a 2-category. Let W_0 be a subclass of the class of unit 2-morphisms for the fully faithful Hom-right-adjoints of $A^{\simeq} \to \mathrm{Lax}(A) \to A$. Let us furthermore denote by W the closure of W_0 under equivalences, vertical and horizontal composition. Then

1. the induced $A^{\simeq} \to \mathrm{Lax}(A)[W^{-1}] \xrightarrow{\exists!} A$ again has fully faithful Hom-right-adjoints for objects coming from $A^{\simeq}$, and furthermore
2. the induced $A^{\simeq} \to \mathrm{Lax}(A)[W^{-1}] \xrightarrow{\exists!} A$ is even initial among those $A^{\simeq} \to X \to A$ in $\mathrm{LocRARI}(A)$ which also have this subclass of units invertible, i. e. more precisely, those for which the uniquely existing 2-functor

$$\begin{tikzcd}
 & A^{\simeq} \arrow[dl, "\iota^{\simeq}"'] \arrow[dr, "U^{\simeq}"] \arrow[dd, bend left] & \\
\mathrm{Lax}(A) \arrow[rr, dashed, "\exists!"] \arrow[dr, "\lambda"'] & & X \arrow[dl, "F"] \\
 & A &
\end{tikzcd}$$

provided by the extended universal property of the lax 2-functor classifier sends this subclass W_0 to invertible 2-morphisms.

Proof. 1. Let us consider the induced triangle

$$\begin{tikzcd}
A^{\simeq} \arrow[d, "\iota^{\simeq}"'] \arrow[dr] & \\
\mathrm{Lax}(A) \arrow[r, "\gamma"] \arrow[d, "\lambda"'] & \mathrm{Lax}(A)[W^{-1}]\,. \arrow[dl, dashed, "\exists!\widetilde{\lambda}"] \\
A &
\end{tikzcd}$$

First we observe by Theorem 3.2.5 that we have a canonical commutative triangle

$$\begin{tikzcd}
 & \mathrm{Lax}(A)(a,\widetilde{a}) \arrow[dl, "\phi"'] \arrow[dr, "\mathrm{ap}_{\gamma}"] & \\
\mathrm{Lax}(A)(a,\widetilde{a})[W_{a,\widetilde{a}}^{-1}] \arrow[rr, dashed, "\exists!", "\simeq"'] & & \mathrm{Lax}(A)[W^{-1}](\gamma(a),\gamma(\widetilde{a}))
\end{tikzcd}$$

so we may apply Lemma 3.3.1 to the lower triangle of

$$\begin{tikzcd}
 & \mathrm{Lax}(A)[W^{-1}](\gamma(a),\gamma(\widetilde{a})) \arrow[dl, "\widetilde{\lambda}_{a,\widetilde{a}}" near end] \\
\mathrm{Lax}(A)(a,\widetilde{a}) \arrow[ur, "\mathrm{ap}_{\gamma}"] \arrow[r, "\phi"] \arrow[d, "\lambda_{a,\widetilde{a}}"] & \mathrm{Lax}(A)(a,\widetilde{a})[W_{a,\widetilde{a}}^{-1}] \arrow[u, dashed, "\exists!"', "\simeq"] \arrow[dl] \\
A(a,\widetilde{a}) &
\end{tikzcd}$$

and the fully faithful right adjoints to $\lambda_{a,\widetilde{a}}$ to conclude that also the $\widetilde{\lambda}_{a,\widetilde{a}}$ also have fully-faithful right adjoints.

2. We want to show that this 2-morphism localisation $A^{\simeq} \to \mathrm{Lax}(A)[W^{-1}] \xrightarrow{\exists!} A$ is in fact the initial such object in $\mathrm{LocRARI}(A)$ with this invertibility property. To this end we consider for any other object $A^{\simeq} \to X \to A$ of $\mathrm{LocRARI}(A)$ such that the underlying 2-functor

$\mathrm{Lax}(A) \to X$ of the unique morphism in $\mathrm{LocRARI}(A)$ inverts the unit 2-morphisms in W_0 the upgraded localisation equivalence

$$\mathrm{Fun}_{/A}^{A^{\simeq}\to A/}((\mathrm{Lax}(A)[W^{-1}], \widetilde{\lambda}, \gamma\iota), (X, F, U)) \xrightarrow[\gamma^*]{\simeq} \mathrm{Fun}_{/A}^{A^{\simeq}\to A/, W\text{-inv}}((\mathrm{Lax}(A), \lambda, \iota), (X, F, U))$$

obtained basically through the cube

$$\begin{tikzcd}
 & A \arrow[rr, equal] \arrow[dd, "\widetilde{\lambda}"' near end] & & A \arrow[dd, "\lambda"] \\
\mathrm{Fun}_{/A}((\mathrm{Lax}(A)[W^{-1}], \widetilde{\lambda}), (X, F)) \arrow[ur] \arrow[rr, "\simeq", "\gamma^*"'] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}_{/A}^{W\text{-inv}}((\mathrm{Lax}(A), \lambda), (X, F)) \arrow[ur] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \\
 & \mathrm{Fun}(\mathrm{Lax}(A)[W^{-1}], A) \arrow[rr, "\simeq", "\gamma^*"'] & & \mathrm{Fun}^{W\text{-inv}}(\mathrm{Lax}(A), A)\,, \\
\mathrm{Fun}(\mathrm{Lax}(A)[W^{-1}], X) \arrow[ur, "F_*"] \arrow[rr, "\simeq", "\gamma^*"'] & & \mathrm{Fun}^{W\text{-inv}}(\mathrm{Lax}(A), X) \arrow[ur, "F_*"'] &
\end{tikzcd}$$

whose left and right faces are pullbacks. To prove the initiality statement it is sufficient to prove that under this equivalence vertically adjointable triangles correspond exactly to vertically adjointable triangles. As we already know that composing adjointable squares gives again adjointable squares we only need to show that for every diagram

$$\begin{tikzcd}
 & A^{\simeq} \arrow[dl, "\iota"'] \arrow[dr, "U"] \arrow[dd, bend left] & \\
\mathrm{Lax}(A) \arrow[rr, "H"] \arrow[dr, "\lambda"'] & & X \arrow[dl, "F"] \\
 & A &
\end{tikzcd}$$

the induced right hand side triangle in the diagram

$$\begin{tikzcd}
 & A^{\simeq} \arrow[dl, "\iota^{\simeq}"'] \arrow[drr, "U^{\simeq}"] & & \\
\mathrm{Lax}(A) \arrow[r, "\gamma"'] \arrow[rrr, bend left, "H"] \arrow[dr, "\lambda"'] & \mathrm{Lax}(A)[W^{-1}] \arrow[rr, dashed, "\exists ! \widetilde{H}"'] \arrow[d, "\widetilde{\lambda}"'] & & X \arrow[dll, "F"] \\
 & A & &
\end{tikzcd}$$

gives a Hom-locally adjointable triangle with respect to the fully faithful Hom-right-adjoints of $\widetilde{\lambda}$ deduced in the first step. But by the first step of this proof this situation is on

Hom-categories exactly of the form

$$\begin{tikzcd}
 & & \mathrm{Lax}(A)[W^{-1}](\gamma(a),\gamma(\widetilde{a})) \arrow[ddd, bend left, "\widetilde{\lambda}"] \arrow[drr, dashed, "\exists!\widetilde{H}"] & & \\
\mathrm{Lax}(A)(a,\widetilde{a}) \arrow[urr, "\gamma"] \arrow[rr, "\phi"'] \arrow[rrrr, bend left, "H"] \arrow[ddrr, "\lambda"'] & & \mathrm{Lax}(A)(a,\widetilde{a})[W_{a,\widetilde{a}}^{-1}] \arrow[u, "\simeq"] \arrow[dd, "\exists!"'] \arrow[rr, dashed, "\exists!"'] & & X(U(a),U(\widetilde{a})) \arrow[ddll, "F"] \\
 & & & & \\
 & & A(a,\widetilde{a}) & &
\end{tikzcd}$$

to which we can apply our Lemma 3.3.2 to deduce the desired adjointability claim.

□

Remark 3.3.4

Note here that the above Theorem 3.3.3 but with respect to the simple universal property Definition 1.3.1 would have required also precise control over how 2-morphism localisations alter the underlying space of objects, which seems to be not only much harder to achieve in general, but also in the examples we want to apply Theorem 3.3.3 to below. The flexibility of just asking for object space sections in Definition 1.1.1 circumvents this difficulty entirely.

3.3.2 Application: The Initial *Strict* Hom-Local Right Adjoint Section

As a first trivial application of Theorem 3.3.3 we prove that the identity id_A on A is the initial object of $\mathrm{LocRARI}(A)$ with *strict* local right adjoint sections, i. e. whose uniquely induced lax Hom-locally right adjoint section, by the extended universal property Definition 1.1.1, is a strict 2-functor.

This will be the main ingredient in proving the reformulation Definition 3.4.13 of the universal property of the lax Gray tensor product in Proposition 3.4.19.

Definition 3.3.5

An object

$$\begin{tikzcd}
C^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr] & X \arrow[d, "F"] \\
 & C
\end{tikzcd}$$

of $\mathrm{LocRARI}(C)$, for a 2-category C, is called strict if it satisfies that the following kinds of unit 2-morphisms for the Hom-adjunctions are invertible

1. for every object c of C the unit at id_c, $\delta_c \colon \mathrm{id}_c \Rightarrow U(\mathrm{id}_c)$
2. for every pair of composable 1-morphisms f and g in C the unit at $U(g) \circ U(f)$,

$$\gamma_{f,g} \colon U(g) \circ U(f) \Rightarrow U(g \circ f) \ .$$

We denote the full subcategory of $\mathrm{LocRARI}(C)$ on strict objects by $\mathrm{LocRARI}_{\mathrm{strict}}(C)$.

Example 3.3.6

For a 2-category C, an object easily seen to be in $\mathrm{LocRARI}_{\mathrm{strict}}(C)$ is

$$\begin{tikzcd} C^{\simeq} \arrow[r, "\gamma"] \arrow[dr, "\gamma"'] & C \arrow[d, equal] \\ & C \end{tikzcd}$$

Corollary 3.3.7

Combining Theorem 3.3.3 with our consistency result Lemma 3.2.10 about *strict* lax 2-functors we deduce that for a 2-category A, (A, id, γ) is the initial object of $\mathrm{LocRARI}_{\mathrm{strict}}(A)$, i. e. the forgetful functor of the slice-under category

$$^{(C,\mathrm{id},\gamma)/}\,\mathrm{LocRARI}_{\mathrm{strict}}(A) \xrightarrow{\simeq} \mathrm{LocRARI}_{\mathrm{strict}}(A)$$

is already an equivalence.

3.3.3 Application: The Localised Universal Property of the Unitary Lax 2-Functor Classifier

As a second direct application of Theorem 3.3.3 we derive a localised synthetic universal property of the *unitary* lax functor classifier.

Definition 3.3.8
1. A lax functor $F\colon A \rightsquigarrow B$ is called unitary, if the strict 2-functor $F\colon \mathrm{Lax}(A) \to B$ inverts all the identity comparison 2-morphisms $\mathrm{id}_a \Rightarrow \iota(\mathrm{id}_a)$ for all objects a of A.
2. Hence we define the unitary lax 2-functor classifier $\mathrm{ULax}(A)$ of A as the 2-morphism localisation $\mathrm{Lax}(A)[\delta^{-1}]$ of $\mathrm{Lax}(A)$ at all those identity comparison 2-morphisms.

Definition 3.3.9

Let A be a 2-category. We can an object

$$\begin{tikzcd} A^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \end{tikzcd}$$

of $\mathrm{LocRARI}(A)$ unitary, if for all objects a of $A^{\simeq}$ the unit 2-morphism of the Hom-adjunction $F_{U(a),U(a)} \dashv U_{a,a}$ at the identity $\mathrm{id}_{U(a)}$, $\delta\colon \mathrm{id}_{U(a)} \Rightarrow U(\mathrm{id}_a)$ is invertible. These unitary objects define a full subcategory $\mathrm{LocRARI}_{\mathrm{unitary}}(A)$ of $\mathrm{LocRARI}(A)$.

Corollary 3.3.10

Let A be a 2-category. Let us denote by W_{unitary} the subclass of unit 2-morphisms of the Hom-right-adjoints of $\lambda\colon \mathrm{Lax}(A) \to A$ consisting of the identity comparison 2-morphisms $\mathrm{id}_a \Rightarrow \iota(\mathrm{id}_a)$ for all a. Then the induced 2-morphism localisation $A^{\simeq} \to \mathrm{ULax}(A) \coloneqq \mathrm{Lax}(A)[W_{\mathrm{unitary}}^{-1}] \xrightarrow{\exists!} A$ together with its induced functors is the initial object in $\mathrm{LocRARI}_{\mathrm{unitary}}(A)$ by Theorem 3.3.3.

Remark 3.3.11

Through Theorem 1.4.5 we know that our synthetic lax functor classifiers agree with the envelope construction in the model of 2-categories given by globular complete Segal cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$. This in turn by, for example, Proposition 1.2.12 is the classifying object for inert-cocartesian functors, a model for lax 2-functors in this 2-category model. The notion of unitary lax functors modelled as certain inert-cocartesian functors as been compared by Abellan in [Abe23b] to a model of unitary lax functors in the model of 2-categories given by so-called *scaled simplicial sets* used in [GHL21]. Hence Corollary 3.3.10 proves that both these models of unitary lax functors agree with the standalone synthetic definition of unitary lax functors as strict 2-functors out of the initial object of $\mathrm{LocRARI}_{\mathrm{unitary}}(A)$. Additionally, Corollary 3.3.10 proves that all these unitary lax functor classifiers admit fully-faithful right adjoints on Hom-categories with respect to the canonical 2-functor $\mathrm{ULax}(A) \to A$.

3.4 Two Reformulations of the Localised Universal Property for the Lax Gray Tensor Product

In this section we will discuss synthetic approaches, more precisely universal properties, of the lax Gray tensor product of 2-categories.

As already hinted at in the introduction of this chapter, the lax Gray tensor product $\otimes_{\mathrm{lax}}$ seems to be one of the fundamental building blocks for almost all lax phenomena in 2-category theory, e. g. you can use it to define lax natural transformations one at a time as 2-functors $\Delta^1 \otimes_{\mathrm{lax}} A \to B$, or look at its adjoint $(-) \otimes_{\mathrm{lax}} A \dashv \mathrm{Fun}_{\mathrm{lax}}(A, -)$ which gives the moduli of all of them, i. e. the 2-category of all 2-functors $A \to B$, all lax natural transformations as well as their morphisms. Historically, the lax Gray tensor product has been constructed by explicit description of the objects, 1-morphisms, 2-morphisms and their composition laws in Gray's [Gra74]. In the realm of higher category theory, it has been constructed for several models of 2- and more generally n-categories as well as all the way up to actual (∞, ∞)-categories. However these constructions either

1. left Kan extend the ordinary Gray tensor product of, nicely behaved, ordinary n-categories along their inclusion into all n-categories, like [Mae21],
2. or give explicit combinatorial models for it based on simplices or cubes like in [Ver08b], [CM23] and [Cam23],
3. or base their definitions on another lax phenomenon, like lax functor classifiers, which are in turn modeled via combinatorial descriptions like in [GR17].

We would like to take a different approach and try to give universal properties for the lax Gray tensor product that neither refer to the ordinary Gray tensor product, nor base themselves on combinatorics. Instead we want to find one that can be formulated, much like the synthetic universal property Definition 1.1.1 of the lax functor classifier, by only refering to elementary aspects of 2-category theory as formulated precisely for example in Remark 1.2.1.

Now one could argue that, by endowing the lax functor classifier with a synthetic universal property which is elementary in the sense of Remark 1.2.1, one can at least take the approach to lax Gray tensor products as 2-morphism localisations of the lax functor classifier, and call this localisation universal property with respect to lax functors a synthetic universal property of the lax Gray product. Whereas one can now hardly argue against this definition being synthetic, it still makes reference to a higher dimensional colimit construction, the inversion of 2-morphisms, an already more complicated aspect of 2-category theory.

But indeed even this approach does not answer the question whether the lax Gray tensor product itself is a *primitive* lax phenomenon, in the sense that it can be given an elementary universal property *not* making any reference to another lax phenomenon. Also, from the practical point of view, just having *any* universal property for a certain object does not a priori mean one already understands everything there is to know about that object. Finding stronger universal properties might reveal different aspects and more properties of the object, like the 2-functors $A \otimes_{\mathrm{lax}} B \to A \times B$ having fully-faithful Hom-right adjoints of a very precise shape, in our case.

3.4.1 The Localised Universal Property of the Lax Gray Tensor Product

In this subsection we give a first realisation of our main goal in this chapter, i. e. to find an elementary synthetic universal property of the lax Gray tensor product. More precisely, we employ Theorem 3.3.3 to descend the extended universal property Definition 1.1.1 along the 2-morphism localisation $\mathrm{Lax}(A \times B) \to \mathrm{Lax}(A \times B)[W_{\mathrm{Gray}}^{-1}]$ that should give the lax Gray tensor product.

As discussed in the introduction of this chapter, this is motivated by the observation that the 2-functor from the ordinary lax Gray tensor product to the ordinary cartesian product has on ordinary Hom-1-categories

$$(A \otimes_{\mathrm{lax}} B)((a,b),(\widetilde{a},\widetilde{b})) \to A(a,\widetilde{a}) \times B(b,\widetilde{b})$$

also fully-faithful right adjoints, which are given by the assignment $(f,g) \mapsto (f,\mathrm{id}) \circ (\mathrm{id},g)$.

Definition 3.4.1

Let A and B be two 2-categories. A commutative triangle of 2-functors

$$\begin{tikzcd} A^{\simeq} \times B^{\simeq} \arrow[r, "U"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \times B\,, \end{tikzcd}$$

where $\gamma\colon A^{\simeq} \times B^{\simeq} \to A \times B$ denotes the inclusion of the underlying space of objects, is said to have local Gray right adjoint sections if

1. it has local right adjoint sections, i. e. satisfies that for every pair of objects $a, \widetilde{a}$ of $A^{\simeq}$ and $b, \widetilde{b}$ of $B^{\simeq}$, the induced functor
$$F_{U(a,b),U(\widetilde{a},\widetilde{b})}\colon X(U(a,b),U(\widetilde{a},\widetilde{b})) \to (A \times B)(F(U(a,b)),F(U(\widetilde{a},\widetilde{b}))) \simeq A(a,\widetilde{a}) \times B(b,\widetilde{b})$$
on Hom-$(\infty,1)$-categories admits a fully-faithful right adjoint $U_{a,\widetilde{a},b,\widetilde{b}}$, by abuse of notation often just written as U,
2. for every pair of objects (a,b) of $A^{\simeq} \times B^{\simeq}$ the unit 2-morphism of $F_{U(a,b),U(a,b)} \dashv U_{a,a,b,b}$ at the identity $\mathrm{id}_{(a,b)}$, $\eta_{\mathrm{id}_{(a,b)}}\colon \mathrm{id}_{(a,b)} \Rightarrow U(\mathrm{id}_a, \mathrm{id}_b)$ is assumed to be invertible,
3. for every composable pair of morphisms f and $\widetilde{f}$ in A and object b in B, the unit 2-morphism at the 1-morphism $U(\widetilde{f}, \mathrm{id}_b) \circ U(f, \mathrm{id}_b)$, $\eta\colon U(\widetilde{f}, \mathrm{id}_b) \circ U(f, \mathrm{id}_b) \Rightarrow U(\widetilde{f} \circ f, \mathrm{id}_b)$, is assumed to be invertible,
4. for every composable pair of morphisms g and $\widetilde{g}$ in B and object a in A, the unit 2-morphism at the 1-morphism $U(\mathrm{id}_a, \widetilde{g}) \circ U(\mathrm{id}_a, g)$, $\eta\colon U(\mathrm{id}_a, \widetilde{g}) \circ U(\mathrm{id}_a, g) \Rightarrow U(\mathrm{id}_a, \widetilde{g} \circ g)$, is assumed to be invertible,
5. for every pair of morphisms f in A and g in B, the unit 2-morphism at the 1-morphism $U(f,\mathrm{id}) \circ U(\mathrm{id},g)$, $\eta\colon U(f,\mathrm{id}) \circ U(\mathrm{id},g) \Rightarrow U(f,g)$, is assumed to be invertible.

These conditions define a full subcategory $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$ of the category $\mathrm{LocRARI}(A \times B)$. A lax Gray tensor product of A and B is an initial object $A^{\simeq} \times B^{\simeq} \to A \otimes_{\mathrm{lax}} B \to A \times B$ in $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$.

Remark 3.4.2

Note here, that while this universal property is indeed elementary, synthetic and can be stated without explicitly mentioning lax 2-functors, one could still argue that it looks like an unpacked version of the extended universal property Definition 1.1.1 of $\mathrm{Lax}(A \times B)$ plus the above mentioned 2-morphism localisation universal property. This is ultimately also the essence of this universal property as we will see later in Corollary 3.4.6. To adhere more to our goal of finding a simple and concise universal property of the lax Gray tensor product, we later give the reformulations Definition 3.4.11 and Definition 3.4.13.

The following lemma states compatibility of $\mathrm{LocRARI}^{\mathrm{Gray}}(-,-)$ with pullback, which gives us functoriality of the lax Gray tensor product.

Lemma 3.4.3

For 2-functors $A \to C$ and $B \to D$ the pullback functor $\mathrm{LocRARI}(C \times D) \to \mathrm{LocRARI}(A \times B)$ restricts to the full subcategories $\mathrm{LocRARI}^{\mathrm{Gray}}(C, D) \to \mathrm{LocRARI}^{\mathrm{Gray}}(A, B)$.

Proof. This is an application of Lemma 1.2.11. □

We can also more generally define n-ary versions of local Gray right adjoint sections and lax Gray tensor products.

Definition 3.4.4

Let $A_1, \dots, A_n$ be n 2-categories. A commutative triangle of 2-functors

$$\begin{tikzcd}
(A_1 \times \cdots \times A_n)^{\simeq} \arrow[r, "U"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\
& A_1 \times \cdots \times A_n
\end{tikzcd}$$

is said to have local Gray right adjoint sections if

1. it has again as above Hom-local right adjoint sections $U_{a_1,\dots,a_n}$,
2. for every n-uple of objects $(a_1, \dots, a_n)$ of $(A_1 \times \cdots \times A_n)^{\simeq}$ the unit 2-morphism at the identity $\mathrm{id}_{(a_1,\dots,a_n)}$, $\eta_{\mathrm{id}_{(a_1,\dots,a_n)}} \colon \mathrm{id}_{(a_1,\dots,a_n)} \Rightarrow U(\mathrm{id}_{a_1}, \dots, \mathrm{id}_{a_n})$ is assumed to be invertible,
3. for all $i \in \{1, \dots, n\}$ and for every composable pair of morphisms f_i and $\widetilde{f}_i$ in A_i the unit 2-morphism at the 1-morphism $U(\mathrm{id}_{a_1}, \dots, \widetilde{f}_i, \dots, \mathrm{id}_{a_n}) \circ U(\mathrm{id}_{a_1}, \dots, f_i, \dots, \mathrm{id}_{a_n})$,
$$\eta \colon U(\mathrm{id}_{a_1}, \dots, \widetilde{f}_i, \dots, \mathrm{id}_{a_n}) \circ U(\mathrm{id}_{a_1}, \dots, f_i, \dots, \mathrm{id}_{a_n}) \Rightarrow U(\mathrm{id}_{a_1}, \dots, \widetilde{f}_i \circ f_i, \dots, \mathrm{id}_{a_n}),$$
is assumed to be invertible,
4. for every n-uple of 1-morphisms $(f_1, \dots, f_n)$ in $A_1 \times \cdots \times A_n$, and for all $1 \leq k \leq n-1$ the unit 2-morphism at the 1-morphism $U(f_1, \dots, f_k, \mathrm{id}, \dots, \mathrm{id}) \circ U(\mathrm{id}, \dots, \mathrm{id}, f_{k+1}, \dots, f_n)$, $\eta \colon U(f_1, \dots, f_k, \mathrm{id}, \dots, \mathrm{id}) \circ U(\mathrm{id}, \dots, \mathrm{id}, f_{k+1}, \dots, f_n) \Rightarrow U(f_1, \dots, f_n)$ is assumed to be invertible.

These conditions define a full subcategory $\mathrm{LocRARI}^{\mathrm{Gray}}(A_1, \dots, A_n)$ of the $\mathrm{LocRARI}(A_1 \times \cdots \times A_n)$. An n-ary lax Gray tensor product of $A_1, \dots, A_n$ is an initial object

$$(A_1 \times \cdots \times A_n)^{\simeq} \to A_1 \otimes_{\mathrm{lax}} \dots \otimes_{\mathrm{lax}} A_n \to A_1 \times \cdots \times A_n$$

in $\mathrm{LocRARI}^{\mathrm{Gray}}(A_1, \dots, A_n)$.

Remark 3.4.5

The universal properties allow us to construct associativity comparison 2-functors like for example $A \otimes_{\mathrm{lax}} B \otimes_{\mathrm{lax}} C \to (A \otimes_{\mathrm{lax}} B) \otimes_{\mathrm{lax}} C$ by observing that

$$\begin{tikzcd}
A^{\simeq} \times B^{\simeq} \times C^{\simeq} \arrow[r] \arrow[ddr] & (A \otimes_{\mathrm{lax}} B) \otimes_{\mathrm{lax}} C \arrow[d, "\lambda_{A \otimes_{\mathrm{lax}} B, C}"] \\
& (A \otimes_{\mathrm{lax}} B) \times C \arrow[d, "\lambda_{A,B} \times C"] \\
& (A \times B) \times C
\end{tikzcd}$$

again has ternary local Gray right adjoint sections. However, we will not discuss synthetic proofs of their invertibility in this work.

We now apply our Theorem 3.3.3 to show that the localised version of the lax Gray tensor product satisfies our universal property Definition 3.4.1.

Corollary 3.4.6

Let A and B be two 2-categories. Let us denote by W_{Gray} the subclass of unit 2-morphisms of the Hom-right-adjoints of $\lambda\colon \mathrm{Lax}(A \times B) \to A \times B$ specified in Definition 3.4.1 in the definition of having local Gray right adjoint sections. Then the induced 2-morphism localisation $A^{\simeq} \times B^{\simeq} \to \mathrm{Lax}(A \times B)[W_{\mathrm{Gray}}^{-1}] \xrightarrow{\exists!} A \times B$ together with its induced functors satisfies the universal property of being the lax Gray tensor product of A and B from Definition 3.4.1. The same is more generally true for the n-ary versions of the lax Gray tensor product.

Proof. We employ Theorem 3.3.3 to the subclass of unit 2-morphisms W_{Gray}. □

Let us quickly comment on how our synthetic universal property of the lax Gray tensor product compares to the approaches mentioned above. By Abellan's comparison [Abe23b] we know that the definition of the lax Gray tensor product in terms of lax functors like in [GR17] agrees with the simplicial models pioneered by Verity in [Ver08b], [Ver08a] and used also in [GHL21] but also with [CKM25], [Mae21] and [CM23]. Hence all of them satisfy our synthetic universal property Definition 3.4.1 by Corollary 3.4.6.

Remark 3.4.7

To relate back to the introduction of this work, lax 2-functors $A \times B \rightsquigarrow C$ inverting W_{Gray} are called cubical.

Remark 3.4.8

One can take the above corollary as an existence statement, i. e. that the lax Gray tensor product as defined synthetically in Definition 3.4.1 *exists*. On the other hand, one can also interpret it as a proposition stating that it has fully-faithful Hom-right adjoints and enjoys a different universal property, i. e. initiality in a different category, contrasted to its universal property as a 2-morphism localisation of $\mathrm{Lax}(A \times B)$, which might shed more light on different aspects of its nature.

As a first synthetic structural result for the lax Gray tensor product we now show that groupoidal inputs will let the lax Gray tensor product degenerate back to the cartesian product.

Lemma 3.4.9

For 2-categories A and B, if *at least one* of them is a space, then every object of the full subcategory $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \subseteq \mathrm{LocRARI}(A\times B)$ already is contained in $\mathrm{LocRARI}_{\mathrm{strict}}(A\times B)$. As $(A\times B, \mathrm{id}, \gamma)$ is the initial object in $\mathrm{LocRARI}_{\mathrm{strict}}(A\times B)$ by Corollary 3.3.7, and also lies in $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$. In particular $(A\times B, \mathrm{id}, \gamma)$ is the lax Gray tensor product of A and B.

Proof. We prove the statement for A being a space, the other case is similiar. Let

$$\begin{tikzcd} A^{\simeq}\times B^{\simeq} \arrow[r, "U^{\simeq}"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A\times B \end{tikzcd}$$

be an object of $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$. By Corollary 3.3.7 we need to show that all the identity comparison units δ and composition comparison units γ, with notation as in Definition 3.3.5, are invertible, and we already have the δ, so we are left with the γ.

1. It suffices to show that $\eta\colon U(\mathrm{id},g)\circ U(e,\mathrm{id}) \Rightarrow U(e,g)$ is invertible, by the following commutative diagram of units.

$$\begin{tikzcd} U(\tilde e,\mathrm{id})\circ U(\mathrm{id},g)\circ U(e,\mathrm{id})\circ U(\mathrm{id},h) \arrow[r, Rightarrow, "\eta\circ\eta"] \arrow[d, Rightarrow, "\mathrm{id}\circ\eta\circ\mathrm{id}"] & U(\tilde e,g)\circ U(e,h) \arrow[d, Rightarrow, "\gamma"'] \\ U(\tilde e,\mathrm{id})\circ U(e,g)\circ U(\mathrm{id},h) \arrow[r, Rightarrow, "\eta"] & U(\tilde e\circ e, g\circ h) \\ U(\tilde e,\mathrm{id})\circ U(e,\mathrm{id})\circ U(\mathrm{id},g)\circ U(\mathrm{id},h) \arrow[r, Rightarrow, "\gamma\circ\gamma"] \arrow[u, Rightarrow, "\mathrm{id}\circ\eta\circ\mathrm{id}"] & U(\tilde e\circ e,\mathrm{id})\circ U(\mathrm{id},g\circ h) \arrow[u, Rightarrow, "\eta"'] \end{tikzcd}$$

   More precisely, we want to show that the upper right hand vertical γ is invertible, and we know by Definition 3.4.1 that the γ in the bottom horizontal, the η in the top, the lower right hand vertical, the lower left hand vertical 2-morphisms are invertible, hence the middle horizontal one too.

2. From Corollary 3.3.7 applied to $A\times\{b\}$ we know that

$$\eta\colon (e,\mathrm{id}) \coloneqq U^{\simeq}(e,\mathrm{id}) \Rightarrow U_{(a,b),(\tilde a,b)}(e,\mathrm{id}) \eqqcolon U(e,\mathrm{id})$$

   for a 1-morphism $e\colon a\simeq \tilde a$ in $A^{\simeq}$, is invertible and by the commutative triangle

$$\begin{tikzcd} U(\mathrm{id},g)\circ(e,\mathrm{id}) \arrow[r, Rightarrow, "\mathrm{id}\circ\eta"] \arrow[dr, Rightarrow, "\eta"'] & U(\mathrm{id},g)\circ U(e,\mathrm{id}) \arrow[d, Rightarrow, "\eta"] \\ & U(e,g) \end{tikzcd}$$

   for a 1-morphism $g\colon b\to \tilde{\tilde b}$ in B, it is thus sufficient to prove that $\eta\colon U(\mathrm{id},g)\circ U(e,\mathrm{id}) \Rightarrow U(e,g)$ is invertible. Now we steal an idea from Definition 3.4.11, and consider the commutative square

$$\begin{tikzcd} X((\tilde a,b),(\tilde a,\tilde{\tilde b})) \arrow[r, "{*(e,\mathrm{id})}", "\simeq"'] \arrow[d, "F"'] & X((a,b),(\tilde a,\tilde{\tilde b})) \arrow[d, "F"] \\ A(\tilde a,\tilde a)\times B(b,\tilde{\tilde b}) \arrow[r, "{*(e,\mathrm{id})}", "\simeq"'] & A(a,\tilde a)\times B(b,\tilde{\tilde b}), \end{tikzcd}$$

whose horizontal functors are equivalences. This in particular implies that the square in a pullback, and hence Lemma 1.2.11 gives us its vertical adjointability for free. If we now evaluate the adjointability, more precisely the implication that the units of the adjunctions commute, at the 1-morphisms $U(\mathrm{id}, g)$ then we get that the whiskered invertible unit $(U(\mathrm{id}, g) = U(\mathrm{id}, g)) \circ (e, \mathrm{id})$ agrees with the unit of the right hand adjunction at $U(\mathrm{id}, g) \circ (e, \mathrm{id})$, which is just the desired $\eta\colon U(\mathrm{id}, g) \circ U(e, \mathrm{id}) \Rightarrow U(e, g)$, hence must be invertible.

□

From this we can synthetically deduce that the lax Gray tensor product has a unit object given by the one-object 2-category Δ^0.

Corollary 3.4.10

From Lemma 3.4.9 we deduce that the terminal 2-category Δ^0 is the unit for the lax Gray tensor product, as we have $\Delta^0 \otimes_{\mathrm{lax}} A \simeq \Delta^0 \times A \simeq A \simeq A \times \Delta^0 \simeq A \otimes_{\mathrm{lax}} \Delta^0$.

3.4.2 A First Reformulation of the Localised Universal Property

We will now give a first reformulation of the localised universal property Definition 3.4.1 that tries to rewrite the individual invertibility properties of Definition 3.4.1 into slightly more conceptual properties. More precisely we will repackage points 3 and 5, as well as point 4 and 5 of Definition 3.4.1 into one single property respectively.

To illustrate this reformulation, let us consider the special case of the lax Gray tensor product of 2-categories $B(\mathcal{V}, \otimes)$ and $B(\mathcal{W}, \otimes)$, where $(\mathcal{V}, \otimes)$ and $(\mathcal{W}, \otimes)$ are monoidal 1-categories and the construction $B(\mathcal{V}, \otimes)$ considers the monoidal 1-category as a 2-category with roughly speaking

1. one object $*$, up to homotopy, i. e. we have an essentially surjective functor $\Delta^0 \to B(\mathcal{V}, \otimes)^{\simeq}$,
2. the single Hom-1-category $B(\mathcal{V}, \otimes)(*, *) := \mathcal{V}$,
3. the identity id_* given by the monoidal unit $\mathbb{1}$ of $\mathcal{V}$,
4. composition given by the tensor product $\otimes$ of $\mathcal{V}$.

Let us also write, for illustration purpose, the images of the Hom-right adjoint sections of

$$\lambda\colon (B(\mathcal{V}, \otimes) \otimes_{\mathrm{lax}} B(\mathcal{W}, \otimes))((*, *), (*, *)) \to B(\mathcal{V}, \otimes)(*, *) \times B(\mathcal{W}, \otimes)(*, *) = \mathcal{V} \times \mathcal{W}$$

on a pair (v, w) as an *elementary tensor* $v \otimes w$, just like in a product of ordinary monoids, whereas our monoidal 1-categories are actually monoids in Cat. Now, it is actually this extra dimension of our monoids being internal to Cat that allows us to consider potential *lax* versions of the usual product of ordinary monoid objects, which in our case would be just $B(\mathcal{V}, \otimes) \times B(\mathcal{W}, \otimes) \simeq B(\mathcal{V} \times \mathcal{W}, \otimes)$. Note here that by the composite of essential surjections

$$\Delta^0 \simeq \Delta^0 \times \Delta^0 \to B(\mathcal{V}, \otimes)^{\simeq} \times B(\mathcal{W}, \otimes)^{\simeq} \to B(\mathcal{V}, \otimes) \otimes_{\mathrm{lax}} B(\mathcal{W}, \otimes)$$

we know that we can view this lax Gray tensor product again as a monoid in Cat. Then points 2, 3, 4 and 5 of Definition 3.4.1 can be read as the the following lax comparison 2-morphisms

$$\begin{aligned} \mathrm{id}_{*,*} &\to (\mathbb{1} \otimes \mathbb{1}) \\ (\widetilde{v} \otimes \mathbb{1}) \circ (v \otimes \mathbb{1}) &\to (\widetilde{v}v \otimes \mathbb{1}) \\ (\mathbb{1} \otimes \widetilde{w}) \circ (\mathbb{1} \otimes w) &\to (\mathbb{1} \otimes \widetilde{w}w) \\ (v \otimes \mathbb{1}) \circ (\mathbb{1} \otimes w) &\to (v \otimes w) \end{aligned}$$

inside $B(\mathcal{V}, \otimes) \otimes_{\mathrm{lax}} B(\mathcal{W}, \otimes)$ to be invertible, where we surpressed writing the individual tensor products of $\mathcal{V}$ and $\mathcal{W}$. The first one could be interpreted as $\mathbb{1} \otimes \mathbb{1}$ being the unit of this lax tensor product, whereas the second and third show that is bi-multiplicative, in each of its arguments seperately. We will now repackage points 3 and 5, as well as points 4 and 5, into the fact that the following two lax comparison 2-morphism

$$(\widetilde{v} \otimes \mathbb{1}) \circ (v \otimes w) \to (\widetilde{v}v \otimes w)$$
$$(v \otimes \widetilde{w}) \circ (\mathbb{1} \otimes w) \to (v \otimes \widetilde{w}w)$$

are invertible, respectively. This means we are allowed to pull multiplication with $\widetilde{v} \otimes \mathbb{1}$ from the left inside, as well as multiplication with $\mathbb{1} \otimes w$ from the right. As in the case for the ordinary product of monoids we could interpret this as defining an actual left $\mathcal{V}$-action and right $\mathcal{W}$-action, i. e. a strict left $B(\mathcal{V}, \otimes)$-action and right $B(\mathcal{W}, \otimes)$-action on our lax tensor product $B(\mathcal{V}, \otimes) \otimes_{\text{lax}} B(\mathcal{W}, \otimes)$. However, the multiplication comparison

$$(\widetilde{v} \otimes \widetilde{w}) \circ (v \otimes w) \to (\widetilde{v}v \otimes \widetilde{w}w)$$

is in general *not* invertible, resulting in something more general than just $B(\mathcal{V} \times \mathcal{W}, \otimes)$.

Going back to general 2-categories, i. e. the many-object versions, such strictness of left and right actions can be reformulated into the adjointability of certain composition induced squares.

Definition 3.4.11

Let A and B be two $(\infty, 2)$-categories. A commutative triangle of 2-functors

$$\begin{tikzcd} A^{\simeq} \times B^{\simeq} \arrow[r, "U"] \arrow[dr, "\gamma"'] & X \arrow[d, "F"] \\ & A \times B \end{tikzcd}$$

is said to have local Gray right adjoint sections if

1. it has local right adjoint sections, i. e. satisfies that for every pair of objects $a, \widetilde{a}$ of $A^{\simeq}$ and $b, \tilde{b}$ of $B^{\simeq}$, the induced functor
$$F_{U(a,b),U(\widetilde{a},\tilde{b})} \colon X(U(a,b), U(\widetilde{a}, \tilde{b})) \to (A \times B)(F(U(a,b)), F(U(\widetilde{a}, \tilde{b}))) \simeq A(a, \widetilde{a}) \times B(b, \tilde{b})$$
on Hom-$(\infty, 1)$-categories admits a fully-faithful right adjoint $U_{a,\widetilde{a},b,\tilde{b}}$, by abuse of notation often just written as U.
2. for every pair of objects (a, b) of $A^{\simeq} \times B^{\simeq}$ the unit 2-morphism of $F_{U(a,b),U(a,b)} \dashv U_{a,a,b,b}$ at the identity $\mathrm{id}_{(a,b)}$, $\eta_{\mathrm{id}_{(a,b)}} \colon \mathrm{id}_{(a,b)} \Rightarrow U(\mathrm{id}_a, \mathrm{id}_b)$ is assumed to be invertible,
3. For every morphism h in A precomposition with $U(h, \mathrm{id})$ makes the induced square
$$\begin{tikzcd} X((\widetilde{a}, b), -) \arrow[r, "U(h,\mathrm{id})_*"] \arrow[d, "F"', shift right] \arrow[d, phantom, "\dashv"] & X((a, b), -) \arrow[d, "F"', shift right] \arrow[d, phantom, "\dashv"] \\ A(\widetilde{a}, -) \times B(b, -) \arrow[r, "(h,\mathrm{id})_*"] \arrow[u, dashed, bend right] & A(a, -) \times B(b, -) \arrow[u, dashed, bend right] \end{tikzcd}$$
adjointable for all objects in the second variable of the Hom-categories.
4. For every morphism g in B postcomposition with $U(\mathrm{id}, g)$ makes the induced square
$$\begin{tikzcd} X(-, (a, b)) \arrow[r, "{}_*U(\mathrm{id},g)"] \arrow[d, "F"', shift right] \arrow[d, phantom, "\dashv"] & X(-, (a, \tilde{b})) \arrow[d, "F"', shift right] \arrow[d, phantom, "\dashv"] \\ A(-, a) \times B(-, b) \arrow[r, "{}_*(\mathrm{id},g)"] \arrow[u, dashed, bend right] & A(-, a) \times B(-, \tilde{b}) \arrow[u, dashed, bend right] \end{tikzcd}$$
adjointable for all objects in the first variable of the Hom-categories.

The proof that this definition agrees with Definition 3.4.1 is just a matter of playing with the naturality of our Hom-adjunction units.

Lemma 3.4.12

The Definition 3.4.1 and Definition 3.4.11 are equivalent, hence resulting in equivalent full subcategories of $\mathrm{LocRARI}(A \times B)$ and hence in equivalent characterisations of the lax Gray tensor product of A and B.

Proof. We will show that points 3 and 5 of Definition 3.4.1 are equivalent to point 3 of Definition 3.4.11. The proof that points 4 and 5 of Definition 3.4.1 are equivalent to point 4 of Definition 3.4.11 is entirely similiar. To get points 3 and 5 of Definition 3.4.1 out of point 3 of Definition 3.4.11 one just needs to inspect the adjointability at hand morphisms of the form (f, id) and (id, g). To arrive at the converse implication we contemplate the following two naturality squares of our Hom-adjunction units.

$$\begin{tikzcd}
U(\widetilde{h}, \mathrm{id}) \circ U(h, g) \arrow[d, Rightarrow, "\eta"'] & U(\widetilde{h}, \mathrm{id}) \circ U(h, \mathrm{id}) \circ U(\mathrm{id}, g) \arrow[l, Rightarrow, "\mathrm{id} \circ \eta"'] \arrow[r, Rightarrow, "\eta \circ \mathrm{id}"] \arrow[d, Rightarrow, "\eta"] & U(\widetilde{h} \circ h, \mathrm{id}) \times U(\mathrm{id}, g) \arrow[d, Rightarrow, "\eta"] \\
U(\widetilde{h} \circ h, g) \arrow[r, equal] & U(\widetilde{h} \circ h, g) \arrow[r, equal] & U(\widetilde{h} \circ h, g)
\end{tikzcd}$$

Point 3 of Definition 3.4.1 tells us that in this diagram the right hand top horizontal 2-morphism is invertible. Point 5 of Definition 3.4.1 in turn tells us that the right most vertical unit 2-morphism as well as the left hand top horizontal 2-morphism is invertible. Hence we may conclude the desired invertibility of the left most vertical unit 2-morphism. □

3.4.3 A Funny Universal Property for the Lax Gray Tensor Product

The last reformulation of the universal property of the lax Gray tensor product will have the upshot of being short and concise, i. e. we will stuff away all the invertibility properties, and in one aspect it is even better than the extended universal property Definition 1.1.1, because we won't need to restrict morphisms as we did for $\mathrm{LocRARI}(A)$.

Ordinary 2-Categorical Inspiration The base idea here is inspired by the universal property for the *pseudo* Gray tensor product for ordinary 2-categories proved in [BG17] by Bourke and Gurski. So placing ourselves in the realm of ordinary 2-categories for this paragraph, let us recall that Gray originally gave his construction of the *ordinary* lax Gray tensor product of ordinary 2-categories by generators and relations in [Gra74]. He also defined the *pseudo* version of his Gray tensor product by forcing its lax commutativity witnesses inside the squares

$$\begin{tikzcd}
(a_0, b_0) \arrow[r, "(\mathrm{id}_{a_0}, g)"] \arrow[d, "(f, \mathrm{id}_{b_0})"'] & (a_0, b_1) \arrow[d, "(f, \mathrm{id}_{b_1})"] \\
(a_1, b_0) \arrow[r, "(\mathrm{id}_{a_1}, g)"'] \arrow[ur, phantom, "\Nearrow \gamma_{f,g}"] & (a_1, b_1)
\end{tikzcd} \tag{3.4.3.1}$$

to be 2-isomorphisms. This pseudo Gray tensor product then becomes 2-equivalent to the cartesian product for ordinary 2-categories, and is the cofibrant replacement of the cartesian product in the folk model structure on ordinary 2-categories [Lac02].

In [BG17], Bourke and Gurski found a clean universal property for this pseudo version by characterising it as a certain unique interpolation between two monoidal structures on the ordinary category of ordinary 2-categories. More precisely, as a starting point they chose the boundaries of the above squares Equation 3.4.3.1, with *no* filling commutativity data and assemble them into an ordinary 2-category $A\Box B$ which can be rigorously constructed as the ordinary pushout of ordinary 2-categories

$$\begin{tikzcd}
\mathrm{Ob}(A)\times\mathrm{Ob}(B) \arrow[r] \arrow[d] & \mathrm{Ob}(A)\times B \arrow[d] \arrow[ddr, bend left] & \\
A\times\mathrm{Ob}(B) \arrow[r] \arrow[drr, bend right] & A\Box B \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!"] & \\
& & A\times B
\end{tikzcd}$$

together with its uniquely induced 2-functor to the strict cartesian product $A \times B$, and where $\mathrm{Ob}(-)$ denotes the *set* of objects. One may think of $A\Box B$ as the version of $A \times B$ where we simply forget that the above squares Equation 3.4.3.1were commutative, and it is called the funny tensor product. Bourke and Gurski now realised in [BG17], that the pseudo Gray tensor product enjoys the universal property of being the initial factorisation

$$A\Box B \to A \otimes_{\mathrm{pseudo}} B \to A \times B$$

sitting between $A \times B$ and $A\Box B$ whose second 2-functor is Hom-locally fully-faithful.

The first essential observation of Bourke and Gurski here is that the Hom-local fully-faithfullness induces the weak commutativity 2-isomorphisms $\gamma_{f,g}$ in Equation 3.4.3.1. Concretely, by looking at the two 1-morphisms $(\mathrm{id}, g) \circ (f, \mathrm{id})$ and $(f, \mathrm{id}) \circ (\mathrm{id}, g)$ through the fully-faithful 1-functor

$$(A \otimes_{\mathrm{pseudo}} B)((a_0, b_0), (a_1, b_1)) \to A(a_0, a_1) \times B(b_0, b_1) \tag{3.4.3.2}$$

we may always lift the identity $(f, g) = (f, g)$ in the target to an isomorphism

$$\gamma_{f,g} \colon (\mathrm{id}, g) \circ (f, \mathrm{id}) \cong (f, \mathrm{id}) \circ (\mathrm{id}, g)\ .$$

The second observation here is that any factorisation $A\Box B \to A \otimes_{\mathrm{pseudo}} B \to A \times B$ automatically provides us with two sections of Equation 3.4.3.2, given by $(-, \mathrm{id}_{b_1}) \circ (\mathrm{id}_{a_0}, -)$ and $(\mathrm{id}_{a_1}, -) \circ (-, \mathrm{id}_{b_0})$. In particular Equation 3.4.3.2 is now fully-faithful and surjective, hence an equivalence, and these Hom-sections become inverses. Equivalently, one can now also rephrase the Hom-local fully-faithfulness of the second 2-functor in such a factorisation

$$A\Box B \to A \otimes_{\mathrm{pseudo}} B \to A \times B$$

into asking for either the canonical Hom-section $(-, \mathrm{id}_{b_1}) \circ (\mathrm{id}_{a_0}, -)$ or $(\mathrm{id}_{a_1}, -) \circ (-, \mathrm{id}_{b_0})$ to be inverse equivalences to Equation 3.4.3.2.

Guided by Definition 3.4.1, we now laxify Bourke and Gurski's idea by asking $(-, \mathrm{id}_{b_1})\circ(\mathrm{id}_{a_0}, -)$ to be just a *right adjoint section.*

Back in the realm of higher category theory we consider again underlying spaces of objects instead of object sets, and define the funny tensor product of 2-categories A and B to be the pushout of the span contained in the square

$$\begin{tikzcd}
A^{\simeq}\times B^{\simeq} \arrow[r] \arrow[d] & A^{\simeq}\times B \arrow[d] \\
A\times B^{\simeq} \arrow[r] & A\times B\,,
\end{tikzcd}$$

whose cogap map induces the canonical 2-functor $\phi\colon A \times B^{\simeq} \underset{A^{\simeq}\times B^{\simeq}}{\cup} A^{\simeq} \times B \to A \times B$ to the cartesian product of 2-categories.

Definition 3.4.13

Let A and B be two 2-categories. A commutative triangle of 2-functors

$$\begin{tikzcd} A \times B^{\simeq} \underset{A^{\simeq}\times B^{\simeq}}{\cup} A^{\simeq} \times B \arrow[r, "U"] \arrow[dr, "\phi"'] & X \arrow[d, "F"] \\ & A \times B \end{tikzcd}$$

is said to be locally Gray right adjoint if for every pair of objects $a, \tilde{a}$ of $A^{\simeq}$ and $b, \tilde{b}$ of $B^{\simeq}$, the induced section of

$$F_{U(a,b),U(\tilde{a},\tilde{b})}\colon X(U(a,b), U(\tilde{a},\tilde{b})) \to (A \times B)(F(U(a,b)), F(U(\tilde{a},\tilde{b}))) \simeq A(a,\tilde{a}) \times B(b,\tilde{b})$$

given by $U(-, \mathrm{id}_{\tilde{b}}) \circ U(\mathrm{id}_a, -)$ constitutes a right adjoint section. We define $\mathrm{LocGrayRAdj}(A, B)$ to be the full subcategory of ${}^{\phi/}(2\text{-}\mathrm{Cat}_{/(A\times B)})$ on those objects that are locally Gray right adjoint. We will call an initial object the lax Gray tensor product of A and B, or more precisely say that it satisfies the funny universal property of the lax Gray product.

As the first step towards the comparison of this definition with Definition 3.4.1, we want to make the claim that we don't need to restrict the morphisms like in $\mathrm{LocGrayRAdj}(A, B)$ to the ones preserving the adjointability properties of the objects, i. e. we didn't restrict to morphisms inducing adjointable triangles on Hom-1-categories.

The reason is that by asking our Hom-adjoint sections to be of this precise form, i. e. built from data we get from the 2-functors $U\colon A \times B^{\simeq} \underset{A^{\simeq}\times B^{\simeq}}{\cup} A^{\simeq} \times B \to X$, already forces the potential mates of the Hom-triangles to get preserved. The precise underlying statement for 1-categories here is the content of the following lemma.

Lemma 3.4.14

For a commutative diagram of natural transformations between functors of 1-categories

$$\begin{tikzcd} & D \arrow[dl, "U"'] \arrow[dr, "V"] \arrow[dd, equal] & \\ C \arrow[r, phantom, "\xRightarrow{\varepsilon}"] \arrow[rr, bend right, "H"] \arrow[dr, "F"'] & & E \arrow[l, phantom, "\xLeftarrow{\tilde{\varepsilon}}"] \arrow[dl, "G"] \\ & D & \end{tikzcd}$$

if we have that the two natural transformations $\varepsilon, \tilde{\varepsilon}$ to the identity in the back upgrade to an adjunction, then we get that the lower triangle is automatically adjointable. More precisely, the invertible natural transformation filling the upper triangle will be equivalent to the mate of the lower triangle's natural transformation under the adjunctions.

Proof. We may paste the unit $\tilde{\eta}\colon \mathrm{id} \Rightarrow VG$ onto the lower right corner of both domain and codomain of the invertible 3-morphism

$$
\begin{tikzcd}
D \arrow[dr, hook, "U"'] \arrow[ddrr, bend right, equal] & & \\
& C \arrow[r, "H"] \arrow[dr, "F"'] \arrow[d, phantom, "\Downarrow\varepsilon"] & E \arrow[d, phantom, "\Leftarrow \simeq\alpha"] \arrow[dl, "G"] \\
& & D
\end{tikzcd}
\simeq
\begin{tikzcd}
& D \arrow[dl, hook, "U"'] \arrow[dr, "V"] \arrow[ddrr, bend left, equal] \arrow[d, phantom, "\Rightarrow \simeq\beta"] & & \\
C \arrow[rr, "H"'] & {} & E \arrow[dr, "G"'] \arrow[r, phantom, "\Rightarrow\widetilde{\varepsilon}"] & {} \\
& & & D
\end{tikzcd}
$$

and employ the triangle identity of $G \dashv V$ to cancel out $\widetilde{\varepsilon}$ on the right hand side. ☐

The following statement now proves all our morphisms in $\mathrm{LocGrayRAdj}(A, B)$ as automatically Hom-triangle adjointable, as state above.

Lemma 3.4.15

In Definition 3.4.13 of $\mathrm{LocGrayRAdj}(A, B)$, the morphisms automatically acquire the Hom-category triangle adjointabilities. More precisely, for every morphism between objects which are locally Gray right adjoint

$$
\begin{tikzcd}
& A \times B^{\simeq} \underset{A^{\simeq} \times B^{\simeq}}{\cup} A^{\simeq} \times B \arrow[dl, "U"'] \arrow[dr, "V"] \arrow[dd, "\phi"'] & \\
X \arrow[rr, "H"] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\
& A \times B &
\end{tikzcd}
$$

we get for free that for all pairs of objects $a, \widetilde{a}$ in $A^{\simeq}$ and $b, \widetilde{b}$ in $B^{\simeq}$ the triangle of actions of the 2-functors on Hom-1-categories

$$
\begin{tikzcd}
X((a,b),(\widetilde{a},\widetilde{b})) \arrow[rr, "H"] \arrow[dr, "F"'] & & Y((a,b),(\widetilde{a},\widetilde{b})) \arrow[dl, "G"] \\
& A(a,\widetilde{a}) \times B(b,\widetilde{b}) &
\end{tikzcd}
$$

is vertically adjointable.

Proof. From the extra 2-functor $U \colon A \times B^{\simeq} \underset{A^{\simeq} \times B^{\simeq}}{\cup} A^{\simeq} \times B \to X$ that objects of $\mathrm{LocGrayRAdj}(A, B)$ have, we in particular get commutative diagrams

$$
\begin{tikzcd}
& A(a,\widetilde{a}) \times \{\mathrm{id}_{\widetilde{b}}\} \arrow[dl, "U(-{,}\mathrm{id}_{\widetilde{b}})"'] \arrow[dr, "V(-{,}\mathrm{id}_{\widetilde{b}})"] \arrow[dd, hook] & \\
X((a,\widetilde{b}),(\widetilde{a},\widetilde{b})) \arrow[rr, "H"] \arrow[dr, "F"'] & & Y((a,\widetilde{b}),(\widetilde{a},\widetilde{b})) \arrow[dl, "G"] \\
& A(a,\widetilde{a}) \times B(\widetilde{b},\widetilde{b}) &
\end{tikzcd}
$$

and similiar diagrams for $U(\mathrm{id}_a, -)$ and $V(\mathrm{id}_a, -)$ Combining these two and using 2-functoriality we get a commutative prism

$$\begin{tikzcd}
 & A(a,\widetilde{a}) \times B(b,\widetilde{b}) \arrow[dl, "U(-{,}\mathrm{id}_{\widetilde{b}})\circ U(\mathrm{id}_a{,}-)"'] \arrow[dr, "V(-{,}\mathrm{id}_{\widetilde{b}})\circ V(\mathrm{id}_a{,}-)"] \arrow[dd, equal] & \\
X((a,b),(\widetilde{a},\widetilde{b})) \arrow[rr, "H"'] \arrow[dr, "F"'] & & Y((a,b),(\widetilde{a},\widetilde{b})) \arrow[dl, "G"] \\
 & A(a,\widetilde{a}) \times B(b,\widetilde{b}) &
\end{tikzcd}$$

to which we may by our assumptions of being locally Gray right adjoint apply Lemma 3.4.14. $\square$

To use this automatic Hom-triangle adjointablity in the comparison of Definition 3.4.1 and Definition 3.4.13 later we give the following global reformulation.

Corollary 3.4.16

The above Lemma 3.4.15 can be given the following more coherent form. Let us define the pullback

$$\begin{tikzcd}
{}^{\gamma/}\,\mathrm{LocGrayRAdj}(A,B) \arrow[r, "\simeq", "\text{forget}"'] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{LocGrayRAdj}(A,B) \arrow[d, hook] \\
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}({}^{\gamma/}(\text{2-Cat}_{/A\times B})) \arrow[r, "\simeq", "\text{forget}"'] & {}^{\phi/}(\text{2-Cat}_{/A\times B})\,.
\end{tikzcd}$$

Then we have the dashed fully-faithful factorisation

$$\begin{tikzcd}
{}^{\gamma/}\,\mathrm{LocGrayRAdj}(A,B) \arrow[r, hook, dashed, "\exists!"] \arrow[d, hook] & \mathrm{LocRARI}(A\times B) \arrow[d] \\
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}({}^{\gamma/}(\text{2-Cat}_{/A\times B})) \arrow[r, "\text{forget}"] & {}^{\gamma/}(\text{2-Cat}_{/A\times B})
\end{tikzcd}$$

by Lemma 3.4.15.

The final technical ingredient we need for the comparison Proposition 3.4.19 is that our pullback functors $M^*\colon {}^{\gamma_L/}(\text{2-Cat}_{/L}) \to {}^{\gamma_K/}(\text{2-Cat}_{/K})$ for 2-functors $M\colon K \to L$ actually have left adjoints, whenever $M^{\simeq}$ is a equivalence.

To this end we employ the following general 1-categorical statement saying that adjunctions also induce adjunctions on the appropriate slice categories.

Lemma 3.4.17

For a left adjoint $F\colon C \to D$ between 1-categories with right adjoint U and unit $\eta\colon \mathrm{id} \Rightarrow UF$ we get that for every object c of C, the induced functor of F on slices under c

$$^{c/}F\colon {}^{c/}C \to {}^{F(c)/}D$$

is again a left adjoint with right adjoint ${}^{\eta/}U$ given by

$$^{F(c)/}D \xrightarrow{^{F(c)/}U} {}^{U(F(c))/}C \xrightarrow{\ast\eta_c} {}^{c/}C\ .$$

Proof. We need to produce a natural equivalence from

$$^{F(c)/}D((^{c/}F)(\check{c}, f), (d, g)) \simeq D(F(\check{c}), d) \times^{({}_*F(f), \mathrm{incl})}_{D(F(c),d)} \{g\}$$

to

$$^{c/}C((\check{c}, f), (^{\eta/}U)(d, g)) \simeq C(\check{c}, U(d)) \times^{({}_*f, \mathrm{incl})}_{C(c,U(d))} \{U(g) \circ \eta_c\} .$$

This can be done by considering the equivalence on cospans

$$\begin{tikzcd}
D(F(\check{c}), d) \arrow[r, "{}_*F(f)"] \arrow[d, "\simeq"] & D(F(c), d) \arrow[d, "\simeq"] & \{g\} \arrow[l, "\mathrm{incl}"'] \arrow[d, "\simeq"'] \\
C(\check{c}, U(d)) \arrow[r, "{}_*f"'] & C(c, U(d)) & \{U(g) \circ \eta_c\} \arrow[l, "\mathrm{incl}"]
\end{tikzcd}$$

and taking its induced equivalence on pullbacks. □

We are now equipped to prove that the desired left adjoints of our pullback functors exists in the situation where we don't change the underlying space of objects.

Corollary 3.4.18

For a 2-functor $M \colon K \to L$ which induces an equivalence on underlying spaces, the postcompose with M pullback along M adjunction

$$\begin{tikzcd}
\text{2-Cat}_{/L} \arrow[r, "M^*"'] \arrow[r, phantom, shift left=2, "\perp"] & \text{2-Cat}_{/K} \arrow[l, bend right, "M_!"']
\end{tikzcd}$$

also induces a left adjoint $^{\gamma_K/}M_!$ of the pullback functor

$$^{\eta/}(M^*) \colon {}^{\gamma_L/}(\text{2-Cat}_{/L}) \to {}^{(L^{\simeq} \times_L K \to K)/}(\text{2-Cat}_{/K}) \xrightarrow{{}_*\eta_{\gamma_K}} {}^{\gamma_K/}(\text{2-Cat}_{/K}) .$$

Proof. We apply Lemma 3.4.17 to the adjunction

$$\begin{tikzcd}
\text{2-Cat}_{/L} \arrow[r, "M^*"'] \arrow[r, phantom, shift left=2, "\perp"] & \text{2-Cat}_{/K} \arrow[l, bend right, "M_!"']
\end{tikzcd}$$

sliced under the object $\gamma_K \colon K^{\simeq} \to K$ and get an adjunction as on the right in

$$\begin{tikzcd}
{}^{(L^{\simeq} \to L)/}(\text{2-Cat}_{/L}) \arrow[r, phantom, "\simeq"] & {}^{(K^{\simeq} \to K \to L)/}(\text{2-Cat}_{/L}) \arrow[r, "^{\eta/}(M^*)"'] \arrow[r, phantom, shift left=2, "\perp"] & {}^{(K^{\simeq} \to K)/}(\text{2-Cat}_{/K}) , \arrow[l, bend right, "^{\gamma_K/}M_!"']
\end{tikzcd}$$

which we can extend to an adjunction to the category on the left as we have by assumption the equivalence

$$\begin{tikzcd}
K^{\simeq} \arrow[r, "M^{\simeq}", "\simeq"'] \arrow[d] & L^{\simeq} \arrow[d] \\
K \arrow[r, "M"] & L .
\end{tikzcd}$$

□

The second main result of this chapter is the following comparison of the localised universal property of the lax Gray tensor product of A and B and its reformulation in terms of its interpolation Definition 3.4.13 between the funny and the cartesian products of 2-categories.

Proposition 3.4.19

For every two 2-categories A and B the 1-categories $\mathrm{LocGrayRAdj}(A,B)$ and $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$ are canonically equivalent, hence also resulting in equivalent universal properties for the lax Gray tensor product.

Proof. 1. In the first step of this proof we enhance $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$ until it becomes comparable to $\mathrm{LocGrayRAdj}(A,B)$. Applying Lemma 3.4.9 to $A^{\simeq}\times B^{\simeq}$, $A\times B^{\simeq}$ and $A^{\simeq}\times B$ we can pullback the equivalences of Corollary 3.3.7 like in the $A^{\simeq}\times B$ case

$$\begin{tikzcd}
{}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[r, "\simeq"] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & \mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[d, "(A^{\simeq}\times B\to A\times B)^*"] \\
& \mathrm{LocRARI}^{\mathrm{Gray}}(A^{\simeq},B) \arrow[d, hook] \\
{}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}\mathrm{LocRARI}_{\mathrm{strict}}(A^{\simeq}\times B) \arrow[r, "\simeq"'] & \mathrm{LocRARI}_{\mathrm{strict}}(A^{\simeq}\times B)
\end{tikzcd} \tag{3.4.3.3}$$

to deduce equivalences as the top horizontal one in the above pullback. Note here that we also have a morphism of pullbacks to the definitional pullback

$$\begin{tikzcd}
{}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}({}^{\gamma/}(\text{2-Cat}_{/A\times B})) \arrow[r, "\mathrm{forget}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {}^{\gamma/}(\text{2-Cat}_{/A\times B}) \arrow[d, "(A^{\simeq}\times B\to A\times B)^*"] \\
{}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}({}^{\gamma/}(\text{2-Cat}_{/A^{\simeq}\times B})) \arrow[r, "\mathrm{forget}"'] & {}^{\gamma/}(\text{2-Cat}_{/A^{\simeq}\times B})\,,
\end{tikzcd}$$

whose transition functors are non-full subcategory inclusions, thus we get an induced non-full subcategory inclusion

$${}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \to {}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}({}^{\gamma/}(\text{2-Cat}_{/A\times B}))$$

of exactly those objects and morphisms which, after forgetting appropriately, induce objects and morphisms of $\mathrm{LocRARI}^{\mathrm{Gray}}(A,B)$. By using the total Hom-space equivalence Lemma A.2.11 for the adjunctions from Corollary 3.4.18, the above pullback category is equivalent to the slice-under category ${}^{(A^{\simeq}\times B,\gamma_A\times\mathrm{id}_B,\gamma)/}({}^{\gamma/}(\text{2-Cat}_{/A\times B}))$ of objects

$$\begin{tikzcd}
A^{\simeq}\times B^{\simeq} \arrow[dr, "\gamma"'] \arrow[drr] \arrow[ddrr, bend right, "\gamma"'] & & \\
& A^{\simeq}\times B \arrow[r] \arrow[dr, "\gamma_A\times\mathrm{id}_B"'] & X \arrow[d] \\
& & A\times B
\end{tikzcd}$$

and their obvious morphisms. Let us denote the composite non-full subcategory inclusion

$$\begin{aligned}
{}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) &\to {}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}({}^{\gamma/}(\text{2-Cat}_{/A\times B})) \\
&\xrightarrow{\simeq} {}^{(A^{\simeq}\times B,\gamma_A\times\mathrm{id}_B,\gamma)/}({}^{\gamma/}(\text{2-Cat}_{/A\times B}))
\end{aligned}$$

by $i_{A^{\simeq}\times B}$. Putting the top horizontal equivalences of Equation 3.4.3.3 for all three combinations $A^{\simeq}\times B^{\simeq}$, $A\times B^{\simeq}$ and $A^{\simeq}\times B$ together we can deduce that the dashed functor in the following top definitional pullback square

$$\begin{tikzcd}
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[r] \arrow[d] \arrow[dd, bend right, dashed, "\simeq"'] \arrow[dr, phantom, "\lrcorner" very near start] & {}^{(A\times B^{\simeq},\mathrm{id},\gamma)/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[d] \arrow[ddl, "\simeq"'] \\
{}^{(A^{\simeq}\times B,\mathrm{id},\gamma)/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[r] \arrow[d, "\simeq"] & {}^{(A^{\simeq}\times B^{\simeq},\mathrm{id},\gamma)/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[dl, "\simeq"] \\
\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) &
\end{tikzcd}$$

is also an equivalence. Now our compatible non-full subcategory inclusions $i_{A^{\simeq}\times B}$, $i_{A\times B^{\simeq}}$ and $i_{A^{\simeq}\times B^{\simeq}}$ induce for us a morphism of pullbacks from the above one to the pullback

$$\begin{tikzcd}
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}({}^{\gamma/}(2\text{-Cat}_{/A\times B})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & {}^{(A\times B^{\simeq},\mathrm{id}_A\times\gamma_B,\gamma)/}({}^{\gamma/}(2\text{-Cat}_{/A\times B})) \arrow[d] \\
{}^{(A^{\simeq}\times B,\gamma_A\times\mathrm{id}_B,\gamma)/}({}^{\gamma/}(2\text{-Cat}_{/A\times B})) \arrow[r] & {}^{(A^{\simeq}\times B^{\simeq},\gamma_A\times\gamma_B,\gamma)/}({}^{\gamma/}(2\text{-Cat}_{/A\times B}))
\end{tikzcd}$$

and we get an induced non-full subcategory inclusion

$$\begin{tikzcd}
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[d, "i"] \\
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}({}^{\gamma/}(2\text{-Cat}_{/A\times B}))\,.
\end{tikzcd}$$

2. Now we can assemble all our constructed functors into the diagram

$$\begin{tikzcd}
\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) & \mathrm{LocGrayRAdj}(A,B) \\
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}\mathrm{LocRARI}^{\mathrm{Gray}}(A,B) \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[d, "i"'] & {}^{\gamma/}\mathrm{LocGrayRAdj}(A,B) \arrow[u, "\simeq"'] \arrow[dl, hook] \\
{}^{(A^{\simeq}\times B \underset{A^{\simeq}\times B^{\simeq}}{\cup} A\times B^{\simeq},\phi,(\gamma,\gamma))/}({}^{\gamma/}(2\text{-Cat}_{/A\times B})) &
\end{tikzcd}$$

and we see that, in order to get an equivalence from the top left corner to the top right corner, it suffices to show that the dashed subcategory factorisation over the diagonal full subcategory inclusion first of all exists, and second of all that it is an equivalence. By property 5 from Definition 3.4.1, we can deduce that the local Gray right adjoint sections are in fact equivalent to the functors $U(-,\mathrm{id}_{\tilde{b}})\circ U(\mathrm{id}_a,-)$ which also agree with the canonical functors out of $A\times B^{\simeq}\underset{A^{\simeq}\times B^{\simeq}}{\cup} A^{\simeq}\times B$ each object gets via the top left vertical equivalence in the above diagram. More precisely, we have the canonical comparison

$$\begin{tikzcd}
A(a,\tilde{a}) \times B(b,\tilde{\tilde{b}}) \arrow[r, "U(-,\mathrm{id}_{\tilde{b}})\circ U(\mathrm{id}_a,-)"] \arrow[dr, equal, "\simeq"] & X((a,b),(\tilde{a},\tilde{\tilde{b}})) \arrow[d, "F"'] \arrow[dr, equal] \arrow[d, phantom, shift left=4ex, "\Swarrow\eta"] & \\
& A(a,\tilde{a}) \times B(b,\tilde{\tilde{b}}) \arrow[r, hook, "U_{(a,b),(\tilde{a},\tilde{b})}"'] & X((a,b),(\tilde{a},\tilde{\tilde{b}}))
\end{tikzcd}$$

and we know that it is componentwise invertible by property 5. Thus we get that the dashed factorisation exists. As we now factored our non-full subcategory inclusion i over this full subcategory inclusion we get for free that our dashed functor is on Hom-spaces fully-faithful.

3. Next we need to show essential surjectivity on Hom-spaces. But by construction our essential image on Hom-spaces are exactly those morphisms which, after forgetting to ${}^{\gamma/}(2\text{-Cat}_{/A\times B})$, are those coming from $\mathrm{LocRARI}(A\times B)$. However, by Corollary 3.4.16 we already know that morphisms between objects of ${}^{\gamma/}\,\mathrm{LocGrayRAdj}(A,B)$ induce such after forgetting to $\mathrm{LocRARI}(A\times B)$.
4. Now all that is left to prove is essential surjectivity of our dashed functor. To see essential surjectivity, we just need to show that objects of $\mathrm{LocGrayRAdj}(A,B)$ also admit local Gray right adjoint sections. The local right adjoint sections are there by definition, but the unit invertibilities can be deduced in the following way. For an adjunction of 1-categories $F \dashv U$ and an object d of the left adjoint's codomain D we get the following pulled back adjunction.

$$\begin{tikzcd}
C_d \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & C \arrow[d, "F"'] \\
\Delta^0 \arrow[r, "d"'] \arrow[u, bend left=60, "U(d)", "\vdash"'] & D \arrow[u, bend right=60, "U"', "\dashv"]
\end{tikzcd}$$

In particular in this situation we have contractibility of the Hom-spaces $C_d(c, U(d)) \simeq *$ for objects c in the fiber C_d. For any object c of C we thus get that the unit morphism $\eta_c\colon c \to UFc$ has to be the unique object of $C_{F(c)}(c, UFc) \simeq *$. If we now choose for the adjunction the appropriate Hom-adjunctions and for the object c we take

(a) $\mathrm{id}_{U(a,b)}$ in $X((a,b),(\tilde{a},\tilde{\tilde{b}}))$ then we have by 2-functoriality of $A\times B^{\simeq} \underset{A^{\simeq}\times B^{\simeq}}{\cup} A^{\simeq}\times B \to X$ a canonical invertible 2-morphism $\mathrm{id}_{U(a,b)} \simeq U(\mathrm{id},\mathrm{id}) \circ U(\mathrm{id},\mathrm{id})$,

(b) $(U(\tilde{f},\mathrm{id}) \circ U(\mathrm{id},\mathrm{id})) \circ (U(f,\mathrm{id}) \circ U(\mathrm{id},\mathrm{id}))$ we have by 2-functoriality the invertible composition $U(\tilde{f},\mathrm{id})\circ U(\mathrm{id},\mathrm{id})\circ U(f,\mathrm{id})\circ U(\mathrm{id},\mathrm{id}) \simeq U(\tilde{f}\circ f,\mathrm{id}) \simeq U(\tilde{f}\circ f,\mathrm{id})\circ U(\mathrm{id},\mathrm{id})$,

(c) $(U(\mathrm{id},\mathrm{id})\circ U(\mathrm{id},\tilde{g},))\circ(U(\mathrm{id},\mathrm{id})\circ U(\mathrm{id},g))$ we have by 2-functoriality the invertible composition $(U(\mathrm{id},\mathrm{id})\circ U(\mathrm{id},\tilde{g},))\circ(U(\mathrm{id},\mathrm{id})\circ U(\mathrm{id},g)) \simeq U(\mathrm{id},\tilde{g}\circ g) \simeq U(\mathrm{id},\mathrm{id})\circ U(\mathrm{id},\tilde{g}\circ g)$,

(d) $(U(f,\mathrm{id}) \circ U(\mathrm{id},\mathrm{id})) \circ (U(\mathrm{id},\mathrm{id}) \circ U(\mathrm{id},g))$ we have by 2-functoriality the canonical invertible composition $(U(f,\mathrm{id}) \circ U(\mathrm{id},\mathrm{id})) \circ (U(\mathrm{id},\mathrm{id}) \circ U(\mathrm{id},g)) \simeq U(f,\mathrm{id}) \circ U(\mathrm{id},g)$,

which have to agree with the unit 2-morphisms be contractibility, hence proving these units to be invertible.

□

Remark 3.4.20

To close this section, let us comment on the asymmetry of our laxification of Bourke and Gurski's universal property from [BG17] as discussed above. As already explained in the introduction of this chapter, the 2-functor from the ordinary lax Gray tensor product to the ordinary cartesian product of ordinary 2-categories has both a left adjoint section as well as a right adjoint section

$$\begin{tikzcd}
(A \otimes_{\mathrm{lax}} B)((a,b),(\widetilde{a},\widetilde{b})) \arrow[d, "\lambda" description] \\
A(a,\widetilde{a}) \times B(b,\widetilde{b})\,. \arrow[u, bend left=60, "(-,\mathrm{id}_{\widetilde{b}})\circ(\mathrm{id}_a,-)"] \arrow[u, bend right=60, "(\mathrm{id}_{\widetilde{a}},-)\circ(-,\mathrm{id}_b)"'] \arrow[u, phantom, bend left=30, "\dashv"] \arrow[u, phantom, bend right=30, "\dashv"]
\end{tikzcd}$$

The methods developed in this chapter only allow us to compare the 2-morphism localisation $\mathrm{Lax}(A \times B)[W_{\mathrm{Gray}}^{-1}]$ with Definition 3.4.1, Definition 3.4.11 and Definition 3.4.13, but do not seem to immediately imply the left adjointness with the left hand Hom-section $(-, \mathrm{id}_{\widetilde{b}}) \circ (\mathrm{id}_a, -)$, so we will not discuss such a comparison in this work.

Note however, that the left adjointness property ceases to exist, when we stop forcing our restrictions $A \times \{b\} \rightsquigarrow A \otimes_{\mathrm{lax}} B$ and $\{a\} \times B \rightsquigarrow A \otimes_{\mathrm{lax}} B$ to be strict 2-functors, i. e. if we throw out points 2, 3 and 4 in Definition 3.4.1. Ordinary 2-category intuition then dictates the resulting tensor product of ordinary 2-categories to classify to lax 2-functors $A \rightsquigarrow \mathrm{LaxFun}^{\mathrm{lax\ nat}}(B, C)$, instead of strict 2-functors $A \to \mathrm{Fun}_{\mathrm{lax}}(B, C)$, where $\mathrm{LaxFun}^{\mathrm{lax\ nat}}(B, C)$ denotes the 2-category of lax functors, lax natural transformations and their morphisms. This in turn implies that this weird tensor product should be computable as $\mathrm{Lax}(A) \otimes_{\mathrm{lax}} \mathrm{Lax}(B)$ and its canonical 2-functor $\mathrm{Lax}(A) \otimes_{\mathrm{lax}} \mathrm{Lax}(B) \to A \times B$ does *not* admit a left adjoint section, as the case $A = \Delta^0 = B$ shows. So as soon as we generalise Gray-type tensor products from strict 2-functors to lax 2-functors, right adjoint sections are all one can hope for.

Appendix A

Background on Synthetic 1-Category Theory

In this appendix we recall some fundamental 1-categorical definitions and prove the 1-categorical statements that the previous sections proofs relied on, namely Corollary 1.2.9, Corollary 1.2.10, Lemma 1.2.13, Proposition 1.3.3, Corollary 1.3.4, Lemma 1.3.5, and Lemma 1.2.11.

In this exposition we try to keep the proofs as elementary and model-independent as possible, based on the definitions and facts we chose to base ourselves on, which are recorded in the first section. This is partly, because we could not find elementary and model-independent proofs for the statements Corollary 1.2.10, Lemma 1.2.13, Lemma 1.3.5, and Lemma 1.2.11. We do not claim any originality for these statements or their proofs. The statement Proposition 1.3.3 can already be deduced from the dual of [Lur17, Proposition 7.3.2.6]. For its extension Corollary 1.3.4 and the directed lifting property of left adjoints against cocartesian fibrations Corollary 1.2.9, we could not at all find a reference in the literature.

A.1 Definitions and Basic Properties

We start by reviewing some definitions and facts that we need for the 1-categorical statements in the previous sections, like adjunctions, (co)cartesian fibrations, (co)cartesian functors between those and the adjointability of commutative squares.

Definition A.1.1

An adjunction $F \dashv U$ between functor $F\colon A \to B$, called the left adjoint, and $U\colon B \to A$, called the right adjoint, consists of two natural transformations $\eta\colon \mathrm{id} \Rightarrow UF$ and $\varepsilon\colon FU \Rightarrow \mathrm{id}$ and two identifications $\mathrm{id}_U \simeq (U\varepsilon) \circ (\eta U)$ and $\mathrm{id}_F \simeq (\varepsilon F) \circ (F\eta)$, called triangle identities. When the unit η is invertible we say that the left adjoint F is fully-faithful. Dually, when the counit ε is invertible we say that the right adjoint U is fully-faithful.

Definition A.1.2

A functor $p\colon X \to B$ is a cocartesian fibration if the functor

$$(\mathrm{ev}_0, p_*)\colon\ \mathrm{Fun}(\Delta^1, X) \to X \times_B \mathrm{Fun}(\Delta^1, B)$$

has a fully-faithful left adjoint. Dually, it is a cartesian fibration if the functor

$$(p_*, \mathrm{ev}_1)\colon \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\Delta^1, B) \times_B X$$

has a fully-faithful right adjoint.

Definition A.1.3

Let

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "V"'] & A \arrow[d, "U"] \\ Y \arrow[r, "K"'] & B \end{tikzcd}$$

be a commutative square for which both the functors U and V are right adjoints. From the data of these two adjunctions, more precisely the counit ε' of the adjunction $G \dashv V$ and the unit η of the adjunction $F \dashv U$, we can form the composed natural transformation

$$\begin{tikzcd} Y \arrow[r, "G"] \arrow[dr, equal] \arrow[dr, phantom, "\varepsilon'"] & X \arrow[r, "H"] \arrow[d, "V"] & A \arrow[d, "U"'] \arrow[dr, equal] \arrow[dr, phantom, "\eta"] & \\ & Y \arrow[r, "K"'] & B \arrow[r, "F"'] & A \end{tikzcd}$$

which is called the vertical mate of this commutative square. Such a commutative square in which both vertical functors are right adjoints is called vertically adjointable if its vertical mate transformation is invertible.

Definition A.1.4

Let

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ A \arrow[r, "G"'] & B \end{tikzcd}$$

be a commutative square such that p and q are cocartesian fibrations. We call the functor F a cocartesian functor over G if the induced commutative square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, X) \arrow[r, "F_*"] \arrow[d, "(\mathrm{ev}_0, p_*)"'] & \mathrm{Fun}(\Delta^1, Y) \arrow[d, "(\mathrm{ev}_0, q_*)"] \\ X \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "F \times_G G_*"'] & Y \times_B \mathrm{Fun}(\Delta^1, B) \end{tikzcd}$$

is vertically adjointable.

We record the following facts about the arrow categories $\mathrm{Fun}(\Delta^1, A)$ and their domain/codomain-fibrations $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, A) \to A$ for later reference.

Proposition A.1.5

For every category A we have

1. $\mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, A) \to A$ is a cocartesian fibration.
2. $\mathrm{ev}_0\colon \mathrm{Fun}(\Delta^1, A) \to A$ is a cartesian fibration.
3. The fibers of $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}(\Delta^1, A) \to A \times A$ are spaces, or equivalently $(\mathrm{ev}_0, \mathrm{ev}_1)$ is a conservative functor.
4. The functor $\Delta_A := (\Delta^1 \to \Delta^0)_*\colon A \simeq \mathrm{Fun}(\Delta^0, A) \to \mathrm{Fun}(\Delta^1, A)$ is both a section to ev_0 and ev_1 and the invertible natural transformations $\mathrm{ev}_1 \circ \Delta_A \simeq \mathrm{id}$ and $\mathrm{id} \simeq \Delta_A \circ \mathrm{ev}_1$ witnessing this are counit, respectively unit for adjunctions $\mathrm{ev}_1 \dashv \Delta_A \dashv \mathrm{ev}_0$.
5. Furthermore, the unit $\mathrm{id} \Rightarrow \Delta_A \circ \mathrm{ev}_1$ is the ev_0-cartesian lift of $\Delta_A \circ \mathrm{ev}_1$ along the canonical natural transformation $\mathrm{ev}_0 \Rightarrow \mathrm{ev}_1$. More precisely the composite
$$\mathrm{Fun}(\Delta^1, A) \xrightarrow{(\mathrm{id}, \Delta_A \mathrm{ev}_1)} \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \xhookrightarrow{\mathrm{cart}_{\mathrm{ev}_0}} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, A)) \quad \text{(A.1.0.1)}$$
is equal to the unit.
6. Dually, the counit $\Delta_A \circ \mathrm{ev}_0 \Rightarrow \mathrm{id}$ is the ev_1-cocartesian lift of $\Delta_A \circ \mathrm{ev}_0$ along the canonical natural transformation $\mathrm{ev}_0 \Rightarrow \mathrm{ev}_1$, i. e. the composite
$$\mathrm{Fun}(\Delta^1, A) \xrightarrow{(\Delta_A \mathrm{ev}_0, \mathrm{id})} \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \xhookrightarrow{\mathrm{cocart}_{\mathrm{ev}_1}} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, A)) \quad \text{(A.1.0.2)}$$
is equal to the counit.
7. The cocartesian pushforward $\mathrm{cocart}_{\mathrm{ev}_1}\colon \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, A))$ of ev_1 is ev_0-vertical, i. e. factors over $\Delta_A\colon A \hookrightarrow \mathrm{Fun}(\Delta^1, A)$ when postcomposed with $(\mathrm{ev}_0)_*$.
8. The cartesian pullback functor $\mathrm{cart}_{\mathrm{ev}_0}\colon \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, A))$ of ev_0 is ev_1-vertical, i. e. factors over $\Delta_A\colon A \hookrightarrow \mathrm{Fun}(\Delta^1, A)$ when postcomposed with $(\mathrm{ev}_1)_*$.

Proof. All the above statements can be justified by inserting certain adjunctions between Δ^0, Δ^1, Δ^2 and $\Delta^1 \times \Delta^1$ into $\mathrm{Fun}(-, A)$ and using the canonical pushout identifications $\Delta^2 \simeq \Delta^1 \cup_{\Delta^0} \Delta^1$ and $\Delta^1 \times \Delta^1 \simeq \Delta^2 \cup_{\Delta^1} \Delta^2$, i. e. that the square can be written as two commutative triangles glued along their composite edges. The first statement follows from looking at the adjunction

$$\begin{tikzcd} \Delta^1 \arrow[r, hook, bend left, "{(0,2)}"] \arrow[r, phantom, "\perp"] & \Delta^2 \arrow[l, bend left, "\sigma_0"] \end{tikzcd} \quad \text{(A.1.0.3)}$$

where the right adjoint σ_0 contracts the edge $0 \to 1$, by applying pushout along $(0,2)\colon \Delta^1 \hookrightarrow \Delta^2$ to get

$$\begin{tikzcd} \Delta^2 \simeq \Delta^2 \cup_{\Delta^1} \Delta^1 \arrow[r, hook, bend left, "{(0,2)}"] \arrow[r, phantom, "\perp"] & \Delta^2 \cup_{\Delta^1} \Delta^2 \simeq \Delta^1 \times \Delta^1 \arrow[l, bend left, "\sigma_0"] \end{tikzcd} \quad \text{(A.1.0.4)}$$

The fourth statement can be proven from the inital and terminal object adjunctions

$$\begin{tikzcd} \Delta^0 \arrow[r, hook, bend left, "0"] \arrow[r, hook, bend right, "1"'] & \Delta^1 \arrow[l, "!" description] \end{tikzcd} \quad \text{(A.1.0.5)}$$

for Δ^1. The counit here for the adjunction $0 \dashv !$ is given by the functor $\Delta^1 \times \Delta^1 \to \Delta^1$ contracting the edges $(0,0) \to (0,1)$ and $(0,0) \to (1,0)$. In this sense one can then also deduce the fifth statement by observing that this counit agrees with the composite

$$\Delta^1 \leftarrow \Delta^2 \simeq \Delta^2 \underset{\Delta^1}{\cup} \Delta^1 \leftarrow \Delta^2 \underset{\Delta^1}{\cup} \Delta^2 \simeq \Delta^1 \times \Delta^1,$$

where the leftmost functor contracts the edge $0 \to 1$. All other statements follow similary or by further inspecting these adjunctions. $\square$

Proposition A.1.6

For every functor $F\colon A \to B$ we have that

1. in the commutative square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A) \arrow[r, "F_*"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Fun}(\Delta^1, B) \arrow[d, "\mathrm{ev}_1"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

   the functor F_* is cocartesian over F and
2. in the commutative square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A) \arrow[r, "F_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}(\Delta^1, B) \arrow[d, "\mathrm{ev}_0"] \\ A \arrow[r, "F"'] & B \end{tikzcd}$$

   the functor F_* is cartesian over F.

Proof. This follows from the fact that inserting the simplicial adjunction from the proof of Proposition A.1.5 in the first variable of $\mathrm{Fun}(-,-)$ and observing that this commutes with plugging in F into the second variable of $\mathrm{Fun}(-,-)$. $\square$

Furthermore we the following fact about commutative squares in categories, they can be decomposed into two commutative triangles that agree on their composite edges.

Proposition A.1.7

For any category A the square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, A)) \arrow[r, "((\mathrm{ev}_0)_*, \mathrm{ev}_1)"] \arrow[d, "(\mathrm{ev}_0, (\mathrm{ev}_1)_*)"'] & \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \arrow[d, "\mathrm{comp}"] \\ \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "\mathrm{comp}"'] & \mathrm{Fun}(\Delta^1, A) \end{tikzcd}$$

commutes and is a pullback.

Proof. We use the following canonical pushout square.

$$\begin{tikzcd} \Delta^1 \arrow[r, "02"'] \arrow[d, "02"] & \Delta^2 \arrow[d] \\ \Delta^2 \arrow[r] & \Delta^1 \times \Delta^1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

□

A.2 Adjunctions, Adjointablility and Cocartesian Fibrations

In this section we collect some general statements about adjunctions, cocartesian fibrations and some elementary stability properties of them.

The first statement is about the functor $\mathrm{Fun}(A,-)$ preserving adjoints. Furthermore this adjointability is in a sense preserved when varying A, i. e. we get adjointable squares.

Lemma A.2.1

Let A be a category and $F\colon X \to Y$ a (fully-faithful) left adjoint. Then $\mathrm{Fun}(A,X) \to \mathrm{Fun}(A,Y)$ is again a (fully-faithful) left adjoint. Furthermore, for any functor $H\colon A \to B$, the commutative square

$$\begin{tikzcd} \mathrm{Fun}(B,X) \arrow[r, "\mathrm{Fun}(H,X)"] \arrow[d, "\mathrm{Fun}(B,F)"'] & \mathrm{Fun}(A,X) \arrow[d, "\mathrm{Fun}(A,F)"] \\ \mathrm{Fun}(B,Y) \arrow[r, "\mathrm{Fun}(H,Y)"'] & \mathrm{Fun}(A,Y) \end{tikzcd} \tag{A.2.0.1}$$

is vertically adjointable. The same is true for right adjoints.

Proof. We have the post-whiskering functor

$$\mathrm{Fun}(A,C) \xrightarrow{\alpha_*} \mathrm{Fun}(A,\mathrm{Fun}(\Delta^1,D)) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1,\mathrm{Fun}(A,D))$$

for any natural transformation $C \xrightarrow{\alpha} \mathrm{Fun}(\Delta^1,D)$, hence we can say that $\mathrm{Fun}(A,-)$ preserves natural transformations. In a similar way it can be seen that $\mathrm{Fun}(A,-)$ preserves compositions and identities of natural transformations. Hence it preserves the data of adjunctions. For adjointablility we now just need to observe that we have

$$\mathrm{Fun}(F,D) \circ \mathrm{Fun}(B,\alpha) = \mathrm{Fun}(A,\alpha) \circ \mathrm{Fun}(F,C)$$

which allows us to rewrite the mate of our square into $\mathrm{Fun}(F, U\varepsilon \circ \eta U)$, which is equivalent to the the identity natural transformation because of the triangle identities. □

The following statement is about $\mathrm{Fun}(\Delta^1,-)$ preserving cocartesian fibrations and its evaluations functors ev_0 and ev_1 constituting cocartesian functors.

Corollary A.2.2

Let $p\colon A \twoheadrightarrow I$ be a cocartesian fibration. Then $p_*\colon \mathrm{Fun}(\Delta^1,A) \to \mathrm{Fun}(\Delta^1,I)$ is also a cocartesian fibration and both evaluation functors ev_0 and ev_1

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A) \arrow[r, "\mathrm{ev}_i"] \arrow[d, "p_*"'] & A \arrow[d, "p"] \\ \mathrm{Fun}(\Delta^1, I) \arrow[r, "\mathrm{ev}_i"'] & I \end{tikzcd} \tag{A.2.0.2}$$

are cocartesian functors.

Proof. As $\mathrm{Fun}(\Delta^1, -)$ preserves pullbacks we have the following commutative triangle.

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, A)) \arrow[d, "(\mathrm{ev}_0,(p_*)_*)"'] \arrow[dr, "(\mathrm{ev}_0,p_*)_*"] & \\ \mathrm{Fun}(\Delta^1, A) \times_{\mathrm{Fun}(\Delta^1,I)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, I)) & \mathrm{Fun}(\Delta^1, A \times_I \mathrm{Fun}(\Delta^1, I)) \arrow[l, "\simeq"] \end{tikzcd}$$

Hence the left hand vertical functor has a fully-faithful left adjoint as the right hand diagonal functor has one, because $\mathrm{Fun}(\Delta^1, -)$ preserves having fully-faithful adjoints by Lemma A.2.1. Using $i\colon \Delta^0 \hookrightarrow \Delta^1$ as H in the same lemma also gives us the needed adjointablility of the square witnessing the desired cocartesianness of the evaluation functors ev_i. □

We now construct for every adjunction defined using unit and counit its action on the appropriate Hom-spaces.

Construction A.2.3

For an adjunction with left adjoint $F\colon B \to A$, right adjoint $U\colon A \to B$, unit $\eta\colon \mathrm{id} \Rightarrow UF$ and counit $\varepsilon\colon FU \to \mathrm{id}$ we can construct the following two functors.

$$\phi\colon B \times_A \mathrm{Fun}(\Delta^1, A) \xrightarrow{\eta \times_U (U_*, \mathrm{ev}_1)} \mathrm{Fun}(\Delta^1, B) \times_B \mathrm{Fun}(\Delta^1, B) \times_B A \xrightarrow{\mathrm{comp} \times_{\mathrm{id}} \mathrm{id}} \mathrm{Fun}(\Delta^1, B) \times_B A$$

$$\psi\colon \mathrm{Fun}(\Delta^1, B) \times_B A \xrightarrow{(\mathrm{ev}_0, F_*) \times_F \varepsilon} B \times_A \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \xrightarrow{\mathrm{id} \times_{\mathrm{id}} \mathrm{comp}} B \times_A \mathrm{Fun}(\Delta^1, A)$$

These functors are called the action of the adjunction on Hom-spaces. By construction these two functors commute with the canonical projections $(\mathrm{ev}_0 \circ \mathrm{pr}_0, \mathrm{pr}_1)\colon \mathrm{Fun}(\Delta^1, B) \times_B A \to B \times A$ and $(\mathrm{pr}_0, \mathrm{ev}_1 \circ \mathrm{pr}_1)\colon B \times_A \mathrm{Fun}(\Delta^1, A) \to B \times A$.

These actions on Hom-spaces are in fact inverse to each other, via the triangle identities of the adjunction. We will later prove that having such a parametrized Hom-equivalence is actually equivalent to constituting an adjunction.

Lemma A.2.4

For an adjunction $(F, U, \eta, \varepsilon)$ the two functors ϕ and ψ constructed in Construction A.2.3 are inverse to each other.

Proof. For $\psi \circ \phi \simeq \mathrm{id}$ we start by looking at the composite commutative diagram

$$\begin{tikzcd}
B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[dd, "\mathrm{id} \times_{\mathrm{id}} \mathrm{id} \times ((FU)_*, \varepsilon\, \mathrm{ev}_1)"'] & & B \times_A \operatorname{Fun}(\Delta^1, A) \arrow[ll, "(\mathrm{id}, F\eta, \mathrm{id})"'] \arrow[d, "\eta \times_U (U_*, \mathrm{ev}_1)"] \\
& & \operatorname{Fun}(\Delta^1, B) \times_B \operatorname{Fun}(\Delta^1, B) \times_B A \arrow[dll, "(\mathrm{ev}_0, F_*) \times_F (F_*, \varepsilon)"'] \arrow[d, "\mathrm{comp} \times_{\mathrm{id}} \mathrm{id}"] \\
B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[dd, "\mathrm{id} \times_{\mathrm{id}} \times_{\mathrm{id}} \mathrm{comp}"'] \arrow[drr, "\mathrm{id} \times_{\mathrm{id}} \mathrm{comp} \times_{\mathrm{id}} \mathrm{id}"] & & \operatorname{Fun}(\Delta^1, B) \times_B A \arrow[d, "(\mathrm{ev}_0, F_*) \times_F \varepsilon"] \\
& & B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[d, "\mathrm{id} \times_{\mathrm{id}} \mathrm{comp}"] \\
B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[rr, "\mathrm{id} \times_{\mathrm{id}} \mathrm{comp}"'] & & B \times_A \operatorname{Fun}(\Delta^1, A)
\end{tikzcd}$$

which on the right hand vertical compostition just unpacks the definitions of $\psi \circ \phi$ and realize that the other composite going around the rectangle can be can reformulated like in the diagram

$$\begin{tikzcd}
& B \times_A \operatorname{Fun}(\Delta^1, A) \arrow[d, "(\mathrm{id}, F\eta, \mathrm{id})"'] \\
& B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[dl, "\mathrm{id} \times_{\mathrm{id}} \mathrm{id} \times_{\mathrm{id}} (\varepsilon\, \mathrm{ev}_0, \mathrm{id})"'] \arrow[dd, "\mathrm{id} \times_{\mathrm{id}} \mathrm{id} \times ((FU)_*, \varepsilon\, \mathrm{ev}_1)"'] \\
B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[dd, "\mathrm{id} \times_{\mathrm{id}} \mathrm{comp} \times_{\mathrm{id}} \mathrm{id}"'] \arrow[ddr, "\mathrm{id} \times_{\mathrm{id}} \mathrm{id} \times_{\mathrm{id}} \mathrm{comp}"] & \\
& B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[d, dashed, "\mathrm{id} \times_{\mathrm{id}} \times_{\mathrm{id}} \mathrm{comp}"] \\
B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[dr, bend right, "\mathrm{id} \times_{\mathrm{id}} \mathrm{comp}"'] & B \times_A \operatorname{Fun}(\Delta^1, A) \times_A \operatorname{Fun}(\Delta^1, A) \arrow[d, "\mathrm{id} \times_{\mathrm{id}} \mathrm{comp}"'] \\
& B \times_A \operatorname{Fun}(\Delta^1, A)
\end{tikzcd}$$

using naturality of ε and associativity. Now we just need to observe that the left most vertical composite is equivalent to $(\mathrm{id}, \varepsilon F \circ F\eta, \mathrm{id})$ which is by the triangle identities itself equivalent to the identity. The equivalence $\phi \circ \psi \simeq \mathrm{id}$ can be seen similarly. $\square$

As a preparation to the functoriality of adjointing natural transformations between adjoints we recall the so-called middle four interchange lemma, which states that the two ways one can compose natural transformations between composable pairs of functors agree.

Lemma A.2.5

For $F, G\colon A \to B$, $H, K\colon B \to C$ and $\alpha\colon F \Rightarrow G$, $\beta\colon H \Rightarrow K$, the middle four interchange holds, i. e. we have $(\beta G)(H\alpha) \simeq (K\alpha)(\beta F)$.

Proof. Let us denote the composite functor of

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(X,Y)) \times \mathrm{Fun}(\Delta^1, \mathrm{Fun}(Y,Z)) \arrow[d, "(\mathrm{pr}_0)_* \times (\mathrm{pr}_1)_*"'] \\
\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Fun}(X,Y)) \times \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Fun}(Y,Z)) \arrow[d, "\simeq"] \\
\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Fun}(X,Y) \times \mathrm{Fun}(Y,Z)) \arrow[d, "\simeq"] \\
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, \mathrm{Fun}(X,Y) \times \mathrm{Fun}(Y,Z)))
\end{tikzcd}$$

by $(-) \times (-)$. Let us also abbreviate $C \coloneqq \mathrm{Fun}(X,Y) \times \mathrm{Fun}(Y,Z)$. Then the middle four interchange follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(X,Y)) \times \mathrm{Fun}(\Delta^1, \mathrm{Fun}(Y,Z)) \arrow[d, "(-)\times(-)"] & \\
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, C)) \arrow[r, "((\mathrm{ev}_0)_*, \mathrm{ev}_1)"] \arrow[d, "(\mathrm{ev}_0, (\mathrm{ev}_1)_*)"'] & \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[d, "\mathrm{comp}"] \\
\mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[r, "\mathrm{comp}"'] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "\mathrm{Fun}(\Delta^1, \circ)"] \\
& \mathrm{Fun}(\Delta^1, \mathrm{Fun}(X,Z))
\end{tikzcd}$$

where we use Proposition A.1.7. □

Using the middle four interchange we can prove the following preliminary result about adjointable squares.

Lemma A.2.6

Let

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\simeq\alpha"] & C \arrow[d, "G"] \\
B \arrow[r, "K"'] & D
\end{tikzcd}$$

be a commutative square with adjunctions $F \dashv U$ and $G \dashv V$, such that the square is vertically adjointable. Then the counits of the two adjunctions commute, i. e. we can construct a commutative square

$$\begin{tikzcd}
B \arrow[r, "K"] \arrow[d, "(U, \varepsilon)"'] & D \arrow[d, "(V, \tilde{\varepsilon})"] \\
A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[r, "H \times_K K_*"'] & C \times_D \mathrm{Fun}(\Delta^1, D)
\end{tikzcd}$$

and the units commute, i. e. we can construct a commutative square

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "{(\eta, F)}"'] & C \arrow[d, "{(\widetilde{\eta}, G)}"] \\
\operatorname{Fun}(\Delta^1, A) \times_A B \arrow[r, "H_* \times_H K"'] & \operatorname{Fun}(\Delta^1, C) \times_C D
\end{tikzcd}$$

where the bottom horizontal functor uses the invertible mate transformation $\mathrm{mate}_\alpha \colon HU \cong VK$.

Proof. The adjointablility of the square means by definition that the mate transformation of α, i. e. the composite $\mathrm{mate}_\alpha \coloneqq (VK\varepsilon) \circ (V\alpha U) \circ (\widetilde{\eta} HU) \colon HU \Rightarrow VK$ is invertible. This tells us diagrammatically, that the following lift exists.

$$\begin{tikzcd}
B \arrow[rr, dashed] \arrow[d, "{(U,\varepsilon)}"] & & D \arrow[dd, hook, "\Delta_V"'] \\
A \times_B^{(F,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, B) \arrow[d, "\mathrm{id} \times_K K_*"] & & \\
A \times_D^{(KF,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \arrow[d, "\alpha_*"] & & \operatorname{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1, V)} D \\
A \times_D^{(GH,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \arrow[r, "H \times_{\mathrm{id}} \mathrm{id}"] & C \times_D^{(G,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \arrow[ur, "\phi"] &
\end{tikzcd}$$

Postcomposing the diagram with $\mathrm{pr}_1 \colon \operatorname{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1, V)} D \to D$ we can see that the lift must be equivalent to K.

Now we can postcompose the inverse ψ to ϕ, by Lemma A.2.4, and obtain

$$\begin{tikzcd}
B \arrow[r, "K"] \arrow[d, "{(U,\varepsilon)}"] & D \arrow[dr, "{(V,\widetilde{\varepsilon})}"] \arrow[dd, hook, "\Delta_V"'] & \\
A \times_B^{(F,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, B) \arrow[d, "\mathrm{id} \times_K K_*"] & & C \times_D^{(G,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \\
A \times_D^{(KF,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \arrow[d, "\alpha_*"] & \operatorname{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1, V)} D \arrow[ur, "\psi"] & \\
A \times_D^{(GH,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \arrow[r, "H \times_{\mathrm{id}} \mathrm{id}"] & C \times_D^{(G,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \arrow[u, "\phi"] \arrow[uur, equal] &
\end{tikzcd}$$

so that after investing $\psi \circ \Delta_V \simeq (V, \widetilde{\varepsilon})$ we have our desired square.

For the second commutative square we proceed in the following similiar way. Using the middle four interchange Lemma A.2.5 several times and then the triangle identities we can see that the mate of mate_α, i. e. the transformation $(\widetilde{\varepsilon} KF) \circ (G \,\mathrm{mate}_\alpha\, F) \circ (GH\eta)$, is equivalent to α via the following equivalences, hence invertible.

$$\begin{aligned}
(\widetilde{\varepsilon} KF) \circ (G \,\mathrm{mate}_\alpha\, F) \circ (GH\eta) &\simeq (\widetilde{\varepsilon} KF) \circ (GVK\varepsilon F) \circ (GV\alpha UF) \circ (G\widetilde{\eta} HUF) \circ (GH\eta) \\
&\simeq (K\varepsilon F) \circ (KF\eta) \circ \alpha \circ (\widetilde{\varepsilon} GH) \circ (G\widetilde{\eta} H) \\
&\simeq \alpha
\end{aligned}$$

This means diagrammatically that we get the dashed lift in the following diagram.

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[d, "{(\eta,F)}"] & C \arrow[dd, hook, "\Delta_G"] \\
\mathrm{Fun}(\Delta^1, A) \times_A^{(\mathrm{ev}_1,U)} B \arrow[d, "H_* \times_H \mathrm{id}"] & \\
\mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,HU)} B \arrow[d, "(\mathrm{mate}_\alpha)_*"] & C \times_D^{(G,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, D) \\
\mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,VK)} B \arrow[r, "\mathrm{id} \times_{\mathrm{id}} K"] & \mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,V)} D \arrow[u, "\psi"]
\end{tikzcd}$$

Postcomposing the diagram with the domain functor $\mathrm{pr}_0\colon C \times_D^{(G,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, D) \to C$ we can see that this lift must be equivalent to H. After postcomposing the diagram with the inverse equivalence

$$\phi\colon C \times_D^{(G,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, D) \to \mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,V)} D \tag{A.2.0.3}$$

of ψ we obtain the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "{(\eta,F)}"] & C \arrow[dd, hook, "\Delta_G"] \arrow[dr, "{(\widetilde{\eta},G)}"] & \\
\mathrm{Fun}(\Delta^1, A) \times_A^{(\mathrm{ev}_1,U)} B \arrow[d, "H_* \times_H \mathrm{id}"] & & \mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,V)} D \\
\mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,HU)} B \arrow[d, "(\mathrm{mate}_\alpha)_*"] & C \times_D^{(G,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, D) \arrow[ur, "\phi"] & \\
\mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,VK)} B \arrow[r, "\mathrm{id} \times_{\mathrm{id}} K"] & \mathrm{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,V)} D \arrow[u, "\psi"] \arrow[uur, equal] &
\end{tikzcd}$$

and investing the fact that $(\widetilde{\eta}, G) \simeq \phi \circ \Delta_G$, we obtain the desired commutative square. $\square$

The following lemma is a preliminary version of the pullback stability of left adjoints in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$, where we restrict to the situation that the corresponding right adjoints are additionally fully-faithful.

Lemma A.2.7

Consider a commutative cube of categories such that the top and bottom faces are pullbacks.

$$
\begin{tikzcd}
A_2 \arrow[dd, "F_2"'] & & A_1 \arrow[ll] \arrow[dd, "F_1"] & \\
& A_0 \arrow[ul] \arrow[dd] & & A \arrow[ll] \arrow[ul] \arrow[dd, "F"] \\
B_2 & & B_1 \arrow[ll, "F_0"'] & \\
& B_0 \arrow[ul] & & B \arrow[ll] \arrow[ul]
\end{tikzcd}
\tag{A.2.0.4}
$$

If the F_i have fully-faithful right adjoints and the back face as well as the left face are horizontally adjointable, then the pullback induced functor F also has a fully-faithful right adjoint and the right and front faces are also horizontally adjointable.

Proof. Let us denote the right adjoints by U_i and their units by η_i. We produce the candidate for the right adjoint U of F, as well as a natural equivalence $\varepsilon\colon FU \simeq \mathrm{id}$, via the horizontally levelwise pullback diagram

$$
\begin{tikzcd}
B_2 \arrow[dd, "U_2"] \arrow[dddd, bend right, equal] & & B_1 \arrow[ll] \arrow[dd, "U_1"] \arrow[dddd, bend right, equal] & \\
& B_0 \arrow[ul] \arrow[dd, "U_0"] \arrow[dddd, bend right, equal] & & B \arrow[ll] \arrow[ul] \arrow[dd, dashed, "\exists! U"] \arrow[dddd, bend left, equal] \\
A_2 \arrow[dd, "F_2"] & & A_1 \arrow[ll] \arrow[dd, "F_1"] & \\
& A_0 \arrow[ul] \arrow[dd, "F_0"] & & A \arrow[ll] \arrow[ul] \arrow[dd, "F"] \\
B_2 & & B_1 \arrow[ll] & \\
& B_0 \arrow[ul] & & B \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

whose composite cube is, as indicated, the vertical identity cube. To produce a candidate for the unit of this adjunction we look at the following cube induced on the back and the left hand vertical faces by Lemma A.2.6 and whose top and bottom faces are pullbacks.

$$
\begin{tikzcd}
A_2 \arrow[dd, "\eta_2"] & & A_1 \arrow[ll] \arrow[dd, "\eta_1"] & \\
& A_0 \arrow[ul] \arrow[dd, "\eta_0"] & & A \arrow[ll] \arrow[ul] \arrow[dd, dashed, "\exists! \eta"] \\
\mathrm{Fun}(\Delta^1, A_2) & & \mathrm{Fun}(\Delta^1, A_1) \arrow[ll] & \\
& \mathrm{Fun}(\Delta^1, A_0) \arrow[ul] & & \mathrm{Fun}(\Delta^1, A) \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

Note that, again by pullback, this natural transformation has the right domain and codomain. To deduce that ηU is invertible we use the following horizontally levelwise pullback diagram.

$$
\begin{tikzcd}
B_2 \arrow[dd, "U_2"] \arrow[dddd, bend right, "U_2\varepsilon_2^{-1}"'] & & B_1 \arrow[ll] \arrow[dd, "U_1"] \arrow[dddd, bend right, "U_1\varepsilon_1^{-1}"'] & \\
& B_0 \arrow[ul] \arrow[dd, "U_0"] \arrow[dddd, bend right, "U_0\varepsilon_0^{-1}"'] & & B \arrow[ul] \arrow[ll] \arrow[dd, "U"] \arrow[dddd, bend left, "U\varepsilon^{-1}"] \\
A_2 \arrow[dd, "\eta_2"] & & A_1 \arrow[ll] \arrow[dd, "\eta_1"] & \\
& A_0 \arrow[ul] \arrow[dd, "\eta_0"] & & A \arrow[ul] \arrow[ll] \arrow[dd, dashed, "\exists!\eta"] \\
\mathrm{Fun}(\Delta^1, A_2) & & \mathrm{Fun}(\Delta^1, A_1) \arrow[ll] & \\
& \mathrm{Fun}(\Delta^1, A_0) \arrow[ul] & & \mathrm{Fun}(\Delta^1, A) \arrow[ul] \arrow[ll]
\end{tikzcd}
$$

The invertibility of $F\eta$ can be proven similarly, via the diagram

$$
\begin{tikzcd}
A_2 \arrow[dd, "\eta_2"] \arrow[dddd, bend right, "\varepsilon_2^{-1}F_2"'] & & A_1 \arrow[ll] \arrow[dd, "\eta_1"] \arrow[dddd, bend right, "\varepsilon_1^{-1}F_1"'] & \\
& A_0 \arrow[ul] \arrow[dd, "\eta_0"] \arrow[dddd, bend right, "\varepsilon_0^{-1}F_0"'] & & A \arrow[ul] \arrow[ll] \arrow[dd, dashed, "\exists!\eta"] \arrow[dddd, bend left, "\varepsilon^{-1}F"] \\
\mathrm{Fun}(\Delta^1, A_2) \arrow[dd, "(F_2)_*"] & & \mathrm{Fun}(\Delta^1, A_1) \arrow[ll] \arrow[dd, "(F_1)_*"] & \\
& \mathrm{Fun}(\Delta^1, A_0) \arrow[ul] \arrow[dd, "(F_0)_*"] & & \mathrm{Fun}(\Delta^1, A) \arrow[ul] \arrow[ll] \arrow[dd, "F_*"] \\
\mathrm{Fun}(\Delta^1, B_2) & & \mathrm{Fun}(\Delta^1, B_1) \arrow[ll] & \\
& \mathrm{Fun}(\Delta^1, B_0) \arrow[ul] & & \mathrm{Fun}(\Delta^1, B) \arrow[ul] \arrow[ll]
\end{tikzcd}
$$

In particular, this proves that η whiskered with the functors $A \to A_i \xrightarrow{F_i} B_i$ is invertible, which also proves the adjointability claim as the counits of all these adjunctions are invertible. $\square$

As a first application we can deduce that some more functors and commutative squares give cocartesian fibrations and functors.

Lemma A.2.8

For any functor $F\colon A \to C$ the functor $\mathrm{ev}_1\colon A \times_C \mathrm{Fun}(\Delta^1, C) \to C$ is a cocartesian fibration.

Furthermore, for any commutative square of functors

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "F"'] & B \arrow[d, "G"] \\
C \arrow[r, "K"'] & D
\end{tikzcd} \tag{A.2.0.5}$$

the functor $H \times_K K_*$ in the commutative square

$$\begin{tikzcd}
A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[r, "H \times_K K_*"] \arrow[d, "\mathrm{ev}_1"'] & C \times_D \mathrm{Fun}(\Delta^1, D) \arrow[d, "\mathrm{ev}_1"] \\
B \arrow[r, "K"'] & D
\end{tikzcd} \tag{A.2.0.6}$$

is a cocartesian functor.

Proof. We know that $\mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, A) \twoheadrightarrow A$ is a cocartesian fibration, and the evaluation functors $\mathrm{ev}_0 \colon \mathrm{Fun}(\Delta^1, C) \to C$ always have fully-faithful left adjoints Δ_C. Thus we can deduce from the commutative cube

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, B) \arrow[dd, "\mathrm{ev}_0"'] & & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \arrow[ll, "\mathrm{ev}_0"'] \arrow[dd] & \\
& \mathrm{Fun}(\Delta^1, A) \arrow[ul, "F_*"'] \arrow[dd, "\mathrm{ev}_0"' near start] & & \mathrm{Fun}(\Delta^1, A \times_B \mathrm{Fun}(\Delta^1, B)) \arrow[ul] \arrow[ll, "{(\mathrm{ev}_0, (\mathrm{ev}_1)_*)}"' near end] \arrow[dd, "{(\mathrm{ev}_0, (\mathrm{pr}_1)_*)}"'] \\
B & & \mathrm{Fun}(\Delta^1, B) \times_B \mathrm{Fun}(\Delta^1, B) \arrow[ll] & \\
& A \arrow[ul, "F"] & & A \times_B \mathrm{Fun}(\Delta^1, B) \times_B \mathrm{Fun}(\Delta^1, B) \arrow[ul] \arrow[ll]
\end{tikzcd}$$

with top and bottom faces pullbacks and Lemma A.2.7 the first statement. For the second statement we proceed in two steps. As a first step we factorize the given square into a triangle and a pullback as in

$$\begin{tikzcd}
A \arrow[r, dashed] \arrow[rr, bend left, "H"] \arrow[dr, "F"'] & C \times_D B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "G"] \\
& C \arrow[r, "K"'] & D
\end{tikzcd}$$

Then for the pullback square we deduce the adjointability claim from Lemma A.2.7 applied to the cube

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, C \times_D B) \times_{\mathrm{Fun}(\Delta^1,C)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, C)) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, C)) \arrow[dr] \arrow[dd] & \\
& \mathrm{Fun}(\Delta^1, B) \times_{\mathrm{Fun}(\Delta^1,D)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, D)) \arrow[rr] \arrow[dd] & & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, D)) \arrow[dd] \\
(C \times_D B) \times_C \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[dr] & \\
& B \times_D \mathrm{Fun}(\Delta^1, D) \times_D \mathrm{Fun}(\Delta^1, D) \arrow[rr] & & \mathrm{Fun}(\Delta^1, D) \times_D \mathrm{Fun}(\Delta^1, D)
\end{tikzcd}$$

with top and bottom faces pullbacks and the fact that the claim holds when $F = \mathrm{id}$ and $G = \mathrm{id}$ by Proposition A.1.6, i. e. the front face is adjointable. For the remaining triangle we can deduce from the cube

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, A) \times_{\mathrm{Fun}(\Delta^1,C)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, C)) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{Fun}(\Delta^1, A) \arrow[dr] \arrow[dd] & \\
& \mathrm{Fun}(\Delta^1, C \times_D B) \times_{\mathrm{Fun}(\Delta^1,C)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, C)) \arrow[rr] \arrow[dd] & & \mathrm{Fun}(\Delta^1, C \times_D B) \arrow[dd] \\
A \times_C \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\
& (C \times_D B) \times_C \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C) \arrow[rr] & & C \times_D B
\end{tikzcd}$$

with top and bottom faces pullbacks in the same way that the back face is adjointable. □

We continue this section with the most basic stability property of right adjoints with invertible counit we will use and from which all the other stability properties we want to invoke can be derived from.

Lemma 1.2.11

Let

$$\begin{tikzcd}
X \arrow[r, "H"] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "U"] \\
Y \arrow[r, "K"'] & B
\end{tikzcd}$$

be a pullback square such that U has a fully-faithful left adjoint F. Then its pullback V also has a fully-faithful left adjoint G, and the pullback square is furthermore horizontally adjointable. Additionally, one can compute the fully-faithful left adjoint G as the pullback of F along H, and the (co)unit of the adjunction $G \dashv V$ as the pullback of the (co)unit of $F \dashv U$ along K, respectively H.

Proof. First let us define the functor G that is supposed to be left adjoint section to V and the invertible unit $\eta' \colon \mathrm{id} \simeq VG$ via the following pullback.

$$\begin{tikzcd}
Y \arrow[r, "K"] \arrow[d, dashed, "\exists! G"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=60, equal, "\simeq \exists! \eta'"'] & B \arrow[d, "F"] \arrow[dd, bend left=60, equal, "\simeq \eta"] \\
X \arrow[r, "H"] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "U"] \\
Y \arrow[r, "K"] & B
\end{tikzcd}$$

Next we define the counit ε' of the adjunction also via the following pullback.

$$\begin{tikzcd}
Y \arrow[rr, "K"] \arrow[dr, "G"] \arrow[dddr, equal] & & B \arrow[dr, "F"] \arrow[dddr, equal] & & \\
& X \arrow[rr, "H"] \arrow[dr, dashed, "\exists! \varepsilon'"'] \arrow[dd, "V"] & & A \arrow[dr, "\varepsilon"] \arrow[dd, "U"] & \\
& & \mathrm{Fun}(\Delta^1, X) \arrow[rr, "H_*"] \arrow[dd, "V_*"] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(\Delta^1, A) \arrow[dd, "U_*"] \\
& Y \arrow[rr, "K"] \arrow[dr, "\eta'"'] & & B \arrow[dr, "\eta"] & \\
& & \mathrm{Fun}(\Delta^1, Y) \arrow[rr, "K_*"'] & & \mathrm{Fun}(\Delta^1, B)
\end{tikzcd}$$

where the bottom square comes from Lemma A.2.6. Now, as εF is equivalent to $F\eta^{-1}$ we also get by pullback that $\varepsilon' G$ is equivalent to $G(\eta')^{-1}$ and thus that G really is a left adjoint to V with counit ε' and invertible unit η'. To see vertical adjointablility of the original pullback square it is sufficient to see that εHG is invertible as the adjunction $G \dashv V$ has invertible unit. But by the above diagram we already know that $\varepsilon HG \simeq H_* \circ \varepsilon' G$ and we just established invertibility of $\varepsilon' G$. $\square$

As a corollary we can immediately deduce that cocartesian fibrations are stable under pullback.

Corollary A.2.9

Let

$$\begin{tikzcd}
Y \arrow[r, "H"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
A \arrow[r, "K"'] & B
\end{tikzcd}$$

be a pullback such that p is a cocartesian fibration. Then its pullback q is also a cocartesian fibration and the functor H is a cocartesian functor over K.

Proof. From the pullback we get another pullback

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, Y) \arrow[r, "H_*"] \arrow[d, "{(\mathrm{ev}_0, q_*)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, X) \arrow[d, "{(\mathrm{ev}_0, p_*)}"] \\
Y \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "H \times_K K_*"'] & X \times_B \mathrm{Fun}(\Delta^1, B)
\end{tikzcd}$$

to which we can apply Lemma 1.2.11 to get the desired statement. □

Another corollary of Lemma 1.2.11 is that the cocartesian fibrations $\mathrm{ev}_1\colon A \times_B \mathrm{Fun}(\Delta^1, B) \to B$ inherit the adjunction $\Delta_B \dashv \mathrm{ev}_0$ from $\mathrm{ev}_1\colon \mathrm{Fun}(\Delta^1, B) \to B$.

Corollary A.2.10

For a functor $F\colon A \to B$ the functor $\mathrm{pr}_0\colon A \times_B \mathrm{Fun}(\Delta^1, B) \to A$ has as a fully-faithful left adjoint the functor $\Delta_F \coloneqq (\mathrm{id}, \Delta_B \circ F)\colon A \to A \times_B \mathrm{Fun}(\Delta^1, B)$.

Proof. We apply Lemma 1.2.11 to the defining pullback

$$\begin{tikzcd}
A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[r, "\mathrm{pr}_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\
\mathrm{Fun}(\Delta^1, B) \arrow[r, "\mathrm{ev}_0"'] & B
\end{tikzcd}$$

□

With these statements we can prove the equivalence of adjunctions given in terms of units/counits and Hom-equivalences.

Lemma A.2.11

Let $F\colon B \to A$ and $U\colon A \to B$ be two functors. Then the following pieces of data are equivalent.

1. A unit natural transformation $\eta\colon \mathrm{id} \Rightarrow UF$, a counit natural transformation $\varepsilon\colon FU \Rightarrow \mathrm{id}$, such that the triangle-identities hold, i. e. $\varepsilon F \circ F\eta \simeq \mathrm{id}$ and $U\varepsilon \circ \eta U \simeq \mathrm{id}$
2. An equivalence

$$\begin{tikzcd}
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}"'] & & \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[dl, "\mathrm{ev}"] \\
& B \times A &
\end{tikzcd} \tag{A.2.0.7}$$

over the canonical functors to $B \times A$.

Proof. We already deduced the second statement from the first in Lemma A.2.4.

For the converse implication we assume to be given inverse equivalences ϕ and ψ. Then we define the unit and the counit as follows.

$$\eta \coloneqq B \xrightarrow{\Delta_F} B \times_A \mathrm{Fun}(\Delta^1, A) \xrightarrow{\phi} \mathrm{Fun}(\Delta^1, B) \times_B A \xrightarrow{\mathrm{pr}_0} \mathrm{Fun}(\Delta^1, B)$$

$$\varepsilon \coloneqq A \xrightarrow{\Delta_U} \mathrm{Fun}(\Delta^1, B) \times_B A \xrightarrow{\psi} B \times_A \mathrm{Fun}(\Delta^1, A) \xrightarrow{\mathrm{pr}_1} \mathrm{Fun}(\Delta^1, A)$$

We know from Lemma A.2.8 that $\mathrm{ev}_1 \colon B \times_A \mathrm{Fun}(\Delta^1, A) \to A$ is a cocartesian fibration. Now we observe that the counit $e \colon \Delta_F \,\mathrm{ev}_0 \Rightarrow \mathrm{id}$ of the adjunction $\Delta_F \dashv \mathrm{ev}_0$, from Corollary A.2.10, is the ev_1-cocartesian lift of

$$\begin{tikzcd}
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "\mathrm{ev}_0"'] \arrow[drr, bend right, "\mathrm{ev}_1"'] & B \arrow[r, "\Delta_F"] \arrow[dr, "F"'] \arrow[d, phantom, "\Downarrow"] & B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[d, "\mathrm{ev}_1"] \\
& {} & A
\end{tikzcd}$$

as it is exactly

$$B \times_A \mathrm{Fun}(\Delta^1, A) \xrightarrow{(\mathrm{ev}_0\, \Delta_F, \mathrm{pr}_1)} B \times_A \mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \xrightarrow{\mathrm{cocart}_{\mathrm{ev}_1}} \mathrm{Fun}(\Delta^1, B \times_A \mathrm{Fun}(\Delta^1, A))$$

just as for $\mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, A) \to A$ in Proposition A.1.5. But as ϕ is an equivalence it is in particular a cocartesian functor between cocartesian fibrations over A as follows.

$$\begin{tikzcd}
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[rr, "\phi \simeq"] \arrow[dr] \arrow[ddr, bend right, "\mathrm{ev}_1"'] & & \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[dl] \arrow[ddl, bend left, "\mathrm{pr}_1"] \\
& B \times A \arrow[d, "\mathrm{pr}_A"] & \\
& A &
\end{tikzcd}$$

Hence it sends the ev_1-cocartesian morphism e to a pr_1-cocartesian morphism ϕe. Thus ϕ itself is a pr_1-cocartesian lift of

$$\begin{tikzcd}
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "\mathrm{pr}_0"] \arrow[drrr, bend right, "\mathrm{ev}_1"'] & B \arrow[r, "\Delta_F"] \arrow[drr, "F"] \arrow[dr, phantom, "\Rightarrow"] & B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "\phi"] \arrow[dr, "\mathrm{ev}_1"] & \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[d, "\mathrm{pr}_1"] \\
& {} & & A
\end{tikzcd}$$

But cocartesian lifts for the pulled back cocartesian fibration $\mathrm{Fun}(\Delta^1, B) \times_B A \to A$ are essentially given by applying U and then composing. Hence ϕ itself is equivalent to the top horizontal composite in the following commutative diagram.

$$\begin{tikzcd}
& \mathrm{Fun}(\Delta^1, B) \times_B A \times_A \mathrm{Fun}(\Delta^1, A) \arrow[dr, "\mathrm{comp} \circ (\mathrm{id} \times_{\mathrm{id}} U_*)"] \arrow[dd, "\mathrm{id} \times_{\mathrm{id}} (U_*, \mathrm{ev}_1)"'] & \\
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[ur, "(\phi \circ \Delta_F) \times_{\mathrm{id}} \mathrm{id}"] \arrow[dr, "\eta \times_U (U_*, \mathrm{ev}_1)"'] & & \mathrm{Fun}(\Delta^1, B) \times_B A \\
& \mathrm{Fun}(\Delta^1, B) \times_B \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[ur, "\mathrm{comp} \times_{\mathrm{id}} \mathrm{id}"'] &
\end{tikzcd}$$

This we can now precompose with $\psi \circ \Delta_U$ to obtain

$$\begin{tikzcd}
A \arrow[r, "\Delta_U"'] \arrow[d, "(U, \varepsilon)"'] & \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[dl, "\psi"'] \arrow[drr, equal] & & \\
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "\eta \times_U (U_*, \mathrm{ev}_1)"'] \arrow[rrr, bend left, "\phi", "\simeq"'] & \mathrm{Fun}(\Delta^1, B) \times_B \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[rr, "\mathrm{comp} \times_{\mathrm{id}} \mathrm{id}"'] & & \mathrm{Fun}(\Delta^1, B) \times_B A
\end{tikzcd}$$

i. e. the desired triangle identity $U\varepsilon \circ \eta U \simeq \mathrm{id}$. The other triangle identity can be proven similarly. $\square$

The following lemma is an explicit characterization of when functors factor through fully-faithful left adjoints, in terms of the adjunction data. In particular it gives a convenient description of the essential image of a fully-faithful functor in the case that it is a left adjoint, if we have a notion of underlying space of objects.

Lemma A.2.12

Let $U\colon A \to B$ be a right adjoint with fully-faithful left adjoint F and counit $\varepsilon\colon FU \Rightarrow \mathrm{id}$, and $T\colon X \to A$ another functor. Then the following are equivalent.

1. The whiskering εT is invertible.
2. There exists a lift

$$\begin{tikzcd} & B \arrow[d, "F"] \\ X \arrow[ur, dashed, "\exists l"] \arrow[r, "T"'] \arrow[ur, phantom, "\simeq\alpha", near end, shift right=3] & A \end{tikzcd} \tag{A.2.0.8}$$

Proof. To prove the second assertion from the first we can just choose $U \circ T$ and the invertible natural transformation $\varepsilon T\colon FUT \simeq T$ to be the lift. If we are given a lift l as in the second assertion, then we proceed in the following way. The triangle identity $\varepsilon F \circ F\eta \simeq \mathrm{id}$ and the invertibility of η give us the invertibility of εF. But as $\varepsilon Fl \simeq \varepsilon T$ by α we deduce that εT is also invertible. By Lemma A.2.11 and the invertibility of the unit we can also deduce from $\alpha\colon Fl \simeq T$ that there is a unique $(U\alpha)(\eta l)\colon l \simeq UFl \simeq UT$, i. e. that the lift must be unique in this situation. $\square$

The next statement is about an alternative characterisation of the adjointablility of squares, more precisely a square between adjunctions is adjointable if and only if the units commute in the following sense.

Lemma A.2.13

Let

$$\begin{tikzcd} A_1 \arrow[r, "H"] \arrow[d, "U_1"'] & A_2 \arrow[d, "U_2"] \\ B_1 \arrow[r, "K"'] & B_2 \end{tikzcd} \tag{A.2.0.9}$$

be commutative square such that we have adjoints $F_1 \dashv U_1$ and $F_2 \dashv U_2$. Then the following two statements are equivalent.

1. The square is adjointable.

2. The composed natural transformation

$$\begin{tikzcd}
B_1 \arrow[rrr, "K"] \arrow[ddd, "{(\eta_1, F_1)}"'] & & & B_2 \arrow[ddd, "{(\eta_2, F_2)}"] \arrow[ddl, "\Delta_{F_2}"] \\
& B_2 \times_{A_2} \mathrm{Fun}(\Delta^1, A_2) \arrow[urr, "\mathrm{ev}_0"] \arrow[dr, equal] \arrow[r, phantom, "\Leftarrow \varepsilon"] & {} & \\
& \mathrm{Fun}(\Delta^1, B_2) \times_{B_2} A_2 \arrow[u, "\psi_2"] \arrow[drr, equal] & B_2 \times_{A_2} \mathrm{Fun}(\Delta^1, A_2) \arrow[dr, "\phi_2"] & \\
\mathrm{Fun}(\Delta^1, B_2) \times_{B_2} A_2 \arrow[ur, "K_* \times_K H"] \arrow[rrr, "K_* \times_K H"'] & & & \mathrm{Fun}(\Delta^1, B_2) \times_{B_2} A_2
\end{tikzcd}$$

is invertible.

Proof. The first statement is by Lemma A.2.11 equivalent to the existence of the dashed functor in the top part of the following diagram.

$$\begin{tikzcd}
B_1 \arrow[rr, dashed, "\exists"'] \arrow[d, "{(\eta_1, F_1)}"] & & B_2 \arrow[d, hook, "\Delta_{F_2}"] \\
\mathrm{Fun}(\Delta^1, B_1) \times_{B_1}^{(\mathrm{ev}_1, U_1)} A_1 \arrow[d, "{(\mathrm{ev}_0, K_*) \times_K \mathrm{id}}"] & & B_2 \times_{A_2}^{(F_2, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, A_2) \arrow[ddd, bend left, start anchor=south east, end anchor=north east] \\
B_1 \times_{B_2}^{(K, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B_2) \times_{B_2}^{(\mathrm{ev}_1, KU_1)} A_1 \arrow[d, "\mathrm{id} \times_{\mathrm{id}} \alpha_* \times_{\mathrm{id}} \mathrm{id}"] \arrow[dd, bend right, start anchor=south west, end anchor=north west] & & \\
B_1 \times_{B_2}^{(K, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B_2) \times_{B_2}^{(\mathrm{ev}_1, U_2 H)} A_1 \arrow[rr, "{(\mathrm{pr}_1, H\, \mathrm{pr}_2)}"'] \arrow[d] & & \mathrm{Fun}(\Delta^1, B_2) \times_{B_2}^{(\mathrm{ev}_1, U_2)} A_2 \arrow[uu, "\psi_2"] \arrow[d] \\
B_1 \times A_1 \arrow[rr, "K \times H"'] & & B_2 \times A_2
\end{tikzcd}$$

By the bottom colored part of the diagram such a lift, if it exists, must be equivalent to the functor K. By Lemma A.2.12 the existence of such dashed lift along the fully-faithful left adjoint Δ_{F_2} is equivalent to the post-whiskering with the counit of $\Delta_{F_2} \dashv \mathrm{ev}_0$ being invertible. As ϕ_2 and ψ_2 are by definition inverse equivalences this invertibility is itself equivalent to the second assertion above. □

An easy corollary of the equivalence of the unit/counit description and the Hom-equivalence description of adjunctions is that a functor F has a right adjoint if and only the cocartesian fibration $\mathrm{ev}_1 \colon B \times_A \mathrm{Fun}(\Delta^1, A) \to A$ admits a fully-faithful right adjoint. This gives a way of going back and forth between general adjunctions and adjunctions with invertible counit.

Lemma A.2.14

For any functor $F \colon B \to A$ the following two statements are equivalent.

1. F has a right adjoint.
2. The functor $\mathrm{ev}_1 \colon B \times_A \mathrm{Fun}(\Delta^1, A) \to A$ has a fully-faithful right adjoint.

Proof. Let us start by assuming the first assertion. By Lemma A.2.11 we have an equivalence

$$\begin{tikzcd}
B \times_A^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, A) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ev}"] \arrow[ddr, "\mathrm{ev}_1"'] & & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[dl, "\mathrm{ev}"'] \arrow[ddl, "\mathrm{pr}_1"] \\
& B \times A \arrow[d, "\mathrm{pr}_A"] & \\
& A &
\end{tikzcd} \tag{A.2.0.10}$$

But we also know that the functor pr_1 above always has the fully-faithful right adjoint Δ_U. Hence ev_1 is the composite of an equivalence and a functor which has a fully-faithful right adjoint, thus it also has one. To show the backwards implication, let us consider the commutative triangle

$$\begin{tikzcd}
B \arrow[rr, "\Delta_F"] \arrow[dr, "F"'] & & B \times_A^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, A) \arrow[dl, "\mathrm{ev}_1"] \\
& A &
\end{tikzcd} \tag{A.2.0.11}$$

Furthermore, it is always true that Δ_F is left adjoint to $\mathrm{pr}_1 \colon B \times_A^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, A) \to B$. Now we have factorized F into two left adjoints, hence it also is one. $\square$

The next lemma is about an instance of the so-called mate correspondence, the passage back and forth between natural transformations $F \to G$ between left adjoints $F \dashv U$, $G \dashv V$ and natural transformations in the reversed directions $V \to U$ between their right adjoints. We first construct the equivalence.

Construction A.2.15

Let $U, V \colon A \to B$ be two right adjoints with left adjoints $F \dashv U$ and $G \dashv V$. Let us denote the Hom-space of $\mathrm{Fun}(A, B)$ from U to V by $U \to V$. We can then use the Hom-equivalences from Lemma A.2.11 of the adjunctions ${}_*U \dashv {}_*F$ and $G_* \dashv V_*$ to get the mate correspondence equivalence matecorr

$$\begin{aligned}
(U \to V) = ({}_*U(\mathrm{id}) \to V) &\simeq (\mathrm{id} \to {}_*F(V)) \\
&= (\mathrm{id} \to VF) \\
&= (\mathrm{id} \to V_*(F)) \simeq (G_*(\mathrm{id}) \to F) = (G \to F)
\end{aligned}$$

Note that by associativity of whiskering and natural transformation composition, we could have also define it by first invoking $G_* \dashv V_*$ and then ${}_*U \dashv {}_*F$.

This mate correspondence equivalence is functorial in the following sense.

Lemma A.2.16

Let $U, V, W \colon A \to B$ be three right adjoints with left adjoints $F \dashv U$, $G \dashv V$ and $H \dashv W$. The mate correspondence equivalences constructed in Construction A.2.15 commute with composition of natural transformations, i. e.

$$\begin{tikzcd}
(U \to V) \times (V \to W) \arrow[r, "\mathrm{comp}"] \arrow[d, "\mathrm{matecorr} \times \mathrm{matecorr}"'] & (U \to W) \arrow[d, "\mathrm{matecorr}"] \\
(G \to F) \times (H \to G) \arrow[r, "\mathrm{comp}"'] & (H \to F)
\end{tikzcd}$$

commutes. Furthermore, they are unital, i. e. the functors $\mathrm{matecorr}\colon (U \to U) \to (F \to F)$ send identities to identities. In particular we have that the following dashed lift always exists.

$$\begin{tikzcd}
(U \simeq V) \arrow[r, dashed, "\mathrm{matecorr}"] \arrow[d] & (G \simeq F) \arrow[d] \\
(U \to V) \arrow[r, "\mathrm{matecorr}"'] & (G \to F)
\end{tikzcd}$$

Proof. This is follows from repeated application of Lemma A.2.5 and the triangle identities of $G \dashv V$. The second claim is an immediate consequence of the triangle identity of $F \dashv U$. The third claim follows from the characterisation of invertible maps as having a section and a retraction. □

The following lemma is a functorial version of the fact that cocartesian lifts are equivalences if and only if their projection to the base category is so.

Lemma A.2.17

Let $p\colon X \twoheadrightarrow B$ be a cocartesian fibration. Let $f\colon x \to \widetilde{x}$ be a morphism in X, or more generally a natural transformation between functors $A \to X$ for some category A, such that f is a p-cocartesian lift. If its image $p(f)$ under p is an equivalence, then f is also an equivalence.

Proof. We have the commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X) \arrow[r, "(\mathrm{ev}_0, p_*)"] \arrow[d, "\mathrm{ev}_0"'] & X \times_B \mathrm{Fun}(\Delta^1, B) \arrow[dl, "\mathrm{pr}_0"] \\
X &
\end{tikzcd}$$

of right adjoints. By Lemma A.2.16 we may pass to left adjoints and obtain the still commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X) & X \times_B \mathrm{Fun}(\Delta^1, B) \arrow[l, "\mathrm{cocart}"'] \\
X \arrow[u, "\Delta_X"] \arrow[ur, "\Delta_p"'] &
\end{tikzcd}$$

from which the desired statement follows immediately. □

The next statement is a functorial version of the diagrammatic left cancellation property of cocartesian morphisms in cocartesian fibrations.

Lemma A.2.18
Let $p\colon X \twoheadrightarrow B$ be a cocartesian fibration. Let

$$\begin{tikzcd} x_0 \arrow[r, "f", "\text{cocart}"'] \arrow[dr, "g\circ f"'] & x_1 \arrow[d, "g"] \\ & x_2 \end{tikzcd}$$

be a commutative triangle of morphisms in X, or more general natural transformations between functors $A \to X$ for some category A, such that f is a p-cocartesian lift. Then g is a cocartesian lift if and only if $g \circ f$ is a cocartesian lift.

Proof. We have the following commutative square.

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, X)) \arrow[d, "(\mathrm{ev}_0, p_*)"'] \arrow[r, "((\mathrm{ev}_0)_*, p_*)"] & \mathrm{Fun}(\Delta^1, X) \times^{(p_*, (\mathrm{ev}_0)_*)}_{\mathrm{Fun}(\Delta^1, B)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \arrow[d, "\mathrm{ev}_0 \times_{\mathrm{ev}_0} \mathrm{id}"] \\ \mathrm{Fun}(\Delta^1, X) \times^{(p_*, \mathrm{ev}_0)}_{\mathrm{Fun}(\Delta^1, B)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \arrow[r, "\mathrm{ev}_0 \times_{\mathrm{ev}_0} \mathrm{id}"'] & X \times_B \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \end{tikzcd}$$

The left hand vertical functor is a right adjoint because $p_*\colon \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\Delta^1, B)$ is a cocartesian fibration. The upper diagonal functor is a right adjoint because it is $\mathrm{Fun}(\Delta^1, -)$ applied to the adjunction witnessing that p is a cocartesian fibration. The lower diagonal functor is right adjoint as it is the pullback of the adjunction witnessing cocartesianness of p, which is an adjunction over $\mathrm{Fun}(\Delta^1, B)$, along $\mathrm{ev}_0\colon \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \to \mathrm{Fun}(\Delta^1, B)$. The right hand vertical functor is right adjoint as it is the pullback of the adjunction witnessing cocartesianness of p, which is an adjunction over $\mathrm{Fun}(\Delta^1, B)$, along $(\mathrm{ev}_0)_*\colon \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \to \mathrm{Fun}(\Delta^1, B)$. Now we now that we have an commutative square consisting of right adjoints, hence Lemma A.2.16 tells us that the fully adjointed square, where we pass to left adjoints both vertically and horizontally, commutes again. Now to deduce our desired statement we just need to take the functor

$$X \times_B \mathrm{Fun}(\Delta^1, B) \times_B \mathrm{Fun}(\Delta^1, B) \xrightarrow{\mathrm{id}_X \times_{\mathrm{id}_B} \mathrm{cocart}_{\mathrm{ev}_1}} X \times_B \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B))$$

and precompose it onto our commutative square of left adjoints to obtain the desired claim. $\square$

In the following lemma we concern ourselves with squares that are horizontally right adjointable as well as vertically left adjointable. We prove that in this case the order of adjointing along the two different orientations does not matter.

Lemma A.2.19

Let

$$\begin{tikzcd}
A \arrow[r, "F_0"] \arrow[d, "V_0"'] \arrow[dr, phantom, "\nearrow\alpha"] & B \arrow[d, "V_1"] \\
C \arrow[r, "F_1"'] & D
\end{tikzcd}$$

be a natural transformation such that we have adjoints $F_0 \dashv U_0$, $F_1 \dashv U_1$, $G_0 \dashv V_0$ and $G_1 \dashv V_1$. Let us furthermore assume that the vertically adjointed mate as below on the left as well as the horizontally adjointed mate as below on the right

$$\begin{tikzcd}
A \arrow[r, "F_0"] \arrow[dr, phantom, "\nwarrow\beta"] & B \\
C \arrow[u, "G_0"] \arrow[r, "F_1"'] & D \arrow[u, "G_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "V_0"'] \arrow[dr, phantom, "\searrow\gamma"] & B \arrow[l, "U_0"'] \arrow[d, "V_1"] \\
C & D \arrow[l, "U_1"]
\end{tikzcd}$$

are invertible. Then the horizontally adjointed mate of β^{-1} and the vertically adjointed mate of γ^{-1} as depicted in

$$\begin{tikzcd}
A \arrow[dr, phantom, "\gamma^{-1}\nearrow \quad \nearrow\beta^{-1}"] & B \arrow[l, "U_0"'] \\
C \arrow[u, "G_0"] & D \arrow[l, "U_1"] \arrow[u, "G_1"']
\end{tikzcd}$$

are equivalent.

Proof. In the diagram

$$\begin{tikzcd}
(F_1V_0 \to V_1F_0) \arrow[d, "\simeq"'] & (V_0U_0 \to U_1V_1) \arrow[l, "\simeq"'] \arrow[dl, "\simeq"] & (V_0U_0 \simeq U_1V_1) \arrow[l, hook'] \arrow[d, hook] \arrow[ddll, "\simeq"] \\
(F_1G_1 \to G_0F_0) & & (U_1V_1 \to V_0U_0) \arrow[d, "\simeq"] \arrow[dl, "\simeq"'] \\
(F_1G_1 \simeq G_0F_0) \arrow[u, hook] \arrow[r, hook] & (G_0F_0 \to F_1G_1) & (G_0U_1 \to U_0G_1) \arrow[l, "\simeq"]
\end{tikzcd}$$

all functors that are decorated by $\simeq$ are instances of mate correspondence functors. Here the upper left hand triangle and the lower right hand triangle commute because of the fact that the mate correspondence for composed adjoints can be decomposed by the mate correspondences for the factor adjoints, because units and counits of composed adjoints can be built from the units and counits of their factor adjoints. The two middle squares of the diagram commute by Lemma A.2.16. Now the proof of the desired statement is a simple diagram chase in this diagram. □

The next statement is a functorial version of cocartesian pushforward preserving cocartesianness of morphisms.

Lemma A.2.20
Let $p\colon X \twoheadrightarrow B$ be a cocartesian fibration. Then

$$\begin{tikzcd}
X \times_B \mathrm{Fun}(\Delta^1, B) \arrow[rr, "\mathrm{cocart}"] \arrow[dr, two heads, "\mathrm{pr}_1"'] & & \mathrm{Fun}(\Delta^1, X) \arrow[dl, two heads, "p_*"] \\
& \mathrm{Fun}(\Delta^1, B) &
\end{tikzcd}$$

is a cocartesian functor between cocartesian fibrations.

Proof. The triangle commutes as cocart is left adjoint to (ev_0, p_*) with invertible unit. pr_1 is a cocartesian fibration as it is the pullback of p. To show that cocart is cocartesian we want to show that

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X \times_B \mathrm{Fun}(\Delta^1, B)) \arrow[r, "(\mathrm{ev}_0, (\mathrm{pr}_1)_*)"] \arrow[d, "\mathrm{cocart}_*"] & X \times_B \mathrm{Fun}(\Delta^1, B) \times_{\mathrm{Fun}(\Delta^1, B)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \arrow[d, "\mathrm{cocart} \times_{\mathrm{id}} \mathrm{id}"] \\
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, X)) \arrow[r, "(\mathrm{ev}_0, p_*)"'] & \mathrm{Fun}(\Delta^1, X) \times^{(p_*, \mathrm{ev}_0)}_{\mathrm{Fun}(\Delta^1, B)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B))
\end{tikzcd}$$

is vertically adjointable. But this is exactly the horizontally adjointed square to

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, X)) \arrow[r, "(\mathrm{ev}_0, p_*)"] \arrow[d, "(\mathrm{ev}_0, p_*)_*"] & \mathrm{Fun}(\Delta^1, X) \times^{(p_*, \mathrm{ev}_0)}_{\mathrm{Fun}(\Delta^1, B)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B)) \arrow[d, "(\mathrm{ev}_0, p_*) \times_{\mathrm{id}} \mathrm{id}"] \\
\mathrm{Fun}(\Delta^1, X \times_B \mathrm{Fun}(\Delta^1, B)) \arrow[r, "(\mathrm{ev}_0, (\mathrm{pr}_1)_*)"'] & X \times_B \mathrm{Fun}(\Delta^1, B) \times_{\mathrm{Fun}(\Delta^1, B)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, B))
\end{tikzcd}$$

As explained in the proof of Lemma A.2.19, to first horizontally adjoint and then vertically adjoint the result is equivalent to fully passing to left adjoints with all the functors. But as the above square is, as explained in the proof of Lemma A.2.18, a commutative square of right adjoints we can deduce as we did there that the fully adjointed square also commutes, which proves our claim. □

A.3 The Directed Lifting Property of Left Adjoints against Cocartesian Fibrations

In this section we will discuss a generalisation of the unique left lifting property of cofinal functors against left fibrations. Informally, we will concern ourselves with laxly commuting squares of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\swArrow\alpha"] & X \arrow[d, two heads, "p"] \\
B \arrow[r] & Y
\end{tikzcd}$$

where F is a left adjoint and p a cocartesian fibration. We will show that in this situation one can always construct a lax lift l of this square, i. e.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "F"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r] \arrow[ur, dashed, "l"'] \arrow[u, phantom, near end, "\widetilde{\alpha}\Swarrow"] & Y
\end{tikzcd}$$

together with an identification $p\widetilde{\alpha} \simeq \alpha$. This is achieved by employing the adjunction $F \dashv U$ and cocartesian lifting for p and letting them work together. Moreover this lax lift will be special in the sense that it will be initial among all lifts that recover the original lax square via postwhiskering with p. More precisely the strategy is to precompose the original lax square with the counit $\varepsilon\colon FU \Rightarrow \mathrm{id}$ to obtain

$$\begin{tikzcd}
B \arrow[r, "U"] \arrow[dr, equal] \arrow[dr, phantom, bend left, "\Swarrow\varepsilon"] & A \arrow[r] \arrow[d, "F"] \arrow[dr, phantom, "\Swarrow\alpha"] & X \arrow[d, two heads, "p"] \\
& B \arrow[r] & Y
\end{tikzcd}$$

and then then cocartesian lift the top horizontal composite functor along this natural transformation to get

$$\begin{tikzcd}
B \arrow[r, "U"] \arrow[dr, equal] \arrow[rr, dashed, bend right, "\exists_! l"'] & A \arrow[r] \arrow[d, phantom, "\Downarrow\exists_!\theta"] & X \arrow[d, two heads, "p"] \\
& B \arrow[r] & Y
\end{tikzcd}$$

This we can then in turn precompose with the unit $\eta\colon\ \mathrm{id} \Rightarrow UF$

$$\begin{tikzcd}
A \arrow[d, "F"'] \arrow[dr, equal] \arrow[dr, phantom, bend right, "\eta\Swarrow"] & & \\
B \arrow[r, "U"] \arrow[dr, equal] \arrow[rr, dashed, bend right, "\exists_! l"'] & A \arrow[r] \arrow[d, phantom, "\Downarrow\exists_!\theta"] & X \arrow[d, two heads, "p"] \\
& B \arrow[r] & Y
\end{tikzcd}$$

to get our lax lift $(l, \widetilde{\alpha})$ as indicated in above.

This directed lifting property itself we be a corollary to the following general lemma applied to the adjointable square obtained by precomposing with F and postcomposing with p.

Lemma A.3.1

Let

$$\begin{tikzcd}
A \arrow[r, "U"] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\simeq\alpha"] & C \arrow[d, "q"] \\
B \arrow[r, "V"'] & D
\end{tikzcd} \tag{A.3.0.1}$$

be a commutative square of functors with adjunctions $F \dashv U$, $G \dashv V$ such that the square is adjointable. Let us furthermore assume that p is a cocartesian fibration. Then the functor

$$(\mathrm{ev}_0, \alpha \circ q_*, p)\colon\ \mathrm{Fun}(\Delta^1, C) \times_C A \to C \times_D \mathrm{Fun}(\Delta^1, D) \times_D B$$

has a fully-faithful left adjoint.

Proof. By definition of cocartesianness we know that

$$(\mathrm{ev}_0, p_*)\colon \operatorname{Fun}(\Delta^1, A) \to A \times_B \operatorname{Fun}(\Delta^1, B)$$

has a fully-faithful left adjoint. Now we pull back along F and obtain

$$\begin{tikzcd}
C \times_A^{(F,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, A) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "{(\mathrm{id}, p_*, p)}"'] & \operatorname{Fun}(\Delta^1, A) \arrow[d, "{(\mathrm{ev}_0, p_*)}"] \\
C \times_B^{(pF,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, B) \arrow[r, "F \times_{\mathrm{id}} \mathrm{id}"'] \arrow[d, "\mathrm{pr}_0"'] & A \times_B^{(p,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, B) \arrow[d, "\mathrm{pr}_0"] \\
C \arrow[r, "F"'] & A
\end{tikzcd} \tag{A.3.0.2}$$

The composite rectangle is a pullback as well as the bottom square, thus by pullback cancellation also the top square. By Lemma 1.2.11 we conclude that the functor (id, p_*, p) also has a fully-faithful left adjoint. Next we observe that the diagram

$$\begin{tikzcd}
C \times_A^{(F,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, A) \arrow[r, "\phi^{F \dashv U}"] \arrow[dr, "{(\mathrm{id},p_*,p)}"'] & \operatorname{Fun}(\Delta^1, C) \times_C^{(\mathrm{ev}_1,U)} A \arrow[r, "{(\mathrm{ev}_0,\alpha\circ q_*,p)}"'] & C \times_D^{(q,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, D) \times_D^{(\mathrm{ev}_1,V)} B \arrow[d, "C \times_D \psi^{G \dashv V}"] \\
& C \times_B^{(pF,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, B) \arrow[r, "_\circ\mathrm{mate}_\alpha"'] & C \times_B^{(Gq,\mathrm{ev}_0)} \operatorname{Fun}(\Delta^1, B)
\end{tikzcd}$$

where the functors denoted ϕ and ψ are the Hom-equivalences given by Lemma A.2.11 from the adjunctions $F \dashv U$ and $G \dashv V$, is commutative by naturality of $G\alpha$ and the counit of $G \dashv V$. As all the other functors are equivalences we deduce that $(\mathrm{ev}_0, \alpha \circ q_*, p)$ also has a fully-faithful left adjoint. $\square$

Corollary 1.2.9

Let $F\colon A \to B$ be a left adjoint with right adjoint U and $p\colon X \twoheadrightarrow Y$ a cocartesian fibration. Then the functor

$$\operatorname{Fun}(\Delta^1, \operatorname{Fun}(A, X)) \underset{\operatorname{Fun}(A,X)}{\times} \operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X) \underset{\operatorname{Fun}(A,Y)}{\times} \operatorname{Fun}(\Delta^1, \operatorname{Fun}(A, Y)) \underset{\operatorname{Fun}(A,Y)}{\times} \operatorname{Fun}(B, Y)$$

defined as in Lemma A.3.1 from the adjointable square

$$\begin{tikzcd}
\operatorname{Fun}(B, X) \arrow[r, "{\operatorname{Fun}(F, X)}"] \arrow[d, "{\operatorname{Fun}(B, p)}"'] & \operatorname{Fun}(A, X) \arrow[d, "{\operatorname{Fun}(A, p)}"] \\
\operatorname{Fun}(B, Y) \arrow[r, "{\operatorname{Fun}(F, Y)}"'] & \operatorname{Fun}(A, Y)
\end{tikzcd} \tag{1.2.2.5}$$

has a fully-faithful left adjoint.

We now discuss special cases of the general directed lifting of left adjoints against cocartesian fibrations, namely what happens if the starting lax square is actually already commutative, i. e. the natural transformation is invertible, and recover an actual non-lax lifting in the case that F is fully-faithful.

Corollary 1.2.10

Let $F\colon A \to B$ be a left adjoint and $p\colon X \twoheadrightarrow Y$ a cocartesian fibration. If we furthermore assume the left adjoint F to be fully-faithful, then the adjunction from Corollary 1.2.9 restricts on both sides to an adjunction

$$\begin{tikzcd}
\mathrm{Fun}(B,X) \arrow[rr, "({}_*F, p_*)"] \arrow[rr, phantom, bend left=20, "\perp"'] \arrow[dr, "p_*"'] & & \mathrm{Fun}(A,X) \times^{(p_*, {}_*F)}_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y)\,, \arrow[ll, bend right=40] \arrow[dl, "\mathrm{pr}_1"] \\
& \mathrm{Fun}(B,Y) &
\end{tikzcd} \tag{1.2.2.6}$$

which even lives over $\mathrm{Fun}(B,Y)$ via the indicated functors in the above diagram.

In this particular situation, by further unpacking the proof of Corollary 1.2.9, we can characterize the essential image of the fully-faithful left adjoint of this adjunction via Lemma A.2.12 as exactly those functors $L\colon B \to X$ whose whiskering $L\varepsilon$, where ε is the counit of the adjunction $F \dashv U$, is a p-cocartesian lift.

Proof. In the situation of Corollary 1.2.9 we have that the square

$$\begin{tikzcd}
\mathrm{Fun}(B,X) \arrow[d, "({}_*F, p_*)"'] \arrow[r, "\Delta_{\mathrm{Fun}(F,X)}"] & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(A,X)) \times_{\mathrm{Fun}(A,X)} \mathrm{Fun}(B,X) \arrow[d] \\
\mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y) \arrow[r, "(\mathrm{id}, \Delta_{\mathrm{Fun}(A,Y)}, \mathrm{id})"'] & \mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(A,Y)) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y)
\end{tikzcd} \tag{A.3.0.3}$$

commutes. By the proof of Lemma A.3.1 and the invertibility of the unit of the adjunction we then deduce that the left adjoint of Corollary 1.2.9 also factors through

$$\Delta_{({}_*F)}\colon \mathrm{Fun}(B,X) \to \mathrm{Fun}(\Delta^1, \mathrm{Fun}(A,X)) \times_{\mathrm{Fun}(A,X)} \mathrm{Fun}(B,X).$$

$\square$

Informally, this adjunction now tells us that we can lift any commutative square

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "F"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "K"'] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.3.0.4}$$

via pre-whiskering with the counit of $F \dashv U$ and then cocartesian lifting along p. But the adjunction tells us even more, the lift constructed in the explained manner is even initial in $\mathrm{Fun}(B, X)$ among all other lifts of this fixed square.

To close this section we restrict further to the left adjoints $\Delta_G \colon C \hookrightarrow C \times_Y \mathrm{Fun}(\Delta^1, Y)$.

Corollary A.3.2

For every 1-morphism $G \colon C \to Y$ and every cocartesian fibration $p \colon X \twoheadrightarrow Y$ we take in the lifting adjunction Corollary 1.2.10, of fully-faithful left adjoints against cocartesian fibrations, F to be $\Delta_G \colon C \to C \times_Y \mathrm{Fun}(\Delta^1, Y)$ from Corollary A.2.10 and pullback this relative adjunction over $\mathrm{Fun}(C \times_Y \mathrm{Fun}(\Delta^1, Y), Y)$ along the object $\mathrm{ev}_1 \colon \Delta^0 \to \mathrm{Fun}(C \times_Y \mathrm{Fun}(\Delta^1, Y), Y)$ to arrive at the lifting adjunction

$$\begin{tikzcd} \mathrm{Fun}_{/Y}(\mathrm{ev}_1, p) \arrow[r, "{}_*(\Delta_G)"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathrm{Fun}_{/Y}(G, p) \arrow[l, bend right, hook'] \end{tikzcd}$$

In this situation we can give an even more useful characterisation of the lifts obtained through this lax lifting procedure.

Corollary A.3.3

If we specialize this lifting situation of Corollary 1.2.10 even further by taking as the left adjoint F the functor $\Delta_G \colon C \hookrightarrow C \times_Y \mathrm{Fun}(\Delta^1, Y)$ for some arbitrary functor $G \colon C \to Y$, and also take K to be the functor $\mathrm{ev}_1 \colon C \times_Y \mathrm{Fun}(\Delta^1, Y) \to Y$, then the prescribed lifting strategy for commutative squares of the form

$$\begin{tikzcd} C \arrow[r, "H"] \arrow[d, "\Delta_G"'] & X \arrow[d, two heads, "p"] \\ C \times_Y \mathrm{Fun}(\Delta^1, Y) \arrow[r, two heads, "\mathrm{ev}_1"'] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.3.0.5}$$

produces even stronger results then the characterisation of the essential image of the lifting left adjoint. Indeed, a lift of such a square is in the essential image of the lifting left adjoint if and only if it is a cocartesian functor between the indicated cocartesian fibrations.

Proof. To prove this let us start with an arbitrary ev_1-cocartesian pushforward of an arbitrary object $(c, f \colon F(c) \to y)$ in $C \times_Y \mathrm{Fun}(\Delta^1, Y)$ along an arbitrary morphism $g \colon y \to y'$ in Y. Such a cocartesian lift is given by

$$\begin{tikzcd} F(c) \arrow[r, equal, "F(\mathrm{id}_c)"] \arrow[d, "f"'] & F(c) \arrow[d, "gf"] \\ y \arrow[r, "g"'] & y' \end{tikzcd}$$

We can precompose this with the component of the counit ε of the adjunction $\Delta_G \dashv \mathrm{ev}_0$ at the object (c, f) to get

$$\begin{tikzcd} F(c) \arrow[r, equal, "F(\mathrm{id}_c)"] \arrow[d, equal, "F(\mathrm{id}_c)"'] & F(c) \arrow[r, equal, "F(\mathrm{id}_c)"] \arrow[d, "f"'] & F(c) \arrow[d, "gf"] \\ F(c) \arrow[r, "f"'] & y \arrow[r, "g"'] & y' \end{tikzcd}$$

This composite is itself the component of ε at the object (b, gf). Furthermore, ε is itself an ev_1-cocartesian lift. Hence if such a functor $C \times_Y \mathrm{Fun}(\Delta^1, Y) \to X$ lifting the square sends the components of ε to cocartesian lifts then by the cancellation property of cocartesian lifts Lemma A.2.18 also arbitrary ev_1-cocartesian lifts. □

Remark A.3.4

The only things that we really need to know in the previous corollary about the functors

$$C \xrightarrow{\Delta_G} C \times_Y \mathrm{Fun}(\Delta^1, Y) \xrightarrow{\mathrm{ev}_1} Y$$

is the following. We need to know that Δ_G has a retraction r, i. e. we have $\eta\colon \mathrm{id} \simeq r \circ \Delta_G$, that we have a natural transformation $\varepsilon\colon \Delta_G \circ r \Rightarrow \mathrm{id}$ such that $\varepsilon\Delta_G \simeq \Delta_G\eta^{-1}$, that ev_1 is a cocartesian fibration, that ε is a cocartesian lift of $\mathrm{ev}_1\,\varepsilon$ and that $\Delta_G \circ r$ sends cocartesian lifts to equivalences. From the last property one can then conclude that $r \simeq r \circ \Delta_G \circ r$ also sends cocartesian lifts to equivalences. As ε is a particular such a cocartesian lift we get that $r\varepsilon$ is invertible. From this one can deduce that Δ_G is left adjoint to r, with invertible unit.

Corollary A.3.5

In particular Corollary A.3.3 proves that we can restrict the adjunction of Corollary A.3.2 to its essential image and obtain an equivalence

$${}_*(\Delta_G)\colon\ \mathrm{Fun}^{\mathrm{cocart}}_{/Y}(\mathrm{ev}_1, p) \xrightarrow{\simeq} \mathrm{Fun}_{/Y}(G, p)$$

where we denote by $\mathrm{Fun}^{\mathrm{cocart}}_{/Y}(\mathrm{ev}_1, p)$ the full subcategory of $\mathrm{Fun}_{/Y}(\mathrm{ev}_1, p)$ on the cocartesian functors over Y, i. e. we proved that $\mathrm{ev}_1 : C \times_Y \mathrm{Fun}(\Delta^1, Y) \twoheadrightarrow Y$ is the free cocartesian fibration on G.

A.4 More Advanced Stability Properties of Adjunctions and Cocartesian Fibrations

In this section we address more intricate stability properties of adjunctions, like Lemma 1.3.5 the stability of pullbacks of left adjoints seen as objects in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$, under the assumption that the cospan legs are adjointable squares. From this one can immediately deduce that pullbacks of cocartesian fibrations seen as objects in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$ are cocartesian fibrations again if the assume that the cospan legs are cocartesian functors. After that we turn our attention to Lemma 1.2.13 the stability of right adjoints under pullback along cocartesian fibrations.

We start of with an extension of the pullback stability of right adjoints which have fully-faithful left adjoints.

Lemma A.4.1

Let

$$\begin{tikzcd} Y \arrow[r, "V"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\ A \arrow[r, "U"'] & B \end{tikzcd}$$

be a pullback square such that U has a fully-faithful left adjoint F and p is a cocartesian fibration. Then its pullback q is also a cocartesian fibration and the functor V is a cocartesian functor over U. Furthermore, the pullback V of U also has a fully-faithful left adjoint G and the pullback square is furthermore horizontally adjointable and in this adjointed square G is a cocartesian functor over F.

Proof. The first part of the statement is just Corollary A.2.9. From the proof of Lemma 1.2.11 we know that the left adjoint is produced by the pullback factorization

$$\begin{tikzcd} X \arrow[r, "G"] \arrow[rr, bend left, equal] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[r, "V"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ B \arrow[r, "F"'] \arrow[rr, bend right, equal] & A \arrow[r, "U"'] & B \end{tikzcd}$$

which in turn induces the pullback factorization

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, X) \arrow[r, "G_*"] \arrow[rr, bend left, equal] \arrow[d, "(\mathrm{ev}_0, p_*)"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, Y) \arrow[r, "V_*"] \arrow[d, "(\mathrm{ev}_0, q_*)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, X) \arrow[d, "(\mathrm{ev}_0, p_*)"] \\ X \times_B \mathrm{Fun}(\Delta^1, B) \arrow[r, "G \times_F F_*"'] \arrow[rr, bend right, equal] & Y \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "V \times_U U_*"'] & X \times_B \mathrm{Fun}(\Delta^1, B) \end{tikzcd}$$

hence both squares are also vertically adjointable by Lemma 1.2.11. In particular, we see that G is cocartesian over F. ☐

In order to prove the full version of the pullback stability of left adjoints in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$ our strategy will be to translate having a right adjoint into the Hom-equivalence characterisation of adjunctions, see Lemma A.2.11, and then use that commutative squares induce cocartesian functors between the so-called free cocartesian fibrations of the left and right hand vertical functors of the square, from Lemma A.2.8.

We first produce the appropriate commutative squares of Hom-equivalences from adjointable squares.

Lemma A.4.2

Let

$$\begin{tikzcd} A_0 \arrow[r, "H"] \arrow[d, "U_0"'] & A_1 \arrow[d, "U_1"] \\ B_0 \arrow[r, "K"'] & B_1 \end{tikzcd} \tag{A.4.0.1}$$

be a commutative square of functors such that the U_i both have left adjoints F_i and the square is adjointable. The by Lemma A.2.13 commutative square

$$\begin{tikzcd} B_0 \arrow[r, "K"] \arrow[d, "{(\eta_0, F_0)}"'] & B_1 \arrow[d, "{(\eta_1, F_1)}"] \\ \mathrm{Fun}(\Delta^1, B_0) \times_{B_0} A_0 \arrow[r, "K_* \times_K H"'] & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \end{tikzcd} \tag{A.4.0.2}$$

can be factorized vertically into commutative squares

$$\begin{tikzcd} B_0 \arrow[r, "K"] \arrow[d, "\Delta_{F_0}"] \arrow[dd, bend right=60, "{(\eta_0, F_0)}"'] & B_1 \arrow[d, "\Delta_{F_0}"] \arrow[dd, bend left=60, "{(\eta_1, F_1)}"] \\ B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[r, "K \times_H H_*"] \arrow[d, "\phi_0"] & B_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[d, "\phi_1"] \\ \mathrm{Fun}(\Delta^1, B_0) \times_{B_0} A_0 \arrow[r, "H \times_K K_*"'] & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \end{tikzcd} \tag{A.4.0.3}$$

Proof. As

$$\begin{tikzcd} B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[r, "H \times_K K_*"] \arrow[d, two heads, "\mathrm{ev}_1"'] & B_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[d, two heads, "\mathrm{ev}_1"] \\ A_0 \arrow[r, "K"'] & A_1 \end{tikzcd} \tag{A.4.0.4}$$

and by pullback also

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, B_0) \times_{B_0} A_0 \arrow[r, "K_* \times_K H"] \arrow[d, two heads, "\mathrm{ev}_1"'] & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \arrow[d, two heads, "\mathrm{ev}_1"] \\ A_0 \arrow[r, "H"'] & A_1 \end{tikzcd} \tag{A.4.0.5}$$

are cocartesian functors by Lemma A.2.8 both composites of the bottom square in Equation A.4.0.3 are cocartesian functors over the functor $H\colon A_0 \to A_1$. As the outer rectangle in Equation A.4.0.3 commutes both composites are lifts of the lower rectangle in

$$\begin{tikzcd}
 & B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[rr, "\phi_0"] \arrow[rr, phantom, shift right=3ex, "\simeq"] & & \mathrm{Fun}(\Delta^1, B_0) \times_{B_0} A_0 \arrow[dr, "K_* \times_K H"] & \\
B_0 \arrow[ur, hook, "\Delta_{F_0}"] \arrow[r, "K"] \arrow[d, hook, "\Delta_{F_0}"'] & B_1 \arrow[r, hook, "\Delta_{F_1}"] & B_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[rr, "\phi_1"] & & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \arrow[d, two heads, "\mathrm{pr}_1"] \\
B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[rrrru, dashed] \arrow[rr, two heads, "\mathrm{ev}_1"'] & & A_0 \arrow[rr, "H"'] & & A_1
\end{tikzcd}$$

that are even cocartesian functors over H. But cocartesian functors in particular preserve the ev_1-cocartesian lift

$$\begin{tikzcd}
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r, "\mathrm{ev}_0"] \arrow[rr, equal, bend right] & B \arrow[r, hook, "\Delta_F"] \arrow[d, phantom, "\Downarrow\mathrm{counit}"] & B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[d, two heads, "\mathrm{ev}_1"] \\
 & {} & A
\end{tikzcd}$$

and by the adjunction of Corollary 1.2.9 there exists a unique such lift, hence both lifts must be equivalent.

Alternatively, one could also argue in the following more structured way. By pulling back along H we can reduce without loss of generality to talking about our two lifts of the bottem left square in

$$\begin{tikzcd}
 & B_1 \arrow[r, hook, "\Delta_{F_1}"] & B_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[dr, "\phi_1"] & \\
B_0 \arrow[ur, "K"] \arrow[rr, "\exists!"] \arrow[d, hook, "\Delta_{F_0}"'] & & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_0 \arrow[r] \arrow[d, two heads, "\mathrm{pr}_0"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \arrow[d, two heads, "\mathrm{pr}_1"] \\
B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[rru, dashed] \arrow[rr, two heads, "\mathrm{ev}_1"'] & & A_0 \arrow[r, "H"'] & A_1
\end{tikzcd}$$

which now have been factorized into cocartesian functors over A_0. But by Corollary 1.2.9 and Corollary A.3.3, there already exists a unique such cocartesian lift, hence our two lifts must be equivalent. $\square$

Now we are prepared to prove that the pullback of a cospan of left adjoints in $\mathrm{Fun}(\Delta^1, \mathrm{Cat})$ with legs being adjointable squares produces again a left adjoint and the pullback structure maps will automatically also be adjointable.

Lemma 1.3.5

Consider the commutative cube of categories such that the top and bottom faces are pullbacks.

$$
\begin{tikzcd}
A_2 \arrow[dd, "F_2"'] & & A_1 \arrow[ll] \arrow[dd, "F_1"] & \\
& A_0 \arrow[ul] \arrow[dd] & & A \arrow[ul] \arrow[ll] \arrow[dd, "F"] \\
B_2 & & B_1 \arrow[ll, "F_0"] & \\
& B_0 \arrow[ul] & & B \arrow[ul] \arrow[ll]
\end{tikzcd}
\tag{1.3.0.5}
$$

If the F_i are left adjoints and the back face as well as the left face are adjointable, then the pullback induced functor F is also a left adjoint and the right and front faces are also adjointable. Additionally, if all the left adjoints F_i, or all the right adjoints U_i respectively, are fully-faithful, then so is the pulled-back one.

Proof. By Lemma A.4.2 we have commutative squares

$$
\begin{tikzcd}
B_0 \arrow[r, "K"] \arrow[d, hook, "\Delta_{F_0}"'] \arrow[dd, bend right=60, "{(\eta_0,F_0)}"'] & B_2 \arrow[d, hook, "\Delta_{F_2}"] \arrow[dd, bend left=60, "{(\eta_2,F_2)}"] & B_1 \arrow[l, "M"'] \arrow[d, hook, "\Delta_{F_1}"] \arrow[dd, bend left=60, "{(\eta_1,F_1)}"] \\
B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[r, "K \times_H H_*"] \arrow[d, "\phi_0"'] & B_2 \times_{A_2} \mathrm{Fun}(\Delta^1, A_2) \arrow[d, "\phi_2"] & B_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[l, "M \times_L L_*"'] \arrow[d, "\phi_1"] \\
\mathrm{Fun}(\Delta^1, B_0) \times_{B_0} A_0 \arrow[r, "K_* \times_K H"'] & \mathrm{Fun}(\Delta^1, B_2) \times_{B_2} A_2 & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \arrow[l, "M_* \times_M L"]
\end{tikzcd}
$$

Hence we can now pull back to obtain the commutative diagram

$$
\begin{tikzcd}
B \arrow[rrr] \arrow[dr] \arrow[dd, hook, dashed, "\exists ! \Delta_F"'] & & & B_1 \arrow[dr, "M"] \arrow[dd, hook, "\Delta_{F_1}"] & & \\
& B_0 \arrow[rrr, "K"] \arrow[dd, hook, "\Delta_{F_0}"'] & & & B_2 \arrow[dd, hook, "\Delta_{F_2}"] & \\
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[rrr] \arrow[dr] \arrow[dd, dashed, "\exists ! \simeq"'] \arrow[dddd, bend right=60, "\exists !"'] & & & B_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[dr, "M \times_L L_*"] \arrow[dd, "\phi_1", "\simeq"'] & & \\
& B_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[rrr, "K \times_H H_*"] \arrow[dd, "\phi_0"', "\simeq"] & & & B_2 \times_{A_2} \mathrm{Fun}(\Delta^1, A_2) \arrow[dd, "\phi_2", "\simeq"'] & \\
\mathrm{Fun}(\Delta^1, B) \times_B A \arrow[rrr] \arrow[dr] \arrow[dd, "\exists !"'] & & & \mathrm{Fun}(\Delta^1, B_1) \times_{B_1} A_1 \arrow[dr, "M_* \times_M L"] \arrow[dd] & & \\
& \mathrm{Fun}(\Delta^1, B_0) \times_{B_0} A_0 \arrow[rrr, "K_* \times_K H"] \arrow[dd] & & & \mathrm{Fun}(\Delta^1, B_2) \times_{B_2} A_2 \arrow[dd] & \\
B \times A \arrow[rrr] \arrow[dr] & & & B_1 \times A_1 \arrow[dr, "M \times L"] & & \\
& B_0 \times A_0 \arrow[rrr, "K \times H"'] & & & B_2 \times A_2 &
\end{tikzcd}
$$

in which each of the horizontal squares is a pullback and the left most vertical functors are the unique ones induces by pullback. In particular we obtain the equivalence

$$\begin{tikzcd}
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[rr, dashed, "\simeq"] \arrow[dr, "\mathrm{ev}"'] & & \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[dl, "\mathrm{ev}"] \\
& B \times A &
\end{tikzcd} \tag{A.4.0.6}$$

proving that F has a right adjoint. As $\phi \circ \Delta_F = (\eta, F)$ we can also deduce invertibility of η if all the η_i are invertible. For the counits one can apply a dual argument reversing the adjunction equivalences in the diagrams above. From the composed commutative diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\Delta_F"] \arrow[dd, bend right=60, "{(\eta, F)}"'] & B_i \arrow[d, "\Delta_{F_i}"] \arrow[dd, bend right=60, "{(\eta_i, F_i)}"'] \arrow[ddr, "\Delta_{F_i}"] & \\
B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[r] \arrow[d, "\phi"] & B_i \times_{A_i} \mathrm{Fun}(\Delta^1, A_i) \arrow[d, "\phi_i"] & \\
\mathrm{Fun}(\Delta^1, B) \times_B A \arrow[r] & \mathrm{Fun}(\Delta^1, B_i) \times_{B_i} A_i \arrow[r, "\psi_i"'] & B_i \times_{A_i} \mathrm{Fun}(\Delta^1, A_i)
\end{tikzcd} \tag{A.4.0.7}$$

one can then also deduce the adjointablility claim as in the proof of Lemma A.2.13. ☐

The following is an immediate corollary of Lemma A.2.7 with our definition of cocartesian fibrations.

Lemma A.4.3

Consider the commutative cube of categories such that the top and bottom faces are pullbacks.

$$\begin{tikzcd}
X_2 \arrow[d, two heads, "F_2"'] & & X_1 \arrow[ll] \arrow[d, two heads, "F_1"] & \\
& X_0 \arrow[ul] \arrow[d, two heads, "F_0"] & & X \arrow[ul] \arrow[ll] \arrow[d, "F"] \\
A_2 & & A_1 \arrow[ll] & \\
& A_0 \arrow[ul] & & A \arrow[ul] \arrow[ll]
\end{tikzcd} \tag{A.4.0.8}$$

If the F_i are cocartesian fibrations and $X_0 \to X_2$ and $X_1 \to X_2$ are cocartesian functors, then the pullback induced functor F is also a cocartesian fibration and $X \to X_0$, $X \to X_1$ are also cocartesian functors.

Proof. The commutative cube gives us the following commutative cube in which the top and bottom faces are again pullbacks.

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X_2) \arrow[dd] & & \mathrm{Fun}(\Delta^1, X_1) \arrow[ll] \arrow[dd] & \\
& \mathrm{Fun}(\Delta^1, X_0) \arrow[ul] \arrow[dd] & & \mathrm{Fun}(\Delta^1, X) \arrow[ul] \arrow[ll] \arrow[dd] \\
X_2 \times_{A_2} \mathrm{Fun}(\Delta^1, A_2) & & X_1 \times_{A_1} \mathrm{Fun}(\Delta^1, A_1) \arrow[ll] & \\
& X_0 \times_{A_0} \mathrm{Fun}(\Delta^1, A_0) \arrow[ul] & & X \times_A \mathrm{Fun}(\Delta^1, A) \arrow[ul] \arrow[ll]
\end{tikzcd}$$

By definition of cocartesian fibrations we know that all the vertical functors except the most right one have fully-faithful left adjoints. By definition of cocartesian functors we have that the back face and the left face are adjointable. Now Lemma 1.2.11 tells us that also the right most vertical functor has a fully-faithful left adjoint and that the front face and right face of the cube are also adjointable. Thus we get that F is a cocartesian fibration and the $A \to A_i$ are cocartesian functors. □

At last we are equipped to prove that general right adjoints are stable under pullbacks along cocartesian fibrations. Indeed, the bare fact that the pullback of the right adjoint is again a right adjoint can be obtained from the adjunction characterisation Lemma A.2.14 and several applications of Lemma 1.2.11. But we actually also recover the additional conclusions from Lemma A.4.1, i. e. that the pullback square is adjointable and that the pulled back right adjoint as well as its left adjoint constitute cocartesian functors.

Lemma 1.2.13

Let

$$\begin{tikzcd}
Y \arrow[r, "V"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "p"] \\
A \arrow[r, "U"'] & B
\end{tikzcd}$$

be a pullback square with p a cocartesian fibration and U has a left adjoint F. Then its pullback V also has a left adjoint, denoted by G, and the square is adjointable. Additionally, V is cocartesian over U and G is also a cocartesian functor over F. Furthermore, if the counit of $F \dashv U$ is invertible, then the counit of $G \dashv V$ will be invertible too.

Proof. From Corollary A.2.9 we already know that V is cocartesian over U. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X) \times_X^{(\mathrm{ev}_1, V)} Y \arrow[r, "\mathrm{pr}_0"] \arrow[d, "{(\mathrm{ev}_0, p_*, q)}"'] \arrow[dd, bend right=60, "p_* \times_p q"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, X) \arrow[d, "{(\mathrm{ev}_0, p_*)}"] \arrow[dd, bend left=60, "p_*"] \\
X \times_B^{(p, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1, U)} A \arrow[r, "{(\mathrm{pr}_0, \mathrm{pr}_1)}"] \arrow[d, "{(\mathrm{pr}_1, \mathrm{pr}_2)}"'] \arrow[dr, phantom, "\lrcorner"] & X \times_B^{(p, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \arrow[d, "\mathrm{pr}_1"] \\
\mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1, U)} A \arrow[r, "\mathrm{pr}_0"'] & \mathrm{Fun}(\Delta^1, B)
\end{tikzcd}$$

in which the bottom square is a pullback by definition and the composed rectangle by the defining pullback of Y. Hence by pullback cancellation we deduce that the top square is also a pullback. But by p being a cocartesian fibration we know that (ev_0, p_*) has a fully-faithful left adjoint. Thus we can infer from Lemma 1.2.11 that (ev_0, p_*, q) also has one. On the other hand, U having a left adjoint gives us by the dual of Lemma A.2.14 that the right hand vertical functor in the pullback

$$\begin{tikzcd}
X \times_B^{(p, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1, U)} A \arrow[r] \arrow[d, "\mathrm{pr}_0"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1, U)} A \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "p"'] & B
\end{tikzcd}$$

has a fully-faithful left adjoint. By Lemma 1.2.11 this means that also its pullback, the left hand vertical functor in the above diagram has a fully-faithful left adjoint. Now the commutative triangle

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X) \times_X^{(\mathrm{ev}_1, V)} Y \arrow[r, "{(\mathrm{ev}_0, p_*, q)}"] \arrow[dr, "\mathrm{ev}_0"'] & X \times_B^{(p, \mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1, U)} A \arrow[d, "\mathrm{pr}_0"] \\
& X
\end{tikzcd} \tag{A.4.0.9}$$

tells us that we factorized ev_0 into two functors that both have fully-faithful left adjoints, hence it itself has one.

By carefully tracing through the these steps one can also conclude that if the counit of $F \dashv U$ is invertible, that the counit of $G \dashv V$ is invertible too.

To show adjointability of the pullback with respect to this newly constructed adjoint it suffices by Lemma A.2.13 to show that

$$
\begin{tikzcd}
X \arrow[rrr, "p"] \arrow[dddd, "(\widetilde{\eta},G)"'] & & & B \arrow[dl, "\Delta_F"] \arrow[dddd, "(\eta,F)"] \\
& B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[urr, "\mathrm{ev}_0"] \arrow[dr, equal] & \arrow[l, phantom, "\Leftarrow \varepsilon"] & \\
& & B \times_A \mathrm{Fun}(\Delta^1, A) \arrow[ddr, "\phi"] & \\
& \mathrm{Fun}(\Delta^1, B) \times_B A \arrow[uu, "\psi"] \arrow[drr, equal] & & \\
\mathrm{Fun}(\Delta^1, X) \times_X Y \arrow[ur, "p_* \times_p q"] \arrow[rrr, "p_* \times_p q"'] & & & \mathrm{Fun}(\Delta^1, B) \times_B A
\end{tikzcd}
\tag{A.4.0.10}
$$

is invertible, where G is the fully-faithful left adjoint of V and $\widetilde{\eta}$ the unit of this adjunction. We furthermore know from the application of Lemma 1.2.11 above that

$$
\begin{tikzcd}
X \arrow[r, "{(\mathrm{id},\eta,F)}"] \arrow[dr, equal] & X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r, "{(\mathrm{pr}_1,\mathrm{pr}_2)}"] \arrow[d, "\mathrm{pr}_0"'] \arrow[dl, phantom, "\Nearrow \overline{\eta}"] & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[d, "\mathrm{ev}_0"] \arrow[dr, equal] \arrow[dr, phantom, "\Nearrow \varepsilon'"'] & \\
& X \arrow[r, "p"'] & B \arrow[r, "{(\eta,F)}"'] & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A
\end{tikzcd}
$$

is invertible, but as $\overline{\eta}$ was invertible to begin with, this is equivalent to the whiskering of ε' being invertible. Note also that the color-highlighted natural transformations in these two diagrams coincide. We also know by Lemma A.2.14 that the functor $(\widetilde{\eta}, G)$ is exactly the fully-faithful left adjoint of $\mathrm{ev}_0\colon \mathrm{Fun}(\Delta^1, X) \times_X^{(\mathrm{ev}_1,V)} Y \to X$. By construction of this left adjoint we have also the commutative diagram

$$
\begin{tikzcd}
X \arrow[d, "(\widetilde{\eta},G)"'] \arrow[dr, "{(\mathrm{id},\eta,F)}"] & & \\
\mathrm{Fun}(\Delta^1, X) \times_X^{(\mathrm{ev}_1,V)} Y \arrow[r, "{(\mathrm{ev}_0,p_*,q)}"'] \arrow[rr, bend right, "p_* \times_p q"'] & X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r, "{(\mathrm{pr}_1,\mathrm{pr}_2)}"'] & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A
\end{tikzcd}
$$

That means we can actually deduce from the invertibility of $\varepsilon'(\mathrm{pr}_1, \mathrm{pr}_2)(\mathrm{id}, \eta, F)$ the desired invertibility of Equation A.4.0.10.

Let us now look into why G is a cocartesian functor. In the first step we produced a fully-faithful left adjoint for (ev_0, p_*, q) by applying Lemma 1.2.11. One can picture the pullback definition of this left adjoint L as in diagram of pullbacks

$$\begin{tikzcd}
X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r, "(\mathrm{pr}_0,\mathrm{pr}_1)"'] \arrow[d, hook, "L"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddd, bend right=60, "(\mathrm{pr}_1,\mathrm{pr}_2)"'] & X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \arrow[d, hook, "\mathrm{cocart}"] \arrow[ddd, bend left=60, "\mathrm{pr}_1"] \\
\mathrm{Fun}(\Delta^1, X) \times_X^{(\mathrm{ev}_1,V)} Y \arrow[r, "\mathrm{pr}_0"] \arrow[d, "(\mathrm{ev}_0,p_*,q)"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right=50, "p_* \times_p q"'] & \mathrm{Fun}(\Delta^1, X) \arrow[d, "(\mathrm{ev}_0,p_*)"] \arrow[dd, bend left=50, "p_*"] \\
X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r, "(\mathrm{pr}_0,\mathrm{pr}_1)"] \arrow[d, "(\mathrm{pr}_1,\mathrm{pr}_2)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \arrow[d, "\mathrm{pr}_1"] \\
\mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r, "\mathrm{pr}_0"'] & \mathrm{Fun}(\Delta^1, B)
\end{tikzcd}$$

But as we proved in Lemma A.2.20, the functor cocart is in fact a cocartesian functor from pr_1 to p_*. Hence by Lemma A.4.3 the functor L is also a cocartesian functor between its pulled back cocartesian fibrations $(\mathrm{pr}_1, \mathrm{pr}_2)$ and $p_* \times_p q$. In the second step we applied Lemma 1.2.11 to the pullback

$$\begin{tikzcd}
X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r] \arrow[d, "\mathrm{pr}_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[d, "\mathrm{ev}_0"] \\
X \arrow[r, "p"'] & B
\end{tikzcd}$$

But as p is a cocartesian fibration we can deduce from Lemma A.4.1 that the pulled back fully-faithful left adjoint T of the functor pr_0 in the pullback above is in fact cocartesian over the left adjoint (η, F) of ev_0 above. Let us now put these two cocartesianness observations together with the way we composed L and T to get $(\widetilde{\eta}, G)$

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, two heads, "p"] \arrow[rrr, bend left=20, "G"] & X \times_B^{(p,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[r, "L"] \arrow[d, two heads] & \mathrm{Fun}(\Delta^1, X) \times_X^{(\mathrm{ev}_1,V)} Y \arrow[r, "\mathrm{pr}_Y"'] \arrow[dl, two heads, "p_* \times_p q"] & Y \arrow[d, two heads, "q"] \\
B \arrow[r, "{(\eta, F)}"'] \arrow[rrr, bend right=20, "F"'] & \mathrm{Fun}(\Delta^1, B) \times_B^{(\mathrm{ev}_1,U)} A \arrow[rr, "\mathrm{pr}_A"'] & & A
\end{tikzcd}$$

to see that G is in fact cocartesian over F, as we can just post compose these to cocartesian functors with the indicated cocartesian functor pr_Y over pr_A. $\square$

A.5 Relative Adjunctions

In this section we give an elementary proof of the statement that a cocartesian functor

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, two heads, "p"'] & & B \arrow[dl, two heads, "q"] \\ & I & \end{tikzcd} \tag{A.5.0.1}$$

between cocartesian fibrations over a base category I has a relative right adjoint U over I if and only if it does so fiberwise in the objects of I, i. e. the functors F_i on the fibers. We will furthermore enhance this result by Corollary 1.3.4, a characterisation of the cocartesianness of the right adjoint U in terms of the squares

$$\begin{tikzcd} A_i \arrow[r, "F_i"] \arrow[d, "k_!"'] & B_i \arrow[d, "k_!"] \\ A_j \arrow[r, "F_j"'] & B_j \end{tikzcd} \tag{A.5.0.2}$$

being adjointable.

The hard part is to show that the fiberwise given right adjoints glue together to give a right adjoint on the total categories, but we can alternatively characterise the existence of both fiberwise right adjoints and the total category right adjoint in terms of their relative slice categories

$$A_{q(b)} \times_{B_{q(b)}} (B_{q(b)})_{/b} \qquad \text{and} \qquad A \times_B B_{/b}$$

having terminal objects. Our strategy will be to exhibit the canonical inclusion functor

$$A_{q(b)} \times_{B_{q(b)}} (B_{q(b)})_{/b} \hookrightarrow A \times_B B_{/b}$$

as a right adjoint, thus preserving terminal objects.

We will obtain this by pulling back, via Lemma 1.2.13, the canonical fully-faithful right adjoint $\Delta_I \colon I \hookrightarrow \mathrm{Fun}(\Delta^1, I)$ along a certain cocartesian fibration that we are going to construct in the following lemma.

Lemma A.5.1

For any cocartesian functor

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & I & \end{tikzcd} \tag{A.5.0.3}$$

between cocartesian fibrations p and q the functor $q_* \circ \mathrm{pr}_1 \colon A \times_B \mathrm{Fun}(\Delta^1, B) \to \mathrm{Fun}(\Delta^1, I)$ is also a cocartesian fibration.

Proof. We can apply Lemma A.4.3 to the cube

$$
\begin{tikzcd}
A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \mathrm{Fun}(\Delta^1, B) \arrow[dr, "\mathrm{ev}_0"] \arrow[dd, two heads, "q_*" near start] & \\
& A \arrow[rr, "F" near start] \arrow[dd, two heads, "p" near start] & & B \arrow[dd, two heads, "q"] \\
\mathrm{Fun}(\Delta^1, I) \arrow[rr, equal] \arrow[dr, "\mathrm{ev}_0"] & & \mathrm{Fun}(\Delta^1, I) \arrow[dr, "\mathrm{ev}_0"] & \\
& I \arrow[rr, equal] & & I
\end{tikzcd}
$$

because we know by assumption that F is a cocartesian functor and by Corollary A.2.2 that q_* is a cocartesian fibration and ev_0 is also a cocartesian functor. □

Corollary A.5.2

Let F be a cocartesian functor

$$
\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& I &
\end{tikzcd}
\tag{A.5.0.4}
$$

between cocartesian fibrations p and q. Applying Lemma 1.2.13 to the cocartesian fibration

$$
A \times_B \mathrm{Fun}(\Delta^1, B) \twoheadrightarrow \mathrm{Fun}(\Delta^1, I)
$$

obtained from Lemma A.5.1 and the adjunction $\mathrm{ev}_1 \dashv \Delta_I$ we can conclude that the top horizontal functor in the pullback

$$
\begin{tikzcd}
A \times_B \mathrm{Fun}_{/I}(\Delta^1, B) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner" very near start] & A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[d, two heads, "q_*"] \\
I \arrow[r, "\Delta_I"'] & \mathrm{Fun}(\Delta^1, I)
\end{tikzcd}
$$

is a fully-faithful right adjoint.

Now the only thing left to do is to exhibit the adjunction we obtained as an adjunction over B, so that we can deduce from it also the fiberwise adjunctions.

Lemma A.5.3

The adjunction

$$
\begin{tikzcd}
A \times_B \mathrm{Fun}_{/I}(\Delta^1, B) \arrow[r, "\mathrm{incl}"'] \arrow[r, phantom, "\perp", shift left=2] & A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[l, bend right=40]
\end{tikzcd}
$$

from Corollary A.5.2 lives over B via the functors

$$\begin{tikzcd}
A \times_B \mathrm{Fun}_{/I}(\Delta^1, B) \arrow[rr, "\mathrm{incl}"] \arrow[dr, "\mathrm{ev}_1"'] & & A \times_B \mathrm{Fun}(\Delta^1, B) \arrow[dl, "\mathrm{ev}_1"] \\
& B &
\end{tikzcd} \tag{A.5.0.5}$$

i. e. the unit and the counit are invertible after postcomposing with the respective functors ev_1.

Proof. The counit is already invertible by construction. Unpacking the proof of Lemma 1.2.13 one sees that the unit of this adjunction is actually the cocartesian lift of

$$\begin{tikzcd}
A \times_B^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \arrow[rr, equal] \arrow[d, two heads] & & A \times_B^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \arrow[dd, two heads] \\
\mathrm{Fun}(\Delta^1, I) \arrow[dr, "\mathrm{ev}_1"'] \arrow[drr, equal, bend left] & \arrow[d, phantom, "\Downarrow\eta"] & \\
& I \arrow[r, hook, "\Delta_I"'] & \mathrm{Fun}(\Delta^1, I)
\end{tikzcd}$$

where η is the unit of the adjunction $\Delta_I \dashv \mathrm{ev}_1$. Note that this whiskered natural transformation becomes invertible after post-whiskering with $\mathrm{ev}_1 \colon \mathrm{Fun}(\Delta^1, I) \to I$. Recall furthermore from the applied Lemma A.4.3 that the top horizontal functors in

$$\begin{tikzcd}
A \times_B^{(F,\mathrm{ev}_0)} \mathrm{Fun}(\Delta^1, B) \arrow[r, "\mathrm{pr}_1"] \arrow[d, two heads, "q_* \circ \mathrm{pr}_1"'] \arrow[rr, bend left, "\mathrm{ev}_1"] & \mathrm{Fun}(\Delta^1, B) \arrow[r, "\mathrm{ev}_1"'] \arrow[d, two heads, "q_*"] & B \arrow[d, two heads, "q"] \\
\mathrm{Fun}(\Delta^1, I) \arrow[r, equal] \arrow[rr, bend right, "\mathrm{ev}_1"'] & \mathrm{Fun}(\Delta^1, I) \arrow[r, "\mathrm{ev}_1"'] & I
\end{tikzcd}$$

are cocartesian. Hence they preserve cocartesian lifts. The statement now follows from Lemma A.2.17. □

Next we make precise what a relative adjunction, i. e. an adjunction over a base category I is.

Definition A.5.4

Let

$$\begin{tikzcd}
A \arrow[rr, shift left, "F"] \arrow[rr, shift right, "G"'] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& I &
\end{tikzcd}$$

be two functors over I. We define a natural transformation over I a to be natural transformation $\alpha\colon F \Rightarrow G$ if as a functor $\alpha\colon A \to \mathrm{Fun}(\Delta^1, B)$ we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[rr, bend left, "\alpha"] \arrow[dr, "p"'] & \mathrm{Fun}_{/I}(\Delta^1, B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, B) \arrow[d, "q_*"] \\ & I \arrow[r, "\Delta_I"] & \mathrm{Fun}(\Delta^1, I) \end{tikzcd}$$

Furthermore, we say that a functor F over I as below on the left is left adjoint over I to a functor U over I as below on the right in

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "U"] \arrow[dr, "q"'] & & A \arrow[dl, "p"] \\ & I & \end{tikzcd}$$

if we have natural transformations $\eta\colon \mathrm{id} \Rightarrow UF$ and $\varepsilon\colon FU \Rightarrow \mathrm{id}$ over I, such that η and ε are unit and counit of an adjunction $F \dashv U$.

The final step is to show that relative adjunctions are in fact stable under changing the base category via pulling back, in order to deduce the desired fiberwise adjunctions.

Lemma A.5.5

Let

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & J & \end{tikzcd} \tag{A.5.0.6}$$

be a functor over a base category J such that it is left adjoint to U over J. Let us also be given another functor $t\colon I \to J$. Then

$$\begin{tikzcd} I \times_J A \arrow[rr, "I \times_J F"] \arrow[dr, "I \times_J p"'] & & I \times_J B \arrow[dl, "I \times_J q"] \\ & I & \end{tikzcd} \tag{A.5.0.7}$$

is left adjoint to $I \times_J U$ over I.

Proof. Let us denote the unit of the adjunction $F \dashv U$ by $\eta\colon A \to \mathrm{Fun}(\Delta^1, A)$ and the counit by $\varepsilon\colon B \to \mathrm{Fun}(\Delta^1, B)$. The fact that η becomes invertible when postwhiskered with the functor p to J can be expressed as the commutativity of the following square.

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "p"'] & \mathrm{Fun}(\Delta^1, A) \arrow[d, "p_*"] \\ J \arrow[r, "\Delta_J"'] & \mathrm{Fun}(\Delta^1, J) \end{tikzcd} \tag{A.5.0.8}$$

By definition of the relative functor category $\mathrm{Fun}_{/J}(\Delta^1, A)$ this induces us by pullback a functor $A \to \mathrm{Fun}_{/J}(\Delta^1, A)$ over J, which we will also call η by abuse of notation. Similarly, we can do this for ε to get a functor $B \to \mathrm{Fun}_{/J}(\Delta^1, B)$. Note that the identity natural transformation between on the functor id_A can also be upgraded to a functor $\Delta_A \colon A \to \mathrm{Fun}_{/J}(\Delta^1, A)$ over J, similarly for B. As morphism composition $\mathrm{Fun}(\Delta^1, A) \times_A \mathrm{Fun}(\Delta^1, A) \to \mathrm{Fun}(\Delta^1, A)$ also pulls back to a functor $\mathrm{Fun}_{/J}(\Delta^1, A) \times_A \mathrm{Fun}_{/J}(\Delta^1, A) \to \mathrm{Fun}_{/J}(\Delta^1, A)$, over J, one can also express the triangle identities of $F \dashv U$ using these functors over J. As we know that in the commutative cube

$$\begin{tikzcd}
I \times_J \mathrm{Fun}_{/J}(\Delta^1, A) \arrow[rr] \arrow[dr, "\exists!"'] \arrow[dd] & & \mathrm{Fun}_{/J}(\Delta^1, A) \arrow[dr] \arrow[dd] & \\
& \mathrm{Fun}(\Delta^1, I \times_J A) \arrow[rr] \arrow[dd] & & \mathrm{Fun}(\Delta^1, A) \arrow[dd, "p_*"] \\
I \arrow[rr, "t"] \arrow[dr, "\Delta_I"'] & & J \arrow[dr, "\Delta_I"'] & \\
& \mathrm{Fun}(\Delta^1, I) \arrow[rr, "t_*"'] & & \mathrm{Fun}(\Delta^1, J)
\end{tikzcd}$$

the back, the right and the front face are pullbacks, we are able to deduce by pullback cancellation that also the left hand face is a pullback, i. e. that we have an equivalence

$$I \times_J \mathrm{Fun}_{/J}(\Delta^1, A) \xrightarrow{\simeq} \mathrm{Fun}_{/I}(\Delta^1, I \times_J A)$$

over I. Hence by pulling back our relative version of η for example we get

$$I \times_J A \xrightarrow{I \times_J \eta} I \times_J \mathrm{Fun}_{/J}(\Delta^1, A) \xrightarrow{\simeq} \mathrm{Fun}_{/I}(\Delta^1, I \times_J A)$$

i. e. a relative natural transformation over I between the functors

$$\mathrm{id}_{(I \times_J A)} \quad \text{and} \quad (I \times_J U)(I \times_J F) \colon I \times_J A \to I \times_J A$$

over I. Now one just needs to contemplate that the all the squares in the commutative prisms

$$\begin{tikzcd}
\mathrm{Fun}_{/I}(\Delta^1, I \times_J A) \times_{(I \times_J A)} \mathrm{Fun}_{/I}(\Delta^1, I \times_J A) \arrow[rr, "\mathrm{comp}"] \arrow[dr] \arrow[dd] & & \mathrm{Fun}_{/I}(\Delta^1, I \times_J A) \arrow[dl] \arrow[dd] \\
& I \arrow[dd, "t"] & \\
\mathrm{Fun}_{/J}(\Delta^1, A) \times_A \mathrm{Fun}_{/J}(\Delta^1, A) \arrow[rr, "\mathrm{comp}"] \arrow[dr] & & \mathrm{Fun}_{/J}(\Delta^1, A) \arrow[dl] \\
& J &
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \times_J A \arrow[rr, "\Delta_{(I \times_J A)}"] \arrow[dd] \arrow[dr] & & \mathrm{Fun}_{/I}(\Delta^1, I \times_J A) \arrow[dl] \arrow[dd] \\
& I \arrow[dd, "t"] & \\
A \arrow[rr, "\Delta_A"] \arrow[dr, "p"'] & & \mathrm{Fun}_{/J}(\Delta^1, A) \arrow[dl] \\
& J &
\end{tikzcd}$$

are in fact pullbacks, to see that compositions of relative natural transformations over J and identity relative natural transformations over J pull back to compositions of and identity natural transformations over I. The same is also true for whiskerings of relative natural transformations, and also if we replace $p\colon A \to J$ by $q\colon B \to J$. Thus also the triangle identities of $F \dashv U$, which live over J pull back to triangle identities for $I \times_J F$ and $I \times_J U$. $\square$

Corollary A.5.6

Let F be a cocartesian functor

$$\begin{tikzcd}
A \arrow[rr, "F"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& I &
\end{tikzcd} \tag{A.5.0.9}$$

Then pulling back the adjunction over B from Lemma A.5.3 for each object b of B tells us that the inclusions

$$A_{q(b)} \times_{B_{q(b)}} (B_{q(b)})_{/b} \hookrightarrow A \times_B B_{/b}$$

are all fully-faithful right adjoints.

Proof. This follows from Lemma A.5.3 and Lemma A.5.5. $\square$

We are now equipped to conclude the fact that relative right adjoints can be given fiberwise.

Proposition 1.3.3

Let $p\colon A \twoheadrightarrow I$, $q\colon B \twoheadrightarrow I$ be two cocartesian fibrations over some category I and $F\colon A \to B$ a cocartesian functor over I with $\alpha\colon p \simeq qF$. Let us denote for every object i of I by $F_i\colon A_i \to B_i$ the restriction of F, A and B to the fiber over i, i. e. the pullbacks along $\Delta^0 \xrightarrow{i} I$. Then the following two statements are equivalent.

1. For all objects i of I the functor F_i has a right adjoint U_i.
2. the functor $F\colon A \to B$ has a right adjoint U, which is also a functor over I, i. e. we have that the canonical mate $q\varepsilon \circ \alpha U\colon pU \Rightarrow qFU \Rightarrow q$ of α is invertible, or equivalently the counit $\varepsilon\colon FU \Rightarrow \mathrm{id}$ of the adjunction is a natural transformation over I, i. e. the whiskering $q\varepsilon$ is invertible.

Proof. We employ the equivalent characterisation of a functor $G\colon X \to Y$ being left adjoint if and only if for all objects y of Y the relative slice categories $X \times_Y Y_{/y}$ have terminal objects, for both our statements. By Corollary A.5.6 now if $A_{q(b)} \times_{B_{q(b)}} (B_{q(b)})_{/b}$ admits a terminal object, then this will be preserved by the right adjoint

$$A_{q(b)} \times_{B_{q(b)}} (B_{q(b)})_{/b} \hookrightarrow A \times_B B_{/b}$$

Thus we have that also $A \times_B B_{/b}$ admits a terminal object. The other way around, if we know that $A \times_B B_{/b}$ has a terminal object and that the counit map ε_b witnessing this terminality actually already lives in the fiber over $q(b)$, then the terminal object already lives in and is terminal in $A_{q(b)} \times_{B_{q(b)}} (B_{q(b)})_{/b}$. □

For our purposes this is actually, as mentioned in the beginning of this section, not enough, we also need to control the cocartesianness of the glued-together right adjoint U. To this end, the following characterisation of cocartesianness of U in terms of just the fiberwise adjunctions $F_i \dashv U_i$ is useful.

Corollary 1.3.4

Let F be a cocartesian functor

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, two heads, "p"'] & & B \arrow[dl, two heads, "q"] \\ & I & \end{tikzcd} \tag{1.3.0.3}$$

between cocartesian fibrations p and q. Let us futhermore assume that F has a right adjoint $U\colon B \to A$ over the base I. Then the following statements are equivalent.

1. U is a cocartesian functor.
2. For every morphism $k\colon i \to j$ in I the square

$$\begin{tikzcd} A_i \arrow[r, "F_i"] \arrow[d, "k_!"'] & B_i \arrow[d, "k_!"] \\ A_j \arrow[r, "F_j"'] & B_j \end{tikzcd} \tag{1.3.0.4}$$

   is horizontally adjointable.

Proof. Cocartesianness of F means by definition that

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A) \arrow[r, "F_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}(\Delta^1, B) \arrow[d, "\mathrm{ev}_0"] \\ A \times_I \mathrm{Fun}(\Delta^1, I) \arrow[r, "F \times_{\mathrm{id}} \mathrm{id}"'] & B \times_I \mathrm{Fun}(\Delta^1, I) \end{tikzcd} \tag{A.5.0.10}$$

is vertically adjointable, i. e. the vertical mate is invertible and thus makes the square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A) \arrow[r, "F_*"] & \mathrm{Fun}(\Delta^1, B) \\ A \times_I \mathrm{Fun}(\Delta^1, I) \arrow[u, "\mathrm{cocart}"] \arrow[r, "F \times_{\mathrm{id}} \mathrm{id}"'] & B \times_I \mathrm{Fun}(\Delta^1, I) \arrow[u, "\mathrm{cocart}"'] \end{tikzcd} \tag{A.5.0.11}$$

commutative. But by the assumption that U is a relative right adjoint over I we also know that its horizontal mate is invertible, i. e. make the square

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A) \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}(\Delta^1, B) \arrow[l, "U_*"'] \arrow[d, "\mathrm{ev}_0"] \\ A \times_I \mathrm{Fun}(\Delta^1, I) & B \times_I \mathrm{Fun}(\Delta^1, I) \arrow[l, "U \times_{\mathrm{id}} \mathrm{id}"] \end{tikzcd} \tag{A.5.0.12}$$

commutative. Now we can apply Lemma A.2.19 to conclude that the horizontal mate of Equation A.5.0.11 is equivalent to the vertical mate of Equation A.5.0.12. But vertical adjointability of Equation A.5.0.12 is by definition cocartesianness of the functor U, i. e. the first statement. On the other hand, the composite diagram

$$\begin{tikzcd} A \arrow[r, "F"] & B \\ \mathrm{Fun}(\Delta^1, A) \arrow[u, "\mathrm{ev}_0"] \arrow[r, "F_*"] & \mathrm{Fun}(\Delta^1, B) \arrow[u, "\mathrm{ev}_0"'] \\ A \times_I \mathrm{Fun}(\Delta^1, I) \arrow[u, "\mathrm{cocart}"] \arrow[r, "F \times_{\mathrm{id}} \mathrm{id}"'] & B \times_I \mathrm{Fun}(\Delta^1, I) \arrow[u, "\mathrm{cocart}"'] \end{tikzcd} \tag{A.5.0.13}$$

is always horizontally adjointable as we have for every cocartesian fibration by invertibility of the unit of the adjunction cocart $\dashv (\mathrm{ev}_0, p_*)$ that

$$\begin{tikzcd} A \times_I \mathrm{Fun}(\Delta^1, I) \arrow[r, "\mathrm{cocart}"] \arrow[dr, equal] & \mathrm{Fun}(\Delta^1, A) \arrow[d, "{(\mathrm{ev}_0, p_*)}"'] \arrow[dr, "\mathrm{ev}_0"] & \\ & A \times_I \mathrm{Fun}(\Delta^1, A) \arrow[r, "\mathrm{pr}_0"'] & A \end{tikzcd} \tag{A.5.0.14}$$

commutes. Now as $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}(\Delta^1, A) \to A \times A$ is conservative, the horizontal adjointability of Equation A.5.0.11 is equivalent to horizontal adjointability of

$$\begin{tikzcd}
A \arrow[rr, "F"] & & B \\
\mathrm{Fun}(\Delta^1, A) \arrow[u, "\mathrm{ev}_1"] \arrow[rr, "F_*"] & & \mathrm{Fun}(\Delta^1, B) \arrow[u, "\mathrm{ev}_1"'] \\
A \times_I \mathrm{Fun}(\Delta^1, I) \arrow[u, "\mathrm{cocart}"] \arrow[rr, "F \times_{\mathrm{id}} \mathrm{id}"'] & & B \times_I \mathrm{Fun}(\Delta^1, I) \arrow[u, "\mathrm{cocart}"']
\end{tikzcd} \tag{A.5.0.15}$$

The equivalence of this and the second statement is due to the fact that we can check invertibility of the mate natural transformation objectwise. ☐

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