Factorial moments in a generalized lattice gas model

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We construct a simple multicomponent lattice gas model in one dimension in which each site can either be empty or occupied by at most one particle of any one of D species. Particles interact with a nearest-neighbor interaction which depends on the species involved. This model is capable of reproducing the relations between factorial moments observed in high-energy scattering experiments for moderate values of D. The factorial moments of the negative binomial distribution can be obtained exactly in the limit as D becomes large, and two suitable prescriptions involving randomly drawn nearest-neighbor interactions are given. These results indicate the need for considerable care in any attempt to extract information regarding possible critical phenomena from empirical factorial moments.

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I. INTRODUCTION

Factorial moments provide a useful tool in the analysis of high-energy scattering data such as obtained in e^+e^- scattering [1], $p\bar{p}$ scattering (at energies up to 900 GeV) [2], or the scattering of protons or heavy ions (e.g., ^{16}O and ^{32}S) by heavy nuclei at a projectile energy of 200 GeV/A [3]. One considers the full range of some variable (usually the rapidity) for which one knows the average multiplicity $\langle n\rangle$ and its dispersion $\langle \Delta n^2\rangle = \langle n^2\rangle - \langle n\rangle^2$. The data are then broken into M equal bins, and one constructs the factorial moments as a suitably normalized average of a combination of moments in each individual bin. Specifically,

$$F_q(M) = \left[\frac{1}{M}\sum_{i=1}^{M}\langle n(n-1)(n-2)\cdots(n-q+1)\rangle_i\right] \Big/ \left[\frac{1}{M}\sum_{i=1}^{M}\langle n\rangle_i\right]^q \quad . \tag{1}$$

Although interest in these forms has been motivated by theoretical considerations [4], factorial moments are merely one way to arrange the experimental data. Evidently, $F_q(M)$ probes certain combinations of the properties of the 1-, 2-, 3-, ..., q-body correlations between particles in each bin. If noninteracting particles are distributed in the various bins in a purely statistical fashion, all factorial moments are 1 for all M. This result is completely inconsistent with the data. Consider, for example, $p\bar{p}$ scattering at 900 GeV. Here, $F_2(M)$ is seen to grow from 1 (for small M) to roughly 1.7 (for the largest values of M considered). Over the same range, $F_3(M)$ ranges from 1 to 4.2, $F_4(M)$ ranges from 1 to 15, and $F_5(M)$ ranges from 1 to 80. This growth in the factorial moments with M is often called "intermittency" in the literature and is frequently regarded as indicating the presence of fluctuations of many different sizes.

It has been observed [5] that the relations *between* the various factorial moments for $p\bar{p}$ scattering (and all of the other physical processes mentioned above) can be reproduced with remarkable accuracy by the negative binomial (NB) distribution for which

$$F_q^{\mathrm{NB}}(M) = (1+cM)(1+2cM)\cdots(1+[q-1]cM) \ , \tag{2}$$

with

$$c = \frac{\langle \Delta n^2\rangle - \langle n\rangle}{\langle n\rangle^2} \quad . \tag{3}$$

Plots of $F_q(M)$ versus M evidently depend on the global averages $\langle n\rangle$ and $\langle \Delta n^2\rangle$. Such plots will be different for the various physical processes considered and are not reproduced by the negative binomial distributions as given by Eq. (2) [6]. However, since F_2^{NB} is simply $(1+cM)$, it is tempting to consider the more "universal" plots of F_q versus F_2 [7]. Such plots no longer depend on the parameter c and invite the comparison of data from very different processes [8]. Such plots have been made over the available range $1 < F_2 < 1.7$. They do reveal universal behavior [9] and are in striking agreement with the curves obtained from the negative binomial distribution.

The dramatic growth of the factorial moments with M led Bialas and Peschanski to note that "an observation of a variation in $\langle F_i\rangle$ (our F_q) with δy (our M) indicates the presence of genuine fluctuations which must have some physical origin" [4]. Many authors have taken up the challenge of describing this physical origin [10]. Some have concentrated on the apparent presence of fluctuations of many different sizes and considered models incorporating a variety of critical phenomena [11]. Others

have studied both schematic and more realistic versions of cascade models [12,5].

In two interesting papers, Chau and Huang have offered a different kind of insight [13]. They imagine that the full range of rapidity corresponds to the N sites of a one-dimensional Ising (or lattice gas) model [14]. This model is exactly solvable. They consider the q-body correlations and the factorial moments which come from a lattice gas model (in the limit $N \to \infty$). The two parameters of the Hamiltonian are determined by fixing the global values of $\langle n \rangle$ and $\langle \Delta n^2 \rangle$. The resulting factorial moments have the form

$$F_q^{\mathrm{LG}}(M) = \sum_{k=0}^{q-1} \frac{q!(q-1)!}{(q-k)!k!(q-1-k)!2^k} (cM)^k \ , \qquad (4)$$

where c is given by Eq. (3). This result is both rather more and rather less than meets the eye.

Let us address the "less" first. While the lattice gas factorial moments are not identical to the corresponding moments of the negative binomial distribution, both share the common small M expansion

$$F_q(M) = 1 + \frac{q(q-1)}{2} cM + O(c^2 M^2) \ . \qquad (5)$$

We have already noted that the leading term is due to "one-body effects." It should come as no surprise that the term of order M is precisely due to two-body correlations. Under the assumption that genuine two-body correlations have a finite range, the q dependence of the term of order M is uniquely determined. The coefficient c, which can be expressed as a suitable integral of the two-body correlation function, is also fixed by the global dispersion, $\langle \Delta n^2 \rangle$. Thus, the observation of Chau and Huang that the factorial moments "given by single negative binomials are almost exactly the same as ours for $F_2 \to 1$" is a trivial consequence of the rules of the game (i.e., the fixing of $\langle n \rangle$ and $\langle \Delta n^2 \rangle$) and the fact that the Ising model predicts many-body correlations of a finite range.

Chau and Huang further note that the negative binomial distributions are more successful than the lattice gas model for values of F_2 larger than 1.5. This is the domain where terms of order M^2 and higher, which are not model independent, play a role. A few comments on the existing data are in order before continuing. Data for 200 GeV/A ^{16}O and ^{32}S scattering on emulsions are limited to the range $1 < F_2 < 1.35$. Thus, these data never really probe the interesting (model-dependent) regions of F_2. (Over this range, the difference between F_3^{LG} and F_3^{NB} is less than 2.8% which is small compared with the experimental uncertainties.) The data for 200 GeV p on emulsion and for 900 GeV $p\bar{p}$ scattering have a somewhat wider range covering $1 < F_2 \leq 1.7$. At the upper end of this range, the difference between lattice gas and negative binomial forms for F_3 grows to 6.2% while the nonlinear terms in the negative binomial form account for 24% of F_3. Uncertainties in the data in this region (both systematic and statistical) are approximately 7.2%. The situation is similar for the existing data on F_4 and F_5. It is possible to find convincing empirical evidence for higher than linear terms in F_q. It is more difficult to distinguish empirically between negative binomial and lattice gas models although the former does provide a better global fit.

We believe that Chau and Huang have provided something very positive in their work. Their model admits an elementary generalization which shows one possible route which leads smoothly from the lattice gas model to the negative binomial distributions [15]. While the nature of this route may be of limited empirical value given the current status of the data, it strikes us as being of some theoretical importance. This generalization, which is the primary focus of this paper, is simply stated: Consider a one-dimensional lattice gas model (again in the limit where the number of sites, N, is infinite) where each site can either be empty or occupied by one particle which can be of any one of D species. Each species has a chemical potential μ_d, and each pair of species has a nearest-neighbor interaction of strength $\epsilon_{dd'}$. Like the ordinary lattice gas (with $D = 1$), this model allows for the analytic determination of the q-body correlation functions and the factorial moments, $F_q(M)$, using the obvious generalization of standard techniques. As usual, this involves the construction and diagonalization of a matrix $\mathcal{M}$ related to the partition function for one pair of adjacent sites.

The factorial moments for this generalized lattice gas model can then be specified exactly in terms of certain (weighted) moments of an elementary function of the eigenvalues of $\mathcal{M}$. We shall show that, for any value of D, these moments can be chosen to reproduce all factorial moments of the ordinary lattice gas model. Further, we shall show that these moments can also be chosen to approach the factorial moments of the negative binomial distribution. In the limit $D \to \infty$, this model permits the exact reproduction of *all* factorial moments of the negative binomial distribution.

Given the specific form of $\mathcal{M}$, there is no guarantee that these conditions on its eigenvalues can be met for any choice of the parameters appearing in the related Hamiltonian. We shall, thus, give two methods for the selection of the various chemical potentials and nearest-neighbor interaction parameters in $\mathcal{M}$ which explicitly reproduce the factorial moments of the negative binomial distribution. This choice is essentially a "random dynamics" model in which the D chemical potentials are set equal and in which the various nearest-neighbor interactions are drawn at random according to a certain distribution. The random nature of this model and the smallness of the dispersion in its factorial moments for D finite and small provides some understanding for the success of cascade model calculations of these processes and for the relative insensitivity of such calculations to the details of input parameters.

The organization of this paper will be as follows. In Sec. II we summarize the results of Chau and Huang. One purpose of this summary is to establish the notation of the lattice gas model (as opposed to the Ising model) which will later be generalized. In the process we shall establish a useful (and more general) theorem regarding the factorial moments and shall explain precisely which

features of the Ising model are responsible for the form of Eq. (4). Along the way, we shall point out why the general form of Eq. (5) is more a matter of definition than of physical content.

In Sec. III we shall establish our generalization to a multicomponent lattice gas model containing D species, outline the techniques for its analytic solution, and present the general form for the resulting factorial moments.

In Sec. IV we shall consider the constraints which must be imposed on this generalized model if it is (i) to reproduce the results of Chau and Huang or (ii) to reproduce the factorial moments of the negative binomial distribution. These constraints will be established analytically as a well-defined distribution of the eigenvalues of a certain $(D+1)$-dimensional matrix. We shall also report the results of numerical studies which allow us to express these constraints in terms of a well-defined but random distribution of the $D(D+1)/2$ nearest-neighbor interaction strengths between the D species.

A variety of conclusions will be drawn in Sec. V.

Two short appendixes are provided to deal with more technical matters.

II. LATTICE GAS MODEL

Here, we follow the arguments of Chau and Huang with the notational difference that we consider a one-dimensional lattice gas rather than an Ising model. We consider N microscopic sites which can have an occupation number of either 0 or 1. Particles occupying adjacent sites will experience a nearest-neighbor interaction of strength ϵ. The system is assumed to be cyclic in the sense that particle N interacts with particle 1 as well as with particle $(N-1)$. The Hamiltonian for the system is simply

$$H = \epsilon[n_1 n_2 + n_2 n_3 + \cdots + n_{N-1} n_N + n_N n_1] \quad . \tag{6}$$

The partition function for this system is [16]

$$\sum \exp\left[-H - \mu \sum_{i=1}^{N} n_i\right] \quad . \tag{7}$$

The external summation here covers the 2^N terms where each n_i has the value 0 or 1. This problem is rendered trivial by constructing a two-dimensional matrix $\mathcal{M}$ such that

$$\mathcal{M}_{n_1 n_2} = \exp\left[-\epsilon n_1 n_2 - \frac{1}{2}\mu n_1 - \frac{1}{2}\mu n_2\right] \quad . \tag{8}$$

It is convenient to let the matrix indices run over 0 and 1.

The partition function for the system is now simply the trace of $\mathcal{M}^N$. This trace is most easily constructed by defining the orthogonal matrix θ which diagonalizes $\mathcal{M}$:

$$\mathcal{M}_d = \theta^T \mathcal{M} \theta \quad . \tag{9}$$

The diagonal form $\mathcal{M}_d$ has two eigenvalues $\lambda_\pm$, and we shall choose θ such that $(\mathcal{M}_d)_{00}$ is the eigenvalue of larger magnitude, λ_+. (Given the form of $\mathcal{M}$, it is clear that $\lambda_+ > 0$.) We then find that

$$\mathrm{Tr}[\mathcal{M}^N] = \mathrm{Tr}[\theta \mathcal{M}_d{}^N \theta^T] = \mathrm{Tr}[\mathcal{M}_d{}^N] = \lambda_+^N + \lambda_-^N \quad . \tag{10}$$

In the limit $N \to \infty$, we can neglect the term λ_-^N at the cost of introducing an error which is exponentially small in N.

In order to calculate the various correlation functions, we introduce the number operator matrix for site i, n_i. Clearly, $(n_i)_{n_1 n_2} = \delta_{1 n_1} \delta_{1 n_2}$, i.e.,

$$n_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{11}$$

The number operator at each site is an idempotent with $n_i^2 = n_i$. Since all sites are equivalent [due to the cyclic form of Eq. (6)], the average number of particles per site is simply

$$\langle n_i \rangle = \mathrm{Tr}\left[n_i \mathcal{M}^N\right] / \mathrm{Tr}\left[\mathcal{M}^N\right] \quad . \tag{12}$$

Given the fact that $\mathrm{Tr}[\mathcal{M}^N] = \lambda_+^N$ in the large N limit, we see that it is useful to replace $\mathcal{M}_d$ by another diagonal matrix $\bar{\mathcal{M}}_d$, in which each diagonal element is simply divided by λ_+. Thus, $(\bar{\mathcal{M}}_d)_{00} = 1$ and $(\bar{\mathcal{M}}_d)_{11} = \bar{\lambda}_-$ where $|\bar{\lambda}_-| < 1$. With this notation,

$$\langle n_i \rangle = \mathrm{Tr}\left[n_i \theta \bar{\mathcal{M}}_d^N \theta^T\right] \quad . \tag{13}$$

Given the trivial structure of n_i,

$$\langle n_i \rangle = (\theta \bar{\mathcal{M}}_d^N \theta^T)_{11} \to (\theta_{10})^2 \quad , \tag{14}$$

where the last form applies in the limit $N \to \infty$. While θ_{10} and $\bar{\lambda}_-$ can be determined in terms of ϵ and μ following an explicit diagonalization of $\mathcal{M}$, this is not necessary. It is our intention to impose global constraints on $\langle n \rangle = N\langle n_i \rangle$ and $\langle \Delta n^2 \rangle$. This can be done with equal ease working with θ_{10} and $\bar{\lambda}_-$.

The two-body correlation function can now be written immediately as

$$\langle n_i n_{i+j} \rangle = (\theta \bar{\mathcal{M}}_d^j \theta^T)_{11} (\theta \bar{\mathcal{M}}_d^{N-j} \theta^T)_{11} \quad . \tag{15}$$

Taking the large N limit and making the usual assumption that $N - j$ is always $O(N)$, we find

$$\langle n_i n_{i+j} \rangle = \langle n_i \rangle [\langle n_i \rangle + (1 - \langle n_i \rangle) \bar{\lambda}_-^j] \quad . \tag{16}$$

Equation (16) is valid for $j \geq 0$. The structure of this two-body correlation function is instructive and very general. For large separations (large j), $\langle n_i n_{i+j} \rangle$ approaches the uncorrelated, statistical value of $\langle n_i \rangle^2$. The approach to this asymptotic value is exponentially fast. For small j, there are significant short-range correlations. For $j = 0$, we find $\langle n_i n_{i+j} \rangle = \langle n_i \rangle$ independent of $\bar{\lambda}_-$ which is merely a reflection of the fact that the number operator is idempotent.

It is elementary to determine the global dispersion $\langle \Delta n^2 \rangle$. Since it is our intention to constrain $\langle \Delta n^2 \rangle$ to

be consistent with data, let us deal with this problem immediately. We first construct

$$\langle n^2\rangle = \sum_{i_1,i_2=1}^{N} \langle n_{i_1} n_{i_2}\rangle = N\langle n_i\rangle + 2\sum_{i_1<i_2}\langle n_{i_1} n_{i_2}\rangle \quad . \tag{17}$$

It is, of course, possible to do the summations in Eq. (17) exactly given the form of Eq. (16). For purposes of later arguments, it is more useful to obtain $\langle n^2\rangle$ approximately by making approximations which neglect terms of order $1/N$. Thus, we write

$$\langle n^2\rangle = \langle n\rangle + 2\langle n_i\rangle^2 \sum_{i_1<i_2} 1 + 2\langle n_i\rangle(1-\langle n_i\rangle)\sum_{i_1<i_2}\bar{\lambda}_-^{i_2-i_1}. \tag{18}$$

The first sum in Eq. (18) will be approximated by $N^2/2$. The second sum is more interesting. Since $\bar{\lambda}_-^j$ converges exponentially with j, we shall allow the i_2 sum to extend over all values $1 \le (i_2 - i_1) \le \infty$. The sum over i_1 then merely introduces a factor of N. Further, we neglect $\langle n_i\rangle$ compared to 1. These approximations each introduce errors of order $1/N$ which are acceptable as $N \to \infty$. We immediately obtain

$$\langle n^2\rangle = \langle n\rangle + \langle n\rangle^2 + 2\langle n\rangle\frac{\bar{\lambda}_-}{1-\bar{\lambda}_-} \quad . \tag{19}$$

We are now able to set $\bar{\lambda}_-$ in order to reproduce any desired dispersion.

The calculation of factorial moments can be simplified significantly when the number operator is idempotent. This issue is addressed in Appendix A where appropriate operator (and ensemble average) identities are established. Using the definition of the factorial moments, Eq. (1), using the simplifying result of Eq. (A6), and adopting the same large N approximations, we can obtain $F_2(M)$. [Now, the sums analogous to those in Eq. (17) extend to an upper limit of N_B where $N_B = N/M$. Since the number of bins, M, is finite, we are also concerned with the limit $N_B \to \infty$.] We find

$$F_2^{\mathrm{LG}}(M) = 1 + cM \ , \tag{20}$$

where

$$c = \frac{2}{\langle n\rangle}\frac{\bar{\lambda}_-}{1-\bar{\lambda}_-} \quad . \tag{21}$$

Returning to Eqs. (15) and (16), it is now elementary to construct the q-body correlation functions as

$$\begin{aligned}\langle n_{i_1} n_{i_1+i_2}\cdots n_{i_1+\cdots+i_q}\rangle = {} & \langle n_i\rangle[\langle n_i\rangle + (1-\langle n_i\rangle)\bar{\lambda}_-^{i_2}] \\ & \times[\langle n_i\rangle + (1-\langle n_i\rangle)\bar{\lambda}_-^{i_3}] \\ & \times\cdots[\langle n_i\rangle + (1-\langle n_i\rangle)\bar{\lambda}_-^{i_q}] \quad .\end{aligned} \tag{22}$$

Equation (22) applies for $i_1, i_2, \ldots, i_q \ge 0$. This form, which is not unique to the lattice gas model, determines the form of the factorial moments and merits comment. The q-body correlation function is completely determined by the two-body correlation function and $\langle n_i\rangle$. More precisely, the q-body correlation function is a product of the $(q-1)$ two-body correlators between adjacent particles. Finally, the individual two-body correlators are given as the sum of a statistical term and a short-range piece. These qualitative features are sufficient to determine the associated factorial moments uniquely without any detailed information about the precise nature of the short-range part of the two-body correlator. To emphasize this independence, we shall make the replacement of $(1-\langle n_i\rangle)\bar{\lambda}_-^j$ by $g(j)$ where the only restriction on $g(j)$ is that it is of short range.

Now let us turn to an arbitrary factorial moment in the limit $N \to \infty$. We again use the operator identity derived in Appendix A, Eq. (A6), and replace sums by integrals to obtain

$$\begin{aligned}F_q^{\mathrm{LG}}(M) = q!\left(\frac{M}{\langle n\rangle}\right)^q \int_0^{N_B} dx_1 \int_0^{N_B-x_1} dx_2 \cdots \int_0^{N_B-x_1-\cdots-x_{q-1}} dx_q \\ \times\frac{\langle n\rangle}{N}\left[\frac{\langle n\rangle}{N} + g(x_2)\right]\left[\frac{\langle n\rangle}{N} + g(x_3)\right]\cdots\left[\frac{\langle n\rangle}{N} + g(x_q)\right] \quad .\end{aligned} \tag{23}$$

The structure of this result is clear. We now imagine expanding the $(q-1)$ square brackets. Since the function $g(x_i)$ is short range, we can extend the x_i integration to infinity for any of those 2^{q-1} terms which contains a factor of $g(x_i)$. Let us introduce the notation

$$G = \int_0^\infty dx\ g(x) \quad . \tag{24}$$

The term $O(M^0)$ can only come from picking the statistical factor in each term. There is one way to do this. We obtain no factors of G. The remaining integrals give a factor of $N_B^q/q!$. Hence, the leading term in $F_q^{\mathrm{LG}}(M)$ is always 1, as expected. The term $O(M)$ comes from picking $(q-1)$ statistical factors and one factor of g. There are $(q-1)$ ways to do this, and we obtain a factor of G. The remaining integrals give a factor of $N_B^{q-1}/(q-1)!$. Hence, the next term in $F_q^{\mathrm{LG}}(M)$ is precisely $(2GM/\langle n\rangle)q(q-1)/2$. The general result is now obvious:

$$F_q^{\mathrm{LG}}(M) = \sum_{k=0}^{q-1}\left[\frac{2GM}{\langle n\rangle}\right]^k \frac{q!(q-1)!}{(q-k)!k!(q-1-k)!2^k} \quad . \tag{25}$$

Comparing Eq. (25) with $q = 2$ with the result of Eq. (20), we see that we can make the identification

$$c = \frac{2G}{\langle n \rangle} \ . \tag{26}$$

Equations (25) and (26) are precisely the forms found by Chau and Huang as quoted in Eq. (4). The present derivation makes it clear that these results *do* depend on the fact that the q-body correlator is a product of the $(q-1)$ consecutive two-body correlators and that they do *not* depend on the specific form of the short-range two-body correlations.

The fact that

$$F_q(M) = 1 + \frac{q(q-1)}{2} cM + O(c^2 M^2) \tag{27}$$

is even more general. It depends only on the fact that the q-body correlation function contains no long-range terms and thus includes all one-dimensional models. For any model we can consider

$$\langle n_{i_1} n_{i_1+i_2} \cdots n_{i_1+\cdots+i_q} \rangle \tag{28}$$

in the limit where one of the indices (e.g., i_k) is small and all others are large. In the absence of long-range correlations, this limit must reduce to

$$\langle n_i \rangle^{q-2} \langle n_i n_{i+i_k} \rangle \ . \tag{29}$$

(Translational invariance ensures that the final term here depends only on i_k.) Such terms are the only ones which contribute to the linear piece of $F_q(M)$. Their q dependence is protected by elementary combinatorics. The coefficient c is protected by the fact that the two-body correlation has been adjusted to fit the empirical global dispersion, $\langle \Delta n^2 \rangle$. In a similar fashion, the terms $O(M^2)$ receive contributions from both two- and three-body correlations. If we had insisted on establishing global constraints on $\langle n^3 \rangle$, these terms would be similarly protected.

Thus, the fact that lattice gas factorial moments agree with those coming from the negative binomial distribution in the limit $cM \to 0$ provides *no* indication that the lattice gas model is correct. It is merely an indication of the fact that there are no long-range correlations and that the "rules of the game" are to consider only models with fixed global $\langle n \rangle$ and $\langle \Delta n^2 \rangle$.

III. A GENERALIZED LATTICE GAS MODEL

As noted in Sec. II, the ordinary lattice gas model does not reproduce the factorial moments of the negative binomial distribution. The fact that there is agreement in the small cM limit is a consequence of general properties of one-dimensional models and is not indicative of any particular merit of the lattice gas model. In this section we shall construct a simple generalization of this lattice gas model which has the capacity to reproduce the negative binomial factorial moments *exactly*. This model is a conceptually simple one-dimensional model which is also exactly solvable (in a sense which is appropriate for constructing factorial moments).

Consider a lattice gas in which site i is either unoccupied or is occupied by at most one particle which can be of any one of D species. Each species has its own chemical potential μ_d. Particles again have a nearest-neighbor interaction. The strength of the interaction between a particle of species d and a particle of species d' is $\epsilon_{dd'}$.

This model can be solved using precisely the standard techniques of Sec. II. The matrix $\mathcal{M}$ now becomes a real, symmetric matrix of dimension $D+1$. The elements in this matrix are

$$\begin{aligned} \mathcal{M}_{00} &= 1 \ , \\ \mathcal{M}_{0d} &= \exp\left[-\mu_d/2\right] \ , \\ \mathcal{M}_{dd'} &= \exp\left[-\epsilon_{dd'} - \mu_d/2 - \mu_{d'}/2\right] \ . \end{aligned} \tag{30}$$

(We allow the matrix indices to run from 0 to D. Again, the inverse temperature, β, has been set equal to 1.) The number operator at site i is also a $(D+1)$-dimensional matrix having the form $n_i = \mathbb{1} - T$ with $T_{dd'} = \delta_{d0}\delta_{d'0}$, i.e.,

$$n_i = \begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \ . \tag{31}$$

Both this number operator and T are idempotent, so that the results of Eqs. (A3) and (A6) remain valid both at the operator level and at the level of ensemble averages.

We again find the transformation θ which diagonalizes $\mathcal{M}$ and places the eigenvalue of largest magnitude in $(\mathcal{M}_d)_{00}$ [17]. As before, we construct the matrix $\bar{\mathcal{M}}_d$ by dividing each element of $\mathcal{M}_d$ by the eigenvalue of largest magnitude. (The form of $\mathcal{M}$ again ensures that this largest eigenvalue will be positive.) Thus, $(\bar{\mathcal{M}}_d)_{00}$ is again 1. While we shall retain this notation, it is not completely necessary in practice. The spirit of the model is, ultimately, to take the limit as $N \to \infty$ for fixed $\langle n \rangle$. In this limit, the average occupation per site, $\langle n_i \rangle$, tends to zero. This limit will be realized by taking the limit as all of the $\mu_d \to +\infty$. In this limit, the coupling of the $D \times D$ submatrix (with $d, d' \neq 0$) to the remaining elements of $\mathcal{M}$ becomes arbitrarily weak. In the limit, it is legitimate to treat this coupling in lowest-order perturbation theory. We shall return to this point in Sec. IV.

With these preliminaries in hand, the construction of the various correlation functions and factorial moments proceeds as before. The average number of particles per site is simply

$$\langle n_i \rangle = \frac{\langle n \rangle}{N} = \mathrm{Tr}[(\mathbb{1} - T)\theta \bar{\mathcal{M}}_d^N \theta^T] \ . \tag{32}$$

In the limit $N \to \infty$, $\bar{\mathcal{M}}_d^N$ may be set equal to T with exponentially small errors. The two terms in the factor $(\mathbb{1} - T)$ must be treated separately [18]. One immediately finds

$$\langle n_i \rangle = 1 - (\theta_{00})^2 \ . \tag{33}$$

As usual, $\langle n \rangle$ and hence $(\theta_{00})^2$ will be fixed by experiment.

The two-body correlator is readily calculated as

$$\langle n_i n_{i+j} \rangle = \mathrm{Tr}[(\mathbb{1} - T)\theta \bar{\mathcal{M}}_d^j \theta^T (\mathbb{1} - T)\theta T \theta^T] \ . \quad (34)$$

Expanding the factors of $(\mathbb{1} - T)$, one immediately obtains

$$\langle n_i n_{i+j} \rangle = \langle n_i \rangle \left[\langle n_i \rangle + (1 - \langle n_i \rangle) \sum_{\ell=1}^{D} (\theta_{0\ell})^2 \bar{\lambda}_\ell^j \right] \ . \quad (35)$$

This form is virtually identical to that of Eq. (16). The first term is again the asymptotic statistical probability of finding one particle (of any species) on each of the two sites. Since $|\bar{\lambda}_\ell| < 1$, the second term here again vanishes exponentially for large j. However, in this case, we are confronted with a sum of exponentials rather than a single term.

Following the arguments from Eqs. (17) to (21) we find that [19]

$$F_2^{\mathrm{GLG}}(M) = 1 + cM \ , \quad (36)$$

with

$$c = \frac{\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle^2} = \frac{2}{\langle n \rangle} \sum_{\ell=1}^{D} b_\ell^2 \frac{\bar{\lambda}_\ell}{1 - \bar{\lambda}_\ell} \ . \quad (37)$$

Here, we have taken the notational liberty of exploiting the fact that θ is an orthogonal matrix and made the substitution

$$b_\ell^2 = \frac{(\theta_{0\ell})^2}{1 - \langle n_i \rangle} \quad \text{with} \quad \sum_{\ell=1}^{D} b_\ell^2 = 1 \ . \quad (38)$$

Again, the similarities between Eqs. (37) and (21) are striking. The only difference is the presence of the sum over ℓ.

Since certain differences arise at the level of the three-body correlation function, this term is worth discussing specifically. Obviously,

$$\langle n_i n_{i+j} n_{i+j+k} \rangle = \mathrm{Tr}[(\mathbb{1} - T)\theta \bar{\mathcal{M}}_d^j \theta^T (\mathbb{1} - T)\theta \bar{\mathcal{M}}_d^k \theta^T (\mathbb{1} - T)\theta T \theta^T] \ . \quad (39)$$

Expanding the factors of $(\mathbb{1} - T)$ and constructing the traces gives rise to the form

$$\langle n_i n_{i+j} n_{i+j+k} \rangle = \langle n_i \rangle^3 + \langle n_i \rangle^2 (1 - \langle n_i \rangle) \left[\sum b_\ell^2 \bar{\lambda}_\ell^j + \sum b_\ell^2 \bar{\lambda}_\ell^k - \sum b_\ell^2 \bar{\lambda}_\ell^j \sum b_\ell^2 \bar{\lambda}_\ell^k \right] + \langle n_i \rangle (1 - \langle n_i \rangle) \sum b_\ell^2 \bar{\lambda}_\ell^{j+k} \ . \quad (40)$$

This result differs from the expression for the ordinary lattice gas. The three-body correlator *cannot* be expressed as a product of consecutive two-body correlators. This is important since it will give us precisely the freedom we need to proceed from the lattice gas factorial moments to those of the negative binomial distribution. Nevertheless, this correlator is still determined by the two-body correlation function. It merely requires the addition of $\langle n_i n_{i+j+k} \rangle$. This fact is also important, since we seek factorial moments which depend on the global properties of the distribution through c only. Note that in the special case of $D = 1$ appropriate for the ordinary lattice gas, the sums collapse to a single term with $b_1^2 = 1$.

It is now straightforward to perform the sums necessary to determine $F_3^{\mathrm{GLG}}(M)$ in the limit as $N \to \infty$. Using Eq. (A6) we find

$$F_3^{\mathrm{GLG}}(M) = 1 + 3cM + \frac{3}{2} d^2 M^2 \ . \quad (41)$$

The presence of the term involving $\langle n_i n_{i+j+k} \rangle$ in the three-body correlation function has forced the introduction of the new term

$$d^2 = \frac{4}{\langle n \rangle^2} \sum_{\ell=1}^{D} b_\ell^2 \left(\frac{\bar{\lambda}_\ell}{1 - \bar{\lambda}_\ell} \right)^2 \ . \quad (42)$$

We note that this third factorial moment will equal that of the original lattice gas model provided that $d^2 = c^2$. There are (at least) two situations where this will happen. One is the case where precisely one of the b_ℓ^2 is equal to 1 (with the remaining b_ℓ^2 equal to 0). The other is the case where all of the $\bar{\lambda}_\ell$ are equal.

We now turn to the general case of $F_q^{\mathrm{GLG}}(M)$. The strategy is now perfectly straightforward and exceptionally tedious. We construct the q-body correlation function as the obvious generalization of Eqs. (34) and (39). We expand the various factors of $(\mathbb{1} - T)$ and perform the requisite traces. Using Eq. (A6), we then perform the requisite bin sums. The final result follows from some algebra and combinatorics. We find

$$F_q^{\mathrm{GLG}}(M) = \sum_{k=0}^{q-1} f_k^{(q)} M^k \ , \quad (43)$$

where

$$f_k^{(q)} = \frac{q!}{k!(q-k)!} \left[\frac{d^k}{dz^k} \left(1 + \sum_{\mu=1}^{\infty} z^\mu s_\mu \right)^{(q-k)} \right]_{z=0} \ . \quad (44)$$

Here, we have introduced the definition

$$s_\mu = \frac{1}{\langle n \rangle^\mu} \sum_{\ell=1}^{D} b_\ell{}^2 \left(\frac{\bar{\lambda}_\ell}{1 - \bar{\lambda}_\ell} \right)^\mu \ , \quad (45)$$

which implies $s_1 = c/2$.

It is also desirable to derive a generating function from which the factorial moments can be obtained by simple differentiation. To this end, we first observe that $f_k^{(q)}$ can be rewritten as

$$f_k^{(q)} = \frac{1}{(q-k)!}\left[\frac{d^q}{dz^q} z^{q-k}\left(1+\sum_{\mu=1}^{\infty} z^\mu s_\mu\right)^{(q-k)}\right]_{z=0} . \tag{46}$$

We now change the summation index in Eq. (43) from k to $(q-k)$ to obtain

$$F_q^{\mathrm{GLG}}(M) = M^q\left[\frac{d^q}{dz^q}\sum_{k=1}^{q}\frac{1}{k!}\left(\frac{z}{M}\left\{1+\sum_{\mu=1}^{\infty} s_\mu z^\mu\right\}\right)^k\right]_{z=0} . \tag{47}$$

The sum over k can be extended to range from 0 to ∞ since the corresponding derivatives are zero. The factorial moments thus become

$$F_q^{\mathrm{GLG}}(M) = M^q\left[\frac{d^q}{dz^q}\exp\left(\frac{z}{M}\left\{1+\sum_{\mu=1}^{\infty} s_\mu z^\mu\right\}\right)\right]_{z=0} . \tag{48}$$

Equations (43)–(45) and (48) represent the final results of our generalized lattice gas model.

The various terms $f_k^{(q)}$ which characterize the factorial moments of our generalized lattice gas model are completely determined by the leading terms $f_{k'}^{(k'+1)}$ for $k' \le k$. This follows from combinatoric arguments and has virtually nothing to do with the underlying microscopic details of the model. It does involve two basic ingredients of the model. First, that each site has an occupancy of either 0 or 1. Second, that the nonstatistical pieces of the various correlations are of short range [20]. It was not *a priori* obvious that any one-dimensional model would yield the factorial moments of the negative binomial distribution. It is now clear, however, that the generalized lattice gas model is capable of reproducing the factorial moments of *any* distribution (including the NB) through a suitable choice of the s_μ. It remains to be demonstrated that there exists a choice of the underlying Hamiltonian which will provide the desired s_μ. Such demonstration is the topic of the next section.

IV. OBTAINING THE FACTORIAL MOMENTS OF THE NEGATIVE BINOMIAL DISTRIBUTION

As we have repeatedly emphasized, we are certainly able to reproduce the factorial moments of the ordinary lattice gas model. We will show in Appendix B that this happens in the special case where

$$s_\mu = \left(\frac{c}{2}\right)^\mu . \tag{49}$$

As noted, this special case can be realized either when the sums over ℓ contain only one term or when all the reduced eigenvalues are equal.

We are also able to pick the various terms s_μ in such a way as to reduce Eqs. (43)–(45) to Eq. (2) exactly. It is readily verified that the choice

$$s_\mu = \frac{c^\mu}{\mu+1} \tag{50}$$

establishes the desired equality. (The demonstration of this fact is given in Appendix B.) With this choice, the factorial moments of our generalized lattice gas model are rendered *identical* to the factorial moments of the negative binomial distribution. This answer poses a question. Is it possible to pick the dimension D (the number of species) and to find a nearest-neighbor Hamiltonian and its related matrix $\mathcal{M}$ such that the constraints of Eq. (50) are satisfied? As we shall see, it is.

As noted in the previous section, we are ultimately interested in the $N \to \infty$ limit. We shall arbitrarily set all chemical potentials equal to μ_0 and realize this limit by taking $\mu_0 \to \infty$. It is now sufficient to consider the D-dimensional submatrix $\tilde{\mathcal{M}}$ obtained by neglecting the zeroth row and column of $\mathcal{M}$. Using first-order perturbation theory, the coupling of $\tilde{\mathcal{M}}$ to the remaining elements of $\mathcal{M}$ can then be treated exactly. This will result in a slightly different expression for s_μ. We obtain

$$s_\mu = \frac{1}{\langle n\rangle^\mu}\sum_{k=1}^{D} a_k^2\left(\frac{\tilde{\lambda}_k}{1-\tilde{\lambda}_k}\right)^\mu , \tag{51}$$

where the $\tilde{\lambda}_k$ are the eigenvalues of $\tilde{\mathcal{M}}$ [21] and the a_k^2 are normalized coefficients which follow from the eigenvectors of $\tilde{\mathcal{M}}$ as

$$a_k = \mathcal{N}\frac{1}{1-\tilde{\lambda}_k}\sum_{i=1}^{D}\tilde{\theta}_{ik} . \tag{52}$$

Here, $\tilde{\theta}$ is the matrix that diagonalizes $\tilde{\mathcal{M}}$, and $\mathcal{N}$ is chosen such that $\sum a_k^2 = 1$.

We now rewrite Eq. (50) as

$$\sum_{k=1}^{D} a_k^2\left(\frac{\tilde{\lambda}_k}{1-\tilde{\lambda}_k}\right)^\mu = \frac{[c\langle n\rangle]^\mu}{\mu+1} . \tag{53}$$

The purpose of this rewriting is to emphasize that, in order to reproduce the negative binomial factorial moments, it is necessary to satisfy an infinite number of constraints, Eq. (53). There are only a finite number of parameters in the Hamiltonian for finite D. Specifically, there are $D(D+1)/2$ interaction parameters to be specified [22]. The satisfaction of an infinite number of constraints will evidently require the limit $D \to \infty$. (We shall return to this point below.)

It is interesting to note that in the $D \to \infty$ limit, we can replace the summation in Eq. (53) by an integral in order to obtain

$$\int_{-1}^{+1} d\lambda\ a(\lambda)^2 \left(\frac{\lambda}{1-\lambda}\right)^{\mu} = \frac{[c\langle n\rangle]^{\mu}}{\mu+1}\ , \tag{54}$$

where the coefficients $a(\lambda)^2$ now play the role of a (non-negative) continuous weighting function. Working with a new variable

$$\eta = \frac{\lambda}{1-\lambda}\ , \tag{55}$$

we thus seek another weighting function $W(\eta)$ such that

$$\int_{-1/2}^{+\infty} d\eta\ W(\eta)\ \eta^{\mu} = \frac{[c\langle n\rangle]^{\mu}}{\mu+1}\ . \tag{56}$$

As it happens, this set of constraints allows us to determine $W(\eta)$ by inspection. We simply obtain a step function:

$$W(\eta) = \theta(c\langle n\rangle - \eta)\theta(\eta)\ . \tag{57}$$

This choice (barring questions of uniform convergence) is unique. It is readily transformed into a statement about the weighting function $a(\lambda)^2$. We find $a(\lambda)^2 = 0$ except

$$a(\lambda)^2 = \frac{1}{c\langle n\rangle(1-\lambda)^2} \quad \text{for} \quad 0 \le \lambda \le \frac{c\langle n\rangle}{1+c\langle n\rangle}\ . \tag{58}$$

This statement of the constraints will be extremely useful in finding a scheme for picking the parameters in the Hamiltonian in order to reproduce the negative binomial moments.

It is clear that the form of $\tilde{\mathcal{M}}$ as given by Eq. (30) (for $d, d' \neq 0$) is somewhat special. It is necessary to demonstrate that there exists at least one scheme for picking the $D(D+1)/2$ parameters of $\tilde{\mathcal{M}}$ in such a way that the various constraints for obtaining the negative binomial distribution are satisfied. Here, we shall simply propose two different prescriptions and demonstrate by numerical example that these prescriptions indeed yield the desired result. Of course, we shall make no claims regarding the uniqueness of our prescriptions. The first prescription is as follows.

Draw D random numbers, x_d, from the interval $[0,1]$. Set $\tilde{\mathcal{M}}_{dd}$ equal to $x_d L$ where $L > 0$. Set all off-diagonal elements of $\tilde{\mathcal{M}}$ equal to zero. For each draw, choose L such that $s_1 = c/2$. (This sets the dispersion to its empirical value for each draw.)

The physical content of this prescription is clear. While identical particles experience a nearest-neighbor interaction which ranges from $\approx -\mu_0$ to $+\infty$, inequivalent particles experience a nearest-neighbor interaction in the range $-\mu_0 \ll \epsilon_{dd'} \le +\infty$. The success of this prescription is guaranteed by a comparison of Eqs. (52) and (58). Choosing $\tilde{\mathcal{M}}$ to be diagonal guarantees that the various sums in Eq. (52) will be independent of k. The λ dependence of Eq. (58) will be obtained from Eq. (52) provided that the eigenvalues are distributed uniformly over a finite, positive range. This is ensured by our prescription.

For our numerical studies, we consider the values of $\langle n\rangle = 20$ and $\langle \Delta n^2\rangle = 110$ appropriate for the description of $p\bar{p}$ scattering at 200 GeV. (For this reaction, empirical values for the factorial moments are readily available. Qualitatively similar results are obtained for other high-energy scattering experiments.) These data indicate that we should set

$$c\langle n\rangle = \frac{\langle \Delta n^2\rangle - \langle n\rangle}{\langle n\rangle} = 4.5\ . \tag{59}$$

We now construct the ratios

$$r_q = \frac{(q+1)\sum_{k=1}^{D} a_k^2 \left(\frac{\tilde{\lambda}_k}{1-\tilde{\lambda}_k}\right)^q}{[c\langle n\rangle]^q}\ . \tag{60}$$

Our only constraint, $s_1 = c/2$, ensures $r_1 = 1$. When we have satisfied the constraints of Eq. (53), all of these ratios should be equal to one [23]. Since there are no parameters to adjust, this is an extremely stringent test of our prescription. It succeeds. In Table I we report the results of our numerical studies for $2 \le q \le 6$ and $D = 1, 2, 4, \ldots, 512$. The notation $\langle\langle r_q\rangle\rangle$ is a reminder that, since our theory is now randomly drawn, we must

TABLE I. The ensemble average and dispersion of r_q according to our first prescription for $2 \le q \le 6$ and various values of the number of particle species, D. The average was taken over 10^5 randomly drawn theories with $c\langle n\rangle = 4.5$ for each draw. The case $D = 1$ corresponds to the ordinary lattice gas (Ising) model [13] and has no dispersion.

D	$\langle\langle r_2\rangle\rangle$	$\langle\langle r_3\rangle\rangle$	$\langle\langle r_4\rangle\rangle$	$\langle\langle r_5\rangle\rangle$	$\langle\langle r_6\rangle\rangle$
1	0.75	0.5	0.3125	0.1875	0.109375
2	0.812 ± 0.023	0.598 ± 0.035	0.417 ± 0.037	0.280 ± 0.033	0.184 ± 0.027
4	0.881 ± 0.046	0.723 ± 0.078	0.569 ± 0.091	0.435 ± 0.091	0.326 ± 0.083
8	0.941 ± 0.067	0.851 ± 0.132	0.751 ± 0.179	0.652 ± 0.207	0.558 ± 0.218
16	0.981 ± 0.076	0.951 ± 0.173	0.916 ± 0.270	0.879 ± 0.360	0.842 ± 0.441
32	0.998 ± 0.069	0.999 ± 0.170	1.005 ± 0.295	1.017 ± 0.442	1.039 ± 0.612
64	1.002 ± 0.050	1.007 ± 0.128	1.019 ± 0.229	1.040 ± 0.355	1.071 ± 0.515
128	1.001 ± 0.035	1.005 ± 0.088	1.013 ± 0.154	1.026 ± 0.233	1.044 ± 0.327
256	1.001 ± 0.024	1.003 ± 0.061	1.007 ± 0.105	1.013 ± 0.155	1.023 ± 0.210
512	1.000 ± 0.017	1.002 ± 0.042	1.004 ± 0.072	1.007 ± 0.106	1.012 ± 0.142

perform an "ensemble average" over theories as well as the usual thermodynamic ensemble average. For each value of D, this ensemble average was performed over 10^5 theories.

Several comments are in order. Our prescription meets the remaining conditions of Eq. (50) with increasing accuracy as $D \to \infty$. For fixed q, the value of $\langle\langle r_q \rangle\rangle$ approaches 1 like $1/D$ as $D \to \infty$. The dispersion also vanishes (like $1/\sqrt{D}$). Thus, as D becomes large, our simple prescription converges to the results of the NB for any fixed q. For fixed D, the error in $\langle\langle r_q \rangle\rangle$ and its dispersion grow as $q \to \infty$. Our point here is that there exists at least one simple prescription for satisfying Eq. (50) exactly.

Given the limited scope and quality of existing data, we do not need particularly large D to fit the relations between measured factorial moments. The value of $D = 16$ results in sufficiently small errors and dispersions that randomly drawn dynamics have a high probability of reproducing the relations between empirical factorial moments for $p\bar{p}$ scattering at 200 GeV within existing experimental uncertainties. This value of $D = 16$ is also sufficient to provide a quantitative description of relations between factorial moments for 900 GeV $p\bar{p}$ scattering and, indeed, of all other high-energy scattering experiments for which factorial moments are known. To demonstrate this, Fig. 1 shows the factorial moments F_3 to F_5 for $p\bar{p}$ scattering at 200, 546, and 900 GeV as a function of F_2.

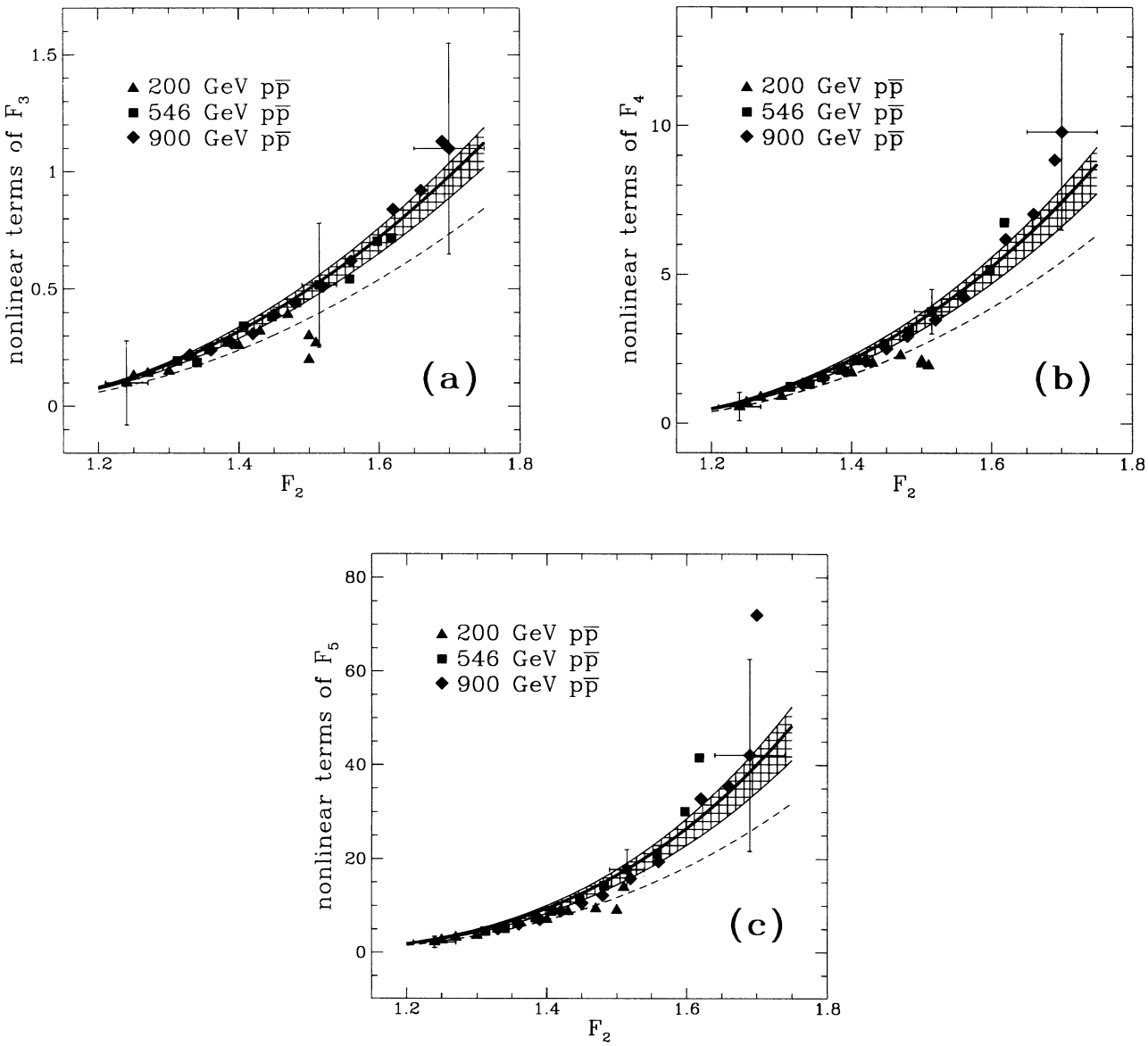

FIG. 1. Plots of $\tilde{F}_q$ as defined in Eq. (61) versus F_2 for $3 \leq q \leq 5$. Data are taken from Ref. [2]. The dashed line corresponds to the predictions of the ordinary lattice gas (Ising) model. The boldface line represents the negative binomial distribution. The shaded area is the prediction of the generalized lattice gas model according to our first prescription for $D = 16$. This "band" around the NB becomes narrower as D increases. The error bars shown are representative for the quoted uncertainties in the data (see the text for further comments on the error bars).

We have suppressed the model-independent constant and linear pieces of these moments; i.e., we plot

$$\tilde{F}_q(F_2) = F_q(F_2) - 1 - \frac{q(q-1)}{2}(F_2 - 1) \quad . \tag{61}$$

The theoretical "bands" shown on each figure correspond to the theoretical dispersion obtained with $D = 16$. (More than 50% of all randomly drawn theories lie inside this band.) These bands were obtained with parameters chosen to reproduce the 200 GeV data. The use of parameters chosen to reproduce the higher-energy data would result in somewhat narrower bands that are slightly shifted downwards.

The error bars in Fig. 1 represent the published uncertainties and have roughly equal statistical and systematic components. Some comment regarding the statistical error is in order. Since $F_2(M)$ and $F_q(M)$ are drawn from the *same* data set, significant statistical correlations can exist. Unfortunately, it is impossible to make a quantitative statement about such correlations without a detailed investigation of the data reduction techniques actually used. We have, however, performed simulations based on a suitably modified form of the negative binomial distribution. The results suggest that the uncertainties shown in Fig. 1 actually represent a significant overestimate of the real uncertainties in the determination of $F_q(F_2)$. This belief is also supported by the observation that the data points appear to have a far smaller spread than their errors would indicate. A reduction of these errors by a factor of approximately 3 would make it possible to provide a convincing discrimination between the negative binomial distribution and the Ising model distribution. Our simulations suggest that an error reduction of this magnitude is not out of the question.

This first prescription has the virtues of simplicity and guaranteed success. It can, however, be criticized on the grounds that it has singled out the species-diagonal interactions ϵ_{dd} for special treatment. We thus consider a second and more democratic prescription in which all pairs of species are treated equivalently.

> Draw the various terms $\epsilon_{dd'}$ at random over the uniform interval
>
> $$\epsilon_0 - \mu_0 \leq \epsilon_{dd'} \leq \epsilon_0 - \mu_0 + \Delta\epsilon \quad .$$
>
> For each choice of $\tilde{\mathcal{M}}$, adjust ϵ_0 in order to reproduce the desired value of c. (This can always be done.) Adjust $\Delta\epsilon$ to reproduce the value $r_2 = 1$. In the event that this cannot be done, discard the draw [24].

This more democratic selection of $\tilde{\mathcal{M}}$ means that we must actually perform the requisite diagonalizations. This means that one must be content with studying a smaller number of samples. More seriously, it is no longer elementary to establish rules which guarantee the desired equivalence between Eqs. (52) and (58). It is for this reason that we have imposed the somewhat artificial constraint that $r_2 = 1$.

The results of numerical studies with this second prescription are summarized in Table II for $D = 32$, 64, 128, and 256. For each value of D we considered 10^3 matrices. In this case, we plot the average values (and the corresponding dispersions) for ϵ_0 and $\Delta\epsilon$ as well as the various ratios r_q for $3 \leq q \leq 6$. The large values of $\Delta\epsilon$ required indicate that we are dealing with sparse matrices. This has two apparently negative consequences. First, there will be a great many eigenvalues close to zero. Second, there will be only a small preference for positive eigenvalues [25]. These facts would appear to make it difficult to satisfy Eq. (58). At the very least, we anticipate much slower convergence (as a function of D) when using this second prescription. On the other hand, the eigenvectors associated with negative eigenvalues necessarily involve terms of mixed signs so that there will be a greater tendency for cancellation in the sums over $\tilde{\theta}_{ik}$ in Eq. (52) for negative eigenvalues.

Somewhat surprisingly, these effects conspire to a remarkable degree. The numerical results of Table II suggest that our second prescription also leads to the factorial moments of the negative binomial distribution in the large D limit. This claim of success is necessarily somewhat tentative. The slower convergence of this prescription suggests the need to study values of D larger than those in Table I. The need to diagonalize many large matrices restricts us to smaller values of D and smaller statistical samples. We have performed other tests which also suggest that this second prescription works. For example, we have made histograms of the a_k^2 of Eq. (52) as a function of the variable η defined above. Except for the expected "Gibbs phenomenon," we find remarkable agreement with the distribution of Eq. (57) as D becomes large. We have also studied the cases $D = 512$ and 1028 with very poor statistics. These results are also consistent with our assertion that the factorial moments of the negative binomial distribution will result in the large D

TABLE II. The ensemble average and dispersion of r_q according to our second prescription for $3 \leq q \leq 6$ and various values of the number of particle species, D. The average was taken over 10^3 randomly drawn theories with $c\langle n\rangle = 4.5$ for each draw. Also shown are average value and dispersion of ϵ_0 and $\Delta\epsilon$.

D	ϵ_0	$\Delta\epsilon$	$\langle\langle r_3\rangle\rangle$	$\langle\langle r_4\rangle\rangle$	$\langle\langle r_5\rangle\rangle$	$\langle\langle r_6\rangle\rangle$
32	0.21 ± 0.37	114 ± 75	0.945 ± 0.025	0.862 ± 0.059	0.769 ± 0.094	0.674 ± 0.124
64	0.47 ± 0.23	149 ± 63	0.959 ± 0.022	0.895 ± 0.054	0.820 ± 0.089	0.741 ± 0.122
128	0.62 ± 0.18	223 ± 71	0.973 ± 0.021	0.930 ± 0.054	0.876 ± 0.093	0.818 ± 0.133
256	0.73 ± 0.15	363 ± 85	0.988 ± 0.020	0.967 ± 0.054	0.941 ± 0.098	0.910 ± 0.148

limit.

The purpose of this second prescription has been to demonstrate the existence of at least one democratic way to reproduce the factorial moments of the negative binomial distribution. More efficient schemes may well exist.

V. DISCUSSIONS AND CONCLUSIONS

We have demonstrated that a simple extension of the one-dimensional lattice gas model to include nearest-neighbor interactions between pairs of D species of particles can reproduce the factorial moments of the negative binomial distribution exactly provided that a number of constraints are met. We have further demonstrated that these constraints can be met by simple models of the microscopic dynamics in which all particles have the same chemical potential and in which the nearest-neighbor interactions are drawn at random according to the prescriptions of the previous section.

Since our model approaches the negative binomial distribution as $D \to \infty$, it will reproduce the relations between empirical factorial moments including those obtained in $p\bar{p}$ scattering, e^+e^- scattering and relativistic heavy ion collisions. The only question is whether the number of species required to provide an adequate fit is sufficiently small to be considered "physically reasonable." Given the results of Fig. 1 for 16 species, we believe that the present model is physically reasonable. These results also offer some understanding for both the success of cascade calculations and the anecdotal observation that the results of calculations are often surprisingly insensitive to the details of the model. According to our picture, any randomly drawn theory would be likely to do as well (at least at the level of the factorial moments).

It has been suggested that the slopes α_q/α_2 in plots of $\ln F_q$ versus $\ln F_2$ can be used to distinguish between cascade systems and systems exhibiting some critical phenomenon such as the transition to a quark-gluon plasma [9]. It has been shown that systems at the critical temperature of a second-order phase transition display monofractal structure with $\alpha_q/\alpha_2 \sim (q-1)$ [11]. In contrast, cascade models lead to a multifractal structure with $\alpha_q/\alpha_2 \sim q(q-1)$ (in Gaussian approximation) [12]. Our model (along with the Ising model) shows that the monofractal structure of a critical phenomenon can also be obtained from a simple but heterogeneous (equilibrium) system. It indicates that some caution is required before claiming insight into the nature of microscopic mechanisms on such a basis. Indeed, the present model may be a useful test case for proposed schemes for the detection of critical phenomena from the study of, e.g., factorial moments.

Before closing, we would like to offer one speculative remark regarding the existence (or possible nonexistence) of signatures which would honestly discriminate between "interesting" critical systems and those "uninteresting" heterogeneous systems considered here with which they can be confused. We suspect that the study of global properties, such as the factorial moments, can only provide convincing tests of criticality for systems whose time evolution can be studied. The present results suggest that it may always be possible to devise equilibrium, heterogeneous models which can simulate the manifestations of critical phenomena to arbitrary accuracy in systems where the only information available is that provided at $t = \infty$. Evidently, this class of systems includes all scattering experiments in nuclear and high-energy physics. The only way to distinguish between these classes of models is to provide *independent* arguments that the heterogeneous alternatives to critical systems are "too artificial" or "too unphysical" to be accepted. Given the present results, we suspect that this case cannot be made. Thus, in physics as in the rest of life, much more information is to be gained from the study of "living" systems than from performing autopsies on "dead" systems.

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APPENDIX A: SIMPLIFYING FACTORIAL MOMENTS

Consider any model for which the number operator at any site i is an idempotent:

$$n_i^2 = n_i \quad . \tag{A1}$$

Such models include either the ordinary lattice gas model of Sec. II or the generalized lattice gas model of Sec. III. Define the number operator for a single bin,

$$n = \sum n_i \quad , \tag{A2}$$

where i runs over every site in the bin.

Now assume that we know that the following operator identity holds for some fixed order q:

$$n(n-1)(n-2)\cdots(n-q+1) = \sum{}' n_{i_1} n_{i_2} n_{i_3} \cdots n_{i_q} \quad . \tag{A3}$$

The sum extends over all sites in a given bin. The prime denotes that no two of the site indices i_1 to i_q are equal. Note that Eq. (A3) applies at the operator level before ensemble averages are carried out. Our aim is to show that, if Eq. (A3) applies for q, it necessarily applies for $q+1$ as well. To see this, multiply Eq. (A3) by the number operator n. This leads to

$$n[n(n-1)(n-2)\cdots(n-q+1)]$$

$$= \sum{}' n_{i_1} n_{i_2} n_{i_3} \cdots n_{i_q} n_{i_{q+1}} + q \sum{}' n_{i_1} n_{i_2} n_{i_3} \cdots n_{i_q} \quad . \tag{A4}$$

The second term here simply corresponds to the fact that the index i_{q+1} can be equal to any one of the q indices i_1 to i_q. Using Eq. (A3), we can move the final term in Eq. (A4) to the left of the equation to obtain

$$n(n-1)(n-2)\cdots(n-q+1)(n-q) = \sum{}' n_{i_1} n_{i_2} n_{i_3} \cdots n_{i_q} n_{i_{q+1}} \quad . \tag{A5}$$

This is the desired extension of the result. Thus, Eq. (A3) will apply for all q if it applies for $q = 1$. It applies trivially for $q = 1$ since the primed sum is of no meaning in this case (there being only one term in the product) and Eq. (A3) collapses to the definition, Eq. (A2).

Given the form in which the various correlation functions can be expressed in our models, one final rearrangement of Eq. (A3) is useful. Specifically,

$$n(n-1)(n-2)\cdots(n-q+1) = q! \sum_{i_1<i_2<\ldots<i_q} n_{i_1} n_{i_2} n_{i_3} \ldots n_{i_q} \quad . \tag{A6}$$

The final result is that Eqs. (A3) and (A6) apply for all q for all problems in which the number operator is idempotent. Further, one can take ensemble averages of these operator identities, and the ensemble averages can be taken under the summations. These results provide a considerable technical simplification in the evaluation of factorial moments.

APPENDIX B: SPECIAL CASES OF THE GENERALIZED LATTICE GAS MODEL

Here, we wish to reduce the factorial moments of our generalized lattice gas model to (i) the factorial moments of the ordinary lattice gas model and (ii) the factorial moments of the negative binomial distribution by simplifying the general expressions, Eqs. (43)–(45), with the use of special assumptions regarding the choice of the terms s_μ.

Consider, first, the ordinary lattice gas results. The purpose here is an exercise in method since we already know the result to be true. Set the various s_μ as

$$s_\mu = \left(\frac{c}{2}\right)^\mu \quad . \tag{B1}$$

We can then perform the sum in the square brackets in Eq. (44) exactly to yield

$$1 + \sum_{\mu=1}^{\infty} z^\mu s_\mu = \frac{1}{1 - zc/2} \quad . \tag{B2}$$

The desired derivative is immediately obtained as

$$\left[\frac{d^k}{dz^k}\left(\frac{1}{1-zc/2}\right)^{(q-k)}\right]_{z=0} = (q-k)(q-k+1)\cdots(q-1)\left(\frac{c}{2}\right)^k = \frac{(q-1)!}{(q-k-1)!}\left(\frac{c}{2}\right)^k \quad . \tag{B3}$$

This gives

$$f_k^{(q)} = \frac{q!(q-1)!}{(q-k)!k!(q-1-k)!2^k} c^k \quad . \tag{B4}$$

Comparison with Eq. (4) reveals the expected agreement.

For the case of the negative binomials, it is more convenient to start with the generalized generating function of Eq. (48). We now wish to set

$$s_\mu = \frac{c^\mu}{\mu + 1} \quad . \tag{B5}$$

The sum over μ can again be performed exactly to yield

$$1 + \sum_{\mu=1}^{\infty} z^\mu s_\mu = -\frac{\ln[1-zc]}{zc} \quad . \tag{B6}$$

Substitution of this expression into Eq. (48) yields

$$\begin{aligned} F_q(M) &= M^q \left[\frac{d^q}{dz^q} \exp\left(-\frac{\ln[1-zc]}{cM}\right)\right]_{z=0} \\ &= (cM)^q \left[\frac{d^q}{dy^q}(1-y)^{-\frac{1}{cM}}\right]_{y=0} \\ &= (cM)^q \left(\frac{1}{cM}\right)\left(\frac{1}{cM}+1\right)\cdots\left(\frac{1}{cM}+[q-1]\right) \\ &= (1+cM)(1+2cM)\cdots(1+[q-1]cM), \end{aligned} \tag{B7}$$

which is seen to be identical with the NB result of Eq. (2).

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[14] This idea was first suggested by Dias de Deus and Seixas who studied the lattice gas model numerically. See J. Dias de Deus and J. C. Seixas, Phys. Lett. B **246**, 506 (1990); J. C. Seixas, in *Intermittency in High Energy Collisions*, edited by F. Cooper, R. C. Hwa, and I. Sarcevic (World Scientific, Singapore, 1991).

[15] For a brief overview, see A. D. Jackson, T. Wettig, and N. L. Balazs (unpublished).

[16] Here, and in the following, we will set the inverse temperature β equal to 1 without loss of generality.

[17] Here, the subscript d means diagonal and is not to be confused with species d.

[18] This is the only real complication relative to the $D = 1$ lattice gas model of the previous section.

[19] Again, we are taking the $N \to \infty$ limit for fixed $\langle n \rangle$ for which it is justified to approximate $1 - \langle n_i \rangle$ by 1.

[20] Given the absence of long-range order in one-dimensional systems, this requirement is readily satisfied.

[21] We consider only the physically interesting case where $|\tilde{\lambda}_k| < 1$ for all k.

[22] The chemical potentials have already been set equal to a common μ_0.

[23] For the ordinary lattice gas, $D = 1$, these ratios are simply $(q+1)/2^q$.

[24] This possibility becomes increasingly unlikely as D becomes large.

[25] Since $\tilde{\mathcal{M}}$ contains only positive elements, we are guaranteed that some preference for positive eigenvalues will remain. Further, if the eigenvalue of largest magnitude is nondegenerate, it will be positive.