Two-phase flows with bulk-surface interaction described by a Navier–Stokes–Cahn–Hilliard model with dynamic boundary conditions

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Jonas Stange

aus Hemer

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Die Arbeit wurde angeleitet von:

PD Dr. Patrik Knopf (Universität Regensburg)

Prüfungsausschuss:

Vorsitzender:	Prof. Dr. Bernd Ammann
Erst-Gutachter:	PD Dr. Patrik Knopf
Zweit-Gutachter:	Prof. Dr. Maurizio Grasselli (Politecnico di Milano)
weiterer Prüfer:	Prof. Dr. Harald Garcke
Ersatzprüfer:	Prof. Dr. Helmut Abels

Abstract

This thesis is devoted to the derivation and analysis of a novel bulk-surface Navier–Stokes–Cahn–Hilliard system describing the evolution of viscous fluid mixtures with dynamic boundary conditions. The model accounts for both bulk and surface phase-separation processes, incorporates convective effects, and allows mass exchange between the bulk and the boundary. In addition, it includes a surface Navier–Stokes equation to describe viscous flow and momentum transport along the boundary, motivated in particular by applications to biological membranes.

The derivation of the model is based on local mass-balance laws, local energy-dissipation principles, and a Lagrange-multiplier approach, yielding a thermodynamically consistent system with coupled bulk-surface dynamics.

The analytical part of the thesis comprises three components: a preliminary part and two main parts. In the preliminary part, we study a broad class of bulk-surface systems, including elliptic bulk-surface systems with nonlinearities or non-constant coefficients, as well as a bulk-surface Stokes-type system. These models appear frequently throughout the thesis as subsystems. We present a thorough well-posedness theory, including higher regularity of weak solutions and the existence of strong solutions.

In the first main part, we study a convective bulk-surface Cahn–Hilliard subsystem with prescribed velocity fields. For this system, we establish the existence, uniqueness, and higher regularity of weak solutions for both regular and singular free energy densities. For regular potentials, we employ a suitable Faedo–Galerkin scheme, while for singular potentials, the proof is based on an approximation scheme using the Yosida approximation. Moreover, we prove the propagation of regularity and separation properties for logarithmic-type potentials and a broad class of velocity fields. In the two-dimensional setting, we further obtain the uniqueness of weak solutions and the existence of global-in-time strong solutions under suitable assumptions on the mobility functions. Moreover, we analyze the long-time behavior of weak solutions, proving the existence of a minimal pullback attractor and convergence to equilibrium.

In the second part, we consider the full bulk-surface Navier–Stokes–Cahn–Hilliard system. Building on the results obtained for the convective bulk-surface Cahn–Hilliard subsystem, we prove the existence of a global-in-time weak solution via an implicit time-discretization scheme. In addition, we establish the existence and uniqueness of global-in-time strong solutions in two spatial dimensions using a semi-Galerkin approximation based on the eigenfunctions of a novel bulk-surface Stokes operator.

Zusammenfassung

Diese Arbeit widmet sich der Herleitung und der Analysis eines neuen Volumen-Oberflächen-Navier–Stokes–Cahn–Hilliard-Systems zur Beschreibung der Dynamik viskoser Flüssigkeitsgemische mit Wechselwirkungen zwischen Volumen und Oberfläche. Das Modell berücksichtigt sowohl die Phasenseparation im Volumen als auch auf der Oberfläche, umfasst konvektive Effekte und ermöglicht den Massenaustausch zwischen Volumen und Oberfläche. Darüber hinaus wird eine Navier–Stokes-Gleichung auf der Oberfläche eingeführt, um viskose Strömungen und den Impulstransport entlang der Oberfläche zu beschreiben, was insbesondere durch Anwendungen in der Biophysik, etwa an Zellmembranen, motiviert wird.

Die Herleitung des Modells basiert auf lokalen Massenbilanzen, lokalen Energiedissipationsgesetzen sowie einem Lagrange-Multiplikator-Ansatz und führt zu einem thermodynamisch konsistenten System mit gekoppelter Volumen-Oberflächen-Dynamik.

Der analytische Teil der Arbeit gliedert sich in drei Komponenten: Einen vorbereitenden Teil sowie zwei Hauptteile. Im vorbereitenden Teil unsersuchen wir eine breite Klasse von Volumen-Oberflächen-Systemen, darunter elliptische Volumen-Oberflächen-Systeme mit Nichtlinearitäten oder nicht-konstanten Koeffizienten, sowie Volumen-Oberflächen-Stokes-Systeme. Diese Modelle treten im Verlauf der Arbeit wiederholt als Teilsysteme auf. Wir entwickeln eine umfassende Wohlgestelltheitstheorie, die sowohl höhere Regularität schwacher Lösungen als auch die Existenz starker Lösungen umfasst.

Im ersten Hauptteil analysieren wir ein konvektives Volumen-Oberflächen-Cahn–Hilliard-Teilsystem mit vorgegebenen Geschwindigkeitsfeldern. Für dieses System zeigen wir Existenz, Eindeutigkeit und höhere Regularität schwacher Lösungen für reguläre sowie singuläre Potentiale sowie für eine breite Klasse von Geschwindigkeitsfeldern. Im Fall regulärer Potentiale verwenden wir ein geeignetes Faedo–Galerkin-Verfahren, während für singuläre Potentiale ein Approximationsverfahren auf Basis der Yosida-Regularisierung eingesetzt wird. Darüber hinaus beweisen wir Regularitätsfortpflanzung und Separationseigenschaften für logarithmische Potentiale. Im zweidimensionalen Fall erhalten wir zudem die Eindeutigkeit schwacher Lösungen sowie die Existenz globaler starker Lösungen unter geeigneten Annahmen an die Mobilitätsfunktionen. Darüber hinaus analysieren wir das Langzeitverhalten schwacher Lösungen und zeigen die Existenz eines Pullback-Attraktors sowie die Konvergenz gegen stationäre Lösungen.

Im zweiten Teil betrachten wir dann das vollständige Volumen-Oberflächen-Navier–Stokes–Cahn–Hilliard-System. Aufbauend auf den Ergebnissen für das konvektive Volumen-Oberflächen-Cahn–Hilliard-Teilsystem zeigen wir die Existenz globaler schwacher Lösungen mittels eines impliziten Zeitdiskretisierungsverfahrens. Außerdem beweisen wir die Existenz und Eindeutigkeit global starker Lösungen im zweidimensionalen Fall mithilfe eines Semi-Galerkin-Verfahrens, das auf den Eigenfunktionen eines Volumen-Oberflächen-Stokes-Operators basiert.

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Für Mama.

Contents

Chapter 1

Introduction

1.1 Phase-field models

The study of mixtures of two (or more) immiscible materials is a central yet challenging topic in fluid mechanics and materials science, with numerous applications across biology, chemistry, and engineering. Typical phenomena include thermocapillary flows, spinodal decomposition, droplet breakup, and moving contact lines. A key difficulty in modeling such systems lies in accurately describing the evolution of the interface separating the different components. Two major approaches have been developed, depending on how the interface separating the different components is represented (see [7, 57, 88, 143] and the references therein): the sharp interface method and the diffuse interface method.

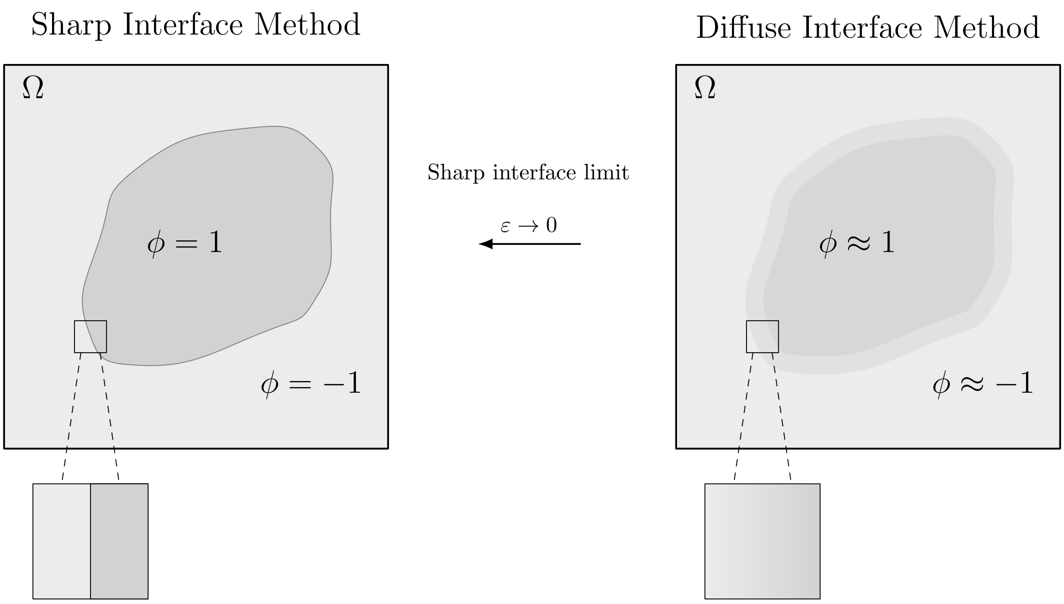

Figure 1.1: Comparison of sharp and diffuse interface methods.

We briefly describe and compare both approaches. To this end, we consider a binary mixture contained in a smooth, bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, over a time interval $(0, T)$, where $T > 0$ is a prescribed final time. The distribution of the two components can be described by their respective concentrations (or volume fractions) $\phi_1, \phi_2 : \Omega \times (0, T) \to \mathbb{R}$, which satisfy $\phi_1 + \phi_2 = 1$. Rather than treating these quantities separately, it is convenient to introduce an order parameter

$$\phi : \Omega \times (0, T) \to \mathbb{R}, \qquad \phi(x, t) \coloneqq \phi_2(x, t) - \phi_1(x, t),$$

which characterizes the local composition of the mixture.

In classical sharp interface models, the two concentrations ϕ_1 and ϕ_2 occupy distinct regions separated by an interface of zero thickness. Accordingly, the concentrations are given by the characteristic functions of the respective regions, and the order parameter exhibits a discontinuous jump between the values -1 and 1 across the interface, as shown in Figure (1.2). The interface itself is described as an evolving hypersurface whose motion is coupled to the governing equations in the bulk regions and suitable interface conditions. Since the interface is part of the solution, such models lead to free boundary problems, which are mathematically intricate and often difficult to analyze. While sharp interface models are highly accurate in many settings, they may fail when interfacial effects occur on small length scales, for instance, during topological changes or near critical points.

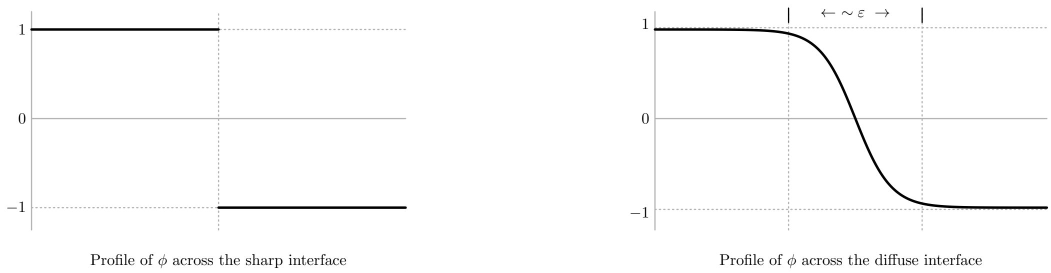

Figure 1.2: Comparison of the order parameter ϕ across the interface.

An alternative framework is provided by the diffuse interface models. In this approach, the sharp interface is replaced by a thin interfacial layer, referred to as the diffuse interface, whose thickness is proportional to a small parameter $\varepsilon > 0$. The order parameter, now commonly called the phase-field, varies smoothly across this region, taking values close to -1 and 1 in the bulk phases and transitioning continuously between them within the interface (see Figure (1.2)). A major advantage of diffuse interface models is that they avoid the explicit tracking of the interface. Instead, the evolution of the system is described by partial differential equations posed on the fixed domain Ω, typically formulated in Eulerian coordinates. Moreover, interfacial effects are naturally incorporated as volumetric contributions concentrated within the diffuse layer. In many situations, the relationship between sharp and diffuse interface descriptions can be rigorously established via the sharp interface limit, in which the interfacial thickness parameter ε tends to zero.

1.2 The Cahn–Hilliard equation

The Cahn–Hilliard equation was proposed in [30] to model phase separation in binary alloys, such as iron–nickel. If a blend of these metals is rapidly quenched below a critical temperature, the resulting state is unstable, and small pockets of relatively pure iron and nickel may soon appear with a characteristic length scale. Over time, these may coarsen into larger pockets, initially quickly and then more slowly. Nowadays, it is frequently used in mathematical models describing phenomena in materials science, life sciences, and image processing. Mathematically, the Cahn–Hilliard equation reads as follows:

$$\partial_t \phi = \operatorname{div}(m_\Omega(\phi)\nabla\mu) \qquad \text{in } Q^T, \tag{1.1a}$$
$$\mu = -\varepsilon\Delta\phi + \tfrac{1}{\varepsilon}F'(\phi) \qquad \text{in } Q^T. \tag{1.1b}$$

Here, $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ is a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$, whose outer unit normal vector field is denoted by $\mathbf{n}$. Moreover, $T > 0$ is a prescribed final time, and we write $Q^T := \Omega \times (0,T)$ and $\Sigma^T := \Gamma \times (0,T)$. The function $\phi : Q^T \to \mathbb{R}$ is the phase-field, and $\mu : Q^T \to \mathbb{R}$ the chemical potential. The Cahn–Hilliard equation (1.1) is usually equipped with homogeneous Neumann boundary conditions on both the phase-field ϕ and the chemical potential μ:

$$\partial_{\mathbf{n}}\phi = 0 \qquad \text{on } \Sigma^T, \tag{1.1c}$$

$$m_\Omega(\phi)\partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \Sigma^T, \tag{1.1d}$$

and supplemented with suitable initial conditions

$$\phi|_{t=0} = \phi_0 \qquad \text{in } \Omega, \tag{1.1e}$$

for a prescribed $\phi_0 : \Omega \to \mathbb{R}$. Here, $\partial_{\mathbf{n}}$ denotes the normal derivative, and ε is a positive constant related to the thickness of the diffuse interface.

The model (1.1) can be interpreted as an H^{-1}-type gradient flow with respect to the free energy functional

$$E_\Omega(\phi) := \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\,\mathrm{d}x, \tag{1.2}$$

also called Ginzburg–Landau energy or Cahn–Hilliard energy. Then, the chemical potential is the first variation of the free energy functional

$$\mu = \frac{\delta E_\Omega(\phi)}{\delta\phi}.$$

The term $\frac{\varepsilon}{2}|\nabla\phi|^2$ is often referred to as the Dirichlet energy density. The function F represents the free energy density and is usually of double-well-type, meaning it is coercive and has two minima near -1 and 1, with a local maximum in between, usually at 0. As pointed out in [127], the Dirichlet energy density was added to the free energy density for a regularizing effect.

For the original applications of the Cahn–Hilliard equation in the context of materials science, the most common choice for F is the so-called Flory–Huggins potential (cf. [68, 111]), which is also referred to as the logarithmic potential. It is given by

$$W_{\log}(s) := \frac{\Theta}{2}\big[(1+s)\ln(1+s) + (1-s)\ln(1-s)\big] - \frac{\Theta_c}{2}s^2 \qquad \text{for } s \in [-1,1]. \tag{1.3}$$

Here, we adopt the convention that $0\ln 0$ is interpreted as zero. In (1.3), $\Theta > 0$ denotes the absolute temperature of the system, while Θ_c denotes the critical temperature, under which phase separation processes of the two materials occur. The function $W_{\log}$ is usually extended to $\mathbb{R}\setminus[-1,1]$ by setting it equal to $+\infty$. It is classified as a singular potential, since

$$W_{\log}'(s) = \frac{\Theta}{2}\big[\ln(1+s) - \ln(1-s)\big] - \Theta_c s \to \pm\infty \qquad \text{as } s \to \pm 1.$$

Another commonly used potential is the double-obstacle potential, given by

$$W_{\mathrm{obst}}(s) := \begin{cases} 1 - s^2 & \text{if } s \in [-1,1], \\ +\infty & \text{if } s \in \mathbb{R}\setminus[-1,1]. \end{cases} \tag{1.4}$$

Since W_{obst} is not differentiable, the corresponding equation (1.1b) must be reformulated as a differential inclusion involving subdifferentials. Due to the analytical difficulties caused by these singular potentials, they are often replaced by smooth double-well potentials. A standard choice is the quartic double-well potential

$$W_{\text{dw}}(s) := \frac{1}{4}(s^2-1)^2 \qquad \text{for } s \in \mathbb{R}. \tag{1.5}$$

Although this potential is mathematically convenient, it does not constrain the phase-field variable ϕ to remain within the physically meaningful interval $[-1,1]$, in contrast to W_{log} and W_{obst}. In this thesis, we focus on potentials of logarithmic-type.

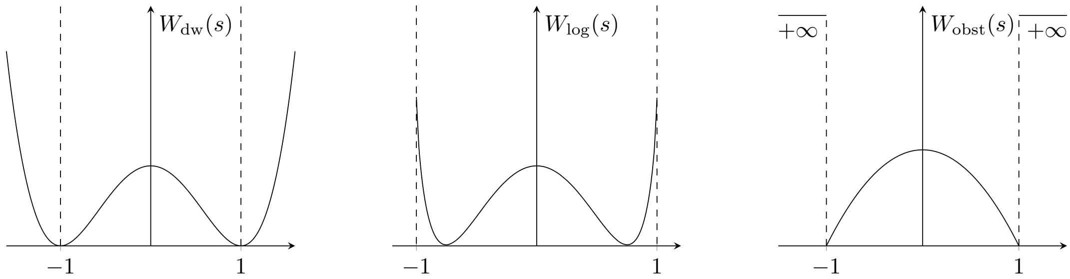

Figure 1.3: The double-well potential W_{dw}, the logarithmic potential W_{log} and the double-obstacle potential W_{obst}.

The function $m_\Omega : [-1,1] \to \mathbb{R}$ denotes the Onsager mobility. It typically depends on the phase-field and models both the spatial distribution and the intensity of diffusion processes. The mobility usually assumes one of the following forms:

(1) $m_\Omega \equiv m_\Omega^0 \in [0,\infty)$ (constant), (1.6)

(2) $m_\Omega \in C([-1,1]),\ 0 < m_* \leq m_\Omega(s) \leq m^*,\ s \in [-1,1]$ (non-degenerate), (1.7)

(3) $m_\Omega(s) = m_\Omega^0(1-s^2),\ s \in [-1,1],\ m_\Omega^0 \in [0,\infty)$ (degenerate). (1.8)

A mobility function that degenerates at the pure phases ± 1 was already proposed in the seminal works [30,31]. As discussed in [32,108,166], the choice (1.8) is physically reasonable. The existence of weak solutions is well established for both constant and non-degenerate mobilities, whereas the uniqueness of weak solutions is only known for constant mobilities in arbitrary dimensions. For non-degenerate mobilities, uniqueness remains open in $d \geq 3$; for $d = 1, 2$, see [21,53]. The analysis of degenerate mobilities is considerably more delicate. Existence results were first obtained in one dimension in [180], and subsequently extended to arbitrary dimensions in [61]. However, the uniqueness remains unknown due to the lack of regularity.

Sufficiently regular solutions to the Cahn–Hilliard system (1.1) satisfy the mass conservation law

$$\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi(0)\,\mathrm{d}x \tag{1.9}$$

for all $t \in [0,T]$. This means that the total mass of the mixture is conserved. Moreover, the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\Omega(\phi(t)) = -\int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x \tag{1.10}$$

holds for all $t \in [0,T]$. As the right-hand side of (1.10) is non-positive, the energy E_Ω is decreasing, and the right-hand side can be interpreted as a dissipation rate. We refer to [14,15,22,39,60,63,103,104,135,138,141,151,162] for a non-exhaustive list of works on the mathematical analysis of (1.1). We further refer to the review paper [177] as well as the book [136] for a general overview of Cahn–Hilliard-type models.

1.3 The Cahn–Hilliard equation with dynamic boundary conditions

(L1) Fixed contact angle. The boundary condition (1.1c) imposes that the diffuse interface separating the fluids always meets the boundary Γ at a right angle of ninety degrees. This arises from the intrinsic structure of the diffuse interface. Namely, across the interface, the phase-field ϕ transitions monotonically between the values close to -1 and values close to 1, following a characteristic profile (cf. the sketched graph on the right of Figure 1.2). Consequently, for this strictly monotone transition to be maintained, the interface must intersect the boundary orthogonally, as otherwise, the profile would induce a non-zero normal derivative $\partial_{\mathbf{n}}\phi$. Naturally, such a condition is quite restrictive and may not hold in practical applications, where the contact angle between the interface and the boundary can differ from ninety degrees and may even evolve dynamically over time.

(L2) No transfer of mass between bulk and surface. The mass conservation law, which is a consequence of the no-flux boundary condition (1.1d), ensures that no mass is allowed to exit the domain. In particular, a transfer of mass between the bulk and the surface cannot be described, which might be required to model phenomena in which the boundary absorbs one of the materials.

Due to these limitations of the classical Cahn–Hilliard equation, and to capture short-range interactions between the bulk Ω and the surface Γ more effectively, mathematicians and physicists [23,66,67,114] proposed augmenting the standard bulk free energy (1.2) with an additional surface contribution of Ginzburg–Landau-type. More precisely, they introduced the surface energy functional

$$E_\Gamma(\psi) := \int_\Gamma \frac{\delta\kappa}{2}|\nabla_\Gamma\psi|^2 + \frac{1}{\delta}G(\psi)\,\mathrm{d}\Gamma.$$

Here, $\psi : \Sigma^T \to \mathbb{R}$ denotes the phase-field on the boundary, and the function G is a double-well potential, such as the classical fourth-order potential (1.5) or the logarithmic Flory–Huggins potential (1.3). The parameter $\delta > 0$ characterizes the interfacial thickness on Γ, and plays a role analogous to ε in the bulk. The coefficient $\kappa \geq 0$ weights the surface Dirichlet energy $\psi \mapsto \int_\Gamma \frac{1}{2}|\nabla_\Gamma\psi|^2\,\mathrm{d}\Gamma$, and induces for $\kappa > 0$ a regularizing effect on the phase separation along the boundary. In the limiting case $\kappa = 0$, the functional E_Γ is closely related to models for moving contact lines, as studied in [172].

The total free energy is then defined as

$$E^0_{\text{free}}(\phi,\psi) := \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\,\mathrm{d}x + \int_\Gamma \frac{\delta\kappa}{2}|\nabla_\Gamma\psi|^2 + \frac{1}{\delta}G(\psi)\,\mathrm{d}\Gamma. \tag{1.11}$$

In many applications, especially when no chemical reactions occur that might transform the materials on the boundary, the surface phase-field ψ is interpreted as the trace of the bulk phase-field. This motivates the coupling condition

$$\phi = \psi \qquad \text{on } \Sigma^T. \tag{1.12}$$

Based on the free energy (1.11), various Cahn–Hilliard and Allen–Cahn models with dynamic boundary conditions have been proposed and analyzed in the literature. Without claiming completeness, we refer, for example, to [35, 40, 41, 45–49, 70, 85, 86, 115–117, 125, 129, 130, 132, 134, 137]. In this thesis, we focus on dynamic boundary conditions of Cahn–Hilliard-type. In particular, we consider the following system on the boundary Γ:

$$\partial_t\psi = \mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\partial_{\mathbf{n}}\mu \qquad \text{on } \Sigma^T, \tag{1.13}$$

$$\theta = -\delta\kappa\Delta_\Gamma\psi + \frac{1}{\delta}G'(\psi) + \varepsilon\partial_{\mathbf{n}}\phi \qquad \text{on } \Sigma^T. \tag{1.14}$$

Here, $\theta : \Sigma^T \to \mathbb{R}$ denotes the chemical potential on the boundary, and m_Γ is the Onsager mobility of the fluid mixture on the boundary, which may, in general, be constant, non-degenerate, or degenerate. The operator Δ_Γ denotes the Laplace–Beltrami operator on Γ. In addition to the trace condition (1.12), a further coupling condition between the bulk and surface chemical potentials μ and θ is required. For constant mobilities, several types of coupling conditions have been proposed:

GMS model. In [71] (for $\kappa = 0$), and in [102] (for $\kappa > 0$), the authors proposed the following Dirichlet-type coupling condition

$$\mu = \beta\theta \qquad \text{on } \Sigma^T \tag{1.15}$$

for the chemical potentials with some constant $\beta \in \mathbb{R}$. This means that the chemical potentials are always in chemical equilibrium. Sufficiently regular solutions to ((1.1), (1.12), (1.13), (1.15)) satisfy the mass conservation law

$$\beta\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \beta\int_\Omega \phi(0)\,\mathrm{d}x + \int_\Gamma \psi(0)\,\mathrm{d}\Gamma \tag{1.16}$$

for all $t \in [0, T]$, as well as the energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\text{free}}(\phi(t), \psi(t)) = -\int_\Omega |\nabla\mu|^2\,\mathrm{d}x - \int_\Gamma |\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma \tag{1.17}$$

for all $t \in [0, T]$. In the case of $\beta > 0$ and for constant mobility functions, the well-posedness of weak solutions was studied in [102], and their long-time behavior was analyzed in [102] and [86]. For more general $\beta \in \mathbb{R}$ with $\beta|\Omega|+|\Gamma| \neq 0$, the well-posedness of weak solutions was recently shown in [121, 122].

LW model. In [126], the authors derived the system ((1.1), (1.12), (1.13)) with homogeneous Neumann boundary condition

$$\partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \Sigma^T \tag{1.18}$$

on the chemical potential through an energetic variational approach based on Onsager's law and the least action principle. In this case, the chemical potentials are not directly coupled. Mechanical interactions of the materials between bulk and surface are, however, still taken into account through the trace relation (1.12). As (1.18) implies that the mass flux between bulk and surface is zero, sufficiently regular solutions to ((1.1), (1.12), (1.13), (1.18)) satisfy the mass conservation laws

$$\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi(0)\,\mathrm{d}x \qquad \text{and} \qquad \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \int_\Gamma \psi(0)\,\mathrm{d}\Gamma \tag{1.19}$$

for all $t \in [0,T]$, as well as the energy dissipation law (1.17). In particular, in this case, the mass in the bulk and on the surface is conserved separately, and consequently, no mass transfer between the bulk and the surface is allowed. The well-posedness of this system was first studied in [126], where the authors required strong assumptions on the volume of the domain and its boundary in the case $\kappa = 0$. These assumptions were later removed in [85].

KLLM model. In [116], the authors proposed the following Robin-type boundary condition

$$L\partial_{\mathbf{n}}\mu = \beta\theta - \mu \qquad \text{on } \Sigma^T \tag{1.20}$$

with $L \in (0,\infty)$ and $\beta \in \mathbb{R}$. The parameter $\frac{1}{L}$ is related to the kinetic rate associated with absorption/desorption processes. Sufficiently regular solutions to ((1.1), (1.12),
(1.13), (1.20)) satisfy the mass conservation law (1.16) and the energy dissipation law

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\text{free}}(\phi(t),\psi(t)) = &-\int_\Omega |\nabla\mu|^2 \,\mathrm{d}x - \int_\Gamma |\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma \\ &- \frac{1}{L}\int_\Gamma (\beta\theta-\mu)^2 \,\mathrm{d}\Gamma\end{aligned} \tag{1.21}$$

for all $t \in [0,T]$. The well-posedness of weak and strong solutions, as well as the long-time behavior, was investigated in [116]. Moreover, the authors showed that the model ((1.1), (1.12), (1.13), (1.20)) can be understood as an interpolation between the GMS model and the LW model. Namely, formally considering the limit $L \to 0$ in (1.20), one obtains the trace relation (1.15) on the chemical potentials. On the other hand, by sending $L \to \infty$, one recovers the homogeneous Neumann boundary condition (1.18) on the chemical potential μ.

Including the case of non-degenerate mobility functions, these boundary conditions can be unified through the boundary condition

$$Lm_\Omega(\phi)\partial_{\mathbf{n}}\mu = \beta\theta - \mu \qquad \text{on } \Sigma^T \tag{1.22}$$

for $L \in [0,\infty]$, which then has to be understood in the sense that

$$\begin{cases} \mu = \beta\theta & \text{if } L = 0, \\ Lm_\Omega(\phi)\partial_{\mathbf{n}}\mu = \beta\theta - \mu & \text{if } L \in (0,\infty), \\ m_\Omega(\phi)\partial_{\mathbf{n}}\mu = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Sigma^T.$$

Then, sufficiently regular solutions to ((1.1), (1.12), (1.13), (1.22)) satisfy the mass conservation laws

$$\begin{cases} \beta\displaystyle\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \beta\int_\Omega \phi(0)\,\mathrm{d}x + \int_\Gamma \psi(0)\,\mathrm{d}\Gamma & \text{if } L \in [0,\infty), \\ \displaystyle\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi(0)\,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \int_\Gamma \psi(0)\,\mathrm{d}\Gamma & \text{if } L = \infty \end{cases} \tag{1.23}$$

for all $t \in [0,T]$, and the energy dissipation law

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E^0_{\text{free}}(\phi(t),\psi(t)) = &-\int_\Omega m_\Omega(\phi)|\nabla\mu|^2 \,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma \\ &- \chi(L)\int_\Gamma (\beta\theta-\mu)^2 \,\mathrm{d}\Gamma\end{aligned} \tag{1.24}$$

for all $t \in [0,T]$. Here, the function χ is defined as

$$\chi : [0,\infty] \to [0,\infty), \qquad \chi(r) := \begin{cases} 0 & \text{if } r \in \{0,\infty\}, \\ \frac{1}{r} & \text{if } r \in (0,\infty), \end{cases} \tag{1.25}$$

and is used to distinguish the different cases corresponding to the choice of $L \in [0,\infty]$.

When the materials on the boundary differ from those in the bulk, for example, if they are transformed through chemical reactions, the trace condition (1.12) is no longer appropriate. To account for this, a more general transmission condition was proposed in [42, 115]:

$$\varepsilon K \partial_{\mathbf{n}} \phi = H(\psi) - \phi \qquad \text{on } \Sigma^T, \tag{1.26}$$

where $K \in [0,\infty]$ and H is a transmission function. This condition has to be understood again in the sense of

$$\begin{cases} \phi = H(\psi) & \text{if } K = 0, \\ \varepsilon K \partial_{\mathbf{n}} \phi = H(\psi) - \phi & \text{if } K \in (0,\infty), \\ \partial_{\mathbf{n}} \phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma^T. \tag{1.27}$$

A commonly used choice is the linear transmission law $H(s) = \alpha s$ for all $s \in \mathbb{R}$ and a constant $\alpha \in \mathbb{R}$. We remark that the Robin-type boundary condition is not only physically meaningful, but also advantageous for the mathematical analysis of the case $K = 0$. Formally, as $K \to 0$, we recover the trace relation $\phi = H(\psi)$ on Σ^T. In the case $H(s) = \alpha s + \beta$ with $\alpha \neq 0$ and $\beta \in \mathbb{R}$, this strategy has been successful for a coupled bulk-surface Allen–Cahn system in [42], and then used for a bulk-surface Cahn–Hilliard system in [115]. Moreover, the limit $K \to \infty$ formally yields the homogeneous Neumann boundary condition $\partial_{\mathbf{n}} \phi = 0$ on Σ^T.

In this thesis, we restrict ourselves to transmission functions of the form $H(s) = \alpha s$ for $\alpha \in \mathbb{R}$. Then (1.27) becomes

$$\begin{cases} \phi = \alpha \psi & \text{if } K = 0, \\ \varepsilon K \partial_{\mathbf{n}} \phi = \alpha \psi - \phi & \text{if } K \in (0,\infty), \\ \partial_{\mathbf{n}} \phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma^T. \tag{1.28}$$

The full bulk-surface Cahn–Hilliard system ((1.1), (1.12), (1.13), (1.22), (1.28)) then takes the form

$$\partial_t \phi = \operatorname{div}(m_\Omega(\phi) \nabla \mu) \qquad \text{in } Q^T, \tag{1.29a}$$

$$\mu = -\varepsilon \Delta \phi + \frac{1}{\varepsilon} F'(\phi) \qquad \text{in } Q^T, \tag{1.29b}$$

$$\partial_t \psi = \operatorname{div}_\Gamma(m_\Gamma(\psi) \nabla_\Gamma \theta) - \beta m_\Omega(\phi) \partial_{\mathbf{n}} \mu \qquad \text{on } \Sigma^T, \tag{1.29c}$$

$$\theta = -\delta \kappa \Delta_\Gamma \psi + \frac{1}{\delta} G'(\psi) + \alpha \varepsilon \partial_{\mathbf{n}} \phi \qquad \text{on } \Sigma^T, \tag{1.29d}$$

$$\begin{cases} \varepsilon K \partial_{\mathbf{n}} \phi = \alpha \psi - \phi & \text{if } K \in [0,\infty), \\ \partial_{\mathbf{n}} \phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma^T, \tag{1.29e}$$

$$\begin{cases} L m_\Omega(\phi) \partial_{\mathbf{n}} \mu = \beta \theta - \mu & \text{if } L \in [0,\infty), \\ m_\Omega(\phi) \partial_{\mathbf{n}} \mu = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Sigma^T, \tag{1.29f}$$

$$\phi|_{t=0} = \phi_0 \qquad \text{in } \Omega, \tag{1.29g}$$

$$\psi|_{t=0} = \psi_0 \qquad \text{on } \Gamma. \tag{1.29h}$$

This model, incorporating all coupling regimes $K, L \in [0, \infty]$, was first proposed and analyzed in a convective setting in [121], and subsequently further investigated in [98, 119, 122, 163, 164].

The energy associated with system (1.29) is given by

$$\begin{aligned} E^K_{\text{free}}(\phi, \psi) &:= \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\,\mathrm{d}x + \int_\Gamma \frac{\delta\kappa}{2}|\nabla_\Gamma\psi|^2 + \frac{1}{\delta}G(\psi)\,\mathrm{d}\Gamma \\ &\quad + \chi(K)\int_\Gamma (\alpha\psi - \phi)^2\,\mathrm{d}\Gamma, \end{aligned} \tag{1.30}$$

where the function χ was defined in (1.25). In contrast to the free energy E^0_{free} defined in (1.11), the definition of E^K_{free} includes an additional term that measures the deviation of the trace $\phi|_\Gamma$ from ψ (cf. [51, 115]). We observe that sufficiently regular solutions to (1.29) satisfy the mass conservation laws (1.23) as well as the following energy dissipation law

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E^K_{\text{free}}(\phi(t), \psi(t)) &= -\int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma \\ &\quad - \chi(L)\int_\Gamma (\beta\theta - \mu)^2\,\mathrm{d}\Gamma \end{aligned} \tag{1.31}$$

for all $t \in [0, T]$ and for all $K \in [0, \infty]$. The explicit dependence on $K \in [0, \infty]$ of the free energy E^K_{free} is usually omitted, and we write E_{free} instead of E^K_{free}.

1.4 Navier–Stokes–Cahn–Hilliard models for two-phase flows

1.4.1 The Model H

One of the most fundamental models describing the motion of two viscous, incompressible fluids with constant densities is Model H. It was originally introduced by Hohenberg and Halperin in [109] to study the critical points of single- and binary-fluid systems. A detailed derivation of the model was proposed in [105] and [165] for the motion of fluids driven by capillary forces. In this model, the fluid densities are assumed to be matched, i.e., the two constant densities are equal. The system consists of an incompressible Navier–Stokes equation coupled to a convective Cahn–Hilliard system, and is given by

$$\rho\,\partial_t\mathbf{v} + \rho\,\mathrm{div}(\mathbf{v}\otimes\mathbf{v}) = \mathrm{div}\,\mathbf{T} \qquad \text{in } Q^T, \tag{1.32a}$$

$$\mathrm{div}\,\mathbf{v} = 0 \qquad \text{in } Q^T, \tag{1.32b}$$

$$\partial_t\phi + \mathrm{div}(\phi\mathbf{v}) = \mathrm{div}(m_\Omega(\phi)\nabla\mu) \qquad \text{in } Q^T, \tag{1.32c}$$

$$\mu = -\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi) \qquad \text{in } Q^T, \tag{1.32d}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \phi|_{t=0} = \phi_0 \qquad \text{in } \Omega. \tag{1.32e}$$

Here, ρ denotes the constant density of the mixture, $\mathbf{v} : Q^T \to \mathbb{R}^d$ the velocity field, $\phi : Q^T \to \mathbb{R}$ the phase-field, and $\mu : Q^T \to \mathbb{R}$ the chemical potential. The functions $\mathbf{v}_0 : \Omega \to \mathbb{R}^d$ and $\phi_0 : \Omega \to \mathbb{R}$ represent prescribed initial data. Furthermore, $\mathbf{T}$ denotes a stress tensor, which is given by

$$\mathbf{T} = \mathbf{S} - p\mathbf{I} - \varepsilon\nabla\phi\otimes\nabla\phi \quad \text{with} \quad \mathbf{S} := 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v} \qquad \text{in } Q^T.$$

Here, $p : Q^T \to \mathbb{R}$ is the pressure, while the coefficient $\nu_\Omega : \mathbb{R} \to \mathbb{R}$ denotes the viscosity of the mixture in the bulk, which in general depends on the phase-field. If the two pure fluids have constant viscosities ν_1 and ν_2, respectively, a common choice is the affine interpolation

$$\nu_\Omega(\phi) = \frac{\nu_2 - \nu_1}{2}\phi + \frac{\nu_2 + \nu_1}{2} \qquad \text{in } Q^T.$$

Moreover, $\mathbf{D}$ denotes the symmetric gradient operator, defined by

$$\mathbf{D}\mathbf{v} := \frac{1}{2}\Big((\nabla \mathbf{v}) + (\nabla \mathbf{v})^\top\Big).$$

Finally, $\varepsilon > 0$ is a parameter related to the thickness of the diffuse interface, m_Ω denotes the mobility function, and F is a double-well potential. We refer to [3, 4, 24, 74, 75, 95, 101] for the mathematical analysis of Model H.

1.4.2 The Abels–Garcke–Grün model (AGG model)

The fundamental assumption in the derivation of Model H is that the density of the mixture is constant, meaning that the fluids have the same constant density ρ. In the past two decades, several works investigated suitable Navier–Stokes–Cahn–Hilliard generalizations of Model H, aiming to describe both incompressible mixtures with unmatched densities and compressible two-phase flows. We refer to [9, 11, 25, 69, 88, 100, 106, 127, 154, 156, 170, 171], without claiming completeness. A very popular generalization of Model H with unmatched densities is the model derived by Abels, Garcke, and Grün in [9], which reads as

$$\begin{aligned}
&\partial_t(\rho(\phi)\mathbf{v}) + \operatorname{div}(\mathbf{v} \otimes (\rho(\phi)\mathbf{v} + \mathbf{J})) = \operatorname{div}\mathbf{T} && \text{in } Q^T, && (1.33a)\\
&\operatorname{div}\mathbf{v} = 0 && \text{in } Q^T, && (1.33b)\\
&\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \operatorname{div}(m_\Omega(\phi)\nabla\mu) && \text{in } Q^T, && (1.33c)\\
&\mu = -\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi) && \text{in } Q^T, && (1.33d)\\
&\mathbf{v}|_{t=0} = \mathbf{v}_0, \quad \phi|_{t=0} = \phi_0 && \text{in } \Omega. && (1.33e)
\end{aligned}$$

Here, in contrast to Model H, the density ρ might vary, and in general depends on the phase-field. It is given by

$$\rho(\phi) = \frac{\tilde\rho_2 - \tilde\rho_1}{2}\phi + \frac{\tilde\rho_2 + \tilde\rho_1}{2} \qquad \text{in } Q^T,$$

where $\tilde\rho_1$ and $\tilde\rho_2$ denote the constant individual densities of the two fluids. As $\rho(\phi)$ is in general not constant, an additional mass flux $\mathbf{J}$ needs to be included in the Navier–Stokes equation to make the model thermodynamically consistent. It is of the form

$$\mathbf{J} = -\frac{\tilde\rho_2 - \tilde\rho_1}{2}m_\Omega(\phi)\nabla\mu \qquad \text{in } Q^T.$$

We note that, in particular, in the case of matched densities, i.e., $\tilde\rho_1 = \tilde\rho_2$, we have

$$\rho(\phi) \equiv \tilde\rho_1 = \tilde\rho_2 \qquad \text{and} \qquad \mathbf{J} \equiv \mathbf{0}.$$

Thus, we observe that the AGG model (1.33) reduces in this case to Model H (1.32).

In the literature, both Model H (1.32) and the AGG model (1.33) have mostly been supplemented with the classical no-slip boundary condition for the velocity field

$\mathbf{v}$ as well as homogeneous Neumann boundary conditions for the phase-field ϕ and the chemical potential μ:

$$\mathbf{v} = 0 \qquad \text{on } \Sigma^T, \tag{1.34}$$

$$\partial_{\mathbf{n}}\phi = 0 \qquad \text{on } \Sigma^T, \tag{1.35}$$

$$m_\Omega(\phi)\partial_{\mathbf{n}}\mu = 0 \qquad \text{on } \Sigma^T. \tag{1.36}$$

The system (1.33) is associated with the total energy

$$E(\mathbf{v}, \phi) := \int_\Omega \frac{1}{2}\rho(\phi)|\mathbf{v}|^2 \,\mathrm{d}x + \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\,\mathrm{d}x,$$

which consists of the kinetic energy

$$E_{\text{kin}}(\mathbf{v}, \phi) := \int_\Omega \frac{1}{2}\rho(\phi)|\mathbf{v}|^2 \,\mathrm{d}x \tag{1.37}$$

and the Ginzburg–Landau energy E_Ω defined in (1.2). As for the standard Cahn–Hilliard equation without convective term, the boundary conditions (1.34) and (1.36) entail that

$$\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi(0)\,\mathrm{d}x$$

for all $t \in [0, T]$, meaning that the mass in the bulk is always conserved. Moreover, sufficiently regular solutions satisfy the energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{v}(t), \phi(t)) = -\int_\Omega 2\nu_\Omega(\phi)|\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x - \int_\Omega m_\Omega(\phi)|\nabla\mu|^2 \,\mathrm{d}x \tag{1.38}$$

for all $t \in [0, T]$.

For analytical results, we refer to [5, 6, 8, 13, 94, 95], and for the sharp interface limit $\varepsilon \to 0$ to [7, 9], where it was shown that formally, the AGG model converges to a two-phase Navier–Stokes free boundary problem.

(L3) Convection-induced motion of the contact line cannot be described very precisely. There are only limited possibilities for accurately describing contact line motion. In many applications involving two-phase flows, it is essential to capture the evolution of the contact line, i.e., the region where the fluid interface meets the boundary. As observed, for example, in [58, 148, 153], the no-slip boundary condition (1.34) is not well suited for modeling moving contact line phenomena, since it enforces the velocity field to vanish at the boundary. In sharp interface models, this limitation is even more pronounced, as contact-line singularities may arise. In such cases, the tangential force exerted by the fluids at the contact line becomes unbounded; see [58, 155] for further discussion of these singularities. In contrast, diffuse interface models do allow for motion of the contact line even under the no-slip boundary condition (1.34), which constitutes a significant advantage over sharp interface approaches. Nevertheless, any such motion can arise only from convection or diffusion processes in the bulk, away from the boundary. In particular, since (1.34) enforces a vanishing velocity field at the boundary, convective effects acting directly at, or in the immediate vicinity of, the boundary are neglected. These effects, however, are expected to play a crucial role in accurately capturing the dynamics of the contact line.

1.5 Navier–Stokes–Cahn–Hilliard models with dynamic boundary conditions for two-phase flows

1.5.1 Dynamic boundary conditions of Allen–Cahn-type

To overcome the aforementioned limitations **(L1)** and **(L3)**, the authors in [148] proposed a new class of generalized Navier slip boundary conditions based on a variational derivation through Onsager's principle of maximal energy dissipation. These boundary conditions read as

$$\mathbf{v} \cdot \mathbf{n} = 0, \qquad [\mathbf{S}\mathbf{n} + \gamma \mathbf{v}]_{\tan} = L(\psi) \nabla_\Gamma \psi \qquad \text{on } \Sigma^T, \tag{1.39a}$$

$$\partial_t \psi + \mathbf{v}_{\tan} \cdot \nabla_\Gamma \psi = -\beta L(\psi) \qquad \text{on } \Sigma^T, \tag{1.39b}$$

$$\partial_{\mathbf{n}} \mu = 0 \qquad \text{on } \Sigma^T, \tag{1.39c}$$

where $\psi = \phi$ on Σ^T as in (1.12), and the operator L is given by

$$L(\psi) \coloneqq -\kappa \Delta_\Gamma \psi + \varepsilon \partial_{\mathbf{n}} \phi + G'(\psi).$$

Here, the subscript tan denotes the tangential component of any vector field $\mathbf{u}$ on Σ^T, i.e.,

$$\mathbf{u}_{\tan} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}.$$

Moreover, ∇_Γ and Δ_Γ denote the tangential gradient and the Laplace–Beltrami operator on Γ, respectively. The phenomenological parameters β and κ are assumed to be positive, and G denotes a surface double-well potential. The surface evolution of the concentration ψ is governed by (1.39b), which takes the form of a convective Allen–Cahn-type equation posed on the surface. The boundary condition (1.39a) is an inhomogeneous Navier slip boundary condition, which involves a slip parameter $\gamma \geq 0$ that may depend on the phase-field ψ. Compared to the no-slip boundary condition (1.34), it provides a much better description of the contact line motion [146–148].

Model H (1.32), supplemented by (1.39), was first analyzed in [76], where the authors established the existence of global-in-time weak solutions. Their analysis covers cases in which both F and G are polynomial-type potentials, as well as the case in which F is singular while G remains polynomial. In the polynomial setting, they further proved the convergence of any weak solution toward a stationary state. For the AGG model (1.33) coupled with the Allen–Cahn-type boundary condition (1.39b), the existence of global-in-time weak solutions was shown in [79]. More recently, a compressible Navier–Stokes–Cahn–Hilliard system with boundary condition (1.39) has been investigated in [38]. In addition, Navier–Stokes–Allen–Cahn and Navier–Stokes–Voigt–Allen–Cahn systems with (1.39) were studied in [77]. We also mention [83], where a stochastic Model H with dynamic boundary conditions of Allen–Cahn-type (1.39) was considered. It is worth emphasizing that none of the aforementioned works establishes the uniqueness of weak solutions, neither in two nor in three dimensions. To date, the only available uniqueness result concerns quasi-strong solutions for the Navier–Stokes–Voigt–Allen–Cahn systems with (1.39), as shown in [77].

1.5.2 Dynamic boundary conditions of Cahn–Hilliard-type

To overcome all the limitations **(L1)**, **(L2)**, and **(L3)**, the authors of [97] derived a new class of thermodynamically consistent Navier–Stokes–Cahn–Hilliard systems with dynamic boundary conditions. Similar to the derivation of the AGG model in [9], the derivation is based on local mass balance laws, local energy dissipation laws, and the Lagrange multiplier approach. The system reads as

$$
\begin{aligned}
&\partial_t(\rho(\phi)\mathbf{v}) + \operatorname{div}(\mathbf{v}\otimes(\rho(\phi)\mathbf{v}+\mathbf{J})) = \operatorname{div}\mathbf{T}, \qquad \operatorname{div}\mathbf{v}=0 && \text{in } Q^T, && (1.40\text{a})\\
&[\mathbf{S}\mathbf{n} + \gamma(\psi)\mathbf{v}]_{\tan} = \big[-\psi\nabla_\Gamma\theta + \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{v}\big]_{\tan} && \text{on } \Sigma^T, && (1.40\text{b})\\
&\mathbf{v}\cdot\mathbf{n} = 0 && \text{on } \Sigma^T, && (1.40\text{c})\\
&\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \operatorname{div}(m_\Omega(\phi)\nabla\mu) && \text{in } Q^T, && (1.40\text{d})\\
&\mu = -\varepsilon\Delta\phi + \tfrac{1}{\varepsilon}F'(\phi) && \text{in } Q^T, && (1.40\text{e})\\
&\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{v}_{\tan}) = \operatorname{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\partial_{\mathbf{n}}\mu && \text{on } \Sigma^T, && (1.40\text{f})\\
&\theta = -\delta\kappa\Delta_\Gamma\psi + \tfrac{1}{\delta}G'(\psi) + \varepsilon\partial_{\mathbf{n}}\phi && \text{on } \Sigma^T, && (1.40\text{g})\\
&\phi=\psi, \qquad \begin{cases} L m_\Omega(\phi)\partial_{\mathbf{n}}\mu = \beta\theta - \mu & \text{if } L\in[0,\infty),\\ m_\Omega(\phi)\partial_{\mathbf{n}}\mu = 0 & \text{if } L=\infty \end{cases} && \text{on } \Sigma^T, && (1.40\text{h})\\
&\mathbf{v}|_{t=0} = \mathbf{v}_0 \qquad \phi|_{t=0} = \phi_0 && \text{in } \Omega, && (1.40\text{i})\\
&\psi|_{t=0} = \psi_0 && \text{on } \Gamma. && (1.40\text{j})
\end{aligned}
$$

In contrast to the dynamic boundary conditions (1.39) that were of Allen–Cahn-type, the dynamic boundary conditions in system (1.40) are of Cahn–Hilliard-type. More precisely, they can be regarded as a convective variant of (1.29) in the special case $K=0$. As discussed in Chapter 1.3, these boundary conditions help to overcome the limitations **(L1)** and **(L2)**. Moreover, (1.40b) is an inhomogeneous Navier slip boundary condition similar to (1.39a), which helps to overcome **(L3)**.

The system (1.40) is associated with the total energy

$$E(\mathbf{v},\phi,\psi) := E_{\text{kin}}(\phi,\mathbf{v}) + E_{\text{free}}(\phi,\psi),$$

where E_{kin} is the kinetic energy as in (1.37), and $E_{\text{free}} = E_\Omega + E_\Gamma$ is the free energy defined in (1.11). Sufficiently regular solutions to (1.40) satisfy the mass conservation laws

$$
\begin{cases}
\beta\displaystyle\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \beta\int_\Omega\phi(0)\,\mathrm{d}x + \int_\Gamma\psi(0)\,\mathrm{d}\Gamma & \text{if } L\in[0,\infty),\\
\displaystyle\int_\Omega\phi(t)\,\mathrm{d}x = \int_\Omega\phi(0)\,\mathrm{d}x \quad\text{and}\quad \int_\Gamma\psi(t)\,\mathrm{d}\Gamma = \int_\Gamma\psi(0)\,\mathrm{d}\Gamma & \text{if } L=\infty
\end{cases}
\tag{1.41}
$$

for all $t\in[0,T]$ as well as the energy dissipation law

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{v}(t),\phi(t),\psi(t)) = &-\int_\Omega 2\nu_\Omega(\phi)|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x - \int_\Gamma\gamma(\phi)|\mathbf{v}|^2\,\mathrm{d}\Gamma - \int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x\\
&-\int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma - \chi(L)\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma,
\end{aligned}
$$

for all $t\in[0,T]$, where χ is the function introduced in (1.25).

Although the model (1.40) allows for unmatched densities, variable contact angles, an improved description of contact line movement via the generalized Navier boundary condition (1.40b), and a transfer of material between bulk and surface (if $L<\infty$), it still exhibits some important limitations.

The first analytical results for the system (1.40) were obtained in [97] for $K=L=0$ and regular potentials. There, the authors established the existence of weak solutions in the case of matched densities, i.e., in the case $\rho(\phi) \equiv \tilde{\rho}_1 = \tilde{\rho}_2$. As in the AGG model, we notice that for matched densities, we have

$$\mathbf{J} \equiv 0,$$

and the Navier–Stokes equation (1.40a) with boundary condition (1.40b) reduces to

$$\begin{aligned}
&\rho\,\partial_t \mathbf{v} + \rho\,\mathrm{div}(\mathbf{v} \otimes \mathbf{v}) - \mathrm{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) + \nabla p = -\varepsilon \mathrm{div}(\nabla\phi \otimes \nabla\phi) && \text{in } Q^T, \\
&[\mathbf{S}\mathbf{n} + \gamma(\psi)\mathbf{v}]_{\tan} = [-\psi\nabla_\Gamma\theta]_{\tan} && \text{on } \Sigma^T.
\end{aligned}$$

Moreover, they proved the uniqueness of weak solutions in the two-dimensional setting, assuming additional regularity on the coefficients and a compatibility assumption on the potentials. We note that singular potentials, like the Flory–Huggins potential, are not admissible. These were covered in the recent work [81]. There, the authors considered more general singular potentials, including the physically relevant logarithmic potential, and established the existence of weak solutions for $K = 0$ and $L \in (0, \infty]$ and non-matched densities. They also studied the case $K = L = 0$, but had to assume matched densities as in [97]. This problem arises from the flux term $\mathbf{J}$ appearing in the generalized Navier slip boundary condition (1.40b), which reads as

$$\mathbf{J} \cdot \mathbf{n} = -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m_\Omega(\phi)\partial_{\mathbf{n}}\mu \qquad \text{on } \Sigma^T.$$

In the case $L \in (0, \infty)$, this term can directly formally be rewritten using the boundary condition (1.40h) as

$$\mathbf{J} \cdot \mathbf{n} = -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2L}(\beta\theta - \mu) \qquad \text{on } \Sigma^T,$$

while in the case $L = \infty$, this term directly vanishes. However, the case $L = 0$ is more delicate, as such a reformulation of the normal derivative $\partial_{\mathbf{n}}\mu$ is not possible. Moreover, in the setting of weak solutions, one usually expects the regularity $\mu \in L^2(0, T; H^1(\Omega))$, which does not provide enough regularity to control this term.

A natural question is whether one can expect higher regularity of weak solutions, or even the existence of global-in-time strong solutions to system (1.40). The main difficulty in this direction is the trace $\mathbf{v}_{\tan}$ of the velocity field, which appears in the boundary convection term $\mathrm{div}_\Gamma(\psi\mathbf{v}_{\tan})$ in (1.40f). At the level of weak solutions, this trace does not have sufficient regularity to develop a satisfactory regularity theory for the convective bulk-surface Cahn–Hilliard system. This difficulty already appears at the level of weak solutions in [81], where the authors had to add an additional regularization term on the boundary. In Chapter 5 and Chapter 6, we therefore study the convective bulk-surface Cahn–Hilliard subsystem (1.40d)–(1.40h) with a general prescribed boundary velocity field $\mathbf{w}$, replacing $\mathbf{v}_{\tan}$. Even in this setting, the regularity requirements for strong solutions are stringent, suggesting that global-in-time strong solutions to the full system (1.40) are not expected in general.

We further point out the recent model derived in [118], where a generalization of (1.40) is considered on an evolving domain. To date, no analytical results are available, which may be attributed to the lack of regularity of the trace of the velocity field.

1.6 A new bulk-surface Navier–Stokes–Cahn–Hilliard model for fluid mixtures

In many applications across materials science, chemistry, engineering, and the life sciences, the materials at the boundary may differ from those in the bulk. In such situations, the rheological properties of the surface materials often need to be taken into account, cf. [152].

An important example is the classical fluid-mosaic model of cell membrane structure, as described in the seminal work [160]. There, biological membranes are conceptualized as laterally incompressible, two-dimensional viscous fluid layers with embedded proteins. This justifies treating the boundary as an active viscous layer that supports both lateral momentum transport and compositional evolution. These concepts were formalized mathematically in [17] as a continuum model, in which lipid bilayers are described by a surface Navier–Stokes equation with positive but finite viscosity, which is expected to dominate relaxation dynamics. These observations motivate the inclusion of a surface Navier–Stokes equation in system (1.40) to account for viscous and convective effects at the boundary.

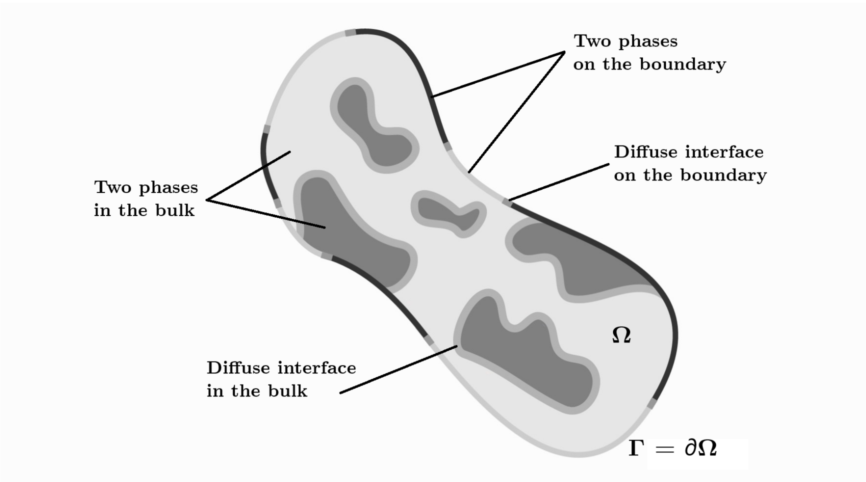

Figure 1.4: Sketch of a domain (representing, e.g., a biological cell) with two phases (fluids) in the bulk and two phases (fluids) on the boundary, each separated by a diffuse interface.

Moreover, as discussed previously, the trace condition $\phi = \psi$ on Σ^T is only reasonable if the materials on the boundary are the same as in the bulk. We therefore also include the boundary conditions (1.28) to accommodate different materials.

Similar to the derivation of (1.33) in [9] and of (1.40) in [97], the derivation of our model, which can be found in Chapter 3, is based on local mass balance laws, both in the bulk and on the surface, with a priori unknown mass fluxes. We further consider local energy-dissipation laws that include additional energy-flux terms. Eventually, we employ the Lagrange multiplier approach, with the chemical potentials μ and θ as Lagrange multipliers, to complete the derivation of our model by identifying the unknown mass and energy fluxes. The new bulk-surface Navier–Stokes–Cahn–Hilliard model we propose reads as follows:

$$\partial_t(\rho(\phi)\mathbf{v}) + \operatorname{div}(\mathbf{v} \otimes (\rho(\phi)\mathbf{v} + \mathbf{J})) = \operatorname{div} \mathbf{T}, \qquad \operatorname{div} \mathbf{v} = 0 \qquad \text{in } Q^T, \quad (1.42a)$$
$$\partial_t(\sigma(\psi)\mathbf{w}) + \operatorname{div}_\Gamma(\mathbf{w} \otimes (\sigma(\psi)\mathbf{w} + \mathbf{K})) = \operatorname{div}_\Gamma \mathbf{T}_\Gamma + \mathbf{Z}, \quad \operatorname{div}_\Gamma \mathbf{w} = 0 \qquad \text{on } \Sigma^T, \quad (1.42b)$$
$$\mathbf{v}|_\Gamma = \mathbf{w}, \qquad \mathbf{v} \cdot \mathbf{n} = 0 \qquad \text{on } \Sigma^T, \quad (1.42c)$$

$$\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \operatorname{div}(m_\Omega(\phi)\nabla\mu) \qquad \text{in } Q^T, \quad (1.42d)$$
$$\mu = -\varepsilon\Delta\phi + \tfrac{1}{\varepsilon}F'(\phi) \qquad \text{in } Q^T, \quad (1.42e)$$
$$\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) = \operatorname{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\partial_\mathbf{n}\mu \qquad \text{on } \Sigma^T, \quad (1.42f)$$
$$\theta = -\delta\kappa\Delta_\Gamma\psi + \tfrac{1}{\delta}G'(\psi) + \alpha\varepsilon\partial_\mathbf{n}\phi \qquad \text{on } \Sigma^T, \quad (1.42g)$$

$$\begin{cases} \varepsilon K \partial_{\mathbf{n}} \phi = \alpha\psi - \phi & \text{if } K \in [0,\infty), \\ \partial_{\mathbf{n}}\phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma^T, \quad (1.42h)$$

$$\begin{cases} L m_\Omega(\phi)\partial_{\mathbf{n}}\mu = \beta\theta - \mu & \text{if } L \in [0,\infty), \\ m_\Omega(\phi)\partial_{\mathbf{n}}\mu = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Sigma^T, \quad (1.42i)$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \qquad \phi|_{t=0} = \phi_0 \qquad \text{in } \Omega, \quad (1.42j)$$

$$\mathbf{w}|_{t=0} = \mathbf{w}_0, \qquad \psi|_{t=0} = \psi_0 \qquad \text{on } \Gamma. \quad (1.42k)$$

Here, viscous dynamics on the boundary are represented by an additional surface velocity field $\mathbf{w} : \Sigma^T \to \mathbb{R}^d$. In this model, we assume that the bulk and surface velocity fields are coupled through the trace relation (1.42c)$_1$. This means that the motions of the bulk materials and the surface materials are directly related. Alternatively, stress-based coupling conditions between $\mathbf{v}$ and $\mathbf{w}$ might also be conceivable and could be discussed in future contributions. As we demand the non-penetration boundary condition (1.42c)$_2$, the trace relation (1.42c)$_1$ further entails that the surface velocity field $\mathbf{w}$ is an evolving tangential vector field on Γ. The bulk phase-field $\phi : Q^T \to \mathbb{R}$ represents the location of the two materials in the bulk, whereas the two materials on the boundary are represented by the surface phase-field $\psi : \Sigma^T \to \mathbb{R}$.

The time evolution of $\mathbf{v}$ and $\mathbf{w}$ is described by the bulk Navier–Stokes equation (1.42a) and the tangential surface Navier–Stokes equation (1.42b), respectively. Both the bulk and surface flows are assumed to be incompressible, i.e., the velocity fields are divergence-free. Here, $\sigma(\psi)$ denotes the phase-field dependent density on the surface, and is given by

$$\sigma(\psi) \coloneqq \frac{\tilde\sigma_2 - \tilde\sigma_1}{2}\psi + \frac{\tilde\sigma_2 + \tilde\sigma_1}{2} \qquad \text{on } \Sigma^T. \quad (1.43)$$

Furthermore, $\mathbf{T}_\Gamma$ is a stress tensor, whereas $\mathbf{Z}$ represents a surface force density that accounts for mechanical interactions between the bulk materials and the surface materials. They are given by

$$\mathbf{T}_\Gamma \coloneqq \mathbf{S}_\Gamma - q\mathbf{I} - \delta\kappa\nabla_\Gamma\psi \otimes \nabla_\Gamma\psi \quad \text{with} \quad \mathbf{S}_\Gamma \coloneqq 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w} \qquad \text{on } \Sigma^T, \quad (1.44)$$

$$\mathbf{Z} \coloneqq -[\mathbf{Sn}]_{\tan} + \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} + \tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} + \alpha\varepsilon\partial_{\mathbf{n}}\phi\,\nabla_\Gamma\psi - \gamma(\phi,\psi)\mathbf{w} \qquad \text{on } \Sigma^T. \quad (1.45)$$

The coefficient $\nu_\Gamma : \mathbb{R} \to \mathbb{R}$ denotes the viscosity of the mixture on the surface. As above, $\gamma(\phi,\psi)$ denotes a slip coefficient associated with tangential friction. It may now depend on both phase-fields, as they are not necessarily coupled via the trace relation as in (1.40h). Moreover,

$$\mathbf{D}_\Gamma\mathbf{w} \coloneqq \frac{1}{2}\big((\nabla_\Gamma\mathbf{w}) + (\nabla_\Gamma\mathbf{w})^\top\big)$$

denotes the surface symmetric gradient of $\mathbf{w}$. The quantity $\mathbf{K}$ denotes a mass flux related to the interfacial motion on the boundary, while $\mathbf{J}_\Gamma$ represents the mass flux corresponding to the transfer of material between bulk and surface. They are given by

$$\mathbf{K} \coloneqq -\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} m_\Gamma(\psi)\nabla_\Gamma\theta \qquad \mathbf{J}_\Gamma \coloneqq -\beta\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} m_\Omega(\phi)\nabla\mu \qquad \text{on } \Sigma^T. \quad (1.46)$$

The system (1.42) is associated with the total energy

$$E_{\text{tot}}(\mathbf{v},\mathbf{w},\phi,\psi) \coloneqq \int_\Omega \frac{1}{2}\rho(\phi)|\mathbf{v}|^2 \,\mathrm{d}x + \int_\Gamma \frac{1}{2}\sigma(\psi)|\mathbf{w}|^2 \,\mathrm{d}\Gamma$$

$$\begin{aligned}&+ \int_\Omega \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi)\,\mathrm{d}x + \int_\Gamma \frac{\delta\kappa}{2}|\nabla_\Gamma\psi|^2 + \frac{1}{\delta}G(\psi)\,\mathrm{d}\Gamma \\ &+ \chi(K)\int_\Gamma \frac{1}{2}(\alpha\psi - \phi)\,\mathrm{d}\Gamma,\end{aligned} \tag{1.47}$$

where χ is the function introduced in (1.25). Sufficiently regular solutions to (1.42) satisfy the mass conservation laws

$$\begin{cases} \beta\displaystyle\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \beta\int_\Omega \phi_0\,\mathrm{d}x + \int_\Gamma \psi_0\,\mathrm{d}\Gamma & \text{if } L \in [0,\infty), \\ \displaystyle\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi_0\,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \int_\Gamma \psi_0\,\mathrm{d}\Gamma & \text{if } L = \infty \end{cases} \tag{1.48}$$

for all $t \in [0,T]$ as well as the energy dissipation law

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{tot}}(\mathbf{v}(t),\mathbf{w}(t),\phi(t),\psi(t)) &= -\int_\Omega 2\nu_\Omega(\phi)|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x - \int_\Gamma 2\nu_\Gamma(\psi)|\mathbf{D}_\Gamma\mathbf{w}|^2\,\mathrm{d}\Gamma \\ &\quad - \int_\Gamma \gamma(\phi,\psi)|\mathbf{w}|^2\,\mathrm{d}\Gamma - \int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x \\ &\quad - \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma - \chi(L)\int_\Gamma (\beta\theta-\mu)^2\,\mathrm{d}\Gamma\end{aligned} \tag{1.49}$$

for all $t \in [0,T]$.

1.7 Structure of the thesis

The thesis is organized as follows.

Chapter 2 introduces the notation and collects the mathematical tools used throughout the thesis.

In Chapter 3, we derive the bulk-surface Navier–Stokes–Cahn–Hilliard model (1.42), which is the central model of this work.

Chapter 4 is devoted to several auxiliary bulk-surface systems that play an important role in the analysis of (1.42), including elliptic bulk-surface systems with logarithmic nonlinearities or non-constant coefficients, and a bulk-surface Stokes system. For these systems, we establish well-posedness of weak solutions, higher regularity, and the existence of strong solutions.

The remainder of the thesis is divided into two parts.

In Part I, we study the convective bulk-surface Cahn–Hilliard subsystem of (1.42). In Chapter 5, we prove existence, uniqueness, and higher regularity of weak solutions for regular potentials. In Chapter 6, we turn to singular potentials and establish existence, uniqueness, and propagation of regularity of weak solutions for a broad class of velocity fields. As the uniqueness and propagation of regularity generally require constant mobility functions, we study the case of non-degenerate mobilities separately in Chapter 7. There, we prove the uniqueness of weak solutions and the existence of strong solutions in the two-dimensional setting. Finally, in Chapter 8, we analyze the long-time behavior of weak solutions, proving the existence of a pullback attractor and convergence to a single equilibrium as $t \to \infty$.

In Part II, we study the full bulk-surface Navier–Stokes–Cahn–Hilliard system (1.42). In Chapter 9, we prove the existence of global-in-time weak solutions by means of an implicit time-discretization scheme, building on the results of the previous chapters, in particular on a characterization of the subdifferential of the convex part of the free energy. In Chapter 10, we establish the existence and uniqueness of global-in-time strong solutions in two spatial dimensions via a suitable semi-Galerkin scheme based on eigenfunctions of a bulk-surface Stokes operator.

1.8 Publications

This thesis mainly consists of the following publications and submitted papers:

[98] A. Giorgini, P. Knopf, and J. Stange, *Regularity of a bulk-surface Cahn–Hilliard model driven by Leray velocity fields.* Preprint: arXiv:2506.18617, 2025.

[119] P. Knopf, A. Poiatti, J. Stange, and S. Yayla, *Long-time dynamics of a bulk-surface convective Cahn–Hilliard system: Pullback attractors and convergence to equilibrium.* Preprint: arXiv:2603.10923, 2026.

[121] P. Knopf, and J. Stange, *Well-posedness of a bulk-surface convective Cahn–Hilliard system with dynamic boundary conditions*, NoDEA Nonlinear Differential Equations Appl., 31 (2024), pp. Paper No. 82, 48.

[122] P. Knopf, and J. Stange, *Strong well-posedness and separation properties for a bulk-surface convective Cahn–Hilliard system with singular potentials*, J. Differential Equations, 439 (2025), pp. Paper No. 113408, 60.

[123] P. Knopf, and J. Stange, *A thermodynamically consistent model for bulk-surface viscous fluid mixtures: Model derivation and mathematical analysis.* Preprint: arXiv:2509.11925, 2025.

[163] J. Stange, *Global well-posedness of strong solutions to a bulk-surface Navier–Stokes–Cahn–Hilliard model with non-degenerate mobilities in two dimensions*, Preprint: arXiv:2511.06847, 2025.

[164] J. Stange, *Well-posedness and long-time behavior of a bulk-surface Cahn–Hilliard model with non-degenerate mobility*, Nonlinear Analysis, 268 (2026), p. 114060.

Chapter 2

Preliminaries

This chapter collects the basic mathematical tools that will be used repeatedly in this thesis.

2.1 Basic notation and function spaces

2.1.1 Basic notation

We write $\mathbb{N}$ to denote the set of natural numbers (excluding zero). For any real Banach space X with $\|\cdot\|_X$, its dual space is denoted by X'. The corresponding duality pairing between $\phi \in X'$ and $\zeta \in X$ is written as $\langle \phi, \zeta \rangle_X$. If X is a Hilbert space, we write $(\cdot,\cdot)_X$ to denote its inner product.

Let $p \in [1,\infty]$ and let $I \subset [0,\infty)$ be an interval. The Bochner space $L^p(I,X)$ consists of all Bochner measurable and p-integrable functions from I into X. We further define $W^{1,p}(I;X)$ as the space of all functions $f \in L^p(I;X)$ whose (vector-valued) distributional time derivative $\partial_t f$ belongs to $L^p(I;X)$. In particular, we set $H^1(I;X) = W^{1,2}(I;X)$. The set of continuous functions $f : I \to X$ is denoted by $C(I;X)$, while $C_w(I;X)$ denotes the space of functions $f : I \to X$ that are continuous on I with respect to the weak topology on X, i.e., the map $I \ni t \mapsto \langle \phi, f(t) \rangle_X$ is continuous for all $\phi \in X'$. Moreover, $BC(I;X)$ and $BC_w(I;X)$ denote the subspaces of bounded functions in $C(I;X)$ and $C_w(I;X)$, respectively.

2.1.2 Function spaces in the bulk and on the surface

Throughout this section, let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded domain with sufficiently smooth boundary $\Gamma \coloneqq \partial\Omega$, and let $\mathbf{n}$ denote the exterior unit normal vector field on Γ. For any $s \geq 0$ and $p \in [1,\infty]$, the Lebesgue and Sobolev–Slobodeckij spaces for functions mapping from Ω to $\mathbb{R}$ are denoted as $L^p(\Omega)$ and $W^{s,p}(\Omega)$, respectively. We write $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{s,p}(\Omega)}$ to denote the standard norms on these spaces. If $p = 2$, we use the notation $H^s(\Omega) = W^{s,2}(\Omega)$, with the convention that $H^0(\Omega)$ is identified with $L^2(\Omega)$. For the Lebesgue and Sobolev–Slobodeckij spaces on the boundary Γ, we use an analogous notation. The corresponding spaces of vector-valued mappings into $\mathbb{R}^d$ or matrix-valued functions mapping into $\mathbb{R}^{d\times d}$ are denoted by boldface letters, namely $\mathbf{L}^p$, $\mathbf{W}^{s,p}$ and $\mathbf{H}^s$. In particular, we write $\mathbf{L}^p_{\tan}(\Gamma)$, $\mathbf{W}^{s,p}_{\tan}(\Gamma)$ and $\mathbf{H}^s_{\tan}(\Gamma)$ to denote the corresponding spaces of tangential vector fields on Γ.

For $p \in [1,\infty]$, we further introduce the first-order homogeneous Sobolev spaces

$$\dot{W}^{1,p}(\Omega) \coloneqq \{u \in L^p_{\mathrm{loc}}(\Omega) \,:\, \nabla u \in \mathbf{L}^p(\Omega)\}/_{\sim}\,,$$
$$\dot{W}^{1,p}(\Gamma) \coloneqq \{v \in L^p(\Gamma) \,:\, \nabla_\Gamma v \in \mathbf{L}^p(\Gamma)\}/_{\sim}\,,$$

where the notation $/_\sim$ indicates that functions differing only by an additive constant are considered equivalent and thus identified with each other. Here, $L^p_{\mathrm{loc}}(\Omega)$ is the collection of all functions, which belong to $L^p(K)$ for every compact subset $K \subset \Omega$. On the boundary, we simply consider $L^p(\Gamma)$, since Γ is itself a compact submanifold. Consequently, the functionals $\|\nabla \cdot\|_{\mathbf{L}^p(\Omega)}$ and $\|\nabla_\Gamma \cdot\|_{\mathbf{L}^p(\Gamma)}$ define norms on the spaces $\dot{W}^{1,p}(\Omega)$ and $\dot{W}^{1,p}(\Gamma)$, respectively.

Furthermore, we write $\mathbf{P}^\Gamma$ to denote the orthogonal projection of $\mathbb{R}^d$ into the tangent space of the submanifold Γ. This means that $\mathbf{P}^\Gamma$ is a linear map, which can be represented as

$$\mathbf{P}^\Gamma = \mathbf{I} - \mathbf{n} \otimes \mathbf{n},$$

where $\mathbf{I} \in \mathbb{R}^{d\times d}$ is the unity matrix. For any vector field $\mathbf{w}$ on Γ, we also use the notation

$$[\mathbf{w}]_{\tan} = \mathbf{P}^\Gamma(\mathbf{w}).$$

2.1.3 Differential operators in the bulk and on the surface

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ with boundary $\Gamma \coloneqq \partial\Omega$. Let $f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$ be scalar functions, let $\mathbf{v} : \Omega \to \mathbb{R}^d$ and $\mathbf{w} : \Gamma \to \mathbb{R}^d$ be vector fields, and let $\mathbf{A} : \Omega \to \mathbb{R}^{d\times d}$ and $\mathbf{B} : \Gamma \to \mathbb{R}^{d\times d}$ be matrix-valued functions. For now, we assume that the boundary Γ and all these functions are sufficiently regular. As usual, ∇f, $\operatorname{div}\mathbf{v}$, and $\Delta f = \operatorname{div}(\nabla f)$ denote the gradient of f, the divergence of $\mathbf{v}$ and the Laplacian of f. We point out that ∇f is understood as a column vector in $\mathbb{R}^d$.

Next, we define the tangential gradient of g as

$$\nabla_\Gamma g \coloneqq \mathbf{P}^\Gamma \nabla \tilde{g},$$

where $\tilde{g}$ denotes a suitable extension of g to a tubular neighborhood U of Γ in $\mathbb{R}^d$. We note that $\nabla_\Gamma g$ does not depend on the concrete extension $\tilde{g}$. We refer to [59] for a more detailed exposition of these materials. Here, $\nabla_\Gamma g$ is also to be understood as a column vector in $\mathbb{R}^d$. Next, we define the surface divergence of $\mathbf{w}$ as

$$\operatorname{div}_\Gamma \mathbf{w} \coloneqq \operatorname{tr}(\nabla_\Gamma \mathbf{w}),$$

where tr denotes the trace of a matrix. The Laplace–Beltrami operator of g is then defined as $\Delta_\Gamma g = \operatorname{div}_\Gamma(\nabla_\Gamma g)$. For bulk vector fields, the gradient is defined as

$$\nabla \mathbf{v} = \begin{pmatrix} (\nabla \mathbf{v}_1)^T \\ \vdots \\ (\nabla \mathbf{v}_d)^T \end{pmatrix} \in \mathbb{R}^{d\times d},$$

where $\mathbf{v}_1, \dots, \mathbf{v}_d$ denote the components of $\mathbf{v}$. In other words, $\nabla\mathbf{v}$ is the Jacobian of $\mathbf{v}$. On the surface Γ, the projective (covariant) gradient of $\mathbf{w}$ is defined as

$$\nabla_\Gamma \mathbf{w} = \mathbf{P}^\Gamma \nabla \widetilde{\mathbf{w}} \mathbf{P}^\Gamma,$$

where $\widetilde{\mathbf{w}}$ now denotes an extension of $\mathbf{w}$ to an open neighborhood of Γ in $\mathbb{R}^d$. If $\mathbf{w}$ is a tangential vector field, i.e., $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ, its gradient can also be expressed as

$$\nabla_\Gamma \mathbf{w} = \mathbf{P}^\Gamma \begin{pmatrix} (\nabla_\Gamma \mathbf{w}_1)^T \\ \vdots \\ (\nabla_\Gamma \mathbf{w}_d)^T \end{pmatrix} \in \mathbb{R}^{d\times d},$$

where $\mathbf{w}_1, \dots, \mathbf{w}_d$ denote the components of $\mathbf{w}$. For the matrix-valued functions $\mathbf{A}$ and $\mathbf{B}$, we further define the divergences

$$\operatorname{div}\mathbf{A} = \begin{pmatrix} \operatorname{div}\mathbf{A}_{1*} \\ \vdots \\ \operatorname{div}\mathbf{A}_{d*} \end{pmatrix}, \qquad \operatorname{div}_\Gamma \mathbf{B} = \mathbf{P}^\Gamma \begin{pmatrix} \operatorname{div}_\Gamma \mathbf{B}_{1*} \\ \vdots \\ \operatorname{div}_\Gamma \mathbf{B}_{d*} \end{pmatrix}.$$

Here, $\mathbf{A}_{1*}, \dots, \mathbf{A}_{d*}$ and $\mathbf{B}_{1*}, \dots, \mathbf{B}_{d*}$ denote the rows of $\mathbf{A}$ and $\mathbf{B}$, respectively.

2.1.4 Bulk-surface product spaces

For any $s \geq 0$ and $p \in [1, \infty]$, we define

$$\mathcal{L}^p := L^p(\Omega) \times L^p(\Gamma), \qquad \mathcal{W}^{s,p} := W^{s,p}(\Omega) \times W^{s,p}(\Gamma).$$

We write $\mathcal{H}^s = \mathcal{W}^{s,2}$ and identify $\mathcal{L}^2$ with $\mathcal{H}^0$. We point out that $\mathcal{H}^s$ is a Hilbert space with respect to the inner product

$$\big((\phi, \psi), (\zeta, \zeta_\Gamma)\big)_{\mathcal{H}^s} := (\phi, \zeta)_{H^s(\Omega)} + (\psi, \zeta_\Gamma)_{H^s(\Gamma)} \qquad \text{for all } (\phi, \psi), (\zeta, \zeta_\Gamma) \in \mathcal{H}^s,$$

and its induced norm $\|\cdot\|_{\mathcal{H}^s} := (\cdot, \cdot)_{\mathcal{H}^s}^{\frac{1}{2}}$. We further recall that the duality pairing on $\mathcal{H}^1$ satisfies

$$\big\langle (\phi, \psi), (\zeta, \zeta_\Gamma) \big\rangle_{\mathcal{H}^1} = (\phi, \zeta)_{L^2(\Omega)} + (\psi, \zeta_\Gamma)_{L^2(\Gamma)}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1$ if $(\phi, \psi) \in \mathcal{L}^2$.

For every $\phi \in H^1(\Omega)'$, we denote by $\langle \phi \rangle_\Omega = |\Omega|^{-1} \langle \phi, 1 \rangle_{H^1(\Omega)}$ its generalized mean value over Ω. If $\phi \in L^1(\Omega)$, its spatial mean can simply be expressed as $\langle \phi \rangle_\Omega = |\Omega|^{-1} \int_\Omega \phi \,\mathrm{d}x$. The spatial mean of any $\psi \in H^1(\Gamma)'$, denoted by $\langle \psi \rangle_\Gamma$, is defined similarly. We further define for any $p \in [2, \infty]$ the space

$$\mathcal{L}^p_{(0)} := L^p_{(0)}(\Omega) \times L^p_{(0)}(\Gamma),$$

where

$$L^p_{(0)}(\Omega) := \{\phi \in L^p(\Omega) : \langle \phi \rangle_\Omega = 0\}, \qquad L^p_{(0)}(\Gamma) := \{\psi \in L^p(\Gamma) : \langle \psi \rangle_\Gamma = 0\}.$$

Now, let $L \in [0, \infty]$ and $\beta \in \mathbb{R}$. We introduce the linear subspace

$$\mathcal{D}_\beta := \{(\phi, \psi) \in \mathcal{H}^1 : \phi = \beta\psi \text{ a.e. on } \Gamma\},$$

and then define

$$\mathcal{H}^1_{L,\beta} := \begin{cases} \mathcal{H}^1 & \text{if } L \in (0, \infty], \\ \mathcal{D}_\beta & \text{if } L = 0. \end{cases}$$

Subject to the inner product $(\cdot, \cdot)_{\mathcal{H}^1_{L,\beta}} := (\cdot, \cdot)_{\mathcal{H}^1}$ and its induced norm, $\mathcal{H}^1_{L,\beta}$ is a Hilbert space. Furthermore, we define the product

$$\big\langle (\phi, \psi), (\zeta, \zeta_\Gamma) \big\rangle_{\mathcal{H}^1_{L,\beta}} = (\phi, \zeta)_{L^2(\Omega)} + (\psi, \zeta_\Gamma)_{L^2(\Gamma)}$$

for all $(\phi, \psi), (\zeta, \zeta_\Gamma) \in \mathcal{L}^2$. Using the Riesz representation theorem, this product can be extended to a duality pairing on $(\mathcal{H}^1_{L,\beta})' \times \mathcal{H}^1_{L,\beta}$, which will also be denoted as $\langle \cdot, \cdot \rangle_{\mathcal{H}^1_{L,\beta}}$.

Next, for $(\phi, \psi) \in (\mathcal{H}^1_{L,\beta})'$, we define the generalized bulk-surface mean

$$\operatorname{mean}(\phi, \psi) := \frac{\big\langle (\phi, \psi), (\beta, 1) \big\rangle_{\mathcal{H}^1_{L,\beta}}}{\beta^2|\Omega| + |\Gamma|},$$

which reduces to

$$\operatorname{mean}(\phi, \psi) = \frac{\beta|\Omega|\langle\phi\rangle_\Omega + |\Gamma|\langle\psi\rangle_\Gamma}{\beta^2|\Omega| + |\Gamma|}$$

if $(\phi, \psi) \in \mathcal{L}^2$. We then define the closed linear subspace

$$\mathcal{V}^1_{L,\beta} = \begin{cases} \{(\phi, \psi) \in \mathcal{H}^1_{L,\beta} : \operatorname{mean}(\phi, \psi) = 0\} & \text{if } L \in [0, \infty), \\ \{(\phi, \psi) \in \mathcal{H}^1 : \langle\phi\rangle_\Omega = \langle\psi\rangle_\Gamma = 0\} & \text{if } L = \infty. \end{cases}$$

Note that $\mathcal{V}^1_{L,\beta}$ is a Hilbert space with respect to the inner product $(\cdot, \cdot)_{\mathcal{H}^1}$. We further introduce the bilinear form

$$\begin{aligned} \big((\phi, \psi), (\zeta, \zeta_\Gamma)\big)_{L,\beta} := &\int_\Omega \nabla\phi \cdot \nabla\zeta \, \mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi \cdot \nabla_\Gamma\zeta_\Gamma \, \mathrm{d}\Gamma \\ &+ \chi(L) \int_\Gamma (\beta\psi - \phi)(\beta\zeta_\Gamma - \zeta) \, \mathrm{d}\Gamma \end{aligned}$$

for all $(\phi, \psi), (\zeta, \zeta_\Gamma) \in \mathcal{H}^1$, where

$$\chi : [0, \infty] \to [0, \infty), \qquad \chi(r) := \begin{cases} \frac{1}{r} & \text{if } r \in (0, \infty), \\ 0 & \text{if } r \in \{0, \infty\}. \end{cases}$$

Moreover, we define

$$\|(\phi, \psi)\|_{L,\beta} := \big((\phi, \psi), (\phi, \psi)\big)^{\frac{1}{2}}_{L,\beta}$$

for all $(\phi, \psi) \in \mathcal{H}^1$. The bilinear form defines an inner product on $\mathcal{V}^1_{L,\beta}$, and $\|\cdot\|_{L,\beta}$ defines a norm on $\mathcal{V}^1_{L,\beta}$, that is equivalent to the norm $\|\cdot\|_{\mathcal{H}^1}$, see, e.g., [117, Corollary A.2]. In particular, $(\mathcal{V}^1_{L,\beta}, (\cdot, \cdot)_{L,\beta}, \|\cdot\|_{L,\beta})$ is a Hilbert space. Finally, we define the space

$$\mathcal{V}^{-1}_{L,\beta} := \begin{cases} \{(\phi, \psi) \in (\mathcal{H}^1_{L,\beta})' : \operatorname{mean}(\phi, \psi) = 0\} & \text{if } L \in [0, \infty), \\ \{(\phi, \psi) \in (\mathcal{H}^1)' : \langle\phi\rangle_\Omega = \langle\psi\rangle_\Gamma = 0\} & \text{if } L = \infty. \end{cases}$$

2.1.5 Spaces of solenoidal vector fields and Helmholtz projections

For $p \in [2, \infty]$, we introduce the spaces

$$\begin{aligned} \mathbf{L}^p_{\mathrm{div}}(\Omega) &:= \{\mathbf{v} \in \mathbf{L}^p(\Omega) : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}, \\ \mathbf{L}^p_{\mathrm{div}}(\Gamma) &:= \{\mathbf{w} \in \mathbf{L}^p(\Gamma) : \operatorname{div}_\Gamma \mathbf{w} = 0, \ \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \Gamma\}, \end{aligned}$$

and we set

$$\boldsymbol{\mathcal{L}}^p_{\mathrm{div}} := \mathbf{L}^p_{\mathrm{div}}(\Omega) \times \mathbf{L}^p_{\mathrm{div}}(\Gamma).$$

Then, for $s \geq 0$ and $p \in [2, \infty]$, we define $\mathbf{W}^{s,p}_{\mathrm{div}}(\Omega) = \mathbf{W}^{s,p}(\Omega) \cap \mathbf{L}^2_{\mathrm{div}}(\Omega)$. Analogously, we define $\mathbf{W}^{s,p}_{\mathrm{div}}(\Gamma) = \mathbf{W}^{s,p}(\Gamma) \cap \mathbf{L}^2_{\mathrm{div}}(\Gamma)$, and then set $\boldsymbol{\mathcal{W}}^{s,p}_{\mathrm{div}} = \mathbf{W}^{s,p}_{\mathrm{div}}(\Omega) \times \mathbf{W}^{s,p}_{\mathrm{div}}(\Gamma)$. Furthermore, for $s > \frac{1}{2}$, we set

$$\boldsymbol{\mathcal{W}}^{s,p}_0 := \{(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{W}}^{s,p} : \mathbf{v}\cdot\mathbf{n} = 0, \ \mathbf{v}|_\Gamma = \mathbf{w} \text{ on } \Gamma\},$$

$$\mathcal{W}^{s,p}_{0,\mathrm{div}} \coloneqq \mathcal{W}^{s,p}_0 \cap \mathcal{W}^{s,p}_{\mathrm{div}}.$$

As before, we use the notation $\mathcal{H}^s_0 = \mathcal{W}^{s,2}_0$ as well as $\mathcal{H}^s_{0,\mathrm{div}} = \mathcal{W}^{s,2}_{0,\mathrm{div}}$.

Let $r \in [2, \infty)$. We define the Helmholtz projection on Ω as

$$\mathbf{P}^{\Omega}_{\mathrm{div}} : \mathbf{L}^r(\Omega) \to \mathbf{L}^r_{\mathrm{div}}(\Omega), \quad \mathbf{P}^{\Omega}_{\mathrm{div}}(\mathbf{v}) = \mathbf{v} - \nabla p, \tag{2.1}$$

where $p \in \dot{W}^{1,r}(\Omega)$ is (up to an additive constant) uniquely determined by

$$\int_\Omega \nabla p \cdot \nabla \zeta \,\mathrm{d}x = \int_\Omega \mathbf{v} \cdot \nabla \zeta \,\mathrm{d}x \quad \text{for all } \zeta \in \dot{W}^{1,r'}(\Omega),$$

cf. [158, Theorem 1.4]. Here, r' denotes the dual Sobolev exponent of r. In a similar manner, the surface Helmholtz projection is defined via

$$\mathbf{P}^{\Gamma}_{\mathrm{div}} : \mathbf{L}^r_{\tan}(\Gamma) \to \mathbf{L}^r_{\mathrm{div}}(\Gamma), \quad \mathbf{P}^{\Gamma}_{\mathrm{div}}(\mathbf{w}) = \mathbf{w} - \nabla_\Gamma q, \tag{2.2}$$

where $q \in \dot{W}^{1,r}(\Gamma)$ is (up to an additive constant) uniquely determined by

$$\int_\Gamma \nabla_\Gamma q \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma = \int_\Gamma \mathbf{w} \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \qquad \text{for all } \zeta_\Gamma \in \dot{W}^{1,r'}(\Gamma), \tag{2.3}$$

cf. [159, Lemma A.1]. Here, r' denotes again the dual Sobolev exponent of r, i.e., $r' = \frac{r}{r-1}$. The existence of this projection is also related to the surface Helmholtz decomposition, which can be found, for example, in [149, Theorem 4.2]. According to [159, Sect. 2], we further have

$$\int_\Gamma \mathbf{P}^{\Gamma}_{\mathrm{div}}(\mathbf{w}) \cdot \mathbf{u} \,\mathrm{d}\Gamma = \int_\Gamma \mathbf{w} \cdot \mathbf{P}^{\Gamma}_{\mathrm{div}}(\mathbf{u}) \,\mathrm{d}\Gamma$$

for all $\mathbf{w} \in \mathbf{L}^r_{\tan}(\Gamma)$ and $\mathbf{u} \in \mathbf{L}^{r'}_{\tan}(\Gamma)$.

2.2 Useful inequalities

2.2.1 Embeddings and interpolation inequalities

In this section, we report some results from the theory of Sobolev spaces, namely, embedding and interpolation inequalities.

First of all, we recall the following interpolation inequality, which can be found in [139].

Lemma 2.2.1. *There exists a positive constant C such that*

$$\|u\|_{L^2(\partial\Omega)} \le C \|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|u\|^{\frac{1}{2}}_{H^1(\Omega)} \qquad \text{for all } u \in H^1(\Omega).$$

We further will need the following embedding results from [16, Theorem 10.9 and Theorem 10.13].

Theorem 2.2.2. *Let $m_1, m_2 \in \mathbb{N}_0$, and let $p_1, p_2 \in [1, \infty)$, $\alpha \in [0, 1]$.*

(i) *If $m_1 - \frac{d}{p_1} \ge m_2 - \frac{d}{p_2}$ and $m_1 \ge m_2$, then we have the continuous embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow W^{m_2,p_2}(\Omega).$$

In particular, there exists a constant $C > 0$, such that for all $f \in W^{m_1,p_1}(\Omega)$ it holds that

$$\|f\|_{W^{m_2,p_2}(\Omega)} \le C \|f\|_{W^{m_1,p_1}(\Omega)}.$$

(ii) *If* $m_1 - \frac{d}{p_1} > m_2 - \frac{d}{p_2}$ *and* $m_1 > m_2$*, then we have the compact embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow W^{m_2,p_2}(\Omega).$$

(iii) *If* $m_1 - \frac{d}{p_1} > m_2 + \alpha$ *with* $\alpha \in (0,1)$*, then we have the continuous embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow C^{m_2,\alpha}(\overline{\Omega}).$$

In particular, for every $f \in W^{m_1,p_1}(\Omega)$*, there exists a unique continuous functions which we will again denote by* f *that agrees almost everywhere in* Ω *with* f*. Moreover, there exists a constant* $C > 0$ *such that for all* $f \in W^{m_1,p_1}(\Omega)$ *it holds that*

$$\|f\|_{C^{m_2,\alpha}(\overline{\Omega})} \le C\|f\|_{W^{m_1,p_1}(\Omega)}.$$

(iv) *If* $m_1 - \frac{d}{p_1} > m_2 + \alpha$*, then we have the compact embedding*

$$W^{m_1,p_1}(\Omega) \hookrightarrow C^{m_2,\alpha}(\overline{\Omega}).$$

We note that similarly results hold for the boundary $\Gamma = \partial\Omega$, which is a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, assuming that it is sufficiently smooth.

Finally, we recall the following result on interpolation between Sobolev spaces.

LEMMA 2.2.3. *Let* $\theta \in (0,1)$ *and* $r, s_0, s_1 \in \mathbb{R}$ *satisfy*

$$r = (1-\theta)s_0 + \theta s_1.$$

Further suppose that Ω *is of class* C^l *with an integer* $l \ge \max\{s_0, s_1\}$*. Then there exist positive constants* C_Ω *and* C_Γ*, depending only on* Ω*,* Γ*,* r, s_0, s_1 *and* θ*, such that the following interpolation inequalities hold:*

$$\|f\|_{H^r(\Omega)} \le C_\Omega \|f\|_{H^{s_0}(\Omega)}^{1-\theta} \|f\|_{H^{s_1}(\Omega)}^{\theta} \qquad \text{for all } f \in H^{\max\{s_0,s_1\}}(\Omega), \tag{2.4}$$

$$\|f_\Gamma\|_{H^r(\Gamma)} \le C_\Gamma \|f_\Gamma\|_{H^{s_0}(\Gamma)}^{1-\theta} \|f_\Gamma\|_{H^{s_1}(\Gamma)}^{\theta} \qquad \text{for all } f_\Gamma \in H^{\max\{s_0,s_1\}}(\Gamma). \tag{2.5}$$

Inequality (2.4) follows from an interpolation result shown in [173, Sec. 4.3.1, Theorem 1 and Remark 1], whereas (2.5) follows from an interpolation result presented in [174, Sec. 7.4.5, Remark 2].

2.2.2 The Gronwall lemma

We report in this section some Gronwall-type results that can be found, e.g., in [27, 56, 169].

LEMMA 2.2.4. *Let* $y : [t_0, \infty) \to \mathbb{R}$ *be an absolutely continuous function, and* g, h *two positive summable functions on* $[t_0, \infty)$ *which satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \le g(t)y(t) + h(t) \qquad \text{for a.e. } t \ge t_0.$$

Then we have

$$y(t) \le y(t_0) \exp\left(\int_{t_0}^t g(r)\,\mathrm{d}r\right) + \int_0^t \exp\left(\int_s^t g(r)\,\mathrm{d}r\right) h(s)\,\mathrm{d}s \qquad \text{for all } t \ge t_0.$$

Lemma 2.2.5. *Let $y : [t_0, \infty) \to \mathbb{R}$ be a continuous function, and g, h two positive summable functions on $[t_0, \infty)$ which satisfy*

$$y(t) \le h(t) + \int_{t_0}^{t} g(s) y(s) \, \mathrm{d}s \qquad \text{for all } t \ge t_0.$$

Then

$$y(t) \le h(t) + \int_{t_0}^{t} g(s) h(s) \exp\Big(\int_{s}^{t} g(r) \, \mathrm{d}r \Big) \, \mathrm{d}s \qquad \text{for all } t \ge t_0.$$

Lemma 2.2.6. *Let $T > 0$. Assume that $y : [0, T] \to \mathbb{R}$ is a continuous function, g a positive summable function and R a non-negative constant, which satisfy*

$$\frac{1}{2} y^2(t) \le \frac{1}{2} R^2 + \int_0^t g(s) y(s) \, \mathrm{d}s \qquad \text{for all } t \in [0, T].$$

Then it holds

$$|y(t)| \le R + \int_0^t g(s) \, \mathrm{d}s \qquad \text{for all } t \in [0, T].$$

Lemma 2.2.7. *Let $y : [t_0, \infty) \to \mathbb{R}$ be an absolutely continuous positive function, and g, h two positive locally summable functions on $[t_0, \infty)$, which satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \le g(t) y(t) + h(t) \qquad \text{for a.e. } t \ge t_0,$$

and

$$\int_t^{t+r} g(s) \, \mathrm{d}s \le a_1, \quad \int_t^{t+r} h(s) \, \mathrm{d}s \le a_2, \quad \int_t^{t+r} y(s) \, \mathrm{d}s \le a_3 \quad \text{for all } t \ge t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then it holds

$$y(t) \le \left(\frac{a_3}{r} + a_2 \right) e^{a_1} \qquad \text{for all } t \ge t_0 + r.$$

We conclude this section by stating another variant of Gronwall's lemma, which follows from [93, Lemma 2.5], see also [113, Corollary A.1].

Lemma 2.2.8. *Let g, h, y be three non-negative locally integrable functions on $[t_0, \infty)$ which satisfy, for some $\gamma > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) + \gamma y(t) \le g(t) y^{\frac{1}{2}}(t) + h(t) \qquad \text{for a.e. } t \ge t_0.$$

Additionally, assume that there exist positive constants A_1 and A_2 such that

$$\int_t^{t+1} g(s) \, \mathrm{d}s \le A_1, \quad \int_t^{t+1} h(s) \, \mathrm{d}s \le A_2 \qquad \text{for all } t \ge t_0.$$

Then it holds that

$$y(t) \le 2 y(t_0) \mathrm{e}^{-\gamma (t - t_0)} + Q(\gamma, A_1, A_2),$$

where

$$Q(\gamma, A_1, A_2) = \Big(\frac{\mathrm{e}^{\frac{\gamma}{2}}}{1 - \mathrm{e}^{-\frac{\gamma}{2}}} A_1 \Big)^2 + \frac{2 \mathrm{e}^{\gamma}}{1 - \mathrm{e}^{-\gamma}} A_2.$$

Remark 2.2.9. Throughout this thesis, references to the Gronwall lemma are understood to mean the standard differential form given in Lemma 2.2.4. Any other variants will be cited explicitly.

2.3 Maximal monotone operators and the Yosida approximation

In this section, we recall some fundamental definitions and results regarding maximal monotone operators. For a more thorough exposition of these materials, we refer to [19, 20, 27, 157]. Throughout this section, X denotes a Hilbert space with inner product $(\cdot,\cdot)_X$ and corresponding norm $\|\cdot\|_X$. However, most of the concepts presented here can be extended to the more general setting of Banach spaces.

Definition 2.3.1. *Let $A : X \to \mathcal{P}(X)$ be a possible multi-valued operator, where $\mathcal{P}(X)$ denotes the power set of X. The domain, range, and graph of A are defined as*

$$\begin{aligned} &D(A) \coloneqq \{x \in X : Ax \neq \emptyset\}, \qquad R(A) \coloneqq \{y \in Ax : x \in D(A)\}, \\ &G(A) \coloneqq \{(x,y) \in X \times X : x \in D(A),\ y \in Ax\}, \end{aligned}$$

respectively. The operator A is said to be monotone if

$$(f-g, x-y)_X \geq 0 \qquad \text{for all } x, y \in D(A), \quad f \in Ax,\ g \in Ay.$$

Moreover, A is said to be maximal monotone if

$$(f-g, x-y)_X \geq 0 \quad \text{for all } (y,g) \in G(A) \qquad \Longrightarrow \qquad x \in D(A),\ f \in Ax.$$

Equivalently, an operator A is maximal monotone if it is not strictly included in any other monotone operator $B : X \to \mathcal{P}(X)$, i.e, $A \subset B$ implies that $A = B$. Here, the inclusion $A \subset B$ is in the sense that $G(A) \subset G(B)$.

Another useful characterization is given by the result below, see [20, Theorem 2.2].

Proposition 2.3.2. *Let $A : X \to X$ be a monotone operator. Then A is maximal monotone if and only if $R(\mathrm{Id} + \lambda A) = X$ for every $\lambda > 0$ (equivalently, for some $\lambda > 0$).*

Moreover, one can show that the graph of a maximal monotone operator is always weakly-strongly closed in $X \times X$ (cf. [20, Proposition 2.1].

Proposition 2.3.3. *Let A be a maximal monotone operator, and consider a sequence $\{(y_n, x_n)\}_{n\in\mathbb{N}} \subset G(A)$ such that*

$$x_n \to x \qquad \text{weakly in } X, \qquad y_n \to y \qquad \text{strongly in } X$$

as $n \to \infty$. Then $(x,y) \in G(A)$.

2.3.1 The Yosida approximation

Maximal monotone operators arise in nonlinear evolution equations and introduce significant technical challenges, as they are generally multi-valued and lack regularity. A key tool for overcoming these difficulties is the Yosida approximation, which, as we will see, is single-valued and Lipschitz continuous.

Let A be a maximal monotone operator. Then, according to Proposition 2.3.2, the operator $\mathrm{Id} + \lambda A$ is surjective. Hence, for every fixed $u \in X$, there exists $(x, f) \in G(A)$ such that

$$x + \lambda f = u, \quad f \in Ax.$$

This allows us to define the resolvent operator J_λ and the associated Yosida approximation A_λ for every $\lambda > 0$ as

$$J_\lambda := (\mathrm{Id} + \lambda A)^{-1}, \quad A_\lambda := \frac{1}{\lambda}(\mathrm{Id} - J_\lambda) = \frac{1}{\lambda}(\mathrm{Id} - (\mathrm{Id} + \lambda A)^{-1}),$$

respectively. It is well-known that both J_λ and A_λ are Lipschitz operators with Lipschitz constant 1 and $\frac{1}{\lambda}$, respectively, see [27, Proposition 2.6, p.26].

Another key result that we will use several times throughout the thesis is the following.

Proposition 2.3.4. *Let A be a maximal monotone operator, and let $\{(x_\lambda, f_\lambda)\}_{\lambda>0} \subset G(A_\lambda)$ be such that*

$$\begin{aligned}
&x_\lambda \to x \quad \textit{strongly in } X, \quad f_\lambda \to f \quad \textit{weakly in } X \quad \textit{as } \lambda \to 0,\\
&\limsup_{\lambda\to 0}(x_\lambda, f_\lambda)_X \le (x, f)_X.
\end{aligned}$$

Then $(x, f) \in G(A)$ and $\lim_{\lambda\to 0}(x_\lambda, f_\lambda)_X = (x, f)_X$.

This result is a generalization of the one presented in [28, Lemma 1.3 (e)] (see also [20, Proposition 2.2 (iv)]), and the proof works in a similar manner. In the special case where A is given as a subdifferential $\partial\varphi$ of a proper, convex, and lower semicontinuous function $\varphi : X \to (-\infty, +\infty]$ (see Section 2.3.2 for its definition), we refer to [87, Section 5.2] for a proof of this result.

2.3.2 The subdifferential of convex functions

An important class of maximal monotone operators is given by the subdifferentials of proper, convex, and lower semicontinuous functions. Subdifferentials arise naturally in convex analysis and play a central role in various applications, including optimization and evolution equations. In the following, we recall their definition and summarize the main properties relevant to our setting.

For a function $\varphi : X \to (-\infty, +\infty]$, we define its effective domain as

$$D(\varphi) := \{x \in X : \varphi(x) < +\infty\}.$$

Then, φ is called proper, if it is not identical to $+\infty$, i.e. $D(\varphi) \neq \emptyset$, and convex, if it satisfies the inequality

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in X$ and $t \in [0, 1]$. We call φ lower semicontinuous on X if

$$\liminf_{y\to x} \varphi(y) \ge \varphi(x) \quad \text{for all } x \in X.$$

Definition 2.3.5. *Let $\varphi : X \to (-\infty, +\infty]$ be a proper and convex function. The subdifferential of φ is the multi-valued operator $\partial\varphi : X \to \mathcal{P}(X)$ defined by*

$$\partial\varphi(x) := \{\xi \in X : \varphi(y) \ge \varphi(x) + (\xi, y - x)_X \ \forall y \in X\}$$

for all $x \in X$.

The following result shows that the subdifferential of a proper, convex and lower semicontinuous function is a maximal monotone operator in the sense of Definition 2.3.1, see [20, Theorem 2.8].

PROPOSITION 2.3.6. *Let $\varphi : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then its subdifferential $\partial\varphi$ is a maximal monotone operator.*

As we will frequently deal with operators of this form, it is important to understand their Yosida approximation.

DEFINITION 2.3.7. *Let $\varphi : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous functions. For every $\lambda > 0$, we define the Moreau–Yosida regularization of φ as*

$$\varphi_\lambda(x) := \inf_{y\in X} \left\{ \frac{\|y-x\|_X^2}{2\lambda} + \varphi(y) \right\} \quad \textit{for all } x \in X.$$

The Moreau–Yosida approximation of φ enjoys a number of useful properties that establish a close connection with the Yosida approximation of $\partial\varphi$ (see [27, Proposition 2.11, p. 39]).

THEOREM 2.3.8. *Let $\varphi : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then the following properties hold:*

(i) *For every $x \in X$ and $\lambda > 0$ it holds $\varphi_\lambda(x) = \frac{\lambda}{2}\|(\partial\varphi)_\lambda(x)\|_X^2 + \varphi(J_\lambda x)$.*

(ii) *For every $\lambda > 0$, φ_λ is well-defined, convex, continuous, and Fréchet differentiable with $\nabla\varphi_\lambda = (\partial\varphi)_\lambda$.*

(iii) *For every $x \in X$ and $\lambda > 0$ it holds $\varphi_\lambda(x) \le \varphi(x)$ and $\varphi_\lambda(x) \nearrow \varphi(x)$ as $\lambda \searrow 0$.*

2.3.3 Approximation of singular potentials

In this thesis, we mainly work with potentials F and G of the form

$$F(s) = F_1(s) + F_2(s) \quad \text{and} \quad G(s) = G_1(s) + G_2(s), \quad \text{for all } s \in [-1, 1],$$

where $F_1, G_1 \in C([-1,1]) \cap C^2(-1,1)$ with

$$\lim_{s\searrow -1} F_1'(s) = \lim_{s\searrow -1} G_1'(s) = -\infty \quad \text{and} \quad \lim_{s\nearrow 1} F_1'(s) = \lim_{s\nearrow 1} G_1'(s) = +\infty.$$

There further exist constants $\Theta_\Omega, \Theta_\Gamma > 0$ such that

$$F_1''(s) \ge \Theta_\Omega \quad \text{and} \quad G_1''(s) \ge \Theta_\Gamma \qquad \text{for all } s \in (-1,1).$$

We extend F_1 and G_1 by defining $F_1(s) = G_1(s) = +\infty$ for $s \notin [-1,1]$ and, without loss of generality, we assume that $F_1(0) = G_1(0) = 0$ and $F_1'(0) = G_1'(0) = 0$.

The derivatives F_1' and G_1' of the singular parts F_1 and G_1, respectively, can then be seen as maximal monotone operators in the sense of Definition 2.3.1. In particular, we can consider their Moreau–Yosida approximations which we will denote with $F_{1,\lambda}$ and $G_{1,\lambda}$, respectively. We collect the following properties, which will be used several times throughout the thesis.

(Y1) For every $\lambda > 0$, $F_{1,\lambda} \in C^{2,1}_{\mathrm{loc}}(\mathbb{R})$ with $F_{1,\lambda}(0) = F_{1,\lambda}'(0) = 0$.

(Y2) $F_{1,\lambda}$ is convex with

$$F_{1,\lambda}''(s) \ge \frac{\Theta_\Omega}{1+\Theta_\Omega} \quad \text{for all } s \in \mathbb{R}.$$

(Y3) For every $\lambda > 0$, $F_{1,\lambda}'$ is Lipschitz continuous on $\mathbb{R}$ with constant $\frac{1}{\lambda}$.

(Y4) There exist $\overline{\lambda} \in (0,1)$ and $C > 0$ such that

$$F_{1,\lambda}(s) \geq \frac{1}{4\overline{\lambda}} s^2 - C \qquad \text{for all } s \in \mathbb{R} \text{ and } \lambda \in (0, \overline{\lambda}).$$

(Y5) For every $\lambda > 0$, it holds $0 \leq F_{1,\lambda}(s) \leq F_1(s)$ for all $r \in [-1,1]$. Moreover, as $\lambda \to 0$, we have $F_{1,\lambda}(s) \to F_1(s)$ for all $s \in [-1,1]$ and $|F'_{1,\lambda}(s)| \to |F'_1(s)|$ for $s \in (-1,1)$.

Analogous properties hold true for $G_{1,\lambda}$ instead of $F_{1,\lambda}$.

Chapter 3

Mathematical models

The content of this chapter is taken from the following work.

[80] P. Knopf, and J. Stange, *A thermodynamically consistent model for bulk-surface viscous fluid mixtures: Model derivation and mathematical analysis.* Preprint: arXiv:2509.11925, 2025.

In this chapter, we derive the full bulk-surface Navier–Stokes–Cahn–Hilliard model (1.42). Following [9,97,118], the derivation is based on local mass balance laws in both the bulk and on the surface, involving a-priori unknown mass fluxes. In addition, we consider local energy dissipation laws that include further energy flux contributions. Finally, we employ a Lagrange multiplier approach, in which the chemical potentials μ and θ act as Lagrange multipliers, to complete the derivation by identifying the unknown mass and energy fluxes.

3.1 Model derivation

3.1.1 Local mass balance laws

We want to describe the time evolution of two fluids, indexed with $j = 1, 2$, in a (sufficiently smooth) domain $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ on a time interval $[0, T]$ with $T > 0$. We write $Q^T = \Omega \times (0, T)$ and $\Sigma^T = \Gamma \times (0, T)$ with $\Gamma = \partial\Omega$. Each bulk material is assumed to have a constant individual density $\tilde{\rho}_j$, $j = 1, 2$, whereas each surface material has a constant individual density $\tilde{\sigma}_j$, $j = 1, 2$. We further assume that the mass densities $\rho_j : Q^T \to \mathbb{R}$, $j = 1, 2$, of the two fluids in the bulk satisfy the following mass balance law:

$$\partial_t \rho_j + \operatorname{div} \widehat{\mathbf{J}}_j = 0, \quad j = 1, 2, \quad \text{in } Q^T. \tag{3.1}$$

Here, $\widehat{\mathbf{J}}_j$ are the mass fluxes that correspond to the motion of the two materials in the bulk. In our model, we also want to allow for a transfer of material between bulk and surface, which is, for instance, needed to describe absorption processes. To this end, we additionally consider functions $\sigma_j : \Sigma^T \to \mathbb{R}$, $j = 1, 2$, representing the mass densities of the two fluids on the surface. In general, the materials on the surface might differ from those in the bulk, which, for example, is the case if the materials are transformed by chemical reactions occurring at the boundary. Therefore, we interpret the densities σ_j as independent functions that are not necessarily the traces of the densities ρ_j, respectively. They are supposed to satisfy the following mass balance law on the boundary:

$$\partial_t \sigma_j + \operatorname{div}_\Gamma \widehat{\mathbf{K}}_j = \widehat{\mathbf{J}}_{\Gamma,j} \cdot \mathbf{n}, \quad j = 1, 2, \quad \text{on } \Sigma^T. \tag{3.2}$$

Here, $\widehat{\mathbf{K}}_j$ stands for the mass flux on the boundary, and the right-hand side $\widehat{\mathbf{J}}_{\Gamma,j}$ describes the transfer of mass between bulk and surface. This exchange of material is assumed to be balanced by the relation

$$\frac{\widehat{\mathbf{J}}_{\Gamma,1}}{\tilde{\sigma}_1}\cdot\mathbf{n} + \frac{\widehat{\mathbf{J}}_{\Gamma,2}}{\tilde{\sigma}_2}\cdot\mathbf{n} = 0 \quad \text{on } \Sigma^T. \tag{3.3}$$

We next assume that the motion of our materials is described by individual velocity fields $\mathbf{v}_j : Q^T \to \mathbb{R}^d$ and $\mathbf{w}_j : \Sigma^T \to \mathbb{R}^d$, $j = 1, 2$, in the bulk and on the surface, respectively. These can be expressed as

$$\mathbf{v}_j = \frac{\widehat{\mathbf{J}}_j}{\rho_j} \quad \text{in } Q^T, \qquad \mathbf{w}_j = \frac{\widehat{\mathbf{K}}_j}{\sigma_j} \quad \text{on } \Sigma^T, \qquad j = 1, 2.$$

Consequently, the mass balance laws (3.1) and (3.2) can be rewritten as

$$\partial_t\rho_j + \operatorname{div}(\rho_j\mathbf{v}_j) = 0, \qquad j = 1, 2, \quad \text{in } Q^T, \tag{3.4}$$
$$\partial_t\sigma_j + \operatorname{div}_\Gamma(\sigma_j\mathbf{w}_j) = \widehat{\mathbf{J}}_{\Gamma,j}, \quad j = 1, 2, \quad \text{on } \Sigma^T. \tag{3.5}$$

We now write $\phi_j : Q^T \to \mathbb{R}$ and $\psi_j : \Sigma^T \to \mathbb{R}$, $j = 1, 2$, to denote the volume fractions of the two fluids in the bulk and on the surface, respectively. Provided that each component has a constant density $\tilde{\rho}_j$ and $\tilde{\sigma}_j$, $j = 1, 2$, respectively, the volume fractions can be identified as

$$\phi_j = \frac{\rho_j}{\tilde{\rho}_j} \quad \text{in } Q^T, \qquad \psi_j = \frac{\sigma_j}{\tilde{\sigma}_j} \quad \text{on } \Sigma^T, \qquad j = 1, 2. \tag{3.6}$$

Assuming that the excess volume is zero, we have

$$\phi_1 + \phi_2 = 1 \quad \text{in } Q^T, \qquad \psi_1 + \psi_2 = 1 \quad \text{on } \Sigma^T. \tag{3.7}$$

We further define the order of parameters

$$\phi = \phi_2 - \phi_1 \quad \text{in } Q^T, \qquad \psi = \psi_2 - \psi_1 \quad \text{on } \Sigma^T. \tag{3.8}$$

Furthermore, let $\mathbf{v} : Q^T \to \mathbb{R}^d$ and $\mathbf{w} : \Sigma^T \to \mathbb{R}^d$ denote the volume-averaged velocity fields associated with the two fluids in the bulk and on the surface, respectively. By means of the order parameters, they can be expressed as

$$\mathbf{v} = \phi_1\mathbf{v}_1 + \phi_2\mathbf{v}_2 \quad \text{in } Q^T, \tag{3.9}$$
$$\mathbf{w} = \psi_1\mathbf{w}_1 + \psi_2\mathbf{w}_2 \quad \text{on } \Sigma^T. \tag{3.10}$$

We assume that both fluids cannot penetrate the boundary Γ, and that the velocity field on the boundary is an extension of the one in the bulk. This leads to the conditions

$$\mathbf{v}\cdot\mathbf{n} = 0, \qquad \mathbf{w} = \mathbf{v}|_\Gamma \qquad \text{on } \Sigma^T. \tag{3.11}$$

As the first relation entails that $\mathbf{v}|_\Gamma$ is a tangential vector field, the same holds for $\mathbf{w}$ because of the second identity. We now introduce

$$\begin{aligned}
\mathbf{J}_j &= \widehat{\mathbf{J}}_j - \rho_j\mathbf{v}, && j = 1, 2, \quad \text{in } Q^T,\\
\mathbf{J}_{\Gamma,j} &= \widehat{\mathbf{J}}_{\Gamma,j} - \sigma_j\mathbf{w}, && j = 1, 2, \quad \text{on } \Sigma^T,\\
\mathbf{K}_j &= \widehat{\mathbf{K}}_j - \sigma_j\mathbf{w}, && j = 1, 2, \quad \text{on } \Sigma^T,
\end{aligned}$$

to denote the mass fluxes relative to the volume-averaged velocities. In view of (3.11), the mass balance equations (3.4) and (3.5) can thus be rewritten as

$$\partial_t \rho_j + \operatorname{div}(\rho_j \mathbf{v}) + \operatorname{div} \mathbf{J}_j = 0, \qquad j = 1,2, \quad \text{in } Q^T, \tag{3.12}$$

$$\partial_t \sigma_j + \operatorname{div}_\Gamma(\sigma_j \mathbf{w}) + \operatorname{div}_\Gamma \mathbf{K}_j = \mathbf{J}_{\Gamma,j} \cdot \mathbf{n}, \quad j = 1,2, \quad \text{on } \Sigma^T. \tag{3.13}$$

We now define the total mass densities

$$\rho = \rho_1 + \rho_2 \quad \text{in } Q^T, \tag{3.14}$$

$$\sigma = \sigma_1 + \sigma_2 \quad \text{on } \Sigma^T. \tag{3.15}$$

By means of (3.12) and (3.13), this leads to the relations

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) + \operatorname{div}(\mathbf{J}_1 + \mathbf{J}_2) = 0 \qquad \text{in } Q^T, \tag{3.16}$$

$$\partial_t \sigma + \operatorname{div}_\Gamma(\sigma \mathbf{w}) + \operatorname{div}_\Gamma(\mathbf{K}_1 + \mathbf{K}_2) = (\mathbf{J}_{\Gamma,1} + \mathbf{J}_{\Gamma,2}) \cdot \mathbf{n} \qquad \text{on } \Sigma^T. \tag{3.17}$$

3.1.2 Local balance laws for the linear momentum

The next step is to consider the local balance laws for the linear momentum:

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = \operatorname{div} \tilde{\mathbf{T}} \qquad \text{in } Q^T, \tag{3.18}$$

$$\partial_t(\sigma \mathbf{w}) + \operatorname{div}_\Gamma(\sigma \mathbf{w} \otimes \mathbf{w}) = \operatorname{div}_\Gamma \tilde{\mathbf{T}}_\Gamma + \mathbf{Z} \quad \text{on } \Sigma^T. \tag{3.19}$$

Here, $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{T}}_\Gamma$ are stress tensors that need to be specified later by means of constitutive assumptions, and $\mathbf{Z}$ is the surface force density that accounts for mechanical interactions between bulk and surface materials. We point out that $\mathbf{Z}$ is a tangential vector field. In the following, we write

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \quad \text{in } Q^T, \qquad \mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2, \quad \mathbf{J}_\Gamma = \mathbf{J}_{\Gamma,1} + \mathbf{J}_{\Gamma,2} \quad \text{on } \Sigma^T. \tag{3.20}$$

By means of (3.16), the balance law (3.18) can be reformulated as

$$\begin{aligned} \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) &= \operatorname{div} \tilde{\mathbf{T}} + (\operatorname{div} \mathbf{J})\mathbf{v} \\ &= \operatorname{div}(\tilde{\mathbf{T}} + \mathbf{v} \otimes \mathbf{J}) - \nabla \mathbf{v}\, \mathbf{J} \quad \text{in } Q^T. \end{aligned} \tag{3.21}$$

Analogously, using (3.17) to reformulate (3.19), we get

$$\begin{aligned} &\sigma(\partial_t \mathbf{w} + \mathbf{w} \cdot \nabla_\Gamma \mathbf{w}) \\ &\quad = \operatorname{div}_\Gamma(\tilde{\mathbf{T}}_\Gamma + \mathbf{w} \otimes \mathbf{K}) - [\nabla_\Gamma \mathbf{w}\, \mathbf{K}]_{\tan} - (\mathbf{J}_\Gamma \cdot \mathbf{n})\mathbf{w} + \mathbf{Z} \quad \text{on } \Sigma^T. \end{aligned} \tag{3.22}$$

Thus, defining the frame indifferent objective tensors

$$\mathbf{T} = \tilde{\mathbf{T}} + \mathbf{v} \otimes \mathbf{J} \quad \text{in } Q^T, \quad \mathbf{T}_\Gamma = \tilde{\mathbf{T}}_\Gamma + \mathbf{w} \otimes \mathbf{K} \quad \text{on } \Sigma^T, \tag{3.23}$$

we can rewrite (3.21) and (3.22) as

$$\rho \partial_t \mathbf{v} + \nabla \mathbf{v}\,(\rho \mathbf{v} + \mathbf{J}) = \operatorname{div} \mathbf{T} \qquad \text{in } Q^T, \tag{3.24}$$

$$\sigma \partial_t \mathbf{w} + [\nabla_\Gamma \mathbf{w}\,(\sigma \mathbf{w} + \mathbf{K})]_{\tan} = \operatorname{div}_\Gamma \mathbf{T}_\Gamma - (\mathbf{J}_\Gamma \cdot \mathbf{n})\mathbf{w} + \mathbf{Z} \quad \text{on } \Sigma^T. \tag{3.25}$$

Next, we multiply the equations (3.4) each by $\frac{1}{\tilde{\rho}_j}$, $j = 1,2$, and add the resulting equations. Recalling (3.6), (3.8) and (3.9), we obtain

$$\operatorname{div} \mathbf{v} = \operatorname{div}\left(\frac{\rho_1}{\tilde{\rho}_1}\mathbf{v}_1 + \frac{\rho_2}{\tilde{\rho}_2}\mathbf{v}_2\right) = -\partial_t\left(\frac{\rho_1}{\tilde{\rho}_1} + \frac{\rho_2}{\tilde{\rho}_2}\right) = -\partial_t 1 = 0 \quad \text{in } Q^T, \tag{3.26}$$

which means that the volume-averaged velocity field is divergence-free. Similarly, multiplying (3.5) by $\frac{1}{\tilde{\sigma}_j}$, $j = 1, 2$, adding the resulting equations, and recalling (3.3), (3.6), (3.8) and (3.10), we deduce

$$\begin{aligned} \mathrm{div}_\Gamma \mathbf{w} &= \mathrm{div}_\Gamma \left(\frac{\sigma_1}{\tilde{\sigma}_1} \mathbf{w}_1 + \frac{\sigma_2}{\tilde{\sigma}_2} \mathbf{w}_2 \right) \\ &= -\mathrm{div}_\Gamma \left(\frac{\sigma_1}{\tilde{\sigma}_1} + \frac{\sigma_2}{\tilde{\sigma}_2} \right) + \left(\frac{\widehat{\mathbf{J}}_{\Gamma,1}}{\tilde{\sigma}_1} + \frac{\widehat{\mathbf{J}}_{\Gamma,2}}{\tilde{\sigma}_2} \right) \cdot \mathbf{n} = 0 \quad \text{on } \Sigma^T. \end{aligned} \tag{3.27}$$

We now define the quantities

$$\tilde{\mathbf{J}}_j = \frac{\mathbf{J}_j}{\tilde{\rho}_j}, \quad j = 1, 2, \qquad \mathbf{J}_\phi = \tilde{\mathbf{J}}_2 - \tilde{\mathbf{J}}_1 \qquad \text{in } Q^T, \tag{3.28}$$

$$\tilde{\mathbf{J}}_{\Gamma,j} = \frac{\mathbf{J}_{\Gamma,j}}{\tilde{\sigma}_j}, \quad j = 1, 2, \qquad \mathbf{J}_\psi = \tilde{\mathbf{J}}_{\Gamma,2} - \tilde{\mathbf{J}}_{\Gamma,1} \qquad \text{on } \Sigma^T, \tag{3.29}$$

$$\tilde{\mathbf{K}}_j = \frac{\mathbf{K}_j}{\tilde{\sigma}_j}, \quad j = 1, 2, \qquad \mathbf{K}_\psi = \tilde{\mathbf{K}}_2 - \tilde{\mathbf{K}}_1 \qquad \text{on } \Sigma^T. \tag{3.30}$$

Then, multiplying the equations (3.12) each by $\frac{1}{\tilde{\rho}_j}$, $j = 1, 2$, and subtracting the resulting equations, we deduce that

$$\partial_t \phi + \mathrm{div}(\phi \mathbf{v}) + \mathrm{div}\, \mathbf{J}_\phi = 0 \quad \text{in } Q^T. \tag{3.31}$$

Furthermore, multiplying the equations (3.12) each by $\frac{1}{\tilde{\rho}_j}$, $j = 1, 2$, adding the resulting equations, and recalling (3.7) and (3.26), we conclude that

$$\mathrm{div}(\tilde{\mathbf{J}}_1 + \tilde{\mathbf{J}}_2) = -\partial_t(\phi_1 + \phi_2) - \mathrm{div}((\phi_1 + \phi_2)\mathbf{v}) = 0 \quad \text{in } Q^T. \tag{3.32}$$

Arguing similarly for (3.13), and recalling (3.7) and (3.27), we obtain

$$\partial_t \psi + \mathrm{div}_\Gamma(\psi \mathbf{w}) + \mathrm{div}_\Gamma\, \mathbf{K}_\psi = \mathbf{J}_\psi \cdot \mathbf{n} \quad \text{on } \Sigma^T, \tag{3.33}$$

and

$$\mathrm{div}_\Gamma(\tilde{\mathbf{K}}_1 + \tilde{\mathbf{K}}_2) = -\partial_t(\psi_1 + \psi_2) - \mathrm{div}_\Gamma((\psi_1 + \psi_2)\mathbf{w}) + (\widehat{\mathbf{J}}_{\Gamma,1} + \widehat{\mathbf{J}}_{\Gamma,2}) \cdot \mathbf{n} = 0 \tag{3.34}$$

on Σ^T. Using (3.28) and (3.32), we infer that

$$\begin{aligned} \mathrm{div}\, \mathbf{J} &= \frac{1}{2} \left[\tilde{\rho}_2 \mathrm{div}\, \tilde{\mathbf{J}}_2 + \tilde{\rho}_1 \mathrm{div}\, \tilde{\mathbf{J}}_1 \right] + \frac{1}{2} \left[\tilde{\rho}_2 \mathrm{div}\, \tilde{\mathbf{J}}_2 + \tilde{\rho}_1 \mathrm{div}\, \tilde{\mathbf{J}}_1 \right] \\ &= \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \mathrm{div}\, \tilde{\mathbf{J}}_2 - \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \mathrm{div}\, \tilde{\mathbf{J}}_2 = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \mathrm{div}\, \mathbf{J}_\phi \qquad \text{in } Q^T. \end{aligned} \tag{3.35}$$

Therefore, the fluxes $\mathbf{J}$ and $\frac{1}{2}(\tilde{\rho}_2 - \tilde{\rho}_1)\mathbf{J}_\phi$ can merely differ by an additive divergence-free function. As in [9, 97], we assume that

$$\mathbf{J} = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \mathbf{J}_\phi \quad \text{in } Q^T. \tag{3.36}$$

Proceeding analogously, we derive the identity

$$\mathrm{div}_\Gamma\, \mathbf{K} = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} \mathrm{div}_\Gamma\, \mathbf{K}_\psi \quad \text{on } \Sigma^T,$$

and assume that

$$\mathbf{K} = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} \mathbf{K}_\psi \quad \text{on } \Sigma^T. \tag{3.37}$$

Moreover, assuming that the phase-fields ϕ and ψ only attain values in the interval $[-1,1]$, we use (3.6), (3.7), (3.8), (3.14) and (3.15) to conclude that the densities $\rho = \rho(\phi)$ and $\sigma = \sigma(\psi)$ are given by

$$\rho(\phi) = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\phi + \frac{\tilde{\rho}_2 + \tilde{\rho}_1}{2} \quad \text{in } Q^T, \tag{3.38}$$

$$\sigma(\psi) = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2}\psi + \frac{\tilde{\sigma}_2 + \tilde{\sigma}_1}{2} \quad \text{on } \Sigma^T. \tag{3.39}$$

Finally, differentiating formula (3.15) with respect to time and employing (3.33) and (3.37), we deduce that $\sigma = \sigma(\psi)$ fulfills the identity

$$\partial_t \sigma + \operatorname{div}_\Gamma(\sigma \mathbf{w}) + \operatorname{div}_\Gamma \mathbf{K} = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2}\mathbf{J}_\psi \cdot \mathbf{n} \quad \text{on } \Sigma^T.$$

Comparing this equation with (3.17) (see also (3.20)), we finally conclude that

$$\mathbf{J}_\Gamma \cdot \mathbf{n} = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2}\mathbf{J}_\psi \cdot \mathbf{n} \quad \text{on } \Sigma^T. \tag{3.40}$$

3.1.3 Local energy dissipation laws

Proceeding as in [9] and [97], we consider the following energy density in the bulk:

$$e_\Omega(\mathbf{v}, \phi, \nabla\phi) = \frac{\rho(\phi)}{2}|\mathbf{v}|^2 + f(\phi, \nabla\phi) \quad \text{in } Q^T. \tag{3.41}$$

Here, the first summand on the right-hand side represents the kinetic energy density while the second summand denotes the free energy density in the bulk. It is assumed to be of Ginzburg–Landau-type, that is,

$$f(\phi, \nabla\phi) = \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}F(\phi) \qquad \text{in } Q^T. \tag{3.42}$$

As in [97], we additionally introduce the following surface energy density

$$e_\Gamma(\mathbf{w}, \phi, \psi, \nabla_\Gamma\psi) = \frac{\sigma(\psi)}{2}|\mathbf{w}|^2 + f_\Gamma(\phi, \psi, \nabla_\Gamma\psi) \quad \text{on } \Sigma^T. \tag{3.43}$$

As in the bulk, the first summand on the right-hand side represents the kinetic energy density on the surface, whereas the second summand stands for the free energy density on the surface. It is also assumed to be of Ginzburg–Landau-type with an additional term accounting for the relation between ϕ and ψ, namely,

$$f_\Gamma(\phi, \psi, \nabla_\Gamma\psi) = \frac{\delta\kappa}{2}|\nabla_\Gamma\psi|^2 + \frac{1}{\delta}G(\psi) + \frac{1}{2}\chi(K)(\alpha\psi - \phi)^2 \qquad \text{on } \Sigma^T, \tag{3.44}$$

for $K \in [0, \infty]$. Here, χ is the function defined in (1.25).

We now consider an arbitrary test volume $V(t) \subset \Omega$, $t \in [0, T]$, which is transported by the flow associated with $\mathbf{v}$. As we assume our physical system to be isothermal, the second law of thermodynamics leads to the dissipation inequality

$$\begin{aligned} 0 \geq \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V(t)} e_\Omega(\mathbf{v}, \psi, \nabla\phi)\,\mathrm{d}x + \int_{\partial V(t)\cap\Gamma} e_\Gamma(\mathbf{w}, \phi, \psi, \nabla_\Gamma\psi)\,\mathrm{d}\Gamma\right] \\ + \int_{\partial V(t)\cap\Omega} \mathbf{J}_e \cdot \mathbf{n}_{\partial V(t)}\,\mathrm{d}\Gamma + \int_{\partial(\partial V(t)\cap\Gamma)} \mathbf{K}_e \cdot \mathbf{n}_{\partial(\partial V(t)\cap\Gamma)}\,\mathrm{d}\Gamma_\Gamma. \end{aligned} \tag{3.45}$$

In this inequality, $\mathbf{J}_e$ and $\mathbf{K}_e$ are energy fluxes that will be specified later, $\mathbf{n}_{\partial V(t)}$ denotes the outer unit normal vector field on $\partial V(t)$, whereas $\mathbf{n}_{\partial(\partial V(t)\cap\Gamma)}$ is the unit conormal vector field on $\partial(\partial V(t)\cap\Gamma)$. Since we consider a thermodynamically closed system, there is no exchange of energy over the boundary Γ, and thus, the domain of the first integral in the second line is just $\partial V(t)\cap\Omega$ instead of $\partial V(t)$. We further point out that $\partial(\partial V(t)\cap\Gamma)\subset\Gamma$ is to be understood as the relative boundary of the set $\partial V(t)\cap\Gamma$ within the submanifold Γ. Applying Gauß's divergence theorem on both integrals in the second line, and recalling that $\mathbf{n}_{\partial V(t)}=\mathbf{n}$ on $\partial V(t)\cap\Gamma$, we reformulate (3.45) as

$$\begin{aligned} 0 \geq \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V(t)} e_\Omega(\mathbf{v},\psi,\nabla\phi)\,\mathrm{d}x + \int_{\partial V(t)\cap\Gamma} e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\,\mathrm{d}\Gamma\right] \\ + \int_{V(t)} \operatorname{div}\mathbf{J}_e\,\mathrm{d}x + \int_{\partial V(t)\cap\Gamma} \operatorname{div}_\Gamma \mathbf{K}_e - \mathbf{J}_e\cdot\mathbf{n}\,\mathrm{d}\Gamma. \end{aligned} \tag{3.46}$$

Using the Reynolds transport theorem (see, e.g., [18, Theorem 2.11.1]), we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} e_\Omega(\mathbf{v},\phi,\nabla\phi)\,\mathrm{d}x = \int_{V(t)} \partial_t e_\Omega(\mathbf{v},\phi,\nabla\phi) + \operatorname{div}(e_\Omega(\mathbf{v},\phi,\nabla\phi)\mathbf{v})\,\mathrm{d}x. \tag{3.47}$$

Similarly, applying the Reynolds transport theorem for evolving hypersurfaces (see, e.g., [18, Theorem 2.10.1]), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial V(t)\cap\Gamma} e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\,\mathrm{d}\Gamma = \int_{V(t)\cap\Gamma} &\partial_t e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi) \\ &+ \operatorname{div}_\Gamma(e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\mathbf{w})\,\mathrm{d}\Gamma. \end{aligned} \tag{3.48}$$

Combining (3.46), (3.47) and (3.48), we thus get

$$\begin{aligned} 0 \geq &\int_{V(t)} \partial_t e_\Omega(\mathbf{v},\phi,\nabla\phi) + \operatorname{div}(e_\Omega(\mathbf{v},\phi,\nabla\phi)\mathbf{v}) + \operatorname{div}\mathbf{J}_e\,\mathrm{d}x \\ &+ \int_{\partial V(t)\cap\Gamma} \partial_t e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi) + \operatorname{div}_\Gamma(e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_e - \mathbf{J}_e\cdot\mathbf{n}\,\mathrm{d}\Gamma. \end{aligned} \tag{3.49}$$

In particular, this inequality holds true for all test volumes $V(t)\subset\Omega$ with $\partial V(t)\cap\Gamma=\emptyset$. Therefore, we infer the local dissipation law in the bulk, which reads as

$$0 \geq -\mathcal{D}_\Omega \coloneqq \partial_t e_\Omega(\mathbf{v},\phi,\nabla\phi) + \operatorname{div}(e_\Omega(\mathbf{v},\phi,\nabla\phi)\mathbf{v}) + \operatorname{div}\mathbf{J}_e \tag{3.50}$$

in Q^T. Let now $c>0$ be arbitrary, and consider a test volume $V(t)\subset\Omega$ with $|V(t)|$ being sufficiently small such that

$$\int_{V(t)} \mathcal{D}_\Omega\,\mathrm{d}x < c.$$

Then, invoking (3.49), we infer that

$$\int_{\partial V(t)\cap\Gamma} \partial_t e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi) + \operatorname{div}_\Gamma(e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_e - \mathbf{J}_e\cdot\mathbf{n}\,\mathrm{d}\Gamma < c.$$

As $c>0$ and the test volume $V(t)$ were arbitrary (except for the above smallness condition on $V(t)$), we conclude the following local dissipation law on the boundary:

$$0 \geq -\mathcal{D}_\Gamma \coloneqq \partial_t e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi) + \operatorname{div}_\Gamma(e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_e - \mathbf{J}_e\cdot\mathbf{n} \tag{3.51}$$

on Σ^T.

3.1.4 Completion of the model derivation via the Lagrange multiplier approach

We now want to identify the still undetermined flux terms using the *Lagrange multiplier approach*. Therefore, we introduce Lagrange multipliers μ and θ, which need to be determined in the course of this subsection. In the final model, μ will represent the *bulk chemical potential*, while θ will represent the *surface chemical potential*. Invoking the identities (3.31) and (3.33), the local energy dissipation laws (3.50) and (3.51) can be expressed as

$$\begin{aligned} 0 \geq -\mathcal{D}_\Omega &= \partial_t e_\Omega(\mathbf{v},\phi,\nabla\phi) + \operatorname{div}(e_\Omega(\mathbf{v},\phi,\nabla\phi)\mathbf{v}) + \operatorname{div}\mathbf{J}_e \\ &\quad - \mu(\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) + \operatorname{div}\mathbf{J}_\phi) && \text{in } Q^T, \quad (3.52) \\ 0 \geq -\mathcal{D}_\Gamma &= \partial_t e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi) + \operatorname{div}_\Gamma(e_\Gamma(\mathbf{w},\phi,\psi,\nabla_\Gamma\psi)\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_e \\ &\quad - \mathbf{J}_e\cdot\mathbf{n} - \theta(\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_\psi - \mathbf{J}_\psi\cdot\mathbf{n}) && \text{on } \Sigma^T. \quad (3.53) \end{aligned}$$

In the following, to provide a cleaner presentation, we will simply write ρ, σ, f and f_Γ instead of $\rho(\phi)$, $\sigma(\psi)$, $f(\phi,\nabla\phi)$ and $f_\Gamma(\phi,\psi,\nabla_\Gamma\psi)$. Invoking the definition of the energy densities e_Ω and e_Γ (see (3.41) and (3.43)), we reformulate (3.52) and (3.53) as

$$\begin{aligned} 0 \geq -\mathcal{D}_\Omega &= \partial_t(\tfrac{1}{2}\rho|\mathbf{v}|^2) + \operatorname{div}(\tfrac{1}{2}\rho|\mathbf{v}|^2\mathbf{v}) + \partial_t f + \operatorname{div}(f\mathbf{v}) + \operatorname{div}\mathbf{J}_e \\ &\quad - \mu(\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) + \operatorname{div}\mathbf{J}_\phi) && \text{in } Q^T, \quad (3.54) \\ 0 \geq -\mathcal{D}_\Gamma &= \partial_t(\tfrac{1}{2}\sigma|\mathbf{w}|^2) + \operatorname{div}_\Gamma(\tfrac{1}{2}\sigma|\mathbf{w}|^2\mathbf{w}) + \partial_t f_\Gamma + \operatorname{div}_\Gamma(f_\Gamma\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_e \\ &\quad - \mathbf{J}_e\cdot\mathbf{n} - \theta(\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) + \operatorname{div}_\Gamma\mathbf{K}_\psi - \mathbf{J}_\psi\cdot\mathbf{n}) && \text{on } \Sigma^T. \quad (3.55) \end{aligned}$$

In the following, for functions $h : Q^T \to \mathbb{R}$ and $h_\Gamma : \Sigma^T \to \mathbb{R}$, their material derivatives are given by

$$\begin{aligned} \partial_t^\bullet h &= \partial_t h + \mathbf{v}\cdot\nabla h && \text{in } Q^T, \\ \partial_t^\circ h_\Gamma &= \partial_t h_\Gamma + \mathbf{w}\cdot\nabla_\Gamma h_\Gamma && \text{on } \Sigma^T. \end{aligned}$$

We point out that in the material derivative ∂_t° on the boundary, only the tangential gradient instead of the full gradient is relevant. This is because we already know from (3.11) that $\mathbf{w}$ is a tangential vector field. With these definitions at hand, we can express the derivatives $\partial_t^\bullet f$ and $\partial_t^\circ f_\Gamma$ through the chain rule as

$$\begin{aligned} \partial_t^\bullet f &= \partial_\phi f\,\partial_t^\bullet\phi + \partial_{\nabla\phi} f\,\partial_t^\bullet\nabla\phi && \text{in } Q^T, && (3.56) \\ \partial_t^\circ f_\Gamma &= \partial_\phi f_\Gamma\,\partial_t^\circ\phi + \partial_\psi f_\Gamma\,\partial_t^\circ\psi + \partial_{\nabla_\Gamma\psi} f_\Gamma\,\partial_t^\circ\nabla_\Gamma\psi && \text{on } \Sigma^T. && (3.57) \end{aligned}$$

Computations in the bulk

In the bulk we proceed exactly as in [9] and [97] to reformulate the inequality (3.54) as

$$\begin{aligned} 0 \geq\ & \operatorname{div}\big[\mathbf{J}_e - \tfrac{1}{2}|\mathbf{v}|^2\mathbf{J} + \mathbf{T}^\top\mathbf{v} - \mu\mathbf{J}_\phi + \partial_{\nabla\phi} f\,\partial_t^\bullet\phi\big] \\ & + \big[\partial_\phi f - \operatorname{div}(\partial_{\nabla\phi} f) - \mu\big]\partial_t^\bullet\phi \\ & - \big[\mathbf{T} + \nabla\phi\otimes\partial_{\nabla\phi} f\big] : \nabla\mathbf{v} + \nabla\mu\cdot\mathbf{J}_\phi \qquad \text{in } Q^T. \end{aligned} \tag{3.58}$$

To ensure that (3.58) holds, we now choose the chemical potential μ and the energy flux $\mathbf{J}_e$ as

$$\mu = \partial_\phi f - \operatorname{div}(\partial_{\nabla\phi} f) \qquad \text{in } Q^T, \tag{3.59}$$

$$\mathbf{J}_e = \tfrac{1}{2}|\mathbf{v}|^2\mathbf{J} - \mathbf{T}^\top\mathbf{v} + \mu\mathbf{J}_\phi - \partial_{\nabla\phi} f \partial_t^\bullet \phi \qquad \text{in } Q^T. \tag{3.60}$$

In view of these choices, the first two lines of the right-hand side of (3.58) vanish. Furthermore, we assume the mass flux $\mathbf{J}_\phi$ to be of Fick's type, which means that

$$\mathbf{J}_\phi = -m_\Omega(\phi)\nabla\mu \qquad \text{in } Q^T. \tag{3.61}$$

Here, $m_\Omega = m_\Omega(\phi)$ is a non-negative function, which represents the mobility in the bulk. Therefore, (3.58) reduces to

$$0 \geq -\big[\mathbf{T} + \nabla\phi \otimes \partial_{\nabla\phi} f\big] : \nabla\mathbf{v} - m_\Omega(\phi)|\nabla\mu|^2 \qquad \text{in } Q^T. \tag{3.62}$$

We next define a tensor

$$\mathbf{S} = \mathbf{T} + p\mathbf{I} + \nabla\phi \otimes \partial_{\nabla\phi} f \qquad \text{in } Q^T.$$

Here, the scalar variable p denotes the pressure in the bulk, and $\mathbf{I}$ stands for the identity. The tensor $\mathbf{S}$ is the *viscous stress tensor* that corresponds to the irreversible changes of the energy caused by friction. For Newtonian fluids, $\mathbf{S}$ is usually assumed to be given by

$$\mathbf{S} = 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v},$$

where the non-negative function $\nu_\Omega = \nu_\Omega(\phi)$ represents the viscosity of the fluids in the bulk, and $\mathbf{D}\mathbf{v}$ is the symmetric gradient of $\mathbf{v}$. This choice, together with (3.62) and the identity $p\mathbf{I} : \nabla\mathbf{v} = p\,\mathrm{div}\ \mathbf{v} = 0$ in Q^T, implies that (3.58) is fulfilled as it reduces to

$$0 \geq -\nu_\Omega(\phi)|\mathbf{D}\mathbf{v}|^2 - m_\Omega(\phi)|\nabla\mu|^2 \qquad \text{in } Q^T. \tag{3.63}$$

In view of (3.42), the chemical potential μ given by (3.59) can be expressed as

$$\mu = -\varepsilon\Delta\phi + \frac{1}{\varepsilon}F'(\phi) \qquad \text{in } Q^T. \tag{3.64}$$

Moreover, the total stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{S} - p\mathbf{I} - \nabla\phi \otimes \partial_{\nabla\phi} f = 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v} - p\mathbf{I} - \varepsilon\nabla\phi \otimes \nabla\phi \qquad \text{in } Q^T. \tag{3.65}$$

Furthermore, plugging (3.61) into (3.31), we obtain the equation

$$\partial_t\phi + \mathrm{div}(\phi\mathbf{v}) = \mathrm{div}(m_\Omega(\phi)\nabla\mu) \qquad \text{in } Q^T. \tag{3.66}$$

Then, (3.18) can be reformulated as

$$\partial_t(\rho(\phi)\mathbf{v}) + \mathrm{div}(\mathbf{v} \otimes (\rho(\phi)\mathbf{v} + \mathbf{J})) - \mathrm{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) + \nabla p = -\varepsilon\mathrm{div}(\nabla\phi \otimes \nabla\phi) \tag{3.67}$$

in Q^T, where $\mathbf{T}$ is given by (3.65) and, due to (3.36) and (3.61), the flux term $\mathbf{J}$ is given by

$$\mathbf{J} = -\frac{\tilde\rho_2 - \tilde\rho_1}{2} m_\Omega(\phi)\nabla\mu \qquad \text{in } Q^T. \tag{3.68}$$

Computations on the surface

Next, we consider the local energy dissipation law (3.55) on the surface. Recalling the formulas for f (see (3.42)) and $\mathbf{J}_e$ (see (3.60)), we deduce from (3.55) that

$$\begin{aligned} 0 \geq\ &\partial_t(\tfrac{1}{2}\sigma|\mathbf{w}|^2) + \mathrm{div}_\Gamma(\tfrac{1}{2}\sigma|\mathbf{w}|^2\mathbf{w}) + \partial_\phi f_\Gamma \partial_t^\circ \phi + \partial_\psi f_\Gamma \partial_t^\circ \psi + \partial_{\nabla_\Gamma \psi} f_\Gamma \partial_t^\circ \nabla_\Gamma \psi \\ &+ \mathrm{div}_\Gamma \mathbf{K}_e - \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w}\cdot\mathbf{w} - \mathbf{Tn}\cdot\mathbf{w} - \mu \mathbf{J}_\phi \cdot \mathbf{n} + \varepsilon(\nabla\phi\cdot\mathbf{n})\partial_t^\circ\phi \\ &- \theta\partial_t^\circ\psi - \mathrm{div}_\Gamma(\mathbf{K}_\psi\theta) + \nabla_\Gamma\theta\cdot\mathbf{K}_\psi + \theta\mathbf{J}_\psi\cdot\mathbf{n} \end{aligned} \tag{3.69}$$

on Σ^T. Here, we have used that

$$\mathbf{v}\cdot\nabla\phi = \mathbf{v}\cdot[\nabla_\Gamma\phi + \mathbf{n}(\nabla\phi\cdot\mathbf{n})] = \mathbf{v}\cdot\nabla_\Gamma\phi = \mathbf{w}\cdot\nabla_\Gamma\phi \qquad \text{on } \Sigma^T,$$

which follows from the definition of the surface gradient and the boundary condition (3.11). Next, a straightforward computation yields

$$\begin{aligned} &\mathrm{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma \partial_t^\circ \psi) \\ &\quad = \mathrm{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma)\partial_t^\circ\psi + \partial_{\nabla_\Gamma\psi} f_\Gamma \cdot \partial_t^\circ \nabla_\Gamma\psi + [\nabla_\Gamma\psi \otimes \partial_{\nabla_\Gamma\psi} f_\Gamma] : \nabla_\Gamma \mathbf{w} \qquad \text{on } \Sigma^T. \end{aligned}$$

We now use this identity along with (3.15), (3.25), (3.33), (3.37) and (3.40) to reformulate (3.69). After a technical but straightforward computation, we arrive at

$$\begin{aligned} 0 \geq\ &\mathrm{div}_\Gamma[\mathbf{K}_e - \tfrac{1}{2}|\mathbf{w}|^2\mathbf{K} + \mathbf{T}_\Gamma^\top \mathbf{w} - \theta\mathbf{K}_\psi + \partial_{\nabla_\Gamma\psi} f_\Gamma \partial_t^\circ\psi + \tfrac{1}{2}\chi(K)(\alpha\psi-\phi)^2\mathbf{w}] \\ &+ [\partial_\phi f_\Gamma + \varepsilon\nabla\phi\cdot\mathbf{n}]\partial_t^\circ\phi + [\partial_\psi f_\Gamma - \mathrm{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) - \theta]\partial_t^\circ\psi \\ &+ [[\mathbf{Tn}]_{\tan} - \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} - \tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} - \tfrac{1}{2}\chi(K)\nabla_\Gamma(\alpha\psi-\phi)^2 + \mathbf{Z}]\cdot\mathbf{w} \\ &- [\mathbf{T}_\Gamma + \nabla_\Gamma\psi\otimes\partial_{\nabla_\Gamma\psi} f_\Gamma] : \nabla_\Gamma\mathbf{w} + \nabla_\Gamma\theta\cdot\mathbf{K}_\psi + \theta\mathbf{J}_\psi\cdot\mathbf{n} - \mu\mathbf{J}_\phi\cdot\mathbf{n} \end{aligned} \tag{3.70}$$

on Σ^T. In order to ensure that (3.70) is fulfilled, we choose the mass flux $\mathbf{K}_\psi$ and the energy flux $\mathbf{K}_e$ as follows:

$$\mathbf{K}_\psi = -m_\Gamma(\psi)\nabla_\Gamma\theta \qquad \text{on } \Sigma^T, \tag{3.71}$$

$$\mathbf{K}_e = \frac{1}{2}|\mathbf{w}|^2\mathbf{K} - \mathbf{T}_\Gamma^\top\mathbf{w} + \theta\mathbf{K}_\psi - \partial_{\nabla_\Gamma\psi} f_\Gamma \partial_t^\circ\psi - \tfrac{1}{2}\chi(K)(\alpha\psi-\phi)^2\mathbf{w} \qquad \text{on } \Sigma^T. \tag{3.72}$$

In (3.71), $m_\Gamma = m_\Gamma(\psi)$ is a non-negative function representing the mobility on the surface. This choice of $\mathbf{K}_e$ ensures that the first line of the right-hand side of (3.70) vanishes. Due to (3.71) and (3.72), inequality (3.70) reduces to

$$\begin{aligned} 0 \geq\ &[\partial_\phi f_\Gamma + \varepsilon\nabla\phi\cdot\mathbf{n}]\partial_t^\circ\phi + [\partial_\psi f_\Gamma - \mathrm{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) - \theta]\partial_t^\circ\psi \\ &+ [[\mathbf{Tn}]_{\tan} - \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} - \tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} - \tfrac{1}{2}\chi(K)\nabla_\Gamma(\alpha\psi-\phi)^2 + \mathbf{Z}]\cdot\mathbf{w} \\ &- [\mathbf{T}_\Gamma + \nabla_\Gamma\psi\otimes\partial_{\nabla_\Gamma\psi} f_\Gamma] : \nabla_\Gamma\mathbf{w} - m_\Gamma(\psi)|\nabla_\Gamma\theta|^2 + \theta\mathbf{J}_\psi\cdot\mathbf{n} - \mu\mathbf{J}_\phi\cdot\mathbf{n} \end{aligned} \tag{3.73}$$

on Σ^T. We further assume that the flux terms $\mathbf{J}_\phi$ and $\mathbf{J}_\psi$ are directly proportional. To be precise, we assume that there exists a real number β such that

$$\mathbf{J}_\psi\cdot\mathbf{n} = \beta\mathbf{J}_\phi\cdot\mathbf{n} = -\beta m_\Omega(\phi)\nabla\mu\cdot\mathbf{n} \qquad \text{on } \Sigma^T. \tag{3.74}$$

The parameter β is related to the transfer of material between bulk and surface and indicates the proportion of bulk material that becomes surface material (or vice versa). This further entails that

$$\mathbf{J}_\Gamma\cdot\mathbf{n} = -\beta\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} m_\Omega(\phi)\nabla\mu\cdot\mathbf{n} \qquad \text{on } \Sigma^T. \tag{3.75}$$

The second equality in (3.74) follows from (3.61). We further make the constitutive assumption

$$\begin{aligned}\mathbf{Z} &= -[\mathbf{Sn}]_{\tan} + \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} + \tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} - \gamma(\phi,\psi)\mathbf{w} + \alpha\varepsilon(\nabla\phi\cdot\mathbf{n})\nabla_\Gamma\psi \\ &= -[\mathbf{Tn}]_{\tan} + \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} + \tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} - \gamma(\phi,\psi)\mathbf{w} + \varepsilon(\nabla\phi\cdot\mathbf{n})\nabla_\Gamma(\alpha\psi-\phi)\end{aligned} \tag{3.76}$$

on Σ^T. Here, $\gamma = \gamma(\phi,\psi)$ is a non-negative slip parameter that is related to friction at the boundary. By this choice of $\mathbf{Z}$, inequality (3.73) further reduces to

$$\begin{aligned}0 \geq\ & [\partial_\phi f_\Gamma + \varepsilon\nabla\phi\cdot\mathbf{n}]\partial_t^\circ\phi + [\partial_\psi f_\Gamma - \mathrm{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) - \theta]\partial_t^\circ\psi \\ & + [\varepsilon(\nabla\phi\cdot\mathbf{n}) - \chi(K)(\alpha\psi-\phi)]\,\nabla_\Gamma(\alpha\psi-\phi)\cdot\mathbf{w} \\ & - [\mathbf{T}_\Gamma + \nabla_\Gamma\psi\otimes\partial_{\nabla_\Gamma\psi} f_\Gamma] : \nabla_\Gamma\mathbf{w} - m_\Gamma(\psi)|\nabla_\Gamma\theta|^2 + \theta\mathbf{J}_\psi\cdot\mathbf{n} - \mu\mathbf{J}_\phi\cdot\mathbf{n}\end{aligned} \tag{3.77}$$

on Σ^T. Analogously as in the bulk, we introduce a tensor

$$\mathbf{S}_\Gamma = \mathbf{T}_\Gamma + q\mathbf{I} + \nabla_\Gamma\psi\otimes\partial_{\nabla_\Gamma\psi} f_\Gamma \qquad \text{on } \Sigma^T.$$

Here, the scalar variable q denotes the pressure on the surface, and the tensor $\mathbf{S}_\Gamma$ is the *viscous stress tensor* on the surface, which is assumed to be given by

$$\mathbf{S}_\Gamma = 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}.$$

Here, $\nu_\Gamma = \nu_\Gamma(\psi)$ is a non-negative function representing the viscosity of the fluids on the surface and $\mathbf{D}_\Gamma\mathbf{w}$ is the symmetric surface gradient of $\mathbf{w}$. Therefore, the total stress tensor on the surface reads as

$$\mathbf{T}_\Gamma = \mathbf{S}_\Gamma - q\mathbf{I} - \delta\kappa\nabla_\Gamma\psi\otimes\nabla_\Gamma\psi \qquad \text{on } \Sigma^T. \tag{3.78}$$

This means that (3.19) can be reformulated as

$$\partial_t(\sigma(\psi)\mathbf{w}) + \mathrm{div}_\Gamma(\mathbf{w}\otimes(\sigma(\psi)\mathbf{w}+\mathbf{K})) = \mathrm{div}_\Gamma\mathbf{T}_\Gamma + \mathbf{Z} \quad \text{on } \Sigma^T \tag{3.79}$$

where, according to (3.37) and (3.71), we have

$$\mathbf{K} = -\frac{\tilde\sigma_2-\tilde\sigma_1}{2} m_\Gamma(\psi)\nabla_\Gamma\theta \quad \text{on } \Sigma^T, \tag{3.80}$$

$\mathbf{T}_\Gamma$ is given by (3.78), and $\mathbf{Z}$ is given by (3.76). Moreover, by means of (3.78) and the identity $q\mathbf{I} : \nabla_\Gamma\mathbf{w} = q\mathrm{div}_\Gamma\ \mathbf{w} = 0$ on Σ^T, (3.73) reduces to

$$\begin{aligned}0 \geq\ & [\partial_\phi f_\Gamma + \varepsilon\nabla\phi\cdot\mathbf{n}]\partial_t^\circ\phi + [\partial_\psi f_\Gamma - \mathrm{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) - \theta]\partial_t^\circ\psi \\ & + [\varepsilon(\nabla\phi\cdot\mathbf{n}) - \chi(K)(\alpha\psi-\phi)]\,\nabla_\Gamma(\alpha\psi-\phi)\cdot\mathbf{w} \\ & - \nu_\Gamma(\psi)|\mathbf{D}_\Gamma\mathbf{w}|^2 - \gamma(\phi,\psi)|\mathbf{w}|^2 - m_\Gamma(\psi)|\nabla_\Gamma\theta|^2 - (\beta\theta-\mu)m_\Omega(\phi)\nabla\mu\cdot\mathbf{n}\end{aligned} \tag{3.81}$$

on Σ^T. The first two terms in the third line of the right-hand side of (3.81) are clearly non-positive. The third term is non-positive if, depending on a parameter $L\in[0,\infty]$, one of the following boundary conditions holds:

$$\beta\theta-\mu = 0 \qquad \text{on } \Sigma^T \text{ if } L=0, \tag{3.82a}$$

$$m_\Omega(\phi)\nabla\mu\cdot\mathbf{n} = \tfrac{1}{L}(\beta\theta-\mu) \qquad \text{on } \Sigma^T \text{ if } L\in(0,\infty), \tag{3.82b}$$

$$m_\Omega(\phi)\nabla\mu\cdot\mathbf{n} = 0 \qquad \text{on } \Sigma^T \text{ if } L=\infty. \tag{3.82c}$$

Similarly, to deal with the first and the second line of (3.81), we assume ϕ and ψ to fulfill one of the boundary conditions

$$\alpha\psi-\phi = 0 \qquad \text{on } \Sigma^T \text{ if } K=0, \tag{3.83a}$$

$$\begin{aligned} \varepsilon\nabla\phi\cdot\mathbf{n} &= \tfrac{1}{K}(\alpha\psi-\phi) && \text{on } \Sigma^T \text{ if } K\in(0,\infty), && (3.83b)\\ \nabla\phi\cdot\mathbf{n} &= 0 && \text{on } \Sigma^T \text{ if } K=\infty, && (3.83c)\end{aligned}$$

where $K \in [0,\infty]$ is the number that was introduced in (3.44). To show that (3.81) is actually fulfilled, we need to distinguish the cases $K=0$, $K\in(0,\infty)$ and $K=\infty$.

The case $K=0$. Without loss of generality, we assume that $\alpha \neq 0$. The case $\alpha = 0$ can be handled similarly but the computations are even easier. Then, we immediately see that the second line in (3.81) vanishes. In the definition of f_Γ, we have $\chi(K)=\chi(0)=0$ and thus $\partial_\phi f_\Gamma = 0$ on Σ^T. Moreover, due to (3.83a), $\alpha^{-1}\phi$ can be interpreted as an extension of ψ in some neighborhood of Σ^T, which entails that $\partial_t^\circ\phi = \alpha\partial_t^\circ\psi$ on Σ^T. Consequently, the first line of the right-hand side of (3.81) can be reformulated as

$$[\partial_\psi f_\Gamma - \operatorname{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) + \alpha\varepsilon\nabla\phi\cdot\mathbf{n} - \theta]\partial_t^\circ\psi \qquad \text{on } \Sigma^T.$$

Choosing

$$\theta = \partial_\psi f_\Gamma - \operatorname{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) + \alpha\varepsilon\nabla\phi\cdot\mathbf{n} \qquad \text{on } \Sigma^T,$$

we thus ensure that the first line of (3.81) also vanishes. This shows that inequality (3.81) is fulfilled.

The case $K\in(0,\infty)$. The boundary condition (3.83b) entails that

$$\begin{aligned} \varepsilon(\nabla\phi\cdot\mathbf{n})[\alpha\nabla_\Gamma\psi - \nabla_\Gamma\phi]\cdot\mathbf{w} &= \frac{1}{K}(\alpha\psi-\phi)(\alpha\nabla_\Gamma\psi - \nabla_\Gamma\phi)\cdot\mathbf{w} \\ &= \frac{1}{2}\chi(K)\nabla_\Gamma(\alpha\psi-\phi)^2\cdot\mathbf{w} \qquad \text{on } \Sigma^T,\end{aligned}$$

and therefore, the second line of (3.81) vanishes. In view of the definition of f_Γ, we further have

$$\begin{aligned} \partial_\phi f_\Gamma &= -\chi(K)(\alpha\psi-\phi) = -\varepsilon\nabla\phi\cdot\mathbf{n} && \text{on } \Sigma^T,\\ \partial_\psi f_\Gamma &= \frac{1}{\delta}G'(\psi) + \alpha\chi(K)(\alpha\psi-\phi) = \frac{1}{\delta}G'(\psi) + \alpha\varepsilon\nabla\phi\cdot\mathbf{n} && \text{on } \Sigma^T.\end{aligned}$$

This implies the first line of (3.81) can be rewritten as

$$[\partial_\psi f_\Gamma - \operatorname{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) - \theta]\partial_t^\circ\psi \qquad \text{on } \Sigma^T.$$

Hence, the choice

$$\theta = \partial_\psi f_\Gamma - \operatorname{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) \qquad \text{on } \Sigma^T,$$

ensures that the first line also vanishes. Consequently, inequality (3.81) is fulfilled.

The case $K=\infty$. Since $\chi(K)=\chi(\infty)=0$, it directly follows from (3.83c) that the second line in (3.81) vanishes. Moreover, since $\partial_\phi f_\Gamma = 0$ on Σ^T, the first line of (3.81) can be reformulated as

$$[\partial_\psi f_\Gamma - \operatorname{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) - \theta]\partial_t^\circ\psi \qquad \text{on } \Sigma^T.$$

Thus, choosing

$$\theta = \partial_\psi f_\Gamma - \operatorname{div}_\Gamma(\partial_{\nabla_\Gamma\psi} f_\Gamma) \qquad \text{on } \Sigma^T,$$

ensures that the first line also vanishes. This shows that inequality (3.81) is fulfilled.

By the definition of f_Γ (see (3.44)), we conclude in all cases $K \in [0,\infty]$ that

$$\theta = -\delta\kappa\Delta_\Gamma\psi + \frac{1}{\delta}G'(\psi) + \alpha\varepsilon\nabla\phi\cdot\mathbf{n} \qquad \text{on } \Sigma^T. \tag{3.84}$$

Eventually, substituting (3.71) and (3.74) into (3.33), we obtain

$$\partial_t\psi + \mathrm{div}_\Gamma(\psi\mathbf{w}) = \mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\nabla\mu\cdot\mathbf{n} \qquad \text{on } \Sigma^T. \tag{3.85}$$

In summary, we have thus derived system (1.42).

Chapter 4

Analysis of bulk-surface systems

4.1 Introduction

During the analysis of the convective bulk-surface Cahn–Hilliard subsystem and the bulk-surface Navier–Stokes–Cahn–Hilliard model (1.42), a variety of auxiliary bulk-surface subsystems naturally arise. These include, for instance, elliptic problems with nonlinearities or non-constant coefficients, as well as bulk-surface Stokes-type systems.

Rather than treating these systems solely as technical tools within the main part of this thesis, it is both natural and advantageous to study them separately and in a more general framework. This approach not only allows for a clearer and more structured presentation of the arguments, but also leads to results of independent interest. In particular, the analysis of these subsystems provides general insights into bulk-surface coupling mechanisms that are not restricted to the specific models considered in this thesis.

Moreover, the results obtained in this chapter can be applied to a broader class of problems involving coupled bulk-surface phenomena. As such, they form a useful toolkit that may be employed in the study of other diffuse interface models and related systems with dynamic boundary conditions.

4.2 A bulk-surface elliptic system

A part of this section is contained in the following publication:

[121] P. Knopf, and J. Stange, *Well-posedness of a bulk-surface convective Cahn–Hilliard system with dynamic boundary conditions*, NoDEA Nonlinear Differential Equations Appl., 31 (2024), pp. Paper No. 82, 48.

We begin by recalling results for a bulk-surface elliptic system that has been studied previously in [117]. To this end, let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a non-empty, bounded domain of class C^2 with boundary $\Gamma = \partial\Omega$. The system reads as

$$-\Delta u = f \qquad \text{in } \Omega, \tag{4.1a}$$
$$-\Delta_\Gamma v + \alpha \partial_{\mathbf{n}} u = f_\Gamma \qquad \text{on } \Gamma, \tag{4.1b}$$
$$K\partial_{\mathbf{n}} u = \alpha v - u \qquad \text{on } \Gamma, \tag{4.1c}$$

for given source terms $(f, f_\Gamma) \in \mathcal{V}^{-1}_{K,\alpha}$, with $K \in [0,\infty]$ and $\alpha \in \mathbb{R}$. We recall that the boundary condition (4.1c) has to be understood in the sense that

$$\begin{cases} u = \alpha v & \text{if } K = 0, \\ K\partial_{\mathbf{n}} u = \alpha v - u & \text{if } K \in (0,\infty), \\ \partial_{\mathbf{n}} u = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Gamma.$$

We call $(u,v) \in \mathcal{H}^1_{K,\alpha}$ a weak solution to (4.1) if

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \zeta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma + \chi(K) \int_\Gamma (\alpha v - u)(\alpha \zeta_\Gamma - \zeta) \,\mathrm{d}\Gamma \\ &\quad = \langle (f, f_\Gamma), (\zeta, \zeta_\Gamma) \rangle_{\mathcal{H}^1_{K,\alpha}} \end{aligned} \tag{4.2}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$. Using the bilinear form $(\cdot,\cdot)_{K,\alpha}$, the weak formulation (4.2) can alternatively be written as

$$\big((u,v),(\zeta,\zeta_\Gamma)\big)_{K,\alpha} = \langle (f,f_\Gamma),(\zeta,\zeta_\Gamma)\rangle_{\mathcal{H}^1_{K,\alpha}}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$.

To prove the well-posedness of weak solutions to (4.1), the following bulk-surface Poincaré inequality is essential.

Lemma 4.2.1. *Let $\beta \in \mathbb{R}$ with $\alpha\beta|\Omega| + |\Gamma| \neq 0$. There exists a constant $C_P > 0$, depending only on K, α, β and Ω, such that*

$$\|(u,v)\|_{\mathcal{H}^1} \le C_P \|(u,v)\|_{K,\alpha} \tag{4.3}$$

for all $(u,v) \in \mathcal{H}^1_{K,\alpha}$ with $\beta|\Omega|\langle u\rangle_\Omega + |\Gamma|\langle v\rangle_\Gamma = 0$.

We obtain the following well-posedness result for weak solutions.

Proposition 4.2.2. *Let $(f, f_\Gamma) \in \mathcal{V}^{-1}_{K,\alpha}$ and $\beta \in \mathbb{R}$ with $\alpha\beta|\Omega| + |\Gamma| \neq 0$. Then there exists a unique solution $(u,v) \in \mathcal{H}^1_{K,\alpha}$ with*

$$\|\mathcal{S}_{K,\alpha}(f,f_\Gamma)\|_{\mathcal{H}^1} \le C\|(f,f_\Gamma)\|_{(\mathcal{H}^1_{K,\alpha})'}. \tag{4.4}$$

In the case $K \in (0,\infty)$, this result was first established in [117]. The case $K = 0$ was also considered there, albeit in a slightly different functional setting, and can be extended to the present framework. If $K = \infty$, the systems (4.1) decouples into the problems

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ \partial_{\mathbf{n}} u &= 0 && \text{on } \Gamma \end{aligned}$$

and

$$-\Delta_\Gamma v = f_\Gamma \qquad \text{on } \Gamma,$$

which can be solved independently in Ω and on Γ, see [54, Lemma A.1] or [4, Lemma 2].

Moreover, the following higher regularity result was shown in [117, Theorem 3.3].

Proposition 4.2.3. *Suppose that Ω is of class C^{k+2} and that $(f,g) \in \mathcal{V}^{-1}_{K,\alpha} \cap \mathcal{H}^k$ for some $k \in \mathbb{N}$. Then it holds that $\mathcal{S}_{K,\alpha}(f,f_\Gamma) \in \mathcal{H}^{k+2} \cap \mathcal{H}^1_{K,\alpha}$ with*

$$\|\mathcal{S}_{K,\alpha}(f,f_\Gamma)\|_{\mathcal{H}^{k+2}} \le C\|(f,f_\Gamma)\|_{\mathcal{H}^k}.$$

In the case $k = 2$, we extend this higher regularity result to right-hand sides in $\mathcal{L}^p$ for $p \in [2, \infty)$.

PROPOSITION 4.2.4. *Let Ω be of class C^2, let $p \in [2, \infty)$ and consider a pair $(f, f_\Gamma) \in \mathcal{V}_{K,\alpha}^{-1} \cap \mathcal{L}^p$. Then there exists a constant $C > 0$ such that*

$$\|\mathcal{S}_{K,\alpha}(f, f_\Gamma)\|_{\mathcal{W}^{2,p}} \leq C\|(f, f_\Gamma)\|_{\mathcal{L}^p}. \tag{4.5}$$

This result was established in [122, Proposition A.1]. We omit the proof here, as the more involved case of non-constant coefficients will be treated in Section 4.4, see Theorem 4.4.4.

4.3 A bulk-surface elliptic system with logarithmic nonlinearity

The content of this section is based on the results of the following preprint:

[98] A. GIORGINI, P. KNOPF, AND J. STANGE, *Regularity of a bulk-surface Cahn–Hilliard model driven by Leray velocity fields.* Preprint: arXiv:2506.18617, 2025.

In this section, we establish well-posedness and regularity results for the bulk-surface elliptic system (4.1) with additional singular nonlinearities. To be precise, we study the following system

$$-\Delta u + F_1'(u) = f \qquad \text{on } \Omega, \tag{4.6a}$$

$$-\Delta_\Gamma v + G_1'(v) + \alpha\partial_{\mathbf{n}} u = f_\Gamma \qquad \text{on } \Gamma, \tag{4.6b}$$

$$K\partial_{\mathbf{n}} u = \alpha v - u \qquad \text{on } \Gamma, \tag{4.6c}$$

where $K \in [0, \infty)$, $\alpha \in [-1, 1]$, $(f, f_\Gamma) \in \mathcal{L}^2$ are given functions and the nonlinearities F_1 and G_1 satisfy the following assumption:

(S) It holds $F_1, G_1 \in C([-1, 1]) \cap C^2(-1, 1)$ with

$$\lim_{s \searrow -1} F_1'(s) = \lim_{s \searrow -1} G_1'(s) = -\infty \quad \text{and} \quad \lim_{s \nearrow 1} F_1'(s) = \lim_{s \nearrow 1} G_1'(s) = +\infty.$$

Moreover, there exist constants $\Theta_\Omega, \Theta_\Gamma > 0$ such that

$$F_1''(s) \geq \Theta_\Omega, \qquad G_1''(s) \geq \Theta_\Gamma \qquad \text{for all } s \in (-1, 1). \tag{4.7}$$

Without loss of generality, we suppose $F_1(0) = G_1(0) = F_1'(0) = G_1'(0) = 0$. Lastly, we assume that the singular part of the boundary potential dominates the singular part of the bulk potential in the following sense: there exist $\kappa_1, \kappa_2 > 0$ such that

$$|F_1'(\alpha s)| \leq \kappa_1 |G_1'(s)| + \kappa_2 \qquad \text{for all } s \in (-1, 1). \tag{4.8}$$

These assumptions are motivated by the physically relevant Flory–Huggins potential (1.3). In particular, we may choose as F_1 and G_1 the convex part of the Flory–Huggins potential, respectively.

We note that, if $F_{1,\lambda}$ and $G_{1,\lambda}$ denote for $\lambda > 0$ the Moreau–Yosida approximations of F_1 and G_1, respectively, then we infer from (4.8) that

$$|F_{1,\lambda}'(\alpha s)| \leq \kappa_1 |G_{1,\lambda}'(s)| + \kappa_2 \qquad \text{for all } s \in \mathbb{R}, \tag{4.9}$$

see [33, Lemma 4.4].

4.3.1 The subdifferential of the convex part of the free energy

Before we look at the system (4.6), we consider the convex part of the free energy E_{free}. To this end, we consider the functional $\widetilde{E}_{\text{free}} : \mathcal{L}^2 \to (-\infty, \infty]$ defined by

$$\widetilde{E}_{\text{free}}(u,v) = \begin{cases} \displaystyle\int_\Omega \frac{1}{2}|\nabla u|^2 + F_1(u)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma v|^2 + G_1(v)\,\mathrm{d}\Gamma & \text{if } (u,v) \in \operatorname{Dom} \widetilde{E}_{\text{free}}, \\ \displaystyle\quad +\chi(K)\int_\Gamma \frac{1}{2}(\alpha v - u)^2\,\mathrm{d}\Gamma & \\ \infty & \text{else,} \end{cases}$$

where

$$\operatorname{Dom} \widetilde{E}_{\text{free}} = \{(u,v) \in \mathcal{H}^1_{K,\alpha} : |u| \leq 1 \text{ a.e. in } \Omega,\ |v| \leq 1 \text{ a.e. on } \Gamma\}.$$

We first observe that the functional $\widetilde{E}_{\text{free}}$ has the following properties.

LEMMA 4.3.1. *The functional $\widetilde{E}_{\text{free}}$ as defined above is proper, lower semicontinuous and convex.*

PROOF. It is clear that $\widetilde{E}_{\text{free}}$ is proper and convex as F_1 and G_1 are convex by (4.7). The lower semicontinuity of $\widetilde{E}_{\text{free}}$ can be shown by proceeding analogously to [14, Lemma 4.1]. □

The main goal of this subsection is establish the following result, which gives an explicit characterization of the subdifferential $\partial\widetilde{E}_{\text{free}}$ of $\widetilde{E}_{\text{free}}$.

PROPOSITION 4.3.2. *The subdifferential $\partial\widetilde{E}_{\text{free}}$ of $\widetilde{E}_{\text{free}}$ is a maximal monotone operator in $\mathcal{L}^2$, whose essential domain is given by*

$$D(\partial\widetilde{E}_{\text{free}}) = \{(u,v) \in \mathcal{H}^2 : (F_1'(u), G_1'(v)) \in \mathcal{L}^2,\ K\partial_{\mathbf{n}}u = \alpha v - u \textit{ a.e. on } \Gamma\}.$$

Moreover, for any $(u,v) \in D(\partial\widetilde{E}_{\text{free}})$ it holds

$$\partial\widetilde{E}_{\text{free}}(u,v) = (-\Delta u + F_1'(u), -\Delta_\Gamma v + G_1'(v) + \alpha\partial_{\mathbf{n}}u). \tag{4.11}$$

PROOF. In view Lemma 4.3.1, Proposition 2.3.6 readily yields that the subdifferential $\partial\widetilde{E}_{\text{free}}$ is a maximal monotone operator. To show the second part, we proceed as in the general scheme presented in [20, Proposition 2.9]. To this end, we consider the operator $\mathcal{A} : \mathcal{L}^2 \to \mathcal{L}^2$ that is defined as

$$\mathcal{A}(u,v) = (-\Delta u + F_1'(u), -\Delta_\Gamma v + G_1'(v) + \alpha\partial_{\mathbf{n}}u) \qquad \text{for all } (u,v) \in \operatorname{Dom}(\mathcal{A}),$$

where the domain of $\mathcal{A}$ is given as

$$\operatorname{Dom}(\mathcal{A}) = \{(u,v) \in \mathcal{H}^2 : (F_1'(u), G_1'(v)) \in \mathcal{L}^2,\ K\partial_{\mathbf{n}}u = \alpha v - u \text{ a.e. on } \Gamma\}.$$

The goal of this proof is now to show that $\mathcal{A} = \partial\widetilde{E}_{\text{free}}$. We start by proving the inclusion $\mathcal{A} \subset \partial\widetilde{E}_{\text{free}}$. To this end, let $(u,v) \in \operatorname{Dom}(\mathcal{A})$ and $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$. Then, taking into account the convexity of F_1 and G_1, we first notice that

$$F_1'(u)(u-\zeta) \geq F_1(u) - F_1(\zeta) \quad \text{and} \quad G_1'(v)(v-\zeta_\Gamma) \geq G_1(v) - G_1(\zeta_\Gamma).$$

Then, recalling the simple algebraic identity

$$\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) = \frac{|\mathbf{a}|^2}{2} - \frac{|\mathbf{b}|^2}{2} + \frac{|\mathbf{a}+\mathbf{b}|^2}{2} \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^d,$$

an application of integration by parts implies

$$
\begin{aligned}
&\big(\mathcal{A}(u,v),(u,v)-(\zeta,\zeta_\Gamma)\big)_{\mathcal{L}^2} \\
&\quad= \int_\Omega \big(-\Delta u + F_1'(u)\big)(u-\zeta)\,\mathrm{d}x + \int_\Gamma \big(-\Delta_\Gamma v + G_1'(v) + \alpha\partial_{\mathbf{n}}u\big)(v-\zeta_\Gamma)\,\mathrm{d}\Gamma \\
&\quad= \int_\Omega \nabla u\cdot\nabla(u-\zeta) + F_1'(u)(u-\zeta)\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma v\cdot\nabla_\Gamma(v-\zeta_\Gamma) + G_1'(v)\,\mathrm{d}\Gamma \\
&\qquad + \chi(K)\int_\Gamma (\alpha v - u)\big(\alpha(v-u) - (\alpha\zeta_\Gamma - \zeta)\big)\,\mathrm{d}\Gamma \\
&\quad\geq \int_\Omega \frac{1}{2}|\nabla u|^2 + F_1'(u)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma v|^2 + G_1(v)\,\mathrm{d}\Gamma + \chi(K)\int_\Gamma \frac{1}{2}(\alpha v - u)^2\,\mathrm{d}\Gamma \\
&\qquad - \int_\Omega \frac{1}{2}|\nabla\zeta|^2 + F_1'(\zeta)\,\mathrm{d}x - \int_\Gamma \frac{1}{2}|\nabla_\Gamma\zeta_\Gamma|^2 + G_1(\zeta_\Gamma)\,\mathrm{d}\Gamma - \chi(K)\int_\Gamma \frac{1}{2}(\alpha\zeta_\Gamma - \zeta)^2\,\mathrm{d}\Gamma \\
&= \widetilde{E}_{\mathrm{free}}(u,v) - \widetilde{E}_{\mathrm{free}}(\zeta,\zeta_\Gamma).
\end{aligned}
$$

This directly implies that $\mathcal{A}(u,v)\in\partial\widetilde{E}_{\mathrm{free}}(u,v)$ for all $(u,v)\in\mathrm{Dom}\,(\mathcal{A})$, which means that $\mathcal{A}\subset\partial\widetilde{E}_{\mathrm{free}}$.

In order to show the reverse inclusion $\partial\widetilde{E}_{\mathrm{free}}\subset\mathcal{A}$, it is enough to prove that $\mathcal{A}$ is a maximal monotone operator in $\mathcal{L}^2$. Thus, by Proposition 2.3.2, it is equivalent to prove that $R(\mathrm{Id}+\mathcal{A})=\mathcal{L}^2$. To this end, we fix $(f,f_\Gamma)\in\mathcal{L}^2$ and consider the following system

$$
\begin{aligned}
u - \Delta u + F_1'(u) &= f && \text{in } \Omega, && (4.12a)\\
v - \Delta_\Gamma v + G_1'(v) + \alpha\partial_{\mathbf{n}}u &= f_\Gamma && \text{on } \Gamma, && (4.12b)\\
K\partial_{\mathbf{n}}u &= \alpha v - u && \text{on } \Gamma. && (4.12c)
\end{aligned}
$$

To show the existence of a suitable solution to (4.12), we approximate the system with

$$
\begin{aligned}
u - \Delta u + F_{1,\lambda}'(u) &= f && \text{in } \Omega, && (4.13a)\\
v - \Delta_\Gamma v + G_{1,\lambda}'(v) + \alpha\partial_{\mathbf{n}}u &= f_\Gamma && \text{on } \Gamma, && (4.13b)\\
K\partial_{\mathbf{n}}u &= \alpha v - u && \text{on } \Gamma, && (4.13c)
\end{aligned}
$$

for any $\lambda>0$, where $F_{1,\lambda}$ and $G_{1,\lambda}$ denote the Moreau–Yosida regularization of F_1 and G_1, respectively. We start by showing that, for any $\lambda>0$, system (4.13) has a unique solution $(u_\lambda,v_\lambda)\in\mathcal{H}^2$. To do so, we fix $\lambda>0$, and consider for any $(u,v)\in\mathcal{L}^2$ the auxiliary problem

$$
\begin{aligned}
\lambda\overline{u} - \lambda\Delta\overline{u} + \overline{u} &= \lambda f + \big(1+\lambda F_1'\big)^{-1}(u) && \text{in } \Omega, && (4.14a)\\
\lambda\overline{v} - \lambda\Delta_\Gamma\overline{v} + \lambda\alpha\partial_{\mathbf{n}}\overline{u} + \overline{v} &= \lambda f_\Gamma + \big(1+\lambda G_1'\big)^{-1}(v) && \text{on } \Gamma, && (4.14b)\\
K\partial_{\mathbf{n}}\overline{u} &= \alpha\overline{v} - \overline{u} && \text{on } \Gamma. && (4.14c)
\end{aligned}
$$

The existence of a unique weak solution $(\overline{u},\overline{v})\in\mathcal{H}^1_{K,\alpha}$ to (4.14) can easily be shown by employing the Lax–Milgram lemma. This means, that $(\overline{u},\overline{v})$ satisfies

$$
\begin{aligned}
&\lambda\int_\Omega \overline{u}\,\zeta\,\mathrm{d}x + \lambda\int_\Gamma \overline{v}\,\zeta_\Gamma\,\mathrm{d}\Gamma + \lambda\int_\Omega \nabla\overline{u}\cdot\nabla\zeta\,\mathrm{d}x + \lambda\int_\Gamma \nabla_\Gamma\overline{v}\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma \\
&\qquad + \lambda\chi(K)\int_\Gamma (\alpha\overline{v}-\overline{u})(\alpha\zeta_\Gamma - \zeta)\,\mathrm{d}\Gamma + \int_\Omega \overline{u}\,\zeta\,\mathrm{d}x + \int_\Gamma \overline{v}\,\zeta_\Gamma\,\mathrm{d}\Gamma \qquad (4.15)\\
&\quad= \lambda\int_\Omega f\,\zeta\,\mathrm{d}x + \lambda\int_\Gamma f_\Gamma\,\zeta_\Gamma\,\mathrm{d}\Gamma + \int_\Omega (1+\lambda F_1')^{-1}(u)\,\zeta\,\mathrm{d}x + \int_\Gamma (1+\lambda G_1')^{-1}(v)\,\zeta_\Gamma\,\mathrm{d}\Gamma
\end{aligned}
$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$. We can then define the operator $T_\lambda : \mathcal{L}^2 \to \mathcal{L}^2$ by $(u, v) \mapsto (\overline{u}, \overline{v})$. Our goal is now to show that the map T_λ is a contraction. Towards this end, let $(u_1, v_1), (u_2, v_2) \in \mathcal{L}^2$ be arbitrary, and consider the corresponding solutions $(\overline{u}_1, \overline{v}_1), (\overline{u}_2, \overline{v}_2) \in \mathcal{H}^1_{K,\alpha}$ to system (4.14). Taking the differences of the corresponding weak formulation and considering the test function $(\zeta, \zeta_\Gamma) = (\overline{u}_1, \overline{v}_2) - (\overline{u}_2, \overline{v}_2) \in \mathcal{H}^1_{K,\alpha}$ shows that

$$\begin{aligned}
&\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|^2_{K,\alpha} + \|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|^2_{\mathcal{L}^2} + \frac{1}{\lambda}\|(\overline{u}_1, \overline{v}_1) - (\overline{u}_2, \overline{v}_2)\|^2_{\mathcal{L}^2} \\
&\quad = \frac{1}{\lambda}\int_\Omega \Big(\big(1 + \lambda F_1'\big)^{-1}(u_1) - \big(1 + \lambda F_1'\big)^{-1}(u_2)\Big)(\overline{u}_1 - \overline{u}_2)\,\mathrm{d}x \\
&\qquad + \frac{1}{\lambda}\int_\Gamma \Big(\big(1 + \lambda G_1'\big)^{-1}(v_1) - \big(1 + \lambda G_1'\big)^{-1}(v_2)\Big)(\overline{v}_1 - \overline{v}_2)\,\mathrm{d}\Gamma.
\end{aligned}$$

Recalling that the resolvent operator of a maximal monotone operator is Lipschitz continuous with Lipschitz constant 1, i.e., it holds

$$\big|\big(1 + \lambda F_1'\big)^{-1}(x) - \big(1 + \lambda F_1'\big)^{-1}(y)\big| \le |x - y| \qquad \text{for all } x, y \in \mathbb{R},$$

with the same estimate holding for $(1 + \lambda G_1')^{-1}$, we infer

$$\|T_\lambda(u_1, v_1) - T_\lambda(u_2, v_2)\|^2_{\mathcal{L}^2} + \frac{1}{\lambda}\|T_\lambda(u_1, v_1) - T_\lambda(u_2, v_2)\|^2_{\mathcal{L}^2} \le \frac{1}{\lambda}\|(u_1, v_1) - (u_2, v_2)\|^2_{\mathcal{L}^2}.$$

Consequently, we deduce

$$\|T_\lambda(u_1, v_1) - T_\lambda(u_2, v_2)\|_{\mathcal{L}^2} \le \frac{1}{\sqrt{1 + \lambda}}\|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{L}^2},$$

which implies that the map T_λ is a contraction. Therefore, applying the Banach fixed-point theorem, we conclude the existence of a pair $(u_\lambda, v_\lambda) \in \mathcal{L}^2$ satisfying $T_\lambda(u_\lambda, v_\lambda) = (u_\lambda, v_\lambda)$. In view of the definition of the Yosida approximation, we readily infer that $(u_\lambda, v_\lambda) \in \mathcal{L}^2$ solves (4.13) in the weak sense, i.e., it holds

$$\begin{aligned}
&\int_\Omega u_\lambda\,\zeta + \nabla u_\lambda \cdot \nabla \zeta + F'_{1,\lambda}(u_\lambda)\,\zeta\,\mathrm{d}x + \int_\Gamma v_\lambda\,\zeta_\Gamma + \nabla_\Gamma v_\lambda \cdot \nabla_\Gamma \zeta_\Gamma + G'_{1,\lambda}(v_\lambda)\,\zeta_\Gamma\,\mathrm{d}\Gamma \\
&\qquad + \chi(K)\int_\Gamma (\alpha v_\lambda - u_\lambda)(\alpha\zeta_\Gamma - \zeta)\,\mathrm{d}\Gamma \qquad\qquad (4.16)\\
&\quad = \int_\Omega f\,\zeta\,\mathrm{d}x + \int_\Gamma f_\Gamma\,\zeta_\Gamma\,\mathrm{d}\Gamma
\end{aligned}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$. Now, using $(\zeta, \zeta_\Gamma) = (u_\lambda, v_\lambda) \in \mathcal{H}^1_{K,\alpha}$ as a test function in (4.16), we have

$$\begin{aligned}
&\|(u_\lambda, v_\lambda)\|^2_{K,\alpha} + \|(u_\lambda, v_\lambda)\|^2_{\mathcal{L}^2} + \int_\Omega F'_{1,\lambda}(u_\lambda)\,u_\lambda\,\mathrm{d}x + \int_\Gamma G'_{1,\lambda}(v_\lambda)\,v_\lambda\,\mathrm{d}\Gamma \\
&\quad = \int_\Omega f\,u_\lambda\,\mathrm{d}x + \int_\Gamma f_\Gamma\,v_\lambda\,\mathrm{d}\Gamma.
\end{aligned}$$

Taking again the convexity of $F_{1,\lambda}$ and $G_{1,\lambda}$ into account similarly to above and the fact that $F_{1,\lambda}(0) = G_{1,\lambda}(0) = 0$, an application of Young's inequality yields

$$\|(u_\lambda, v_\lambda)\|_{K,\alpha} + \|(u_\lambda, v_\lambda)\|_{\mathcal{L}^2} + \|(F_{1,\lambda}(u_\lambda), G_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^1} \le \|(f, f_\Gamma)\|_{\mathcal{L}^2}. \tag{4.17}$$

In the next step, we aim to show that (u_λ, v_λ) belongs to $\mathcal{H}^2$ and satisfies the estimate

$$\|(u_\lambda, v_\lambda)\|_{\mathcal{H}^2} + \|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^2} \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big) \tag{4.18}$$

with a constant $C > 0$ independent of $\lambda > 0$. To this end, the letter C will denote in the following generic positive constants that are independent of λ. To establish the estimate (4.18), we have to make a case distinction and treat the cases $K = 0$ and $K \in (0, \infty)$ separately.

The case $K \in (0, \infty)$. In this case, we may use $(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda)) \in \mathcal{H}^1$ as a test function in (4.16) and obtain

$$\begin{aligned}
&\|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|^2_{\mathcal{L}^2} + \int_\Omega F'_{1,\lambda}(u_\lambda)\, u_\lambda \,\mathrm{d}x + \int_\Gamma G'_{1,\lambda}(v_\lambda)\, v_\lambda \,\mathrm{d}\Gamma \\
&\quad + \int_\Omega F''_{1,\lambda}(u_\lambda)|\nabla u_\lambda|^2 \,\mathrm{d}x + \int_\Gamma G''_{1,\lambda}(v_\lambda)|\nabla_\Gamma v_\lambda|^2 \,\mathrm{d}\Gamma \\
&\quad + \frac{1}{K}\int_\Gamma (\alpha v_\lambda - u_\lambda)(\alpha G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma \\
&= \int_\Omega f\, F'_{1,\lambda}(u_\lambda) \,\mathrm{d}x + \int_\Gamma f_\Gamma\, G'_{1,\lambda}(v_\lambda) \,\mathrm{d}\Gamma.
\end{aligned}$$

Employing again the convexity of $F_{1,\lambda}$ and $G_{1,\lambda}$, we can argue similarly as for (4.17) to infer

$$\begin{aligned}
&\|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|^2_{\mathcal{L}^2} \\
&\quad \le \|(f, f_\Gamma)\|^2_{\mathcal{L}^2} - \frac{2}{K}\int_\Gamma (\alpha v_\lambda - u_\lambda)(\alpha G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma.
\end{aligned} \tag{4.19}$$

To estimate the last term on the right-hand side of (4.19), we use the monotonicity of $F'_{1,\lambda}$ as well as the domination property (4.9) to deduce that

$$\begin{aligned}
&-\frac{2}{K}\int_\Gamma (\alpha v_\lambda - u_\lambda)(\alpha G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma \\
&\quad = -\frac{2}{K}\int_\Gamma (\alpha v_\lambda - u_\lambda)(F'_{1,\lambda}(\alpha v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma \\
&\qquad - \frac{2}{K}\int_\Gamma (\alpha v_\lambda - u_\lambda)(\alpha G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma \\
&\quad \le -\frac{2}{K}\int_\Gamma (\alpha v_\lambda - u_\lambda)(\alpha G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma \\
&\quad \le \frac{1}{2}\|G'_{1,\lambda}(v_\lambda)\|^2_{L^2(\Gamma)} + C\|(u_\lambda, v_\lambda)\|^2_{\mathcal{H}^1}.
\end{aligned}$$

Thus, recalling (4.17), we readily infer from (4.19) that

$$\|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^2} \le C\|(f, f_\Gamma)\|_{\mathcal{L}^2}.$$

Finally, regularity theory for elliptic systems with bulk-surface coupling as presented in Theorem 4.2.4 implies that

$$\begin{aligned}
\|(u_\lambda, v_\lambda)\|_{\mathcal{H}^2} &\le C\Big(1 + \|(f - u_\lambda - F'_{1,\lambda}(u_\lambda), f_\Gamma - v_\lambda - G'_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^2}\Big) \\
&\le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big).
\end{aligned}$$

The case $K = 0$. This case requires a different argument, since we cannot use $(\zeta, \zeta_\Gamma) = (F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))$ as a test function in (4.16) as it does not necessarily satisfy the required trace relation. Nonetheless, applying directly regularity theory for elliptic systems with bulk-surface coupling as presented in Section 4.2, we obtain that $(u_\lambda, v_\lambda) \in \mathcal{H}^2$ and therefore, the equations in (4.16) hold almost everywhere on their respective domains. This enables us to multiply (4.13a) with $F'_{1,\lambda}(u_\lambda)$ and (4.13b) with $G'_{1,\lambda}(v_\lambda)$. We then integrate the equations over Ω and Γ, respectively, and use integration by parts. Adding the resultants leads to

$$\begin{aligned}
&\|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^2} + \int_\Omega F'_{1,\lambda}(u_\lambda)\, u_\lambda \,\mathrm{d}x + \int_\Gamma G'_{1,\lambda}(v_\lambda)\, v_\lambda \,\mathrm{d}\Gamma \\
&\quad + \int_\Omega F''_{1,\lambda}(u_\lambda)|\nabla u_\lambda|^2 \,\mathrm{d}x + \int_\Gamma G''_{1,\lambda}(v_\lambda)|\nabla_\Gamma v_\lambda|^2 \,\mathrm{d}\Gamma \\
&\quad + \int_\Gamma \partial_\mathbf{n} u_\lambda(\alpha G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma \\
&= \int_\Omega f\, F'_{1,\lambda}(u_\lambda) \,\mathrm{d}x + \int_\Gamma f_\Gamma\, G'_{1,\lambda}(v_\lambda) \,\mathrm{d}\Gamma.
\end{aligned} \tag{4.20}$$

Using the trace relation $u_\lambda = \alpha v_\lambda$ a.e. on Γ in combination with the domination property (4.9), we obtain

$$\begin{aligned}
\int_\Gamma \partial_\mathbf{n} u_\lambda(G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(u_\lambda)) \,\mathrm{d}\Gamma &= \int_\Gamma \partial_\mathbf{n} u_\lambda(G'_{1,\lambda}(v_\lambda) - F'_{1,\lambda}(\alpha v_\lambda)) \,\mathrm{d}\Gamma \\
&\le \frac{1}{2}\|G_{1,\lambda}(v_\lambda)\|_{L^2(\Gamma)} + C\Big(1 + \|\partial_\mathbf{n} u_\lambda\|_{L^2(\Gamma)}\Big).
\end{aligned}$$

For the other terms appearing in (4.20), we can argue as in the case $K \in (0,\infty)$ and get

$$\|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^2} \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2} + \|\partial_\mathbf{n} u_\lambda\|_{L^2(\Gamma)}\Big). \tag{4.21}$$

To handle the normal derivative $\partial_\mathbf{n} u$ appearing on the right-hand side of (4.21), we recall that by the trace theorem [29, Chapter 2, Theorem 2.24], we find $s \in (\frac{3}{2}, 2)$ and a constant $C = C(s) > 0$ such that

$$\|\partial_\mathbf{n} u_\lambda\|_{L^2(\Gamma)} \le C\|u_\lambda\|_{H^s(\Omega)}. \tag{4.22}$$

Thus, using the compact embeddings $H^2(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^2(\Omega)$ in combination with Ehrling's lemma, we deduce that

$$\begin{aligned}
\|u_\lambda\|_{H^2(\Omega)} + \|\partial_\mathbf{n} u_\lambda\|_{L^2(\Gamma)} &\le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2} + \|u_\lambda\|_{H^s(\Omega)}\Big) \\
&\le \frac{1}{2}\|u_\lambda\|_{H^2(\Omega)} + C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big).
\end{aligned} \tag{4.23}$$

Thus, coming back to (4.21), we find that

$$\|(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda))\|_{\mathcal{L}^2} \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big), \tag{4.24}$$

and therefore, we deduce from elliptic regularity theory for systems with bulk-surface coupling that

$$\|(u_\lambda, v_\lambda)\|_{\mathcal{H}^2} \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big). \tag{4.25}$$

Consequently, according to the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma, there exist functions u, v, ξ and ξ_Γ such that

$$\begin{aligned}
(u_\lambda, v_\lambda) \to (u, v) \qquad &\text{weakly in } \mathcal{H}^2, \\
&\text{strongly in } \mathcal{H}^1_{K,\alpha},
\end{aligned}$$

$$(F'_{1,\lambda}(u_\lambda), G'_{1,\lambda}(v_\lambda)) \to (\xi, \xi_\Gamma) \qquad \text{weakly in } \mathcal{L}^2,$$

as $\lambda \to 0$ along a non-relabeled subsequence. These convergences readily entail that (u, v) satisfies

$$\begin{aligned} u - \Delta u + \xi &= f && \text{a.e. in } \Omega, && (4.26a) \\ v - \Delta_\Gamma v + \xi_\Gamma + \alpha \partial_{\mathbf{n}} u &= f_\Gamma && \text{a.e. on } \Gamma, && (4.26b) \\ K \partial_{\mathbf{n}} u &= \alpha v - u && \text{a.e. on } \Gamma. && (4.26c) \end{aligned}$$

To conclude that (u, v) is indeed a solution to (4.12), we have to make sure that

$$\begin{cases} u(x) \in (-1, 1) & \text{for a.e. } x \in \Omega, \qquad \xi = F'_1(u) \quad \text{a.e. in } \Omega, \\ v(z) \in (-1, 1) & \text{for a.e. } z \in \Gamma, \qquad \xi_\Gamma = G'_1(v) \quad \text{a.e. on } \Gamma \end{cases} \tag{4.27}$$

holds true. To this end, we first notice that the convergences along with the weak-strong convergence principle imply that

$$\lim_{\lambda \to 0} \int_\Omega F'_{1,\lambda}(u_\lambda)\, u_\lambda \, \mathrm{d}x = \int_\Omega \xi\, u \, \mathrm{d}x, \qquad \lim_{\lambda \to 0} \int_\Gamma G'_{1,\lambda}(v_\lambda)\, v_\lambda \, \mathrm{d}\Gamma = \int_\Gamma \xi_\Gamma\, v \, \mathrm{d}\Gamma.$$

The claim (4.27) then follows directly from Proposition 2.3.4 since F'_1 and G'_1 can be regarded as maximal monotone operators. Finally, by means of weak lower semicontinuity, we conclude from (4.18) and the aforementioned convergences that

$$\|(u, v)\|_{\mathcal{H}^2} + \|(F'_1(u), G'_1(v))\|_{\mathcal{L}^2} \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big).$$

We have thus shown that (u, v) is a solution to (4.12), and because $(f, f_\Gamma) \in \mathcal{L}^2$ was arbitrary, we infer that $\mathcal{A} = \partial \widetilde{E}_{\text{free}}$, which finishes the proof. □

4.3.2 Well-posedness of weak solutions and higher regularity

After having proven the characterization of the subdifferential of the convex part of the free energy, we can turn our attention to the study of the system (4.6). Our first result is concerned with the existence of solutions, which is a direct consequence of the identification of $\partial \widetilde{E}_{\text{free}}$ proven in Proposition 4.3.2.

PROPOSITION 4.3.3. *Let $(f, f_\Gamma) \in \mathcal{L}^2$. Then there exists a solution (u, v) to system (4.10) such that $(u, v) \in \mathcal{H}^2 \cap \mathcal{H}^1_{K,\alpha}$, $(F'_1(u), G'_1(v)) \in \mathcal{L}^2$, and the equations (4.10) are satisfied almost everywhere on their respective domains. Moreover, there exists a constant $C > 0$, depending only on $\Omega, K, \alpha, \Theta_\Omega, \Theta_\Gamma, \kappa_1$ and κ_2 such that*

$$\|(u, v)\|_{\mathcal{H}^2} + \|(F'_1(u), G'_1(v))\|_{\mathcal{L}^2} \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big). \tag{4.28}$$

PROOF. In view of the identification (4.11) of the subdifferential $\partial \widetilde{E}_{\text{free}}$ from Proposition 4.3.2, and recalling (4.7), we readily deduce by partial integration that $\partial \widetilde{E}_{\text{free}}$ is coercive. Namely, it holds that

$$\big(\partial \widetilde{E}_{\text{free}}(u, v) - \partial \widetilde{E}_{\text{free}}(\zeta, \zeta_\Gamma), (u, v) - (\zeta, \zeta_\Gamma)\big)_{\mathcal{L}^2} \ge \min\{\Theta_\Omega, \Theta_\Gamma\} \|(u, v) - (\zeta, \zeta_\Gamma)\|^2_{\mathcal{L}^2}$$

for all $(u, v), (\zeta, \zeta_\Gamma) \in D(\partial \widetilde{E}_{\text{free}})$. In turn, this implies that $\partial \widetilde{E}_{\text{free}}$ is surjective on $\mathcal{L}^2$. Thus, the existence of a solution (u, v) with the desired properties is a direct consequence of the characterization of the subdifferential $\partial \widetilde{E}_{\text{free}}$ in Proposition 4.3.2. Lastly, the estimate (4.28) can be verified by performing similar computations to the one of the proof of Proposition 4.3.2, in particular those in the derivation of (4.18). □

Next, we prove that the solution to (4.6), whose existence is guaranteed by Proposition 4.3.3, is unique and satisfies a stability estimate.

PROPOSITION 4.3.4. *Let $(f_1, f_{\Gamma 1}), (f_2, f_{\Gamma 2}) \in \mathcal{L}^2$, and consider corresponding solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{H}^2 \cap \mathcal{H}^1_{K,\alpha}$ to system (4.6). Then there exists a constant $C > 0$, depending only on Θ_Ω and Θ_Γ, such that*

$$\|(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^1} \leq C \|(f_1 - f_2, f_{\Gamma 1} - f_{\Gamma 2})\|_{\mathcal{L}^2}. \tag{4.29}$$

In particular, this entails that the solution to system (4.6) is unique.

PROOF. For brevity, we define

$$(\tilde{u}, \tilde{v}, \tilde{f}, \tilde{f}_\Gamma) = (u_1 - u_2, v_1 - v_2, f_1 - f_2, f_{\Gamma 1} - f_{\Gamma 2}).$$

Then, $(\tilde{u}, \tilde{v})$ satisfies

$$-\Delta \tilde{u} + F_1'(u_1) - F_1'(u_2) = \tilde{f} \qquad \text{a.e. in } \Omega, \tag{4.30a}$$

$$-\Delta_\Gamma \tilde{v} + G_1'(v_1) - G_1'(v_2) + \alpha \partial_{\mathbf{n}} \tilde{u} = \tilde{f}_\Gamma \qquad \text{a.e. on } \Gamma, \tag{4.30b}$$

$$K \partial_{\mathbf{n}} \tilde{u} = \alpha \tilde{v} - \tilde{u} \qquad \text{a.e. on } \Gamma. \tag{4.30c}$$

Now, testing (4.30a) with $\tilde{u}$ and (4.30b) with $\tilde{v}$, an application of integration by parts in combination with the boundary condition (4.30c) yields

$$\begin{aligned} &\int_\Omega |\nabla \tilde{u}|^2 \,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma \tilde{v}|^2 \,\mathrm{d}\Gamma + \chi(K) \int_\Gamma (\alpha \tilde{v} - \tilde{u})^2 \,\mathrm{d}\Gamma \\ &+ \int_\Omega (F_1'(u_1) - F_1'(u_2))\, \tilde{u} \,\mathrm{d}x + \int_\Gamma (G_1'(v_1) - G_1'(v_2))\, \tilde{v} \,\mathrm{d}\Gamma = \int_\Omega \tilde{f}\, \tilde{u} \,\mathrm{d}x + \int_\Gamma \tilde{f}_\Gamma\, \tilde{v} \,\mathrm{d}\Gamma. \end{aligned}$$

Then, exploiting the convexity assumption on the potentials (4.7) as well as Hölder's and Young's inequality, we obtain

$$\|(\tilde{u}, \tilde{v})\|_{K,\alpha}^2 + \frac{\min\{\Theta_\Omega, \Theta_\Gamma\}}{2} \|(\tilde{u}, \tilde{v})\|_{\mathcal{L}^2}^2 \leq \frac{1}{2 \min\{\Theta_\Omega, \Theta_\Gamma\}} \|(\tilde{f}, \tilde{f}_\Gamma)\|_{\mathcal{L}^2}^2,$$

from which the desired stability estimate (4.29) readily follows. □

Next, we establish some higher regularity results for the unique solution (u, v) to (4.6).

PROPOSITION 4.3.5. *Let $(f, f_\Gamma) \in \mathcal{L}^p$ with $2 \leq p \leq \infty$, and consider the corresponding unique solution $(u, v) \in \mathcal{H}^2 \cap \mathcal{H}^1_{K,\alpha}$ to system (4.6). Then $(u, v) \in \mathcal{W}^{2,p}$ and $(F_1'(u), G_1'(v)) \in \mathcal{L}^p$. Moreover, there exists a constant $C_p > 0$ such that*

$$\|(u, v)\|_{\mathcal{W}^{2,p}} + \|(F_1'(u), G_1'(v))\|_{\mathcal{L}^p} \leq C_p \Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^p}\Big). \tag{4.31}$$

Additionally, if $(f, f_\Gamma) \in \mathcal{L}^\infty$, there exists $\delta \in (0, 1]$, depending on $\|(f, f_\Gamma)\|_{\mathcal{L}^\infty}$, such that

$$|u(x)| \leq 1 - \delta \quad \text{for all } x \in \overline{\Omega}, \qquad |v(z)| \leq 1 - \delta \quad \text{for all } z \in \Gamma. \tag{4.32}$$

Remark 4.3.6. Under the assumptions of Proposition 4.3.5 one can additionally show that

$$\|F_1'(v)\|_{L^p(\Gamma)} \leq C_p (1 + \|(f, f_\Gamma)\|_{\mathcal{L}^p}). \tag{4.33}$$

This can be done using a different testing procedure than the one we use in the proof of Proposition 4.3.5. We refer to [70, 132], where these computations were carried out in detail directly on a solution to a Cahn–Hilliard model with dynamic boundary conditions.

Proof. In this proof, we assume without loss of generality that $\alpha \neq 0$. The case $\alpha = 0$ can be handled in a similar manner and the computations are even easier. Moreover, we can assume that $p \in (2, \infty]$, as the case $p = 2$ is already covered by Proposition 4.3.3. Now, for $k \in \mathbb{N}$, we introduce the truncation

$$h_k : \mathbb{R} \to \mathbb{R}, \quad h_k(s) := \begin{cases} -1 + \frac{1}{k} & \text{if } s < -1 + \frac{1}{k}, \\ s & \text{if } s \in [-1 + \frac{1}{k}, 1 - \frac{1}{k}], \\ 1 - \frac{1}{k} & \text{if } s > 1 - \frac{1}{k}, \end{cases}$$

which is globally Lipschitz continuous. Then, we define $u_k := \alpha h_k \circ (\alpha^{-1} u)$ and $v_k := h_k \circ v$. Since h_k is globally Lipschitz continuous and $(u, v) \in \mathcal{H}^1$, the chain rule for the combination of Lipschitz and Sobolev functions (see, e.g., [183, Corollary 3.2]) implies that $(u_k, v_k) \in \mathcal{H}^1$ with

$$\nabla u_k = \nabla u \, \mathbf{1}_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(u), \qquad \nabla_\Gamma v_k = \nabla_\Gamma v \, \mathbf{1}_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(v)$$

for any $k \in \mathbb{N}$, where $\mathbf{1}_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}$ denotes the indicator function of the set $[-1 + \frac{1}{k}, 1 - \frac{1}{k}]$, defined as

$$\mathbf{1}_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(s) := \begin{cases} 0 & \text{if } s < -1 + \frac{1}{k}, \\ 1 & \text{if } s \in [-1 + \frac{1}{k}, 1 - \frac{1}{k}], \\ 0 & \text{if } s > 1 - \frac{1}{k}, \end{cases}$$

for any $k \in \mathbb{N}$ and $s \in [-1, 1]$. Moreover, for any $k \in \mathbb{N}$ and $p \in (2, \infty)$, we see that $|F_1'(u_k)|^{p-2} F_1'(u_k) \in H^1(\Omega)$ and $|G_1'(v_k)|^{p-2} G_1'(v_k) \in H^1(\Gamma)$ are well-defined with

$$\begin{aligned} \nabla \big(|F_1'(u_k)|^{p-2} F_1'(u_k) \big) &= (p-1) |F_1'(u_k)|^{p-2} F_1''(u_k) \nabla u_k, \\ \nabla_\Gamma \big(|G_1'(v_k)|^{p-2} G_1'(v_k) \big) &= (p-1) |G_1'(v_k)|^{p-2} G_1''(v_k) \nabla_\Gamma v_k. \end{aligned}$$

The proof is divided into several steps, as we have to handle certain cases of $p \in (0, \infty]$ and $K \in [0, \infty)$ differently. In the following, we denote with the letter C always non-negative constants independent of k, whose values may change from line to line.

The case $p \in (2, \infty)$ and $K \in (0, \infty)$. We start by multiplying (4.6a) with $|F_1'(u_k)|^{p-2} F_1'(u_k)$ and (4.6b) with $|G_1'(v_k)|^{p-2} G_1'(v_k)$. Integrating the resulting equations over Ω and Γ, respectively, we apply integration by parts. Adding the resultants leads to

$$\begin{aligned} &\int_\Omega F_1'(u) F_1'(u_k) |F_1'(u_k)|^{p-2} \, \mathrm{d}x + \int_\Gamma G_1'(v) G_1'(v_k) |G_1'(v_k)|^{p-2} \, \mathrm{d}\Gamma \\ &\quad = -(p-1) \int_\Omega \nabla u \cdot \nabla u_k |F_1'(u_k)|^{p-2} F_1''(u_k) \, \mathrm{d}x \\ &\qquad - (p-1) \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma v_k |G_1'(v_k)|^{p-2} G_1''(v_k) \, \mathrm{d}\Gamma \\ &\qquad + \int_\Omega f \, F_1'(u_k) |F_1'(u_k)|^{p-2} \, \mathrm{d}x + \int_\Gamma f_\Gamma \, G_1'(v_k) |G_1'(v_k)|^{p-2} \, \mathrm{d}\Gamma \\ &\qquad - \frac{1}{K} \int_\Gamma (\alpha v - u) \Big(\alpha G_1'(v_k) |G_1'(v_k)|^{p-2} - F_1'(u_k) |F_1'(u_k)|^{p-2} \Big) \, \mathrm{d}\Gamma. \end{aligned} \tag{4.34}$$

We first notice that in view of the assumption **(S)**, the first two terms on the right-hand side of (4.34) are non-negative. Then, for the next two terms, a simple application of

Hölder's and Young's inequality yields

$$\begin{aligned}
&\int_\Omega f\, F_1'(u_k)|F_1'(u_k)|^{p-2}\,\mathrm{d}x + \int_\Gamma f_\Gamma\, G_1'(v_k)|G_1'(v_k)|^{p-2}\,\mathrm{d}\Gamma \\
&\quad \le \|f\|_{L^p(\Omega)}\|F_1'(u_k)\|_{L^p(\Omega)}^{p-1} + \|f_\Gamma\|_{L^p(\Gamma)}\|G_1'(v_k)\|_{L^p(\Gamma)}^{p-1}.
\end{aligned} \tag{4.35}$$

Next, noting again on assumption **(S)**, we see that $F_1'(s)$ and $F_1'(\alpha h_k(\alpha^{-1}s))$ as well as $G_1'(s)$ and $G_1'(h_k(s))$ have the same sign for all $s \in (-1,1)$. This readily implies that

$$\begin{aligned}
F_1'(u_k)^2 &\le F_1'(u)F_1'(u_k) \qquad \text{a.e. in } \Omega,\\
G_1'(v_k)^2 &\le G_1'(v)G_1'(v_k) \qquad \text{a.e. on } \Gamma.
\end{aligned}$$

As a consequence, we find that

$$\begin{aligned}
&\int_\Omega |F_1'(u_k)|^p\,\mathrm{d}x + \int_\Gamma |G_1'(v_k)|^p\,\mathrm{d}\Gamma \\
&\quad \le \int_\Omega |F_1'(u_k)|^{p-2}F_1'(u_k)F_1'(u)\,\mathrm{d}x + \int_\Gamma |G_1'(v_k)|^{p-2}G_1'(v_k)G_1'(v)\,\mathrm{d}\Gamma.
\end{aligned} \tag{4.36}$$

For the last term on the right-hand side of (4.34), we first notice that the function $\mathbb{R} \ni s \mapsto |F_1'(\alpha h_k(s))|^{p-2}F_1'(\alpha h_k(s))$ is monotonically increasing for $\alpha > 0$ and monotonically decreasing for $\alpha < 0$. Thus, recalling the definition of u_k and exploiting the domination property (4.8), we deduce

$$\begin{aligned}
&-\frac{1}{K}\int_\Gamma (\alpha v - u)\Big(\alpha|G_1'(v_k)|^{p-2}G_1'(v_k) - |F_1'(u_k)|^{p-2}F_1'(u_k)\Big)\,\mathrm{d}\Gamma \\
&\quad = -\frac{\alpha}{K}\int_\Gamma (v - \alpha^{-1}u)\Big(|F_1'(\alpha v_k)|^{p-2}F_1'(\alpha v_k) - |F_1'(u_k)|^{p-2}F_1'(u_k)\Big)\,\mathrm{d}\Gamma \\
&\qquad - \frac{1}{K}\int_\Gamma (\alpha v - u)\Big(\alpha|G_1'(v_k)|^{p-2}G_1'(v_k) - |F_1'(\alpha v_k)|^{p-2}F_1'(\alpha v_k)\Big)\,\mathrm{d}\Gamma \\
&\quad \le \frac{2}{K}\bigg(\int_\Gamma |G_1'(v_k)|^{p-1}\,\mathrm{d}\Gamma + 2^{p-2}\kappa_1^{p-2}\int_\Gamma |G_1'(v_k)|^{p-1}\,\mathrm{d}\Gamma + 2^{p-2}\kappa_2^{p-1}|\Gamma|\bigg).
\end{aligned} \tag{4.37}$$

Here, we additionally used Minkowski's inequality

$$|a+b|^{p-1} \le 2^{p-2}\Big(|a|^{p-1} + |b|^{p-1}\Big) \qquad \text{for any } a, b \in \mathbb{R}.$$

Hence, by collecting the estimates (4.35)-(4.37), we infer from (4.34) that

$$\begin{aligned}
&\|F_1'(u_k)\|_{L^p(\Omega)}^p + \|G_1'(v_k)\|_{L^p(\Gamma)}^p \\
&\quad \le \|f\|_{L^p(\Omega)}\|F_1'(u_k)\|_{L^p(\Omega)}^{p-1} + \Big(\|f_\Gamma\|_{L^p(\Gamma)} + K^{-1}(2 + 2^{p-1}\kappa_1^{p-1})\Big)\|G_1'(v_k)\|_{L^p(\Gamma)}^{p-1} \\
&\qquad + K^{-1}2^{p-1}\kappa_2^{p-1}|\Gamma| \\
&\quad \le \frac{1}{2}\|F_1'(u_k)\|_{L^p(\Omega)}^p + \frac{1}{2}\|G_1'(v_k)\|_{L^p(\Gamma)}^p + \frac{1}{p}\left(\frac{2(p-1)}{p}\right)^{p-1}\|f\|_{L^p(\Omega)}^p \\
&\qquad + \frac{1}{p}\left(\frac{2(p-1)}{p}\right)^{p-1}\Big(\|f_\Gamma\|_{L^p(\Gamma)} + K^{-1}(2 + 2^{p-1}\kappa_1^{p-1})\Big)^p + K^{-1}2^{p-1}\kappa_2^{p-1}|\Gamma|.
\end{aligned}$$

Thus, absorbing the respective terms and taking the p-th rot, we find

$$\begin{aligned} &\|F_1'(u_k)\|_{L^p(\Omega)} + \|G_1'(v_k)\|_{L^p(\Gamma)} \\ &\quad \le 2^{\frac{p+1}{p}} \Bigg[\frac{1}{p^{\frac{1}{p}}} \Big(\frac{2(p-1)}{p} \Big)^{\frac{p-1}{p}} \Big(\|(f, f_\Gamma)\|_{\mathcal{L}^p} + K^{-1}(2 + 2^{p-1}\kappa_1^{p-1}) \Big) \\ &\qquad + K^{-\frac{1}{p}} 2^{\frac{p-1}{p}} \kappa_2^{\frac{p-1}{p}} |\Gamma|^{\frac{1}{p}} \Bigg] \\ &\quad \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^p}\Big) \end{aligned} \tag{4.38}$$

with a constant $C = C(p) > 0$. Now, since

$$F_1'(u_k) \to F_1'(u) \qquad \text{a.e. in } \Omega, \qquad G_1'(v_k) \to G_1'(v) \qquad \text{a.e. on } \Gamma \tag{4.39}$$

as $k \to \infty$, we apply Fatou's lemma to derive the estimate

$$\begin{aligned} &\int_\Omega |F_1'(u)|^p \,\mathrm{d}x + \int_\Gamma |G_1'(v)|^p \,\mathrm{d}\Gamma \\ &\quad = \int_\Omega \liminf_{k\to\infty} |F_1'(u_k)|^p \,\mathrm{d}x + \int_\Gamma \liminf_{k\to\infty} |G_1'(v_k)|^p \,\mathrm{d}\Gamma \\ &\quad \le \liminf_{k\to\infty} \int_\Omega |F_1'(u_k)|^p \,\mathrm{d}x + \liminf_{k\to\infty} \int_\Gamma |G_1'(v_k)|^p \,\mathrm{d}\Gamma \\ &\quad \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^p}^p\Big), \end{aligned}$$

which concludes the proof in this case after another application of elliptic regularity theory for systems with bulk-surface coupling as presented in Section 4.2.

The case $p \in (2, \infty)$ and $K = 0$. We start by considering the case $p \in (2, 4]$. By using the same testing procedure as in the case $K \in (0, \infty)$ and exploiting the convexity assumption (4.7) on the potentials F_1 and G_1, we arrive at

$$\begin{aligned} &\int_\Omega |F_1'(u_k)|^p \,\mathrm{d}x + \int_\Gamma |G_1'(v_k)|^p \,\mathrm{d}\Gamma \\ &\quad \le \int_\Omega f\, |F_1'(u_k)|^{p-2} F_1'(u_k) \,\mathrm{d}x + \int_\Gamma f_\Gamma\, |G_1'(v_k)|^{p-2} G_1'(v_k) \,\mathrm{d}\Gamma \\ &\qquad - \int_\Gamma \partial_\mathbf{n} u \Big(\alpha |G_1'(v_k)|^{p-2} G_1'(v_k) - |F_1'(u_k)|^{p-2} F_1'(u_k) \Big) \,\mathrm{d}\Gamma. \end{aligned} \tag{4.40}$$

We note that this inequality differs from the one for $K \in (0, \infty)$ only in the last term on the right-hand side, which now contains the normal derivative $\partial_\mathbf{n} u$, and thus, has to be treated differently. First, due to the regularity $u \in H^2(\Omega)$, the trace embedding $H^1(\Omega) \hookrightarrow L^4(\Gamma)$ implies that $\partial_\mathbf{n} u \in L^4(\Gamma)$. In particular, it holds that

$$\|\partial_\mathbf{n} u\|_{L^4(\Gamma)} \le C\|u\|_{H^2(\Omega)} \le C\|(f, f_\Gamma)\|_{\mathcal{L}^2} \tag{4.41}$$

in view (4.28). Then, utilizing the trace relation $u_k = \alpha v_k$ a.e. on Γ in combination with the domination property (4.8), we bound the last term on the right-hand side of (4.40) as

$$\begin{aligned} &- \int_\Gamma \partial_\mathbf{n} u \Big(\alpha |G_1'(v_k)|^{p-2} G_1'(v_k) - |F_1'(u_k)|^{p-2} F_1'(u_k) \Big) \,\mathrm{d}\Gamma \\ &\quad = - \int_\Gamma \partial_\mathbf{n} u \Big(\alpha |G_1'(v_k)|^{p-2} G_1'(v_k) - |F_1'(\alpha v_k)|^{p-2} F_1'(\alpha v_k) \Big) \,\mathrm{d}\Gamma \end{aligned}$$

$$\leq \int_\Gamma |\partial_{\mathbf{n}} u| \Big((1 + 2^{p-2}\kappa_1^{p-1}) |G_1'(v_k)|^{p-1} + 2^{p-2}\kappa_2^{p-1} \Big) \, \mathrm{d}\Gamma.$$

Consequently, we infer

$$\begin{aligned}
&\|F_1'(u_k)\|_{L^p(\Omega)}^p + \|G_1'(v_k)\|_{L^p(\Gamma)}^p \\
&\quad \leq \|f\|_{L^p(\Omega)} \|F_1'(u_k)\|_{L^p(\Omega)}^{p-1} + \Big(\|f_\Gamma\|_{L^p(\Gamma)} + (1 + 2^{p-2}\kappa_1^{p-1}) \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)} \Big) \|G_1'(v_k)\|_{L^p(\Gamma)}^{p-1} \\
&\qquad + 2^{p-2}\kappa_2^{p-1} |\Gamma|^{\frac{p-1}{p}} \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)} \\
&\quad \leq \frac{1}{2} \|F_1'(u_k)\|_{L^p(\Omega)}^p + \frac{1}{2} \|G_1'(v_k)\|_{L^p(\Gamma)}^p + \frac{1}{p} \Big(\frac{2(p-1)}{p} \Big)^{p-1} \|f\|_{L^p(\Omega)}^p \\
&\qquad + \frac{1}{p} \Big(\frac{2(p-1)}{p} \Big)^{p-1} \Big(\|f_\Gamma\|_{L^p(\Gamma)} + (1 + 2^{p-2}\kappa_1^{p-1}) \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)} \Big)^p \\
&\qquad + 2^{p-2}\kappa_2^{p-1} |\Gamma|^{\frac{p-1}{p}} \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)}.
\end{aligned}$$

Thus, absorbing the respective terms and taking the p-th root, we deduce with the help of (4.41) that

$$\begin{aligned}
&\|F_1'(u_k)\|_{L^p(\Omega)} + \|G_1'(v_k)\|_{L^p(\Gamma)} \\
&\quad \leq 2^{\frac{p+1}{p}} \bigg[\frac{1}{p^{\frac{1}{p}}} \Big(\frac{2(p-1)}{p} \Big)^{\frac{p-1}{p}} \big(\|(f, f_\Gamma)\|_{\mathcal{L}^p} + (1 + 2^{p-2}\kappa_1^{p-1}) \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)} \big) \\
&\qquad + 2^{\frac{p-2}{p}} \kappa_2^{\frac{p-1}{p}} |\Gamma|^{\frac{p-1}{p^2}} \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)}^{\frac{1}{p}} \bigg] \\
&\quad \leq C \Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^p} \Big)
\end{aligned} \tag{4.42}$$

with a constant $C = C(p) > 0$. The desired claim follows again by Fatou's lemma and elliptic regularity theory. Consider now $p \in (4, \infty)$. In particular, it holds that $(f, f_\Gamma) \in \mathcal{L}^4$. By our previous computations, this implies $(u, v) \in \mathcal{W}^{2,4}$ with

$$\|(u, v)\|_{\mathcal{W}^{2,4}} \leq C \|(f, f_\Gamma)\|_{\mathcal{L}^4}.$$

Consequently, as we have with the trace theorem and the Sobolev inequality the embeddings $W^{1,4}(\Omega) \hookrightarrow W^{\frac{3}{4},4}(\Gamma) \hookrightarrow L^\infty(\Gamma)$, we find

$$\|\partial_{\mathbf{n}} u\|_{L^\infty(\Gamma)} \leq C \|u\|_{W^{2,4}(\Omega)} \leq C \|(f, f_\Gamma)\|_{\mathcal{L}^4} \leq C \|(f, f_\Gamma)\|_{\mathcal{L}^p}.$$

This allows us to take $p \in (4, \infty)$ in our previous calculations, which finishes the proof in this case after applying again elliptic regularity theory.

We still have to prove the claim for $p = \infty$. To this end, consider $(f, f_\Gamma) \in \mathcal{L}^\infty$. We cannot directly use the estimates (4.38) and (4.42) shown in the previous cases as the constant C therein is not independent of p. In fact, it is unbounded as $p \to \infty$. Nonetheless, the computation are similar, but we have to show the corresponding estimates for $G_1'(v)$ and $F_1'(u)$ separately. To be precise, we first show the estimate for $G_1'(v)$, which will then be a key ingredient in the proof of the estimate for $F_1'(u)$.

To this end, we first notice that our previous calculations show that $(u, v) \in \mathcal{W}^{2,r}$ and $(F_1'(u), G_1'(v)) \in \mathcal{L}^r$ for any $r \in (2, \infty)$. As seen above, this implies in particular that $\partial_{\mathbf{n}} u \in L^\infty(\Gamma)$ together with the estimate

$$\|\partial_{\mathbf{n}} u\|_{L^\infty(\Gamma)} \leq C_E \|(f, f_\Gamma)\|_{\mathcal{L}^\infty}. \tag{4.43}$$

Note that C_E does not depend on $r \in (2,\infty)$.

We start by showing the corresponding estimate for $G_1'(v_k)$. We fix an arbitrary $r \in (2,\infty)$ and multiply equation (4.6b) again with $|G_1'(v_k)|^{r-2}G_1'(v_k)$. Integrating over Γ, we proceed similarly as in the case $p \in (2,\infty)$ above. In particular, we obtain

$$\|G_1'(v_k)\|_{L^r(\Gamma)}^r \le \Big(\|f_\Gamma\|_{L^r(\Gamma)} + \|\partial_{\mathbf{n}} u\|_{L^r(\Gamma)}\Big)\|G_1'(v_k)\|_{L^r(\Gamma)}^{r-1}.$$

Recalling (4.43), this directly implies

$$\begin{aligned}\|G_1'(v_k)\|_{L^r(\Gamma)} &\le (1+|\Gamma|)\Big(\|f_\Gamma\|_{L^\infty(\Gamma)} + \|\partial_{\mathbf{n}} u\|_{L^\infty(\Gamma)}\Big)\\ &\le C_{E,*}(1+|\Gamma|)\|(f,f_\Gamma)\|_{\mathcal{L}^\infty},\end{aligned}$$

where $C_{E,*} = \max\{1, C_E\}$. As the right-hand side is independent of r, we may consider the limit $r \to \infty$ to deduce

$$\|G_1'(v_k)\|_{L^\infty(\Gamma)} \le C_{E,*}(1+|\Gamma|)\|(f,f_\Gamma)\|_{\mathcal{L}^\infty}.$$

Finally, in view (4.39), letting $k \to \infty$ yields

$$\|G_1'(v)\|_{L^\infty(\Gamma)} \le C_{E,*}(1+|\Gamma|)\|(f,f_\Gamma)\|_{\mathcal{L}^\infty}. \tag{4.44}$$

Now, for the corresponding estimate of $F_1'(u)$, we multiply (4.6a) with $|F_1'(u_k)|^{r-2}F_1'(u_k)$, integrate over Ω and integrate by parts to find

$$\int_\Omega |F_1'(u_k)|^r \,\mathrm{d}x \le \int_\Omega f\,|F_1'(u_k)|^{r-2}F_1'(u_k)\,\mathrm{d}x + \int_\Gamma \partial_{\mathbf{n}} u\,|F_1'(u_k)|^{r-2}F_1'(u_k)\,\mathrm{d}\Gamma. \tag{4.45}$$

The rest of the proof is again divided into the two parts $K \in (0,\infty)$ and $K = 0$.

The case $p = \infty$ and $K \in (0,\infty)$. In this case, we can use the boundary condition (4.6c) to rewrite the normal derivative $\partial_{\mathbf{n}} u$. Thus, using the same monotonicity argument as in the case $p \in (2,\infty)$ and domination property, we have

$$\begin{aligned}&\int_\Gamma \partial_{\mathbf{n}} u\,|F_1'(u_k)|^{r-2}F_1'(u_k)\,\mathrm{d}\Gamma\\ &\quad= \frac{1}{K}\int_\Gamma (\alpha v - u)|F_1'(u_k)|^{r-2}F_1'(u_k)\,\mathrm{d}\Gamma\\ &\quad= \frac{1}{K}\int_\Gamma (\alpha v - u)|F_1'(\alpha v_k)|^{r-2}F_1'(\alpha v_k)\,\mathrm{d}\Gamma\\ &\qquad - \frac{1}{K}\int_\Gamma (\alpha v - u)\Big(|F_1'(\alpha v_k)|^{r-2}F_1'(\alpha v_k) - |F_1'(u_k)|^{r-2}F_1'(u_k)\Big)\,\mathrm{d}\Gamma\\ &\quad\le \frac{2}{K}\Big(2^{r-2}\kappa_1^{r-1}\|G_1'(u_k)\|_{L^r(\Gamma)}^{r-1} + 2^{r-2}\kappa_2^{r-1}|\Gamma|\Big)\\ &\quad\le \frac{2}{K}\Big(2^{r-2}\kappa_1^{r-1}|\Gamma|^{\frac{r-1}{r}}(1+|\Gamma|)^{r-1}C_{E,*}^{r-1}\|(f,f_\Gamma)\|_{\mathcal{L}^\infty}^{r-1} + 2^{r-2}\kappa_2^{r-1}|\Gamma|\Big).\end{aligned}$$

Thus, we readily infer

$$\begin{aligned}\|F_1'(u_k)\|_{L^r(\Omega)}^r &\le \frac{1}{2}\|F_1'(u_k)\|_{L^r(\Omega)}^r + \frac{1}{r}\left(\frac{2(r-1)}{r}\right)^{r-1}\|f\|_{L^r(\Omega)}^r\\ &\qquad + \frac{2}{K}\Big(2^{r-2}\kappa_1^{r-1}|\Gamma|^{\frac{r-1}{r}}(1+|\Gamma|)^{r-1}C_{E,*}^{r-1}\|(f,f_\Gamma)\|_{\mathcal{L}^\infty}^{r-1} + 2^{r-2}\kappa_2^{r-1}|\Gamma|\Big).\end{aligned}$$

After another absorption argument, we take the r-th root and get

$$\|F_1'(u_k)\|_{L^r(\Omega)} \le 2^{\frac{1}{r}}\left[\frac{1}{r^{\frac{1}{r}}}\left(\frac{2(r-1)}{r}\right)^{\frac{r-1}{r}}(1+|\Omega|)\|f\|_{L^\infty(\Omega)}\right.$$

$$
\begin{aligned}
&\quad + \frac{2}{K}\Big(2^{\frac{r-2}{r}} \kappa_1^{\frac{r-1}{r}} |\Gamma|^{\frac{r-1}{r^2}} (1+|\Gamma|)^{\frac{r-1}{r}} C_{E,*}^{\frac{r-1}{r}} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}^{\frac{r-1}{r}} + 2^{\frac{r-2}{r}} \kappa_2^{\frac{r-1}{r}} |\Gamma|\Big)\Big] \\
&\leq C\Big(1 + \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}\Big)
\end{aligned}
$$

with a constant $C > 0$ independent of r. As done above for G_1', we let first $r \to \infty$ and then $k \to \infty$ to deduce

$$
\|F_1'(u)\|_{L^\infty(\Omega)} \leq C\Big(1 + \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}\Big).
$$

An application of elliptic regularity theory finishes the proof in this case.

The case $p = \infty$ and $K = 0$. In this case, the trace relation $\alpha v_k = u_k$ a.e. on Γ allows us to directly rewrite the second term on the right-hand side. Then, using again the domination property, we obtain as before

$$
\begin{aligned}
&\int_\Gamma \partial_{\mathbf{n}} u\, |F_1'(u_k)|^{r-2} F_1'(u_k)\,\mathrm{d}\Gamma \\
&\quad = \int_\Gamma \partial_{\mathbf{n}} u\, |F_1'(\alpha v_k)|^{r-2} F_1'(\alpha v_k)\,\mathrm{d}\Gamma \\
&\quad \leq \int_\Gamma |\partial_{\mathbf{n}} u| \Big(2^{r-2}\kappa_1^{r-1} |G_1'(v_k)|^{r-1} + 2^{r-1}\kappa_2^{r-1}\Big)\,\mathrm{d}\Gamma \\
&\quad \leq |\Gamma| \|\partial_{\mathbf{n}} u\|_{L^\infty(\Gamma)} \Big(2^{r-2}\kappa_1^{r-1} \|G_1'(v_k)\|_{L^\infty(\Gamma)}^{r-1} + 2^{r-2}\kappa_2^{r-1}\Big) \\
&\quad \leq |\Gamma| \Big(2^{r-2}\kappa_1^{r-1} C_{E,*}^r (1+|\Gamma|)^{r-1} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}^r + 2^{r-1}\kappa_2^{r-1} C_{E,*} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}\Big).
\end{aligned}
$$

Therefore, using again Hölder's and Young's inequalities for the first term on the right-hand side of (4.45), we obtain

$$
\begin{aligned}
\|F_1'(u_k)\|_{L^r(\Omega)}^r &\leq \frac{1}{2}\|F_1'(u_k)\|_{L^r(\Omega)}^r + \frac{1}{r}\Big(\frac{2(r-1)}{r}\Big)^{r-1} \|f\|_{L^r(\Omega)}^r \\
&\quad + |\Gamma| \Big(2^{r-2}\kappa_1^{r-1} C_{E,*}^r (1+|\Gamma|)^{r-1} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}^r \\
&\quad + 2^{r-1}\kappa_2^{r-1} C_{E,*} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}\Big).
\end{aligned}
$$

Thus, absorbing the corresponding term and taking the r-th root, we end up with

$$
\begin{aligned}
\|F_1'(u_k)\|_{L^r(\Omega)} &\leq 2^{\frac{1}{r}} \Bigg[\frac{1}{r^{\frac{1}{r}}}\Big(\frac{2(r-1)}{r}\Big)^{\frac{r-1}{r}} \|f\|_{L^r(\Omega)} \\
&\quad + |\Gamma|^{\frac{1}{r}} \Big(2^{\frac{r-2}{r}} \kappa_1^{\frac{r-1}{r}} C_{E,*} (1+|\Gamma|)^{\frac{r-1}{r}} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty} \\
&\quad + 2^{\frac{r-1}{r}} \kappa_2^{\frac{r-1}{r}} C_{E,*}^{\frac{1}{r}} \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}^{\frac{1}{r}}\Big)\Bigg] \\
&\leq C\Big(1 + \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}\Big)
\end{aligned}
$$

where the constant $C > 0$ is independent of r. Arguing as above readily yields

$$
\|F_1'(u)\|_{L^\infty(\Omega)} \leq C\Big(1 + \|(f,f_\Gamma)\|_{\mathcal{L}^\infty}\Big).
$$

Finally, elliptic regularity provides the desired estimate on (u,v).

In summary, this means that the estimate (4.31) has been established for all cases $p \in [2,\infty]$ and $K \in [0,\infty)$. Finally, since $u \in H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ and $v \in H^2(\Gamma) \hookrightarrow C(\Gamma)$, the separation property (4.32) follows from assumption **(S)** combined with the regularity $(F_1'(u), G_1'(v)) \in \mathcal{L}^\infty$. The proof is complete. $\square$

COROLLARY 4.3.7. *Let* $(f, f_\Gamma) \in \mathcal{H}^1$*, and let* $(u, v) \in \mathcal{H}^2 \cap \mathcal{H}^1_{K,\alpha}$ *be the corresponding unique solution to system* (4.6)*. Then, there exists a constant* $C > 0$*, depending on* Ω, K, α, F_1 *and* G_1 *such that the following inequalities holds:*

(i) *If* $K = 0$*, then*

$$
\begin{aligned}
&\|(-\Delta u, -\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u)\|^2_{\mathcal{L}^2} \\
&\quad \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{H}^1}\Big)\|\nabla u\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)}\|\nabla u\|^{\frac{1}{2}}_{\mathbf{H}^1(\Omega)} \\
&\qquad + \|(\nabla f, \nabla_\Gamma f_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla u, \nabla_\Gamma v)\|_{\boldsymbol{\mathcal{L}}^2}.
\end{aligned}
$$

(ii) *If* $K \in (0, \infty)$*, then*

$$
\begin{aligned}
&\|(-\Delta u, -\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u)\|^2_{\mathcal{L}^2} \\
&\quad \le C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{H}^1}\Big) + \|(\nabla f, \nabla_\Gamma f_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla u, \nabla_\Gamma v)\|_{\boldsymbol{\mathcal{L}}^2}.
\end{aligned}
$$

PROOF. We consider again the truncations u_k and v_k of u and v, respectively, as defined in the proof of Proposition 4.3.5. We multiply (4.6a) by $-\Delta u$ and integrate over Ω, and (4.6b) by $-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u$ and integrate over Γ. Adding the resulting equations leads to

$$
\begin{aligned}
&\int_\Omega (\Delta u)^2 \,\mathrm{d}x + \int_\Gamma (-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u)^2 \,\mathrm{d}\Gamma \\
&\quad = \int_\Omega f(-\Delta u) \,\mathrm{d}x + \int_\Gamma f_\Gamma(-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u) \,\mathrm{d}\Gamma \\
&\qquad - \int_\Omega F_1'(u_k)(-\Delta u) \,\mathrm{d}x - \int_\Omega (F_1'(u) - F_1'(u_k))(-\Delta u) \,\mathrm{d}x \\
&\qquad - \int_\Gamma G_1'(v_k)(-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u) \,\mathrm{d}\Gamma - \int_\Gamma (G_1'(v) - G_1'(v_k))(-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u) \,\mathrm{d}\Gamma.
\end{aligned} \tag{4.46}
$$

Arguing as previously done in the proof of Proposition 4.3.5 using monotonicity, we find that

$$
\begin{aligned}
&- \int_\Omega F_1'(u_k)(-\Delta u) \,\mathrm{d}x - \int_\Gamma G_1'(v_k)(-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u) \,\mathrm{d}\Gamma \\
&\quad = - \int_\Omega \nabla u \cdot \nabla u_k F_1''(u_k) \,\mathrm{d}x - \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma v_k G_1''(v_k) \,\mathrm{d}\Gamma \\
&\qquad - \int_\Gamma (\alpha G_1'(v_k) - F_1'(u_k))\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma \\
&\quad \le - \int_\Gamma (\alpha G_1'(v_k) - F_1'(u_k))\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma.
\end{aligned}
$$

Then, using integration by parts on the first two terms on the right-hand side of (4.46), we obtain

$$
\begin{aligned}
&\int_\Omega (\Delta u)^2 \,\mathrm{d}x + \int_\Gamma (-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u)^2 \,\mathrm{d}\Gamma \\
&\quad \le \int_\Omega \nabla f \cdot \nabla u \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma f_\Gamma \cdot \nabla_\Gamma v \,\mathrm{d}\Gamma + \int_\Gamma (\alpha f_\Gamma - f)\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma \\
&\qquad - \int_\Omega (F_1'(u) - F_1'(u_k))(-\Delta u) \,\mathrm{d}x - \int_\Gamma (G_1'(v) - G_1'(v_k))\big(-\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u\big) \,\mathrm{d}\Gamma \\
&\qquad - \int_\Gamma (\alpha G_1'(v_k) - F_1'(u_k))\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma.
\end{aligned}
$$

Since $(-\Delta u, -\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u) \in \mathcal{L}^2$, we readily notice that the third and fourth term vanish as $k \to \infty$. Moreover, the first two terms on the right-hand side can be simply estimated by Hölder's inequality as

$$\int_\Omega \nabla f \cdot \nabla u \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma f_\Gamma \cdot \nabla_\Gamma v \,\mathrm{d}\Gamma \leq \|(\nabla f, \nabla_\Gamma f_\Gamma)\|_{\mathcal{L}^2} \|(\nabla u, \nabla_\Gamma v)\|_{\mathcal{L}^2}.$$

For the remaining terms, we handle the cases $K = 0$ and $K \in (0, \infty)$ separately.

The case $K = 0$. Using the trace relation $u_k = \alpha v_k$ a.e. on Γ, we combine the domination property (4.8), Lemma 2.2.1 and Proposition 2.2.1 to infer that

$$\begin{aligned}
&- \int_\Gamma (\alpha G_1'(v_k) - F_1'(u_k))\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma \\
&\leq \Big(\|G_1'(v_k)\|_{L^2(\Gamma)} + \|F_1'(\alpha v_k)\|_{L^2(\Gamma)}\Big)\|\partial_{\mathbf{n}} u\|_{L^2(\Gamma)} \\
&\leq C\Big(1 + \|G_1'(v_k)\|_{L^2(\Gamma)}\Big)\|\nabla u\|_{\mathbf{L}^2(\Gamma)} \\
&\leq C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big)\|\nabla u\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}}\|\nabla u\|_{\mathbf{H}^1(\Omega)}^{\frac{1}{2}},
\end{aligned}$$

if k is sufficiently large. Analogously, using again Lemma 2.2.1, we deduce

$$\int_\Gamma (\alpha f_\Gamma - f)\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma \leq C\|(f, f_\Gamma)\|_{\mathcal{H}^1}\|\nabla u\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}}\|\nabla u\|_{\mathbf{H}^1(\Omega)}^{\frac{1}{2}}.$$

The case $K \in (0, \infty)$. Since $K \in (0, \infty)$, we may use the boundary condition (4.6c) to rewrite the normal derivative $\partial_{\mathbf{n}} u$ as $\frac{1}{K}(\alpha v - u)$ a.e. on Γ. Then, exploiting again the monotonicity of the function $\mathbb{R} \ni s \mapsto F_1'(\alpha h_k(s))$ as well as the domination property (4.8), we infer that

$$\begin{aligned}
&- \int_\Gamma (\alpha G_1'(v_k) - F_1'(u_k))\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma \\
&= -\frac{1}{K}\int_\Gamma (\alpha G_1'(v_k) - F_1'(u_k))(\alpha v - u) \,\mathrm{d}\Gamma \\
&= -\frac{1}{K}\int_\Gamma (F_1'(\alpha v_k) - F_1'(v_k))(\alpha v - u) \,\mathrm{d}\Gamma \\
&\quad - \frac{1}{K}\int_\Gamma (\alpha G_1'(v_k) - F_1'(\alpha v_k))(\alpha v - u) \,\mathrm{d}\Gamma \\
&\leq \frac{2}{K}\Big(\|G_1'(v_k)\|_{L^2(\Gamma)} + \|F_1'(\alpha v_k)\|_{L^2(\Gamma)}\Big) \\
&\leq C(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2})
\end{aligned}$$

if k is sufficiently large. Here, we have additionally used that $|u| \leq 1$ a.e. in Ω and $|u| \leq 1$ a.e. on Γ. Lastly, due to the trace embedding and $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ we have

$$\int_\Gamma (\alpha f_\Gamma - f)\partial_{\mathbf{n}} u \,\mathrm{d}\Gamma = \frac{1}{K}\int_\Gamma (\alpha f_\Gamma - f)(\alpha v - u) \,\mathrm{d}\Gamma \leq C\|(f, f_\Gamma)\|_{\mathcal{H}^1}.$$

Combining the previous estimates finishes the proof. □

PROPOSITION 4.3.8. *Let $(f, f_\Gamma) \in \mathcal{H}^1$ and consider the corresponding solution $(u, v) \in \mathcal{H}^2 \cap \mathcal{H}^1_{K,\alpha}$ to system (4.6). Then there exists a constant $C_p > 0$ such that*

$$\|(u, v)\|_{\mathcal{W}^{2,p}} + \|(F_1'(u), G_1'(v))\|_{\mathcal{L}^p} \leq C_p\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{H}^1}\Big) \tag{4.47}$$

where $p = 6$ if $d = 3$ and any $p \in [2, \infty)$ if $d = 2$. Now, let Ω be of class of C^{k+2} and $(f, f_\Gamma) \in \mathcal{H}^k \cap \mathcal{L}^\infty$ for $k \in \{1, 2\}$. Moreover, assume that $F_1, G_1 \in C^3(-1, 1)$ if $k = 2$. Then we additionally have $(u, v) \in \mathcal{H}^{k+2}$, and there exists a constant $C_k > 0$ such that

$$\|(u, v)\|_{\mathcal{H}^{k+2}} \leq C_k\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{H}^k}\Big). \tag{4.48}$$

Proof. The estimate (4.47) readily follows from (4.31) in view of the Sobolev embedding $\mathcal{H}^1 \hookrightarrow \mathcal{L}^p$ for any $p \in [2, 6]$ if $d = 3$ and any $p \in [2, \infty)$ if $d = 2$.

To prove the second assertion, let $(f, f_\Gamma) \in \mathcal{H}^k \cap \mathcal{L}^\infty$ for $k \in \{1, 2\}$. Then, according to Proposition 4.3.5, u and v satisfy the separation property (4.32) due to $(f, f_\Gamma) \in \mathcal{L}^\infty$. Thus, as $(u, v) \in \mathcal{H}^2$, and $F_1, G_1 \in C^{k+1}(-1, 1)$, we infer that $(F_1'(u), G_1'(v)) \in \mathcal{H}^k$ with

$$\|(F_1'(u), G_1'(v))\|_{\mathcal{H}^k} \leq C\|(u, v)\|_{\mathcal{H}^k} \leq C\|(f, f_\Gamma)\|_{\mathcal{L}^2}. \tag{4.49}$$

Consequently, as $(f, f_\Gamma) \in \mathcal{H}^k$, regularity theory for elliptic systems with bulk-surface coupling yields $(u, v) \in \mathcal{H}^{k+2}$ along with the estimate

$$\|(u, v)\|_{\mathcal{H}^{k+2}} \leq C\Big(1 + \|(f, f_\Gamma)\|_{\mathcal{H}^k} + \|(F_1'(u), G_1'(v))\|_{\mathcal{H}^k}\Big).$$

The readily verifies (4.48) for any $k \in \{1, 2\}$ in view of (4.49), which completes the proof. □

Remark 4.3.9. It is straightforward to check that the statements of Proposition 4.3.3, Proposition 4.3.4, Proposition 4.3.5 except for the separation property (4.32), Corollary 4.3.7 and Proposition 4.3.8 remain true when the potentials F and G are replaced by their respective Moreau–Yosida approximations $F_{1,\lambda}$ and $G_{1,\lambda}$ with $\lambda > 0$ (cf. Section 4.3.1). Although the separation property (4.32) cannot be obtained when using the Moreau–Yosida approximations, the corresponding proofs can be carried out similarly. As the potentials $F_{1,\lambda}$ and $G_{1,\lambda}$ are regular, the proofs do not require any truncation of the functions u and v. It is crucial that for $F_{1,\lambda}$ and $G_{1,\lambda}$, the constant in the estimate (4.31) is still independent of λ.

4.4 Bulk-surface elliptic system with non-constant coefficients

The results presented in this section are contained in the following publication:

[164] J. Stange, *Well-posedness and long-time behavior of a bulk-surface Cahn–Hilliard model with non-degenerate mobility*, Nonlinear Analysis, 268 (2026), p. 114060.

In this section, we establish well-posedness and regularity results for an elliptic system with bulk-surface coupling and non-constant coefficients. These regularity results will be of crucial importance in the proofs of Theorem 7.3.1 and Theorem 10.2.1. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded with C^1-boundary $\Gamma := \partial\Omega$. Then we study the following system:

$$\begin{aligned} -\mathrm{div}(m_\Omega(\phi)\nabla u) &= f && \text{in } \Omega, \\ -\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v) + m_\Omega(\phi)\partial_\mathbf{n} u &= f_\Gamma && \text{on } \Gamma, \\ L m_\Omega(\phi)\partial_\mathbf{n} u &= \beta v - u && \text{on } \Gamma, \end{aligned}$$

where $(f, f_\Gamma) \in (\mathcal{H}^1_{L,\beta})'$, and $\phi : \Omega \to \mathbb{R}$ and $\psi : \Gamma \to \mathbb{R}$ are given measurable functions with $|\phi| \leq 1$ a.e. in Ω and $|\psi| \leq 1$ a.e. on Γ. Moreover, the coefficients $m_\Omega, m_\Gamma : [-1,1] \to \mathbb{R}$ are supposed to satisfy the following condition:

(M) We require $m_\Omega, m_\Gamma \in C([-1,1])$, and the existence of constants $m_*, m^* > 0$ such that

$$0 < m_* \leq m_\Omega(s), m_\Gamma(s) \leq m^* \qquad \text{for all } s \in [-1,1].$$

In (4.50) we allow for $L \in [0,\infty]$, meaning, that the boundary conditions is again understood as

$$\begin{cases} u = \beta v & \text{if } L = 0, \\ m_\Omega(\phi)\partial_{\mathbf{n}} u = \beta v - u & \text{if } L \in (0,\infty), \\ m_\Omega(\phi)\partial_{\mathbf{n}} u = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Gamma.$$

We call $(u, v) \in \mathcal{H}^1_{L,\beta}$ a weak solution to system (4.50) if

$$\begin{aligned} &\int_\Omega m_\Omega(\phi)\nabla u \cdot \nabla\eta \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma v \cdot \nabla_\Gamma \eta_\Gamma \,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta v - u)(\beta\eta_\Gamma - \eta)\,\mathrm{d}\Gamma \\ &\quad = \langle (f, f_\Gamma), (\eta, \eta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} \end{aligned} \tag{4.51}$$

for all $(\eta, \eta_\Gamma) \in \mathcal{H}^1_{L,\beta}$.

4.4.1 Well-posedness of weak solutions

Our first result is concerned with the well-posedness of weak solutions to (4.50).

THEOREM 4.4.1. *Let $(f, f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta}$. Then there exists a unique weak solution $(u, v) \in \mathcal{H}^1_{L,\beta}$ to (4.50). Moreover, there exists a non-negative constant C, depending only on Ω, L, β and m_* such that*

$$\|(u, v)\|_{L,\beta} \leq C\|(f, f_\Gamma)\|_{(\mathcal{H}^1_{L,\beta})'}. \tag{4.52}$$

PROOF. We consider the bilinear form $\mathcal{B}^{\phi,\psi}_{L,\beta}$ on $\mathcal{H}^1_{L,\beta}$ defined as

$$\begin{aligned} \mathcal{B}^{\phi,\psi}_{L,\beta}((\zeta, \zeta_\Gamma), (\xi, \xi_\Gamma)) \coloneqq &\int_\Omega m_\Omega(\phi)\nabla\zeta \cdot \nabla\xi \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma\zeta_\Gamma \cdot \nabla_\Gamma\xi_\Gamma \,\mathrm{d}\Gamma \\ &+ \chi(L)\int_\Gamma (\beta\zeta_\Gamma - \zeta)(\beta\xi_\Gamma - \xi)\,\mathrm{d}\Gamma \end{aligned}$$

for $(\zeta, \zeta_\Gamma), (\xi, \xi_\Gamma) \in \mathcal{H}^1_{L,\beta}$. In view of the uniform positivity of the mobility functions, see (M), we have

$$\begin{aligned} \mathcal{B}^{\phi,\psi}_{L,\beta}((u, v), (u, v)) &\geq m_*\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + m_*\|\nabla_\Gamma v\|^2_{\mathbf{L}^2(\Gamma)} + \chi(L)\|\beta v - u\|^2_{L^2(\Gamma)} \\ &\geq \min\{1, m_*\}\|(u, v)\|^2_{L,\beta} \end{aligned} \tag{4.53}$$

for all $(u, v) \in \mathcal{H}^1_{L,\beta}$. Recalling that $\|\cdot\|_{L,\beta}$ defines a norm on $\mathcal{V}^1_{L,\beta}$, the estimate (4.53) shows that $\mathcal{B}^{\phi,\psi}_{L,\beta}$ defines a coercive bilinear form on $\mathcal{V}^1_{L,\beta}$. Moreover, due to the boundedness of the mobility functions, $\mathcal{B}^{\phi,\psi}_{L,\beta}$ is clearly continuous on $\mathcal{V}^1_{L,\beta}$. Next, for all $(\overline{\zeta}, \overline{\zeta}_\Gamma) \in \mathcal{V}^1_{L,\beta}$ it holds that

$$\langle (f, f_\Gamma), (\overline{\zeta}, \overline{\zeta}_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} \leq \|(f, f_\Gamma)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\overline{\zeta}, \overline{\zeta}_\Gamma)\|_{\mathcal{H}^1}$$

$$\leq C\|(f,f_\Gamma)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\overline{\zeta},\overline{\zeta}_\Gamma)\|_{L,\beta}$$

by employing the bulk-surface Poincaré inequality from Lemma 4.2.1. This proves that the map $\mathcal{V}^1_{L,\beta} \ni (\overline{\zeta},\overline{\zeta}_\Gamma) \mapsto \langle (f,f_\Gamma),(\overline{\zeta},\overline{\zeta}_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} \in \mathbb{R}$ is a continuous linear functional and therefore is an element of $(\mathcal{V}^1_{L,\beta})'$. This allows us to apply the Lemma of Lax–Milgram, which provides a unique pair

$$(u,v) = (u_{(f,f_\Gamma)}, v_{(f,f_\Gamma)}) \in \mathcal{V}^1_{L,\beta}$$

such that

$$\mathcal{B}^{\phi,\psi}_{L,\beta}((u,v),(\overline{\zeta},\overline{\zeta}_\Gamma)) = \langle (f,f_\Gamma),(\overline{\zeta},\overline{\zeta}_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} \qquad \text{for all } (\overline{\zeta},\overline{\zeta}_\Gamma) \in \mathcal{V}^1_{L,\beta}. \tag{4.54}$$

We note that the solution (u,v) moreover depends on the mobility functions m_Ω and m_Γ as well as on ϕ and ψ. To show that (u,v) is a weak solution to (4.50), it remains to prove that (4.54) holds for all $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$. To this end, let $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$ be arbitrary and consider

$$(\overline{\zeta},\overline{\zeta}_\Gamma) \coloneqq (\zeta - \beta\mathrm{mean}(\zeta,\zeta_\Gamma), \zeta_\Gamma - \mathrm{mean}(\zeta,\zeta_\Gamma)).$$

Then $(\overline{\zeta},\overline{\zeta}_\Gamma)$ belongs to $\mathcal{V}^1_{L,\beta}$ and is therefore an admissible test function in (4.54). Thus, plugging $(\overline{\zeta},\overline{\zeta}_\Gamma)$ into (4.54) and using the fact that $(f,f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta}$, we find

$$\mathcal{B}^{\phi,\psi}_{L,\beta}((u,v),(\zeta,\zeta_\Gamma)) = \langle (f,f_\Gamma),(\zeta,\zeta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}}.$$

As $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$ was arbitrarily chosen, this shows that (u,v) is a weak solution to (4.50). Lastly, exploiting again the positivity of the mobility functions, the estimate (4.52) follows from the simple computation

$$\begin{aligned}
\min\{1,m_*\}\|(u,v)\|^2_{L,\beta} &\leq \mathcal{B}^{\phi,\psi}_{L,\beta}((u,v),(u,v)) \\
&= \langle (f,f_\Gamma),(u,v)\rangle_{\mathcal{H}^1_{L,\beta}} \\
&\leq \|(f,f_\Gamma)\|_{(\mathcal{H}^1_{L,\beta})'}\|(u,v)\|_{\mathcal{H}^1} \\
&\leq C\|(f,f_\Gamma)\|_{(\mathcal{H}^1_{L,\beta})'}\|(u,v)\|_{L,\beta},
\end{aligned}$$

which finishes the proof. □

In view of Theorem 4.4.1, we can define a solution operator

$$\mathcal{S}^{\phi,\psi}_{L,\beta} : \mathcal{V}^{-1}_{L,\beta} \to \mathcal{V}^1_{L,\beta}, \quad (f,f_\Gamma) \mapsto \mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma) = (\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f,f_\Gamma), \mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(f,f_\Gamma)),$$

where $\mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma)$ is the unique weak solution to (4.50). Moreover, we can define an inner product and its induced norm on $\mathcal{V}^1_{L,\beta}$ by

$$\begin{aligned}
((f,f_\Gamma),(\zeta,\zeta_\Gamma))_{L,\beta,\phi,\psi} &\coloneqq \int_\Omega m_\Omega(\phi)\nabla f \cdot \nabla\zeta \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma f_\Gamma \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \\
&\qquad + \chi(L)\int_\Gamma (\beta f_\Gamma - f)(\beta\zeta_\Gamma - \zeta)\,\mathrm{d}\Gamma, \\
\|(f,f_\Gamma)\|_{L,\beta,\phi,\psi} &\coloneqq ((f,f_\Gamma),(f,f_\Gamma))^{\frac{1}{2}}_{L,\beta,\phi,\psi}
\end{aligned}$$

for all $(f,f_\Gamma),(\zeta,\zeta_\Gamma) \in \mathcal{V}^1_{L,\beta}$. One readily sees that

$$\min\{1,\sqrt{m_*}\}\|(f,f_\Gamma)\|_{L,\beta,\phi,\psi} \leq \|(f,f_\Gamma)\|_{L,\beta} \leq \max\{1,\sqrt{m^*}\}\|(f,f_\Gamma)\|_{L,\beta,\phi,\psi} \tag{4.55}$$

for all $(f, g) \in \mathcal{H}^1_{L,\beta}$. In particular, the norms $\|\cdot\|_{L,\beta}$ and $\|\cdot\|_{L,\beta,\phi,\psi}$ are equivalent on $\mathcal{V}^1_{L,\beta}$.

Next, we define an inner product and its induced norm on $\mathcal{V}^{-1}_L$ by

$$\begin{aligned} \big((f, f_\Gamma), (\zeta, \zeta_\Gamma)\big)_{L,\beta,\phi,\psi,*} &:= \big(\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f, f_\Gamma), \mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(\zeta, \zeta_\Gamma)\big)_{L,\beta,\phi,\psi}, \\ \|(f, f_\Gamma)\|_{L,\beta,\phi,\psi,*} &:= \big((f, f_\Gamma), (f, f_\Gamma)\big)^{\frac{1}{2}}_{L,\beta,\phi,\psi,*} \end{aligned}$$

for all $(f, f_\Gamma), (\zeta, \zeta_\Gamma) \in \mathcal{V}^{-1}_{L,\beta}$. Using the respective weak formulation satisfied by the solution operators $\mathcal{S}_{L,\beta}$ and $\mathcal{S}^{\phi,\psi}_{L,\beta}$, one can readily check that the norms $\|\cdot\|_{L,\beta,\phi,\psi,*}$ and $\|\cdot\|_{L,\beta,*}$ are equivalent on $\mathcal{V}^{-1}_{L,\beta}$ with

$$\begin{aligned} \min\{1, \sqrt{m_*}\}\|(f, f_\Gamma)\|_{L,\beta,\phi,\psi,*} &\le \|(f, f_\Gamma)\|_{L,\beta,*} \\ &\le \max\{1, \sqrt{m^*}\}\|(f, f_\Gamma)\|_{L,\beta,\phi,\psi,*} \end{aligned} \tag{4.56}$$

for all $(f, f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta}$. In particular, we also obtain that $\|\cdot\|_{L,\beta,\phi,\psi,*}$ and $\|\cdot\|_{(\mathcal{H}^1_{L,\beta})'}$ are equivalent on $\mathcal{V}^{-1}_{L,\beta}$.

4.4.2 Existence of strong solutions and higher regularity

After having proven the well-posedness of weak solutions to (4.50), we turn our attention to the existence of strong solutions. In particular, we show that every weak solution is a strong solution. Moreover, we show higher regularity estimates for various right-hand sides (f, f_Γ).

Proposition 4.4.2. *Let Ω be of class C^2, let $(\phi, \psi) \in \mathcal{W}^{1,\infty}$, $m_\Omega, m_\Gamma \in C^1([-1, 1])$, and consider a pair $(f, f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta} \cap \mathcal{L}^2$. Then, we have $\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma) \in \mathcal{H}^2$, along with the existence of a constant $C > 0$ such that*

$$\begin{aligned} &\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|_{\mathcal{H}^2} \\ &\quad \le C\Big(\|(f, f_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\phi \cdot \nabla\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f, f_\Gamma), \nabla_\Gamma\psi \cdot \nabla_\Gamma\mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(f, f_\Gamma))\|_{\mathcal{L}^2}\Big). \end{aligned} \tag{4.57}$$

Proof. For the sake of brevity, we denote again with (u, v) the unique weak solution $\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)$ provided by Theorem 4.4.1. To show $(u, v) \in \mathcal{H}^2$, we have to make a case distinction according to the value of the parameter $L \in [0, \infty]$.

The case $L = 0$. We start by choosing $\eta_\Gamma = 0$ in the weak formulation (4.51) to obtain

$$\int_\Omega m_\Omega(\phi)\nabla u \cdot \nabla\eta \,\mathrm{d}x = \int_\Omega f\,\eta \,\mathrm{d}x \tag{4.58}$$

for all $\eta \in H^1_0(\Omega)$. Then, given $\overline{\eta} \in C^\infty_c(\Omega)$, we can use $\eta = \frac{\overline{\eta}}{m_\Omega(\phi)} \in H^1_0(\Omega)$ as a test function in (4.58) since $\phi \in W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$. Consequently, it holds that

$$\int_\Omega \nabla u \cdot \nabla\overline{\eta} \,\mathrm{d}x = \int_\Omega \frac{1}{m_\Omega(\phi)}\Big(f + \nabla m_\Omega(\phi) \cdot \nabla u\Big)\,\overline{\eta} \,\mathrm{d}x$$

for all $\overline{\eta} \in C^\infty_c(\Omega)$. In particular, as $\overline{\eta}$ was arbitrary, we deduce that the distributional derivative Δu belongs to $L^2(\Omega)$ and satisfies

$$-\Delta u = \frac{1}{m_\Omega(\phi)}\Big(f + \nabla m_\Omega(\phi) \cdot \nabla u\Big) \qquad \text{a.e. in } \Omega.$$

As we additionally know that $u|_\Gamma = \beta v \in H^1(\Gamma)$, we may apply elliptic regularity theory for the Poisson–Dirichlet problem (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) to conclude that $u \in H^{\frac{3}{2}}(\Omega)$ with

$$\begin{aligned} \|u\|_{H^{\frac{3}{2}}(\Omega)} &\leq C\Big(\|f\|_{L^2(\Omega)} + \|\nabla m_\Omega(\phi)\cdot\nabla u\|_{L^2(\Omega)} + \|v\|_{H^1(\Gamma)}\Big) \\ &\leq C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|\nabla\phi\cdot\nabla u\|_{L^2(\Omega)}\Big). \end{aligned} \tag{4.59}$$

Now, since $\Delta u \in L^2(\Omega)$ and $u \in H^{\frac{3}{2}}(\Omega)$, we can use a variant of the elliptic trace theorem (see, e.g., [29, Theorem 2.27] or [42, Theorem A.1]) to deduce that $\partial_{\mathbf{n}} u \in L^2(\Gamma)$ together with the estimate

$$\|\partial_{\mathbf{n}} u\|_{L^2(\Gamma)} \leq C\|u\|_{H^{\frac{3}{2}}(\Omega)}. \tag{4.60}$$

Consequently, an application of integration by parts gives

$$\int_\Omega m_\Omega(\phi)\nabla u\cdot\nabla\eta \,\mathrm{d}x = \int_\Omega f\,\eta\,\mathrm{d}x + \int_\Gamma m_\Omega(\phi)\partial_{\mathbf{n}} u\,\eta\,\mathrm{d}x \tag{4.61}$$

for all $\eta \in H^1(\Omega)$. Consider now an arbitrary function $\eta_\Gamma \in H^1(\Gamma)$. According to the inverse trace theorem (see, e.g., [110, Theorem 4.2.3]), there exists a function $\hat\eta \in H^{\frac{3}{2}}(\Omega)$ such that $\hat\eta|_\Gamma = \eta_\Gamma$ a.e. on Γ. Thus, choosing $\eta = \beta\hat\eta$, we see that $(\eta,\eta_\Gamma) \in \mathcal{H}^1_{L,\beta}$ is an admissible test function in (4.51), and in view of (4.61), we obtain

$$\int_\Gamma m_\Gamma(\psi)\nabla_\Gamma v\cdot\nabla_\Gamma\eta_\Gamma \,\mathrm{d}\Gamma = \int_\Gamma \left(f_\Gamma - \beta m_\Omega(\phi)\partial_{\mathbf{n}} u\right)\eta_\Gamma\,\mathrm{d}\Gamma.$$

As $\eta_\Gamma \in H^1(\Gamma)$ was arbitrarily chosen, we can infer, similar to above, that v is a weak solution to the surface elliptic equation

$$-\Delta_\Gamma v = \frac{1}{m_\Gamma(\psi)}\Big(f_\Gamma - \beta m_\Omega(\phi)\partial_{\mathbf{n}} u + \nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\Big) \qquad \text{on } \Gamma.$$

Since $\psi \in W^{2,4}(\Gamma) \hookrightarrow W^{1,\infty}(\Gamma)$, and m_Γ' is bounded by **(M)**, the Sobolev inequality yields that $\nabla_\Gamma m_\Gamma(\psi) \in \mathbf{L}^\infty(\Gamma)$, which, in combination with $v \in H^1(\Gamma)$, implies that $\nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v \in L^2(\Gamma)$. Thus, as m_Ω is bounded, and $\partial_{\mathbf{n}} u \in L^2(\Gamma)$ by (4.60), we find that

$$\frac{1}{m_\Gamma(\psi)}\Big(f_\Gamma - \beta m_\Omega(\phi)\partial_{\mathbf{n}} u + \nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\Big) \in L^2(\Gamma).$$

Therefore, recalling that Γ is a compact submanifold of class C^2 without boundary, we may apply regularity theory for elliptic equations on submanifolds (see, e.g., [167, s.5, Theorem 1.3]) to deduce $v \in H^2(\Gamma)$ together with the estimate

$$\begin{aligned} \|v\|_{H^2(\Gamma)} &\leq C\Big(\|f_\Gamma\|_{L^2(\Gamma)} + \|\partial_{\mathbf{n}} u\|_{L^2(\Gamma)} + \|\nabla_\Gamma\, m_\Gamma(\psi)\cdot\nabla_\Gamma v\|_{L^2(\Gamma)}\Big) \\ &\leq C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^2}\Big). \end{aligned} \tag{4.62}$$

Here, we additionally made use of (4.52), (4.59) and (4.60). Then, since $u|_\Gamma = \beta v$ a.e. on Γ, we further deuce that $u|_\Gamma \in H^2(\Gamma)$, and recalling $-\Delta u \in L^2(\Omega)$, we eventually conclude from elliptic regularity theory for the Poisson–Dirichlet problem (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) that $u \in H^2(\Omega)$ with

$$\begin{aligned} \|u\|_{H^2(\Omega)} &\leq C\Big(\|f\|_{L^2(\Omega)} + \|\nabla m_\Omega(\phi)\cdot\nabla u\|_{L^2(\Omega)} + \|v\|_{H^2(\Gamma)}\Big) \\ &\leq C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^2}\Big). \end{aligned} \tag{4.63}$$

Finally, combining (4.62) and (4.63) leads to

$$\|(u,v)\|_{\mathcal{H}^2} \le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^2}\Big).$$

The case $L \in (0,\infty)$. In this case, we start by taking $\eta = 0$ in (4.51). Then, the weak formulation reduces to

$$\int_\Gamma m_\Gamma(\psi)\nabla_\Gamma v\cdot\nabla\eta_\Gamma \,\mathrm{d}\Gamma + \frac{1}{L}\int_\Gamma (\beta v - u)\,\beta\eta_\Gamma \,\mathrm{d}\Gamma = \int_\Gamma f_\Gamma\,\eta_\Gamma\,\mathrm{d}\Gamma$$

for all $\eta_\Gamma \in H^1(\Gamma)$. By arguing as in the case for $L = 0$, we obtain that v is a weak solution of the surface elliptic problem

$$-\Delta_\Gamma v = \frac{1}{m_\Gamma(\psi)}\Big(f_\Gamma - \frac{\beta}{L}(\beta v - u) + \nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\Big) \qquad \text{on } \Gamma.$$

Then, since $u \in H^1(\Omega)$, we have by the trace theorem $\beta v - u|_\Gamma \in H^{\frac{1}{2}}(\Gamma)$, and thus, in particular, $\beta v - u|_\Gamma \in L^2(\Gamma)$. Therefore, we are again in a position to apply elliptic regularity theory on submanifolds to deduce $v \in H^2(\Gamma)$ together with the estimate

$$\begin{aligned}\|v\|_{H^2(\Gamma)} &\le C\Big(\|f_\Gamma\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)} + \|u\|_{L^2(\Gamma)} + \|\nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\|_{L^2(\Gamma)}\Big)\\ &\le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|\nabla_\Gamma\psi\cdot\nabla_\Gamma v\|_{L^2(\Gamma)}\Big).\end{aligned} \tag{4.64}$$

Next, we choose $\eta_\Gamma = 0$ in the weak formulation (4.51), which entails

$$\int_\Omega m_\Omega(\phi)\nabla u\cdot\nabla\eta\,\mathrm{d}x + \frac{1}{L}\int_\Gamma (\beta v - u)\,\eta\,\mathrm{d}\Gamma = \int_\Omega f\,\eta\,\mathrm{d}x$$

for all $\eta \in H^1(\Omega)$. This allows us to conclude similarly to the case $L = 0$ that u is a weak solution to the Poisson–Neumann problem

$$\begin{aligned}-\Delta u &= \frac{1}{m_\Omega(\phi)}\Big(f + \nabla m_\Omega(\phi)\cdot\nabla u\Big) && \text{in } \Omega,\\ \partial_{\mathbf{n}} u &= \frac{1}{L m_\Omega(\phi)}(\beta v - u) && \text{on } \Gamma.\end{aligned}$$

Applying elliptic regularity theory for Poisson's equation with inhomogeneous Neumann boundary condition (see, e.g., [167, s.5, Proposition 7.7]), we deduce $u \in H^2(\Omega)$ with

$$\begin{aligned}\|u\|_{H^2(\Omega)} &\le C\Big(\|f\|_{L^2(\Omega)} + \|v\|_{H^{\frac{1}{2}}(\Gamma)} + \|u\|_{H^{\frac{1}{2}}(\Gamma)} + \|u\|_{L^2(\Omega)} + \|\nabla m_\Omega(\phi)\cdot\nabla u\|_{L^2(\Omega)}\Big)\\ &\le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|\nabla\phi\cdot\nabla u\|_{L^2(\Omega)}\Big).\end{aligned} \tag{4.65}$$

Finally, combining (4.64) and (4.65), we conclude that

$$\|(u,v)\|_{\mathcal{H}^2} \le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^2}\Big).$$

The case $L = \infty$. In this case, the system (4.50) decouples to the non-homogeneous Poisson problem with a homogeneous Neumann boundary condition

$$-\mathrm{div}(m_\Omega(\phi)\nabla u) = f \qquad \text{in } \Omega, \tag{4.66a}$$

$$m_\Omega(\phi)\partial_{\mathbf{n}} u = 0 \qquad \text{on } \Gamma, \tag{4.66b}$$

and the non-homogeneous surface elliptic problem

$$-\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v) = f_\Gamma \qquad \text{on } \Gamma. \tag{4.67}$$

Applying elliptic regularity theory for Poisson's equation with a homogeneous Neumann boundary condition (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) and regularity theory for the Laplace–Beltrami equation (see, e.g., [176, Lemma B.1]), respectively, we directly deduce the desired regularities $u \in H^2(\Omega)$ and $v \in H^2(\Gamma)$ along with the estimates

$$\begin{aligned}
\|u\| &\le C\Big(\|f\|_{L^2(\Omega)} + \|\nabla m_\Omega(\phi)\cdot\nabla u\|_{L^2(\Omega)}\Big),\\
\|v\| &\le C\Big(\|f_\Gamma\|_{L^2(\Gamma)} + \|\nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\|_{L^2(\Gamma)}\Big).
\end{aligned}$$

□

Assuming that the right-hand side (f, f_Γ) and the mobility functions are more regular, we even can prove $\mathcal{H}^3$-regularity of the corresponding solution $\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)$.

Corollary 4.4.3. *Let Ω be of class C^3, let $(\phi,\psi) \in \mathcal{W}^{2,4}$, and assume that $m_\Omega, m_\Gamma \in C^2([-1,1])$ and $(f, f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta} \cap \mathcal{H}^1$. Then $\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma) \in \mathcal{H}^3$. Moreover, there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|_{\mathcal{H}^3}\\
&\quad\le C\Big(1 + \mathbf{1}_{\{0\}}(L)\|(\phi,\psi)\|_{\mathcal{H}^2}\Big)\\
&\qquad\times\Bigg(\Big\|\Big(\frac{f}{m_\Omega(\phi)}, \frac{f_\Gamma}{m_\Gamma(\psi)}\Big)\Big\|_{\mathcal{H}^1}\\
&\qquad\quad+\Big\|\Big(\frac{m_\Omega'(\phi)\nabla\phi\cdot\nabla\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f, f_\Gamma)}{m_\Omega(\phi)}, \frac{m_\Gamma'(\psi)\nabla_\Gamma\psi\cdot\nabla_\Gamma\mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(f, f_\Gamma)}{m_\Gamma(\psi)}\Big)\Big\|_{\mathcal{H}^1}\Bigg),
\end{aligned} \tag{4.68}$$

where $\mathbf{1}_{\{0\}}(\cdot)$ denotes the indicator function of the set $\{0\}$.

Proof. Abbreviating again $(u, v) = \mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)$, we can move along the lines from the proof of Proposition 4.4.2 and show that $(u, v) \in \mathcal{H}^3$ in combination with the estimate

$$\begin{aligned}
&\|(u, v)\|_{\mathcal{H}^3}\\
&\quad\le C\Bigg(\Big\|\Big(\frac{f}{m_\Omega(\phi)}, \frac{f_\Gamma}{m_\Gamma(\psi)}\Big)\Big\|_{\mathcal{H}^1} + \Big\|\Big(\frac{m_\Omega'(\phi)\nabla\phi\cdot\nabla u}{m_\Omega(\phi)}, \frac{m_\Gamma'(\psi)\nabla_\Gamma\psi\cdot\nabla_\Gamma v}{m_\Gamma(\psi)}\Big)\Big\|_{\mathcal{H}^1}\\
&\qquad+\Big\|\frac{m_\Omega(\phi)\partial_{\mathbf{n}}u}{m_\Gamma(\psi)}\Big\|_{H^1(\Gamma)}\Bigg).
\end{aligned}$$

In the following, we will show that the last summand on the right-hand side can be bounded in terms of the first two. To do so, we only have to take a closer look at the case $L = 0$. Indeed, if $L = \infty$, this term clearly vanishes, and for $L \in (0,\infty)$, we make use of the boundary condition $Lm_\Omega(\phi)\partial_{\mathbf{n}}u = \beta v - u$ a.e. on Γ to find that

$$\begin{aligned}
\Big\|\frac{m_\Omega(\phi)\partial_{\mathbf{n}}u}{m_\Gamma(\psi)}\Big\|_{H^1(\Gamma)} = \frac{1}{L}\Big\|\frac{\beta v - u}{m_\Gamma(\psi)}\Big\|_{H^1(\Gamma)} &\le C\|(u, v)\|_{\mathcal{H}^2}\\
&\le C\Big(\|(f, f_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^2}\Big)
\end{aligned}$$

in view of (4.57). Lastly, if $L = 0$, it holds that

$$\nabla_\Gamma\left(\frac{m_\Omega(\phi)\partial_{\mathbf{n}}u}{m_\Gamma(\psi)}\right) = \frac{m_\Omega'(\phi)\partial_{\mathbf{n}}u\nabla_\Gamma\phi}{m_\Gamma(\psi)^2} + \frac{m_\Omega(\phi)\nabla_\Gamma\partial_{\mathbf{n}}u}{m_\Gamma(\psi)^2} - \frac{m_\Omega(\phi)m_\Gamma'(\psi)\partial_{\mathbf{n}}u\nabla_\Gamma\psi}{m_\Gamma(\psi)^2}$$

a.e. on Γ. For the first term, we use the trace theorem together with the Sobolev inequality and find

$$\|\partial_{\mathbf{n}}u\nabla_\Gamma\phi\|_{\mathbf{L}^2(\Gamma)} \le \|\partial_{\mathbf{n}}u\|_{L^4(\Gamma)}\|\nabla_\Gamma\phi\|_{\mathbf{L}^4(\Gamma)} \le C\|\phi\|_{H^2(\Omega)}\|u\|_{H^2(\Omega)}.$$

Next, for the second term, we argue as in the beginning of the proof of Proposition 4.4.2, in particular as for (4.59) and (4.60). To be precise, we obtain

$$\begin{aligned}\|\nabla_\Gamma\partial_{\mathbf{n}}u\|_{\mathbf{L}^2(\Gamma)} &\le \|\partial_{\mathbf{n}}u\|_{H^1(\Gamma)}\\ &\le C\|u\|_{H^{\frac{5}{2}}(\Omega)}\\ &\le C\left(\left\|\frac{f}{m_\Omega(\phi)}\right\|_{H^1(\Omega)} + \left\|\frac{m_\Omega'(\phi)\nabla\phi\cdot\nabla u}{m_\Omega(\phi)}\right\|_{H^1(\Omega)} + \|v\|_{H^2(\Gamma)}\right).\end{aligned}$$

Finally, for the last term, we combine the trace theorem again with the Sobolev inequality to deduce that

$$\|\partial_{\mathbf{n}}u\nabla_\Gamma\psi\|_{\mathbf{L}^2(\Gamma)} \le \|\partial_{\mathbf{n}}u\|_{L^4(\Gamma)}\|\nabla_\Gamma\psi\|_{\mathbf{L}^4(\Gamma)} \le C\|\psi\|_{H^2(\Gamma)}\|u\|_{H^2(\Omega)}.$$

Thus, from the estimates above and (4.57), we conclude (4.68), which finishes the proof. □

Next, we aim to generalize the result from Proposition 4.4.2 to the case of L^p-regularity theory.

Proposition 4.4.4. *Let Ω be of class C^2, let $p \in [2,\infty)$ and consider a pair $(f, f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta} \cap \mathcal{L}^p$. We further assume that $(\phi,\psi) \in \mathcal{W}^{2,4}$ such that*

$$\left(\nabla\phi\cdot\nabla\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f,f_\Gamma), \nabla_\Gamma\psi\cdot\nabla_\Gamma\mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(f,f_\Gamma)\right) \in \mathcal{L}^p. \tag{4.69}$$

Then, $\mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma) \in \mathcal{W}^{2,p}$. Furthermore, there exists a constant $C > 0$ such that

$$\begin{aligned}&\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma)\|_{\mathcal{W}^{2,p}}\\ &\quad\le C\left(\|(f,f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi\cdot\nabla\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f,f_\Gamma), \nabla_\Gamma\psi\cdot\nabla_\Gamma\mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(f,f_\Gamma))\|_{\mathcal{L}^p}\right).\end{aligned} \tag{4.70}$$

Proof. Since $(f,f_\Gamma) \in \mathcal{V}^{-1}_{L,\beta} \cap \mathcal{L}^2$, we already know from Proposition 4.4.2 that $(u,v) := \mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma) \in \mathcal{H}^2$, and that

$$-\Delta u = \frac{1}{m_\Omega(\phi)}\Big(f + \nabla m_\Omega(\phi)\cdot\nabla u\Big) \qquad \text{a.e. in } \Omega, \tag{4.71a}$$

$$L\partial_{\mathbf{n}}u = \frac{1}{m_\Omega(\phi)}(\beta v - u) \qquad \text{a.e. on } \Gamma, \tag{4.71b}$$

as well as

$$-\Delta_\Gamma v = \frac{1}{m_\Gamma(\psi)}\Big(f_\Gamma - \beta m_\Omega(\phi)\partial_{\mathbf{n}}u + \nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\Big) \qquad \text{a.e. on } \Gamma \tag{4.72}$$

hold. As in the proof of Proposition 4.4.2, we will handle the cases $L = 0, L \in (0,\infty)$ and $L = \infty$ separately.

The case $L = 0$. Since the boundary Γ is a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, Sobolev's embedding theorem implies that

$$u|_\Gamma = \beta v \in W^{t,p}(\Gamma) \qquad \text{with } t = \frac{5}{2} + \frac{d-1}{p} - \frac{d}{2}.$$

As by assumption $f, \nabla\phi \cdot \nabla u \in L^p(\Omega)$, we may apply elliptic regularity theory for Poisson's equation with an inhomogeneous Dirichlet boundary condition (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) to deduce

$$u \in W^{s,p}(\Omega) \qquad \text{with } s = \min\Big\{2, \frac{5}{2} + \frac{d}{p} - \frac{d}{2}\Big\} \geq 1 + \frac{2}{p}$$

together with the estimate

$$\begin{aligned} \|u\|_{W^{s,p}(\Omega)} &\leq C\Big(\|f\|_{L^p(\Omega)} + \|\nabla m_\Omega(\phi) \cdot \nabla u\|_{L^p(\Omega)} + \|v\|_{W^{t,p}(\Gamma)}\Big) \\ &\leq C\Big(\|f\|_{L^p(\Omega)} + \|\nabla \phi \cdot \nabla u\|_{L^p(\Omega)} + \|v\|_{H^2(\Gamma)}\Big) \\ &\leq C\Big(\|(f, f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi \cdot \nabla u, \nabla_\Gamma \psi \cdot \nabla_\Gamma v)\|_{\mathcal{L}^p}\Big). \end{aligned} \tag{4.73}$$

This readily yields

$$\nabla u \in \mathbf{W}^{s-1,p}(\Omega) \hookrightarrow \mathbf{W}^{\frac{2}{p},p}(\Omega).$$

Then, since $\frac{2}{p} - \frac{1}{p} = \frac{1}{p}$ is positive and not an integer, the trace theorem $W^{\frac{2}{p},p}(\Omega) \hookrightarrow W^{\frac{1}{p},p}(\Gamma)$ in combination with (4.73) implies that

$$\|\partial_{\mathbf{n}} u\|_{W^{\frac{1}{p},p}(\Gamma)} \leq C\|\nabla u\|_{\mathbf{W}^{\frac{2}{p},p}(\Omega)} \leq C\Big(\|(f, f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi \cdot \nabla u, \nabla_\Gamma \psi \cdot \nabla_\Gamma v)\|_{\mathcal{L}^p}\Big).$$

Consequently, the assumptions $f_\Gamma, \nabla_\Gamma\psi \cdot \nabla_\Gamma v \in L^p(\Gamma)$ enable us to apply regularity theory for the Laplace–Beltrami equation (see, e.g., [176, Lemma B.1]), which entails $v \in W^{2,p}(\Gamma)$ with

$$\begin{aligned} \|v\|_{W^{2,p}(\Gamma)} &\leq C\Big(\|f_\Gamma\|_{L^p(\Gamma)} + \|\partial_{\mathbf{n}} u\|_{L^p(\Gamma)} + \|\nabla_\Gamma m_\Gamma(\psi) \cdot \nabla_\Gamma v\|_{L^p(\Gamma)}\Big) \\ &\leq C\Big(\|(f, f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi \cdot \nabla u, \nabla_\Gamma \psi \cdot \nabla_\Gamma v)\|_{\mathcal{L}^p}\Big). \end{aligned} \tag{4.74}$$

Then, since $u|_\Gamma = \beta v \in W^{2,p}(\Gamma)$, elliptic regularity theory for Poisson's equation with an inhomogeneous Dirichlet boundary condition (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) entails that $u \in W^{2,p}(\Omega)$ along with the estimate

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\leq C\Big(\|f\|_{L^p(\Omega)} + \|\nabla m_\Omega(\phi) \cdot \nabla u\|_{L^p(\Omega)} + \|v\|_{W^{2,p}(\Gamma)}\Big) \\ &\leq C\Big(\|(f, f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi \cdot \nabla u, \nabla_\Gamma \psi \cdot \nabla_\Gamma v)\|_{\mathcal{L}^p}\Big). \end{aligned} \tag{4.75}$$

Combining (4.74) and (4.75), we eventually obtain (4.70), which finishes the proof in the case $L = 0$.

The case $L \in (0, \infty)$. Since $u \in H^2(\Omega)$, the trace theorem combined with Sobolev's embedding theorem yields that

$$u \in H^{\frac{3}{2}}(\Gamma) \hookrightarrow W^{t,p}(\Gamma) \qquad \text{with } t = 2 + \frac{d-1}{p} - \frac{d}{2}.$$

Then, as $\phi \in W^{2,4}(\Omega)$, the trace theorem and the Sobolev embedding theorem further imply that

$$\nabla\phi \in \mathbf{W}^{1,4}(\Omega) \hookrightarrow \mathbf{W}^{\frac{3}{4},4}(\Gamma) \hookrightarrow \mathbf{L}^{\infty}(\Gamma).$$

Consequently, recalling the assumption (M) on the mobility function m_Ω, we readily deduce that $\frac{1}{m_\Omega(\phi)} \in W^{1,\infty}(\Gamma)$. Hence, as $v \in H^2(\Gamma) \hookrightarrow W^{t,p}(\Gamma)$, we find that

$$\partial_{\mathbf{n}} u = \frac{1}{Lm_\Omega(\phi)}(\beta v - u) \in W^{t,p}(\Gamma).$$

Therefore, as by assumption $f_\Gamma, \nabla_\Gamma\psi \cdot \nabla_\Gamma v \in L^p(\Gamma)$, we infer from regularity theory for the Laplace–Beltrami equation (see, e.g., [176, Lemma B.1]) that

$$\begin{aligned}
\|v\|_{W^{2,p}(\Gamma)} &\le C\Big(\|f_\Gamma\|_{L^p(\Gamma)} + \|\partial_{\mathbf{n}} u\|_{W^{t,p}(\Gamma)} + \|\nabla_\Gamma m_\Gamma(\psi)\cdot\nabla_\Gamma v\|_{L^p(\Gamma)}\Big)\\
&\le C\Big(\|f_\Gamma\|_{L^p(\Gamma)} + \|\nabla_\Gamma\psi\cdot\nabla_\Gamma v\|_{L^p(\Gamma)} + \|(u,v)\|_{\mathcal{H}^2}\Big)\\
&\le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^p}\Big).
\end{aligned} \tag{4.76}$$

Then, since $f, \nabla\phi\cdot\nabla u \in L^p(\Omega)$, elliptic regularity theory for Poisson's equation with an inhomogeneous Neumann boundary condition (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) provides that

$$u \in W^{s,p}(\Omega) \qquad \text{with } s = \min\Big\{2, 3 + \frac{d}{p} - \frac{d}{2}\Big\} \ge 1 + \frac{2}{p}$$

together with the estimate

$$\begin{aligned}
\|u\|_{W^{s,p}(\Omega)} &\le C\Big(\|f\|_{L^p(\Omega)} + \|\nabla m_\Omega(\phi)\cdot\nabla u\|_{L^p(\Omega)} + \|\partial_{\mathbf{n}} u\|_{W^{t,p}(\Gamma)}\Big)\\
&\le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^p}\Big).
\end{aligned}$$

Observing that $1 + \frac{2}{p} - \frac{1}{p} = 1 + \frac{1}{p}$ is positive and not an integer, the trace theorem shows that

$$u|_\Gamma \in W^{1+\frac{1}{p},p}(\Gamma),$$

from which we deduce that

$$\partial_{\mathbf{n}} u = \frac{1}{Lm_\Omega(\phi)}(\beta v - u) \in W^{1+\frac{1}{p},p}(\Gamma).$$

Finally, using once more elliptic regularity theory for Poisson's equation with an inhomogeneous Neumann boundary condition (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]), we conclude that $u \in H^2(\Omega)$ along with

$$\begin{aligned}
\|u\|_{W^{2,p}(\Omega)} &\le C\Big(\|f\|_{L^p(\Omega)} + \|\nabla m_\Omega(\phi)\cdot\nabla u\|_{L^p(\Omega)} + \|\partial_{\mathbf{n}} u\|_{W^{1+\frac{1}{p},p}(\Gamma)}\Big)\\
&\le C\Big(\|(f,f_\Gamma)\|_{\mathcal{L}^p} + \|(\nabla\phi\cdot\nabla u, \nabla_\Gamma\psi\cdot\nabla_\Gamma v)\|_{\mathcal{L}^p}\Big).
\end{aligned} \tag{4.77}$$

Combining the estimates (4.76) and (4.77) immediately yields (4.70).

The case $L = \infty$. In this case, both the Poisson–Neumann problem (4.71) and the Laplace–Beltrami equation (4.72) are completely decoupled. Hence, we can apply elliptic regularity theory for Poisson's equation with a homogeneous Neumann boundary condition (see, e.g., [29, Theorem 3.2] or [42, Theorem A.2]) and regularity theory for the Laplace–Beltrami equation (see, e.g., [176, Lemma B.1]), respectively, and directly reach the conclusion with the respective estimates, which finishes the proof. □

We conclude this chapter with the following interpolation inequality.

PROPOSITION 4.4.5. *Let $K \in (0,\infty]$ and $\alpha \in \mathbb{R}$ such that $\alpha\beta|\Omega| + |\Gamma| \neq 0$. Then there exists a constant $C > 0$ such that*

$$\|(f, f_\Gamma)\|_{\mathcal{L}^2} \leq C\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta}\|(f, f_\Gamma)\|^{\frac{1}{2}}_{K,\alpha} + \mathbf{1}_{\{\infty\}}(K)C\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|_{L,\beta} \tag{4.78}$$

for all $(f, f_\Gamma) \in \mathcal{V}^1_{L,\beta}$. Here, $\mathbf{1}_{\{\infty\}}(\cdot)$ denotes the characteristic function of the set $\{\infty\}$.

PROOF. Let $(f, f_\Gamma) \in \mathcal{V}^1_{L,\beta}$. Then, in view of the weak formulation (4.51) it holds that

$$\|(f, f_\Gamma)\|_{\mathcal{L}^2} = \sqrt{\big(\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma), (f, f_\Gamma)\big)_{L,\beta}} \leq \|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta}\|(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta}.$$

If $K \in (0,\infty)$, we can simply use the bulk-surface Poincaré inequality from Lemma 4.2.1 to compute

$$\|(f, f_\Gamma)\|_{L,\beta} \leq C\|(f, f_\Gamma)\|_{\mathcal{H}^1} \leq C\|(f, f_\Gamma)\|_{K,\alpha},$$

and the claim readily follows. In the case $K = \infty$, the bulk-surface Poincaré inequality is not available, and instead we use

$$\begin{aligned}\|(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta} &\leq C\|(f, f_\Gamma)\|^{\frac{1}{2}}_{\mathcal{H}^1} \leq C\Big(\|(f, f_\Gamma)\|^{\frac{1}{2}}_{\mathcal{L}^2} + \|(\nabla f, \nabla_\Gamma f_\Gamma)\|^{\frac{1}{2}}_{\mathcal{L}^2}\Big)\\ &\leq C\Big(\|(f, f_\Gamma)\|^{\frac{1}{2}}_{\mathcal{L}^2} + \|(f, f_\Gamma)\|^{\frac{1}{2}}_{K,\alpha}\Big).\end{aligned}$$

This leads to

$$\begin{aligned}\|(f, f_\Gamma)\|_{\mathcal{L}^2} &\leq C\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta}\Big(\|(f, f_\Gamma)\|^{\frac{1}{2}}_{\mathcal{L}^2} + \|(f, f_\Gamma)\|^{\frac{1}{2}}_{K,\alpha}\Big)\\ &\leq \frac{1}{2}\|(f, f_\Gamma)\|_{\mathcal{L}^2} + C\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|_{L,\beta} + C\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta}\|(f, f_\Gamma)\|^{\frac{1}{2}}_{L,\beta},\end{aligned}$$

and we obtain the corresponding estimate in the case $K = \infty$ after an absorption argument. □

4.5 Bulk-surface Stokes system

This section is based on the works in:

[123] P. KNOPF, AND J. STANGE, *A thermodynamically consistent model for bulk-surface viscous fluid mixtures: Model derivation and mathematical analysis.* Preprint: arXiv:2509.11925, 2025.

[163] J. STANGE, *Global well-posedness of strong solutions to a bulk-surface Navier–Stokes–Cahn–Hilliard model with non-degenerate mobilities in two dimensions*, Preprint: arXiv:2511.06847, 2025.

In this last section, we turn our attention to the study of the following bulk-surface Stokes system:

$$-\operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{4.79a}$$

$$\operatorname{div}\mathbf{v} = g \qquad \text{in } \Omega, \tag{4.79b}$$

$$-\mathrm{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}) + 2\nu_\Omega(\phi)[\mathbf{D}\mathbf{v}\,\mathbf{n}]_{\tan} + \nabla_\Gamma\, q + \gamma(\phi,\psi)\mathbf{w} = \mathbf{f}_\Gamma \qquad \text{on } \Gamma, \tag{4.79c}$$
$$\mathrm{div}_\Gamma\,\mathbf{w} = g_\Gamma \qquad \text{on } \Gamma, \tag{4.79d}$$
$$\mathbf{v}|_\Gamma = \mathbf{w},\ \ \mathbf{v}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{4.79e}$$

Here, $(\mathbf{f},\mathbf{f}_\Gamma) \in (\boldsymbol{\mathcal{H}}^1_0)'$ and $(g, g_\Gamma) \in \mathcal{L}^2_{(0)}$. Moreover, $\phi : \Omega \to \mathbb{R}$ and $\psi : \Gamma \to \mathbb{R}$ are given measurable functions with $|\phi| \le 1$ a.e. in Ω and $|\psi| \le 1$ a.e. on Γ, whereas the coefficients $\nu_\Omega, \nu_\Gamma : [-1,1] \to \mathbb{R}$ and $\gamma : [-1,1]^2 \to \mathbb{R}$ satisfy the following condition:

(V1) It holds $\nu_\Omega, \nu_\Gamma \in C([-1,1]), \gamma \in C([-1,1]^2)$, and there exist positive constants ν_*, ν^*, γ_* and γ^* such that

$$0 < \nu_* \le \nu_\Omega(s), \nu_\Gamma(s) \le \nu^* \qquad \text{for all } s \in [-1,1], \tag{4.80}$$

as well as

$$0 < \gamma_* \le \gamma(s,r) \le \gamma^* \qquad \text{for all } (s,r) \in [-1,1]^2. \tag{4.81}$$

We call $(\mathbf{v},\mathbf{w},p,q) \in \boldsymbol{\mathcal{H}}^1_0 \times \mathcal{L}^2_{(0)}$ a weak solution to system (4.79) if

$$\begin{aligned}
&\int_\Omega 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w} : \mathbf{D}_\Gamma\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi,\psi)\mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&\quad - \int_\Omega p\,\mathrm{div}\,\widehat{\mathbf{v}}\,\mathrm{d}x - \int_\Gamma q\,\mathrm{div}_\Gamma\,\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&= \big\langle(\mathbf{f},\mathbf{f}_\Gamma),(\widehat{\mathbf{v}},\widehat{\mathbf{w}})\big\rangle_{\boldsymbol{\mathcal{H}}^1_0}
\end{aligned} \tag{4.82}$$

for all $(\widehat{\mathbf{v}},\widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_0$, and

$$\mathrm{div}\,\mathbf{v} = g \qquad \text{a.e. in } \Omega, \qquad \mathrm{div}_\Gamma\,\mathbf{w} = g_\Gamma \qquad \text{a.e. on } \Gamma.$$

4.5.1 Well-posedness of weak solutions

To prove the existence of a unique weak solution to (4.79), we start by establishing the following bulk-surface Korn-type inequality, which yields the coercivity of the bilinear form corresponding to the problem.

LEMMA 4.5.1. *There exists a constant $C_K > 0$ such that*

$$\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}^1} \le C_K\Big(\|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{D}_\Gamma\mathbf{w}\|_{\mathbf{L}^2(\Gamma)} + \|\mathbf{w}\|_{\mathbf{L}^2(\Gamma)}\Big)$$

for all $(\mathbf{v},\mathbf{w}) \in \boldsymbol{\mathcal{H}}^1_0$.

PROOF. First, we recall the surface Korn inequality: There exists a constant $C_\Gamma > 0$ such that

$$\|\widetilde{\mathbf{w}}\|_{\mathbf{H}^1(\Gamma)} \le C_\Gamma\Big(\|\mathbf{D}_\Gamma\widetilde{\mathbf{w}}\|_{\mathbf{L}^2(\Gamma)} + \|\widetilde{\mathbf{w}}\|_{\mathbf{L}^2(\Gamma)}\Big) \tag{4.83}$$

for all $\widetilde{\mathbf{w}} \in \mathbf{H}^1(\Gamma)$, see, e.g. [112, Formula (4.7)]. Furthermore, the standard Korn inequality (see [26, Remark IV.7.3.]) states that there exists a constant $C_\Omega > 0$ such that

$$\|\widetilde{\mathbf{v}}\|_{\mathbf{H}^1(\Omega)} \le C_\Omega\|\mathbf{D}\widetilde{\mathbf{v}}\|_{\mathbf{L}^2(\Omega)} \tag{4.84}$$

for all $\widetilde{\mathbf{v}} \in \mathbf{H}^1(\Omega)$ with $\widetilde{\mathbf{v}}|_\Gamma = \mathbf{0}$ a.e. on Γ.

Let now $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}_0^1$ be arbitrary. According to the inverse trace theorem (see [110, Theorem 4.2.3]), there exists a function $\widehat{\mathbf{w}} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$ such that $\widehat{\mathbf{w}} = \mathbf{w}$ a.e. on Γ and there is a constant $C_{\mathrm{tr}} > 0$ independent of $\widehat{\mathbf{w}}$ and $\mathbf{w}$ such that

$$\|\widehat{\mathbf{w}}\|_{\mathbf{H}^{\frac{3}{2}}(\Omega)} \leq C_{\mathrm{tr}} \|\mathbf{w}\|_{\mathbf{H}^1(\Gamma)}. \tag{4.85}$$

In particular, we have $(\mathbf{v} - \widehat{\mathbf{w}})|_\Gamma = 0$ a.e. on Γ. Hence, using Korn's inequality (4.84) and the inverse trace estimate (4.85), we get

$$\begin{aligned} \|\mathbf{v} - \widehat{\mathbf{w}}\|_{\mathbf{H}^1(\Omega)} &\leq C_\Omega \Big(\|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{D}\widehat{\mathbf{w}}\|_{\mathbf{L}^2(\Omega)} \Big) \\ &\leq C_\Omega \Big(\|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + C_{\mathrm{tr}} \|\mathbf{w}\|_{\mathbf{H}^1(\Gamma)} \Big). \end{aligned}$$

Finally, using the surface Korn inequality (4.83), we conclude that

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} &\leq \|\widehat{\mathbf{w}}\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{v} - \widehat{\mathbf{w}}\|_{\mathbf{H}^1(\Omega)} \\ &\leq C_\Omega \|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + C_{\mathrm{tr}}(1 + C_\Omega) \|\mathbf{w}\|_{\mathbf{H}^1(\Gamma)} \\ &\leq C_K \Big(\|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{D}_\Gamma \mathbf{w}\|_{\mathbf{L}^2(\Gamma)} + \|\mathbf{w}\|_{\mathbf{L}^2(\Gamma)} \Big) \end{aligned}$$

for some constant $C_K > 0$ that is independent of $(\mathbf{v}, \mathbf{w})$. This proves the claim. □

We are now in a position to present the following well-posedness result.

THEOREM 4.5.2. *Let $(\mathbf{f}, \mathbf{f}_\Gamma) \in (\boldsymbol{\mathcal{H}}_0^1)'$. Then there exists a unique weak solution $(\mathbf{v}, \mathbf{w}, p, q)$ to (4.79) consisting of a pair of velocity fields $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}_0^1$ and an associated pressure pair $(p, q) \in \mathcal{L}^2_{(0)}$. Moreover, there exists a constant $C > 0$ such tat*

$$\|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{H}}_0^1} + \|(p, q)\|_{\mathcal{L}^2} \leq C \|(\mathbf{f}, \mathbf{f}_\Gamma)\|_{(\boldsymbol{\mathcal{H}}_0^1)'}. \tag{4.86}$$

PROOF. The proof is divided into two steps. First, we consider the case of incompressible fluids, i.e., $g = 0$ and $g_\Gamma = 0$. Then, in a next step, we study the general case $(g, g_\Gamma) \in \mathcal{L}^2_{(0)}$.

Step 1: $g = 0, g_\Gamma = 0$. We start by defining the bilinear form $\boldsymbol{\mathcal{B}} : \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}} \times \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}} \to \mathbb{R}$ as

$$\begin{aligned} \boldsymbol{\mathcal{B}}\big((\mathbf{v}, \mathbf{w}), (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})\big) = &\int_\Omega 2\nu_\Omega(\phi) \mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi) \mathbf{D}_\Gamma \mathbf{w} : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\ &+ \int_\Gamma \gamma(\phi, \psi) \mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \end{aligned}$$

for $(\mathbf{v}, \mathbf{w}), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$. Then, the weak formulation (4.82) can be reformulated as

$$\boldsymbol{\mathcal{B}}\big((\mathbf{v}, \mathbf{w}), (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})\big) = \big\langle (\mathbf{f}, \mathbf{f}_\Gamma), (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}_0^1} \tag{4.87}$$

for all $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$. In view of Lemma 4.5.1 as well as the assumption **(V1)**, we infer that $\boldsymbol{\mathcal{B}}$ is coercive. Consequently, as the right-hand side of (4.87) defines a continuous linear functional on $\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$, the Lax–Milgram theorem implies that, for every $(\mathbf{f}, \mathbf{f}_\Gamma) \in (\boldsymbol{\mathcal{H}}_0^1)'$, there exists a unique pair $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$ which satisfies (4.87) for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$. Moreover, we have

$$\|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{H}}^1} \leq \frac{C_K}{\min\{2\nu_*, \gamma_*\}} \|(\mathbf{f}, \mathbf{f}_\Gamma)\|_{(\boldsymbol{\mathcal{H}}_0^1)'}.$$

Next, to reconstruct the pressure pair (p, q), we need the following lemma, which is an adaptation of [124, Lemma 3.2].

LEMMA 4.5.3. *The operator*

$$\underline{\operatorname{div}} : \boldsymbol{\mathcal{H}}_0^1/\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1 \to \mathcal{L}_{(0)}^2, \qquad (\mathbf{v},\mathbf{w}) \mapsto (\operatorname{div}\mathbf{v}, \operatorname{div}_\Gamma \mathbf{w}), \tag{4.88}$$

is an isomorphism. In particular, there exists a constant $C_{\text{iso}} > 0$ *such that*

$$\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}_0^1/\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1} \le C_{\text{iso}} \|(\operatorname{div}\mathbf{v}, \operatorname{div}_\Gamma \mathbf{w})\|_{\mathcal{L}^2}. \tag{4.89}$$

PROOF. We start by considering the operator

$$\underline{\widetilde{\operatorname{div}}} : \boldsymbol{\mathcal{H}}_0^1 \to \mathcal{L}_{(0)}^2, \qquad (\mathbf{v},\mathbf{w}) \mapsto (\operatorname{div}\mathbf{v}, \operatorname{div}_\Gamma \mathbf{w}).$$

This operator is clearly well-defined with $\ker \underline{\widetilde{\operatorname{div}}} = \boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1$. We now show that $\underline{\widetilde{\operatorname{div}}}$ is surjective. To this end, let $(g, g_\Gamma) \in \mathcal{L}_{(0)}^2$ be arbitrary. Then, let $\mathbf{w} \in \mathbf{H}^1(\Gamma)$ with $\mathbf{w}\cdot\mathbf{n} = 0$ on Γ be a solution of

$$\operatorname{div}_\Gamma \mathbf{w} = g_\Gamma \qquad \text{on } \Gamma.$$

The existence of such a solution $\mathbf{w}$ can be shown by employing L^2-theory for the Laplace–Beltrami operator, noticing that the right-hand side of mean-value free, see, e.g., [59]. Next, utilizing L^2-theory of the divergence operator in Ω (see, e.g, [26, 82]), we find $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that

$$\begin{aligned} \operatorname{div}\mathbf{v} &= g &&\text{in } \Omega,\\ \mathbf{v} &= \mathbf{w} &&\text{on } \Gamma. \end{aligned}$$

We have therefore shown the existence of $(\mathbf{v},\mathbf{w}) \in \boldsymbol{\mathcal{H}}_0^1$ satisfying

$$\underline{\widetilde{\operatorname{div}}}(\mathbf{v},\mathbf{w}) = (g, g_\Gamma),$$

which proves that $\underline{\widetilde{\operatorname{div}}} : \boldsymbol{\mathcal{H}}_0^1 \to \mathcal{L}_{(0)}^2$ is a surjective homomorphism. Consequently, by the fundamental theorem of homomorphisms, we conclude that $\underline{\operatorname{div}} : \boldsymbol{\mathcal{H}}_0^1/\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1 \to \mathcal{L}_{(0)}^2$ is an isomorphism. $\square$

As a corollary, we obtain the following (see also [124, Corollary 3.3]):

COROLLARY 4.5.4. *The operator* $\underline{\nabla} : \mathcal{L}_{(0)}^2 \to (\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1)^\perp \subset (\boldsymbol{\mathcal{H}}_0^1)'$*, defined by*

$$\underline{\nabla}(p,q)(\mathbf{v},\mathbf{w}) = -\int_\Omega p \operatorname{div}\mathbf{v} \, \mathrm{d}x - \int_\Gamma q \operatorname{div}_\Gamma \mathbf{w} \, \mathrm{d}\Gamma, \qquad (\mathbf{v},\mathbf{w}) \in \boldsymbol{\mathcal{H}}_0^1,$$

is an isomorphism. Moreover, we have

$$\|(p,q)\|_{\mathcal{L}^2} \le C_{\text{iso}} \|\underline{\nabla}(p,q)\|_{(\boldsymbol{\mathcal{H}}_0^1)'}, \tag{4.90}$$

where C_{iso} *is the same constant as in* (4.89).

PROOF. The claim readily follows from Lemma 4.5.3, the fact that

$$\underline{\nabla} = -\underline{\operatorname{div}}' : (\mathcal{L}_{(0)}^2)' \to (\boldsymbol{\mathcal{H}}_0^1/\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1)'$$

is an isomorphism, form the isometry $(\boldsymbol{\mathcal{H}}_0^1/\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1)' \simeq (\boldsymbol{\mathcal{H}}_{0,\operatorname{div}}^1)^\perp$, and by Riesz' isometry $\mathcal{L}_{(0)}^2 \simeq (\mathcal{L}_{(0)}^2)'$. Recall also that forming the adjoint preserves the operator norm. $\square$

Now, Corollary 4.5.4 allows us to reconstruct the pressure pair (p, q). Namely, considering the unique weak solution $(\mathbf{v}, \mathbf{w})$ that was constructed above by the Lax–Milgram theorem, Corollary 4.5.4 yields the existence of a unique pair $(p, q) \in \mathcal{L}^2_{(0)}$ such that

$$\begin{aligned} \underline{\nabla}(p,q)(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = \big\langle (\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} &- \int_\Omega 2\nu_\Omega(\phi) \mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x \\ &- \int_\Gamma 2\nu_\Gamma(\psi) \mathbf{D}_\Gamma \mathbf{w} : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma - \int_\Gamma \gamma(\phi,\psi) \mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \end{aligned}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_0$. Recalling the weak formulation (4.87), we readily deduce that

$$\begin{aligned} &\int_\Omega 2\nu_\Omega(\phi) \mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi) \mathbf{D}_\Gamma \mathbf{w} : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi,\psi) \mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\ &\qquad - \int_\Omega p \operatorname{div} \widehat{\mathbf{v}} \,\mathrm{d}x - \int_\Gamma q \operatorname{div}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\ &= \big\langle (\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} \end{aligned}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_0$.

Step 2: $(g, g_\Gamma) \in \mathcal{L}^2_{(0)}$. In this step, we now consider general $(g, g_\Gamma) \in \mathcal{L}^2_{(0)}$. Then, according to Lemma 4.5.3, there exists a unique pair $(\mathbf{v}_0, \mathbf{w}_0) \in \boldsymbol{\mathcal{H}}^1_0$ such that

$$\underline{\operatorname{div}}(\mathbf{v}_0, \mathbf{w}_0) = (g, g_\Gamma).$$

Next, we define the continuous and linear functional $(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_\Gamma) \in (\boldsymbol{\mathcal{H}}^1_0)'$ via

$$\begin{aligned} \big\langle (\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} := \big\langle (\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} &- \int_\Omega 2\nu_\Omega(\phi) \mathbf{D}\mathbf{v}_0 : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x \\ &- \int_\Gamma 2\nu_\Gamma(\psi) \mathbf{D}_\Gamma \mathbf{w}_0 : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma - \int_\Gamma \gamma(\phi,\psi) \mathbf{w}_0 \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \end{aligned}$$

for $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_0$. Employing again the bulk-surface Korn inequality from Lemma 4.5.1 and recalling the assumption **(V1)** on the coefficients, the Lax–Milgram theorem provides the existence of a unique pair $(\mathbf{v}_1, \mathbf{w}_1) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$ such that

$$\begin{aligned} &\int_\Omega 2\nu_\Omega(\phi) \mathbf{D}\mathbf{v}_1 : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi) \mathbf{D}_\Gamma \mathbf{w}_1 : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi,\psi) \mathbf{w}_1 \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\ &\quad = \big\langle (\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} \end{aligned}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_0$. Then, by Corollary 4.5.4, there exists a unique pair $(p, q) \in \mathcal{L}^2_{(0)}$ satisfying

$$\begin{aligned} \underline{\nabla}(p,q)(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = \big\langle (\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} &- \int_\Omega 2\nu_\Omega(\phi) \mathbf{D}\mathbf{v}_1 : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x \\ &- \int_\Gamma 2\nu_\Gamma(\psi) \mathbf{D}_\Gamma \mathbf{w}_1 : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma - \int_\Gamma \gamma(\phi,\psi) \mathbf{w}_1 \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \end{aligned}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_0$. Now, we define $(\mathbf{v}, \mathbf{w}) := (\mathbf{v}_0 + \mathbf{v}_1, \mathbf{w}_0 + \mathbf{w}_1)$. Then

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \operatorname{div} \mathbf{v}_0 + \operatorname{div} \mathbf{v}_1 = g && \text{in } \Omega, \\ \operatorname{div}_\Gamma \mathbf{w} &= \operatorname{div}_\Gamma \mathbf{w}_0 + \operatorname{div}_\Gamma \mathbf{w}_1 = g_\Gamma && \text{on } \Gamma, \end{aligned}$$

and, by construction of (p, q) and the definition of $(\tilde{\mathbf{f}}, \tilde{\mathbf{f}}_\Gamma)$, we readily notice that $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}_0^1$ solves

$$\begin{aligned} &\int_\Omega 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w} : \mathbf{D}_\Gamma\widehat{\mathbf{w}} \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi,\psi)\mathbf{w}\cdot\widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\ &\qquad - \int_\Omega p \operatorname{div}\widehat{\mathbf{v}} \,\mathrm{d}x - \int_\Gamma q \operatorname{div}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\ &\quad = \langle (\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \rangle_{\boldsymbol{\mathcal{H}}_0^1} \end{aligned} \tag{4.91}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}_0^1$. In particular, $(\mathbf{v}, \mathbf{w}, p, q)$ is a weak solution to (4.79). □

4.5.2 Existence of strong solutions and higher regularity for constant coefficients

In the next step, assuming that the coefficients ν_Ω, ν_Γ and γ are positive constants, we prove that system (4.79) admits a unique strong solution. Furthermore, we restrict ourselves to the case of incompressible fluids, as it is the more relevant case for our subsequent analysis of system (1.42). The proof of the existence of a unique strong solution relies on a fixed-point argument. To this end, we start by considering the surface Stokes equation and the corresponding surface Stokes operator.

Let $r \in [2, \infty)$ be arbitrary. We define the surface Stokes operator by

$$\mathbf{A}^\Gamma(\mathbf{w}) = -\mathbf{P}_{\mathrm{div}}^\Gamma \operatorname{div}_\Gamma(2\nu_\Gamma \mathbf{D}_\Gamma \mathbf{w}) \qquad \text{for all } \mathbf{w} \in D(\mathbf{A}^\Gamma) = \mathbf{W}_{\mathrm{tan}}^{2,r}(\Gamma) \cap \mathbf{L}_{\mathrm{div}}^r(\Gamma).$$

We further refer to [144, 159] for the surface Stokes operator on more general surfaces, to [10] for the surface Stokes operator on evolving surfaces, and to the survey article [107] for the Stokes operator in various other geometric settings.

The following theorem ensures the existence of a unique strong solution to the surface Stokes problem.

Lemma 4.5.5. *Let $\mathbf{f}_\Gamma \in \mathbf{L}_{\mathrm{tan}}^r(\Gamma)$. Then there exists a unique strong solution $(\mathbf{w}, q)$ with $\mathbf{w} \in \mathbf{W}_{\mathrm{tan}}^{2,r}(\Gamma) \cap \mathbf{L}_{\mathrm{div}}^r(\Gamma)$ and $q \in W^{1,r}(\Gamma) \cap L_{(0)}^r(\Gamma)$ of the surface Stokes problem*

$$-\operatorname{div}_\Gamma(2\nu_\Gamma \mathbf{D}_\Gamma \mathbf{w}) + \nabla_\Gamma q + \gamma \mathbf{w} = \mathbf{f}_\Gamma \qquad \text{on } \Gamma, \tag{4.92}$$

$$\operatorname{div}_\Gamma \mathbf{w} = 0 \qquad \text{on } \Gamma. \tag{4.93}$$

Moreover, there exists a constant $C > 0$, independent of $\mathbf{f}_\Gamma$, such that

$$\|\mathbf{w}\|_{\mathbf{W}^{2,r}(\Gamma)} + \|q\|_{W^{1,r}(\Gamma)} \le C\|\mathbf{f}_\Gamma\|_{\mathbf{L}^r(\Gamma)}. \tag{4.94}$$

Proof. The definition of the projection $\mathbf{P}_{\mathrm{div}}^\Gamma$ implies that there exists a function $q_1 \in \dot{W}^{1,r}(\Gamma)$ such that

$$\mathbf{f}_\Gamma = \mathbf{P}_{\mathrm{div}}^\Gamma \mathbf{f}_\Gamma + \nabla_\Gamma q_1 \quad \text{a.e. on } \Gamma. \tag{4.95}$$

We further know from [144, Corollary 3.4] that the operator $\mathbf{A}^\Gamma$ has maximal L^p-regularity. Hence, $\{z \in \mathbb{C} : \operatorname{Re} z < 0\} \subset \rho(\mathbf{A}^\Gamma)$, and there exists a constant $M \ge 1$ such that for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, $z + \mathbf{A}^\Gamma$ is invertible with

$$\|(z + \mathbf{A}^\Gamma)^{-1}\|_{\mathrm{op}} \le \frac{M}{1 + |z|}, \tag{4.96}$$

see, e.g., [143, Proposition 3.5.2]. Consequently, we find $\mathbf{w} \in D(\mathbf{A}^\Gamma)$ such that

$$\gamma \mathbf{w} + \mathbf{A}^\Gamma(\mathbf{w}) = \mathbf{P}_{\mathrm{div}}^\Gamma \mathbf{f}_\Gamma \qquad \text{a.e. on } \Gamma. \tag{4.97}$$

In view of (4.96), it holds that

$$\|\mathbf{w}\|_{\mathbf{W}^{2,r}(\Gamma)} \leq \frac{M}{1+\gamma}\|\mathbf{f}_\Gamma\|_{\mathbf{L}^r(\Gamma)}.$$

As $\mathbf{w} \in \mathbf{W}^{2,r}_{\tan}(\Gamma) \cap \mathbf{L}^r_{\mathrm{div}}(\Gamma)$, we further have

$$-\mathrm{div}_\Gamma(2\nu_\Gamma \mathbf{D}_\Gamma \mathbf{w}),\ \mathbf{A}^\Gamma(\mathbf{w}) \in \mathbf{L}^r(\Gamma) \qquad \text{and} \qquad \mathrm{div}_\Gamma\, \mathbf{w} = 0 \quad \text{a.e. on } \Gamma.$$

Hence, by the definition of the projection $\mathbf{P}^\Gamma_{\mathrm{div}}$, we find a function $q_2 \in \dot{W}^{1,r}(\Gamma)$ such that

$$-\mathrm{div}_\Gamma(2\nu_\Gamma \mathbf{D}_\Gamma \mathbf{w}) = \mathbf{A}^\Gamma(\mathbf{w}) - \nabla_\Gamma q_2 \quad \text{a.e. on } \Gamma. \tag{4.98}$$

Defining $q \coloneqq q_1 + q_2$ and combining (4.95), (4.97) and (4.98), we conclude that

$$-\mathrm{div}_\Gamma(2\nu_\Gamma \mathbf{D}_\Gamma \mathbf{w}) + \gamma\mathbf{w} + \nabla_\Gamma q = \mathbf{A}^\Gamma(\mathbf{w}) + \gamma\mathbf{w} + \nabla_\Gamma q_1 = \mathbf{f}_\Gamma \quad \text{a.e. on } \Gamma. \tag{4.99}$$

Moreover, the choice of q is unique if we additionally demand $\langle q\rangle_\Gamma = 0$. Then, the Poincaré–Wirtinger inequality yields $q \in W^{1,r}(\Gamma)$. In this way, $(\mathbf{w}, q)$ is the unique strong solution of (4.92) with the desired regularities. □

We are now in a position to prove the existence of a strong solution to (4.79).

THEOREM 4.5.6. *Assume that the coefficients ν_Ω, ν_Γ and γ are constant, and let $(\mathbf{f}, \mathbf{f}_\Gamma) \in \mathbf{L}^r(\Omega) \times \mathbf{L}^r_{\tan}(\Gamma)$. Then the unique weak solution to (4.79) satisfies $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{W}}^{2,r}_{0,\mathrm{div}}$ and $(p, q) \in \mathcal{W}^{1,r} \cap \mathcal{L}^r_{(0)}$. Moreover, there exists a constant $C > 0$ such that*

$$\|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{W}}^{2,r}} + \|(p, q)\|_{\mathcal{W}^{1,r}} \leq C\|(\mathbf{f}, \mathbf{f}_\Gamma)\|_{\boldsymbol{\mathcal{L}}^r}. \tag{4.100}$$

PROOF. Let $\widetilde{\mathbf{w}} \in \mathbf{W}^{2-\frac{1}{r},r}_{\mathrm{div}}(\Gamma)$ be arbitrary. Then, according to [168, Proposition 2.2 and Proposition 2.3], there exists a unique strong solution $\mathbf{v} \in \mathbf{W}^{2,r}_{\mathrm{div}}(\Omega)$ and $p \in W^{1,r}(\Omega) \cap L^r_{(0)}(\Omega)$ of the bulk Stokes problem

$$-\mathrm{div}(2\nu_\Omega \mathbf{D}\mathbf{v}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{4.101a}$$

$$\mathrm{div}\, \mathbf{v} = 0 \qquad \text{in } \Omega, \tag{4.101b}$$

$$\mathbf{v} = \widetilde{\mathbf{w}} \qquad \text{on } \Gamma. \tag{4.101c}$$

Furthermore, [168, Proposition 2.2] provides the estimate

$$\|\mathbf{v}\|_{\mathbf{W}^{2,r}(\Omega)} + \|p\|_{W^{1,r}(\Omega)} \leq C\Big(\|\mathbf{f}\|_{\mathbf{L}^r(\Omega)} + \|\widetilde{\mathbf{w}}\|_{\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)}\Big). \tag{4.102}$$

By the trace theorem, we have $\mathbf{D}\mathbf{v} \in L^r(\Gamma; \mathbb{R}^{d\times d})$ and thus,

$$\mathbf{f}_\Gamma - 2\nu_\Omega\big[\mathbf{D}\mathbf{v}\,\mathbf{n}\big]_{\tan} \in \mathbf{L}^r_{\tan}(\Gamma).$$

Hence, according to Lemma 4.5.5, there exists a unique strong solution $\mathbf{w} \in \mathbf{W}^{2,r}_{\mathrm{div}}(\Gamma)$ and $q \in W^{1,r}(\Gamma) \cap L^r_{(0)}$ of the surface Stokes problem

$$-\mathrm{div}_\Gamma(2\nu_\Gamma \mathbf{D}_\Gamma \mathbf{w}) + \nabla_\Gamma q + \gamma\mathbf{w} = \mathbf{f}_\Gamma - 2\nu_\Omega\big[\mathbf{D}\mathbf{v}\,\mathbf{n}\big]_{\tan} \qquad \text{on } \Gamma,$$

$$\mathrm{div}_\Gamma\, \mathbf{w} = 0 \qquad \text{on } \Gamma.$$

This defines an operator $T : \mathbf{W}^{2-\frac{1}{r},r}_{\mathrm{div}}(\Gamma) \to \mathbf{W}^{2-\frac{1}{r},r}_{\mathrm{div}}(\Gamma)$, $\widetilde{\mathbf{w}} \mapsto T\widetilde{\mathbf{w}} = \mathbf{w}$ with $T\widetilde{\mathbf{w}} \in \mathbf{W}^{2,r}_{\mathrm{div}}(\Gamma)$ for all $\widetilde{\mathbf{w}} \in \mathbf{W}^{2-\frac{1}{r},r}_{\mathrm{div}}(\Gamma)$. Using the estimates (4.94) and (4.102), we deduce that

$$\begin{aligned} \|T\widetilde{\mathbf{w}}_1 - T\widetilde{\mathbf{w}}_2\|_{\mathbf{W}^{2,r}(\Gamma)} &\le C\|\mathbf{D}\mathbf{v}_1\,\mathbf{n} - \mathbf{D}\mathbf{v}_2\,\mathbf{n}\|_{\mathbf{L}^r(\Gamma)} \le C\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{2,r}(\Omega)} \\ &\le C\|\widetilde{\mathbf{w}}_1 - \widetilde{\mathbf{w}}_2\|_{\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)}. \end{aligned} \tag{4.103}$$

In particular, this implies that T is continuous. Furthermore, since $\mathbf{W}^{2,r}(\Gamma)$ is compactly embedded in $\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)$, T is compact. To apply the Leray–Schauder fixed-point theorem, we are left to show that the set

$$\Big\{\widetilde{\mathbf{w}} \in \mathbf{W}^{2-\frac{1}{r},r}_{\mathrm{div}}(\Gamma) \;:\; \widetilde{\mathbf{w}} = \lambda T\widetilde{\mathbf{w}} \text{ for some } 0 \le \lambda \le 1\Big\}$$

is bounded. To this end, let $\widetilde{\mathbf{w}} \in \mathbf{W}^{2-\frac{1}{r},r}_{\mathrm{div}}(\Gamma)$ such that $\widetilde{\mathbf{w}} = \lambda T\widetilde{\mathbf{w}}$ for some $\lambda \in [0,1]$. Without loss of generality, we assume that $\lambda > 0$ since otherwise $\widetilde{\mathbf{w}} = \mathbf{0}$. By definition of T, it then holds

$$-\mathrm{div}_\Gamma\Big(2\nu_\Gamma \mathbf{D}_\Gamma \frac{\widetilde{\mathbf{w}}}{\lambda}\Big) + \nabla_\Gamma q + \gamma\frac{\widetilde{\mathbf{w}}}{\lambda} = \mathbf{f}_\Gamma - 2\nu_\Omega\big[\mathbf{D}\mathbf{v}\,\mathbf{n}\big]_{\tan} \qquad \text{a.e. on } \Gamma, \tag{4.104a}$$

$$\mathrm{div}_\Gamma \frac{\widetilde{\mathbf{w}}}{\lambda} = 0 \qquad \text{a.e. on } \Gamma. \tag{4.104b}$$

Moreover, we have

$$-\mathrm{div}(2\nu_\Omega \mathbf{D}\mathbf{v}) + \nabla p = \mathbf{f} \qquad \text{a.e. in } \Omega, \tag{4.105a}$$

$$\mathrm{div}\,\mathbf{v} = 0 \qquad \text{a.e. in } \Omega, \tag{4.105b}$$

$$\mathbf{v} = \widetilde{\mathbf{w}} \qquad \text{a.e. on } \Gamma. \tag{4.105c}$$

Now, we multiply (4.105a) by $\lambda\mathbf{v}$ and (4.104a) by $\lambda\widetilde{\mathbf{w}}$ and integrate the resulting equations. After integrating by parts, we get

$$\begin{aligned} &\lambda\int_\Omega 2\nu_\Omega|\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma|\mathbf{D}_\Gamma\widetilde{\mathbf{w}}|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma|\widetilde{\mathbf{w}}|^2 \,\mathrm{d}\Gamma \\ &\quad = \lambda\int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x + \lambda\int_\Gamma \mathbf{f}_\Gamma\cdot\widetilde{\mathbf{w}}\,\mathrm{d}\Gamma. \end{aligned}$$

Since $\lambda \in (0,1]$, it directly follows that

$$\begin{aligned} &\int_\Omega 2\nu_\Omega|\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma|\mathbf{D}_\Gamma\widetilde{\mathbf{w}}|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma|\widetilde{\mathbf{w}}|^2 \,\mathrm{d}\Gamma \\ &\quad \le \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x + \int_\Gamma \mathbf{f}_\Gamma\cdot\widetilde{\mathbf{w}}\,\mathrm{d}\Gamma. \end{aligned}$$

Hence, employing the bulk-surface Korn inequality from Lemma 4.5.1, we deduce that

$$\|(\mathbf{v},\widetilde{\mathbf{w}})\|_{\boldsymbol{\mathcal{H}}^1} \le \frac{C_K}{\sqrt{\min\{2\nu_\Omega, 2\nu_\Gamma, \gamma\}}}\|(\mathbf{f},\mathbf{f}_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2}.$$

Then, using the Gagliardo–Nirenberg inequality and arguing as in (4.103), we obtain

$$\begin{aligned} \|\widetilde{\mathbf{w}}\|_{\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)} &\le C\|\widetilde{\mathbf{w}}\|^{1-\theta}_{\mathbf{H}^1(\Gamma)}\|\widetilde{\mathbf{w}}\|^{\theta}_{\mathbf{W}^{2,r}(\Gamma)} \\ &\le C\|(\mathbf{f},\mathbf{f}_\Gamma)\|^{1-\theta}_{\boldsymbol{\mathcal{L}}^2}\Big(\|\mathbf{f}\|_{\mathbf{L}^r(\Omega)} + \|\widetilde{\mathbf{w}}\|_{\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)}\Big)^{\theta} \end{aligned}$$

$$\leq C\Big(1 + \|\widetilde{\mathbf{w}}\|^{\theta}_{\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)}\Big)$$

for some $\theta \in (0,1)$, which readily yields the bound

$$\|\widetilde{\mathbf{w}}\|_{\mathbf{W}^{2-\frac{1}{r},r}(\Gamma)} \leq C.$$

This allows us to apply the Leray–Schauder fixed-point theorem, which implies that T has a fixed-point $\mathbf{w}_* \in \mathbf{W}^{2,r}_{\mathrm{div}}(\Gamma)$ with $T\mathbf{w}_* = \mathbf{w}_*$.

Finally, we write q_* to denote the pressure associated with $\mathbf{w}_*$, and we define $(\mathbf{v}_*, p_*)$ as the corresponding unique strong solution to the bulk Stokes problem (4.101). In this way, we have constructed functions

$$(\mathbf{v}_*, \mathbf{w}_*) \in \boldsymbol{\mathcal{W}}^{2,r}_{0,\mathrm{div}}, \qquad (p_*, q_*) \in \mathcal{W}^{1,r} \cap \mathcal{L}^r_{(0)},$$

which are a strong solution to the bulk-surface Stokes system (4.79). As every strong solution is also a weak solution, we conclude from Theorem 4.5.2 that the strong solution is unique. As a last step, we have to verify the inequality (4.100). To this end, we recall the trace embedding $W^{\frac{3}{4},r}(\Omega) \hookrightarrow L^r(\Gamma)$ (see, e.g., [29, Part I, Chapter 2, Theorem 2.24]), which holds since $\frac{3}{4} > \frac{1}{2} \geq \frac{1}{r}$. Using this embedding along with Sobolev embeddings, Ehrling's lemma and estimate (4.86), we obtain

$$\begin{aligned}
\|\mathbf{D}\mathbf{v}_*\|_{\mathbf{L}^r(\Gamma)} &\leq C\,\|\mathbf{D}\mathbf{v}_*\|_{\mathbf{W}^{\frac{3}{4},r}(\Omega)} \\
&\leq C\,\|\mathbf{v}_*\|_{\mathbf{W}^{\frac{7}{4},r}(\Omega)} \\
&\leq \varepsilon\|\mathbf{v}_*\|_{\mathbf{W}^{2,r}(\Omega)} + C_\varepsilon\|\mathbf{v}_*\|_{\mathbf{H}^1(\Omega)} \\
&\leq \varepsilon\|\mathbf{v}_*\|_{\mathbf{W}^{2,r}(\Omega)} + C_\varepsilon\, C\,\|(\mathbf{f}, \mathbf{f}_\Gamma)\|_{\boldsymbol{\mathcal{L}}^r}
\end{aligned} \tag{4.106}$$

for any $\varepsilon > 0$ and some constant $C_\varepsilon > 0$ that may depend on ε and the same quantities as the constants denoted by C. Fixing a sufficiently small $\varepsilon > 0$, the desired inequality (4.100) follows by combining (4.106) with the estimates (4.94) and (4.102). This completes the proof. ☐

4.5.3 Existence of strong solutions for non-constant coefficients

In this section, we handle the more delicate case of variable coefficients ν_Ω, ν_Γ and γ, and prove the existence of a unique strong solution to (4.79) in this case. To do so, we need the following stronger assumption on the viscosities:

(V2) It holds $\nu_\Omega, \nu_\Gamma \in C^2([-1,1]), \gamma \in C([-1,1]^2)$, and there exist positive constants ν_*, ν^*, γ_* and γ^* such that

$$0 < \nu_* \leq \nu_\Omega(s), \nu_\Gamma(s) \leq \nu^* \qquad \text{for all } s \in [-1,1] \tag{4.107}$$

as well as

$$0 < \gamma_* \leq \gamma(s,r) \leq \gamma^* \qquad \text{for all } (s,r) \in [-1,1]^2. \tag{4.108}$$

The assumption **(V2)** differs from **(V1)** in the regularity of the viscosities, namely, in **(V1)**, it was only assumed that $\nu_\Omega, \nu_\Gamma \in C([-1,1])$. As for the case of constant coefficients, the proof of the existence of a unique strong solution to (4.79) assuming **(V2)** is based on a fixed-point argument. To this end, we start by establishing the existence of a unique strong solution to the bulk and the surface Stokes equation, respectively.

The following lemma provides the existence of a unique strong solution to the non-homogeneous Stokes equation on Ω with non-constant viscosity.

LEMMA 4.5.7. *Let* $\nu_\Omega \in C^2([-1,1])$ *satisfy* (4.107), $\phi \in W^{1,\infty}(\Omega)$, $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *and* $\mathbf{w}_* \in \mathbf{H}^{\frac{3}{2}}_{\tan}(\Gamma)$. *Then, there exists a unique solution* $(\mathbf{v}, p) \in \mathbf{H}^2(\Omega) \times (H^1(\Omega) \cap L^2_{(0)}(\Omega))$ *to*

$$-\operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{4.109a}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \Omega, \tag{4.109b}$$

$$\mathbf{v} = \mathbf{w}_* \qquad \text{on } \Gamma. \tag{4.109c}$$

Furthermore, there exists a constant $C > 0$ *such that*

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)} \le C\Big(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{w}_*\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)}\Big). \tag{4.110}$$

PROOF. **Uniqueness.** According to the Lemma of Lax–Milgram, the corresponding homogeneous problem has a unique weak solution. As the zero solution trivially solves the homogeneous problem, the uniqueness of (4.109) readily follows.
Existence. We first consider the system

$$\begin{aligned} -\operatorname{div}(2\mathbf{D}\mathbf{v}) + \nabla p &= \mathbf{f} && \text{in } \Omega,\\ \operatorname{div} \mathbf{v} &= 0 && \text{in } \Omega,\\ \mathbf{v} &= \mathbf{w}_* && \text{on } \Gamma. \end{aligned}$$

The existence of a unique solution $(\mathbf{v}_1, p_1) \in \mathbf{H}^2(\Omega) \times (H^1(\Omega) \cap L^2_{(0)}(\Omega))$ has been proven in [34] for $d = 3$, and in [168] for $d = 2$, along with the existence of a constant $C_1 > 0$ such that

$$\|\mathbf{v}_1\|_{\mathbf{H}^2(\Omega)} + \|p_1\|_{H^1(\Omega)} \le C_1\Big(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{w}_*\|_{\mathbf{H}^{\frac{3}{2}}(\Gamma)}\Big).$$

Now, we consider

$$-\operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) + \nabla p = \mathbf{f} + \operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_1) \qquad \text{in } \Omega, \tag{4.111a}$$

$$\operatorname{div} \mathbf{v} = 0 \qquad \text{in } \Omega, \tag{4.111b}$$

$$\mathbf{v} = 0 \qquad \text{on } \Gamma. \tag{4.111c}$$

As the right-hand side of (4.111a) lies in $\mathbf{L}^2(\Omega)$ by assumption on ν_Ω and the regularity of $\mathbf{v}_1$, using the Lemma of Lax–Milgram, one can show the existence of a unique $\mathbf{v}_2 \in \mathbf{H}^1_0(\Omega)$ satisfying

$$\int_\Omega 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_2 : \mathbf{D}\hat{\mathbf{v}} \,\mathrm{d}x = \int_\Omega (\mathbf{f} + \operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_1)) \cdot \hat{\mathbf{v}} \,\mathrm{d}x$$

for all $\hat{\mathbf{v}} \in \mathbf{H}^1_{0,\mathrm{div}}(\Omega)$. Hence, according to [12, Section 4, Lemma 4], we additionally find that $\mathbf{v}_2 \in \mathbf{H}^2(\Omega)$. Then, applying integration by parts, for all $\hat{\mathbf{v}} \in \mathbf{H}^1_{0,\mathrm{div}}(\Omega)$ it holds that

$$\int_\Omega \Big(-\operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_2) - \mathbf{f} - \operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_1)\Big) \cdot \hat{\mathbf{v}} \,\mathrm{d}x = 0.$$

Consequently, [161, Lemma II.2.2.2] yields the existence of $p_2 \in L^2_{(0)}(\Omega)$ such that

$$\nabla p_2 = -\operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_2) - \mathbf{f} - \operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}_1) \tag{4.112}$$

in the sense of distributions. As the right-hand side of (4.112) is an element of $\mathbf{L}^2(\Omega)$, we conclude $\nabla p_2 \in \mathbf{L}^2(\Omega)$, and thus, $(\mathbf{v}_2, p_2)$ is a strong solution to (4.111). Moreover, we have the following estimate

$$\|\mathbf{v}_2\|_{\mathbf{H}^2(\Omega)} + \|p_2\|_{H^1(\Omega)} \le C_2\Big(\|\mathbf{v}_1\|_{\mathbf{H}^2(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\Big)$$

for a constant $C_2 > 0$. Finally, defining $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ and $p = p_2$, we infer that $(\mathbf{v}, p)$ is a solution to (4.109) with the desired regularities and satisfies the estimate (4.110). □

Next, we consider the surface Stokes equation, and have the following result concerning the existence of a unique strong solution.

LEMMA 4.5.8. *Let* $\nu_\Gamma \in C^2([-1,1])$ *and* $\gamma \in C([-1,1]^2)$ *satisfy* (4.107) *and* (4.108), *respectively, and let* $\psi \in W^{1,\infty}(\Gamma)$ *and* $\mathbf{f}_\Gamma \in \mathbf{L}^2_{\mathrm{tan}}(\Gamma)$. *Then, there exists a unique solution* $(\mathbf{w}, q) \in \mathbf{H}^2_{\mathrm{div}}(\Gamma) \times (H^1(\Gamma) \cap L^2_{(0)}(\Gamma))$ *to*

$$-\mathrm{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}) + \gamma(\phi,\psi)\mathbf{w} + \nabla_\Gamma q = \mathbf{f}_\Gamma \qquad \text{on } \Gamma, \tag{4.113a}$$

$$\mathrm{div}_\Gamma\, \mathbf{w} = 0 \qquad \text{on } \Gamma. \tag{4.113b}$$

Additionally, there exists a constant $C > 0$ *such that*

$$\|\mathbf{w}\|_{\mathbf{H}^2(\Gamma)} + \|q\|_{H^1(\Gamma)} \le C\|\mathbf{f}_\Gamma\|_{\mathbf{L}^2(\Gamma)}. \tag{4.114}$$

PROOF. Recalling the surface Korn inequality (4.83), and noting on the uniform positivity of both ν_Γ and γ , the Lemma of Lax–Milgram guarantees the existence of a unique $\mathbf{w} \in \mathbf{H}^1_{\mathrm{div}}(\Gamma)$ satisfying

$$\int_\Gamma 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w} : \mathbf{D}_\Gamma\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi,\psi)\mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma = \int_\Gamma \mathbf{f}_\Gamma\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \tag{4.115}$$

for all $\widehat{\mathbf{w}} \in \mathbf{H}^1_{\mathrm{div}}(\Gamma)$ along with a constant $C > 0$ such that

$$\|\mathbf{w}\|_{\mathbf{H}^1(\Gamma)} \le C\|\mathbf{f}_\Gamma\|_{\mathbf{L}^2(\Gamma)}. \tag{4.116}$$

Next, we observe that (4.115) can be reformulated as

$$\int_\Gamma 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w} : \mathbf{D}_\Gamma\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \int_\Gamma \mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma = \int_\Gamma \mathbf{F}_\Gamma\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \tag{4.117}$$

for all $\widehat{\mathbf{w}} \in \mathbf{H}^1_{\mathrm{div}}(\Gamma)$, where $\mathbf{F}_\Gamma = \mathbf{f}_\Gamma + (1-\gamma(\phi,\psi))\mathbf{w} \in \mathbf{L}^2(\Gamma)$. In view of (4.108) and (4.116) , it holds $\|\mathbf{F}_\Gamma\|_{\mathbf{L}^2(\Gamma)} \le C\|\mathbf{f}_\Gamma\|_{\mathbf{L}^2(\Gamma)}$. Consequently, utilizing regularity theory for (4.117) (see [8, Lemma 7.4]), we deduce $\mathbf{w} \in \mathbf{H}^2(\Gamma)$ with

$$\|\mathbf{w}\|_{\mathbf{H}^2(\Gamma)} \le C\|\mathbf{F}_\Gamma\|_{\mathbf{L}^2(\Gamma)} \le C\|\mathbf{f}_\Gamma\|_{\mathbf{L}^2(\Gamma)}$$

for some constant $C > 0$. Finally, as $\mathbf{w} \in \mathbf{H}^2(\Gamma)$, an application of integration by parts yields that

$$\int_\Gamma \mathbf{P}^\Gamma_{\mathrm{div}}\big(-\mathrm{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}) + \gamma(\phi,\psi)\mathbf{w} - \mathbf{f}_\Gamma\big)\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma = 0$$

for all $\widehat{\mathbf{w}} \in \mathbf{H}^1(\Gamma)$. Thus, by definition of the projection $\mathbf{P}^\Gamma_{\mathrm{div}}$, we find $q \in \dot{H}^1(\Gamma)$ such that

$$\nabla_\Gamma q = -\mathrm{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}) + \gamma(\phi,\psi)\mathbf{w} - \mathbf{f}_\Gamma \qquad \text{a.e. on } \Gamma. \tag{4.118}$$

Moreover, the choice of q is unique if we additionally require that $\langle q\rangle_\Gamma = 0$. Then, by the Poincaré–Wirtinger inequality, we obtain $q \in H^1(\Gamma)$. In this way, $(\mathbf{w}, q)$ is the unique solution to (4.113) satisfying the desired regularities as well as the estimate (4.114). $\square$

Finally, we can present the main theorem regarding the existence of strong solutions in the case of non-constant coefficients.

THEOREM 4.5.9. *Assume that (V2) holds, let* $(\phi, \psi) \in \mathcal{W}^{1,\infty}$, *and let* $(\mathbf{f}, \mathbf{f}_\Gamma) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2_{\tan}(\Gamma)$. *Then, the unique weak solution* $(\mathbf{v}, \mathbf{w}, p, q)$ *to* (4.79) *with right-hand side* $(\mathbf{f}, \mathbf{f}_\Gamma)$ *has the additional regularity* $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ *and* $(p, q) \in \mathcal{H}^1 \cap \mathcal{L}^2_{(0)}$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{H}}^2} + \|(p, q)\|_{\mathcal{H}^1} \leq C\|(\mathbf{f}, \mathbf{f}_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2}.$$

In particular, $(\mathbf{v}, \mathbf{w}, p, q)$ *is the unique strong solution to* (4.79).

We do not present the proof of Theorem 4.5.9, as it works analogously to the one of Theorem 4.5.6 by combining Lemma 4.5.7 and Lemma 4.5.8.

4.5.4 A bulk-surface Stokes operator: Regularity and spectral theory

In this last section, we define and study a new bulk-surface Stokes operator. To this end, we consider the situation of constant coefficients, and assume that $\nu_\Omega = \nu_\Gamma \equiv 1$ as well as $\gamma \equiv 1$. Then, we define the bulk-surface Stokes operator as

$$\begin{aligned}
&\boldsymbol{\mathcal{A}} : D(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\mathcal{L}}^2_{\mathrm{div}} \to \boldsymbol{\mathcal{L}}^2_{\mathrm{div}},\\
&\qquad (\mathbf{v}, \mathbf{w}) \mapsto \big(-\mathbf{P}^\Omega_{\mathrm{div}}(\mathrm{div}(2\mathbf{D}\mathbf{v})), -\mathbf{P}^\Gamma_{\mathrm{div}}(\mathrm{div}_\Gamma(2\mathbf{D}_\Gamma\mathbf{w}) + 2[\mathbf{D}\mathbf{v}\,\mathbf{n}]_{\tan} + \mathbf{w})\big),
\end{aligned}$$

where $D(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$.

The next result collects some first properties of the operator $\boldsymbol{\mathcal{A}}$ and its inverse.

LEMMA 4.5.10. *The operator* $\boldsymbol{\mathcal{A}}$ *is invertible and its inverse*

$$\boldsymbol{\mathcal{S}} : \boldsymbol{\mathcal{L}}^2_{\mathrm{div}} \to \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}, \quad \boldsymbol{\mathcal{S}}(\mathbf{f}, \mathbf{f}_\Gamma) = \boldsymbol{\mathcal{A}}^{-1}(\mathbf{f}, \mathbf{f}_\Gamma),$$

which maps $(\mathbf{f}, \mathbf{f}_\Gamma)$ *onto the corresponding strong solution of the bulk-surface Stokes problem* (4.109), *is a linear and bounded operator that is injective, compact, and self-adjoint with respect to the inner product of* $\boldsymbol{\mathcal{L}}^2$.

PROOF. The invertibility of $\boldsymbol{\mathcal{A}}$ and the existence of the inverse Stokes operator $\boldsymbol{\mathcal{S}}$ are a direct consequence of Theorem 4.5.6. As the Stokes operator $\boldsymbol{\mathcal{A}}$ is linear and invertible, it is clear that $\boldsymbol{\mathcal{S}}$ is linear and injective. Combining the estimate (4.100) with the trace embedding $\mathbf{H}^2(\Omega) \hookrightarrow \mathbf{L}^2(\Gamma)$, we deduce that $\boldsymbol{\mathcal{S}}$ is bounded. Moreover, as $\boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{L}}^2_{\mathrm{div}}) = D(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$, we conclude from the compact embedding $\boldsymbol{\mathcal{H}}^2 \hookrightarrow \boldsymbol{\mathcal{L}}^2$ that $\boldsymbol{\mathcal{S}}$ is compact. It remains to prove that $\boldsymbol{\mathcal{S}}$ is self-adjoint. To this end, let $(\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{f}}, \widehat{\mathbf{f}}_\Gamma) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ be arbitrary. Choosing $(\mathbf{v}, \mathbf{w}) = \boldsymbol{\mathcal{S}}(\mathbf{f}, \mathbf{f}_\Gamma) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ and $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = \boldsymbol{\mathcal{S}}(\widehat{\mathbf{f}}, \widehat{\mathbf{f}}_\Gamma) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ in the weak formulation (4.82), we conclude that

$$\begin{aligned}
\big(\boldsymbol{\mathcal{S}}(\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{f}}, \widehat{\mathbf{f}}_\Gamma)\big)_{\boldsymbol{\mathcal{L}}^2} &= \big\langle (\widehat{\mathbf{f}}, \widehat{\mathbf{f}}_\Gamma), (\mathbf{v}, \mathbf{w}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0}\\
&= \int_\Omega 2\mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\mathbf{D}_\Gamma\mathbf{w} : \mathbf{D}_\Gamma\widehat{\mathbf{w}} \,\mathrm{d}\Gamma + \int_\Gamma \mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\\
&= \big\langle (\mathbf{f}, \mathbf{f}_\Gamma), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \big\rangle_{\boldsymbol{\mathcal{H}}^1_0} = \big((\mathbf{f}, \mathbf{f}_\Gamma), \boldsymbol{\mathcal{S}}(\widehat{\mathbf{f}}, \widehat{\mathbf{f}}_\Gamma)\big)_{\boldsymbol{\mathcal{L}}^2}.
\end{aligned} \tag{4.119}$$

This proves that $\boldsymbol{\mathcal{S}}$ is self-adjoint with respect to the inner product of $\boldsymbol{\mathcal{L}}^2$, and thus, the proof is complete. □

THEOREM 4.5.11. *The operator* $\boldsymbol{\mathcal{A}}$ *has countably many eigenvalues, and each of them has a finite-dimensional eigenspace. Repeating each eigenvalue according to its multiplicity, we can interpret them as a sequence* $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ *with*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \qquad \text{and} \qquad \lambda_k \to \infty \qquad \text{as } k \to \infty.$$

Moreover, there exists an orthonormal basis $\{(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)\}_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ *of* $\boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ *such that for every* $k \in \mathbb{N}$, $(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)$ *is an eigenfunction of* $\boldsymbol{\mathcal{A}}$ *to the eigenvalue* λ_k. *This means that* $(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)$ *is non-trivial and satisfies*

$$\boldsymbol{\mathcal{A}}(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k) = \lambda_k(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k) \quad \Leftrightarrow \quad (\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k) = \lambda_k \boldsymbol{\mathcal{S}}(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k).$$

In particular, any pair $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ *can be expressed as*

$$(\mathbf{v}, \mathbf{w}) = \sum_{k=1}^{\infty} \big((\mathbf{v}, \mathbf{w}), (\tilde{\mathbf{w}}_k, \tilde{\mathbf{w}}_k)\big)_{\boldsymbol{\mathcal{L}}^2} (\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)$$

in the sense that the series converges in $\boldsymbol{\mathcal{L}}^2$.

Proof. Due to the properties established in Lemma 4.5.10, the assertions directly follow by applying the spectral theorem for compact normal operators to the solution operator $\boldsymbol{\mathcal{S}}$ (see, e.g., [16, Sect. 2.12] and also [64, Secs. D.5–D.6]). $\square$

Corollary 4.5.12. *The bulk-surface Helmholtz projection*

$$\boldsymbol{\mathcal{P}}_{\mathrm{div}} : \boldsymbol{\mathcal{L}}^2 \to \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}, \quad \boldsymbol{\mathcal{P}}_{\mathrm{div}}(\mathbf{v}, \mathbf{w}) = (\mathbf{P}^{\Omega}_{\mathrm{div}}(\mathbf{v}), \mathbf{P}^{\Gamma}_{\mathrm{div}}(\mathbf{w}))$$

can be represented as

$$\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\mathbf{v}, \mathbf{w}) = \sum_{k=1}^{\infty} \big((\mathbf{v}, \mathbf{w}), (\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)\big)_{\boldsymbol{\mathcal{L}}^2} (\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k),$$

for all $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{L}}^2$, *in the sense that the series converges in* $\boldsymbol{\mathcal{L}}^2$.

Proof. As $\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$, the claim directly follows from the fact that, according to Theorem 4.5.11, the eigenfunctions $\{(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)\}_{k\in\mathbb{N}}$ form an orthonormal basis of $\boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$. $\square$

Corollary 4.5.13. *The eigenfunctions* $\{(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)\}_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ *form a complete orthogonal system of* $D(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$.

Proof. First note that $\boldsymbol{\mathcal{A}}$ is self-adjoint, which can be shown by performing similar computations as for $\boldsymbol{\mathcal{S}}$. The claim then follows as in the proof of [26, Theorem II.6.6]. $\square$

Corollary 4.5.14. *The inclusion* $\boldsymbol{\mathcal{W}}^{2,r}_{0,\mathrm{div}} \hookrightarrow \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ *is dense for every* $r \in (2, \infty)$.

Proof. Let $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ be arbitrary. Since $\boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}} \hookrightarrow \boldsymbol{\mathcal{L}}^r_{\mathrm{div}}$, Theorem 4.5.6 implies that

$$(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k) = \lambda_k \boldsymbol{\mathcal{S}}(\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k) \in \boldsymbol{\mathcal{W}}^{2,r}_{0,\mathrm{div}}$$

for all $k \in \mathbb{N}$. Hence, for $N \in \mathbb{N}$, we have

$$\boldsymbol{\mathcal{W}}^{2,r}_{0,\mathrm{div}} \ni \sum_{k=1}^{N} \big((\mathbf{v}, \mathbf{w}), (\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k)\big)_{\boldsymbol{\mathcal{L}}^2} (\tilde{\mathbf{v}}_k, \tilde{\mathbf{w}}_k) \to (\mathbf{v}, \mathbf{w}) \quad \text{strongly in } \boldsymbol{\mathcal{L}}^2$$

as $N \to \infty$, which proves the claim. $\square$

To conclude this chapter, we note that by construction of the bulk-surface Stokes operator $\boldsymbol{\mathcal{A}}$, it holds that

$$\|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{H}}^2} \le C \|\boldsymbol{\mathcal{A}}(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \qquad \text{for all } (\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}} \tag{4.120}$$

for all $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ in view of (4.100). In particular, $\|\boldsymbol{\mathcal{A}}(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}$ is an equivalent norm on $\boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$.

Part I

The convective bulk-surface Cahn–Hilliard subsystem

Chapter 5

Existence and uniqueness of weak solutions for regular potentials

The content of this chapter is based on the following publication.

[122] P. Knopf, and J. Stange, *Well-posedness of a bulk-surface convective Cahn–Hilliard system with dynamic boundary conditions*, NoDEA Nonlinear Differential Equations Appl., 31 (2024), pp. Paper No. 82, 48.

5.1 Introduction

We begin the first part of this thesis by studying the following convective bulk-surface Cahn–Hilliard model

$$\begin{aligned}
&\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \operatorname{div}(m_\Omega(\phi)\nabla\mu) && \text{in } Q, && (5.1a)\\
&\mu = -\Delta\phi + F'(\phi) && \text{in } Q, && (5.1b)\\
&\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) = \operatorname{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\partial_\mathbf{n}\mu && \text{on } \Sigma, && (5.1c)\\
&\theta = -\Delta_\Gamma\psi + G'(\psi) + \alpha\partial_\mathbf{n}\phi && \text{on } \Sigma, && (5.1d)\\
&\begin{cases} K\partial_\mathbf{n}\phi = \alpha\psi - \phi & \text{if } K\in[0,\infty),\\ \partial_\mathbf{n}\phi = 0 & \text{if } K = \infty \end{cases} && \text{on } \Sigma, && (5.1e)\\
&\begin{cases} Lm_\Omega(\phi)\partial_\mathbf{n}\mu = \beta\theta - \mu & \text{if } L\in[0,\infty),\\ m_\Omega(\phi)\partial_\mathbf{n}\mu = 0 & \text{if } L = \infty \end{cases} && \text{on } \Sigma, && (5.1f)\\
&\phi|_{t=0} = \phi_0 && \text{in } \Omega, && (5.1g)\\
&\psi|_{t=0} = \psi_0 && \text{on } \Gamma. && (5.1h)
\end{aligned}$$

This system can be interpreted as a subsystem of the full bulk-surface Navier–Stokes–Cahn–Hilliard system (1.42) which we derived in Chapter 3. In contrast to the full model, the velocity fields $\mathbf{v}$ and $\mathbf{w}$ are prescribed here, whereas in (1.42), they are determined by a coupled Navier–Stokes system. Moreover, variants of this system arise in several related models, including a Navier–Stokes–Cahn–Hilliard model with Allen–Cahn-type dynamic boundary condition in [76, 77, 79], a Cahn–Hilliard–Brinkman model with Cahn–Hilliard-type dynamic boundary conditions [52], a Navier–Stokes–Cahn–Hilliard model with Cahn–Hilliard-type dynamic boundary conditions [80, 97], and more recently in Navier–Stokes–Cahn–Hilliard systems with Cahn–Hilliard-type dynamic boundary conditions on evolving surfaces [118].

The model (5.1) was first introduced in [121] as a unifying framework combining several previously studied Cahn–Hilliard-type models with dynamic boundary condition, see, for instance, [102, 115, 116, 126].

Our primary objective is to treat singular potentials such as the Flory–Huggins potential (1.3). Due to its singular nature, it is a standard approach to approximate it by regular potentials, for example the classical double-well potential (1.5). Therefore, as a first step, we consider regular potentials, allowing for more general growth conditions than in the quartic case (1.5).

The aim of this chapter is to establish existence and uniqueness of weak solutions to (5.1) in the case of regular potentials. In addition, we derive several higher regularity properties of these weak solutions.

In absence of convection, i.e., if $\mathbf{v} \equiv \mathbf{0}$ and $\mathbf{w} \equiv \mathbf{0}$, several techniques are available to prove the existence of weak solutions. For instance, the gradient flow structure of the system has been exploited in [85,115,116] for several coupling conditions K and L, typically via an implicit time discretization scheme. However, due to the presence of convective terms, the gradient flow structure of (5.1) is lost in the present setting. An alternative strategy is based on viscous regularizations, as employed in [41,70,129–131] for various coupling conditions K and L. The resulting systems are then treated using abstract results for doubly nonlinear evolution inclusions combined with fixed-point arguments. Via an asymptotic limit, one then recovers the existence of a weak solution to the original system. We also mention [120, 128, 131] for the analysis of non-local variants of (5.1). We further mention [178, 179], where the authors established the existence of weak solutions for degenerate mobility functions.

In the presence of convection, fewer results are available. In [49], a viscous model with convection acting only in the bulk, i.e., $\mathbf{w} = \mathbf{0}$, was studied using a Faedo–Galerkin approximation. Well-posedness and optimal control problems were further investigated in [51], while the non-viscous case was considered in [91], including asymptotic analysis.

In this chapter, we consider general velocity fields $\mathbf{v}$ and $\mathbf{w}$, which are not required to satisfy any coupling condition such as $\mathbf{w} = \mathbf{v}_{\tan}$. Furthermore, we treat all coupling regimes $K, L \in [0, \infty]$. Our strategy is as follows. We first establish the existence of weak solutions in the case $K, L \in (0, \infty)$ by means of a Faedo–Galerkin scheme. Due to the dependence of admissible test function spaces on the coupling parameters, this approach cannot be applied directly to all limiting cases. We then rigorously show the existence of weak solutions to the remaining models by performing asymptotic limits as $K \to 0, K \to \infty, L \to 0$ and $L \to \infty$. This approach has been successfully employed in several related works. It was first introduced in [42] for an Allen–Cahn model with dynamic boundary conditions for the limit $K \to 0$, and subsequently applied to Cahn–Hilliard-type systems in [115, 116, 130, 132]. This asymptotic approach has also proven effective for more complex coupled systems. For instance, the limit $K \to 0$ has been studied for the Cahn–Hilliard–Brinkman model in [52], while the limit $L \to 0$ has been analyzed for the bulk-surface Navier–Stokes–Cahn–Hilliard system (1.42) in [123]. In addition, we employ this strategy in Chapter 9 to establish the existence of weak solutions to (1.42) in the case $K = 0$.

Then, assuming constant mobility functions, we further show the uniqueness of weak solutions together with a continuous dependence estimate with respect to the initial data and the velocity fields. Moreover, we show higher regularity of weak solutions assuming more restrictive growth assumptions on the potentials.

5.2 Additional preliminaries

We present a simple interpolation inequality, that will be frequently used in our mathematical analysis.

LEMMA 5.2.1. *Let $K \in [0, \infty]$ and $\alpha \in \mathbb{R}$. Then, it holds*

$$\|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^2}^2 \le 2\|(\zeta, \zeta_\Gamma)\|_{(\mathcal{H}^1_{K,\alpha})'}\|(\nabla\zeta, \nabla_\Gamma\zeta_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2} + \|(\zeta, \zeta_\Gamma)\|^2_{(\mathcal{H}^1_{K,\alpha})'}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$.

PROOF. Let $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$ be arbitrary. Recalling the definition of $\mathcal{H}^1_{K,\alpha}$, we have

$$\|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1_{K,\alpha}} = \|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1} \le \|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^2} + \|(\nabla\zeta, \nabla_\Gamma\zeta_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2}.$$

Using Young's inequality, we deduce

$$\begin{aligned}
\|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^2}^2 &= \langle(\zeta, \zeta_\Gamma), (\zeta, \zeta_\Gamma)\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&\le \|(\zeta, \zeta_\Gamma)\|_{(\mathcal{H}^1_{K,\alpha})'}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1_{K,\alpha}} \\
&\le \|(\zeta, \zeta_\Gamma)\|_{(\mathcal{H}^1_{K,\alpha})'}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^2} + \|(\zeta, \zeta_\Gamma)\|_{(\mathcal{H}^1_{K,\alpha})'}\|(\nabla\zeta, \nabla_\Gamma\zeta_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\le \frac{1}{2}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^2}^2 + \frac{1}{2}\|(\zeta, \zeta_\Gamma)\|^2_{(\mathcal{H}^1_{K,\alpha})'} + \|(\zeta, \zeta_\Gamma)\|_{(\mathcal{H}^1_{K,\alpha})'}\|(\nabla\zeta, \nabla_\Gamma\zeta_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2}.
\end{aligned}$$

Hence, the claim directly follows. □

Moreover, we will need the following chain rule, which is an adaption from the one established in [52, Proposition A.1.(b)].

PROPOSITION 5.2.2. *Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ be a non-empty, bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$, and let $K, L \in [0, \infty]$ and $\alpha, \beta \in \mathbb{R}$ be arbitrary with $\alpha\beta|\Omega| + |\Gamma| \neq 0$. Moreover, let $a, b \in \mathbb{R}$ with $a < b$. Let (u, v) be a pair such that*

$$\begin{aligned}
(\partial_t u, \partial_t v) &\in L^2(a, b; (\mathcal{H}^1_{L,\beta})'), \\
(u, v) &\in L^\infty(a, b; \mathcal{H}^1_{K,\alpha}) \cap L^2(a, b; \mathcal{H}^3).
\end{aligned}$$

Moreover, we assume that

$$\begin{cases} K\partial_{\mathbf{n}} u = \alpha v - u & \text{if } K \in [0, \infty), \\ \partial_{\mathbf{n}} u = 0 & \text{if } K = \infty \end{cases} \quad \text{a.e. on } \Gamma \times (a, b), \tag{5.2}$$

and

$$(-\Delta u, -\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u) \in L^2(a, b; \mathcal{H}^1_{L,\beta}).$$

Then, the continuity property $(u, v) \in C([a, b]; \mathcal{H}^1_{K,\alpha})$ holds, the mapping

$$[a, b] \ni t \mapsto \|(u(t), v(t))\|^2_{K,\alpha}$$

is absolutely continuous, and we have the chain rule formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(u(t), v(t))\|^2_{K,\alpha} = \langle(\partial_t u, \partial_t v)(t), (-\Delta u, -\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u)(t)\rangle_{\mathcal{H}^1_{L,\beta}} \tag{5.3}$$

for almost all $t \in [a, b]$.

Proof of Proposition 5.2.2. In the cases $(K, L) \in [0, \infty) \times \{\infty\}$, a proof was already given in [52, Proposition A.1.(b)]. For completeness, we present the proof here for all cases $K, L \in [0, \infty]$, by adapting the argument of [52]. To this end, we first note that in view of the regularity assumptions on (u, v) and their time derivative, the Aubin–Lions–Simon lemma readily entails that $(u, v) \in C([a, b]; \mathcal{L}^2)$. We can thus extend the functions u and v onto $[2a - b, a]$ by defining $u(t)$ and $v(t)$ by reflection for all $t < a$.

Let $\pi \in C_c^\infty(\mathbb{R})$ be non-negative with $\operatorname{supp} \pi \subset (0, 1)$ and $\|\pi\|_{L^1(\mathbb{R})} = 1$. For any $k \in \mathbb{N}$, we set

$$\pi_k(s) := k\pi(ks) \qquad \text{for all } s \in \mathbb{R}.$$

Then, for any Banach space X and any function $f \in L^2(a - 1, b; X)$, we define

$$f_k(t) := (\pi_k * f)(t) = \int_{t-\frac{1}{k}}^{t} \pi_k(t - s) f(s) \,\mathrm{d}s \qquad \text{for all } t \in [a, b] \quad \text{and} \quad k \in \mathbb{N}.$$

By this construction, we have $f_k \in C^\infty([a, b]; X)$ with $f_k \to f$ strongly in $L^2(a, b; X)$ as $k \to \infty$.

Now, for any $k \in \mathbb{N}$, we use $X = H^3(\Omega)$ to define u_k and $X = H^3(\Gamma)$ to define v_k as described above. By this construction, it holds that $\partial_t u_k = (\partial_t u)_k$ and $\partial_t \nabla u_k = \nabla \partial_t u_k$ a.e. in $\Omega \times (a, b)$ as well as $\partial_t v_k = (\partial_t v)_k$ and $\partial_t \nabla_\Gamma v_k = \nabla_\Gamma \partial_t v_k$ a.e. on $\Gamma \times (a, b)$ for all $k \in \mathbb{N}$. Moreover, we have

$$\begin{aligned}
u_k &\to u && \text{strongly in } L^2(a, b; H^3(\Omega)), && (5.4)\\
v_k &\to v && \text{strongly in } L^2(a, b; H^3(\Gamma)), && (5.5)\\
(u_k, v_k) &\to (u, v) && \text{strongly in } L^2(a, b; \mathcal{H}^1_{K,\alpha}), && (5.6)\\
(\partial_t u_k, \partial_t v_k) &\to (\partial_t u, \partial_t v) && \text{strongly in } L^2(a, b; (\mathcal{H}^1_{L,\beta})') && (5.7)
\end{aligned}$$

as $k \to \infty$. Furthermore, we readily see that

$$\begin{aligned}
\|u_k\|_{L^\infty(a,b;H^1(\Omega))} &\le \|u\|_{L^\infty(a,b;H^1(\Omega))},\\
\|v_k\|_{L^\infty(a,b;H^1(\Gamma))} &\le \|v\|_{L^\infty(a,b;H^1(\Gamma))}
\end{aligned} \tag{5.8}$$

for all $k \in \mathbb{N}$. In the following, we denote with C generic positive constants independent of $k \in \mathbb{N}$, which may change their value from line to line. Now, for any $k \in \mathbb{N}$, we derive the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|(u_k, v_k)\|_{K,\alpha}^2 = \langle (\partial_t u_k, \partial_t v_k), (-\Delta u_k, -\Delta_\Gamma v_k + \alpha \partial_\mathbf{n} u_k) \rangle_{\mathcal{H}^1_{L,\beta}} \tag{5.9}$$

on $[a, b]$ by differentiating under the integral sign and applying integration by parts. Similarly, for $j, k \in \mathbb{N}$, it holds

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|(u_j - u_k, v_j - v_k)\|_{K,\alpha}^2\\
&\quad = \langle (\partial_t (u_j - u_k), \partial_t (v_j - v_k)), (-\Delta(u_j - u_k), -\Delta_\Gamma (v_j - v_k) + \alpha \partial_\mathbf{n} (u_j - u_k)) \rangle_{\mathcal{H}^1_{L,\beta}}\\
&\quad \le C\Big(\|(\partial_t (u_j - u_k), \partial_t (v_j - v_k))\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(u_j - u_k, v_j - v_k)\|_{\mathcal{H}^3}^2 \Big)
\end{aligned} \tag{5.10}$$

on $[a, b]$. Here, we have used the embedding $H^3(\Omega) \hookrightarrow H^2(\Gamma)$ resulting from the trace theorem, which yields

$$\|\partial_\mathbf{n} (u_j - u_k)\|_{H^1(\Gamma)} \le \|u_j - u_k\|_{H^2(\Gamma)} \le C \|u_j - u_k\|_{H^3(\Omega)}$$

on $[a,b]$. Let now $s,t \in [a,b]$ be arbitrary with $s \le t$. We then integrate inequality (5.10) with respect to time from s to t. This yields

$$\begin{aligned} &\|((u_j-u_k),(v_j-v_k))(t)\|_{K,\alpha}^2 \\ &\quad \le \|((u_j-u_k),(v_j-v_k))(s)\|_{K,\alpha}^2 \\ &\qquad + C\int_s^t \Big(\|(\partial_t(u_j-u_k),\partial_t(v_j-v_k))\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(u_j-u_k,v_j-v_k)\|_{\mathcal{H}^3}^2 \Big)\,\mathrm{d}r. \end{aligned} \tag{5.11}$$

In light of the convergences (5.4)-(5.5), we can fix $s \in [a,t]$ such that $(u_k(s),v_k(s)) \to (u(s),v(s))$ strongly in $\mathcal{H}^3$ along a non-relabeled subsequence $k \to \infty$. Recalling (5.4)-(5.7), we can thus deduce that the right-hand side of (5.11) tends to zero as $j,k \to \infty$. Consequently, we infer that $(\nabla u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([a,b];\mathbf{L}^2(\Omega))$ and $(\nabla_\Gamma v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([a,b];\mathbf{L}^2(\Gamma))$. Thus,

$$\nabla u_k \to \nabla u \qquad \text{strongly in } C([a,b];\mathbf{L}^2(\Omega)), \tag{5.12}$$

$$\nabla_\Gamma v_k \to \nabla_\Gamma v \qquad \text{strongly in } C([a,b];\mathbf{L}^2(\Gamma)) \tag{5.13}$$

as $k \to \infty$. In view of the already established regularity $(u,v) \in C([a,b];\mathcal{L}^2)$, we readily deduce that $(u,v) \in C([a,b];\mathcal{H}^1)$.

Let now $s,t \in [a,b]$ be arbitrary with $s \le t$. We then integrate (5.9) in time over $[s,t]$ and find

$$\begin{aligned} &\frac12 \|(u_k(t),v_k(t))\|_{K,\alpha}^2 \\ &\quad = \frac12 \|(u_k(s),v_k(s))\|_{K,\alpha}^2 + \int_s^t \langle (\partial_t u_k,\partial_t v_k),(-\Delta u_k, -\Delta_\Gamma v_k + \alpha\partial_{\mathbf{n}} u_k)\rangle_{\mathcal{H}^1_{L,\beta}}\,\mathrm{d}r. \end{aligned}$$

In view of the convergences (5.4)-(5.7) and (5.12)-(5.13), we may pass to the limit $k \to \infty$ in this identity and obtain

$$\begin{aligned} &\frac12 \|(u(t),v(t))\|_{K,\alpha}^2 \\ &\quad = \frac12 \|(u(s),v(s))\|_{K,\alpha}^2 + \int_s^t \langle (\partial_t u,\partial_t v),(-\Delta u, -\Delta_\Gamma v + \alpha\partial_{\mathbf{n}} u)\rangle_{\mathcal{H}^1_{L,\beta}}\,\mathrm{d}r. \end{aligned}$$

As the integrand of the integral on the right-hand side is an element of $L^1(a,b)$, we conclude that the mapping $[a,b] \ni t \mapsto \|(u(t),v(t))\|_{K,\alpha}^2$ is absolutely continuous on $[a,b]$. It is thus differentiable almost everywhere on $[a,b]$ and its derivative satisfies the chain rule formula (5.3), which completes the proof. □

5.3 Main results

5.3.1 Assumptions

We begin by stating our main assumptions used throughout this chapter.

(A1) Ω is a non-empty, bounded C^1-domain in $\mathbb{R}^d$ with $d \in \{2,3\}$, whose boundary is denoted by $\Gamma := \partial\Omega$. Moreover, $T > 0$ denotes an arbitrary final time and for brevity, we use the notation

$$Q^T := \Omega \times (0,T), \qquad \Sigma^T := \Gamma \times (0,T).$$

(A2) The constants in system (5.1) satisfy $\alpha,\beta \in \mathbb{R}$ with $\alpha\beta|\Omega| + |\Gamma| \neq 0$.

(A3) The mobility functions $m_\Omega : \mathbb{R} \to \mathbb{R}$ and $m_\Gamma : \mathbb{R} \to \mathbb{R}$ are bounded, continuous and uniformly positive. This means that there exist constants $m_*, m^* > 0$ such that

$$0 < m_* \leq m_\Omega(s), m_\Gamma(s) \leq m^* \qquad \text{for all } s \in \mathbb{R}.$$

(A4) The potentials $F : \mathbb{R} \to [0,\infty)$ and $G : \mathbb{R} \to [0,\infty)$ are continuously differentiable and there exist exponents p and q satisfying

$$p \in \begin{cases} [2,\infty) & \text{if } d = 2, \\ [2,6] & \text{if } d = 3, \end{cases} \quad \text{and} \quad q \in [2,\infty)$$

as well as constants $c_{F'}, c_{G'} \geq 0$ such that the first-order derivatives satisfy the growth conditions

$$|F'(s)| \leq c_{F'}(1 + |s|^{p-1}), \tag{5.14}$$
$$|G'(s)| \leq c_{G'}(1 + |s|^{q-1}) \tag{5.15}$$

for all $s \in \mathbb{R}$. These assumptions already imply the existence of constants $c_F, c_G \geq 0$ such that F and G satisfy the growth conditions

$$F(s) \leq c_F(1 + |s|^p), \tag{5.16}$$
$$G(s) \leq c_G(1 + |s|^q) \tag{5.17}$$

for all $s \in \mathbb{R}$.

(A5) There exist constants $a_F, a_G > 0$ and $b_F, b_G \geq 0$ such that the potentials F and G introduced in **(A4)** satisfy

$$F(s) \geq a_F|s|^2 - b_F, \tag{5.18}$$
$$G(s) \geq a_G|s|^2 - b_G \tag{5.19}$$

for all $s \in \mathbb{R}$.

(A6) The potentials F and G introduced in **(A4)** are twice continuously differentiable, and there exist constants $c_{F''}, c_{G''} \geq 0$ such that the second-order derivatives of F and G satisfy the growth conditions

$$|F''(s)| \leq c_{F''}(1 + |s|^{p-2}), \tag{5.20}$$
$$|G''(s)| \leq c_{G''}(1 + |s|^{q-2}) \tag{5.21}$$

for all $s \in \mathbb{R}$. We point out that these assumptions already imply the growth conditions (5.14)–(5.17) stated in **(A4)**.

(A7) For the velocity fields we assume

$$(\mathbf{v}, \mathbf{w}) \in L^2(0,T; \mathbf{L}^3_{\mathrm{div}}(\Omega) \times \mathbf{L}^{2+\omega}_{\mathrm{tan}}(\Gamma))$$

for some $\omega > 0$.

5.3.2 Existence of weak solutions

We start by introducing the notion of a weak solution to (5.1).

Definition 5.3.1. *Suppose that assumptions* **(A1)–(A4)** *and* **(A7)** *hold. Let* $K, L \in [0,\infty]$ *and let* $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ *be an arbitrary initial datum. The quadruplet* $(\phi, \psi, \mu, \theta)$ *is called a weak solution of the system* (5.1) *on* $[0,T]$ *if the following properties hold:*

(i) *The functions ϕ, ψ, μ and θ have the following regularity:*

$$(\phi,\psi) \in C([0,T];\mathcal{L}^2) \cap H^1(0,T;(\mathcal{H}^1_{L,\beta})') \cap L^\infty(0,T;\mathcal{H}^1_{K,\alpha}), \tag{5.22a}$$

$$(\mu,\theta) \in L^2(0,T,\mathcal{H}^1_{L,\beta}). \tag{5.22b}$$

(ii) *The functions ϕ and ψ satisfy the initial conditions*

$$\phi|_{t=0} = \phi_0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \psi|_{t=0} = \psi_0 \quad \text{a.e. on } \Gamma. \tag{5.23}$$

(iii) *The functions ϕ, ψ, μ and θ satisfy the weak formulation*

$$\begin{aligned} &\langle(\partial_t\phi,\partial_t\psi),(\zeta,\zeta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} - \int_\Omega \phi\mathbf{v}\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\ &\quad = -\int_\Omega m_\Omega(\phi)\nabla\mu\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\ &\qquad - \chi(L)\int_\Gamma(\beta\theta-\mu)(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma, \end{aligned} \tag{5.24a}$$

$$\begin{aligned} &\int_\Omega \mu\,\eta \,\mathrm{d}x + \int_\Gamma \theta\,\eta_\Gamma \,\mathrm{d}\Gamma \\ &\quad = \int_\Omega \nabla\phi\cdot\nabla\eta + F'(\phi)\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi\cdot\nabla_\Gamma\eta_\Gamma + G'(\psi)\eta_\Gamma \,\mathrm{d}\Gamma \\ &\qquad + \chi(K)\int_\Gamma(\alpha\psi-\phi)(\alpha\eta_\Gamma-\eta)\,\mathrm{d}\Gamma, \end{aligned} \tag{5.24b}$$

a.e. on $[0,T]$ for all $(\zeta,\zeta_\Gamma)\in\mathcal{H}^1_{L,\beta}$, $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$.

(iv) *The functions ϕ and ψ satisfy the mass conservation laws*

$$\begin{cases} \beta\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \beta\int_\Omega \phi_0\,\mathrm{d}x + \int_\Gamma \psi_0\,\mathrm{d}\Gamma & \text{if } L\in[0,\infty), \\ \int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi_0\,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \int_\Gamma \psi_0\,\mathrm{d}\Gamma & \text{if } L=\infty \end{cases} \tag{5.25}$$

for all $t\in[0,T]$.

(v) *The energy inequality*

$$\begin{aligned} &E_{\text{free}}(\phi(t),\psi(t)) + \int_0^t\int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &\quad + \chi(L)\int_0^t\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &\quad - \int_0^t\int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &\le E_{\text{free}}(\phi_0,\psi_0) \end{aligned} \tag{5.26}$$

holds for all $t\in[0,T]$.

Now, we are ready to formulate the main results of this chapter. The first main result provides the existence of a weak solution to system (5.1) in all cases $K,L\in[0,\infty]$.

THEOREM 5.3.2. *Suppose that the assumptions* **(A1)**–**(A4)** *hold. Let $K,L\in[0,\infty]$, and let $(\phi_0,\psi_0)\in\mathcal{H}^1_{K,\alpha}$ be arbitrary initial data. In the case $K\in\{0,\infty\}$, we further* *assume that* **(A5)** *holds. Then, there exists a weak solution* $(\phi, \psi, \mu, \theta)$ *of the system* (5.1) *in the sense of Definition* 5.3.1*, which has the additional regularity*

$$(\mu, \theta) \in L^4(0,T;\mathcal{L}^2) \quad \text{if } K \in (0,\infty]. \tag{5.27}$$

Let us now assume that **(A6)** *holds, that the domain* Ω *is of class* C^k *for* $k \in \{2,3\}$*, and in the case* $d = 3$*, we further assume that* **(A4)** *holds with* $p \le 4$*. Then, we additionally have*

$$(\phi, \psi) \in L^4(0,T;\mathcal{H}^2) \qquad \text{if } K \in (0,\infty], \tag{5.28}$$
$$(\phi, \psi) \in L^2(0,T;\mathcal{H}^k) \qquad \text{for } k = 2,3, \tag{5.29}$$
$$(\phi, \psi) \in C([0,T];\mathcal{H}^1) \qquad \text{if } (K,L) \in [0,\infty] \times (0,\infty] \text{ and } k = 3, \tag{5.30}$$

and the equations

$$\begin{aligned}
&\mu = -\Delta\phi + F'(\phi) && \text{a.e. in } Q^T,\\
&\theta = -\Delta_\Gamma\psi + G'(\psi) + \alpha\partial_{\mathbf{n}}\phi && \text{a.e. on } \Sigma^T,\\
&\begin{cases} K\partial_{\mathbf{n}}\phi = \alpha\psi - \phi & \text{if } K \in [0,\infty),\\ \partial_{\mathbf{n}}\phi = 0 & \text{if } K = \infty \end{cases} && \text{a.e. on } \Sigma^T
\end{aligned}$$

are satisfied.

The proof of Theorem 5.3.2 is divided into two parts. In Section 5.4, we first prove the existence of weak solutions in all cases $K, L \in [0,\infty]$. Then, in Section 5.6, we establish the higher regularity of weak solutions.

Remark 5.3.3. The continuity property (5.30) also holds in the case $K = L = 0$ provided that $\alpha \neq 0$, $\beta \neq 0$ and the potentials F and G satisfy the compatibility condition

$$F(\alpha s) = \alpha\beta G(s) \quad \text{for all } s \in \mathbb{R}.$$

In that case, we have $(F'(\phi), G'(\psi)) \in \mathcal{H}^1_{0,\beta} = \mathcal{D}_\beta$, which allows us to use Proposition 5.2.2 similarly as in Section 5.6.

5.3.3 Continuous dependence and uniqueness of weak solutions

Our second main result for regular potentials shows the continuous dependence of weak solutions on the velocity fields and the initial data in the case of constant mobilities. As a direct consequence, this entails uniqueness of the weak solution.

Theorem 5.3.4. (Continuous dependence and uniqueness) *Suppose that the assumptions* **(A1)**–**(A4)** *hold, and that the mobility functions* m_Ω *and* m_Γ *are constant. If* $d = 3$*, we additionally assume that* **(A4)** *holds with* $p < 6$*. Let* $K, L \in [0,\infty]$*, let* $(\phi_0^1, \psi_0^1), (\phi_0^2, \psi_0^2) \in \mathcal{H}^1_{K,\alpha}$ *be two pairs of initial data, which satisfy*

$$\begin{cases} \operatorname{mean}(\phi_0^1, \psi_0^1) = \operatorname{mean}(\phi_0^2, \psi_0^2) & \text{if } L \in [0,\infty),\\ \langle\phi_0^1\rangle_\Omega = \langle\phi_0^2\rangle_\Omega \quad \text{and} \quad \langle\psi_0^1\rangle_\Gamma = \langle\psi_0^2\rangle_\Gamma & \text{if } L = \infty, \end{cases} \tag{5.31}$$

and let $(\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \in L^2(0,T;\mathbf{L}^3_{\mathrm{div}}(\Omega)) \times \mathbf{L}^{2+\omega}_{\mathrm{tan}}(\Gamma))$ *for some* $\omega > 0$ *be given velocity fields. Suppose that* $(\phi_1, \psi_1, \mu_1, \theta_1)$ *and* $(\phi_2, \psi_2, \mu_2, \theta_2)$ *are weak solutions to* (5.1) *in the sense of Definition* 5.3.1 *originating from* (ϕ_0^1, ψ_0^1) *and* (ϕ_0^2, ψ_0^2) *with velocity fields* $(\mathbf{v}_1, \mathbf{w}_1)$ *and* $(\mathbf{v}_2, \mathbf{w}_2)$*, respectively. Then, the continuous dependence estimate*

$$\|(\phi_1(t) - \phi_2(t), \psi_1(t) - \psi_2(t))\|^2_{L,\beta,*}$$

$$
\begin{aligned}
&\leq \|(\phi_0^1-\phi_0^2,\psi_0^1-\psi_0^2)\|_{L,\beta,*}^2 \exp\left(\int_0^t P(r)\,\mathrm{d}r\right) \qquad (5.32)\\
&\quad + \int_0^t \|(\mathbf{v}_1(s)-\mathbf{v}_2(s),\mathbf{w}_1(s)-\mathbf{w}_2(s))\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}^2 \exp\left(\int_s^t P(r)\,\mathrm{d}r\right)\mathrm{d}s
\end{aligned}
$$

holds for almost all $t \in [0,T]$, where $P(\cdot) := C(1+\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)})$, and the constant $C>0$ depends only on Ω, the parameters of the system, the initial data, the final time $T>0$ and ω.

In particular, if $(\phi_0^1,\psi_0^1) = (\phi_0^2,\psi_0^2)$ a.e. in $\Omega\times\Gamma$, $(\mathbf{v}_1,\mathbf{w}_1) = (\mathbf{v}_2,\mathbf{w}_2)$ a.e. in $Q^T\times\Sigma^T$, estimate (5.32) ensures uniqueness of the corresponding weak solution.

5.4 Existence of weak solutions

As discussed in the introduction, the proof of Theorem 5.3.2 is divided into two parts. We start by establishing the existence of weak solutions in the case $K, L \in (0,\infty)$ by means of a Faedo–Galerkin scheme.

5.4.1 Existence of weak solutions for $K, L \in (0,\infty)$

THEOREM 5.4.1. *Suppose that the assumptions* **(A1)–(A4)** *and* **(A7)** *hold. Let $K, L\in(0,\infty)$, and let $(\phi_0,\psi_0)\in\mathcal{H}^1$ be arbitrary initial data. Then there exists a weak solution (ϕ,ψ,μ,θ) of the system (5.1) in the sense of Definition 5.3.1. Additionally, it holds that*

$$(\mu,\theta)\in L^4(0,T;\mathcal{L}^2).$$

PROOF. **Step 1: Discretization via a Faedo–Galerkin scheme.** It is well known that the problems

$$
\begin{cases} -\Delta\zeta = \lambda\zeta, & \text{in } \Omega,\\ \partial_{\mathbf{n}}\zeta = 0, & \text{on } \Gamma,\end{cases} \qquad -\Delta_\Gamma\zeta_\Gamma = \lambda_\Gamma\zeta_\Gamma \quad \text{on } \Gamma,
$$

have countable many eigenvalues, which can be written as increasing sequences $\{\lambda^j\}_{j\in\mathbb{N}}$ and $\{\lambda_\Gamma^j\}_{j\in\mathbb{N}}$, respectively. The eigenfunctions $\{\zeta_j\}_{j\in\mathbb{N}}\subset H^1(\Omega)$ associated to $\{\lambda^j\}_{j\in\mathbb{N}}$ form an orthonormal Schauder of $L^2(\Omega)$, while the eigenfunctions $\{\zeta_{\Gamma j}\}_{j\in\mathbb{N}}\subset H^1(\Gamma)$ associated to $\{\lambda_\Gamma^j\}_{j\in\mathbb{N}}$ form an orthonormal Schauder basis of $L^2(\Gamma)$. In particular, we fix the eigenfunctions associated to the first eigenvalues $\lambda_\Omega^1 = \lambda_\Gamma^1 = 0$ as $\zeta_1\equiv|\Omega|^{-1/2}$ and $\zeta_{\Gamma 1}\equiv|\Gamma|^{-1/2}$. Moreover, the eigenfunctions $\{\zeta_j\}_{j\in\mathbb{N}}$ and $\{\zeta_{\Gamma j}\}_{j\in\mathbb{N}}$ also form an orthogonal Schauder basis of $H^1(\Omega)$ and $H^1(\Gamma)$, respectively.

For any $k\in\mathbb{N}$, we introduce the finite-dimensional subspaces

$$V_k = \operatorname{span}\{\zeta_1,\dots,\zeta_k\}\subset H^1(\Omega), \qquad V_{\Gamma k} = \operatorname{span}\{\zeta_{\Gamma 1},\dots,\zeta_{\Gamma k}\}\subset H^1(\Gamma)$$

along with the orthogonal $L^2(\Omega)$-projection $P_{V_k}: L^2(\Omega)\to V_k$, and the orthogonal $L^2(\Gamma)$-projection $P_{V_{\Gamma k}}: L^2(\Gamma)\to V_{\Gamma k}$. Moreover, we define $\mathcal{V}_k = V_k\times V_{\Gamma k}$. In particular, there exist constants $C_\Omega, C_\Gamma>0$ depending only on Ω and Γ, respectively, such that for all $\zeta\in H^1(\Omega)$ and $\zeta_\Gamma\in H^1(\Gamma)$,

$$\|P_{V_k}\zeta\|_{H^1(\Omega)}\le C_\Omega\|\zeta\|_{H^1(\Omega)}, \qquad \|P_{V_{\Gamma k}}\zeta_\Gamma\|_{H^1(\Gamma)}\le C_\Gamma\|\zeta_\Gamma\|_{H^1(\Gamma)}.$$

For any $k\in\mathbb{N}$, and $t\in[0,T]$, we make the ansatz

$$\phi_k(t) := \sum_{j=1}^k a_j^k(t)\,\zeta_j \quad \text{a.e. in } \Omega, \qquad \psi_k(t) := \sum_{j=1}^k b_j^k(t)\,\zeta_{\Gamma j} \quad \text{a.e. on } \Gamma,$$

$$\mu_k(t) := \sum_{j=1}^{k} c_j^k(t)\,\zeta_j \quad \text{a.e. in } \Omega, \qquad \theta_k(t) := \sum_{j=1}^{k} d_j^k(t)\,\zeta_{\Gamma j} \quad \text{a.e. on } \Gamma.$$

Here, the scalar, time-dependent coefficients $a_j^k, b_j^k, c_j^k, d_j^k$, $j = 1, \dots, k$, are assumed to be continuously differentiable functions that are not yet determined. They need to be designed in a way such that the discretized weak formulation

$$\begin{aligned}
&\langle (\partial_t \phi_k, \partial_t \psi_k), (\zeta, \zeta_\Gamma) \rangle_{\mathcal{H}^1} - \int_\Omega \phi_k \mathbf{v} \cdot \nabla \zeta \,\mathrm{d}x - \int_\Gamma \psi_k \mathbf{w} \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \\
&\quad = - \int_\Omega m_\Omega(\phi_k) \nabla \mu_k \cdot \nabla \zeta \,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi_k) \nabla_\Gamma \theta_k \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \qquad (5.34\text{a}) \\
&\qquad - L^{-1} \int_\Gamma (\beta \theta_k - \mu_k)(\beta \zeta_\Gamma - \zeta) \,\mathrm{d}\Gamma,
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \mu_k \,\eta \,\mathrm{d}x + \int_\Gamma \theta_k \,\eta_\Gamma \,\mathrm{d}\Gamma \\
&\quad = \int_\Omega \nabla \phi_k \cdot \nabla \eta + F'(\phi_k) \eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma \psi_k \cdot \nabla_\Gamma \eta_\Gamma + G'(\psi_k) \eta_\Gamma \,\mathrm{d}\Gamma \qquad (5.34\text{b}) \\
&\qquad + K^{-1} \int_\Gamma (\alpha \psi_k - \phi_k)(\alpha \eta_\Gamma - \eta) \,\mathrm{d}\Gamma
\end{aligned}$$

holds on $[0, T]$ for all $(\zeta, \zeta_\Gamma), (\eta, \eta_\Gamma) \in \mathcal{V}_k$, supplemented with the initial conditions

$$\phi_k(0) = P_{V_k}(\phi_0) \quad \text{a.e. in } \Omega, \qquad \psi_k(0) = P_{V_{\Gamma k}}(\psi_0) \quad \text{a.e. on } \Gamma. \qquad (5.35)$$

We consider now the corresponding coefficients vectors

$$\begin{aligned}
\mathbf{a}^k &= (a_1^k, \dots, a_k^k) : [0, T] \to \mathbb{R}^k, \qquad & \mathbf{b}^k &= (b_1^k, \dots, b_k^k) : [0, T] \to \mathbb{R}^k, \\
\mathbf{c}^k &= (c_1^k, \dots, c_k^k) : [0, T] \to \mathbb{R}^k, \qquad & \mathbf{d}^k &= (d_1^k, \dots, d_k^k) : [0, T] \to \mathbb{R}^k.
\end{aligned}$$

Testing (5.34a) with $(\zeta_1, \zeta_{\Gamma 1}), \dots, (\zeta_k, \zeta_{\Gamma k})$, we conclude that $(\mathbf{a}^k, \mathbf{b}^k)^\top$ is determined by a system of $2k$ ordinary differential equations, whose right-hand side depends continuously on $\mathbf{a}^k, \mathbf{b}^k, \mathbf{c}^k$ and $\mathbf{d}^k$. In view of (5.35), this system is subject to the initial conditions

$$\begin{aligned}
[\mathbf{a}^k]_j(0) &= a_j^m(0) = (\phi_0, \zeta_j)_{L^2(\Omega)} \qquad && \text{for all } j \in \{1, \dots, k\}, \\
[\mathbf{b}^k]_j(0) &= b_j^m(0) = (\psi_0, \zeta_{\Gamma j})_{L^2(\Gamma)} \qquad && \text{for all } j \in \{1, \dots, k\}.
\end{aligned}$$

Moreover, testing (5.34b) with $(\zeta_1, \zeta_{\Gamma 1}), \dots, (\zeta_k, \zeta_{\Gamma k})$, we infer that $(\mathbf{c}^k, \mathbf{d}^k)^\top$ is explicitly given by a system of $2k$ algebraic equations, whose right-hand side depends continuously on $\mathbf{a}^k$ and $\mathbf{b}^k$. Thus, replacing $(\mathbf{c}^k, \mathbf{d}^k)^\top$ in the right-hand side of the aforementioned ODE system by this algebraic description, we obtain a closed $2k$-dimensional ODE system for $(\mathbf{a}^k, \mathbf{b}^k)^\top$, whose right-hand side depends continuously on $(\mathbf{a}^k, \mathbf{b}^k)^\top$. We can thus apply the Cauchy–Peano theorem to obtain a local solution $(\mathbf{a}^k, \mathbf{b}^k)^\top : [0, T_k^*) \cap [0, T] \to \mathbb{R}^{2k}$ with $T_k^* > 0$ to the corresponding initial value problem. Without loss of generality, we assume that $T_k^* \le T$ and that $(\mathbf{a}^k, \mathbf{b}^k)^\top$ is non-extendable, i.e., T_k^* is chosen as large as possible. Now, we can reconstruct $(\mathbf{c}^k, \mathbf{d}^k)^\top : [0, T_k^*) \to \mathbb{R}^{2k}$ by the aforementioned $2k$-dimensional system of algebraic equations. In view of (5.33), we thus have shown the existence of functions

$$(\phi_k, \psi_k) \in C^1([0, T_k^*); \mathcal{H}^1), \qquad (\mu_k, \theta_k) \in C^1([0, T_k^*); \mathcal{H}^1)$$

solving (5.34) on the time interval $[0, T_k^*)$ subject to the initial conditions (5.35).

Step 2: Uniform estimates. We establish suitable estimates for each approximate solution $(\phi_k, \psi_k, \mu_k, \theta_k)$, which are uniform with respect to the index k. To this end, let $k \in \mathbb{N}$, and consider an arbitrary $T_k < T_k^*$. In the following, let C denote a generic non-negative constant depending only on the initial data and the constants introduced in **(A3)**–**(A4)** including the final time T, but not on k or T_k.

Testing (5.34a) with (μ_k, θ_k), (5.34b) with $-(\partial_t\phi_k, \partial_t\psi_k)$, adding the resulting equations, and integrating with respect to time from 0 to t, we derive the discrete energy identity

$$\begin{aligned}
&E_{\mathrm{free}}(\phi_k(t), \psi_k(t)) + \int_0^t\int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad + L^{-1}\int_0^t\int_\Gamma (\beta\theta_k - \mu_k)^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \qquad (5.36)\\
&\quad = E_{\mathrm{free}}(\phi_k(0), \psi_k(0)) + \int_0^t\int_\Omega \phi_k\mathbf{v}\cdot\nabla\mu_k \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma \psi_k\mathbf{w}\cdot\nabla_\Gamma\theta_k \,\mathrm{d}\Gamma\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0, T_k]$. Now, due to the growth conditions on F, G (see **(A4)**), the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^q(\Gamma)$, the trace embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$, and the properties of the projections P_{V_k} and $P_{V_{\Gamma k}}$, we use the initial conditions (5.35) to infer

$$\begin{aligned}
&E_{\mathrm{free}}(\phi_k(0), \psi_k(0)) \\
&\le \frac{1}{2}\|\nabla P_{V_k}(\phi_0)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2}\|\nabla_\Gamma P_{V_{\Gamma k}}(\psi_0)\|_{\mathbf{L}^2(\Gamma)}^2 + c_F\Big(|\Omega| + \|P_{V_k}(\phi_0)\|_{L^p(\Omega)}^p\Big) \qquad (5.37)\\
&\quad + c_G\Big(|\Gamma| + \|P_{V_{\Gamma k}}(\psi_0)\|_{L^q(\Gamma)}^q\Big) + \frac{1}{2K}\Big(|\alpha|\|P_{V_{\Gamma k}}(\psi_0)\|_{L^2(\Gamma)} + \|P_{V_k}(\phi_0)\|_{L^2(\Gamma)}\Big)^2 \\
&\le C\|(\phi_0, \psi_0)\|_{\mathcal{H}^1}^2 + C\Big(1 + \|\phi_0\|_{H^1(\Omega)}^p + \|\psi_0\|_{H^1(\Gamma)}^q\Big).
\end{aligned}$$

To obtain a suitable bound on the energy and the integral terms on the left-hand side of (5.36), we have to deal with the convective terms appearing on the right-hand side. To this end, we start by deriving for all $t \in [0, T_k]$ the estimate

$$\begin{aligned}
&\Big|\int_0^t\int_\Omega \phi_k\mathbf{v}\cdot\nabla\mu_k \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma \psi_k\mathbf{w}\cdot\nabla_\Gamma\theta_k \,\mathrm{d}\Gamma\,\mathrm{d}s\Big| \\
&\quad \le \frac{m_*}{2}\int_0^t\int_\Omega |\nabla\mu_k|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{m_*}{2}\int_0^t\int_\Gamma |\nabla_\Gamma\theta_k|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \qquad (5.38)\\
&\quad\quad + C\int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2 \|(\phi_k, \psi_k)\|_{\mathcal{H}^1}^2 \,\mathrm{d}s
\end{aligned}$$

using again the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^{\frac{2+\omega}{1+\omega}}(\Gamma)$. Here, we have additionally introduced the notation $\boldsymbol{\mathcal{X}}_\omega := \mathbf{L}^3(\Omega) \times \mathbf{L}^{2+\omega}(\Gamma)$ out of brevity. Recalling that $F, G \ge 0$, and that the mobility functions m_Ω and m_Γ are uniformly positive (see **(A3)**), we infer from (5.37) and (5.38) that

$$\begin{aligned}
&\frac{1}{2}\|(\phi_k, \psi_k)\|_{K,\alpha}^2 + \frac{m_*}{2}\int_0^t\int_\Omega |\nabla\mu_k|^2 \,\mathrm{d}x\,\mathrm{d}s + \frac{m_*}{2}\int_0^t\int_\Gamma |\nabla_\Gamma\theta_k|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad + L^{-1}\int_0^t\int_\Gamma (\beta\theta_k - \mu_k)^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad \le C + C\int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2 \|(\phi_k, \psi_k)\|_{\mathcal{H}^1}^2 \,\mathrm{d}s \qquad (5.39)\\
&\quad \le C + C\int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2 \|(\phi_k, \psi_k)\|_{K,\alpha}^2 \,\mathrm{d}s
\end{aligned}$$

for all $t \in [0, T_k]$. Here, the last inequality follows from the bulk-surface Poincaré inequality as presented in Lemma 4.2.1 whose usage is justified by the following reasoning. First, by the aforementioned bulk-surface Poincaré inequality we have

$$\|(\phi_k - \beta\mathrm{mean}(\phi_k, \psi_k), \psi_k - \mathrm{mean}(\phi_k, \psi_k))\|_{\mathcal{H}^1} \leq C_P \|(\phi_k, \psi_k)\|_{K,\alpha} \tag{5.40}$$

on $[0, T_k]$. Then, testing (5.34a) with $(\beta, 1)$ yields the discrete mass conservation law

$$\beta \int_\Omega \phi_k(t) \,\mathrm{d}x + \int_\Gamma \psi_k(t) \,\mathrm{d}\Gamma = \beta \int_\Omega \phi_0 \,\mathrm{d}x + \int_\Gamma \psi_0 \,\mathrm{d}\Gamma \tag{5.41}$$

for all $t \in [0, T_k]$, where we used that

$$\begin{aligned} \mathrm{mean}(P_{V_k}(\phi_0), P_{V_{\Gamma k}}(\psi_0)) &= \frac{\big((P_{V_k}(\phi_0), P_{V_{\Gamma k}}(\psi_0)), (\beta, 1)\big)_{\mathcal{L}^2}}{\beta^2|\Omega| + |\Gamma|} \\ &= \frac{\big((\phi_0, \psi_0), (\beta, 1)\big)_{\mathcal{L}^2}}{\beta^2|\Omega| + |\Gamma|} \\ &= \mathrm{mean}(\phi_0, \psi_0). \end{aligned}$$

Consequently, we infer from (5.40) with (5.41) that

$$\|(\phi_k, \psi_k)\|_{\mathcal{H}^1} \leq C(1 + \|(\phi_k, \psi_k)\|_{K,\alpha})$$

on $[0, T_k]$. Thus, using Gronwall's lemma, we conclude from (5.39) that

$$\|(\phi_k(t), \psi_k(t))\|_{K,\alpha}^2 \leq C \exp\left(C \int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2 \,\mathrm{d}s\right) \leq C \exp\left(C \int_0^T \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2 \,\mathrm{d}s\right)$$

for all $t \in [0, T_k]$. We thus readily deduce that

$$\|(\phi_k, \psi_k)\|_{L^\infty(0,T_k;\mathcal{H}^1)} + K^{-\frac{1}{2}} \|\alpha\psi_k - \phi_k\|_{L^\infty(0,T_k;L^2(\Gamma))} \leq C \tag{5.42}$$

after another application of the bulk-surface Poincaré inequality. Then, integrating (5.39) in time from 0 to T, we infer with (5.42) that

$$\|(\nabla\mu_k, \nabla_\Gamma\theta_k)\|_{L^2(0,T_k;\boldsymbol{\mathcal{L}}^2)} + L^{-\frac{1}{2}} \|\beta\theta_k - \mu_k\|_{L^2(0,T_k;L^2(\Gamma))} \leq C. \tag{5.43}$$

Next, we derive a uniform estimate for (μ_k, θ_k) in the full $\mathcal{H}^1$-norm. To this end, let $(\eta, \eta_\Gamma) \in \mathcal{H}^1$ and consider $(\overline{\eta}, \overline{\eta}_\Gamma) = (P_{V_k}(\eta), P_{V_{\Gamma k}}(\eta_\Gamma))$. By the growth conditions (5.14) and (5.15) from **(A4)**, the Sobolev embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^{2q}(\Gamma)$, and the trace theorem $H^1(\Omega) \hookrightarrow L^2(\Gamma)$, we have

$$\begin{aligned} |\langle(\mu_k, \theta_k), (\eta, \eta_\Gamma)\rangle_{\mathcal{H}^1}| &= |\langle(\mu_k, \theta_k), (\overline{\eta}, \overline{\eta}_\Gamma)\rangle_{\mathcal{H}^1}| \\ &\leq \|\nabla\phi_k\|_{\mathbf{L}^2(\Omega)} \|\nabla\eta\|_{\mathbf{L}^2(\Omega)} + \|F'(\phi_k)\|_{L^{\frac{6}{5}}(\Omega)} \|\eta\|_{L^6(\Omega)} \\ &\quad + \|\nabla_\Gamma\psi_k\|_{\mathbf{L}^2(\Gamma)} \|\nabla_\Gamma\eta_\Gamma\|_{\mathbf{L}^2(\Gamma)} + \|G'(\psi_k)\|_{L^2(\Gamma)} \|\eta_\Gamma\|_{L^2(\Gamma)} \\ &\quad + K^{-1} \|\alpha\psi_k - \phi_k\|_{L^2(\Gamma)} \|\alpha\eta_\Gamma - \eta\|_{L^2(\Gamma)} \\ &\leq C\Big(1 + \|\phi_k\|_{H^1(\Omega)} + \|\phi_k\|_{L^6(\Omega)}^5\Big) \|\eta\|_{H^1(\Omega)} \\ &\quad + C\Big(1 + \|\psi_k\|_{H^1(\Gamma)} + \|\psi_k\|_{L^{2q}(\Gamma)}^q\Big) \|\eta_\Gamma\|_{H^1(\Gamma)} \\ &\leq C\left(1 + \|\phi_k\|_{H^1(\Omega)}^5 + \|\psi_k\|_{H^1(\Gamma)}^q\right) \|(\eta, \eta_\Gamma)\|_{\mathcal{H}^1} \end{aligned} \tag{5.44}$$

on $[0, T_k]$. Taking the supremum over all $(\eta, \eta_\Gamma) \in \mathcal{H}^1$ with $\|(\eta, \eta_\Gamma)\|_{\mathcal{H}^1} \leq 1$, and exploiting (5.42), we conclude that

$$\|(\mu_k, \theta_k)\|_{L^\infty(0,T_k;(\mathcal{H}^1)')} \leq C. \tag{5.45}$$

Now, due to Lemma 5.2.1, we have

$$\|(\mu_k, \theta_k)\|_{\mathcal{L}^2}^2 \leq 2\|(\mu_k, \theta_k)\|_{(\mathcal{H}^1)'}\|(\nabla\mu_k, \nabla_\Gamma\theta_k)\|_{\mathcal{L}^2} + \|(\mu_k, \theta_k)\|_{(\mathcal{H}^1)'}^2 \tag{5.46}$$

on $[0, T_k]$. Utilizing (5.43) and (5.45), we deduce from (5.46) that

$$\|(\mu_k, \theta_k)\|_{L^4(0,T_k;\mathcal{L}^2)} \leq C. \tag{5.47}$$

In view of (5.43) we additionally infer

$$\|(\mu_k, \theta_k)\|_{L^2(0,T_k;\mathcal{H}^1)} \leq C. \tag{5.48}$$

Lastly, we derive a uniform estimate of the time derivatives. Therefore, we consider again $(\overline{\zeta}, \overline{\zeta}_\Gamma) \coloneqq (P_{V_k}(\zeta), P_{V_{\Gamma k}}(\zeta_\Gamma))$ for an arbitrary pair $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1$. Recalling that the mobility functions m_Ω and m_Γ are bounded (see **(A3)**), we use Hölder's inequality as well as the continuous embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^{\frac{2+\omega}{1+\omega}}(\Gamma)$ to infer that

$$\begin{aligned}
|\langle(\partial_t\phi_k, \partial_t\psi_k), (\zeta, \zeta_\Gamma)\rangle_{\mathcal{H}^1}| &= |\langle(\partial_t\phi_k, \partial_t\psi_k), (\overline{\zeta}, \overline{\zeta}_\Gamma)\rangle_{\mathcal{H}^1}| \\
&\leq C\|(\phi_k, \psi_k)\|_{\mathcal{H}^1}\|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1} \\
&\quad + C\|(\mu_k, \theta_k)\|_{L,\beta}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1}
\end{aligned}$$

on $[0, T_k]$. Hence, after taking the supremum over all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1$ with $\|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1} \leq 1$, taking the square of the resulting inequality and integrating in time over $[0, T_k]$, we conclude with the uniform estimates (5.42) and (5.48) as well as the regularity of the velocity fields that

$$\|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^2(0,T_k;(\mathcal{H}^1)')} \leq C. \tag{5.49}$$

Step 3: Extension of the approximate solution onto $[0, T]$. Using the definition of the approximate solution (5.33) and the uniform estimate (5.42), we obtain for any $T_k \in [0, T_k^*)$, all $t \in [0, T_k]$ and $j \in \{1, \dots, k\}$,

$$\begin{aligned}
|a_j^k(t)| + |b_j^k(t)| &= |(\phi_k(t), \zeta_j)_{L^2(\Omega)}| + |(\psi_k, \zeta_{\Gamma j})_{L^2(\Gamma)}| \\
&\leq \|\phi_k\|_{L^\infty(0,T_k;L^2(\Omega))} + \|\psi_k\|_{L^\infty(0,T_k;L^2(\Gamma))} \leq C.
\end{aligned}$$

This shows that the solution $(\mathbf{a}^k, \mathbf{b}^k)^\top$ is bounded on the time interval $[0, T_k^*)$ by a constant that is independent of k and T_k^*. Hence, by classical ODE theory, this allows us to extend the solution beyond the time T_k^*. However, as $(\mathbf{a}^k, \mathbf{b}^k)^\top$ was assumed to be non-extendable, this is a contradiction. We thus infer that the solution $(\mathbf{a}^k, \mathbf{b}^k)^\top$ actually exists on $[0, T]$. Since the coefficients $(\mathbf{c}^k, \mathbf{d}^k)^\top$ can be reconstructed from $(\mathbf{a}^k, \mathbf{b}^k)^\top$, we further conclude that the coefficients $(\mathbf{c}^k, \mathbf{d}^k)^\top$ also exist on $[0, T]$. This automatically entails that the approximate solution $(\phi_k, \psi_k, \mu_k, \theta_k)$ exists on $[0, T]$ for all $k \in \mathbb{N}$ and satisfies the discretized weak formulation (5.34) on $[0, T]$. Additionally, as the particular choice of T_k did not play any role in the derivation of the uniform estimates established in Step 2, we infer that the estimates (5.42), (5.43), (5.47), (5.48) and (5.49) remain true after replacing T_k with T. In summary, we conclude that for each $k \in \mathbb{N}$, the approximate solution $(\phi_k, \psi_k, \mu_k, \theta_k)$ satisfies the uniform estimate

$$\begin{aligned}
&\|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^2(0,T;(\mathcal{H}^1)')} + \|(\phi_k, \psi_k)\|_{L^\infty(0,T;\mathcal{H}^1)} \\
&\quad + \|(\mu_k, \theta_k)\|_{L^2(0,T;\mathcal{H}^1)} + \|(\mu_k, \theta_k)\|_{L^4(0,T;\mathcal{L}^2)} \leq C.
\end{aligned} \tag{5.50}$$

Step 4: Convergence to a weak solution. In view of the uniform estimate (5.50), the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma imply the existence of functions ϕ, ψ, μ and θ such that

$$
\begin{aligned}
(\partial_t\phi_k, \partial_t\psi_k) &\to (\partial_t\phi, \partial_t\psi) && \text{weakly in } L^2(0,T;(\mathcal{H}^1)'), && (5.51)\\
(\phi_k, \psi_k) &\to (\phi, \psi) && \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1), &&\\
& && \text{strongly in } C([0,T];\mathcal{H}^s) \text{ for all } s \in [0,1), && (5.52)\\
(\mu_k, \theta_k) &\to (\mu, \theta) && \text{weakly in } L^4(0,T;\mathcal{L}^2) \cap L^2(0,T;\mathcal{H}^1), && (5.53)
\end{aligned}
$$

as $k \to \infty$, along a non-relabeled subsequence. From these convergences, we readily obtain the desired regularities (5.22a)–(5.22b) of the functions ϕ, ψ, μ and θ. Hence, Definition 5.3.1(i) is fulfilled.

Using Sobolev's embedding theorem, we infer from (5.52) that

$$
\begin{aligned}
\phi_k &\to \phi && \text{strongly in } C([0,T];L^r(\Omega)) \text{ for all } r \in [2,6), \text{ and a.e. in } Q^T, && (5.54)\\
\psi_k &\to \psi && \text{strongly in } C([0,T];L^r(\Gamma)) \text{ for all } r \in [2,\infty), \text{ and a.e. on } \Sigma^T, && (5.55)
\end{aligned}
$$

as $k \to \infty$, after another subsequence extraction. Further, by the trace theorem, (5.52) and (5.53) imply that

$$\alpha\psi_k - \phi_k \to \alpha\psi - \phi \qquad \text{strongly in } C([0,T];L^2(\Gamma)) \tag{5.56}$$

as well as

$$\beta\theta_k - \mu_k \to \beta\theta - \mu \qquad \text{weakly in } L^2(0,T;L^2(\Gamma)), \tag{5.57}$$

respectively, as $k \to \infty$, after another subsequence extraction. Moreover, due to the growth conditions on F' from **(A4)**, the uniform estimate (5.42) and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we deduce that $F'(\phi_k)$ is bounded in $L^{\frac{6}{5}}(Q^T)$ uniformly in $k \in \mathbb{N}$. Hence, there exists a function $f \in L^{\frac{6}{5}}(Q^T)$ such that $F'(\phi_k) \to f$ weakly in $L^{\frac{6}{5}}(Q^T)$ as $k \to \infty$. Considering that $F'(\phi_k) \to F'(\phi)$ a.e. in Q^T as $k \to \infty$ due to (5.54), we infer from a convergence principle based on Egorov's theorem (see, e.g., [55, Proposition 2.1c, p.173]) that $F'(\phi) = f$. We thus conclude that

$$F'(\phi_k) \to F'(\phi) \qquad \text{weakly in } L^{\frac{6}{5}}(Q^T), \text{ and a.e. in } Q^T \tag{5.58}$$

as $k \to \infty$. Recalling the growth conditions on G and G' (see **(A4)**), the convergence in (5.55) in combination with Lebesgue's general convergence theorem yields

$$
\begin{aligned}
G(\psi_k) &\to G(\psi) && \text{strongly in } L^1(\Sigma^T), \text{ and a.e. on } \Sigma^T, && (5.59)\\
G'(\psi_k) &\to G'(\psi) && \text{strongly in } L^2(\Sigma^T), \text{ and a.e. on } \Sigma^T, && (5.60)
\end{aligned}
$$

as $k \to \infty$. Furthermore, by means of Lebesgue's dominated convergence theorem, it follows from (5.54), (5.55) and **(A3)** that

$$
\begin{aligned}
m_\Omega(\phi_k) &\to m_\Omega(\phi) && \text{strongly in } L^r(Q^T), \text{ and a.e. in } Q^T, && (5.61)\\
m_\Gamma(\psi_k) &\to m_\Gamma(\psi) && \text{strongly in } L^r(\Sigma^T), \text{ and a.e. on } \Sigma^T, && (5.62)
\end{aligned}
$$

for all $r \in [2,\infty)$, as $k \to \infty$. Moreover, as the mobility functions m_Ω and m_Γ are bounded (see **(A3)**), we use Lebesgue's general convergence theorem along with the weak-strong convergence principle to find that

$$\sqrt{m_\Omega(\phi_k)}\nabla\mu_k \to \sqrt{m_\Omega(\phi)}\nabla\mu \qquad \text{weakly in } \mathbf{L}^2(Q^T), \tag{5.63}$$

$$\sqrt{m_\Gamma(\psi_k)}\nabla_\Gamma\theta_k \to \sqrt{m_\Gamma(\psi)}\nabla_\Gamma\theta \qquad \text{weakly in } \mathbf{L}^2(\Sigma^T), \tag{5.64}$$
$$m_\Omega(\phi_k)\nabla\mu_k \to m_\Omega(\phi)\nabla\mu \qquad \text{weakly in } \mathbf{L}^2(Q^T), \tag{5.65}$$
$$m_\Gamma(\psi_k)\nabla_\Gamma\theta_k \to m_\Gamma(\psi)\nabla_\Gamma\theta \qquad \text{weakly in } \mathbf{L}^2(\Sigma^T), \tag{5.66}$$

as $k \to \infty$. Lastly, exploiting the convergences (5.53), (5.54) and (5.55), and using again the weak-strong convergence principle, we infer

$$\phi_k\mathbf{v}\cdot\nabla\mu_k \to \phi\mathbf{v}\cdot\nabla\mu \qquad \text{strongly in } L^1(Q^T), \text{ and a.e. in } Q^T, \tag{5.67}$$
$$\psi_k\mathbf{w}\cdot\nabla_\Gamma\theta_k \to \psi\mathbf{w}\cdot\nabla_\Gamma\theta \qquad \text{strongly in } L^1(\Sigma^T), \text{ and a.e. on } \Sigma^T, \tag{5.68}$$

as $k \to \infty$.

We now multiply both equations of the discretized weak formulation (5.34) with an arbitrary test function $\pi \in C_c^\infty([0,T])$ and integrate in time from 0 to T. Using the convergences (5.51)–(5.68) established above, we may pass to the limit $k \to \infty$ to infer that

$$\begin{aligned}
&\int_0^T \langle(\partial_t\phi,\partial_t\psi)(\zeta_j,\zeta_{\Gamma j})\rangle_{\mathcal{H}^1}\,\pi\,\mathrm{d}t - \int_0^T\int_\Omega \phi\mathbf{v}\cdot\nabla\zeta_j\,\pi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\zeta_{\Gamma j}\,\pi\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad - \int_0^T\int_\Omega m_\Omega(\phi)\nabla\mu\cdot\nabla\zeta_j\,\pi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Gamma m_\Gamma(\psi)\nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta_{\Gamma j}\,\pi\,\mathrm{d}\Gamma\,\mathrm{d}t \qquad (5.69\text{a})\\
&\quad - L^{-1}\int_0^T\int_\Gamma(\beta\theta-\mu)(\beta\zeta_{\Gamma j}-\zeta_j)\,\pi\,\mathrm{d}\Gamma\,\mathrm{d}t,
\end{aligned}$$

$$\begin{aligned}
&\int_0^T\int_\Omega \mu\,\eta_j\,\pi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Gamma \theta\,\eta_{\Gamma j}\,\pi\,\mathrm{d}\Gamma\,\mathrm{d}t \qquad (5.69\text{b})\\
&= \int_0^T\int_\Omega \nabla\phi\cdot\nabla\zeta_j\,\pi + F'(\phi)\zeta_j\,\pi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Gamma \nabla_\Gamma\psi\cdot\nabla_\Gamma\zeta_{\Gamma j}\,\pi + G'(\psi)\zeta_{\Gamma j}\,\pi\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad + K^{-1}\int_0^T\int_\Gamma(\alpha\psi-\phi)(\alpha\zeta_{\Gamma j}-\zeta_j)\,\pi\,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned}$$

hold for all $j \in \mathbb{N}$ and all test functions $\pi \in C_c^\infty([0,T])$. Since $\operatorname{span}\{(\zeta_j,\zeta_{\Gamma j}) : j \in \mathbb{N}\}$ is dense in $\mathcal{H}^1$, this proves that the quadruplet (ϕ,ψ,μ,θ) satisfies the weak formulation (5.24) for all test functions $(\zeta,\zeta_\Gamma),(\eta,\eta_\Gamma) \in \mathcal{H}^1$. We thus conclude that Definition 5.3.1(iii) is fulfilled.

Since the approximate solutions (ϕ_k,ψ_k) satisfy the discrete mass conservation law (see (5.41)) for all $k \in \mathbb{N}$, we can use the convergence (5.52) to deduce that (ϕ,ψ) satisfies the mass conservation law (5.25) as well. Thus, Definition 5.3.1(iv) is fulfilled. Concerning the initial conditions, we infer from (5.35), due to the convergence properties of the projections, that

$$(\phi_k(0),\psi_k(0)) \to (\phi_0,\psi_0) \quad \text{strongly in } \mathcal{L}^2$$

as $k \to \infty$. Additionally, the convergences (5.54) and (5.55) yield

$$(\phi_k(0),\psi_k(0)) \to (\phi(0),\psi(0)) \quad \text{strongly in } \mathcal{L}^2$$

as $k \to \infty$. We deduce that $\phi(0) = \phi_0$ a.e. in Ω and $\psi(0) = \psi_0$ a.e. on Γ. This proves that Definition 5.3.1(ii) is fulfilled.

To verify the energy inequality (5.26), we consider an arbitrary non-negative test function $\pi \in C_c^\infty(0,T)$. Multiplying (5.36) with π and integrating in time from 0 to T yields

$$\int_0^T E_{\mathrm{free}}(\phi_k(t),\psi_k(t))\,\pi(t)\,\mathrm{d}t$$

$$
\begin{aligned}
&+ \int_0^T \int_0^t \int_\Omega \Big(m_\Omega(\phi_k)|\nabla \mu_k|^2 - \phi_k \mathbf{v} \cdot \nabla \mu_k \Big) \pi(t) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t \\
&+ \int_0^T \int_0^t \int_\Gamma \Big(m_\Gamma(\psi_k)|\nabla_\Gamma \theta_k|^2 - \psi_k \mathbf{w} \cdot \nabla_\Gamma \theta_k \Big) \pi(t) \, \mathrm{d}\Gamma \, \mathrm{d}s \, \mathrm{d}t \\
&+ L^{-1} \int_0^T \int_0^t \int_\Gamma (\beta\theta_k - \mu_k)^2 \pi(t) \, \mathrm{d}\Gamma \, \mathrm{d}s \, \mathrm{d}t \\
= &\int_0^T E_{\mathrm{free}}(\phi_k(0), \psi_k(0)) \, \pi(t) \, \mathrm{d}t.
\end{aligned}
$$

As $k \to \infty$, we infer from (5.59) that

$$
\int_0^T \Big(\int_\Gamma G(\psi_k) \, \mathrm{d}\Gamma \Big) \pi(t) \, \mathrm{d}t \to \int_0^T \Big(\int_\Gamma G(\psi) \, \mathrm{d}\Gamma \Big) \pi(t) \, \mathrm{d}t. \tag{5.70}
$$

Additionally, using (5.54) and **(A4)**, Fatou's lemma implies

$$
\int_0^T \Big(\int_\Omega F(\phi) \, \mathrm{d}x \Big) \pi(t) \, \mathrm{d}t \le \liminf_{k\to\infty} \int_0^T \Big(\int_\Omega F(\phi_k) \, \mathrm{d}x \Big) \pi(t) \, \mathrm{d}t. \tag{5.71}
$$

Moreover, from the strong convergence in (5.56) we obtain

$$
\int_0^T \Big(\int_\Gamma (\alpha\psi_k - \phi_k)^2 \, \mathrm{d}\Gamma \Big) \pi(t) \, \mathrm{d}t \to \int_0^T \Big(\int_\Gamma (\alpha\psi - \phi)^2 \, \mathrm{d}\Gamma \Big) \pi(t) \, \mathrm{d}t \tag{5.72}
$$

as $k \to \infty$. Lastly, in view of (5.52) and the weak lower semicontinuity of norms, we have

$$
\begin{aligned}
&\int_0^T \Big(\frac{1}{2} \|\nabla \phi\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2} \|\nabla_\Gamma \psi\|^2_{\mathbf{L}^2(\Gamma)} \Big) \pi(t) \, \mathrm{d}t \\
&\quad \le \liminf_{k\to\infty} \int_0^T \Big(\frac{1}{2} \|\nabla \phi_k\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2} \|\nabla_\Gamma \psi_k\|^2_{\mathbf{L}^2(\Gamma)} \Big) \pi(t) \, \mathrm{d}t.
\end{aligned} \tag{5.73}
$$

Combining (5.70)–(5.73) yields

$$
\int_0^T E_{\mathrm{free}}(\phi(t), \psi(t)) \, \pi(t) \, \mathrm{d}t \le \liminf_{k\to\infty} \int_0^T E_{\mathrm{free}}(\phi_k(t), \psi_k(t)) \, \pi(t) \, \mathrm{d}t. \tag{5.74}
$$

As a consequence of (5.63), (5.64) and the weak lower semicontinuity of norms with respect to the weak convergence, another application of Fatou's lemma entails that

$$
\begin{aligned}
&\int_0^T \int_0^t \int_\Omega m_\Omega(\phi) |\nabla \mu|^2 \, \pi(t) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t + \int_0^T \int_0^t \int_\Gamma m_\Gamma(\psi) |\nabla_\Gamma \theta|^2 \, \pi(t) \, \mathrm{d}\Gamma \, \mathrm{d}s \, \mathrm{d}t \\
&\le \liminf_{k\to\infty} \Big(\int_0^T \int_0^t \int_\Omega m_\Omega(\phi_k) |\nabla \mu_k|^2 \, \pi(t) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t \\
&\qquad + \int_0^T \int_0^t \int_\Gamma m_\Gamma(\psi_k) |\nabla_\Gamma \theta_k|^2 \, \pi(t) \, \mathrm{d}\Gamma \, \mathrm{d}s \, \mathrm{d}t \Big).
\end{aligned} \tag{5.75}
$$

Further, due to the convergences (5.67) and (5.68), we find that

$$
\begin{aligned}
&\int_0^T \int_0^t \int_\Omega \phi_k \mathbf{v} \cdot \nabla \mu_k \, \pi(t) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t + \int_0^T \int_0^t \int_\Gamma \psi_k \mathbf{w} \cdot \nabla_\Gamma \theta_k \, \pi(t) \, \mathrm{d}\Gamma \, \mathrm{d}s \, \mathrm{d}t \\
&\quad \longrightarrow \int_0^T \int_0^t \int_\Omega \phi \mathbf{v} \cdot \nabla \mu \, \pi(t) \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t + \int_0^T \int_0^t \int_\Gamma \psi \mathbf{w} \cdot \nabla_\Gamma \theta \, \pi(t) \, \mathrm{d}\Gamma \, \mathrm{d}s \, \mathrm{d}t
\end{aligned} \tag{5.76}
$$

as $k \to \infty$. Additionally, as convex and continuous functionals are weakly lower semicontinuous, we conclude from (5.57) that

$$\begin{aligned}\int_0^T \int_0^t \int_\Gamma (\beta\theta - \mu)^2\, \pi(t) \,\mathrm{d}\Gamma \,\mathrm{d}s \,\mathrm{d}t \\ \le \liminf_{k\to\infty} \int_0^T \int_0^t \int_\Gamma (\beta\theta_k - \mu_k)^2\, \pi(t) \,\mathrm{d}\Gamma \,\mathrm{d}s \,\mathrm{d}t.\end{aligned} \tag{5.77}$$

Lastly, recalling the growth conditions on F and G (see **(A4)**), we use the initial conditions (5.35) as well as the convergence properties of the projections P_{V_k} and $P_{V_{\Gamma k}}$ together with Lebesgue's general convergence theorem to infer that

$$\lim_{k\to\infty} E_{\text{free}}(\phi_k(0), \psi_k(0)) = E_{\text{free}}(\phi_0, \psi_0). \tag{5.78}$$

Collecting (5.75)–(5.78) finally yields

$$\begin{aligned}&E_{\text{free}}(\phi(t), \psi(t)) + \int_0^t \int_\Omega m_\Omega(\phi)|\nabla\mu|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma \,\mathrm{d}s \\ &\quad + L^{-1} \int_0^t \int_\Gamma (\beta\theta - \mu)^2 \,\mathrm{d}\Gamma \,\mathrm{d}s \\ &\quad - \int_0^t \int_\Omega \phi\mathbf{v} \cdot \nabla\mu \,\mathrm{d}x \,\mathrm{d}s - \int_0^t \int_\Gamma \psi\mathbf{w} \cdot \nabla_\Gamma\theta \,\mathrm{d}\Gamma \,\mathrm{d}s \\ &\le E_{\text{free}}(\phi_0, \psi_0)\end{aligned} \tag{5.79}$$

for almost every $t \in [0, T]$. To prove that (5.79) holds for every $t \in [0, T]$, first note that all integral terms in (5.79) depend continuously on time. Furthermore, due to $(\phi, \psi) \in C([0, T]; \mathcal{L}^2)$, we deduce that the following functions

$$t \mapsto \|\nabla\phi(t)\|^2_{\mathbf{L}^2(\Omega)}, \quad t \mapsto \|\nabla_\Gamma\psi(t)\|^2_{\mathbf{L}^2(\Gamma)}, \quad t \mapsto \int_\Omega F(\phi(t)) \,\mathrm{d}x, \quad t \mapsto \int_\Gamma G(\psi(t)) \,\mathrm{d}\Gamma$$

are lower semicontinuous on $[0, T]$. For the first two functions, this is a consequence of the lower semicontinuity of the respective norms, while for the last two functions, it follows from Fatou's lemma. This already entails that the weak solution $(\phi, \psi, \mu, \theta)$ satisfies the energy inequality for all times $t \in [0, T]$. Thus, Definition 5.3.1(v) is fulfilled.

We have therefore shown that the quadruplet $(\phi, \psi, \mu, \theta)$ is a weak solution of system (5.1) in the sense of Definition 5.3.1. □

Remark 5.4.2. This proof should work similarly in the cases $K = L = 0$ and $K = L = \infty$. In the first case, one uses a Faedo–Galerkin scheme as in [97] based on eigenfunctions of a suitable bulk-surface elliptic problem with Dirichlet-type coupling condition (cf. [117, Theorem 3.3]). In the second case, the bulk-surface Cahn–Hilliard equation (5.1) reduces to two, uncoupled Cahn–Hilliard equations, one in Ω and one on Γ. Therefore, a Faedo–Galerkin basis as in the proof of Theorem 5.4.1 can be used. Also note that, while we do not have the bulk-surface Poincaré inequality at our disposal anymore if $K = \infty$, we can use the standard Poincaré inequality for functions with vanishing mean, since the approximate solutions satisfy the mass conservation law (5.25) for $L = \infty$.

5.4.2 Asymptotic limits and existence of weak solutions to the limit models

In this section, we investigate the asymptotic limits $K \to 0$ and $K \to \infty$, and $L \to 0$ and $L \to \infty$ of the system (5.1).

Remark 5.4.3. In this section, we will need the additional assumption **(A5)** when approaching the limit cases $K \in \{0, \infty\}$. This is because the precise dependence of the constant C_P from the Poincaé inequality on K is not known, and therefore, we cannot rely on it. Instead, we use **(A5)** to directly obtain uniform bounds from the energy functional.

The asymptotic limit $K \to 0$

THEOREM 5.4.4. (Asymptotic limit $K \to 0$) *Suppose that the assumptions* **(A1)**–**(A5)** *and* **(A7)** *hold. Let $L \in (0, \infty)$ and let $(\phi_0, \psi_0) \in \mathcal{D}_\alpha$ be arbitrary initial data. For any $K \in (0, \infty)$, let $(\phi_K, \psi_K, \mu_K, \theta_K)$ denote a weak solution to system* (5.1) *in the sense of Definition 5.3.1 with initial data (ϕ_0, ψ_0). Then, there exists a quadruplet $(\phi_*, \psi_*, \mu_*, \theta_*)$ with $\phi_* = \alpha\psi_*$ a.e. on Σ^T such that*

$$
\begin{aligned}
(\partial_t \phi_K, \partial_t \psi_K) &\to (\partial_t \phi_*, \partial_t \psi_*) && \textit{weakly in } L^2(0,T;(\mathcal{H}^1)'),\\
(\phi_K, \psi_K) &\to (\phi_*, \psi_*) && \textit{weakly-star in } L^\infty(0,T;\mathcal{H}^1),\\
& && \textit{strongly in } C([0,T];\mathcal{H}^s) \textit{ for all } s \in [0,1),\\
(\mu_K, \theta_K) &\to (\mu_*, \theta_*) && \textit{weakly in } L^2(0,T;\mathcal{H}^1),\\
\alpha\psi_K - \phi_K &\to 0 && \textit{strongly in } L^\infty(0,T,L^2(\Gamma)),
\end{aligned}
$$

as $K \to 0$, up to subsequence extraction, with

$$
\|\alpha\psi_K - \phi_K\|_{L^\infty(0,T;L^2(\Gamma))} \le C\sqrt{K}, \tag{5.80}
$$

and the limit $(\phi_*, \psi_*, \mu_*, \theta_*)$ is a weak solution to the system (5.1) *in the sense of Definition 5.3.1 with $K = 0$.*

Remark 5.4.5. (a) As the right-hand side in (5.80) tends to zero as $K \to 0$, this explains why the Dirichlet-type boundary condition $\phi_* = \alpha\psi_*$ a.e. on Σ^T appears in the limit model corresponding to $K = 0$.

(b) The result of Theorem 5.4.4 remains valid if we replace the initial data $(\phi_0, \psi_0) \in \mathcal{D}_\alpha$ by a sequence $(\phi_{0K}, \psi_{0K}) \in \mathcal{H}^1$ satisfying

$$
E^K_{\text{free}}(\phi_{0K}, \psi_{0K}) \to E^0_{\text{free}}(\phi_0, \psi_0) \quad \text{as } K \to 0
$$

for some pair $(\phi_0, \psi_0) \in \mathcal{H}^1$ (see also [115, Theorem 2.3]). Here, the energy E^0_{free} is given as

$$
E^0_{\text{free}}(\eta, \eta_\Gamma) = \int_\Omega \frac{1}{2}|\nabla\eta|^2 + F(\eta)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma \eta_\Gamma|^2 + G(\eta_\Gamma)\,\mathrm{d}\Gamma
$$

for $(\eta, \eta_\Gamma) \in \mathcal{H}^1$.

In this case, one can show that $\phi_0 = \alpha\psi_0$ a.e. on Γ, and that the limiting quadruplet $(\phi_*, \psi_*, \mu_*, \theta_*)$ is a weak solution to system (5.1) corresponding to the initial data (ϕ_0, ψ_0).

Proof of Theorem 5.4.4. We consider an arbitrary sequence $(K_m)_{m\in\mathbb{N}} \subset (0,\infty)$ such that $K_m \to 0$ as $m \to \infty$ and a corresponding sequence of weak solutions $(\phi_{K_m}, \psi_{K_m}, \mu_{K_m}, \theta_{K_m})$ to the initial data (ϕ_0, ψ_0). In this proof, we use the letter C to denote generic positive constants independent of K_m and m. In order to prove suitable uniform estimates, we make use of the energy inequality and the additional growth assumptions made on F and G (see **(A5)**). Let now $m \in \mathbb{N}$ be arbitrary. First, note that $\phi_0 = \alpha\psi_0$ a.e. on Γ directly implies

$$E_{\text{free}}^{K_m}(\phi_0, \psi_0) = E_{\text{free}}^{0}(\phi_0, \psi_0) \le C. \tag{5.81}$$

Next, due to the growth condition **(A5)**, we find that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla\phi_{K_m}|^2 \,\mathrm{d}x + \int_\Omega F(\phi_{K_m}) \,\mathrm{d}x + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\psi_{K_m}|^2 \,\mathrm{d}\Gamma + \int_\Gamma G(\psi_{K_m}) \,\mathrm{d}\Gamma \\
&\ge \frac{1}{2}\int_\Omega |\nabla\phi_{K_m}|^2 \,\mathrm{d}x + a_F\int_\Omega \phi_{K_m}^2 \,\mathrm{d}x - b_F|\Omega| \\
&\qquad + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\psi_{K_m}|^2 \,\mathrm{d}\Gamma + a_G\int_\Gamma \psi_{K_m}^2 \,\mathrm{d}\Gamma - b_G|\Gamma| \\
&\ge c\|(\phi_{K_m}, \psi_{K_m})\|_{\mathcal{H}^1}^2 - b_F|\Omega| - b_G|\Gamma|
\end{aligned}$$

a.e. on $[0,T]$, where $c = \min\{\frac{1}{2}, a_F, a_G\}$. Then, using the energy inequality (5.26), and bounding the convective terms as in (5.38), we deduce that

$$\begin{aligned}
&c\|(\phi_{K_m}, \psi_{K_m})\|_{\mathcal{H}^1}^2 + \frac{1}{2K_m}\int_\Gamma (\alpha\psi_{K_m} - \phi_{K_m})^2 \,\mathrm{d}\Gamma + \frac{m_*}{2}\int_0^t\int_\Omega |\nabla\mu_{K_m}|^2 \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \frac{m_*}{2}\int_0^t\int_\Gamma |\nabla_\Gamma\theta_{K_m}|^2 \,\mathrm{d}\Gamma \,\mathrm{d}s + \frac{1}{L}\int_0^t\int_\Gamma (\beta\theta_{K_m} - \mu_{K_m})^2 \,\mathrm{d}\Gamma \,\mathrm{d}s \\
&\le C + C\int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2 \|(\phi_{K_m}, \psi_{K_m})\|_{\mathcal{H}^1}^2 \,\mathrm{d}s
\end{aligned}$$

a.e. on $[0,T]$. Thus, using Gronwall's lemma, we readily infer that

$$\begin{aligned}
&\|(\phi_{K_m}, \psi_{K_m})\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(\nabla\mu_{K_m}, \nabla_\Gamma\theta_{K_m})\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} \\
&\quad + \|\beta\theta_{K_m} - \mu_{K_m}\|_{L^2(0,T;L^2(\Gamma))} \le C,
\end{aligned} \tag{5.82}$$

as well as

$$\|\alpha\psi_{K_m} - \phi_{K_m}\|_{L^\infty(0,T;L^2(\Gamma))} \le C\sqrt{K_m}. \tag{5.83}$$

This already entails (5.80). We now test the weak formulation (5.24b) for $(\mu_{K_m}, \theta_{K_m})$ with $(\eta, \eta_\Gamma) \in \mathcal{D}_\alpha$. As $\eta = \alpha\eta_\Gamma$ a.e. on Γ, the corresponding boundary term involving K_m vanishes. Therefore, arguing as in the proof of Theorem 5.3.2, we deduce

$$\|(\mu_{K_m}, \theta_{K_m})\|_{L^\infty(0,T;\mathcal{D}_\alpha')} \le C. \tag{5.84}$$

Now, using Lemma 5.2.1, we find that

$$\|(\mu_{K_m}, \theta_{K_m})\|_{\mathcal{L}^2}^2 \le 2\|(\mu_{K_m}, \theta_{K_m})\|_{\mathcal{D}_\alpha'}\|(\nabla\mu_{K_m}, \nabla_\Gamma\theta_{K_m})\|_{\boldsymbol{\mathcal{L}}^2} + \|(\mu_{K_m}, \theta_{K_m})\|_{\mathcal{D}_\alpha'}^2.$$

In combination with (5.82) and (5.84), this yields

$$\|(\mu_{K_m}, \theta_{K_m})\|_{L^2(0,T;\mathcal{H}^1)} \le C. \tag{5.85}$$

Lastly, proceeding similarly as in the derivation of (5.49) in the proof of Theorem 5.4.1, it follows from the weak formulation (5.24a) and estimate (5.82) that

$$\|(\partial_t\phi_{K_m}, \partial_t\psi_{K_m})\|_{L^2(0,T;(\mathcal{H}^1)')} \le C. \tag{5.86}$$

In summary, combining (5.82), (5.85) and (5.86), we have thus shown that

$$\begin{aligned} &\|(\partial_t\phi_{K_m}, \partial_t\psi_{K_m})\|_{L^2(0,T;(\mathcal{H}^1)')} + \|(\phi_{K_m}, \psi_{K_m})\|_{L^\infty(0,T;\mathcal{H}^1)} \\ &\quad + \|(\mu_{K_m}, \theta_{K_m})\|_{L^2(0,T;\mathcal{H}^1)} + \|\beta\theta_{K_m} - \mu_{K_m}\|_{L^2(0,T;L^2(\Gamma))} \leq C. \end{aligned} \tag{5.87}$$

Using the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma, we deduce the existence of functions $(\phi_*, \psi_*, \mu_*, \theta_*)$ such that

$$(\partial_t\phi_{K_m}, \partial_t\psi_{K_m}) \to (\partial_t\phi_*, \partial_t\psi_*) \qquad \text{weakly in } L^2(0,T;(\mathcal{H}^1)'), \tag{5.88a}$$

$$\begin{aligned} (\phi_{K_m}, \psi_{K_m}) \to (\phi_*, \psi_*) \qquad &\text{weakly-star in } L^\infty(0,T;\mathcal{H}^1), \\ &\text{strongly in } C([0,T];\mathcal{H}^s) \text{ for all } s \in [0,1), \end{aligned} \tag{5.88b}$$

$$(\mu_{K_m}, \theta_{K_m}) \to (\mu_*, \theta_*) \qquad \text{weakly in } L^2(0,T;\mathcal{H}^1), \tag{5.88c}$$

as $m \to \infty$, along a non-relabeled subsequence. Additionally, due to (5.88b) and the trace theorem, we infer

$$\alpha\psi_{K_m} - \phi_{K_m} \to \alpha\psi_* - \phi_* \qquad \text{strongly in } C([0,T];L^2(\Gamma)) \tag{5.89}$$

as $m \to \infty$, which, in combination with (5.83), already entails $\phi_* = \alpha\psi_*$ a.e. on Σ^T. It is then clear that from the convergence properties (5.88a)–(5.88c) the quadruplet $(\phi_*, \psi_*, \mu_*, \theta_*)$ has the desired regularity as stated in Definition 5.3.1(i).

Further, due to (5.88b), we infer that

$$(\phi_{K_m}(0), \psi_{K_m}(0)) \to (\phi_*(0), \psi_*(0)) \qquad \text{strongly in } \mathcal{L}^2 \tag{5.90}$$

as $m \to \infty$, and therefore $\phi_*(0) = \phi_0$ a.e. in Ω and $\psi_*(0) = \psi_0$ a.e. on Γ. Thus, Definition 5.3.1(ii) is fulfilled.

Regarding the weak formulation, we consider $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1$ and $(\eta, \eta_\Gamma) \in \mathcal{D}_\alpha$ as test functions, multiply the resulting equations with $\pi \in C_c^\infty([0,T])$ and integrate in time from 0 to T. We obtain

$$\begin{aligned} &\int_0^T \langle(\partial_t\phi_{K_m}, \partial_t\psi_{K_m}), (\zeta, \zeta_\Gamma)\rangle_{\mathcal{H}^1} \pi \,\mathrm{d}t - \int_0^T \int_\Omega \phi_{K_m} \mathbf{v} \cdot \nabla\zeta \,\pi \,\mathrm{d}x \,\mathrm{d}t \\ &\quad - \int_0^T \int_\Gamma \psi_{K_m} \mathbf{w} \cdot \nabla_\Gamma\zeta_\Gamma \,\pi \,\mathrm{d}\Gamma \,\mathrm{d}t \\ &= - \int_0^T \int_\Omega m_\Omega(\phi_{K_m}) \nabla\mu_{K_m} \cdot \nabla\zeta \,\pi \,\mathrm{d}x \,\mathrm{d}t \\ &\quad - \int_0^T \int_\Gamma m_\Gamma(\psi_{K_m}) \nabla_\Gamma\theta_{K_m} \cdot \nabla_\Gamma\zeta_\Gamma \,\pi \,\mathrm{d}\Gamma \,\mathrm{d}t \\ &\quad - L^{-1} \int_0^T \int_\Gamma (\beta\theta_{K_m} - \mu_{K_m})(\beta\zeta_\Gamma - \zeta) \,\pi \,\mathrm{d}\Gamma \,\mathrm{d}t, \end{aligned} \tag{5.91a}$$

$$\begin{aligned} &\int_0^T \int_\Omega \mu_{K_m} \eta \,\pi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Gamma \theta_{K_m} \eta_\Gamma \,\pi \,\mathrm{d}\Gamma \,\mathrm{d}t \\ &= \int_0^T \int_\Omega \nabla\phi_{K_m} \cdot \nabla\eta \,\pi + F'(\phi_{K_m})\eta \,\pi \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_\Gamma \nabla_\Gamma\psi_{K_m} \cdot \nabla_\Gamma\eta_\Gamma \,\pi + G'(\psi_{K_m})\eta_\Gamma \,\pi \,\mathrm{d}\Gamma \,\mathrm{d}t. \end{aligned} \tag{5.91b}$$

Proceeding similarly as in the proof of Theorem 5.3.2, the convergences (5.88a)–(5.88c) allow us to pass to the limit $m \to \infty$, and deduce that $(\phi_*, \psi_*, \mu_*, \theta_*)$ satisfies the desired weak formulation (5.24). Thus, Definition 5.3.1(iii) is fulfilled.

The mass conservation law follows by passing to the limit $m \to \infty$ in the mass conservation law for (ϕ_{K_m}, ψ_{K_m}). Alternatively, one can test the weak formulation (5.24a) with $(\beta, 1) \in \mathcal{H}^1$, integrate with respect to time from 0 to t, and employ the fundamental theorem of calculus to infer (5.25). We conclude that Definition 5.3.1(iv) is satisfied.

Concerning the energy inequality (5.26), we note that

$$E^0_{\text{free}}(\phi_{K_m}(t), \psi_{K_m}(t)) \le E^{K_m}_{\text{free}}(\phi_{K_m}(t), \psi_{K_m}(t)) \tag{5.92}$$

for all $t \in [0, T]$ since the boundary term involving K_m is non-negative. Thus, we find

$$\begin{aligned}\int_0^T E^0_{\text{free}}(\phi_*(t), \psi_*(t))\, \pi(t)\, \mathrm{d}t &\le \liminf_{m\to\infty} \int_0^T E^0_{\text{free}}(\phi_{K_m}(t), \psi_{K_m}(t))\, \pi(t)\, \mathrm{d}t \\ &\le \liminf_{m\to\infty} \int_0^T E^{K_m}_{\text{free}}(\phi_{K_m}(t), \psi_{K_m}(t))\, \pi(t)\, \mathrm{d}t\end{aligned} \tag{5.93}$$

for all non-negative test functions $\pi \in C_c^\infty(0, T)$. Here, the first inequality follows by proceeding similarly as in the proof of Theorem 5.3.2. We now use the energy inequality (5.26) written for $(\phi_{K_m}, \psi_{K_m}, \mu_{K_m}, \theta_{K_m})$ to further bound the right-hand side of (5.93). Using lower semicontinuity arguments (similar to those in the proof of Theorem 5.3.2), and recalling (5.81), passing to the limit $m \to \infty$ leads to the corresponding energy inequality for $(\phi_*, \psi_*, \mu_*, \theta_*)$.

This proves that the quadruplet $(\phi_*, \psi_*, \mu_*, \theta_*)$ is a weak solution of (5.1) in the sense of Definition 5.3.1 with $K = 0$. ☐

The asymptotic limit $K \to \infty$

THEOREM 5.4.6. (Asymptotic limit $K \to \infty$) *Suppose that the assumptions* **(A1)**–**(A5)** *and* **(A7)** *hold. Let $L \in (0, \infty)$ and let $(\phi_0, \psi_0) \in \mathcal{H}^1$ be arbitrary initial data. For any $K \in (0, \infty)$, let $(\phi^K, \psi^K, \mu^K, \theta^K)$ denote a weak solution to system* (5.1) *in the sense of Definition 5.3.1 with initial data (ϕ_0, ψ_0). Then, there exists a quadruplet $(\phi^*, \psi^*, \mu^*, \theta^*)$ such that*

$$\begin{aligned}
(\partial_t\phi^K, \partial_t\psi^K) &\to (\partial_t\phi^*, \partial_t\psi^*) && \text{weakly in } L^2(0,T;(\mathcal{H}^1)'),\\
(\phi^K, \psi^K) &\to (\phi^*, \psi^*) && \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1),\\
& && \text{strongly in } C([0,T];\mathcal{H}^s) \text{ for all } s \in [0,1),\\
(\mu^K, \theta^K) &\to (\mu^*, \theta^*) && \text{weakly in } L^4(0,T;\mathcal{L}^2) \cap L^2(0,T;\mathcal{H}^1),\\
\frac{1}{K}(\alpha\psi^K - \phi^K) &\to 0 && \text{strongly in } L^\infty(0,T,L^2(\Gamma)),
\end{aligned}$$

as $K \to \infty$, up to subsequence extraction, with

$$\frac{1}{K}\|\alpha\psi^K - \phi^K\|_{L^\infty(0,T;L^2(\Gamma))} \le \frac{C}{\sqrt{K}}, \tag{5.94}$$

and the limit $(\phi^*, \psi^*, \mu^*, \theta^*)$ is a weak solution to the system (5.1) *in the sense of Definition 5.3.1 with $K = \infty$, which additionally satisfies $(\mu^*, \theta^*) \in L^4(0, T; \mathcal{L}^2)$.*

Remark 5.4.7. Suppose that for any $K \in (0, \infty)$, the phase-field ϕ^K is sufficiently regular such that the boundary condition (5.1e) holds in the strong sense, that is

$$K\partial_{\mathbf{n}}\phi^K = \alpha\psi^K - \phi^K \quad \text{a.e. on } \Sigma^T.$$

This is actually fulfilled under additional assumptions on the regularity of Γ and the parameter p (if $d = 3$), see Theorem 5.6.1. Then, estimate (5.94) can be reformulated as

$$\|\partial_{\mathbf{n}}\phi^K\|_{L^\infty(0,T;L^2(\Gamma))} \le \frac{C}{\sqrt{K}}.$$

As the right-hand side of this inequality tends to zero as $K \to \infty$, this explains why the homogeneous Neumann boundary condition $\partial_{\mathbf{n}}\phi^* = 0$ a.e. on Σ^T appears in the limit model corresponding to $K = \infty$.

Proof of Theorem 5.4.6. We consider an arbitrary sequence $(K_m)_{m\in\mathbb{N}} \subset (0,\infty)$ such that $K_m \to \infty$ as $m \to \infty$. Without loss of generality, we assume that $K_m \in [1,\infty)$ for all $m \in \mathbb{N}$. For any $m \in \mathbb{N}$, let $(\phi^{K_m}, \psi^{K_m}, \mu^{K_m}, \theta^{K_m})$ denote a weak solution of the system (5.1) in the sense of Definition 5.3.1 with initial data (ϕ_0, ψ_0) corresponding to the parameter K_m. In this proof, we use the letter C to denote generic positive constants independent of K_m and m. Let now $m \in \mathbb{N}$ be arbitrary.

As we have seen in the proof of Theorem 5.3.2 (see (5.37)), the initial energy satisfies

$$E^{K_m}_{\text{free}}(\phi^{K_m}(0), \psi^{K_m}(0)) \le C(1 + K_m^{-1}) \le C \tag{5.95}$$

since $K_m \ge 1$ for all $m \in \mathbb{N}$. This allows us to use the same argumentation as in the proof of Theorem 5.4.4, and we infer

$$\begin{aligned} &\|(\phi^{K_m}, \psi^{K_m})\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(\nabla\mu^{K_m}, \nabla_\Gamma\theta^{K_m})\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} \\ &\quad + \|\beta\theta^{K_m} - \mu^{K_m}\|_{L^2(0,T;L^2(\Gamma))} \le C \end{aligned} \tag{5.96}$$

as well as

$$\frac{1}{K_m}\|\alpha\psi^{K_m} - \phi^{K_m}\|_{L^\infty(0,T;L^2(\Gamma))} \le \frac{C}{\sqrt{K_m}}. \tag{5.97}$$

This already implies (5.94). We now test the weak formulation (5.24b) with $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1$. Using again that $K_m \ge 1$, we find that

$$\|(\mu^{K_m}, \theta^{K_m})\|_{L^\infty(0,T;(\mathcal{H}^1)')} \le C. \tag{5.98}$$

This estimate allows us to apply Lemma 5.2.1, and using (5.96) and (5.98), we conclude

$$\|(\mu^{K_m}, \theta^{K_m})\|_{L^4(0,T;\mathcal{L}^2)} \le C. \tag{5.99}$$

In combination with (5.96) we thus deduce

$$\|(\mu^{K_m}, \theta^{K_m})\|_{L^2(0,T;\mathcal{H}^1)} \le C. \tag{5.100}$$

Lastly, exploiting (5.96), we conclude from the weak formulation (5.24a) by a comparison argument that

$$\|(\partial_t\phi^{K_m}, \partial_t\psi^{K_m})\|_{L^2(0,T;(\mathcal{H}^1)')} \le C. \tag{5.101}$$

In summary, combining (5.96) and (5.99)–(5.101), we thus have shown that

$$\begin{aligned} &\|(\partial_t\phi^{K_m}, \partial_t\psi^{K_m})\|_{L^2(0,T;(\mathcal{H}^1)')} + \|(\phi^{K_m}, \psi^{K_m})\|_{L^\infty(0,T;\mathcal{H}^1)} \\ &\quad + \|(\mu^{K_m}, \theta^{K_m})\|_{L^4(0,T;\mathcal{L}^2)} + \|(\mu^{K_m}, \theta^{K_m})\|_{L^2(0,T;\mathcal{H}^1)} \le C. \end{aligned} \tag{5.102}$$

In view of the uniform estimate (5.102), the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma imply the existence of functions $(\phi^*, \psi^*, \mu^*, \theta^*)$ such that

$$(\partial_t \phi^{K_m}, \partial_t \psi^{K_m}) \to (\partial_t \phi^*, \partial_t \psi^*) \qquad \text{weakly in } L^2(0,T;(\mathcal{H}^1)'), \tag{5.103a}$$

$$\begin{aligned}(\phi^{K_m}, \psi^{K_m}) \to (\phi^*, \psi^*) \qquad & \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1), \\ & \text{strongly in } C([0,T];\mathcal{H}^s) \text{ for all } s \in [0,1),\end{aligned} \tag{5.103b}$$

$$(\mu^{K_m}, \theta^{K_m}) \to (\mu^*, \theta^*) \qquad \text{weakly in } L^4(0,T;\mathcal{L}^2) \cap L^2(0,T;\mathcal{H}^1), \tag{5.103c}$$

as $m \to \infty$, along a non-relabeled subsequence. Additionally, thanks to (5.97) and the trace theorem, we infer

$$\frac{1}{K_m}(\alpha\psi^{K_m} - \phi^{K_m}) \to 0 \qquad \text{strongly in } L^\infty(0,T;L^2(\Gamma)) \tag{5.104}$$

as $m \to \infty$. We readily deduce that Definition 5.3.1(i) is satisfied.

Using the Sobolev embedding $\mathcal{H}^s \hookrightarrow \mathcal{L}^2$ for $s \in (0,1)$ along with (5.103b), we find that

$$(\phi^{K_m}(0), \psi^{K_m}(0)) \to (\phi^*(0), \psi^*(0)) \qquad \text{strongly in } \mathcal{L}^2, \tag{5.105}$$

as $m \to \infty$. In view of the initial conditions satisfied by (ϕ^{K_m}, ψ^{K_m}), we deduce that Definition 5.3.1(ii) is fulfilled.

As in the proof of Theorem 5.4.4, the convergences (5.103a)–(5.104) are sufficient to pass to the limit in the weak formulations (5.24) associated with K_m to conclude that the limit $(\phi^*, \psi^*, \mu^*, \theta^*)$ fulfills the weak formulation (5.24) associated with $K = \infty$. Thus, we infer that Definition 5.3.1(iii) is satisfied.

The mass conservation law as stated in Definition 5.3.1(iv) also follows with the same reasoning as in the proof of Theorem 5.4.4.

For the energy inequality, note that for all non-negative $\pi \in C_c^\infty(0,T)$, we have

$$\begin{aligned}\int_0^T E_{\text{free}}^\infty(\phi^*(t), \psi^*(t))\, \pi(t) \,\mathrm{d}t &\leq \liminf_{m\to\infty} \int_0^T E_{\text{free}}^\infty(\phi^{K_m}(t), \psi^{K_m}(t))\, \pi(t) \,\mathrm{d}t \\ &\leq \liminf_{m\to\infty} \int_0^T E_{\text{free}}^{K_m}(\phi^{K_m}(t), \psi^{K_m}(t))\, \pi(t) \,\mathrm{d}t.\end{aligned} \tag{5.106}$$

The first inequality follows the same argumentation that we already have seen before, whereas the second inequality is due to (5.104). We finish the proof by mimicking the proof of the energy inequality in Theorem 5.3.2, which is based on the convergence results (5.103a)–(5.103c) and the assumptions **(A3)**–**(A4)**.

We thus have shown that the quadruplet $(\phi^*, \psi^*, \mu^*, \theta^*)$ is a weak solution to the system (5.1) in the sense of Definition 5.3.1 for $K = \infty$. ☐

The asymptotic limit $L \to 0$

In this section, we investigate the asymptotic limits $L \to 0$ and $L \to \infty$ of the system (5.1) for fixed $K \in [0,\infty]$.

THEOREM 5.4.8. (Asymptotic limit $L \to 0$) *Suppose that the assumptions* **(A1)**–**(A4)** *and* **(A7)** *hold. Let $K \in [0,\infty]$ and let $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ be arbitrary initial data. In the case $K \in \{0,\infty\}$, we further assume that* **(A5)** *holds. For any $L \in (0,\infty)$,* *let $(\phi_L, \psi_L, \mu_L, \theta_L)$ denote a weak solution of the system (5.1) in the sense of Definition 5.3.1 with initial data (ϕ_0, ψ_0). Then, there exists a quadruplet $(\phi_\star, \psi_\star, \mu_\star, \theta_\star)$ with $\mu_\star = \beta\theta_\star$ a.e. on Σ^T such that*

$$\begin{aligned}
(\partial_t\phi_L, \partial_t\psi_L) &\to (\phi_\star, \psi_\star) && \textit{weakly in } L^2(0,T;\mathcal{D}'_\beta),\\
(\phi^L, \psi^L) &\to (\phi_\star, \psi_\star) && \textit{weakly-star in } L^\infty(0,T;\mathcal{H}^1_{K,\alpha}),\\
& && \textit{strongly in } C([0,T];\mathcal{H}^s) \textit{ for all } s\in[0,1),\\
(\mu_L, \theta_L) &\to (\mu_\star, \theta_\star) && \textit{weakly in } L^2(0,T;\mathcal{H}^1),\\
& && \textit{weakly in } L^4(0,T;\mathcal{L}^2) \textit{ if } K\in(0,\infty],\\
\beta\theta_L - \mu_L &\to 0 && \textit{strongly in } L^2(0,T;L^2(\Gamma)),
\end{aligned}$$

as $L \to 0$, up to subsequence extraction, with

$$\|\beta\theta_L - \mu_L\|_{L^2(0,T;L^2(\Gamma))} \le C\sqrt{L}, \tag{5.107}$$

and the quadruplet $(\phi_\star, \psi_\star, \mu_\star, \theta_\star)$ is a weak solution to the system (5.1) in the sense of Definition 5.3.1 with $L = 0$, which additionally satisfies $(\mu_\star, \theta_\star) \in L^4(0,T;\mathcal{L}^2)$ in the case $K \in (0,\infty]$.

Remark 5.4.9. As the right-hand side in (5.107) tends to zero as $L \to 0$, this explains why the Dirichlet-type boundary condition $\mu_\star = \beta\theta_\star$ a.e. on Σ^T appears in the limit model corresponding to $L = 0$.

Proof of Theorem 5.4.8. We consider an arbitrary sequence $(L_m)_{m\in\mathbb{N}} \subset (0,\infty)$ such that $L_m \to 0$ as $m \to \infty$ and a corresponding weak solution $(\phi_{L_m}, \psi_{L_m}, \theta_{L_m}, \mu_{L_m})$ to the system (5.1) in the sense of Definition 5.3.1 to the initial data (ϕ_0, ψ_0). In this proof, we denote by C arbitrary positive constants independent of L_m and m, which may change their value from line to line. Let now $m \in \mathbb{N}$ be arbitrary.

If $K \in [0,\infty)$, we conclude from the energy inequality (5.26) that

$$\|(\phi_{L_m}, \psi_{L_m})\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(\nabla\mu_{L_m}, \nabla_\Gamma\theta_{L_m})\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} \le C, \tag{5.108}$$

as well as

$$\|\beta\theta_{L_m} - \mu_{L_m}\|_{L^2(0,T;L^2(\Gamma))} \le C\sqrt{L_m}. \tag{5.109}$$

If $K = \infty$, we do not have the bulk-surface Poincaré inequality at our disposal. Instead, we have to argue as in the proof of Theorem 5.4.4 and make use of the additional assumption **(A5)** to obtain the estimate (5.108). In particular, (5.109) yields (5.107). Arguing as in the proof of Theorem 5.3.2, we additionally infer that

$$\|(\mu_{L_m}, \theta_{L_m})\|_{L^\infty(0,T;(\mathcal{H}^1_{K,\alpha})')} \le C. \tag{5.110}$$

If $K \in (0,\infty]$, we may use Lemma 5.2.1 and further obtain

$$\|(\mu_{L_m}, \theta_{L_m})\|_{L^4(0,T;\mathcal{L}^2)} \le C, \tag{5.111}$$

which in combination with (5.108) yields

$$\|(\mu_{L_m}, \theta_{L_m})\|_{L^2(0,T;\mathcal{H}^1)} \le C. \tag{5.112}$$

In the case $K = 0$, we argue analogously as in the proof of Theorem 5.4.4 to deduce that

$$\|(\mu_{L_m}, \theta_{L_m})\|_{L^2(0;T;\mathcal{L}^2)} \le C, \tag{5.113}$$

from which we obtain (5.112) as well.

For the time derivatives, we again proceed similarly as in the proof of Theorem 5.3.2 but choose the test function space $\mathcal{D}_\beta$ instead of $\mathcal{H}^1$. We then obtain due to the weak formulation (5.24a) and the uniform bounds (5.108) and (5.109) that

$$\|(\partial_t \phi_{L_m}, \partial_t \psi_{L_m})\|_{L^2(0,T;\mathcal{D}'_\beta)} \le C. \tag{5.114a}$$

In view of the uniform estimates (5.108), (5.111), (5.112) and (5.114a), the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma imply the existence of functions $\phi_\star, \psi_\star, \mu_\star$ and $\theta_\star$ such that

$$(\partial_t \phi_{L_m}, \partial_t \psi_{L_m}) \to (\partial_t \phi_\star, \partial_t \psi_\star) \qquad \text{weakly in } L^2(0,T;\mathcal{D}'_\beta), \tag{5.114b}$$

$$\begin{aligned} (\phi_{L_m}, \psi_{L_m}) \to (\phi_\star, \psi_\star) \qquad & \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1_{K,\alpha}), \\ & \text{strongly in } C([0,T];\mathcal{H}^s) \text{ for all } s \in [0,1), \end{aligned} \tag{5.114c}$$

$$\begin{aligned} (\mu_{L_m}, \theta_{L_m}) \to (\mu_\star, \theta_\star) \qquad & \text{weakly in } L^2(0,T;\mathcal{H}^1), \\ & \text{weakly in } L^4(0,T;\mathcal{L}^2) \text{ if } K \in (0,\infty], \end{aligned} \tag{5.114d}$$

as $m \to \infty$, along a non-relabeled subsequence. Furthermore, we conclude from (5.114d) and the trace theorem that

$$\beta\theta_{L_m} - \mu_{L_m} \to \beta\theta_\star - \mu_\star \qquad \text{weakly in } L^2(0,T;L^2(\Gamma)) \tag{5.115}$$

as $m \to \infty$, which, in combination with (5.109), already entails $\mu_\star = \beta\theta_\star$ a.e. on Σ^T, due to the uniqueness of the limit. Proceeding similarly as in the case $K \to 0$ (see the proof of Theorem 5.4.4), we eventually show that the quadruplet $(\phi_\star, \psi_\star, \mu_\star, \theta_\star)$ is a weak solution of the system (5.1) in the sense of Definition 5.3.1 for $L = 0$. ☐

The asymptotic limit $L \to \infty$

THEOREM 5.4.10. (Asymptotic limit $L \to \infty$) *Suppose that the assumptions* **(A1)**–**(A4)** *and* **(A7)** *hold. Let $K \in [0,\infty]$, and let $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ be arbitrary initial data. In the case $K \in \{0,\infty\}$, we further assume that* **(A5)** *holds. For any $L \in (0,\infty)$, let $(\phi^L, \psi^L, \mu^L, \theta^L)$ denote a weak solution to the system (5.1) in the sense of Definition 5.3.1 with initial data (ϕ_0, ψ_0). Then there exists a quadruplet $(\phi^\star, \psi^\star, \mu^\star, \theta^\star)$ such that*

$$\begin{aligned} (\partial_t \phi^L, \partial_t \psi^L) \to (\partial_t \phi^\star, \partial_t \psi^\star) \qquad & \textit{weakly in } L^2(0,T;(\mathcal{H}^1)'), \\ (\phi^L, \psi^L) \to (\phi^\star, \psi^\star) \qquad & \textit{weakly-star in } L^\infty(0,T;\mathcal{H}^1_{K,\alpha}), \\ & \textit{strongly in } C([0,T];\mathcal{H}^s) \textit{ for all } s \in [0,1), \\ (\mu^L, \theta^L) \to (\mu^\star, \theta^\star) \qquad & \textit{weakly in } L^2(0,T;\mathcal{H}^1), \\ & \textit{weakly in } L^4(0,T;\mathcal{L}^2) \textit{ if } K \in (0,\infty], \\ \frac{1}{L}(\beta\theta^L - \mu^L) \to 0 \qquad & \textit{strongly in } L^2(0,T;L^2(\Gamma)), \end{aligned}$$

as $L \to \infty$, up to subsequence extraction, with

$$\frac{1}{L}\|\beta\theta^L - \mu^L\|_{L^2(0,T;L^2(\Gamma))} \le \frac{C}{\sqrt{L}}, \tag{5.116}$$

and the quadruplet $(\phi^\star, \psi^\star, \mu^\star, \theta^\star)$ is a weak solution to the system (5.1) in the sense of Definition 5.3.1 for $L = \infty$, which additionally satisfies $(\mu^\star, \theta^\star) \in L^4(0,T;\mathcal{L}^2)$ in the case $K \in (0,\infty]$.

Remark 5.4.11. Assuming that for any $L \in (0,\infty)$, the chemical potential μ^L is sufficiently regular such that the boundary condition (5.1f) holds in the strong sense, that is

$$Lm_\Omega(\phi^L)\partial_{\mathbf{n}}\mu^L = \beta\theta^L - \mu^L \quad \text{a.e. on } \Sigma^T,$$

estimate (5.116) can be reformulated as

$$\|\partial_{\mathbf{n}}\mu^L\|_{L^2(0,T;L^2(\Gamma))} \le \frac{C}{\sqrt{L}}.$$

As the right-hand side of this inequality tends to zero as $L \to \infty$, this explains why the homogeneous Neumann boundary condition $\partial_{\mathbf{n}}\mu^\star = 0$ a.e. on Σ^T appears in the limit model corresponding to $L = \infty$.

PROOF OF THEOREM 5.4.10. We consider an arbitrary sequence $(L_m) \subset (0,\infty)$ such that $L_m \to \infty$ as $m \to \infty$ and a corresponding weak solution $(\phi^{L_m}, \psi^{L_m}, \theta^{L_m}, \mu^{Lm})$ to (5.1) in the sense of Definition 5.3.1 to the initial data (ϕ_0, ψ_0). Since $L_m \to \infty$ as $m \to \infty$, we can assume without loss of generality that $L_m \in [1,\infty)$ for all $m \in \mathbb{N}$. In this proof, we use the letter C to denote generic positive constants independent of L_m and m, which may change their value from line to line. Let now $m \in \mathbb{N}$ be arbitrary.

Using the same argumentation as in the proof of Theorem 5.4.8, we infer

$$\|(\phi^{L_m}, \psi^{L_m})\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(\nabla\mu^{L_m}, \nabla_\Gamma\theta^{L_m})\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} \le C \tag{5.117}$$

as well as

$$\frac{1}{L_m}\|\beta\theta^{L_m} - \mu^{L_m}\|_{L^2((0,T;L^2(\Gamma))} \le \frac{C}{\sqrt{L_m}}. \tag{5.118}$$

In particular, (5.118) already entails (5.116). If $K \in (0,\infty]$, we again derive the estimate

$$\|(\mu^{L_m}, \theta^{L_m})\|_{L^4(0,T;\mathcal{L}^2)} \le C, \tag{5.119}$$

while in the case $K = 0$, we merely obtain

$$\|(\mu^{L_m}, \theta^{L_m})\|_{L^2(0,T;\mathcal{L}^2)} \le C.$$

In both cases, we thus get

$$\|(\mu^{L_m}, \theta^{L_m})\|_{L^2(0,T;\mathcal{H}^1)} \le C. \tag{5.120}$$

For the time derivatives, we proceed similarly as in the proof of Theorem 5.3.2 and obtain due to the weak formulation (5.24a) and the uniform bounds (5.117) that

$$\|(\partial_t\phi^{L_m}, \partial_t\psi^{L_m})\|_{L^2(0,T;(\mathcal{H}^1)')} \le C\left(1 + \frac{1}{\sqrt{L_m}}\right) \le C. \tag{5.121}$$

Here we additionally used $L_m \ge 1$ for all $m \in \mathbb{N}$. In view of the uniform estimates (5.117), (5.119)–(5.120) and (5.121), the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma imply the existence of functions $\phi^\star, \psi^\star\mu^\star$ and $\theta^\star$ such that

$$(\partial_t\phi^{L_m}, \partial_t\psi^{L_m}) \to (\partial_t\phi^\star, \partial_t\psi^\star) \qquad \text{weakly in } L^2(0,T;(\mathcal{H}^1)'), \tag{5.122a}$$

$$(\phi^{L_m}, \psi^{L_m}) \to (\phi^\star, \psi^\star) \qquad \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1_{K,\alpha}),$$

$$\text{strongly in } C([0,T];\mathcal{H}^s) \text{ for all } s \in [0,1), \tag{5.122b}$$

$$(\mu^{L_m}, \theta^{L_m}) \to (\mu^\star, \theta^\star) \qquad \begin{aligned} &\text{weakly in } L^2(0,T;\mathcal{H}^1), \\ &\text{weakly in } L^4(0,T;\mathcal{L}^2) \text{ if } K \in (0,\infty], \end{aligned} \tag{5.122c}$$

as $m \to \infty$, along a non-relabeled subsequence. Due to (5.117), we additionally have

$$\frac{1}{L_m}(\beta\theta^{L_m} - \mu^{L_m}) \to 0 \qquad \text{strongly in } L^2(0,T;L^2(\Gamma)) \tag{5.123}$$

as $m \to \infty$. Arguing further as in the case $K \to \infty$ (see the proof of Theorem 5.4.6), we eventually show that the quadruplet $(\phi^\star, \psi^\star, \mu^\star, \theta^\star)$ satisfies Definition 5.3.1(i)–5.3.1(iii) and 5.3.1(v). For the mass conservation law, simply note that the weak formulations for $\phi^\star$ and $\psi^\star$, as well as the test functions, are not coupled, which allows us to use $\zeta \equiv 1$ and $\zeta_\Gamma \equiv 1$ as test functions, respectively. Integrating the resulting equations in time from 0 to t, and employing the fundamental theorem of calculus, we infer

$$\int_\Omega \phi^\star(t) \,\mathrm{d}x = \int_\Omega \phi_0 \,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi^\star(t) \,\mathrm{d}\Gamma = \int_\Gamma \psi_0 \,\mathrm{d}\Gamma$$

for all $t \in [0,T]$, which shows that $(\phi^\star, \psi^\star, \mu^\star, \theta^\star)$ is a solution of the system (5.1) in the sense of Definition 5.3.1 for $L = \infty$. □

We conclude this section with the following remark.

Remark 5.4.12. If the uniqueness of the respective limiting models is known (e.g. if the mobility functions are constant, cf. Theorem 5.3.4), we even obtain convergence of the *full* sequence (not only a subsequence) in Theorem 5.4.4, Theorem 5.4.6, Theorem 5.4.8 and Theorem 5.4.10, respectively, by means of a standard subsequence argument.

5.5 Continuous dependence and uniqueness of weak solutions

In this section, we present the continuous dependence and uniqueness of weak solutions to the system (5.1) as stated in Theorem 5.3.4.

Proof of Theorem 5.3.4. As the mobility functions m_Ω and m_Γ are constant, we assume, without loss of generality, that $m_\Omega \equiv 1$ and $m_\Gamma \equiv 1$. In the following, we use the letter C to denote generic positive constants depending only on Ω, the parameters of the system (5.1), the initial data and the prescribed velocity fields.

We consider two weak solutions $(\phi_1, \psi_1, \mu_1, \theta_1)$ and $(\phi_2, \psi_2, \mu_2, \theta_2)$ originating from the initial data $(\phi_0^1, \psi_0^1), (\phi_0^2, \psi_0^2) \in \mathcal{H}^1_{K,\alpha}$ with velocity fields $(\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \in L^2(0,T;\mathbf{L}^3_{\mathrm{div}}(\Omega)) \times \mathbf{L}^{2+\omega}_{\tan}(\Gamma))$, respectively, and set

$$(\phi, \psi, \mu, \theta, \mathbf{v}, \mathbf{w}) \coloneqq (\phi_1 - \phi_2, \psi_1 - \psi_2, \mu_1 - \mu_2, \theta_1 - \theta_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{w}_1 - \mathbf{w}_2).$$

Then, due to their respective weak formulations (5.24), the quadruplet $(\phi, \psi, \mu, \theta)$ satisfies

$$\begin{aligned} &\langle (\partial_t\phi, \partial_t\psi), (\zeta, \zeta_\Gamma) \rangle_{\mathcal{H}^1_{L,\beta}} - \int_\Omega \phi_1 \mathbf{v} \cdot \nabla\zeta \,\mathrm{d}x - \int_\Omega \phi\mathbf{v}_2 \cdot \nabla\zeta \,\mathrm{d}x \\ &\quad - \int_\Gamma \psi_1 \mathbf{w} \cdot \nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma - \int_\Gamma \psi\mathbf{w}_2 \cdot \nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \end{aligned} \tag{5.124a}$$

$$
\begin{aligned}
&= -\int_\Omega \nabla\mu\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma \nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma - \chi(L)\int_\Gamma (\beta\theta-\mu)(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma,\\
&\int_\Omega \mu\,\eta \,\mathrm{d}x + \int_\Gamma \theta\,\eta_\Gamma \,\mathrm{d}\Gamma\\
&= \int_\Omega (F'(\phi_1)-F'(\phi_2))\eta\,\mathrm{d}x + \int_\Gamma (G'(\psi_1)-G'(\psi_2))\eta_\Gamma\,\mathrm{d}\Gamma \qquad (5.124\text{b})\\
&\quad + \int_\Omega \nabla\phi\cdot\nabla\eta\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi\cdot\nabla_\Gamma\eta_\Gamma\,\mathrm{d}\Gamma + \chi(K)\int_\Gamma (\alpha\psi-\phi)(\alpha\eta_\Gamma-\eta)\,\mathrm{d}\Gamma,
\end{aligned}
$$

a.e. on $[0,T]$ for all $(\zeta,\zeta_\Gamma)\in\mathcal{H}^1_{L,\beta}$ and $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$. Testing (5.124a) wit $(\zeta,\zeta_\Gamma) = (\beta,1)\in\mathcal{H}^1_{L,\beta}$, we readily notice that

$$\mathrm{mean}(\partial_t\phi,\partial_t\psi) = 0 \qquad \text{a.e. on } [0,T].$$

Moreover, recalling the mass conservation law (5.25), we have

$$\mathrm{mean}(\phi,\psi) = 0 \qquad \text{a.e. on } [0,T].$$

In particular, both $(\partial_t\phi(t),\partial_t\psi(t))$ and $(\phi(t),\psi(t))$ are an admissible right-hand side in the elliptic problem with bulk-surface coupling (4.1) for almost all $t\in[0,T]$. In addition, due to the linearity of the solution operator $\mathcal{S}_{L,\beta}$,

$$\partial_t\mathcal{S}_{L,\beta}(\phi,\psi) = \mathcal{S}_{L,\beta}(\partial_t\phi,\partial_t\psi) \qquad \text{in } \mathcal{H}^1_{L,\beta} \text{ a.e. on } [0,T].$$

Thus, choosing $(\zeta,\zeta_\Gamma) = \mathcal{S}_{L,\beta}(\phi,\psi)\in\mathcal{H}^1_{L,\beta}$ in (5.124a), we derive the identity

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|^2_{L,\beta,*} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathcal{S}_{L,\beta}(\phi,\psi)\|^2_{L,\beta}\\
&\quad = \big(\mathcal{S}_{L,\beta}(\partial_t\phi,\partial_t\psi),\mathcal{S}_{L,\beta}(\phi,\psi)\big)_{L,\beta}\\
&\quad = -\langle(\partial_t\phi,\partial_t\psi),\mathcal{S}_{L,\beta}(\phi,\psi)\rangle_{\mathcal{H}^1_{L,\beta}}\\
&\quad = \big((\mu,\theta),\mathcal{S}_{L,\beta}(\phi,\psi)\big)_{L,\beta} \qquad (5.125)\\
&\qquad - \big((\phi_1\mathbf{v}+\phi\mathbf{v}_2,\psi_1\mathbf{w}+\psi\mathbf{w}_2),(\nabla\mathcal{S}^\Omega_{L,\beta}(\phi,\psi),\nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi,\psi))\big)_{\mathcal{L}^2}\\
&\quad = -((\mu,\theta),(\phi,\psi))_{\mathcal{L}^2}\\
&\qquad - \big((\phi_1\mathbf{v}+\phi\mathbf{v}_2,\psi_1\mathbf{w}+\psi\mathbf{w}_2),(\nabla\mathcal{S}^\Omega_{L,\beta}(\phi,\psi),\nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi,\psi))\big)_{\mathcal{L}^2}
\end{aligned}
$$

a.e. on $[0,T]$. For the first term on the right-hand side, we use (5.124b) to find that

$$
\begin{aligned}
-\big((\mu,\theta),(\phi,\psi)\big)_{\mathcal{L}^2} &= -\|(\phi,\psi)\|^2_{K,\alpha} - \int_\Omega (F'(\phi_1)-F'(\phi_2))\,\phi\,\mathrm{d}x\\
&\qquad - \int_\Gamma (G'(\psi_1)-G'(\psi_2))\,\psi\,\mathrm{d}\Gamma
\end{aligned} \qquad (5.126)
$$

a.e. on $[0,T]$. Now, exploiting the growth condition on F'' (see **(A4)**) with $p<6$ and using the fundamental theorem of calculus as well as Hölder's inequality, we compute

$$
\begin{aligned}
&\Big|\int_\Omega (F'(\phi_2)-F'(\phi_1))\,\phi\,\mathrm{d}x\Big| = \Big|\int_\Omega\int_0^1 F''(s\phi_2+(1-s)\phi_1)\,\mathrm{d}s\,(\phi_2-\phi_1)\,\phi\,\mathrm{d}x\Big|\\
&\quad\le C\int_\Omega \Big(1+|\phi_1|^{p-2}+|\phi_2|^{p-2}\Big)\,|\phi|^2\,\mathrm{d}x \qquad (5.127)\\
&\quad\le C(1+\|\phi_1\|^{p-2}_{L^6(\Omega)}+\|\phi_2\|^{p-2}_{L^6(\Omega)})\|\phi\|^2_{L^{\frac{12}{8-p}}(\Omega)} \le C\|\phi\|^2_{L^{\frac{12}{8-p}}(\Omega)}
\end{aligned}
$$

a.e. on $[0,T]$. Recalling $H^1(\Gamma) \hookrightarrow L^q(\Gamma)$, we analogously infer

$$\Big| \int_\Gamma (G'(\psi_1) - G'(\psi_2))\, \psi \,\mathrm{d}\Gamma \Big| \leq C\|\psi\|^2_{L^{\frac{12}{8-p}}(\Gamma)} \tag{5.128}$$

a.e. on $[0,T]$. Combining (5.127)–(5.128) yields

$$\Big| \int_\Omega (F'(\phi_1) - F'(\phi_2))\, \phi \,\mathrm{d}x \Big| + \Big| \int_\Gamma (G'(\psi_1) - G'(\psi_2))\, \psi \,\mathrm{d}\Gamma \Big| \leq C\|(\phi,\psi)\|^2_{\mathcal{L}^{\frac{12}{8-p}}}$$

a.e. on $[0,T]$. Next, recalling again that $p < 6$, which entails the compact embedding $\mathcal{H}^1 \hookrightarrow \mathcal{L}^{\frac{12}{8-p}}$, we deduce with Ehrling's lemma and the bulk-surface Poincaré inequality if $K \in [0,\infty)$ that

$$C\|(\phi,\psi)\|^2_{\mathcal{L}^{\frac{12}{8-p}}} \leq \frac{1}{4}\|(\phi,\psi)\|^2_{K,\alpha} + C\|(\phi,\psi)\|^2_{L,\beta,*} \tag{5.129}$$

a.e. on $[0,T]$. In the case $K = \infty$, we first use Ehrling's Lemma to find that

$$\begin{aligned}
\|(\phi,\psi)\|_{\mathcal{L}^2} &\leq \frac{1}{2}\|(\phi,\psi)\|_{\mathcal{H}^1} + C\|(\phi,\psi)\|_{L,\beta,*} \\
&\leq \frac{1}{2}\|(\phi,\psi)\|_{\mathcal{L}^2} + \frac{1}{2}\|(\phi,\psi)\|_{K,\alpha} + C\|(\phi,\psi)\|_{L,\beta,*}
\end{aligned}$$

a.e. on $[0,T]$. Consequently, we obtain

$$\|(\phi,\psi)\|_{\mathcal{H}^1} \leq 2\|(\phi,\psi)\|_{K,\alpha} + C\|(\phi,\psi)\|_{L,\beta,*} \tag{5.130}$$

a.e. on $[0,T]$. Thus, we make use of Ehrling's lemma and find with (5.130) that

$$\begin{aligned}
C\|(\phi,\psi)\|^2_{\mathcal{L}^{\frac{12}{8-p}}} &\leq \frac{1}{20}\|(\phi,\psi)\|^2_{\mathcal{H}^1} + C\|(\phi,\psi)\|^2_{L,\beta,*} \\
&\leq \frac{1}{4}\|(\phi,\psi)\|^2_{K,\alpha} + C\|(\phi,\psi)\|^2_{L,\beta,*}.
\end{aligned} \tag{5.131}$$

Then, for the second term on the right-hand side of (5.125) we have

$$\begin{aligned}
&\big((\phi_1\mathbf{v} + \phi\mathbf{v}_2, \psi_1\mathbf{w} + \psi\mathbf{w}_2), (\nabla\mathcal{S}^\Omega_{L,\beta}(\phi,\psi), \nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi,\psi))\big)_{\boldsymbol{\mathcal{L}}^2} \\
&\leq \|(\phi_1\mathbf{v} + \phi\mathbf{v}_2, \psi_1\mathbf{w} + \psi\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla\mathcal{S}^\Omega_{L,\beta}(\phi,\psi), \nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi,\psi))\|_{\boldsymbol{\mathcal{L}}^2} \\
&\leq \Big(\|(\phi_1\mathbf{v}, \psi_1\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} + \|(\phi\mathbf{v}_2, \psi\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\|\mathcal{S}_{L,\beta}(\phi,\psi)\|_{L,\beta} \\
&\leq \Big(C\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega} + \|(\phi,\psi)\|_{L^6(\Omega)\times L^{\frac{2+\omega}{1+\omega}}(\Gamma)}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}\Big)\|(\phi,\psi)\|_{L,\beta,*}
\end{aligned} \tag{5.132}$$

a.e. on $[0,T]$, where, as before, $\boldsymbol{\mathcal{X}}_\omega = \mathbf{L}^3(\Omega) \times \mathbf{L}^{2+\omega}(\Gamma)$. Now, if $K \in [0,\infty)$, we use Young's inequality in combination with the Sobolev embedding $\mathcal{H}^1 \hookrightarrow L^6(\Omega)\times L^{\frac{2+\omega}{1+\omega}}(\Gamma)$ and the bulk-surface Poincaré inequality to obtain

$$\begin{aligned}
&\|(\phi,\psi)\|_{L^6(\Omega)\times L^{\frac{2+\omega}{1+\omega}}(\Gamma)}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}\|(\phi,\psi)\|_{L,\beta,*} \\
&\leq \frac{1}{4}\|(\phi,\psi)\|^2_{K,\alpha} + C\|(\mathbf{v}_2,\mathbf{w}_2)\|^2_{\boldsymbol{\mathcal{X}}_\omega}\|(\phi,\psi)\|^2_{L,\beta,*}
\end{aligned} \tag{5.133}$$

a.e. on $[0,T]$. As we have already seen above, the case $K = \infty$ needs a slightly different argument. In this case, we use (5.130) with another application of Young's inequality and the aforementioned Sobolev embedding to obtain

$$\|(\phi,\psi)\|_{L^6(\Omega)\times L^{\frac{2+\omega}{1+\omega}}(\Gamma)}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}\|(\phi,\psi)\|_{L,\beta,*}$$

$$\begin{aligned}&\leq C\Big(\|(\phi,\psi)\|_{K,\alpha}+\|(\phi,\psi)\|_{L,\beta,*}\Big)\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}\|(\phi,\psi)\|_{L,\beta,*}\\&\leq \frac{1}{4}\|(\phi,\psi)\|_{K,\alpha}^2+C\Big(1+\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}^2\Big)\|(\phi,\psi)\|_{L,\beta,*}^2\end{aligned}\tag{5.134}$$

a.e. on $[0,T]$. Another application of Young's inequality yields

$$C\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}\|(\phi,\psi)\|_{L,\beta,*}\leq\frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2+C\|(\phi,\psi)\|_{L,\beta,*}^2\tag{5.135}$$

a.e. on $[0,T]$. Plugging (5.133) and (5.135) back into (5.132), we infer

$$\begin{aligned}&\big((\phi_1\mathbf{v}+\phi\mathbf{v},\psi_1\mathbf{w}+\psi\mathbf{w}_2),(\nabla\mathcal{S}_{L,\beta}^\Omega(\phi,\psi),\nabla_\Gamma\mathcal{S}_{L,\beta}^\Gamma(\phi,\psi))\big)_{\boldsymbol{\mathcal{L}}^2}\\&\leq\frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2+\frac{1}{4}\|(\phi,\psi)\|_{K,\alpha}^2+C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}^2\|(\phi,\psi)\|_{L,\beta,*}^2\end{aligned}\tag{5.136}$$

a.e. on $[0,T]$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{L,\beta,*}^2+\frac{1}{2}\|(\phi,\psi)\|_{K,\alpha}^2\leq\frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{X}}_\omega}^2+P\|(\phi,\psi)\|_{L,\beta,*}^2$$

a.e. on $[0,T]$, where $P(\cdot)\coloneqq C\Big(1+\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{X}}_\omega}^2\Big)$. As $P\in L^1(0,T)$, Gronwall's lemma directly implies

$$\begin{aligned}\|(\phi(t),\psi(t))\|_{L,\beta,*}^2&\leq\|(\phi(0),\psi(0))\|_{L,\beta,*}^2\exp\Big(\int_0^t P(r)\,\mathrm{d}r\Big)\\&\quad+\int_0^t\|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{X}}_\omega}^2\exp\Big(\int_s^t P(r)\,\mathrm{d}r\Big)\,\mathrm{d}s\end{aligned}$$

for almost all $t\in[0,T]$, which proves the desired continuous dependence estimate (5.32). As a consequence, if $(\phi_0^1,\psi_0^1)=(\phi_0^2,\psi_0^2)$ a.e. in $\Omega\times\Gamma$ and $(\mathbf{v}_1,\mathbf{w}_1)=(\mathbf{v}_2,\mathbf{w}_2)$ a.e. in $Q^T\times\Sigma^T$, the uniqueness of the corresponding weak solution immediately follows. Thus, the proof is complete. □

5.6 Higher regularity of weak solutions

The higher regularity of weak solutions as stated in Theorem 5.3.2 is guaranteed by the following theorem.

THEOREM 5.6.1. *Suppose that the assumptions* **(A1)**–**(A7)** *hold. Let* $K,L\in[0,\infty]$, *and let* $(\phi_0,\psi_0)\in\mathcal{H}_{K,\alpha}^1$ *be arbitrary initial data. Suppose that the domain* Ω *is of class* C^k *for* $k\in\{2,3\}$, *and in the case* $d=3$, *we further assume that* **(A4)** *holds with* $p\leq 4$. *Then, if a weak solution* (ϕ,ψ,μ,θ) *of the system* (5.1) *in the sense of Definition 5.3.1 exists, it additionally satisfies*

$$(\phi,\psi)\in L^4(0,T;\mathcal{H}^2)\qquad\text{if } K\in(0,\infty],\tag{5.137}$$

$$(\phi,\psi)\in L^2(0,T;\mathcal{H}^k)\qquad\text{for } k\in\{2,3\},\tag{5.138}$$

$$(\phi,\psi)\in C([0,T];\mathcal{H}^1)\qquad\text{if } (K,L)\in[0,\infty]\times(0,\infty] \text{ and } k=3,\tag{5.139}$$

as well as

$$\mu=-\Delta\phi+F'(\phi)\qquad\text{a.e. in } Q^T,\tag{5.140}$$

$$\theta=-\Delta_\Gamma\psi+G'(\psi)+\alpha\partial_{\mathbf{n}}\phi\qquad\text{a.e. on } \Sigma^T,\tag{5.141}$$

$$\begin{cases}K\partial_{\mathbf{n}}\phi=\alpha\psi-\phi & \text{if } K\in[0,\infty),\\ \partial_{\mathbf{n}}\phi=0 & \text{if } K=\infty\end{cases}\qquad\text{a.e. on } \Sigma^T.\tag{5.142}$$

PROOF. Let $(\phi, \psi, \mu, \theta)$ be a weak solution to the system (5.1) in the sense of Definition 5.3.1. We infer from the weak formulation (5.24b) that

$$\begin{aligned}&\int_\Omega \nabla\phi(t)\cdot\nabla\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi(t)\cdot\nabla_\Gamma\eta_\Gamma \,\mathrm{d}\Gamma + \chi(K)\int_\Gamma (\alpha\psi(t)-\phi(t))(\alpha\eta_\Gamma-\eta)\,\mathrm{d}\Gamma \\ &\quad = \int_\Omega (\mu(t)-F'(\phi(t)))\eta\,\mathrm{d}x + \int_\Gamma (\theta(t)-G'(\psi(t)))\eta_\Gamma\,\mathrm{d}\Gamma \end{aligned} \tag{5.143}$$

for almost all $t\in[0,T]$ and all $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$. This means that the pair $(\phi(t),\psi(t))$ is a weak solution of the bulk-surface elliptic problem

$$\begin{aligned}
&-\Delta\phi(t) = f(t) && \text{in } \Omega,\\
&-\Delta_\Gamma\psi(t) + \alpha\partial_{\mathbf{n}}\phi(t) = f_\Gamma(t) && \text{on } \Gamma,\\
&\begin{cases} K\partial_{\mathbf{n}}\phi(t) = \alpha\psi(t)-\phi(t) & \text{if } K\in[0,\infty),\\ \partial_{\mathbf{n}}\phi(t) = 0 & \text{if } K=\infty \end{cases} && \text{on } \Gamma,
\end{aligned}$$

for almost all $t\in[0,T]$, in the sense of [117, Definition 3.1], where

$$f(t) = \mu(t) - F'(\phi(t)), \quad \text{and} \quad f_\Gamma(t) = \theta(t) - G'(\psi(t)).$$

Recalling that $p\le 4$, we use the growth conditions on F' and G' (see **(A4)**), the Sobolev embeddings $H^1(\Omega)\hookrightarrow L^6(\Omega)$ and $H^1(\Gamma)\hookrightarrow L^{2(q-1)}(\Gamma)$ and the fact that $(\phi,\psi)\in L^\infty(0,T;\mathcal{H}^1_{K,\alpha})$ to derive the estimate

$$\|F'(\phi(t))\|_{L^2(\Omega)} \le C\Big(1+\|\phi(t)\|^{p-1}_{L^{2(p-1)}(\Omega)}\Big) \le C, \tag{5.145}$$

as well as

$$\|G'(\psi(t))\|_{L^2(\Gamma)} \le C\Big(1+\|\psi(t)\|^{q-1}_{L^{2(q-1)}(\Gamma)}\Big) \le C, \tag{5.146}$$

for almost all $t\in[0,T]$. In particular, the estimates (5.145) and (5.146) imply $(f(t),f_\Gamma(t))\in\mathcal{L}^2$ with

$$\|(f(t),f_\Gamma(t))\|_{\mathcal{L}^2} \le C\Big(1+\|(\mu(t),\theta(t))\|_{\mathcal{L}^2}\Big) \tag{5.147}$$

for almost all $t\in[0,T]$.

We first consider the case $k=2$. We then deduce from regularity theory for elliptic problems with bulk-surface coupling as presented in Chapter 4 that $(\phi(t),\psi(t))\in\mathcal{H}^2$ with

$$\begin{aligned}\|(\phi(t),\psi(t))\|_{\mathcal{H}^2} &\le C\|(f(t),f_\Gamma(t))\|_{\mathcal{L}^2}\\ &\le C\Big(\|(\mu(t),\theta(t))\|_{\mathcal{L}^2} + \|(F'(\phi(t)),G'(\psi(t)))\|_{\mathcal{L}^2}\Big)\\ &\le C(1+\|(\mu(t),\theta(t))\|_{\mathcal{L}^2})\end{aligned}$$

for almost all $t\in[0,T]$. After squaring this inequality, we integrate in time from 0 to T and use the regularity of (μ,θ) to infer that $(\phi,\psi)\in L^2(0,T;\mathcal{H}^2)$. In the case $K\in(0,\infty]$, we even obtain $(\phi,\psi)\in L^4(0,T;\mathcal{H}^2)$ since $(\mu,\theta)\in L^4(0,T;\mathcal{L}^2)$.

Let us now consider the case $k=3$. Since $p\le 4$, we use Hölder's inequality and the Sobolev embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ to derive the estimate

$$\begin{aligned}\|F''(\phi(t))\nabla\phi(t)\|_{\mathbf{L}^2(\Omega)} &\le C\|\nabla\phi(t)\|_{\mathbf{L}^2(\Omega)} + C\||\phi(t)|^4\nabla\phi(t)\|_{\mathbf{L}^2(\Omega)}\\ &\le C\|\nabla\phi(t)\|_{\mathbf{L}^2(\Omega)} + C\|\phi(t)\|^2_{L^6(\Omega)}\|\nabla\phi(t)\|_{\mathbf{L}^6(\Omega)}\\ &\le C\Big(1+\|\phi(t)\|_{H^2(\Omega)}\Big)\end{aligned}$$

for almost all $t \in [0,T]$. In combination with (5.145) this yields

$$\begin{aligned} \|F'(\phi(t))\|_{H^1(\Omega)} &\le \|F'(\phi(t))\|_{L^2(\Omega)} + \|F''(\phi(t))\nabla\phi(t)\|_{\mathbf{L}^2(\Omega)} \\ &\le C\Big(1 + \|\phi(t)\|_{H^2(\Omega)}\Big) \end{aligned} \tag{5.148}$$

for almost all $t \in [0,T]$. Proceeding similarly, we derive the estimate

$$\|G''(\psi(t))\nabla_\Gamma\psi(t)\|_{\mathbf{L}^2(\Gamma)} \le C\Big(1 + \|\psi(t)\|_{H^2(\Gamma)}\Big),$$

which leads to

$$\|G'(\psi(t))\|_{H^1(\Gamma)} \le C\Big(1 + \|\psi(t)\|_{H^2(\Gamma)}\Big) \tag{5.149}$$

for almost all $t \in [0,T]$. We thus obtain $(f(t), f_\Gamma(t)) \in \mathcal{H}^1$, and may use again regularity theory for elliptic problems with bulk-surface coupling to infer $(\phi(t),\psi(t)) \in \mathcal{H}^3$ with

$$\begin{aligned} \|(\phi(t),\psi(t))\|_{\mathcal{H}^3} &\le C\Big(1 + \|(f(t), f_\Gamma(t))\|_{\mathcal{H}^1}\Big) \\ &\le C\Big(1 + \|(\mu(t),\theta(t))\|_{\mathcal{H}^1} + \|(F'(\phi(t)), G'(\psi(t)))\|_{\mathcal{H}^1}\Big) \\ &\le C\Big(1 + \|(\mu(t),\theta(t))\|_{\mathcal{H}^1} + \|(\phi(t),\psi(t))\|_{\mathcal{H}^2}\Big) \\ &\le C\Big(1 + \|(\mu(t),\theta(t))\|_{\mathcal{H}^1}\Big) \end{aligned}$$

for almost all $t \in [0,T]$. First squaring, and then integrating this inequality in time over $[0,T]$ yields $(\phi,\psi) \in L^2(0,T;\mathcal{H}^3)$.

This means that (5.137) and (5.138) are established. If $(K,L) \in [0,\infty] \times (0,\infty]$ and $k = 3$, the additional continuity property (5.139) follows from Proposition 5.2.2. Therefore, the proof is complete. □

Remark 5.6.2. In the other cases, where the boundary conditions involving K and L are of the same type (i.e., $K = L = 0$ or $K = L = \infty$), it should also be possible to obtain the regularities (5.137) and (5.138) even for $d = 3$ and $p \in (4,6)$. This is because in these cases, the system (5.1) can be discretized by a Faedo–Galerkin scheme (cf. Remark 5.4.2) and therefore, the higher order regularity estimates can be performed on the level of the approximate solutions. In the case $K = L = 0$, we refer the reader to the proof of [97, Theorem 3.3], where this line of argument is carried out in detail.

Chapter 6

Existence, uniqueness and propagation of regularity of weak solutions for singular potentials

This chapter is based on the works in:

[98] A. Giorgini, P. Knopf, and J. Stange, *Regularity of a bulk-surface Cahn–Hilliard model driven by Leray velocity fields.* Preprint: arXiv:2506.18617, 2025.

[119] P. Knopf, A. Poiatti, J. Stange, and S. Yayla, *Long-time dynamics of a bulk-surface convective Cahn–Hilliard system: Pullback attractors and convergence to equilibrium.* Preprint: arXiv:2603.10923, 2026.

[122] P. Knopf, and J. Stange, *Strong well-posedness and separation properties for a bulk-surface convective Cahn–Hilliard system with singular potentials*, J. Differential Equations, 439 (2025), pp. Paper No. 113408, 60.

6.1 Introduction

In this chapter, we continue the study of the following convective bulk-surface Cahn-Hilliard model:

$$\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \operatorname{div}(m_\Omega(\phi)\nabla\mu) \qquad \text{in } Q, \tag{6.1a}$$

$$\mu = -\Delta\phi + F'(\phi) \qquad \text{in } Q, \tag{6.1b}$$

$$\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) = \operatorname{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\partial_\mathbf{n}\mu \qquad \text{on } \Sigma, \tag{6.1c}$$

$$\theta = -\Delta_\Gamma\psi + G'(\psi) + \alpha\partial_\mathbf{n}\phi \qquad \text{on } \Sigma, \tag{6.1d}$$

$$\begin{cases} K\partial_\mathbf{n}\phi = \alpha\psi - \phi & \text{if } K \in [0,\infty), \\ \partial_\mathbf{n}\phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma, \tag{6.1e}$$

$$\begin{cases} Lm_\Omega(\phi)\partial_\mathbf{n}\mu = \beta\theta - \mu & \text{if } L \in [0,\infty), \\ m_\Omega(\phi)\partial_\mathbf{n}\mu = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Sigma, \tag{6.1f}$$

$$\phi|_{t=0} = \phi_0 \qquad \text{in } \Omega, \tag{6.1g}$$

$$\psi|_{t=0} = \psi_0 \qquad \text{on } \Gamma. \tag{6.1h}$$

In Chapter 5, we analyzed this system in the case of regular potentials and established existence, and, assuming constant mobility functions, uniqueness and higher regularity of weak solutions. The present chapter is devoted to extending these results to the case of singular potentials, in particular to logarithmic-type potentials such as the Flory–Huggins potential $W_{\log}$ introduced in (1.3).

The main difficulty stems from the singular nature of the potential. To overcome this, we employ the Moreau–Yosida approximation introduced in Section 2.3. This approximation yields a family of regularized potentials with Lipschitz continuous derivatives, allowing us to apply the results obtained in Chapter 5. More precisely, we construct approximate solutions corresponding to the regularized system and derive uniform estimates with respect to the approximation parameter. Passing to the limit in the weak formulation and the associated energy inequality then yields the existence of weak solutions for the original system with singular potentials. Furthermore, in the case of constant mobilities, we obtain uniqueness and continuous dependence estimates with respect to the initial data and the velocity fields.

In a second step, we investigate the propagation of regularity. We show that the unique weak solution regularizes instantaneously, provided the velocity fields satisfy suitable regularity assumptions. In particular, we consider two classes of admissible velocity fields: one with enhanced regularity in time, and another with stronger spatial regularity, namely the class of Leray-type velocity fields. While the former assumption is primarily of technical nature, it plays a crucial role in establishing the propagation of regularity for the physically more relevant Leray-type setting. Notably, such regularity arises when the velocity fields are not prescribed but instead determined by additional evolution equations, such as the bulk-surface Navier–Stokes–Cahn–Hilliard system (1.42).

Finally, in the case of two spatial dimensions and under additional growth assumptions on the potential F, we establish the strict separation property for the phase fields. This means that the order parameter remains uniformly separated from the pure phases ± 1. As expected, such a result is currently out of reach in three spatial dimensions, even for the classical Cahn–Hilliard equation with homogeneous Neumann boundary conditions. However, due to the fact that Γ is a $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, we are able to obtain a strict separation property for the surface phase field ψ even in the three-dimensional setting, by adapting techniques from the two-dimensional bulk case.

For related results in the non-convective setting, we refer to [43–45,70,129,131,137], where strong solutions and separation properties have been established.

6.2 Main results

We start by stating our main assumptions needed in this section.

6.2.1 Assumptions

(A1) We consider a non-empty, bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, with C^2-boundary $\Gamma := \partial\Omega$.

(A2) The constants $\alpha, \beta \in \mathbb{R}$ appearing in system (6.1) are supposed to satisfy $\alpha \in [-1,1]$ as well as $\alpha\beta|\Omega| + |\Gamma| \neq 0$.

(A3) For the mobility functions we require $m_\Omega, m_\Gamma \in C([-1,1])$. Furthermore, we assume the existence of constants $m_*, m^* > 0$ such that

$$0 < m_* \leq m_\Omega(s), m_\Gamma(s) \leq m^* \qquad \text{for all } s \in [-1,1]. \tag{6.2}$$

(A4) For the potentials, we assume that $F, G : \mathbb{R} \to \mathbb{R}$ are of the form

$$F(s) = F_1(s) + F_2(s), \qquad G(s) = G_1(s) + G_2(s),$$

where $F_1, G_1 \in C([-1,1]) \cap C^2(-1,1)$ such that $F_1(0) = F_1'(0) = G_1(0) = G_1'(0) = 0$,

$$\lim_{s\searrow -1} F_1'(s) = \lim_{s\searrow -1} G_1'(s) = -\infty \quad \text{and} \quad \lim_{s\nearrow 1} F_1'(s) = \lim_{s\nearrow 1} G_1'(s) = +\infty,$$

and there exist constants $\Theta_\Omega, \Theta_\Gamma > 0$ such that

$$F_1''(s) \geq \Theta_\Omega \quad \text{and} \quad G_1''(s) \geq \Theta_\Gamma \qquad \text{for all } s \in (-1,1). \tag{6.3}$$

We extend F_1 and G_1 on $\mathbb{R}$ by defining $F_1(s) = G_1(s) = +\infty$ for $s \notin [-1,1]$. For F_2 and G_2, we assume $F_2, G_2 \in C^1(\mathbb{R})$ such that their derivatives are globally Lipschitz continuous. Lastly, we require that the singular part of the boundary potential dominates the singular part of the bulk potential in the sense that there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$|F_1'(\alpha s)| \leq \kappa_1 |G_1'(s)| + \kappa_2 \qquad \text{for all } s \in (-1,1). \tag{6.4}$$

(A5) We assume that one of the following conditions hold:

(A5.1) There exist constants $C_\sharp > 0$ and $\gamma_\sharp \in [1,2)$ such that

$$F_1''(s) \leq C_\sharp \mathbf{e}^{C_\sharp |F_1'(s)|^{\gamma_\sharp}} \qquad \text{for all } s \in (-1,1). \tag{6.5}$$

(A5.2) As $\delta \searrow 0$, for some $\kappa > \frac{1}{2}$, it holds that

$$\frac{1}{F_1'(1-2\delta)} = O\left(\frac{1}{|\ln\delta|^\kappa}\right), \qquad \frac{1}{|F_1'(-1+2\delta)|} = O\left(\frac{1}{|\ln\delta|^\kappa}\right). \tag{6.6}$$

(A6) We assume that

$$(\mathbf{v}, \mathbf{w}) \in L^2(0,\infty; \mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma)).$$

(A7) At least one of the following conditions is satisfied:

(A7.1) $(\mathbf{v}, \mathbf{w})$ has the regularities

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) \in H^1_{\mathrm{uloc}}([0,\infty); \mathbf{L}^{\frac{6}{5}}(\Omega) \times \mathbf{L}^{1+\omega}(\Gamma)) \cap L^2(0,\infty; \boldsymbol{\mathcal{L}}^2)\\ \cap L^\infty(0,\infty; \mathbf{L}^3_{\mathrm{div}}(\Omega) \times \mathbf{L}^{2+\omega}_{\mathrm{div}}(\Gamma))\end{aligned}$$

for some $\omega > 0$.

(A7.2) $(\mathbf{v}, \mathbf{w})$ has the regularities

$$(\mathbf{v}, \mathbf{w}) \in L^2(0,\infty; \boldsymbol{\mathcal{H}}^1) \cap L^\infty(0,\infty; \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}),$$

and it holds

$$\mathbf{v}|_\Gamma = \mathbf{w} \qquad \text{a.e. on } \Sigma \quad \text{if } K = 0.$$

Remark 6.2.1. The assumption (6.4) was first made in [33] and has since then been used in various contributions in the literature (see, e.g., [45, 70, 122, 130]). We also refer to [89], where an opposite inequality was postulated. Namely, they assumed that the bulk potential dominates the surface potential.

We observe that the choice $F = G = W_{\log}$ with the logarithmic potential defined in (1.3) is admissible.

6.2.2 Existence of weak solutions

As for regular potentials, we first present the definition of a weak solution to (6.1).

DEFINITION 6.2.2. *Suppose that the assumptions* **(A1)**-**(A4)** *and* **(A6)** *hold. Let* $K, L \in [0,\infty]$, *and let* $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ *be an arbitrary initial datum satisfying*

$$F_1(\phi_0) \in L^1(\Omega), \quad G_1(\psi_0) \in L^1(\Gamma). \tag{6.7a}$$

Furthermore, in the case $L \in [0,\infty)$ *we assume that*

$$\beta \operatorname{mean}(\phi_0,\psi_0),\ \operatorname{mean}(\phi_0,\psi_0) \in (-1,1) \qquad \text{if } L \in [0,\infty) \tag{6.7b}$$

while in the case $L = \infty$ *we assume that*

$$\langle \phi_0 \rangle_\Omega,\ \langle \psi_0 \rangle_\Gamma \in (-1,1) \qquad \text{if } L = \infty. \tag{6.7c}$$

The quadruplet (ϕ,ψ,μ,θ) *is called a weak solution of the system* (6.1) *on* $[0,\infty)$ *if the following properties hold:*

(i) *The functions* ϕ, ψ, μ *and* θ *have the regularities*

$$(\phi,\psi) \in C([0,\infty);\mathcal{L}^2) \cap L^\infty(0,\infty;\mathcal{H}^1_{K,\alpha}), \tag{6.8a}$$
$$(\partial_t\phi,\partial_t\psi) \in L^2(0,\infty;(\mathcal{H}^1_{L,\beta})'), \tag{6.8b}$$
$$(\mu,\theta) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1_{L,\beta}), \tag{6.8c}$$
$$(F'(\phi),G'(\psi)) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^2), \tag{6.8d}$$

and it holds

$$|\phi| < 1 \quad \text{a.e. in } Q \quad \text{and} \quad |\psi| < 1 \quad \text{a.e. on } \Sigma. \tag{6.9}$$

(ii) *The initial conditions are satisfied in the following sense:*

$$\phi|_{t=0} = \phi_0 \quad \text{a.e. in } \Omega, \quad \text{and} \quad \psi|_{t=0} = \psi_0 \quad \text{a.e. on } \Gamma. \tag{6.10}$$

(iii) *The variational formulation*

$$\begin{aligned} &\big\langle (\partial_t\phi,\partial_t\psi),(\zeta,\zeta_\Gamma)\big\rangle_{\mathcal{H}^1_{L,\beta}} - \int_\Omega \phi\mathbf{v}\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\ &= -\int_\Omega m_\Omega(\phi)\nabla\mu\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\ &\quad - \chi(L)\int_\Gamma (\beta\theta-\mu)(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma, \end{aligned} \tag{6.11a}$$

$$\begin{aligned} &\int_\Omega \mu\,\eta \,\mathrm{d}x + \int_\Gamma \theta\,\eta_\Gamma \,\mathrm{d}\Gamma \\ &= \int_\Omega \nabla\phi\cdot\nabla\eta + F'(\phi)\,\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi\cdot\nabla_\Gamma\eta_\Gamma + G'(\psi)\,\eta_\Gamma \,\mathrm{d}\Gamma \\ &\quad + \chi(K)\int_\Gamma (\alpha\psi-\phi)(\alpha\eta_\Gamma-\eta)\,\mathrm{d}\Gamma \end{aligned} \tag{6.11b}$$

holds a.e. on $(0,\infty)$ *for all* $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$ *and* $(\eta,\eta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$.

(iv) *The functions* ϕ *and* ψ *satisfy the mass conservation laws*

$$\begin{cases} \beta\displaystyle\int_\Omega \phi(t)\,\mathrm{d}x + \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \beta\int_\Omega \phi_0\,\mathrm{d}x + \int_\Gamma \psi_0\,\mathrm{d}\Gamma & \text{if } L \in [0,\infty), \\ \displaystyle\int_\Omega \phi(t)\,\mathrm{d}x = \int_\Omega \phi_0\,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi(t)\,\mathrm{d}\Gamma = \int_\Gamma \psi_0\,\mathrm{d}\Gamma & \text{if } L = \infty \end{cases} \tag{6.12}$$

for all $t \in [0,\infty)$.

(v) *The energy inequality*

$$\begin{aligned}
&E_{\text{free}}(\phi(t),\psi(t)) + \int_0^t \int_\Omega m_\Omega(\phi)|\nabla\mu|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad + \chi(L)\int_0^t \int_\Gamma (\beta\theta-\mu)^2 \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \phi\mathbf{v}\cdot\nabla\mu \,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta \,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\le E_{\text{free}}(\phi_0,\psi_0)
\end{aligned} \tag{6.13}$$

holds for all $t\in[0,\infty)$.

Our first result establishes the existence of weak solutions.

THEOREM 6.2.3. *Suppose that the assumptions* **(A1)**-**(A4)** *and* **(A6)** *hold. Let* $K,L\in[0,\infty]$, *and let* $(\phi_0,\psi_0)\in\mathcal{H}^1_{K,\alpha}$ *be an arbitrary initial datum satisfying* (6.7). *Then, there exists at least one global-in-time weak solution* (ϕ,ψ,μ,θ) *to* (6.1) *in the sense of Definition 6.2.2. Moreover, it holds that*

$$(\phi,\psi)\in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,p}), \qquad (F'(\phi),G'(\psi))\in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^p) \tag{6.14}$$

for $p=6$ *if* $d=3$ *and* $p\in[2,\infty)$ *if* $d=2$, *as well as*

$$\begin{aligned}
&(\phi,\psi)\in L^4_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2) && \text{if } K\in(0,\infty], \\
&(\phi,\psi)\in L^3_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2) && \text{if } K=0.
\end{aligned} \tag{6.15}$$

Furthermore, the equations

$$\mu = -\Delta\phi + F'(\phi) \qquad \text{a.e. in } Q, \tag{6.16a}$$

$$\theta = -\Delta_\Gamma\psi + G'(\psi) + \alpha\partial_{\mathbf{n}}\phi \qquad \text{a.e. on } \Sigma, \tag{6.16b}$$

$$\begin{cases} K\partial_{\mathbf{n}}\phi = \alpha\psi-\phi & \text{if } K\in[0,\infty), \\ \partial_{\mathbf{n}}\phi = 0 & \text{if } K=\infty \end{cases} \qquad \text{a.e. on } \Sigma \tag{6.16c}$$

hold. Finally, if $L\in(0,\infty]$, *any weak solution satisfies the energy identity, i.e.,* (6.13) *holds with an equality.*

Remark 6.2.4. We remark on the assumptions imposed in Theorem 6.2.3.

(a) The domination property (6.4) is only needed in the case $K\in[0,\infty)$, while if $K\in(0,\infty]$, it is sufficient to assume that Ω is a Lipschitz domain. The reason for this is that if $K=0$, we have to apply elliptic regularity theory for systems with bulk-surface coupling to obtain higher regularity on the level of approximate solutions. Otherwise we cannot handle the normal derivative $\partial_{\mathbf{n}}\phi$, nor is it defined in a strong sense.

(b) The existence of a weak solution can be established under much weaker assumptions on the potentials. Namely, for the singular parts F_1,G_1, it is enough to assume that they are non-negative, proper, lower semicontinuous and convex with $F_1(0)=G_1(0)=0$. In this case, the subdifferentials

$$f_1 \coloneqq \partial F_1, \qquad g_1 \coloneqq \partial G_1$$

are maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$ in the sense of Definition 2.3.1 according to Proposition 2.3.6. Then, the weak formulation (6.11b) has to be reformulated as

$$\int_\Omega \mu\,\eta \,\mathrm{d}x + \int_\Gamma \theta\,\eta_\Gamma \,\mathrm{d}\Gamma$$

$$= \int_\Omega \nabla\phi \cdot \nabla\eta + \xi + F_2'(\phi)\,\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi \cdot \nabla_\Gamma\eta_\Gamma + \xi_\Gamma + G_2'(\psi)\,\eta_\Gamma \,\mathrm{d}\Gamma$$
$$+ \chi(K) \int_\Gamma (\alpha\psi - \phi)(\alpha\eta_\Gamma - \eta)\,\mathrm{d}\Gamma$$

a.e. on $(0,\infty)$, for all $(\eta,\eta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$, where $(\xi,\xi_\Gamma) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^2)$ is such that

$$\phi(x,t) \in D(f_1) \quad \text{for a.e. } (x,t) \in Q, \qquad \psi(z,t) \in D(g_1) \quad \text{for a.e. } (z,t) \in \Sigma,$$

and

$$\xi \in f_1(\phi) \quad \text{a.e. in } Q, \qquad \xi_\Gamma \in g_1(\psi) \quad \text{a.e. on } \Sigma.$$

Here, $D(f_1)$ and $D(g_1)$ denote the effective domains of the maximal monotone operators f_1 and g_1, respectively. Then, under superquadratic growth conditions on F_1 and G_1, and a more abstract domination property, the proof of Theorem 6.2.3 we present here works verbatim even for this more general class of potentials, see [122, Theorem 3.4]. The main drawback of these potentials is, while uniqueness is available under additional assumptions, further regularity properties of the corresponding weak solutions seems limited.

6.2.3 Continuous dependence and uniqueness of weak solutions

Our second result shows the continuous dependence of the phase-fields on the velocity fields and the initial data in the case of constant mobilities.

THEOREM 6.2.5. *Suppose that the assumptions* **(A1)**-**(A6)** *and that the mobility functions m_Ω and m_Γ are constant. Let $K, L \in [0,\infty]$, let $(\phi_0^1,\psi_0^1), (\phi_0^2,\psi_0^2) \in \mathcal{H}^1_{K,\alpha}$ be two pairs of initial data satisfying* (6.7) *as well as*

$$\begin{cases} \mathrm{mean}(\phi_0^1,\psi_0^1) = \mathrm{mean}(\phi_0^2,\psi_0^2) & \text{if } L \in [0,\infty), \\ \langle\phi_0^1\rangle_\Omega = \langle\phi_0^2\rangle_\Omega \quad \text{and} \quad \langle\psi_0^1\rangle_\Gamma = \langle\psi_0^2\rangle_\Gamma & \text{if } L = \infty, \end{cases} \tag{6.17}$$

and let

$$(\mathbf{v}_1,\mathbf{w}_1) \in L^2(0,\infty;\mathbf{L}^2_{\mathrm{div}}(\Omega)\times\mathbf{L}^2_{\mathrm{tan}}(\Gamma)) \quad \text{and} \quad (\mathbf{v}_2,\mathbf{w}_2) \in L^2(0,\infty;\mathbf{L}^3_{\mathrm{div}}(\Omega)\times\mathbf{L}^{2+\omega}_{\mathrm{tan}}(\Gamma))$$

for some $\omega > 0$ be prescribed velocity fields. Consider two weak solutions $(\phi_1,\psi_1,\mu_1,\theta_1)$ and $(\phi_2,\psi_2,\mu_2,\theta_2)$ to system (6.1) *in the sense of Definition 6.2.2 originating from (ϕ_0^1,ψ_0^1) and (ϕ_0^2,ψ_0^2) with velocity fields $(\mathbf{v}_1,\mathbf{w}_1)$ and $(\mathbf{v}_2,\mathbf{w}_2)$, respectively. Then, for any $T > 0$, there exists a constant $C > 0$ such that the continuous dependence estimate*

$$\begin{aligned} &\|(\phi_1(t)-\phi_2(t),\psi_1(t)-\psi_2(t))\|^2_{L,\beta,*} + \int_0^t \|(\phi_1(r)-\phi_2(r),\psi_1(r)-\psi_2(r))\|^2_{K,\alpha}\,\mathrm{d}r \\ &\le C\|(\phi_0^1-\phi_0^2,\psi_0^1-\psi_0^2)\|^2_{L,\beta,*}\exp\left(\int_0^t P(r)\,\mathrm{d}r\right) \\ &\quad + C\int_0^t \|(\mathbf{v}_1(s)-\mathbf{v}_2(s),\mathbf{w}_1(s)-\mathbf{w}_2(s))\|^2_{\mathcal{L}^2}\exp\left(\int_s^t P(r)\,\mathrm{d}r\right)\mathrm{d}s \end{aligned} \tag{6.18}$$

holds for almost all $t \in [0,T]$, where $P := C(1 + \|(\mathbf{v}_2,\mathbf{w}_2)\|^2_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)})$, and the constant $C > 0$ depends only on Ω, the parameters of the system, the initial data, the final time $T > 0$, and ω.

In particular, if $(\phi_0^1,\psi_0^1) = (\phi_0^2,\psi_0^2)$ a.e. in $\Omega\times\Gamma$ and $(\mathbf{v}_1,\mathbf{w}_1) = (\mathbf{v}_2,\mathbf{w}_2)$ a.e. in $Q\times\Sigma$, (6.18) *yields uniqueness of the phase-fields of the corresponding weak solution.*

Remark 6.2.6. In three spatial dimensions, uniqueness of weak solutions to (6.1) is currently limited to the case of constant mobility functions. However, in the two-dimensional setting, uniqueness of weak solutions can be achieved, and will be subject of Chapter 7.

6.2.4 Propagation of regularity and separation properties for weak solutions

Our next main result is concerned with establishing higher regularity properties of weak solutions. Assuming that the mobility functions are constant and that the potentials as well as the velocity fields are more regular, we prove that the unique weak solution to (6.1) regularizes instantaneously in time.

THEOREM 6.2.7. *In addition to the assumptions* **(A1)**–**(A7)**, *we assume that the domain* Ω *is at least of class* C^3, *and that the mobility functions* m_Ω *and* m_Γ *are constant. Let* $K, L \in [0,\infty]$ *and let* $(\phi_0,\psi_0) \in \mathcal{H}^1_{K,\alpha}$ *be an arbitrary initial datum satisfying* (6.7). *Let* (ϕ,ψ,μ,θ) *be the corresponding unique solution in the sense of Definition* 6.2.2 *on* $(0,\infty)$. *Then, for any* $\varsigma > 0$, (ϕ,ψ,μ,θ) *has the following regularity:*

$$(\partial_t\phi, \partial_t\psi) \in L^\infty(\varsigma,\infty;(\mathcal{H}^1_{L,\beta})') \cap L^2(\varsigma,\infty;\mathcal{H}^1), \tag{6.19a}$$

$$(\phi,\psi) \in L^\infty(\varsigma,\infty;\mathcal{W}^{2,6}) \cap \Big(BC(\overline{Q_\varsigma}) \times BC(\overline{\Sigma_\varsigma})\Big), \tag{6.19b}$$

$$(\mu,\theta) \in L^\infty(\varsigma,\infty;\mathcal{H}^1) \cap L^2(\varsigma,\infty;\mathcal{H}^2), \tag{6.19c}$$

$$(F'(\phi), G'(\psi)) \in L^2(\varsigma,\infty;\mathcal{L}^\infty) \cap L^\infty(\varsigma,\infty;\mathcal{L}^6). \tag{6.19d}$$

In (6.19) we used the notation

$$Q_\varsigma := \Omega \times (\varsigma,\infty), \quad \Sigma_\varsigma := \Gamma \times (\varsigma,\infty)$$

for any $\varsigma > 0$.

Remark 6.2.8. As for Remark 6.2.6, the propagation of regularity of weak solutions can be established for $d = 2$ even for non-degenerate mobility functions, see Theorem 7.3.3. Moreover, in that case, the regularity assumptions on the velocity fields in **(A7.1)** can be relaxed to

$$(\mathbf{v},\mathbf{w}) \in H^1_{\mathrm{uloc}}([0,\infty);\mathbf{L}^{1+\omega}(\Omega) \times \mathbf{L}^1(\Gamma)) \cap L^2(0,\infty;\boldsymbol{\mathcal{L}}^2) \cap L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2_{\mathrm{div}})$$

for some $\omega > 0$. This is due to the better Sobolev embeddings in two dimensions.

Remark 6.2.9. Theorem 6.2.7 only establishes the propagation of regularity for the unique weak solution to (6.1). Under additional assumptions on the initial data (see Theorem 7.3.3), one can even prove that the unique weak solution is in fact a strong solution, i.e., the regularity properties (6.19) hold for $\varsigma = 0$, and the system (6.1) is satisfied almost everywhere. For $d = 3$, and if we suppose **(A7.2)**, this, however, only works in the case $L \in (0,\infty]$, as the proof relies on an approximation procedure of the initial data, and a certain trace relation cannot be guaranteed, which is necessary in the case $L = 0$. We refer to [98, Theorem 3.6 and Remark 3.7] for a more detailed reasoning. Nonetheless, the proof works for $d = 2$ without this additional approximation procedure, as in this case, the separation property is known, see Theorem 6.2.10 and Theorem 7.3.3.

The proof of Theorem 6.2.10 can be found in Section 6.5.

Finally, we can show the instantaneous separation property for weak solutions in two dimensions.

THEOREM 6.2.10. *Let $d = 2$. Suppose that the assumptions from Theorem 6.2.7 hold, and consider the corresponding unique weak solution $(\phi, \psi, \mu, \theta)$. Then, for any $\varsigma > 0$, there exists $\delta = \delta(\varsigma) > 0$, additionally depending on the norms of the initial data, such that*

$$\|\phi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \|\psi(t)\|_{L^\infty(\Gamma)} \leq 1 - \delta \qquad \textit{for all } t \geq \varsigma. \tag{6.20}$$

6.3 Existence of weak solutions

In this section, we present the proof of Theorem 6.2.3. The proof is based on constructing suitable approximation solutions. This is done by approximating the singular potentials by their respective Moreau–Yosida regularization, and considering more regular velocity fields. Then, the approximative system is solved using Theorem 5.3.2. The main difficulty lies then in proving suitable uniform bounds on the approximative solution, which allows us to conclude the existence of a weak solution to the original system using compactness arguments.

PROOF OF THEOREM 6.2.3.
Step 1: Definition of an approximate solution. We first construct a sequence of approximate solutions on any time interval $(0, T)$ for $T > 0$. To this end, let $T > 0$ be arbitrary and let us first assume that $(\mathbf{v}, \mathbf{w}) \in L^2(0, T; \mathbf{L}^3_{\text{div}}(\Omega) \times \mathbf{L}^3_{\tan}(\Gamma))$. We would like to point out that it is sufficient to assume $(\mathbf{v}, \mathbf{w}) \in L^2(0, T; \mathbf{L}^3_{\text{div}}(\Omega) \times \mathbf{L}^{2+\omega}_{\tan}(\Gamma))$ for some $\omega > 0$ as in Theorem 5.3.2, see [122, Theorem 3.4]. Next, for any $\lambda \in (0, \frac{1}{2})$, we consider the Moreau–Yosida approximations $F_{1,\lambda}$ and $G_{1,\lambda}$ of F and G, respectively, and set

$$F_\lambda \coloneqq F_{1,\lambda} + F_2, \qquad G_\lambda \coloneqq G_{1,\lambda} + G_2.$$

Then we consider the regularized problem

$$\partial_t \phi_\lambda + \operatorname{div}(\phi_\lambda \mathbf{v}) = \operatorname{div}(m_\Omega(\phi_\lambda)\nabla \mu_\lambda) \qquad \text{in } Q^T, \tag{6.21a}$$

$$\mu = -\Delta \phi_\lambda + F_\lambda'(\phi_\lambda) \qquad \text{in } Q^T, \tag{6.21b}$$

$$\partial_t \psi_\lambda + \operatorname{div}_\Gamma(\psi_\lambda \mathbf{w}) = \operatorname{div}_\Gamma(m_\Gamma(\psi_\lambda)\nabla_\Gamma \theta_\lambda) - \beta m_\Omega(\phi_\lambda)\partial_{\mathbf{n}}\mu_\lambda \qquad \text{on } \Sigma^T, \tag{6.21c}$$

$$\theta_\lambda = -\Delta_\Gamma \psi_\lambda + G_\lambda'(\psi_\lambda) + \alpha \partial_{\mathbf{n}}\phi_\lambda \qquad \text{on } \Sigma^T, \tag{6.21d}$$

$$\begin{cases} K\partial_{\mathbf{n}}\phi_\lambda = \alpha\psi_\lambda - \phi_\lambda & \text{if } K \in [0, \infty), \\ \partial_{\mathbf{n}}\phi_\lambda = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma^T, \tag{6.21e}$$

$$\begin{cases} Lm_\Omega(\phi_\lambda)\partial_{\mathbf{n}}\mu_\lambda = \beta\theta_\lambda - \mu_\lambda & \text{if } L \in [0, \infty), \\ m_\Omega(\phi_\lambda)\partial_{\mathbf{n}}\mu_\lambda = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Sigma^T, \tag{6.21f}$$

$$\phi_\lambda|_{t=0} = \phi_0 \qquad \text{in } \Omega, \tag{6.21g}$$

$$\psi_\lambda|_{t=0} = \psi_0 \qquad \text{on } \Gamma. \tag{6.21h}$$

In view of Theorem 5.3.2, there exists a weak solution $(\phi_\lambda, \psi_\lambda, \mu_\lambda, \theta_\lambda)$ in the sense of Definition 5.3.1, where in the weak formulation the potentials F and G are replaced by F_λ and G_λ, respectively. Moreover, the energy inequality reads now as

$$\begin{aligned} E_{\text{free},\lambda}(\phi_\lambda, \psi_\lambda) &+ \int_0^t \int_\Omega m_\Omega(\phi_\lambda)|\nabla \mu_\lambda|^2 \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_\Gamma m_\Gamma(\psi_\lambda)|\nabla_\Gamma \theta_\lambda|^2 \,\mathrm{d}\Gamma \,\mathrm{d}s \\ &+ \chi(L) \int_0^t \int_\Gamma (\beta\theta_\lambda - \mu_\lambda)^2 \,\mathrm{d}\Gamma \,\mathrm{d}s \end{aligned}$$

$$= E_{\mathrm{free},\lambda}(\phi_0, \psi_0) + \int_0^t \int_\Omega \phi_\lambda \mathbf{v} \cdot \nabla \mu_\lambda \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Gamma \psi_\lambda \mathbf{w} \cdot \nabla_\Gamma \theta_\lambda \, \mathrm{d}\Gamma \, \mathrm{d}s$$

on $[0, T]$, where the regularized energy is given by

$$\begin{aligned} E_{\mathrm{free},\lambda}(\eta, \eta_\Gamma) = &\int_\Omega \frac{1}{2}|\nabla \eta|^2 + F_\lambda(\eta) \, \mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma \eta_\Gamma|^2 + G_\lambda(\eta_\Gamma) \, \mathrm{d}\Gamma \\ &+ \chi(K) \int_\Gamma \frac{1}{2}(\alpha\eta_\Gamma - \eta)^2 \, \mathrm{d}\Gamma \end{aligned} \tag{6.22}$$

for $(\eta, \eta_\Gamma) \in \mathcal{H}^1_{K,\alpha}$. Additionally, as Ω is of class C^2, Theorem 5.3.2 provides that $(\phi_\lambda, \psi_\lambda) \in L^2(0, T; \mathcal{H}^2)$, and the equations

$$\mu_\lambda = -\Delta\phi_\lambda + F'_\lambda(\phi_\lambda) \qquad \text{a.e. in } Q^T, \tag{6.23a}$$

$$\theta_\lambda = -\Delta_\Gamma \psi_\lambda + G'_\lambda(\psi_\lambda) + \alpha\partial_\mathbf{n}\phi_\lambda \qquad \text{a.e. on } \Sigma^T, \tag{6.23b}$$

$$\begin{cases} K\partial_\mathbf{n}\phi_\lambda = \alpha\psi_\lambda - \phi_\lambda & \text{if } K \in [0, \infty), \\ \partial_\mathbf{n}\phi_\lambda = 0 & \text{if } K = \infty \end{cases} \qquad \text{a.e. on } \Sigma^T \tag{6.23c}$$

are satisfied.

Step 2: Uniform estimates of the approximate solution. We now establish uniform bounds on the approximate solution that are based on the energy inequality. In the following, we denote with C positive constant that may depend on the parameters of the system but are independent of λ. First, we notice that by definition of the approximate energy $E_{\mathrm{free},\lambda}$ it holds that

$$\begin{aligned} &\frac{1}{2}\|\nabla\phi_\lambda(t)\|^2_{L^2(\Omega)} + \int_\Omega F_\lambda(\phi_\lambda(t)) \, \mathrm{d}x + \frac{1}{2}\|\nabla_\Gamma\psi_\lambda(t)\|^2_{L^2(\Gamma)} + \int_\Gamma G_\lambda(\psi_\lambda(t)) \, \mathrm{d}\Gamma \\ &\quad + \chi(K) \int_\Gamma \frac{1}{2}(\alpha\psi_\lambda(t) - \phi_\lambda(t))^2 \, \mathrm{d}\Gamma + \int_0^t \int_\Omega m_\Omega(\phi_\lambda)|\nabla\mu_\lambda|^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\quad + \int_0^t \int_\Gamma m_\Gamma(\psi_\lambda)|\nabla_\Gamma\theta_\lambda|^2 \, \mathrm{d}\Gamma \, \mathrm{d}s + \chi(L) \int_0^t \int_\Gamma (\beta\theta_\lambda - \mu_\lambda)^2 \, \mathrm{d}\Gamma \, \mathrm{d}s \\ &\leq E_{\mathrm{free},\lambda}(\phi_0, \psi_0) + \int_0^t \int_\Omega \phi_\lambda \mathbf{v} \cdot \nabla\mu_\lambda \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Gamma \psi_\lambda \mathbf{w} \cdot \nabla_\Gamma\theta_\lambda \, \mathrm{d}\Gamma \, \mathrm{d}s \end{aligned} \tag{6.24}$$

for all $t \in [0, T]$. Now, we observe on account of Theorem 2.3.8 and the assumption on the initial data (6.7a) that

$$\int_\Omega F_\lambda(\phi_0) \, \mathrm{d}x + \int_\Gamma G_\lambda(\psi_0) \, \mathrm{d}\Gamma \leq \int_\Omega F(\phi_0) \, \mathrm{d}x + \int_\Gamma G(\psi_0) \, \mathrm{d}\Gamma \leq C,$$

which entails that $E_{\mathrm{free},\lambda}(\phi_0, \psi_0) \leq C$. On the other hand, for the convective terms on the right-hand side of (6.24), we use again Hölder's and Young's inequality to obtain the estimate

$$\begin{aligned} &\Big|\int_0^t \int_\Omega \phi_\lambda \mathbf{w} \cdot \nabla\mu_\lambda \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Gamma \psi_\lambda \mathbf{w} \cdot \nabla_\Gamma\theta_\lambda \, \mathrm{d}\Gamma \, \mathrm{d}s\Big| \\ &\quad\leq \frac{m_*}{2} \int_0^t \int_\Omega |\nabla\mu_\lambda|^2 \, \mathrm{d}x \, \mathrm{d}s + \frac{m_*}{2} \int_0^t \int_\Gamma |\nabla_\Gamma\theta_\lambda|^2 \, \mathrm{d}\Gamma \, \mathrm{d}s \\ &\quad\quad + C \int_0^t \|(\mathbf{v}, \mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^3} \|(\phi_\lambda, \psi_\lambda)\|^2_{\mathcal{H}^1} \, \mathrm{d}s \end{aligned} \tag{6.25}$$

for all $t \in [0, T]$. Next, in view of **(Y4)**, we note that

$$\int_\Omega F_\lambda(\phi_\lambda) \, \mathrm{d}x + \int_\Gamma G_\lambda(\psi_\lambda) \, \mathrm{d}\Gamma \geq \frac{1}{2}\int_\Omega \phi_\lambda^2 \, \mathrm{d}x + \frac{1}{2}\int_\Gamma \psi_\lambda^2 \, \mathrm{d}\Gamma - C \tag{6.26}$$

on $[0,T]$ for all $\lambda \in (0,\frac{1}{2})$. Then, by (6.26) and the definition of $E_{\text{free},\lambda}$, it readily follows that

$$\begin{aligned} \frac{1}{2}\|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{H}^1} &\le C + \frac{1}{2}\|(\phi_\lambda,\psi_\lambda)\|_{K,\alpha}^2 + \int_\Omega F_\lambda(\phi_\lambda)\,\mathrm{d}x + \int_\Gamma G_\lambda(\psi_\lambda)\,\mathrm{d}\Gamma \\ &= C + E_{\text{free},\lambda}(\phi_\lambda,\psi_\lambda) \end{aligned} \tag{6.27}$$

a.e.on $[0,T]$. Hence, combining (6.24), (6.25) and (6.27), we obtain after an application of Gronwall's lemma that

$$\begin{aligned} &\|(\phi_\lambda,\psi_\lambda)\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(F_{1,\lambda}(\phi_\lambda),G_{1,\lambda}(\psi_\lambda))\|_{L^\infty(0,T;\mathcal{L}^1)} \\ &\quad + \|(\nabla\mu_\lambda,\nabla_\Gamma\theta_\lambda)\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} + \chi(L)^{\frac{1}{2}}\|\beta\theta_\lambda - \mu_\lambda)\|_{L^2(0,T;L^2(\Gamma))} \le C. \end{aligned} \tag{6.28}$$

Here, we additionally used that the Moreau–Yosida approximations $F_{1,\lambda}$ and $G_{1,\lambda}$ are non-negative, see **(Y5)**. Next, to derive uniform estimate of the time derivative of the phase-fields, we test (6.11a) with an arbitrary test function $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$. Then, recalling that the mobility functions m_Ω and m_Γ are bounded according to (6.2), we use Hölder's inequality in combination with the continuous embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Gamma) \hookrightarrow L^r(\Gamma)$ for all $r \in [1,\infty)$ to infer

$$\begin{aligned} |\langle(\partial_t\phi_\lambda,\partial_t\psi_\lambda),(\zeta,\zeta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}}| &\le \|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{L}^6}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^3}\|(\nabla\zeta,\nabla_\Gamma\zeta_\Gamma)\|_{\boldsymbol{\mathcal{L}}^2} \\ &\quad + \|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}\|(\zeta,\zeta_\Gamma)\|_{L,\beta}. \end{aligned}$$

a.e. on $[0,T]$. Thus, after taking the supremum over all $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$ with $\|(\zeta,\zeta_\Gamma)\|_{\mathcal{H}^1} \le 1$, we take the square of the resulting inequality and integrate in time from 0 to T to obtain

$$\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|^2_{(\mathcal{H}^1_{L,\beta})'} \le C\Big(\|(\phi_\lambda,\psi_\lambda)\|^2_{\mathcal{H}^1}\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^3} + \|(\mu_\lambda,\theta_\lambda)\|^2_{L,\beta}\Big)$$

a.e. on $[0,T]$. In view of the uniform estimate (6.28) and the regularity of the velocity fields $\mathbf{v}$ and $\mathbf{w}$, we deduce that

$$\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{L^2(0,T;(\mathcal{H}^1_{L,\beta})')} \le C. \tag{6.29}$$

Step 4: Uniform L^1-estimate on $F'_{1,\lambda}(\phi_\lambda)$ and $G'_{1,\lambda}(\psi_\lambda)$. We next establish L^1-bounds on $F'_{1,\lambda}(\phi_\lambda)$ and $G'_{1,\lambda}(\psi_\lambda)$. In the following, we will only consider the case $L \in [0,\infty)$. The proof in the case $L = \infty$ works similarly, but one uses the means $\langle\phi_0\rangle_\Omega$ and $\langle\psi_0\rangle_\Gamma$ as well as $\langle\mu_\lambda\rangle_\Omega$ and $\langle\theta_\lambda\rangle_\Gamma$ instead of the bulk-surface means $\mathrm{mean}(\phi_0,\psi_0)$ and $\mathrm{mean}(\mu_\lambda,\theta_\lambda)$. This distinction is related to the mass conservation law (6.12) and the conditions on the initial data (6.7b) and (6.7c). We refer to [52] for more details if $K \in [0,\infty)$ and $L = \infty$. The case $K = L = \infty$ is similar, but easier. We now test (6.11b) with

$$(\eta,\eta_\Gamma) := \begin{cases} (\phi_\lambda - \alpha\beta\mathrm{mean}(\phi_0,\psi_0), \psi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0)) & \text{if } K = 0, \\ (\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0), \psi_\lambda - \mathrm{mean}(\phi_0,\psi_0)) & \text{if } K \in (0,\infty], \end{cases}$$

which is an element of $\mathcal{H}^1_{K,\alpha}$. After some rearrangements, as well as adding and subtracting $\beta\mathrm{mean}(\phi_0,\psi_0)$ and $\mathrm{mean}(\phi_0,\psi_0)$ multiple times in the case $K = 0$, we obtain

$$\begin{aligned} &\|\nabla\phi_\lambda\|^2_{L^2(\Omega)} + \int_\Omega F'_{1,\lambda}(\phi_\lambda)(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x \\ &\quad + \|\nabla_\Gamma\psi_\lambda\|^2_{L^2(\Gamma)} + \int_\Gamma G'_{1,\lambda}(\psi_\lambda)(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega (\mu_\lambda - \beta\mathrm{mean}(\mu_\lambda,\theta_\lambda))(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x \\
&\quad + \int_\Gamma (\theta_\lambda - \mathrm{mean}(\mu_\lambda,\theta_\lambda))(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma \\
&\quad - \chi(K)\int_\Gamma (\alpha\psi_\lambda - \phi_\lambda)(\alpha(\psi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0)) - (\phi_\lambda - \mathrm{mean}(\phi_0,\psi_0)))\,\mathrm{d}\Gamma \\
&\quad - \int_\Omega F_2'(\phi_\lambda)(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x - \int_\Gamma G_2'(\psi_\lambda)(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma \\
&\quad + \mathbf{1}_{\{0\}}(K)\bigg(\int_\Omega \mu_\lambda(\beta\mathrm{mean}(\phi_\lambda,\psi_\lambda) - \alpha\beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x \\
&\qquad\qquad + \int_\Gamma \theta_\lambda(\mathrm{mean}(\phi_0,\psi_0) - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma \\
&\qquad\qquad - \int_\Omega F_\lambda'(\phi_\lambda)(\beta\mathrm{mean}(\phi_\lambda,\psi_\lambda) - \alpha\beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x \\
&\qquad\qquad - \int_\Gamma G_\lambda'(\psi_\lambda)(\mathrm{mean}(\phi_0,\psi_0) - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma\bigg)
\end{aligned} \tag{6.30}
$$

a.e. on $[0,T]$. Here, $\mathbf{1}_{\{0\}}$ denotes again the indicator function of the set $\{0\}$. Note hat the subtracted mean values $\beta\mathrm{mean}(\mu_\lambda,\theta_\lambda)$ and $\mathrm{mean}(\mu_\lambda,\theta_\lambda)$ in the first two summands on the right-hand side of (6.30) do not change the values of the respective integrals, since

$$
\beta|\Omega|\langle\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0)\rangle_\Omega + |\Gamma|\langle\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0)\rangle_\Gamma = 0
$$

a.e. on $[0,T]$ due to the mass conservation law (6.12).

For the left-hand side of (6.30), we can use the Miranville–Zelik inequality (see [138, Appendix A.1] or [89, p. 908]) due to the assumption that $\beta\mathrm{mean}(\phi_0,\psi_0), \mathrm{mean}(\phi_0,\psi_0) \in (-1,1)$, see (6.7b). Thus, there exist positive constants c_1, c_2 and a non-negative constant c_3 such that

$$
\begin{aligned}
&c_1\|F_{1,\lambda}'(\phi_\lambda)\|_{L^1(\Omega)} + c_2\|G_{1,\lambda}'(\psi_\lambda)\|_{L^1(\Gamma)} - c_3 \\
&\quad\le \int_\Omega F_{1,\lambda}'(\phi_\lambda)(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x + \int_\Gamma G_{1,\lambda}'(\psi_\lambda)(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma
\end{aligned} \tag{6.31}
$$

a.e. on $[0,T]$. On the other hand, we use the bulk-surface Poincaré inequality for the first two terms on the right-hand side of (6.30) to find that

$$
\begin{aligned}
&\int_\Omega (\mu_\lambda - \beta\mathrm{mean}(\mu_\lambda,\theta_\lambda))(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x \\
&\quad + \int_\Gamma (\theta_\lambda - \mathrm{mean}(\mu_\lambda,\theta_\lambda))(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma \\
&\quad \le \|(\mu_\lambda - \beta\mathrm{mean}(\mu_\lambda,\theta_\lambda), \theta_\lambda - \mathrm{mean}(\mu_\lambda,\theta_\lambda))\|_{\mathcal{L}^2} \\
&\qquad \times \|(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0), \psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\|_{\mathcal{L}^2} \\
&\quad \le C(1 + \|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{L}^2})\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}
\end{aligned} \tag{6.32}
$$

a.e. on $[0,T]$. Furthermore, in view of the Lipschitz continuity of F_2' and G_2', we infer that

$$
\begin{aligned}
&- \int_\Omega F_2'(\phi_\lambda)(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x - \int_\Gamma G_2'(\psi_\lambda)(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}\Gamma \\
&\quad = - \int_\Omega (F_2'(\phi_\lambda) - F_2'(\beta\mathrm{mean}(\phi_0,\psi_0)))(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \int_\Gamma (G_2'(\psi_\lambda) - G_2'(\mathrm{mean}(\phi_0,\psi_0)))(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}\Gamma \\
&- \int_\Omega F_2'(\beta\mathrm{mean}(\phi_0,\psi_0))(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}x \\
&- \int_\Gamma G_2'(\mathrm{mean}(\phi_0,\psi_0))(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}\Gamma \\
\le\ & C + C\int_\Omega (\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0))^2 \,\mathrm{d}x + C\int_\Gamma (\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0))^2 \,\mathrm{d}\Gamma \\
\le\ & C(1 + \|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{L}^2}^2)
\end{aligned}
$$

a.e. on $[0,T]$, and thus

$$
\begin{aligned}
&- \chi(K)\int_\Gamma (\alpha\psi_\lambda - \phi_\lambda)(\alpha(\psi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0)) - (\phi_\lambda - \mathrm{mean}(\phi_0,\psi_0))) \,\mathrm{d}\Gamma \\
&\quad - \int_\Omega F_2'(\phi_\lambda)(\phi_\lambda - \beta\mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}x - \int_\Gamma G_2'(\psi_\lambda)(\psi_\lambda - \mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}\Gamma \\
&\quad \le C(1 + \|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{H}^1}^2)
\end{aligned} \tag{6.33}
$$

a.e. on $[0,T]$. For the remaining terms on the right-hand side of (6.30), which are only present if $K = 0$, we make use of the fact that Ω is of class C^2. Thus, we know that $(\phi_\lambda,\psi_\lambda) \in L^2(0,T;\mathcal{H}^2)$ and that the equations (6.23) are satisfied. Hence, after integration by parts, we deduce that

$$
\begin{aligned}
&\int_\Omega (F_\lambda'(\phi_\lambda) - \mu_\lambda)(\alpha\beta\mathrm{mean}(\phi_0,\psi_0) - \beta\mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}x \\
&\qquad + \int_\Gamma (G_\lambda'(\psi_\lambda) - \theta_\lambda)(\beta\mathrm{mean}(\phi_0,\psi_0) - \mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}\Gamma \\
&\quad = \int_\Omega \Delta\phi_\lambda(\alpha\beta\mathrm{mean}(\phi_0,\psi_0) - \beta\mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}x \\
&\qquad + \int_\Gamma (\Delta_\Gamma\psi_\lambda - \alpha\partial_{\mathbf{n}}\phi_\lambda)(\beta\mathrm{mean}(\phi_0,\psi_0) - \mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}\Gamma \\
&\quad = \int_\Gamma \partial_{\mathbf{n}}\phi_\lambda(\alpha\mathrm{mean}(\phi_0,\psi_0) - \beta\mathrm{mean}(\phi_0,\psi_0)) \,\mathrm{d}\Gamma \\
&\quad \le C\|\partial_{\mathbf{n}}\phi_\lambda\|_{L^2(\Gamma)}
\end{aligned} \tag{6.34}
$$

a.e. on $[0,T]$. Collecting (6.30)-(6.34) and recalling (6.26), we find that

$$
\|(F_{1,\lambda}'(\phi_\lambda), G_{1,\lambda}'(\psi_\lambda))\|_{\mathcal{L}^1} \le C(1 + \|(\mu_\lambda,\theta_\lambda)\|_{L,\beta} + \mathbf{1}_{\{0\}}(K)\|\partial_{\mathbf{n}}\phi_\lambda(t)\|_{L^2(\Gamma)}) \tag{6.35}
$$

a.e. on $[0,T]$.

To eventually pass to the limit $\lambda \to 0$ in the weak formulation (6.11), estimate (6.35) is not sufficient. To this end, we prove L^2-bounds for the terms $F_{1,\lambda}'(\phi_\lambda)$ and $G_{1,\lambda}'(\psi_\lambda)$. For this, we take advantage of the domination property (6.4) in the case $K \in [0,\infty)$ as well as the boundary regularity for $K = 0$. This leads to the fact that the related analysis has to be performed differently for the cases $K = 0$ and $K \in (0,\infty]$.

Step 5: Uniform L^2-estimate of the potentials for $K \in (0,\infty]$. Since $K \in (0,\infty]$, we have $\mathbf{1}_{\{0\}}(K) = 0$, and thus, due to (6.35), it holds

$$
\|(F_{1,\lambda}'(\phi_\lambda), G_{1,\lambda}'(\psi_\lambda))\|_{\mathcal{L}^1} \le C(1 + \|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}) \tag{6.36}
$$

a.e. on $[0,T]$. Furthermore, since both F_2' and G_2' are globally Lipschitz continuous, exploiting the estimate (6.26) yields

$$\|(F_\lambda'(\phi_\lambda), G_\lambda'(\psi_\lambda))\|_{\mathcal{L}^1} \leq C(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}) \tag{6.37}$$

a.e. on $[0,T]$. Now, since $K \in (0,\infty]$, we have $\mathcal{H}^1_{K,\alpha} = \mathcal{H}^1$ and may test (6.11b) with $\left(\beta^2|\Omega| + |\Gamma|\right)^{-1}(\beta, 1)$. Hence, (6.26) and (6.37) imply that

$$|\text{mean}(\mu_\lambda, \theta_\lambda)| \leq C(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}) \tag{6.38}$$

a.e. on $[0,T]$. Whence, using (6.26) and the bulk-surface Poincaré inequality once again, we infer

$$\|(\mu_\lambda, \theta_\lambda)\|_{\mathcal{H}^1} \leq C(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}) \tag{6.39}$$

a.e. on $[0,T]$. Consequently, from the proof of Proposition 4.3.5 (note also Remark 4.3.9), we obtain

$$\|(F_{1,\lambda}'(\phi_\lambda), G_{1,\lambda}'(\psi_\lambda))\|_{\mathcal{L}^2} \leq C(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}) \tag{6.40}$$

a.e. on $[0,T]$. For the case $K = \infty$, the corresponding computations can be repeated analogously, but are even easier, as the corresponding bulk-surface elliptic system is completely uncoupled since $\partial_\mathbf{n}\phi_\lambda = 0$ a.e. on Σ^T. For the corresponding estimate on the bulk we also refer to [54, Lemma A.1] or [4, Lemma 2]. The computations on the surface work in a similar manner.

Step 6: Uniform L^2-estimate of the potentials in the case $K = 0$. In this case, we additionally have to deal with the normal derivative $\partial_\mathbf{n}\phi_\lambda$ appearing on the right-hand side of (6.35). First, we multiply (6.23a) and (6.23b) by $\left(\beta^2|\Omega| + |\Gamma|\right)^{-1}\beta$ and $\left(\beta^2|\Omega|+|\Gamma|\right)^{-1}$ and integrate over Ω and Γ, respectively. After adding the resulting equations and applying integration by parts, we infer

$$|\text{mean}(\mu_\lambda, \theta_\lambda)| \leq C\Big(\|F_\lambda'(\phi_\lambda)\|_{L^1(\Omega)} + \|G_\lambda'(\psi_\lambda)\|_{L^1(\Gamma)}\Big) \tag{6.41}$$

a.e. on $[0,T]$. Exploiting again the global Lipschitz continuity of F_2' and G_2', respectively, as well as (6.26) and (6.35), we deduce from (6.41) that

$$|\text{mean}(\mu_\lambda, \theta_\lambda)| \leq C\Big(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta} + \|\partial_\mathbf{n}\phi_\lambda\|_{L^2(\Gamma)}\Big) \tag{6.42}$$

a.e. on $[0,T]$. Thus, after another application of the bulk-surface Poincaré inequality, we arrive at

$$\|(\mu_\lambda, \theta_\lambda)\|_{\mathcal{H}^1} \leq C\Big(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta} + \|\partial_\mathbf{n}\phi_\lambda\|_{L^2(\Gamma)}\Big) \tag{6.43}$$

a.e. on $[0,T]$. Then, since Ω is of class C^2, another application of Proposition 4.3.5 (see also Remark 4.3.9) along with the estimate (6.43) leads to

$$\begin{aligned} &\|(\phi_\lambda, \psi_\lambda)\|_{\mathcal{H}^2} + \|(F_{1,\lambda}'(\phi_\lambda), G_{1,\lambda}'(\psi_\lambda))\|_{\mathcal{L}^2} \\ &\quad \leq C(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta} + \|\partial_\mathbf{n}\phi_\lambda\|_{L^2(\Gamma)}) \end{aligned} \tag{6.44}$$

a.e. on $[0,T]$. Then, exploiting again the trace theorem together with Ehrling's lemma, we readily obtain

$$\begin{aligned} &\|(\phi_\lambda, \psi_\lambda)\|_{\mathcal{H}^2} + \|(F_{1,\lambda}'(\phi_\lambda), G_{1,\lambda}'(\psi_\lambda))\|_{\mathcal{L}^2} \\ &\quad \leq C(1 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}), \end{aligned} \tag{6.45}$$

see (4.22)–(4.25). In particular, integrating (6.40) and (6.45) in time, readily yields

$$\|(F'_{1,\lambda}(\phi_\lambda), G'_{1,\lambda}(\psi_\lambda))\|_{L^2(0,T;\mathcal{L}^2)} \leq C \tag{6.46}$$

in all cases $K \in [0, \infty]$.

Step 7: Passage to the limit $\lambda \to 0$. We eventually pass to the limit $\lambda \to 0$. In view of the estimates (6.28), (6.29), (6.39) and (6.46), the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma imply the existence of functions $\phi, \psi, \mu, \theta, \xi$ and ξ_Γ such that

$$\begin{aligned}
(\partial_t\phi_\lambda, \partial_t\psi_\lambda) &\to (\partial_t\phi, \partial_t\psi) && \text{weakly in } L^2(0,T;(\mathcal{H}^1_{L,\beta})'), && \text{(6.47a)}\\
(\phi_\lambda, \psi_\lambda) &\to (\phi, \psi) && \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1_{K,\alpha}), && \text{(6.47b)}\\
& && \text{strongly in } C([0,T];\mathcal{L}^2), && \text{(6.47c)}\\
(\mu_\lambda, \theta_\lambda) &\to (\mu, \theta) && \text{weakly in } L^2(0,T;\mathcal{H}^1_{L,\beta}), && \text{(6.47d)}\\
(F'_{1,\lambda}(\phi_\lambda), G'_{1,\lambda}(\psi_\lambda)) &\to (\xi, \xi_\Gamma) && \text{weakly in } L^2(0,T;\mathcal{L}^2), && \text{(6.47e)}
\end{aligned}$$

along a subsequence $\lambda \to 0$, which will not be relabeled. Arguing as in Theorem 5.3.2, we can easily show that the above weak and strong convergences suffice to pass to the limit in the weak formulation (6.11a)-(6.11b) written for $F = F_\lambda$ and $G = G_\lambda$. Regarding Definition 6.2.2(iii), we have to make sure that the inclusions

$$\begin{cases}
\phi(x,t) \in (-1,1) \quad \text{for a.e. } (x,t) \in Q^T, & \xi = F'_1(\phi) \quad \text{a.e. in } Q^T,\\
\psi(z,t) \in (-1,1) \quad \text{for a.e. } (z,t) \in \Sigma^T, & \xi_\Gamma = G'_1(\psi) \quad \text{a.e. on } \Sigma^T
\end{cases} \tag{6.48}$$

hold true. To this end, first note that due to the convergence (6.47c)-(6.47e) along with the weak-strong convergence principle, we infer

$$\begin{aligned}
\lim_{\lambda\to 0} \int_0^T \int_\Omega F'_{1\lambda}(\phi_\lambda)\, \phi_\lambda \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \xi\, \phi \,\mathrm{d}x\,\mathrm{d}t,\\
\lim_{\lambda\to 0} \int_0^T \int_\Gamma G'_{1,\lambda}(\psi_\lambda)\, \psi_\lambda \,\mathrm{d}\Gamma\,\mathrm{d}t &= \int_0^T \int_\Gamma \xi_\Gamma\, \psi \,\mathrm{d}\Gamma\,\mathrm{d}t.
\end{aligned}$$

As F'_1 and G'_1 can be regarded as maximal monotone operators, the claim (6.48) follows from Proposition 2.3.4. We also refer to [19, Proposition 1.1, p. 42] and [87, Seciont 5.2]. Furthermore, it is straightforward to check that the mass conservation law as in Definition 6.2.2(iv) holds, due to the aforementioned strong convergences of ϕ_λ and ψ_λ. Lastly, the energy inequality in Definition 6.2.2(v) can be established using the above convergences and lower semicontinuity arguments as in the proof of Theorem 5.3.2.

Step 8: Existence of weak solutions for $(\mathbf{v}, \mathbf{w}) \in L^2(0,\infty;\mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma))$. We are left to show the existence of a weak solution to (6.1) for any pair of velocity fields $(\mathbf{v}, \mathbf{w}) \in L^2(0,\infty;\mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma))$. For this purpose, let $(\mathbf{v}, \mathbf{w}) \in L^2(0,\infty;\mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma))$, and consider a sequence $\{(\mathbf{v}_k, \mathbf{w}_k)\}_{k\in\mathbb{N}} \subset L^2(0,\infty;\mathbf{L}^3_{\mathrm{div}}(\Omega) \times \mathbf{L}^3_{\mathrm{tan}}(\Gamma))$ such that

$$(\mathbf{v}_k, \mathbf{w}_k) \to (\mathbf{v}, \mathbf{w}) \qquad \text{strongly in } L^2(0,\infty;\boldsymbol{\mathcal{L}}^2) \tag{6.49}$$

as $k \to \infty$ and

$$\|(\mathbf{v}_k, \mathbf{w}_k)\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} \leq \|(\mathbf{v}, \mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} \tag{6.50}$$

for all $k \in \mathbb{N}$.

Let $T > 0$ and $k \in \mathbb{N}$. Then, owing to the first part of the proof, there exists a quadruplet $(\phi_k, \psi_k, \mu_k, \theta_k)$ solving (6.1) in the sense of Definition 6.2.2 on $(0,T)$ with $(\mathbf{v}, \mathbf{w})$ replaced by $(\mathbf{v}_k, \mathbf{w}_k)$. In the following, we denote with the letter C positive constant that are independent of $k \in \mathbb{N}$. Now, as we additionally know that $|\phi_k| < 1$ a.e. in Q^T and $|\psi_k| < 1$ a.e. on Σ^T, we can go back to the energy inequality (6.13) and obtain using Young's inequality and (6.50) that

$$\begin{aligned} E_{\mathrm{free}}(\phi_k(t), \psi_k(t)) + \min\left\{1, \frac{m_*}{2}\right\} \int_0^t \|(\mu,\theta)\|_{L,\beta}^2 \,\mathrm{d}s &\le \frac{1}{2m_*} \int_0^t \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 \,\mathrm{d}s \\ &\le \frac{1}{2m_*} \int_0^\infty \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 \,\mathrm{d}s \end{aligned}$$

for a.e. $t \in [0,T]$. In particular, we immediately obtain that

$$\begin{aligned} &\|(\phi_k, \psi_k)\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(\nabla\mu_k, \nabla_\Gamma\theta_k)\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} \\ &\quad + \chi(L)^{\frac{1}{2}} \|\beta\theta_k - \mu_k\|_{L^2(0,T;L^2(\Gamma))} \le C, \end{aligned} \tag{6.51}$$

and a comparison argument in (6.11a) leads to

$$\|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^2(0,T;(\mathcal{H}^1_{L,\beta})'} \le C. \tag{6.52}$$

Moreover, repeating the argumentation from Step 4–Step 6, we have

$$\|(F'(\phi_k), G'(\psi_k))\|_{\mathcal{L}^2} + |\mathrm{mean}(\mu_k, \theta_k)| \le C(1 + \|(\mu_k, \theta_k)\|_{L,\beta}) \tag{6.53}$$

a.e. on $[0,T]$. We note that constants denoted by C only depend on the $\mathcal{H}^1$-norm of the initial data (ϕ_0, ψ_0) and the $L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)$-norm of the velocity fields $(\mathbf{v}, \mathbf{w})$, and are in particular independent of the final time $T > 0$. This allows us to extend the approximate solutions $(\phi_k, \psi_k, \mu_k, \theta_k)$ to $[0,\infty)$ for any $k \in \mathbb{N}$. Then, the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma imply the existence of functions $\phi, \psi, \mu, \theta, \xi$ and ξ_Γ, which exist on $[0,\infty)$, such that, for any $T > 0$,

$$\begin{aligned} (\phi_k, \psi_k) &\to (\phi, \psi) && \text{weakly-star in } L^\infty(0,T;\mathcal{H}^1_{K,\alpha}), && \text{(6.54a)} \\ & && \text{weakly in } H^1(0,T;(\mathcal{H}^1_{L,\beta})'), && \text{(6.54b)} \\ & && \text{strongly in } C([0,T];\mathcal{L}^2), && \text{(6.54c)} \\ (\mu_k, \theta_k) &\to (\mu, \theta) && \text{weakly in } L^2(0,T;\mathcal{H}^1_{L,\beta}), && \text{(6.54d)} \\ (F_1'(\phi_k), G_1'(\psi_k)) &\to (\xi, \xi_\Gamma) && \text{weakly in } L^2(0,T;\mathcal{L}^2), && \text{(6.54e)} \end{aligned}$$

as $k \to \infty$, up to a subsequence extraction. As seen above, from these convergences and the properties of F_1 and G_1, we readily infer that

$$|\phi| < 1 \quad \text{a.e. in } Q^T \quad \text{and} \quad |\psi| < 1 \quad \text{a.e. on } \Sigma^T$$

together with

$$F_1'(\phi) = \xi \quad \text{a.e. in } Q^T \quad \text{and} \quad G_1'(\psi) = \xi_\Gamma \quad \text{a.e. on } \Sigma^T.$$

One can then readily check $(\phi, \psi, \mu, \theta)$ is a weak solution to (6.1) in the sense of Definition 6.2.2 as seen in the previous step. It remains to establish the desired regularities (6.14) and (6.15). To this end, we notice that we can repeat the above arguments, now working directly on the solution itself. In particular, the energy inequality shows that

$$\begin{aligned} &\|(\phi, \psi)\|_{L^\infty(0,\infty;\mathcal{H}^1)} + \|(\nabla\mu, \nabla_\Gamma\theta)\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} \\ &\quad + \chi(L)^{\frac{1}{2}} \|\beta\theta - \mu\|_{L^2(0,\infty;L^2(\Gamma))} \le C, \end{aligned} \tag{6.55}$$

and, after another comparison argument, we readily find that

$$\|(\partial_t\phi, \partial_t\psi)\|_{L^2(0,\infty;(\mathcal{H}^1_{L,\beta})')} \leq C. \tag{6.56}$$

Then, we can prove that

$$|\mathrm{mean}(\mu,\theta)| \leq C(1 + \|(\mu,\theta)\|_{L,\beta}) \tag{6.57}$$

a.e. on $[0,\infty)$, which entails in combination with Proposition 4.3.5 that

$$\|(\phi,\psi)\|_{\mathcal{W}^{2,p}} + \|(F'(\phi), G'(\psi))\|_{\mathcal{L}^p} \leq C_p(1 + \|(\mu,\theta)\|_{L,\beta}) \tag{6.58}$$

a.e. on $[0,\infty)$ for $p = 6$ if $d = 3$ and any $p \in [2,\infty)$ if $d = 2$. In particular, it holds that

$$(\phi,\psi) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,p}), \qquad (F'(\phi), G'(\psi)) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^p)$$

for the aforementioned values of p. Moreover, utilizing the bulk-surface Poincaré inequality, we obtain from (6.55) and (6.57) that

$$(\mu,\theta) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1).$$

Finally, by the Lipschitz continuity of F_2' and G_2', an application of Corollary 4.3.7 yields that

$$\begin{aligned}\|(-\Delta\phi, -\Delta_\Gamma\psi + \alpha\partial_{\mathbf{n}}\phi)\|^2_{\mathcal{L}^2} &\leq C\Big(1 + \|(\mu - F_2'(\phi), \theta - G_2'(\psi))\|_{\mathcal{H}^1}\Big) \\ &\leq C\Big(1 + \|(\mu,\theta)\|_{L,\beta}\Big)\end{aligned} \tag{6.59}$$

a.e. on $[0,\infty)$ in the case $K \in (0,\infty]$, which entails

$$(\phi,\psi) \in L^4_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2) \tag{6.60}$$

in view of (6.55). On the other hand, if $K = 0$, Corollary 4.3.7 implies that

$$(\phi,\psi) \in L^3_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2).$$

Step 9: Energy identity in the case $L \in (0,\infty]$. In a last step we prove that the energy identity holds for weak solutions in the case $L \in (0,\infty]$. To this end, we consider again the convex part $\widetilde{E}_{\mathrm{free}}$ of the free energy E_{free} as defined in Section 4.3.1, which is a proper, convex and lower semicontinuous functional on $\mathcal{L}^2$ according to Lemma 4.3.1. Then, owing to [150, Lemma 4.1] in combination with Proposition 4.3.2, we deduce that $[0,\infty) \ni t \mapsto \widetilde{E}_{\mathrm{free}}(\phi(t),\psi(t))$ is absolutely continuous and it holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_{\mathrm{free}}(\phi,\psi) &= \big\langle(\partial_t\phi,\partial_t\psi), (-\Delta\phi + F_1'(\phi), -\Delta_\Gamma\psi + G_1'(\psi) + \alpha\partial_{\mathbf{n}}\phi)\big\rangle_{\mathcal{H}^1} \\ &= \big\langle(\partial_t\phi,\partial_t\psi), (\mu - F_2'(\phi), \theta - G_2'(\psi))\big\rangle_{\mathcal{H}^1}\end{aligned}$$

a.e. on $(0,\infty)$. As a consequence, since F is bounded by assumption **(A4)** and $\phi \in BC([0,\infty);L^2(\Omega))$, Lebesgue's convergence theorem entails that $\int_\Omega F(\phi(\cdot))\,\mathrm{d}x \in BC([0,\infty);L^2(\Omega))$, which, in turn, implies that $\phi \in BC([0,\infty);H^1(\Omega))$. Analogous reasoning provides that $\psi \in BC([0,\infty);H^1(\Gamma))$. Hence, testing (6.11a) with (μ,θ) and exploiting the standard chain rule in $L^2(0,\infty;\mathcal{H}^1) \cap H^1(0,\infty;(\mathcal{H}^1)')$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{free}}(\phi,\psi) + \int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta - \mu)^2\,\mathrm{d}\Gamma \\ &\quad - \int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma = 0\end{aligned}$$

a.e. on $(0,\infty)$, from which the energy equality follows. $\square$

6.4 Continuous dependence and uniqueness of weak solutions

In this section, we present the proof of the result on continuous dependence of weak solutions to the system (6.1), which in particular implies their uniqueness, as stated in Theorem 6.2.5.

Proof of Theorem 6.2.5. As the mobilities m_Ω and m_Γ are supposed to be constant, we assume, without loss of generality, that $m_\Omega \equiv m_\Gamma \equiv 1$. In the following, C denotes generic positive constants that may change their value from line to line and only depend on Ω, the parameters of the system (6.1), and the norms of the initial data.
We consider two weak solutions $(\phi_1, \psi_1, \mu_1, \theta_1)$ and $(\phi_2, \psi_2, \mu_2, \theta_2)$ originating from the initial data $(\phi_0^1, \psi_0^1), (\phi_0^2, \psi_0^2) \in \mathcal{H}^1_{K,\alpha}$ with velocity fields

$$(\mathbf{v}_1, \mathbf{w}_1) \in L^2(0,\infty; \mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma)) \;\text{ and }\; (\mathbf{v}_2, \mathbf{w}_2) \in L^2(0,\infty; \mathbf{L}^3_{\mathrm{div}}(\Omega) \times \mathbf{L}^{2+\omega}_{\mathrm{tan}}(\Gamma))$$

for some $\omega > 0$, respectively. Then we set

$$(\phi, \psi, \mu, \theta, \mathbf{v}, \mathbf{w}) := (\phi_1 - \phi_2, \psi_1 - \psi_2, \mu_1 - \mu_2, \theta_1 - \theta_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{w}_1 - \mathbf{w}_2).$$

Repeating the line of argument of Theorem 5.3.4, we can show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|^2_{L,\beta,*} \\
&\quad = -\big((\mu,\theta),(\phi,\psi)\big)_{\mathcal{L}^2} \\
&\qquad - \big((\phi_1\mathbf{v} + \phi\mathbf{v}_2, \psi_1\mathbf{w} + \psi\mathbf{w}_2), (\nabla\mathcal{S}^\Omega_{L,\beta}(\phi,\psi), \nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi,\psi))\big)_{\mathcal{L}^2}
\end{aligned} \tag{6.61}$$

a.e. on $(0,\infty)$. For the last term on the right-hand side of (6.61), we use the weak formulation (6.11b) to obtain

$$\begin{aligned}
-\big((\mu,\theta),(\phi,\psi)\big)_{\mathcal{L}^2} = -\|(\phi,\psi)\|^2_{K,\alpha} &- \int_\Omega (F'(\phi_2) - F'(\phi_1))\,\phi \,\mathrm{d}x \\
&- \int_\Gamma (G'(\psi_2) - G'(\psi_1))\,\psi \,\mathrm{d}\Gamma
\end{aligned}$$

a.e. on $(0,\infty)$. Then, by monotonicity of F_1' and G_1', it readily follows that

$$-\int_\Omega (F_1'(\phi_2) - F_1'(\phi_1))\,\phi \,\mathrm{d}x - \int_\Gamma (G_1'(\psi_2) - G_1'(\psi_1))\,\psi \,\mathrm{d}\Gamma \le 0 \tag{6.62}$$

a.e. on $(0,\infty)$. On the other, since F_2' and G_2' are both globally Lipschitz continuous, we infer with Young's inequality that

$$-\int_\Omega (F_2'(\phi_2) - F_2'(\phi_1))\,\phi \,\mathrm{d}x - \int_\Gamma (G_2'(\psi_2) - G_2'(\psi_1))\,\psi \,\mathrm{d}\Gamma \le C\|(\phi,\psi)\|^2_{\mathcal{L}^2} \tag{6.63}$$

a.e. on $(0,\infty)$. To control the $\mathcal{L}^2$-norm of (ϕ,ψ), we make a case distinction between $K \in [0,\infty)$ and $K = \infty$. If $K \in [0,\infty)$, we can simply use the Ehrling lemma and the bulk-surface Poincaré inequality to obtain

$$\|(\phi,\psi)\|^2_{\mathcal{L}^2} \le \frac{1}{6}\|(\phi,\psi)\|^2_{K,\alpha} + C\|(\phi,\psi)\|^2_{L,\beta,*}. \tag{6.64}$$

In the case $K = \infty$, we first use Ehrling's lemma to find

$$\|(\phi,\psi)\|^2_{\mathcal{L}^2} \le \frac{1}{7}\|(\phi,\psi)\|^2_{\mathcal{H}^1} + C\|(\phi,\psi)\|^2_{L,\beta,*}$$

$$\leq \frac{1}{7}\|(\phi,\psi)\|_{\mathcal{L}^2}^2 + \frac{1}{7}\|(\nabla\phi,\nabla_\Gamma\psi)\|_{\mathcal{L}^2}^2 + C\|(\phi,\psi)\|_{L,\beta,*}^2.$$

Consequently, absorbing the corresponding term into the left-hand side, we deduce that

$$\|(\phi,\psi)\|_{\mathcal{L}^2}^2 \leq \frac{1}{6}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\phi,\psi)\|_{L,\beta,*}^2. \tag{6.65}$$

Now, exploiting the facts that $|\phi| < 1$ a.e. in Q and $|\psi| < 1$ a.e. on Σ, we can estimate the last term on the right-hand side of (6.61) with the help of (6.64)–(6.65) and the Sobolev inequality as

$$\begin{aligned}
&\Big|\int_\Omega (\phi_1\mathbf{v} + \phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_{L,\beta}^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma (\psi_1\mathbf{w} + \psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_{L,\beta}^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big| \\
&\quad\leq C\Big(\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|_{\mathcal{H}^1}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}\Big)\|\mathcal{S}_{L,\beta}(\phi,\psi)\|_{L,\beta} \qquad (6.66)\\
&\quad\leq \frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\phi,\psi)\|_{\mathcal{H}^1}^2 + C\Big(1 + \|(\mathbf{w}_2,\mathbf{w}_2)\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}^2\Big)\|(\phi,\psi)\|_{L,\beta}^2 \\
&\quad\leq \frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + \frac{1}{4}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(1 + \|(\mathbf{w}_2,\mathbf{w}_2)\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}^2\Big)\|(\phi,\psi)\|_{L,\beta}^2
\end{aligned}$$

a.e. on $(0,\infty)$. Consequently, combining (6.62)-(6.66), we deduce from (6.61) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{L,\beta,*}^2 + \frac{1}{2}\|(\phi,\psi)\|_{K,\alpha}^2 \\
&\qquad\leq \frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\Big(1 + \|(\mathbf{v}_1,\mathbf{w}_1)\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}^2\Big)\|(\phi,\psi)\|_{L,\beta,*}^2
\end{aligned}$$

a.e. on $(0,\infty)$, and an application of Gronwall's lemma readily entails the continuous dependence estimate (6.18). ☐

6.5 Propagation of regularity and separation properties

In this section, we present the proof of Theorem 6.2.7 concerning the propagation of regularity of weak solutions. Before giving the full proof, we briefly describe the main idea. The proof is divided into two parts.

In the first part, we consider velocity fields $(\mathbf{v},\mathbf{w})$ satisfying the regularity assumptions stated in **(A7.1)**. Under these assumptions, higher-order regularity estimates can be derived by applying difference quotients in time and performing suitable energy estimates. This allows us to prove the propagation of regularity for the corresponding weak solution.

In the second part, we treat velocity fields satisfying the assumption **(A7.2)**. In this case, the velocity fields lack sufficient time regularity, and therefore the computations based on difference quotients in time used in the first part cannot be justified directly. To overcome this difficulty, we introduce an approximation procedure. More precisely, we approximate the singular potentials by their Moreau–Yosida regularization and simultaneously approximate the velocity fields by a sequence $\{(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\}_{\lambda>0}$ such that $(\mathbf{v}_\lambda,\mathbf{w}_\lambda)$ satisfies the assumption **(A7.1)** used in the first part. For the approximate problem, the results from the first part yield the propagation of regularity for the corresponding approximate weak solution. Using **(A7.2)**, we then show suitable higher-order estimates that are uniform with respect to the approximation parameter λ. Finally, a compactness argument allows us to pass to the limit $\lambda \to 0$ and conclude the propagation of regularity for the original weak solution.

Proof of Theorem 6.2.7. **The case with assumption (A7.1).**

Step 1. We start by showing $(\phi, \psi) \in W^{1,\infty}(\varsigma, \infty; (\mathcal{H}^1_{L,\beta})') \cap H^1(\varsigma, \infty; \mathcal{H}^1)$ and $(\nabla\mu, \nabla_\Gamma\theta) \in L^\infty(\varsigma, \infty; \boldsymbol{\mathcal{L}}^2)$ for any $\varsigma > 0$.

For any function $f : [0, \infty) \to X$, where X is a Banach space, $t \in [0, \infty)$ and $h \in (0, 1)$, we denote by

$$\partial_t^h f(t) := \frac{f(t+h) - f(t)}{h}$$

the forward-in-time difference quotient of f at time t.

Now, let $h \in (0, 1)$. Then, owing to (6.11a), we have that

$$\begin{aligned}
&\langle (\partial_t \partial_t^h \phi, \partial_t \partial_t^h \psi), (\zeta, \zeta_\Gamma) \rangle_{\mathcal{H}^1_{L,\beta}} \\
&\quad = \big(\partial_t^h \phi\, \mathbf{v}, \nabla\zeta\big)_{\mathbf{L}^2(\Omega)} + \big(\partial_t^h \mathbf{v}\, \phi(\cdot + h), \nabla\zeta\big)_{\mathbf{L}^2(\Omega)} \\
&\qquad + \big(\partial_t^h \psi\, \mathbf{w}, \nabla_\Gamma\zeta_\Gamma\big)_{\mathbf{L}^2(\Gamma)} + \big(\partial_t^h \mathbf{w}\, \psi(\cdot + h), \nabla_\Gamma\zeta_\Gamma\big)_{\mathbf{L}^2(\Gamma)} - \big((\partial_t^h \mu, \partial_t^h \theta), (\zeta, \zeta_\Gamma)\big)_{L,\beta}
\end{aligned} \tag{6.67}$$

a.e. on $(0, \infty)$ for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$, where

$$\partial_t^h \mu = -\Delta \partial_t^h \phi + \partial_t^h F'(\phi) \qquad \text{a.e. in } Q, \tag{6.68a}$$

$$\partial_t^h \theta = -\Delta_\Gamma \partial_t^h \psi + \partial_t^h G'(\psi) + \alpha \partial_\mathbf{n} \partial_t^h \phi \qquad \text{a.e. on } \Sigma, \tag{6.68b}$$

$$\begin{cases} K \partial_\mathbf{n} \partial_t^h \phi = \alpha \partial_t^h \psi - \partial_t^h \phi & \text{if } K \in [0, \infty), \\ \partial_\mathbf{n} \partial_t^h \phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{a.e. on } \Sigma. \tag{6.68c}$$

Recalling that by definition of the solution operator $\mathcal{S}_{L,\beta}$ (see Section 4.2) we have

$$\big(\mathcal{S}_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi), (\partial_t^h \mu, \partial_t^h \theta)\big)_{L,\beta} = \big((\partial_t^h \phi, \partial_t^h \psi), (\partial_t^h \mu, \partial_t^h \theta)\big)_{\mathcal{L}^2}$$

a.e. on $(0, \infty)$, as well as

$$\operatorname{mean}(\mathcal{S}^\Omega_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi), \mathcal{S}^\Gamma_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi)) = 0$$

a.e. on $(0, \infty)$, we test (6.67) with $(\zeta, \zeta_\Gamma) := \mathcal{S}_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi)$ and find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|(\partial_t^h \phi, \partial_t^h \psi)\|^2_{L,\beta,*} + \big((\partial_t^h \mu, \partial_t^h \theta), (\partial_t^h \phi, \partial_t^h \psi)\big)_{\mathcal{L}^2} \\
&\quad = -\big(\partial_t^h \phi\, \mathbf{v}, \nabla \mathcal{S}^\Omega_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi)\big)_{\mathbf{L}^2(\Omega)} - \big(\partial_t^h \mathbf{v}\, \phi(\cdot + h), \nabla \mathcal{S}^\Omega_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi)\big)_{\mathbf{L}^2(\Omega)} \\
&\qquad - \big(\partial_t^h \psi\, \mathbf{w}, \nabla_\Gamma \mathcal{S}^\Gamma_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi)\big)_{\mathbf{L}^2(\Gamma)} - \big(\partial_t^h \mathbf{w}\, \psi(\cdot + h), \nabla_\Gamma \mathcal{S}^\Gamma_{L,\beta}(\partial_t^h \phi, \partial_t^h \psi)\big)_{\mathbf{L}^2(\Gamma)} \\
&\quad =: \sum_{k=1}^4 J_k
\end{aligned} \tag{6.69}$$

a.e. on $(0, \infty)$, see also the proof of Theorem 6.2.5, where similar calculations where performed. Next, exploiting (6.68), we have

$$\begin{aligned}
-\big((\partial_t^h \mu, \partial_t^h \theta), (\partial_t^h \phi, \partial_t^h \psi)\big)_{\mathcal{L}^2} &= -\|(\partial_t^h \phi, \partial_t^h \psi)\|^2_{K,\alpha} - \int_\Omega \partial_t^h F'(\phi)\, \partial_t^h \phi \,\mathrm{d}x \\
&\quad - \int_\Gamma \partial_t^h G'(\psi)\, \partial_t^h \psi \,\mathrm{d}\Gamma
\end{aligned} \tag{6.70}$$

a.e. on $(0, \infty)$. Combining (6.69) and (6.70), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|(\partial_t^h \phi, \partial_t^h \psi)\|^2_{L,\beta,*} + \|(\partial_t^h \phi, \partial_t^h \psi)\|^2_{K,\alpha} + \int_\Omega \partial_t^h F_1'(\phi)\, \partial_t^h \phi \,\mathrm{d}x + \int_\Gamma \partial_t^h G_1'(\psi)\, \partial_t^h \psi \,\mathrm{d}\Gamma$$

$$= -\int_\Omega \partial_t^h F_2'(\phi)\, \partial_t^h \phi \,\mathrm{d}x - \int_\Gamma \partial_t^h G_2'(\psi)\, \partial_t^h \psi \,\mathrm{d}\Gamma + \sum_{k=1}^4 J_k \tag{6.71}$$

a.e. on $(0,\infty)$. Arguing as in the proof of Theorem 6.2.5 by using the monotonicity of F_1' and G_1', respectively, and the global Lipschitz continuity of F_2' and G_2', respectively, we obtain from (6.71) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2 + \|(\partial_t^h\phi,\partial_t^h\psi)\|_{K,\alpha}^2 \le C\|(\partial_t^h\phi,\partial_t^h\psi)\|_{\mathcal{L}^2}^2 + \sum_{k=1}^4 J_k \tag{6.72}$$

a.e. on $(0,\infty)$. Then, for the first term on the right-hand side of (6.72), we argue as for (6.64) and (6.65), and obtain with Young's inequality that

$$C\|(\partial_t^h\phi,\partial_t^h\psi)\|_{\mathcal{L}^2}^2 \le \frac{1}{4}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{K,\alpha}^2 + C\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2 \tag{6.73}$$

a.e. on $(0,\infty)$. Next, we intend to bound the terms J_k, $k=1,\dots,4$. To this end, we recall that the velocity fields $\mathbf{v}$ and $\mathbf{w}$ are divergence-free, and use integration by parts, Hölder's inequality, Young's inequality and Sobolev embeddings, and the bulk-surface Poincaré inequality to obtain

$$\begin{aligned}
|J_1+J_3| &\le \|\nabla\partial_t^h\phi\|_{\mathbf{L}^2(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^3(\Omega)}\|\mathcal{S}_{L,\beta}^\Omega(\partial_t^h\phi,\partial_t^h\psi)\|_{L^6(\Omega)} \\
&\qquad + \|\nabla_\Gamma\partial_t^h\psi\|_{\mathbf{L}^2(\Gamma)}\|\mathbf{w}\|_{\mathbf{L}^{2+\omega}(\Gamma)}\|\mathcal{S}_{L,\beta}^\Gamma(\partial_t^h\phi,\partial_t^h\psi)\|_{L^{\frac{2(2+\omega)}{\omega}}(\Gamma)} \\
&\le \frac{1}{4}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{K,\alpha}^2 + C\|(\mathbf{v},\mathbf{w})\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}^2\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2
\end{aligned} \tag{6.74}$$

a.e. on $(0,\infty)$, as well as

$$\begin{aligned}
|J_2+J_4| &\le \|\nabla\phi(\cdot+h)\|_{\mathbf{L}^\infty(\Omega)}\|\partial_t^h\mathbf{v}\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)}\|\mathcal{S}_{L,\beta}^\Omega(\partial_t^h\phi,\partial_t^h\psi)\|_{L^6(\Omega)} \\
&\qquad + \|\nabla_\Gamma\psi(\cdot+h)\|_{\mathbf{L}^\infty(\Gamma)}\|\partial_t^h\mathbf{w}\|_{\mathbf{L}^{1+\omega}(\Gamma)}\|\mathcal{S}_{L,\beta}^\Gamma(\partial_t^h\phi,\partial_t^h\psi)\|_{L^{\frac{1+\omega}{\omega}}(\Gamma)} \\
&\le \frac{1}{2}\|(\partial_t^h\mathbf{v},\partial_t^h\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2 \\
&\qquad + C\|(\phi(\cdot+h),\psi(\cdot+h))\|_{\mathcal{W}^{1,\infty}}^2\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2
\end{aligned} \tag{6.75}$$

a.e. on $(0,\infty)$. Thus, collecting (6.73)-(6.75), we deduce from (6.72) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2 + \frac{1}{2}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{K,\alpha}^2 \\
&\qquad \le \frac{1}{2}\|(\partial_t^h\mathbf{v},\partial_t^h\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2 + P_h\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2
\end{aligned} \tag{6.76}$$

a.e. on $(0,\infty)$, where

$$P_h(\cdot) \coloneqq C\Big(1 + \|(\mathbf{v},\mathbf{w})\|_{\mathbf{L}^3(\Omega)\times\mathbf{L}^{2+\omega}(\Gamma)}^2 + \|(\phi(\cdot+h),\psi(\cdot+h))\|_{\mathcal{W}^{1,\infty}}^2\Big) \in L^1_{\mathrm{loc}}([0,\infty)).$$

Now, we fix an arbitrary $r\in(0,1]$ and let $\varsigma>0$. Then, using a standard estimate for difference quotients, which can be found, e.g., in [70, p. 317] (see also [92, Lemma 7.23]), we have

$$\int_t^{t+r}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2\,\mathrm{d}s \le \int_t^{t+r+h}\|(\partial_t\phi,\partial_t\psi)\|_{L,\beta,*}^2\,\mathrm{d}s$$

for a.e. $t \in (0,\infty)$. Thus, a comparison argument in the weak formulation (6.11a) in combination with $|\phi| \leq 1$ a.e. in Q and $|\psi| \leq 1$ a.e. on Σ as seen before shows that

$$\begin{aligned}\sup_{t\geq 0}\int_t^{t+r} &\|(\partial_t^h\phi,\partial_t^h\psi)\|_{L,\beta,*}^2\,\mathrm{d}s \\ &\leq C\bigg(\int_0^\infty \|(\mu,\theta)\|_{L,\beta}^2\,\mathrm{d}s + \sup_{t\geq 0}\int_t^{t+1}\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2\,\mathrm{d}s\bigg) =: a_3.\end{aligned}$$

In a similar manner, it holds

$$\int_t^{t+r}\|(\partial_t^h\mathbf{v},\partial_t^h\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2\,\mathrm{d}s \leq \int_t^{t+r+h}\|(\partial_t\mathbf{v},\partial_t\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2\,\mathrm{d}s$$

and thus

$$\begin{aligned}\sup_{t\geq 0}\int_t^{t+r}&\|(\partial_t^h\mathbf{v},\partial_t^h\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2\,\mathrm{d}s \\ &\leq \sup_{t\geq 0}\int_t^{t+1}\|(\partial_t\mathbf{v},\partial_t\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2\,\mathrm{d}s =: a_2.\end{aligned}$$

Lastly,

$$\begin{aligned}\sup_{t\geq 0}\int_t^{t+r} & P_h\,\mathrm{d}s \\ &\leq \sup_{t\geq 0}\int_t^{t+1} C\Big(1+\|(\mathbf{v},\mathbf{w})\|_{\mathbf{L}^{\frac{6}{5}}(\Omega)\times\mathbf{L}^{1+\omega}(\Gamma)}^2 + \|(\phi(\cdot+h),\psi(\cdot+h))\|_{\mathcal{W}^{2,6}}^2\Big)\,\mathrm{d}s \\ &=: a_1.\end{aligned}$$

We are now in a position to apply the uniform Gronwall lemma as stated in Lemma 2.2.7 which implies

$$\|(\partial_t^h\phi(t+r),\partial_t^h\psi(t+r))\|_{L,\beta,*}^2 \leq \Big(\frac{a_3}{r}+a_2\Big)\mathbf{e}^{a_1} \qquad \text{for all } t\geq 0.$$

Consequently, for any $s \geq \varsigma$, choosing $r =: \min\{1,\varsigma\}$ and $t = s-r$ yields

$$\|(\partial_t^h\phi(s),\partial_t^h\psi(s))\|_{L,\beta,*}^2 \leq \Big(\frac{a_3}{\min\{1,\varsigma\}}+a_2\Big)\mathbf{e}^{a_1}.$$

Next, we integrate (6.76) in time over $[t,t+1]$ for an arbitrary $t\geq\varsigma$, and find that

$$\sup_{t\geq\varsigma}\int_t^{t+1}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{K,\alpha}^2\,\mathrm{d}s \leq a_2+(1+a_1)\Big(\frac{a_3}{\min\{1,\varsigma\}}+a_2\Big)\mathbf{e}^{a_1}.$$

Thus, as the estimates are both uniform in $h\in(0,1)$, we can consider the limit $h\to 0$ to conclude that

$$(\partial_t\phi,\partial_t\psi) \in L^\infty(\varsigma,\infty;(\mathcal{H}^1_{L,\beta})')\cap L^2_{\mathrm{uloc}}([\varsigma,\infty);\mathcal{H}^1). \tag{6.77}$$

Then, recalling the inequality

$$\|(\mu,\theta)\|_{L,\beta}^2 \leq C(\|(\partial_t\phi,\partial_t\psi)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2) \tag{6.78}$$

a.e. on $(0,\infty)$, which follows again from a comparison argument in (6.11a), we readily obtain the regularity

$$t\mapsto \|(\mu(t),\theta(t))\|_{L,\beta} \in L^\infty(\varsigma,\infty) \tag{6.79}$$

in view of $(\partial_t\phi, \partial_t\psi) \in L^\infty(\varsigma,\infty;(\mathcal{H}^1_{L,\beta})')$ and $(\mathbf{v},\mathbf{w}) \in L^\infty(0,\infty;\mathcal{L}^2)$.

Step 2: We next show $(\phi,\psi) \in L^\infty(\varsigma,\infty;\mathcal{W}^{2,6}) \cap \big(BC(\overline{Q_\varsigma}) \times BC(\overline{\Sigma_\varsigma})\big)$, $(\mu,\theta) \in L^\infty(\varsigma,\infty;\mathcal{H}^1)$ and $(F'(\phi), G'(\psi)) \in L^\infty(\varsigma,\infty;\mathcal{L}^6)$ for any $\varsigma > 0$.

To prove these regularities, we restrict ourselves to the case $L \in [0,\infty)$. If $L = \infty$, we can argue analogously (with obvious modifications).

First, note that, in view of Theorem 6.2.3, we have

$$
\begin{aligned}
-\Delta\phi + F_1'(\phi) &= \mu - F_2'(\phi) && \text{a.e. in } Q, && (6.80a)\\
-\Delta_\Gamma\psi + G_1'(\psi) + \alpha\partial_{\mathbf{n}}\phi &= \theta - G_2'(\psi) && \text{a.e. on } \Sigma, && (6.80b)\\
K\partial_{\mathbf{n}}\phi &= \alpha\psi - \phi && \text{a.e. on } \Sigma. && (6.80c)
\end{aligned}
$$

Then, by following verbatim the proof of Theorem 6.2.3, in particular Steps 4 - 6, see also Step 9, and thereby using Proposition 4.3.5, we obtain the estimates

$$
|\mathrm{mean}(\mu,\theta)| \le C(1 + \|(\mu,\theta)\|_{L,\beta}) \tag{6.81}
$$

a.e. on $(0,\infty)$, as well as

$$
\|(\phi,\psi)\|_{\mathcal{W}^{2,6}} + \|(F_1'(\phi), G_1'(\psi))\|_{\mathcal{L}^6} \le C(1 + \|(\mu,\theta)\|_{L,\beta}) \tag{6.82}
$$

a.e. on $(0,\infty)$. Finally, by (6.79), we find

$$
(\phi,\psi) \in L^\infty(\varsigma,\infty;\mathcal{W}^{2,6}), \qquad (F'(\phi), G'(\psi)) \in L^\infty(\varsigma,\infty;\mathcal{L}^6). \tag{6.83}
$$

Lastly, as we know from Step 1 that $\partial_t\phi \in L^2_{\mathrm{uloc}}([\varsigma,\infty);H^1(\Omega))$, we use the compact embedding $H^2(\Omega) \hookrightarrow H^{\frac{7}{4}}(\Omega)$, the continuous embeddings $H^{\frac{7}{4}}(\Omega) \hookrightarrow H^1(\Omega)$ and $H^{\frac{7}{4}}(\Omega) \hookrightarrow C(\overline{\Omega})$, and the Aubin–Lions–Simon lemma to deduce that

$$
\phi \in C([\varsigma,T];H^{\frac{7}{4}}(\Omega)) \hookrightarrow C([\varsigma,T];C(\overline{\Omega})) \cong C(\overline{Q_\varsigma^T})
$$

for any $T > 0$. As T is arbitrary, we in particular directly obtain $\phi \in BC(\overline{Q_\varsigma})$. Analogously, we infer $\psi \in BC(\overline{\Sigma_\varsigma})$.

Step 3: By means of elliptic regularity theory we now show $(\mu,\theta) \in L^2_{\mathrm{uloc}}([\varsigma,\infty);\mathcal{H}^2)$ for any $\varsigma > 0$.

In view of the regularities established in Step 1, and using that $\mathbf{v}$ and $\mathbf{w}$ are divergence-free in Ω and on Γ, respectively, we find that

$$
\begin{aligned}
&\int_\Omega \nabla\mu\cdot\nabla\zeta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta\theta-\mu)(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma\\
&\quad = \int_\Omega \big(-\partial_t\phi - \nabla\phi\cdot\mathbf{v}\big)\zeta \,\mathrm{d}x + \int_\Gamma \big(-\partial_t\psi - \nabla_\Gamma\psi\cdot\mathbf{w}\big)\zeta_\Gamma \,\mathrm{d}\Gamma
\end{aligned} \tag{6.84}
$$

a.e. on (ς,∞) for all $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$. This means that the pair (μ,θ) is a weak solution of the bulk-surface elliptic problem

$$
\begin{aligned}
&-\Delta\mu = -\partial_t\phi - \nabla\phi\cdot\mathbf{v} && \text{in } Q_\varsigma,\\
&-\Delta_\Gamma\theta + \beta\partial_{\mathbf{n}}\mu = -\partial_t\psi - \nabla_\Gamma\psi\cdot\mathbf{w} && \text{on } \Sigma_\varsigma,\\
&\begin{cases} L\partial_{\mathbf{n}}\mu = \beta\theta - \mu & \text{if } L \in [0,\infty),\\ \partial_{\mathbf{n}}\mu = 0 & \text{if } L = \infty \end{cases} && \text{on } \Sigma_\varsigma.
\end{aligned}
$$

Thus, by the very definition of the solution operator $\mathcal{S}_{L,\beta}$, it holds that

$$
(\mu - \beta\mathrm{mean}(\mu,\theta), \theta - \mathrm{mean}(\mu,\theta)) = -\mathcal{S}_{L,\beta}(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w}).
$$

This allows us to apply regularity theory for elliptic problems with bulk-surface coupling to find that

$$\begin{aligned}
&\|(\mu - \beta \mathrm{mean}(\mu,\theta), \theta - \mathrm{mean}(\mu,\theta))\|_{\mathcal{H}^2} \\
&\quad = \|\mathcal{S}_{L,\beta}(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w})\|_{\mathcal{H}^2} \\
&\quad \le C\|(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w})\|_{\mathcal{L}^2} \\
&\quad \le C\|(\partial_t\phi, \partial_t\psi)\|_{\mathcal{L}^2} + C\|(\nabla\phi, \nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad \le C\|(\partial_t\phi, \partial_t\psi)\|_{\mathcal{L}^2} + C\|(\phi,\psi)\|_{\mathcal{W}^{2,6}}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}
\end{aligned}$$

a.e. on (ς,∞). By **(A7)**, (6.38), (6.77) and (6.83), the latter entails that

$$(\mu,\theta) \in L^2_{\mathrm{uloc}}([\varsigma,\infty);\mathcal{H}^2).$$

Note that, while estimate (6.38) was only proven for the approximate solution $(\mu_\lambda, \theta_\lambda)$, the same estimate holds for (μ,θ) by performing the same computation now directly with the solution itself.

Step 4. We show that $(F'(\phi), G'(\psi)) \in L^2(\varsigma,\infty;\mathcal{L}^\infty)$ for any $\varsigma > 0$.

The Sobolev embedding $\mathcal{H}^2 \hookrightarrow \mathcal{L}^\infty$ implies that $(\mu,\theta) \in L^2(\varsigma,\infty;\mathcal{L}^\infty)$. This allows us to apply Proposition 4.3.5 for $p = \infty$ to deduce

$$\|(F_1'(\phi), G_1'(\psi))\|_{\mathcal{L}^\infty} \le C\big(1 + \|(\mu - F_2'(\phi), \theta - G_2'(\psi))\|_{\mathcal{L}^\infty}\big)$$

a.e. on (ς,∞). Thus, as F_2' and G_2' are globally Lipschitz, we readily obtain with (6.83) that

$$(F'(\phi), G'(\psi)) \in L^2(\varsigma,\infty;\mathcal{L}^\infty).$$

The case with assumption (A7.2).

Step 1: Definition of an approximate system. As in the proof of Theorem 6.2.3, we consider the Moreau–Yosida regularizations $F_{1,\lambda}$ and $G_{1,\lambda}$ of F_1 and G_1, respectively, and define

$$F_\lambda = F_{1,\lambda} + F_2, \qquad G_\lambda = G_{1,\lambda} + G_2.$$

Moreover, we regularize the velocity fields $\mathbf{v}$ and $\mathbf{w}$. To this end, we start by extending $\mathbf{v}$ and $\mathbf{w}$ in time by zero to obtain $(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in L^2(\mathbb{R};\boldsymbol{\mathcal{H}}^1) \cap L^\infty(\mathbb{R};\boldsymbol{\mathcal{L}}^2_{\mathrm{div}})$. Then, we choose a mollifier $\pi \in C_c^\infty(\mathbb{R};[0,\infty))$ with $\|\pi\|_{L^1(\mathbb{R})} = 1$, and we set $\pi_\lambda := \lambda^{-1}\pi(\lambda^{-1}\cdot)$ for any $\lambda \in (0,\frac{1}{2})$. We further define $\mathbf{v}_\lambda := \tilde{\mathbf{v}} *_t \pi_\lambda$ and $\mathbf{w}_\lambda := \tilde{\mathbf{w}} *_t \pi_\lambda$, where $*_t$ denotes the convolution with respect to the time variable. This provides a sequence $\{(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\}_{\lambda\in(0,\frac{1}{2})} \subset C^\infty(\mathbb{R};\boldsymbol{\mathcal{H}}^1 \cap \boldsymbol{\mathcal{L}}^2_{\mathrm{div}})$ such that

$$(\mathbf{v}_\lambda, \mathbf{w}_\lambda) \to (\mathbf{v}_\lambda, \mathbf{w}_\lambda) \qquad \text{strongly in } L^2(0,\infty;\boldsymbol{\mathcal{H}}^1), \tag{6.86}$$

$$\text{weakly-star in } L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2) \tag{6.87}$$

as $\lambda \to 0$, up to a subsequence extraction. For more details, we refer to [16, S. 4.13-4.15]. In particular, according to [16, S. 4.13 (2)], we have

$$\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)\cap L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)} \le \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)\cap L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)} \tag{6.88}$$

for all $\lambda \in (0,\frac{1}{2})$. We further point out that the approximate velocity fields are still divergence-free and preserve the trace relation $\mathbf{v}_\lambda|_\Gamma = \mathbf{w}_\lambda$ a.e. on Σ if $K = 0$ as the convolution only affects the time variable.

Then, according to Theorem 5.3.2 and Theorem 5.3.4, there exists a unique quadruplet $(\phi_\lambda, \psi_\lambda, \mu_\lambda, \theta_\lambda)$ such that

$$
\begin{aligned}
(\partial_t\phi_\lambda, \partial_t\psi_\lambda) &\in L^2(0,\infty;(\mathcal{H}^1_{L,\beta})'),\\
(\phi_\lambda, \psi_\lambda) &\in L^\infty(0,\infty;\mathcal{H}^1_{K,\alpha}) \cap L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,6}),\\
(\mu_\lambda, \theta_\lambda) &\in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1_{L,\beta}),
\end{aligned}
$$

which solves

$$
\begin{aligned}
&\partial_t\phi_\lambda + \operatorname{div}(\phi_\lambda \mathbf{v}_\lambda) = \Delta\mu_\lambda && \text{in } Q, && (6.89a)\\
&\mu_\lambda = -\Delta\phi_\lambda + F_\lambda'(\phi_\lambda) && \text{in } Q, && (6.89b)\\
&\partial_t\psi_\lambda + \operatorname{div}_\Gamma(\psi_\lambda \mathbf{w}_\lambda) = \Delta_\Gamma\theta_\lambda && \text{on } \Sigma, && (6.89c)\\
&\theta_\lambda = -\Delta_\Gamma\psi_\lambda + G_\lambda'(\psi_\lambda) + \alpha\partial_\mathbf{n}\phi_\lambda && \text{on } \Sigma, && (6.89d)\\
&\begin{cases} K\partial_\mathbf{n}\phi_\lambda = \alpha\psi_\lambda - \phi_\lambda & \text{if } K\in[0,\infty),\\ \partial_\mathbf{n}\phi_\lambda = 0 & \text{if } K=\infty \end{cases} && \text{on } \Sigma, && (6.89e)\\
&\begin{cases} L\partial_\mathbf{n}\mu_\lambda = \beta\theta_\lambda - \mu_\lambda & \text{if } L\in[0,\infty),\\ \partial_\mathbf{n}\mu_\lambda = 0 & \text{if } L=\infty \end{cases} && \text{on } \Sigma, && (6.89f)\\
&\phi_\lambda|_{t=0} = \phi_0 && \text{in } \Omega, && (6.89g)\\
&\psi_\lambda|_{t=0} = \psi_0 && \text{on } \Gamma && (6.89h)
\end{aligned}
$$

in the sense of Definition 5.3.1 with F and G replaced by F_λ and G_λ, respectively, and $\mathbf{v}$ and $\mathbf{w}$ replaced by $\mathbf{v}_\lambda$ and $\mathbf{w}_\lambda$, respectively. In particular, it holds

$$
\begin{aligned}
&\langle(\partial_t\phi_\lambda, \partial_t\psi_\lambda), (\zeta, \zeta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} - \int_\Omega \phi_\lambda \mathbf{v}_\lambda\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma \psi_\lambda \mathbf{w}_\lambda\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma\\
&= -\int_\Omega m_\Omega(\phi_\lambda)\nabla\mu_\lambda\cdot\nabla\zeta\,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi_\lambda)\nabla_\Gamma\theta_\lambda\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma \qquad (6.90)\\
&\quad - \chi(L)\int_\Gamma(\beta\theta_\lambda - \mu_\lambda)(\beta\zeta_\Gamma - \zeta)\,\mathrm{d}\Gamma
\end{aligned}
$$

a.e. on $(0,\infty)$ for all $(\zeta,\zeta_\Gamma)\in\mathcal{H}^1_{L,\beta}$, and

$$
\begin{aligned}
&\mu_\lambda = -\Delta\phi_\lambda + F_\lambda'(\phi_\lambda) && \text{a.e. in } Q, && (6.91a)\\
&\theta_\lambda = -\Delta_\Gamma\psi_\lambda + G_\lambda'(\psi_\lambda) + \alpha\partial_\mathbf{n}\phi_\lambda && \text{a.e. on } \Sigma, && (6.91b)\\
&\begin{cases} K\partial_\mathbf{n}\phi_\lambda = \alpha\psi_\lambda - \phi_\lambda & \text{if } K\in[0,\infty),\\ \partial_\mathbf{n}\phi_\lambda = 0 & \text{if } K=\infty \end{cases} && \text{a.e. on } \Sigma. && (6.91c)
\end{aligned}
$$

Moreover, the energy inequality reads now as

$$
\begin{aligned}
&E_{\mathrm{free},\lambda}(\phi_\lambda(t), \psi_\lambda(t)) + \int_0^t\int_\Omega |\nabla\mu_\lambda|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma |\nabla_\Gamma\theta_\lambda|^2\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&\quad + \chi(L)\int_0^t\int_\Gamma (\beta\theta_\lambda - \mu_\lambda)^2\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&\leq E_{\mathrm{free},\lambda}(\phi_0,\psi_0) + \int_0^t\int_\Omega \phi_\lambda\mathbf{v}_\lambda\cdot\nabla\mu_\lambda\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma \psi_\lambda\mathbf{w}_\lambda\cdot\nabla_\Gamma\theta_\lambda\,\mathrm{d}\Gamma\,\mathrm{d}s
\end{aligned}
$$

for all $t\in[0,\infty)$, where the approximate energy $E_{\mathrm{free},\lambda}$ is again given by

$$
E_{\mathrm{free},\lambda}(\eta,\eta_\Gamma) := \int_\Omega \frac{1}{2}|\nabla\eta|^2 + F_\lambda(\eta)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma\eta_\Gamma|^2 + G_\lambda(\eta_\Gamma)\,\mathrm{d}\Gamma
$$

$$+\chi(K)\int_\Gamma \frac{1}{2}(\alpha\eta_\Gamma-\eta)^2\,\mathrm{d}\Gamma$$

for all $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$. In view of (6.88), we can argue as in Theorem 6.2.3 and readily obtain

$$\begin{aligned}&\|(\phi_\lambda,\psi_\lambda)\|_{L^\infty(0,\infty;\mathcal{H}^1)}+\|(\nabla\mu_\lambda,\nabla_\Gamma\theta_\lambda)\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)}+\chi(L)^{\frac12}\|\beta\theta_\lambda-\mu_\lambda\|_{L^2(0,\infty;L^2(\Gamma))}\\&\quad+\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{L^2(0,\infty;(\mathcal{H}^1_{L,\beta})')}\le C\end{aligned}\tag{6.92}$$

as well as

$$\|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{W}^{2,6}}+\|(F'_{1,\lambda}(\phi_\lambda),G'_{1,\lambda}(\psi_\lambda))\|_{\mathcal{L}^6}\le C(1+\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta})\tag{6.93}$$

a.e. on $(0,\infty)$. We further recall the following inequality

$$|\mathrm{mean}(\mu_\lambda,\theta_\lambda)|\le C(1+\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta})\tag{6.94}$$

a.e. on $(0,\infty)$, see (6.38) and (6.42).

Step 2. We recall that the approximative velocity fields $(\mathbf{v}_\lambda,\mathbf{w}_\lambda)$ satisfy the assumption **(A7.1)**. Thus, for any $\hat\varsigma>0$, in view of the properties of the Yosida approximations $F'_{1,\lambda}$ and $G'_{1,\lambda}$, it is straightforward to adapt the arguments from the first part of this proof to show that

$$(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\in L^\infty(\hat\varsigma,\infty;(\mathcal{H}^1_{L,\beta})')\cap L^2_{\mathrm{uloc}}([\hat\varsigma,\infty);\mathcal{H}^1),\tag{6.95a}$$

$$(\phi_\lambda,\psi_\lambda)\in L^\infty(\hat\varsigma,\infty;\mathcal{W}^{2,6})\cap(BC(\overline{Q_{\hat\varsigma}})\times BC(\overline{\Sigma_{\hat\varsigma}})),\tag{6.95b}$$

$$(\mu_\lambda,\theta_\lambda)\in L^\infty(\hat\varsigma,\infty;\mathcal{H}^1_{L,\beta})\cap L^2_{\mathrm{uloc}}([\hat\varsigma,\infty);\mathcal{H}^2).\tag{6.95c}$$

We now prove several higher regularity results, which are not uniform with respect to $\lambda\in(0,\frac12)$, but are crucial for our analysis to prove the desired uniform estimates.
Step 2.1: First, we show that $(\partial_t\mu_\lambda,\partial_t\theta_\lambda)$ exists in the weak sense and belongs to $L^2_{\mathrm{uloc}}([\hat\varsigma,\infty),(\mathcal{H}^1_{K,\alpha})')$. To this end, we take the forward-in-time difference quotient of (6.91a)–(6.91c) . Then we have

$$\partial_t^h\mu_\lambda=-\Delta\partial_t^h\phi_\lambda+\partial_t^hF'_\lambda(\phi_\lambda)\qquad\text{a.e. in } Q,\tag{6.96a}$$

$$\partial_t^h\theta_\lambda=-\Delta_\Gamma\partial_t^h\psi_\lambda+\partial_t^hG'_\lambda(\psi_\lambda)+\alpha\partial_{\mathbf{n}}\partial_t^h\phi_\lambda\qquad\text{a.e. on }\Sigma,\tag{6.96b}$$

$$\begin{cases}K\partial_{\mathbf{n}}\partial_t^h\phi_\lambda=\alpha\partial_t^h\psi_\lambda-\partial_t^h\phi_\lambda & \text{if } K\in[0,\infty),\\ K\partial_{\mathbf{n}}\partial_t^h\phi_\lambda=0 & \text{if } K=\infty\end{cases}\qquad\text{a.e. on }\Sigma.\tag{6.96c}$$

Let now $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$. Then, multiplying (6.96a) and (6.96b) by η and η_Γ, and integrating over Ω and Γ, respectively, an application of integration by parts yields under consideration of (6.96c) that

$$\begin{aligned}&\langle(\partial_t^h\mu_\lambda,\partial_t^h\theta_\lambda),(\eta,\eta_\Gamma)\rangle_{\mathcal{H}^1_{K,\alpha}}=\int_\Omega\partial_t^h\mu_\lambda\,\eta\,\mathrm{d}x+\int_\Gamma\partial_t^h\theta_\lambda\,\eta_\Gamma\,\mathrm{d}\Gamma\\&=\int_\Omega\nabla\partial_t^h\phi_\lambda\cdot\nabla\eta+\partial_t^hF'_\lambda(\phi_\lambda)\,\eta\,\mathrm{d}x\\&\quad+\int_\Gamma\nabla_\Gamma\partial_t^h\psi_\lambda\cdot\nabla_\Gamma\eta_\Gamma+\partial_t^hG'_\lambda(\psi_\lambda)\,\eta_\Gamma\,\mathrm{d}\Gamma\\&\quad+\chi(K)\int_\Gamma(\alpha\partial_t^h\psi_\lambda-\partial_t^h\phi_\lambda)(\alpha\eta_\Gamma-\eta)\,\mathrm{d}\Gamma\end{aligned}\tag{6.97}$$

a.e. on $(0,\infty)$. Consequently, employing Hölder's inequality we have

$$
\begin{aligned}
&|\langle(\partial_t^h\mu_\lambda,\partial_t^h\theta_\lambda),(\eta,\eta_\Gamma))\rangle_{\mathcal{H}^1_{K,\alpha}}| \\
&\quad\le C\Big(\|(\partial_t^h\phi_\lambda,\partial_t^h\psi_\lambda)\|_{K,\alpha}+\|(\partial_t^hF_\lambda'(\phi_\lambda),\partial_t^hG_\lambda'(\psi_\lambda))\|_{\mathcal{L}^2}\Big)\|(\eta,\eta_\Gamma)\|_{\mathcal{H}^1}
\end{aligned}
$$

a.e. on $(0,\infty)$. Now, taking the supremum over all $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$ with $\|(\eta,\eta_\Gamma)\|_{\mathcal{H}^1}\le 1$ yields

$$
\|(\partial_t^h\mu_\lambda,\partial_t^h\theta_\lambda)\|_{(\mathcal{H}^1_{K,\alpha})'}\le C\Big(\|(\partial_t^h\phi_\lambda,\partial_t^h\psi_\lambda)\|_{K,\alpha}+\|(\partial_t^hF_\lambda'(\phi_\lambda),\partial_t^hG_\lambda'(\psi_\lambda))\|_{\mathcal{L}^2}\Big)
$$

a.e. on $(0,\infty)$. Moreover, we note again that

$$
\|(\partial_t^h\phi_\lambda,\partial_t^h\psi_\lambda)\|_{K,\alpha}\le C\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{K,\alpha}
$$

a.e. on $(\hat\varsigma,\infty)$, while in view of the Lipschitz continuity of the Yosida approximations and F_2' and G_2', we infer

$$
\|(\partial_t^hF_\lambda'(\phi_\lambda),\partial_t^hG_\lambda'(\psi_\lambda))\|_{\mathcal{L}^2}\le C(1+\tfrac{1}{\lambda})\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{\mathcal{L}^2}
$$

a.e. on $(\hat\varsigma,\infty)$. Thus, we obtain the bound

$$
\|(\partial_t^h\mu_\lambda,\partial_t^h\theta_\lambda)\|_{(\mathcal{H}^1_{K,\alpha})'}\le C(1+\tfrac{1}{\lambda})\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{\mathcal{L}^2}+C\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{K,\alpha} \tag{6.98}
$$

a.e. on $(\hat\varsigma,\infty)$. In particular, as the right-hand side of (6.98) is independent of $h\in(0,1)$, we are allowed to pass to the limit $h\to 0$ which yields

$$
(\partial_t\mu_\lambda,\partial_t\theta_\lambda)\in L^2_{\mathrm{uloc}}([\hat\varsigma,\infty);(\mathcal{H}^1_{K,\alpha})'). \tag{6.99}
$$

Furthermore, passing to the limit $h\to 0$ in (6.97), we conclude that

$$
\begin{aligned}
\langle(\partial_t\mu_\lambda,\partial_t\theta_\lambda),(\eta,\eta_\Gamma)\rangle_{\mathcal{H}^1_{K,\alpha}}&=\int_\Omega\nabla\partial_t\phi_\lambda\cdot\nabla\eta+F_\lambda''(\phi_\lambda)\partial_t\phi_\lambda\,\eta\,\mathrm{d}x\\
&\quad+\int_\Gamma\nabla_\Gamma\partial_t\psi_\lambda\cdot\nabla_\Gamma\eta_\Gamma+G_\lambda''(\psi_\lambda)\partial_t\psi_\lambda\,\eta_\Gamma\,\mathrm{d}\Gamma\\
&\quad+\chi(K)\int_\Gamma(\alpha\partial_t\psi_\lambda-\partial_t\phi_\lambda)(\alpha\eta_\Gamma-\eta)\,\mathrm{d}\Gamma
\end{aligned}
\tag{6.100}
$$

a.e. on $(\hat\varsigma,\infty)$ for all $(\eta,\eta_\Gamma)\in\mathcal{H}^1_{K,\alpha}$.

Step 2.2 Next, we show that $(\phi_\lambda,\psi_\lambda)\in L^2_{\mathrm{uloc}}([\hat\varsigma,\infty);\mathcal{W}^{3,6})$. To this end, we apply regularity theory for bulk-surface elliptic systems and get

$$
\begin{aligned}
\|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{W}^{3,6}}&\le C\Big(1+\|(\mu_\lambda-F_\lambda'(\phi_\lambda),\theta_\lambda-G_\lambda'(\psi_\lambda))\|_{\mathcal{W}^{1,6}}\Big)\\
&\le C\Big(1+\|(\mu_\lambda,\theta_\lambda)\|_{\mathcal{H}^2}+\|(F_\lambda'(\phi_\lambda),G_\lambda'(\psi_\lambda))\|_{\mathcal{W}^{1,6}}\Big)
\end{aligned}
$$

a.e. on $(\hat\varsigma,\infty)$. Here, we have already made use of the Sobolev embedding $\mathcal{H}^2\hookrightarrow\mathcal{W}^{1,6}$. For the term involving the potentials, we first remark that $(F_\lambda'(\phi_\lambda),G_\lambda'(\psi_\lambda))\in L^\infty(\hat\varsigma,\infty;\mathcal{L}^6)$. In addition, noting again on the Lipschitz continuity of F_λ' and G_λ', it holds that

$$
|\nabla F_\lambda'(\phi_\lambda)|=|F_\lambda''(\phi_\lambda)\nabla\phi_\lambda|\le C(1+\tfrac{1}{\lambda})|\nabla\phi_\lambda|
$$

a.e. on $(\hat{\varsigma}, \infty)$, with an analogous estimate for the surface gradient of $G'_\lambda(\psi_\lambda)$. Hence, $(\nabla F'_\lambda(\phi_\lambda), \nabla_\Gamma G'_\lambda(\psi_\lambda)) \in L^\infty(\hat{\varsigma}, \infty; \boldsymbol{\mathcal{L}}^6)$ and we conclude

$$(F'_\lambda(\phi_\lambda), G'_\lambda(\psi_\lambda)) \in L^\infty(\hat{\varsigma}, \infty; \mathcal{W}^{1,6}).$$

As additionally $(\mu_\lambda, \theta_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{H}^2)$, we readily deduce that

$$(\phi_\lambda, \psi_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{W}^{3,6}).$$

Step 2.3: Lastly, we prove that $(\mu_\lambda, \theta_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{H}^3) \cap BC([\hat{\varsigma}, \infty); \mathcal{H}^1_{L,\beta})$ for all $\hat{\varsigma} > 0$. Once again, by elliptic regularity theory for bulk-surface elliptic systems and the Sobolev embedding, we obtain

$$\begin{aligned}
&\|(\mu_\lambda - \beta\mathrm{mean}(\mu_\lambda, \theta_\lambda), \theta_\lambda - \mathrm{mean}(\mu_\lambda, \theta_\lambda))\|_{\mathcal{H}^3} \\
&\quad \le C\|(\partial_t\phi_\lambda - \nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \partial_t\psi_\lambda - \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\|_{\mathcal{H}^1} \\
&\quad \le C\Big(\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|_{\mathcal{H}^1} + \|(\nabla\phi_\lambda, \nabla_\Gamma\psi_\lambda)\|_{\boldsymbol{\mathcal{W}}^{1,\infty}}\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}\Big) \\
&\quad \le C\Big(\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|_{\mathcal{H}^1} + \|(\phi_\lambda, \psi_\lambda)\|_{\mathcal{W}^{3,6}}\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}\Big)
\end{aligned}$$

a.e. on $(\hat{\varsigma}, \infty)$. In light of (6.88), (6.94) and the previous step, the latter entails that $(\mu_\lambda, \theta_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{H}^3)$. Moreover, in view of the regularities (6.95c) and (6.99), the Aubin–Lions–Simon lemma implies that $(\mu_\lambda, \theta_\lambda) \in BC([\hat{\varsigma}, \infty); \mathcal{L}^2)$. Then, if $K = 0$, and recalling that in this case we have $\mathbf{v}_\lambda|_\Gamma = \mathbf{w}_\lambda$ a.e. on Σ, we compute

$$\nabla\phi_\lambda \cdot \mathbf{v}_\lambda = \big[\nabla\phi_\lambda - \mathbf{n}(\nabla\phi_\lambda \cdot \mathbf{n})\big] \cdot \mathbf{v}_\lambda = \alpha\nabla_\Gamma\psi_\lambda \cdot \mathbf{v}_\lambda = \alpha\nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda \qquad \text{a.e. on } \Sigma,$$

from which we obtain

$$\begin{aligned}
&(-\Delta\mu_\lambda, -\Delta_\Gamma\theta_\lambda + \beta\partial_{\mathbf{n}}\mu_\lambda) \\
&\quad = -(\partial_t\phi_\lambda + \nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \partial_t\psi_\lambda + \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{H}^1_{K,\alpha}).
\end{aligned}$$

This allows us to apply Proposition 5.2.2, which yields

$$(\mu_\lambda, \theta_\lambda) \in BC([\hat{\varsigma}, \infty); \mathcal{H}^1_{L,\beta})$$

along with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\mu_\lambda, \theta_\lambda)\|^2_{L,\beta} = \langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), (-\Delta\mu_\lambda, -\Delta_\Gamma\theta_\lambda + \beta\partial_{\mathbf{n}}\mu_\lambda)\rangle_{\mathcal{H}^1_{K,\alpha}} \tag{6.101}$$

a.e. on $(\hat{\varsigma}, \infty)$.

Step 3: Higher order uniform estimates. Now, we establish the main estimates for the regularity argument, which will be uniform with respect to $\lambda \in (0, \frac{1}{2})$. To this end, we test (6.100) with $(\partial_t\phi_\lambda, \partial_t\psi_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{H}^1_{K,\alpha})$ obtaining

$$\begin{aligned}
&\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), (\partial_t\phi_\lambda, \partial_t\psi_\lambda)\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&= \int_\Omega |\nabla\partial_t\phi_\lambda|^2 + F''_\lambda(\phi_\lambda)\,|\partial_t\phi_\lambda|^2 \,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma\partial_t\psi_\lambda|^2 + G''_\lambda(\psi_\lambda)\,|\partial_t\psi_\lambda|^2 \,\mathrm{d}\Gamma \\
&\quad + \chi(K)\int_\Gamma (\alpha\partial_t\psi_\lambda - \partial_t\phi_\lambda)^2 \,\mathrm{d}\Gamma \\
&= \|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|^2_{K,\alpha} + \int_\Omega F''_\lambda(\phi_\lambda)\,|\partial_t\phi_\lambda|^2 \,\mathrm{d}x + \int_\Gamma G''_\lambda(\psi_\lambda)\,|\partial_t\psi_\lambda|^2 \,\mathrm{d}\Gamma
\end{aligned} \tag{6.102}$$

a.e. on $(\hat{\varsigma}, \infty)$. On the other hand, exploiting the identities ((6.89a), (6.89c)) for $(\partial_t\phi_\lambda, \partial_t\psi_\lambda)$, which hold on a.e. in $Q_{\hat{\varsigma}}$ and a.e. on $\Sigma_{\hat{\varsigma}}$, respectively, as well as the chain rule (6.101), we find

$$
\begin{aligned}
&\big\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), (\partial_t\phi_\lambda, \partial_t\psi_\lambda)\big\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&\quad = \big\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), (\Delta\mu_\lambda - \nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \Delta_\Gamma\theta_\lambda - \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda - \beta\partial_\mathbf{n}\mu_\lambda)\big\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&\quad = -\big\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), (-\Delta\mu_\lambda, -\Delta_\Gamma\theta_\lambda + \beta\partial_\mathbf{n}\mu_\lambda)\big\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&\qquad - \big\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), \nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\mu_\lambda, \theta_\lambda)\|^2_{L,\beta} - \big\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), \nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big\rangle_{\mathcal{H}^1_{K,\alpha}}
\end{aligned} \tag{6.103}
$$

a.e. on $(\hat{\varsigma}, \infty)$. Now, for the second term on the right-hand side of (6.103), we recall that $(\nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda) \in L^2_{\mathrm{uloc}}([\hat{\varsigma}, \infty); \mathcal{H}^1_{K,\alpha})$. This allows us to choose $(\eta, \eta_\Gamma) = (\nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)$ as a test function in the weak formulation (6.100). In this way, we obtain

$$
\begin{aligned}
&\big\langle(\partial_t\mu_\lambda, \partial_t\theta_\lambda), \nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big\rangle_{\mathcal{H}^1_{K,\alpha}} \\
&\quad = \big((\partial_t\phi_\lambda, \partial_t\psi_\lambda), (\nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big)_{K,\alpha} \\
&\qquad + \int_\Omega F''_\lambda(\phi_\lambda)\partial_t\phi_\lambda\nabla\phi_\lambda \cdot \mathbf{v}_\lambda \,\mathrm{d}x + \int_\Gamma G''_\lambda(\psi_\lambda)\partial_t\psi_\lambda\nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda \,\mathrm{d}\Gamma \\
&\quad = \big((\partial_t\phi_\lambda, \partial_t\psi_\lambda), (\nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big)_{K,\alpha} \\
&\qquad + \int_\Omega \nabla(F'_\lambda(\phi_\lambda)) \cdot \mathbf{v}_\lambda\, \partial_t\phi_\lambda \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma(G'_\lambda(\psi_\lambda)) \cdot \mathbf{w}_\lambda\, \partial_t\psi_\lambda \,\mathrm{d}\Gamma \\
&\quad = \big((\partial_t\phi_\lambda, \partial_t\psi_\lambda), (\nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big)_{K,\alpha} \\
&\qquad - \int_\Omega F'_\lambda(\phi_\lambda)\, \nabla\partial_t\phi_\lambda \cdot \mathbf{v}_\lambda \,\mathrm{d}x - \int_\Gamma G'_\lambda(\psi_\lambda)\, \nabla_\Gamma\partial_t\psi_\lambda \cdot \mathbf{w}_\lambda \,\mathrm{d}\Gamma
\end{aligned} \tag{6.104}
$$

a.e. on $(\hat{\varsigma}, \infty)$. Here we used integration by parts along with the facts that $\operatorname{div} \mathbf{v}_\lambda = 0$ a.e. in Q, $\operatorname{div}_\Gamma \mathbf{w}_\lambda = 0$ and $\mathbf{v}_\lambda \cdot \mathbf{n} = \mathbf{w}_\lambda \cdot \mathbf{n} = 0$ a.e. on Σ. Thus, collecting (6.102)–(6.104), we find that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\mu_\lambda, \theta_\lambda)\|^2_{L,\beta} + \|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|^2_{K,\alpha} + \int_\Omega F''_{1,\lambda}(\phi_\lambda)|\partial_t\phi_\lambda|^2 \,\mathrm{d}x + \int_\Gamma G''_{1,\lambda}(\psi_\lambda)|\partial_t\psi_\lambda|^2 \,\mathrm{d}\Gamma \\
&\quad = -\big((\partial_t\phi_\lambda, \partial_t\psi_\lambda), (\nabla\phi_\lambda \cdot \mathbf{v}_\lambda, \nabla_\Gamma\psi_\lambda \cdot \mathbf{w}_\lambda)\big)_{K,\alpha} + \int_\Omega F'_\lambda(\phi_\lambda)\, \nabla\partial_t\phi_\lambda \cdot \mathbf{v}_\lambda \,\mathrm{d}x \\
&\qquad + \int_\Gamma G'_\lambda(\psi_\lambda)\, \nabla_\Gamma\partial_t\psi_\lambda \cdot \mathbf{w}_\lambda - \int_\Gamma F''_2(\phi_\lambda)\, |\partial_t\phi_\lambda|^2 \,\mathrm{d}x - \int_\Gamma G''_2(\psi_\lambda)\, |\partial_t\psi_\lambda|^2 \,\mathrm{d}\Gamma
\end{aligned} \tag{6.105}
$$

a.e. on $(\hat{\varsigma}, \infty)$. Due to the convexity of the Moreau–Yosida approximations, we immediately notice that the last two terms on the left-hand side of (6.105) are non-negative. Then, recalling that F'_2 and G'_2 are both Lipschitz continuous, we use (6.64)-(6.65) and a comparison argument in (6.90) to infer

$$
\begin{aligned}
&\Big| -\int_\Gamma F''_2(\phi_\lambda)\, |\partial_t\phi_\lambda|^2 \,\mathrm{d}x - \int_\Gamma G''_2(\psi_\lambda)\, |\partial_t\psi_\lambda|^2 \,\mathrm{d}\Gamma \Big| \\
&\quad \le C\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|^2_{\mathcal{L}^2} \\
&\quad \le \frac{1}{6}\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|^2_{K,\alpha} + C\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|^2_{L,\beta,*} \\
&\quad \le \frac{1}{6}\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|^2_{K,\alpha} + C\Big(\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\mu_\lambda, \theta_\lambda)\|^2_{L,\beta}\Big)
\end{aligned} \tag{6.106}
$$

a.e. on $(\hat{\varsigma},\infty)$. We recall that by the definition of the bilinear form $(\cdot,\cdot)_{K,\alpha}$ we have

$$\begin{aligned} &-\big((\partial_t\phi_\lambda,\partial_t\psi_\lambda),(\nabla\phi_\lambda\cdot\mathbf{v}_\lambda,\nabla_\Gamma\psi_\lambda\cdot\mathbf{w}_\lambda)\big)_{K,\alpha}\\ &\quad=-\int_\Omega\nabla\partial_t\phi_\lambda\cdot\nabla(\nabla\phi_\lambda\cdot\mathbf{v}_\lambda)\,\mathrm{d}x-\int_\Gamma\nabla_\Gamma\partial_t\psi_\lambda\cdot\nabla_\Gamma(\nabla_\Gamma\psi_\lambda\cdot\mathbf{w}_\lambda)\,\mathrm{d}\Gamma\\ &\qquad-\chi(K)\int_\Gamma(\alpha\partial_t\psi_\lambda-\partial_t\phi_\lambda)(\alpha\nabla_\Gamma\psi_\lambda\cdot\mathbf{w}_\lambda-\nabla\phi_\lambda\cdot\mathbf{v}_\lambda)\,\mathrm{d}\Gamma \end{aligned}\tag{6.107}$$

a.e. on $(\hat{\varsigma},\infty)$, and we now intend to bound the terms on the right-hand side of (6.107) step by step. For the first term, we use the Sobolev embedding $W^{1,6}(\Omega)\hookrightarrow L^\infty(\Omega)$, (6.93) and Young's inequality to compute

$$\begin{aligned} &\Big|\int_\Omega\nabla\partial_t\phi_\lambda\cdot\nabla(\nabla\phi_\lambda\cdot\mathbf{v}_\lambda)\,\mathrm{d}x\Big|\\ &\quad\le\Big(\|\phi_\lambda\|_{W^{2,6}(\Omega)}\|\mathbf{v}_\lambda\|_{\mathbf{L}^3(\Omega)}+\|\nabla\phi_\lambda\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{v}_\lambda\|_{\mathbf{H}^1(\Omega)}\Big)\|\nabla\partial_t\phi_\lambda\|_{\mathbf{L}^2(\Omega)}\\ &\quad\le\frac16\|\nabla\partial_t\phi_\lambda\|_{\mathbf{L}^2(\Omega)}^2+C\|\phi_\lambda\|_{W^{2,6}(\Omega)}^2\|\mathbf{v}_\lambda\|_{\mathbf{H}^1(\Omega)}^2\\ &\quad\le\frac16\|\nabla\partial_t\phi_\lambda\|_{\mathbf{L}^2(\Omega)}^2+C\|\mathbf{v}_\lambda\|_{\mathbf{H}^1(\Omega)}^2+C\|\mathbf{v}_\lambda\|_{\mathbf{H}^1(\Omega)}^2\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}^2 \end{aligned}\tag{6.108}$$

a.e. on $(\hat{\varsigma},\infty)$. In a similar manner, exploiting again (6.93) and the embedding $W^{1,6}(\Gamma)\hookrightarrow L^\infty(\Gamma)$, we have

$$\begin{aligned} &\Big|\int_\Gamma\nabla_\Gamma\partial_t\psi_\lambda\cdot\nabla_\Gamma(\nabla_\Gamma\psi_\lambda\cdot\mathbf{w}_\lambda)\,\mathrm{d}\Gamma\Big|\\ &\quad\le\frac16\|\nabla_\Gamma\partial_t\psi_\lambda\|_{\mathbf{L}^2(\Gamma)}^2+C\|\mathbf{w}_\lambda\|_{\mathbf{H}^1(\Gamma)}^2+C\|\mathbf{w}_\lambda\|_{\mathbf{H}^1(\Gamma)}^2\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}^2 \end{aligned}\tag{6.109}$$

a.e. on $(\hat{\varsigma},\infty)$. Then, for the last term, we use the trace theorem in combination with (6.93) to infer that

$$\begin{aligned} &\Big|\chi(K)\int_\Gamma(\alpha\partial_t\psi_\lambda-\partial_t\phi_\lambda)(\alpha\nabla_\Gamma\psi_\lambda\cdot\mathbf{w}_\lambda-\nabla\phi_\lambda\cdot\mathbf{v}_\lambda)\,\mathrm{d}\Gamma\Big|\\ &\le\chi(K)\Big(\|\nabla_\Gamma\psi_\lambda\|_{\mathbf{L}^4(\Gamma)}\|\mathbf{w}_\lambda\|_{\mathbf{L}^4(\Gamma)}+\|\nabla\phi_\lambda\|_{\mathbf{L}^4(\Gamma)}\|\mathbf{v}_\lambda\|_{\mathbf{L}^4(\Gamma)}\Big)\|\alpha\partial_t\psi_\lambda-\partial_t\phi_\lambda\|_{L^2(\Gamma)}\\ &\le\frac16\chi(K)\|\alpha\partial_t\psi_\lambda-\partial_t\phi_\lambda\|_{L^2(\Gamma)}^2+C\chi(K)\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\phi_\lambda,\psi_\lambda)\|_{\mathcal{H}^2}^2\\ &\le\frac16\chi(K)\|\alpha\partial_t\psi_\lambda-\partial_t\phi_\lambda\|_{L^2(\Gamma)}^2+C\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2+C\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}^2 \end{aligned}\tag{6.110}$$

a.e. on $(\hat{\varsigma},\infty)$. Thus, we combine (6.108)-(6.110) to conclude with (6.107) that

$$\begin{aligned} &\big|-\big((\partial_t\phi_\lambda,\partial_t\psi_\lambda),(\nabla\phi_\lambda\cdot\mathbf{v}_\lambda,\nabla_\Gamma\psi_\lambda\cdot\mathbf{w}_\lambda)\big)_{K,\alpha}\big|\\ &\le\frac16\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{K,\alpha}^2+C\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2+C\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\mu_\lambda,\theta_\lambda)\|_{L,\beta}^2 \end{aligned}\tag{6.111}$$

a.e. on $(\hat{\varsigma},\infty)$. Furthermore, by means of (6.93), the second and third term on the right-hand side of (6.105) can be estimated as

$$\begin{aligned} &\Big|\int_\Omega F_\lambda'(\phi_\lambda)\,\nabla\partial_t\phi_\lambda\cdot\mathbf{v}_\lambda\,\mathrm{d}x+\int_\Gamma G_\lambda'(\psi_\lambda)\,\nabla_\Gamma\partial_t\psi_\lambda\cdot\mathbf{w}_\lambda\,\mathrm{d}\Gamma\Big|\\ &\quad\le\|(F_\lambda'(\phi_\lambda)G_\lambda'(\psi_\lambda))\|_{\mathcal{L}^6}\|(\nabla\partial_t\phi_\lambda,\nabla_\Gamma\partial_t\psi_\lambda)\|_{\boldsymbol{\mathcal{L}}^2}\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{L}}^3}\\ &\quad\le\frac16\|(\partial_t\phi_\lambda,\partial_t\psi_\lambda)\|_{K,\alpha}^2+C\|(F_\lambda'(\phi_\lambda),G_\lambda'(\psi_\lambda))\|_{\mathcal{L}^6}^2\|(\mathbf{v}_\lambda,\mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2 \end{aligned}\tag{6.112}$$

$$\leq \frac{1}{6}\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|_{K,\alpha}^2 + C\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2 + C\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}^2$$

a.e. on $(\hat{\varsigma}, \infty)$. Finally, collecting the estimates (6.106), (6.111) and (6.112), we end up with the differential inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}^2 + \frac{1}{2}\|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|_{K,\alpha}^2 \\ &\quad \leq C\Big(\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}^2\Big) + C\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}^2\end{aligned} \tag{6.113}$$

a.e. on $(\hat{\varsigma}, \infty)$. The aim is again to apply for uniform Gronwall inequality from Lemma 2.2.7. To this end, fix $r \in (0, 1]$. Then, by (6.88) and (6.92), we clearly have the estimates

$$\begin{aligned}&\sup_{t\geq 0}\int_t^{t+r} C\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\mathcal{H}^1}^2 \,\mathrm{d}s \leq C\int_0^\infty \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{H}}^1}^2 \,\mathrm{d}s =: a_1, \\ &\sup_{t\geq 0}\int_t^{t+r} C\Big(\|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{H}}^1}^2 + \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}^2\Big) \,\mathrm{d}s \leq C\left(1 + \int_0^\infty \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{H}}^1}^2 \,\mathrm{d}s\right) =: a_2, \\ &\sup_{t\geq 0}\int_t^{t+r} \|(\mu_\lambda, \theta_\lambda)\|_{L,\beta}^2 \,\mathrm{d}s \leq \int_0^\infty \|(\mathbf{v}_\lambda, \mathbf{w}_\lambda)\|_{\boldsymbol{\mathcal{L}}^2}^2 \,\mathrm{d}s \leq \int_0^\infty \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 \,\mathrm{d}s =: a_3.\end{aligned}$$

This allows us to apply Lemma 2.2.7 with $t_0 = \hat{\varsigma}$, which yields

$$\|(\mu_\lambda(t+r), \theta_\lambda(t+r))\|_{L,\beta}^2 \leq \Big(\frac{a_3}{\hat{\varsigma}} + a_2\Big)\mathbf{e}^{a_1} \qquad \text{for all } t \geq \hat{\varsigma}.$$

Now, let $\varsigma > 0$ be arbitrary and take $s \geq \varsigma$. Choosing $\hat{\varsigma} := \frac{\varsigma}{2}$, $r := \min\{1, \frac{\varsigma}{2}\}$ and $t = s - r$, we infer

$$\|(\mu_\lambda(s), \theta_\lambda(s))\|_{L,\beta}^2 \leq \Big(\frac{2a_3}{\varsigma} + a_2\Big)\mathbf{e}^{a_1} \qquad \text{for all } s \geq \varsigma. \tag{6.114}$$

Then integrating (6.113) in time from t to $t+1$ for $t \geq \varsigma$, yields under consideration of (6.114) that

$$\sup_{t\geq\varsigma}\int_t^{t+1} \|(\partial_t\phi_\lambda, \partial_t\psi_\lambda)\|_{K,\alpha}^2 \,\mathrm{d}s \leq (2a_1 + 1)\Big(\frac{2a_3}{\varsigma} + a_2\Big)\mathbf{e}^{a_1} + 2a_1 + 2a_3.$$

Now, letting $\lambda \to 0$, we conclude that

$$t \mapsto \|(\mu(t), \theta(t))\|_{L,\beta} \in L^\infty(\varsigma, \infty), \tag{6.115}$$

$$t \mapsto \|(\partial_t\phi(t), \partial_t\psi(t))\|_{K,\alpha} \in L^2_{\mathrm{uloc}}([\varsigma, \infty)) \tag{6.116}$$

for any $\varsigma > 0$. The, as previously seen, we obtain from a comparison argument in ((6.1a), (6.1c)) that

$$\|(\partial_t\phi, \partial_t\psi)\|_{(\mathcal{H}^1_{L,\beta})'} \leq C\Big(\|(\mu, \theta)\|_{L,\beta} + \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\Big)$$

a.e. on (ς, ∞). In view of the regularities (6.115) and $(\mathbf{v}, \mathbf{w}) \in L^\infty(0, \infty; \boldsymbol{\mathcal{L}}^2)$, the latter entails that $(\partial_t\phi, \partial_t\psi) \in L^\infty(\varsigma, \infty; (\mathcal{H}^1_{L,\beta})')$ for any $\varsigma > 0$. The remaining regularity properties can be verified as in the first case. This finishes the proof of Theorem 6.2.7.

□

As a consequence of Theorem 6.2.7, we can improve the energy inequality (6.13) to be an energy equality, see also [70, 129, 132]. Observe that the energy equality was already proven in Theorem 6.2.3 assuming that $L \in (0, \infty]$. However, assuming constant mobilities and additional regularity on the velocity fields, it can be proven to hold also in the case $L = 0$.

PROPOSITION 6.5.1. *Suppose the assumptions from Theorem 6.2.7 hold, and consider unique global weak solution* $(\phi, \psi, \mu, \theta)$. *Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{free}}(\phi(t),\psi(t)) + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta\theta-\mu)^2\,\mathrm{d}\Gamma \\ - \int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma = 0\end{aligned} \tag{6.117}$$

for a.e. $t > 0$, *and*

$$\begin{aligned}&E_{\text{free}}(\phi(t),\psi(t)) + \int_0^t\int_\Omega |\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma |\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &\quad + \chi(L)\int_0^t\int_\Gamma (\beta\theta-\mu)^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &\quad - \int_0^t\int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma\,\mathrm{d}s \\ &= E_{\text{free}}(\phi_0,\psi_0)\end{aligned} \tag{6.118}$$

for all $t \geq 0$.

PROOF. We have already seen in Step 9 of the proof of Theorem 6.2.3 that $[0,\infty) \ni t \mapsto \widetilde{E}_{\text{free}}(\phi(t),\psi(t))$ is absolutely continuous, where $\widetilde{E}_{\text{free}}$ denotes the convex part of the free energy E_{free} as defined in Section 4.3. Moreover, owing to [150, Lemma 4.1] in combination with Proposition 4.3.2, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_{\text{free}}(\phi,\psi) &= \langle(\partial_t\phi,\partial_t\psi),(-\Delta\phi+F_1'(\phi),-\Delta_\Gamma\psi+G_1'(\psi)+\alpha\partial_{\mathbf{n}}\phi)\rangle_{\mathcal{H}^1_{L,\beta}} \\ &= \langle(\partial_t\phi,\partial_t\psi),(\mu-F_2'(\phi),\theta-G_2'(\psi))\rangle_{\mathcal{H}^1_{L,\beta}} \\ &= \int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma \\ &\quad - \int_\Omega F_2'(\phi)\partial_t\phi\,\mathrm{d}x - \int_\Gamma G_2'(\psi)\partial_t\psi\,\mathrm{d}\Gamma \\ &\quad - \int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma\end{aligned}$$

a.e. on (ς,∞) for any $\varsigma > 0$. Consequently,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\text{free}}(\phi,\psi) &= \int_\Omega |\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma |\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma \\ &\quad - \int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma\end{aligned} \tag{6.119}$$

a.e. on (ς,∞). Since $\varsigma > 0$ was arbitrary, we readily obtain (6.117). Then, integrating (6.119) over (s,t) for $s,t > \varsigma$ with $s \leq t$, we infer

$$\begin{aligned}&E_{\text{free}}(\phi(t),\psi(t)) - E_{\text{free}}(\phi(s),\psi(s)) \\ &= \int_s^t\int_\Omega |\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_\Gamma |\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma\,\mathrm{d}r + \chi(L)\int_s^t\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma\,\mathrm{d}r \\ &\quad - \int_s^t\int_\Omega \phi\mathbf{v}\cdot\nabla\mu\,\mathrm{d}x\,\mathrm{d}r - \int_s^t\int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta\,\mathrm{d}\Gamma\,\mathrm{d}r.\end{aligned} \tag{6.120}$$

It follows from (6.13) that $\limsup_{s\to 0} E_{\text{free}}(\phi(s),\psi(s)) \leq E_{\text{free}}(\phi(0),\psi(0))$. On the other hand, by weak lower semicontinuity of norms and Lebesgue's dominated convergence theorem, we have $\liminf_{s\to 0} E_{\text{free}}(\phi(s),\psi(s)) \geq E_{\text{free}}(\phi(0),\psi(0))$. As a result, it holds that $\lim_{s\to 0} E_{\text{free}}(\phi(s),\psi(s)) = E_{\text{free}}(\phi(0),\psi(0))$. This allows us to pass to the limit $s \to 0$ in (6.120) and conclude the energy identity (6.118). $\square$

Finally, we can prove the separation property in the two-dimensional setting.

Proof of Theorem 6.2.10. **The case with (A5.1).** Let $(\phi, \psi, \mu, \theta)$ be the unique weak solution to (6.1), and let $\varsigma > 0$. First, recall that from **(A5.1)** there exist constants $C_\sharp > 0$ and $\gamma_\sharp \in [1, 2)$ such that

$$F_1''(s) \leq C_\sharp e^{C_\sharp |F_1'(s)|^{\gamma_\sharp}} \quad \text{for all } s \in (-1, 1). \tag{6.121}$$

Next, since $d = 2$, we know that for all $u \in H^1(\Omega)$ and $r \in [2, \infty)$ it holds

$$\|u\|_{L^r(\Omega)} \leq C_\Omega \sqrt{r} \|u\|_{H^1(\Omega)}$$

for some constant C_Ω depending only on Ω (see, e.g., [175, p. 479]). Consequently, we infer from Proposition 4.3.5 with the Lipschitz continuity of F_2' and G_2' that

$$\begin{aligned}\|F_1'(\phi)\|_{L^r(\Omega)} &\leq C\sqrt{r}\Big(\|(\mu, \theta)\|_{L^\infty(\varsigma,\infty;\mathcal{H}^1)} + \|(\phi, \psi)\|_{L^\infty(\varsigma,\infty;\mathcal{H}^1)}\Big)\\ &\leq C\sqrt{r}\end{aligned} \tag{6.122}$$

a.e. on (ς, ∞) for all $r \in [2, \infty)$. This inequality allows us to argue similarly to the proof of [73, Theorem 3.1], and we deduce that

$$\int_\Omega |e^{C_\sharp |F_1'(\phi)|^{\gamma_\sharp}}|^m \, \mathrm{d}x \leq C(m) \tag{6.123}$$

for all $m \in \mathbb{N}$ and some constant $C(m)$ depending on m and the constant C from (6.122). In view of (6.121), inequality (6.123) with $m = 6$ immediately entails that

$$\|F_1''(\phi)\|_{L^\infty(\varsigma,\infty;L^6(\Omega))} \leq C. \tag{6.124}$$

Thus, utilizing the chain rule as well as the continuous embedding $H^2(\Omega) \hookrightarrow W^{1,6}(\Omega)$ we obtain

$$\begin{aligned}\|\nabla F_1'(\phi)\|_{L^\infty(\varsigma,\infty;\mathbf{L}^3(\Omega))} &\leq \|F_1''(\phi)\|_{L^\infty(\varsigma,\infty;L^6(\Omega))}\|\nabla\phi\|_{L^\infty(\varsigma,\infty;\mathbf{L}^6(\Omega))}\\ &\leq C\|\phi\|_{L^\infty(\varsigma,\infty;H^2(\Omega))} \leq C.\end{aligned} \tag{6.125}$$

Finally, using (6.122) written for $r = 3$, and exploiting the continuous embedding $W^{1,3}(\Omega) \hookrightarrow C(\overline{\Omega})$, we infer from (6.125) that

$$\|F_1'(\phi(t))\|_{L^\infty(\Omega)} \leq \|F_1'(\phi)\|_{L^\infty(\varsigma,\infty;W^{1,3}(\Omega))} \leq C =: C_\star \tag{6.126}$$

for a.e. $t \in (\varsigma, \infty)$. Since Theorem 6.2.7 implies that $\phi \in C(\overline{Q_\varsigma})$, the inequality (6.126) holds true even for every $t \in (\varsigma, \infty)$. Lastly, taking $\delta := 1 - (F_1')^{-1}(C_\star)$, leads to the desired conclusion. The separation property for ψ can be established similarly by relying on the estimate (4.33) from Remark 4.3.6, see also [70].

The case with (A5.2). In this case, we exploit the dissipative structure of (6.1) in combination with a De Giorgi-type iteration scheme, following the recent approach developed in [81]. This method has already been successfully adapted to dynamic boundary conditions for certain parameter choices of K and L, see, for instance, [129, 132]. The extension to all $K, L \in [0, \infty]$ is straightforward, and we omit the details here. □

Chapter 7

Uniqueness of weak solutions and existence of strong solutions for non-degenerate mobility functions and singular potentials

The results presented in this chapter are largely based on the following works:

[163] J. Stange, *Well-posedness and long-time behavior of a bulk-surface Cahn–Hilliard mdoel with non-degenerate mobility*, Nonlinear Analysis, 268 (2026), p. 114060.

[164] J. Stange, *Global well-posedness of strong solutions to a bulk-surface Navier–Stokes–Cahn–Hilliard model with non-degenerate mobilities in two dimensions*, Preprint: arXiv:2511.06847, 2025.

7.1 Introduction

In this chapter, we continue the analysis of weak solutions to system (6.1). In Chapter 6, we established the existence of weak solution and, under the assumption of constant mobility functions, proved the uniqueness and propagation of regularity. The restriction on the mobilities is standard and appears frequently in the literature, including in the non-convective setting and even for Cahn–Hilliard systems with homogeneous Neumann boundary conditions, see, for instance, [8, 21, 70, 129–131].

However, this assumption represents a significant limitation, since in many physical applications the diffusion intensity may vary spatially and is not expected to be uniform throughout the domain. In this chapter, we extend the aforementioned results to the case of non-degenerate mobility functions in the two-dimensional setting. This restriction appears to be necessary, as the analysis relies on the special Ladyzhenskaya inequality for $d = 2$.

For the classical Cahn–Hilliard equation with homogeneous Neumann boundary conditions, the recent work [53] establishes the uniqueness and the propagation of regularity of weak solutions for non-degenerate mobility functions in two dimensions, and thus improved the state of the art dating back to the work [21]. We also mention the recent work [62], where the authors established a weak-strong uniqueness principle for the Cahn–Hilliard equation on an evolving surface.

Building on the approach developed in [53], we extend the uniqueness theory to the convective Cahn–Hilliard equation with dynamic boundary conditions (6.1). To the best of our knowledge, this is the first result addressing uniqueness and propagation of regularity in the setting of dynamic boundary conditions with non-degenerate mobilities.

Our uniqueness proof relies on two main-ingredients: a novel well-posedness and regularity theory for a bulk-surface elliptic system with non-constant coefficients eveloped in Section 4.4, as well as the higher regularity properties for weak solutions established in Theorem 6.2.3, in particular (6.15). In contrast to [53], additional boundary terms arising from the intricate nature of the dynamic boundary conditions must be handled, which requires further technical work.

The uniqueness proof, in particular, the key differential inequality established therein, allows us to show the existence of a unique strong solution for a wide range of velocity fields. As a consequence, we also obtain the instantaneous separation property.

7.2 Additional preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a non-empty, bounded domain with C^2-boundary. Then we have the following interpolation inequality.

Lemma 7.2.1. *There exists a constant $C > 0$, such that for all $2 \leq r < \infty$ it holds that*

$$\|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^r} \leq C\sqrt{r}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{L}^2}^{\frac{2}{r}}\|(\zeta, \zeta_\Gamma)\|_{\mathcal{H}^1}^{\frac{r-2}{r}} \qquad \text{for all } (\zeta, \zeta_\Gamma) \in \mathcal{H}^1. \tag{7.1}$$

Since Γ is a 1-dimensional submanifold of $\mathbb{R}$, we can readily obtain (7.1) as an extension of the following Gagliardo–Nirenberg–Sobolev inequality (see [140, Proposition and Remark 1]): there exists a constant $C > 0$, such that for all $2 \leq r < \infty$ it holds that

$$\|\zeta\|_{L^r(\Omega)} \leq C\sqrt{r}\|\zeta\|_{L^2(\Omega)}^{\frac{2}{r}}\|\zeta\|_{H^1(\Omega)}^{\frac{r-2}{r}} \qquad \text{for all } \zeta \in H^1(\Omega).$$

Then, based on (7.1), we can provide some further estimates of the solution operator $\mathcal{S}_{L,\beta}^{\phi,\psi}$ which was introduced in Section 4.4 in two spatial dimensions.

To this end, consider $(f, f_\Gamma) \in \mathcal{V}_{L,\beta}^{-1} \cap \mathcal{L}^2$ and $(\phi, \psi) \in \mathcal{W}^{2,4}$ with $|\phi| \leq 1$ a.e. in Ω and $|\psi| \leq 1$ a.e. on Γ. We further assume that the mobility functions m_Ω, m_Γ satisfy the assumption **(A3)**.

Then, in view of (4.57) and (7.1), we readily find that

$$\begin{aligned}
&\|\mathcal{S}_{L,\beta}^{\phi,\psi}(f, f_\Gamma)\|_{\mathcal{H}^2} \\
&\quad \leq C\Big(\|(\nabla\phi \cdot \nabla\mathcal{S}_{L,\beta}^{\phi,\psi,\Omega}(f, f_\Gamma), \nabla_\Gamma\psi \cdot \nabla_\Gamma\mathcal{S}_{L,\beta}^{\phi,\psi,\Gamma}(f, f_\Gamma))\|_{\mathcal{L}^2} + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big) \\
&\quad \leq C\Big(\|(\nabla\phi, \nabla_\Gamma\psi)\|_{\mathcal{L}^4}\|(\nabla\mathcal{S}_{L,\beta}^{\phi,\psi,\Omega}(f, f_\Gamma), \nabla_\Gamma\mathcal{S}_{L,\beta}^{\phi,\psi,\Gamma}(f, f_\Gamma))\|_{\mathcal{L}^4} + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big) \\
&\quad \leq C\Big(\|(\nabla\phi, \nabla_\Gamma\psi)\|_{\mathcal{L}^2}^{\frac{1}{2}}\|(\phi, \psi)\|_{\mathcal{H}^2}^{\frac{1}{2}}\|\mathcal{S}_{L,\beta}^{\phi,\psi}(f, f_\Gamma)\|_{L,\beta}^{\frac{1}{2}}\|\mathcal{S}_{L,\beta}^{\phi,\psi}(f, f_\Gamma)\|_{\mathcal{H}^2}^{\frac{1}{2}} \\
&\qquad + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big).
\end{aligned}$$

Consequently, we infer from Young's inequality that

$$\begin{aligned}
&\|\mathcal{S}_{L,\beta}^{\phi,\psi}(f, f_\Gamma)\|_{\mathcal{H}^2} \\
&\quad \leq C\Big(\|(\nabla\phi, \nabla_\Gamma\psi)\|_{\mathcal{L}^2}\|(\phi, \psi)\|_{\mathcal{H}^2}\|\mathcal{S}_{L,\beta}^{\phi,\psi}(f, f_\Gamma)\|_{L,\beta} + \|(f, f_\Gamma)\|_{\mathcal{L}^2}\Big).
\end{aligned} \tag{7.2}$$

Furthermore, if $(f, g) \in \mathcal{V}_{L,\beta}^{-1} \cap \mathcal{L}^4$, we can choose $p = 4$ in (4.70), and exploiting (7.1), we find

$$\|\mathcal{S}_{L,\beta}^{\phi,\psi}(f, f_\Gamma)\|_{\mathcal{W}^{2,4}}$$

$$
\begin{aligned}
&\leq C\Big(\|(\nabla\phi,\nabla_\Gamma\psi)\|_{\mathcal{L}^8}\|(\nabla\mathcal{S}^{\phi,\psi,\Omega}_{L,\beta}(f,f_\Gamma),\nabla_\Gamma\mathcal{S}^{\phi,\psi,\Gamma}_{L,\beta}(f,f_\Gamma))\|_{\mathcal{L}^8}+\|(f,f_\Gamma)\|_{\mathcal{L}^4}\Big) \qquad (7.3)\\
&\leq C\Big(\|(\nabla\phi,\nabla_\Gamma\psi)\|^{\frac14}_{\mathcal{L}^2}\|(\phi,\psi)\|^{\frac34}_{\mathcal{H}^2}\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma)\|^{\frac14}_{L,\beta}\|\mathcal{S}^{\phi,\psi}_{L,\beta}(f,f_\Gamma)\|^{\frac34}_{\mathcal{H}^2}+\|(f,f_\Gamma)\|_{\mathcal{L}^4}\Big).
\end{aligned}
$$

Lastly, we present a chain rule, which is a generalization of Proposition 5.2.2 to the case of non-degenerate mobility functions.

PROPOSITION 7.2.2. *Let $\Omega\subset\mathbb{R}^d$, $d=2,3$, be an open bounded domain with C^3-boundary, let $a,b\in\mathbb{R}$, $a<b$, and $(\phi,\psi)\in H^1(a,b;\mathcal{L}^3)\cap L^\infty(a,b;\mathcal{W}^{2,4})$ with $|\phi|\leq 1$ a.e. in $\Omega\times(a,b)$ and $|\psi|\leq 1$ a.e. on $\Gamma\times(a,b)$. Furthermore, let $m_\Omega,m_\Gamma\in C^2([-1,1])$ satisfy (7.17). Consider a pair (u,v) such that*

$$
\begin{aligned}
(\partial_t u,\partial_t v)&\in L^2(a,b;(\mathcal{H}^1_{K,\alpha})'),\\
(u,v)&\in L^\infty(a,b;\mathcal{H}^1_{L,\beta})\cap L^2(a,b;\mathcal{H}^3).
\end{aligned}
$$

Moreover, we assume that

$$
\begin{cases}
Lm_\Omega(\phi)\partial_{\mathbf{n}}u=\beta v-u & \text{if } L\in[0,\infty),\\
m_\Omega(\phi)\partial_{\mathbf{n}}u=0 & \text{if } L=\infty
\end{cases}
\qquad \text{a.e. on } \Gamma\times(a,b),
$$

and

$$
(\mathrm{div}(m_\Omega(\phi)\nabla u),\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v)-\beta m_\Omega(\phi)\partial_{\mathbf{n}}u)\in L^2(a,b;\mathcal{H}^1_{K,\alpha}).
$$

Then, the continuity property $(u,v)\in C([a,b];\mathcal{H}^1_{L,\beta})$ holds, the mapping

$$
t\mapsto\int_\Omega m_\Omega(\phi(t))|\nabla u(t)|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma(\psi(t))|\nabla_\Gamma v(t)|^2\,\mathrm{d}\Gamma+\chi(L)\int_\Gamma(\beta v(t)-u(t))^2\,\mathrm{d}\Gamma
$$

is absolutely continuous on $[a,b]$, and the chain rule formula

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac12\Big(\int_\Omega m_\Omega(\phi)|\nabla u|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma v|^2\,\mathrm{d}\Gamma+\chi(L)\int_\Gamma(\beta v-u)^2\,\mathrm{d}\Gamma\Big)\\
&\quad=\langle(\partial_t u,\partial_t v),(-\mathrm{div}(m_\Omega(\phi)\nabla u),-\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v)+\beta m_\Omega(\phi)\partial_{\mathbf{n}}u)\rangle_{\mathcal{H}^1_{K,\alpha}} \qquad (7.4)\\
&\qquad+\int_\Omega m_\Omega'(\phi)\partial_t\phi|\nabla u|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma'(\psi)\partial_t\psi|\nabla_\Gamma v|^2\,\mathrm{d}\Gamma
\end{aligned}
$$

holds a.e. on $[a,b]$.

PROOF. We adapt the proof of Proposition 5.2.2. In the present setting, additional care is required to handle the terms arising from the non-degenerate mobility, which necessitates further technical considerations. First, we note that in view of the regularity assumptions on (u,v), we have $(u,v)\in C([a,b];\mathcal{L}^2)$, and can thus extend the functions u and v onto $[2a-b,a]$ by reflection for all $t<a$.

Let $\pi\in C_c^\infty(\mathbb{R})$ be non-negative with $\mathrm{supp}\,\pi\subset(0,1)$ and $\|\pi\|_{L^1(\mathbb{R})}=1$. For any $k\in\mathbb{N}$, we set

$$
\pi_k(s):=k\pi(ks) \qquad \text{for all } s\in\mathbb{R}.
$$

Then, for any Banach space X and any function $f\in L^2(a-1,b;X)$, we define

$$
f_k(t):=(\pi_k*f)(t)=\int_{t-\frac1k}^t\pi_k(t-s)f(s)\,\mathrm{d}s \qquad \text{for all } t\in[a,b] \quad \text{and} \quad k\in\mathbb{N}.
$$

By this construction, we have $f_k \in C^\infty([a,b];X)$ with $f_k \to f$ strongly in $L^2(a,b;X)$ as $k \to \infty$.

Now, for any $k \in \mathbb{N}$, we use $X = H^3(\Omega)$ to define u_k and $X = H^3(\Gamma)$ to define v_k as described above. By this construction, it holds that $\partial_t u_k = (\partial_t u)_k$ and $\partial_t \nabla u_k = \nabla \partial_t u_k$ a.e. in $\Omega \times (a,b)$ as well as $\partial_t v_k = (\partial_t v)_k$ and $\partial_t \nabla_\Gamma v_k = \nabla_\Gamma \partial_t v_k$ a.e. on $\Gamma \times (a,b)$ for all $k \in \mathbb{N}$. Moreover, we have

$$
\begin{aligned}
u_k &\to u && \text{strongly in } L^2(a,b;H^3(\Omega)), && (7.5)\\
v_k &\to v && \text{strongly in } L^2(a,b;H^3(\Gamma)), && (7.6)\\
(u_k,v_k) &\to (u,v) && \text{strongly in } L^2(a,b;\mathcal{H}^1_{L,\beta}), && (7.7)\\
(\partial_t u_k,\partial_t v_k) &\to (\partial_t u,\partial_t v) && \text{strongly in } L^2(a,b;(\mathcal{H}^1_{K,\alpha})') && (7.8)
\end{aligned}
$$

as $k \to \infty$. Furthermore, we readily see that

$$
\begin{aligned}
\|u_k\|_{L^\infty(a,b;H^1(\Omega))} &\le \|u\|_{L^\infty(a,b;H^1(\Omega))},\\
\|v_k\|_{L^\infty(a,b;H^1(\Gamma))} &\le \|v\|_{L^\infty(a,b;H^1(\Gamma))}
\end{aligned}
\tag{7.9}
$$

for all $k \in \mathbb{N}$. In the following, we will denote with C generic positive constants independent of $k \in \mathbb{N}$, which may change their value from line to line. Now, for any $k \in \mathbb{N}$, we derive the identity

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega m_\Omega(\phi)|\nabla u_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma v_k|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta v_k - u_k)^2\,\mathrm{d}\Gamma\Big)\\
&\quad = \big\langle(\partial_t u_k,\partial_t v_k),(-\mathrm{div}(m_\Omega(\phi)\nabla u_k), -\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v_k) + \beta m_\Omega(\phi)\partial_\mathbf{n} u_k)\big\rangle_{\mathcal{H}^1_{K,\alpha}}\\
&\qquad + \int_\Omega m_\Omega'(\phi)\partial_t\phi|\nabla u_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma'(\psi)\partial_t\psi|\nabla_\Gamma v_k|^2\,\mathrm{d}\Gamma
\end{aligned}
\tag{7.10}
$$

a.e. on $[a,b]$ by differentiating under the integral sign and applying integration by parts. Similarly, for $j,k \in \mathbb{N}$, we calculate

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega m_\Omega(\phi)|\nabla(u_j-u_k)|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma(v_j-v_k)|^2\,\mathrm{d}\Gamma\\
&\qquad + \chi(L)\int_\Gamma\big(\beta(v_j-v_k)-(u_j-u_k)\big)^2\,\mathrm{d}\Gamma\Big)\\
&= \big\langle(\partial_t(u_j-u_k),\partial_t(v_j-v_k)),\\
&\qquad(-\mathrm{div}(m_\Omega(\phi)\nabla(u_j-u_k)), -\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma(v_j-v_k)) + \beta m_\Omega(\phi)\partial_\mathbf{n}(u_j-u_k))\big\rangle_{\mathcal{H}^1_{K,\alpha}}\\
&\qquad + \int_\Omega m_\Omega'(\phi)\partial_t\phi|\nabla(u_j-u_k)|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma'(\psi)\partial_t\psi|\nabla_\Gamma(v_j-v_k)|^2\,\mathrm{d}\Gamma \qquad (7.11)\\
&\le \|(\partial_t(u_j-u_k),\partial_t(v_j-v_k))\|_{(\mathcal{H}^1_{K,\alpha})'}\\
&\qquad\times\|(\mathrm{div}(m_\Omega(\phi)\nabla(u_j-u_k)),\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma(v_j-v_k)) - \beta m_\Omega(\phi)\partial_\mathbf{n}(u_j-u_k))\|_{\mathcal{H}^1}\\
&\qquad + C\|(\partial_t\phi,\partial_t\psi)\|_{\mathcal{L}^3}\|(\nabla(u_j-u_k),\nabla_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^6}\|(\nabla(u_j-u_k),\nabla_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^2}
\end{aligned}
$$

a.e. on $]a,b]$. To estimate the right-hand side suitably, we use standard computations together with the Sobolev inequality to find that

$$
\begin{aligned}
&\|(\mathrm{div}(m_\Omega(\phi)\nabla(u_j-u_k)),\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma(v_j-v_k)) - \beta m_\Omega(\phi)\partial_\mathbf{n}(u_j-u_k))\|_{\mathcal{H}^1}\\
&\quad\le \|(\mathrm{div}(m_\Omega(\phi)\nabla(u_j-u_k)),\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma(v_j-v_k)) - \beta m_\Omega(\phi)\partial_\mathbf{n}(u_j-u_k))\|_{\mathcal{L}^2}\\
&\qquad + \|(\nabla\big(\mathrm{div}(m_\Omega(\phi)\nabla(u_j-u_k))\big),\\
&\qquad\qquad \nabla_\Gamma\big(\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma(v_j-v_k)) - \beta m_\Omega(\phi)\partial_\mathbf{n}(u_j-u_k)\big))\|_{\boldsymbol{\mathcal{H}}^1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|(m_\Omega(\phi)\Delta(u_j-u_k), m_\Gamma(\psi)\Delta_\Gamma(v_j-v_k))\|_{\mathcal{L}^2} \\
&\quad + \|(m_\Omega'(\phi)\nabla\phi\cdot\nabla(u_j-u_k), m_\Gamma'(\psi)\nabla_\Gamma\psi\cdot\nabla_\Gamma(v_j-v_k))\|_{\mathcal{L}^2} \\
&\quad + \|(m_\Omega(\phi)\nabla\Delta(u_j-u_k), m_\Gamma(\psi)\nabla_\Gamma\Delta_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad + \|(m_\Omega'(\phi)\Delta(u_j-u_k)\nabla\phi, m_\Gamma'(\psi)\Delta_\Gamma(v_j-v_k)\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad + \|(m_\Omega''(\phi)(\nabla\phi\cdot\nabla(u_j-u_k))\nabla\phi, m_\Gamma''(\psi)(\nabla_\Gamma\psi\cdot\nabla_\Gamma(v_j-v_k))\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad + \|(m_\Omega'(\phi)D^2\phi\nabla(u_j-u_k), m_\Gamma'(\psi)D_\Gamma^2\psi\nabla_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad + \|(m_\Omega'(\phi)D^2(u_j-u_k)\nabla\phi, m_\Gamma'(\psi)D_\Gamma^2(v_j-v_k)\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad + \|\beta m_\Omega(\phi)\partial_{\mathbf{n}}(u_j-u_k)\|_{L^2(\Gamma)} + \|\beta m_\Omega'(\phi)\nabla_\Gamma\phi\partial_{\mathbf{n}}(u_j-u_k)\|_{\mathbf{L}^2(\Gamma)} \\
&\quad + \|\beta m_\Omega(\phi)\nabla_\Gamma\partial_{\mathbf{n}}(u_j-u_k)\|_{\mathbf{L}^2(\Gamma)} \\
&\leq C\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^2} + C\|(\nabla\phi,\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^1} \\
&\quad + C\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^3} + C\|(\nabla\phi,\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^2} \\
&\quad + C\|(\nabla\phi,\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^\infty}^2\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^1} \\
&\quad + C\|(\phi,\psi)\|_{\mathcal{W}^{2,6}}\|(\nabla(u_j-u_k), \nabla_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^3} \\
&\quad + C\|(\nabla\phi,\nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^2} + C\|u_j-u_k\|_{H^2(\Omega)} \\
&\quad + C\|\phi\|_{W^{2,4}(\Omega)}\|(u_j-u_k)\|_{\mathcal{H}^2} + C\|u_j-u_k\|_{H^3(\Omega)} \\
&\leq C\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^3}
\end{aligned}
$$

a.e. on $[a,b]$. Here, we have additionally used the embedding $H^3(\Omega)\hookrightarrow H^2(\Gamma)$ yielding

$$\|\partial_{\mathbf{n}}(u_j-u_k)\|_{H^1(\Gamma)} \leq \|u_j-u_k\|_{H^2(\Gamma)} \leq C\|u_j-u_k\|_{H^3(\Omega)}$$

a.e. on $[a,b]$, whereas the embedding $W^{2,4}(\Omega)\hookrightarrow W^{1,\infty}(\Gamma)$ shows that

$$\|\nabla_\Gamma\phi\|_{L^\infty(\Gamma)} \leq \|\phi\|_{W^{1,\infty}(\Gamma)} \leq C\|\phi\|_{W^{2,4}(\Omega)}$$

a.e. on $[a,b]$. Next, employing (7.9), we have

$$
\begin{aligned}
&\|(\nabla(u_j-u_k), \nabla_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^6}\|(\nabla(u_j-u_k), \nabla_\Gamma(v_j-v_k))\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad \leq C\|(u,v)\|_{L^\infty(I;\mathcal{H}^1)}\|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^3}
\end{aligned}
$$

a.e. on $[a,b]$. Collecting our previous estimates, we conclude from (7.11) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega m_\Omega(\phi)|\nabla(u_j-u_k)|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma(v_j-v_k)|^2\,\mathrm{d}\Gamma \\
&\qquad + \chi(L)\int_\Gamma \big(\beta(v_j-v_k)-(u_j-u_k)\big)^2\,\mathrm{d}\Gamma\Big) \\
&\quad \leq C\Big(\|(\partial_t(u_j-u_k), \partial_t(v_j-v_k))\|_{(\mathcal{H}^1_{K,\alpha})'} + \|(\partial_t\phi,\partial_t\psi)\|_{\mathcal{L}^3}\Big) \\
&\qquad\quad \times \|(u_j-u_k, v_j-v_k)\|_{\mathcal{H}^3}
\end{aligned}
\tag{7.12}
$$

a.e. on $[a,b]$. Now, let $s,t\in[a,b]$ be arbitrary with $s\leq t$. We then integrate (7.12) with respect to time over $[s,t]$, and obtain

$$
\begin{aligned}
&\int_\Omega m_\Omega(\phi(t))|\nabla(u_j-u_k)(t)|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(t))|\nabla_\Gamma(v_j-v_k)(t)|^2\,\mathrm{d}\Gamma \\
&\quad + \chi(L)\int_\Gamma \big(\beta(v_j-v_k)(t)-(u_j-u_k)(t)\big)^2\,\mathrm{d}\Gamma \\
&\leq \int_\Omega m_\Omega(\phi(s))|\nabla(u_j-u_k)(s)|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(s))|\nabla_\Gamma(v_j-v_k)(s)|^2\,\mathrm{d}\Gamma
\end{aligned}
\tag{7.13}
$$

$$
\begin{aligned}
&+ \chi(L) \int_\Gamma \big(\beta(v_j - v_k)(s) - (u_j - u_k)(s)\big)^2 \,\mathrm{d}\Gamma \\
&+ C \int_s^t \|(\partial_t(u_j - u_k), \partial_t(v_j - v_k))\|^2_{(\mathcal{H}^1_{K,\alpha})'} + \|(u_j - u_k, v_j - v_k)\|^2_{\mathcal{H}^3} \,\mathrm{d}r \\
&+ C \int_s^t \|(\partial_t\phi, \partial_t\psi)\|_{\mathcal{H}^1} \|(u_j - u_k, v_j - v_k)\|_{\mathcal{H}^2} \,\mathrm{d}r.
\end{aligned}
$$

In light of the convergences (7.5)-(7.6), we can fix $s \in [a,t]$ such that $(u_k(s), v_k(s)) \to (u(s), v(s))$ strongly in $\mathcal{H}^3$ along a non-relabeled subsequence $k \to \infty$. Recalling (7.5)-(7.8), we thus deduce that the right-hand side of (7.13) tends to zero as $j, k \to \infty$. As m_Ω and m_Γ are uniformly positive according to (7.17), we infer that $(\nabla u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([a,b]; \mathbf{L}^2(\Omega))$ and $(\nabla_\Gamma v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([a,b]; \mathbf{L}^2(\Gamma))$. Consequently,

$$
\begin{aligned}
\nabla u_k &\to \nabla u && \text{strongly in } C([a,b]; \mathbf{L}^2(\Omega)), && (7.14)\\
\nabla_\Gamma v_k &\to \nabla_\Gamma v && \text{strongly in } C([a,b]; \mathbf{L}^2(\Gamma)) && (7.15)
\end{aligned}
$$

as $k \to \infty$. Since we have already shown that $(u,v) \in C([a,b]; \mathcal{L}^2)$, we readily deduce that $(u,v) \in C([a,b]; \mathcal{H}^1)$.

Let now $s, t \in [a,b]$ be arbitrary with $s \le t$. We then integrate (7.10) in time from s to t and find

$$
\begin{aligned}
&\int_\Omega m_\Omega(\phi(t)) |\nabla u_k(t)|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(t)) |\nabla_\Gamma v_k(t)|^2 \,\mathrm{d}\Gamma + \chi(L) \int_\Gamma (\beta v_k(t) - u_k(t))^2 \,\mathrm{d}\Gamma \\
&\quad = \int_\Omega m_\Omega(\phi(s)) |\nabla u_k(s)|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(s)) |\nabla_\Gamma v_k(s)|^2 \,\mathrm{d}\Gamma \\
&\qquad + \chi(L) \int_\Gamma (\beta v_k(s) - u_k(s))^2 \,\mathrm{d}\Gamma \qquad (7.16)\\
&\qquad + 2 \int_s^t \big\langle (\partial_t u_k, \partial_t v_k), \\
&\qquad\qquad (-\mathrm{div}(m_\Omega(\phi)\nabla u_k), -\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v_k) + \beta m_\Omega(\phi)\partial_\mathbf{n} u_k)\big\rangle_{\mathcal{H}^1_{K,\alpha}} \,\mathrm{d}r \\
&\qquad + 2 \int_s^t \left(\int_\Omega m_\Omega'(\phi)\partial_t\phi |\nabla u_k|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma'(\psi)\partial_t\psi |\nabla_\Gamma v_k|^2 \,\mathrm{d}\Gamma \right) \mathrm{d}r.
\end{aligned}
$$

It is now clear from the convergences (7.5)-(7.8) and (7.14)-(7.15) to pass to the limit in all terms except the last one on the right-hand side. Here, we notice that

$$
\begin{aligned}
&\Big| \int_s^t \int_\Omega m_\Omega'(\phi)\partial_t\phi |\nabla u_k|^2 \,\mathrm{d}x\,\mathrm{d}r - \int_s^t \int_\Omega m_\Omega'(\phi)\partial_t\phi |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}r \Big| \\
&\quad = \Big| \int_s^t \int_\Omega m_\Omega'(\phi)\partial_t\phi (\nabla u_k + \nabla u) \cdot (\nabla u_k - \nabla u) \,\mathrm{d}x\,\mathrm{d}r \Big| \\
&\quad \le \|m_\Omega'\|_{L^\infty(-1,1)} \|\partial_t\phi\|_{L^2(a,b;L^3(\Omega))} \|\nabla u_k + \nabla u\|_{L^2(a,b;\mathbf{L}^6(\Omega))} \|\nabla u_k - \nabla u\|_{C([a,b];\mathbf{L}^2(\Omega))} \\
&\quad \le 2\|m_\Omega'\|_{L^\infty(-1,1)} \|\partial_t\phi\|_{L^2(a,b;L^3(\Omega))} \|u\|_{L^2(a,b;H^3(\Omega))} \|\nabla u_k - \nabla u\|_{C([a,b];\mathbf{L}^2(\Omega))} \\
&\quad \longrightarrow 0
\end{aligned}
$$

as $k \to \infty$ in light of (7.14). A similar computation shows that the corresponding term on the boundary also converges. All together, we are now able to pass to the limit $k \to \infty$ in (7.16) and conclude that

$$
\int_\Omega m_\Omega(\phi(t)) |\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(t)) |\nabla_\Gamma v(t)|^2 \,\mathrm{d}\Gamma + \chi(L) \int_\Gamma (\beta v(t) - u(t))^2 \,\mathrm{d}\Gamma
$$

$$
\begin{aligned}
&= \int_\Omega m_\Omega(\phi(s))|\nabla u(s)|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(s))|\nabla_\Gamma v(s)|^2 \,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta v(s) - u(s))^2 \,\mathrm{d}\Gamma \\
&\quad + 2\int_s^t \langle(\partial_t u, \partial_t v), (-\mathrm{div}(m_\Omega(\phi)\nabla u), -\mathrm{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma v) + \beta m_\Omega(\phi)\partial_{\mathbf{n}} u)\rangle_{\mathcal{H}^1_{K,\alpha}} \,\mathrm{d}r \\
&\quad + 2\int_s^t \Big(\int_\Omega m_\Omega'(\phi)\partial_t\phi|\nabla u|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma'(\psi)\partial_t\psi|\nabla_\Gamma v|^2 \,\mathrm{d}\Gamma \Big) \,\mathrm{d}r.
\end{aligned}
$$

By our previous considerations, we readily see that the integrands on the right-hand side belong to $L^1(a,b)$, and thus, that the mapping

$$
t \mapsto \int_\Omega m_\Omega(\phi(t))|\nabla u(t)|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi(t))|\nabla_\Gamma v(t)|^2 \,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta v(t) - u(t))^2 \,\mathrm{d}\Gamma
$$

is absolutely continuous on $[a,b]$. It is therefore differentiable almost everywhere on $[a,b]$ and its derivative satisfies the formula (7.4). This finishes the proof. $\square$

7.3 Main results

We start by stating our main assumptions that will be used throughout this chapter.

7.3.1 Assumptions

(A1) We consider a non-empty, bounded domain $\Omega \subset \mathbb{R}^2$ with C^3-boundary $\Gamma := \partial\Omega$.

(A2) The constants $\alpha, \beta \in \mathbb{R}$ appearing in system (6.1) are supposed to satisfy $\alpha \in [-1,1]$ as well as $\alpha\beta|\Omega| + |\Gamma| \neq 0$.

(A3) For the mobility functions we require $m_\Omega, m_\Gamma \in C^2([-1,1])$. Furthermore, we assume the existence of constants $m_*, m^* > 0$ such that

$$
0 < m_* \leq m_\Omega(s), m_\Gamma(s) \leq m^* \qquad \text{for all } s \in [-1,1]. \tag{7.17}
$$

(A4) For the potentials, we assume that $F, G : \mathbb{R} \to \mathbb{R}$ are of the form

$$
F(s) = F_1(s) + F_2(s), \qquad G(s) = G_1(s) + G_2(s),
$$

where $F_1, G_1 \in C([-1,1]) \cap C^2(-1,1)$ such that $F_1(0) = F_1'(0) = G_1(0) = G_1'(0) = 0$,

$$
\lim_{s\searrow -1} F_1'(s) = \lim_{s\searrow -1} G_1'(s) = -\infty \quad \text{and} \quad \lim_{s\nearrow 1} F_1'(s) = \lim_{s\nearrow 1} G_1'(s) = +\infty,
$$

and there exist constants $\Theta_\Omega, \Theta_\Gamma > 0$ such that

$$
F_1''(s) \geq \Theta_\Omega \quad \text{and} \quad G_1''(s) \geq \Theta_\Gamma \qquad \text{for all } s \in (-1,1). \tag{7.18}
$$

We extend F_1 and G_1 on $\mathbb{R}$ by defining $F_1(s) = G_1(s) = +\infty$ for $s \notin [-1,1]$. For F_2 and G_2, we assume $F_2, G_2 \in C^1(\mathbb{R})$ such that their derivatives are globally Lipschitz continuous. Lastly, we require that the singular part of the boundary potential dominates the singular part of the bulk potential in the sense that there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$
|F_1'(\alpha s)| \leq \kappa_1 |G_1'(s)| + \kappa_2 \qquad \text{for all } s \in (-1,1). \tag{7.19}
$$

(A5) We assume that one of the following conditions hold:

(A5.1) There exist constants $C_\sharp > 0$ and $\gamma_\sharp \in [1,2)$ such that

$$F_1''(s) \le C_\sharp \mathbf{e}^{C_\sharp |F_1'(s)|^{\gamma_\sharp}} \qquad \text{for all } s \in (-1,1). \tag{7.20}$$

(A5.2) As $\delta \searrow 0$, for some $\kappa > \frac{1}{2}$, it holds that

$$\frac{1}{F_1'(1-2\delta)} = O\left(\frac{1}{|\ln\delta|^\kappa}\right), \qquad \frac{1}{|F_1'(-1+2\delta)|} = O\left(\frac{1}{|\ln\delta|^\kappa}\right). \tag{7.21}$$

Moreover, for the velocity fields, we need the following regularity assumptions:

(A6) We assume that

$$(\mathbf{v},\mathbf{w}) \in L^2(0,\infty;\mathbf{L}^2_{\mathrm{div}}(\Omega)\times\mathbf{L}^2_{\mathrm{tan}}(\Gamma)).$$

(A7) At least one of the following conditions is satisfied:

(A7.1) $(\mathbf{v},\mathbf{w})$ has the regularities

$$(\mathbf{v},\mathbf{w}) \in H^1_{\mathrm{uloc}}([0,\infty);\mathbf{L}^{1+\omega}(\Omega)\times\mathbf{L}^1(\Gamma)) \cap L^2(0,\infty;\boldsymbol{\mathcal{L}}^2) \cap L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2_{\mathrm{div}})$$

for some $\omega > 0$.

(A7.2) $(\mathbf{v},\mathbf{w})$ has the regularities

$$(\mathbf{v},\mathbf{w}) \in L^2(0,\infty;\boldsymbol{\mathcal{H}}^1) \cap L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2_{\mathrm{div}}).$$

7.3.2 Continuous dependence and uniqueness of weak solutions

Our first main result is concerned with the uniqueness of weak solutions, which improves the result obtained in Theorem 6.2.5 by allowing for non-degenerate mobility functions.

THEOREM 7.3.1. *Suppose that the assumption* **(A1)**-**(A4)** *and* **(A6)** *hold. Let* $K, L \in (0,\infty]$. *Let* $(\phi_0^1,\psi_0^1), (\phi_0^2,\psi_0^2) \in \mathcal{H}^1$ *satisfy* (6.7) *as well as*

$$\begin{cases} \mathrm{mean}(\phi_0^1,\psi_0^1) = \mathrm{mean}(\psi_0^2,\psi_0^2) & \text{if } L \in [0,\infty), \\ \langle\phi_0^1\rangle_\Omega = \langle\phi_0^2\rangle_\Omega \quad \text{and} \quad \langle\psi_0^1\rangle_\Gamma = \langle\psi_0^2\rangle_\Gamma & \text{if } L = \infty, \end{cases} \tag{7.22}$$

and let $(\mathbf{v}_1,\mathbf{w}_1), (\mathbf{v}_2,\mathbf{w}_2) \in L^2(0,\infty;\mathbf{L}^2_{\mathrm{div}}(\Omega)\times\mathbf{L}^2_{\mathrm{tan}}(\Gamma))$ *be prescribed velocity fields. We consider two weak solutions* $(\phi_1,\psi_1,\mu_1,\theta_1)$ *and* $(\phi_2,\psi_2,\mu_2,\theta_2)$ *originating from* (ϕ_0^1,ψ_0^1) *and* (ϕ_0^2,ψ_0^2) *with velocity fields* $(\mathbf{v}_1,\mathbf{w}_1)$ *and* $(\mathbf{v}_2,\mathbf{w}_2)$*, respectively. Then, for any* $T > 0$*, there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\|(\phi_1(t)-\phi_2(t),\psi_1(t)-\psi_2(t))\|^2_{(\mathcal{H}^1_{L,\beta})'} + \int_0^t \|(\phi_1(s)-\phi_2(s),\psi_1(s)-\psi_2(s))\|^2_{K,\alpha}\,\mathrm{d}s \\ &\quad\le \|(\phi_0^1-\phi_0^2,\psi_0^1-\psi_0^2)\|^2_{(\mathcal{H}^1_{L,\beta})'} \exp\left(\int_0^t Q(r)\,\mathrm{d}r\right) \\ &\qquad + \int_0^t \|(\mathbf{v}_1(s)-\mathbf{v}_2(s),\mathbf{w}_1(s)-\mathbf{w}_2(s))\|_{\mathbf{L}^{1+\omega}(\Omega)\times\mathbf{L}^1(\Gamma)} \exp\left(\int_s^t Q(r)\,\mathrm{d}r\right)\mathrm{d}s \end{aligned} \tag{7.23}$$

for almost all $t \in [0,T]$ *and any* $\omega \in (0,1]$*, where* $Q \in L^1(0,T)$ *is given by*

$$Q(\cdot) := C\Big(1 + \|(\mathbf{v}_2,\mathbf{w}_2)\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\mu_2,\theta_2)\|^2_{L,\beta} + \|(\partial_t\phi_1,\partial_t\psi_1)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi_1,\psi_1)\|^4_{\mathcal{H}^2}\Big),$$

and the constant $C > 0$ *depends only on* Ω*, the initial data, the parameters of the systems and* ω*.*

Remark 7.3.2. The restriction $K \in (0, \infty]$ in Theorem 7.3.1 is necessary, as only in this case we are able to prove that

$$(\phi, \psi) \in L^4_{\mathrm{uloc}}([0, \infty); \mathcal{H}^2),$$

which is an essential ingredient in the proof of the uniqueness of weak solutions. If $K = 0$, one only has

$$(\phi, \psi) \in L^3_{\mathrm{uloc}}([0, \infty); \mathcal{H}^2),$$

see Theorem 6.2.3, which appears insufficient to establish the uniqueness of weak solutions to (6.1) for non-degenerate mobility functions. Moreover, the case $L = 0$ currently appears out of reach, since the key interpolation result in Proposition 4.4.5 is no longer applicable. The case $K = L = 0$, however, might be tractable under suitable compatibility assumptions on the potentials.

7.3.3 Existence of strong solutions and separation properties

Next, based on the continuous dependence estimate (7.23), we can prove the existence of strong solutions.

THEOREM 7.3.3. *Suppose that the assumptions* **(A1)**-**(A7)** *hold. Additionally assume that Ω is of class C^3. Let $K, L \in (0, \infty]$. Let $(\phi_0, \psi_0) \in \mathcal{H}^1$ satisfy* (6.7) *and the following compatibility condition:*

(C) *There exists $(\mu_0, \theta_0) \in \mathcal{H}^1_{L,\beta}$ such that for all $(\eta, \eta_\Gamma) \in \mathcal{H}^1$ it holds*

$$\begin{aligned}
&\int_\Omega \mu_0\, \eta \,\mathrm{d}x + \int_\Gamma \theta_0\, \eta_\Gamma \,\mathrm{d}\Gamma \\
&= \int_\Omega \nabla\phi_0 \cdot \nabla\eta + F'(\phi_0)\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi_0 \cdot \nabla_\Gamma\eta_\Gamma + G'(\psi_0)\eta_\Gamma \,\mathrm{d}\Gamma \\
&\quad + \chi(K) \int_\Gamma (\alpha\psi_0 - \phi_0)(\alpha\eta_\Gamma - \eta) \,\mathrm{d}\Gamma.
\end{aligned}$$

Then the unique weak solution $(\phi, \psi, \mu, \theta)$ to (6.1) *additionally satisfies the following regularities:*

$$(\partial_t\phi, \partial_t\psi) \in L^\infty(0, \infty; (\mathcal{H}^1_{L,\beta})') \cap L^2_{\mathrm{uloc}}([0, \infty); \mathcal{H}^1), \tag{7.24a}$$
$$(\phi, \psi) \in L^\infty(0, \infty; \mathcal{H}^3), \tag{7.24b}$$
$$(\mu, \theta) \in L^\infty(0, \infty; \mathcal{H}^1_{L,\beta}) \cap L^4_{\mathrm{uloc}}([0, \infty); \mathcal{H}^2), \tag{7.24c}$$
$$(F'(\phi), G'(\psi)) \in L^\infty(0, \infty; \mathcal{L}^p), (F''(\phi), G''(\psi)) \in L^\infty(0, \infty, \mathcal{L}^p) \tag{7.24d}$$

for any $2 \le p < \infty$. Moreover, the equations (6.1a)-(6.1b) *are satisfied almost everywhere in Q, while* (6.1c)-(6.1d) *and the boundary conditions* (6.1e)-(6.1f) *are satisfied almost everywhere on Σ. In particular, $(\phi, \psi, \mu, \theta)$ is the unique strong solution to system* (6.1). *Moreover, if* **(A7.2)** *hold, then $(\mu, \theta) \in L^2_{\mathrm{uloc}}([0, \infty); \mathcal{H}^3)$.*

Remark 7.3.4. We comment on the compatibility assumption **(C)** imposed in Theorem 7.3.3.

(a) This condition implies that (ϕ_0, ψ_0) is a weak solution to

$$\begin{aligned}
-\Delta\phi_0 + F_1'(\phi_0) &= \mu_0 - F_2'(\phi_0) && \text{in } \Omega, \\
-\Delta_\Gamma\psi_0 + G_1'(\psi_0) + \alpha\partial_{\mathbf{n}}\phi_0 &= \theta_0 - G_2'(\psi_0) && \text{on } \Gamma, \\
K\partial_{\mathbf{n}}\phi_0 &= \alpha\psi_0 - \phi_0 && \text{on } \Gamma.
\end{aligned}$$

The results presented in Section 4.2 show that (ϕ_0, ψ_0) is actually a strong solution of this system and additionally satisfies

$$(\phi_0, \psi_0) \in \mathcal{W}^{2,p}, \qquad (F_1'(\phi_0), G_1'(\psi_0)) \in \mathcal{L}^p$$

for any $2 \leq p < \infty$.

(b) If $L \in (0, \infty]$, it is quite straightforward to find suitable initial data (ϕ_0, ψ_0) which satisfy the compatibility condition **(C)**. Indeed, let $(\phi_0, \psi_0) \in \mathcal{H}^3$ satisfy (6.7), and additionally assume that

$$(F'(\phi_0), G'(\psi_0)) \in \mathcal{H}^1 \tag{7.25}$$

and

$$\begin{cases} K\partial_{\mathbf{n}}\phi_0 = \alpha\psi_0 - \phi_0 & \text{if } K \in [0, \infty), \\ \partial_{\mathbf{n}}\phi_0 = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Gamma.$$

We then define $(\mu_0, \theta_0) \in \mathcal{H}^1 = \mathcal{H}^1_{L,\beta}$ via

$$\begin{aligned} \mu_0 &:= -\Delta\phi_0 + F'(\phi_0) && \text{in } \Omega, \\ \theta_0 &:= -\Delta_\Gamma\psi_0 + G'(\psi_0) + \alpha\partial_{\mathbf{n}}\phi_0 && \text{on } \Gamma. \end{aligned}$$

Via integration by parts it is now easy to see that condition **(C)** is fulfilled.

Clearly, (7.25) is satisfied if there exists $\delta_0 \in (0, 1]$ such that, for all $x \in \Omega$ and $z \in \Gamma$ it holds

$$|\phi_0(x)| \leq 1 - \delta_0 \quad \text{and} \quad |\psi_0(z)| \leq 1 - \delta_0,$$

i.e., if ϕ_0 and ψ_0 are strictly separated from ± 1.

If $L = 0$, the situation is more involved, as the above procedure does in general not work, since the resulting chemical potential μ_0 and θ_0 do not necessarily satisfy the trace relation $\mu_0 = \beta\theta_0$ on Γ that is demanded in **(C)**.

Remark 7.3.5. In addition to the regularities stated in (7.24a)-(7.24d), one can show that

$$F'(\psi) \in L^\infty(0, \infty; L^p(\Gamma)), \tag{7.26}$$

see [129, 132], and also Remark 4.3.6.

Remark 7.3.6. In the setting of Theorem 7.3.3, assuming that the velocity fields satisfy **(A7.2)**, it is possible to show the existence of a strong solution to (6.1) assuming that the mobility functions m_Ω, m_Γ are only in $C^1([-1, 1])$. This can be achieved by a suitable approximation of the mobility functions. We refer to [164] for more details, where this procedure was carried out in detail for the non-convective bulk-surface Cahn–Hilliard model, i.e., for $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0})$. In the convective case, the proof can be carried out similarly by combining with the arguments of [163], see also [163, Remark A.4]. Note that, while it is possible to show the uniqueness of strong solutions (see the computations made in the proof of Theorem 10.2.3), Theorem 7.3.1 does not provide the uniqueness of weak solutions.

As a consequence, we can prove that the unique strong solution from Theorem 7.3.3 satisfies the instantaneous separation property.

THEOREM 7.3.7. *Suppose that the assumptions from Theorem 7.3.3 hold, and consider the corresponding unique strong solution $(\phi, \psi, \mu, \theta)$. Then, there exists $\delta > 0$, additionally depending on the norms of the initial data, such that*

$$\|\phi(t)\|_{L^\infty(\Omega)} \le 1 - \delta, \quad \|\psi(t)\|_{L^\infty(\Gamma)} \le 1 - \delta \qquad \textit{for all } t \ge 0. \tag{7.27}$$

The proof of Theorem 7.3.7 will not be presented here, as the argumentation is the same as in the case of constant mobility functions as presented in Theorem 6.2.10 using the regularity properties established in Theorem 7.3.3.

7.4 Continuous dependence and uniqueness of weak solutions

In this section, we present the proof of Theorem 7.3.1 regarding the uniqueness of weak solutions of system (6.1).

PROOF OF THEOREM 7.3.1. Let $(\phi_0^1, \psi_0^1), (\phi_0^2, \psi_0^2) \in \mathcal{H}^1$ be two admissible pairs of initial data satisfying (6.7) as well as (7.22), and consider given velocity fields $(\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \in L^2(0, \infty; \mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma))$. We consider the corresponding weak solutions $(\phi_1, \psi_1, \mu_1, \theta_1)$ and $(\phi_2, \psi_2, \mu_2, \theta_2)$ originating from (ϕ_0^1, ψ_0^1) and (ϕ_0^2, ψ_0^2) with velocity fields $(\mathbf{v}_1, \mathbf{w}_1)$ and $(\mathbf{v}_2, \mathbf{w}_2)$. Then, defining

$$(\phi, \psi, \mu, \theta, \mathbf{v}, \mathbf{w}) := (\phi_1 - \phi_2, \psi_1 - \psi_2, \mu_1 - \mu_2, \theta_1 - \theta_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{w}_1 - \mathbf{w}_2),$$

we have in view of (6.11a) and (6.16) that

$$\begin{aligned}
&\langle (\partial_t\phi, \partial_t\psi), (\zeta, \zeta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} \\
&\quad = \int_\Omega (\phi_1 \mathbf{v} + \phi \mathbf{v}_2) \cdot \nabla\zeta \,\mathrm{d}x + \int_\Gamma (\psi_1 \mathbf{w} + \psi \mathbf{w}_2) \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \\
&\qquad - \int_\Omega m_\Omega(\phi_1)\nabla\mu \cdot \nabla\zeta \,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi_1)\nabla_\Gamma\theta \cdot \nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\
&\qquad - \chi(L)\int_\Gamma (\beta\theta - \mu)(\beta\zeta_\Gamma - \zeta)\,\mathrm{d}\Gamma - \int_\Omega (m_\Omega(\phi_1) - m_\Omega(\phi_1))\nabla\mu_2 \cdot \nabla\zeta \,\mathrm{d}x \\
&\qquad - \int_\Gamma (m_\Gamma(\psi_1) - m_\Gamma(\psi_2))\nabla_\Gamma\theta_2 \cdot \nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma
\end{aligned} \tag{7.28}$$

a.e. on $(0, \infty)$ for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$, as well as

$$-\Delta\phi + F'(\phi_1) - F'(\phi_2) = \mu \qquad \text{a.e. in } Q, \tag{7.29a}$$
$$-\Delta_\Gamma\psi + G'(\psi_1) - G'(\psi_2) + \alpha\partial_{\mathbf{n}}\phi = \theta \qquad \text{a.e. on } \Sigma, \tag{7.29b}$$
$$K\partial_{\mathbf{n}}\phi = \alpha\psi - \phi \qquad \text{a.e. on } \Sigma. \tag{7.29c}$$

Now, we multiply (7.29a) with ϕ and (7.29b) with ψ, integrate over Ω and Γ, respectively, and perform integration by parts. Adding the resulting equations leads to

$$\begin{aligned}
&\|(\phi, \psi)\|^2_{K,\alpha} + \int_\Omega (F_1'(\phi_1) - F_1'(\phi_2))\phi \,\mathrm{d}x + \int_\Gamma (G_1'(\psi_1) - G_1'(\psi_2))\psi \,\mathrm{d}\Gamma \\
&\quad - \int_\Omega \mu\,\phi \,\mathrm{d}x - \int_\Gamma \theta\,\psi \,\mathrm{d}\Gamma \\
&\quad = \int_\Omega (F_2'(\phi_1) - F_2'(\phi_2))\phi \,\mathrm{d}x + \int_\Gamma (G_2'(\psi_1) - G_2'(\psi_2))\psi \,\mathrm{d}\Gamma
\end{aligned} \tag{7.30}$$

a.e. on $(0,\infty)$. Next, we want to rewrite the integrals involving the chemical potentials in terms of the solution operator $\mathcal{S}_{L,\beta}^{\phi_1,\psi_1}(\partial_t\phi,\partial_t\psi)$. To this end, to make the following computations more readable, we use the abbreviation $\mathcal{S}_1(f,f_\Gamma)=\mathcal{S}_{L,\beta}^{\phi_1,\psi_1}(f,f_\Gamma)$ as well as $(\cdot,\cdot)_1=(\cdot,\cdot)_{L,\beta,\phi_1,\psi_1}$. In particular, with this notation,

$$\|(\zeta,\zeta_\Gamma)\|_1^2=\int_\Omega m_\Omega(\phi_1)|\nabla\zeta|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma(\psi_1)|\nabla_\Gamma\zeta_\Gamma|^2\,\mathrm{d}\Gamma+\chi(L)\int_\Gamma(\beta\zeta_\Gamma-\zeta)^2\,\mathrm{d}\Gamma$$

is a norm which is equivalent to $\|\cdot\|_{\mathcal{H}^1}$ on $\mathcal{V}_{L,\beta}^1$. Similarly, we use $\|\cdot\|_{1,*}=\|\cdot\|_{L,\beta,\phi_1,\psi_1,*}$, which is an equivalent norm on $\mathcal{V}_{L,\beta}^{-1}$. We then find

$$\begin{aligned}
&-\int_\Omega\mu\,\phi\,\mathrm{d}x-\int_\Gamma\theta\,\psi\,\mathrm{d}\Gamma\\
&\quad=-\big(\mathcal{S}_1(\phi,\psi),(\mu_1-\mu_2,\theta_1-\theta_2)\big)_1\\
&\quad=\big\langle(\partial_t\phi,\partial_t\psi),\mathcal{S}_1(\phi,\psi)\big\rangle_{\mathcal{H}^1_{L,\beta}}-\int_\Omega(m_\Omega(\phi_1)-m_\Omega(\phi_2))\nabla\mu_2\cdot\nabla\mathcal{S}_1(\phi,\psi)\,\mathrm{d}x\\
&\qquad-\int_\Gamma(m_\Gamma(\psi_1)-m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\qquad+\int_\Omega(\phi_1\mathbf{v}+\phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x+\int_\Gamma(\psi_1\mathbf{w}+\psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\quad=\big(\mathcal{S}_1(\partial_t\phi,\partial_t\psi),\mathcal{S}_1(\phi,\psi)\big)_1-\int_\Omega(m_\Omega(\phi_1)-m_\Omega(\phi_2))\nabla\mu_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x \qquad (7.31)\\
&\qquad-\int_\Gamma(m_\Gamma(\psi_1)-m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\qquad+\int_\Omega(\phi_1\mathbf{v}+\phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x+\int_\Gamma(\psi_1\mathbf{w}+\psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\quad=\big((\phi,\psi),\mathcal{S}_1(\partial_t\phi,\partial_t\psi)\big)_{\mathcal{L}^2}-\int_\Omega(m_\Omega(\phi_1)-m_\Omega(\phi_2))\nabla\mu_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x\\
&\qquad-\int_\Gamma(m_\Gamma(\psi_1)-m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\qquad+\int_\Omega(\phi_1\mathbf{v}+\phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x+\int_\Gamma(\psi_1\mathbf{w}+\psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma
\end{aligned}$$

a.e. on $(0,\infty)$, using, in this order, the weak formulations for $\mathcal{S}_1(\phi,\psi)$, for $(\partial_t\phi,\partial_t\psi)$ (see (7.28)), for $\mathcal{S}_1(\partial_t\phi,\partial_t\psi)$ and again for $\mathcal{S}_1(\phi,\psi)$. Plugging the identity (7.31) back into (7.30) yields

$$\begin{aligned}
&\|(\phi,\psi)\|_{K,\alpha}^2+\int_\Omega(F_1'(\phi_1)-F_1'(\phi_2))\phi\,\mathrm{d}x+\int_\Gamma(G_1'(\psi_1)-G_1'(\psi_2))\psi\,\mathrm{d}\Gamma\\
&\qquad+\int_\Omega\mathcal{S}_1(\partial_t\phi,\partial_t\psi)\phi\,\mathrm{d}x+\int_\Gamma\mathcal{S}_1(\partial_t\phi,\partial_t\psi)\psi\,\mathrm{d}\Gamma\\
&\qquad-\int_\Omega(m_\Omega(\phi_1)-m_\Omega(\phi_2))\nabla\mu_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x\\
&\qquad-\int_\Gamma(m_\Gamma(\psi_1)-m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\qquad+\int_\Omega(\phi_1\mathbf{v}+\phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x+\int_\Gamma(\psi_1\mathbf{w}+\psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\\
&\quad=\int_\Omega(F_2'(\phi_1)-F_2'(\phi_2))\phi\,\mathrm{d}x+\int_\Gamma(G_2'(\psi_1)-G_2'(\psi_2))\psi\,\mathrm{d}\Gamma
\end{aligned}$$

a.e. on $(0,\infty)$. Now, we claim that

$$\int_\Omega\mathcal{S}_1^\Omega(\partial_t\phi,\partial_t\psi)\phi\,\mathrm{d}x+\int_\Gamma\mathcal{S}_1^\Gamma(\partial_t\phi,\partial_t\psi)\psi\,\mathrm{d}\Gamma$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{1,*}^2 \\
&\quad + \frac{1}{2}\Big(\mathcal{S}_1(\partial_t\phi_1,\partial_t\psi_1),\big(m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2\big)\Big)_{L,\beta} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{1,*}^2 \\
&\quad + \frac{1}{2}\int_\Omega \nabla\mathcal{S}_1^\Omega(\partial_t\phi_1,\partial_t\psi_1)\cdot m_\Omega''(\phi_1)\nabla\phi_1|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2 \,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Gamma \nabla_\Gamma\mathcal{S}_1^\Gamma(\partial_t\phi_1,\partial_t\psi_1)\cdot m_\Gamma''(\psi_1)\nabla_\Gamma\psi_1|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 \,\mathrm{d}\Gamma \\
&\quad + \int_\Omega \nabla\mathcal{S}_1^\Omega(\partial_t\phi_1,\partial_t\psi_1)\cdot m_\Omega'(\phi_1)D^2\mathcal{S}_1^\Omega(\phi,\psi)\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x \\
&\quad + \int_\Gamma \nabla_\Gamma\mathcal{S}_1^\Gamma(\partial_t\phi_1,\partial_t\psi_1)\cdot m_\Gamma'(\psi_1)D_\Gamma^2\mathcal{S}_1^\Gamma(\phi,\psi)\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma \\
&\quad + \frac{1}{2}\chi(L)\int_\Gamma \big(\beta\mathcal{S}_1^\Gamma(\partial_t\phi_1,\partial_t\psi_1) - \mathcal{S}_1^\Omega(\partial_t\phi_1,\partial_t\psi_1)\big) \\
&\qquad\qquad \times\big(\beta m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 - m_\Omega'(\phi_1)|\nabla_\Gamma\mathcal{S}_1^\Omega(\phi,\psi)|^2\big)\,\mathrm{d}\Gamma
\end{aligned}
\tag{7.32}
$$

a.e. on $(0,\infty)$. Here, $D^2 f$ and $D_\Gamma^2 f_\Gamma$ denote the Hessian and the surface Hessian of f and f_Γ, respectively. Then, having (7.32) at hand, we exploit the monotonicity of F_1' and G_1' as well as the Lipschitz continuity of F_2' and G_2', respectively, which leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{1,*}^2 + \|(\phi,\psi)\|_{K,\alpha}^2 \le C\|(\phi,\psi)\|_{\mathcal{L}^2}^2 + I_1 + I_2 + I_3 \tag{7.33}
$$

a.e. on $(0,\infty)$, where

$$
\begin{aligned}
I_1 &= -\frac{1}{2}\Big(\mathcal{S}_1(\partial_t\phi_1,\partial_t\psi_1),\big(m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2\big)\Big)_{L,\beta}, \\
I_2 &= \int_\Omega (m_\Omega(\phi_1)-m_\Omega(\phi_2))\nabla\mu_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x \\
&\quad + \int_\Gamma (m_\Gamma(\psi_1)-m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma,
\end{aligned}
$$

and

$$
I_3 = \int_\Omega (\phi_1\mathbf{v}+\phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma (\psi_1\mathbf{w}+\psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma.
$$

The rest of the proof now consists of justifying the chain-rule formula (7.32) as well as estimating the nonlinear terms I_1, I_2 and I_3.

Proof of (7.32). Let $\pi \in C_c^\infty(\mathbb{R})$ be non-negative with $\operatorname{supp}\pi \subset (0,1)$ and $\|\pi\|_{L^1(\mathbb{R})} = 1$. For $k\in\mathbb{N}$, we set

$$
\pi_k(s) := k\pi(ks), \qquad s\in\mathbb{R}.
$$

Then, for any Banach space X and any function $f\in L^p(-1,T;X)$ with $2\le p<\infty$, we define

$$
f_k(t) := (\pi_k * f)(t) = \int_{t-\frac{1}{k}}^t \pi_k(t-s)f(s)\,\mathrm{d}s \tag{7.34}
$$

for all $t\in[0,T]$ and all $k\in\mathbb{N}$. By this construction, we have $f_k\in C^\infty([0,T];X)$ with $f_k\to f$ strongly in $L^p(0,T;X)$ as $k\to\infty$, see [16, S. 4.13-4.15].

Now, let $T>0$, $k\in\mathbb{N}$ and $p=4$. We choose $X=H^2(\Omega)$ to define ϕ_1^k and $X=H^2(\Gamma)$ to define ψ_1^k as described above. By this construction, we then have $\partial_t\phi_1^k=(\partial_t\phi_1)^k$ and $\partial_t\nabla\phi_1^k=\nabla\partial_t\phi_1^k$ a.e. in Q^T as well as $\partial_t\psi_1^k=(\partial_t\psi_1)^k$ and $\partial_t\nabla_\Gamma\psi_1^k=\nabla_\Gamma\partial_t\psi_1^k$ a.e. on Σ^T for all $k\in\mathbb{N}$. Moreover, as $k\to\infty$,

$$\phi_1^k\to\phi_1 \qquad \text{strongly in } L^4(0,T;H^2(\Omega)), \tag{7.35}$$
$$\psi_1^k\to\psi_1 \qquad \text{strongly in } L^4(0,T;H^2(\Gamma)), \tag{7.36}$$
$$(\partial_t\phi_1^k,\partial_t\psi_1^k)\to(\partial_t\phi_1,\partial_t\psi_1^k) \qquad \text{strongly in } L^2(0,T;(\mathcal{H}^1_{L,\beta})'). \tag{7.37}$$

In particular, along a non-relabeled subsequence, as $k\to\infty$,

$$\phi_1^k\to\phi_1,\quad \nabla\phi_1^k\to\nabla\phi_1 \qquad \text{a.e. in } Q^T, \tag{7.38}$$
$$\psi_1^k\to\psi_1,\quad \nabla_\Gamma\psi_1^k\to\nabla_\Gamma\psi_1 \qquad \text{a.e. on } \Sigma^T. \tag{7.39}$$

Additionally, we find that

$$\begin{aligned}\|\phi_1^k\|_{L^\infty(0,T;H^1(\Omega))}&\le\|\phi_1\|_{L^\infty(0,T;H^1(\Omega))},\\ \|\psi_1^k\|_{L^\infty(0,T;H^1(\Gamma))}&\le\|\psi_1\|_{L^\infty(0,T;H^1(\Gamma))},\end{aligned} \tag{7.40}$$

as well as

$$|\phi_k|\le 1 \quad \text{a.e. in } Q^T, \qquad |\psi_k|\le 1 \quad \text{a.e. on } \Sigma^T \tag{7.41}$$

for all $k\in\mathbb{N}$. We further point out that in light of the convergences (7.35)-(7.36) we infer the existence of $k_*\in\mathbb{N}$ such that

$$\|(\phi_1^k,\psi_1^k)\|_{L^4(0,T;\mathcal{H}^2)}\le 1+\|(\phi_1,\psi_1)\|_{L^4(0,T;\mathcal{H}^2)} \qquad \text{for all } k\ge k_*. \tag{7.42}$$

In the following, we assume that $k\in\mathbb{N}$ satisfies $k\ge k_*$. Furthermore, we denote by the letter C a generic positive constant whose value may change from line to line, but is independent of the parameter $k\in\mathbb{N}$. We now define, for $k\in\mathbb{N}$, the sequence

$$\mathcal{S}^k(\phi,\psi)\coloneqq\mathcal{S}^{\phi_1^k,\psi_1^k}_{L,\beta}(\phi,\psi).$$

It readily follows from (4.52) and (7.41) that

$$\|\mathcal{S}^k(\phi,\psi)\|_{L^\infty(0,T;\mathcal{H}^1)}\le C. \tag{7.43}$$

Next, in view of (7.40) and (7.43), an application of (7.2) entails that

$$\|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{H}^2}\le C\Big(1+\|(\phi_1^k,\psi_1^k)\|_{\mathcal{H}^2}\Big)$$

a.e. on $(0,T)$, which implies that

$$\int_0^T\|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{H}^2}^4\,\mathrm{d}s\le CT+C\int_0^T\|(\phi_1^k,\psi_1^k)\|_{\mathcal{H}^2}^4\,\mathrm{d}s\le C, \tag{7.44}$$

due to (7.42). Furthermore, employing the estimate (7.3) in combination with (7.40) and (7.43), we deduce that

$$\begin{aligned}&\|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{W}^{2,4}}\\ &\quad\le C\Big(\|(\nabla\phi_1^k,\nabla_\Gamma\psi_1^k)\|_{\mathcal{L}^2}^{\frac14}\|(\phi_1^k,\psi_1^k)\|_{\mathcal{H}^2}^{\frac34}\|\mathcal{S}^k(\phi,\psi)\|_{L,\beta}^{\frac14}\|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{H}^2}^{\frac34}+\|(\phi,\psi)\|_{\mathcal{L}^4}\Big)\\ &\quad\le C\Big(1+\|(\phi_1^k,\psi_1^k)\|_{\mathcal{H}^2}^{\frac34}\|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{H}^2}^{\frac34}\Big)\end{aligned}$$

a.e. on $(0,T)$. Hence, integrating the foregoing inequality in time over $(0,T)$ and using Young's inequality together with (7.42) and (7.44), we have

$$\int_0^T \|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{W}^{2,4}}^{\frac{8}{3}} \,\mathrm{d}s \leq CT + C\int_0^T \|(\phi_1^k,\psi_1^k)\|_{\mathcal{H}^2}^4 + \|\mathcal{S}^k(\phi,\psi)\|_{\mathcal{H}^2}^4 \,\mathrm{d}s \leq C. \tag{7.45}$$

Now, we study the convergence properties of the operator $\mathcal{S}^k(\phi,\psi)$ as $k\to\infty$. To this end, let $(f,f_\Gamma)\in L^2(0,T;\mathcal{V}_{L,\beta}^{-1})$. Then, by definition of $\mathcal{S}_1$ and $\mathcal{S}^k$, we know that

$$\begin{aligned}
&\int_\Omega m_\Omega(\phi_1)\nabla\mathcal{S}_1^\Omega(f,f_\Gamma)\cdot\nabla\zeta \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_1)\nabla_\Gamma\mathcal{S}_1^\Gamma(f,f_\Gamma)\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma\\
&\qquad + \chi(L)\int_\Gamma (\beta\mathcal{S}_1^\Gamma(f,f_\Gamma) - \mathcal{S}_1^\Omega(f,f_\Gamma))(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma\\
&\quad = \int_\Omega m_\Omega(\phi_1^k)\nabla\mathcal{S}^{k,\Omega}(f,f_\Gamma)\cdot\nabla\zeta\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_1^k)\nabla_\Gamma\mathcal{S}^{k,\Gamma}(f,f_\Gamma)\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma\\
&\qquad + \chi(L)\int_\Gamma (\beta\mathcal{S}^{k,\Gamma}(f,f_\Gamma) - \mathcal{S}^{k,\Omega}(f,f_\Gamma))(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma
\end{aligned}$$

a.e. on $(0,T)$ for all $(\zeta,\zeta_\Gamma)\in\mathcal{H}^1_{L,\beta}$, which implies that

$$\begin{aligned}
&\int_\Omega m_\Omega(\phi_1)\nabla(\mathcal{S}_1^\Omega(f,f_\Gamma) - \mathcal{S}^{k,\Omega}(f,f_\Gamma))\cdot\nabla\zeta\,\mathrm{d}x\\
&\qquad + \int_\Gamma m_\Gamma(\psi_1)\nabla_\Gamma(\mathcal{S}_1^\Gamma(f,f_\Gamma) - \mathcal{S}^{k,\Gamma}(f,f_\Gamma))\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma\\
&\qquad + \chi(L)\int_\Gamma (\beta(\mathcal{S}_1^\Gamma(f,f_\Gamma) - \mathcal{S}^{k,\Gamma}(f,f_\Gamma)) - (\mathcal{S}_1^\Omega(f,f_\Gamma) - \mathcal{S}^{k,\Omega}(f,f_\Gamma)))(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma\\
&\quad = \int_\Omega (m_\Omega(\phi_1^k) - m_\Omega(\phi_1))\nabla\mathcal{S}^{k,\Omega}(f,f_\Gamma)\cdot\nabla\zeta\,\mathrm{d}x\\
&\qquad + \int_\Gamma (m_\Gamma(\psi_1^k) - m_\Gamma(\psi_1))\nabla_\Gamma\mathcal{S}^{k,\Gamma}(f,f_\Gamma)\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma
\end{aligned}\tag{7.46}$$

a.e. on $(0,T)$ for all $(\zeta,\zeta_\Gamma)\in\mathcal{H}^1_{L,\beta}$. Therefore, taking $(\zeta,\zeta_\Gamma) = \mathcal{S}_1(f,f_\Gamma) - \mathcal{S}^k(f,f_\Gamma)\in\mathcal{H}^1_{L,\beta}$ as a test function in (7.46), we find with (4.55) that

$$\begin{aligned}
&\int_0^T \|\mathcal{S}_1(f,f_\Gamma) - \mathcal{S}^k(f,f_\Gamma)\|_{L,\beta}^2\,\mathrm{d}s\\
&\quad\leq \frac{1}{\min\{1,m^*\}^2}\int_0^T \|(m_\Omega(\phi_1^k) - m_\Omega(\phi_1))\nabla\mathcal{S}^{k,\Omega}(f,f_\Gamma)\|_{\mathbf{L}^2(\Omega)}^2\,\mathrm{d}s\\
&\qquad + \frac{1}{\min\{1,m^*\}^2}\int_0^T \|(m_\Gamma(\psi_1^k) - m_\Gamma(\psi_1))\nabla_\Gamma\mathcal{S}^{k,\Gamma}(f,f_\Gamma)\|_{\mathbf{L}^2(\Gamma)}^2\,\mathrm{d}s\\
&\quad\leq \frac{1}{\min\{1,m^*\}^2}\int_0^T \|m_\Omega(\phi_1^k) - m_\Omega(\phi_1)\|_{L^\infty(\Omega)}^2\|\nabla\mathcal{S}^{k,\Omega}(f,f_\Gamma)\|_{\mathbf{L}^2(\Omega)}^2\,\mathrm{d}s\\
&\qquad + \frac{1}{\min\{1,m^*\}^2}\int_0^T \|m_\Gamma(\psi_1^k) - m_\Gamma(\psi_1)\|_{L^\infty(\Gamma)}^2\|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(f,f_\Gamma)\|_{\mathbf{L}^2(\Gamma)}^2\,\mathrm{d}s.
\end{aligned}\tag{7.47}$$

We aim to pass to the limit $k\to\infty$ in (7.47) by applying the dominated convergence theorem. Toward this end, we first notice by (4.56) that

$$\begin{aligned}
&\|m_\Omega(\phi_1^k) - m_\Omega(\phi_1)\|_{L^\infty(\Omega)}^2\|\nabla\mathcal{S}^{k,\Omega}(f,f_\Gamma)\|_{\mathbf{L}^2(\Omega)}^2\\
&\qquad + \|m_\Gamma(\psi_1^k) - m_\Gamma(\psi_1)\|_{L^\infty(\Gamma)}^2\|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(f,f_\Gamma)\|_{\mathbf{L}^2(\Gamma)}^2
\end{aligned}$$

$$\leq C\|\mathcal{S}(f, f_\Gamma)\|_{L,\beta}^2 \in L^1(0,T).$$

On the other hand, since m_Ω and m_Γ are Lipschitz continuous on $[-1,1]$, the Sobolev inequality shows that

$$\begin{aligned}
&\|m_\Omega(\phi_1^k) - m_\Omega(\phi_1)\|_{L^\infty(\Omega)}^2 + \|m_\Gamma(\psi_1^k) - m_\Gamma(\psi_1)\|_{L^\infty(\Gamma)}^2 \\
&\quad \leq C\|\phi_1^k - \phi_1\|_{L^\infty(\Omega)}^2 + C\|\psi_1^k - \psi_1\|_{L^\infty(\Gamma)}^2 \\
&\quad \leq C\|\phi_1^k - \phi_1\|_{H^2(\Omega)}^2 + C\|\psi_1^k - \psi_1\|_{H^2(\Gamma)}^2
\end{aligned}$$

a.e. on $(0,T)$. Thus, in light of (7.35)-(7.36), we obtain that

$$\begin{aligned}
&\|m_\Omega(\phi_1^k) - m_\Omega(\phi_1)\|_{L^\infty(\Omega)}^2 \|\nabla\mathcal{S}^{k,\Omega}(f, f_\Gamma)\|_{\mathbf{L}^2(\Omega)}^2 \\
&\qquad + \|m_\Gamma(\psi_1^k) - m_\Gamma(\psi_1)\|_{L^\infty(\Gamma)}^2 \|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(f, f_\Gamma)\|_{\mathbf{L}^2(\Gamma)}^2 \\
&\quad \leq C\|(\phi_1^k - \phi_1, \psi_1^k - \psi_1)\|_{\mathcal{H}^2}^2 \|\mathcal{S}(f, f_\Gamma)\|_{L,\beta}^2 \to 0
\end{aligned}$$

up to a subsequence $k \to \infty$ a.e. on $(0,T)$. Consequently, we can use (7.47) and the Lebesgue convergence theorem to conclude that

$$\|\mathcal{S}^k(f, f_\Gamma) - \mathcal{S}_1(f, f_\Gamma)\|_{L,\beta} \to 0 \qquad \text{strongly in } L^2(0,T)$$

as $k \to \infty$. In particular, for $(f, f_\Gamma) = (\partial_t\phi, \partial_t\psi)$ and $(f, f_\Gamma) = (\phi, \psi)$, we find that

$$\mathcal{S}^k(\partial_t\phi, \partial_t\psi) \to \mathcal{S}_1(\partial_t\phi, \partial_t\psi) \qquad \text{strongly in } L^2(0,T;\mathcal{H}^1), \tag{7.48}$$

as well as

$$\mathcal{S}^k(\phi, \psi) \to \mathcal{S}_1(\phi, \psi) \qquad \text{strongly in } L^2(0,T;\mathcal{H}^1), \tag{7.49}$$

as $k \to \infty$, respectively. Furthermore, in view of the uniform estimates (7.44) and (7.45), we deduce that

$$\mathcal{S}^k(\phi, \psi) \to \mathcal{S}_1(\phi, \psi) \qquad \text{weakly in } L^4(0,T;\mathcal{H}^2) \tag{7.50}$$

and

$$\mathcal{S}^k(\phi, \psi) \to \mathcal{S}_1(\phi, \psi) \qquad \text{weakly in } L^{\frac{8}{3}}(0,T;\mathcal{W}^{2,4}), \tag{7.51}$$

as $k \to \infty$, respectively. Next, for any $\varepsilon \in (0, \frac{1}{2})$, by Lemma 2.2.3, there exists a constant $C = C(\varepsilon) > 0$ such that

$$\begin{aligned}
\|f\|_{H^{2-\varepsilon}(\Omega)} &\leq C\|f\|_{H^1(\Omega)}^{\varepsilon}\|f\|_{H^2(\Omega)}^{1-\varepsilon} &&\text{for all } f \in H^2(\Omega), \\
\|f_\Gamma\|_{H^{2-\varepsilon}(\Gamma)} &\leq C\|f_\Gamma\|_{H^1(\Gamma)}^{\varepsilon}\|f_\Gamma\|_{H^2(\Gamma)}^{1-\varepsilon} &&\text{for all } f_\Gamma \in H^2(\Gamma).
\end{aligned}$$

Thus, combining these two estimates, by Young's inequality, we deduce

$$\|(f, f_\Gamma)\|_{L^{\frac{4}{1+\varepsilon}}(0,T;\mathcal{H}^{2-\varepsilon})} \leq C\|(f, f_\Gamma)\|_{L^2(0,T;\mathcal{H}^1)}\|(f, f_\Gamma)\|_{L^4(0,T;\mathcal{H}^2)} \tag{7.52}$$

for all $(f, f_\Gamma) \in L^4(0,T;\mathcal{H}^2)$. Choosing $(f, f_\Gamma) = \mathcal{S}^k(\phi, \psi) - \mathcal{S}_1(\phi, \psi) \in L^4(0,T;\mathcal{H}^2)$ in (7.52), the convergences (7.49) and (7.50) lead to

$$\mathcal{S}^k(\phi, \psi) \to \mathcal{S}_1(\phi, \psi) \qquad \text{strongly in } L^{\frac{4}{1+\varepsilon}}(0,T;\mathcal{H}^{2-\varepsilon}) \tag{7.53}$$

as $k \to \infty$. Now, we claim that for any $\sigma \in C_c^\infty(0,T)$ it holds that

$$\begin{aligned}
&\int_0^T \big(\mathcal{S}^k(\partial_t\phi,\partial_t\psi),(\phi,\psi)\big)_{\mathcal{L}^2}\sigma \,\mathrm{d}s \\
&\quad = -\frac{1}{2}\int_0^T \|(\phi,\psi)\|_{1,*}^2\partial_t\sigma \,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^T \big(\partial_t\phi_1^k, m_\Omega(\phi_1^k)\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\cdot\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\big)_{\mathcal{L}^2}\sigma \,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^T \big(\partial_t\psi_1^k, m_\Gamma(\psi_1^k)\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)\cdot\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)\big)_{\mathcal{L}^2}\sigma \,\mathrm{d}s.
\end{aligned}\tag{7.54}$$

To this end, for almost every $t \in (0,T)$, let $c(\cdot,t)$ be measurable in Ω, $c_\Gamma(\cdot,t)$ be measurable on Γ with $(\partial_t c(\cdot,t),\partial_t c_\Gamma(\cdot,t)) \in \mathcal{L}^2$, and $(f(\cdot,t),f_\Gamma(\cdot,t)) \in \mathcal{V}_{L,\beta}^{-1}\cap\mathcal{L}^2$ with $(\partial_t f(\cdot,t),\partial_t f_\Gamma(\cdot,t)) \in \mathcal{L}^2$ be given. Then, differentiating the weak formulation (4.51) that is satisfied by $\mathcal{S}_{L,\beta}^{c,c_\Gamma}(f,f_\Gamma)$ and taking $(\zeta,\zeta_\Gamma) = \mathcal{S}_{L,\beta}^{c,c_\Gamma}(f,f_\Gamma) \in \mathcal{H}_{L,\beta}^1$ as a test function, we obtain that

$$\begin{aligned}
&\int_\Omega \partial_t f\, \mathcal{S}_{L,\beta}^{c,c_\Gamma,\Omega}(f,f_\Gamma)\,\mathrm{d}x + \int_\Gamma \partial_t g\, \mathcal{S}_{L,\beta}^{c,c_\Gamma,\Gamma}(f,f_\Gamma)\,\mathrm{d}\Gamma \\
&= \int_\Omega m_\Omega'(c)\partial_t c\,\big|\nabla\mathcal{S}_{L,\beta}^{c,c_\Gamma,\Omega}(f,f_\Gamma)\big|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma'(c_\Gamma)\partial_t c_\Gamma\,\big|\nabla_\Gamma\mathcal{S}_{L,\beta}^{c,c_\Gamma,\Gamma}(f,f_\Gamma)\big|^2\,\mathrm{d}\Gamma \\
&\quad + \int_\Omega f\,\partial_t\mathcal{S}_{L,\beta}^{c,c_\Gamma,\Omega}(f,f_\Gamma)\,\mathrm{d}x + \int_\Gamma f_\Gamma\,\partial_t\mathcal{S}_{L,\beta}^{c,c_\Gamma,\Gamma}(f,f_\Gamma)\,\mathrm{d}\Gamma
\end{aligned}$$

a.e. on $(0,T)$. Hence, noting again on (4.51), we find that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(f,f_\Gamma)\|_{L,\beta,c,c_\Gamma,*}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\mathcal{S}_{L,\beta}^{c,c_\Gamma}(f,f_\Gamma)\|_{L,\beta,c,c_\Gamma}^2 \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{S}_{L,\beta}^{c,c_\Gamma}(f,f_\Gamma),(f,f_\Gamma)\big)_{\mathcal{L}^2} \\
&\quad = \big(\partial_t\mathcal{S}_{L,\beta}^{c,c_\Gamma}(f,f_\Gamma),(f,f_\Gamma)\big)_{\mathcal{L}^2} + \big(\mathcal{S}_{L,\beta}^{c,c_\Gamma}(f,f_\Gamma),(\partial_t f,\partial_t f_\Gamma)\big)_{\mathcal{L}^2} \\
&\quad = 2\big(\mathcal{S}_{L,\beta}^{c,c_\Gamma}(\partial_t f,\partial_t f_\Gamma),(f,f_\Gamma)\big)_{\mathcal{L}^2} - \int_\Omega m_\Omega'(c)\partial_t c\,\big|\nabla\mathcal{S}_{L,\beta}^{c,c_\Gamma,\Omega}(f,f_\Gamma)\big|^2\,\mathrm{d}x \\
&\qquad - \int_\Gamma m_\Gamma'(c_\Gamma)\partial_t c_\Gamma\,\big|\nabla_\Gamma\mathcal{S}_{L,\beta}^{c,c_\Gamma,\Gamma}(f,f_\Gamma)\big|^2\,\mathrm{d}\Gamma
\end{aligned}\tag{7.55}$$

a.e. on $(0,T)$. As (c,c_Γ) and (f,f_Γ) were arbitrary, we can simply take $(c,c_\Gamma) = (\phi_1^k,\psi_1^k)$ and $(f,f_\Gamma) = (\phi,\psi)$ in (7.55) obtaining (7.54).

Now, we wish to take the limit $k \to \infty$ in (7.54). To this end, we first notice that from the convergences (7.48) and (7.49) we easily conclude that

$$\lim_{k\to\infty}\int_0^T \big(\mathcal{S}^k(\partial_t\phi,\partial_t\psi),(\phi,\psi)\big)_{\mathcal{L}^2}\sigma\,\mathrm{d}s = \int_0^T \big(\mathcal{S}_1(\partial_t\phi,\partial_t\psi),(\phi,\psi)\big)_{\mathcal{L}^2}\sigma\,\mathrm{d}s \tag{7.56}$$

as well as

$$\begin{aligned}
\lim_{k\to\infty}\frac{1}{2}\int_0^T \|(\phi,\psi)\|_{k,*}^2\partial_t\sigma\,\mathrm{d}s &= \lim_{k\to\infty}\frac{1}{2}\int_0^T \big(\mathcal{S}^k(\phi,\psi),(\phi,\psi)\big)_{\mathcal{L}^2}\partial_t\sigma\,\mathrm{d}s \\
&= \frac{1}{2}\int_0^T \big(\mathcal{S}_1(\phi,\psi),(\phi,\psi)\big)_{\mathcal{L}^2}\partial_t\sigma\,\mathrm{d}s \\
&= \frac{1}{2}\int_0^T \|(\phi,\psi)\|_{1,*}^2\partial_t\sigma\,\mathrm{d}s,
\end{aligned}\tag{7.57}$$

respectively. To handle the last two terms on the right-hand side of (7.54), we show that there exists a constant $C > 0$, independent of $k \in \mathbb{N}$, such that

$$\int_0^T \|(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2)\|_{L,\beta}^2 \,\mathrm{d}s \le C. \tag{7.58}$$

In fact, by (7.1), we have

$$\begin{aligned}
&\|\nabla(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2)\|_{\mathbf{L}^2(\Omega)} \\
&\quad\le \|m_\Omega''(\phi_1^k)\nabla\phi_1^k|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2\|_{\mathbf{L}^2(\Omega)} + 2\|m_\Omega'(\phi_1^k)D^2\mathcal{S}^{k,\Omega}(\phi,\psi)\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^2(\Omega)} \\
&\quad\le C\Big(\|\nabla\phi_1^k\|_{\mathbf{L}^6(\Omega)}\|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^6(\Omega)}^2 + \|D^2\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^4(\Omega)}\|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^4(\Omega)}\Big) \\
&\quad\le C\Big(\|\nabla\phi_1^k\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{3}}\|\phi_1^k\|_{H^2(\Omega)}^{\frac{2}{3}}\|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^2(\Omega)}^{\frac{2}{3}}\|\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{H^2(\Omega)}^{\frac{4}{3}} \\
&\qquad + \|D^2\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^4(\Omega)}\|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}}\|\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{H^2(\Omega)}^{\frac{1}{2}}\Big)
\end{aligned}$$

a.e. on $(0,T)$. Hence, applying the estimates (7.40) and (7.43), we find

$$\begin{aligned}
&\|\nabla(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2)\|_{\mathbf{L}^2(\Omega)} \\
&\quad\le C\Big(\|\phi_1^k\|_{H^2(\Omega)}^{\frac{2}{3}}\|\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{H^2(\Omega)}^{\frac{4}{3}} + \|D^2\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^4(\Omega)}\|\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{H^2(\Omega)}^{\frac{1}{2}}\Big) \\
&\quad\le C\Big(\|\phi_1^k\|_{H^2(\Omega)}^2 + \|\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{H^2(\Omega)}^2 + \|\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{W^{2,4}(\Omega)}^{\frac{4}{3}}\Big)
\end{aligned} \tag{7.59}$$

a.e. on $(0,T)$. In a similar manner, we derive the estimate

$$\begin{aligned}
&\|\nabla_\Gamma(m_\Gamma(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2)\|_{\mathbf{L}^2(\Gamma)} \\
&\quad\le C\Big(\|\psi_1^k\|_{H^2(\Gamma)}^2 + \|\mathcal{S}^{k,\Gamma}(\phi,\psi)\|_{H^2(\Gamma)}^2 + \|\mathcal{S}^{k,\Gamma}(\phi,\psi)\|_{W^{2,4}(\Gamma)}^{\frac{4}{3}}\Big)
\end{aligned} \tag{7.60}$$

a.e. on $(0,T)$. Lastly, we use the trace theorem and the Sobolev inequality to obtain

$$\begin{aligned}
&\|\beta m_\Gamma'(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2 - m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2\|_{L^2(\Gamma)} \\
&\quad\le C\|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)\|_{\mathbf{L}^4(\Gamma)}^2 + C\|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)\|_{\mathbf{L}^4(\Gamma)}^2 \\
&\quad\le C\|\mathcal{S}_k(\phi,\psi)\|_{\mathcal{H}^2}^2
\end{aligned} \tag{7.61}$$

a.e. on $(0,T)$. Thus, in conclusion, combining the estimates (7.59)-(7.61) yields

$$\begin{aligned}
&\|(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2)\|_{L,\beta} \\
&\quad\le C\Big(\|(\phi_1^k,\psi_1^k)\|_{\mathcal{H}^2}^2 + \|\mathcal{S}_k(\phi,\psi)\|_{\mathcal{H}^2}^2 + \|\mathcal{S}_k(\phi,\psi)\|_{\mathcal{W}^{2,4}}^{\frac{4}{3}}\Big)
\end{aligned} \tag{7.62}$$

a.e. on $(0,T)$, and the desired claim (7.58) readily follows in light of the estimates (7.42), (7.44) and (7.45). We then compute

$$\begin{aligned}
&\Big((\partial_t\phi_1^k,\partial_t\psi_1^k),(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2)\Big)_{\mathcal{L}^2} \\
&\quad= \Big(\mathcal{S}(\partial_t\phi_1^k,\partial_t\psi_1^k),(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2)\Big)_{L,\beta} \\
&\qquad+ \Big(\mathcal{S}(\partial_t(\phi_1^k-\phi_1),\partial_t(\psi_1^k-\psi_1)), \\
&\qquad\qquad\times(m_\Omega'(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2)\Big)_{L,\beta}
\end{aligned}$$

a.e. on $(0,T)$. Thus, multiplying the above identity by $\sigma \in C_c^\infty(0,T)$ and integrating in time over $(0,T)$, we take the limit $k \to \infty$ and obtain using (7.35)-(7.36) and (7.58) that

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^T \Big((\partial_t \phi_1^k, \partial_t \psi_1^k), (m_\Omega'(\phi_1^k)|\nabla \mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi)|^2) \Big)_{\mathcal{L}^2} \sigma \,\mathrm{d}s \qquad (7.63)\\
&= \lim_{k\to\infty} \int_0^T \Big(\mathcal{S}(\partial_t \phi_1^k, \partial_t \psi_1^k), (m_\Omega'(\phi_1^k)|\nabla \mathcal{S}^{k,\Omega}(\phi,\psi)|^2, m_\Gamma'(\psi_1^k)|\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi)|^2) \Big)_{L,\beta} \sigma \,\mathrm{d}s.
\end{aligned}$$

To compute the limit of the right-hand side of (7.63), we first note that by our above computations, specifically (7.59)-(7.61), there exist functions $h_1, \dots, h_5$ such that, along a non-relabeled subsequence $k \to \infty$,

$$m_\Omega''(\phi_1^k)|\nabla \mathcal{S}^{k,\Omega}(\phi,\psi)|^2 \to h_1 \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{7.64a}$$

$$2m_\Omega'(\phi_1^k) D^2\mathcal{S}^{k,\Omega}(\phi,\psi)\nabla \mathcal{S}^{k,\Omega}(\phi,\psi) \to h_2 \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)), \tag{7.64b}$$

$$m_\Gamma''(\psi_1^k)|\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi)|^2 \to h_3 \quad \text{weakly in } L^2(0,T;L^2(\Gamma)), \tag{7.64c}$$

$$2m_\Gamma'(\psi_1^k) D_\Gamma^2\mathcal{S}^{k,\Gamma}(\phi,\psi)\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi) \to h_4 \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\Gamma)), \tag{7.64d}$$

$$\beta m_\Gamma'(\psi_1^k)|\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi)|^2 - m_\Omega'(\phi_1^k)|\nabla \mathcal{S}^{k,\Omega}(\phi,\psi)|^2 \to h_5 \quad \text{weakly in } L^2(0,T;L^2(\Gamma)). \tag{7.64e}$$

As a last step, we are left with identifying the weak limits $h_1, \dots, h_5$. First, using (7.53) with $\varepsilon = \frac{1}{4}$, we find that

$$\mathcal{S}_k(\phi,\psi) \to \mathcal{S}_1(\phi,\psi) \qquad \text{strongly in } L^{\frac{16}{5}}(0,T;\mathcal{H}^{\frac{7}{4}})$$

as $k \to \infty$. In particular, this implies that

$$\big(|\nabla \mathcal{S}^{k,\Omega}(\phi,\psi)|^2, |\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi)|^2\big) \to \big(|\nabla \mathcal{S}_1^\Omega(\phi,\psi)|^2, |\nabla_\Gamma \mathcal{S}_1^\Gamma(\phi,\psi)|^2\big) \quad \text{strongly in } L^{\frac{16}{5}}(0,T;\mathcal{L}^3)$$

as $k \to \infty$. Next, due to (7.35)-(7.36), we have

$$(\nabla \phi_1^k, \nabla_\Gamma \psi_1^k) \to (\nabla \phi_1, \nabla_\Gamma \psi_1) \qquad \text{strongly in } L^4(0,T;\boldsymbol{\mathcal{L}}^3)$$

as $k \to \infty$. Finally, since $m_\Omega'', m_\Gamma'' \in C([-1,1])$, (7.38) implies that

$$(m_\Omega''(\phi_1^k), m_\Gamma''(\psi_1^k)) \to (m_\Omega''(\phi_1), m_\Gamma''(\psi_1)) \qquad \text{strongly in } L^8(0,T;\mathcal{L}^{12})$$

as $k \to \infty$. Thus, we readily infer that

$$m_\Omega''(\phi_1^k)\nabla\phi_1^k|\nabla \mathcal{S}^{k,\Omega}(\phi,\psi)|^2 \to m_\Omega''(\phi_1)\nabla\phi_1|\nabla \mathcal{S}_1^\Omega(\phi,\psi)|^2 \quad \text{weakly in } L^1(0,T;\mathbf{L}^1(\Omega)),$$

$$m_\Gamma''(\psi_1^k)\nabla_\Gamma\psi_1^k|\nabla_\Gamma \mathcal{S}^{k,\Gamma}(\phi,\psi)|^2 \to m_\Gamma''(\psi_1)\nabla_\Gamma\psi_1|\nabla_\Gamma \mathcal{S}_1^\Gamma(\phi,\psi)|^2 \quad \text{weakly in } L^1(0,T;\mathbf{L}^1(\Gamma))$$

as $k \to \infty$, from which we deduce that

$$h_1 = m''_\Omega(\phi_1)\nabla\phi_1|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2,$$

as well as

$$h_3 = m''_\Gamma(\psi_1)\nabla_\Gamma\psi_1|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2.$$

Similarly, owing to (7.49) and (7.51), and noting on

$$(m'_\Omega(\phi_1^k), m'_\Gamma(\psi_1^k)) \to (m'_\Omega(\phi_1), m'_\Gamma(\psi_1)) \qquad \text{strongly in } L^8(0,T;\mathcal{L}^4)$$

as $k \to \infty$, we obtain

$$\begin{aligned} 2m'_\Omega(\phi_1^k)D^2\mathcal{S}^{k,\Omega}(\phi,\psi)\nabla\mathcal{S}^{k,\Omega}(\phi,\psi) &\to 2m'_\Omega(\phi_1)D^2\mathcal{S}_1^\Omega(\phi,\psi)\nabla\mathcal{S}_1^\Omega(\phi,\psi) \\ &\qquad \text{weakly in } L^1(0,T;\mathbf{L}^1(\Omega)), \\ 2m'_\Gamma(\psi_1^k)D_\Gamma^2\mathcal{S}^{k,\Gamma}(\phi,\psi)\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi) &\to 2m'_\Gamma(\psi_1)D_\Gamma^2\mathcal{S}_1^\Gamma(\phi,\psi)\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi) \\ &\qquad \text{weakly in } L^1(0,T;\mathbf{L}^1(\Gamma)) \end{aligned}$$

as $k \to \infty$. This entails that

$$h_2 = 2m'_\Omega(\phi_1)D^2\mathcal{S}_1^\Omega(\phi,\psi)\nabla\mathcal{S}_1^\Omega(\phi,\psi)$$

and

$$h_4 = 2m'_\Gamma(\psi_1)D_\Gamma^2\mathcal{S}_1^\Gamma(\phi,\psi)\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi).$$

Lastly, to identify h_5 we note that given the aforementioned convergences and the trace theorem, we find that

$$\begin{aligned} |\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|_\Gamma|^2 &\to |\nabla\mathcal{S}_1^\Omega(\phi,\psi)|_\Gamma|^2 &&\text{strongly in } L^{\frac{16}{3}}(0,T;L^{\frac{3}{2}}(\Gamma)), \\ |\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2 &\to |\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 &&\text{strongly in } L^{\frac{16}{3}}(0,T;L^{\frac{3}{2}}(\Gamma)) \end{aligned}$$

as $k \to \infty$, and thus, readily deduce that

$$\begin{aligned} &\beta m'_\Gamma(\psi_1^k)|\nabla_\Gamma\mathcal{S}^{k,\Gamma}(\phi,\psi)|^2 - m'_\Omega(\phi_1^k)|\nabla\mathcal{S}^{k,\Omega}(\phi,\psi)|_\Gamma|^2 \\ &\quad \to \beta m'_\Gamma(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 - m'_\Omega(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|_\Gamma|^2 \qquad \text{weakly in } L^1(0,T;L^1(\Gamma)) \end{aligned}$$

as $k \to \infty$. This shows

$$h_5 = \beta m'_\Gamma(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 - m'_\Omega(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2.$$

Therefore, by exploiting (7.64a)-(7.64e) in (7.63), we conclude that

$$\begin{aligned} &\int_0^T \big(\mathcal{S}_1(\partial_t\phi,\partial_t\psi)(\phi,\psi)\big)_{\mathcal{L}^2}\sigma \,\mathrm{d}s \\ &\quad = \frac{1}{2}\int_0^T \|(\phi,\psi)\|_{1,*}\partial_t\sigma \,\mathrm{d}s \\ &\qquad - \frac{1}{2}\int_0^T \Big(\mathcal{S}(\partial_t\phi_1,\partial_t\psi_1), (m'_\Omega(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m'_\Gamma(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2)\Big)_{L,\beta}\sigma \,\mathrm{d}s \end{aligned}$$

for any $\sigma \in C_c^\infty(0,T)$. The latter readily implies the desired conclusion (7.32).

Estimates of the nonlinear terms. In the rest of the proof, the letter C denotes a generic positive constant that may change its value from line to line, and which depends on the parameter of the system and the initial energies $E_{\text{free}}(\phi_1^0, \psi_1^0)$ and $E_{\text{free}}(\phi_2^0, \psi_2^0)$. We now intend to bound the terms I_1, I_2 and I_3.

Regarding I_1, we use (4.4) to initially find

$$\begin{aligned} |I_1| &= \big(\mathcal{S}(\partial_t\phi_1, \partial_t\psi_1)), (m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2)\big)_{L,\beta} \\ &\le C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2)\|_{L,\beta} \end{aligned}$$

a.e. on $(0,\infty)$. For the second term on the right-hand side, we note that

$$\begin{aligned} &\|(m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2)\|_{L,\beta} \\ &\quad\le \|(m_\Omega''(\phi_1)\nabla\phi_1|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, m_\Gamma''(\psi_1)\nabla_\Gamma\psi_1|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2)\|_{\mathcal{L}^2} \\ &\qquad + \|(m_\Omega'(\phi_1)D^2\mathcal{S}_1^\Omega(\phi,\psi)\nabla\mathcal{S}_1^\Omega(\phi,\psi), m_\Gamma'(\psi_1)D_\Gamma^2\mathcal{S}_1^\Gamma(\phi,\psi)\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^2} \\ &\qquad + \chi(L)\|\beta m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 - m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2\|_{L^2(\Gamma)} \end{aligned}$$

a.e. on $(0,\infty)$, which entails

$$\begin{aligned} |I_1| &\le C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\nabla\phi_1|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2, \nabla_\Gamma\psi_1|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2)\|_{\mathcal{L}^2} \\ &\quad + C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(D^2\mathcal{S}_1^\Omega(\phi,\psi)\nabla\mathcal{S}_1^\Omega(\phi,\psi), D_\Gamma^2\mathcal{S}_1^\Gamma(\phi,\psi)\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^2} \\ &\quad + C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'} \\ &\qquad \times \chi(L)\|\beta m_\Gamma'(\psi_1)|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)|^2 - m_\Omega'(\phi_1)|\nabla\mathcal{S}_1^\Omega(\phi,\psi)|^2\|_{L^2(\Gamma)} \\ &=: J_1 + J_2 + J_3 \end{aligned}$$

a.e. on $(0,\infty)$. Utilizing (7.2) and owing to (4.78) in the case $K \in (0,\infty)$, we see that

$$\begin{aligned} &\|\mathcal{S}_1(\phi,\psi)\|_{\mathcal{H}^2} \\ &\quad\le C\Big(\|(\nabla\phi_1, \nabla_\Gamma\psi_1)\|_{\mathcal{L}^2}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta} + \|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac12}\|(\phi,\psi)\|_{K,\alpha}^{\frac12}\Big) \end{aligned} \tag{7.65}$$

a.e. on $(0,\infty)$. Then, concerning J_1, we employ the estimates (7.1) and (7.65) to find

$$\begin{aligned} J_1 &\le C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\nabla\phi_1, \nabla_\Gamma\psi_1)\|_{\mathcal{L}^6}\|(\nabla\mathcal{S}_1^\Omega(\phi,\psi), \nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^6}^2 \\ &\le C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\nabla\phi_1, \nabla_\Gamma\psi_1)\|_{\mathcal{L}^2}^{\frac13}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac23}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac23}\|\mathcal{S}_1(\phi,\psi)\|_{\mathcal{H}^2}^{\frac43} \\ &\le C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac23}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac23} \\ &\qquad \times \Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta} + \|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac12}\|(\phi,\psi)\|_{L,\beta}^{\frac12}\Big)^{\frac43} \\ &\le C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^2\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\ &\qquad + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac23}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac43}\|(\phi,\psi)\|_{K,\alpha}^{\frac23}\Big) \\ &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^2\|\mathcal{S}(\phi,\psi)\|_{L,\beta}^2 \\ &\qquad + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2\Big) \\ &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \end{aligned} \tag{7.66}$$

a.e. on $(0,\infty)$. On the other hand, if $K = \infty$, a similar argumentation with (4.78) leads to the estimate

$$J_1 \le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(1 + \|(\partial_t\phi_1, \partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2$$

a.e. on $(0,\infty)$, since (4.78) only contains a good Gronwall term. In the following we restrict ourselves to the case $K\in(0,\infty)$ out of brevity. Nevertheless, as seen above, the case $K=\infty$ can be handled in a similar manner. Next, we consider J_2. Due to (7.1) and (7.65) we have

$$\begin{aligned}
&\|(\nabla\mathcal{S}_1^\Omega(\phi,\psi),\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^4}\\
&\quad\leq C\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{\mathcal{H}^2}^{\frac12}\\
&\quad\leq C\Big(\|(\nabla\phi_1,\nabla_\Gamma\psi_1)\|_{\mathcal{L}^2}^{\frac12}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}+\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac34}\|(\phi,\psi)\|_{K,\alpha}^{\frac14}\Big)\\
&\quad\leq C\Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}+\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac34}\|(\phi,\psi)\|_{K,\alpha}^{\frac34}\Big)
\end{aligned}\tag{7.67}$$

a.e. on $(0,\infty)$. Then, exploiting

$$\begin{aligned}
\|(\phi,\psi)\|_{\mathcal{L}^4}&\leq C\|(\phi,\psi)\|_{\mathcal{L}^2}^{\frac12}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12}\\
&\leq C\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac14}\|(\phi,\psi)\|_{K,\alpha}^{\frac14}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12}\\
&\leq C\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac14}\|(\phi,\psi)\|_{K,\alpha}^{\frac34}
\end{aligned}\tag{7.68}$$

a.e. on $(0,\infty)$, which follows from the bulk-surface Poincaré inequality, (4.78) and (7.1), we obtain similarly from (7.3), using again (7.65), that

$$\begin{aligned}
&\|\mathcal{S}_1(\phi,\psi)\|_{\mathcal{W}^{2,4}}\\
&\quad\leq C\Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac14}\|\mathcal{S}_1(\phi,\psi)\|_{\mathcal{H}^2}^{\frac34}+\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac14}\|(\phi,\psi)\|_{K,\alpha}^{\frac34}\Big)\\
&\quad\leq C\Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac32}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}+\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac34}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac58}\|(\phi,\psi)\|_{L,\beta}^{\frac38}\\
&\qquad+\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac14}\|(\phi,\psi)\|_{K,\alpha}^{\frac34}\Big)
\end{aligned}\tag{7.69}$$

a.e. on $(0,\infty)$. Consequently, noticing that

$$\begin{aligned}
J_2&\leq C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(D^2\mathcal{S}_1^\Omega(\phi,\psi),D^2_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^4}\\
&\quad\times\|(\nabla\mathcal{S}_1^\Omega(\phi,\psi),\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^4}
\end{aligned}$$

a.e. on $(0,\infty)$, we deduce from (7.67) and (7.69) that

$$\begin{aligned}
J_2&\leq C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^2\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2\\
&\quad+C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac32}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac74}\|(\phi,\psi)\|_{K,\alpha}^{\frac14}\\
&\quad+C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac54}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac{13}{8}}\|(\phi,\psi)\|_{K,\alpha}^{\frac38}\\
&\quad+C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac34}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac{11}{8}}\|(\phi,\psi)\|_{K,\alpha}^{\frac58}\\
&\quad+C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac54}\|(\phi,\psi)\|_{K,\alpha}^{\frac34}\\
&\quad+C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}\|(\phi,\psi)\|_{K,\alpha}\\
&=:K_1+\ldots+K_6
\end{aligned}$$

a.e. on $(0,\infty)$. Next, we control the terms $K_1,\ldots,K_6$ separately. Applying Young's inequality, we readily find that

$$K_1\leq C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2+\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2,$$

$$
\begin{aligned}
K_2 &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^{\frac{8}{7}}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac{12}{7}}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2, \\
K_3 &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^{\frac{16}{13}}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac{20}{13}}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2, \\
K_4 &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^{\frac{16}{11}}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac{12}{11}}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2, \\
K_5 &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^{\frac{8}{5}}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac{4}{5}}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2, \\
K_6 &\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2
\end{aligned}
$$

a.e. on $(0,\infty)$, respectively. Therefore, we conclude that

$$
J_2 \le \frac{5}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \tag{7.70}
$$

a.e. on $(0,\infty)$. For J_3, we can simply use the trace theorem and the Sobolev inequality in combination with (4.52) and (7.65) to find that

$$
\begin{aligned}
J_3 &= C\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\chi(L)\|\beta m_\Gamma'(\psi_1)\big|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\big|^2 - m_\Omega'(\phi_1)\big|\nabla\mathcal{S}_1^\Omega(\phi,\psi)\big|^2\|_{L^2(\Gamma)} \\
&\le C\chi(L)\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\Big(\|\nabla\mathcal{S}_1^\Omega(\phi,\psi)\|_{\mathbf{L}^4(\Gamma)} + \|\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathbf{L}^4(\Gamma)}^2\Big) \\
&\le C\chi(L)\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|\mathcal{S}_1(\phi,\psi)\|_{\mathcal{H}^2}^2 \qquad (7.71) \\
&\le C\chi(L)\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^2\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 + \|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}\|(\phi,\psi)\|_{K,\alpha}\Big) \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\chi(L)\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^2\Big) \\
&\quad \times \|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\chi(L)\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2
\end{aligned}
$$

a.e. on $(0,\infty)$. Collecting the the estimates (7.66), (7.70) and (7.71), we obtain the following bound for I_1

$$
|I_1| \le \frac{7}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}^1_{L,\beta})'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \tag{7.72}
$$

a.e. on $(0,\infty)$. Next, concerning I_2, we employ again (7.1), (7.67) and (7.68), and deduce

$$
\begin{aligned}
|I_2| &= \Big| -\int_\Omega (m_\Omega(\phi_1) - m_\Omega(\phi_2))\nabla\mu_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x \\
&\qquad - \int_\Gamma (m_\Gamma(\psi_1) - m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big| \\
&\le \|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\mathcal{L}^2}\|(m_\Omega(\phi_1) - m_\Omega(\phi_2), m_\Gamma(\psi_1) - m_\Gamma(\psi_2))\|_{\mathcal{L}^4} \\
&\qquad \times \|(\nabla\mathcal{S}_1^\Omega(\phi,\psi),\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\mathcal{L}^4}
\end{aligned}
$$

$$\begin{aligned}
&\le C\|(\nabla\mu_2, \nabla_\Gamma\theta_2)\|_{\mathcal{L}^2}\|(\phi,\psi)\|_{\mathcal{L}^4}\|(\nabla\mathcal{S}_1^\Omega(\phi,\psi), \nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\boldsymbol{\mathcal{L}}^4} \\
&\le C\|(\mu_2,\theta_2)\|_{L,\beta}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac14}\|(\phi,\psi)\|_{K,\alpha}^{\frac34} \qquad (7.73)\\
&\quad \times \Big(\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta} + \|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac34}\|(\phi,\psi)\|_{K,\alpha}^{\frac14}\Big) \\
&\le C\|(\mu_2,\theta_2)\|_{L,\beta}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac54}\|(\phi,\psi)\|_{K,\alpha}^{\frac34} \\
&\quad + C\|(\mu_2,\theta_2)\|_{L,\beta}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}\|(\phi,\psi)\|_{K,\alpha} \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\mu_2,\theta_2)\|_{L,\beta}^{\frac85}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac45} + \|(\mu_2,\theta_2)\|_{L,\beta}^2\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \\
&\le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(\|(\mu_2,\theta_2)\|_{L,\beta}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2
\end{aligned}$$

a.e. on $(0,\infty)$. Finally, for I_3, we first observe that

$$\begin{aligned}
|I_3| &= \Big|\int_\Omega (\phi_1\mathbf{v} + \phi\mathbf{v}_2)\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma (\psi_1\mathbf{w} + \psi\mathbf{w}_2)\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big| \\
&\le \Big|\int_\Omega \nabla\phi_1\cdot\mathbf{v}\,\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi_1\cdot\mathbf{w}\,\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big| \qquad (7.74)\\
&\quad + \Big|\int_\Omega \phi\mathbf{v}_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big|
\end{aligned}$$

a.e. on $(0,\infty)$. For the first term on the right-hand side of (7.74), we use Hölder's, Sobolev's and the bulk-surface Poincaré inequality to deduce that

$$\begin{aligned}
&\Big|\int_\Omega \nabla\phi_1\cdot\mathbf{v}\,\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi_1\cdot\mathbf{w}\,\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big| \\
&\le \|\nabla\phi_1\|_{\mathbf{L}^{\frac{2(1+\omega)}{\omega}}(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^{1+\omega}(\Omega)}\|\mathcal{S}_1^\Omega(\phi,\psi)\|_{L^{\frac{2(1+\omega)}{\omega}}(\Omega)} \\
&\qquad + \|\nabla_\Gamma\psi_1\|_{\mathbf{L}^\infty(\Gamma)}\|\mathbf{w}\|_{\mathbf{L}^1(\Gamma)}\|\mathcal{S}_1^\Gamma(\phi,\psi)\|_{L^\infty(\Gamma)} \\
&\le C\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{Y}}_\omega}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta} \\
&\le \frac12\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{Y}}_\omega}^2 + C_\omega\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^2\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2
\end{aligned}$$

a.e. on $(0,\infty)$ for any $\omega > 0$, where $\boldsymbol{\mathcal{Y}}_\omega = \mathbf{L}^{1+\omega}(\Omega)\times\mathbf{L}^1(\Gamma)$, and $C_\omega > 0$ is a constant with the same dependencies as C but additionally depends on ω. For the second term on the right-hand side of (7.74), we use a similar argument to [96, Lemma 5.1], see also [94]. However, since in our case we have non-degenerate mobility functions, additional terms are coming from (7.2) that have to be handled suitably. To be precise, exploiting the bulk-surface Poincaré inequality, (4.78), (7.1) and (7.2), we find that

$$\begin{aligned}
&\Big|\int_\Omega \phi\mathbf{v}_2\cdot\nabla\mathcal{S}_1^\Omega(\phi,\psi)\,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}_2\cdot\nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi)\,\mathrm{d}\Gamma\Big| \\
&\le \|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\|(\phi,\psi)\|_{\mathcal{L}^4}\|(\nabla\mathcal{S}_1^\Omega(\phi,\psi), \nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\boldsymbol{\mathcal{L}}^4} \\
&\le C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\|(\phi,\psi)\|_{\mathcal{L}^2}^{\frac12}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12}\|(\nabla\mathcal{S}_1^\Omega(\phi,\psi), \nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\boldsymbol{\mathcal{L}}^2}^{\frac12} \\
&\quad \times \|(\nabla\mathcal{S}_1^\Omega(\phi,\psi), \nabla_\Gamma\mathcal{S}_1^\Gamma(\phi,\psi))\|_{\boldsymbol{\mathcal{H}}^1}^{\frac12} \\
&\le C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\|(\phi,\psi)\|_{\mathcal{L}^2}^{\frac12}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac12} \\
&\quad \times \Big(\|(\nabla\phi_1,\nabla_\Gamma\psi_1)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac12}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac12} + \|(\phi,\psi)\|_{\mathcal{L}^2}^{\frac12}\Big) \\
&\le C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\Big(\|(\phi,\psi)\|_{\mathcal{L}^2}^{\frac12}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12}\|(\nabla\phi_1,\nabla_\Gamma\psi_1)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac12}\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta} \\
&\quad + \|(\phi,\psi)\|_{\mathcal{L}^2}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^{\frac12}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\mathcal{L}^2}\Big(1+\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^{\frac{1}{2}}\Big)\|(\phi,\psi)\|_{\mathcal{H}^1}\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}\\
&\le \frac{1}{8}\|(\phi,\psi)\|_{K,\alpha}^2 + C\Big(1+\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\mathcal{L}^2}^2+\|(\phi_1,\psi_1)\|_{\mathcal{H}^2}\Big)\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2
\end{aligned}
$$

a.e. on $(0,\infty)$. Lastly, utilizing Proposition 4.4.5 along with Young's inequality, we find

$$
C\|(\phi,\psi)\|_{\mathcal{L}^2} \le \frac{1}{18}\|(\phi,\psi)\|_{K,\alpha}^2 + C\|\mathcal{S}_1(\phi,\psi)\|_{L,\beta}^2 \tag{7.75}
$$

a.e. on $(0,\infty)$. Recalling the equivalence of the respective norms on $\mathcal{V}_{L,\beta}^{-1}$, we combine the differential inequality (7.33) with the estimates (7.72), (7.73) and (7.75), and end up with

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{1,*}^2 + \frac{1}{2}\|(\phi,\psi)\|_{K,\alpha}^2 \le \frac{1}{2}\|(\mathbf{v},\mathbf{w})\|_{\mathfrak{Y}_\omega}^2 + Q(t)\|(\phi,\psi)\|_{1,*}^2 \tag{7.76}
$$

a.e. on $(0,\infty)$, where

$$
Q(\cdot) = C\Big(1+\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\mathcal{L}^2}^2 + \|(\mu_2,\theta_2)\|_{L,\beta}^2 + \|(\partial_t\phi_1,\partial_t\psi_1)\|_{(\mathcal{H}_{L,\beta}^1)'}^2 + \|(\phi_1,\psi_1)\|_{\mathcal{H}^2}^4\Big).
$$

Since $Q\in L^1(0,T)$ for any $T>0$, and noting again the equivalence of the respective norms on $\mathcal{V}_{L,\beta}^{-1}$, an application of Gronwall's lemma entails for any $T>0$ that

$$
\begin{aligned}
&\|(\phi_1(t)-\phi_2(t),\psi_1(t)-\psi_2(t))\|_{(\mathcal{H}_{L,\beta}^1)'}^2\\
&\quad\le \|(\phi_1^0-\phi_2^0,\psi_1^0-\psi_2^0)\|_{(\mathcal{H}_{L,\beta}^1)'}^2\exp\Big(\int_0^t Q(r)\,\mathrm{d}r\Big)\\
&\qquad+\int_0^t\|(\mathbf{v}_1(s)-\mathbf{v}_2(s),\mathbf{w}_1(s)-\mathbf{w}_2(s)\|_{\mathfrak{Y}_\omega}^2\exp\Big(\int_s^t Q(r)\,\mathrm{d}r\Big)\,\mathrm{d}s
\end{aligned}
$$

for all $t\in[0,T]$. Consequently, if $\phi_1^0=\phi_2^0$ a.e. in Ω and $\psi_1^0=\psi_2^0$ a.e. on Γ, the above inequality yields the uniqueness of weak solutions. $\square$

7.5 Existence of strong solutions and separation properties

In this section, we prove Theorem 7.3.3 regarding the existence of strong solutions.

Similarly to the proof of Theorem 6.2.7, the proof of Theorem 7.3.3 is two-fold. First, for velocity fields satisfying the assumption **(A7.1)**, the existence of a strong solution is based on the differential inequality (7.76) using time difference quotients. Then, for the assumption **(A7.2)**, we use again a suitable approximation of these.

Proof of Theorem 7.3.3. **The case with assumption (A7.1).**

To prove the existence of strong solution, we make use of the differential inequality (7.76) proven in the uniqueness part of Theorem 7.3.1. First, for brevity, we use the notation

$$
\begin{aligned}
&\mathcal{S}_h(f,g) = \mathcal{S}_{L,\beta}^{\phi(\cdot+h),\psi(\cdot+h)}(f,g),\\
&\|\cdot\|_h = \|\cdot\|_{L,\beta}^{\phi(\cdot+h),\psi(\cdot+h)},\\
&\|\cdot\|_{h,*} = \|\cdot\|_{L,\beta,*}^{\phi(\cdot+h),\psi(\cdot+h)}
\end{aligned}
$$

for $h \in [0,1)$. Then for any $h \in (0,1)$, we apply the estimate (7.76) to $(\phi_1,\psi_1) = (\phi(\cdot+h),\psi(\cdot+h))$ and $(\phi_2,\psi_2) = (\phi,\psi)$ with corresponding velocity fields $(\mathbf{v}_1,\mathbf{w}_1) = (\mathbf{v}(\cdot+h),\mathbf{w}(\cdot+h))$ and $(\mathbf{v}_2,\mathbf{w}_2) = (\mathbf{v},\mathbf{w})$, respectively. Dividing the resulting inequality by h^2, and denoting the difference quotient in time of a function f again by $\partial_t^h f(t) = \frac{1}{h}(f(t+h)-f(t))$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{h,*}^2 + \frac{1}{2}\|(\partial_t^h\phi,\partial_t^h\psi)\|_{K,\alpha}^2 \\ &\quad\le \frac{1}{2}\|(\partial_t^h\mathbf{v},\partial_t^h\mathbf{w})\|_{\boldsymbol{\mathcal{Y}}_\omega}^2 + Q_h(t)\|(\partial_t^h\phi,\partial_t^h\psi)\|_{h,*}^2\end{aligned}\tag{7.77}$$

a.e. on $(0,\infty)$, where, now,

$$\begin{aligned}Q_h(\cdot) = C\Big(&1 + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\mu,\theta)\|_{L,\beta}^2 + \|(\partial_t\phi(\cdot+h),\partial_t\psi(\cdot+h))\|_{(\mathcal{H}^1_{L,\beta})'}^2 \\ &+ \|(\phi(\cdot+h),\psi(\cdot+h))\|_{\mathcal{H}^2}^4\Big) \in L^1(0,T) \quad \text{for all } T>0.\end{aligned}$$

Before applying the uniform Gronwall lemma to (7.77), we aim to control the initial data. To this end, we can argue similarly as for (7.33). Now, instead, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi-\phi_0,\psi-\psi_0)\|_{0,*}^2 + \|(\phi-\phi_0,\psi-\psi_0)\|_{K,\alpha}^2 \\ &\quad\le C\|(\phi-\phi_0,\psi-\psi_0)\|_{\mathcal{L}^2}^2 + \big((\mu_0,\theta_0),\mathcal{S}_0(\phi-\phi_0,\psi-\psi_0)\big)_{L,\beta,\phi,\psi} \\ &\qquad + \int_\Omega \phi\mathbf{w}\cdot\nabla\mathcal{S}_0^\Omega(\phi-\phi_0,\psi-\psi_0)\,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\mathcal{S}_0^\Gamma(\phi-\phi_0,\psi-\psi_0)\,\mathrm{d}\Gamma \\ &\qquad - \frac{1}{2}\Big(\mathcal{S}_0(\partial_t\phi,\partial_t\psi), \\ &\qquad\qquad (m_\Omega'(\phi)|\nabla\mathcal{S}_0^\Omega(\phi-\phi_0,\psi-\psi_0)|^2, m_\Gamma'(\psi)|\nabla_\Gamma\mathcal{S}_0^\Gamma(\phi-\phi_0,\psi-\psi_0)|^2)\Big)_{L,\beta}\end{aligned}\tag{7.78}$$

a.e. on $(0,\infty)$. For the first term on the right-hand side of (7.78), we simply use Proposition 4.4.5 and get

$$C\|(\phi-\phi_0,\psi-\psi_0)\|_{\mathcal{L}^2}^2 \le \frac{1}{4}\|(\phi-\phi_0,\psi-\psi_0)\|_{K,\alpha}^2 + C\|(\phi-\phi_0,\psi-\psi_0)\|_{0,*}^2$$

a.e. on $(0,\infty)$, while for the second term we directly have

$$\big|\big((\mu_0,\theta_0),\mathcal{S}_0(\phi-\phi_0,\psi-\psi_0)\big)_{L,\beta,\phi,\psi}\big| \le C\|(\mu_0,\theta_0)\|_{L,\beta}\|(\phi-\phi_0,\psi-\psi_0)\|_{L,\beta,*}$$

a.e. on $(0,\infty)$. Next, for the third term on the right-hand side of (7.78), we exploit the facts $|\phi|<1$ a.e. in Q and $|\psi|<1$ a.e. on Σ, and obtain with Hölder's inequality

$$\begin{aligned}&\Big|\int_\Omega \phi\mathbf{w}\cdot\nabla\mathcal{S}_0^\Omega(\phi-\phi_0,\psi-\psi_0)\,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\mathcal{S}_0^\Gamma(\phi-\phi_0,\psi-\psi_0)\,\mathrm{d}\Gamma\Big| \\ &\quad\le \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\|(\phi-\phi_0,\psi-\psi_0)\|_{L,\beta,*}\end{aligned}$$

a.e. on $(0,\infty)$, whereas the last term can be estimated as I_1 in the proof of Theorem 7.3.1, see (7.72). Namely, we have

$$\begin{aligned}&\Big| -\frac{1}{2}\Big(\mathcal{S}_0(\partial_t\phi,\partial_t\psi), \\ &\qquad (m_\Omega'(\phi)|\nabla\mathcal{S}_0^\Omega(\phi-\phi_0,\psi-\psi_0)|^2, m_\Gamma'(\psi)|\nabla_\Gamma\mathcal{S}_0^\Gamma(\phi-\phi_0,\psi-\psi_0)|^2)\Big)_{L,\beta}\Big| \\ &\quad\le \frac{1}{4}\|(\phi-\phi_0,\psi-\psi_0)\|_{K,\alpha}^2\end{aligned}$$

$$+ C\Big(\|(\partial_t\phi,\partial_t\psi)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi,\psi)\|^4_{\mathcal{H}^2}\Big)\|(\phi-\phi_0,\psi-\psi_0)\|^2_{L,\beta,*}$$

a.e. on $(0,\infty)$. Collecting these estimates leads to the following differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi-\phi_0,\psi-\psi_0)\|^2_{0,*} \le R(t)\|(\phi-\phi_0,\psi-\psi_0)\|_{0,*}$$

for a.e. $t\in(0,\infty)$, where

$$\begin{aligned} R(\cdot) \coloneqq C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta} + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} + \Big(\|(\partial_t\phi,\partial_t\psi)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi,\psi)\|^4_{\mathcal{H}^2}\Big) \\ \times \|(\phi-\phi_0,\psi-\psi_0)\|_{0,*}\Big). \end{aligned}$$

An application of a quadratic variant of the Gronwall lemma as presented in Lemma 2.2.6 yields

$$\begin{aligned} &\|(\phi-\phi_0,\psi-\psi_0)\|_{0,*} \\ &\quad\le C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta} + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}\Big)t \\ &\qquad + C\int_0^t \Big(\|(\partial_t\phi,\partial_t\psi)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi,\psi)\|^4_{\mathcal{H}^2}\Big)\|(\phi-\phi_0,\psi-\psi_0)\|_{0,*}\,\mathrm{d}s \end{aligned}$$

a.e. on $(0,\infty)$. Then, taking $t=h$ and dividing the resulting inequality by h, we obtain

$$\begin{aligned} &\|(\partial_t^h\phi(0),\partial_t^h\psi(0))\|_{0,*} \\ &\quad\le C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta} + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}\Big) \\ &\qquad + C\int_0^h \Big(\|(\partial_t\phi,\partial_t\psi)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi,\psi)\|^4_{\mathcal{H}^2}\Big)\|(\partial_t^h\phi(0),\partial_t^h\psi(0))\|_{0,*}\,\mathrm{d}s. \end{aligned}$$

This enables us to apply the integrated version of the Gronwall inequality from Lemma 2.2.5 to deduce

$$\begin{aligned} &\|(\partial_t^h\phi(0),\partial_t^h\psi(0))\|_{0,*} \\ &\quad\le C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta} + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}\Big) \\ &\qquad \times\Bigg(1 + \int_0^h \Big(\|(\partial_t\phi,\partial_t\psi)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi,\psi)\|^4_{\mathcal{H}^2}\Big) \\ &\qquad\qquad \times \exp\Big(\int_s^h \Big(\|(\partial_t\phi,\partial_t\psi)\|^2_{(\mathcal{H}^1_{L,\beta})'} + \|(\phi,\psi)\|^4_{\mathcal{H}^2}\Big)\,\mathrm{d}r\Big)\,\mathrm{d}s\Bigg). \end{aligned} \tag{7.79}$$

Now that we have a suitable control of the initial data, we use again the Gronwall lemma for the differential inequality (7.77) and get

$$\begin{aligned} \|(\partial_t^h\phi,\partial_t^h\psi)\|^2_{h,*} &\le \|(\partial_t^h\phi(0),\partial_t^h\psi(0))\|^2_{h,*}\exp\Big(C\int_0^t Q_h(r)\,\mathrm{d}r\Big) \\ &\quad + C\int_0^t \|(\partial_t^h\mathbf{v},\partial_t^h\mathbf{w})\|^2_{\boldsymbol{\mathcal{Y}}_\omega}\exp\Big(C\int_s^t Q_h(r)\,\mathrm{d}r\Big)\,\mathrm{d}s \end{aligned} \tag{7.80}$$

a.e. on $(0,\infty)$. Then, noting on

$$\begin{aligned} &\|(\partial_t^h\mathbf{v}(t),\partial_t^h\mathbf{w}(t))\|_{\boldsymbol{\mathcal{Y}}_\omega} \le \frac{1}{h}\int_t^{t+h}\|(\partial_t\mathbf{v}(s),\partial_t\mathbf{w}(s))\|_{\boldsymbol{\mathcal{Y}}_\omega}\,\mathrm{d}s, \\ &\lim_{h\to0}\frac{1}{h}\int_t^{t+h}\|(\partial_t\mathbf{v}(s),\partial_t\mathbf{w}(s))\|_{\boldsymbol{\mathcal{Y}}_\omega}\,\mathrm{d}s = \|(\partial_t\mathbf{v}(t),\partial_t\mathbf{w}(t))\|_{\boldsymbol{\mathcal{Y}}_\omega} \end{aligned} \tag{7.81}$$

for a.e. $t \in (0,\infty)$, and (7.79), we observe that the right-hand side of (7.80) is uniformly bounded in $h \in (0,1)$. Moreover, by (6.55), (6.56) and (6.60), for any $h \in (0,1)$, we have

$$\int_0^t Q_h(r)\,\mathrm{d}r \le C\int_0^t 1 + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2\,\mathrm{d}r \tag{7.82}$$

for a.e. $t \in (0,\infty)$. Consequently, taking the limit $h \to 0$ in (7.80), and noting on (7.79), (7.81) and (7.82) yields

$$\begin{aligned}
&\|(\partial_t\phi,\partial_t\psi)\|_{L,\beta,*}^2 \\
&\quad\le C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta}^2 + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2 + \int_0^t \|(\partial_t\mathbf{v},\partial_t\mathbf{w})\|_{\boldsymbol{\mathcal{Y}}_\omega}^2\,\mathrm{d}r\Big) \\
&\qquad\times \exp\Big(C\int_0^t 1 + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2\,\mathrm{d}r\Big)
\end{aligned}\tag{7.83}$$

a.e. on $(0,\infty)$. Thus, we find a constant $C_0 > 0$, independent of h, such that

$$\|(\partial_t\phi(t),\partial_t\psi(t))\|_{(\mathcal{H}^1_{L,\beta})'}^2 \le C_0 \tag{7.84}$$

for a.e. $t \in (0,1)$. To derive a global control in time, we intend to use again the uniform Gronwall lemma as in Lemma 2.2.7. To this end, utilizing again (6.55), (6.56) and (6.59) and the assumptions on the velocity fields **(A6)**–**(A7.1)**, we derive similarly to the proof of Theorem 6.2.7 the estimates

$$\begin{aligned}
&\sup_{t\ge0}\int_t^{t+1} P_h(s)\,\mathrm{d}s \le C\big(1 + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2\big), \\
&\sup_{t\ge0}\int_t^{t+1} \|(\partial_t^h\phi(s),\partial_t^h\psi(s))\|_{h,*}^2\,\mathrm{d}s \le C\big(1 + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2\big), \\
&\sup_{t\ge0}\int_t^{t+1} \|(\partial_t^h\mathbf{v}(s),\partial_t^h\mathbf{w}(s))\|_{\boldsymbol{\mathcal{Y}}_\omega}^2\,\mathrm{d}s \le \|(\partial_t\mathbf{v},\partial_t\mathbf{w})\|_{L^2_{\mathrm{uloc}}([0,\infty);\boldsymbol{\mathcal{Y}}_\omega)}^2.
\end{aligned}\tag{7.85}$$

Consequently, an application of Lemma 2.2.7 with $t_0 = 0$ and $r = 1$ shows that

$$\begin{aligned}
&\|(\partial_t\phi(t),\partial_t\psi(t))\|_{L,\beta,*}^2 \\
&\quad\le C\Big(1 + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2 + \|(\partial_t\mathbf{v},\partial_t\mathbf{w})\|_{L^2_{\mathrm{uloc}}([0,\infty);\boldsymbol{\mathcal{Y}}_\omega)}^2\Big) \\
&\qquad\times \exp\Big(C\big(1 + \|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2\big)\Big)
\end{aligned}\tag{7.86}$$

for a.e. $t \ge 1$. Here, we have already taken the limit $h \to 0$ and used again the norm equivalence on $\mathcal{V}^{-1}_{L,\beta}$ (see (4.56)). The latter implies together with (7.84) that

$$(\partial_t\phi,\partial_t\psi) \in L^\infty(0,\infty;(\mathcal{H}^1_{L,\beta})'), \tag{7.87}$$

and the estimate (7.86) holds for a.e. $t \in (0,\infty)$. Next, we integrate (7.77) over the time interval $[t,t+1]$ for $t \ge 0$, and utilize the estimates (7.85) and (7.86). By passing to the limit $h \to 0$ in the corresponding estimate, we deduce that

$$\begin{aligned}
&\sup_{t\ge0}\int_t^{t+1} \|(\partial_t\phi(s),\partial_t\psi(s))\|_{K,\alpha}^2\,\mathrm{d}s \\
&\quad\le C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta}^2 + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2 + \|(\partial_t\mathbf{v},\partial_t\mathbf{w})\|_{L^2_{\mathrm{uloc}}([0,\infty);\boldsymbol{\mathcal{Y}}_\omega)}^2\Big) \\
&\qquad\times \Big(1 + \|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2\Big)\exp\Big(C\big(1 + \|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)}^2\big)\Big),
\end{aligned}\tag{7.88}$$

which gives, employing (4.78), (7.87), that

$$(\partial_t\phi, \partial_t\psi) \in L^2_{\mathrm{uloc}}([0,\infty); \mathcal{H}^1)$$

Then, a comparison argument in (6.11a) together with (7.86) and the regularity $(\mathbf{v}, \mathbf{w}) \in L^\infty(0,\infty; \boldsymbol{\mathcal{L}}^2)$ further shows that

$$\|(\mu,\theta)\|_{L,\beta} \le C\Big(\|(\partial_t\phi, \partial_t\psi)\|_{L^\infty(0,\infty;(\mathcal{H}^1_{L,\beta})')} + \|(\mathbf{v},\mathbf{w})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)}\Big) \tag{7.89}$$

a.e. on $(0,\infty)$. Thus, in light of (6.57) and (7.86), we have

$$(\mu,\theta) \in L^\infty(0,\infty; \mathcal{H}^1)$$

by exploiting the bulk-surface Poincaré inequality. By (6.58), we learn that

$$(\phi,\psi) \in L^\infty(0,\infty; \mathcal{W}^{2,p}), \qquad (F'(\phi), G'(\psi)) \in L^\infty(0,\infty; \mathcal{L}^p) \tag{7.90}$$

for any $2 \le p < \infty$. Moreover, by arguing as for Theorem 6.2.10, we find

$$(F''(\phi), G''(\psi)) \in L^\infty(0,\infty; \mathcal{L}^p)$$

for any $2 \le p < \infty$ (see also [129, 132]), from which we deduce that

$$(\phi,\psi) \in L^\infty(0,\infty; \mathcal{H}^3)$$

by elliptic regularity theory for systems with bulk-surface coupling. Lastly, to derive higher-order estimates for the chemical potentials, we first notice that

$$(\mu - \beta\mathrm{mean}(\mu,\theta), \theta - \mathrm{mean}(\mu,\theta)) = -\mathcal{S}_0(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w}). \tag{7.91}$$

Then, by (4.52), (4.78), (7.2), (7.89) and (7.90), we have

$$\begin{aligned}
&\|(\mu - \beta\mathrm{mean}(\mu,\theta), \theta - \mathrm{mean}(\mu,\theta))\|_{\mathcal{H}^2} \\
&\quad = \|\mathcal{S}_0(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w})\|_{\mathcal{H}^2} \\
&\quad \le C(\|(\nabla\phi, \nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^2}\|(\phi,\psi)\|_{\mathcal{H}^2}\|\mathcal{S}_0(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w})\|_{L,\beta} \\
&\qquad + \|(\partial_t\phi + \nabla\phi\cdot\mathbf{v}, \partial_t\psi + \nabla_\Gamma\psi\cdot\mathbf{w})\|_{\mathcal{L}^2}) \\
&\quad \le C\Big(\|(\mu,\theta)\|_{L,\beta} + \|(\partial_t\phi, \partial_t\psi)\|_{\mathcal{L}^2} + \|(\nabla\phi\cdot\mathbf{v}, \nabla_\Gamma\psi\cdot\mathbf{w})\|_{\mathcal{L}^2}\Big) \\
&\quad \le C\Big(1 + \|(\partial_t\phi, \partial_t\psi)\|_{\mathcal{L}^2} + \|(\nabla\phi, \nabla_\Gamma\psi)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\Big) \\
&\quad \le C\Big(1 + \|\mathcal{S}_0(\partial_t\phi, \partial_t\psi)\|_{L,\beta}^{\frac12}\|(\partial_t\phi, \partial_t\psi)\|_{K,\alpha}^{\frac12} + \|\mathcal{S}_0(\partial_t\phi, \partial_t\psi)\|_{L,\beta} + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\Big) \\
&\quad \le C\Big(1 + \|(\partial_t\phi, \partial_t\psi)\|_{K,\alpha}^{\frac12}\Big)
\end{aligned} \tag{7.92}$$

a.e. on $(0,\infty)$. Integrating the previous estimate in time from t to $t+1$ for $t \ge 0$, it follows that

$$\sup_{t\ge0}\int_t^{t+1} \|(\mu - \beta\mathrm{mean}(\mu,\theta), \theta - \mathrm{mean}(\mu,\theta))\|_{\mathcal{H}^2}^4 \,\mathrm{d}s \le C. \tag{7.93}$$

In light of (6.57), the latter entails that $(\mu,\theta) \in L^4_{\mathrm{uloc}}([0,\infty); \mathcal{H}^2)$.

The case with assumption (A7.2).

Now, we assume that the velocity fields $(\mathbf{v}, \mathbf{w})$ satisfy the assumptions stated in **(A7.2)**. Then, as we have already seen in the proof of Theorem 6.2.7, we can construct a sequence $\{(\mathbf{v}_k, \mathbf{w}_k)\}_{k\in\mathbb{N}} \subset C_c^\infty(0,\infty;\boldsymbol{\mathcal{H}}^1 \cap \boldsymbol{\mathcal{L}}^2_{\mathrm{div}})$ such that

$$(\mathbf{v}_k, \mathbf{w}_k) \to (\mathbf{v}, \mathbf{w}) \qquad \begin{aligned} &\text{strongly in } L^2(0,\infty;\boldsymbol{\mathcal{H}}^1), \\ &\text{weakly-star in } L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2) \end{aligned} \tag{7.94}$$

as $k \to \infty$. Moreover, it holds that

$$\|(\mathbf{v}_k, \mathbf{w}_k)\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)\cap L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)} \le \|(\mathbf{v}, \mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)\cap L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)} \tag{7.95}$$

for all $k \in \mathbb{N}$.

Now, owing to the first part of the proof, there exists a unique global weak solution $(\phi_k, \psi_k, \mu_k, \theta_k)$ to (6.1) with velocity fields $(\mathbf{v}_k, \mathbf{w}_k)$, which satisfies the additional regularities

$$(\partial_t\phi_k, \partial_t\psi_k) \in L^\infty(0,\infty;(\mathcal{H}^1_{L,\beta})') \cap L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1), \tag{7.96a}$$
$$(\phi_k, \psi_k) \in L^\infty(0,\infty;\mathcal{H}^3), \tag{7.96b}$$
$$(\mu_k, \theta_k) \in L^\infty(0,\infty;\mathcal{H}^1_{L,\beta}) \cap L^4_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2), \tag{7.96c}$$
$$(F'(\phi_k), G'(\psi_k)) \in L^\infty(0,\infty;\mathcal{L}^p), (F''(\phi_k), G''(\psi_k)) \in L^\infty(0,\infty;\mathcal{L}^p) \tag{7.96d}$$

for any $2 \le p < \infty$ and all $k \in \mathbb{N}$. With these regularities at hand, one can deduce similarly to the proof of Theorem 6.2.7 the existence of the time derivative $(\partial_t\mu_k, \partial_t\theta_k)$ for all $k \in \mathbb{N}$ in the sense that $(\partial_t\mu_k, \partial_t\theta_k) \in L^2_{\mathrm{uloc}}([0,\infty);(\mathcal{H}^1)')$, and it satisfies

$$\begin{aligned} \langle(\partial_t\mu_k, \partial_t\theta_k), (\eta, \eta_\Gamma)\rangle_{\mathcal{H}^1} = &\int_\Omega \nabla\partial_t\phi_k \cdot \nabla\eta + F''(\phi_k)\partial_t\phi_k\eta \,\mathrm{d}x \\ &+ \int_\Gamma \nabla_\Gamma\partial_t\psi_k \cdot \nabla_\Gamma\eta_\Gamma + G''(\psi_k)\partial_t\psi_k\eta_\Gamma \,\mathrm{d}\Gamma \\ &+ \chi(K)\int_\Gamma (\alpha\partial_t\psi_k - \partial_t\phi_k)(\alpha\eta_\Gamma - \eta) \,\mathrm{d}\Gamma \end{aligned} \tag{7.97}$$

a.e. on $(0,\infty)$ for all $(\eta, \eta_\Gamma) \in \mathcal{H}^1$. In the following, the letter C will denote generic positive constants that are independent of k.

Now, as we have already seen in the proof of Theorem 6.2.3, see in particular Step 9, it follows from (7.95) that

$$\begin{aligned} &\|(\phi_k, \psi_k)\|_{L^\infty(0,\infty;\mathcal{H}^1)} + \|(\nabla\mu_k, \nabla_\Gamma\theta_k)\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} + \chi(L)^{\frac12}\|\beta\theta_k - \mu_k\|_{L^2(0,\infty;L^2(\Gamma))} \\ &\quad + \|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^2(0,\infty;(\mathcal{H}^1_{L,\beta})')} + \|(\phi_k, \psi_k)\|_{L^4_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2)} \le C \end{aligned} \tag{7.98}$$

as well as

$$\begin{aligned} &\|(\phi_k, \psi_k)\|_{\mathcal{W}^{2,p}} + \|(F'(\phi_k), G'(\psi_k))\|_{\mathcal{L}^p} + |\mathrm{mean}(\mu_k, \theta_k)| \\ &\quad \le C\Big(1 + \|(\mu_k, \theta_k)\|_{L,\beta}\Big) \end{aligned} \tag{7.99}$$

a.e. on $(0,\infty)$ for any $2 \le p < \infty$. To establish the higher regularity estimates, we aim to apply the chain rule established in Proposition 7.2.2. To this end, we first have to show that $(\mu_k, \theta_k) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^3)$. Recalling

$$(\mu_k - \beta\mathrm{mean}(\mu_k, \theta_k), \theta_k - \mathrm{mean}(\mu_k, \theta_k))$$

$$= -\mathcal{S}_{L,\beta}^{\phi_k,\psi_k}(\partial_t\phi_k + \nabla\phi_k \cdot \mathbf{v}_k, \partial_t\psi_k + \nabla_\Gamma\psi_k \cdot \mathbf{w}_k),$$

the estimate (4.68) entails that

$$\begin{aligned}
&\|(\mu_k - \beta\mathrm{mean}(\mu_k,\theta_k), \theta_k - \mathrm{mean}(\mu_k,\theta_k))\|_{\mathcal{H}^3} \\
&\quad\le C\Big(1 + \mathbf{1}_{\{0\}}(L)\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}\Big)\bigg(\Big\|\Big(\frac{\partial_t\phi_k + \nabla\phi_k\cdot\mathbf{v}_k}{m_\Omega(\phi_k)}, \frac{\partial_t\psi_k + \nabla_\Gamma\psi_k\cdot\mathbf{w}_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\qquad + \Big\|\Big(\frac{m_\Omega'(\phi_k)\nabla\phi_k\cdot\nabla\mu_k}{m_\Omega(\phi_k)}, \frac{m_\Gamma'(\psi_k)\nabla_\Gamma\psi_k\cdot\nabla_\Gamma\theta_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1}\bigg) \\
&\quad\le C\Big(1 + \mathbf{1}_{\{0\}}(L)\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}\Big)\Big\|\Big(\frac{\partial_t\phi_k}{m_\Omega(\phi_k)}, \frac{\partial_t\psi_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\qquad + C\Big(1 + \mathbf{1}_{\{0\}}(L)\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}\Big)\Big\|\Big(\frac{\nabla\phi_k\cdot\mathbf{v}_k}{m_\Omega(\phi_k)}, \frac{\nabla_\Gamma\psi_k\cdot\mathbf{w}_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\qquad + C\Big(1 + \mathbf{1}_{\{0\}}(L)\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}\Big)\Big\|\Big(\frac{m_\Omega'(\phi_k)\nabla\phi_k\cdot\nabla\mu}{m_\Omega(\phi_k)}, \frac{m_\Gamma'(\psi_k)\nabla_\Gamma\psi_k\cdot\nabla_\Gamma\theta_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\quad =: I_1 + I_2 + I_3
\end{aligned} \tag{7.100}$$

a.e. on $(0,\infty)$. By standard computations based on Hölder's inequality, and using in particular the boundedness of the mobility functions, we can estimate

$$\begin{aligned}
&\Big\|\Big(\frac{\partial_t\phi_k}{m_\Omega(\phi_k)}, \frac{\partial_t\psi_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\quad\le \Big\|\Big(\frac{\partial_t\phi_k}{m_\Omega(\phi_k)}, \frac{\partial_t\psi_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{L}^2} + \Big\|\Big(\frac{m_\Omega'(\phi_k)\partial_t\phi_k\nabla\phi_k}{m_\Omega(\phi_k)^2}, \frac{m_\Gamma'(\psi_k)\partial_t\psi_k\nabla_\Gamma\psi_k}{m_\Gamma(\psi_k)^2}\Big)\Big\|_{\mathcal{L}^2} \\
&\qquad + \Big\|\Big(\frac{\nabla\partial_t\phi_k}{m_\Omega(\phi_k)}, \frac{\nabla_\Gamma\partial_t\psi_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{L}^2} \\
&\quad\le C\Big(\|(\partial_t\phi_k,\partial_t\psi_k)\|_{\mathcal{L}^2} + \|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{\mathcal{L}^2} + \|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}\Big)
\end{aligned} \tag{7.101}$$

a.e. on $(0,\infty)$,

$$\begin{aligned}
&\Big\|\Big(\frac{\nabla\phi_k\cdot\mathbf{v}_k}{m_\Omega(\phi_k)}, \frac{\nabla_\Gamma\psi_k\cdot\mathbf{w}_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\quad\le \Big\|\Big(\frac{\nabla\phi_k\cdot\mathbf{v}_k}{m_\Omega(\phi_k)}, \frac{\nabla_\Gamma\psi\cdot\mathbf{w}_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{L}^2} \\
&\qquad + \Big\|\Big(\frac{m_\Omega'(\phi_k)(\nabla\phi_k\cdot\mathbf{v}_k)\nabla\phi_k}{m_\Omega(\phi_k)^2}, \frac{m_\Gamma'(\psi_k)(\nabla_\Gamma\psi_k\cdot\mathbf{w}_k)\nabla_\Gamma\psi_k}{m_\Gamma(\psi_k)^2}\Big)\Big\|_{\boldsymbol{\mathcal{L}}^2} \\
&\qquad + \Big\|\Big(\frac{\nabla(\nabla\phi_k\cdot\mathbf{v}_k)}{m_\Omega(\phi_k)}, \frac{\nabla_\Gamma(\nabla_\Gamma\psi_k\cdot\mathbf{w}_k)}{m_\Gamma(\psi_k)}\Big)\Big\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad\le C\Big(\|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} + \|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}^2\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\qquad + \|(\phi_k,\psi_k)\|_{\mathcal{W}^{2,4}}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^3} + \|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1}\Big) \\
&\quad\le C\|(\phi_k,\psi_k)\|_{\mathcal{W}^{2,4}}\Big(1 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1} + \|(\phi_k,\psi_k)\|_{\mathcal{W}^{2,4}}\Big)
\end{aligned} \tag{7.102}$$

a.e. on $(0,\infty)$, as well as

$$\begin{aligned}
&\Big\|\Big(\frac{m_\Omega'(\phi_k)\nabla\phi_k\cdot\nabla\mu_k}{m_\Omega(\phi_k)}, \frac{m_\Gamma'(\psi_k)\nabla_\Gamma\psi_k\cdot\nabla_\Gamma\theta_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{H}^1} \\
&\quad\le \Big\|\Big(\frac{m_\Omega'(\phi_k)\nabla\phi_k\cdot\nabla\mu_k}{m_\Omega(\phi_k)}, \frac{m_\Gamma'(\psi_k)\nabla_\Gamma\psi_k\cdot\nabla_\Gamma\theta_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\mathcal{L}^2}
\end{aligned}$$

$$
\begin{aligned}
&+\Big\|\Big(\frac{(m_\Omega''(\phi_k)m_\Omega(\phi_k)-m_\Omega'(\phi_k)^2)(\nabla\phi_k\cdot\nabla\mu_k)\nabla\phi_k}{m_\Omega(\phi_k)^2},\\
&\qquad\frac{(m_\Gamma''(\psi_k)m_\Gamma(\psi_k)-m_\Gamma'(\psi_k)^2)(\nabla_\Gamma\psi_k\cdot\nabla_\Gamma\theta_k)\nabla_\Gamma\psi_k}{m_\Gamma(\psi_k)^2}\Big)\Big\|_{\boldsymbol{\mathcal{L}}^2}\\
&+\Big\|\Big(\frac{m_\Omega'(\phi_k)D^2\phi_k\nabla\mu_k}{m_\Omega(\phi_k)},\frac{m_\Gamma'(\psi_k)D_\Gamma^2\psi_k\nabla_\Gamma\theta_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\boldsymbol{\mathcal{L}}^2} \qquad (7.103)\\
&+\Big\|\Big(\frac{m_\Omega'(\phi_k)D^2\mu_k\nabla\phi_k}{m_\Omega(\phi_k)},\frac{m_\Gamma'(\psi_k)D_\Gamma^2\theta_k\nabla_\Gamma\psi_k}{m_\Gamma(\psi_k)}\Big)\Big\|_{\boldsymbol{\mathcal{L}}^2}\\
&\le C\Big(\|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^2}+\|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}^2\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^2}\\
&+\|(\phi_k,\psi_k)\|_{\mathcal{W}^{2,4}}\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^4}\\
&+\|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^\infty}\|(\mu_k-\beta\mathrm{mean}(\mu_k,\theta_k),\theta_k-\mathrm{mean}(\mu_k,\theta_k))\|_{\mathcal{H}^2}\Big)
\end{aligned}
$$

a.e. on $(0,\infty)$. Recalling the Sobolev embedding $\mathcal{W}^{2,4}\hookrightarrow\mathcal{W}^{1,\infty}$, and exploiting (7.95) and (7.96), the estimates (7.99) and (7.101)–(7.103) entail that $(\mu_k,\theta_k)\in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^3)$. We stress that the corresponding estimate on the $\mathcal{H}^3$-norm is not yet uniform in $k\in\mathbb{N}$. In view of this regularity, we are in a position to apply Proposition 7.2.2 which yields $(\mu_k,\theta_k)\in BC([0,\infty);\mathcal{H}^1_{L,\beta})$, and the following chain rule

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma+\chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma\Big)\\
&=\big\langle(\partial_t\mu_k,\partial_t\theta_k),(-\mathrm{div}(m_\Omega(\phi_k)\nabla\mu_k),-\mathrm{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\psi_k)+\beta m_\Omega(\phi_k)\partial_{\mathbf{n}}\mu_k)\big\rangle_{\mathcal{H}^1}\\
&\quad+\int_\Omega m_\Omega'(\phi_k)\partial_t\phi_k|\nabla\mu_k|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma'(\psi_k)\partial_t\psi_k|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma \qquad (7.104)
\end{aligned}
$$

holds a.e. on $(0,\infty)$. As $(\phi_k,\psi_k,\mu_k,\theta_k)$ is a strong solution to the corresponding convective Cahn–Hilliard equation, the dual pairing on the right-hand side of (7.104) can be rewritten, under the consideration of (7.97) as

$$
\begin{aligned}
&\big\langle(\partial_t\mu_k,\partial_t\theta_k),(-\mathrm{div}(m_\Omega(\phi_k)\nabla\mu_k),-\mathrm{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\psi_k)+\beta m_\Omega(\phi_k)\partial_{\mathbf{n}}\mu_k)\big\rangle_{\mathcal{H}^1}\\
&\quad=-\big\langle(\partial_t\mu_k,\partial_t\theta_k),(\partial_t\phi_k+\mathrm{div}(\phi_k\mathbf{v}_k),\partial_t\psi_k+\mathrm{div}_\Gamma(\psi_k\mathbf{w}_k))\big\rangle_{\mathcal{H}^1}\\
&\quad=-\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2-\int_\Omega F''(\phi_k)|\partial_t\phi_k|^2\,\mathrm{d}x-\int_\Gamma G''(\psi_k)|\partial_t\psi_k|^2\,\mathrm{d}\Gamma \qquad (7.105)\\
&\qquad-\big\langle(\partial_t\mu_k,\partial_t\theta_k),(\mathrm{div}(\phi_k\mathbf{v}_k),\mathrm{div}_\Gamma(\psi_k\mathbf{w}_k))\big\rangle_{\mathcal{H}^1}
\end{aligned}
$$

a.e. on $(0,\infty)$. Hence, exploiting the assumptions on the potentials and recalling (6.104), it follows from (7.104) and (7.105) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma+\chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma\Big)\\
&\quad+\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2\\
&\le C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{\mathcal{L}^2}+\int_\Omega m_\Omega'(\phi_k)\partial_t\phi_k|\nabla\mu_k|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma'(\psi_k)\partial_t\psi_k|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma\\
&\quad-\big\langle(\partial_t\mu_k,\partial_t\theta_k),(\mathrm{div}(\phi_k\mathbf{v}_k),\mathrm{div}_\Gamma(\psi_k\mathbf{w}_k))\big\rangle_{\mathcal{H}^1}+\int_\Omega F'(\phi_k)\nabla\partial_t\phi_k\cdot\mathbf{v}_k\,\mathrm{d}x \qquad (7.106)\\
&\quad+\int_\Gamma G'(\psi_k)\nabla_\Gamma\partial_t\psi_k\cdot\mathbf{w}_k\,\mathrm{d}\Gamma
\end{aligned}
$$

a.e. on $(0,\infty)$. For the first term on the right-hand side of (7.106), we have already seen in the first part of the proof that this can be estimated as

$$
C\|(\phi_k,\psi_k)\|_{\mathcal{L}^2}^2\le\frac{1}{6}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+C\Big(\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2+\|(\mu_k,\theta_k)\|_{L,\beta}^2\Big). \qquad (7.107)
$$

Then, to control the next two terms on the right-hand side of (7.106), we first compute as for (7.92)

$$\begin{aligned}&\|(\mu_k-\beta\mathrm{mean}(\mu_k,\theta_k),\theta_k-\mathrm{mean}(\mu_k,\theta_k))\|_{\mathcal{H}^2}\\&\quad\le C\Big(1+\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\|(\mu_k,\theta_k)\|_{L,\beta}\\&\qquad+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac12}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac12}\Big)\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^{\frac12}+C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\end{aligned}\tag{7.108}$$

a.e. on $(0,\infty)$. Now, we use (7.1), the previous inequality and another comparison argument to compute

$$\begin{aligned}&\Big|\int_\Omega m_\Omega'(\phi_k)\partial_t\phi_k|\nabla\mu_k|^2\,\mathrm{d}x+\int_\Gamma m_\Gamma'(\psi_k)\partial_t\psi_k|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma\Big|\\&\quad\le C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{\mathcal{L}^2}\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^4}^2\\&\quad\le C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{\mathcal{L}^2}\|(\mu_k,\theta_k)\|_{L,\beta}\|(\mu_k-\beta\mathrm{mean}(\mu_k,\theta_k),\theta_k-\mathrm{mean}(\mu_k,\theta_k))\|_{\mathcal{H}^2}\\&\quad\le C\Big(\|\mathcal{S}_{L,\beta}^{\phi_k,\psi_k}(\partial_t\phi_k,\partial_t\psi_k)\|_{L,\beta}^{\frac12}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^{\frac12}+\|\mathcal{S}_{L,\beta}^{\phi_k,\psi_k}(\partial_t\phi_k,\partial_t\psi_k)\|_{L,\beta}\Big)\\&\qquad\qquad\times\|(\mu_k,\theta_k)\|_{L,\beta}\|(\mu_k-\beta\mathrm{mean}(\mu_k,\theta_k),\theta_k-\mathrm{mean}(\mu_k,\theta_k))\|_{\mathcal{H}^2}\\&\quad\le C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^{\frac12}\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac52}\Big(1+\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\\&\qquad+C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}\Big(\|(\mu_k,\theta_k)\|_{L,\beta}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\|(\mu_k,\theta_k)\|_{L,\beta}\\&\qquad+C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^{\frac12}\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac12}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac12}\Big)\|(\mu_k,\theta_k)\|_{L,\beta}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\\&\qquad+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\|(\mu_k,\theta_k)\|_{L,\beta}^2\\&\qquad\qquad\times\Big(1+\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\\&\qquad+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac32}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac32}\Big)\|(\mu_k,\theta_k)\|_{L,\beta}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^{\frac12}\\&\qquad+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\|(\mu_k,\theta_k)\|_{L,\beta}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\\&\quad=:R_1+\ldots+R_6\end{aligned}\tag{7.109}$$

a.e. on $(0,\infty)$. Next, we control the terms $R_1,\ldots,R_6$ separately. Applying Young's inequality, we readily find that

$$\begin{aligned}R_1&\le\frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+C\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac{10}{3}}\Big(1+\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}^{\frac43}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac43}\Big)\\&\le\frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+\Big(1+(\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}^{\frac43}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^{\frac43})\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac43}\Big)\\&\qquad\times\|(\mu_k,\theta_k)\|_{L,\beta}^2\\&\le\frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+\Big(1+\|(\phi_k,\psi_k)\|_{\mathcal{H}^2}^4+\|(\mu_k,\theta_k)\|_{L,\beta}^2+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^4\Big)\\&\qquad\times\|(\mu_k,\theta_k)\|_{L,\beta}^2,\\R_2&\le\frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^2+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\Big)\|(\mu_k,\theta_k)\|_{L,\beta}^2,\\R_3&\le\frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\\&\qquad\times\|(\mu_k,\theta_k)\|_{L,\beta}^2\\&\le\frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2+C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2+C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^2+\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}\Big)\\&\qquad\times\|(\mu_k,\theta_k)\|_{L,\beta}^2,\end{aligned}$$

$$
\begin{aligned}
R_4 &\le C\Big(1 + \|(\phi_k,\psi_k)\|_{\mathcal{H}^2}^4 + \|(\mu_k,\theta_k)\|_{L,\beta}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\Big)\|(\mu_k,\theta_k)\|_{L,\beta}^2,\\
R_5 &\le \frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2 + C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\Big)\|(\mu_k,\theta_k)\|_{L,\beta}^{\frac{4}{3}}\\
&\le \frac{1}{30}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2 + C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\Big)\\
&\qquad \times \|(\mu_k,\theta_k)\|_{L,\beta}^2,\\
R_6 &\le C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} + C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\Big)\|(\mu_k,\theta_k)\|_{L,\beta}^2
\end{aligned}
$$

a.e. on $(0,\infty)$. Therefore, we conclude that

$$
\begin{aligned}
&\Big|\int_\Omega m_\Omega'(\phi_k)\partial_t\phi_k|\nabla\mu_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma'(\psi_k)\partial_t\psi_k|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma\Big|\\
&\quad \le \frac{1}{4}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2 + C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\\
&\qquad + C\Big(1 + \|(\phi_k,\psi_k)\|_{\mathcal{H}^2}^4 + \|(\mu_k,\theta_k)\|_{L,\beta}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^4\Big)\|(\mu_k,\theta_k)\|_{L,\beta}^2
\end{aligned}
\tag{7.110}
$$

a.e. on $(0,\infty)$. Then, for the last term on the right-hand side of (7.106), we have already seen in the second part of the proof of Theorem 6.2.7, see (6.111) and (6.112), that

$$
\begin{aligned}
&\big|\big((\partial_t\phi_k,\partial_t\psi_k),(\mathrm{div}(\phi_k\mathbf{v}_k),\mathrm{div}_\Gamma(\psi_k\mathbf{w}_k))\big)_{K,\alpha}\big|\\
&\qquad + \Big|\int_\Omega F'(\phi_k)\nabla\partial_t\phi_k\cdot\mathbf{v}_k\,\mathrm{d}x + \int G'(\psi_k)\nabla_\Gamma\partial_t\psi_k\cdot\mathbf{w}_k\,\mathrm{d}\Gamma\Big|\\
&\quad \le \frac{1}{6}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2 + C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1}^2 + C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\mu_k,\theta_k)\|_{L,\beta}^2
\end{aligned}
\tag{7.111}
$$

a.e. on $(0,\infty)$. Thus, collecting the estimates (7.107), (7.110) and (7.111), we end up with the differential inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma\Big)\\
&\qquad + \frac{1}{2}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2\\
&\quad \le C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1}^2\\
&\qquad + Q_k(t)\Big(\int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma\Big)
\end{aligned}
\tag{7.112}
$$

a.e. on $(0,\infty)$, where

$$
Q_k(\cdot) \coloneqq C\Big(1 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^4 + \|(\phi_k,\psi_k)\|_{\mathcal{H}^2}^4 + \|(\mu_k,\theta_k)\|_{L,\beta}^2\Big).
$$

Applying Gronwall's lemma yields

$$
\begin{aligned}
\|(\mu_k,\theta_k)\|_{L,\beta}^2 &\le C\|(\mu_k(0),\theta_k(0))\|_{L,\beta}^2\exp\Big(\int_0^t Q_k(r)\,\mathrm{d}r\Big)\\
&\qquad + C\int_0^t \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^1}^2\exp\Big(\int_s^t Q_k(r)\,\mathrm{d}r\Big)\,\mathrm{d}s
\end{aligned}
$$

a.e. on $(0,\infty)$. As $(\mu_k(0),\theta_k(0)) = (\mu_0,\theta_0)$ for all $k\in\mathbb{N}$, we readily infer from (7.95) and (7.98) that

$$
\begin{aligned}
\sup_{t\in[0,1]}\|(\mu_k(t),\theta_k(t)\|_{L,\beta}^2 &\le C\Big(1 + \|(\mu_0,\theta_0)\|_{L,\beta}^2 + \|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2\Big)\\
&\qquad \times \exp\Big(C(1 + \|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2)\Big).
\end{aligned}
\tag{7.113}
$$

To derive a global control in time, we aim to use again the uniform Gronwall inequality from Lemma 2.2.7. In view of (7.95) and (7.98), we deduce similarly to the first part of this proof that

$$\begin{aligned}\sup_{t\geq 1}\|(\mu_k(t),\theta_k(t))\|_{L,\beta}^2 &\leq C\Big(1+\|(\mu_0,\theta_0)\|_{L,\beta}^2+\|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2\Big)\\ &\qquad\qquad \times\exp\Big(C\big(1+\|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2\big)\Big).\end{aligned} \tag{7.114}$$

Next, integrating (7.112) in time from t to $t+1$ for any $t\geq 0$, we readily deduce on account of (7.113), and (7.114) that

$$\begin{aligned}&\sup_{t\geq 0}\int_t^{t+1}\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2\,\mathrm{d}s\\ &\quad\leq C\Big(1+\|(\mu_0,\theta_0)\|_{L,\beta}^2+\|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2\Big)\\ &\qquad\quad\times\Big(C\big(1+\|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2\big)\Big)\exp\Big(C\big(1+\|(\mathbf{v},\mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)}^2\big)\Big).\end{aligned} \tag{7.115}$$

By (4.78), (7.95), (7.113), (7.114) and (7.115), we then conclude

$$\|(\partial_t\phi_k,\partial_t\psi_k)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1)}\leq C.$$

Now, based on the estimates (7.113) and (7.114), it follows from (7.99) that

$$\|(\phi_k,\psi_k)\|_{L^\infty(0,\infty;\mathcal{W}^{2,p})}+\|(F'(\phi_k),G'(\psi_k))\|_{L^\infty(0,\infty;\mathcal{L}^p)}\leq C_p \tag{7.116}$$

for any $2\leq p<\infty$. Moreover, by arguing as in the proof of Theorem 6.2.10, we have

$$\|(F''(\phi_k),G''(\psi_k))\|_{L^\infty(0,\infty;\mathcal{L}^p)}+\|(\phi_k,\psi_k)\|_{L^\infty(0,\infty;\mathcal{H}^3)}\leq C_p'.$$

Finally, coming back to (7.100) and to (7.108) with the estimates (7.101)–(7.103), we deduce with (7.95), (7.99), (7.113), (7.114) and (7.115) that

$$\|(\mu_k,\theta_k)\|_{L^4_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2)}+\|(\mu_k,\theta_k)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^3)}\leq C.$$

By standard compactness arguments, these estimates guarantee the existence of a limit quadruple (ϕ,ψ,μ,θ) solving (6.1) and satisfying the desired regularity properties. By Theorem 7.3.1, (ϕ,ψ,μ,θ) is thus the unique strong solution to (6.1), which finishes the proof. □

Chapter 8

Existence of a pullback attractor and convergence to equilibrium for weak solutions

This chapter is based on the work in:

[119] P. Knopf, A. Poiatti, J. Stange, and S. Yayla, *Long-time dynamics of a bulk-surface convective Cahn–Hilliard system: Pullback attractors and convergence to equilibrium.* Preprint: arXiv:2603.10923, 2026.

8.1 Introduction

In the previous chapters, we analyzed the following bulk-surface Cahn–Hilliard system:

$$\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \Delta\mu \qquad \text{in } Q, \tag{8.1a}$$

$$\mu = -\Delta\phi + F'(\phi) \qquad \text{in } Q, \tag{8.1b}$$

$$\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) = \Delta_\Gamma\theta - \beta\partial_\mathbf{n}\mu \qquad \text{on } \Sigma, \tag{8.1c}$$

$$\theta = -\Delta_\Gamma\psi + G'(\psi) + \alpha\partial_\mathbf{n}\phi \qquad \text{on } \Sigma, \tag{8.1d}$$

$$\begin{cases} K\partial_\mathbf{n}\phi = \alpha\psi - \phi & \text{if } K \in [0,\infty), \\ \partial_\mathbf{n}\phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{on } \Sigma, \tag{8.1e}$$

$$\begin{cases} L\partial_\mathbf{n}\mu = \beta\theta - \mu & \text{if } L \in [0,\infty), \\ \partial_\mathbf{n}\mu = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Sigma, \tag{8.1f}$$

$$\phi|_{t=\tau} = \phi_\tau \qquad \text{in } \Omega, \tag{8.1g}$$

$$\psi|_{t=\tau} = \psi_\tau \qquad \text{on } \Gamma. \tag{8.1h}$$

For this system, we established the existence of weak solutions, uniqueness, and the propagation of regularity. In the present chapter, we complete the analysis by studying the long-time behavior of weak solutions.

For bulk-surface Cahn–Hilliard models without convection, the long-time dynamics are by now well understood, and a comprehensive body of literature is available, see, for instance, [70, 86, 126, 129, 131, 132, 137, 163]. We also refer to [40, 72, 90, 145] for the long-time behavior of further Cahn–Hilliard-type systems with dynamic boundary conditions.

In contrast, only few results are available for convective bulk-surface Cahn–Hilliard systems. In [50], a viscous convective model with dynamic boundary conditions was studied, and it was shown that the corresponding ω-limit set is non-empty and consists of stationary solutions. Another contribution is [182], where the long-time behavior of a convective Cahn–Hilliard model in two spatial dimensions was analyzed. However, the convective term considered there has a very specific structure and differs significantly from the one appearing in (8.1).

A major difficulty in the analysis of the long-time behavior of weak solutions to (8.1) arises from the fact that the underlying dynamical system is non-autonomous. Unless the velocity fields are time-independent, the evolution cannot be described by a semigroup. For this reason, we need to move from the framework of semigroups and global attractors to the more general theory of two-parameter processes and pullback attractors. In addition, the presence of convection prevents the total free energy E_{free} in general from being a Lyapunov functional, as it is no longer monotonically decreasing along trajectories. This significantly complicates the analysis of the asymptotic behavior, since classical arguments based on energy dissipation are no longer directly applicable.

To address these challenges, we consider the system starting from an arbitrary initial time $\tau \in \mathbb{R}$. Our first objective is to show that the dynamical system of the underlying system (8.1) can be interpreted as a continuous two-parameter process. In the non-convective case, i.e, when $(\mathbf{v}, \mathbf{w}) \equiv (\mathbf{0}, \mathbf{0})$, this process reduces to a continuous semigroup, reflecting the autonomous nature of the problem. Using abstract results from the theory of non-autonomous dynamical systems, we then prove that this process admits a minimal pullback attractor. We also refer to [113], where a similar approach was used to establish the existence of a minimal pullback attractor for a non-autonomous Cahn–Hilliard–Darcy system with mass source.

In the second part of this chapter, we investigate the convergence to equilibrium. We show that every weak solution to (8.1) converges, as $t \to \infty$, to a single steady state. In particular, the ω-limit set of each trajectory is a singleton. The proof relies on a Łojasiewicz–Simon inequality and the fact that stationary solutions are uniformly separated from the pure phases. Due to the non-autonomous nature of the system and the lack of a Lyapunov structure, it is necessary to derive suitable decay estimates tailored to the problem. In this context, a key role plays the asymptotic behavior of the velocity fields $(\mathbf{v}, \mathbf{w})$, which are required to satisfy a suitable decay condition as $t \to \infty$, see **(A9)**.

Finally, we remark that convergence to equilibrium in non-autonomous Cahn-Hilliard-type systems has also been studied in [113] for a two-dimensional Cahn–Hilliard–Darcy model with mass source. However, the setting there differs from the present one, as the non-autonomous effects enter through a source term rather than through convection.

8.2 Additional preliminaries

In this section, we introduce additional notions needed to study the long-time behavior of weak solutions to (8.1)

First, let (X, d) be a metric space. Then for any sets $A, B \subset X$, the *Hausdorff semidistance* is defined as

$$\operatorname{dist}_X(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b)\,.$$

Next, we introduce the phase space, on which our process will be defined. To this end, let $\alpha, \beta \in \mathbb{R}$, $K, L \in [0, \infty]$ as well as $m \in \mathbb{R}$ if $L \in [0, \infty)$ and $m = (m_1, m_2) \in \mathbb{R}^2$

if $L = \infty$. We define

$$\mathcal{W}^1_{K,L,\alpha,\beta,m} := \begin{cases} \{(\phi,\psi) \in \mathcal{H}^1_{K,\alpha} \,|\, \mathrm{mean}(\phi,\psi) = m\} & \text{if } L \in [0,\infty), \\ \{(\phi,\psi) \in \mathcal{H}^1_{K,\alpha} \,|\, \langle\phi\rangle_\Omega = m_1, \ \langle\psi\rangle_\Gamma = m_2\} & \text{if } L = \infty, \end{cases}$$

which is a closed subset of $\mathcal{H}^1$. Hence, it is a complete metric space with respect to the norm-induced metric on $\mathcal{H}^1$. For any $s > 1$, we further set

$$\mathcal{W}^s_{K,L,\alpha,\beta,m} := \mathcal{W}^1_{K,L,\alpha,\beta,m} \cap \mathcal{H}^s.$$

Additionally, we define

$$\mathcal{X}_{K,L,\alpha,\beta,m} := \{(\phi,\psi) \in \mathcal{W}^1_{K,L,\alpha,\beta,m} : \|\phi\|_{L^\infty(\Omega)} \leq 1, \ \|\psi\|_{L^\infty(\Gamma)} \leq 1\}.$$

The metric $\mathrm{d}_{\mathcal{X}_{K,L,\alpha,\beta,m}}$ is defined as follows:

$$\begin{aligned} \mathrm{d}_{\mathcal{X}_{K,L,\alpha,\beta,m}}((\phi,\psi),(\zeta,\zeta_\Gamma)) := \|(\phi,\psi) - (\zeta,\zeta_\Gamma)\|_{\mathcal{H}^1} + \Big| \int_\Omega F_1(\phi) \,\mathrm{d}x - \int_\Omega F_1(\zeta) \,\mathrm{d}x \Big| \\ + \Big| \int_\Gamma G_1(\psi) \,\mathrm{d}\Gamma - \int_\Gamma G_1(\zeta_\Gamma) \,\mathrm{d}\Gamma \Big|, \end{aligned}$$

and $(\mathcal{X}_{K,L,\alpha,\beta,m}, \mathrm{d}_{\mathcal{X}_{K,L,\alpha,\beta,m}}(\cdot,\cdot))$ is thus a complete metric space.

8.3 Main results

8.3.1 Assumptions

We start by stating our main assumptions needed throughout this chapter.

(A1) Ω is a non-empty, bounded domain in $\mathbb{R}^d$ with $d \in \{2,3\}$, whose boundary $\Gamma := \partial\Omega$ is at least of class C^3. Moreover, $\tau \in \mathbb{R}$ is a prescribed initial time, and we use the notation

$$Q_\tau := \Omega \times (\tau,\infty), \qquad \Sigma_\tau := \Gamma \times (\tau,\infty).$$

(A2) The constants in system (8.1) satisfy $K, L \in [0,\infty]$, $\alpha \in [-1,1]$ and $\beta \in \mathbb{R}$ with $\alpha\beta|\Omega| + |\Gamma| \neq 0$.

(A3) For the mobility functions, we assume that both m_Ω and m_Γ are constant. Without loss of generality, we thus simply assume that $m_\Omega = m_\Gamma \equiv 1$.

(A4) It holds $F_1, G_1 \in C([-1,1]) \cap C^2(-1,1)$ and

$$\lim_{s\searrow -1} F_1'(s) = \lim_{s\searrow -1} G_1'(s) = -\infty \quad \text{and} \quad \lim_{s\nearrow 1} F_1'(s) = \lim_{s\nearrow 1} G_1'(s) = +\infty.$$

Moreover, there exists a constant $\Theta > 0$ such that

$$F_1''(s) \geq \Theta \quad \text{and} \quad G_1''(s) \geq \Theta \quad \text{for all } s \in (-1,1).$$

As usual, we extend $F_1(s) = G_1(s) = +\infty$ for all $s \notin [-1,1]$ Without loss of generality, we assume $F_1(0) = G_1(0) = 0$ and $F_1'(0) = G_1'(0) = 0$. In particular, those entail

$$F_1(s) \geq 0 \quad \text{and} \quad G_1(s) \geq 0 \quad \text{for all } s \in [-1,1].$$

For F_2 and G_2 we assume $F_2, G_2 \in C^1(\mathbb{R})$ such that their derivatives are globally Lipschitz continuous. Lastly, we require that the singular part of the boundary potential dominates the singular part of the bulk potential in the sense that there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$|F_1'(\alpha s)| \leq \kappa_1 |G_1'(s)| + \kappa_2 \quad \text{for all } s \in (-1,1). \tag{8.2}$$

Next, we have to fix the mean value of the initial data to define the underlying process on a suitable phase space. To this end, we suppose the following.

(A5) If $L \in [0, \infty)$, let $m \in \mathbb{R}$ satisfy

$$\beta m,\ m \in (-1, 1),$$

while in the case $L = \infty$, let $m = (m_1, m_2) \in \mathbb{R}^2$ be given such that

$$m_1,\ m_2 \in (-1, 1).$$

For the velocity fields, we need the following assumptions, that guarantee the existence of a unique weak solution with the propagation of regularity.

(A6) We assume that

$$(\mathbf{v}, \mathbf{w}) \in L^2(\mathbb{R}; \mathbf{L}^2_{\mathrm{div}}(\Omega) \times \mathbf{L}^2_{\mathrm{tan}}(\Gamma)).$$

(A7) At least one of the following conditions is satisfied:

(A7.1) $(\mathbf{v}, \mathbf{w})$ has the regularities

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) \in H^1_{\mathrm{uloc}}(\mathbb{R}; \mathbf{L}^{\frac{6}{5}}(\Omega) \times \mathbf{L}^{1+\omega}(\Gamma)) \cap L^2(\mathbb{R}; \boldsymbol{\mathcal{L}}^2)\\ \cap L^\infty(\mathbb{R}; \mathbf{L}^3_{\mathrm{div}}(\Omega) \times \mathbf{L}^{2+\omega}_{\mathrm{div}}(\Gamma))\end{aligned}$$

for some $\omega > 0$.

(A7.2) $(\mathbf{v}, \mathbf{w})$ has the regularities

$$(\mathbf{v}, \mathbf{w}) \in L^2(\mathbb{R}; \boldsymbol{\mathcal{H}}^1) \cap L^\infty(\mathbb{R}; \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}),$$

and it holds

$$\mathbf{v}|_\Gamma = \mathbf{w} \qquad \text{a.e. on } \Gamma \times \mathbb{R} \quad \text{if } K = 0.$$

We note that, in contrast to Chapter 6, the velocity fields are now defined on $\mathbb{R}$ instead of $[0, \infty)$. This is necessary since the initial time $\tau \in \mathbb{R}$ is arbitrary. Clearly, the results presented in Chapter 6 remain true when the initial time 0 is replaced by τ. Moreover, for the existence of a pullback attractor, we require the following.

(A8) The velocity fields have the additional regularity

$$(\mathbf{v}, \mathbf{w}) \in L^2_{\mathrm{uloc}}(\mathbb{R}; \boldsymbol{\mathcal{L}}^6),$$

and if $K = 0$, we demand that $\alpha = \beta$.

We observe that the additional regularity assumption on the velocity fields assumed in **(A8)** is only needed if **(A7.1)** is assumed, as otherwise, this is already included in **(A7.2)**.

Lastly, in order to show the converges to a single equilibrium, we need an additional decay condition on the velocity fields.

(A9) There exist $T_{\mathrm{dec}} > 0$ and $a \geq 0$ such that the mapping

$$(T_{\mathrm{dec}}, \infty) \ni t \mapsto \big\|(\mathbf{v}(t), \mathbf{w}(t))\big\|_{\boldsymbol{\mathcal{L}}^2}$$

is non-increasing, and the integral

$$\int_{T_{\mathrm{dec}}}^\infty \mathrm{e}^{as} \big\|(\mathbf{v}(s), \mathbf{w}(s))\big\|_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}s$$

is finite.

Remark 8.3.1. We point out that that condition **(A9)** is fulfilled, for instance, if $\|(\mathbf{v}(t), \mathbf{w}(t))\|_{\mathcal{L}^2}$ decays exponentially fast as $t \to \infty$. If $a > 0$, thanks to Hölder's inequality, condition **(A9)** further implies that for any $\gamma \in (0, 1)$, it holds that

$$\begin{aligned}&\int_{T_{\mathrm{dec}}}^{\infty} \|(\mathbf{v}(s), \mathbf{w}(s))\|_{\mathcal{L}^2}^{\gamma} \,\mathrm{d}s \\ &\quad\le \left(\int_{T_{\mathrm{dec}}}^{\infty} \mathrm{e}^{as}\|(\mathbf{v}(s), \mathbf{w}(s))\|_{\mathcal{L}^2} \,\mathrm{d}s\right)^{\gamma} \left(\int_{T_{\mathrm{dec}}}^{\infty} \mathrm{e}^{-a\frac{\gamma}{1-\gamma}s} \,\mathrm{d}s\right)^{1-\gamma} < \infty.\end{aligned}$$

8.3.2 Existence of a pullback attractor

In view of Theorem 6.2.3, for given $(\phi_\tau, \psi_\tau) \in \mathcal{X}_{K,L,\alpha,\beta,m}$, we can define the mapping

$$\begin{aligned}&S_m^{K,L}(t, \tau) : \mathcal{X}_{K,L,\alpha,\beta,m} \to \mathcal{X}_{K,L,\alpha,\beta,m}, \\ &S_m^{K,L}(t, \tau)(\phi_\tau, \psi_\tau) := (\phi^{K,L}, \psi^{K,L})(t; \tau, \phi_\tau, \psi_\tau),\end{aligned} \tag{8.3}$$

where $(\phi^{K,L}, \psi^{K,L}, \mu^{K,L}, \theta^{K,L})(\cdot; \tau, \phi_\tau, \psi_\tau) \in \mathcal{X}_{K,L,\alpha,\beta,m}$ is the unique weak solution to (8.1) on $[\tau, \infty)$ corresponding to the velocity fields $(\mathbf{v}, \mathbf{w})$ with the initial condition

$$(\phi^{K,L}, \psi^{K,L})(\tau; \tau, \phi_\tau, \psi_\tau) = (\phi_\tau, \psi_\tau).$$

We show in Lemma 8.4.2, that the family $\{S_m^{K,L}(t, \tau)\}_{t \ge \tau}$ defines a continuous process on $\mathcal{X}_{K,L,\alpha,\beta,m}$ in the sense of Definition 8.4.1.

Our first main result regarding the long-time behavior of weak solutions to (8.1) establishes the existence of a minimal pullback attractor for the family $\{S_m^{K,L}(t, \tau)\}_{t \ge \tau}$.

THEOREM 8.3.2. *Suppose that* **(A1)**–**(A8)** *are fulfilled. Then the continuous process* $\{S_m^{K,L}(t, \tau)\}_{t \ge \tau}$ *associated with system* (8.1) *admits a minimal pullback attractor* $\mathcal{A}$ *in the sense of Definition 8.4.5, which is given by*

$$\mathcal{A}_{\mathcal{D}}(t) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} S_m^{K,L}(t, \tau) D_0}^{\mathcal{X}_{K,L,\alpha,\beta,m}} \subset \overline{D_0}^{\mathcal{X}_{K,L,\alpha,\beta,m}} \quad \text{for all } t \in \mathbb{R}, \tag{8.4}$$

where $\widehat{D}_0 = \{D_0(t) \equiv D_0\}_{t \in \mathbb{R}}$ *is the compact pullback absorbing set of Proposition 8.4.9.*

We refer to Definition 8.4.5 to the notion of a (minimal) pullback attractor.

The proof of Theorem 8.3.2 will be shown in Section 8.4.

8.3.3 Convergence to equilibrium

In the second part of this chapter, we study the behavior of weak solutions as $t \to \infty$. To this end, we assume without loss of generality that $\tau = 0$. Then we have the following result.

THEOREM 8.3.3. *Suppose that the assumptions* **(A1)**–**(A7)** *hold. Moreover, suppose that* **(A9)** *holds with* $a > 0$. *In addition, we assume that* F_1, G_1 *are real analytic on* $(-1, 1)$ *while* F_2, G_2 *are real analytic on* $\mathbb{R}$. *Let* $(\phi_0, \psi_0) \in \mathcal{X}_{K,L,\alpha,\beta,m}$, *and let* $(\phi, \psi, \mu, \theta)$ *denote the corresponding unique weak solution to* (8.1) *on* $[0, \infty)$. *Then it holds*

$$\lim_{t \to \infty} \|(\phi(t) - \phi_\infty, \psi(t) - \psi_\infty)\|_{\mathcal{H}^2} = 0, \tag{8.5}$$

where $(\phi_\infty, \psi_\infty) \in \mathcal{H}^2$ *is a solution to the stationary bulk-surface Cahn–Hilliard equation*

$$\begin{aligned} -\Delta\phi_\infty + F'(\phi_\infty) &= \mu_\infty && \text{in } \Omega, \\ -\Delta_\Gamma\psi_\infty + G'(\psi_\infty) + \alpha\partial_{\mathbf{n}}\phi_\infty &= \theta_\infty && \text{on } \Gamma, \\ K\partial_{\mathbf{n}}\phi_\infty &= \alpha\psi_\infty - \phi_\infty && \text{on } \Gamma, \end{aligned}$$

with

$$\begin{cases} \beta \displaystyle\int_\Omega \phi_\infty \,\mathrm{d}x + \int_\Gamma \psi_\infty \,\mathrm{d}\Gamma = \beta \int_\Omega \phi_0 \,\mathrm{d}x + \int_\Gamma \psi_0 \,\mathrm{d}\Gamma & \text{if } L \in [0,\infty), \\ \displaystyle\int_\Omega \phi_\infty \,\mathrm{d}x = \int_\Omega \phi_0 \,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi_\infty \,\mathrm{d}\Gamma = \int_\Gamma \psi_0 \,\mathrm{d}\Gamma & \text{if } L = \infty. \end{cases}$$

The constants μ_∞ *and* θ_∞ *appearing in* (8.6) *are given by*

$$\mu_\infty = \beta\theta_\infty = \frac{\beta}{\alpha\beta|\Omega| + |\Gamma|}\left(\alpha \int_\Omega F'(\phi_\infty)\,\mathrm{d}x + \int_\Gamma G'(\psi_\infty)\,\mathrm{d}\Gamma\right) \quad \text{if } L \in [0,\infty),$$

$$\begin{cases} \mu_\infty = \dfrac{1}{|\Omega|}\left(\displaystyle\int_\Omega F'(\phi_\infty)\,\mathrm{d}x - \int_\Gamma \partial_{\mathbf{n}}\phi_\infty \,\mathrm{d}\Gamma\right), \\ \theta_\infty = \dfrac{1}{|\Gamma|}\left(\displaystyle\int_\Gamma G'(\psi_\infty) + \alpha\partial_{\mathbf{n}}\phi_\infty \,\mathrm{d}\Gamma\right) \end{cases} \quad \text{if } L = \infty.$$

The proof of Theorem 8.3.3 is presented in Section 8.5

8.4 Existence of a pullback attractor

The aim of this section is to prove the existence of a minimal pullback attractor. However, we have to start by introducing some necessary concepts and present an abstract result, which will used to prove Theorem 8.3.2.

8.4.1 Formulation as a process

We first introduce the concept of a two-parameter continuous process, which generalizes the notion of semigroups.

Definition 8.4.1 (Continuous process). *A **continuous process** on a metric space* X *is a family* $\{S(t,\tau)\}_{t\geq\tau}$ *of continuous mappings* $S(t,\tau) : X \to X$ *with the following properties:*

(i) *For all* $\tau \in \mathbb{R}$ *and* $x \in X$*, it holds* $S(\tau,\tau)x = x$.

(ii) *For all* $t \geq s \geq \tau$*, it holds* $S(t,\tau) = S(t,s)\,S(s,\tau)$.

(iii) *The mapping*

$$\{(t,\tau) \in \mathbb{R}^2 \,:\, t \geq \tau\} \times X \ni (t,\tau,x) \mapsto S(t,\tau)x \in X$$

is continuous.

The following lemma shows that the dynamical system associated with (8.1) can actually be formulated as a process.

Lemma 8.4.2. *Suppose that* **(A1)**–**(A7)** *are fulfilled. Then, the family* $\{S_m^{K,L}(t,\tau)\}_{t\geq\tau}$ *defined in* (8.3) *is a continuous process on* $\mathcal{X}_{K,L,\alpha,\beta,m}$.

PROOF. The family is well-defined by Proposition 6.2.3. It is straightforward to check that conditions (i) and (ii) in Definition 8.4.1 are satisfied. We now show that the map

$$[\tau,\infty)\times\mathbb{R}\times\mathcal{X}_{K,L,\alpha,\beta,m}\ni(t;\tau,\phi_\tau,\psi_\tau)\to S_m^{K,L}(t,\tau)(\phi_\tau,\psi_\tau)\in\mathcal{X}_{K,L,\alpha,\beta,m}$$

is continuous. Continuity with respect to τ and (ϕ_*,ψ_*) follows from the continuous dependence estimate (6.18). We further need to show that

$$t\mapsto(\phi^{K,L},\psi^{K,L})(t;\tau,\phi_\tau,\psi_\tau)\in C([\tau,\infty);\mathcal{H}^1).$$

In view of the regularities established in Theorem 6.2.7, we use the Aubin–Lions–Simon lemma to infer that $(\phi^{K,L},\psi^{K,L})(\,\cdot\,;\tau,\phi_\tau,\psi_\tau)\in C(\tau,\infty;\mathcal{H}^1)$. To prove continuity at $t=\tau$, we consider any sequence $\{t_k\}_{k\in\mathbb{N}}\subset(\tau,\infty)$ with $t_k\to\tau$ as $k\to\infty$. As the sequence $\{(\phi^{K,L},\psi^{K,L})(t_k)\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{H}^1$, we deduce that, after subsequence extraction,

$$(\phi^{K,L},\psi^{K,L})(t_k)\to(\phi_\tau,\psi_\tau)\quad\text{weakly in }\mathcal{H}^1\text{ , strongly in }\mathcal{L}^2\text{, and a.e. in }\Omega\times\Gamma\tag{8.7}$$

as $k\to\infty$. Moreover, by definition of the free energy, we have

$$\begin{aligned}&\frac12\|(\phi^{K,L}(t_k),\psi^{K,L}(t_k))\|_{K,\alpha}^2-\frac12\|(\phi_\tau,\psi_\tau)\|_{K,\alpha}^2\\&\quad=E_{\text{free}}(\phi^{K,L}(t_k),\psi^{K,L}(t_k))-E_{\text{free}}(\phi_\tau,\psi_\tau)\\&\qquad+\int_\Omega F(\phi_\tau)-F(\phi^{K,L}(t_k))\,\mathrm{d}x+\int_\Gamma G(\psi_\tau)-G(\psi^{K,L}(t_k))\,\mathrm{d}\Gamma.\end{aligned}$$

Since F and G are both continuous on $[-1,1]$, the corresponding integrals converge to zero as $k\to\infty$ by the dominated convergence theorem. Furthermore, it is easy to see that

$$E_{\text{free}}(\phi_\tau,\psi_\tau)\le\liminf_{k\to\infty}E_{\text{free}}(\phi^{K,L}(t_k),\psi^{K,L}(t_k))$$

as all terms in the energy are either convex (and thus weakly lower semicontinuous) or continuous (by dominated convergence). We refer to the proof of Theorem 5.3.2, where a similar argument was carried out in detail. On the other and, by taking the limit superior in the energy inequality (6.13), we infer that

$$\limsup_{k\to\infty}E_{\text{free}}(\phi^{K,L}(t_k),\psi^{K,L}(t_k))\le E_{\text{free}}(\phi_\tau,\psi_\tau).$$

Thus, we conclude that $E_{\text{free}}(\phi^{K,L}(t_k),\psi^{K,L}(t_k))\to E_{\text{free}}(\phi_*,\psi_*)$ as $k\to\infty$, and this directly yields

$$\frac12\|(\phi^{K,L}(t_k),\psi^{K,L}(t_k))\|_{K,\alpha}^2\to\frac12\|(\phi_\tau,\psi_\tau)\|_{K,\alpha}^2$$

as $k\to\infty$. Together with (8.7), this implies that

$$(\nabla\phi^{K,L}(t_k),\nabla_\Gamma\psi^{K,L}(t_k))\to(\nabla\phi_\tau,\nabla_\Gamma\psi_\tau)\quad\text{strongly in }\boldsymbol{\mathcal{L}}^2$$

as $k\to\infty$. As the limit is independent of the previously extracted subsequence, this convergence actually remains true for the whole sequence $\{t_k\}_{k\in\mathbb{N}}$. Hence, we conclude that $(\nabla\phi^{K,L},\nabla_\Gamma\psi^{K,L})\in C([\tau,\infty);\boldsymbol{\mathcal{L}}^2)$. In combination with $(\phi^{K,L},\psi^{K,L})\in C([\tau,\infty);\mathcal{L}^2)$, this proves the claim. ☐

In the special case of constant velocity fields, the continuous process $\{S_m^{K,L}(t,\tau)\}_{t\geq\tau}$ in fact coincides with a continuous semigroup. However, in view of our assumptions, the only choice of constant velocity fields that agrees with **(A7)** is $(\mathbf{v},\mathbf{w}) \equiv (\mathbf{0},\mathbf{0})$.

COROLLARY 8.4.3. *Suppose that the assumptions of Lemma 8.4.2 hold with* $(\mathbf{v},\mathbf{w}) \equiv (\mathbf{0},\mathbf{0})$. *Then, for all* $s \in \mathbb{R}$ *and* $t \geq s$, *we have* $S_m^{K,L}(t,s) = S_m^{K,L}(t-s,0)$. *This means that the process is autonomous. For* $t \geq 0$, *we define*

$$T_m^{K,L}(t) \coloneqq S_m^{K,L}(t,0) : \mathcal{X}_{K,L,\alpha,\beta,m} \to \mathcal{X}_{K,L,\alpha,\beta,m}.$$

This family $\{T_m^{K,L}(t)\}_{t\geq 0}$ *has the following properties:*

(i) $T_m^{K,L}(0)$ *is the identity map on* $\mathcal{X}_{K,L,\alpha,\beta,m}$.

(ii) *For all* $t, s \geq 0$, $T_m^{K,L}(t+s) = T_m^{K,L}(t)\, T_m^{K,L}(s)$.

(iii) *The mapping*

$$[0,\infty) \times \mathcal{X}_{K,L,\alpha,\beta,m} \ni (t,\phi_0,\psi_0) \to T_m^{K,L}(t)(\phi_0,\psi_0) \in \mathcal{X}_{K,L,\alpha,\beta,m}$$

is continuous.

This means that the family $\{T_m^{K,L}(t)\}_{t\geq 0}$ *defines a continuous semigroup on* $\mathcal{X}_{K,L,\alpha,\beta,m}$.

PROOF. Since the time evolution is independent of the initial time, we readily see that the identity $S_m^{K,L}(t,s) = S_m^{K,L}(t-s,0)$ holds. Then, property (i) follows directly from property (i) in the Definition 8.4.1. Then, property (ii) holds since

$$\begin{aligned} T_m^{K,L}(t+s) &= S_m^{K,L}(t+s,0) = S_m^{K,L}(t+s,s)\, S_m^{K,L}(s,0) \\ &= S_m^{K,L}(t+s-s,0)\, S_m^{K,L}(s,0) = T_m^{K,L}(t)\, T_m^{K,L}(s). \end{aligned}$$

Finally, the continuity in (iii) follows from the continuity of $S_m^{K,L}$ as previously shown in Lemma 8.4.2. □

8.4.2 Existence of a unique pullback attractor

Now, we are going to investigate the long-time behavior of the non-autonomous dynamical system $(S_m^{K,L}(t,\tau), \mathcal{X}_{K,L,\alpha,\beta,m})$ through the theory of pullback attractors. As a preliminary step, we first recall the fundamental framework, including the basic definitions and a key theorem concerning pullback attractors, whose assumptions will be verified for our system in the subsequent analysis. In the following, for any metric space X, we write $\mathcal{P}(X)$ to denote the family of all non-empty subsets of X.

We start by recalling the main concepts related to pullback dynamics that will be used to establish our main result (see [133]).

DEFINITION 8.4.4 (Pullback absorbing sets). *Let* $\{S(t,\tau)\}_{t\geq\tau}$ *be a continuous process on a metric space* X. *A family* $\widehat{D}_0 = \{D_0(t)\}_{t\in\mathbb{R}}$ *of non-empty sets is called **pullback absorbing** for the process* $\{S(t,\tau)\}_{t\geq\tau}$, *if for any bounded set* $D \in \mathcal{P}(X)$ *and any* $t \in \mathbb{R}$, *there exists a time* $\tau_0(t,D) \leq t$ *such that*

$$S(t,\tau)D \subset D_0(t) \quad \textit{for all } \tau \leq \tau_0(t,D).$$

In this case, $\tau_0(\cdot,\cdot)$ *is referred to as the corresponding **pullback absorption time** for the absorbing family* $\widehat{D}$.

DEFINITION 8.4.5 (Pullback attractor). *Let* $\{S(t,\tau)\}_{t\geq\tau}$ *be a continuous process on a metric space* X. *A family* $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *of non-empty subsets of* X *is called a **pullback attractor** for the process* $\{S(t,\tau)\}_{t\geq\tau}$ *in* X *if*

(i) *$\mathcal{A}(t)$ is compact in X for any $t \in \mathbb{R}$,*

(ii) *$\mathcal{A}$ is invariant, i.e., $S(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \geq \tau$,*

(iii) *$\mathcal{A}$ is pullback attracting, i.e., for any $t \in \mathbb{R}$ and any bounded set D it holds*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X\big(S(t,\tau)D, \mathcal{A}(t)\big) = 0.$$

A pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is referred to as ***minimal****, if for every family $\widehat{C} = \{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{P}(X)$ of closed sets, which satisfies*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X\big(S(t,\tau)D, C(t)\big) = 0 \qquad \text{for all bounded sets } D,$$

it holds that $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Remark 8.4.6. In the case of autonomous dynamical systems, the definition of a global attractor reduces to that of a compact, invariant, and attracting set. These properties are sufficient to ensure that the attractor is both minimal among closed attracting sets and maximal among invariant sets.

In contrast, the situation is more subtle in the non-autonomous setting. The notion of a pullback attractor defined above does not automatically enjoy analogous minimality or maximality properties. In particular, it is not guaranteed to be the maximal invariant set attracting all bounded subsets, nor the minimal closed attracting set. A key difficulty is that one cannot, in general, deduce the self-attracting property

$$\lim_{\tau \to -\infty} \operatorname{dist}_X\big(S(t,\tau)\mathcal{A}(\tau), \mathcal{A}(t)\big) = 0 \qquad \text{for all } t \in \mathbb{R}.$$

For this, minimality must be imposed *a priori* in the definition, leading to the concept of a *minimal pullback attractor*. In particular, the minimality requirement ensures the uniqueness of the attractor.

To recover minimality or maximality as a consequence of the defining properties, one needs to strengthen the notion of attraction by considering time-dependent families of sets belonging to a suitable universe. This leads to alternative notions of pullback attractors, which may in general be larger than the one considered here, see, for example, [133].

We will use the following proposition to prove the existence of a pullback attractor for the process $S_m^{K,L}$ (see [133, Theorem 7] and [84, Proposition 3.7]).

Proposition 8.4.7. *Let $\{S(t,\tau)\}_{t\geq\tau}$ be a continuous process on a metric space X. Assume that there exists a compact attracting family of sets $\widehat{K} = \{K(t)\}_{t\in\mathbb{R}}$ for $\{S(t,\tau)\}_{t\geq\tau}$, where $K(t)$ is compact in X for every $t \in \mathbb{R}$, such that*

$$\lim_{\tau \to -\infty} \operatorname{dist}_X\big(S(t,\tau)D, K(t)\big) = 0,$$

for any bounded set $D \in \mathcal{P}(X)$. Then there exists a minimal pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in X, which is given by

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{D\in\mathcal{P}(X)\\ \text{bounded}}} \Lambda(D,t)}^{X},$$

where

$$\Lambda(D,t) = \bigcap_{s\leq t} \overline{\bigcup_{\tau\leq s} S(t,\tau)D}^{X} \qquad \text{for any bounded set } D \in \mathcal{P}(X).$$

Additionally, if there exists a family $\widehat{D}_0 = \{D_0(t)\}_{t\in\mathbb{R}} \subset \mathcal{P}(X)$ *such that* $\widehat{D}_0$ *is pullback absorbing for* $\{S(t,\tau)\}_{t\geq\tau}$ *and if* $D_0(t) \equiv D_0$ *for all* $t \in \mathbb{R}$*, then*

$$\mathcal{A}(t) = \Lambda(D_0, t) \subset \overline{D_0}^X \quad \textit{for all } t \in \mathbb{R}. \tag{8.8}$$

Remark 8.4.8. Observe that, if the pullback absorbing set is time-independent and the attractor $\mathcal{A}$ exists, then (8.8) implies that the dissipative nature of the system ensures that the pullback attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is uniformly bounded in time, a property which is not guaranteed in general.

To apply Proposition 8.4.7, we start by proving the existence of a pullback absorbing family of sets for the process $\{S_m^{K,L}(t,\tau)\}_{t\geq\tau}$.

Proposition 8.4.9. *Suppose that* **(A1)–(A8)** *are fulfilled. Then there is a family* $\widehat{D}_0 = \{D_0(t)\}_{t\in\mathbb{R}}$ *of compact sets in* X *that is pullback absorbing for the process* $\{S_m^{K,L}(t,\tau)\}_{t\geq\tau}$ *associated with system* (8.1) *with pullback absorption time* $\tau_0(D,t) \leq t-2$ *for all bounded sets* D *and* $t \in \mathbb{R}$*. More precisely, the family* $\widehat{D}_0$ *is constant in time, and there exists a compact set* D_0 *such that* $D_0(t) \equiv D_0$ *for any* $t \in \mathbb{R}$*.*

Proof. In this proof, let C denote a generic positive constant depending only on the prescribed velocity fields, the quantities specified in **(A1)** and **(A2)**, F and G, which may change its value from line to line. We know from Proposition 6.5.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{free}}(\phi,\psi) + \|(\mu,\theta)\|_{L,\beta}^2 = \int_\Omega \phi\mathbf{v}\cdot\nabla\mu \,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta \,\mathrm{d}\Gamma \tag{8.9}$$

a.e. on (τ,∞). Using the Ladyzhenskaya inequality and the bulk-surface Poincaré inequality, we find that

$$\begin{aligned}
&\left|\int_\Omega \phi\mathbf{v}\cdot\nabla\mu \,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta \,\mathrm{d}\Gamma\right| \\
&\quad\leq \|(\mu,\theta)\|_{L,\beta}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^6}\|(\phi,\psi)\|_{\mathcal{L}^3} \\
&\quad\leq C\|(\mu,\theta)\|_{L,\beta}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^6}\|(\phi,\psi)\|_{\mathcal{L}^2}^{\frac12}\|(\phi,\psi)\|_{\mathcal{H}^1}^{\frac12} \\
&\quad\leq \frac12\|(\mu,\theta)\|_{L,\beta}^2 + C\Big(1+\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^6}^2\Big)\|(\phi,\psi)\|_{K,\alpha}
\end{aligned}$$

a.e. on (τ,∞). Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{free}}(\phi,\psi) + \frac12\|(\mu,\theta)\|_{L,\beta}^2 \leq C\Big(1+\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^6}^2\Big)\|(\phi,\psi)\|_{K,\alpha} \tag{8.10}$$

a.e. on (τ,∞). For the following computations, we assume that $L \in [0,\infty)$. The case $L=\infty$ can be handled similarly. Setting $(\phi^*,\psi^*) = (\phi - \beta\mathrm{mean}(\phi_\tau,\psi_\tau), \psi - \mathrm{mean}(\phi_\tau,\psi_\tau))$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac12\|(\phi^*,\psi^*)\|_{L,\beta,*}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\frac12\|\mathcal{S}_{L,\beta}(\phi^*,\psi^*)\|_{L,\beta}^2 \\
&\quad= \big(\mathcal{S}_{L,\beta}(\partial_t\phi^*,\partial_t\psi^*),\mathcal{S}_{L,\beta}(\phi^*,\psi^*)\big)_{L,\beta} \\
&\quad= -\langle(\partial_t\phi^*,\partial_t\psi^*),\mathcal{S}_{L,\beta}(\phi^*,\psi^*)\rangle_{\mathcal{H}^1_{L,\beta}} \\
&\quad= \big((\mu,\theta),\mathcal{S}_{L,\beta}(\phi^*,\psi^*)\big)_{L,\beta} - \big(\phi\mathbf{v},\nabla\mathcal{S}^\Omega_{L,\beta}(\phi^*,\psi^*)\big)_{\mathbf{L}^2(\Omega)} - \big(\psi\mathbf{w},\nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi^*,\psi^*)\big)_{\mathbf{L}^2(\Gamma)} \\
&\quad= -\big((\mu,\theta),(\phi^*,\psi^*)\big)_{\mathcal{L}^2} - \big(\phi\mathbf{v},\nabla\mathcal{S}^\Omega_{L,\beta}(\phi^*,\psi^*)\big)_{\mathbf{L}^2(\Omega)} - \big(\psi\mathbf{w},\nabla_\Gamma\mathcal{S}^\Gamma_{L,\beta}(\phi^*,\psi^*)\big)_{\mathbf{L}^2(\Gamma)}
\end{aligned} \tag{8.11}$$

a.e. on (τ,∞). Then, using the weak formulation (6.11b), it follows that

$$\begin{aligned}-\big((\mu,\theta),(\phi^*,\psi^*)\big)_{\mathcal{L}^2} = -\|(\phi,\psi)\|_{K,\alpha}^2 - \int_\Omega F'(\phi)\,\phi^*\,\mathrm{d}x - \int_\Gamma G'(\psi)\,\psi^*\,\mathrm{d}\Gamma \\ + \mathbf{1}_{\{0\}}(K)(\alpha-\beta)\mathrm{mean}(\phi_\tau,\psi_\tau)\int_\Gamma(\alpha\psi-\phi)\,\mathrm{d}\Gamma\end{aligned} \tag{8.12}$$

a.e. on (τ,∞), where $\mathbf{1}_{\{0\}}(\cdot)$ denotes the characteristic function of the set $\{0\}$, which is added to deal with all the cases $K\in[0,\infty]$ simultaneously. In view of assumption **(A8)**, where $\alpha=\beta$ if $K=0$ was demanded, we readily see that the last term on the right-hand side of (8.12) vanishes in all cases $K\in[0,\infty]$. Now, exploiting the convexity of F_1 and G_1, respectively, as well as the Lipschitz continuity of F_2' and G_2', respectively, we find that

$$\begin{aligned}&\int_\Omega F_1'(\phi)(\phi-\beta\mathrm{mean}(\phi_\tau,\psi_\tau))\,\mathrm{d}x + \int_\Gamma G_1'(\psi)(\psi-\mathrm{mean}(\phi_\tau,\psi_\tau))\,\mathrm{d}\Gamma \\ &\quad\ge \int_\Omega F_1(\phi)-F_1(\beta\mathrm{mean}(\phi_\tau,\psi_\tau))\,\mathrm{d}x + \int_\Gamma G_1(\psi)-G_1(\mathrm{mean}(\phi_\tau,\psi_\tau))\,\mathrm{d}\Gamma \\ &\quad\ge \int_\Omega F_1(\phi)\,\mathrm{d}x + \int_\Gamma G_1(\psi)\,\mathrm{d}\Gamma - |\Omega|F_1(\beta\mathrm{mean}(\phi_\tau,\psi_\tau)) - |\Gamma|G_1(\mathrm{mean}(\phi_\tau,\psi_\tau)),\end{aligned} \tag{8.13}$$

and

$$\int_\Omega F_2(\phi)\,\phi^*\,\mathrm{d}x + \int_\Gamma G_2(\psi)\,\psi^*\,\mathrm{d}x \le C \tag{8.14}$$

a.e. on (τ,∞). For the convection-related terms on the right-hand side of (8.11), we use again the Ladyzhenskaya inequality to obtain

$$\begin{aligned}&\left|\int_\Omega \phi\mathbf{v}\cdot\nabla\mathcal{S}_{L,\beta}^\Omega(\phi^*,\psi^*)\,\mathrm{d}x + \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\mathcal{S}_{L,\beta}^\Gamma(\phi^*,\psi^*)\,\mathrm{d}\Gamma\right| \\ &\quad= \left|\int_\Omega \mathcal{S}_{L,\beta}^\Omega(\phi^*,\psi^*)\mathbf{v}\cdot\nabla\phi\,\mathrm{d}x + \int_\Gamma \mathcal{S}_{L,\beta}^\Gamma(\phi^*,\psi^*)\mathbf{w}\cdot\nabla_\Gamma\psi\,\mathrm{d}\Gamma\right| \\ &\quad\le \|\mathcal{S}_{L,\beta}(\phi^*,\psi^*)\|_{\mathcal{L}^3}\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^6}\|(\phi,\psi)\|_{K,\alpha} \\ &\quad\le C\|\mathcal{S}_{L,\beta}(\phi^*,\psi^*)\|_{\mathcal{L}^2}^{\frac12}\|(\phi^*,\psi^*)\|_{L,\beta,*}^{\frac12}\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^6}\|(\phi,\psi)\|_{K,\alpha} \\ &\quad\le \frac12\|(\phi,\psi)\|_{K,\alpha}^2 + C\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^6}^2\|(\phi^*,\psi^*)\|_{L,\beta,*}\end{aligned} \tag{8.15}$$

a.e. on (τ,∞). Thus, collecting the estimates (8.13)-(8.15) to bound the right-hand side of (8.11), we deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac12\|(\phi^*,\psi^*)\|_{L,\beta,*}^2 + \frac12\|(\phi,\psi)\|_{K,\alpha}^2 + \int_\Omega F_1(\phi)\,\mathrm{d}x + \int_\Gamma G_1(\psi)\,\mathrm{d}\Gamma \\ &\quad\le C + C\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^6}^2\|(\phi^*,\psi^*)\|_{L,\beta,*}\end{aligned} \tag{8.16}$$

a.e. on (τ,∞). Consequently, adding (8.10) and (8.16), and using again the fact that F_2' and G_2' are both Lipschitz continuous, leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(E_{\mathrm{free}}(\phi,\psi)+\frac12\|(\phi^*,\psi^*)\|_{L,\beta,*}^2\Big) + E_{\mathrm{free}}(\phi,\psi) + \frac12\|(\mu,\theta)\|_{L,\beta}^2 + \frac12\|(\phi,\psi)\|_{K,\alpha}^2 \\ &\quad\le C + C\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^6}^2\Big(\|(\phi,\psi)\|_{K,\alpha} + \|(\phi^*,\psi^*)\|_{L,\beta,*}\Big)\end{aligned} \tag{8.17}$$

a.e. on (τ,∞). Recalling the definition of E_{free} and the fact that F and G are bounded from below on $[-1,1]$, it easy to see that there exist constants C' and C'' (which may depend on the same quantities as C) such that

$$\frac12\|(\phi,\psi)\|_{K,\alpha}^2 + \frac12\|(\phi^*,\psi^*)\|_{L,\beta,*}^2 - C' \le E_{\mathrm{free}}(\phi,\psi) + \frac12\|(\phi^*,\psi^*)\|_{L,\beta,*}^2 \tag{8.18}$$

$$\leq \frac{1}{2}\|(\phi,\psi)\|_{K,\alpha}^2 + \frac{1}{2}\|(\phi^*,\psi^*)\|_{L,\beta,*}^2 + C''$$

a.e. on (τ,∞). Therefore, defining $\Upsilon(t) \coloneqq C' + E_{\text{free}}(\phi,\psi) + \frac{1}{2}\|(\phi^*,\psi^*)\|_{L,\beta,*}^2$, we readily infer from Minkowski's inequality that

$$\Upsilon^{\frac{1}{2}}(t) \geq \frac{1}{2}\Big(\|(\phi,\psi)\|_{K,\alpha} + \|(\phi^*,\psi^*)\|_{L,\beta,*}\Big)$$

a.e. on (τ,∞). Thus, we obtain from (8.17) the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(t) + \Upsilon(t) + \frac{1}{2}\|(\mu,\theta)\|_{L,\beta}^2 \leq C + C\|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^6}^2\Upsilon^{\frac{1}{2}}(t) \tag{8.19}$$

a.e. on (τ,∞). Hence, applying the variant of Gronwall's inequality presented in Lemma 2.2.8, we find that

$$\Upsilon(t) \leq 2\Upsilon(\tau)\mathrm{e}^{-\frac{1}{2}(t-\tau)} + \Big(C\|(\mathbf{v},\mathbf{w})\|_{L^2_{\mathrm{uloc}}(\mathbb{R};\mathcal{L}^6)}^2\Big)^2 + C \tag{8.20}$$

for a.e. $t \in (\tau,\infty)$. Observe that, by continuity $(\phi,\psi) \in C([\tau,\infty);\mathcal{H}^1)$, (8.20) even holds for all $t \in [\tau,\infty)$. Now, we recall that $(\phi(s),\psi(s)) \in \mathcal{X}_{K,L,\alpha,\beta,m}$ for all $s \geq \tau$. If $K \in [0,\infty)$, we use the bulk-surface Poincaré inequality to deduce that

$$\|(\phi(s),\psi(s))\|_{\mathcal{H}^1} \leq C_1\|(\phi(s),\psi(s))\|_{K,\alpha} \quad \text{for all } s \geq \tau, \tag{8.21}$$

where C_1 and C_2 are positive constants depending only on K, α, β and Ω. If $K=\infty$, we use (6.65) instead, which yields

$$\|(\phi(s),\psi(s))\|_{\mathcal{H}^1} \leq C_1\Big(\|(\phi(s),\psi(s))\|_{K,\alpha} + \|(\phi(s),\psi(s))\|_{L,\beta,*}\Big) \qquad \text{for all } s \geq \tau.$$

This allows us to conclude that

$$\|(\phi(t),\psi(t))\|_{\mathcal{H}^1}^2 \leq C_*\Big(1 + \|(\phi_\tau,\psi_\tau)\|_{\mathcal{H}^1}\Big)\mathrm{e}^{\frac{1}{2}(\tau-t)} + C_{**} \tag{8.22}$$

for all $t \geq \tau$, where C_* and C_{**} are positive constants which may depend on the same quantities as constants denoted by C. In particular, this means that the constant C_* and C_{**} are independent of τ and (ϕ_τ,ψ_τ).

Now, let $t \in \mathbb{R}$ and $D \subset \mathcal{X}_{K,L,\alpha,\beta,m}$ be bounded. Then, there exists $R = R(D) > 0$ such that

$$\|(\tilde{\phi},\tilde{\psi})\|_{\mathcal{H}^1} \leq R \quad \text{for all } (\tilde{\phi},\tilde{\psi}) \in D.$$

We define

$$\tau_0(D,t) \coloneqq t - 2(1+R) \leq t-2.$$

By this definition, for any $\tau \leq \tau_0(D,t)$, we have $\mathrm{e}^{\frac{1}{2}(\tau-t)} \leq (1+R)^{-1}$. In view of the estimates (8.20)–(8.22), this implies that

$$\|(\phi(t),\psi(t))\|_{\mathcal{H}^1}^2 \leq C_* + C_{**} \quad \text{and} \quad \Upsilon(t) \leq C \tag{8.23}$$

for all $\tau \leq \tau_0(D,t)$ and initial datum $(\phi_\tau,\psi_\tau) \in D$. Now, we set $\rho \coloneqq C_* + C_{**}$,

$$D_0 \coloneqq \Big\{(\tilde{\phi},\tilde{\psi}) \in \mathcal{X}_{K,L,\alpha,\beta,m} \,:\, \|(\tilde{\phi},\tilde{\psi})\|_{\mathcal{H}^1}^2 \leq \rho\Big\}.$$

Then, for any $\tau \le \tau_0(D,t)$, it follows from (8.23) that

$$S_m^{K,L}(t,\tau)D \subset D_0 \qquad \text{for all } \tau \le \tau_0(D,t). \tag{8.24}$$

Since D was assumed to be an arbitrary bounded subset of the phase space, this shows that the (constant in time) family $\widehat{D}_0 := \{D_0(t) \equiv D_0\}_{t\in\mathbb{R}}$ is pullback absorbing for the process $\{S_m^{K,L}(t,\tau)\}_{t\ge\tau}$. We now aim to show that this set can be improved to be a compact set of the phase space. To this end, given $t \in \mathbb{R}$, let us consider $\tau_0(t) := \tau_0(D_0,t) \le t-2$ as to be the pullback absorbing time of D_0 into itself. Let now $r \in [t-1,t]$, $t \in \mathbb{R}$, be arbitrary. After integrating (8.17) in time from $r-1$ to r, we use (8.23) along with the definition of ρ to obtain, for any $\tau \le \tau_0(t)$ and initial datum $(\phi_\tau,\psi_\tau) \in D_0$,

$$\begin{aligned}
&\int_{r-1}^{r} \|(\mu,\theta)\|_{L,\beta}^2 \,\mathrm{d}s \\
&\quad\le C + E_{\text{free}}(\phi(r-1),\psi(r-1)) + \frac{1}{2}\|(\phi^*(r-1),\psi^*(r-1))\|_{L,\beta,*}^2 \\
&\qquad + C\int_{r-1}^{r} \|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{L}}^6}^2 \Big(\|(\phi(s),\psi(s))\|_{K,\alpha} + \|(\phi^*(s),\psi^*(s))\|_{L,\beta,*}\Big) \,\mathrm{d}s \\
&\quad\le C + C\rho^2 + C\rho\,\|(\mathbf{v},\mathbf{w})\|_{L^2_{\mathrm{uloc}}(\mathbb{R};\boldsymbol{\mathcal{L}}^6)}^2 =: \rho' .
\end{aligned}$$

Taking the supremum over all $r \in [t-1,t]$ yields

$$\sup_{r\in[t-1,t]} \int_{r-1}^{r} \|(\mu,\theta)\|_{L,\beta}^2 \,\mathrm{d}s \le \rho'. \tag{8.25}$$

Now, using the estimates (8.23) and (8.25), we can follow the argument of the proof of Theorem 6.2.7 to verify that, for any $\tau \le \tau_0(t)$ and initial datum $(\phi_\tau,\psi_\tau) \in D_0$,

$$\sup_{r\in[t-2,t]} \|(\phi(r),\psi(r))\|_{\mathcal{H}^2}^2 + \sup_{r\in[t-1,t]} \int_{r-1}^{r} \|(\partial_t\phi,\partial_t\psi)\|_{\mathcal{L}^2}^2 \,\mathrm{d}s \le \rho'', \tag{8.26}$$

where $\rho'' > 0$ is a constant with the same dependencies of ρ'. We omit the full details for the sake of brevity. We can then introduce the compact set

$$\mathcal{D}_0 := \Big\{(\tilde\phi,\tilde\psi) \in \mathcal{X}_{K,L,\alpha,\beta,m} \,:\, \|(\tilde\phi,\tilde\psi)\|_{\mathcal{H}^2}^2 \le \rho''\Big\}.$$

From (8.26) we thus conclude that

$$S_m^{K,L}(t,\tau)D \subset \mathcal{D}_0 \qquad \text{for all } \tau \le \tau_0(t). \tag{8.27}$$

As a consequence, since D_0 is a pullback absorbing set for the dynamical system, also $\mathcal{D}_0$, which is of course compact in $\mathcal{X}_{K,L,\alpha,\beta,m}$, is pullback absorbing, concluding the proof. $\square$

We are now in a position to straightforwardly prove Theorem 8.3.2, which establishes the existence of a minimal pullback attractor for the processes $\{S_m^{K,L}(t,\tau)\}_{t\ge\tau}$.

Proof of Theorem 8.3.2. In view of Proposition 8.4.9, we can apply the abstract result provided by Proposition 8.4.7, since the dynamical system admits a compact pullback attracting (in this case even absorbing) set. We thus conclude that the process $\{S_m^{K,L}(t,\tau)\}_{t\ge\tau}$ admits a minimal pullback attractor $\mathcal{A} = \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ in $\mathcal{X}_{K,L,\alpha,\beta,m}$, which is given by (8.4) since the time-indexed family $\widehat{\mathcal{D}}_0$ is constant in time. $\square$

8.5 Convergence to a stationary point

Finally, we want to prove that any weak solution of system (8.1) converges as $t \to \infty$ to a single equilibrium solution. In the following, we simply set $\tau = 0$. This does not mean any loss of generality as the concrete value of τ does not have any impact on the mathematical analysis of the longtime behavior of trajectories. As a first step, we fix an arbitrary $\sigma \in (\frac{d}{2}, 2)$ and recall the definition of the ω-limit set.

Definition 8.5.1 (ω-limit set). *Suppose that* **(A1)–(A7)** *are fulfilled and consider arbitrary initial data* $(\phi_0, \psi_0) \in \mathcal{X}_{K,L,\alpha,\beta,m}$. *Then the set*

$$\omega^{K,L}(\phi_0,\psi_0) := \left\{ (\phi_*,\psi_*) \in \mathcal{W}^\sigma_{K,L,\alpha,\beta,m} \,\middle|\, \begin{matrix} \exists \{t_k\}_{k\in\mathbb{N}} \subset [0,\infty) \text{ with } t_k \to \infty \text{ such that} \\ S^{K,L}_m(t_k,0)(\phi_0,\psi_0) \to (\phi_*,\psi_*) \text{ strongly in } \mathcal{H}^\sigma \end{matrix} \right\},$$

is called the ***ω-limit*** *set to the initial data* (ϕ_0, ψ_0).

Lemma 8.5.2. *Suppose that* **(A1)–(A7)** *are fulfilled and let* $(\phi_0, \psi_0) \in \mathcal{X}_{K,L,\alpha,\beta,m}$ *be arbitrary. Then* $\omega^{K,L}(\phi_0,\psi_0)$ *is a non-empty, compact and connected subset of* $\mathcal{W}^\sigma_{K,L,\alpha,\beta,m}$. *Moreover, it holds that*

$$\operatorname{dist}_{\mathcal{W}^\sigma_{K,L,\alpha,\beta,m}}\big(S^{K,L}_m(t,0)(\phi_0,\psi_0), \omega^{K,L}(\phi_0,\psi_0)\big) \to 0 \qquad \text{as } t \to \infty. \tag{8.28}$$

Proof. The claim follows directly by employing the propagation of regularity from Theorem 6.2.7 as well as [36, Theorem 9.1.8]. □

Definition 8.5.3 (Set of stationary points). *Suppose that* **(A1)–(A5)** *are fulfilled, and let* $\{T^{K,L}_m\}_{t\ge 0}$ *be the strongly continuous semigroup on* $\mathcal{X}_{K,L,\alpha,\beta,m}$ *corresponding to trivial velocity fields* $(\mathbf{v}_\infty, \mathbf{w}_\infty) = (\mathbf{0}, \mathbf{0})$, *which was introduced in Corollary 8.4.3. Then the set*

$$\mathcal{N}^{K,L}_m = \Big\{ (\phi,\psi) \in \mathcal{W}^1_{K,L,\alpha,\beta,m} : T^{K,L}_m(t)(\phi,\psi) = (\phi,\psi) \text{ for all } t \ge 0 \Big\}$$

is called the ***set of stationary points***.

The set of stationary points $\mathcal{N}^{K,L}_m$ owes its name to the fact that its elements are stationary solutions of system (8.1). This is shown by the following lemma.

Lemma 8.5.4. *Suppose that* **(A1)–(A7)** *are fulfilled and let* $(\phi_\infty, \psi_\infty) \in \mathcal{N}^{K,L}_m$. *Then there exist constants* μ_∞ *and* θ_∞ *given by*

$$\mu_\infty = \beta\theta_\infty = \frac{\beta}{\alpha\beta|\Omega| + |\Gamma|}\left(\alpha \int_\Omega F'(\phi_\infty)\,\mathrm{d}x + \int_\Gamma G'(\psi_\infty)\,\mathrm{d}\Gamma \right) \quad \text{if } L \in [0,\infty), \tag{8.29}$$

$$\begin{cases} \mu_\infty = \dfrac{1}{|\Omega|}\left(\displaystyle\int_\Omega F'(\phi_\infty)\,\mathrm{d}x - \int_\Gamma \partial_\mathbf{n}\phi_\infty\,\mathrm{d}\Gamma \right), \\ \theta_\infty = \dfrac{1}{|\Gamma|}\left(\displaystyle\int_\Gamma G'(\psi_\infty) + \alpha\partial_\mathbf{n}\phi_\infty\,\mathrm{d}\Gamma \right) \end{cases} \quad \text{if } L = \infty, \tag{8.30}$$

such that the quadruplet $(\phi_\infty, \psi_\infty, \mu_\infty, \theta_\infty)$, *where* μ_∞ *and* θ_∞ *are now interpreted as constant functions, is the unique weak solution of system (8.1) on the interval* $[0,\infty)$ *corresponding to the initial data* $(\phi_\infty, \psi_\infty)$. *In particular, this solution is stationary as each of the functions* ϕ_∞, ψ_∞, μ_∞, *and* θ_∞ *is constant in time.*

Furthermore, the pair $(\phi_\infty, \psi_\infty)$ *belongs to* $\mathcal{W}^{2,6} \cap \mathcal{W}^\sigma_{K,L,\alpha,\beta,m}$ *and is a strong solution of the semilinear bulk-surface elliptic system*

$$-\Delta\phi_\infty + F'(\phi_\infty) = \mu_\infty \qquad \text{in } \Omega, \tag{8.31a}$$

$$-\Delta_\Gamma \psi_\infty + G'(\psi_\infty) + \alpha \partial_{\mathbf{n}} \phi_\infty = \theta_\infty \qquad \textit{on } \Gamma, \tag{8.31b}$$

$$K \partial_{\mathbf{n}} \phi_\infty = \alpha \psi_\infty - \phi_\infty \qquad \textit{on } \Gamma. \tag{8.31c}$$

PROOF. Let $(\phi_\infty, \psi_\infty) \in \mathcal{N}_m^{K,L}$ be a stationary point of the semigroup $\{T_m^{K,L}(t)\}_{t\geq 0}$. This means that, in particular, $T_m^{K,L}(t)(\phi_\infty, \psi_\infty) = (\phi_\infty, \psi_\infty)$ for all $t \geq 0$. Hence, by the definition of the semigroup $\{T_m^{K,L}\}_{t\geq 0}$, there exist chemical potentials $(\mu_\infty, \theta_\infty) \in L^2_{\mathrm{uloc}}([0,\infty); \mathcal{H}^1_{L,\beta})$ such that the quadruplet $(\phi_\infty, \psi_\infty, \mu_\infty, \theta_\infty)$ is the weak solution of system (8.1) with trivial velocity fields on the interval $[0,\infty)$ associated with the initial data $(\phi_\infty, \psi_\infty)$.

Moreover, the energy inequality corresponding to $(\phi_\infty, \psi_\infty, \mu_\infty, \theta_\infty)$ reads as

$$\begin{aligned} &E_{\mathrm{free}}(\phi_\infty, \psi_\infty) + \int_s^t \int_\Omega |\nabla \mu_\infty|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_s^t \int_\Gamma |\nabla_\Gamma \theta_\infty|^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\ &\quad + \chi(L) \int_s^t \int_\Gamma (\beta\theta_\infty - \mu_\infty)^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\ &\leq E_{\mathrm{free}}(\phi_\infty, \psi_\infty) \end{aligned} \tag{8.32}$$

for almost all $s \geq 0$ and all $t \geq s$. This directly implies that

$$\int_\Omega |\nabla \mu_\infty|^2 \, \mathrm{d}x + \int_\Gamma |\nabla_\Gamma \theta_\infty|^2 \, \mathrm{d}\Gamma + \chi(L) \int_\Gamma (\beta\theta_\infty - \mu_\infty)^2 \, \mathrm{d}\Gamma = 0$$

a.e. in $[0,\infty)$. Hence, in all cases of $L \in [0,\infty]$, it follows that μ_∞ and θ_∞ are constant. If $L \in [0,\infty)$, we additionally have the relation $\mu_\infty = \beta\theta_\infty$.

Furthermore, by means of Theorem 6.2.7, we deduce that $(\phi_\infty, \psi_\infty) \in \mathcal{W}^{2,6} \cap \mathcal{W}^\sigma_{K,L,\alpha,\beta,m}$. Consequently, $(\phi_\infty, \psi_\infty, \mu_\infty, \theta_\infty)$ actually satisfies the equations (8.1b), (8.1d) and (8.1e) even in the strong sense. This means precisely that the pair $(\phi_\infty, \psi_\infty)$ is a strong solution of system (8.31). $\square$

Next, we show that every element of the ω-limit set is actually a stationary point of $T_m^{K,L}$. In particular, this implies that the set of stationary points $\mathcal{N}_m^{K,L}$ is non-empty.

LEMMA 8.5.5. *Suppose that* **(A1)**–**(A7)** *and* **(A9)** *are fulfilled and let* $(\phi_0, \psi_0) \in \mathcal{X}_{K,L,\alpha,\beta,m}$ *be arbitrary. Then it holds that*

$$\omega^{K,L}(\phi_0, \psi_0) \subset \mathcal{N}_m^{K,L},$$

which means that $\omega^{K,L}(\phi_0, \psi_0)$ *consists only of stationary points. Moreover, there exists* $E_* \in \mathbb{R}$ *such that*

$$E_{\mathrm{free}}(\phi_*, \psi_*) = E_* \quad \textit{for all } (\phi_*, \psi_*) \in \omega^{K,L}(\phi_0, \psi_0).$$

PROOF. Our proof is largely inspired by the line of argument in [14,78,142]. However, compared to these previous results, we have to overcome additional difficulties since, due to the involved velocity fields, our energy functional E_{free} is usually not decreasing along trajectories.

Let $(\phi, \psi, \mu, \theta)$ denote the unique weak solution to (8.1) on $[0,\infty)$ originating from the initial data (ϕ_0, ψ_0) with velocity fields $(\mathbf{v}, \mathbf{w})$. Let $(\phi_*, \psi_*) \in \omega^{K,L}(\phi_0, \psi_0)$, and let $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$ be a corresponding sequence with

$$(\phi(t_n), \psi(t_n)) \to (\phi_*, \psi_*) \quad \text{strongly in } \mathcal{H}^\sigma \text{ as } n \to \infty. \tag{8.33}$$

Without loss of generality, we assume that $t_n > 1$ for all $n \in \mathbb{N}$. Now, for any $n \in \mathbb{N}$ and any $t \in [0, \infty)$, we define

$$\begin{aligned}(\phi_n(t), \psi_n(t), \mu_n(t), \theta_n(t)) &\coloneqq (\phi(t + t_n), \psi(t + t_n), \mu(t + t_n), \theta(t + t_n)),\\ (\mathbf{v}_n(t), \mathbf{w}_n(t)) &\coloneqq (\mathbf{v}(t + t_n), \mathbf{w}(t + t_n)).\end{aligned}$$

This means that, for any $n \in \mathbb{N}$, the quadruplet $(\phi_n, \psi_n, \mu_n, \theta_n)$ is a solution to system (8.1), where the velocity fields $(\mathbf{v}, \mathbf{w})$ are replaced by $(\mathbf{v}_n, \mathbf{w}_n)$ and the initial conditions (ϕ_0, ψ_0) are replaced by $(\phi(t_n), \psi(t_n))$.

As strongly convergent sequences are bounded, we deduce that $\{(\phi(t_n), \psi(t_n))\}_{n\in\mathbb{N}} \subset \mathcal{W}^\sigma_{K,L,\alpha,\beta,m}$ is bounded. Recalling Theorem 6.2.3, we infer that there exists a constant $C_* > 0$ (which may depend on the system parameters, $E_{\mathrm{free}}(\phi_0, \psi_0)$ and $\|(\mathbf{v}, \mathbf{w})\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)}$, but not on n) such that

$$\begin{aligned}&\|(\phi_n, \psi_n)\|_{H^1(0,\infty;(\mathcal{H}^1_{L,\beta})')} + \|(\phi_n, \psi_n)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,6})}\\ &\quad + \|(F'(\phi_n), G'(\psi_n))\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^6)} + \|(\mu_n, \theta_n)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1_{L,\beta})} \le C_*\end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, thanks to assumption **(A6)**, we have

$$\|(\mathbf{v}_n, \mathbf{w}_n)\|^2_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} = \int_{t_n}^\infty \|(\mathbf{v}(s), \mathbf{w}(s))\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}s \to 0 \qquad \text{as } n \to \infty. \tag{8.34}$$

Thus, proceeding similarly to the proof of Theorem 6.2.3, we employ the Banach–Alaoglu theorem to find functions $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{\mu}, \widetilde{\theta})$ such that

$$\begin{aligned}
(\phi_n, \psi_n) &\to (\widetilde{\phi}, \widetilde{\psi}) && \text{weakly in } H^1(0, \infty; (\mathcal{H}^1_{L,\beta})'), && (8.35a)\\
& && \text{weakly-star in } L^\infty(0, \infty; \mathcal{H}^1_{K,\alpha}), && (8.35b)\\
& && \text{weakly in } L^2_{\mathrm{uloc}}([0, \infty); \mathcal{W}^{2,6}), && (8.35c)\\
(\mu_n, \theta_n) &\to (\widetilde{\mu}, \widetilde{\theta}) && \text{weakly in } L^2_{\mathrm{uloc}}([0, \infty); \mathcal{H}^1_{L,\beta}), && (8.35d)\\
(F'(\phi_n), G'(\psi_n)) &\to (F'(\widetilde{\phi}), G'(\widetilde{\psi})) && \text{weakly in } L^2_{\mathrm{uloc}}([0, \infty); \mathcal{L}^6), && (8.35e)
\end{aligned}$$

as $n \to \infty$, along a non-relabeled subsequence. Moreover, in view of (8.33) and (8.34), the quadruplet $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{\mu}, \widetilde{\theta})$ is the unique weak solution to (8.1) with velocity fields $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) = (\mathbf{0}, \mathbf{0})$ and initial data (ϕ_*, ψ_*).

Now, we consider the function

$$\begin{aligned}H : [0, \infty) \to \mathbb{R}, \quad H(t) :=& E_{\mathrm{free}}(\phi(t), \psi(t)) - \int_0^t \int_\Omega \phi\mathbf{v} \cdot \nabla\mu \,\mathrm{d}x \,\mathrm{d}r\\ &- \int_0^t \int_\Gamma \psi\mathbf{w} \cdot \nabla_\Gamma\theta \,\mathrm{d}\Gamma \,\mathrm{d}r.\end{aligned}$$

It directly follows from the energy inequality (6.13) that H is non-increasing. Moreover, recalling that $|\phi| < 1$ a.e. in Q and $|\psi| < 1$ a.e. on Σ, we use Hölder's inequality and Young's inequality to deduce that

$$\begin{aligned}H(t) &\ge E_{\mathrm{free}}(\phi(t), \psi(t)) - \int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \|(\nabla\mu, \nabla_\Gamma\theta)\|_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r\\ &\ge -\|(\mu, \theta)\|_{L^\infty(T_{\mathrm{dec}},\infty;\mathcal{H}^1)} \int_{T_{\mathrm{dec}}}^\infty \mathrm{e}^{ar} \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r\\ &\quad - \frac{1}{2} \int_0^{T_{\mathrm{dec}}} \|(\mathbf{v}, \mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r - \frac{1}{2} \int_0^{T_{\mathrm{dec}}} \|(\mu, \theta)\|^2_{\mathcal{H}^1} \,\mathrm{d}r\end{aligned} \tag{8.36}$$

for all $t \in [0, \infty)$. Here, $a \geq 0$ and $T_{\text{dec}} > 0$ are as introduced in **(A9)**. In view of Theorem 6.2.3 and Theorem 6.2.7, we have

$$(\mu, \theta) \in L^2(0, T_{\text{dec}}; \mathcal{H}^1) \cap L^\infty(T_{\text{dec}}, \infty; \mathcal{H}^1).$$

Together with the assumptions **(A6)** and **(A9)** on the velocity fields, we conclude from (8.36) that H is bounded from below. Consequently, there exists $E_* \in \mathbb{R}$ such that $H(t) \to E_*$ as $t \to \infty$. In particular, for any $t \in [0, \infty)$, this entails that

$$H(t + t_n) \to E_* \quad \text{as } n \to \infty.$$

Using the convergences (8.34) and (8.35a)–(8.35d) along with the weak-strong convergence principle, we find that

$$\int_0^t \int_\Omega \phi_n \mathbf{v}_n \cdot \nabla \mu_n \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma \psi_n \mathbf{w}_n \cdot \nabla_\Gamma \theta_n \, \mathrm{d}\Gamma \, \mathrm{d}r \to 0 \qquad \text{as } n \to \infty. \tag{8.37}$$

By the definition of (ϕ_n, ψ_n), we infer that for every $t \geq 0$,

$$\begin{aligned}
&E_{\text{free}}(\phi_n(t), \psi_n(t)) \\
&\quad = H(t + t_n) + \int_0^t \int_\Omega \phi_n \mathbf{v}_n \cdot \nabla \mu_n \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma \psi_n \mathbf{w}_n \cdot \nabla_\Gamma \theta_n \, \mathrm{d}\Gamma \, \mathrm{d}r \\
&\quad \longrightarrow E_*
\end{aligned} \tag{8.38}$$

as $n \to \infty$. Next, we recall that $(\phi_n, \psi_n, \mu_n, \theta_n)$ satisfies the energy inequality

$$\begin{aligned}
&E_{\text{free}}(\phi_n(t), \psi_n(t)) + \int_0^t \int_\Omega |\nabla \mu_n|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma |\nabla_\Gamma \theta_n|^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\
&\quad + \chi(L) \int_0^t \int_\Gamma (\beta\theta_n - \mu_n)^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\
&\leq E_{\text{free}}(\phi_n(0), \psi_n(0)) + \int_0^t \int_\Omega \phi_n \mathbf{v}_n \cdot \nabla \mu_n \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma \psi_n \mathbf{w}_n \cdot \nabla_\Gamma \theta_n \, \mathrm{d}\Gamma \, \mathrm{d}r
\end{aligned} \tag{8.39}$$

for all $t \in [0, \infty)$. Using the convergences (8.35d), (8.37) and (8.38), we use the weak lower semicontinuity of the involved norms to conclude that

$$\begin{aligned}
&E_* + \int_0^t \int_\Omega |\nabla \widetilde{\mu}|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma |\nabla_\Gamma \widetilde{\theta}|^2 \, \mathrm{d}\Gamma \, \mathrm{d}r + \chi(L) \int_0^t \int_\Gamma (\beta\widetilde{\theta} - \widetilde{\mu})^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\
&\quad \leq \lim_{n\to\infty} E_{\text{free}}(\phi_n(t), \psi_n(t)) \\
&\qquad + \liminf_{n\to\infty} \Bigg[\int_0^t \int_\Omega |\nabla \mu_n|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma |\nabla_\Gamma \theta_n|^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\
&\qquad\qquad + \chi(L) \int_0^t \int_\Gamma (\beta\theta_n - \mu_n)^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \Bigg] \\
&\quad \leq \limsup_{n\to\infty} \Bigg[E_{\text{free}}(\phi_n(t), \psi_n(t)) + \int_0^t \int_\Omega |\nabla \mu_n|^2 \, \mathrm{d}x \, \mathrm{d}r + \int_0^t \int_\Gamma |\nabla_\Gamma \theta_n|^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \\
&\qquad\qquad + \chi(L) \int_0^t \int_\Gamma (\beta\theta_n - \mu_n)^2 \, \mathrm{d}\Gamma \, \mathrm{d}r \Bigg] \\
&\quad \leq \limsup_{n\to\infty} E_{\text{free}}(\phi_n(0), \psi_n(0)) = E_* \, .
\end{aligned}$$

Consequently,

$$\int_0^t \int_\Omega |\nabla\widetilde{\mu}|^2 \,\mathrm{d}x\,\mathrm{d}r + \int_0^t \int_\Gamma |\nabla_\Gamma\widetilde{\theta}|^2 \,\mathrm{d}\Gamma\,\mathrm{d}r + \chi(L)\int_0^t \int_\Gamma (\beta\widetilde{\theta} - \widetilde{\mu})^2 \,\mathrm{d}\Gamma\,\mathrm{d}r \le 0$$

for all $t \in [0,\infty)$. This readily entails that $\widetilde{\mu}$ is spatially constant in Ω and $\widetilde{\theta}$ is spatially constant on Γ. As $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{\mu}, \widetilde{\theta})$ is the unique weak solution of system (8.1) with trivial velocity fields to the initial data (ϕ_*, ψ_*), it immediately follows that

$$(\widetilde{\phi}(t), \widetilde{\psi}(t)) = (\phi_*, \psi_*) \quad \text{for all } t \ge 0,$$

and thus $\widetilde{\mu}$ and $\widetilde{\theta}$ are constant also in time by comparison. Consequently, $(\phi_*, \psi_*) \in \mathcal{N}_m^{K,L}$. Finally, recalling that $\sigma > \frac{d}{2}$, we use (8.33) to deduce that

$$E(\phi_n(0), \psi_n(0)) = E(\phi(t_n), \psi(t_n)) \to E(\phi_*, \psi_*)$$

as $n \to \infty$. Because of (8.38), it follows that

$$E(\phi_*, \psi_*) = E_*$$

due to the uniqueness of the limit. We point out that E_* may depend on (ϕ_0, ψ_0), but is independent of the choice of $(\phi_*, \psi_*) \in \omega^{K,L}(\phi_0, \psi_0)$. Hence, the proof is complete. ☐

The next lemma shows that every stationary solution of (8.1) satisfies a strict separation property.

LEMMA 8.5.6. *Suppose that the assumptions* **(A1)**–**(A4)** *and* **(A5)** *are fulfilled. Then for any $(\phi, \psi) \in \mathcal{N}_m^{K,L}$ there exists $\delta_\star = \delta_\star(\phi, \psi) \in (0,1)$ such that*

$$\|\phi\|_{C(\overline{\Omega})} \le 1 - \delta_\star, \qquad \|\psi\|_{C(\Gamma)} \le 1 - \delta_\star. \tag{8.40}$$

PROOF. As any $(\phi, \psi) \in \mathcal{N}_m^{K,L}$ is a solution to the semilinear bulk-surface elliptic system (8.31), the strict separation property (8.40) follows from Proposition 4.3.5 ☐

To show that any weak solution of system (8.1) converges to a stationary point, the following lemma is crucial.

LEMMA 8.5.7 (Łojasiewicz–Simon inequality). *Suppose that the assumptions* **(A1)**–**(A5)** *are fulfilled. In addition, we assume that F_1, G_1 are real analytic functions on $(-1,1)$, and F_2, G_2 are real analytic functions on $\mathbb{R}$. Then for any $(\phi_\infty, \psi_\infty) \in \omega^{K,L}(\phi_0, \psi_0)$, there exist constants $\varpi^* \in (0, \frac{1}{2})$, $b^* > 0$, and $C > 0$, such that*

$$C\left\|\mathbf{P}_L\begin{pmatrix} -\Delta\zeta + F'(\zeta), \\ -\Delta_\Gamma\zeta_\Gamma + G'(\zeta_\Gamma) + \alpha\partial_\mathbf{n}\zeta \end{pmatrix}\right\|_{\mathcal{L}^2} \ge |E_{\text{free}}(\zeta, \zeta_\Gamma) - E_{\text{free}}(\phi_\infty, \psi_\infty)|^{1-\varpi^*} \tag{8.41}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{W}^2_{K,L,\alpha,\beta,m}$ satisfying $\|(\zeta - \phi_\infty, \zeta_\Gamma - \psi_\infty)\|_{\mathcal{H}^2} \le b^*$. Here, $\mathbf{P}_L$ denotes the projection of $\mathcal{L}^2$ onto

$$\begin{cases} \{(\phi,\psi) \in \mathcal{L}^2 : \beta|\Omega|\langle\phi\rangle_\Omega + |\Gamma|\langle\psi\rangle_\Gamma = 0\} & \textit{if } L \in [0,\infty), \\ \{(\phi,\psi) \in \mathcal{L}^2 : \langle\phi\rangle_\Omega = \langle\psi\rangle_\Gamma = 0\} & \textit{if } L = \infty. \end{cases}$$

PROOF. For $K = 0$, $L \in [0,\infty]$ and $\alpha = 1$, the result can be found in [129, Lemma 5.2], and for $K \in [0,\infty)$, $L = \infty$ and $\alpha = 1$, it was established in [132, Lemma 5.2]. For general $K \in [0,\infty]$ and $\alpha \in [-1,1]$ as specified by **(A1)**, the statement can be verified analogously. ☐

We are finally in a position to present the proof of Theorem 8.3.3 concerning the convergence to equilibrium.

Proof of Theorem 8.3.3. Since $\omega^{K,L}(\phi_0,\psi_0)$ is a compact subset of $\mathcal{W}^{\sigma}_{K,L,\alpha,\beta,m}$ with $\sigma\in(\frac{d}{2},2)$ (see Lemma 8.5.2), the embedding $\mathcal{W}^{\sigma}_{K,L,\alpha,\beta,m}\hookrightarrow C(\overline{\Omega})\times C(\Gamma)$ implies that $\omega^{K,L}(\phi_0,\psi_0)$ is also a compact subset of $C(\overline{\Omega})\times C(\Gamma)$. Then, as $\omega^{K,L}(\phi_0,\psi_0)\subset\mathcal{N}^{K,L}_m$ and any pair $(\phi,\psi)\in\mathcal{N}^{K,L}_m$ (see Lemma 8.5.5) is strictly separated from the pure phases (see Lemma 8.5.6), we can argue by contradiction (see, for instance, [14] or [78, Proof of Lemma 3.11]) to show that there exists $\delta\in(0,1)$ such that

$$\|\phi_*\|_{C(\overline{\Omega})}\le 1-\delta,\quad \|\psi_*\|_{C(\Gamma)}\le 1-\delta\qquad\text{for all } (\phi_*,\psi_*)\in\omega^{K,L}(\phi_0,\psi_0).$$

Here, it is crucial that the width of the separation layer δ is independent of $(\phi_*,\psi_*)\in\omega^{K,L}(\phi_0,\psi_0)$.

Moreover, we know from Lemma 8.5.2 that

$$\operatorname{dist}_{\mathcal{W}^{\sigma}_{K,L,\alpha,\beta,m}}\big((\phi(t),\psi(t)),\omega^{K,L}(\phi_0,\psi_0)\big)\to 0\qquad\text{as } t\to\infty,\tag{8.42}$$

with $\sigma\in(\frac{d}{2},2)$.

As the embedding $\mathcal{W}^{s}_{K,L,\alpha,\beta,m}\hookrightarrow C(\overline{\Omega})\times C(\Gamma)$ is compact, there $T_S>0$ such that

$$\|(\phi(t),\psi(t))\|_{C(\overline{\Omega})\times C(\Gamma)}\le 1-\frac{\delta}{2},\qquad\text{for all } t\ge T_S.\tag{8.43}$$

Thanks to this uniform separation property, we obtain by regularity theory for elliptic systems with bulk-surface coupling that

$$\|(\phi,\psi)\|_{L^\infty(T_S,\infty;\mathcal{H}^3)}\le C.\tag{8.44}$$

In particular, this entails that $\omega^{K,L}(\phi_0,\psi_0)$ is a bounded subset of $\mathcal{H}^3$.

Now, to show that $\omega^{K,L}(\phi_0,\psi_0)$ is a singleton, we need to make use of the additional assumption **(A9)**. Using Remark 8.3.1, we deduce that

$$\begin{aligned}
&\int_{T_{\mathrm{dec}}}^\infty \|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{L}}^2}^{\frac{1-2\varpi}{1-\varpi}}\,\mathrm{d}s\\
&\quad\le\left(\int_{T_{\mathrm{dec}}}^\infty \mathrm{e}^{as}\|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{L}}^2}\,\mathrm{d}s\right)^{\frac{1-2\varpi}{1-\varpi}}\left(\int_{T_{\mathrm{dec}}}^\infty \mathrm{e}^{-a\frac{1-2\varpi}{\varpi}s}\,\mathrm{d}s\right)^{\frac{\varpi}{1-\varpi}}\\
&\quad\le\left(\frac{\varpi}{a(1-2\varpi)}\right)^{\frac{\varpi}{1-\varpi}}\left(\int_{T_{\mathrm{dec}}}^\infty \mathrm{e}^{as}\|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{L}}^2}\,\mathrm{d}s\right)^{\frac{1-2\varpi}{1-\varpi}}
\end{aligned}$$

for all $\varpi\in(0,\frac12)$. This entails that

$$\|(\mathbf{v},\mathbf{w})\|_{L^{\frac{1-2\varpi}{1-\varpi}}(T_{\mathrm{dec}},\infty;\boldsymbol{\mathcal{L}}^2)}\le\left(\frac{\varpi}{a(1-2\varpi)}\right)^{\frac{\varpi}{1-2\varpi}}\int_{T_{\mathrm{dec}}}^\infty \mathrm{e}^{as}\|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{L}}^2}\,\mathrm{d}s\tag{8.45}$$

for all $\varpi\in(0,\frac12)$. Moreover, we observe that $(\mathbf{v}(t),\mathbf{w}(t))\to(\mathbf{0},\mathbf{0})$ in $\boldsymbol{\mathcal{L}}^2$ as $t\to\infty$ by assumptions **(A7)** and **(A9)**.

Now, thanks to (8.42), (8.44) and the compact embedding $\mathcal{H}^3\hookrightarrow\mathcal{H}^2$, we have

$$\operatorname{dist}_{\mathcal{H}^2}\big((\phi(t),\psi(t)),\omega^{K,L}(\phi_0,\psi_0)\big)\to 0\qquad\text{as } t\to\infty.\tag{8.46}$$

Furthermore, as $\omega^{K,L}(\phi_0, \psi_0)$ is a compact subset of $\mathcal{W}^{\sigma}_{K,L,\alpha,\beta,m}$ (see Lemma 8.5.2) and a bounded subset of $\mathcal{H}^3$, we conclude by interpolation that $\omega^{K,L}(\phi_0, \psi_0)$ is a compact subset of $\mathcal{H}^2$.

Now, for any $(\phi_*, \psi_*) \in \omega^{K,L}(\phi_0, \psi_0)$, let $\varpi^* \in (0, \frac{1}{2})$ and $b^* > 0$ be as obtained by Lemma 8.5.7. Then, it clearly holds

$$\omega^{K,L}(\phi_0, \psi_0) \subset \bigcup_{(\phi_*,\psi_*)\in\omega^{K,L}(\phi_0,\psi_0)} B_{b^*}(\phi_*, \psi_*),$$

where $B_{b^*}(\phi_*, \psi_*)$ denotes the open ball in $\mathcal{H}^2$ with center (ϕ_*, ψ_*) and radius b_*. As $\omega^{K,L}(\phi_0, \psi_0)$ is a compact subset of $\mathcal{H}^2$, we find finitely many points $\{(\phi_i^*, \psi_i^*)\}_{i=1,...,M}$ with corresponding radii $\{b_i^*\}_{i=1,...,M}$ such that

$$\omega^{K,L}(\phi_0, \psi_0) \subset \bigcup_{i=1}^{M} B_{b_i^*}(\phi_i^*, \psi_i^*)$$

and the Łojasiewicz–Simon inequality is fulfilled on each of these balls with an exponent $\varpi_i^* \in (0, \frac{1}{2})$. This means that for every $i \in \{1, ..., M\}$, there exists a constant $C_i > 0$ such that

$$C_i \left\| \mathbf{P}_L \begin{pmatrix} -\Delta\zeta + F'(\zeta) \\ -\Delta_\Gamma\zeta_\Gamma + G'(\zeta_\Gamma) + \alpha\partial_\mathbf{n}\zeta \end{pmatrix} \right\|_{\mathcal{L}^2} \geq |E_{\text{free}}(\zeta, \zeta_\Gamma) - E_{\text{free}}(\phi_0, \psi_0)|^{1-\varpi_i^*} \tag{8.47}$$

for all $(\zeta, \zeta_\Gamma) \in B_{b_i^*}(\phi_i^*, \psi_i^*)$. Now, we define

$$C := \max_{i=1,...,M} C_i > 0 \quad \text{and} \quad \varpi^* := \min_{i=1,...,M} \varpi_i^* \in (0, \tfrac{1}{2}).$$

As the balls $B_{b_i^*}(\phi_i^*, \psi_i^*)$, $i = 1, ..., M$, are open, there further exists

$$0 < b < \min_{i=1,...,M} b_i^*$$

such that for all $(\zeta, \zeta_\Gamma) \in \mathcal{W}^2_{K,L,\alpha,\beta,m}$, it holds that

$$(\zeta, \zeta_\Gamma) \in \bigcup_{i=1}^{M} B_{b_i^*}(\phi_i^*, \psi_i^*) \quad \text{if} \quad \operatorname{dist}_{\mathcal{H}^2}\big((\zeta, \zeta_\Gamma), \omega^{K,L}(\phi_0, \psi_0)\big) \leq b.$$

Hence, in view of (8.47), and recalling that $E_{\text{free}}|_{\omega^{K,L}(\phi_0,\psi_0)} = E_*$ by Lemma 8.5.2, we conclude that

$$C \left\| \mathbf{P}_L \begin{pmatrix} -\Delta\zeta + F'(\zeta) \\ -\Delta_\Gamma\zeta_\Gamma + G'(\zeta_\Gamma) + \alpha\partial_\mathbf{n}\zeta \end{pmatrix} \right\|_{\mathcal{L}^2} \geq |E_{\text{free}}(\zeta, \zeta_\Gamma) - E_*|^{1-\varpi^*} \tag{8.48}$$

for all $(\zeta, \zeta_\Gamma) \in \mathcal{W}^2_{K,L,\alpha,\beta,m}$ with

$$\operatorname{dist}_{\mathcal{H}^2}\big((\zeta, \zeta_\Gamma), \omega^{K,L}(\phi_0, \psi_0)\big) \leq b.$$

Recalling Lemma 8.5.5 and the convergence (8.46), we conclude that there exists a time $T_F > \max\{T_S, T_{\text{dec}}\}$ such that

$$\operatorname{dist}_{\mathcal{H}^2}\big((\phi(t), \psi(t)), \omega^{K,L}(\phi_0, \psi_0)\big) \leq b$$

for all $t \geq T_F$. Thus, employing (8.48), we obtain

$$
\begin{aligned}
|E_{\text{free}}(\phi(t), \psi(t)) - E_*|^{1-\varpi^*} &\leq C \left\| \mathbf{P}_L \begin{pmatrix} -\Delta\phi(t) + F'(\phi(t)), \\ -\Delta_\Gamma\psi(t) + G'(\psi(t)) + \alpha\partial_{\mathbf{n}}\phi(t) \end{pmatrix} \right\|_{\mathcal{L}^2} \\
&\leq C\|(\mu(t), \theta(t))\|_{L,\beta}
\end{aligned}
\tag{8.49}
$$

for all $t \geq T_F$. Now, using [65, Lemma 7.1], the assertion can be established by following the line of argument in [103]. To this end, let $s, t \geq T_F$ be arbitrary. Then, recalling (8.43), we use the energy inequality (6.13) to deduce that

$$
\begin{aligned}
&\int_s^t \|(\mu,\theta)\|_{L,\beta}^2 \, \mathrm{d}r + \frac{1}{2}\int_s^t \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \\
&\quad\leq E_{\text{free}}(\phi(s),\psi(s)) - E_{\text{free}}(\phi(t),\psi(t)) \\
&\qquad + \int_s^t \int_\Omega \phi\mathbf{v}\cdot\nabla\mu \, \mathrm{d}x \, \mathrm{d}r + \int_s^t \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\theta \, \mathrm{d}\Gamma \, \mathrm{d}r + \frac{1}{2}\int_s^t \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \\
&\quad\leq E_{\text{free}}(\phi(s),\psi(s)) - E_{\text{free}}(\phi(t),\psi(t)) \\
&\qquad + \int_s^t \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r + \frac{1}{2}\int_s^t \|(\mu,\theta)\|_{L,\beta}^2 \, \mathrm{d}r.
\end{aligned}
$$

After rearranging some terms and elevating to the power $2(1-\varpi^*) > 0$, we obtain

$$
\begin{aligned}
&\left(\frac{1}{2}\right)^{2(1-\varpi^*)} \left(\int_s^t \|(\mu,\theta)\|_{L,\beta}^2 \, \mathrm{d}r + \int_s^t \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)} \\
&\quad\leq \left(E_{\text{free}}(\phi(s),\psi(s)) - E_{\text{free}}(\phi(t),\psi(t)) + \int_s^t \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)} \\
&\quad\leq 2^{2(\frac{1}{2}-\varpi^*)} |E_{\text{free}}(\phi(s),\psi(s)) - E_*|^{2(1-\varpi^*)} + 2^{2(\frac{1}{2}-\varpi^*)} |E_{\text{free}}(\phi(t),\psi(t)) - E_*|^{2(1-\varpi^*)} \\
&\qquad + 2^{2(\frac{1}{2}-\varpi^*)} \left(\int_s^t \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)}.
\end{aligned}
$$

Now, we send $t \to \infty$. Recalling that $E_{\text{free}}(\phi(t),\psi(t)) \to E_*$ as $t \to \infty$, we apply (8.49) to infer that

$$
\begin{aligned}
&\left(\frac{1}{2}\right)^{2(1-\varpi^*)} \left(\int_s^\infty \|(\mu,\theta)\|_{L,\beta}^2 \, \mathrm{d}r + \int_s^\infty \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)} \\
&\quad\leq 2^{2(\frac{1}{2}-\varpi^*)} |E_{\text{free}}(\phi(s),\psi(s)) - E_*|^{2(1-\varpi^*)} + 2^{2(\frac{1}{2}-\varpi^*)} \left(\int_s^\infty \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)} \\
&\quad\leq 2^{2(\frac{1}{2}-\varpi^*)} C\|(\mu(s),\theta(s))\|_{L,\beta}^2 + 2^{2(\frac{1}{2}-\varpi^*)} \left(\int_s^\infty \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)}.
\end{aligned}
$$

Since $T_F > T_{\text{dec}}$, the mapping $(T_F, \infty) \ni t \mapsto \|(\mathbf{v}(t),\mathbf{w}(t))\|_{\mathcal{L}^2}$ is non-increasing according to assumption **(A9)**. Thus, using the estimate (8.45), we deduce that

$$
\begin{aligned}
&2^{2(\frac{1}{2}-\varpi^*)} \left(\int_s^\infty \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^2 \, \mathrm{d}r \right)^{2(1-\varpi^*)} \\
&\quad\leq 2^{2(\frac{1}{2}-\varpi^*)} \left(\|(\mathbf{v}(s),\mathbf{w}(s))\|_{\mathcal{L}^2}^{\frac{1}{1-\varpi^*}} \int_s^\infty \|(\mathbf{v},\mathbf{w})\|_{\mathcal{L}^2}^{\frac{1-2\varpi^*}{1-\varpi^*}} \, \mathrm{d}r \right)^{2(1-\varpi^*)} \\
&\quad\leq 2^{2(\frac{1}{2}-\varpi^*)} C\|(\mathbf{v}(s),\mathbf{w}(s))\|_{\mathcal{L}^2}^2.
\end{aligned}
$$

Hence, we conclude that

$$\left(\int_s^\infty \|(\mu,\theta)\|_{L,\beta}^2 \,\mathrm{d}r + \int_s^\infty \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 \,\mathrm{d}r \right)^{2(1-\varpi^*)} \\ \leq C \left(\|(\mu(s),\theta(s))\|_{L,\beta}^2 + \|(\mathbf{v}(s),\mathbf{w}(s))\|_{\boldsymbol{\mathcal{L}}^2}^2 \right)$$

for almost all $s \geq T_F$. According to [65, Lemma 7.1], this yields

$$\int_{T_*}^\infty \|(\mu,\theta)\|_{L,\beta} \,\mathrm{d}r \leq C$$

for some $T_* > 0$. Recalling that $(\mathbf{v}(t),\mathbf{w}(t)) \to (\mathbf{0},\mathbf{0})$ in $\boldsymbol{\mathcal{L}}^2$ as $t \to \infty$, we infer by a comparison argument in (6.11a) that

$$(\partial_t\phi, \partial_t\psi) \in L^1(T_*,\infty;(\mathcal{H}_{L,\beta}^1)').$$

Hence, using the fundamental theorem of calculus, we conclude that, as $t \to \infty$,

$$(\phi(t),\psi(t)) \to (\phi(T_*),\psi(T_*)) + \int_{T_*}^\infty (\partial_t\phi,\partial_t\psi) \,\mathrm{d}r =: (\phi_\infty,\psi_\infty) \quad \text{in } (\mathcal{H}_{L,\beta}^1)'.$$

This allows us to identify $\omega^{K,L}(\phi_0,\psi_0) = \{(\phi_\infty,\psi_\infty)\}$, which completes the proof, since (8.5) directly comes from (8.46). $\square$

Part II

The bulk-surface Navier–Stokes–Cahn–Hilliard model

Chapter 9

Existence of weak solutions

9.1 Introduction

In the second part of this thesis, we study the bulk-surface Navier–Stokes–Cahn–Hilliard model (1.42) which we derived in Chapter 3. Setting $\varepsilon = \delta = \kappa = 1$, the system (1.42) reads as

$$
\begin{aligned}
&\partial_t(\rho(\phi)\mathbf{v}) + \operatorname{div}\big(\mathbf{v}\otimes(\rho(\phi)\mathbf{v}+\mathbf{J})\big) - \operatorname{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) + \nabla p \\
&\quad = -\operatorname{div}(\nabla\phi\otimes\nabla\phi) && \text{in } Q, && (9.1a)\\
&\operatorname{div}\,\mathbf{v} = 0 && \text{in } Q, && (9.1b)\\
&\partial_t(\sigma(\psi)\mathbf{w}) + \operatorname{div}_\Gamma\big(\mathbf{w}\otimes(\sigma(\psi)\mathbf{w}+\mathbf{K})\big) - \operatorname{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}) + \nabla_\Gamma q \\
&\quad = -\operatorname{div}_\Gamma(\nabla_\Gamma\psi\otimes\nabla_\Gamma\psi) - 2\nu_\Omega(\phi)\big[\mathbf{D}\mathbf{v}\,\mathbf{n}\big]_{\tan} + \tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} \\
&\qquad + \tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} + \alpha\partial_\mathbf{n}\phi\nabla_\Gamma\psi - \gamma(\phi,\psi)\mathbf{w} && \text{on } \Sigma, && (9.1c)\\
&\operatorname{div}_\Gamma\,\mathbf{w} = 0 && \text{on } \Sigma, && (9.1d)\\
&\mathbf{v}|_\Gamma = \mathbf{w}, \qquad \mathbf{v}\cdot\mathbf{n} = 0 && \text{on } \Sigma, && (9.1e)\\
&\partial_t\phi + \operatorname{div}(\phi\mathbf{v}) = \operatorname{div}(m_\Omega(\phi)\nabla\mu) && \text{in } Q, && (9.1f)\\
&\mu = -\Delta\phi + F'(\phi) && \text{in } Q, && (9.1g)\\
&\partial_t\psi + \operatorname{div}_\Gamma(\psi\mathbf{w}) = \operatorname{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) - \beta m_\Omega(\phi)\partial_\mathbf{n}\mu && \text{on } \Sigma, && (9.1h)\\
&\theta = -\Delta_\Gamma\psi + G'(\psi) + \alpha\partial_\mathbf{n}\phi && \text{on } \Sigma, && (9.1i)\\
&\begin{cases} K\partial_\mathbf{n}\phi = \alpha\psi - \phi & \text{if } K\in[0,\infty), \\ \partial_\mathbf{n}\phi = 0 & \text{if } K = \infty \end{cases} && \text{on } \Sigma, && (9.1j)\\
&\begin{cases} Lm_\Omega(\phi)\partial_\mathbf{n}\mu = \beta\theta - \mu & \text{if } L\in[0,\infty), \\ m_\Omega(\phi)\partial_\mathbf{n}\mu = 0 & \text{if } L = \infty \end{cases} && \text{on } \Sigma, && (9.1k)\\
&\mathbf{v}|_{t=0} = \mathbf{v}_0, \qquad \phi|_{t=0} = \phi_0 && \text{in } \Omega, && (9.1l)\\
&\mathbf{w}|_{t=0} = \mathbf{w}_0, \qquad \psi|_{t=0} = \psi_0 && \text{on } \Gamma. && (9.1m)
\end{aligned}
$$

We recall that the densities and flux terms are defined as

$$
\begin{aligned}
\rho(\phi) &:= \frac{\tilde\rho_2-\tilde\rho_1}{2}\phi + \frac{\tilde\rho_2+\tilde\rho_1}{2} && \text{in } Q,\\
\sigma(\psi) &:= \frac{\tilde\sigma_2-\tilde\sigma_1}{2}\psi + \frac{\tilde\sigma_2+\tilde\sigma_1}{2} && \text{on } \Sigma,\\
\mathbf{J} &:= -\frac{\tilde\rho_2-\tilde\rho_1}{2}m_\Omega(\phi)\nabla\mu && \text{in } Q,\\
\mathbf{K} &:= -\frac{\tilde\sigma_2-\tilde\sigma_1}{2}m_\Gamma(\psi)\nabla_\Gamma\theta && \text{on } \Sigma,
\end{aligned}
$$

$$\mathbf{J}_\Gamma := -\beta \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} m_\Omega(\phi)\nabla\mu \qquad \text{on } \Sigma.$$

The goal of this chapter is to establish the existence of a global-in-time weak solution to (9.1), see Theorem 9.3.3. In the case of constant mobility functions, this result was first proved in [123] by means of a semi-Galerkin scheme combined with Schauder's fixed point theorem. More precisely, the velocity fields were discretized using eigenfunctions of the bulk-surface Stokes operator $\boldsymbol{\mathcal{A}}$, which was defined and analyzed in Section 4.5. We will also use this approach later to prove the existence of a global-in-time strong solution in two space dimensions in Chapter 10, see Theorem 10.2.1.

The semi-Galerkin approach can also be used to establish the existence of weak solutions in the two-dimensional setting for non-degenerate mobility functions. The reason is that in two space dimensions, a continuous dependence result for weak solutions to the convective bulk-surface Cahn–Hilliard system (6.1) is available, see Theorem 7.3.1. This result is essential to the fixed-point argument used in the semi-Galerkin scheme. In three space dimensions, however, no such continuous dependence result is currently known. For this reason, the semi-Galerkin approach does not appear suitable for proving the existence of weak solutions in the three-dimensional setting with non-degenerate mobility functions.

To overcome this difficulty, we present an alternative proof in this chapter based on an implicit time discretization scheme. The argument developed here has not appeared in the literature so far. This method was previously used in [5] to prove the existence of weak solutions for the AGG model. The key ingredient of this approach is a characterization of the convex part of the Ginzburg–Landau energy $\widetilde{E}_\Omega : L^2(\Omega) \to (-\infty, \infty]$, defined by

$$\widetilde{E}_\Omega(u) = \begin{cases} \int_\Omega \frac{1}{2}|\nabla u|^2 + F_0(u) \,\mathrm{d}x & \text{if } u \in \operatorname{dom} \widetilde{E}_\Omega, \\ \infty & \text{else,} \end{cases}$$

with F_0 as in **(A5)**, and

$$\operatorname{dom} \widetilde{E}_\Omega = \{u \in H^1(\Omega) : |u| \leq 1 \text{ a.e. in } \Omega\},$$

see [1]. We also refer to [14], where the ambient space was chosen as $L^2_{(m)}(\Omega)$ for some $m \in (-1, 1)$.

Implicit time discretization schemes of this type have been successfully adapted to two-phase flow models with dynamic boundary conditions; see, for example, [79, 80]. However, since a corresponding characterization of the subdifferential of the convex part of the bulk-surface free energy $\widetilde{E}_{\text{free}}$ was not available, the works relied on several approximation and regularization procedures.

This gap is closed by our characterization of the subdifferential of the convex part of the bulk-surface free energy established in Proposition 4.3.2. Thanks to this result, we do not need to rely on additional approximation schemes but can instead adapt the approach of [5] directly to the bulk-surface setting. The main additional difficulties arise from the intricate structure of the dynamic boundary conditions.

We expect this approach, and in particular the characterization of the subdifferential, to also apply to a broader class of diffuse interface models with dynamic boundary conditions, such as those considered in [37, 77, 79, 80, 97].

Before presenting the main results, we show that the bulk-surface Navier–Stokes equations can be rewritten in a way that is more useful for our analysis later on. To this end, we start by noting the standard decompositions

$$
\begin{aligned}
-\mathrm{div}(\nabla\phi\otimes\nabla\phi) &= -\nabla\Big(\frac{1}{2}|\nabla\phi|^2+F(\phi)\Big)+\mu\nabla\phi && \text{in } Q,\\
-\mathrm{div}_\Gamma(\nabla_\Gamma\psi\otimes\nabla_\Gamma\psi) &= -\nabla_\Gamma\Big(\frac{1}{2}|\nabla_\Gamma\psi|^2+G(\psi)\Big)+\theta\nabla_\Gamma\psi-\alpha\partial_{\mathbf{n}}\phi\nabla_\Gamma\psi && \text{on } \Sigma,
\end{aligned}
$$

which follow from (9.1g) and (9.1i), respectively. Then, defining

$$
\overline{p} := p+\frac{1}{2}|\nabla\phi|^2+F(\phi), \qquad \overline{q} := q+\frac{1}{2}|\nabla_\Gamma\psi|^2+G(\psi),
$$

we can rewrite (9.1a)-(9.1b) as

$$
\begin{aligned}
&\partial_t(\rho(\phi)\mathbf{v})+\mathrm{div}\,(\mathbf{v}\otimes(\rho(\phi)\mathbf{v}+\mathbf{J}))+\mathrm{div}\,(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v})+\nabla\overline{p}=\mu\nabla\phi && \text{in } Q, && (9.2a)\\
&\partial_t(\sigma(\psi)\mathbf{w})+\mathrm{div}_\Gamma\,(\mathbf{w}\otimes(\sigma(\psi)\mathbf{w}+\mathbf{K}))+\mathrm{div}_\Gamma\,(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w})+\nabla\overline{q}\\
&\quad=\theta\nabla_\Gamma\psi-2\nu_\Omega(\phi)[\mathbf{D}\mathbf{v}\,\mathbf{n}]_{\tan}+\tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w}+\tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w}-\gamma(\phi,\psi)\mathbf{w} && \text{on } \Sigma. && (9.2b)
\end{aligned}
$$

Multiplying (9.1f) with $\frac{\tilde\rho_2-\tilde\rho_1}{2}$ and (9.1h) with $\frac{\tilde\sigma_2-\tilde\sigma_1}{2}$, we readily obtain that the time evolution of $\rho(\phi)$ and $\sigma(\psi)$ is given by

$$
\begin{aligned}
\partial_t\rho(\phi)+\mathrm{div}(\rho(\phi)\mathbf{v}) &= -\mathrm{div}\,\mathbf{J} && \text{in } Q, && (9.3a)\\
\partial_t\sigma(\psi)+\mathrm{div}_\Gamma(\sigma(\psi)\mathbf{w}) &= -\mathrm{div}_\Gamma\,\mathbf{K}+\mathbf{J}_\Gamma\cdot\mathbf{n} && \text{on } \Sigma, && (9.3b)
\end{aligned}
$$

see also (3.16)-(3.17). Then, using (9.3), we can rewrite (9.2) in the following non-conservative form

$$
\begin{aligned}
&\rho(\phi)\partial_t\mathbf{v}+\rho(\phi)(\mathbf{v}\cdot\nabla)\mathbf{v}+(\mathbf{J}\cdot\nabla)\mathbf{v}-\mathrm{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v})+\nabla\overline{p}=\mu\nabla\phi && \text{in } Q, && (9.4a)\\
&\sigma(\psi)\partial_t\mathbf{w}+\sigma(\psi)(\mathbf{w}\cdot\nabla_\Gamma)\mathbf{w}+(\mathbf{K}\cdot\nabla_\Gamma)\mathbf{w}-\mathrm{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w})+\nabla_\Gamma\overline{q}\\
&\quad=\theta\nabla_\Gamma\psi-2\nu_\Omega(\phi)[\mathbf{D}\mathbf{v}\,\mathbf{n}]_{\tan}+\tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w}-\tfrac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w}-\gamma(\phi,\psi)\mathbf{w} && \text{on } \Sigma. && (9.4b)
\end{aligned}
$$

9.2 Additional preliminaries

In this section, we present some additional preliminaries that are needed to show the existence of weak solutions to (9.1).

Lemma 9.2.1. *Let X, Y be Banach spaces with $Y \hookrightarrow X$ and $X' \hookrightarrow Y'$ densely. Then $L^\infty(I;Y)\cap BUC(I;X)\hookrightarrow BC_w(I;Y)$.*

For a proof, see, e.g., [2, Lemma 4.1].

Lemma 9.2.2. *Let $E:[0,T)\to[0,\infty)$, $0<T\le\infty$, be a lower semicontinuous function and let $D:(0,T)\to[0,\infty)$ be an integrable function. Then*

$$
E(0)\pi(0)+\int_0^T E(t)\pi'(t)\,\mathrm{d}t \ge \int_0^T D(t)\pi(t)\,\mathrm{d}t
$$

holds for all $\pi\in W^{1,1}(0,T)$ with $\pi\ge 0$ and $\pi(T)=0$ if and only if

$$
E(t)+\int_s^t D(\tau)\,\mathrm{d}\tau \le E(s)
$$

holds for all $s\le t<T$ and almost all $0\le s<T$ including $s=0$.

For a proof, see [2, Lemma 4.3].

Lastly, for any $(f, f_\Gamma) \in \mathcal{L}^2$ and $(\phi, \psi) \in \mathcal{H}^2$, we consider the system

$$-\operatorname{div}(m_\Omega(\phi)\nabla\mu) + \int_\Omega \mu \,\mathrm{d}x = f \qquad \text{in } \Omega, \tag{9.5a}$$

$$-\operatorname{div}_\Gamma(m_\Gamma(\psi)\nabla_\Gamma\theta) + \beta m_\Omega(\phi)\partial_{\mathbf{n}}\mu + \int_\Gamma \theta \,\mathrm{d}\Gamma = f_\Gamma \qquad \text{on } \Gamma, \tag{9.5b}$$

$$\begin{cases} L m_\Omega(\phi)\partial_{\mathbf{n}}\mu = \beta\theta - \mu & \text{if } L \in [0,\infty), \\ m_\Omega(\phi)\partial_{\mathbf{n}}\mu = 0 & \text{if } L = \infty \end{cases} \qquad \text{on } \Gamma. \tag{9.5c}$$

Then, the following well-posedness result holds.

Proposition 9.2.3. *Let $(f, f_\Gamma) \in \mathcal{L}^2$ and $(\phi, \psi) \in \mathcal{H}^2$, and assume that m_Ω, m_Γ satisfy the assumptions stated in* **(A4)**. *Then, there exists a unique pair $(\mu, \theta) \in \mathcal{H}^2 \cap \mathcal{H}^1_{L,\beta}$ solving (9.5) in the sense that the variational formulation*

$$\begin{aligned}
&\int_\Omega m_\Omega(\phi)\nabla\mu \cdot \nabla\zeta \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma\theta \cdot \nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\
&\qquad + \chi(L)\int_\Gamma (\beta\theta - \mu)(\beta\zeta_\Gamma - \zeta)\,\mathrm{d}\Gamma + \int_\Omega \mu \,\mathrm{d}x \int_\Omega \zeta \,\mathrm{d}x + \int_\Gamma \theta \,\mathrm{d}\Gamma \int_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \\
&= \int_\Omega f\,\zeta \,\mathrm{d}x + \int_\Gamma f_\Gamma\,\zeta_\Gamma \,\mathrm{d}\Gamma
\end{aligned}$$

holds for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$. Furthermore, there exists $C > 0$ such that

$$\|(\mu, \theta)\|_{\mathcal{H}^2} \le C\|(f, g)\|_{\mathcal{L}^2},$$

where the constant C may depend on $\|(\phi, \psi)\|_{\mathcal{H}^2}, m_*, m^*, L, \Omega$ and Γ.

For a proof of this statement, we refer to [80, Lemma 3.1].

Finally, we present an approximation result that will be needed in the proof of the existence of weak solutions to approximate the initial data appropriately.

Lemma 9.2.4. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a non-empty, bounded domain with C^2-boundary $\Gamma := \partial\Omega$. Let $(\phi_0, \psi_0) \in \mathcal{H}^1$ with $|\phi_0| \le 1$ a.e. in Ω and $|\psi_0| \le 1$ a.e. on Γ be given. Then there exists a sequence $\{(\phi_0^N, \psi_0^N)\}_{N\in\mathbb{N}} \subset \mathcal{H}^2$ with $|\phi_0^N| \le 1$ a.e. in Ω, $|\psi_0^N| \le 1$ a.e. on Γ, and $(\phi_0^N, \psi_0^N) \to (\phi_0, \psi_0)$ strongly in $\mathcal{H}^1$ as $N \to \infty$.*

Proof. Let u be a solution to

$$\begin{aligned}
\partial_t u - \Delta u &= 0 && \text{in } Q^T, \\
\partial_{\mathbf{n}} u &= 0 && \text{on } \Sigma^T, \\
u|_{t=0} &= \phi_0 && \text{in } \Omega,
\end{aligned}$$

and choose v as a solution to

$$\begin{aligned}
\partial_t v - \Delta_\Gamma v &= 0 && \text{on } \Sigma^T, \\
v|_{t=0} &= \psi_0 && \text{on } \Gamma.
\end{aligned}$$

Then it holds

$$(u, v) \in C((0, T]; \mathcal{H}^2) \cap C([0, T]; \mathcal{H}^1).$$

Now, setting $(\phi_0^N, \psi_0^N) := (u, v)|_{t=\frac{1}{N}}$ for any $N \in \mathbb{N}$ ensures $(\phi_0^N, \psi_0^N) \in \mathcal{H}^2$, $|\phi_0^N| \le 1$ a.e. in Ω, $|\psi_0^N| \le 1$ a.e. on Γ as well as $(\phi_0^N, \psi_0^N) \to (\phi_0, \psi_0)$ strongly in $\mathcal{H}^1$ as $N \to \infty$. $\square$

9.3 Main results

9.3.1 Main assumptions

We begin by stating our main assumption for this chapter.

(A1) The set $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a non-empty, bounded domain with C^2-boundary $\Gamma := \partial\Omega$.

(A2) The constants occurring in system (9.1) satisfy $\alpha \in [-1,1]$ and $\beta \in \mathbb{R}$ with $\alpha\beta|\Omega| + |\Gamma| \neq 0$.

(A3) The density functions ρ and σ are given by

$$\rho(s) = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}s + \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2},$$
$$\sigma(s) = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2}s + \frac{\tilde{\sigma}_1 + \tilde{\sigma}_2}{2}$$

for all $s \in [-1,1]$, respectively. Here, $\tilde{\rho}_1, \tilde{\rho}_2 > 0$ and $\tilde{\sigma}_1, \tilde{\sigma}_2 > 0$ are the specific densities of the two fluid components in the bulk and on the surface, respectively. Moreover, we use the notation

$$\rho_* := \min\{\tilde{\rho}_1, \tilde{\rho}_2\}, \qquad \rho^* := \max\{\tilde{\rho}_1, \tilde{\rho}_2\},$$
$$\sigma_* := \min\{\tilde{\sigma}_1, \tilde{\sigma}_2\}, \qquad \sigma^* := \max\{\tilde{\sigma}_1, \tilde{\sigma}_2\}.$$

By this definition, we clearly have

$$0 < \rho_* \leq \rho(s) \leq \rho^* \quad \text{and} \quad 0 < \sigma_* \leq \sigma(s) \leq \sigma^* \qquad \text{for all } s \in [-1,1].$$

(A4) The mobility functions $m_\Omega, m_\Gamma : [-1,1] \to \mathbb{R}$, the viscosity functions $\nu_\Omega, \nu_\Gamma : [-1,1] \to \mathbb{R}$ and the friction coefficient $\gamma : [-1,1]^2 \to \mathbb{R}$ are continuous, bounded and uniformly positive. In particular, there exist constants $m_*, m^*, \nu_*, \nu^*, \gamma_*$ and γ^* such that

$$0 < m_* \leq m_\Omega(s), m_\Gamma(s) \leq m^*, \quad 0 < \nu_* \leq \nu_\Omega(s), \nu_\Gamma(s) \leq \nu^* \quad \text{for all } s \in [-1,1],$$

and

$$0 < \gamma_* \leq \gamma(s,r) \leq \gamma^* \quad \text{for all } (s,r) \in [-1,1]^2.$$

(A5) We assume that the potentials $F, G : [-1,1] \to \mathbb{R}$ are of the form

$$F(s) = F_0(s) - \frac{c_F}{2}s^2, \qquad G(s) = G_0(s) - \frac{c_G}{2}s^2 \qquad \text{for } s \in [-1,1]$$

with constants $c_F, c_G > 0$, where $F_0, G_0 \in C([-1,1]) \cap C^2(-1,1)$ are such that

$$\lim_{s \searrow -1} F_0'(s) = \lim_{s \searrow -1} G_0'(s) = -\infty \quad \text{and} \quad \lim_{s \nearrow 1} F_0'(s) = \lim_{s \nearrow 1} G_0'(s) = +\infty.$$

Moreover, we assume that

$$F_0''(s) \geq c_F, \qquad G_0''(s) \geq c_G \qquad \text{for all } s \in (-1,1).$$

Without loss of generality, we assume that $F_0(0) = G_0(0) = 0$ and $F_0'(0) = G_0'(0) = 0$. Lastly, we require the singular part of the boundary potential to dominate the singular part of the bulk potential in the sense that there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$|F_0'(\alpha s)| \leq \kappa_1 |G_0'(s)| + \kappa_2 \qquad \text{for all } s \in (-1,1). \tag{9.6}$$

9.3.2 Existence of weak solutions

Let us introduce a suitable notion of a weak solution to (9.1).

DEFINITION 9.3.1. *Suppose that the assumptions* **(A1)-(A5)** *hold. Let* $K, L \in [0,\infty]$, *and let* $(\mathbf{v}_0, \mathbf{w}_0) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ *and* $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ *be initial data such that*

$$\|\phi_0\|_{L^\infty(\Omega)} \le 1, \qquad \|\psi_0\|_{L^\infty(\Gamma)} \le 1. \tag{9.7a}$$

In addition, we further assume that

$$\beta \mathrm{mean}(\phi_0, \psi_0),\ \mathrm{mean}(\phi_0, \psi_0) \in (-1, 1) \qquad \text{if } L \in [0, \infty), \tag{9.7b}$$

and

$$\langle \phi_0 \rangle_\Omega,\ \langle \psi_0 \rangle_\Gamma \in (-1, 1) \qquad \text{if } L = \infty. \tag{9.7c}$$

The sextuplet $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ *is called a weak solution to system* (9.1) *on* $[0,\infty)$ *if the following properties hold:*

(i) *The functions* $\mathbf{v}, \mathbf{w}, \phi, \psi, \mu$ *and* θ *have the regularities*

$$\begin{aligned}
(\mathbf{v}, \mathbf{w}) &\in BC_w([0,\infty); \boldsymbol{\mathcal{H}}^1_{\mathrm{div}}) \cap L^2(0,\infty; \boldsymbol{\mathcal{H}}^1_{\mathrm{div}}), && (9.8a)\\
(\phi, \psi) &\in BC_w([0,\infty); \mathcal{H}^1_{K,\alpha}) \cap L^2_{\mathrm{uloc}}([0,\infty); \mathcal{H}^2), && (9.8b)\\
(F'(\phi), G'(\psi)) &\in L^2_{\mathrm{uloc}}([0,\infty); \mathcal{L}^2), && (9.8c)\\
(\mu, \theta) &\in L^2_{\mathrm{uloc}}([0,\infty); \mathcal{H}^1_{L,\beta}), && (9.8d)
\end{aligned}$$

and it holds

$$|\phi| < 1 \quad \text{a.e. in } Q \quad \text{and} \quad |\psi| < 1 \quad \text{a.e. on } \Sigma.$$

(ii) *The initial conditions are satisfied in the following sense:*

$$(\mathbf{v}, \phi)|_{t=0} = (\mathbf{v}_0, \phi_0) \quad \text{in } \Omega, \qquad (\mathbf{w}, \psi)|_{t=0} = (\mathbf{w}_0, \psi_0) \quad \text{on } \Gamma. \tag{9.9}$$

(iii) *The variational formulations*

$$\begin{aligned}
&- \int_0^\infty \int_\Omega \rho(\phi)\mathbf{v} \cdot \partial_t \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + - \int_0^\infty \int_\Gamma \sigma(\psi)\mathbf{w} \cdot \partial_t \hat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad - \int_0^\infty \int_\Omega \mathrm{div}(\rho(\phi)\mathbf{v} \otimes \mathbf{v}) \cdot \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Gamma \mathrm{div}_\Gamma(\sigma(\psi)\mathbf{w} \otimes \mathbf{w}) \cdot \hat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_0^\infty \int_\Omega (\mathbf{v} \otimes \mathbf{J}) : \nabla \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Gamma (\mathbf{w} \otimes \mathbf{K}) : \nabla_\Gamma \mathbf{w} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_0^\infty \int_\Omega 2\nu_\Omega(\phi)\mathbf{D}\mathbf{v} : \mathbf{D}\hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Gamma 2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w} : \mathbf{D}_\Gamma\hat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_0^\infty \int_\Gamma \gamma(\phi,\psi)\mathbf{w} \cdot \hat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \qquad\qquad (9.10a)\\
&= \int_0^\infty \int_\Omega \mu \nabla\phi \cdot \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Gamma \theta \nabla_\Gamma \psi \cdot \hat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \frac{\chi(L)}{2}\Big(\beta\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big) \int_0^\infty \int_\Gamma (\beta\theta - \mu)\mathbf{w} \cdot \hat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t,
\end{aligned}$$

and

$$- \int_0^\infty \int_\Omega \phi\, \partial_t \zeta \,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \int_\Gamma \psi\, \partial_t \zeta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t$$

$$
\begin{aligned}
&- \int_0^\infty \int_\Omega \phi \mathbf{v} \cdot \nabla \zeta \,\mathrm{d}x \,\mathrm{d}t - \int_0^\infty \int_\Gamma \psi \mathbf{w} \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \,\mathrm{d}t \qquad (9.10b)\\
&= - \int_0^\infty \int_\Omega m_\Omega(\phi) \nabla \mu \cdot \nabla \zeta \,\mathrm{d}x \,\mathrm{d}t - \int_0^\infty \int_\Gamma m_\Gamma(\psi) \nabla_\Gamma \theta \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma \,\mathrm{d}t\\
&\quad - \chi(L) \int_0^\infty \int_\Gamma (\beta\theta - \mu)(\beta\zeta_\Gamma - \zeta) \,\mathrm{d}\Gamma \,\mathrm{d}t
\end{aligned}
$$

hold for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in C_c^\infty(0,\infty; \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}})$ *and all* $(\zeta, \zeta_\Gamma) \in C_c^\infty(0,\infty; \mathcal{H}^1_{L,\beta})$*, where*

$$
\begin{aligned}
&\mu = -\Delta\phi + F'(\phi) && \text{a.e. in } Q, && (9.11a)\\
&\theta = -\Delta_\Gamma \psi + G'(\psi) + \alpha \partial_{\mathbf{n}} \phi && \text{a.e. on } \Sigma, && (9.11b)\\
&\begin{cases} K \partial_{\mathbf{n}} \phi = \alpha\psi - \phi & \text{if } K \in [0,\infty), \\ \partial_{\mathbf{n}} \phi = 0 & \text{if } K = \infty \end{cases} && \text{a.e. on } \Sigma, && (9.11c)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(\phi) &= \frac{\tilde\rho_2 - \tilde\rho_1}{2}\phi + \frac{\tilde\rho_1 + \tilde\rho_2}{2} && \text{a.e. in } Q,\\
\sigma(\psi) &= \frac{\tilde\sigma_2 - \tilde\sigma_1}{2}\psi + \frac{\tilde\sigma_1 + \tilde\sigma_2}{2} && \text{a.e. on } \Sigma,\\
\mathbf{J} &= -\frac{\tilde\rho_2 - \tilde\rho_1}{2} m_\Omega(\phi) \nabla\mu && \text{a.e. in } Q,\\
\mathbf{K} &= -\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} m_\Gamma(\psi) \nabla_\Gamma\theta && \text{a.e. on } \Sigma.
\end{aligned}
$$

(iv) *The functions* ϕ *and* ψ *satisfy the mass conservation law*

$$
\begin{cases}
\beta \int_\Omega \phi(t) \,\mathrm{d}x + \int_\Gamma \psi(t) \,\mathrm{d}\Gamma = \beta \int_\Omega \phi_0 \,\mathrm{d}x + \int_\Gamma \psi_0 \,\mathrm{d}\Gamma & \text{if } L \in [0,\infty),\\
\int_\Omega \phi(t) \,\mathrm{d}x = \int_\Omega \phi_0 \,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi(t) \,\mathrm{d}\Gamma = \int_\Gamma \psi_0 \,\mathrm{d}\Gamma & \text{if } L = \infty
\end{cases}
\tag{9.12}
$$

for all $t \in [0,\infty)$*.*

(v) *The energy inequality*

$$
\begin{aligned}
&E_{\mathrm{tot}}(\mathbf{v}(t), \mathbf{w}(t), \phi(t), \psi(t)) + \int_s^t \int_\Omega 2\nu_\Omega(\phi) |\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x \,\mathrm{d}r\\
&\quad + \int_s^t \int_\Gamma 2\nu_\Gamma(\psi) |\mathbf{D}_\Gamma \mathbf{w}|^2 \,\mathrm{d}\Gamma \,\mathrm{d}r + \int_s^t \int_\Gamma \gamma(\phi,\psi) |\mathbf{w}|^2 \,\mathrm{d}\Gamma \,\mathrm{d}r\\
&\quad + \int_s^t \int_\Omega m_\Omega(\phi) |\nabla\mu|^2 \,\mathrm{d}x \,\mathrm{d}r + \int_s^t \int_\Gamma m_\Gamma(\psi) |\nabla_\Gamma \theta|^2 \,\mathrm{d}\Gamma \,\mathrm{d}r \qquad (9.13)\\
&\quad + \chi(L) \int_s^t \int_\Gamma (\beta\theta - \mu)^2 \,\mathrm{d}\Gamma \,\mathrm{d}r\\
&\le E_{\mathrm{tot}}(\mathbf{v}(s), \mathbf{w}(s), \phi(s), \psi(s))
\end{aligned}
$$

holds for all $t \in [s,\infty)$ *and almost all* $s \in [0,\infty)$ *including* $s = 0$*.*

We recall that the total energy associated with the system (9.1) is given by

$$
\begin{aligned}
E_{\mathrm{tot}}(\mathbf{v}, \mathbf{w}, \phi, \psi) &:= \int_\Omega \frac{1}{2}\rho(\phi) |\mathbf{v}|^2 \,\mathrm{d}x + \int_\Gamma \frac{1}{2}\sigma(\psi) |\mathbf{w}|^2 \,\mathrm{d}\Gamma + \int_\Omega \frac{1}{2} |\nabla\phi|^2 + F(\phi) \,\mathrm{d}x\\
&\quad + \int_\Gamma \frac{1}{2} |\nabla_\Gamma \psi|^2 + G(\psi) \,\mathrm{d}\Gamma + \chi(K) \int_\Gamma \frac{1}{2} (\alpha\psi - \phi)^2 \,\mathrm{d}\Gamma.
\end{aligned}
$$

Remark 9.3.2. In view of the regularities of a weak solution $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$, it is straightforward to show with the weak formulation (9.10b) that

$$(\partial_t\phi, \partial_t\psi) \in L^2(0, \infty; (\mathcal{H}^1_{L,\beta})').$$

Moreover, it holds that

$$\begin{aligned}
&\langle(\partial_t\phi, \partial_t\psi), (\zeta, \zeta_\Gamma)\rangle_{\mathcal{H}^1_{L,\beta}} - \int_\Omega \phi\mathbf{v}\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma \psi\mathbf{w}\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\
&= -\int_\Omega m_\Omega(\phi)\nabla\mu\cdot\nabla\zeta \,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi)\nabla_\Gamma\theta\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma \\
&\quad - \chi(L)\int_\Gamma (\beta\theta - \mu)(\beta\zeta_\Gamma - \zeta) \,\mathrm{d}\Gamma
\end{aligned}$$

a.e. on $(0, \infty)$ for all $(\zeta, \zeta_\Gamma) \in \mathcal{H}^1_{L,\beta}$. In particular, $(\phi, \psi, \mu, \theta)$ is a weak solution to the convective bulk-surface Cahn–Hilliard equation (6.1) with initial datum (ϕ_0, ψ_0) and velocity fields $(\mathbf{v}, \mathbf{w})$ in the sense of Definition 6.2.2.

We are now in a position to state our main theorem of this chapter, which establishes the existence of a global-in-time weak solution to system (9.1).

THEOREM 9.3.3. *Suppose that the assumptions* **(A1)**-**(A5)** *hold. Let* $K, L \in [0, \infty]$, $(\mathbf{v}_0, \mathbf{w}_0) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$, *and let* $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ *satisfy* (9.7)*. If* $L = 0$*, we additionally assume that*

$$\beta(\tilde\sigma_2 - \tilde\sigma_1) = -(\tilde\rho_2 - \tilde\rho_1). \tag{9.14}$$

Then, there exists at least one global-in-time weak solution $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ *to system* (9.1) *in the sense of Definition 9.3.1.*

We remark the following regarding the compatibility assumption (9.14).

Remark 9.3.4. The compatibility condition (9.14) in the case $L = 0$ formally ensures that

$$\frac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} + \frac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} = \Big(\beta\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} + \frac{\tilde\rho_2 - \tilde\rho_1}{2}\Big)\partial_\mathbf{n}\mu\,\mathbf{w} = 0 \qquad \text{on } \Sigma, \tag{9.15}$$

and thus the corresponding term in the weak formulation (9.10a) vanishes, which is crucial to handle the case $L = 0$. The condition (9.14) is also related to the compatibility condition $\tilde\rho_1 = \tilde\rho_2$ that was imposed in [80] for the analysis of system (1.40) from [97] with $L = 0$. From the viewpoint of mathematical analysis, this is an advantage of the new model (9.1) compared to (1.40) since even in the case $L = 0$ we can allow for unmatched densities of the fluids.

9.4 Existence of weak solutions

In this section, we present the proof of Theorem 9.3.3, which is based on an implicit time discretization scheme. However, this approach cannot be applied directly for $K = 0$ because it requires an approximation of the initial data with higher regularity. Indeed, the core part of the proof requires $\mathcal{H}^2$-regular initial data, whereas we only assume that $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ and that (9.7) holds.

To overcome this difficulty, we employ Lemma 9.2.4, which provides a sequence of admissible initial data $\{(\phi_0^N, \psi_0^N)\}_{N\in\mathbb{N}} \subset \mathcal{H}^2$ such that $(\phi_0^N, \psi_0^N) \to (\phi_0, \psi_0)$ strongly in $\mathcal{H}^1$ as $N \to \infty$, and satisfying (9.7). In particular, it holds

$$|\phi_0^N| \le 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad |\psi_0^N| \le 1 \quad \text{a.e. on } \Gamma. \tag{9.16}$$

Nevertheless, in the case $K = 0$, i.e., when $\phi_0 = \alpha\psi_0$ a.e. on Γ, this relation is in general not preserved by the approximation, so that $\phi_0^N = \alpha\psi_0^N$ a.e. on Γ cannot be guaranteed.

A natural alternative would be to consider a coupled bulk-surface system instead of two uncoupled heat equations as in the proof of Lemma 9.2.4. For example, one could consider the system

$$\begin{aligned} \partial_t u - \Delta u &= 0 && \text{in } Q^T, \\ \partial_t v - \Delta_\Gamma v + \alpha \partial_{\mathbf{n}} u &= 0 && \text{on } \Sigma^T, \\ u &= \alpha v && \text{on } \Sigma^T, \\ (u,v)|_{t=0} &= (\phi_0, \psi_0) && \text{in } \Omega \times \Gamma. \end{aligned}$$

While this preserves the coupling condition, it only yields the estimates

$$|u| \le 1 \quad \text{a.e. in } Q^T \quad \text{and} \quad |\alpha||v| \le 1 \quad \text{a.e. on } \Sigma^T.$$

Since $|\alpha| \le 1$, the latter bound is insufficient to deduce $|v| \le 1$ a.e. on Σ^T, which is essential for the analysis. We remark that such a system was previously considered in [79] to approximate the initial data. However, in that work, the authors first studied the case of regular potentials, and therefore did not require the bound $|v| \le 1$ a.e. on Γ at that stage of the argument.

For this reason, the proof proceeds in two steps. First, we establish the existence of weak solutions in the case $K \in (0, \infty]$ and $L \in [0, \infty]$ by means of an implicit time discretization scheme. In a second step, we treat the case $K = 0$ by passing to the asymptotic limit $K \to 0$, following a similar strategy used in [123], where we considered the asymptotic limit $L \to 0$ to show the existence of a weak solution to (9.1) for constant mobilities in the case $K \in [0, \infty]$ and $L = 0$.

9.4.1 Existence of weak solutions for $K \in (0, \infty]$

In this section, we prove the following theorem, which provides the existence of a weak solution to (9.1) in the case $K \in (0, \infty]$ and $L \in [0, \infty]$.

THEOREM 9.4.1. *Suppose that the assumptions* **(A1)-(A5)** *hold. Let* $K \in (0, \infty]$, $L \in [0, \infty]$, $(\mathbf{v}_0, \mathbf{w}_0) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$, *and let* $(\phi_0, \psi_0) \in \mathcal{H}^1_{K,\alpha}$ *satisfy* (9.7). *If* $L = 0$, *we additionally assume that*

$$\beta(\tilde{\sigma}_2 - \tilde{\sigma}_1) = -(\tilde{\rho}_2 - \tilde{\rho}_1). \tag{9.17}$$

Then, there exists at least one global-in-time weak solution $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ *to system* (9.1) *in the sense of Definition 9.3.1.*

Before presenting the proof of Theorem 9.4.1, we set up a time discretization scheme and show the existence of approximate solutions to the time-discrete system.

An implicit time discretization scheme

We fix $N \in \mathbb{N}$ and denote the time-step size by $h = \frac{1}{N}$. For $k \in \mathbb{N}_0$, let $(\mathbf{v}_k, \mathbf{w}_k) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ and $(\phi_k, \psi_k) \in \mathcal{H}^1$ with $(F_0'(\phi_k), G_0'(\psi_k)) \in \mathcal{L}^2$ be given, and set $(\rho_k, \sigma_k) = (\rho(\phi_k), \sigma(\psi_k))$. Then, we construct

$$(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta) = (\mathbf{v}_{k+1}, \mathbf{w}_{k+1}, \phi_{k+1}, \psi_{k+1}, \mu_{k+1}, \theta_{k+1})$$

as a solution of the nonlinear system (9.18), where

$$\mathbf{J} = \mathbf{J}_{k+1} := -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m_\Omega(\phi_k) \nabla \mu_{k+1} = -\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} m_\Omega(\phi_k) \nabla \mu,$$

$$\mathbf{K} = \mathbf{K}_{k+1} := -\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} m_\Gamma(\psi_k)\nabla_\Gamma \theta_{k+1} = -\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} m_\Gamma(\psi_k)\nabla_\Gamma \theta,$$
$$\mathbf{J}_\Gamma = \mathbf{J}_{\Gamma,k+1} := -\beta\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} m_\Omega(\phi_k)\nabla \mu_{k+1} = -\beta\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} m_\Omega(\phi_k)\nabla \mu.$$

Our aim is to find $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ with $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$, $(\phi, \psi) \in \mathrm{dom}\,(\partial \widetilde{E}_{\mathrm{free}})$ and $(\mu, \theta) \in \mathcal{H}^2_{L,\beta}$ such that

$$\begin{aligned}
&\Big(\frac{\rho\mathbf{v} - \rho_k \mathbf{v}_k}{h}, \widehat{\mathbf{v}}\Big)_{\mathbf{L}^2(\Omega)} + \Big(\frac{\sigma\mathbf{w} - \sigma_k \mathbf{w}_k}{h}, \widehat{\mathbf{w}}\Big)_{\mathbf{L}^2(\Gamma)} \\
&\quad + \big(\mathrm{div}(\rho_k \mathbf{v}\otimes\mathbf{v}), \widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(\mathrm{div}_\Gamma(\sigma_k \mathbf{w}\otimes\mathbf{w}), \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} \\
&\quad + \big(2\nu_\Omega(\phi_k)\mathbf{D}\mathbf{v}, \mathbf{D}\widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma\mathbf{w}, \mathbf{D}_\Gamma\widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} + \big(\gamma(\phi_k,\psi_k)\mathbf{w}, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} \\
&\quad + \big(\mathrm{div}(\mathbf{v}\otimes\mathbf{J}), \widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(\mathrm{div}_\Gamma(\mathbf{w}\otimes\mathbf{K}), \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} \\
&= \big(\mu\nabla\phi, \widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(\theta\nabla_\Gamma\psi, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} + \frac{1}{2}\big((\mathbf{J}\cdot\mathbf{n})\mathbf{w}, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} + \frac{1}{2}\big((\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w}, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)}
\end{aligned} \tag{9.18a}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$, as well as

$$\frac{\phi - \phi_k}{h} + \mathbf{v}\cdot\nabla\phi_k = \mathrm{div}(m_\Omega(\phi_k)\nabla\mu) \qquad \text{a.e. in } \Omega, \tag{9.18b}$$
$$\mu + \frac{c_F}{2}(\phi + \phi_k) = -\Delta\phi + F_0'(\phi) \qquad \text{a.e. in } \Omega, \tag{9.18c}$$
$$\frac{\psi - \psi_k}{h} + \mathbf{w}\cdot\nabla_\Gamma\psi_k = \mathrm{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\theta) - \beta m_\Omega(\phi_k)\partial_\mathbf{n}\mu \qquad \text{a.e. on } \Gamma, \tag{9.18d}$$
$$\theta + \frac{c_G}{2}(\psi + \psi_k) = -\Delta_\Gamma\psi + G_0'(\psi) + \alpha\partial_\mathbf{n}\phi \qquad \text{a.e. on } \Gamma, \tag{9.18e}$$
$$\begin{cases} K\partial_\mathbf{n}\phi = \alpha\psi - \phi & \text{if } K \in (0,\infty), \\ \partial_\mathbf{n}\phi = 0 & \text{if } K = \infty \end{cases} \qquad \text{a.e. on } \Gamma, \tag{9.18f}$$
$$\begin{cases} L m_\Omega(\phi_k)\partial_\mathbf{n}\phi = \beta\theta - \mu & \text{if } L \in [0,\infty), \\ m_\Omega(\phi_k)\partial_\mathbf{n}\phi = 0 & \text{if } L = \infty \end{cases} \qquad \text{a.e. on } \Gamma. \tag{9.18g}$$

Here, and in the following, we use for brevity the notation

$$\rho := \rho(\phi), \qquad \sigma := \sigma(\psi),$$

and the space

$$\mathcal{H}^2_{L,\beta} = \begin{cases} \{(\zeta,\zeta_\Gamma) \in \mathcal{H}^2 : \zeta = \beta\zeta_\Gamma \text{ on } \Gamma\} & \text{if } L = 0, \\ \{(\zeta,\zeta_\Gamma) \in \mathcal{H}^2 : L m_\Omega(\phi_k)\partial_\mathbf{n}\zeta = \beta\zeta_\Gamma - \zeta \text{ on } \Gamma\} & \text{if } L \in (0,\infty), \\ \{(\zeta,\zeta_\Gamma) \in \mathcal{H}^2 : \partial_\mathbf{n}\zeta = 0 \text{ on } \Gamma\} & \text{if } L = \infty. \end{cases}$$

Remark 9.4.2. Multiplying (9.18b) with $-\frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}$ and (9.18d) with $-\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2}$ leads to

$$-\frac{\rho - \rho_k}{h} - \mathbf{v}\cdot\nabla\rho_k = \mathrm{div}\,\mathbf{J} \qquad \text{a.e. in } \Omega,$$
$$-\frac{\sigma - \sigma_k}{h} - \mathbf{w}\cdot\nabla_\Gamma\sigma_k = \mathrm{div}_\Gamma\,\mathbf{K} - \mathbf{J}_\Gamma\cdot\mathbf{n} \qquad \text{a.e. on } \Gamma.$$

Using the identities $\mathrm{div}(\mathbf{v}\otimes\mathbf{J}) = (\mathrm{div}\,\mathbf{J})\mathbf{v} + (\mathbf{J}\cdot\nabla)\mathbf{v}$ and $\mathrm{div}_\Gamma(\mathbf{w}\otimes\mathbf{K}) = (\mathrm{div}_\Gamma\,\mathbf{K})\mathbf{w} + (\mathbf{K}\cdot\nabla_\Gamma)\mathbf{w}$ leads to the following equivalent version of (9.18a):

$$\Big(\frac{\rho\mathbf{v} - \rho_k \mathbf{v}_k}{h}, \widehat{\mathbf{v}}\Big)_{\mathbf{L}^2(\Omega)} + \Big(\frac{\sigma\mathbf{w} - \sigma_k \mathbf{w}_k}{h}, \widehat{\mathbf{w}}\Big)_{\mathbf{L}^2(\Gamma)}$$

$$
\begin{aligned}
&+ \big(\operatorname{div}(\rho_k \mathbf{v} \otimes \mathbf{v}), \widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(\operatorname{div}_\Gamma(\sigma_k \mathbf{w} \otimes \mathbf{w}), \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} \\
&+ \big(2\nu_\Omega(\phi_k)\mathbf{D}\mathbf{v}, \mathbf{D}\widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma \mathbf{w}, \mathbf{D}_\Gamma \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} + \big(\gamma(\phi_k, \psi_k)\mathbf{w}, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} \\
&+ \frac{1}{2}\Big(\Big(\operatorname{div} \mathbf{J} - \frac{\rho - \rho_k}{h} - \mathbf{v} \cdot \nabla \rho_k\Big)\mathbf{v}, \widehat{\mathbf{v}}\Big)_{\mathbf{L}^2(\Omega)} \qquad (9.20) \\
&+ \frac{1}{2}\Big(\Big(\operatorname{div}_\Gamma \mathbf{K} - \frac{\sigma - \sigma_k}{h} - \mathbf{w} \cdot \nabla_\Gamma \sigma_k\Big)\mathbf{w}, \widehat{\mathbf{w}}\Big)_{\mathbf{L}^2(\Gamma)} \\
= &\big(\mu \nabla \phi, \widehat{\mathbf{v}}\big)_{\mathbf{L}^2(\Omega)} + \big(\theta \nabla_\Gamma \psi, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)} + \frac{1}{2}\big((\mathbf{J} \cdot \mathbf{n})\mathbf{w}, \widehat{\mathbf{w}}\big)_{\mathbf{L}^2(\Gamma)}
\end{aligned}
$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$.

Remark 9.4.3. Integrating (9.18b) and (9.18d) with respect to the spatial variable, using the fact that $(\mathbf{v}, \mathbf{w}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$ and (9.18g), we obtain

$$
\begin{cases}
\beta \displaystyle\int_\Omega \phi \,\mathrm{d}x + \int_\Gamma \psi \,\mathrm{d}\Gamma = \beta \int_\Omega \phi_k \,\mathrm{d}x + \int_\Gamma \psi_k \,\mathrm{d}\Gamma & \text{if } L \in [0, \infty), \\
\displaystyle\int_\Omega \phi \,\mathrm{d}x = \int_\Omega \phi_k \,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi \,\mathrm{d}\Gamma = \int_\Gamma \psi_k \,\mathrm{d}\Gamma & \text{if } L = \infty.
\end{cases}
$$

In particular, these observations imply that

$$
\begin{cases}
\beta \displaystyle\int_\Omega \phi_k \,\mathrm{d}x + \int_\Gamma \psi_k \,\mathrm{d}\Gamma = \beta \int_\Omega \phi_0 \,\mathrm{d}x + \int_\Gamma \psi_0 \,\mathrm{d}\Gamma & \text{if } L \in [0, \infty), \\
\displaystyle\int_\Omega \phi_k \,\mathrm{d}x = \int_\Omega \phi_0 \,\mathrm{d}x \quad \text{and} \quad \int_\Gamma \psi_k \,\mathrm{d}\Gamma = \int_\Gamma \psi_0 \,\mathrm{d}\Gamma & \text{if } L = \infty
\end{cases}
$$

for all $k \in \mathbb{N}_0$. As a result, the mass conservation property is preserved at the discrete level.

LEMMA 9.4.4. *Let* $(\phi_k, \psi_k) \in \mathcal{H}^2$ *with*

$$
|\phi_k| \le 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad |\psi_k| \le 1 \quad \text{a.e. on } \Gamma, \tag{9.21}
$$

and further assume

$$
\beta \operatorname{mean}(\phi_k, \psi_k),\ \operatorname{mean}(\phi_k, \psi_k) \in (-1, 1) \qquad \text{if } L \in [0, \infty),
$$

and

$$
\langle \phi_k \rangle_\Omega,\ \langle \psi_k \rangle_\Gamma \in (-1, 1) \qquad \text{if } L = \infty.
$$

Let $(\phi, \psi, \mu, \theta) \in \operatorname{dom}(\partial \widetilde{E}_{\mathrm{free}}) \times \mathcal{H}^1$ *be a quadruple solving* ((9.18c), (9.18e), (9.18f)) *with*

$$
\begin{cases}
\operatorname{mean}(\phi, \psi) = \operatorname{mean}(\phi_k, \psi_k) & \text{if } L \in [0, \infty), \\
\langle \phi \rangle_\Omega = \langle \phi_k \rangle_\Omega \quad \text{and} \quad \langle \psi \rangle_\Gamma = \langle \psi_k \rangle_\Gamma & \text{if } L = \infty.
\end{cases} \tag{9.23}
$$

Then, there exists a constant $C = C(\operatorname{mean}(\phi_k, \psi_k)) > 0$ *such that*

$$
\|(\phi, \psi)\|_{\mathcal{H}^2} + \|(F_0'(\phi), G_0'(\psi))\|_{\mathcal{L}^2} + |\operatorname{mean}(\mu, \theta)| \le C(1 + \|(\mu, \theta)\|_{L,\beta}), \tag{9.24}
$$

$$
\|\partial \widetilde{E}_{\mathrm{free}}(\phi, \psi)\|_{\mathcal{L}^2} \le C(1 + \|(\mu, \theta)\|_{\mathcal{L}^2}). \tag{9.25}
$$

PROOF. Defining $(f, f_\Gamma) \coloneqq (\mu + \frac{c_F}{2}(\phi + \phi_k), \theta + \frac{c_G}{2}(\phi + \psi_k)$, we notice that (ϕ, ψ) is a solution to

$$
-\Delta \phi + F_0'(\phi) = f \qquad \text{in } \Omega,
$$

$$
\begin{aligned}
-\Delta_\Gamma \psi + G_0'(\psi) + \alpha \partial_{\mathbf{n}} \phi &= f_\Gamma && \text{on } \Gamma, \\
K \partial_{\mathbf{n}} \phi &= \alpha \psi - \phi && \text{on } \Gamma.
\end{aligned}
$$

Thus, since $(f, f_\Gamma) \in \mathcal{L}^2$, an application of Proposition 4.3.5 readily yields

$$
\begin{aligned}
\|(\phi,\psi)\|_{\mathcal{H}^2} + \|(F_0'(\phi), G_0'(\psi))\|_{\mathcal{L}^2} &\le C(1 + \|(f, f_\Gamma)\|_{\mathcal{L}^2}) \\
&\le C(1 + \|(\mu,\theta)\|_{\mathcal{L}^2}).
\end{aligned}
$$

Here, we additionally exploited the assumption (9.21) as well as $|\phi| \le 1$ a.e. in Ω and $|\psi| \le 1$ a.e. on Γ, which follows from $(\phi,\psi) \in \operatorname{dom}(\partial \widetilde{E}_{\text{free}})$. Then, recalling (6.57), an application of the bulk-surface Poincaré inequality entails

$$
\|(\phi,\psi)\|_{\mathcal{H}^2} + \|(F_0'(\phi), G_0'(\psi))\|_{\mathcal{L}^2} \le C(1 + \|(\mu,\theta)\|_{L,\beta}).
$$

Moreover, we then readily obtain from (6.57) that

$$
|\operatorname{mean}(\mu,\theta)| \le C(1 + \|(\mu,\theta)\|_{L,\beta}),
$$

which proves (9.24). Finally, the estimate (9.25) on the subdifferential follows immediately in view of the identity

$$
\partial \widetilde{E}_{\text{free}}(\phi,\psi) = \Big(\mu + \frac{c_F}{2}(\phi + \phi_k), \theta + \frac{c_G}{2}(\psi + \psi_k)\Big),
$$

and the above mentioned estimates on (ϕ,ψ) and (ϕ_k,ψ_k). □

Existence of solutions of the time-discrete scheme

Next, we prove that solutions to the discrete system (9.18a)-(9.18g), if they exist, satisfy a suitable discrete energy inequality, which will provide suitable bounds later on.

LEMMA 9.4.5. *Let $(\mathbf{v}_k, \mathbf{w}_k) \in \boldsymbol{\mathcal{L}}^2_{\text{div}}$ and $(\phi_k,\psi_k) \in \mathcal{H}^2$ be given, and set $\rho_k = \rho(\phi_k)$, $\sigma_k = \sigma(\psi_k)$. Then, a solution $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta) \in \boldsymbol{\mathcal{H}}^1_{0,\text{div}} \times \operatorname{dom}(\partial \widetilde{E}_{\text{free}}) \times \mathcal{H}^2_{L,\beta}$ to the time-discrete system (9.18a)-(9.18g) satisfies the following energy inequality*

$$
\begin{aligned}
&E_{\text{tot}}(\mathbf{v},\mathbf{w},\phi,\psi) + \int_\Omega \frac{1}{2}\rho_k|\mathbf{v} - \mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \frac{1}{2}\sigma_k|\mathbf{w} - \mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Omega \frac{1}{2}|\nabla\phi - \nabla\phi_k|^2 \,\mathrm{d}x \\
&\quad + \int_\Gamma \frac{1}{2}|\nabla_\Gamma\psi - \nabla_\Gamma\psi_k|^2 \,\mathrm{d}\Gamma + \chi(K)\int_\Gamma \frac{1}{2}\big(\alpha(\psi - \psi_k) - (\phi - \phi_k)\big)^2 \,\mathrm{d}\Gamma \qquad (9.26) \\
&\quad + h\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x + h\int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}|^2 \,\mathrm{d}\Gamma + h\int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}|^2 \,\mathrm{d}\Gamma \\
&\quad + h\int_\Omega m_\Omega(\phi_k)|\nabla\mu|^2 \,\mathrm{d}x + h\int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma + h\chi(L)\int_\Gamma (\beta\theta - \mu)^2 \,\mathrm{d}\Gamma \\
&\le E_{\text{tot}}(\mathbf{v}_k,\mathbf{w}_k,\phi_k,\psi_k).
\end{aligned}
$$

PROOF. The idea of this proof is to use suitable test functions for the time-discrete system. However, before testing (9.20) with $(\mathbf{v},\mathbf{w})$, we recall the identities

$$
\begin{aligned}
&\int_\Omega \Big(\operatorname{div}(\rho_k\mathbf{v}\otimes\mathbf{v}) - \frac{1}{2}(\mathbf{v}\cdot\nabla\rho_k)\mathbf{v}\Big)\cdot\mathbf{v} \,\mathrm{d}x = \int_\Omega \frac{1}{2}\operatorname{div}(\rho_k\mathbf{v}|\mathbf{v}|^2) \,\mathrm{d}x = 0, \\
&\int_\Omega \Big(\frac{1}{2}(\operatorname{div}\mathbf{J})\mathbf{v} + (\mathbf{J}\cdot\nabla)\mathbf{v}\Big)\cdot\mathbf{v} \,\mathrm{d}x = \int_\Omega \frac{1}{2}\operatorname{div}(\mathbf{J}|\mathbf{v}|^2) \,\mathrm{d}x = \frac{1}{2}\int_\Gamma (\mathbf{J}\cdot\mathbf{n})|\mathbf{v}|^2 \,\mathrm{d}\Gamma,
\end{aligned}
$$

which have been established in [5, Lemma 4.3]. Similarly, we derive

$$
\int_\Gamma \Big(\operatorname{div}_\Gamma(\sigma\mathbf{w}\otimes\mathbf{w}) - \frac{1}{2}(\mathbf{v}\cdot\nabla_\Gamma\sigma_k)\mathbf{w}\Big)\cdot\mathbf{v} \,\mathrm{d}\Gamma = \int_\Gamma \frac{1}{2}\operatorname{div}_\Gamma(\sigma_k\mathbf{w}|\mathbf{w}|^2) \,\mathrm{d}\Gamma = 0,
$$

$$\int_\Gamma \Big(\frac{1}{2}(\mathrm{div}_\Gamma\, \mathbf{K})\mathbf{w} + (\mathbf{K}\cdot\nabla_\Gamma)\mathbf{w}\Big)\cdot\mathbf{w}\ \mathrm{d}\Gamma = \int_\Gamma \frac{1}{2}\mathrm{div}_\Gamma(\mathbf{K}|\mathbf{w}|^2)\ \mathrm{d}\Gamma = 0.$$

Furthermore, the algebraic identity

$$\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) = \frac{|\mathbf{a}|^2}{2} - \frac{|\mathbf{b}|^2}{2} + \frac{|\mathbf{a}-\mathbf{b}|^2}{2} \qquad \text{for } \mathbf{a},\mathbf{b}\in\mathbb{R}^d$$

yields that

$$\begin{aligned}\frac{1}{h}(\rho\mathbf{v}-\rho_k\mathbf{v}_k)\cdot\mathbf{v} &= \frac{1}{2h}\Big(\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2\Big) + \frac{1}{2h}(\rho-\rho_k)|\mathbf{v}|^2 + \frac{1}{2h}\rho_k|\mathbf{v}-\mathbf{v}_k|^2,\\ \frac{1}{h}(\sigma\mathbf{v}-\sigma_k\mathbf{w}_k)\cdot\mathbf{w} &= \frac{1}{2h}\Big(\rho|\mathbf{w}|^2 - \sigma_k|\mathbf{w}_k|^2\Big) + \frac{1}{2h}(\sigma-\sigma_k)|\mathbf{w}|^2 + \frac{1}{2h}\sigma_k|\mathbf{w}-\mathbf{w}_k|^2.\end{aligned}$$

Now, testing (9.20) with $(\mathbf{v},\mathbf{w})$ and using the equations above, leads us to

$$\begin{aligned}0 &= \int_\Omega \frac{1}{2h}\Big(\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2\Big)\ \mathrm{d}x + \int_\Gamma \frac{1}{2h}\Big(\sigma|\mathbf{w}|^2 - \sigma_k|\mathbf{w}_k|^2\Big)\ \mathrm{d}\Gamma + \int_\Omega \frac{1}{2h}\rho_k|\mathbf{v}-\mathbf{v}_k|^2\ \mathrm{d}x\\ &\quad + \int_\Gamma \frac{1}{2h}\sigma_k|\mathbf{w}-\mathbf{w}_k|^2\ \mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}|^2 + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}|^2\\ &\quad + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}|^2\ \mathrm{d}\Gamma - \int_\Omega \mu\nabla\phi_k\cdot\mathbf{v}\ \mathrm{d}x - \int_\Gamma \theta\nabla_\Gamma\psi_k\cdot\mathbf{w}\ \mathrm{d}\Gamma.\end{aligned}$$

Next, taking μ and θ to test with (9.18b) and (9.18d), respectively, further gives

$$\begin{aligned}0 &= \int_\Omega \frac{1}{h}(\phi-\phi_k)\mu\ \mathrm{d}x + \int_\Gamma \frac{1}{h}(\psi-\psi_k)\theta\ \mathrm{d}\Gamma + \int_\Omega \mathbf{v}\cdot\nabla\phi_k\mu\ \mathrm{d}x + \int_\Gamma \mathbf{w}\cdot\nabla_\Gamma\psi_k\theta\ \mathrm{d}\Gamma\\ &\quad + \int_\Omega m_\Omega(\phi_k)|\nabla\mu|^2\ \mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta|^2\ \mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta-\mu)^2\ \mathrm{d}\Gamma,\end{aligned}$$

while testing (9.18c) and (9.18e) with $\frac{1}{h}(\phi-\phi_k)$ and $\frac{1}{h}(\psi-\psi_k)$, respectively, results in

$$\begin{aligned}0 &= \frac{1}{h}\int_\Omega \nabla\phi\cdot(\nabla\phi-\nabla\phi_k)\ \mathrm{d}x + \frac{1}{h}\int_\Gamma \nabla_\Gamma\psi\cdot(\nabla_\Gamma\psi-\nabla_\Gamma\psi_k)\ \mathrm{d}\Gamma\\ &\quad + \frac{1}{h}\chi(K)\int_\Gamma(\alpha\psi-\phi)(\alpha(\psi-\psi_k)-(\phi-\phi_k))\ \mathrm{d}\Gamma + \frac{1}{h}\int_\Omega F_0'(\phi)(\phi-\phi_k)\ \mathrm{d}x\\ &\quad + \frac{1}{h}\int_\Gamma G_0'(\psi)(\psi-\psi_k)\ \mathrm{d}\Gamma - \frac{1}{h}\int_\Omega \mu(\phi-\phi_k)\ \mathrm{d}x - \frac{1}{h}\int_\Gamma \theta(\psi-\psi_k)\ \mathrm{d}\Gamma\\ &\quad - \frac{1}{h}\int_\Omega \frac{c_F}{2}(\phi^2-\phi_k^2)\ \mathrm{d}x - \frac{1}{h}\int_\Gamma \frac{c_G}{2}(\psi^2-\psi_k^2)\ \mathrm{d}\Gamma.\end{aligned}$$

Taking into account the convexity of F_0 and G_0 to estimate

$$F_0'(\phi)(\phi-\phi_k) \geq F_0(\phi) - F_0(\phi_k), \qquad G_0'(\psi)(\psi-\psi_k) \geq G_0(\psi) - G_0(\psi_k),$$

we sum these identities and deduce

$$\begin{aligned}0 &\geq \frac{1}{h}\int_\Omega \frac{1}{2}\Big(\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2\Big)\ \mathrm{d}x + \frac{1}{h}\int_\Gamma \frac{1}{2}\Big(\sigma|\mathbf{w}|^2 - \sigma_k|\mathbf{w}_k|^2\Big)\ \mathrm{d}\Gamma\\ &\quad + \frac{1}{h}\int_\Omega \frac{1}{2}\rho_k|\mathbf{v}-\mathbf{v}_k|^2\ \mathrm{d}x + \frac{1}{h}\int_\Gamma \frac{1}{2}\sigma_k|\mathbf{w}-\mathbf{w}_k|^2\ \mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}|^2\ \mathrm{d}x\\ &\quad + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}|^2\ \mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}|^2\ \mathrm{d}\Gamma + \int_\Omega m_\Omega(\phi_k)|\nabla\mu|^2\ \mathrm{d}x\\ &\quad + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta|^2\ \mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta-\mu)^2\ \mathrm{d}\Gamma + \frac{1}{h}\int_\Omega F_0(\phi)-F_0(\phi_k)\ \mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{h} \int_\Omega \frac{c_F}{2}(\phi^2 - \phi_k^2) + \frac{1}{h} \int_\Gamma G_0(\psi) - G_0(\psi_k) \,\mathrm{d}\Gamma - \frac{1}{h} \int_\Gamma \frac{c_G}{2}(\psi^2 - \psi_k^2) \,\mathrm{d}\Gamma \\
&+ \frac{1}{h} \int_\Omega \frac{1}{2} |\nabla \phi - \nabla \phi_k|^2 \,\mathrm{d}x + \frac{1}{h} \int_\Gamma \frac{1}{2} |\nabla_\Gamma \psi - \nabla_\Gamma \psi_k|^2 \,\mathrm{d}\Gamma \\
&+ \frac{1}{h} \chi(K) \int_\Gamma \frac{1}{2} \big(\alpha(\psi - \psi_k) - (\phi - \phi_k)\big)^2 \,\mathrm{d}\Gamma + \frac{1}{h} \int_\Omega \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} |\nabla \phi_k|^2 \,\mathrm{d}x \\
&+ \frac{1}{h} \int_\Gamma \frac{1}{2} |\nabla_\Gamma \psi|^2 - \frac{1}{2} |\nabla_\Gamma \psi_k|^2 \,\mathrm{d}\Gamma + \frac{1}{h} \chi(K) \int_\Gamma \frac{1}{2} (\alpha \psi - \phi)^2 - \frac{1}{2} (\alpha \psi_k - \phi_k)^2 \,\mathrm{d}\Gamma.
\end{aligned}
$$

By definition of $E_{\mathrm{tot}}(\mathbf{v}, \mathbf{w}, \phi, \psi)$, we have therefore established the time-discrete energy inequality (9.26) as claimed. □

In the next step, we show that the time-discrete system (9.18a)-(9.18g) admits at least one solution.

Lemma 9.4.6. *Let $(\mathbf{v}_k, \mathbf{w}_k) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$ and $(\phi_k, \psi_k) \in \mathcal{H}^2_{K,\alpha}$ be given, and set $(\rho_k, \sigma_k) = (\rho(\phi_k,), \sigma(\psi_k))$. Then, there exists a solution $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}} \times \mathrm{dom}\,(\partial \widetilde{E}_{\mathrm{free}}) \times \mathcal{H}^2_{L,\beta}$ to the time-discrete system (9.18a)-(9.18g).*

Proof. The central concept of this proof is based on the application of the Leray–Schauder principle. To this end, we define two nonlinear operators $\mathcal{M}_k, \mathcal{F}_k : X_{K,L} \to Y$, where

$$
X_{K,L} \coloneqq \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}} \times \mathrm{dom}\,(\partial \widetilde{E}_{\mathrm{free}}) \times \mathcal{H}^2_{L,\beta}, \qquad Y \coloneqq (\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}})' \times \mathcal{L}^2 \times \mathcal{L}^2.
$$

These are constructed in a way such that a sextuple $\mathbf{z} = (\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta) \in X_{K,L}$ solves the system (9.18a)-(9.18g) if and only if the identity

$$
\mathcal{M}_k(\mathbf{z}) = \mathcal{F}_k(\mathbf{z})
$$

holds. For $\mathbf{z} = (\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta) \in X_{K,L}$, we define

$$
\mathcal{M}_k(\mathbf{z}) \coloneqq \begin{pmatrix} \mathcal{T}_k(\mathbf{v}, \mathbf{w}) \\ \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \partial \widetilde{E}_{\mathrm{free}}(\phi, \psi) \\ \mathfrak{S}_{L,\beta} \begin{pmatrix} \mu \\ \theta \end{pmatrix} \end{pmatrix},
$$

where

$$
\begin{aligned}
\langle \mathcal{T}_k(\mathbf{v}, \mathbf{w}), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \rangle \coloneqq &\int_\Omega 2\nu_\Omega(\phi_k) \mathbf{D}\mathbf{v} : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k) \mathbf{D}_\Gamma \mathbf{w} : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\
&+ \int_\Gamma \gamma(\phi_k, \psi_k) \mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma
\end{aligned}
$$

for all $(\mathbf{v}, \mathbf{w}), (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$, and

$$
\mathfrak{S}_{L,\beta} \begin{pmatrix} \mu \\ \theta \end{pmatrix} \coloneqq \begin{pmatrix} -\mathrm{div}(m_\Omega(\phi_k) \nabla \mu) + \int_\Omega \mu \,\mathrm{d}x \\ -\mathrm{div}_\Gamma(m_\Gamma(\psi_k) \nabla_\Gamma \theta) + \beta m_\Omega(\phi_k) \partial_{\mathbf{n}} \mu + \int_\Gamma \theta \,\mathrm{d}\Gamma \end{pmatrix}
$$

for all $(\mu, \theta) \in \mathcal{H}^2_{L,\beta}$. Next, for any $\mathbf{z} = (\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta) \in X_{K,L}$, we define

$$
\mathcal{F}_k(\mathbf{z}) \coloneqq \begin{pmatrix} \boldsymbol{\mathcal{S}} \\ \phi + \mu + \frac{c_F}{2}(\phi + \phi_k) \\ \psi + \theta + \frac{c_G}{2}(\psi + \psi_k) \\ -\frac{\phi - \phi_k}{h} - \mathbf{v} \cdot \nabla \phi_k + \int_\Omega \mu \,\mathrm{d}x \\ -\frac{\psi - \psi_k}{h} - \mathbf{w} \cdot \nabla_\Gamma \psi_k + \int_\Gamma \theta \,\mathrm{d}\Gamma \end{pmatrix},
$$

where $\boldsymbol{\mathcal{S}}$ in the first line of $\mathcal{F}_k$ is defined as

$$\begin{aligned}\boldsymbol{\mathcal{S}} := &-\frac{\rho\mathbf{v}-\rho_k\mathbf{v}_k}{h}-\frac{\sigma\mathbf{w}-\sigma_k\mathbf{w}_k}{h}-\operatorname{div}(\rho_k\mathbf{v}\otimes\mathbf{v})-\operatorname{div}_\Gamma(\sigma_k\mathbf{w}\otimes\mathbf{w})-(\mathbf{J}\cdot\nabla)\mathbf{v}\\ &-(\mathbf{K}\cdot\nabla_\Gamma)\mathbf{w}-\frac{1}{2}\Big(\operatorname{div}\mathbf{J}-\frac{\rho-\rho_k}{2}-\mathbf{v}\cdot\nabla\rho_k\Big)\mathbf{v}\\ &-\frac{1}{2}\Big(\operatorname{div}_\Gamma\mathbf{K}-\frac{\sigma-\sigma_k}{2}-\mathbf{w}\cdot\nabla_\Gamma\sigma_k\Big)\mathbf{w}+\mu\nabla\phi+\theta\nabla_\Gamma\psi+\frac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w}\end{aligned}$$

for all $(\mathbf{v},\mathbf{w}),(\widehat{\mathbf{v}},\widehat{\mathbf{w}})\in\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$.

Using a similar argument as in [79, Section 3.1], one can show that $\mathcal{T}_k:\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}\to(\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}})'$ is invertible with continuous inverse $\mathcal{T}_k^{-1}$ (see also [80, Lemma 3.1]). Moreover, from Lemma 9.2.3, we infer that $\mathfrak{S}_{L,\beta}:\mathcal{H}^2_{L,\beta}\to\mathcal{L}^2$ is invertible with continuous inverse $\mathfrak{S}^{-1}_{L,\beta}$. In what follows, we examine the second line of the $\mathcal{M}_k$. Since $\partial\widetilde{E}_{\mathrm{free}}$ is a maximal monotone operator according to Proposition 4.3.2, the operator

$$\mathrm{Id}+\partial\widetilde{E}_{\mathrm{free}}:\operatorname{dom}(\partial\widetilde{E}_{\mathrm{free}})\to\mathcal{L}^2$$

is invertible. The inverse operator, considered as a mapping

$$(\mathrm{Id}+\partial\widetilde{E}_{\mathrm{free}})^{-1}:\mathcal{L}^2\to\mathcal{H}^{2-s},\qquad s\in(0,\tfrac{1}{4}),$$

is continuous and compact. Indeed, let $(f_l,f_{\Gamma l})\to(f,f_\Gamma)$ in $\mathcal{L}^2$ as $l\to\infty$ with $(f_l,f_{\Gamma l})=(u_l,v_l)+\partial\widetilde{E}_{\mathrm{free}}(u_l,v_l)$ for $l\in\mathbb{N}$ and $(f,f_\Gamma)=(u,v)+\partial\widetilde{E}_{\mathrm{free}}(u,v)$ be given. Then we have $(u_l,v_l)\to(u,v)$ in $\mathcal{H}^1$ as $l\to\infty$ since

$$\begin{aligned}&\|(u_l-u,v_l-v)\|^2_{\mathcal{L}^2}+\|(u_l-u,v_l-v)\|^2_{K,\alpha}\\ &\quad\le\|(u_l-u,v_l-v)\|^2_{\mathcal{L}^2}+\big(\partial\widetilde{E}_{\mathrm{free}}(u_l,v_l)-\partial\widetilde{E}_{\mathrm{free}}(u,v),(u,v)-(u_l,v_l)\big)_{\mathcal{L}^2}\\ &\quad\le\|(u_l,v_l)+\partial\widetilde{E}_{\mathrm{free}}(u_l,v_l)-((u,v)+\partial\widetilde{E}_{\mathrm{free}}(u,v))\|_{\mathcal{L}^2}\|(u,v)-(u_l,v_l)\|_{\mathcal{L}^2}\\ &\quad\le\frac{1}{2}\|(f_l,f_{\Gamma l})-(f,f_\Gamma)\|^2_{\mathcal{L}^2}+\frac{1}{2}\|(u_l,v_l)-(u,v)\|^2_{\mathcal{L}^2}\end{aligned}\tag{9.27}$$

for all $l\in\mathbb{N}$. Moreover, in view of Proposition 4.3.3, we have the estimate

$$\|(u_l,v_l)\|_{\mathcal{H}^2}\le C(1+\|\partial\widetilde{E}_{\mathrm{free}}(u_l,v_l)\|_{\mathcal{L}^2})\le C(\|(u_l,v_l)\|_{\mathcal{L}^2}+\|(f_l,f_{\Gamma l})\|_{\mathcal{L}^2})\tag{9.28}$$

for all $l\in\mathbb{N}$, and due to the strong convergence of $\{(f_l,f_{\Gamma l})\}_{l\in\mathbb{N}}$ in $\mathcal{L}^2$ and the estimate (9.27), we immediately obtain that $\{(u_l,v_l)\}_{l\in\mathbb{N}}$ is bounded in $\mathcal{H}^2$. Thus, exploiting the compact embedding $\mathcal{H}^2\hookrightarrow\mathcal{H}^{2-s}$ yields that $(u_l,v_l)\to(u,v)$ strongly in $\mathcal{H}^{2-s}$ as $l\to\infty$ for any $s\in(0,\frac{1}{4})$.

Altogether we obtain that $\mathcal{M}_k:X_{K,L}\to Y$ is invertible with inverse $\mathcal{M}_k^{-1}:Y\to X_{K,L}$. Note that $X_{K,L}$ is not a Banach space since $\operatorname{dom}(\partial\widetilde{E}_{\mathrm{free}})$ includes inequality constraints. To get a continuous and even compact operator, for any $s\in(0,\frac{1}{4})$, we introduce the Banach spaces

$$\widetilde{X}_{K,L}:=\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}\times\mathcal{H}^{2-s}\times\mathcal{H}^2_{L,\beta},\ s\in(0,\tfrac{1}{4}),\qquad\widetilde{Y}:=\boldsymbol{\mathcal{L}}^{\frac{3}{2}}\times\mathcal{H}^1\times\mathcal{W}^{1,\frac{3}{2}}.$$

In this way, we have the continuous embedding $X_{K,L}\hookrightarrow\widetilde{X}$ and the compact embedding $\widetilde{Y}\hookrightarrow Y$. Eventually, $\mathcal{M}_k^{-1}:\widetilde{Y}\to\widetilde{X}_{K,L}$ is a compact operator.

We proceed by showing that $\mathcal{F}_k:\widetilde{X}_{K,L}\to\widetilde{Y}$ is continuous and maps bounded sets into bounded sets. In the following, we denote with C_k constants that may depend on the parameters of the system and on $k \in \mathbb{N}$. Then, it was shown in [5, (i)-(vi), pp. 467] that the following estimates

$$\begin{aligned}
&\|\rho \mathbf{v}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \leq C\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}(1+\|\phi\|_{L^2(\Omega)}), && \|\operatorname{div}(\rho_k \mathbf{v}\otimes\mathbf{v})\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \leq C_k\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)},\\
&\|\mu\nabla\phi_k\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \leq C_k\|\mu\|_{L^2(\Omega)}, && \|(\operatorname{div}\mathbf{J})\mathbf{v}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \leq C_k\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}\|\mu\|_{H^2(\Omega)},\\
&\|(\mathbf{J}\cdot\nabla)\mathbf{v}\|_{\mathbf{L}^{\frac{3}{2}}(\Omega)} \leq C\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}\|\mu\|_{H^2(\Omega)}, && \|\mathbf{v}\cdot\nabla\phi_k\|_{W^{1,\frac{3}{2}}(\Omega)} \leq C_k\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)},
\end{aligned}$$

hold. As Γ is a $(d-1)$-dimensional submanifold on $\mathbb{R}^d$, we can repeat the argument presented therein and show that analogous estimates hold for the corresponding boundary terms. Lastly, by the trace theorem, we have

$$\|\tfrac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w}\|_{\mathbf{L}^{\frac{3}{2}}(\Gamma)} \leq C\|\mu\|_{H^2(\Omega)}\|\mathbf{w}\|_{\mathbf{H}^1(\Gamma)}.$$

We thus deduce that

$$\|\mathcal{F}_k(\mathbf{z})\|_{\widetilde{Y}} \leq C_k\Big(1+\|\mathbf{z}\|^2_{\widetilde{X}_{K,L}}\Big) \tag{9.29}$$

for all $\mathbf{z}\in\widetilde{X}_{K,L}$. In particular, $\mathcal{F}_k:\widetilde{X}_{K,L}\to\widetilde{Y}$ maps bounded sets into bounded sets. Now, we aim to apply the Leray–Schauder principle (see, e.g., [181, Theorem 6.A]). To this end, we notice that finding a solution $\mathbf{z}\in X_{K,L}$ to the time-discrete problem (9.18a)-(9.18g), i.e., a solution to $\mathcal{M}_k(\mathbf{z})=\mathcal{F}_k(\mathbf{z})$, is equivalent to solving the equation

$$\mathcal{K}_k(\mathbf{f})=\mathbf{f}, \qquad \text{where } \mathcal{K}_k \coloneqq \mathcal{F}_k\circ\mathcal{M}_k^{-1},\ \mathbf{f}=\mathcal{M}_k(\mathbf{z}).$$

Hence, our goal is to find a fixed point of $\mathcal{K}_k$ in $\widetilde{Y}$. It follows from the considerations above that the mapping $\mathcal{K}_k:\widetilde{Y}\to\widetilde{Y}$ is compact. To apply the Leray–Schauder principle, it remains to show that the set

$$\{\mathbf{f}\in\widetilde{Y}\ :\ \mathbf{f}=\lambda\mathcal{K}_k(\mathbf{f}) \text{ for some } 0\leq\lambda\leq 1\} \tag{9.30}$$

is bounded. So, let us consider $\mathbf{f}\in\widetilde{Y}$ and $0\leq\lambda\leq 1$ satisfying $\mathbf{f}=\lambda\mathcal{K}_k(\mathbf{f})$. Defining $\mathbf{z}\coloneqq\mathcal{M}_k^{-1}(\mathbf{f})\in X_{K,L}$ leads to the equation

$$\mathcal{M}_k(\mathbf{z})=\lambda\mathcal{F}_k(\mathbf{z}), \tag{9.31}$$

which is equivalent to the following weak formulation

$$\begin{aligned}
&\int_\Omega 2\nu_\Omega(\phi_k)\mathbf{D}\mathbf{v}:\mathbf{D}\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma\mathbf{w}:\mathbf{D}_\Gamma\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)\mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\\
&\quad + \lambda\int_\Omega \frac{\rho\mathbf{v}-\rho_k\mathbf{v}_k}{h}\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \lambda\int_\Gamma \frac{\sigma\mathbf{w}-\sigma_k\mathbf{w}_k}{h}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \lambda\int_\Omega \operatorname{div}(\rho_k\mathbf{v}\otimes\mathbf{v})\cdot\widehat{\mathbf{v}}\,\mathrm{d}x\\
&\quad + \lambda\int_\Gamma \operatorname{div}_\Gamma(\sigma_k\mathbf{w}\otimes\mathbf{w})\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \lambda\int_\Omega (\mathbf{J}\cdot\nabla)\mathbf{v}\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \lambda\int_\Gamma (\mathbf{K}\cdot\nabla_\Gamma)\mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\\
&\quad + \lambda\int_\Omega \frac{1}{2}\Big(\operatorname{div}\mathbf{J} - \frac{\rho-\rho_k}{2} - \mathbf{v}\cdot\nabla\rho_k\Big)\mathbf{v}\cdot\widehat{\mathbf{v}}\,\mathrm{d}x\\
&\quad + \lambda\int_\Gamma \frac{1}{2}\Big(\operatorname{div}_\Gamma\mathbf{K} - \frac{\sigma-\sigma_k}{2} - \mathbf{w}\cdot\nabla_\Gamma\sigma_k\Big)\mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\\
&= \lambda\int_\Omega \mu\nabla\phi_k\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \lambda\int_\Gamma \theta\nabla_\Gamma\psi_k\cdot\widehat{\mathbf{w}} + \lambda\int_\Gamma \frac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w}\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma
\end{aligned} \tag{9.32}$$

for all $(\widehat{\mathbf{v}},\widehat{\mathbf{w}})\in\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$, and the pointwise identities

$$\lambda\phi+\lambda\mu+\lambda\frac{c_F}{2}(\phi+\phi_k) = \phi-\Delta\phi+F_0'(\phi_0) \qquad \text{a.e. in } \Omega, \tag{9.33a}$$

$$\lambda\psi + \lambda\theta + \lambda\frac{c_G}{2}(\psi+\psi_k) = \psi - \Delta_\Gamma\psi + G_0'(\psi_0) + \alpha\partial_{\mathbf{n}}\phi \qquad \text{a.e. on } \Gamma, \tag{9.33b}$$

$$-\operatorname{div}(m_\Omega(\phi_k)\nabla\mu) + \int_\Omega \mu \,\mathrm{d}x = -\lambda\frac{\phi-\phi_k}{h} - \lambda\mathbf{v}\cdot\nabla\phi_k + \lambda\int_\Omega \mu\,\mathrm{d}x \qquad \text{a.e. in } \Omega, \tag{9.33c}$$

$$-\operatorname{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\theta) + \beta m_\Omega(\phi_k)\partial_{\mathbf{n}}\mu + \int_\Gamma \theta\,\mathrm{d}\Gamma = -\lambda\frac{\psi-\psi_k}{h} - \lambda\mathbf{w}\cdot\nabla_\Gamma\psi_k + \lambda\int_\Gamma \theta\,\mathrm{d}\Gamma \qquad \text{a.e. on } \Gamma, \tag{9.33d}$$

$$\begin{cases} K\partial_{\mathbf{n}}\phi = \alpha\psi - \phi & \text{if } K\in(0,\infty),\\ \partial_{\mathbf{n}}\phi = 0 & \text{if } K=\infty \end{cases} \qquad \text{a.e. on } \Gamma, \tag{9.33e}$$

$$\begin{cases} L m_\Omega(\phi_k)\partial_{\mathbf{n}}\mu = \beta\theta - \mu & \text{if } L\in[0,\infty),\\ m_\Omega(\phi_k)\partial_{\mathbf{n}}\mu = 0 & \text{if } L=\infty \end{cases} \qquad \text{a.e. on } \Gamma. \tag{9.33f}$$

Proceeding analogously to the proof of Lemma 9.4.5, we now test (9.32) with $(\mathbf{v},\mathbf{w})$, (9.33a) with $\frac{1}{h}(\phi-\phi_k)$, (9.33b) with $\frac{1}{h}(\psi-\psi_k)$, (9.33c) with μ, and (9.33d) with θ. This yields

$$\begin{aligned}
&\lambda\int_\Omega \frac{1}{2h}(\rho|\mathbf{v}|^2 - \rho_k|\mathbf{v}_k|^2)\,\mathrm{d}x + \lambda\int_\Gamma \frac{1}{2h}(\sigma|\mathbf{w}|^2 - \sigma_k|\mathbf{w}_k|^2)\,\mathrm{d}\Gamma + \lambda\int_\Omega \frac{1}{2h}\rho_k|\mathbf{v}-\mathbf{v}_k|^2\,\mathrm{d}x\\
&\quad + \lambda\int_\Gamma \frac{1}{2h}\sigma_k|\mathbf{w}-\mathbf{w}_k|^2\,\mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}|^2\,\mathrm{d}\Gamma\\
&\quad + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}|^2\,\mathrm{d}\Gamma + \int_\Omega m_\Omega(\phi_k)|\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma\\
&\quad + \chi(L)\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma + (1-\lambda)\frac{1}{h}\int_\Omega \phi(\phi-\phi_k)\,\mathrm{d}x + (1-\lambda)\frac{1}{h}\int_\Gamma \psi(\psi-\psi_k)\,\mathrm{d}\Gamma\\
&\quad + \frac{1}{h}\int_\Omega \nabla\phi\cdot(\nabla\phi-\nabla\phi_k)\,\mathrm{d}x + \frac{1}{h}\int_\Gamma \nabla_\Gamma\psi\cdot(\nabla_\Gamma\psi-\nabla_\Gamma\psi_k)\,\mathrm{d}\Gamma\\
&\quad + \frac{1}{h}\chi(K)\int_\Gamma(\alpha\psi-\phi)(\alpha(\psi-\psi_k)-(\phi-\phi_k))\,\mathrm{d}\Gamma + \frac{1}{h}\int_\Omega F_0'(\phi)(\phi-\phi_k)\,\mathrm{d}x\\
&\quad + (1-\lambda)\Big(\int_\Omega \mu\,\mathrm{d}x\Big)^2 + \frac{1}{h}\int_\Gamma G_0'(\psi)(\psi-\psi_k)\,\mathrm{d}\Gamma + (1-\lambda)\Big(\int_\Gamma \theta\,\mathrm{d}\Gamma\Big)^2\\
&= \lambda\int_\Omega \frac{c_F}{2h}(\phi^2-\phi_k^2)\,\mathrm{d}x + \lambda\int_\Gamma \frac{c_G}{2h}(\psi^2-\psi_k^2)\,\mathrm{d}\Gamma.
\end{aligned}$$

Next, note that due to $\mathbf{z} = (\mathbf{v},\mathbf{w},\phi,\psi,\mu,\theta) = \mathcal{M}_k^{-1}(\mathbf{f}) \in X_{K,L}$, it holds that $(\phi,\psi)\in \operatorname{dom}(\partial\widetilde{E}_{\text{free}})$, and therefore $|\phi|\le 1$ a.e. in Ω as well as $|\psi|\le 1$ a.e. on Γ. Consequently, it holds that $\rho\ge 0$ a.e. in Ω and $\sigma\ge 0$ a.e. on Γ. Furthermore, we recall the convexity of F_0 and G_0, respectively, as well as the identity

$$\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) = \frac{1}{2}|\mathbf{a}|^2 - \frac{1}{2}|\mathbf{b}|^2 + \frac{1}{2}|\mathbf{a}+\mathbf{b}|^2 \qquad \text{for all } \mathbf{a},\mathbf{b}\in\mathbb{R}^d.$$

Then, dropping any non-essential non-negative term on the left-hand side, we deduce the following inequality

$$\begin{aligned}
&h\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x + h\int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}|^2\,\mathrm{d}\Gamma + h\int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}|^2\,\mathrm{d}\Gamma\\
&\quad + h\int_\Omega m_\Omega(\phi_k)|\nabla\mu|^2\,\mathrm{d}x + h\int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma + h\chi(L)\int_\Gamma(\beta\theta-\mu)^2\,\mathrm{d}\Gamma
\end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega \frac{1}{2}|\nabla\phi|^2 \,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma\psi|^2 \,\mathrm{d}\Gamma + \chi(K)\int_\Gamma \frac{1}{2}(\alpha\psi-\phi)^2 \,\mathrm{d}\Gamma \\
&+ (1-\lambda)h\Big(\int_\Omega \mu \,\mathrm{d}x\Big)^2 + (1-\lambda)h\Big(\int_\Gamma \theta \,\mathrm{d}\Gamma\Big)^2 \\
\le\ & \int_\Omega \frac{1}{2}\rho_k|\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \frac{1}{2}\sigma_k|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Omega \frac{1}{2}|\nabla\phi_k|^2 \,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma\psi_k|^2 \,\mathrm{d}\Gamma \qquad (9.34)\\
&+ \chi(K)\int_\Gamma \frac{1}{2}(\alpha\psi_k-\phi_k)^2 \,\mathrm{d}\Gamma + \int_\Omega F_0(\phi_k) \,\mathrm{d}x + \int_\Gamma G_0(\psi_k) \,\mathrm{d}\Gamma \\
&+ \lambda\int_\Omega \frac{c_F}{2}\phi_k^2 \,\mathrm{d}x + \lambda\int_\Gamma \frac{c_G}{2}\psi_k^2 \,\mathrm{d}\Gamma + \Big|\int_\Omega F_0(\phi) \,\mathrm{d}x\Big| + \Big|\int_\Gamma G_0(\psi) \,\mathrm{d}\Gamma\Big| \\
\le\ & C_k,
\end{aligned}
$$

for a constant C_k that may depend on k and h but is independent of λ, and whose value may change between the lines. Thus, recalling again that $|\phi|, |\phi_k| < 1$ a.e. in Ω and $|\psi|, |\psi_k| < 1$ a.e. on Γ, and estimating λ by 1, we employ the bulk-surface Poincarè inequality and obtain

$$
\begin{aligned}
&\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}^1} + \|(\mu,\theta)\|_{L,\beta} + \|(\phi,\psi)\|_{\mathcal{H}^1} + (1-\lambda)^{\frac{1}{2}}\Big|\int_\Omega \mu \,\mathrm{d}x\Big| + (1-\lambda)^{\frac{1}{2}}\Big|\int_\Gamma \theta \,\mathrm{d}\Gamma\Big| \\
&\le C_k \qquad (9.35)
\end{aligned}
$$

In a subsequent step, we intend to control the $\mathcal{L}^2$-norm of (μ,θ), for which we consider two distinct cases. For $\lambda \in [0, \frac{1}{2})$, a direct use of (9.35) implies

$$
\Big|\int_\Omega \mu \,\mathrm{d}x\Big| + \Big|\int_\Gamma \theta \,\mathrm{d}\Gamma\Big| \le C_k, \tag{9.36}
$$

which enables us to get a control $\|(\mu,\theta)\|_{\mathcal{L}^2}$ in view of the bulk-surface Poincaré inequality. On the other hand, for $\lambda \in [\frac{1}{2}, 1]$, we proceed as in Lemma 9.4.4, with (9.18c) and (9.18e) replaced by (9.33c) and (9.33d), to get the estimate

$$
\frac{1}{2}\Big|\int_\Omega \mu \,\mathrm{d}x\Big| + \frac{1}{2}\Big|\int_\Gamma \theta \,\mathrm{d}\Gamma\Big| \le \lambda\Big|\int_\Omega \mu \,\mathrm{d}x\Big| + \lambda\Big|\int_\Gamma \theta \,\mathrm{d}\Gamma\Big| \le C_k. \tag{9.37}
$$

Thus, employing once again the bulk-surface Poincaré inequality, we obtain from (9.35) and (9.36)-(9.37) the estimate

$$
\|(\mu,\theta)\|_{\mathcal{H}^1} \le C_k. \tag{9.38}
$$

Then, repeating the arguments from the proof of Lemma 9.4.4 and utilizing (9.35), we find

$$
\|(\phi,\psi)\|_{\mathcal{H}^2} + \|\partial \widetilde{E}_{\mathrm{free}}(\phi,\psi)\|_{\mathcal{L}^2} \le C_k. \tag{9.39}
$$

Moreover, incorporating Lemma 9.2.3, where now

$$
(f, f_\Gamma) = \Big(-\lambda\frac{\phi-\phi_k}{h} - \lambda\mathbf{v}\cdot\nabla\phi_k + \lambda\int_\Omega \mu \,\mathrm{d}x, -\lambda\frac{\psi-\psi_k}{h} - \lambda\mathbf{w}\cdot\nabla_\Gamma\psi_k + \lambda\int_\Gamma \theta \,\mathrm{d}\Gamma\Big),
$$

implies in combination with (9.35) and (9.39) that

$$
\|(\mu,\theta)\|_{\mathcal{H}^2} \le C\|(f,f_\Gamma)\|_{\mathcal{L}^2} \le C_k\Big(1 + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^4}\|(\nabla\phi_k,\nabla_\Gamma\psi_k)\|_{\boldsymbol{\mathcal{L}}^4}\Big) \le C_k. \tag{9.40}
$$

Ultimately, combining (9.35), (9.39) and (9.40), we obtain that $\mathbf{z} = (\mathbf{v},\mathbf{w},\phi,\psi,\mu,\theta) \in \widetilde{X}_{K,L} = \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}} \times \mathcal{H}^{2-s} \times \mathcal{H}^2_{L,\beta}$ satisfies the estimate

$$
\|\mathbf{z}\|_{\widetilde{X}_{K,L}} \le C_k,
$$

where

$$\|\mathbf{z}\|_{\widetilde{X}_{K,L}} = \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}^1} + \|(\phi,\psi)\|_{\mathcal{H}^{2-s}} + \|(\mu,\theta)\|_{\mathcal{H}^2}.$$

Eventually, since $\mathbf{f} \in \widetilde{Y}$ fulfills $\mathbf{f} = \lambda\mathcal{K}_k(\mathbf{f}) = \lambda\mathcal{F}_k(\mathbf{z})$, and $\mathcal{F}_k : \widetilde{X}_{K,L} \to \widetilde{Y}$ maps bounded sets into bounded sets by (9.29), we conclude

$$\|\mathbf{f}\|_{\widetilde{Y}} = \|\lambda\mathcal{F}_k(\mathbf{z})\|_{\widetilde{Y}} \le C_k(1 + \|\mathbf{z}\|^2_{\widetilde{X}_{K,L}}) \le C_k,$$

where the constant $C_k > 0$ is independent of $\lambda \in [0,1]$ and $\mathbf{f}$. The latter shows that the set defined in (9.30) is bounded. Consequently, we are in a position to apply the Leray–Schauder fixed-point theorem, which entails the existence of a fixed point $\mathbf{f} \in \widetilde{Y} \hookrightarrow Y$ of the operator $\mathcal{K}_k$. Thus, $\mathbf{z} = \mathcal{M}_k^{-1}(\mathbf{f}) \in \widetilde{X}_{K,L}$ is a solution to $\mathcal{M}_k(\mathbf{z}) = \mathcal{F}_k(\mathbf{z})$. We have therefore shown, for any $k \in \mathbb{N}$, the existence of a solution $\mathbf{z} = (\mathbf{v},\mathbf{w},\phi,\psi,\mu,\theta) = (\mathbf{v}_{k+1},\mathbf{w}_{k+1},\phi_{k+1},\psi_{k+1},\mu_{k+1},\theta_{k+1}) \in X_{K,L}$ to the time-discrete system (9.18a)-(9.18g), which completes the proof of Lemma 9.4.6. $\square$

We are now in a position to present the proof of Theorem 9.4.1.

Proof of Theorem 9.4.1. Let $N \in \mathbb{N}$ and let $h = \frac{1}{N}$ be the time-step size. We consider the initial data $(\mathbf{v}_0,\mathbf{w}_0) \in \boldsymbol{\mathcal{L}}^2_{\text{div}}$ and $(\phi_0,\psi_0) \in \mathcal{H}^1$ satisfying (9.7). Then, according to Lemma 9.2.4, there exists a sequence $\{(\widetilde{\phi}_0^N,\widetilde{\psi}_0^N)\}_{N\in\mathbb{N}} \subset \mathcal{H}^2$ such that $(\widetilde{\phi}_0^N,\widetilde{\psi}_0^N) \to (\phi_0,\psi_0)$ strongly in $\mathcal{H}^1$ as $N \to \infty$ such that

$$|\widetilde{\phi}_0^N| \le 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad |\widetilde{\psi}_0^N| \le 1 \quad \text{a.e. on } \Gamma \tag{9.41a}$$

for all $N \in \mathbb{N}$. In particular, in view of the strong convergence, there exists $m \in [0,1)$ and $\overline{N} \in \mathbb{N}$ such that, for all $N \ge \overline{N}$, it holds

$$|\beta\text{mean}(\widetilde{\phi}_0^N,\widetilde{\psi}_0^N)| \le \overline{m} < 1, \qquad |\text{mean}(\widetilde{\phi}_0^N,\widetilde{\psi}_0^N)| \le \overline{m} < 1 \tag{9.41b}$$

if $L \in [0,\infty)$, and

$$|\langle\widetilde{\phi}_0^N\rangle_\Omega| \le \overline{m} < 1, \qquad |\langle\widetilde{\psi}_0^N\rangle_\Gamma| \le \overline{m} < 1 \tag{9.41c}$$

if $L = \infty$. Thus, redefining $(\phi_0^N,\psi_0^N) = (\widetilde{\phi}_0^{N+\overline{N}},\widetilde{\psi}_0^{N+\overline{N}})$ for $N \in \mathbb{N}$, we infer that (ϕ_0^N,ψ_0^N) is an admissible pair of initial data for all $N \in \mathbb{N}$. Then, according to Lemma 9.4.5 and Lemma 9.4.6, for any $k \in \mathbb{N}$, we can choose iteratively a solution $(\mathbf{v}_{k+1},\mathbf{w}_{k+1},\phi_{k+1},\psi_{k+1},\mu_{k+1},\theta_{k+1})$ to the time-discrete system (9.18a)-(9.18g) with initial data $(\mathbf{v}_0,\mathbf{w}_0,\phi_0^N,\psi_0^N)$, which additionally satisfies the discrete energy inequality (9.26).

As in [5, Section 5], we define $f^N(t)$ on $[-h,\infty)$ through

$$f^N(t) \coloneqq f_k \qquad \text{for } t \in [(k-1)h, kh),\ k \in \mathbb{N},$$

where $f \in \{\mathbf{v},\mathbf{w},\phi,\psi,\mu,\theta\}$. Moreover, we set

$$\rho^N \coloneqq \rho(\phi^N), \qquad \sigma^N \coloneqq \sigma(\psi^N).$$

With these definitions at hand, it in particular holds that

$$f^N((k-1)h) = f_k, \qquad f^N(kh) = f_{k+1}, \qquad f^N(t) = f_{k+1} \qquad \text{for } t \in [kh,(k+1)h).$$

Moreover, we define $f_h(t) \coloneqq f(t-h)$ and

$$(\Delta_h^+ f)(t) \coloneqq f(t+h) - f(t), \qquad \partial_{t,h}^+ f(t) \coloneqq \tfrac{1}{h}(\Delta_h^+ f)(t),$$

$$(\Delta_h^- f)(t) := f(t) - f(t-h), \qquad \partial_{t,h}^- f(t) := \tfrac{1}{h}(\Delta_h^- f)(t).$$

Then, for $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in C_0^\infty(0,\infty; \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}})$, we take $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) = \int_{kh}^{(k+1)h} (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \,\mathrm{d}t$ as a test function in (9.18a) and sum over $k \in \mathbb{N}$. This yields

$$\begin{aligned}
&- \int_Q \rho^N \mathbf{v}^N \cdot \partial_{t,h}^+ \widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t - \int_\Sigma \sigma^N \mathbf{w}^N \cdot \partial_{t,h}^+ \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_Q \mathrm{div}(\rho_h^N \mathbf{v}^N \otimes \mathbf{v}^N) \cdot \widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \mathrm{div}_\Gamma(\sigma_h^N \mathbf{w}^N \otimes \mathbf{w}^N) \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_Q 2\nu_\Omega(\phi_h^N) \mathbf{D}\mathbf{v}^N : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma 2\nu_\Gamma(\psi_h^N) \mathbf{D}_\Gamma \mathbf{w}^N : \mathbf{D}_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \int_\Sigma \gamma(\phi_h^N, \psi_h^N) \mathbf{w}^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t - \int_Q (\mathbf{v}^N \otimes \mathbf{J}^N) : \nabla \widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_\Sigma (\mathbf{w}^N \otimes \mathbf{K}^N) : \nabla_\Gamma \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&= \int_Q \mu^N \nabla \phi_h^N \cdot \widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \theta^N \nabla_\Gamma \psi_h^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad + \frac{1}{2} \int_\Sigma (\mathbf{J}^N - \mathbf{J}_\Gamma^N) \cdot \mathbf{n}\, \mathbf{w}^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t,
\end{aligned} \tag{9.42}$$

where we used the notation

$$\begin{aligned}
&\mathbf{J}^N := -\frac{\tilde\rho_2 - \tilde\rho_1}{2} m_\Omega(\phi_h^N) \nabla \mu^N, \qquad \mathbf{K}^N := -\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} m_\Gamma(\psi_h^N) \nabla_\Gamma \theta^N, \\
&\mathbf{J}_\Gamma^N := -\beta \frac{\tilde\sigma_2 - \tilde\sigma_1}{2} m_\Omega(\phi_h^N) \nabla \mu^N.
\end{aligned}$$

In (9.42), we additionally made use of the following integration by parts formula

$$\begin{aligned}
&\int_Q \partial_{t,h}^-(\rho^N \mathbf{v}^N) \cdot \widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \partial_{t,h}^-(\sigma^N \mathbf{w}^N) \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\qquad = - \int_Q \rho^N \mathbf{v}^N \cdot \partial_{t,h}^+ \widehat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t - \int_\Sigma \sigma^N \mathbf{w}^N \cdot \partial_{t,h}^+ \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t.
\end{aligned}$$

In a similar manner, we derive from (9.18b),(9.18d) and (9.18f) that

$$\begin{aligned}
&\int_Q (\partial_{t,h}^- \phi^N)\, \zeta \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma (\partial_{t,h}^- \psi^N)\, \zeta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&= \int_Q \phi_h^N \mathbf{v}^N \cdot \nabla \zeta \,\mathrm{d}x\,\mathrm{d}t + \int_\Sigma \psi_h^N \mathbf{w}^N \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad - \int_Q m_\Omega(\phi_h^N) \nabla \mu^N \cdot \nabla \zeta \,\mathrm{d}x\,\mathrm{d}t - \int_\Sigma m_\Gamma(\psi_h^N) \nabla_\Gamma \theta^N \cdot \nabla_\Gamma \zeta_\Gamma \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad - \chi(L) \int_\Sigma (\beta\theta^N - \mu^N)(\beta\zeta_\Gamma - \zeta) \,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned} \tag{9.43}$$

for all $(\zeta, \zeta_\Gamma) \in C_0^\infty(0,\infty; \mathcal{H}^1_{L,\beta})$, while we infer from (9.18c), (9.18e) and (9.18g) that

$$\mu^N + \frac{c_F}{2}(\phi + \phi^N) = -\Delta\phi^N + F_0'(\phi^N) \qquad \text{a.e. in } Q, \tag{9.44a}$$

$$\theta^N + \frac{c_G}{2}(\psi + \psi^N) = -\Delta_\Gamma \psi^N + G_0'(\psi^N) + \alpha \partial_\mathbf{n} \phi^N \qquad \text{a.e. on } \Sigma, \tag{9.44b}$$

$$\begin{cases} K\partial_\mathbf{n}\phi^N = \alpha\psi^N - \phi^N & \text{if } K \in (0,\infty), \\ \partial_\mathbf{n}\phi^N = 0 & \text{if } K = \infty \end{cases} \qquad \text{a.e. on } \Sigma. \tag{9.44c}$$

Let $E^N(t)$ be the piecewise linear interpolant of $E_{\mathrm{tot}}(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k)$ at $t_k = kh$, i.e., $E^N(t)$ is defined as

$$E^N(t) := \frac{(k+1)h - t}{h} E_{\mathrm{tot}}(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k) + \frac{t - kh}{h} E_{\mathrm{tot}}(\mathbf{v}_{k+1}, \mathbf{w}_{k+1}, \phi_{k+1}, \psi_{k+1})$$

for $t \in (t_k, t_{k+1})$. For all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$, we also define the dissipation

$$\begin{aligned}
D^N(t) := &\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_{k+1}|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma \mathbf{w}_{k+1}| \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k, \psi_k)|\mathbf{w}_{k+1}|^2 \,\mathrm{d}\Gamma \\
&+ \int_\Omega m_\Omega(\phi_k)|\nabla \mu_{k+1}| \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma \theta_{k+1}|^2 \,\mathrm{d}\Gamma \\
&+ \chi(L) \int_\Gamma (\beta\theta_{k+1} - \mu_{k+1})^2 \,\mathrm{d}\Gamma.
\end{aligned}$$

Then, integrating (9.75) in time directly leads to

$$\begin{aligned}
&E_{\mathrm{tot}}(\mathbf{v}^N(t), \mathbf{w}^N(t), \phi^N(t), \psi^N(t)) + \int_s^t \int_\Omega 2\nu_\Omega(\phi_h^N)|\mathbf{D}\mathbf{v}^N|^2 \,\mathrm{d}x \,\mathrm{d}r \\
&\quad + \int_s^t \int_\Gamma 2\nu_\Gamma(\psi_h^N)|\mathbf{D}_\Gamma \mathbf{w}^N|^2 \,\mathrm{d}\Gamma \,\mathrm{d}r + \int_s^t \int_\Gamma \gamma(\phi_h^N, \psi_h^N)|\mathbf{w}^N|^2 \,\mathrm{d}\Gamma \,\mathrm{d}r \\
&\quad + \int_s^t \int_\Omega m_\Omega(\phi_h^N)|\nabla \mu^N|^2 \,\mathrm{d}x \,\mathrm{d}r + \int_s^t \int_\Gamma m_\Gamma(\psi_h^N)|\nabla_\Gamma \theta^N|^2 \,\mathrm{d}\Gamma \,\mathrm{d}r \\
&\quad + \chi(L) \int_s^t \int_\Gamma (\beta\theta^N - \mu^N)^2 \,\mathrm{d}\Gamma \,\mathrm{d}r \\
&\le E_{\mathrm{tot}}(\mathbf{v}^N(s), \mathbf{w}^N(s), \phi^N(s), \psi^N(s))
\end{aligned} \tag{9.45}$$

for all $0 \le s \le t < \infty$ with $s, t \in h\mathbb{N}$. Now, by (9.41a), we have that $F_0(\phi_0^N)$ and $G_0(\psi_0^N)$ are uniformly bounded in $L^1(\Omega)$ and $L^1(\Gamma)$, respectively. Then, invoking the bulk-surface Korn inequality and, recalling that the discrete solution (ϕ_k, ψ_k) satisfies the mass conservation law (9.23), the bulk-surface Poincaré inequality, we find a constant $C > 0$, independent of $N \in \mathbb{N}$, such that

$$\|(\mathbf{v}^N, \mathbf{w}^N)\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)} + \|(\mathbf{v}^N, \mathbf{w}^N)\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)} \le C, \tag{9.46}$$

$$\|(\phi^N, \psi^N)\|_{L^\infty(0,\infty;\mathcal{H}^1)} \le C, \tag{9.47}$$

$$\|(\nabla \mu^N, \nabla_\Gamma \theta^N)\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} + \chi(L)^{\frac{1}{2}} \|\beta\theta^N - \mu^N\|_{L^2(0,\infty;L^2(\Gamma))} \le C. \tag{9.48}$$

Furthermore, in view Lemma 9.4.4, we find that

$$\int_0^T |\mathrm{mean}(\mu^N, \theta^N)| \,\mathrm{d}t \le C\Big(T + \int_0^T \|(\mu^N, \theta^N)\|_{L,\beta} \,\mathrm{d}t\Big)$$

for all $T > 0$. Thus, employing again the bulk-surface Poincaré inequality, for any $T > 0$, there exists a constant $C = C(T)$, independent of N, such that

$$\|(\mu^N, \theta^N)\|_{L^2(0,T;\mathcal{H}^1)} \le C. \tag{9.49}$$

Using these uniform bounds, we can apply the Banach–Alaoglu theorem to conclude the existence of a sextuplet $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ such that

$$\begin{aligned}
(\mathbf{v}^N, \mathbf{w}^N) &\to (\mathbf{v}, \mathbf{w}) && \text{weakly in } L^2(0,\infty;\boldsymbol{\mathcal{H}}^1), && (9.50a)\\
& && \text{weakly-star in } L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2), && (9.50b)\\
(\phi^N, \psi^N) &\to (\phi, \psi) && \text{weakly-star in } L^\infty(0,\infty;\mathcal{H}^1), && (9.50c)
\end{aligned}$$

$$(\nabla\mu^N, \nabla_\Gamma\theta^N) \to (\nabla\mu, \nabla_\Gamma\theta) \qquad \text{weakly in } L^2(0,\infty;\boldsymbol{\mathcal{L}}^2), \tag{9.50d}$$

$$\beta\theta^N - \mu^N \to \beta\theta - \mu \qquad \text{weakly in } L^2(0,\infty;L^2(\Gamma)), \tag{9.50e}$$

$$(\mu^N, \theta^N) \to (\mu, \theta) \qquad \text{weakly in } L^2(0,T;\mathcal{H}^1), \tag{9.50f}$$

for all $T > 0$, along a non-relabeled subsequence $N \to \infty$. To pass to the limit in the nonlinearities, we need additional strong convergences of (ϕ^N, ψ^N) and $(\mathbf{v}^N, \mathbf{w}^N)$.

To this end, let $\widetilde{\phi}^N := \frac{1}{h}\mathbf{1}_{[0,h]} *_t \phi^N$ be the piecewise linear interpolant of $\phi^N(t_k)$ where $t_k = kh$ and $k \in \mathbb{N}$. Here, the convolution is only taken with respect to the time variable. Similarly, we define $\widetilde{\psi}^N := \frac{1}{h}\mathbf{1}_{[0,h]} *_t \phi^N$. Then,

$$\partial_t\widetilde{\phi}^N = \partial_{t,h}^-\phi^N, \qquad \partial_t\widetilde{\psi}^N = \partial_{t,h}^-\psi^N,$$

and

$$\|(\widetilde{\phi}^N - \phi^N, \widetilde{\psi}^N - \psi^N)\|_{(\mathcal{H}^1_{L,\beta})'} \le h\|(\partial_t\widetilde{\phi}^N, \partial_t\widetilde{\psi}^N)\|_{(\mathcal{H}^1_{L,\beta})'}. \tag{9.51}$$

Due to the aforementioned uniform bounds (9.46)-(9.48), we infer, using a comparison argument in (9.43), that

$$\|(\partial_t\widetilde{\phi}^N, \partial_t\widetilde{\psi}^N)\|_{L^2(0,\infty;(\mathcal{H}^1_{L,\beta})')} \le C. \tag{9.52}$$

Next, note that the boundedness of (ϕ^N, ψ^N) in $L^\infty(0,\infty;\mathcal{H}^1)$ implies the boundedness of $(\widetilde{\phi}^N, \widetilde{\psi}^N)$ in the same space. Combining this fact with the estimate (9.52), we get, with the help of the Aubin–Lions–Simon lemma, for any $T > 0$, the strong convergence

$$(\widetilde{\phi}^N, \widetilde{\psi}^N) \to (\widetilde{\phi}, \widetilde{\psi}) \quad \text{strongly in } C([0,T];\mathcal{H}^{1-s}) \text{ for all } s \in (0, \tfrac{1}{2}) \tag{9.53}$$

along a subsequence $N \to \infty$ for some $(\widetilde{\phi}, \widetilde{\psi}) \in L^\infty(0,\infty;\mathcal{H}^1)$. In particular, it holds that $\widetilde{\phi}^N \to \widetilde{\phi}$ a.e. in Q as well as $\widetilde{\psi}^N \to \widetilde{\psi}$ a.e. on Σ. Furthermore, the estimates (9.51) and (9.52) entail that

$$(\widetilde{\phi}^N - \phi^N, \widetilde{\psi}^N - \psi^N) \to (0,0) \qquad \text{strongly in } L^2(0,\infty;(\mathcal{H}^1_{L,\beta})')$$

as $N \to \infty$, and consequently, $(\widetilde{\phi}, \widetilde{\psi}) = (\phi, \psi)$. Moreover, the convergences (9.50c) and (9.53) combined with Lemma 9.2.1 yield

$$(\phi, \psi) \in BC_w([0,\infty);\mathcal{H}^1).$$

Then, to verify the initial conditions for ϕ and ψ, we first notice that, in view of (9.53), we have

$$(\widetilde{\phi}^N(0), \widetilde{\psi}^N(0)) \to (\phi(0), \psi(0)) \qquad \text{strongly in } \mathcal{L}^2$$

as $N \to \infty$. Besides, it holds that

$$(\widetilde{\phi}^N(0), \widetilde{\psi}^N(0)) = (\phi_0^N, \psi_0^N) \to (\phi_0, \psi_0) \qquad \text{strongly in } \mathcal{H}^1$$

as $N \to \infty$. We deduce that $(\phi, \psi)|_{t=0} = (\phi_0, \psi_0)$ a.e. in $\Omega \times \Gamma$.

To pass to the limit in (9.44a)-(9.44b), we first point out that the right-hand side is given by $\partial\widetilde{E}_{\text{free}}(\phi^N, \psi^N)$, which is bounded according to Lemma 9.4.4 in $L^2(0,T;\mathcal{L}^2)$ for all $T > 0$. Furthermore, the left-hand side converges weakly in $L^2(0,T;\mathcal{L}^2)$ as $N \to \infty$ to

$$(f, f_\Gamma) := (\mu + c_F\phi, \theta + c_G\psi)$$

for all $T > 0$, which means that

$$\partial\widetilde{E}_{\text{free}}(\phi^N, \psi^N) \to (f, f_\Gamma) \qquad \text{weakly in } L^2(0,T;\mathcal{L}^2)$$

as $N \to \infty$ for all $T > 0$. Consequently, we infer together with the strong convergence (9.53) that

$$\langle \partial\widetilde{E}_{\text{free}}(\phi^N, \psi^N), (\phi^N, \psi^N)\rangle_{\mathcal{L}^2} \to \langle (\mu + c_F\phi, \theta + c_G\psi), (\phi, \psi)\rangle_{\mathcal{L}^2}$$

as $N \to \infty$. With $\partial\widetilde{E}_{\text{free}}$ being a maximal monotone operator, we infer that $(\phi, \psi) \in \operatorname{dom}(\partial\widetilde{E}_{\text{free}})$ and $\partial\widetilde{E}_{\text{free}}(\phi, \psi) = (f, f_\Gamma)$ by Proposition 2.3.3, thereby establishing (9.11). This identity additionally entails that $\partial\widetilde{E}_{\text{free}}(\phi, \psi) \in L^2_{\text{uloc}}([0,\infty);\mathcal{H}^1)$, and thus, with Proposition 4.3.8, implies the desired regularities

$$(\phi, \psi) \in L^2_{\text{uloc}}([0,\infty);\mathcal{W}^{2,p}), \qquad (F_0'(\phi), G_0'(\psi)) \in L^2_{\text{uloc}}([0,\infty);\mathcal{L}^p)$$

for $p = 6$ if $d = 3$ and any $p \in [2,\infty)$ if $d = 2$.

To get a better strong convergence result for (ϕ^N, ψ^N), we first recall that by Lemma 9.4.4, $\{(\phi^N, \psi^N)\}_{N\in\mathbb{N}}$ is bounded in $L^2(0,T;\mathcal{H}^2)$ for any $T > 0$, which, together with (9.53) and suitable interpolation inequalities further implies that, up to a subsequence,

$$(\phi^N, \psi^N) \to (\phi, \psi) \qquad \text{strongly in } L^2(0,T;\mathcal{H}^{2-s}) \quad \text{for all } s \in (0,1) \tag{9.54}$$

as $N \to \infty$ for any $T > 0$.

In the next step, we show the strong convergence of $\{(\mathbf{v}^N, \mathbf{w}^N)\}_{N\in\mathbb{N}}$ in $L^2(0,T;\boldsymbol{\mathcal{L}}^2)$ for any $T > 0$. To this end, we denote with $\widetilde{\rho\mathbf{v}}^N$ and $\widetilde{\sigma\mathbf{w}}^N$ the piecewise linear interpolant of $(\rho^N\mathbf{v}^N)(t_k)$ and $(\sigma^N\mathbf{w}^N)(t_k)$, respectively, where $t_k = kh$, $k \in \mathbb{N}$, and let $T > 0$. As previously seen, it then holds that

$$\partial_t(\widetilde{\rho\mathbf{v}}^N) = \partial_{t,h}^-(\rho^N\mathbf{v}^N), \qquad \partial_t(\widetilde{\sigma\mathbf{w}}^N) = \partial_{t,h}^-(\sigma^N\mathbf{w}^N).$$

Now, proceeding similarly to the line of argument in [5, Section 5], we use the uniform estimates (9.46)-(9.48) along with standard interpolation results to show that

$$\begin{aligned}
\{(\rho_h^N\mathbf{v}^N \otimes \mathbf{v}^N, \sigma_h^N\mathbf{w}^N \otimes \mathbf{w}^N)\}_{N\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;\boldsymbol{\mathcal{L}}^{\frac{3}{2}}),\\
\{(\mathbf{v}^N \otimes \nabla\mu^N, \mathbf{w}^N \otimes \nabla_\Gamma\theta^N)\}_{N\in\mathbb{N}} &\text{ is uniformly bounded in } L^{\frac{8}{7}}(0,T;\boldsymbol{\mathcal{L}}^{\frac{4}{3}}),\\
\{(\mu^N\nabla\phi_h^N, \theta^N\nabla_\Gamma\psi_h^N)\}_{N\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;\boldsymbol{\mathcal{L}}^{\frac{3}{2}}),\\
\{(\beta\theta^N - \mu^N)\mathbf{w}^N\}_{N\in\mathbb{N}} &\text{ is uniformly bounded in } L^{\frac{8}{7}}(0,T;\mathbf{L}^{\frac{4}{3}}(\Gamma)).
\end{aligned} \tag{9.55}$$

Moreover, via the trace embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$, we infer from (9.46) that

$$\begin{aligned}
\{(\mathbf{D}\mathbf{v}^N, \mathbf{D}_\Gamma\mathbf{w}^N)\}_{N\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;\boldsymbol{\mathcal{L}}^2),\\
\{\mathbf{w}^N\}_{N\in\mathbb{N}} &\text{ is uniformly bounded in } L^2(0,T;\mathbf{L}^2(\Gamma)).
\end{aligned} \tag{9.56}$$

Regarding the last term on the left-hand side of (9.42), we note that, if $L \in (0,\infty]$, it holds that

$$\begin{aligned}
&\frac{1}{2}\int_\Sigma (\mathbf{J}^N - \mathbf{J}_\Gamma^N)\cdot\mathbf{n}\,\mathbf{w}^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad = \frac{1}{2}\Big(\beta\frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big)\int_\Sigma m_\Omega(\phi_h^N)\partial_{\mathbf{n}}\mu^N\mathbf{w}^N \cdot \widehat{\mathbf{w}}\,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned}$$

$$= \frac{\chi(L)}{2}\Big(\beta\frac{\tilde\sigma_2-\tilde\sigma_1}{2}+\frac{\tilde\rho_2-\tilde\rho_1}{2}\Big)\int_\Sigma(\beta\theta^N-\mu^N)\mathbf{w}^N\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\,\mathrm{d}t,$$

whereas in the case $L=0$, we have

$$\frac{1}{2}\int_\Sigma(\mathbf{J}^N-\mathbf{J}_\Gamma^N)\cdot\mathbf{n}\mathbf{w}^N\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\,\mathrm{d}t=0$$

as $\beta(\tilde\sigma_2-\tilde\sigma_1)=-(\tilde\rho_2-\tilde\rho_1)$ by assumption (9.14). Thus, due to this observation and the uniform bounds (9.55) and (9.56), any $(\widehat{\mathbf{v}},\widehat{\mathbf{w}})\in L^8(0,T;\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})$ is an admissible test function in (9.42). Hence, by a comparison argument, we infer that $\{\partial_t\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\widetilde{\rho\mathbf{v}}^N,\widetilde{\sigma\mathbf{w}}^N)\}_{N\in\mathbb{N}}$ is bounded in $L^{\frac{8}{7}}(0,T;(\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})')$. In addition, as we know that $\{\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\widetilde{\rho\mathbf{v}}^N,\widetilde{\sigma\mathbf{w}}^N)\}_{N\in\mathbb{N}}$ is bounded in $L^2(0,T;\boldsymbol{\mathcal{H}}^1)$, we deduce from the Aubin–Lions lemma the strong convergence

$$\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\widetilde{\rho\mathbf{v}}^N,\widetilde{\sigma\mathbf{w}}^N)\to(\mathbf{v}_*,\mathbf{w}_*)\qquad\text{strongly in } L^2(0,T;\boldsymbol{\mathcal{L}}^2)\tag{9.57}$$

as $N\to\infty$, up to a subsequence, for some $(\mathbf{v}_*,\mathbf{w}_*)\in L^\infty(0,T;\boldsymbol{\mathcal{L}}^2)$. Then, exploiting the weak continuity of $\boldsymbol{\mathcal{P}}_{\mathrm{div}}$ we conclude from the weak convergence $(\widetilde{\rho\mathbf{v}}^N,\widetilde{\sigma\mathbf{w}}^N)\to(\rho\mathbf{v},\sigma\mathbf{w})$ in $L^2(0,T;\boldsymbol{\mathcal{L}}^2)$ as $N\to\infty$ that $(\mathbf{v}_*,\mathbf{w}_*)=\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\rho\mathbf{v},\sigma\mathbf{w})$.

Next, due to (9.50a) and (9.57), we have

$$\begin{aligned}
&\int_0^T\int_\Omega\rho^N|\mathbf{v}^N|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Gamma\sigma^N|\mathbf{w}^N|^2\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad=\int_0^T\int_\Omega\mathbf{P}^\Omega_{\mathrm{div}}(\rho^N\mathbf{v}^N)\cdot\mathbf{v}^N\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Gamma\mathbf{P}^\Gamma_{\mathrm{div}}(\sigma^N\mathbf{w}^N)\cdot\mathbf{w}^N\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad\longrightarrow\int_0^T\int_\Omega\mathbf{P}^\Omega_{\mathrm{div}}(\rho\mathbf{v})\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Gamma\mathbf{P}^\Gamma_{\mathrm{div}}(\sigma\mathbf{w})\cdot\mathbf{w}\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad=\int_0^T\int_\Omega\rho|\mathbf{v}|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Gamma\sigma|\mathbf{w}|^2\,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned}$$

as $N\to\infty$. Moreover, it holds that

$$\begin{aligned}
&\int_0^T\int_\Omega((\rho^N)^{\frac12}\mathbf{v}^N-\rho^{\frac12}\mathbf{v})\cdot\widehat{\mathbf{v}}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Gamma((\sigma^N)^{\frac12}\mathbf{w}^N-\sigma^{\frac12}\mathbf{w})\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad=\int_0^T\int_\Omega((\rho^N)^{\frac12}-\rho^{\frac12})\widehat{\mathbf{v}}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Gamma\rho^{\frac12}(\mathbf{v}^N-\mathbf{v})\cdot\widehat{\mathbf{v}}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^T\int_\Gamma((\sigma^N)^{\frac12}-\sigma^{\frac12})\widehat{\mathbf{w}}\cdot\mathbf{w}^N\,\mathrm{d}\Gamma\,\mathrm{d}t+\int_0^T\int_\Gamma\sigma^{\frac12}(\mathbf{w}^N-\mathbf{w})\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\,\mathrm{d}t\\
&\quad\longrightarrow 0
\end{aligned}$$

as $N\to\infty$. Here, we have used the strong convergence

$$((\rho^N)^{\frac12}\widehat{\mathbf{v}},(\sigma^N)^{\frac12}\widehat{\mathbf{w}})\to(\rho^{\frac12}\widehat{\mathbf{v}},\sigma^{\frac12}\widehat{\mathbf{w}})\qquad\text{strongly in } L^2(0,T;\boldsymbol{\mathcal{L}}^2)$$

as $N\to\infty$ in combination with the weak convergence (9.50a). Consequently, as we have weak convergence and the convergence of norms, we readily conclude that

$$((\rho^N)^{\frac12}\mathbf{v}^N,(\sigma^N)^{\frac12}\mathbf{w}^N)\to(\rho^{\frac12}\mathbf{v},\sigma^{\frac12}\mathbf{w})\qquad\text{strongly in } L^2(0,T;\boldsymbol{\mathcal{L}}^2)$$

as $N\to\infty$. Lastly, since $\rho^N\to\rho$ a.e. in Q, and $\rho\geq\rho_*>0$, we use Lebesgue's dominated convergence theorem to deduce that

$$\mathbf{v}^N=(\rho^N)^{-\frac12}((\rho^N)^{\frac12}\mathbf{v}^N)\to\mathbf{v}\qquad\text{strongly in } L^2(0,T;\mathbf{L}^2(\Omega))\tag{9.58}$$

as $N \to \infty$. Similarly, we find that

$$\mathbf{w}^N = (\sigma^N)^{-\frac{1}{2}}\big((\sigma^N)^{\frac{1}{2}}\mathbf{w}^N\big) \to \mathbf{w} \qquad \text{strongly in } L^2(0,T;\mathbf{L}^2(\Gamma)) \tag{9.59}$$

as $N \to \infty$.

Now, to pass to the limit in (9.42), (9.43) and (9.44), we first notice that

$$(\phi^N, \psi^N) \to (\phi, \psi) \qquad \text{strongly in } L^4(0,T;\mathcal{H}^1) \tag{9.60}$$

as $N \to \infty$, which follows by interpolation using (9.47) and (9.54). We thus readily get

$$\begin{aligned}
&\int_{Q^T} \mu^N \nabla \phi_h^N \cdot \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma^T} \theta^N \nabla_\Gamma \psi_h^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad = -\int_{Q_T} (\nabla \mu^N \phi_h^N) \cdot \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t - \int_{\Sigma^T} (\nabla_\Gamma \theta^N \psi_h^N) \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad \longrightarrow -\int_{Q^T} (\nabla \mu \phi) \cdot \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t - \int_{\Sigma^T} (\nabla_\Gamma \theta \psi) \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad = \int_{Q^T} \mu \nabla \phi \cdot \hat{\mathbf{v}} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma^T} \theta \nabla_\Gamma \psi \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned}$$

as $N \to \infty$ for all $(\hat{\mathbf{v}}, \widehat{\mathbf{w}}) \in C_c^\infty(0,T;\boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}})$. Moreover, from (9.50e) and (9.59), we readily infer in the case $L \in (0,\infty]$ that

$$\begin{aligned}
&\frac{1}{2}\int_{\Sigma^T} (\mathbf{J}^N - \mathbf{J}_\Gamma^N) \cdot \mathbf{w}^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad = \frac{\chi(L)}{2}\Big(\beta \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big) \int_{\Sigma^T} (\beta\theta^N - \mu^N)\mathbf{w}^N \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t \\
&\quad \to \frac{\chi(L)}{2}\Big(\beta \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big) \int_{\Sigma^T} (\beta\theta - \mu)\mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma\,\mathrm{d}t
\end{aligned}$$

as $N \to \infty$ for all $(\hat{\mathbf{v}}, \widehat{\mathbf{w}}) \in C_c^\infty(0,T;\boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}})$. In the case $L = 0$, we have already seen that this term disappears in view of the assumption (9.14). Then, recalling the convergences (9.50a)-(9.50f), (9.54) and (9.58)-(9.59), as well as the uniform bounds (9.55)-(9.56), we can pass to the limit $N \to \infty$ in (9.42) and (9.43) to get (9.10a) and (9.10b).

Verification of the initial condition for $(\mathbf{v}, \mathbf{w})$

In this step, we show that the initial conditions for $(\mathbf{v}, \mathbf{w})$ are fulfilled. To this end, we begin with the following lemma, which has been adapted from [5, Section 5.2] to the current setting.

LEMMA 9.4.7. *Let* $(\mathbf{v}, \mathbf{w}), (\tilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$ *and* $(\overline{\rho}, \overline{\sigma}) \in \mathcal{L}^\infty$ *with* $\overline{\rho} \geq \overline{\rho}_0 > 0$ *and* $\overline{\sigma} \geq \overline{\sigma}_0 > 0$ *such that*

$$\int_\Omega \overline{\rho}\mathbf{v} \cdot \hat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma \overline{\sigma}\,\mathbf{w} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma = \int_\Omega \overline{\rho}\tilde{\mathbf{v}} \cdot \hat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma \overline{\sigma}\widetilde{\mathbf{w}} \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \tag{9.61}$$

for all $(\hat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$. *Then it holds* $(\mathbf{v}, \mathbf{w}) = (\tilde{\mathbf{v}}, \widetilde{\mathbf{w}})$ *a.e. in* $\Omega \times \Gamma$.

PROOF. According to Corollary 4.5.14, there exists a sequence $\{(\mathbf{v}_N, \mathbf{w}_N)\}_{N\in\mathbb{N}} \subset \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$ such that

$$(\mathbf{v}_N, \mathbf{w}_N) \to (\mathbf{v} - \tilde{\mathbf{v}}, \mathbf{w} - \widetilde{\mathbf{w}}) \quad \text{strongly in } \boldsymbol{\mathcal{L}}^2$$

as $N \to \infty$. We now test (9.61) with $(\mathbf{v}_N, \mathbf{w}_N)$. Passing to the limit $N \to \infty$, we conclude that

$$\int_\Omega \overline{\rho}|\mathbf{v} - \widetilde{\mathbf{v}}|^2 \,\mathrm{d}x + \int_\Gamma \overline{\sigma}|\mathbf{w} - \widetilde{\mathbf{w}}|^2 \,\mathrm{d}\Gamma = 0.$$

Since $\overline{\rho}$ and $\overline{\sigma}$ are uniformly positive, we infer $\mathbf{v} = \widetilde{\mathbf{v}}$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{w} = \widetilde{\mathbf{w}}$ in $\mathbf{L}^2(\Gamma)$, which proves the claim. □

At first, we show the weak continuity in time for the projection of $(\rho(\phi)\mathbf{v}, \sigma(\psi)\mathbf{w})$ onto the divergence-free vector fields. To this end, we consider

$$(\mathbf{v}_\star^N, \mathbf{w}_\star^N) \coloneqq \boldsymbol{\mathcal{P}}_{\mathrm{div}}(\widetilde{\rho\mathbf{v}}^N, \widetilde{\sigma\mathbf{w}}^N)$$

and deduce from the arguments above that

$$\begin{aligned}
\{(\mathbf{v}_\star^N, \mathbf{w}_\star^N)\}_{N\in\mathbb{N}} \quad &\text{is bounded in} \quad W^{1,\frac{8}{7}}(0,\infty; (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})') \\
&\qquad\qquad\qquad \hookrightarrow BUC([0,\infty); (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})'), \\
\{(\mathbf{v}_\star^N, \mathbf{w}_\star^N)\}_{N\in\mathbb{N}} \quad &\text{is bounded in} \quad L^\infty(0,\infty; \boldsymbol{\mathcal{L}}^2).
\end{aligned}$$

In view of these bounds, we already showed in (9.57) that $(\mathbf{v}_N^\star, \mathbf{w}_N^\star) \to (\mathbf{v}^\star, \mathbf{w}^\star)$ strongly in $L^2(0,T;\boldsymbol{\mathcal{L}}^2)$ as $N \to \infty$. This implies that

$$\begin{aligned}
(\mathbf{v}_N^\star, \mathbf{w}_N^\star) \to (\mathbf{v}^\star, \mathbf{w}^\star) \quad &\text{weakly in } W^{1,\frac{8}{7}}(0,\infty; (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})') \\
&\text{weakly-star in } L^\infty(0,\infty; \boldsymbol{\mathcal{L}}^2)
\end{aligned} \tag{9.62}$$

as $N \to \infty$. These convergences are first obtained after subsequence extraction, but as the limit is independent of the extracted subsequence, the convergences hold for the whole sequence. In particular, using Lemma 9.2.1, we infer that $(\mathbf{v}^\star, \mathbf{w}^\star) \in BC_w([0,\infty); \boldsymbol{\mathcal{L}}^2)$. Moreover, in view of the continuous embedding

$$W^{1,\frac{8}{7}}(0,\infty; (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})') \hookrightarrow BUC([0,\infty); (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})'),$$

we infer that the evaluation operator

$$\mathrm{tr}_0 : W^{1,\frac{8}{7}}(0,\infty; (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})') \to (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})', \quad (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}) \mapsto (\widetilde{\mathbf{v}}(0), \widetilde{\mathbf{w}}(0))$$

is continuous. Consequently, in combination with the weak convergence (9.62), we obtain

$$(\mathbf{v}_N^\star(0), \mathbf{w}_N^\star(0)) \to (\mathbf{v}^\star(0), \mathbf{w}^\star(0)) \quad \text{weakly in } (\boldsymbol{\mathcal{W}}^{1,4}_{0,\mathrm{div}})' \tag{9.63}$$

as $N \to \infty$.

Now, for $t \in [0,\infty)$, we consider the auxiliary problems

$$-\mathrm{div}\left(\frac{1}{\rho(\phi(t))}\nabla p(t)\right) = \mathrm{div}\left(\frac{1}{\rho(\phi(t))}\mathbf{v}^\star(t)\right) \qquad \text{in } \Omega, \tag{9.64a}$$

$$\partial_{\mathbf{n}} p(t) = 0 \qquad \text{on } \Gamma, \tag{9.64b}$$

and

$$-\mathrm{div}_\Gamma\left(\frac{1}{\sigma(\psi(t))}\nabla_\Gamma q(t)\right) = \mathrm{div}_\Gamma\left(\frac{1}{\sigma(\psi(t))}\mathbf{w}^\star(t)\right) \qquad \text{on } \Gamma. \tag{9.65}$$

The corresponding weak formulations read, for all $t \in [0, \infty)$, as

$$\int_\Omega \frac{1}{\rho(\phi(t))} \nabla p(t) \cdot \nabla \zeta \, \mathrm{d}x = - \int_\Omega \frac{1}{\rho(\phi(t))} \mathbf{v}^\star(t) \cdot \nabla \zeta \, \mathrm{d}x \tag{9.66}$$

for all $\zeta \in H^1(\Omega)$ and

$$\int_\Gamma \frac{1}{\sigma(\psi(t))} \nabla_\Gamma q(t) \cdot \nabla_\Gamma \zeta_\Gamma \, \mathrm{d}\Gamma = - \int_\Gamma \frac{1}{\sigma(\psi(t))} \mathbf{w}^\star(t) \cdot \nabla_\Gamma \zeta_\Gamma \, \mathrm{d}\Gamma \tag{9.67}$$

for all $\zeta_\Gamma \in H^1(\Gamma)$, respectively. By the Lax–Milgram theorem, we find unique weak solutions $p \in L^2(0, \infty; H^1(\Omega) \cap L^2_{(0)}(\Omega))$ and $q \in L^2(0, \infty; H^1(\Gamma) \cap L^2_{(0)}(\Gamma))$, which satisfy

$$\|\nabla p(t)\|_{\mathbf{L}^2(\Omega)} \le C \|\mathbf{v}^\star(t)\|_{\mathbf{L}^2(\Omega)} \tag{9.68}$$

and

$$\|\nabla_\Gamma q(t)\|_{\mathbf{L}^2(\Gamma)} \le C \|\mathbf{w}^\star(t)\|_{\mathbf{L}^2(\Gamma)} \tag{9.69}$$

for all $t \in [0, \infty)$ with a constant $C > 0$ independent of t, respectively. Here, we used the fact that both $\rho \in L^\infty(Q)$ and $\sigma \in L^\infty(\Sigma)$ are bounded from above and from below by positive constants.

We now claim that $(\nabla p, \nabla_\Gamma q) \in BC_w([0, \infty); \boldsymbol{\mathcal{L}}^2)$. As the equations for p and q are completely uncoupled, we only prove that $\nabla p \in C_w([0, \infty); \mathbf{L}^2(\Omega))$, while the other assertion follows by similar arguments. To prove the claim, let $t \in [0, \infty)$ be arbitrary, and let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence with $t_n \to t$ as $n \to \infty$. From $\mathbf{v}^\star \in C_w([0, \infty); \mathbf{L}^2(\Omega))$ we infer in particular that $\{\mathbf{v}^\star(t_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^2(\Omega)$. Thus, due to (9.68), also $\{\nabla p(t_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbf{L}^2(\Omega)$. Since $\{\nabla \tilde{p} : \tilde{p} \in H^1(\Omega) \cap L^2_{(0)}(\Omega)\}$ is a closed subspace of $\mathbf{L}^2(\Omega)$, we infer that there exists $\tilde{p} \in H^1(\Omega) \cap L^2_{(0)}(\Omega)$ such that, up to subsequence extraction, $\nabla p(t_n) \to \nabla \tilde{p}$ weakly in $\mathbf{L}^2(\Omega)$ as $n \to \infty$. By passing to the limit in the weak formulation (9.68) written at t_n, we notice that $\tilde{p}$ is a weak solution to (9.66). Because of uniqueness, this readily yields $\nabla \tilde{p} = \nabla p(t)$. In particular, this means that the weak limit is independent of the extracted subsequence. Thus, the convergence $\nabla p(t_n) \to \nabla \tilde{p}$ weakly in $\mathbf{L}^2(\Omega)$, as $n \to \infty$, remains true for the whole sequence. This shows that $\nabla p \in BC_w([0, \infty); \mathbf{L}^2(\Omega))$. The statement $\nabla_\Gamma q \in BC_w([0, \infty); \mathbf{L}^2(\Gamma))$ can be verified analogously. Thus, the regularity $(\nabla p, \nabla_\Gamma q) \in BC_w([0, \infty); \boldsymbol{\mathcal{L}}^2)$ is established.

As seen before, we know that $(\mathbf{v}^\star, \mathbf{w}^\star) = \boldsymbol{\mathcal{P}}_{\mathrm{div}}(\rho(\phi)\mathbf{v}, \sigma(\psi)\mathbf{w})$. In particular, it holds

$$(\mathbf{v}^\star(t), \mathbf{w}^\star(t)) = \boldsymbol{\mathcal{P}}_{\mathrm{div}}(\rho(\phi(t))\mathbf{v}(t), \sigma(\psi(t))\mathbf{w}(t))$$

for a.e. $t \in (0, \infty)$. By the characterization of the projection $\boldsymbol{\mathcal{P}}_{\mathrm{div}} = (\mathbf{P}^\Omega_{\mathrm{div}}, \mathbf{P}^\Gamma_{\mathrm{div}}) : \boldsymbol{\mathcal{L}}^2 \to \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$, the last identity entails that

$$(\mathbf{v}^\star(t), \mathbf{w}^\star(t)) = (\rho(\phi(t))\mathbf{v}(t) - \nabla \hat{p}(t), \sigma(\psi(t))\mathbf{w}(t) - \nabla_\Gamma \hat{q}(t)) \tag{9.70}$$

for a.e. $t \in (0, \infty)$, where $\hat{p}(t)$ can be fixed as the weak solution to the Poisson–Neumann problem

$$\begin{aligned} &\Delta \hat{p}(t) = \mathrm{div}(\rho(\phi(t))\mathbf{v}(t)) && \text{in } \Omega, \\ &\partial_{\mathbf{n}} \hat{p}(t) = 0 && \text{on } \Gamma, \end{aligned}$$

whereas $\hat{q}(t)$ can be fixed as the weak solution of the inhomogeneous Laplace–Beltrami equation

$$\Delta_\Gamma \hat{q}(t) = \operatorname{div}_\Gamma\big(\sigma(\psi(t))\mathbf{w}(t)\big) \qquad \text{on } \Gamma.$$

Dividing the first component and second component in (9.70) by $\rho(\phi(t))$ and $\sigma(\psi(t))$, respectively, we infer

$$\mathbf{v}(t) = \frac{1}{\rho(\phi(t))}\mathbf{v}^\star(t) + \frac{1}{\rho(\phi(t))}\nabla\hat{p}(t), \tag{9.71}$$

$$\mathbf{w}(t) = \frac{1}{\sigma(\psi(t))}\mathbf{w}^\star(t) + \frac{1}{\sigma(\psi(t))}\nabla_\Gamma\hat{q}(t) \tag{9.72}$$

for a.e. $t \in (0,\infty)$. Hence, for almost all $t \in (0,\infty)$, we deduce that

$$\int_\Omega \frac{1}{\rho(\phi(t))}\nabla\hat{p}(t)\cdot\nabla\zeta \,\mathrm{d}x = -\int_\Omega \frac{1}{\rho(\phi(t))}\mathbf{v}^\star(t)\cdot\nabla\zeta\,\mathrm{d}x$$

for all $\zeta \in H^1(\Omega)$, and

$$\int_\Gamma \frac{1}{\sigma(\psi(t))}\nabla_\Gamma\hat{q}(t)\cdot\nabla_\Gamma\zeta_\Gamma \,\mathrm{d}\Gamma = -\int_\Gamma \frac{1}{\sigma(\psi(t))}\mathbf{w}^\star(t)\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma$$

for all $\zeta_\Gamma \in H^1(\Gamma)$. This means that $\hat{p}$ and $\hat{q}$ are weak solutions of (9.64) and (9.65), respectively. Due to the uniqueness of those weak solutions, we infer $\nabla\hat{p}(t) = \nabla p(t)$ as well as $\nabla_\Gamma\hat{q}(t) = \nabla_\Gamma q(t)$ for almost every $t \in (0,\infty)$. Thus, by redefinition of $(\nabla\hat{p}, \nabla_\Gamma\hat{q})$ on a set of measure zero, we get $(\nabla\hat{p}, \nabla_\Gamma\hat{q}) \in BC_w([0,\infty);\boldsymbol{\mathcal{L}}^2)$. Again, with a redefinition of $(\mathbf{v},\mathbf{w})$ on a set of measure zero, we finally conclude from (9.71)-(9.72) that

$$(\mathbf{v},\mathbf{w}) \in BC_w([0,\infty);\boldsymbol{\mathcal{L}}^2). \tag{9.73}$$

Here, we also used the fact that $(\rho(\phi),\sigma(\psi)) \in BC([0,\infty);\mathcal{L}^2)$ with $\rho \geq \rho_* > 0$ and $\sigma \geq \sigma_* > 0$.

It remains to show that the pair $(\mathbf{v},\mathbf{w})$ fulfills the initial condition (9.9). To this end, as we have by definition $\boldsymbol{\mathcal{P}}_{\mathrm{div}}(\widetilde{\rho\mathbf{v}}^N(0), \widetilde{\sigma\mathbf{w}}^N(0)) = \boldsymbol{\mathcal{P}}_{\mathrm{div}}(\rho_0\mathbf{v}_0, \sigma_0\mathbf{w}_0)$, we infer from (9.63) that

$$\begin{aligned}
&\int_\Omega \rho(\phi_0)\mathbf{v}(0)\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma \sigma(\psi_0)\mathbf{w}(0)\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&\quad= \int_\Omega \mathbf{P}^\Omega_{\mathrm{div}}(\rho(\phi_0)\mathbf{v}(0))\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma \mathbf{P}^\Gamma_{\mathrm{div}}(\sigma(\psi_0)\mathbf{w}(0))\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&\quad= \lim_{N\to\infty}\left(\int_\Omega \mathbf{P}^\Omega_{\mathrm{div}}(\widetilde{\rho\mathbf{v}}^N(0))\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma \mathbf{P}^\Gamma_{\mathrm{div}}(\widetilde{\sigma\mathbf{w}}^N(0))\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma\right) \\
&\quad= \int_\Omega \mathbf{P}^\Omega_{\mathrm{div}}(\rho_0\mathbf{v}_0)\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma \mathbf{P}^\Gamma_{\mathrm{div}}(\sigma_0\mathbf{w}_0)\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&\quad= \int_\Omega \rho(\phi_0)\mathbf{v}_0\cdot\mathbf{P}^\Omega_{\mathrm{div}}(\widehat{\mathbf{v}})\,\mathrm{d}x + \int_\Gamma \sigma(\psi_0)\mathbf{w}_0\cdot\mathbf{P}^\Gamma_{\mathrm{div}}(\widehat{\mathbf{w}})\,\mathrm{d}\Gamma \\
&\quad= \int_\Omega \rho(\phi_0)\mathbf{v}_0\cdot\widehat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma \sigma(\psi_0)\mathbf{w}_0\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma
\end{aligned} \tag{9.74}$$

for all $(\widehat{\mathbf{v}},\widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}$. We are thus in the setting of Lemma 9.4.7, and we finally conclude that (9.9) is fulfilled.

Verification of the energy inequality

To show that the energy inequality (9.13) holds, we first notice that the discrete energy estimate (9.26) implies

$$-\frac{\mathrm{d}}{\mathrm{d}t}E^N(t) = \frac{E_{\mathrm{tot}}(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k) - E_{\mathrm{tot}}(\mathbf{v}_{k+1}, \mathbf{w}_{k+1}, \phi_{k+1}, \psi_{k+1})}{h} \geq D^N(t) \tag{9.75}$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$. Now, let $\pi \in W^{1,1}(0,\infty)$ with $\pi \geq 0$. Then, we multiply (9.75) with π, integrate in time, and apply integration by parts. We thus find

$$E_{\mathrm{tot}}(\mathbf{v}_0, \mathbf{w}_0, \phi_0^N, \psi_0^N)\pi(0) + \int_0^\infty E^N(t)\pi'(t)\,\mathrm{d}t \geq \int_0^\infty D^N(t)\pi(t)\,\mathrm{d}t. \tag{9.76}$$

Due to the strong convergences (9.54),(9.58), and (9.59) we readily find that

$$\begin{aligned} (\mathbf{v}^N(t), \mathbf{w}^N(t)) &\to (\mathbf{v}(t), \mathbf{w}(t)) && \text{strongly in } \boldsymbol{\mathcal{L}}^2, \\ (\phi^N(t), \psi^N(t)) &\to (\phi(t), \psi(t)) && \text{strongly in } C(\overline{\Omega}) \times C(\Gamma) \end{aligned}$$

for a.e. $t \in (0,\infty)$ as $N \to \infty$ up to a subsequence extraction. These convergences readily entail that

$$E^N(t) \to E_{\mathrm{tot}}(\mathbf{v}(t), \mathbf{w}(t), \phi(t), \psi(t))$$

for a.e. $t \in (0,\infty)$ as $N \to \infty$. On the other hand, by lower semicontinuity of norms and the almost everywhere convergence of (ϕ^N, ψ^N) to (ϕ, ψ), we find

$$\liminf_{N\to\infty} \int_0^\infty D^N(t)\pi(t)\,\mathrm{d}t \geq \int_0^\infty D(t)\pi(t)\,\mathrm{d}t,$$

where

$$\begin{aligned} D(t) := &\int_\Omega 2\nu_\Omega(\phi)|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi)|\mathbf{D}_\Gamma\mathbf{w}|^2\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi,\psi)|\mathbf{w}|^2\,\mathrm{d}\Gamma \\ &+ \int_\Omega m_\Omega(\phi)|\nabla\mu|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi)|\nabla_\Gamma\theta|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta\theta - \mu)^2\,\mathrm{d}\Gamma. \end{aligned}$$

We also refer to the proof of Theorem 5.3.2 or to [97, Proof of Theorem 3.3, Step 4], where similar arguments were carried out in detail to show an energy inequality. Hence, passing to the limit $N \to \infty$ in (9.76), we deduce

$$E_{\mathrm{tot}}(\mathbf{v}_0, \mathbf{w}_0, \phi_0, \psi_0)\pi(0) + \int_0^\infty E_{\mathrm{tot}}(\mathbf{v}(t), \mathbf{w}(t), \phi(t), \psi(t))\pi'(t)\,\mathrm{d}t \geq \int_0^\infty D(t)\pi(t)\,\mathrm{d}t.$$

Since $\pi \in W^{1,1}(0,\infty)$ with $\pi \geq 0$ was arbitrary, we can apply Lemma 9.2.2, and conclude that the energy inequality (9.13) holds true.

Lastly, the mass conservation law (9.12) easily follows by testing the variational formulation (9.10b) with $(\zeta, \zeta_\Gamma) \equiv (\beta, 1)$ if $L \in [0,\infty)$ or with $(\zeta, \zeta_\Gamma) \equiv (1, 0)$ and $(\zeta, \zeta_\Gamma) \equiv (0, 1)$ if $L = \infty$.

This means that all conditions 9.3.1(i)-9.3.1(v) from Definition 9.3.1 are verified. Consequently, the sextuplet $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ is a weak solution in the sense of Definition 9.3.1 and therefore, the proof of Theorem 9.3.3 is complete. ☐

9.4.2 Asymptotic limit $K \to 0$

Lastly, in this section, we finish the proof of Theorem 9.3.3 by proving the existence of a weak solution to (9.1) in the case $K = 0$.

THEOREM 9.4.8. *Suppose that the assumptions* **(A1)**-**(A5)** *hold. Let* $L \in [0,\infty]$, $(\mathbf{v}_0, \mathbf{w}_0) \in \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$, *and let* $(\phi_0, \psi_0) \in \mathcal{D}_\alpha$ *satisfy* (9.7). *In the case* $L = 0$, *we additionally assume that*

$$\beta(\tilde{\sigma}_2 - \tilde{\sigma}_1) = -(\tilde{\rho}_2 - \tilde{\rho}_1).$$

For any $K \in (0,\infty)$, *let* $(\mathbf{v}_K, \mathbf{w}_K, \phi_K, \psi_K, \mu_K, \theta_K)$ *denote a weak solution of the system* (9.1) *in the sense of Definition 9.3.1 with initial data* $(\mathbf{v}_0, \mathbf{w}_0, \phi_0, \psi_0)$, *whose existence is guaranteed by Theorem 9.4.8. Then, there exists a sextuplet* $(\mathbf{v}_*, \mathbf{w}_*, \phi_*, \psi_*, \mu_*, \theta_*)$ *such that, for any* $T > 0$,

$$\begin{aligned}
(\mathbf{v}_K, \mathbf{w}_K) \to (\mathbf{v}_*, \mathbf{w}_*) \qquad & \textit{weakly-star in } L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2),\\
& \textit{weakly in } L^2(0,\infty;\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}),\\
& \textit{strongly in } L^2(0,T;\boldsymbol{\mathcal{L}}^2),\\
(\phi_K, \psi_K) \to (\phi_*, \psi_*) \qquad & \textit{weakly-star in } L^\infty(0,\infty;\mathcal{H}^1),\\
& \textit{strongly in } C([0,T];\mathcal{H}^{1-s}) \textit{ for all } s \in (0,\tfrac{1}{2}),\\
(\mu_K, \theta_K) \to (\mu_*, \theta_*) \qquad & \textit{weakly in } L^2(0,T;\mathcal{H}^1),
\end{aligned}$$

as $K \to 0$, *along a non-relabeled subsequence, with*

$$\|\alpha\psi_K - \phi_K\|_{L^\infty(0,\infty;L^2(\Gamma))} \le C\sqrt{K}, \tag{9.77}$$

and the sextuplet $(\mathbf{v}_*, \mathbf{w}_*, \phi_*, \psi_*, \mu_*, \theta_*)$ *is a weak solution to the system* (9.1) *in the sense of Definition 9.3.1 for* $K = 0$. *Additionally, it holds that*

$$(\phi_*, \psi_*) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,p}), \qquad (F'(\phi_*), G'(\psi_*)) \in L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^p)$$

for $p = 6$ *if* $d = 3$ *and any* $p \in [2,\infty)$ *if* $d = 2$.

In analogy to Remark 5.4.5, we remark the following.

Remark 9.4.9. (a) As the right-hand side in (9.77) tends to zero as $K \to 0$, this explains why the Dirichlet type boundary condition $\phi_* = \alpha\psi_*$ a.e. on Σ appears in the limit model corresponding to $K = 0$.

(b) The result of Theorem 9.4.8 remains valid if we replace the initial data $(\phi_0, \psi_0) \in \mathcal{D}_\alpha$ by a sequence $(\phi_{0K}, \psi_{0K}) \in \mathcal{H}^1$ satisfying

$$E^K_{\mathrm{free}}(\phi_{0K}, \psi_{0K}) \to E^0_{\mathrm{free}}(\phi_0, \psi_0) \qquad \text{as } K \to 0$$

for some pair $(\phi_0, \psi_0) \in \mathcal{H}^1$, see also [115, Theorem 2.3]. In this case, one can show that $\phi_0 = \alpha\psi_0$ a.e. on Σ, and that $(\phi_*, \psi_*)|_{t=0} = (\phi_0, \psi_0)$ a.e. in $\Omega \times \Gamma$.

PROOF. We consider an arbitrary sequence $\{K_m\}_{m\in\mathbb{N}} \subset (0,\infty)$ such that $K_m \to 0$ as $m \to \infty$. Let $\{(\mathbf{v}_{K_m}, \mathbf{w}_{K_m}, \phi_{K_m}, \psi_{K_m}, \mu_{K_m}, \theta_{K_m})\}_{m\in\mathbb{N}}$ be the sequence of corresponding unique weak solutions originating from the initial data $(\mathbf{v}_0, \mathbf{w}_0, \phi_0, \psi_0)$. In the following, we use the letter C to denote generic positive constants independent of K_m and m. We first note that, in view of the trace relation $\phi_0 = \alpha\psi_0$ a.e. on Γ, the total energy of the initial data is bounded, i.e.,

$$E^{K_m}_{\mathrm{tot}}(\mathbf{v}_0, \mathbf{w}_0, \phi_0, \psi_0) = E^0_{\mathrm{tot}}(\mathbf{v}_0, \mathbf{w}_0, \phi_0, \psi_0) \le C,$$

since

$$\frac{1}{K_m}\int_\Gamma \frac{1}{2}(\alpha\psi_0 - \phi_0)^2 \,\mathrm{d}\Gamma = 0.$$

Let now $m \in \mathbb{N}$ be arbitrary. Then, in view of the energy inequality (9.13), we directly obtain that

$$\begin{aligned} &\|(\mathbf{v}_{K_m}, \mathbf{w}_{K_m})\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)} + \|(\mathbf{v}_{K_m}, \mathbf{w}_{K_m})\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)} \\ &\quad + \|(\phi_{K_m}, \psi_{K_m})\|_{L^\infty(0,\infty;\mathcal{H}^1)} + \|(\nabla\mu_{K_m}, \nabla_\Gamma\theta_{K_m})\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} \\ &\quad + \chi(L)^{\frac{1}{2}}\|\beta\theta_{K_m} - \mu_{K_m}\|_{L^2(0,\infty;L^2(\Gamma))} \leq C, \end{aligned} \tag{9.78}$$

as well as

$$\|\alpha\psi_{K_m} - \phi_{K_m}\|_{L^2(0,\infty;L^2(\Gamma))} \leq C\sqrt{K_m}. \tag{9.79}$$

Then, the estimate

$$\begin{aligned} &\|(\phi_{K_m}, \psi_{K_m})\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,6})} + \|(F'(\phi_{K_m}), G'(\psi_{K_m}))\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^6)} \\ &\quad + \|(\mu_{K_m}, \theta_{K_m})\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^2)} \leq C \end{aligned} \tag{9.80}$$

can be derived as in the proof of Theorem 9.4.1. Moreover, from a comparison argument, we can show, as in the proof of Theorem 6.2.3, that

$$\|(\partial_t\phi_{K_m}, \partial_t\psi_{K_m})\|_{L^2(0,\infty;(\mathcal{H}^1_{L,\beta})')} \leq C.$$

Consequently, by the Banach–Alaoglu theorem and the Aubin–Lions–Simon lemma, there exists a sextuplet $(\mathbf{v}_*, \mathbf{w}_*, \phi_*, \psi_*, \mu_*, \theta_*)$ such that, for any $T > 0$,

$$\begin{aligned} (\mathbf{v}_{K_m}, \mathbf{w}_{K_m}) &\to (\mathbf{v}_*, \mathbf{w}_*) && \text{weakly-star in } L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2), && \text{(9.81a)} \\ & && \text{weakly in } L^2(0,\infty;\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}), && \text{(9.81b)} \\ (\phi_{K_m}, \psi_{K_m}) &\to (\phi_*, \psi_*) && \text{weakly-star in } L^\infty(0,\infty;\mathcal{H}^1), && \text{(9.81c)} \\ & && \text{strongly in } C([0,T];\mathcal{H}^{1-s}) \text{ for all } s \in (0,\tfrac{1}{2}), && \text{(9.81d)} \\ (\mu_{K_m}, \theta_{K_m}) &\to (\mu_*, \theta_*) && \text{weakly in } L^2(0,T;\mathcal{H}^1), && \text{(9.81e)} \end{aligned}$$

as $m \to \infty$ along a non-relabeled subsequence. Additionally, due to (9.81d) and the trace theorem, we infer

$$\alpha\psi_{K_m} - \phi_{K_m} \to \alpha\psi_* - \phi_* \qquad \text{strongly in } C([0,T];L^2(\Gamma))$$

as $m \to \infty$ for any $T > 0$, which, in combination with (9.79), readily entails $\phi_* = \alpha\psi_*$ a.e. on Σ.

In view of these convergences and the uniform estimates (9.78)-(9.80), we can follow the line of argument of the proof of Theorem 9.4.1 to verify that the limit sextuplet $(\mathbf{v}_*, \mathbf{w}_*, \phi_*, \psi_*, \mu_*, \theta_*)$ is indeed a weak solution to to (9.1) in the sense of Definition 9.3.1 in the case $K = 0$. $\square$

Chapter 10

Global well-posedness of strong solutions for $d = 2$

This chapter is based on the work in:

[163] J. Stange, *Global well-posedness of strong solutions to a bulk-surface Navier–Stokes–Cahn–Hilliard model with non-degenerate mobilities in two dimensions*, Preprint: arXiv:2511.06847, 2025.

10.1 Introduction

In this chapter, we continue the study of the bulk-surface Navier–Stokes–Cahn–Hilliard system (9.1). In Chapter 9, we showed under very general assumptions on the data the existence of a global-in-time weak solution. We now turn our attention to the existence and uniqueness of global-in-time strong solutions.

To this end, we restrict our analysis to the two-dimensional setting, i.e., we assume that $\Omega \subset \mathbb{R}^2$. The three-dimensional analysis differs in several fundamental aspects. First, one can generally expect only local-in-time strong solutions, since even for the classical Navier–Stokes equations, the existence of global-in-time strong solutions under general assumptions on the data remains an open problem in three dimensions. Moreover, in the two-dimensional setting, we are able to include non-degenerate mobility functions by relying on the results obtained for the convective bulk-surface Cahn–Hilliard equation with non-degenerate mobility functions in Chapter 7. Finally, the analysis relies crucially on the separation property of the phase-fields. For the class of singular potentials considered in this work, the separation property is still an open problem in three dimensions, even in the case of standard homogeneous Neumann boundary conditions. To overcome this difficulty in three dimensions, one would likely need to employ several regularization and approximation procedures, as done, for example, in [95] for the AGG model.

For related diffuse interface models without dynamic boundary conditions, such as Model H or the AGG model, the well-posedness of strong solutions has been extensively studied. We refer, for instance, to [8, 13, 94, 95] as well as [99, 100]. In contrast, for two-phase flow models with dynamic boundary conditions, the literature has so far mainly focused on the existence of weak solutions, see, for example, [37, 79, 80, 97, 123]. Progress towards stronger notions of solutions has recently been made in [78], where the authors proved the existence of so-called quasi-strong solutions for an Allen–Cahn–Navier–Stokes–Voigt system with dynamic boundary conditions. To the best of our knowledge, the only uniqueness result so far for Navier–Stokes–Cahn–Hilliard systems with dynamic boundary conditions was established in [97] in the two-dimensional setting for weak solutions to (1.40). However, that result holds only for regular potentials and matched densities, and additionally required a suitable compatibility assumption on the potentials.

10.2 Main results

10.2.1 Assumptions

We begin by stating the main assumptions required to establish the existence and uniqueness of a global-in-time strong solution.

(A1) We consider a bounded domain $\emptyset \neq \Omega \subset \mathbb{R}^2$ with C^3-boundary $\Gamma := \partial\Omega$. We further use the notation

$$Q := \Omega \times (0, \infty), \quad \Sigma := \Gamma \times (0, \infty).$$

(A2) The constants occurring in the system (9.1) satisfy $\alpha \in [-1, 1]$ and $\beta \in \mathbb{R}$ with $\alpha\beta|\Omega| + |\Gamma| \neq 0$.

(A3) The density functions ρ and σ are given by

$$\rho(s) = \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} s + \frac{\tilde{\rho}_1 + \tilde{\rho}_2}{2} \quad \text{and} \quad \sigma(s) = \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} s + \frac{\tilde{\sigma}_1 + \tilde{\sigma}_2}{2}$$

for $s \in [-1, 1]$. Here, $\tilde{\rho}_1, \tilde{\rho}_2 > 0$ and $\tilde{\sigma}_1, \tilde{\sigma}_2 > 0$ are the specific densities of the two fluid components in the bulk and on the surface, respectively. Moreover, we use the notation

$$\begin{aligned} \rho_* &:= \min\{\tilde{\rho}_1, \tilde{\rho}_2\}, & \rho^* &:= \max\{\tilde{\rho}_1, \tilde{\rho}_2\}, \\ \sigma_* &:= \min\{\tilde{\sigma}_1, \tilde{\sigma}_2\}, & \sigma^* &:= \max\{\tilde{\sigma}_1, \tilde{\sigma}_2\}. \end{aligned}$$

By this definition, we clearly have

$$0 < \rho_* \leq \rho(s) \leq \rho^* \quad \text{and} \quad 0 < \sigma_* \leq \sigma(s) \leq \sigma^*$$

for all $s \in [-1, 1]$.

(A4) For the coefficients we assume $m_\Omega, m_\Gamma \in C^2([-1, 1])$, $\nu_\Omega, \nu_\Gamma \in C^2([-1, 1])$, $\gamma \in C^{0,1}([-1, 1]^2)$. Moreover, we postulate the existence of positive constants $m_*, m^*, \nu_*, \nu^*, \gamma_*, \gamma^*$ such that for all $s \in [-1, 1]$,

$$0 < m_* \leq m_\Omega(s), m_\Gamma(s) \leq m^* \quad \text{and} \quad 0 < \nu_* \leq \nu_\Omega(s), \nu_\Gamma(s) \leq \nu^*, \tag{10.1}$$

and, for all $(s, r) \in [-1, 1]^2$,

$$0 < \gamma_* \leq \gamma(s, r) \leq \gamma^*.$$

(A5) We assume that the potentials $F, G : [-1, 1] \to \mathbb{R}$ are of the form

$$F(s) = F_0(s) - \frac{c_F}{2} s^2, \qquad G(s) = G_0(s) - \frac{c_G}{2} s^2 \qquad \text{for } s \in [-1, 1]$$

with constants $c_F, c_G > 0$, where $F_0, G_0 \in C([-1, 1]) \cap C^3(-1, 1)$ are such that

$$\lim_{s \searrow -1} F_0'(s) = \lim_{s \searrow -1} G_0'(s) = -\infty \quad \text{and} \quad \lim_{s \nearrow 1} F_0'(s) = \lim_{s \nearrow 1} G_0'(s) = +\infty.$$

Moreover, we assume that

$$F_0''(s) \geq c_F, \qquad G_0''(s) \geq c_G \qquad \text{for all } s \in (-1,1). \tag{10.2}$$

Without loss of generality, we assume that $F_0(0) = G_0(0) = 0$ and $F_0'(0) = G_0'(0) = 0$. Lastly, we require the singular part of the boundary potential to dominate the singular part of the bulk potential in the sense that there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$|F_0'(\alpha s)| \leq \kappa_1 |G_0'(s)| + \kappa_2 \qquad \text{for all } s \in (-1,1).$$

(A6) We assume that one of the following conditions hold:

(A.6.1) There exist constants $C_\sharp > 0$ and $\gamma_\sharp \in [1,2)$ such that

$$F_0''(s) \leq C_\sharp \mathbf{e}^{C_\sharp |F_0'(s)|^{\gamma_\sharp}} \qquad \text{for all } s \in (-1,1).$$

(A.6.2) As $\delta \searrow 0$, for some $\kappa > \frac{1}{2}$, it holds that

$$\frac{1}{F_0'(1-2\delta)} = O\Big(\frac{1}{|\ln \delta|^\kappa}\Big), \qquad \frac{1}{|F_0'(-1+2\delta)|} = O\Big(\frac{1}{|\ln \delta|^\kappa}\Big).$$

10.2.2 Existence of strong solutions

We now state our first main result on the existence of a global-in-time strong solution.

THEOREM 10.2.1. *Let the assumptions* **(A1)**-**(A6)** *hold, and let* $K, L \in (0,\infty]$. *Assume that* $(\mathbf{v}_0, \mathbf{w}_0) \in \boldsymbol{\mathcal{H}}^1_{\text{div}}$, *and let* $(\phi_0, \psi_0) \in \mathcal{H}^1$ *satisfy* (9.7) *and let the following compatibility assumption hold:*

(C) *There exists* $(\mu_0, \theta_0) \in \mathcal{H}^1$ *such that for all* $(\eta, \eta_\Gamma) \in \mathcal{H}^1$ *it holds*

$$\begin{aligned}&\int_\Omega \mu_0 \eta \,\mathrm{d}x + \int_\Gamma \theta_0 \eta_\Gamma \,\mathrm{d}\Gamma \\ &= \int_\Omega \nabla \phi_0 \cdot \nabla \eta + F'(\phi_0)\eta \,\mathrm{d}x + \int_\Gamma \nabla_\Gamma \psi_0 \cdot \nabla_\Gamma \eta_\Gamma + G'(\psi_0)\eta_\Gamma \,\mathrm{d}\Gamma \\ &\quad + \chi(K) \int_\Gamma (\alpha\psi_0 - \phi_0)(\alpha\eta_\Gamma - \eta) \,\mathrm{d}\Gamma.\end{aligned}$$

Then, there exists a global strong solution $(\mathbf{v}, \mathbf{w}, \phi, \psi, \mu, \theta)$ *to* (9.1) *with the regularities*

$$\begin{aligned}(\mathbf{v}, \mathbf{w}) &\in BC([0,\infty); \boldsymbol{\mathcal{H}}^1_{0,\text{div}}) \cap L^2_{\text{uloc}}([0,\infty); \boldsymbol{\mathcal{H}}^2) \cap H^1_{\text{uloc}}([0,\infty); \boldsymbol{\mathcal{L}}^2_{\text{div}}), \\ (p, q) &\in L^2_{\text{uloc}}([0,\infty); \mathcal{H}^1 \cap \mathcal{L}^2_{(0)}), \\ (\partial_t \phi, \partial_t \psi) &\in L^\infty(0,\infty; (\mathcal{H}^1)') \cap L^2_{\text{uloc}}([0,\infty); \mathcal{H}^1), \\ (\phi, \psi) &\in L^\infty(0,\infty; \mathcal{H}^3), \\ (\mu, \theta) &\in BC([0,\infty); \mathcal{H}^1) \cap L^2_{\text{uloc}}([0,\infty); \mathcal{H}^3) \cap H^1_{\text{uloc}}([0,\infty); (\mathcal{H}^1)'), \\ (F'(\phi), G'(\psi)),&\ (F''(\phi), G''(\psi)) \in L^\infty(0,\infty; \mathcal{L}^p)\end{aligned}$$

for any $2 \leq p < \infty$, *in the sense that the equations* (9.1a)-(9.1b), (9.1f)-(9.1g) *are satisfied a.e. in* Q, *while the equations* (9.1c)-(9.1d) *and* (9.1h)-(9.1i) *and the boundary conditions* (9.1e) *and* (9.1j)-(9.1k) *are satisfied a.e. on* Σ. *Moreover, the solution is such that* $(\mathbf{v}, \mathbf{w})|_{t=0} = (\mathbf{v}_0, \mathbf{w}_0)$ *a.e. in* $\Omega \times \Gamma$ *and* $(\phi, \psi)|_{t=0} = (\phi_0, \psi_0)$ *a.e. in* $\Omega \times \Gamma$.

Remark 10.2.2. As for the convective bulk-surface Cahn–Hilliard equation with non-degenerate mobility functions studied in Section 7, the cases $K = 0$ and $L = 0$ are not admissible. However, in the case of constant mobility functions, say $m_\Omega = m_\Gamma \equiv 1$, the proof of Theorem 10.2.3 can be carried out in a similar manner. Indeed, for constant mobility functions, one can establish the existence of a strong solution with the additional regularity

$$\begin{aligned}
(\mathbf{v}, \mathbf{w}) &\in L^2(0, \infty; \boldsymbol{\mathcal{H}}^2) \cap H^1(0, \infty; \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}), \\
(p, q) &\in L^2(0, \infty; \mathcal{H}^1 \cap \mathcal{L}^2_{(0)}), \\
(\partial_t \phi, \partial_t \psi) &\in L^2(0, \infty; \mathcal{H}^1), \\
(\nabla \mu, \nabla_\Gamma \theta) &\in L^2(0, \infty; \boldsymbol{\mathcal{H}}^2), \qquad (\mu, \theta) \in H^1(0, \infty; (\mathcal{H}^1_{K,\alpha})').
\end{aligned}$$

These enhanced regularities can be achieved by using the regularity theory for the convective bulk-surface Cahn–Hilliard equation with constant mobility functions as established in Theorem 6.2.7, see also Remark 6.2.9.

10.2.3 Uniqueness of strong solutions

Finally, we can prove the uniqueness of strong solutions to (9.1).

THEOREM 10.2.3. *Let the assumptions from Theorem 10.2.1 hold. Then, the global strong solution to* (9.1) *is unique.*

10.3 Existence of strong solutions

In this section, we present the proof of Theorem 10.2.1 on the existence of a global strong solution to (9.1). Since the convective bulk-surface Cahn–Hilliard subsystem has already been investigated in detail in Chapter 6 and Chapter 7, the central idea of the proof is to employ a semi-Galerkin scheme in which the velocity fields are approximated on suitable finite-dimensional subspaces. We would like to mention that the same strategy was already used in [123] to show the existence of global-in-time weak solutions to (9.1).

For standard boundary conditions, such as no-slip or Navier-slip, the approximation is typically based on eigenfunctions of the Stokes operator subject to the corresponding boundary conditions; see, e.g., [94,95,97]. However, in the present setting, the dynamic boundary conditions (9.1a)-(9.1d), and in particular the surface divergence condition (9.1b), render these eigenfunctions unsuitable. To overcome this difficulty, we rely on the analysis presented in Section 4.5 on the newly introduced bulk-surface Stokes operator. The eigenfunctions of this operator provide a natural basis for approximating the velocity fields $\mathbf{v}$ and $\mathbf{w}$.

The construction of approximate solutions is based on a fixed-point argument. More precisely, for given velocity fields $(\mathbf{v}_*, \mathbf{w}_*)$, we solve the corresponding convective Cahn–Hilliard equation and obtain a unique strong solution

$$\boldsymbol{\Pi}(\mathbf{v}_*, \mathbf{w}_*) = (\phi_{(\mathbf{v}_*, \mathbf{w}_*)}, \psi_{(\mathbf{v}_*, \mathbf{w}_*)}, \mu_{(\mathbf{v}_*, \mathbf{w}_*)}, \theta_{(\mathbf{v}_*, \mathbf{w}_*)}).$$

Then, for given $(\phi, \psi, \mu, \theta)$, we solve the Galerkin approximation of the Navier–Stokes system and obtain a corresponding solution

$$\boldsymbol{\Xi}(\phi, \psi, \mu, \theta) = (\mathbf{v}_{(\phi,\psi,\mu,\theta)}, \mathbf{w}_{(\phi,\psi,\mu,\theta)}).$$

This allows us to define the operator

$$\boldsymbol{\Lambda} := \boldsymbol{\Xi} \circ \boldsymbol{\Pi}, \qquad (\mathbf{v}_*, \mathbf{w}_*) \mapsto \big(\mathbf{v}_{\Pi(\mathbf{v}_*,\mathbf{w}_*)}, \mathbf{w}_{\Pi(\mathbf{v}_*,\mathbf{w}_*)}\big).$$

A fixed point of $\boldsymbol{\Lambda}$ then yields a solution to the approximate coupled system. The existence of such a fixed-point is obtained by Schauder's fixed-point theorem.

The main part of the proof consists of deriving higher-order a priori estimates for these approximate solutions that are uniform with respect to the approximation parameter. To this end, we rely on the regularity results for the convective bulk-surface Cahn–Hilliard system with non-degenerate mobility as well as on regularity theory for the bulk-surface Stokes system with non-constant coefficients, as presented in Section 7 and Section 4.5, respectively. These uniform estimates then allow us to pass to the limit in the corresponding weak formulation of the approximate problem.

Proof of Theorem 10.2.1.
Step 1: Definition of the approximate problem. We consider the bulk-surface Stokes operator

$$\begin{aligned}
&\boldsymbol{\mathcal{A}} : D(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\mathcal{L}}^2_{\mathrm{div}} \to \boldsymbol{\mathcal{L}}^2_{\mathrm{div}}, \\
&\qquad\qquad (\mathbf{v}, \mathbf{w}) \mapsto \big(-\mathbf{P}^\Omega_{\mathrm{div}}(\mathrm{div}(2\mathbf{D}\mathbf{v})), -\mathbf{P}^\Gamma_{\mathrm{div}}(\mathrm{div}_\Gamma(2\mathbf{D}_\Gamma\mathbf{w}) + 2[\mathbf{D}\mathbf{v}\,\mathbf{n}]_{\tan} + \mathbf{w})\big),
\end{aligned}$$

that was introduced in Section 4.5. Here, $\mathbf{P}^\Omega_{\mathrm{div}}$ and $\mathbf{P}^\Gamma_{\mathrm{div}}$ are the Leray projections on Ω and Γ, respectively. We further recall that the domain $\mathcal{D}(\boldsymbol{\mathcal{A}})$ is given by

$$D(\boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{H}}^2_{0,\mathrm{div}}.$$

It was shown in Theorem 4.5.11 that $\boldsymbol{\mathcal{A}}$ has countably many positive eigenvalues $\{\widetilde{\lambda}_j\}_{j\in\mathbb{N}}$ with corresponding eigenfunctions $\{(\widetilde{\mathbf{v}}_j, \widetilde{\mathbf{w}}_j)\}_{j\in\mathbb{N}}$. These eigenfunctions are determined by

$$\begin{aligned}
&\int_\Omega 2\mathbf{D}\widetilde{\mathbf{v}}_j : \mathbf{D}\widehat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma 2\mathbf{D}_\Gamma\widetilde{\mathbf{w}}_j : \mathbf{D}_\Gamma\widehat{\mathbf{w}} \,\mathrm{d}\Gamma + \int_\Gamma \widetilde{\mathbf{w}}_j \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma \\
&\qquad = \widetilde{\lambda}_j \int_\Omega \widetilde{\mathbf{v}}_j \cdot \widehat{\mathbf{v}} \,\mathrm{d}x + \widetilde{\lambda}_j \int_\Gamma \widetilde{\mathbf{w}}_j \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma
\end{aligned}$$

for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}$, and can be chosen in such a way that they form an orthonormal basis of $\boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$. In addition, the eigenfunctions $\{(\widetilde{\mathbf{v}}_j, \widetilde{\mathbf{w}}_j)\}_{j\in\mathbb{N}}$ form a complete orthogonal system of $D(\boldsymbol{\mathcal{A}})$, see Corollary 4.5.13. For any $k \in \mathbb{N}$, we introduce the finite-dimensional subspace

$$\boldsymbol{\mathcal{V}}_k := \mathrm{span}\,\{(\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{w}}_1), \dots, (\widetilde{\mathbf{v}}_k, \widetilde{\mathbf{w}}_k)\} \subset \boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}},$$

and denote with $\boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}$ the $\boldsymbol{\mathcal{L}}^2_{\mathrm{div}}$-orthogonal projection onto $\boldsymbol{\mathcal{V}}_k$. Since Ω is of class C^3, regularity theory for the bulk-surface Stokes operator from Theorem 4.5.6 yields that $(\widetilde{\mathbf{v}}_j, \widetilde{\mathbf{w}}_j) \in \boldsymbol{\mathcal{W}}^{2,r}$ for all $r \in [2,\infty)$ and all $j \in \mathbb{N}$. As a consequence, the following inverse Sobolev embedding inequalities hold for all $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{V}}_k$:

$$\begin{aligned}
&\|(\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\|_{\boldsymbol{\mathcal{H}}^1} \le C_k \|(\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\|_{\boldsymbol{\mathcal{L}}^2}, \quad \|(\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\|_{\boldsymbol{\mathcal{H}}^2} \le C_k \|(\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\|_{\boldsymbol{\mathcal{L}}^2}, \\
&\|(\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\|_{\boldsymbol{\mathcal{W}}^{2,r}} \le C_{k,r} \|(\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\|_{\boldsymbol{\mathcal{L}}^2}.
\end{aligned} \tag{10.3}$$

Now, fix $T > 0$. For any $k \in \mathbb{N}$, we seek an approximate solution $(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k, \mu_k, \theta_k)$ to system (9.1) such that

$$(\mathbf{v}_k, \mathbf{w}_k) \in C^1([0,T]; \boldsymbol{\mathcal{V}}_k), \tag{10.4a}$$

$$(\phi_k, \psi_k) \in L^\infty(0,T;\mathcal{W}^{2,p}), \tag{10.4b}$$
$$(\partial_t\phi_k, \partial_t\psi_k) \in L^\infty(0,T;(\mathcal{H}^1)') \cap L^2(0,T;\mathcal{H}^1), \tag{10.4c}$$
$$(\mu_k, \theta_k) \in C([0,T];\mathcal{H}^1) \cap L^2(0,T;\mathcal{H}^3) \cap H^1(0,T;(\mathcal{H}^1)'), \tag{10.4d}$$
$$(F'(\phi_k), G'(\psi_k)),\ (F''(\phi_k), G''(\psi_k)) \in L^\infty(0,T;\mathcal{L}^p), \tag{10.4e}$$

for any $2 \le p < \infty$, with $|\phi_k| < 1$ a.e. in Q^T and $|\psi_k| < 1$ a.e. on Σ^T, in the sense that, a.e. on $[0,T]$, it holds that

$$\begin{aligned}
&\int_\Omega \rho(\phi_k)\partial_t\mathbf{v}_k \cdot \hat{\mathbf{v}} \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)\partial_t\mathbf{w}_k \cdot \widehat{\mathbf{w}} \,\mathrm{d}\Gamma + \int_\Omega \rho(\phi_k)(\mathbf{v}_k\cdot\nabla)\mathbf{v}_k\cdot\hat{\mathbf{v}}\,\mathrm{d}x \\
&\quad + \int_\Gamma \sigma(\psi_k)(\mathbf{w}_k\cdot\nabla_\Gamma)\mathbf{w}_k\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\int_\Omega m_\Omega(\phi_k)(\nabla\mu_k\cdot\nabla)\mathbf{v}_k\cdot\hat{\mathbf{v}}\,\mathrm{d}x \\
&\quad + \frac{\tilde{\sigma}_2-\tilde{\sigma}_1}{2}\int_\Gamma m_\Gamma(\psi_k)(\nabla_\Gamma\theta_k\cdot\nabla_\Gamma)\mathbf{w}_k\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\phi_k)\mathbf{D}\mathbf{v}_k : \mathbf{D}\hat{\mathbf{v}}\,\mathrm{d}x \\
&\quad + \int_\Gamma 2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma\mathbf{w}_k : \mathbf{D}_\Gamma\widehat{\mathbf{w}}\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)\mathbf{w}_k\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&= \int_\Omega \mu_k\nabla\phi_k\cdot\hat{\mathbf{v}}\,\mathrm{d}x + \int_\Gamma \theta_k\nabla_\Gamma\psi_k\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma \\
&\quad + \frac{\chi(L)}{2}\Big(\beta\frac{\tilde{\sigma}_2-\tilde{\sigma}_1}{2} - \frac{\tilde{\rho}_2-\tilde{\rho}_1}{2}\Big)\int_\Gamma(\beta\theta_k-\mu_k)\mathbf{w}_k\cdot\widehat{\mathbf{w}}\,\mathrm{d}\Gamma
\end{aligned} \tag{10.5}$$

for all $(\hat{\mathbf{v}}, \widehat{\mathbf{w}}) \in \boldsymbol{\mathcal{V}}_k$, and

$$\begin{aligned}
&\big\langle(\partial_t\phi_k,\partial_t\psi_k),(\zeta,\zeta_\Gamma)\big\rangle_{\mathcal{H}^1} - \int_\Omega \phi_k\mathbf{v}_k\cdot\nabla\zeta\,\mathrm{d}x - \int_\Gamma \psi_k\mathbf{w}_k\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma \\
&\quad = -\int_\Omega m_\Omega(\phi_k)\nabla\mu_k\cdot\nabla\zeta\,\mathrm{d}x - \int_\Gamma m_\Gamma(\psi_k)\nabla_\Gamma\theta_k\cdot\nabla_\Gamma\zeta_\Gamma\,\mathrm{d}\Gamma \\
&\quad\quad - \chi(L)\int_\Gamma(\beta\theta_k-\mu_k)(\beta\zeta_\Gamma-\zeta)\,\mathrm{d}\Gamma
\end{aligned} \tag{10.6}$$

for all $(\zeta,\zeta_\Gamma) \in \mathcal{H}^1$, where

$$\mu_k = -\Delta\phi_k + F'(\phi_k) \qquad \text{a.e. in } Q^T, \tag{10.7a}$$
$$\theta_k = -\Delta_\Gamma\psi_k + G'(\psi_k) + \alpha\partial_\mathbf{n}\phi_k \qquad \text{a.e. on } \Sigma^T, \tag{10.7b}$$
$$\begin{cases} K\partial_\mathbf{n}\phi_k = \alpha\psi_k - \phi_k & \text{if } K\in(0,\infty), \\ \partial_\mathbf{n}\phi_k = 0 & \text{if } K = \infty \end{cases} \qquad \text{a.e. on } \Sigma^T, \tag{10.7c}$$

together with the initial conditions

$$(\mathbf{v}_k,\mathbf{w}_k)|_{t=0} = \boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0,\mathbf{w}_0), \qquad (\phi_k,\psi_k)|_{t=0} = (\phi_0,\psi_0) \qquad \text{a.e.in } \Omega\times\Gamma. \tag{10.8}$$

We recall the notation

$$Q^T \coloneqq \Omega\times(0,T), \qquad \Sigma^T \coloneqq \Gamma\times(0,T)$$

for any $T > 0$. We further note that the weak formulation corresponding to (10.7) reads as

$$\begin{aligned}
&\int_\Omega \mu_k\eta\,\mathrm{d}x + \int_\Gamma \theta_k\eta_\Gamma\,\mathrm{d}\Gamma \\
&\quad = \int_\Omega \nabla\phi_k\cdot\nabla\eta + F'(\phi_k)\eta\,\mathrm{d}x + \int_\Gamma \nabla_\Gamma\psi_k\cdot\nabla_\Gamma\eta_\Gamma + G'(\psi_k)\eta_\Gamma\,\mathrm{d}\Gamma
\end{aligned} \tag{10.9}$$

$$+ \chi(K) \int_\Gamma (\alpha\psi_k - \phi_k)(\alpha\eta_\Gamma - \eta)\, \mathrm{d}\Gamma$$

a.e. on $[0,T]$ for all $(\eta, \eta_\Gamma) \in \mathcal{H}^1$.

To show the existence of an approximate solution satisfying (10.4)-(10.8), we perform a fixed-point argument. Toward this aim, let $(\mathbf{v}_*, \mathbf{w}_*) \in H^1(0,T;\boldsymbol{\mathcal{V}}_k)$. We consider the convective bulk-surface Cahn–Hilliard equation

$$\begin{aligned}
&\partial_t\phi_k + \operatorname{div}(\phi_k\mathbf{v}_*) = \operatorname{div}(m_\Omega(\phi_k)\nabla\mu_k) && \text{in } Q^T, && (10.10a)\\
&\mu = -\Delta\phi_k + F'(\phi_k) && \text{in } Q^T, && (10.10b)\\
&\partial_t\psi_k + \operatorname{div}_\Gamma(\psi_k\mathbf{w}_*) = \operatorname{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\theta_k) - \beta m_\Omega(\phi_k)\partial_\mathbf{n}\mu_k && \text{on } \Sigma^T, && (10.10c)\\
&\theta_k = -\Delta_\Gamma\psi_k + G'(\psi_k) + \alpha\partial_\mathbf{n}\phi_k && \text{on } \Sigma^T, && (10.10d)\\
&\begin{cases} K\partial_\mathbf{n}\phi_k = \alpha\psi_k - \phi_k & \text{if } K \in (0,\infty),\\ \partial_\mathbf{n}\phi_k = 0 & \text{if } K = \infty \end{cases} && \text{on } \Sigma^T, && (10.10e)\\
&\begin{cases} Lm_\Omega(\phi_k)\partial_\mathbf{n}\mu_k = \beta\theta_k - \mu_k & \text{if } L \in (0,\infty),\\ m_\Omega(\phi_k)\partial_\mathbf{n}\mu_k = 0 & \text{if } L = \infty \end{cases} && \text{on } \Sigma^T, && (10.10f)\\
&\phi_k|_{t=0} = \phi_0 && \text{in } \Omega, && (10.10g)\\
&\psi_k|_{t=0} = \psi_0 && \text{on } \Gamma. && (10.10h)
\end{aligned}$$

In view of Theorem 7.3.3, there exists a unique strong solution to (10.10) such that

$$\begin{aligned}
(\phi_k, \psi_k) &\in L^\infty(0,T;\mathcal{H}^3), && (10.11a)\\
(\partial_t\phi_k, \partial_t\psi_k) &\in L^\infty(0,T;(\mathcal{H}^1)') \cap H^1(0,T;\mathcal{H}^1), && (10.11b)\\
(\mu_k, \theta_k) &\in C([0,T];\mathcal{H}^1) \cap L^2(0,T;\mathcal{H}^3) \cap H^1(0,T;(\mathcal{H}^1)'), && (10.11c)\\
(F'(\phi_k), G'(\psi_k)),\ (F''(\phi_k)&, G''(\psi_k)) \in L^\infty(0,T;\mathcal{L}^p), && (10.11d)
\end{aligned}$$

for any $2 \le p < \infty$, and it holds

$$|\phi_k| < 1 \quad \text{a.e. in } Q^T \quad \text{and} \quad |\psi_k| < 1 \quad \text{a.e. on } \Sigma^T. \tag{10.12}$$

Furthermore, the energy inequality (6.13) in combination with (10.12) yields that

$$\begin{aligned}
&E_{\text{free}}(\phi(t), \psi(t)) + \frac{\min\{1, m_*\}}{2} \int_0^t \|(\mu, \theta)\|_{L,\beta}^2 \,\mathrm{d}s\\
&\quad\le E_{\text{free}}(\phi_0, \psi_0) + \frac{1}{2\min\{1, m_*\}} \int_0^t \|(\mathbf{v}_*, \mathbf{w}_*)\|_{\boldsymbol{\mathcal{L}}^2}^2 \,\mathrm{d}s
\end{aligned} \tag{10.13}$$

for all $t \in [0,T]$. Now, for any $t \in [0,T]$, we make the ansatz

$$(\mathbf{v}_k(t), \mathbf{w}_k(t)) = \sum_{j=1}^k a_j^k(t)(\widetilde{\mathbf{v}}_j, \widetilde{\mathbf{w}}_j) \qquad \text{a.e. in } \Omega \times \Gamma,$$

where the scalar, time-dependent coefficients a_j^k, $j = 1, \dots, k$, are assumed to be continuously differentiable functions, that have to be designed in a way such that the Galerkin approximation of (10.5), that reads as

$$\begin{aligned}
&\int_\Omega \rho(\phi_k)\partial_t\mathbf{v}_k \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)\partial_t\mathbf{w}_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Omega \rho(\phi_k)(\mathbf{v}_* \cdot \nabla)\mathbf{v}_k \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x\\
&\quad + \int_\Gamma \sigma(\psi_k)(\mathbf{w}_* \cdot \nabla_\Gamma)\mathbf{w}_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma - \frac{\tilde\rho_2 - \tilde\rho_1}{2} \int_\Omega m_\Omega(\phi_k)(\nabla\mu_k \cdot \nabla)\mathbf{v}_k \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} \int_\Gamma m_\Gamma(\psi_k)(\nabla_\Gamma \theta_k \cdot \nabla_\Gamma)\mathbf{w}_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\phi_k)\mathbf{D}\mathbf{v}_k : \mathbf{D}\widetilde{\mathbf{v}}_l \,\mathrm{d}x \\
&+ \int_\Gamma 2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma \mathbf{w}_k : \mathbf{D}_\Gamma \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k, \psi_k)\mathbf{w}_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma \\
= &\int_\Omega \mu_k \nabla \phi_k \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \theta_k \nabla_\Gamma \psi_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma \\
&+ \frac{\chi(L)}{2}\Big(\beta \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} - \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big) \int_\Gamma (\beta\theta_k - \mu_k)\mathbf{w}_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma,
\end{aligned} \tag{10.14}
$$

holds a.e. on $[0, T]$ for all $l = 1, \dots, k$. Moreover, the following initial condition

$$
(\mathbf{v}_k, \mathbf{w}_k)|_{t=0} = \boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0, \mathbf{w}_0) \qquad \text{a.e. in } \Omega \times \Gamma \tag{10.15}
$$

needs to be satisfied. Setting $\mathbf{A}_k(t) \coloneqq (a_1^k(t), \dots, a_k^k(t))^\top$, the system (10.14) is equivalent to the system of differential equations

$$
\mathbf{M}^k(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^k(t) + \mathbf{L}^k(t)\mathbf{A}^k(t) = \mathbf{G}^k(t), \tag{10.16}
$$

where

$$
\begin{aligned}
(\mathbf{M}^k(t))_{l,j} &\coloneqq \int_\Omega \rho(\phi_k)\widetilde{\mathbf{v}}_j \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)\widetilde{\mathbf{w}}_j \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma, \\
(\mathbf{L}^k(t))_{l,j} &\coloneqq \int_\Omega \rho(\phi_k)(\mathbf{v}_* \cdot \nabla)\widetilde{\mathbf{v}}_j \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)(\mathbf{w}_* \cdot \nabla_\Gamma)\widetilde{\mathbf{w}}_j \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma \\
&\quad - \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \int_\Omega m_\Omega(\phi_k)(\nabla \mu_k \cdot \nabla)\widetilde{\mathbf{v}}_j \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x \\
&\quad - \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} \int_\Gamma m_\Gamma(\psi_k)(\nabla_\Gamma \theta_k \cdot \nabla_\Gamma)\widetilde{\mathbf{w}}_j \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\phi_k)\mathbf{D}\widetilde{\mathbf{v}}_j : \mathbf{D}\widetilde{\mathbf{v}}_l \,\mathrm{d}x \\
&\quad + \int_\Gamma 2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma \widetilde{\mathbf{w}}_j : \mathbf{D}_\Gamma \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k, \psi_k)\widetilde{\mathbf{w}}_j \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma \\
&\quad - \frac{\chi(L)}{2}\Big(\beta \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} - \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big) \int_\Gamma (\beta\theta_k - \mu_k)\widetilde{\mathbf{w}}_j \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma, \\
(\mathbf{G}^k(t))_l &\coloneqq \int_\Omega \mu_k \nabla \phi_k \cdot \widetilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \theta_k \nabla_\Gamma \psi_k \cdot \widetilde{\mathbf{w}}_l \,\mathrm{d}\Gamma,
\end{aligned}
$$

and the initial condition reads as

$$
\mathbf{A}^k(0)_j = \big(\boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0, \mathbf{w}_0), (\widetilde{\mathbf{v}}_j, \widetilde{\mathbf{w}}_j)\big)_{\boldsymbol{\mathcal{L}}^2}, \qquad j = 1, \dots, k. \tag{10.17}
$$

By (10.11) we have $(\phi_k, \psi_k) \in C([0, T]; \mathcal{H}^2)$, which in turn implies that

$$
\rho(\phi_k), \nu_\Omega(\phi_k), m_\Omega(\phi_k) \in C(\overline{\Omega} \times [0, T])
$$

as well as

$$
\sigma(\psi_k), \nu_\Gamma(\psi_k), m_\Gamma(\psi_k), \gamma(\phi_k, \psi_k) \in C(\Gamma \times [0, T]).
$$

Moreover, as we know that $(\mathbf{v}_*, \mathbf{w}_*) \in C([0, T]; \boldsymbol{\mathcal{L}}^2_{\mathrm{div}})$ and $(\mu_k, \theta_k) \in C([0, T]; \mathcal{H}^1)$, it follows that $\mathbf{M}^k, \mathbf{L}^k \in C([0, T]; \mathbb{R}^{k \times k})$ and $\mathbf{G}^k \in C([0, T]; \mathbb{R}^k)$. Furthermore, the matrix $\mathbf{M}^k(\cdot)$ is positive definite on $[0, T]$, which entails that $\det(\mathbf{M}^k(\cdot))$ is continuous and uniformly positive on $[0, T]$. Consequently, according to Cramer's rule for the inverse, we also have $(\mathbf{M}^k)^{-1} \in C([0, T]; \mathbb{R}^{k \times k})$. Therefore, the classical existence and uniqueness theorem for systems of linear ODEs entails that there exists a unique vector

$\mathbf{A}^k \in C^1([0,T];\mathbb{R}^k)$ solving (10.16) together with the initial condition (10.17). This shows that there exists a unique solution $(\mathbf{v}_k, \mathbf{w}_k) \in C^1([0,T];\boldsymbol{\mathcal{V}}_k)$ to (10.14)-(10.15).

Now, we multiply (10.14) by a_l^k and sum over $l = 1, \dots, k$ to find

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \rho(\phi_k)\partial_t|\mathbf{v}_k|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Gamma \sigma(\psi_k)\partial_t|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\quad + \frac{1}{2}\int_\Omega \rho(\phi_k)\mathbf{v}_* \cdot \nabla|\mathbf{v}_k|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Gamma \sigma(\psi_k)\mathbf{w}_* \cdot \nabla_\Gamma|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\quad + \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\quad - \frac{1}{2}\frac{\tilde\rho_2 - \tilde\rho_1}{2}\int_\Omega m_\Omega(\phi_k)\nabla\mu_k \cdot \nabla|\mathbf{v}_k|^2 \,\mathrm{d}x - \frac{1}{2}\frac{\tilde\sigma_2 - \tilde\sigma_1}{2}\int_\Gamma m_\Gamma(\psi_k)\nabla_\Gamma\theta_k \cdot \nabla_\Gamma|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&= \int_\Omega \mu_k\nabla\phi_k \cdot \mathbf{v}_k \,\mathrm{d}x + \int_\Gamma \theta_k\nabla_\Gamma\psi_k \cdot \mathbf{w}_k \,\mathrm{d}\Gamma \\
&\quad + \frac{\chi(L)}{2}\Big(\beta\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} - \frac{\tilde\rho_2 - \tilde\rho_1}{2}\Big)\int_\Gamma (\beta\theta_k - \mu_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma
\end{aligned}$$

on $[0,T]$. Then, using integration by parts, we observe that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega \rho(\phi_k)|\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma\Big) \\
&\qquad - \frac{1}{2}\int_\Omega \Big(\partial_t\rho(\phi_k) + \mathrm{div}(\rho(\phi_k)\mathbf{v}_*)\Big)|\mathbf{v}_k|^2 \,\mathrm{d}x \\
&\qquad - \frac{1}{2}\int_\Gamma \Big(\partial_t\sigma(\psi_k) + \mathrm{div}_\Gamma(\sigma(\psi_k)\mathbf{w}_*)\Big)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\qquad + \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\qquad + \frac{1}{2}\frac{\tilde\rho_2 - \tilde\rho_1}{2}\int_\Omega \mathrm{div}(m_\Omega(\phi_k)\nabla\mu_k)|\mathbf{v}_k|^2 \,\mathrm{d}x \\
&\qquad + \frac{1}{2}\frac{\tilde\sigma_2 - \tilde\sigma_1}{2}\int_\Gamma \Big(\mathrm{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\theta_k) - \beta m_\Omega(\phi_k)\partial_\mathbf{n}\mu_k\Big)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\quad = \int_\Omega \mu_k\nabla\phi_k \cdot \mathbf{v}_k \,\mathrm{d}x + \int_\Gamma \theta_k\nabla_\Gamma\psi_k \cdot \mathbf{w}_k \,\mathrm{d}\Gamma
\end{aligned}$$

on $[0,T]$. Then, as $\mathbf{v}_*$ and $\mathbf{w}_*$ are both divergence-free on their respective domains, we find that

$$\begin{aligned}
&-\frac{1}{2}\int_\Omega \Big(\partial_t\rho(\phi_k) + \mathrm{div}(\rho(\phi_k)\mathbf{v}_*)\Big)|\mathbf{v}_k|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Gamma \Big(\partial_t\sigma(\psi_k) + \mathrm{div}_\Gamma(\sigma(\psi_k)\mathbf{w}_*)\Big)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&= -\frac{1}{2}\frac{\tilde\rho_2 - \tilde\rho_1}{2}\int_\Omega \Big(\partial_t\phi_k + \mathbf{v}_* \cdot \nabla\phi_k\Big)|\mathbf{v}_k|^2 \,\mathrm{d}x \\
&\quad - \frac{1}{2}\frac{\tilde\sigma_2 - \tilde\sigma_1}{2}\int_\Gamma \Big(\partial_t\psi_k + \mathbf{w}_* \cdot \nabla_\Gamma\psi_k\Big)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&= -\frac{1}{2}\frac{\tilde\rho_2 - \tilde\rho_1}{2}\int_\Omega \mathrm{div}(m_\Omega(\phi_k)\nabla\mu_k)|\mathbf{v}_k|^2 \,\mathrm{d}x \\
&\quad - \frac{1}{2}\frac{\tilde\sigma_2 - \tilde\sigma_1}{2}\int_\Gamma \Big(\mathrm{div}_\Gamma(m_\Gamma(\psi_k)\nabla_\Gamma\theta_k) - \beta m_\Omega(\phi_k)\partial_\mathbf{n}\mu_k\Big)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma
\end{aligned}$$

on $[0,T]$, see also (9.3). Thus, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega \rho(\phi_k)|\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma\Big)$$

$$
\begin{aligned}
&+ \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma \mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&= -\int_\Omega \phi_k \nabla\mu_k \cdot \mathbf{v}_k \,\mathrm{d}x - \int_\Gamma \psi_k \nabla_\Gamma \theta_k \cdot \mathbf{w}_k \,\mathrm{d}\Gamma
\end{aligned} \tag{10.18}
$$

on $[0,T]$. For the right-hand side of (10.18), we use

$$
\begin{aligned}
&\Big| -\int_\Omega \phi_k \nabla\mu_k \cdot \mathbf{v}_k \,\mathrm{d}x - \int_\Gamma \psi_k \nabla_\Gamma \theta_k \cdot \mathbf{w}_k \,\mathrm{d}\Gamma \Big| \\
&\le \|\nabla\mu_k\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}_k\|_{\mathbf{L}^2(\Omega)} + \|\nabla_\Gamma\theta_k\|_{\mathbf{L}^2(\Gamma)} \|\mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\le \frac{\min\{2\nu_*,\gamma_*\}}{2} \|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0} + \frac{1}{2\min\{2\nu_*,\gamma_*\}} \|(\mu_k,\theta_k)\|^2_{L,\beta}
\end{aligned}
$$

on $[0,T]$, which leads to the differential inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega \rho(\phi_k)|\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \Big) + \frac{\min\{2\nu_*,\gamma_*\}}{2} \|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0} \\
&\le \frac{\min\{2\nu_*,\gamma_*\}}{2} \|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0} + \frac{1}{2\min\{2\nu_*,\gamma_*\}} \|(\mu_k,\theta_k)\|^2_{L,\beta}
\end{aligned}
$$

on $[0,T]$. Integrating the above inequality over $[0,s]$ for some $s \in [0,T]$, and using (10.13), it follows that

$$
\begin{aligned}
&\int_\Omega \frac{\rho_*}{2}|\mathbf{v}_k(s)|^2 \,\mathrm{d}x + \int_\Gamma \frac{\sigma_*}{2}|\mathbf{w}_k(s)|^2 \,\mathrm{d}\Gamma \\
&\quad \le \int_\Omega \frac{\rho^*}{2}|\mathbf{P}^\Omega_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0)|^2 \,\mathrm{d}x + \int_\Gamma \frac{\sigma^*}{2}|\mathbf{P}^\Gamma_{\boldsymbol{\mathcal{V}}_k}(\mathbf{w}_0)|^2 \,\mathrm{d}\Gamma \\
&\qquad + \frac{1}{\min\{2\nu_*,\gamma_*\}\min\{1,m_*\}} E_{\mathrm{free}}(\phi_0,\psi_0) \\
&\qquad + \frac{1}{2\min\{2\nu_*,\gamma_*\}\min\{1,m_*\}} \int_0^s \|(\mathbf{v}_*(r),\mathbf{w}_*(r))\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r,
\end{aligned}
$$

which, in turn, entails that

$$
\begin{aligned}
\|(\mathbf{v}_k(s),\mathbf{w}_k(s)\|^2_{\boldsymbol{\mathcal{L}}^2} &\le c_*(\rho^* + \sigma^*)\|(\mathbf{v}_0,\mathbf{w}_0)\|^2_{\boldsymbol{\mathcal{L}}^2} \\
&\quad + \frac{2c_*}{\min\{2\nu_*,\gamma_*\}\min\{1,m_*\}} E_{\mathrm{free}}(\phi_0,\psi_0) \\
&\quad + \frac{c_*}{\min\{2\nu_*,\gamma_*\}\min\{1,m_*\}} \int_0^s \|(\mathbf{v}_*(r),\mathbf{w}_*(r))\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r,
\end{aligned} \tag{10.19}
$$

where $c_* := \frac{1}{\rho_*} + \frac{1}{\sigma_*}$. At this point, setting

$$
\begin{aligned}
&R_1 = c_*(\rho^* + \sigma^*)\|(\mathbf{v}_0,\mathbf{w}_0)\|^2_{\boldsymbol{\mathcal{L}}^2} + \frac{2c_*}{\min\{2\nu_*,\gamma_*\}\min\{1,m_*\}} E_{\mathrm{free}}(\phi_0,\psi_0), \\
&R_2 = \frac{c_*}{\min\{2\nu_*,\gamma_*\}\min\{1,m_*\}}, \qquad R_3 = R_1 T,
\end{aligned}
$$

and assuming

$$
\int_0^t \|(\mathbf{v}_*(r),\mathbf{w}_*(r))\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r \le R_3 \mathbf{e}^{R_2 t} \qquad \text{for all } t \in [0,T], \tag{10.20}
$$

we deduce from (10.19) that

$$
\begin{aligned}
&\int_0^t \|(\mathbf{v}_k(s),\mathbf{w}_k(s))\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}s \\
&\quad \le R_3 + R_2 \int_0^t \int_0^s \|(\mathbf{v}_*(r),\mathbf{w}_*(r))\|^2_{\boldsymbol{\mathcal{L}}^2} \,\mathrm{d}r \,\mathrm{d}s \le R_3 \mathbf{e}^{R_2 t}
\end{aligned} \tag{10.21}
$$

for all $t \in [0,T]$, and furthermore that

$$\sup_{t\in[0,T]} \|(\mathbf{v}_k(t), \mathbf{w}_k(t))\|_{\mathcal{L}^2} \le \big(R_1 + R_3 R_2 \mathbf{e}^{R_2 T}\big)^{\frac{1}{2}} =: M_0. \tag{10.22}$$

Next, we aim to control the time derivative $(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)$. To this end, we multiply (10.14) by $\frac{\mathrm{d}}{\mathrm{d}t} a_l^k$ and sum over $l = 1, \dots, k$. This yields

$$\begin{aligned}
&\int_\Omega \rho(\phi_k) |\partial_t \mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k) |\partial_t \mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&\quad = - \int_\Omega \rho(\phi_k)(\mathbf{v}_* \cdot \nabla)\mathbf{v}_k \cdot \partial_t \mathbf{v}_k \,\mathrm{d}x - \int_\Gamma \sigma(\psi_k)(\mathbf{w}_* \cdot \nabla_\Gamma)\mathbf{w}_k \cdot \partial_t \mathbf{w}_k \,\mathrm{d}\Gamma \\
&\qquad - \int_\Omega 2\nu_\Omega(\phi_k) \mathbf{D}\mathbf{v}_k : \mathbf{D}\partial_t \mathbf{v}_k \,\mathrm{d}x - \int_\Gamma 2\nu_\Gamma(\psi_k) \mathbf{D}_\Gamma \mathbf{w}_k : \mathbf{D}_\Gamma \partial_t \mathbf{w}_k \,\mathrm{d}\Gamma \\
&\qquad - \int_\Gamma \gamma(\phi_k, \psi_k) \mathbf{w}_k \cdot \partial_t \mathbf{w}_k \,\mathrm{d}\Gamma + \frac{\tilde\rho_2 - \tilde\rho_1}{2} \int_\Omega m_\Omega(\phi_k)(\nabla \mu_k \cdot \nabla)\mathbf{v}_k \cdot \partial_t \mathbf{v}_k \,\mathrm{d}x \\
&\qquad + \frac{\tilde\sigma_2 - \tilde\sigma_1}{2} \int_\Gamma m_\Gamma(\psi_k)(\nabla_\Gamma \theta_k \cdot \nabla_\Gamma)\mathbf{w}_k \cdot \partial_t \mathbf{w}_k \,\mathrm{d}\Gamma - \int_\Omega \phi_k \nabla \mu_k \cdot \partial_t \mathbf{v}_k \,\mathrm{d}x \\
&\qquad - \int_\Gamma \psi_k \nabla_\Gamma \theta_k \cdot \partial_t \mathbf{w}_k \,\mathrm{d}\Gamma \\
&\qquad + \frac{\chi(L)}{2} \Big(\beta \frac{\tilde\sigma_2 - \tilde\sigma_1}{2} - \frac{\tilde\rho_2 - \tilde\rho_1}{2}\Big) \int_\Gamma (\beta\theta_k - \mu_k)\mathbf{w}_k \cdot \partial_t \mathbf{w}_k \,\mathrm{d}\Gamma
\end{aligned}$$

on $[0,T]$. By exploiting (10.3), we use Hölder's inequality and the Sobolev embedding to find that

$$\begin{aligned}
&\rho_* \|\partial_t \mathbf{v}_k\|_{\mathbf{L}^2(\Omega)}^2 + \sigma_* \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)}^2 \\
&\quad \le \rho^* \|\mathbf{v}_*\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{v}_k\|_{\mathbf{L}^\infty(\Omega)} \|\partial_t \mathbf{v}_k\|_{\mathbf{L}^2(\Omega)} + \sigma^* \|\mathbf{w}_*\|_{\mathbf{L}^2(\Gamma)} \|\nabla_\Gamma \mathbf{w}_k\|_{\mathbf{L}^\infty(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\qquad + 2\nu^* \|\mathbf{D}\mathbf{v}_k\|_{\mathbf{L}^2(\Omega)} \|\mathbf{D}\partial_t \mathbf{v}_k\|_{\mathbf{L}^2(\Omega)} + 2\nu^* \|\mathbf{D}_\Gamma \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \|\mathbf{D}_\Gamma \partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\qquad + \gamma^* \|\mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} + \Big|\frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big| m^* \|\nabla \mu_k\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{v}_k\|_{\mathbf{L}^\infty(\Omega)} \|\partial_t \mathbf{v}_k\|_{\mathbf{L}^2(\Omega)} \\
&\qquad + \Big|\frac{\tilde\sigma_1 - \tilde\sigma_2}{2}\Big| m^* \|\nabla_\Gamma \theta_k\|_{\mathbf{L}^2(\Gamma)} \|\nabla_\Gamma \mathbf{w}_k\|_{\mathbf{L}^\infty(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\qquad + \|\nabla \mu_k\|_{\mathbf{L}^2(\Omega)} \|\partial_t \mathbf{v}_k\|_{\mathbf{L}^2(\Omega)} + \|\nabla_\Gamma \theta_k\|_{\mathbf{L}^2(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\qquad + \tfrac{1}{2}\chi(L) \Big|\beta \frac{\tilde\sigma_2 - \tilde\sigma_1}{2} - \frac{\tilde\rho_2 - \tilde\rho_1}{2}\Big| \|\beta\theta_k - \mu_k\|_{L^2(\Gamma)} \|\mathbf{w}_k\|_{\mathbf{L}^\infty(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\quad \le C(\rho^* + \sigma^*) \|(\mathbf{v}_*, \mathbf{w}_*)\|_{\mathcal{L}^2} \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\mathcal{W}^{2,4}} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2} \\
&\qquad + \nu^* C_k^2 \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\mathcal{L}^2} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2} + \gamma^* \|\mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \qquad (10.23) \\
&\qquad + C m^* \Big(\Big|\frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big| + \Big|\frac{\tilde\sigma_1 - \tilde\sigma_2}{2}\Big|\Big) \|(\mu_k, \theta_k)\|_{L,\beta} \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\mathcal{W}^{2,4}} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2} \\
&\qquad + C \|(\mu_k, \theta_k)\|_{L,\beta} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2} \\
&\qquad + \chi(L) C \Big|\beta \frac{\tilde\sigma_2 - \tilde\sigma_1}{2} - \frac{\tilde\rho_2 - \tilde\rho_1}{2}\Big| \|\beta\theta - \mu_k\|_{L^2(\Gamma)} \|\mathbf{w}_k\|_{\mathbf{H}^2(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\quad \le C_k(\rho^* + \sigma^*) \|(\mathbf{v}_*, \mathbf{w}_*)\|_{\mathcal{L}^2} \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\mathcal{L}^2} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2} \\
&\qquad + \nu^* C_k^2 \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\mathcal{L}^2} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2} + \gamma^* \|\mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\qquad + C_k m^* \Big(\Big|\frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big| + \Big|\frac{\tilde\sigma_1 - \tilde\sigma_2}{2}\Big|\Big) \|(\mu_k, \theta_k)\|_{L,\beta} \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\mathcal{L}^2} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\mathcal{L}^2}
\end{aligned}$$

$$
\begin{aligned}
&+ C_k \|(\mu_k, \theta_k)\|_{L,\beta} \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \\
&+ \chi(L) C_k \Big|\beta \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{2} - \frac{\tilde{\rho}_1 - \tilde{\rho}_2}{2}\Big| \|\beta\theta_k - \mu_k\|_{L^2(\Gamma)} \|(\mathbf{v}_k, \mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \|\partial_t \mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)}
\end{aligned}
$$

on $[0, T]$. Eventually, in view of (10.13) and (10.20)-(10.22), we infer that

$$
\begin{aligned}
&\int_0^T \|(\partial_t \mathbf{v}_k, \partial_t \mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 \, \mathrm{d}r \\
&\quad \leq 3\Big(C_k(\rho^* + \sigma^*)\Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big) M_0\Big)^2 \int_0^T \|(\mathbf{v}_*, \mathbf{w}_*)\|_{\boldsymbol{\mathcal{L}}^2}^2 \, \mathrm{d}r \\
&\qquad + 3\Big(\nu^* C_k^2 \Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big)\Big)^2 R_4 \mathbf{e}^{R_2 T} + 3\Big(\gamma^* \Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big)\Big)^2 R_4 \mathbf{e}^{R_2 T} \\
&\qquad + 3\Big(C_k m^* \Big(\Big|\frac{\tilde{\rho}_1 - \tilde{\rho}_2}{2}\Big| + \Big|\frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{2}\Big|\Big)\Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big) M_0 + M_0\Big)^2 \int_0^T \|(\mu_k, \theta_k)\|_{L,\beta}^2 \, \mathrm{d}r \\
&\qquad + \Big(C_k \Big|\beta \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{2} - \frac{\tilde{\rho}_1 - \tilde{\rho}_2}{2}\Big| \Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big) M_0\Big)^2 \chi(L)^2 \int_0^T \|\beta\theta_k - \mu_k\|_{L^2(\Gamma)}^2 \, \mathrm{d}r \\
&\quad \leq 3\Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big)^2 \Big(C_k(\rho^* + \sigma^*) M_0 + C_k^2 \nu^* + \gamma^*\Big)^2 R_4 \mathbf{e}^{R_2 T} \qquad (10.24) \\
&\qquad + \Bigg(3\Big(C_k m^* \Big(\Big|\frac{\tilde{\rho}_1 - \tilde{\rho}_2}{2}\Big| + \Big|\frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{2}\Big|\Big)\Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big) M_0 + M_0\Big)^2 \\
&\qquad\qquad + \Big(C_k \Big|\beta \frac{\tilde{\sigma}_1 - \tilde{\sigma}_2}{2} - \frac{\tilde{\rho}_1 - \tilde{\rho}_2}{2}\Big| \Big(\frac{1}{\rho_*} + \frac{1}{\sigma_*}\Big) M_0\Big)^2\Bigg) \Big(2E_{\text{free}}(\phi_0, \psi_0) + R_4 \mathbf{e}^{R_2 T}\Big) \\
&\quad =: M_1^2,
\end{aligned}
$$

where M_1 only depends on $\rho_*, \rho^*, \sigma_*, \sigma^*, \gamma^*, E_{\text{free}}(\phi_0, \psi_0), L, T, \Omega, m_*, m^*$ and k.

We now define the setting for the fixed-point argument. We introduce the set

$$
\begin{aligned}
\boldsymbol{\mathcal{M}} = \Big\{(\mathbf{v}, \mathbf{w}) \in H^1(0, T; \boldsymbol{\mathcal{V}}_k) : \int_0^t \|(\mathbf{v}, \mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 \, \mathrm{d}r \leq R_4 \mathbf{e}^{R_2 t} \text{ for all } t \in [0, T], \\
\|(\partial_t \mathbf{v}, \partial_t \mathbf{w})\|_{L^2(0,T;\boldsymbol{\mathcal{V}}_k)} \leq M_1\Big\},
\end{aligned}
$$

which is a subset of $L^2(0, T; \boldsymbol{\mathcal{V}}_k)$, and define the map

$$
\boldsymbol{\Lambda} : \boldsymbol{\mathcal{M}} \to L^2(0, T; \boldsymbol{\mathcal{V}}_k), \qquad \boldsymbol{\Lambda}(\mathbf{v}_*, \mathbf{w}_*) = (\mathbf{v}_k, \mathbf{w}_k),
$$

where $(\mathbf{v}_k, \mathbf{w}_k)$ is the solution to the system (10.14). By (10.21) and (10.24) it follows immediately that $\boldsymbol{\Lambda}(\boldsymbol{\mathcal{M}}) \subset \boldsymbol{\mathcal{M}}$. We readily notice that $\boldsymbol{\mathcal{M}}$ is convex and closed. Moreover, as a subset of $L^2(0, T; \boldsymbol{\mathcal{V}}_k)$, $\boldsymbol{\mathcal{M}}$ is compact.

We are left to prove the continuity of the map $\boldsymbol{\Lambda}$. To this end, let us consider a sequence $\{(\widehat{\mathbf{v}}_n, \widehat{\mathbf{w}}_n)\}_{n \in \mathbb{N}} \subset \boldsymbol{\mathcal{M}}$ such that $(\widehat{\mathbf{v}}_n, \widehat{\mathbf{w}}_n) \to (\widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_*)$ in $L^2(0, T; \boldsymbol{\mathcal{V}}_k)$ as $n \to \infty$. Thus, in view of Theorem 6.2.3, we find a sequence $(\widehat{\phi}_n, \widehat{\psi}_n, \widehat{\mu}_n, \widehat{\theta}_n)$, $n \in \mathbb{N}$, and $(\widehat{\phi}_*, \widehat{\psi}_*, \widehat{\mu}_*, \widehat{\theta}_*)$ that solve the convective bulk-surface Cahn–Hilliard equation (10.10), where $(\mathbf{v}_*, \mathbf{w}_*)$ is replaced by $(\widehat{\mathbf{v}}_n, \widehat{\mathbf{w}}_n)$ and $(\widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_*)$, respectively. Observing that $(\widehat{\phi}_n(0), \widehat{\psi}_n(0)) = (\widehat{\phi}_*(0), \widehat{\psi}_*(0))$ for all $n \in \mathbb{N}$, an application of the continuous dependence estimate (7.23) shows that

$$
\begin{aligned}
&\|((\widehat{\phi}_n - \widehat{\phi}_*, \widehat{\psi}_n - \widehat{\psi}_*\|_{(\mathcal{H}^1)'}^2 + \int_0^T \|(\widehat{\phi}_n - \widehat{\phi}_*, \widehat{\psi}_n - \widehat{\psi}_*)\|_{K,\alpha}^2 \, \mathrm{d}t \\
&\quad \leq \exp\Big(\int_0^T P_n \, \mathrm{d}t\Big) \int_0^T \|(\widehat{\mathbf{v}}_n - \widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_n - \widehat{\mathbf{w}}_*)\|_{\boldsymbol{\mathcal{L}}^2}^2 \, \mathrm{d}t
\end{aligned}
\qquad (10.25)
$$

on $[0,T]$ for all $n \in \mathbb{N}$, where

$$P_n(\cdot) = C\big(1 + \|(\widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_*)\|_{\mathcal{L}^2}^2 + \|(\widehat{\mu}_*, \widehat{\theta}_*)\|_{L,\beta}^2 + \|(\partial_t\widehat{\phi}_n, \partial_t\widehat{\psi}_n)\|_{(\mathcal{H}^1)'}^2 + \|(\widehat{\phi}_n, \widehat{\psi}_n)\|_{\mathcal{H}^2}^4\big),$$

and the constant $C > 0$ is independent of $n \in \mathbb{N}$. To pass to the limit in (10.25), we have to show that $\{P_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(0,T)$. To do so, we first notice that, since $\{(\widehat{\mathbf{v}}_n, \widehat{\mathbf{w}}_n)\}_{n\in\mathbb{N}}$ belongs to $\boldsymbol{\mathcal{M}}$, it is especially uniformly bounded in $L^2(0,T;\boldsymbol{\mathcal{L}}^2)$. Thus, in view of the regularities provided by Theorem 6.2.3, we find a constant $C = C(p) > 0$ such that

$$\begin{aligned}
&\|(\widehat{\phi}_n, \widehat{\psi}_n)\|_{L^\infty(0,T;\mathcal{H}^1)} + \|(\partial_t\widehat{\phi}_n, \partial_t\widehat{\psi}_n)\|_{L^2(0,T;(\mathcal{H}^1)')} + \|(\nabla\widehat{\mu}_n, \nabla_\Gamma\widehat{\theta}_n)\|_{L^2(0,T;\boldsymbol{\mathcal{L}}^2)} \\
&\quad + \chi(L)^{\frac{1}{2}}\|\beta\widehat{\theta}_n - \widehat{\mu}_n\|_{L^2(0,T;L^2(\Gamma))} + \|(\widehat{\phi}_n, \widehat{\psi}_n)\|_{L^2(0,T;\mathcal{W}^{2,p})} \\
&\quad + \|(F'(\widehat{\phi}_n), G'(\widehat{\psi}_n))\|_{L^2(0,T;\mathcal{L}^p)} + \|(\widehat{\phi}_n, \widehat{\psi}_n)\|_{L^4(0,T;\mathcal{H}^2)} \le C
\end{aligned} \tag{10.26}$$

on $[0,T]$ for any $2 \le p < \infty$. Consequently, it follows that

$$\exp\Big(\int_0^T P_n \,\mathrm{d}t\Big) \le C.$$

We remark that, in view of the assumptions on the initial data, we are even in a position to apply Theorem 7.3.3, which entails that $(\widehat{\phi}_n, \widehat{\psi}_n, \widehat{\mu}_n, \widehat{\theta}_n)$ is in fact a strong solution to (10.10). However, the estimate (10.26) is sufficient for our intention. Then, utilizing the estimate (4.78) in combination with (4.56) and Young's inequality, we have

$$\begin{aligned}
&\|(\widehat{\phi}_n - \widehat{\phi}_*, \widehat{\psi}_n - \widehat{\psi}_*)\|_{\mathcal{L}^2}^2 \\
&\quad \le C\|(\widehat{\phi}_n - \widehat{\phi}_*, \widehat{\psi}_n - \widehat{\psi}_*)\|_{K,\alpha}^2 + C\|(\widehat{\phi}_n - \widehat{\phi}_*, \widehat{\psi}_n - \widehat{\psi}_*)\|_{(\mathcal{H}^1)'}^2
\end{aligned}$$

on $[0,T]$. Thus, we readily infer from (10.25) that

$$(\widehat{\phi}_n, \widehat{\psi}_n) \to (\widehat{\phi}_*, \widehat{\psi}_*) \qquad \text{strongly in } L^\infty(0,T;(\mathcal{H}^1)') \cap L^2(0,T;\mathcal{H}^1), \tag{10.27}$$

$$\text{a.e. in } Q^T \times \Sigma^T, \tag{10.28}$$

as $n \to \infty$, up to extraction of a subsequence. Then, as shown in the proof of Theorem 6.2.3, it holds that

$$|\mathrm{mean}(\widehat{\mu}_n, \widehat{\theta}_n)| \le C(1 + \|(\widehat{\mu}_n, \widehat{\theta}_n)\|_{L,\beta}) \tag{10.29}$$

on $[0,T]$, see (6.57). Consequently, on account of the bulk-surface Poincaré inequality, we infer from (10.29) with (10.26) that

$$\|(\widehat{\mu}_n, \widehat{\theta}_n)\|_{L^2(0,T;\mathcal{H}^1)} \le C. \tag{10.30}$$

Then, in view of (10.26), we find that

$$(F'(\widehat{\phi}_n), G'(\widehat{\psi}_n)) \to (F'(\widehat{\phi}_*), G'(\widehat{\psi}_*)) \qquad \text{weakly in } L^2(0,T;\mathcal{L}^2). \tag{10.31}$$

Here, we have additionally used the fact that F_0' and G_0' are maximal monotone operators to identify the limit functions. We refer to Step 7 of the proof of Theorem 6.2.3 for more details. Next, in view of the weak formulation (10.9) and the convergences (10.27) and (10.31), we obtain

$$(\widehat{\mu}_n, \widehat{\theta}_n) \to (\widehat{\mu}_*, \widehat{\theta}_*) \qquad \text{weakly in } L^2(0,T;(\mathcal{H}^1)')$$

as $n \to \infty$. As we additionally have the uniform bound (10.30), it readily follows that

$$(\widehat{\mu}_n, \widehat{\theta}_n) \to (\widehat{\mu}_*, \widehat{\theta}_*) \qquad \text{weakly in } L^2(0,T;\mathcal{H}^1)$$

as $n \to \infty$.

Now, to show the continuity of $\mathbf{\Lambda}$, we have to prove that

$$\mathbf{\Lambda}(\widehat{\mathbf{v}}_n, \widehat{\mathbf{v}}_n) \to \mathbf{\Lambda}(\widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_*) \qquad \text{strongly in } L^2(0,T;\boldsymbol{\mathcal{V}}_k)$$

as $n \to \infty$. To this end, using the abbreviation $(\mathbf{u}_n^\Omega, \mathbf{u}_n^\Gamma) = \mathbf{\Lambda}(\widehat{\mathbf{v}}_n, \widehat{\mathbf{w}}_n)$, we recall that by definition of $\mathbf{\Lambda}$, $(\mathbf{u}_n^\Omega, \mathbf{u}_n^\Gamma)$ satisfies the variational formulation

$$\begin{aligned}
&\int_\Omega \rho(\widehat{\phi}_n)\partial_t \mathbf{u}_n^\Omega \cdot \tilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \sigma(\widehat{\psi}_n)\partial_t \mathbf{u}_n^\Gamma \cdot \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Omega \rho(\widehat{\phi}_n)(\widehat{\mathbf{v}}_n \cdot \nabla)\mathbf{u}_n^\Omega \cdot \tilde{\mathbf{v}}_l \,\mathrm{d}x \\
&\quad + \int_\Gamma \sigma(\widehat{\psi}_n)(\widehat{\mathbf{w}}_n \cdot \nabla_\Gamma)\mathbf{u}_n^\Gamma \cdot \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2} \int_\Omega m_\Omega(\widehat{\phi}_n)(\nabla \widehat{\mu}_n \cdot \nabla)\mathbf{u}_n^\Omega \cdot \tilde{\mathbf{v}}_l \,\mathrm{d}x \\
&\quad + \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} \int_\Gamma m_\Gamma(\widehat{\psi}_n)(\nabla_\Gamma \widehat{\theta}_n \cdot \nabla_\Gamma)\mathbf{u}_n^\Gamma \cdot \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Omega 2\nu_\Omega(\widehat{\phi}_n)\mathbf{D}\mathbf{u}_n^\Omega : \mathbf{D}\tilde{\mathbf{v}}_l \,\mathrm{d}x \\
&\quad + \int_\Gamma 2\nu_\Gamma(\widehat{\psi}_n)\mathbf{D}_\Gamma \mathbf{u}_n^\Gamma : \mathbf{D}_\Gamma \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\widehat{\phi}_n, \widehat{\psi}_n)\mathbf{u}_n^\Gamma \cdot \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma \\
&= \int_\Omega \widehat{\mu}_n \nabla \widehat{\phi}_n \cdot \tilde{\mathbf{v}}_l \,\mathrm{d}x + \int_\Gamma \widehat{\theta}_n \nabla_\Gamma \widehat{\psi}_n \cdot \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma \\
&\quad + \frac{\chi(L)}{2}\Big(\beta \frac{\tilde{\sigma}_2 - \tilde{\sigma}_1}{2} - \frac{\tilde{\rho}_2 - \tilde{\rho}_1}{2}\Big) \int_\Gamma (\beta \widehat{\theta}_n - \widehat{\mu}_n)\mathbf{u}_n^\Gamma \cdot \tilde{\mathbf{w}}_l \,\mathrm{d}\Gamma
\end{aligned} \tag{10.32}$$

on $[0,T]$ for all $l = 1, \dots, k$. Now, we notice that, since $\{(\mathbf{u}_n^\Omega, \mathbf{u}_n^\Gamma)\}_{n\in\mathbb{N}} \subset \boldsymbol{\mathcal{M}}$, there exists $(\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma) \in H^1(0,T;\boldsymbol{\mathcal{V}}_k)$ such that

$$(\mathbf{u}_n^\Omega, \mathbf{u}_n^\Gamma) \to (\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma) \qquad \text{weakly in } H^1(0,T;\boldsymbol{\mathcal{V}}_k) \tag{10.33}$$

as $n \to \infty$. In light of the compact embedding $H^1(0,T;\boldsymbol{\mathcal{V}}_k) \hookrightarrow C([0,T];\boldsymbol{\mathcal{V}}_k)$ and (10.3), we readily deduce from (10.33) and by compactness, along a non-relabeled subsequence, that

$$\begin{aligned}
(\mathbf{u}_n^\Omega, \mathbf{u}_n^\Gamma) &\to (\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma) && \text{strongly in } C([0,T];\boldsymbol{\mathcal{V}}_k), && (10.34)\\
(\nabla \mathbf{u}_n^\Omega, \nabla_\Gamma \mathbf{u}_n^\Gamma) &\to (\nabla \mathbf{u}_*^\Omega, \nabla_\Gamma \mathbf{u}_*^\Gamma) && \text{strongly in } C([0,T];\boldsymbol{\mathcal{L}}^2) && (10.35)
\end{aligned}$$

as $n \to \infty$. Now, the convergences (10.33)-(10.35) allow us to pass to the limit $n \to \infty$ in (10.32). As $(\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma)$ further satisfies the initial condition $(\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma)|_{t=0} = \boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0, \mathbf{w}_0)$ a.e. in $\Omega \times \Gamma$, which follows readily from (10.34), we infer that $(\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma)$ is a solution to (10.14)-(10.15). In view of the uniqueness of solutions to (10.14)-(10.15), this implies $(\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma) = \mathbf{\Lambda}(\widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_*)$, which entails

$$\mathbf{\Lambda}(\widehat{\mathbf{v}}_n, \widehat{\mathbf{w}}_n) = (\mathbf{u}_n^\Omega, \mathbf{u}_n^\Gamma) \to (\mathbf{u}_*^\Omega, \mathbf{u}_*^\Gamma) = \mathbf{\Lambda}(\widehat{\mathbf{v}}_*, \widehat{\mathbf{w}}_*) \qquad \text{strongly in } L^2(0,T;\boldsymbol{\mathcal{V}}_k)$$

as $n \to \infty$. This proves that the map $\mathbf{\Lambda}$ is continuous. Finally, we are in a position to apply the Schauder fixed-point theorem and conclude that the map $\mathbf{\Lambda}$ has a fixed-point in $\boldsymbol{\mathcal{M}}$, which gives the existence of the approximate solution $(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k, \mu_k, \theta_k)$ satisfying (10.5)-(10.8) for any $k \in \mathbb{N}$.

Step 2: A priori Estimates for the Approximate Solutions. We first observe that, as $T > 0$ was arbitrarily chosen, the approximate solutions $(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k, \mu_k, \theta_k)$ exist on $(0,\infty)$ for any $k \in \mathbb{N}$. In the following, we denote with the letter C generic positive constants that are independent of $k \in \mathbb{N}$. We begin by establishing a-priori estimates based on an energy estimate for the approximate solutions. To this end, we take $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = (\mathbf{v}_k, \mathbf{w}_k)$ in (10.5) and perform integration by parts. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega \rho(\phi_k)|\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma\Big)$$

$$
\begin{aligned}
&+\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \\
&= -\int_\Omega \phi_k\nabla\mu_k\cdot\mathbf{v}_k \,\mathrm{d}x - \int_\Gamma \psi_k\nabla_\Gamma\theta_k\cdot\mathbf{w}_k \,\mathrm{d}\Gamma
\end{aligned}
\tag{10.36}
$$

a.e. on $(0,\infty)$, see (10.18). Then, taking $(\zeta,\zeta_\Gamma) = (\mu_k,\theta_k)$ in (10.6) and $(\eta,\eta_\Gamma) = -(\partial_t\phi_k,\partial_t\psi_k)$ in (10.9), we find with another application of integration by parts that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla\phi_k|^2 + F(\phi_k)\,\mathrm{d}x + \int_\Gamma \frac{1}{2}|\nabla_\Gamma\psi_k|^2 + G(\psi_k)\,\mathrm{d}\Gamma + \chi(K)\int_\Gamma \frac{1}{2}(\alpha\psi_k-\phi_k)^2\,\mathrm{d}\Gamma\Big) \\
&\quad + \int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma + \chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma \\
&= \int_\Omega \phi_k\nabla\mu_k\cdot\mathbf{v}_k\,\mathrm{d}x + \int_\Gamma \psi_k\nabla_\Gamma\theta_k\cdot\mathbf{w}_k
\end{aligned}
\tag{10.37}
$$

a.e. on $(0,\infty)$. By summming (10.36) and (10.37), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{tot}}(\mathbf{v}_k,\mathbf{w}_k,\phi_k,\psi_k) + \int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2\,\mathrm{d}x \\
&\quad + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2\,\mathrm{d}\Gamma \\
&\quad + \int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma \\
&\quad + \chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma = 0
\end{aligned}
$$

a.e. on $(0,\infty)$. Integrating in time from 0 to t for any $t\in(0,\infty)$, we have

$$
\begin{aligned}
&E_{\mathrm{tot}}(\mathbf{v}_k(t),\mathbf{w}_k(t),\phi_k(t),\psi_k(t)) + \int_0^t\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2\,\mathrm{d}\Gamma\,\mathrm{d}s + \int_0^t\int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad + \int_0^t\int_\Omega m_\Omega(\phi_k)|\nabla\mu_k|^2\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Gamma m_\Gamma(\psi_k)|\nabla_\Gamma\theta_k|^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&\quad + \chi(L)\int_\Gamma(\beta\theta_k-\mu_k)^2\,\mathrm{d}\Gamma\,\mathrm{d}s \\
&= E_{\mathrm{tot}}(\boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0,\mathbf{w}_0),\phi_0,\psi_0)
\end{aligned}
$$

for a.e. $t\in(0,\infty)$. As the initial energy can be estimated as

$$
E_{\mathrm{tot}}(\boldsymbol{\mathcal{P}}_{\boldsymbol{\mathcal{V}}_k}(\mathbf{v}_0,\mathbf{w}_0),\phi_0,\psi_0) \le \frac{\max\{\rho_*,\sigma_*\}}{2}\|(\mathbf{v}_0,\mathbf{w}_0)\|_{\boldsymbol{\mathcal{L}}^2} + E_{\mathrm{free}}(\phi_0,\psi_0),
$$

it follows using $|\phi_k|<1$ a.e. in Q, $|\psi_k|<1$ a.e. on Σ and the bulk-surface Korn inequality that

$$
\|(\mathbf{v}_k,\mathbf{w}_k)\|_{L^\infty(0,\infty;\boldsymbol{\mathcal{L}}^2)} + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{L^2(0,\infty;\boldsymbol{\mathcal{H}}^1)} \le C, \tag{10.38}
$$

$$
\|(\phi_k,\psi_k)\|_{L^\infty(0,\infty;\mathcal{H}^1)} \le C, \tag{10.39}
$$

$$
\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{L^2(0,\infty;\boldsymbol{\mathcal{L}}^2)} + \chi(L)^{\frac{1}{2}}\|\beta\theta_k-\mu_k\|_{L^2(0,\infty;L^2(\Gamma))} \le C. \tag{10.40}
$$

Then, recalling (10.29), we employ the bulk-surface Poincaré inequality to deduce that

$$
\|(\mu_k,\theta_k)\|_{\mathcal{H}^1} \le C(1+\|(\mu_k,\theta_k)\|_{L,\beta}) \tag{10.41}
$$

a.e. on $(0, \infty)$. Then, by Proposition 4.3.5 and another application of the bulk-surface Poincaré inequality, we obtain

$$\begin{aligned} \|(\phi_k, \psi_k)\|_{\mathcal{W}^{2,p}} + \|(F'(\phi_k), G'(\psi_k))\|_{\mathcal{L}^p} &\leq C_p(1 + \|(\mu_k, \theta_k)\|_{\mathcal{H}^1}) \\ &\leq C_p(1 + \|(\mu_k, \theta_k)\|_{L,\beta}) \end{aligned} \tag{10.42}$$

a.e. on $(0, \infty)$ for any $2 \leq p < \infty$. By comparison in (10.6), it holds that

$$\|(\partial_t\phi_k, \partial_t\psi_k)\|_{(\mathcal{H}^1)'} \leq C(\|(\mathbf{v}_k, \mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} + \|(\mu_k, \theta_k)\|_{L,\beta}) \tag{10.43}$$

a.e. on $(0, \infty)$. Here, we have additionally made use of (10.12) and the boundedness of the mobility functions. Lastly, since $K \in (0, \infty]$, it follows from Corollary 4.3.7 in combination with (10.39) and (10.41) that

$$\|(-\Delta\phi_k, -\Delta_\Gamma\psi_k + \alpha\partial_{\mathbf{n}}\phi_k)\|_{\mathcal{L}^2}^2 \leq C(1 + \|(\mu_k, \theta_k)\|_{L,\beta})$$

a.e. on $(0, \infty)$. Thus, by elliptic regularity theory for systems with bulk-surface coupling, we obtain

$$\|(\phi_k, \psi_k)\|_{\mathcal{H}^2}^4 \leq C(1 + \|(\mu_k, \theta_k)\|_{L,\beta}^2) \tag{10.44}$$

a.e. on $(0, \infty)$. Consequently, collecting the above estimates, we infer from (10.38) and (10.40) that

$$\begin{aligned} &\|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^2(0,\infty;(\mathcal{H}^1)')} + \|(\phi_k, \psi_k)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{W}^{2,p})} + \|(\phi_k, \psi_k)\|_{L^4_{\mathrm{uloc}}([0,\infty);\mathcal{H}^2)} \\ &\quad + \|(\mu_k, \theta_k)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^2)} + \|(F'(\phi_k), G'(\psi_k))\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{L}^p)} \leq C_p \end{aligned} \tag{10.45}$$

for any $2 \leq p < \infty$.

Step 3: Higher Order Estimates for the Approximate Solutions. Next, we establish higher-order estimates for the approximate solutions $(\mathbf{v}_k, \mathbf{w}_k, \phi_k, \psi_k, \mu_k, \theta_k)$. Based on the estimates (10.38)-(10.40) and (10.45), we can use Theorem 7.3.3, which entails the uniform estimates

$$\begin{aligned} &\|(\phi_k, \psi_k)\|_{L^\infty(0,\infty;\mathcal{W}^{2,p})} + \|(F'(\phi_k), G'(\psi_k))\|_{L^\infty(0,\infty;\mathcal{L}^p)} \\ &\quad + \|(F''(\phi_k), G''(\psi_k))\|_{L^\infty(0,\infty;\mathcal{L}^p)} \leq K_{1,p} \end{aligned} \tag{10.46}$$

for any $2 \leq p \leq \infty$, as well as

$$\begin{aligned} &\|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^\infty(0,\infty;(\mathcal{H}^1)')} + \|(\partial_t\phi_k, \partial_t\psi_k)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^1)} \\ &\quad + \|(\mu_k, \theta_k)\|_{L^\infty(0,\infty;\mathcal{H}^1)} + \|(\mu_k, \theta_k)\|_{L^2_{\mathrm{uloc}}([0,\infty);\mathcal{H}^3)} \leq K_2. \end{aligned} \tag{10.47}$$

Moreover, as a consequence of (10.46), we have

$$\|(\phi_k, \psi_k)\|_{L^\infty(0,\infty;\mathcal{H}^3)} + \|(F'''(\phi_k), G'''(\psi_k))\|_{L^\infty(0,\infty;\mathcal{L}^p)} \leq K_{3,p} \tag{10.48}$$

for any $2 \leq p < \infty$. Next, we aim to establish higher-order estimates of the approximate velocity fields $(\mathbf{v}_k, \mathbf{w}_k)$. To this end, we take $(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = (\partial_t\mathbf{v}_k, \partial_t\mathbf{w}_k)$ as a test function in (10.5) and find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k, \psi_k)|\mathbf{w}_k|^2 \,\mathrm{d}\Gamma\Big) \\ &\quad + \int_\Omega \rho(\phi_k)|\partial_t\mathbf{v}_k|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_k)|\partial_t\mathbf{w}_k|^2 \,\mathrm{d}\Gamma \end{aligned}$$

$$\begin{aligned}
&= -\int_\Omega \rho(\phi_k)(\mathbf{v}_k \cdot \nabla)\mathbf{v}_k \cdot \partial_t \mathbf{v}_k \, \mathrm{d}x - \int_\Gamma \sigma(\psi_k)(\mathbf{w}_k \cdot \nabla_\Gamma)\mathbf{w}_k \cdot \partial_t \mathbf{w}_k \, \mathrm{d}\Gamma \\
&\quad + \int_\Omega 2\nu_\Omega'(\phi_k)\partial_t\phi_k |\mathbf{D}\mathbf{v}_k|^2 \, \mathrm{d}x + \int_\Gamma 2\nu_\Gamma'(\psi_k)\partial_t\psi_k |\mathbf{D}_\Gamma \mathbf{w}_k|^2 \, \mathrm{d}\Gamma \\
&\quad + \int_\Gamma (\partial_1\gamma(\phi_k,\psi_k)\partial_t\phi_k + \partial_2\gamma(\phi_k,\psi_k)\partial_t\psi_k)|\mathbf{w}_k|^2 \, \mathrm{d}\Gamma \\
&\quad + \frac{\tilde\rho_2 - \tilde\rho_1}{2} \int_\Omega m_\Omega(\phi_k)(\nabla\mu_k \cdot \nabla)\mathbf{v}_k \cdot \partial_t \mathbf{v}_k \, \mathrm{d}x \\
&\quad + \frac{\tilde\sigma_2 - \tilde\sigma_1}{2} \int_\Gamma m_\Gamma(\psi_k)(\nabla_\Gamma\theta_k \cdot \nabla_\Gamma)\mathbf{w}_k \cdot \partial_t \mathbf{w}_k \, \mathrm{d}\Gamma \\
&\quad + \frac{\chi(L)}{2}\Big(\beta\frac{\tilde\sigma_2 - \tilde\sigma_1}{2} - \frac{\tilde\rho_2 - \tilde\rho_1}{2}\Big) \int_\Gamma (\beta\theta_k - \mu_k)\mathbf{w}_k \cdot \partial_t \mathbf{w}_k \, \mathrm{d}\Gamma \\
&\quad - \int_\Omega \phi_k \nabla\mu_k \cdot \partial_t \mathbf{v}_k \, \mathrm{d}x - \int_\Gamma \psi_k \nabla_\Gamma\theta_k \cdot \partial_t \mathbf{w}_k \, \mathrm{d}\Gamma \\
&=: \sum_{j=1}^{10} R_j
\end{aligned} \tag{10.49}$$

a.e. on $(0,\infty)$. We now intend to find suitable bounds for the terms R_j, $j = 1,\dots,10$, by utilizing the estimates (10.46)-(10.48). Let ϖ be a positive constant whose precise value will be determined later. Exploiting (7.1), (4.120), and the bulk-surface Korn inequality, we find

$$\begin{aligned}
|R_1 + R_2| &\le \max\{\rho^*,\sigma^*\} \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^4} \|(\nabla\mathbf{v}_k,\nabla_\Gamma\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^4} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\le \frac{\min\{\rho_*,\sigma_*\}}{8} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2 \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2} \\
&\le \frac{\min\{\rho_*,\sigma_*\}}{8} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + \frac{\varpi}{2}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + C\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^4
\end{aligned} \tag{10.50}$$

a.e. on $(0,\infty)$. Then, in view of the assumptions on the coefficients, we have by the bulk-surface Poincaré inequality, (4.120), (7.1) and (10.43) that

$$\begin{aligned}
|R_3 + R_4 + R_5| &\le C\|\partial_t\phi_k\|_{L^2(\Omega)} \|\mathbf{D}\mathbf{v}_k\|_{\mathbf{L}^4(\Omega)}^2 + C\|\partial_t\psi_k\|_{L^2(\Gamma)} \|\mathbf{D}_\Gamma\mathbf{w}_k\|_{\mathbf{L}^4(\Gamma)}^2 \\
&\qquad + C(\|\partial_t\phi_k\|_{L^2(\Gamma)} + \|\partial_t\psi_k\|_{L^2(\Gamma)})\|\mathbf{w}_k\|_{\mathbf{L}^4(\Gamma)}^2 \\
&\le C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{\mathcal{L}^2} \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1} \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2} \\
&\le \frac{\varpi}{2}\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + C\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2 \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2
\end{aligned} \tag{10.51}$$

a.e. on $(0,\infty)$. Next, using interpolation of Sobolev spaces as in Lemma 2.2.3, (4.120) and (10.41), we get

$$\begin{aligned}
|R_6 + R_7| &\le C\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^\infty} \|(\nabla\mathbf{v}_k,\nabla_\Gamma\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\le \frac{\min\{\rho_*,\sigma_*\}}{8} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2
\end{aligned} \tag{10.52}$$

a.e. on $(0,\infty)$. By the trace theorem in combination with (10.41) we find that

$$\begin{aligned}
|R_8| &\le C\|\partial_{\mathbf{n}}\mu_k\|_{L^4(\Gamma)} \|\mathbf{w}_k\|_{\mathbf{L}^4(\Gamma)} \|\partial_t\mathbf{w}_k\|_{\mathbf{L}^2(\Gamma)} \\
&\le \frac{\min\{\rho_*,\sigma_*\}}{8} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2
\end{aligned} \tag{10.53}$$

a.e. on $(0,\infty)$. Lastly, in view of (10.12), we get

$$\begin{aligned}
|R_9 + R_{10}| &\le \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^2} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\le \frac{\min\{\rho_*,\sigma_*\}}{8} \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^2}^2
\end{aligned} \tag{10.54}$$

a.e. on $(0, \infty)$. Combining the above estimates (10.50)-(10.54), we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2\,\mathrm{d}\Gamma\Big)\\
&\qquad + \frac{\min\{\rho_*,\sigma_*\}}{2}\|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\\
&\qquad \le \varpi\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + C\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 \qquad (10.55)\\
&\qquad + C\Big(\|(\partial_t\phi_k,\partial_t\psi_k)\|_{K,\alpha}^2 + \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2\Big)\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2
\end{aligned}
$$

a.e. on $(0,\infty)$, where $C > 0$ is a constant that depends on ϖ, but is independent of k. We are left to control the term $\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}$. This is done by employing regularity theory for the bulk-surface Stokes system as developed in Section 4.5. Thus, arguing as in Section 4.5, for any $k \in \mathbb{N}$ and a.e. $t \in (0,\infty)$, there exists a pressure pair $(p_k(t), q_k(t)) \in \mathcal{H}^1 \cap \mathcal{L}^2_{(0)}$ such that

$$
\begin{aligned}
&-\operatorname{div}(2\nu_\Omega(\phi_k)\mathbf{D}\mathbf{v}_k) + \nabla p_k\\
&\quad = \mu_k\nabla\phi_k - \rho(\phi_k)\partial_t\mathbf{v}_k - \rho(\phi_k)(\mathbf{v}_k\cdot\nabla)\mathbf{v}_k - (\mathbf{J}_k\cdot\nabla)\mathbf{v}_k && \text{a.e. in } Q,\\
&-\operatorname{div}_\Gamma(2\nu_\Gamma(\psi_k)\mathbf{D}_\Gamma\mathbf{w}_k) + 2\nu_\Omega(\phi_k)\big[\mathbf{D}\mathbf{v}_k\,\mathbf{n}\big]_{\tan} + \nabla_\Gamma q_k + \gamma(\phi_k,\psi_k)\mathbf{w}_k\\
&\quad = \theta_k\nabla_\Gamma\psi_k - \sigma(\psi_k)\partial_t\mathbf{w}_k - \sigma(\psi_k)(\mathbf{w}_k\cdot\nabla_\Gamma)\mathbf{w}_k - (\mathbf{K}_k\cdot\nabla_\Gamma)\mathbf{w}_k\\
&\qquad + \tfrac{1}{2}(\mathbf{J}_k\cdot\mathbf{n})\mathbf{w}_k - \tfrac{1}{2}(\mathbf{J}_{\Gamma k}\cdot\mathbf{n})\mathbf{w}_k && \text{a.e. on } \Sigma.
\end{aligned}
$$

Here, we have used the shorthand notation

$$
\begin{aligned}
\mathbf{J}_k &= -\frac{\tilde\rho_2-\tilde\rho_1}{2}m_\Omega(\phi_k)\nabla\mu_k, \qquad \mathbf{K}_k = -\frac{\tilde\sigma_2-\tilde\sigma_1}{2}m_\Gamma(\psi_k)\nabla_\Gamma\theta_k,\\
\mathbf{J}_{\Gamma k} &= -\beta\frac{\tilde\sigma_2-\tilde\sigma_1}{2}m_\Omega(\phi_k)\nabla\mu_k.
\end{aligned}
$$

Consequently, in view of Theorem 4.5.9, we find that

$$
\begin{aligned}
&\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + \|(p_k,q_k)\|_{\mathcal{H}^1}^2\\
&\quad \le C\Big(\|(\mu_k\nabla\phi_k - \rho(\phi_k)\partial_t\mathbf{v}_k - \rho(\phi_k)(\mathbf{v}_k\cdot\nabla)\mathbf{v}_k - (\mathbf{J}_k\cdot\nabla)\mathbf{v}_k, \theta_k\nabla_\Gamma\psi_k - \sigma(\psi_k)\partial_t\mathbf{w}_k\\
&\qquad\qquad - \sigma(\psi_k)(\mathbf{w}_k\cdot\nabla_\Gamma)\mathbf{w}_k - (\mathbf{K}_k\cdot\nabla_\Gamma)\mathbf{w}_k + \tfrac{1}{2}(\mathbf{J}_k\cdot\mathbf{n})\mathbf{w}_k - \tfrac{1}{2}(\mathbf{J}_{\Gamma k}\cdot\mathbf{n})\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2\Big)\\
&\quad \le C\Big(\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^4}^2\|(\nabla\mathbf{v}_k,\nabla_\Gamma\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^4}^2\\
&\qquad + \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{L}}^\infty}^2\|(\nabla\mathbf{v}_k,\nabla_\Gamma\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|\partial_\mathbf{n}\mu_k\|_{L^4(\Gamma)}^2\|\mathbf{w}_k\|_{\boldsymbol{\mathcal{L}}^4}^2\Big)\\
&\quad \le C\Big(\|(\mu_k,\theta_k)\|_{L,\beta}^2 + \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}\\
&\qquad + \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{H}}^2}^2\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2 + \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{H}}^1}^2\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2\Big)
\end{aligned}
$$

a.e. on $(0,\infty)$. Thus, by Young's inequality, we have

$$
\begin{aligned}
\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + \|(p_k,q_k)\|_{\mathcal{H}^1}^2 &\le C_0\|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|_{\boldsymbol{\mathcal{L}}^2}^2 + C_0\|(\mu_k,\theta_k)\|_{L,\beta}^2 \qquad (10.56)\\
&\quad + C_0\Big(\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|_{\boldsymbol{\mathcal{H}}^2}^2 + \|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2\Big)\|(\mathbf{v}_k,\mathbf{w}_k)\|_{\boldsymbol{\mathcal{H}}_0^1}^2
\end{aligned}
$$

a.e. on $(0,\infty)$ with a constant $C_0 > 0$ independent of $k \in \mathbb{N}$. Let us now set

$$
\varepsilon = \frac{\min\{\rho_*,\sigma_*\}}{4C_0}, \qquad \varpi = \frac{\min\{\rho_*,\sigma_*\}}{8C_0}.
$$

Then, multiplying (10.56) by ε and summing the resulting inequality to (10.55), we deduce the differential inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega 2\nu_\Omega(\phi_k)|\mathbf{D}\mathbf{v}_k|^2\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_k)|\mathbf{D}_\Gamma\mathbf{w}_k|^2\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_k,\psi_k)|\mathbf{w}_k|^2\,\mathrm{d}\Gamma\Big)\\
&\quad + \frac{\min\{\rho_*,\sigma_*\}}{8C_0}\|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^2} + \frac{\min\{\rho_*,\sigma_*\}}{4}\|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{L}}^2}\\
&\quad + \frac{\min\{\rho_*,\sigma_*\}}{4C_0}\|(p_k,q_k)\|^2_{\mathcal{H}^1} \qquad (10.57)\\
&\le C\|(\mu_k,\theta_k)\|^2_{L,\beta} + C\Big(\|(\partial_t\phi_k,\partial_t\psi_k)\|^2_{K,\alpha} + \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|^2_{\boldsymbol{\mathcal{H}}^2} + \|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0}\Big)\\
&\quad\times\|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0}
\end{aligned}$$

a.e. on $(0,\infty)$. Observing that

$$\|(\mathbf{v}_k(0),\mathbf{w}_k(0))\|_{\boldsymbol{\mathcal{H}}^1_0} \le C\|(\mathbf{v}_0,\mathbf{w}_0)\|_{\boldsymbol{\mathcal{H}}^1_0},$$

we infer from the Gronwall lemma together with (10.38) and (10.47) that

$$\sup_{0\le t\le 1}\|(\mathbf{v}_k(t),\mathbf{w}_k(t))\|^2_{\boldsymbol{\mathcal{H}}^1_0} \le C(1+\|(\mathbf{v}_0,\mathbf{w}_0)\|^2_{\boldsymbol{\mathcal{H}}^1_0}). \qquad (10.58)$$

To derive a global control independent of time, we use a uniform variant of the Gronwall lemma as stated in Lemma 2.2.7. Noting on (10.38) and (10.47), which entail

$$\sup_{t\ge0}\int_t^{t+1}\|(\partial_t\phi_k,\partial_t\psi_k)\|^2_{K,\alpha} + \|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|^2_{\boldsymbol{\mathcal{H}}^2} + \|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0}\,\mathrm{d}s \le C_1, \qquad (10.59)$$

$$\sup_{t\ge0}\int_t^{t+1}\|(\nabla\mu_k,\nabla_\Gamma\theta_k)\|^2_{\boldsymbol{\mathcal{H}}^2}\,\mathrm{d}s \le C_2, \qquad (10.60)$$

$$\sup_{t\ge0}\int_t^{t+1}\|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^1_0}\,\mathrm{d}s \le C_3, \qquad (10.61)$$

an application of Lemma 2.2.7 with $r=1$ and $t_0=0$ yields that

$$\sup_{t\ge1}\|(\mathbf{v}_k(t),\mathbf{w}_k(t))\|^2_{\boldsymbol{\mathcal{H}}^1_0} \le (C_3+C_2)\mathbf{e}^{C_1}. \qquad (10.62)$$

Then, for any $t\ge0$, we integrate (10.57) in time from t to $t+1$. Thanks to (10.59)-(10.61), it follows from (10.58) and (10.62) that

$$\sup_{t\ge0}\int_t^{t+1}\|(\mathbf{v}_k,\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{H}}^2} + \|(\partial_t\mathbf{v}_k,\partial_t\mathbf{w}_k)\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(p_k,q_k)\|^2_{\mathcal{H}^1}\,\mathrm{d}s \le C. \qquad (10.63)$$

Step 4: Passage to the Limit and Existence of global Strong Solutions. We are in a position to pass to the limit $k\to\infty$. More precisely, in light of the estimates (10.46)-(10.48), (10.58) and (10.62)-(10.63), the Banach–Alaoglu theorem implies the existence of an octuple $(\mathbf{v},\mathbf{w},p,q,\phi,\psi,\mu,\theta)$, such that, for any $T>0$, it holds

$$\begin{aligned}
(\mathbf{v}_k,\mathbf{w}_k)\to(\mathbf{v},\mathbf{w}) &\quad \text{weakly-star in } L^\infty(0,\infty;\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}), && (10.64a)\\
&\quad \text{weakly in } L^2(0,T;\boldsymbol{\mathcal{H}}^2)\cap H^1(0,T;\boldsymbol{\mathcal{L}}^2), && (10.64b)\\
(p_k,q_k)\to(p,q) &\quad \text{weakly in } L^2(0,T;\mathcal{H}^1\cap\mathcal{L}^2_{(0)}), && (10.64c)\\
(\phi_k,\psi_k)\to(\phi,\psi) &\quad \text{weakly-star in } L^\infty(0,\infty;\mathcal{H}^3), && (10.64d)
\end{aligned}$$

$$
(\mu_k, \theta_k) \to (\mu, \theta) \qquad
\begin{aligned}
&\text{weakly in } H^1(0,T;\mathcal{H}^1), && (10.64\text{e})\\
&\text{weakly-star in } L^\infty(0,\infty;\mathcal{H}^1), && (10.64\text{f})\\
&\text{weakly in } L^2(0,T;\mathcal{H}^3)\cap H^1(0,T;(\mathcal{H}^1)'), && (10.64\text{g})
\end{aligned}
$$

up to a non-relabeled subsequence $k \to \infty$. As a consequence of the Aubin–Lions–Simon lemma, for any $T>0$, we further have

$$
\begin{aligned}
(\mathbf{v}_k, \mathbf{w}_k) &\to (\mathbf{v}, \mathbf{w}) && \text{strongly in } L^2(0,T;\boldsymbol{\mathcal{H}}^1_{0,\mathrm{div}}), && (10.65\text{a})\\
(\phi_k, \psi_k) &\to (\phi, \psi) && \text{strongly in } C([0,T];\mathcal{W}^{2,p}) \text{ for all } p\in[2,\infty), && (10.65\text{b})\\
(\mu_k, \theta_k) &\to (\mu, \theta) && \text{strongly in } L^2(0,T;\mathcal{H}^2), && (10.65\text{c})
\end{aligned}
$$

as $k\to\infty$, after another subsequence extraction. Moreover, by Minty's trick, we have

$$
(F'(\phi_k), G'(\psi_k)) \to (F'(\phi), G'(\psi)) \qquad \text{weakly-star in } L^\infty(0,\infty;\mathcal{L}^p)
$$

for $k\to\infty$ for any $2\le p<\infty$. Furthermore, on account of the assumptions on the coefficients (see **(A3)**-**(A4)**), we infer from (10.65b) that

$$
\begin{aligned}
&\rho(\phi_k)\to\rho(\phi), && m_\Omega(\phi_k)\to m_\Omega(\phi), && \nu_\Omega(\phi_k)\to \nu_\Omega(\phi) && \text{strongly in } C(\overline{\Omega}\times[0,T]),\\
&\sigma(\psi_k)\to\sigma(\psi), && m_\Gamma(\psi_k)\to m_\Gamma(\psi), && \nu_\Gamma(\psi_k)\to \nu_\Gamma(\psi) && \text{strongly in } C(\Gamma\times[0,T]),
\end{aligned}
$$

and

$$
\gamma(\phi_k,\psi_k)\to\gamma(\phi,\psi) \qquad \text{strongly in } C(\Gamma\times[0,T])
$$

for $k\to\infty$ and for any $T>0$. The above properties entail the convergence of the nonlinear terms in (10.5), which allows us to pass to the limit as $k\to\infty$ in (10.5), (10.6), and (10.9) (see, e.g., [123] for the limit in the Galerkin formulation). Lastly, we observe that the pressure pair (p,q) satisfies

$$
\nabla p = \rho(\phi)\partial_t\mathbf{v} + \rho(\phi)(\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{J}\cdot\nabla)\mathbf{v} - \mathrm{div}(2\nu_\Omega(\phi)\mathbf{D}\mathbf{v}) - \mu\nabla\phi \qquad \text{a.e. in } Q
$$

and

$$
\begin{aligned}
\nabla_\Gamma q = {}& \sigma(\psi)\partial_t\mathbf{w} + \sigma(\psi)(\mathbf{w}\cdot\nabla_\Gamma)\mathbf{w} + (\mathbf{K}\cdot\nabla_\Gamma)\mathbf{w} - \mathrm{div}_\Gamma(2\nu_\Gamma(\psi)\mathbf{D}_\Gamma\mathbf{w}) - \theta\nabla_\Gamma\psi\\
&+ 2\nu_\Omega(\phi)[\mathbf{D}\mathbf{v}\,\mathbf{n}]_{\tan} + \frac{1}{2}(\mathbf{J}\cdot\mathbf{n})\mathbf{w} - \frac{1}{2}(\mathbf{J}_\Gamma\cdot\mathbf{n})\mathbf{w} + \gamma(\phi,\psi)\mathbf{w} \qquad \text{a.e. on } \Sigma,
\end{aligned}
$$

which finishes the proof. □

10.4 Uniqueness of strong solutions

In this section, we prove Theorem 10.2.3. Namely, we show the uniqueness of strong solutions and their continuous dependence on the initial data.

Proof of Theorem 10.2.3. Let $(\mathbf{v}_0^1, \mathbf{w}_0^1, \phi_0^1, \psi_0^1)$ and $(\mathbf{v}_0^2, \mathbf{w}_0^2, \phi_0^2, \psi_0^2)$ be admissible initial data, and consider two global strong solutions $(\mathbf{v}_1, \mathbf{w}_1, p_1, q_1, \phi_1, \psi_1, \mu_1, \theta_1)$ and $(\mathbf{v}_2, \mathbf{w}_2, p_2, q_2, \phi_2, \psi_2, \mu_2, \theta_2)$ originating from $(\mathbf{v}_0^1, \mathbf{w}_0^1, \phi_0^1, \psi_0^1)$ and $(\mathbf{v}_0^2, \mathbf{w}_0^2, \phi_0^2, \psi_0^2)$, respectively. We set

$$
\begin{aligned}
(\mathbf{v}, \mathbf{w}, p, q) &\coloneqq (\mathbf{v}_1-\mathbf{v}_2, \mathbf{w}_1-\mathbf{w}_2, p_1-p_2, q_1-q_2),\\
(\phi, \psi, \mu, \theta) &\coloneqq (\phi_1-\phi_2, \psi_1-\psi_2, \mu_1-\mu_2, \theta_1-\theta_2).
\end{aligned}
$$

It is clear that

$$\begin{aligned}
&\rho(\phi_1)\partial_t \mathbf{v} + \big(\rho(\phi_1) - \rho(\phi_2)\big)\partial_t \mathbf{v}_2 + \rho(\phi_1)(\mathbf{v}\cdot\nabla)\mathbf{v}_2 \\
&\qquad - \big(\rho(\phi_1) - \rho(\phi_2)\big)(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2 - \frac{\tilde\rho_1 - \tilde\rho_2}{2} m_\Omega(\phi_1)(\nabla\mu\cdot\nabla)\mathbf{v}_2 \\
&\qquad - \frac{\tilde\rho_1 - \tilde\rho_2}{2}\big(m_\Omega(\phi_1) - m_\Omega(\phi_2)\big)(\nabla\mu_2\cdot\nabla)\mathbf{v}_2 - \operatorname{div}(2\nu_\Omega(\phi_1)\mathbf{D}\mathbf{v}) \\
&\qquad - \operatorname{div}\big(2(\nu_\Omega(\phi_1) - \nu_\Omega(\phi_2))\mathbf{D}\mathbf{v}_2\big) + \nabla p \\
&\quad = \mu\nabla\phi_1 + \mu_2\nabla\phi \qquad\qquad \text{a.e. in } Q, \qquad (10.66\text{a})
\end{aligned}$$

$$\begin{aligned}
&\sigma(\psi_1)\partial_t \mathbf{w} + \big(\sigma(\psi_1) - \sigma(\psi_2)\big)\partial_t \mathbf{w}_2 + \sigma(\psi_1)(\mathbf{w}\cdot\nabla_\Gamma)\mathbf{w}_2 \\
&\qquad - \big(\sigma(\psi_1) - \sigma(\psi_2)\big)(\mathbf{w}_2\cdot\nabla_\Gamma)\mathbf{w}_2 - \frac{\tilde\sigma_1 - \tilde\sigma_2}{2} m_\Gamma(\psi_1)(\nabla_\Gamma\theta\cdot\nabla_\Gamma)\mathbf{w}_2 \\
&\qquad - \frac{\tilde\sigma_1 - \tilde\sigma_2}{2}\big(m_\Gamma(\psi_1) - m_\Gamma(\psi_2)\big)(\nabla_\Gamma\theta_2\cdot\nabla_\Gamma)\mathbf{w}_2 - \operatorname{div}_\Gamma(2\nu_\Gamma(\psi_1)\mathbf{D}_\Gamma\mathbf{w}) \\
&\qquad - \operatorname{div}_\Gamma\big(2(\nu_\Gamma(\psi_1) - \nu_\Gamma(\psi_2))\mathbf{D}_\Gamma\mathbf{w}_2\big) + \nabla_\Gamma q \\
&\quad = \theta\nabla_\Gamma\psi_1 + \theta_2\nabla_\Gamma\psi - 2\nu_\Omega(\phi_1)\mathbf{D}\mathbf{v}\,\mathbf{n} - (2\nu_\Omega(\phi_1) - 2\nu_\Omega(\phi_2))\mathbf{D}\mathbf{v}_2\,\mathbf{n} \\
&\qquad + \Big(\beta\frac{\tilde\sigma_1 - \tilde\sigma_2}{2} - \frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big) m_\Omega(\phi_1)\partial_{\mathbf{n}}\mu_1 \mathbf{w} \\
&\qquad + \Big(\beta\frac{\tilde\sigma_1 - \tilde\sigma_2}{2} - \frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big)\big(m_\Omega(\phi_1)\partial_{\mathbf{n}}\mu_1 - m_\Omega(\phi_2)\partial_{\mathbf{n}}\mu_2\big)\mathbf{w}_2 \\
&\qquad - \gamma(\phi_1,\psi_1)\mathbf{w} + \big(\gamma(\phi_1,\psi_1) - \gamma(\phi_2,\psi_2)\big)\mathbf{w}_2 \qquad\qquad \text{a.e. on } \Sigma, \qquad (10.66\text{b})
\end{aligned}$$

as well as

$$\begin{aligned}
&\partial_t\phi + \mathbf{v}_1\cdot\nabla\phi + \mathbf{v}\cdot\nabla\phi_2 \\
&\quad = \operatorname{div}(m_\Omega(\phi_1)\nabla\mu) + \operatorname{div}\big((m_\Omega(\phi_1) - m_\Omega(\phi_2))\nabla\mu_2\big) && \text{a.e. in } Q, && (10.66\text{c}) \\
&\mu = -\Delta\phi + F'(\phi_1) - F'(\phi_2) && \text{a.e. in } Q, && (10.66\text{d}) \\
&\partial_t\psi + \mathbf{w}_1\cdot\nabla_\Gamma\psi + \mathbf{w}\cdot\nabla_\Gamma\psi_2 \\
&\quad = \operatorname{div}_\Gamma(m_\Gamma(\psi_1)\nabla_\Gamma\theta) + \operatorname{div}_\Gamma\big((m_\Gamma(\psi_1) - m_\Gamma(\psi_2))\nabla_\Gamma\theta_2\big) \\
&\qquad - \beta m_\Omega(\phi_1)\partial_{\mathbf{n}}\mu - \beta\big(m_\Omega(\phi_1) - m_\Omega(\phi_2)\big)\partial_{\mathbf{n}}\mu_2 && \text{a.e. on } \Sigma, && (10.66\text{e}) \\
&\theta = -\Delta_\Gamma\psi + G'(\psi_1) - G'(\psi_2) + \alpha\partial_{\mathbf{n}}\phi && \text{a.e. on } \Sigma. && (10.66\text{f})
\end{aligned}$$

Then, we multiply (10.66a) and (10.66b) by $\mathbf{v}$ and $\mathbf{w}$, respectively, and integrate the respective equations over Ω and Γ. Adding the resulting equations leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega \rho(\phi_1)|\mathbf{v}|^2\,\mathrm{d}x + \int_\Gamma \sigma(\psi_1)|\mathbf{w}|^2\,\mathrm{d}\Gamma\Big) \\
&\quad + \int_\Omega 2\nu_\Omega(\phi_1)|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_1)|\mathbf{D}_\Gamma\mathbf{w}|^2\,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_1,\psi_1)|\mathbf{w}|^2\,\mathrm{d}\Gamma \\
&= -\int_\Omega \big(\rho(\phi_1) - \rho(\phi_2)\big)\partial_t\mathbf{v}_2\cdot\mathbf{v}\,\mathrm{d}x - \int_\Gamma \big(\sigma(\psi_1) - \sigma(\psi_2)\big)\partial_t\mathbf{w}_2\cdot\mathbf{w}\,\mathrm{d}\Gamma \\
&\quad - \int_\Omega \rho(\phi_1)(\mathbf{v}\cdot\nabla)\mathbf{v}_2\cdot\mathbf{v}\,\mathrm{d}x - \int_\Gamma \sigma(\psi_1)(\mathbf{w}\cdot\nabla_\Gamma)\mathbf{w}_2\cdot\mathbf{w}\,\mathrm{d}\Gamma \\
&\quad - \int_\Omega \big(\rho(\phi_1) - \rho(\phi_2)\big)(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2\cdot\mathbf{v}\,\mathrm{d}x - \int_\Gamma \big(\sigma(\psi_1) - \sigma(\psi_2)\big)(\mathbf{w}_2\cdot\nabla_\Gamma)\mathbf{w}_2\cdot\mathbf{w}\,\mathrm{d}\Gamma \\
&\quad + \frac{\tilde\rho_1 - \tilde\rho_2}{2}\int_\Omega m_\Omega(\phi_1)(\nabla\mu\cdot\nabla)\mathbf{v}_2\cdot\mathbf{v}\,\mathrm{d}x + \frac{\tilde\sigma_1 - \tilde\sigma_2}{2}\int_\Gamma m_\Gamma(\psi_1)(\nabla_\Gamma\theta\cdot\nabla_\Gamma)\mathbf{w}_2\cdot\mathbf{w}\,\mathrm{d}\Gamma \\
&\quad + \frac{\chi(L)}{2}\Big(\beta\frac{\tilde\sigma_1 - \tilde\sigma_2}{2} - \frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big)\int_\Gamma (\beta\theta - \mu)\mathbf{w}_2\cdot\mathbf{w}\,\mathrm{d}\Gamma \qquad\qquad (10.67)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\tilde\rho_1 - \tilde\rho_2}{2} \int_\Omega (m_\Omega(\phi_1) - m_\Omega(\phi_2))(\nabla\mu_2 \cdot \nabla)\mathbf{v}_2 \cdot \mathbf{v} \,\mathrm{d}x \\
&\quad + \frac{\tilde\sigma_1 - \tilde\sigma_2}{2} \int_\Gamma (m_\Gamma(\psi_1) - m_\Gamma(\psi_2))(\nabla_\Gamma\theta_2 \cdot \nabla_\Gamma)\mathbf{w}_2 \cdot \mathbf{w} \,\mathrm{d}\Gamma \\
&\quad + \int_\Omega 2(\nu_\Omega(\phi_1) - \nu_\Omega(\phi_2))\mathbf{D}\mathbf{v}_2 : \mathbf{D}\mathbf{v} \,\mathrm{d}x + \int_\Gamma 2(\nu_\Gamma(\psi_1) - \nu_\Gamma(\psi_2))\mathbf{D}_\Gamma\mathbf{w}_2 : \mathbf{D}_\Gamma\mathbf{w} \,\mathrm{d}\Gamma \\
&\quad + \int_\Gamma (\gamma(\phi_1,\psi_1) - \gamma(\phi_2,\psi_2))\mathbf{w}_2 \cdot \mathbf{w} \,\mathrm{d}\Gamma + \int_\Omega (\mu_2\nabla\phi + \mu\nabla\phi_1) \cdot \mathbf{v} \,\mathrm{d}x \\
&\quad + \int_\Gamma (\theta_2\nabla_\Gamma\psi + \theta\nabla_\Gamma\psi_1) \cdot \mathbf{w} \,\mathrm{d}\Gamma \\
&=: \sum_{j=1}^{16} W_j
\end{aligned}
$$

a.e. on $(0,\infty)$. Here, we made use of

$$
\begin{aligned}
&- \frac{1}{2}\int_\Omega \partial_t\rho(\phi_1)|\mathbf{v}|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Gamma \partial_t\sigma(\psi_1)|\mathbf{w}|^2 \,\mathrm{d}\Gamma \\
&\qquad + \frac{1}{2}\int_\Omega \rho(\phi_1)\mathbf{v}_1 \cdot \nabla|\mathbf{v}|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Gamma \sigma(\psi_1)\mathbf{w}_1 \cdot \nabla_\Gamma|\mathbf{w}|^2 \,\mathrm{d}\Gamma \\
&\qquad - \frac{1}{2}\frac{\tilde\rho_1 - \tilde\rho_2}{2}\int_\Omega m_\Omega(\phi_1)\nabla\mu_1 \cdot \nabla|\mathbf{v}|^2 \,\mathrm{d}x - \frac{1}{2}\frac{\tilde\sigma_1 - \tilde\sigma_2}{2}\int_\Gamma m_\Gamma(\psi_1)\nabla_\Gamma\theta_1 \cdot \nabla_\Gamma|\mathbf{w}|^2 \,\mathrm{d}\Gamma \\
&\qquad - \frac{\chi(L)}{2}\Big(\beta\frac{\tilde\sigma_1 - \tilde\sigma_2}{2} - \frac{\tilde\rho_1 - \tilde\rho_2}{2}\Big)\int_\Gamma (\beta\theta_1 - \mu_1)|\mathbf{w}|^2 \,\mathrm{d}\Gamma \\
&\qquad = 0
\end{aligned}
$$

a.e. on $(0,\infty)$, which follows from the same considerations as in the proof of Theorem 9.3.3 (see also (9.3)). Next, noting on

$$
L m_\Omega(\phi_1)\partial_{\mathbf{n}}\mu = \beta\theta - \mu + \Big(1 - \frac{m_\Omega(\phi_1)}{m_\Omega(\phi_2)}\Big)(\beta\theta_2 - \mu_2) \qquad \text{a.e. on } \Sigma
$$

for $L \in (0,\infty)$, a straightforward calculation shows in view of (10.66c)-(10.66f) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{K,\alpha}^2 + \int_\Omega m_\Omega(\phi_1)|\nabla\mu|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_1)|\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma \\
&\qquad + \chi(L)\int_\Gamma (\beta\theta - \mu)^2 \,\mathrm{d}\Gamma \\
&\qquad = -\big\langle(\partial_t\phi,\partial_t\psi),(F'(\phi_1) - F'(\phi_2), G'(\psi_1) - G'(\psi_2))\big\rangle_{\mathcal{H}^1} \\
&\qquad - \int_\Omega (m_\Omega(\phi_1) - m_\Omega(\phi_2))\nabla\mu_2 \cdot \nabla\mu \,\mathrm{d}x - \int_\Gamma (m_\Gamma(\psi_1) - m_\Gamma(\psi_2))\nabla_\Gamma\theta_2 \cdot \nabla_\Gamma\theta \,\mathrm{d}\Gamma \\
&\qquad + \int_\Omega (\phi_1\mathbf{v} + \phi\mathbf{v}_2) \cdot \nabla\mu \,\mathrm{d}x + \int_\Gamma (\psi_1\mathbf{w} + \psi\mathbf{w}_2) \cdot \nabla_\Gamma\theta \,\mathrm{d}\Gamma \\
&\qquad =: \sum_{j=17}^{21} W_j
\end{aligned} \tag{10.68}
$$

a.e. on $(0,\infty)$. Hence, summing (10.67) and (10.68), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\Big(\int_\Omega \rho(\phi_1)|\mathbf{v}|^2 \,\mathrm{d}x + \int_\Gamma \sigma(\psi_1)|\mathbf{w}|^2 \,\mathrm{d}\Gamma + \|(\phi,\psi)\|_{K,\alpha}^2\Big) \\
&\qquad + \int_\Omega m_\Omega(\phi_1)|\nabla\mu|^2 \,\mathrm{d}x + \int_\Gamma m_\Gamma(\psi_1)|\nabla_\Gamma\theta|^2 \,\mathrm{d}\Gamma + \chi(L)\int_\Gamma (\beta\theta - \mu)^2 \,\mathrm{d}\Gamma
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_\Omega 2\nu_\Omega(\phi_1)|\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x + \int_\Gamma 2\nu_\Gamma(\psi_1)|\mathbf{D}_\Gamma\mathbf{w}|^2 \,\mathrm{d}\Gamma + \int_\Gamma \gamma(\phi_1,\psi_1)|\mathbf{w}|^2 \,\mathrm{d}\Gamma \\
&= \sum_{j=1}^{21} W_j
\end{aligned}
$$

a.e. on $(0,\infty)$. We recall from the proof of Theorem 7.3.1 that

$$
\|(\phi,\psi)\|_{\mathcal{H}^1}^2 \le C\Big(\|(\phi,\psi)\|_{K,\alpha}^2 + \|(\phi,\psi)\|_{L,\beta,*}^2\Big) \tag{10.69}
$$

a.e. on $(0,\infty)$, see (7.75). To be precise, the second term on the right-hand side of (10.69) only appears in the case $K=\infty$ (and if $L\in(0,\infty)$). Indeed, if $K\in(0,\infty)$, the bulk-surface Poincaré inequality readily yields the claimed estimate. We further recall the differential inequality (7.76), which now reads as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|(\phi,\psi)\|_{L,\beta,\phi_1,\psi_1,*}^2 \le C\Big(\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + C\|(\phi,\psi)\|_{L,\beta,*}^2\Big) \tag{10.70}
$$

a.e. on $(0,\infty)$. To control the terms W_j, $j=1,\dots,21$, we start by exploiting the bulk-surface Korn inequality as well as the regularity of strong solutions to obtain

$$
\begin{aligned}
|W_1+W_2| &\le C\|(\phi,\psi)\|_{\mathcal{L}^6}\|(\partial_t\mathbf{v}_2,\partial_t\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^3} \\
&\le \frac{\min\{2\nu_*,\gamma_*\}}{8}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}_0^1}^2 + C\|(\partial_t\mathbf{v}_2,\partial_t\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^2}^2\|(\phi,\psi)\|_{\mathcal{H}^1}^2,
\end{aligned} \tag{10.71}
$$

$$
\begin{aligned}
|W_3+W_4| &\le C\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^3}\|(\nabla\mathbf{v}_2,\nabla_\Gamma\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^6}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \\
&\le \frac{\min\{2\nu_*,\gamma_*\}}{8}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}_0^1}^2 + C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{H}}^2}^2\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2,
\end{aligned} \tag{10.72}
$$

$$
\begin{aligned}
|W_5+W_6| &\le C\|(\phi,\psi)\|_{\mathcal{L}^6}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^6}\|(\nabla\mathbf{v}_2,\nabla_\Gamma\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^6}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \\
&\le C\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{H}}^2}^2\Big(\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\phi,\psi)\|_{\mathcal{H}^1}^2\Big)
\end{aligned} \tag{10.73}
$$

a.e. on $(0,\infty)$. Then, using additionally (7.1), we find that

$$
\begin{aligned}
&|W_7+W_8+W_9| \\
&\quad\le C\|(m_\Omega(\phi_1)\nabla\mu, m_\Gamma(\psi_1)\nabla_\Gamma\theta)\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla\mathbf{v}_2,\nabla_\Gamma\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^4}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^4} \\
&\qquad + C\|\beta\theta-\mu\|_{L^2(\Gamma)}\|\mathbf{w}_2\|_{\mathbf{L}^4(\Gamma)}\|\mathbf{w}\|_{\mathbf{L}^4(\Gamma)} \\
&\quad\le \frac{\min\{1,m_*\}}{10}\|(\mu,\theta)\|_{L,\beta}^2 + C\|(\nabla\mathbf{v}_2,\nabla_\Gamma\mathbf{w}_2)\|_{\boldsymbol{\mathcal{H}}^1}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}_0^1} \qquad (10.74) \\
&\qquad + C\|\mathbf{w}\|_{\mathbf{L}^2(\Gamma)}\|\mathbf{w}\|_{\mathbf{H}^1(\Gamma)} \\
&\quad\le \frac{\min\{1,m_*\}}{10}\|(\mu,\theta)\|_{L,\beta}^2 + \frac{\min\{2\nu_*,\gamma_*\}}{8}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{H}}_0^1}^2 \qquad (10.75) \\
&\qquad + C\big(1+\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{H}}^2}^2\big)\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2
\end{aligned}
$$

a.e. on $(0,\infty)$. Moreover, we have

$$
\begin{aligned}
&|W_{10}+W_{11}| \\
&\quad\le C\|(\phi,\psi)\|_{\mathcal{L}^6}\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\boldsymbol{\mathcal{L}}^6}\|(\nabla\mathbf{v}_2,\nabla_\Gamma\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^6}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad\le C\Big(\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\boldsymbol{\mathcal{H}}^1}^2 + \|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{H}}^2}^2\Big)\Big(\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\phi,\psi)\|_{\mathcal{H}^1}^2\Big)
\end{aligned} \tag{10.76}
$$

a.e. on $(0,\infty)$, and by the trace theorem it holds

$$
|W_{12}+W_{13}+W_{14}|
$$

$$
\begin{aligned}
&\le C\|(\phi,\psi)\|_{\mathcal{L}^6}\|(\mathbf{D}\mathbf{v}_2,\mathbf{D}_\Gamma\mathbf{w}_2)\|_{\mathcal{L}^3}\|(\mathbf{D}\mathbf{v},\mathbf{D}_\Gamma\mathbf{w})\|_{\mathcal{L}^2}\\
&\quad + C(\|\phi\|_{L^4(\Gamma)} + \|\psi\|_{L^4(\Gamma)})\|\mathbf{w}_2\|_{\mathbf{L}^4(\Gamma)}\|\mathbf{w}\|_{\mathbf{L}^2(\Gamma)}\\
&\le \frac{\min\{2\nu_*,\gamma_*\}}{8}\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{H}}^1_0} + C\|(\phi,\psi)\|^2_{\mathcal{H}^1}
\end{aligned} \tag{10.77}
$$

a.e. on $(0,\infty)$. Furthermore, it holds that

$$
\begin{aligned}
&|W_{15}+W_{16}|\\
&\quad\le \|(\phi,\psi)\|_{\mathcal{L}^4}\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\boldsymbol{\mathcal{L}}^4} + \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla\mu,\nabla_\Gamma\theta)\|_{\boldsymbol{\mathcal{L}}^2}\\
&\quad\le \frac{\min\{1,m_*\}}{10}\|(\mu,\theta)\|^2_{L,\beta} + C\Big(1 + \|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|^2_{\boldsymbol{\mathcal{H}}^1}\Big)\\
&\qquad\quad \times\Big(\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|^2_{\mathcal{H}^1}\Big)
\end{aligned} \tag{10.78}
$$

a.e. on $(0,\infty)$. Next,

$$
|W_{17}| \le \|(\partial_t\phi,\partial_t\psi)\|_{(\mathcal{H}^1)}\|(F'(\phi_1)-F'(\phi_2),G'(\psi_1)-G'(\psi_2))\|_{\mathcal{H}^1} \tag{10.79}
$$

a.e. on $(0,\infty)$. By means of a comparison argument in (10.66c)-(10.66e), we observe that

$$
\begin{aligned}
&\|(\partial_t\phi,\partial_t\psi)\|_{(\mathcal{H}^1)'}\\
&= \sup_{\|(\zeta,\zeta_\Gamma)\|_{\mathcal{H}^1}\le 1} |\langle(\partial_t\phi,\partial_t\psi),(\zeta,\zeta_\Gamma)\rangle_{\mathcal{H}^1}|\\
&\le C(\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|_{\mathcal{L}^4}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^4} + \|(\mu,\theta)\|_{L,\beta}\\
&\qquad + \|(\phi,\psi)\|_{\mathcal{L}^4}\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\boldsymbol{\mathcal{L}}^4})
\end{aligned} \tag{10.80}
$$

a.e. on $(0,\infty)$. For the second term on the right-hand side of (10.79), we recall the existence of $\delta\in(0,1]$ such that

$$
|\phi_j| \le 1-\delta \qquad \text{a.e. in } Q, \qquad |\psi_j| \le 1-\delta \qquad \text{a.e. on } \Sigma \tag{10.81}
$$

for $j=1,2$, see Theorem 7.3.7. In particular, for any $s\in[0,1]$, it holds that

$$
|s\phi_1 + (1-s)\phi_2| \le s(1-\delta) + (1-s)(1-\delta) = 1-\delta \qquad \text{a.e. in } Q.
$$

Similarly, we find that

$$
|s\psi_1 + (1-s)\psi_2| \le 1-\delta \qquad \text{a.e. on } \Sigma
$$

for any $s\in[0,1]$. Noting on the decomposition $F(s) = F_0(s) - \frac{c_F}{2}s^2$ with $F_0 \in C^3(-1,1)$, we use the fundamental theorem of calculus to infer

$$
\begin{aligned}
&\|F'(\phi_1)-F'(\phi_2)\|^2_{H^1(\Omega)}\\
&\quad\le \int_\Omega |F_0'(\phi_1)-F_0'(\phi_2)|^2 \,\mathrm{d}x + \int_\Omega |F_0''(\phi_1)|^2|\nabla\phi|^2\,\mathrm{d}x\\
&\qquad + \int_\Omega |F_0''(\phi_1)-F_0''(\phi_2)|^2|\nabla\phi_2|^2\,\mathrm{d}x + c_F^2\int_\Omega \phi^2\,\mathrm{d}x + c_F^2\int_\Omega|\nabla\phi|^2\,\mathrm{d}x\\
&\quad\le \int_\Omega\Big|\int_0^1 F_0''(s\phi_1+(1-s)\phi_2)^2\,\mathrm{d}s\Big|^2\phi^2\,\mathrm{d}x + C\|\nabla\phi\|^2_{\mathbf{L}^2(\Omega)}\\
&\qquad + \int_\Omega\Big|\int_0^1 F_0'''(s\phi_1+(1-s)\phi_2)^2\,\mathrm{d}s\Big|^2\phi^2|\nabla\phi_2|^2\,\mathrm{d}x + c_F^2\|\phi\|^2_{L^2(\Omega)} + c_F^2\|\nabla\phi\|^2_{\mathbf{L}^2(\Omega)}\\
&\quad\le \sup_{s\in[-1+\delta,1-\delta]}|F_1''(s)|^2\|\phi\|^2_{L^2(\Omega)} + C\|\phi\|^2_{H^1(\Omega)}
\end{aligned} \tag{10.82}
$$

$$
\begin{aligned}
&+ \sup_{s\in[-1+\delta,1-\delta]} |F_1'''(s)|^2 \|\phi\|^2_{L^4(\Omega)} \|\nabla\phi_2\|^2_{\mathbf{L}^4(\Omega)} \\
&\le C\|(\phi,\psi)\|^2_{\mathcal{H}^1}
\end{aligned}
$$

a.e. on $(0,\infty)$. Performing similar computations on the boundary, we deduce

$$
\|G'(\psi_1) - G'(\psi_2)\|^2_{H^1(\Gamma)} \le C\|(\phi,\psi)\|^2_{\mathcal{H}^1} \tag{10.83}
$$

a.e. on $(0,\infty)$. Combining (10.82) and (10.83) leads to

$$
\|(F'(\phi_1) - F'(\phi_2), G'(\psi_1) - G'(\psi_2))\|_{\mathcal{H}^1} \le C\|(\phi,\psi)\|_{\mathcal{H}^1} \tag{10.84}
$$

a.e. on $(0,\infty)$. Thus, going back to (10.79), the estimates (10.80) and (10.84) entail that

$$
\begin{aligned}
|W_{17}| &\le C\Big(\|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|_{\mathcal{L}^4}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^4} + \|(\mu,\theta)\|_{L,\beta,\phi_1,\psi_1} \\
&\quad + \|(\phi,\psi)\|_{\mathcal{L}^4}\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\boldsymbol{\mathcal{L}}^4}\Big)\|(\phi,\psi)\|_{\mathcal{H}^1} \\
&\le \frac{\min\{1,m_*\}}{10}\|(\mu,\theta)\|^2_{L,\beta} + C(1 + \|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|^2_{\boldsymbol{\mathcal{H}}^1}) \\
&\qquad \times \left(\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|^2_{\mathcal{H}^1}\right)
\end{aligned} \tag{10.85}
$$

a.e. on $(0,\infty)$. Lastly, for the remaining terms, we have

$$
\begin{aligned}
&|W_{18} + W_{19}| \\
&\quad \le C\|(\phi,\psi)\|_{\mathcal{L}^4}\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|_{\boldsymbol{\mathcal{L}}^4}\|(\nabla\mu,\nabla_\Gamma\theta)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad \le \frac{\min\{1,m_*\}}{10}\|(\mu,\theta)\|^2_{L,\beta} + C\|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|^2_{\boldsymbol{\mathcal{H}}^1}\|(\phi,\psi)\|^2_{\mathcal{H}^1}
\end{aligned} \tag{10.86}
$$

a.e. on $(0,\infty)$, and

$$
\begin{aligned}
&|W_{20} + W_{21}| \\
&\quad \le \|(\mathbf{v},\mathbf{w})\|_{\boldsymbol{\mathcal{L}}^2}\|(\nabla\mu,\nabla_\Gamma\theta)\|_{\boldsymbol{\mathcal{L}}^2} + C\|(\phi,\psi)\|_{\mathcal{L}^4}\|(\mathbf{v}_2,\mathbf{w}_2)\|_{\boldsymbol{\mathcal{L}}^4}\|(\nabla\mu,\nabla_\Gamma\theta)\|_{\boldsymbol{\mathcal{L}}^2} \\
&\quad \le \frac{\min\{1,m_*\}}{10}\|(\mu,\theta)\|^2_{L,\beta} + C(1 + \|(\mathbf{v}_2,\mathbf{w}_2)\|^2_{\boldsymbol{\mathcal{H}}^1}) \\
&\qquad \times \left(\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|^2_{\mathcal{H}^1}\right)
\end{aligned} \tag{10.87}
$$

a.e. on $(0,\infty)$. Therefore, by (10.70), (10.71)-(10.78) and (10.85)-(10.87), we obtain the differential inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\bigg(\int_\Omega \rho(\phi_1)|\mathbf{v}|^2\,\mathrm{d}x + \int_\Gamma \sigma(\psi_1)|\mathbf{w}|^2\,\mathrm{d}\Gamma + \|(\phi,\psi)\|^2_{K,\alpha} + \|(\phi,\psi)\|^2_{L,\beta,\phi_1,\psi_1,*}\bigg) \\
&\quad + \frac{\min\{1,m_*\}}{2}\|(\mu,\theta)\|^2_{L,\beta} + \frac{\min\{2\nu_*,\gamma_*\}}{2}\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{H}}^1_0} \\
&\le C\Big(1 + \|(\partial_t\mathbf{v}_2,\partial_t\mathbf{w}_2)\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\mathbf{v}_2,\mathbf{w}_2)\|^2_{\boldsymbol{\mathcal{H}}^2} + \|(\nabla\mu_2,\nabla_\Gamma\theta_2)\|^2_{\boldsymbol{\mathcal{H}}^1}\Big) \\
&\qquad \times \left(\|(\mathbf{v},\mathbf{w})\|^2_{\boldsymbol{\mathcal{L}}^2} + \|(\phi,\psi)\|^2_{\mathcal{H}^1} + \|(\phi,\psi)\|^2_{L,\beta,\phi_1,\psi_1,*}\right).
\end{aligned}
$$

a.e. on $(0,\infty)$. Noting on $\rho \ge \rho_* > 0$ a.e. in Q and $\sigma \ge \sigma_* > 0$ a.e. on Σ, as well as on (10.69), we deduce with the help of the Gronwall lemma that, for any $T > 0$,

$$
\|(\mathbf{v}(t),\mathbf{w}(t))\|_{\boldsymbol{\mathcal{L}}^2} + \|(\phi(t),\psi(t))\|_{\mathcal{H}^1}
$$

$$
\begin{aligned}
&\le C\Big(\|(\mathbf{v}(0), \mathbf{w}(0))\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\phi(0), \psi(0))\|_{\mathcal{H}^1}^2\Big) \\
&\quad \times \exp\left(C\int_0^T 1 + \|(\partial_t \mathbf{v}_2(r), \partial_t \mathbf{w}_2(r))\|_{\boldsymbol{\mathcal{L}}^2}^2 + \|(\mathbf{v}_2(r), \mathbf{w}_2(r))\|_{\boldsymbol{\mathcal{H}}^2}^2 \right. \\
&\qquad\quad \left. + \|(\nabla\mu_2(r), \nabla_\Gamma\theta_2(r))\|_{\boldsymbol{\mathcal{H}}^1}^2 \,\mathrm{d}r\right)
\end{aligned}
$$

for all $t \in [0, T]$, which implies the uniqueness of strong solutions. $\square$

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