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Global 2-rings and genuine refinements

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David Gepner^{id}, Sil Linskens^{id} and Luca Pol

ABSTRACT

We introduce the notion of a naive global 2-ring: a functor from the opposite of the ∞ -category of global spaces to presentably symmetric monoidal stable ∞ -categories. By passing to global sections, every naive global 2-ring decategorifies to a multiplicative cohomology theory on global spaces, i.e. a naive global ring. We suggest when a naive global 2-ring deserves to be called *genuine*. As evidence, we associate to such a global 2-ring a family of equivariant cohomology theories which satisfy a version of the change-of-group axioms introduced by Ginzburg, Kapranov, and Vasserot [V. Ginzburg, M. Kapranov and E. Vasserot, *Elliptic algebras and equivariant elliptic cohomology*, Preprint (1995), [arXiv:q-alg/9505012](https://arxiv.org/abs/q-alg/9505012)]. We further show that the decategorified multiplicative global cohomology theory associated to a genuine global 2-ring canonically refines to an $\mathbb{E}_\infty$ -ring object in global spectra. As we show, two interesting examples of genuine global 2-rings are given by quasi-coherent sheaves on the torsion points of an oriented spectral elliptic curve and Lurie's theory of tempered local systems. In particular, we obtain global spectra representing equivariant elliptic cohomology and tempered cohomology.

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1. Introduction

The study of cohomology theories for equivariant spaces has long been a key aspect of algebraic topology. However, in recent years, it has become increasingly clear that many of the most interesting and important cohomology theories defined on equivariant spaces are more properly understood as cohomology theories on topological stacks. Often, this perspective highlights the geometric or algebraic structure underlying such a cohomology theory in a way which the restriction to any group does not. As some examples, we note that Borel cohomology, complex K-theory, (stable) bordism, and stable cohomotopy all admit the additional functoriality which characterizes a cohomology theory defined on topological stacks. More recent additions to the list are tempered cohomology and global elliptic cohomology, as defined by [Lur19] and [GM23], respectively. Such cohomology theories have come to be known as *global cohomology theories*. Before we can make concrete definitions, we first have to introduce a suitable homotopy theory of topological stacks.

1.1 The homotopy theory of topological stacks

We follow [GH07] in considering the ∞ -category $\mathcal{S}_{\text{gl}}$ of *global spaces* (there called *orbispaces*). Global spaces are defined in analogy to (Bredon) G -spaces, the ∞ -category of which, by the theorem of Elmendorf, is equivalent to the presheaf ∞ -category on the G -orbit category. Similarly, global spaces are defined to be a presheaf ∞ -category on the ∞ -category Glo , which one should interpret as an ∞ -category of orbit stacks, with objects given by $\mathbf{B}G$ for a compact Lie group G . Up to homotopy, morphisms in Glo from $\mathbf{B}H \rightarrow \mathbf{B}G$ are given by conjugacy classes of group homomorphisms $\alpha: H \rightarrow G$. A map $f: \mathbf{B}H \rightarrow \mathbf{B}G$ is called *faithful* if it is represented by a monomorphism of groups. The ∞ -categories of G -spaces and global spaces are closely related. There is a fully faithful colimit-preserving functor

$$-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\text{gl}/\mathbf{B}G},$$

whose essential image is given by those global spaces over $\mathbf{B}G$ whose reference map is faithful.

It is often useful to only consider global spaces with isotropy restricted to some family $\mathcal{E}$ of compact Lie groups, which one denotes $\mathcal{S}_{\mathcal{E}\text{-gl}}$ and calls $\mathcal{E}$ -global spaces. For example, tempered cohomology and global elliptic cohomology are most naturally defined as a cohomology theory for global spaces with isotropy in finite abelian groups and compact abelian Lie groups, respectively.

1.2 Multiplicative global cohomology theories

Having introduced our homotopy theory of stacks, one can simply define a *multiplicative $\mathcal{E}$ -global cohomology theory* as a limit-preserving functor

$$E: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}.$$

Taking homotopy groups, we obtain a functor E^* taking values in graded commutative rings and satisfying analogs of the Eilenberg–Steenrod axioms. By higher Brown representability, the ∞ -category of cohomology theories are equivalent to (*commutative*) *naive global rings*, i.e. commutative algebra objects in the ∞ -category of spectrum objects in $\mathcal{E}$ -global spaces. However, as is typically the case in equivariant homotopy theory, we are more interested in ‘genuine’ multiplicative global cohomology theories, which are represented by (*commutative*) *global rings*, i.e. commutative algebra objects in the ∞ -category of $\mathcal{E}$ -global spectra in the sense of [Sch18]. Global spectra can be organized into a symmetric monoidal stable ∞ -category $\text{Sp}_{\mathcal{E}\text{-gl}}$, which admits a suspension spectrum functor $\Sigma_+^\infty: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \text{Sp}_{\mathcal{E}\text{-gl}}$. In particular, any commutative global ring $X \in \text{CAlg}(\text{Sp}_{\mathcal{E}\text{-gl}})$ defines a multiplicative global cohomology theory (and hence a commutative naive global ring) via the assignment

$$\mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}, \quad Y \mapsto \text{map}_{\text{Sp}_{\mathcal{E}\text{-gl}}}(\Sigma_+^\infty Y, X).$$

However, not all multiplicative global cohomology theories E arise in this way. In fact, a global ring contains significantly more structure. For this reason, we say that a global ring X is a *genuine refinement* of the multiplicative global cohomology theory E if there exists a natural equivalence

$$E(-) \simeq \text{map}_{\text{Sp}_{\mathcal{E}\text{-gl}}}(\Sigma_+^\infty(-), X).$$

Remark. As noted before, a genuine refinement endows E with significantly more structure. For example, for any compact Lie group G , and any G -space X , one obtains an $RO(G)$ -graded abelian group $E^*(X)$ together with dimension-shifting transfer maps

$$\text{tr}_H^G: E_H^{*+L}(X) \rightarrow E_G^*(X),$$

for any subgroup $H \subset G$, where L is the tangent representation of G/H . Moreover, this structure is highly compatible as you change the group G , the subgroup H , and the space X .

Most of the cohomology theories recalled above are all canonically ‘genuine’. For example, global spectra representing Borel cohomology, K-theory, (stable) bordism, and cohomotopy were constructed by Schwede [Sch18]. However, neither tempered cohomology nor elliptic cohomology has so far been given a genuine refinement, despite the fact that considerable evidence for such a refinement is contained in [Lur19] and [Lur09a, GM23] for tempered and elliptic cohomology, respectively.

1.3 Naive global 2-rings

Therefore, there is a use for general procedures which construct genuine refinements of multiplicative global cohomology theories. As such, the main concern of this article is the following.

Question. How can one construct genuine refinements of a multiplicative global cohomology theory?

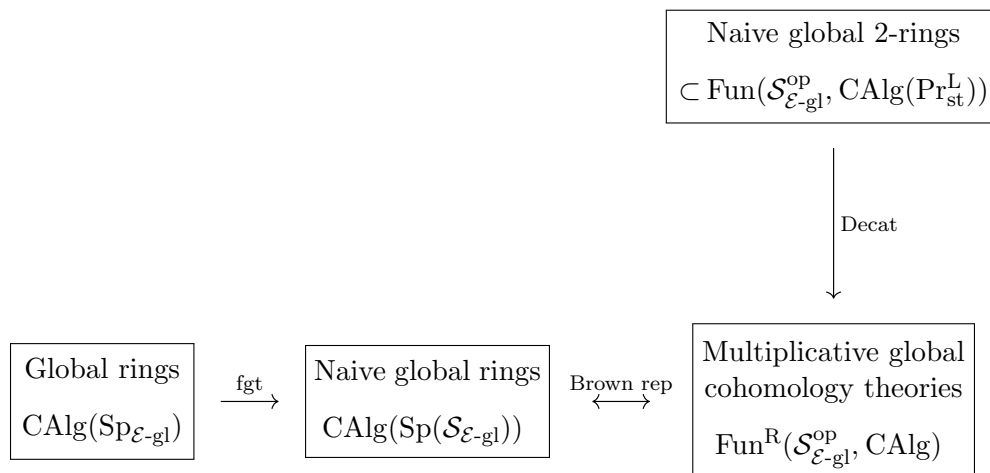
We provide an answer via the process of categorification. One key definition of this article is the following.

DEFINITION. Let $\mathcal{E}$ be a family of compact Lie groups. A *naive $(\mathcal{E})$ -global 2-ring* is a functor

$$\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}), \quad \mathcal{X} \mapsto \mathcal{R}_{\mathcal{X}}, \quad f \mapsto f^*,$$

from global spaces to the ∞ -category of presentably symmetric monoidal stable ∞ -categories, which satisfies two additional conditions that we explain throughout this introduction. We write f_* for the right adjoint to the functor f^* .

Given a functor $\mathcal{R}$ as above, we can produce a functor $\text{H}_{\text{gl}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}$ by taking endomorphism rings of the various unit objects $\mathbb{1}_{\mathcal{R}_{\mathcal{X}}}$. The first condition which makes $\mathcal{R}$ a naive $\mathcal{E}$ -global 2-ring is that $\text{H}_{\text{gl}}(-, \mathcal{R})$ is a multiplicative global cohomology theory. Thus, we think of $\mathcal{R}$ as a *categorification* of $\text{H}_{\text{gl}}(-, \mathcal{R})$. We summarize the situation in the following diagram.



Our interest in considering this categorification of a multiplicative global cohomology theory comes from the fact that it endows its decategorification with significantly more structure, as we explain now.

1.4 Unraveling into families of equivariant cohomology theories

Recall that, in the literature on equivariant elliptic cohomology theories associated to an elliptic curve E , one rarely views equivariant elliptic cohomology as a global cohomology theory valued in spectra. Instead, one restricts to a cohomology theory on G -spaces for some G . This has the benefit that G -equivariant elliptic cohomology then canonically lifts to a functor

$$\mathcal{E}ll_G(-): \mathcal{S}_G^{\text{op}} \rightarrow \text{QCoh}(E[\hat{G}])$$

valued in the stable ∞ -category of quasi-coherent sheaves on the $\hat{G}$ -torsion points of the elliptic curve (here, $\hat{G}$ denotes the Pontryagin dual of the abelian compact Lie group G). However, a priori there is a downside to this perspective: it fails to capture the global nature of equivariant elliptic cohomology.

There is a way, however, to obtain the best of both worlds. One can capture the global nature of equivariant elliptic cohomology by equipping the family of functors $\mathcal{E}ll_G(-)$ with a suitable collection of *change-of-group transformations*. Given a group homomorphism $\alpha: H \rightarrow G$ and a G -space X , these relate the value of $\mathcal{E}ll_H$ on α^*X , the restriction of X to an H -space, and the value of $\mathcal{E}ll_G$ on X . One can find this structure emphasized in the approach to equivariant elliptic cohomology suggested by [GKV95]. As the first step of our approach, we show that any naive global 2-ring canonically induces these data in an extremely coherent way. The next result combines Proposition 8.1 with the results of § 9.

THEOREM A. *Let $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ be a functor. Then, for all $G \in \mathcal{E}$, there exists a lax symmetric monoidal functor*

$$\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\text{op}} \rightarrow \mathcal{R}_{BG}$$

whose composite with the functor $\mathcal{R}_{BG} \rightarrow \text{Sp}$, $X \mapsto \text{map}_{\mathcal{R}_{BG}}(\mathbb{1}, X)$, agrees with $H_{\text{gl}}(-//G, \mathcal{R})$. Moreover, for every group homomorphism $\alpha: H \rightarrow G$, there exists a natural transformation filling the following square.

$$\begin{array}{ccc} \mathcal{S}_G^{\text{op}} & \xrightarrow{\alpha^*} & \mathcal{S}_H^{\text{op}} \\ \mathbb{H}_G(-, \mathcal{R}) \downarrow & \nearrow_{Q_\alpha} & \downarrow \mathbb{H}_H(-, \mathcal{R}) \\ \mathcal{R}_{BG} & \xrightarrow{\alpha^*} & \mathcal{R}_{BH} \end{array} \quad (1.1)$$

Furthermore, these data are coherently functorial. This is encoded in the existence of a functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}}^{\text{op}} \rightarrow \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})$$

extending the assignment above.

In fact, the second condition of being a naive global 2-ring is precisely that the functors $\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\text{op}} \rightarrow \mathcal{R}_{BG}$ are all limit-preserving. In this way, we see that naive global 2-rings also categorify families of equivariant cohomology theories.

1.5 The Ginzburg–Kapranov–Vasserot axioms

The previous result is only useful to the extent to which we are able to control the family of cohomology theories one obtains by unraveling. To isolate a case where this is possible,

we introduce the notion of a *genuine* global 2-ring. Let $\mathcal{T} \subset \mathcal{E}$ be some subset of groups in the family $\mathcal{E}$.

DEFINITION. We say that a naive global 2-ring $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ is $\mathcal{T}$ -*genuine* if the following hold.

- (1) A functor $\mathcal{R}$ satisfies base-change with respect to faithful morphisms $\mathbf{B}H \rightarrow \mathbf{B}G$ with target in $\mathcal{T}$.
- (2) For any faithful morphism $f \in \text{Glo}_{\mathcal{E}}$, the adjunction (f^*, f_*) satisfies the projection formula.
- (3) For all $G \in \mathcal{E}$ and every irreducible G -representation V , the object $\mathbb{H}_G(S^V, \mathcal{R}) \in \mathcal{R}_{\mathbf{B}G}$ is invertible.

In the previous definition, $\mathcal{T} \subset \mathcal{E}$ will typically be some subset of *enough injective objects*, a concept which abstracts the properties of the collection of tori inside all abelian compact Lie groups (see Definition 4.2). This concept ensures that one can effectively control the values at all groups in $\mathcal{E}$ by only remembering the value of G -equivariant cohomology theories for $G \in \mathcal{T}$. The additional flexibility of specifying such a $\mathcal{T}$ is crucial for applications. Specifically, the example of equivariant elliptic cohomology will only be genuine with respect to the tori.

As mentioned, the axioms of a genuine global 2-ring ensure that the unraveling $\mathbb{H}_{\bullet}(-, \mathcal{R})$ of a genuine global 2-ring is particularly well-behaved. In short, points (1) and (2) of the definition above ensure that the family $\mathbb{H}_{\bullet}(-, \mathcal{R})$ of equivariant cohomology theories transforms correctly as we vary the group. We then combine this with point (3) to show that we can extend $\mathbb{H}_G(-, \mathcal{R})$ to a family of $\text{RO}(G)$ -graded theories. More precisely, we show, using points (1) and (2), that $\mathbb{H}_{\bullet}(-, \mathcal{R})$ satisfies the following analogs of the axiom of [GKV95].

THEOREM B. *Let $\mathcal{R}$ be a $\mathcal{T}$ -genuine global 2-ring. Then the unraveling $\mathbb{H}_{\bullet}(-, \mathcal{R})$ satisfies the following axioms.*

- (1) Induction. *Let $\alpha: G \rightarrow G/N$ be a surjective group homomorphism and let X be a G -space such that the action of N on X is free. Then, there is a natural equivalence*

$$\mathbb{H}_{G/N}(\alpha_!(X/N), \mathcal{R}) \xrightarrow{\sim} \alpha_* \mathbb{H}_G(X, \mathcal{R}).$$

- (2) Base-change. *Let $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ be a map in $\text{Glo}_{\mathcal{E}}$ such that $\mathbf{B}G \in \mathcal{T}$. Then, the natural transformation*

$$Q_{\alpha}: \alpha^* \mathbb{H}_G(X, \mathcal{R}) \rightarrow \mathbb{H}_H(\alpha^* X, \mathcal{R})$$

from Theorem A is an equivalence for all compact G -spaces X .

- (3) Künneth. *Let G and H be two groups in $\mathcal{T}$, X a compact G -space, and Y a compact H -space. Then, there is an equivalence*

$$\pi_G^* \mathbb{H}_G(X, \mathcal{R}) \otimes \pi_H^* \mathbb{H}_H(Y, \mathcal{R}) \simeq \mathbb{H}_{G \times H}(X \times Y, \mathcal{R}),$$

where π_H and π_G denote the two projections $G \times H \rightarrow H, G$.

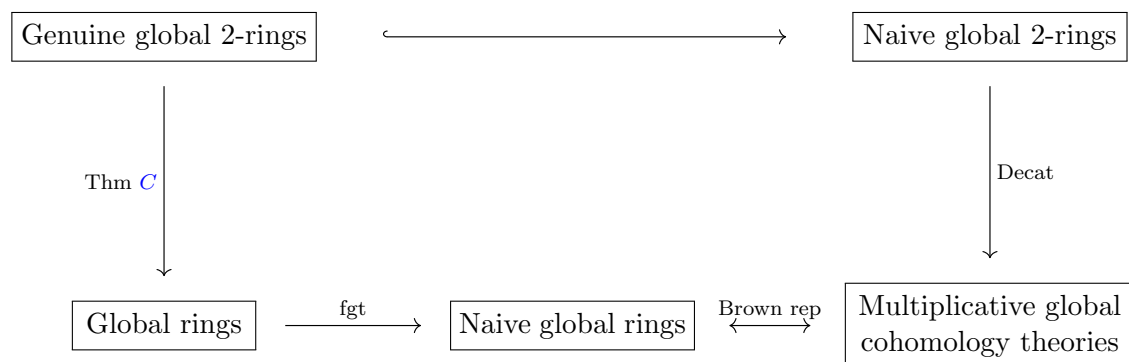
1.6 A genuine refinement of $\mathbf{H}_{\text{gl}}(-, \mathcal{R})$

Using the previous result and the universal property of G -spectra, we are then able to coherently refine the equivariant cohomology theories $\mathbb{H}_G(-, \mathcal{R})$ to cohomology theories on genuine G -spectra. Taking global sections, we obtain a family of equivariant cohomology theories with values in Sp . An application of higher Brown representability yields a compatible collection of G -spectra, which in turn define a global spectrum by the main theorem of [LNP25]. Finally,

we check that this global spectrum is a genuine refinement of $H_{\text{gl}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{Sp}$. This gives the main theorem of this article (see Theorem 12.7 and Proposition 12.8).

THEOREM C. *Let $\mathcal{E}$ be a multiplicative global family of compact Lie groups and let $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ be a genuine global 2-ring. Then, its associated decategorification $H_{\text{gl}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}$ admits a canonical genuine refinement $\Gamma_{\text{gl}}(\mathcal{R}) \in \text{CAlg}(\text{Sp}_{\mathcal{E}\text{-gl}})$. Moreover, this assignment improves to a functor from a suitable category of genuine global 2-rings to global rings.*

One benefit of categorification which is often highlighted is that it has the ability to turn structure into a property. We note that another example of this phenomenon is provided by naive global 2-rings: while a genuine refinement of a multiplicative global cohomology theory is structure, it is a property for a naive global 2-ring to be genuine. We can therefore complete our diagram as follows.



In the body of the article, we improve this picture to an actual commutative diagram of categories and functors.

1.7 Applications: elliptic and tempered cohomology

Our first application is to equivariant elliptic cohomology. By [GM23], a preoriented strict abelian group object $\mathbf{G}$ in a suitable ∞ -category $\mathcal{X}$ canonically induces a functor $\text{Glo}_{\text{ab}} \rightarrow \mathcal{X}$ from the global orbit ∞ -category with isotropy in all compact abelian Lie groups. If the objects of $\mathcal{X}$ admit a sufficiently well-behaved notion of quasi-coherent sheaves, one can obtain a naive global 2-ring by postcomposition. Applying this to an oriented elliptic curve $\mathbf{E}$ in spectral Deligne–Mumford stacks over $\mathbf{S}$, we obtain a naive global 2-ring

$$\mathcal{Q}^{\mathbf{E}}: \mathcal{S}_{\text{ab-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}), \quad X \mapsto \mathcal{Q}_X^{\mathbf{E}}.$$

Applying $\mathcal{Q}^{\mathbf{E}}$ to $\mathbf{B}A$ for a compact abelian group A gives $\text{QCoh}(\mathbf{E}[\widehat{A}])$, the ∞ -category of quasi-coherent sheaves on the $\widehat{A}$ -torsion points of $\mathbf{E}$, where $\widehat{A}$ is the Pontryagin dual of A . Heavily relying on [GM23], we prove the following.

THEOREM D. *Let $\mathbf{E}$ be an oriented elliptic curve in spectral Deligne–Mumford stacks over $\mathbf{S}$. Then, $\mathcal{Q}^{\mathbf{E}}$ is a genuine global 2-ring and so $H_{\text{gl}}(-, \mathcal{Q}^{\mathbf{E}})$ admits a genuine refinement $\Gamma_{\mathbf{E}}(\mathbf{S}, \mathcal{O}_{\mathbf{S}}) := \Gamma_{\text{gl}}(\mathcal{Q}^{\mathbf{E}}) \in \text{CAlg}(\text{Sp}_{\text{ab-gl}})$.*

By construction, the underlying spectrum of $\Gamma_{\mathbf{E}}(\mathbf{S}, \mathcal{O}_{\mathbf{S}})$ is simply $\Gamma(\mathbf{S}, \mathcal{O}_{\mathbf{S}})$, the global sections of $\mathbf{S}$. In particular, we obtain an integral globally equivariant elliptic cohomology spectrum. Further specializing, when $\mathbf{S}$ is $\mathcal{M}_{\text{ell}}^{\text{or}}$, the moduli stack of oriented elliptic curves, we obtain a global spectrum of topological modular forms TMF_{gl} .

Our second application is to tempered cohomology [Lur19]. In this case, we restrict to the family of *finite abelian* groups and write $\mathrm{Glo}_{\mathrm{fab}}$ and $\mathcal{S}_{\mathrm{fab-gl}}$ for the associated global orbit ∞ -category and ∞ -category of global spaces. Given an oriented $\mathbf{P}$ -divisible group $\mathbf{G}$ over a commutative ring spectrum R , Lurie constructs a multiplicative global cohomology theory

$$R_{\mathbf{G}}^{\bullet} : \mathcal{S}_{\mathrm{fab-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg},$$

referred to as *tempered cohomology*. Furthermore, Lurie [Lur19] introduces the notion of tempered $\mathbf{G}$ -local systems on a fab-global space. As shown there, this functions as an extremely well-behaved categorification of tempered cohomology. We capture some of this richness by the following theorem.

THEOREM E. *Suppose $\mathbf{G}$ is an oriented $\mathbf{P}$ -divisible group over R . Then, the functor*

$$\mathrm{LocSys}_{\mathbf{G}}(-) : \mathcal{S}_{\mathrm{fab-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}), \quad X \mapsto \mathrm{LocSys}_{\mathbf{G}}(X),$$

is a genuine global 2-ring and so admits a genuine refinement $\underline{R}_{\mathbf{G}}^{\mathrm{gl}} := \Gamma_{\mathrm{gl}}(\mathrm{LocSys}_{\mathbf{G}}) \in \mathrm{CAlg}(\mathrm{Sp}_{\mathrm{fab-gl}})$.

In particular, restricting to any finite abelian group A , one obtains a genuine A -spectrum $\underline{R}_{\mathbf{G}}^A$. Using the technology developed in this article, we can, in fact, go further and actually identify $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ with the ∞ -category of $\underline{R}_{\mathbf{G}}^A$ -modules in A -spectra. The following theorem combines Theorem 15.27 and Proposition 15.29.

THEOREM F. *There exists a natural equivalence*

$$\mathbb{F}_{\bullet} : \mathrm{LocSys}_{\mathbf{G}}(\bullet) \rightarrow \mathrm{Mod}_{\underline{R}_{\mathbf{G}}}(\mathrm{Sp}_{\bullet})$$

of functors $\mathrm{Glo}_{\mathrm{fab}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$.

1.8 Conventions and notation

- (1) We use capital letters to refer to large categories, and bold font to refer to $(\infty, 2)$ -categories.
- (2) We use Cat for the ∞ -category of ∞ -categories and Cat_2 for the ∞ -category of $(\infty, 2)$ -categories.
- (3) Given a symmetric monoidal ∞ -category $\mathcal{C}$, we write $\mathrm{CAlg}(\mathcal{C})$ for the ∞ -category of commutative algebras ($\mathbb{E}_{\infty}$ -algebras) in $\mathcal{C}$. When $\mathcal{C}$ is the ∞ -category of spectra, we abbreviate this by CAlg . As another key example, we obtain $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$, the ∞ -category of presentably symmetric monoidal stable ∞ -categories and symmetric monoidal left-adjoint functors between them.
- (4) Given an ∞ -category $\mathcal{C}$, we write $\mathrm{Map}_{\mathcal{C}}(-, -)$ for the mapping spaces in $\mathcal{C}$. If $\mathcal{C}$ is stable, then we write $\mathrm{map}_{\mathcal{C}}(-, -)$ for the mapping spectra in $\mathcal{C}$. Finally, if $\mathcal{C}$ is closed symmetric monoidal, then we write $\underline{\mathrm{Hom}}_{\mathcal{C}}(-, -)$ for the internal Hom in $\mathcal{C}$.
- (5) We write $\Delta(\mathcal{C}) : \mathcal{I} \rightarrow \mathrm{Cat}$ for the constant functor on an ∞ -category $\mathcal{C}$.
- (6) We assume all (nonconnective) spectral Deligne–Mumford stacks to be locally noetherian, i.e. they are étale locally of the form $\mathrm{Spec}(A)$ with $\pi_0(A)$ noetherian and $\pi_i(A)$ a finitely generated $\pi_0(A)$ -module for $i \geq 0$. Moreover, we assume (nonconnective) spectral Deligne–Mumford stacks to be quasi-separated, i.e. the fiber product of any two affines over such a stack is quasi-compact again, and all iterated diagonals are quasi-separated (i.e. n -quasi-separated in the sense of [GM23, Definition D.1] for all $n \geq 1$).

PART I. Recollections on global homotopy theory

In this part, we provide some definitions and results in global homotopy theory. We emphasize the perspective on global homotopy provided by the results of [LNP25], which utilizes the notion of a partially lax limit. Therefore, we begin with a section on lax natural transformations and provide a definition of partially lax limits. In the following section, we define the ∞ -category of global spaces and explain its close relationship to equivariant homotopy theory. In the final section of the part, we discuss the ∞ -category of global spectra.

2. Lax natural transformations

In this section, we recall the ∞ -categories of functors and (op)lax natural transformations, and discuss (op)lax slice categories and the mate equivalence.

DEFINITION 2.1. Let $\mathcal{I}$ be an ∞ -category and $\mathbf{C}$ an $(\infty, 2)$ -category. We define the $(\infty, 2)$ -categories

$$\mathbf{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{C}) \quad \text{and} \quad \mathbf{Fun}^{\mathrm{oplax}}(\mathcal{I}, \mathbf{C})$$

as the right-adjoint objects for the left and right Gray tensor product by $\mathcal{I}$, respectively, in the sense of [GHL21], so that

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cat}_2}(\mathbf{A} \boxtimes \mathcal{I}, \mathbf{C}) &\simeq \mathrm{Map}_{\mathrm{Cat}_2}(\mathbf{A}, \mathbf{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{C})), \\ \mathrm{Map}_{\mathrm{Cat}_2}(\mathcal{I} \boxtimes \mathbf{A}, \mathbf{C}) &\simeq \mathrm{Map}_{\mathrm{Cat}_2}(\mathbf{A}, \mathbf{Fun}^{\mathrm{oplax}}(\mathcal{I}, \mathbf{C})), \end{aligned}$$

for every $(\infty, 2)$ -category $\mathbf{A}$. The underlying ∞ -categories will be denoted $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{C})$ and $\mathrm{Fun}^{\mathrm{oplax}}(\mathcal{I}, \mathbf{C})$, respectively.

Example 2.2. The objects of $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{C})$ agree with the objects of $\mathrm{Fun}(\mathcal{I}, \mathbf{C})$. A morphism in $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{C})$ is precisely the data of a lax natural transformation. Informally, a lax natural transformation $\eta: F \Rightarrow G$ between functors $F, G: \mathcal{I} \rightarrow \mathbf{C}$ is a coherent collection of squares as follows, together with 2-morphisms $\epsilon: G(f) \circ \eta_i \Rightarrow \eta_j \circ F(f)$.

$$\begin{array}{ccc} F(i) & \xrightarrow{F(f)} & F(j) \\ \eta_i \downarrow & \nearrow \epsilon & \downarrow \eta_j \\ G(i) & \xrightarrow{G(f)} & G(j) \end{array}$$

If the 2-morphisms ϵ are all equivalences, then η defines a natural transformation. Dually, an oplax natural transformation $\eta: F \Rightarrow G$ is informally a coherent collection of oplaxly commuting squares.

$$\begin{array}{ccc} F(i) & \xrightarrow{F(f)} & F(j) \\ \eta_i \downarrow & \nwarrow \epsilon & \downarrow \eta_j \\ G(i) & \xrightarrow{G(f)} & G(j) \end{array}$$

DEFINITION 2.3. Consider two functors $F, G: \mathcal{I} \rightarrow \mathbf{Cat}$. We write $\mathrm{Nat}^{\mathrm{lax}}(F, G)$ for the mapping categories in $\mathbf{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{Cat})$ from F to G . Dually, we write $\mathrm{Nat}^{\mathrm{oplax}}(F, G)$ for the mapping ∞ -category in $\mathbf{Fun}^{\mathrm{oplax}}(\mathcal{I}, \mathbf{Cat})$.

We now give a definition of partially (op)lax limits, first introduced in [Ber20].

DEFINITION 2.4. Let $F: \mathcal{I} \rightarrow \mathbf{Cat}$ be a functor. We define the *lax limit* of F ,

$$\mathrm{laxlim}_{\mathcal{I}} F := \mathbf{Nat}^{\mathrm{lax}}(\Delta(*), F),$$

to be the ∞ -category of *lax cones*: lax natural transformations from the constant functor on the terminal ∞ -category to F . Here, $\Delta: \mathbf{Cat} \rightarrow \mathbf{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{Cat})$ denotes the diagonal which, under the identification $\mathbf{Cat} \simeq \mathbf{Fun}^{\mathrm{lax}}(*, \mathbf{Cat})$, is precomposition with the projection $\mathcal{I} \rightarrow *$.

Suppose $\mathcal{I}$ is marked by a subcategory $\mathcal{W}$. Then, we define the *partially lax limit* of F ,

$$\mathrm{laxlim}_{\mathcal{I}, \mathcal{W}}^{\dagger} F \subset \mathrm{laxlim}_{\mathcal{I}} F,$$

to be the full subcategory of $\mathrm{laxlim}_{\mathcal{I}} F$ spanned by those lax cones whose restriction to $\mathcal{W} \subset \mathcal{I}$ is strictly natural.

Dually, we define the *oplax limit* $\mathrm{oplaxlim}_{\mathcal{I}} F$ of F to be the ∞ -category $\mathbf{Nat}^{\mathrm{oplax}}(\Delta(*), F)$ of *oplax cones* and $\mathrm{oplaxlim}_{\mathcal{I}, \mathcal{W}}^{\dagger} F \subset \mathrm{oplaxlim}_{\mathcal{I}} F$ to be the full subcategory spanned by those oplax cones whose restriction to $\mathcal{W}$ is strictly natural.

We now introduce an (un)straightening equivalence for the ∞ -category of functors and lax natural transformations. This will be a crucial tool for working with partially lax limits, and will also allow us to connect the definition given above to the definitions of [Ber20] and [LNP25].

DEFINITION 2.5. Write $\mathbf{Cocart}^{\mathrm{lax}}(\mathcal{I})$ and $\mathbf{Cart}^{\mathrm{oplax}}(\mathcal{I})$ for the full subcategories of the $(\infty, 2)$ -category $\mathbf{Cat}_{/\mathcal{I}}$ spanned by the cocartesian and cartesian fibrations, respectively.

THEOREM 2.6 [HHL⁺23a, Theorem E]. *The (un)straightening equivalences of Lurie extend to natural equivalences*

$$\mathbf{Cocart}^{\mathrm{lax}}(\mathcal{I}) \simeq \mathbf{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{Cat}) \quad \text{and} \quad \mathbf{Cart}^{\mathrm{oplax}}(\mathcal{I}) \simeq \mathbf{Fun}^{\mathrm{oplax}}(\mathcal{I}^{\mathrm{op}}, \mathbf{Cat}). \quad (2.1)$$

Remark 2.7. The previous equivalence gives an equivalence

$$\mathbf{Nat}^{\mathrm{lax}}(F, G) \simeq \mathbf{Fun}_{\mathcal{I}}(\mathrm{Un}^{\mathrm{co}}(F), \mathrm{Un}^{\mathrm{co}}(G))$$

when restricted to the mapping ∞ -category from F to G . In particular, because the identity functor on $\mathcal{I}$ is the unstraightening of $\Delta(*)$, we obtain an equivalence

$$\mathrm{laxlim}_{\mathcal{I}} F \simeq \mathbf{Fun}_{\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)).$$

That is, the lax limit of F is equivalent to the ∞ -category of sections of the cocartesian unstraightening of F . Furthermore, if $\mathcal{I}$ is marked by $\mathcal{W}$, then a lax cone is in $\mathrm{laxlim}_{\mathcal{I}, \mathcal{W}}^{\dagger} F$ if and only if the associated section $\mathcal{I} \rightarrow \mathrm{Un}^{\mathrm{co}}(F)$ sends maps in $\mathcal{W}$ to cocartesian edges in $\mathrm{Un}^{\mathrm{co}}(F)$. This follows immediately from the fact that Equation (2.1) restricts to the usual (un)straightening equivalence.

We conclude, by [Ber20, Theorem 4.4], that our definition agrees with the definition of partially lax limits given there, and used in [LNP25]. Finally, we note that all of the observations above dualize and imply analogous statements for oplax limits.

Example 2.8. A particularly degenerate consequence of the previous observations is already interesting. Recall that the cocartesian unstraightening of the constant functor $\Delta(\mathcal{C}): \mathcal{I} \rightarrow \mathbf{Cat}$ is given by the projection $\mathcal{C} \times \mathcal{I} \rightarrow \mathcal{I}$. Therefore, we may compute that

$$\mathrm{laxlim}_{\mathcal{I}} \Delta(\mathcal{C}) \simeq \mathbf{Fun}_{\mathcal{I}}(\mathcal{I}, \mathcal{C}) \simeq \mathbf{Fun}(\mathcal{I}, \mathcal{C}).$$

We conclude that functors $F: \mathcal{I} \rightarrow \mathcal{C}$ are equivalent to lax cones over the constant functor on $\mathcal{C}$. Dually, functors $F: \mathcal{I}^{\mathrm{op}} \rightarrow \mathcal{C}$ are equivalent to oplax cones over the constant functor on $\mathcal{C}$.

While not immediately clear from the description above, we note that if $F: \mathcal{I} \rightarrow \mathbf{Cat}$ lifts to a diagram in $\mathbf{Cat}^{\otimes}$, the ∞ -category of symmetric monoidal ∞ -categories and symmetric monoidal functors, then the partially lax limit $\mathrm{laxlim}^{\dagger} F$ is canonically symmetric monoidal.

PROPOSITION 2.9. *Let $F: \mathcal{I} \rightarrow \mathbf{Cat}^{\otimes}$ be a functor. Then $\mathrm{laxlim}^{\dagger} F$ admits a symmetric monoidal structure such that, for any symmetric monoidal ∞ -category $\mathcal{C}$, we have a natural equivalence*

$$\mathrm{Fun}^{\otimes-\mathrm{lax}}(\mathcal{C}, \mathrm{laxlim}^{\dagger} F) \simeq \mathrm{laxlim}^{\dagger} \mathrm{Fun}^{\otimes-\mathrm{lax}}(\mathcal{C}, F(i))$$

between lax symmetric monoidal functors. In particular,

$$\mathbf{CAlg}(\mathrm{laxlim}^{\dagger} F) \simeq \mathrm{laxlim}^{\dagger} \mathbf{CAlg}(F(i)).$$

Proof. This follows from [LNP25, Remark 5.1]. $\square$

In the remainder of this section, we recall some technical material which we will use in later sections.

2.1 Lax slice categories

In this subsection, we introduce the definition of (op)lax slice categories, and use this to make some identifications of lax limits which will be useful later.

DEFINITION 2.10. Given an $(\infty, 2)$ -category $\mathbf{C}$ with underlying ∞ -category $\mathcal{C}$ and an object X in $\mathbf{C}$, we define $\mathbf{C} \downarrow^{\mathrm{opl}} X$, the *oplax slice category over X* , via the following pullback.

$$\begin{array}{ccc} \mathbf{C} \downarrow^{\mathrm{opl}} X & \longrightarrow & \mathrm{Fun}^{\mathrm{oplax}}([1], \mathbf{C}) \\ \downarrow & & \downarrow \\ \mathcal{C} \times \{X\} & \longrightarrow & \mathcal{C} \times \mathcal{C} \end{array}$$

Analogously, we can define the notion of lax slice category $\mathbf{C} \downarrow^{\mathrm{lax}} X$.

Our interest in oplax slice categories comes from the following theorem.

THEOREM 2.11. *Let $\mathbf{C}$ be an $(\infty, 2)$ -category with underlying ∞ -category $\mathcal{C}$, and fix $X \in \mathbf{C}$.*

- (1) *The forgetful functor $\mathrm{fgt}: \mathbf{C} \downarrow^{\mathrm{opl}} X \rightarrow \mathcal{C}$ is a cartesian fibration and classifies the mapping functor $\mathrm{Map}_{\mathbf{C}}(-, X): \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Cat}$.*
- (2) *The forgetful functor $\mathrm{fgt}: (\mathbf{C} \downarrow^{\mathrm{lax}} X)^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$ is a cocartesian fibration and classifies the same functor as above.*

Proof. Part (1) is proved in [HHL⁺23b, Theorem 7.21]. Part (2) follows by applying part (1) to $\mathbf{C}^{2-\mathrm{op}}$, the $(\infty, 2)$ -category obtained from $\mathbf{C}$ by passing to opposites on hom-categories. $\square$

Applying this to $\mathbf{C} = \mathbf{Cat}$, we obtain another formulation of the (un)straightening equivalence. We write $\mathbf{Cart}$ for the subcategory of $\mathbf{Ar}(\mathbf{Cat})$ spanned by cartesian fibrations and maps of cartesian fibrations (see Definition 7.1).

COROLLARY 2.12. *There exists an equivalence*

$$\mathrm{Un}: \mathbf{Cat} \downarrow^{\mathrm{opl}} \mathbf{Cat} \xrightarrow{\sim} \mathbf{Cart}.$$

Proof. The equivalence is induced by unstraightening the natural equivalence

$$\mathrm{Fun}(-, \mathbf{Cat}) \simeq \mathbf{Cart}((-)^{\mathrm{op}}).$$

$\square$

CONSTRUCTION 2.13. Suppose we have a functor

$$F: \mathcal{I} \rightarrow \mathbf{Cat} \downarrow^{\text{opl}} \mathcal{C}, \quad i \mapsto [\phi_i: F(i) \rightarrow \mathcal{C}].$$

We abuse notation and write F also for the functor $F: \mathcal{I} \rightarrow \mathbf{Cat}$ obtained by forgetting the functor to $\mathcal{C}$. By Theorem 2.11(1), the forgetful functor $\mathbf{Cat} \downarrow^{\text{opl}} \mathcal{C} \rightarrow \mathbf{Cat}$ is the cartesian unstraightening of the functor $\text{Fun}(-, \mathcal{C}): \mathbf{Cat}^{\text{op}} \rightarrow \mathbf{Cat}$. By pulling back along $F: \mathcal{I} \rightarrow \mathbf{Cat}$, this implies that we can interpret F as a section of the cartesian fibration

$$q: \text{Un}^{\text{ct}}(\text{Fun}(F(-), \mathcal{C})) \rightarrow \mathcal{I}^{\text{op}}.$$

By the dual of Remark 2.7, this equivalently defines an object of $\text{oplaxlim}_{\mathcal{I}^{\text{op}}} \text{Fun}(F(-), \mathcal{C})$. Summarizing, we have exhibited, for any $F: \mathcal{I} \rightarrow \mathbf{Cat}$, an equivalence between $\text{oplaxlim} \text{Fun}(F(-), \mathcal{C})$ and $\text{Fun}_{\mathbf{Cat}}(\mathcal{I}, \mathbf{Cat} \downarrow^{\text{opl}} \mathcal{C})$. The previous argument in fact applies to any $(\infty, 2)$ -category $\mathbf{C}$ and an object $X \in \mathbf{C}$, and gives an identification of

$$F: \mathcal{I} \rightarrow \mathbf{C} \downarrow^{\text{opl}} X$$

and objects of $\text{oplaxlim} \mathbf{C}(F(-), X)$. We later apply this also to $\mathbf{Cat}^{\otimes, \text{lax}}$.

2.2 The mate equivalence

We now introduce the mate equivalence, which in fact takes two forms. To state them, we introduce some notation.

DEFINITION 2.14. Let $\mathcal{I}$ be an ∞ -category and let $\mathbf{Cat}$ be the (very large) $(\infty, 2)$ -category of large ∞ -categories. We write

$$\mathbf{Fun}_{\mathbf{L}}^{\text{lax}}(\mathcal{I}, \mathbf{Cat}) \quad \text{and} \quad \mathbf{Fun}_{\mathbf{R}}^{\text{oplax}}(\mathcal{I}, \mathbf{Cat})$$

for the full subcategory of $\mathbf{Fun}^{\text{lax}}(\mathcal{I}, \mathbf{Cat})$ and $\mathbf{Fun}^{\text{oplax}}(\mathcal{I}, \mathbf{Cat})$, respectively, spanned by those functors $\mathcal{I} \rightarrow \mathbf{Cat}$ which send each morphism in $\mathcal{I}$ to a left and right adjoint, respectively.

THEOREM 2.15 [HHL⁺23a, Theorem 5.3.6]. *Let $\mathcal{I}$ be an ∞ -category. Then, there is an equivalence*

$$\mathbf{Fun}_{\mathbf{L}}^{\text{lax}}(\mathcal{I}, \mathbf{Cat}) \simeq \mathbf{Fun}_{\mathbf{R}}^{\text{oplax}}(\mathcal{I}^{\text{op}}, \mathbf{Cat})$$

which sends a diagram $F: \mathcal{I} \rightarrow \mathbf{Cat}^L$ to the corresponding diagram $G: \mathcal{I}^{\text{op}} \rightarrow \mathbf{Cat}^R$ of right adjoints.

Example 2.16. Suppose $F, F': \mathcal{I} \rightarrow \mathbf{Cat}$ are two objects of $\mathbf{Fun}_{\mathbf{L}}^{\text{lax}}(\mathcal{I}, \mathbf{Cat})$, and write G and G' , respectively, for the associated $\mathcal{I}^{\text{op}}$ -shaped diagrams of right adjoints. Consider a lax natural transformation $\gamma: F \Rightarrow F'$, given at a map $f: i \rightarrow j$ in $\mathcal{I}$ by the following laxly commuting square.

$$\begin{array}{ccc} F(i) & \xrightarrow{F(f)} & F(j) \\ \gamma_i \downarrow & \nearrow \alpha & \downarrow \gamma_j \\ F'(i) & \xrightarrow{F'(f)} & F'(j) \end{array} \quad (2.2)$$

By [HHL⁺23a, Proposition 3.2.7], the first mate equivalence sends γ to the oplax natural transformation $\gamma: G \Rightarrow G'$ with squares of the following form.

$$\begin{array}{ccc} G(j) & \xrightarrow{G(f)} & G(i) \\ \gamma_j \downarrow & \swarrow \beta & \downarrow \gamma_i \\ G'(j) & \xrightarrow{G'(f)} & G'(i) \end{array} \quad (2.3)$$

The oplax square above is filled by the *Beck–Chevalley transformation* β of α , defined to be the composite

$$\gamma_j G(f) \xRightarrow{\eta} G'(f) F'(f) \gamma_i G(f) \xRightarrow{\alpha} G'(f) \gamma_j F(f) G(f) \xRightarrow{\epsilon} G'(f) \gamma_i.$$

For the second mate equivalence, we again introduce some notation.

DEFINITION 2.17. Let $\mathcal{I}$ be an ∞ -category. We write

$$\mathrm{Fun}^{\mathrm{R}, \mathrm{lax}}(\mathcal{I}, \mathbf{Cat}) \quad \text{and} \quad \mathrm{Fun}^{\mathrm{L}, \mathrm{op}, \mathrm{lax}}(\mathcal{I}, \mathbf{Cat})$$

for the wide subcategories of $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{I}, \mathbf{Cat})$ and $\mathrm{Fun}^{\mathrm{op}, \mathrm{lax}}(\mathcal{I}, \mathbf{Cat})$, respectively, spanned on morphisms by those (op)lax natural transformations $\eta: F \Rightarrow G$ such that each component η_i is a right/left adjoint, respectively.

THEOREM 2.18 [HHL⁺23a, Theorem 5.3.5]. *Let $\mathcal{I}$ be an ∞ -category. Then, there is an equivalence*

$$\mathbf{Fun}^{\mathrm{R}, \mathrm{lax}}(\mathcal{I}, \mathbf{Cat}) \simeq \mathbf{Fun}^{\mathrm{L}, \mathrm{op}, \mathrm{lax}}(\mathcal{I}, \mathbf{Cat})^{(1,2)\text{-op}},$$

where $(-)^{(1,2)\text{-op}}$ denotes the functor obtained by passing to opposites on the level of both 1- and 2-morphisms.

Example 2.19. Given the square (2.2), let γ_i^{L} and γ_j^{L} denote the left adjoints of γ_i and γ_j , respectively. Then, the second mate equivalence sends the square (2.2) to the following oplaxly commuting square.

$$\begin{array}{ccc} F'(i) & \xrightarrow{F'(f)} & F'(j) \\ \gamma_i^{\mathrm{L}} \downarrow & \swarrow \beta & \downarrow \gamma_j^{\mathrm{L}} \\ F(i) & \xrightarrow{F(f)} & F(j) \end{array}$$

Here β is the Beck–Chevalley transformation of α , defined to be the composite

$$\gamma_j^{\mathrm{L}} F'(f) \xRightarrow{\eta} \gamma_j^{\mathrm{L}} F'(f) \gamma_i^{\mathrm{L}} \xRightarrow{\alpha} \gamma_j^{\mathrm{L}} \gamma_j F(f) \gamma_i^{\mathrm{L}} \xRightarrow{\epsilon} F(f) \gamma_i^{\mathrm{L}}$$

Example 2.20. Specializing the second mate equivalence to the case $\mathcal{I} \simeq *$, we obtain an equivalence $\mathbf{Cat}^{\mathrm{R}} \simeq (\mathbf{Cat}^{\mathrm{L}})^{(1,2)\text{-op}}$.

As a first example of the mate equivalence, we observe that one can identify certain lax and oplax limits taken over adjoint diagrams.

PROPOSITION 2.21. *Let $G: \mathcal{I} \rightarrow \mathbf{Cat}^{\mathrm{R}}$ be a diagram of right adjoints. Write $F: \mathcal{I}^{\mathrm{op}} \rightarrow \mathbf{Cat}^{\mathrm{L}}$ for the diagram of left adjoints. Then, there exists an equivalence $\mathrm{op}\mathrm{lax}\mathrm{lim}_{\mathcal{I}} G \simeq \mathrm{lax}\mathrm{lim}_{\mathcal{I}^{\mathrm{op}}} F$.*

Proof. On categories, this follows immediately from the following chain of equivalences:

$$\mathrm{oplaxlim} G = \mathrm{Nat}^{\mathrm{oplax}}(\Delta(*), G) \simeq \mathrm{Nat}^{\mathrm{lax}}(\Delta(*), F) = \mathrm{laxlim} F,$$

where the first and third are by definition, and the middle one follows from the definition of adjoint diagrams in higher category theory. $\square$

3. Unstable global homotopy theory

In this section, we recall the necessary background on global homotopy theory following [GH07, Rez14, LNP25].

To begin, recall that a *global family* is a collection of compact Lie groups $\mathcal{E}$ which is closed under isomorphisms, passage to subgroups, and quotients. A global family is said to be *multiplicative* if, in addition, it is closed under finite products.

DEFINITION 3.1. Let $\mathcal{E}$ be a global family of compact Lie group.

- (1) We let $\mathrm{Glo}_{\mathcal{E}}$ denote the *global orbit ∞ -category* of [GH07] with isotropy in $\mathcal{E}$, whose objects are given by $\mathbf{B}G$ for $G \in \mathcal{E}$. Up to homotopy, morphisms $\mathbf{B}H \rightarrow \mathbf{B}G$ in $\mathrm{Glo}_{\mathcal{E}}$ are given by conjugacy classes of continuous group homomorphisms $\alpha: H \rightarrow G$. More precisely, by [LNP25, Proposition 6.3], we have

$$\mathrm{Glo}_{\mathcal{E}}(\mathbf{B}H, \mathbf{B}G) \simeq \coprod_{[\alpha]} BC(\alpha), \quad (3.1)$$

where $[\alpha]$ runs through the set of conjugacy classes of continuous group homomorphisms from H to G , and $C(\alpha)$ denotes the centralizer of the image of α .

- (2) We denote by $\mathrm{Orb}_{\mathcal{E}} \subseteq \mathrm{Glo}_{\mathcal{E}}$ the wide subcategory spanned by those morphisms which are represented by an injective group homomorphism.
- (3) The ∞ -category of $\mathcal{E}$ -*global spaces* $\mathcal{S}_{\mathcal{E}\text{-gl}} := \mathrm{Fun}(\mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}}, \mathcal{S})$ is the presheaf ∞ -category on $\mathrm{Glo}_{\mathcal{E}}$. We identify $\mathbf{B}G$ with its image under the Yoneda embedding.

Notation 3.2. When $\mathcal{E}$ is the global family of all compact Lie groups, we omit the $\mathcal{E}$ and simply write Glo , Orb , and $\mathcal{S}_{\mathrm{gl}}$ for these ∞ -categories. When $\mathcal{E}$ is the global family of abelian compact Lie groups, we simplify the notation to $\mathrm{Glo}_{\mathrm{ab}}$, $\mathrm{Orb}_{\mathrm{ab}}$, and $\mathcal{S}_{\mathrm{ab}}$. Similarly, when we restrict to finite abelian groups, we write $\mathrm{Glo}_{\mathrm{fab}}$, $\mathrm{Orb}_{\mathrm{fab}}$, and $\mathcal{S}_{\mathrm{fab}}$.

Notation 3.3. When it is clear from the context, we often simply call an object of $\mathcal{S}_{\mathcal{E}\text{-gl}}$ a global space, leaving the family $\mathcal{E}$ implicit. We also do this with all other ‘global’ objects we consider in this article.

Remark 3.4. By [GM23, Remark 2.14], the ∞ -category Glo_* of pointed objects in Glo is equivalent to CptLie , the topologically enriched category of compact Lie groups. In particular, we obtain a functor $\mathbf{B}(-): \mathrm{CptLie} \rightarrow \mathrm{Glo}$ which ‘forgets the basepoint’.

Remark 3.5. If $\mathcal{E}$ is a multiplicative global family, then $\mathrm{Glo}_{\mathcal{E}}$ has finite products inherited from CptLie . This follows, for example, from a simple computation using [LNP25, Proposition 6.4].

Crucial to the story of global homotopy theory is its close relationship to equivariant homotopy theory. It follows from [LNP25, Lemma 6.13] that, for any $G \in \mathcal{E}$, the slice ∞ -category $(\mathrm{Orb}_{\mathcal{E}})_{/\mathbf{B}G}$ can be identified with Orb_G , the orbit ∞ -category of the compact Lie group G . Restricting along the functor $(\mathrm{Orb}_{\mathcal{E}})_{/\mathbf{B}G} \rightarrow \mathrm{Glo}_{\mathcal{E}}$, $(\mathbf{B}H \rightarrow \mathbf{B}G) \mapsto \mathbf{B}H$, and using Elmendorf’s

theorem yields a *restriction* functor

$$\mathrm{res}_G: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathcal{S}_G$$

into the ∞ -category of G -spaces. The right Kan extension along $\mathrm{Orb}_{\mathcal{E}}/\mathbf{B}G \rightarrow \mathrm{Glo}_{\mathcal{E}}$ defines a right adjoint to res_G . We record the following special case.

Notation 3.6. We denote the right adjoint of $\mathrm{res}_e: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathcal{S}$ by $X \mapsto X^{(-)}$. One verifies that this right adjoint is in fact fully faithful.

Example 3.7. For any compact abelian Lie group K , we have $\mathbf{B}K = BK^{(-)}$ (see the discussion in [Sch20, Theorem 1.2.32]).

The restriction functor res_G also admits a left adjoint $-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}$, which is given by left Kan extension. It is important to record the following special case.

Notation 3.8. The functor $-//e: \mathcal{S} \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}$ is fully faithful and we refer to a global space in its image as a *constant* global space. Given a space X , we identify X with its constant global space $X//e$.

One verifies that $\mathrm{pt}//G \simeq \mathbf{B}G$, so we obtain an induced functor $-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}/\mathbf{B}G$. To describe the essential image of this functor, we introduce the following definition.

DEFINITION 3.9. A morphism of global spaces $f: \mathcal{X} \rightarrow \mathcal{Y}$ is *faithful* (or *representable*) if, for every $\mathbf{B}G \in \mathrm{Glo}_{\mathcal{E}}$ and normal subgroup $N \triangleleft G$, the following diagram is a pullback.

$$\begin{array}{ccc} \mathcal{X}(\mathbf{B}G/N) & \longrightarrow & \mathcal{X}(\mathbf{B}G) \\ \downarrow & & \downarrow \\ \mathcal{Y}(\mathbf{B}G/N) & \longrightarrow & \mathcal{Y}(\mathbf{B}G) \end{array}$$

We distinguish faithful maps by writing $\mathcal{X} \hookrightarrow \mathcal{Y}$.

Example 3.10. A morphism $\mathbf{B}H \rightarrow \mathbf{B}G$ in Glo is faithful if and only if it can be represented by an injective continuous group homomorphism $H \rightarrow G$.

Example 3.11. By Equation (3.1), we see that $\mathbf{B}e \simeq \mathrm{pt}$, and $\mathrm{Map}_{\mathrm{Glo}_{\mathcal{E}}}(\mathrm{pt}, \mathbf{B}G) \simeq BG$. Therefore, we obtain a map $\nu: BG \rightarrow \mathbf{B}G$, given by the counit of the adjunction between spaces and global spaces. Using Equation (3.1) again, it is not hard to see that ν is faithful.

Remark 3.12. By an application of the Yoneda lemma, a map $f: \mathcal{X} \rightarrow \mathcal{Y}$ is faithful if and only if it is right orthogonal to every map $\mathbf{B}p: \mathbf{B}G \rightarrow \mathbf{B}G/N$ between representables such that p is a surjective group homomorphism. In particular, faithful morphisms are closed under pullback.

DEFINITION 3.13. By the dual of [Lur09b, Proposition 5.5.5.7], the faithful maps are the right class of a factorization system on $\mathcal{S}_{\mathcal{E}\text{-gl}}$. We call morphisms in the associated left class *quotient maps* and denote them by $f: \mathcal{X} \twoheadrightarrow \mathcal{Y}$.

Example 3.14. By Remark 3.12, a map $f: \mathbf{B}H \rightarrow \mathbf{B}G$ is a quotient map if and only if it is represented by a surjective group homomorphism.

THEOREM 3.15. Let $\mathcal{E}$ be a global family and consider $G \in \mathcal{E}$. The functor

$$-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}/\mathbf{B}G$$

is fully faithful and an object $f: \mathcal{X} \rightarrow \mathbf{B}G$ is in its essential image if and only if f is faithful.

Proof. The claim for the global family of all compact Lie groups is proved in [GM23, Proposition 2.19]. We can obtain the general case from this one by noting that there is a factorization

$$-//G: \mathcal{S}_G \rightarrow (\mathcal{S}_{\mathcal{E}\text{-gl}})_{/\mathbf{BG}} \rightarrow (\mathcal{S}_{\text{gl}})_{/\mathbf{BG}},$$

where both functors are fully faithful. $\square$

Remark 3.16. As observed in [Rez14, § 5.3], the functor $-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}/\mathbf{BG}}$ admits a left adjoint, and so the ∞ -category of G -spaces is a Bousfield localization of global spaces over $\mathbf{BG}$. We may describe this left adjoint as follows. Given a map of global spaces $f: \mathcal{X} \rightarrow \mathbf{BG}$, we may factor f as a quotient map followed by a faithful map $\mathcal{X} \twoheadrightarrow \mathcal{X}' \hookrightarrow \mathbf{BG}$. The left adjoint applied to f is the map $\mathcal{X}' \hookrightarrow \mathbf{BG}$, while the map $\mathcal{X} \twoheadrightarrow \mathcal{X}'$ gives the unit of the adjunction.

Remark 3.17. We can also give an interpretation of the functoriality of equivariant spaces using the previous result. Given a group homomorphism $\alpha: H \rightarrow G$, pullback and postcomposition by $B\alpha: \mathbf{BH} \rightarrow \mathbf{BG}$ induces an adjunction

$$\mathbf{B}\alpha_!: \mathcal{S}_{\mathcal{E}\text{-gl}/\mathbf{BH}} \rightleftarrows \mathcal{S}_{\mathcal{E}\text{-gl}/\mathbf{BG}} : \mathbf{B}\alpha^*.$$

As observed in Remark 3.12, pulling back preserves faithful maps. Therefore, the right adjoint above restricts to a functor

$$(\mathbf{B}\alpha)^*: (\mathcal{S}_{\mathcal{E}\text{-gl}})_{/\mathbf{BG}}^{\text{fth}} \rightarrow (\mathcal{S}_{\mathcal{E}\text{-gl}})_{/\mathbf{BH}}^{\text{fth}}$$

between faithful morphisms. By [LNP25, Proposition 6.17], this functor agrees under the equivalences of Theorem 3.15 with the standard restriction functoriality $\alpha^*: \mathcal{S}_G \rightarrow \mathcal{S}_H$ of equivariant spaces.

It follows, by standard arguments, that the left adjoint of $\mathbf{B}\alpha^*$, restricted to global spaces over $\mathbf{BG}$ and $\mathbf{BH}$ with faithful structure map, is computed by first applying $\mathbf{B}\alpha_!$ and then reflecting back into $(\mathcal{S}_{\mathcal{E}\text{-gl}})_{/\mathbf{BG}}^{\text{fth}}$. This agrees, by uniqueness of adjoints, with the induction functor on equivariant spaces. In particular, the counit of the adjunction $\alpha_! \dashv \alpha^*$ is given by factoring the counit $\alpha^* X // H \simeq (\mathbf{B}\alpha)^*(X // G) \rightarrow X // G$ of the pullback–postcomposition adjunction into a quotient map followed by a faithful map, as in the following diagram.

$$\begin{array}{ccccc} \alpha^* X // H & \twoheadrightarrow & \alpha_! \alpha^* X // G & \xhookrightarrow{\epsilon} & X // G \\ \downarrow & \lrcorner & & & \downarrow \\ \mathbf{BH} & \xrightarrow{\mathbf{B}\alpha} & & & \mathbf{BG} \end{array}$$

We also briefly discuss the identification of free G -spaces under the equivalence of Theorem 3.15. To establish notation, we recall that $\text{End}_{\text{Orb}_G}(G/e) \simeq G$, and so we obtain a restriction functor

$$\mathcal{S}_G := \text{Fun}(\text{Orb}_G, \mathcal{S}) \xrightarrow{\text{ev}_{G/e}} \text{Fun}(BG, \mathcal{S}) \simeq \mathcal{S}_{/BG}.$$

We call G -spaces in the image of the fully faithful left adjoint, *free* G -spaces.

PROPOSITION 3.18. *A map $\mathcal{X} \rightarrow BG$ of global spaces is faithful if and only if it is in the image of the constant functor. Now, consider the map $\nu: BG \rightarrow \mathbf{BG}$ from Example 3.11. Both functors in the adjunction*

$$\nu_!: \mathcal{S}_{\mathcal{E}\text{-gl}/BG} \rightleftarrows \mathcal{S}_{\mathcal{E}\text{-gl}/\mathbf{BG}} : \nu^*$$

preserve objects with a faithful structure map, and the following square commutes.

$$\begin{array}{ccc} (\mathcal{S}_{\mathcal{E}\text{-gl}})^{\text{fth}}/\mathbf{BG} & \xrightarrow{\nu^*} & (\mathcal{S}_{\mathcal{E}\text{-gl}})^{\text{fth}}/BG \\ \text{res}_G \downarrow \sim & & \sim \downarrow \text{res}_e \\ \mathcal{S}_G & \xrightarrow{\text{ev}_{G/e}} & \mathcal{S}/BG \end{array}$$

In particular, passing to left adjoints, we conclude that an object $\mathcal{X} \rightarrow \mathbf{BG}$ of $\mathcal{S}_{\mathcal{E}\text{-gl}}/\mathbf{BG}$ corresponds to a free G -space if and only if $\mathcal{X}$ is a constant global space and the map to $\mathbf{BG}$ is faithful.

Proof. Suppose $f: \mathcal{X} \rightarrow BG$ is a faithful map of global spaces. For any $\mathbf{BH}$, the following square is a pullback square.

$$\begin{array}{ccc} \mathcal{X}(\text{pt}) & \longrightarrow & \mathcal{X}(\mathbf{BH}) \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\sim} & BG \end{array}$$

Thus, we conclude that the map $\mathcal{X}(\text{pt}) \rightarrow \mathcal{X}(\mathbf{BH})$ is an equivalence, and so $\mathcal{X}$ is constant. For the second statement, we first note that faithful maps are closed under pullback, and so ν^* clearly preserves faithful maps. Since composition of faithful maps is again faithful, and $\nu_!$ is given by postcomposing with ν , it follows that $\nu_!$ preserves faithful maps. The commutativity of the diagram follows from the observation that res_e (which agrees with evaluation at pt) preserves pullback squares, so we obtain the first equivalence in the following sequence:

$$\text{res}_e \nu^* \mathcal{X} \simeq \text{res}_e X \simeq \text{ev}_{G/e} \circ \text{res}_G \mathcal{X}. \quad \square$$

The connection between global and equivariant homotopy theory, expressed by Theorem 3.15, can be extended to give a different perspective on global homotopy theory. Recall from [LNP25, § 6] that there exists a functor $\mathcal{S}_\bullet: \text{Glo}_{\mathcal{E}}^{\text{op}} \rightarrow \text{Cat}$, which sends $\mathbf{BG}$ to $\mathcal{S}_G$, and sends a morphism $\mathbf{B}\alpha: \mathbf{BH} \rightarrow \mathbf{BG}$ to the restriction-inflation functor $\alpha^*: \mathcal{S}_G \rightarrow \mathcal{S}_H$. We recall the following result.

THEOREM 3.19 [LNP25, Theorem 6.18]. *Let $\mathcal{E}$ be a global family of compact Lie groups. Then, the restriction functors induce an equivalence of ∞ -categories*

$$\mathcal{S}_{\mathcal{E}\text{-gl}} \simeq \text{laxlim}_{\text{Glo}_{\mathcal{E}}^{\text{op}}, \text{Orb}_{\mathcal{E}}^{\text{op}}}^{\dagger} \mathcal{S}_\bullet,$$

where we view $\text{Glo}_{\mathcal{E}}^{\text{op}}$ as a marked ∞ -category via the inclusion $\text{Orb}_{\mathcal{E}}^{\text{op}} \subset \text{Glo}_{\mathcal{E}}^{\text{op}}$.

Remark 3.20. In this article, we have many occasions to take a partially (op)lax limit over the marked ∞ -category $(\text{Glo}_{\mathcal{E}}, \text{Orb}_{\mathcal{E}})$, as well as subcategories and opposite categories thereof. We therefore make the global¹ convention that any category derived from Glo is marked by the collection of faithful edges.

Remark 3.21. Using the previous remark, we can informally summarize Theorem 3.19 as follows. A global space $\mathcal{X}$ is equivalent to the data of:

- a G -space $\text{res}_G \mathcal{X}$ for each group $G \in \mathcal{E}$;
- an H -equivariant map $f_\alpha: \alpha^* \text{res}_G \mathcal{X} \rightarrow \text{res}_H \mathcal{X}$ for each continuous group homomorphism $\alpha: H \rightarrow G$;

¹Pun intended.

- the maps f_α are functorial in the sense that there are given homotopies $f_{\beta \circ \alpha} \simeq f_\beta \circ \beta^*(f_\alpha)$ for all composable maps α and β , and $f_{\text{id}} \simeq \text{id}$;
- the map f_α is an equivalence for every continuous *injective* homomorphism α ;
- a homotopy between the map f_{c_g} induced by the conjugation isomorphism and the map $l_g: c_g^* \text{res}_G X \rightarrow \text{res}_G X$ given by left multiplication by g ;
- higher coherences for the homotopies.

Here, we record a consequence of this description for constructing functors out of global spaces.

PROPOSITION 3.22. *Let $\mathcal{C}$ be an ∞ -category. Restriction along the functors $-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}$ induces a functor*

$$\Phi: \text{Fun}(\mathcal{S}_{\mathcal{E}\text{-gl}}, \mathcal{C}) \rightarrow \text{oplaxlim}_{\text{Glo}_{\mathcal{E}}^{\text{op}}}^{\dagger} \text{Fun}(\mathcal{S}_{\bullet}, \mathcal{C}),$$

where the diagram $\text{Fun}(\mathcal{S}_{\bullet}, \mathcal{C})$ is functorial in $\text{Glo}_{\mathcal{E}}^{\text{op}}$ as follows: a morphism $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ in $\text{Glo}_{\mathcal{E}}$ is sent to restriction along the induction functor $\alpha_!: \mathcal{S}_H \rightarrow \mathcal{S}_G$. Furthermore, suppose $\mathcal{C}$ admits all small colimits. Then Φ restricts to an equivalence

$$\text{Fun}^{\text{L}}(\mathcal{S}_{\mathcal{E}\text{-gl}}, \mathcal{C}) \simeq \text{oplaxlim}_{\text{Glo}_{\mathcal{E}}^{\text{op}}}^{\dagger} \text{Fun}^{\text{L}}(\mathcal{S}_{\bullet}, \mathcal{C}).$$

Proof. Recall that $\mathcal{S}_{\mathcal{E}\text{-gl}} \simeq \text{laxlim}^{\dagger} \mathcal{S}_G$. In particular, we obtain a universal partially lax cone as follows.

$$\begin{array}{ccc} & & \mathcal{S}_H \\ & \nearrow \text{res}_H & \uparrow \alpha^* \\ \mathcal{S}_{\mathcal{E}\text{-gl}} & \rightleftarrows & \\ & \searrow \text{res}_G & \downarrow \end{array}$$

Passing to left adjoints (by which we mean using the equivalence $\mathbf{Cat}^{\text{R}} \simeq (\mathbf{Cat}^{\text{L}})^{(1,2)\text{-op}}$, see Example 2.20), we obtain a partially oplax cocone as follows.

$$\begin{array}{ccc} \mathcal{S}_H & & \\ \alpha_! \downarrow & \searrow -//H & \\ & \mathcal{S}_{\mathcal{E}\text{-gl}} & \\ \uparrow -//G & \nearrow & \\ \mathcal{S}_G & & \end{array} \quad (3.2)$$

Given a functor $F: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathcal{C}$, we may precompose by this cocone to obtain a partially oplax cocone with target $\mathcal{C}$, encoded by a diagram $\text{Glo}_{\mathcal{E}} \rightarrow \mathbf{Cat} \downarrow^{\text{opl}} \mathcal{C}$. This, in turn, gives an object of $\text{oplaxlim}^{\dagger} \text{Fun}(\mathcal{S}_G, \mathcal{C})$ (see Construction 2.13).

For the final statement we first observe that, because each functor $-//G$ preserves colimits, the functor constructed above restricts appropriately. To see that it is an equivalence, we compute

$$\begin{aligned} \text{Fun}^{\text{L}}(\mathcal{S}_{\mathcal{E}\text{-gl}}, \mathcal{C}) &\simeq \text{Fun}(\text{Glo}_{\mathcal{E}}, \mathcal{C}) \simeq \text{Fun}(\text{oplaxcolim}^{\dagger} \text{Orb}_{\bullet}, \mathcal{C}) \\ &\simeq \text{oplaxlim}^{\dagger} \text{Fun}(\text{Orb}_{\bullet}, \mathcal{C}) \simeq \text{oplaxlim}^{\dagger} \text{Fun}^{\text{L}}(\mathcal{S}_{\bullet}, \mathcal{C}), \end{aligned}$$

where the second and third equivalences are justified by the proof of Theorem 6.18 and Proposition 4.15 of [LNP25], respectively. $\square$

Remark 3.23. Unwinding the proof of the previous proposition, we find that the image of $F: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathcal{C}$ under the functor of Proposition 3.22 and the equivalence of Theorem 3.15 is given by the family of functors $\{F_G(X) = F(X//G)\}$. The oplax structure map $F_G(\alpha_!(-)) \rightarrow F_H(-)$ is given at an object X by applying F to the unique quotient map $X//G \twoheadrightarrow \alpha_!X//H$, for which the following square commutes.

$$\begin{array}{ccc} X//G & \twoheadrightarrow & \alpha_!X//H \\ \downarrow i & & \downarrow \alpha_!(i) \\ \mathbf{B}G & \xrightarrow{f} & \mathbf{B}H \end{array}$$

4. Stable global homotopy theory

Global spectra were defined in [Sch18] as an ∞ -category representing objects for genuine cohomology theories on global spaces. Just as in the unstable setting, one can describe global spectra as a laxly compatible collection of genuine equivariant spectra.

THEOREM 4.1 [LNP25, Theorem 11.10]. *Let $\mathcal{E}$ be a multiplicative global family of compact Lie groups. Then, there exists a functor*

$$\mathrm{Sp}_\bullet: \mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}), \quad \mathbf{B}G \mapsto \mathrm{Sp}_G$$

which sends $\mathbf{B}G$ to the ∞ -category of genuine G -spectra and an equivalence of symmetric monoidal ∞ -categories

$$\mathrm{Sp}_{\mathcal{E}\text{-gl}} \simeq \mathrm{laxlim}^\dagger \mathrm{Sp}_\bullet.$$

In this section, we refine this result and show that, in certain cases, a simpler description of global spectra is possible.

DEFINITION 4.2. Let $\mathcal{E}$ be a multiplicative global family. We say a full subcategory $\mathcal{T} \subset \mathrm{Glo}_{\mathcal{E}}$ is a subcategory of *enough injectives* if:

- (1) $\mathcal{T}$ is closed under products in $\mathrm{Glo}_{\mathcal{E}}$; and
- (2) given any object $\mathbf{B}G$, there exists a faithful map $\mathbf{B}G \hookrightarrow \mathbf{B}K$ for some $\mathbf{B}K \in \mathcal{T}$.

We view $\mathcal{T}$ as a marked ∞ -category by once again marking the subcategory of faithful maps.

Remark 4.3. Recall from Remark 3.5 that $\mathrm{Glo}_{\mathcal{E}}$ has finite products, so condition (1) of the previous definition is well-defined.

Example 4.4. The full subcategory $\Sigma \subset \mathrm{Glo}_{\mathrm{fin}}$ of the global orbit ∞ -category of finite groups spanned by those $\mathbf{B}G$ such that G is a finite product of symmetric groups is a subcategory of enough injective objects. This is a consequence of Cayley's theorem.

Example 4.5. The full subcategory $\mathrm{Tori} \subset \mathrm{Glo}_{\mathrm{ab}}$ of the global orbit ∞ -category of abelian compact Lie groups spanned by those $\mathbf{B}G$ such that G is a torus is a subcategory of enough injective objects.

Example 4.6. The full subcategory $\mathrm{U} \subset \mathrm{Glo}$ of the global orbit ∞ -category of compact Lie groups spanned by those $\mathbf{B}G$ such that G is a finite product of unitary groups is a subcategory of enough injective objects. This follows from the fact that any compact Lie group admits a faithful unitary representation, by an application of the Peter–Weyl theorem.

The variant of Theorem 4.1 which we prove is as follows.

THEOREM 4.7. *Let $\mathcal{E}$ be a multiplicative global family of compact Lie groups, and suppose $\mathcal{T} \subset \text{Glo}_{\mathcal{E}}$ is a family of enough injective objects. Then, there exists an equivalence of symmetric monoidal ∞ -categories*

$$\text{Sp}_{\mathcal{E}\text{-gl}} \simeq \text{laxlim}_{\mathcal{T}^{\text{op}}}^{\dagger} \text{Sp}_{\bullet}.$$

These equivalences follow from the fact that the inclusion $\mathcal{T}^{\text{op}} \subset \text{Glo}_{\mathcal{E}}^{\text{op}}$ is marked final, a concept from [Abe22] which we recall now.

PROPOSITION 4.8. *Let $F: \mathcal{I} \rightarrow \mathcal{J}$ be a functor of marked ∞ -categories. The following are equivalent.*

- (1) *Given any functor $G: \mathcal{J} \rightarrow \text{Cat}$, $\text{laxlim}^{\dagger} G \simeq \text{laxlim}^{\dagger} GF$ whenever either exist.*
- (2) *Given any functor $G: \mathcal{J} \rightarrow \text{Cat}$, $\text{oplaxlim}^{\dagger} G \simeq \text{oplaxlim}^{\dagger} GF$ whenever either exist.*

Proof. This follows from the fact that $(-)^{\text{op}}$ is an auto-equivalence of Cat which sends lax limits to oplax limits and vice versa. $\square$

DEFINITION 4.9. We say a functor $F: \mathcal{I} \rightarrow \mathcal{J}$ of marked ∞ -categories is *marked final* if it satisfies the equivalent conditions of the previous proposition. We say F is *marked cofinal* if F^{op} is marked final.

Remark 4.10. A functor F is marked cofinal if and only if it preserves partially (op)lax colimits, in the sense of Proposition 4.8. Therefore, our definition agrees with [Abe22, Definition 5.4], and we may freely cite their results.

Notation 4.11. Given a functor $F: \mathcal{I} \rightarrow \mathcal{J}$ such that $\mathcal{J}$ is a marked ∞ -category and $j \in \mathcal{J}$, we write $\mathcal{I}_{j/}$ for the comma category $\{j\} \downarrow F$. We consider this as a marked ∞ -category by marking all the edges whose projection to $\mathcal{J}$ is marked.

Given a marked ∞ -category $\mathcal{C}$, we write $\mathcal{L}(\mathcal{C})$ for the localization of $\mathcal{C}$ at the marked edges in $\mathcal{C}$.

LEMMA 4.12. *Let $\mathcal{C}$ be a marked ∞ -category which admits finite products. Suppose that the functor $X \times (-): \mathcal{C} \rightarrow \mathcal{C}$ is a marked functor for every object $X \in \mathcal{C}$. Then, $\mathcal{L}(\mathcal{C})$ admits finite products and the localization functor $\mathcal{C} \rightarrow \mathcal{L}(\mathcal{C})$ preserves them.*

Proof. The product functor $- \times -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is determined by being right adjoint to the diagonal functor $\Delta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$. Our assumptions imply that the product derives to a functor $\mathcal{L}(- \times -): \mathcal{L}(\mathcal{C} \times \mathcal{C}) \rightarrow \mathcal{L}(\mathcal{C})$, which is right adjoint to $\mathcal{L}(\Delta_{\mathcal{C}})$. Here, we consider $\mathcal{C} \times \mathcal{C}$ as a marked ∞ -category by taking the product in marked ∞ -categories; explicitly, this is given by marking an edge (f, g) whenever both f and g are marked. One can compute that $\mathcal{L}(\mathcal{C} \times \mathcal{C}) \simeq \mathcal{L}(\mathcal{C}) \times \mathcal{L}(\mathcal{C})$; that is, $\mathcal{L}$ preserves products. Therefore, we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xleftarrow[\text{---} \times \text{---}]{\Delta_{\mathcal{C}}} & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{C} \times \mathcal{C}) & \xleftarrow[\mathcal{L}(- \times -)]{\mathcal{L}(\Delta_{\mathcal{C}})} & \mathcal{L}(\mathcal{C}) \\ \sim \downarrow & & \downarrow = \\ \mathcal{L}(\mathcal{C}) \times \mathcal{L}(\mathcal{C}) & \xleftarrow[\text{---} \times \text{---}]{\Delta_{\mathcal{L}(\mathcal{C})}} & \mathcal{L}(\mathcal{C}) \end{array}$$

The dotted arrow defines a product in $\mathcal{L}(\mathcal{C})$. The commutativity of the above diagram also shows that the localization functor preserves products. $\square$

It will be helpful to keep the following example in mind.

Example 4.13. Let $\mathcal{T} \subset \text{Glo}_{\mathcal{E}}$ be a subcategory of enough injectives. For any $\mathbf{BG} \in \text{Glo}_{\mathcal{E}}$, we can form the comma category $\mathcal{T}_{\mathbf{BG}/}$. A morphism as follows in this comma category is marked precisely when f is represented by an injective group homomorphism.

$$\begin{array}{ccc} & \mathbf{BG} & \\ \swarrow & & \searrow \\ \mathbf{BK} & \xrightarrow{f} & \mathbf{BJ} \end{array}$$

Note that $\mathcal{T}_{\mathbf{BG}/}$ has finite products, since $\mathcal{T}$ is closed in $\text{Glo}_{\mathcal{E}}$ under finite products. Furthermore, taking products with a fixed object of $\mathcal{T}_{\mathbf{BG}/}$ preserves faithful maps, and therefore Lemma 4.12 implies that $\mathcal{L}(\mathcal{T}_{\mathbf{BG}/})$ has finite products and that the localization $\mathcal{T}_{\mathbf{BG}/} \rightarrow \mathcal{L}(\mathcal{T}_{\mathbf{BG}/})$ preserves them.

The key ingredient for the proof of Theorem 4.7 is the following criterion.

THEOREM 4.14 [Abe22, Theorem 5.10]. *A functor $F: I \rightarrow J$ is marked cofinal if and only if the following hold.*

- (1) *For every $j \in J$, there exists an object $g: j \rightarrow F(i)$ of $I_{j/}$ which is marked when viewed as a morphism in J .*
- (2) *Any object of $I_{j/}$ of the form above is an initial object in $\mathcal{L}(I_{j/})$.*
- (3) *Given a marked edge $j \rightarrow j'$ of J , the induced map $\mathcal{L}(I_{j'/}) \rightarrow \mathcal{L}(I_{j/})$ preserves initial objects.*

In order to verify condition (2) of the previous criterion, we need the following consequence of the axioms of a subcategory of enough injectives.

LEMMA 4.15. *Let $\mathcal{T} \subset \text{Glo}_{\mathcal{E}}$ be a subcategory of enough injective objects. Given any map $f: \mathbf{BH} \rightarrow \mathbf{BG}$ in $\text{Glo}_{\mathcal{E}}$, there is a commutative square as follows in $\text{Glo}_{\mathcal{E}}$, such that the vertical maps are faithful and $\mathbf{BK}, \mathbf{BJ} \in \mathcal{T}$.*

$$\begin{array}{ccc} \mathbf{BK} & \xrightarrow{\tilde{f}} & \mathbf{BJ} \\ \uparrow & & \uparrow \\ \mathbf{BH} & \xrightarrow{f} & \mathbf{BG} \end{array}$$

Proof. By condition (2) for $\mathcal{T}$, there are faithful maps $i: \mathbf{BH} \hookrightarrow \mathbf{BH}'$ and $j: \mathbf{BG} \hookrightarrow \mathbf{BG}'$ with $\mathbf{BH}', \mathbf{BG}' \in \mathcal{T}$. We can then form the following commutative square.

$$\begin{array}{ccc} \mathbf{BH}' \times \mathbf{BG}' & \xrightarrow{\text{pr}_2} & \mathbf{BG}' \\ \uparrow i \times (j \circ f) & & \uparrow j \\ \mathbf{BH} & \xrightarrow{f} & \mathbf{BG} \end{array}$$

It is only left to note that $\mathbf{BH}' \times \mathbf{BG}' \in \mathcal{T}$ by condition (1), and that the map $i \times (j \circ f)$ is faithful since i is so. $\square$

THEOREM 4.16. *Let $\mathcal{T} \subset \text{Glo}_{\mathcal{E}}$ be a subcategory of enough injective objects. Then, the inclusion $\mathcal{T}^{\text{op}} \rightarrow \text{Glo}_{\mathcal{E}}^{\text{op}}$ is marked final.*

Proof. We apply Theorem 4.14 to the inclusion $\mathcal{T} \rightarrow \text{Glo}_{\mathcal{E}}$. Condition (1) follows from Definition 4.2(2). Note that, if we assume condition (2), then condition (3) is immediate from

the fact that a composite of faithful maps is faithful. Therefore, all that remains is condition (2). Fix an object $\mathbf{B}G \in \mathbf{Glo}_{\mathcal{E}}$, and consider the ∞ -category $\mathcal{T}_{\mathbf{B}G/}$. As discussed in Example 4.13, the comma category $\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})$ has finite products and the localization $\mathcal{T}_{\mathbf{B}G/} \rightarrow \mathcal{L}(\mathcal{T}_{\mathbf{B}G/})$ preserves them. However, given an object $\alpha: \mathbf{B}G \rightarrow X$ in $\mathcal{T}_{\mathbf{B}G/}$, the following composite in $\mathcal{T}_{\mathbf{B}G/}$ exhibits pr_1 as an equivalence in $\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})$.

$$\begin{array}{ccccc} & & \mathbf{B}G & & \\ & \swarrow \alpha & \downarrow \alpha \times \alpha & \searrow \alpha & \\ \mathbf{B}K & \xrightarrow{\Delta} & \mathbf{B}K \times K & \xrightarrow{\mathrm{pr}_1} & \mathbf{B}K \\ & \searrow \mathrm{id} & & \swarrow & \end{array}$$

That is, every object is equivalent to its product. On mapping spaces we obtain that

$$\mathrm{pr}_1: \mathrm{Map}_{\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})}(\beta, \alpha) \times \mathrm{Map}_{\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})}(\beta, \alpha) \rightarrow \mathrm{Map}_{\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})}(\beta, \alpha)$$

is an equivalence for every two objects α, β of $\mathcal{T}_{\mathbf{B}G/}$, which implies that $\mathrm{Map}_{\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})}(\beta, \alpha)$ is always either empty or contractible. Condition (2) of Theorem 4.14 requires that any faithful map $i: \mathbf{B}G \hookrightarrow \mathbf{B}J$ is initial in $\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})$. First, we note that the following commutative diagram shows that any two faithful maps $i: \mathbf{B}G \hookrightarrow \mathbf{B}J$ and $j: \mathbf{B}G \hookrightarrow \mathbf{B}K$ are isomorphic in $\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})$.

$$\begin{array}{ccccc} & & \mathbf{B}G & & \\ & \swarrow i & \downarrow i \times j & \searrow j & \\ \mathbf{B}J & \xrightarrow{\quad} & \mathbf{B}J \times K & \xleftarrow{\quad} & \mathbf{B}K \end{array}$$

Combining this with the observation before, we see that it suffices to show that for any $\beta \in \mathcal{T}_{\mathbf{B}G/}$, there exists a faithful map $i: \mathbf{B}G \hookrightarrow \mathbf{B}J$ such that $\mathrm{Map}_{\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})}(i, \beta)$ is non-empty. For this, we apply Lemma 4.15 to the map β to obtain the following diagram.

$$\begin{array}{ccc} \mathbf{B}J & \xrightarrow{\bar{\beta}} & \mathbf{B}L \\ \uparrow i & & \uparrow j \\ \mathbf{B}G & \xrightarrow{\beta} & \mathbf{B}K \end{array}$$

Observe that we may interpret this as a zig-zag

$$i \xrightarrow{\bar{\beta}} \bar{\beta}i \xleftarrow{j} \beta$$

in $\mathcal{T}_{\mathbf{B}G/}$, where the second map is faithful. This proves that $\mathrm{Map}_{\mathcal{L}(\mathcal{T}_{\mathbf{B}G/})}(i, \beta)$ is non-empty. $\square$

Proof of Theorem 4.7. Since the forgetful functor $\mathrm{Cat}^{\otimes} \rightarrow \mathrm{Cat}$ is conservative and preserves partially lax limits (see [LNP25, Remark 5.1]), it suffices to verify that the induced functors on underlying ∞ -categories are equivalences. This now follows from combining Theorems 4.1 and 4.16. $\square$

PART II. Naive global 2-rings

In this part, we introduce naive global and equivariant ring spectra. We give a precise definition of a genuine refinement for such a ring, and introduce the notion of a naive global 2-ring, the central concept of this work. We then discuss how a naive global 2-ring decategorifies to give

a naive global ring and list some examples. Finally, we spend the remainder of the part on the proof of Theorem A and some related material.

5. Naive global rings and genuine refinements

In this section, we introduce the naive analogs of global and equivariant ring spectra. We then discuss multiplicative cohomology theories and introduce the notion of genuine refinements.

Before diving into the main definitions of this section, we review some background on spectrum objects in an ∞ -category. Recall that, given any presentable ∞ -category $\mathcal{B}$, one can define the category $\mathrm{Sp}(\mathcal{B})$ of spectrum objects in $\mathcal{B}$. Combining [Lur17, Proposition 4.8.1.17 and Example 4.8.1.23], we deduce that

$$\mathrm{Sp}(\mathcal{B}) \simeq \mathcal{B} \otimes \mathrm{Sp} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp}).$$

If $\mathcal{B}$ is presentably symmetric monoidal, then $\mathrm{Sp}(\mathcal{B})$ acquires a symmetric monoidal structure uniquely determined by the following universal property: for any stable and presentably symmetric monoidal ∞ -category $\mathcal{C}$, precomposition with the suspension functor $\Sigma_+^\infty: \mathcal{B} \rightarrow \mathrm{Sp}(\mathcal{B})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}, \otimes}(\mathrm{Sp}(\mathcal{B}), \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}, \otimes}(\mathcal{B}, \mathcal{C})$$

(see [GGN15, Theorem 5.1]).

Remark 5.1. For a stable ∞ -category $\mathcal{C}$ and two objects $X, Y \in \mathcal{C}$, we write $\mathrm{map}_{\mathcal{C}}(X, Y)$ for the spectrum of maps from X to Y . Moreover, we let

$$y: \mathcal{C} \rightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}), \quad X \mapsto \mathrm{map}_{\mathcal{C}}(-, X),$$

denote the spectral Yoneda embedding. For an arbitrary presentably symmetric monoidal ∞ -category $\mathcal{B}$, the equivalence $\mathrm{Sp}(\mathcal{B}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp})$ is concretely given by the assignment $X \mapsto \mathrm{map}_{\mathrm{Sp}(\mathcal{B})}(\Sigma_+^\infty(-), X)$.

We now specialize to our cases of interest.

DEFINITION 5.2. We call $\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})$ and $\mathrm{Sp}(\mathcal{S}_G)$ the ∞ -categories of *naive global spectra* and *naive G -spectra*, respectively. Recall that $\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})$ and $\mathrm{Sp}(\mathcal{S}_G)$ both inherit symmetric monoidal structures from the cartesian monoidal structure on $\mathcal{S}_{\mathcal{E}\text{-gl}}$ and $\mathcal{S}_G$. We call objects in $\mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}}))$ and $\mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G))$ (*commutative*) *naive global rings* and (*commutative*) *naive G -rings*, respectively.

Remark 5.3. Since, in this article, we only consider commutative (i.e. $\mathbb{E}_\infty$ -)rings, we typically drop this from the terminology.

The terminology above is intended to distinguish such objects from their genuine analogs, which contain substantially more structure.

DEFINITION 5.4. We call an object of $\mathrm{CAlg}(\mathrm{Sp}_G)$ a (*commutative*) *G -ring* and an object of $\mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$ a (*commutative*) *global ring*. If we wish to emphasize the distinction with naive equivariant/global rings, we may call them *genuine*.

DEFINITION 5.5. Let G be a compact Lie group and recall that there exists a symmetric monoidal colimit-preserving suspension functor $\Sigma_G^\infty: \mathcal{S}_G \rightarrow \mathrm{Sp}_G$. By the universal property of stabilization, this lifts to a strong monoidal left adjoint $\mathrm{Sp}(\mathcal{S}_G) \rightarrow \mathrm{Sp}_G$, whose right adjoint we denote by $\mathbb{U}_G: \mathrm{Sp}_G \rightarrow \mathrm{Sp}(\mathcal{S}_G)$. As the right adjoint of a symmetric monoidal functor, $\mathbb{U}_G$

is canonically lax symmetric monoidal. In particular, we obtain a functor $\mathbb{U}_G: \mathrm{CAlg}(\mathrm{Sp}_G) \rightarrow \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G))$ from G -rings to naive G -rings.

Remark 5.6. Intuitively, the functor $\mathbb{U}_G: \mathrm{CAlg}(\mathrm{Sp}_G) \rightarrow \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G))$ forgets the additional deloopings for representation spheres with nontrivial G -action encoded by the original naive G -ring. Alternatively, once restricted to the heart of Sp_G , this functor forgets from Green functors to coefficient systems rings. Therefore, in general, one may think of a naive G -ring as lacking the additive transfers contained in a (genuine) G -ring.

It has been crucial in equivariant homotopy theory to build and exploit the additional structure contained in a genuine G -spectrum, over and above that of a naive G -spectrum. To systematically consider this, we make the following definition.

DEFINITION 5.7. Let X be a naive G -ring. We say a G -ring $\tilde{X} \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is a *genuine refinement* of X if there is an equivalence $\mathbb{U}_G(\tilde{X}) \simeq X$. The space of genuine refinements of X is the following pullback in Cat .

$$\begin{array}{ccc} \mathrm{GenRef}(X) & \longrightarrow & \mathrm{CAlg}(\mathrm{Sp}_G) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{\{X\}} & \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G)) \end{array}$$

Because $\mathbb{U}_G: \mathrm{CAlg}(\mathrm{Sp}_G) \rightarrow \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G))$ is conservative, $\mathrm{GenRef}(X)$ is a space.

Remark 5.8. We emphasize again that the space $\mathrm{GenRef}(X)$ of genuine refinements of a naive G -ring is not necessarily contractible. For a concrete example we note that, from the discussion in Remark 5.6, it suffices to give two different Green functor structures on the same coefficient system of rings. Consider the following C_2 -coefficient system R .

$$\begin{array}{ccc} R(C_2/C_2) & & \mathbb{F}_2[x]/x^2 \\ \mathrm{res} \left(\downarrow \begin{array}{c} \curvearrowright \\ \mathrm{tr} \end{array} \right) & = & x \mapsto 0 \left(\downarrow \begin{array}{c} \curvearrowright \\ \mathrm{tr} \end{array} \right) \\ R(C_2/1) & & \mathbb{F}_2. \\ \downarrow \begin{array}{c} \curvearrowright \\ \gamma \end{array} & & \downarrow \begin{array}{c} \curvearrowright \\ \mathrm{id} \end{array} \end{array}$$

Note that there are different choices of transfer maps $\mathrm{tr}: \mathbb{F}_2 \rightarrow \mathbb{F}_2[x]/x^2$ such that R is a Green functor; for example, $a \mapsto 0$ and $a \mapsto ax$.

We may also consider a global analog of the previous definition. To do so, we first note that the equivariant suspension spectrum functors assemble to give a natural transformation $\Sigma_\bullet^\infty: \mathcal{S}_\bullet \Rightarrow \mathrm{Sp}_\bullet$ of symmetric monoidal left-adjoint functors defined over $\mathrm{Glo}_\mathcal{E}^{\mathrm{op}}$ (see [LNP25, Proposition 10.5]).

DEFINITION 5.9. We define the *global suspension* functor

$$\Sigma_{\mathrm{gl}}^\infty: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathrm{Sp}_{\mathcal{E}\text{-gl}}$$

as the functor induced on partially lax limits by the natural transformation $\Sigma_\bullet^\infty: \mathcal{S}_\bullet \rightarrow \mathrm{Sp}_\bullet$.

WARNING 5.10. In [Sch18, Construction 4.1.7], Schwede also constructs a suspension spectrum functor $\Sigma_+^\infty: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathrm{Sp}_{\mathcal{E}\text{-gl}}$. We warn the reader that we have not shown that this functor agrees with the functor defined above, but we do expect this to be the case. In fact, we strongly suspect that the methods of [LNP25] suffice to prove this statement. We use the definition above

because its connection to the equivariant suspension spectrum functor is more immediate and, importantly for us, coherent by definition.

DEFINITION 5.11. We write $\Omega_{\bullet}^{\infty}: \mathrm{Sp}_{\bullet} \Rightarrow \mathcal{S}_{\bullet}$ for the lax natural transformation associated to $\Sigma_{\bullet}^{\infty}$ under the second mate equivalence (2.18). In particular, the component $\Omega_G^{\infty}: \mathrm{Sp}_G \rightarrow \mathcal{S}_G$ of $\Omega_{\bullet}^{\infty}$ at G is the right adjoint of $\Sigma_G^{\infty}: \mathcal{S}_G \rightarrow \mathrm{Sp}_G$.

LEMMA 5.12. *The global suspension functor $\Sigma_{\mathrm{gl}}^{\infty}$ is strong monoidal, and admits a right adjoint $\Omega_{\mathrm{gl}}^{\infty} := \mathrm{laxlim}^{\dagger} \Omega_{\bullet}^{\infty}$.*

Proof. As a partially lax limit of strong monoidal functors, $\Sigma_{\mathrm{gl}}^{\infty}$ is automatically strong monoidal. To construct $\Omega_{\mathrm{gl}}^{\infty}$ and exhibit it as the right adjoint of $\Sigma_{\mathrm{gl}}^{\infty}$, we apply the criteria of [Lin24, Proposition 3.22]. For this, it suffices to show that for any faithful morphism $f: \mathbf{B}H \rightarrow \mathbf{B}G$, the Beck–Chevalley natural transformation

$$f^* \Omega_G^{\infty} \Rightarrow \Omega_H^{\infty} f^*$$

is an equivalence. In other words, we have to check that $\Omega_{\bullet}^{\infty}$ is a natural transformation when restricted to $\mathrm{Orb}_{\mathcal{E}}^{\mathrm{op}} \subset \mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}}$. To check this, we can pass to total mates and instead verify that taking suspension spectra commutes with induction. This is a well-known fact, which could, for instance, be checked in one's preferred model of equivariant spectra. $\square$

DEFINITION 5.13. As before, the strong monoidal left adjoint $\Sigma_{\mathcal{E}\text{-gl}}^{\infty}: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \mathrm{Sp}_{\mathcal{E}\text{-gl}}$ lifts to a symmetric monoidal functor out of the stabilization of global spaces, and we write $\mathbb{U}_{\mathrm{gl}}: \mathrm{Sp}_{\mathcal{E}\text{-gl}} \rightarrow \mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})$ for its right adjoint. This is again lax symmetric monoidal, and so induces a functor from global rings to naive global rings.

Given this, we can discuss the notion of a genuine refinement of a naive global ring.

DEFINITION 5.14. Let X be a naive global ring. We say a global ring $\tilde{X} \in \mathrm{CAlg}(\mathcal{S}_{\mathcal{E}\text{-gl}})$ is a *genuine refinement* of X if there is an equivalence $\mathbb{U}_{\mathrm{gl}}(\tilde{X}) \simeq X$. Once again, we can define the space of genuine refinements.

5.1 Multiplicative cohomology theories

We typically construct and compare naive global and equivariant rings via the cohomology theories they represent.

DEFINITION 5.15. Let $\mathcal{B}$ be a presentably symmetric monoidal ∞ -category and let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞ -category. Then, a *multiplicative $\mathcal{C}$ -valued cohomology theory* on $\mathcal{B}$ is a lax symmetric monoidal limit-preserving functor $F: \mathcal{B}^{\mathrm{op}} \rightarrow \mathcal{C}$. We write $\mathrm{Fun}^{\mathrm{R}, \otimes - \mathrm{lax}}(\mathcal{B}^{\mathrm{op}}, \mathcal{C})$ for the category of multiplicative cohomology theories. We are most interested in the following two cases.

- (1) If $\mathcal{E}$ is a global family and $\mathcal{B} = \mathcal{S}_{\mathcal{E}\text{-gl}}$, we refer to the above functor as a multiplicative $\mathcal{E}$ -global $\mathcal{C}$ -valued cohomology theory.
- (2) If G is a compact Lie group and $\mathcal{B} = \mathcal{S}_G$, we refer to the above functor as a multiplicative G -equivariant $\mathcal{C}$ -valued cohomology theory.

PROPOSITION 5.16. *Let $\mathcal{B}$ be a presentably symmetric monoidal ∞ -category. The assignment $X \mapsto \mathrm{map}_{\mathrm{Sp}(\mathcal{B})}(\Sigma_+^{\infty}(-), X)$ defines an equivalence*

$$\mathrm{CAlg}(\mathrm{Sp}(\mathcal{B})) \simeq \mathrm{Fun}^{\mathrm{R}, \otimes - \mathrm{lax}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp}).$$

Proof. By Theorem A.3, there is a symmetric monoidal equivalence $\mathcal{B} \otimes \mathrm{Sp} \simeq \mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp})$. Here, the right-hand side is symmetric monoidal by localizing the Day convolution symmetric monoidal structure, and the left-hand side is symmetric monoidal as it is the tensor product of two presentably symmetric monoidal ∞ -categories. As mentioned, there is an equivalence $\mathrm{Sp}(\mathcal{B}) \simeq \mathcal{B} \otimes \mathrm{Sp}$. We observe that, after precomposing with the suspension functor, it agrees with $\mathcal{B} \otimes \Sigma_+^\infty: \mathcal{B} \otimes \mathcal{S} \rightarrow \mathcal{B} \otimes \mathrm{Sp}$, and so is canonically symmetric monoidal. Therefore, by the universal property of $\mathrm{Sp}(\mathcal{B})$, the equivalence above is also symmetric monoidal. All in all, we have constructed a symmetric monoidal equivalence

$$\mathrm{Sp}(\mathcal{B}) \xrightarrow{\sim} \mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp}),$$

which, by Remark 5.1, is given as in the proposition. In particular, we may compute

$$\mathrm{CAlg}(\mathrm{Sp}(\mathcal{B})) \simeq \mathrm{CAlg}(\mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp})) \simeq \mathrm{Fun}^{R, \otimes - \mathrm{lax}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Sp}),$$

where the second equivalence utilizes the universal property of Day convolution [Lur17, Example 2.2.6.9], and the fact that the category of commutative algebras in a symmetric monoidal Bousfield localization is equivalent to commutative algebras whose underlying object is local. $\square$

Suppose $\mathcal{B}$ is equipped with the cartesian symmetric monoidal structure. By [Lur17, Theorem 2.4.3.18], we obtain an equivalence

$$\mathrm{Fun}^{\otimes - \mathrm{lax}}(\mathcal{B}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C}))$$

between lax symmetric monoidal functors $F: \mathcal{B}^{\mathrm{op}} \rightarrow \mathcal{C}$ and ordinary functors $\tilde{F}: \mathcal{B}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathcal{C})$. Moreover, because the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \rightarrow \mathcal{C}$ preserves and detects limits, F is limit-preserving if and only if $\tilde{F}$ is. Therefore, the equivalence above restricts to an equivalence

$$\mathrm{Fun}^{R, \otimes - \mathrm{lax}}(\mathcal{B}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}^R(\mathcal{B}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{C})),$$

which gives an alternative definition of multiplicative $\mathcal{C}$ -valued cohomology theories on cartesian monoidal $\mathcal{B}$. When $\mathcal{B}$ is the ∞ -category of G -spaces or of global spaces and $\mathcal{C}$ is the ∞ -category of spectra, we may combine this with the previous proposition to deduce the following result, where we write CAlg for $\mathrm{CAlg}(\mathrm{Sp})$.

PROPOSITION 5.17. *For any compact Lie group G , the assignment $X \mapsto \mathrm{map}_{\mathrm{Sp}(\mathcal{S}_G)}(\Sigma_+^\infty(-), X)$ defines an equivalence*

$$\mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G)) \simeq \mathrm{Fun}^R(\mathcal{S}_G^{\mathrm{op}}, \mathrm{CAlg}).$$

Similarly, for any global family $\mathcal{E}$, the assignment $X \mapsto \mathrm{map}_{\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})}(\Sigma_+^\infty(-), X)$ defines an equivalence

$$\mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})) \simeq \mathrm{Fun}^R(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\mathrm{op}}, \mathrm{CAlg}).$$

Now that we have discussed naive rings and multiplicative cohomology theories, we can further investigate the consequences of having a genuine refinement.

PROPOSITION 5.18.

(1) *The composite*

$$\mathbb{U}_G: \mathrm{CAlg}(\mathrm{Sp}_G) \rightarrow \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G)) \simeq \mathrm{Fun}^R(\mathcal{S}_G^{\mathrm{op}}, \mathrm{CAlg})$$

corresponds to the functor which sends a G -ring X to the functor $\mathrm{map}(\Sigma_G^\infty(-), X)$. The analogous statement holds in the global case.

(2) *Precomposition by the lax cocone $-//G: \mathcal{S}_G^{\mathrm{op}} \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}^{\mathrm{op}}$ defines an equivalence*

$$\Phi: \mathrm{Fun}^R(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\mathrm{op}}, \mathrm{CAlg}) \xrightarrow{\sim} \mathrm{laxlim}^\dagger \mathrm{Fun}^R(\mathcal{S}_G^{\mathrm{op}}, \mathrm{CAlg}).$$

- (3) The restriction functor induces an equivalence $F: \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})) \xrightarrow{\sim} \mathrm{laxlim}^\dagger \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G))$ which fits into the following commutative squares, where the top horizontal equivalence uses Proposition 2.9.

$$\begin{array}{ccc}
 \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}}) & \xrightarrow{\sim} & \mathrm{laxlim}^\dagger \mathrm{CAlg}(\mathrm{Sp}_G) \\
 \mathbb{U}_{\mathrm{gl}} \downarrow & & \downarrow \mathrm{laxlim}^\dagger \mathbb{U}_G \\
 \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})) & \xrightarrow[\sim]{F} & \mathrm{laxlim}^\dagger \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G)) \\
 \textcolor{blue}{5.17} \downarrow \sim & & \sim \downarrow \textcolor{blue}{5.17} \\
 \mathrm{Fun}^R(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\mathrm{op}}, \mathrm{CAlg}) & \xrightarrow[\sim]{\Phi} & \mathrm{laxlim}^\dagger \mathrm{Fun}^R(\mathcal{S}_G^{\mathrm{op}}, \mathrm{CAlg})
 \end{array}$$

Proof. Part (1) follows from Proposition 5.17 combined with the adjunction equivalence

$$\mathrm{map}_{\mathrm{Sp}(\mathcal{S}_G)}(\Sigma_+^\infty(-), \mathbb{U}_G(X)) \simeq \mathrm{map}_{\mathrm{Sp}_G}(\Sigma_G^\infty(-), X).$$

The proof of the global statement is completely analogous. The dual argument of Proposition 3.22 gives part (2). For part (3), we start by constructing a lax symmetric monoidal functor $F: \mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}}) \rightarrow \mathrm{laxlim}^\dagger \mathrm{Sp}(\mathcal{S}_G)$. Using the universal property of the partially lax limits (see [LNP25, Remark 5.1]), it suffices to define a partially lax family of lax symmetric monoidal functors $F_G: \mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}}) \rightarrow \mathrm{Sp}(\mathcal{S}_G)$. We can then invoke [Nik16] to see that there exists a unique lax symmetric monoidal functor F_G making the following diagram commute.

$$\begin{array}{ccc}
 \mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}}) & \xrightarrow{F_G} & \mathrm{Sp}(\mathcal{S}_G) \\
 \Omega^\infty \downarrow & & \downarrow \Omega^\infty \\
 \mathcal{S}_{\mathcal{E}\text{-gl}} & \xrightarrow{\mathrm{res}_G} & \mathcal{S}_G
 \end{array}$$

Passing to commutative algebras and using Proposition 2.9, we obtain a functor

$$F = \{F_G\}: \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})) \rightarrow \mathrm{laxlim}^\dagger \mathrm{CAlg}(\mathrm{Sp}(\mathcal{S}_G)).$$

We claim that F makes the bottom square in part (3) commute. This, in particular, shows that F is an equivalence. The commutativity follows from the series of equivalences

$$\begin{aligned}
 \mathrm{Map}_{\mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})}(\Sigma_+^\infty(-//G), X) &\simeq \mathrm{Map}_{\mathrm{Sp}_G}(-, \mathrm{res}_G \Omega_-^\infty X) \\
 &\simeq \mathrm{Map}_{\mathrm{Sp}_G}(-, \Omega_-^\infty F_G X) \\
 &\simeq \mathrm{Map}_{\mathrm{Sp}(\mathcal{S}_G)}(\Sigma_+^\infty(-), F_G X).
 \end{aligned}$$

It remains to prove that the top square in part (3) commutes. Unraveling the definitions, we see that the commutativity of the square reduces to proving an equivalence $\mathbb{U}_G \mathrm{res}_G X \simeq F_G \mathbb{U}_{\mathrm{gl}}(X)$. Consider the following diagram.

$$\begin{array}{ccc}
 \mathrm{Sp}_{\mathcal{E}\text{-gl}} & \xrightarrow{\mathrm{res}_G} & \mathrm{Sp}_G \\
 \mathbb{U}_{\mathrm{gl}} \downarrow & & \downarrow \mathbb{U}_G \\
 \mathrm{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}}) & \xrightarrow{F_G} & \mathrm{Sp}(\mathcal{S}_G) \\
 \Omega_{\mathrm{gl}}^\infty \downarrow & & \downarrow \Omega^\infty \\
 \mathcal{S}_{\mathcal{E}\text{-gl}} & \xrightarrow{\mathrm{res}_G} & \mathcal{S}_G
 \end{array}$$

We note that the left and right triangles commute. We can check this after passing to left adjoints, where it holds by definition. Moreover, the other square also commutes by the definition of $\Omega_{\text{gl}}^\infty$. Then, the commutativity of the top square follows by the universal property of $\text{Sp}(\mathcal{S}_G)$; we can check this after precomposition with Ω_-^∞ . $\square$

COROLLARY 5.19. *Suppose $\tilde{X} \in \text{Sp}_{\mathcal{E}\text{-gl}}$ is a genuine refinement of a naive global ring X . Then, $\text{res}_G X$ is a genuine refinement of the naive G -ring $\text{res}_G X$.*

Proof. This follows from Proposition 5.18(3). $\square$

6. Naive global 2-rings

In this section, we introduce the notion of a naive global 2-ring, which is the central concept of this work, and list some key examples.

The starting point of our discussion is a functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$. We can now perform two different constructions to $\mathcal{R}$.

DEFINITION 6.1. The *decategorification* of $\mathcal{R}$ is the composite

$$\mathbb{H}_{\text{gl}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \xrightarrow{\mathcal{R}} \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}) \xrightarrow{\text{End}_\bullet(\mathbb{1})} \text{CAlg},$$

where the global section functor is given by $\mathcal{C} \mapsto \text{map}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$.

The second construction that we can perform is given by the following result, which we state here but defer proving until later sections, as its proof requires substantial additional work.

THEOREM 6.2. *Consider a functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$. Then, for all $G \in \mathcal{E}$, there exists a functor as described in Definition 6.3,*

$$\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\text{op}} \rightarrow \text{CAlg}(\mathcal{R}_{\mathbf{B}G}),$$

such that, for every group homomorphism $\alpha: H \rightarrow G$, there exists a natural transformation filling the following square.

$$\begin{array}{ccc} \mathcal{S}_G^{\text{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \text{CAlg}(\mathcal{R}_{\mathbf{B}G}) \\ \alpha^* \downarrow & \nearrow Q_\alpha & \downarrow \alpha^* \\ \mathcal{S}_H^{\text{op}} & \xrightarrow{\mathbb{H}_H(-, \mathcal{R})} & \text{CAlg}(\mathcal{R}_{\mathbf{B}H}) \end{array} \quad (6.1)$$

These data are coherently functorial: this is encoded in the existence of a functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}}^{\text{op}} \rightarrow \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})$$

extending the assignment above.

We call the output $\mathbb{H}_\bullet(-, \mathcal{R})$ of the previous theorem the *unraveling* of $\mathcal{R}$. Our consideration of such families is inspired by the perspective on equivariant elliptic cohomology given in [GKV95].

DEFINITION 6.3. The construction of these data are simple to explain in an incoherent manner.

- (1) Given a group $G \in \mathcal{E}$ and a G -space X , which we regard as a global space $f: X//G \rightarrow \mathbf{B}G$ equipped with a faithful map to $\mathbf{B}G$, we define $\mathbb{H}_G(X, \mathcal{R}) := f_* \mathbb{1}_{X//G}$ to be the pushforward of the unit of $\mathcal{R}_{X//G}$ to $\mathcal{R}_{\mathbf{B}G}$.

- (2) Given a map of G -spaces $X \rightarrow Y$, encoded by the following commutative diagram in $\mathcal{S}_{\mathcal{E}\text{-gl}}$, we define $\mathbb{H}_G(h, \mathcal{R})$ to be the map

$$g_* \mathbb{1}_{\mathcal{R}_{Y//G}} \xrightarrow{g_*(\text{lax}_h)} g_* h_* \mathbb{1}_{\mathcal{R}_{X//G}} \simeq f_* \mathbb{1}_{\mathcal{R}_{X//G}},$$

where lax_h is the lax unit map of the lax monoidal functor h_* .

$$\begin{array}{ccc} X//G & \xrightarrow{h} & Y//G \\ & \searrow f & \swarrow g \\ & \mathbf{B}G & \end{array}$$

- (3) Finally, given a group homomorphism $\alpha: H \rightarrow G$, we define the transformation

$$Q_\alpha: \alpha^* \mathbb{H}_G(-, \mathcal{R}) \Rightarrow \mathbb{H}_H(\alpha^*(-), \mathcal{R})$$

to be the mate of a natural transformation

$$T_\alpha: \mathbb{H}_G(\alpha_!(-), \mathcal{R}) \Rightarrow \alpha_* \mathbb{H}_H(-, \mathcal{R}).$$

We define T_α at an H -space Z , corresponding to a faithful map $f: Z//H \hookrightarrow \mathbf{B}H$, by considering the factorization of αf into a quotient map h followed by an injection $\alpha_! f$.

$$\begin{array}{ccc} Z//H & \xrightarrow{h} & \alpha_! Z//H \\ f \downarrow & & \downarrow \alpha_! f \\ \mathbf{B}H & \xrightarrow{\mathbf{B}\alpha} & \mathbf{B}G \end{array}$$

We then define T_α as the map

$$\begin{aligned} \mathbb{H}_G(\alpha_! Z//H, \mathcal{R}) &\simeq (\alpha_! f)_* (\mathbb{1}_{\mathcal{R}(\alpha_! Z//H)}) \xrightarrow{(\alpha_! f)_*(\text{lax}_h)} (\alpha_! f)_* h_* \mathbb{1}_{\mathcal{R}_{Z//H}} \\ &\simeq \alpha_* f_* \mathbb{1}_{\mathcal{R}_{Z//H}} \simeq \alpha_* \mathbb{H}_H(Z, \mathcal{R}). \end{aligned}$$

It will not surprise the reader that carefully dealing with all of the coherences implicit in this construction is a nontrivial task, which we carry out in §§ 7 and 8.

We want to think of the functor $\mathcal{R}$ as the categorification of the naive global ring $H_{\text{gl}}(-, \mathcal{R})$, and we want to think of the unraveling $\mathbb{H}_\bullet(-, \mathcal{R})$ as a family of equivariant cohomology theories equipped with change-of-group transformations which are canonically associated to $\mathcal{R}$. However, in this generality, the functors $\mathbb{H}_G(-, \mathcal{R})$ and $H_{\text{gl}}(-, \mathcal{R})$ will not be limit-preserving, so we can fix this with the following definition.

DEFINITION 6.4. A *naive global 2-ring* is a functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ whose associated decategorifications $H_{\text{gl}}(-, \mathcal{R})$ and $\mathbb{H}_G(-, \mathcal{R})$ are limit-preserving for all $G \in \mathcal{E}$. A morphism of naive global 2-rings is a symmetric monoidal natural transformation $F: \mathcal{R} \rightarrow \mathcal{R}'$. The ∞ -category of naive global 2-rings is denoted by $2\text{CAlg}_{\mathcal{E}\text{-gl}}^{\text{naive}}$.

WARNING 6.5. We note that the terminology 2-ring is often reserved for small stable categories (see, e.g., [Mat16, § 2.2]). We find it simpler to work in a context with arbitrary colimits, but we make no essential use of this fact. For example, we could always index cohomology theories over finite (equivariant or global) spaces, removing the necessity for filtered colimits in the target entirely.

Example 6.6. By definition, the decategorification of a naive global 2-ring defines a functor

$$2\text{CAlg}_{\mathcal{E}\text{-gl}}^{\text{naive}} \rightarrow \text{Fun}^{\text{R}}(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}}, \text{CAlg}) \simeq \text{CAlg}(\text{Sp}(\mathcal{S}_{\mathcal{E}\text{-gl}})), \quad \mathcal{R} \mapsto H_{\text{gl}}(-, \mathcal{R})$$

to naive global rings, where we used Proposition 5.17.

A good source of examples is given by the following result.

LEMMA 6.7. Any limit-preserving functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ defines a naive global 2-ring.

Proof. Since the global section functor preserves limits, it is clear that the decategorification $\mathbb{H}_{\text{gl}}(-, \mathcal{R})$ is limit-preserving. Let us now argue that $\mathbb{H}_G(-, \mathcal{R})$ preserves limits, which we can do after forgetting commutative algebra structures. Now, consider X in $\mathcal{S}_G$ and suppose X is equivalent to the colimit of G -spaces X_i . Because $-//G: \mathcal{S}_G \rightarrow \mathcal{S}_{\mathcal{E}\text{-gl}}$ preserves colimits, we find that $Y := X//G$ is a colimit of the global spaces $Y_i := X_i//G$. By our assumption on $\mathcal{R}$, we have an equivalence $\mathcal{R}_Y \simeq \lim \mathcal{R}_{Y_i}$. In particular, writing $\pi_i: Y_i \rightarrow Y$ for the components of the universal cocone, given any object B in $\mathcal{R}_{X//G}$, we obtain an equivalence

$$B \simeq \lim_i (\pi_i)_* \pi_i^* B$$

in $\mathcal{R}_Y$. To see this we may, for example, compare universal properties:

$$\begin{aligned} \text{Map}_{\mathcal{R}_Y}(A, B) &\simeq \lim_i \text{Map}_{\mathcal{R}_{Y_i}}(\pi_i^* A, \pi_i^* B) \\ &\simeq \lim_i \text{Map}_{\mathcal{R}_Y}(A, (\pi_i)_* \pi_i^* B) \\ &\simeq \text{Map}_{\mathcal{R}_Y}(A, \lim_i (\pi_i)_* \pi_i^* B). \end{aligned}$$

Finally, note that the canonical faithful map $f: Y \rightarrow \mathbf{BG}$ is induced by the universal property of a colimit by the maps $f_i: Y_i \rightarrow \mathbf{BG}$. We can now calculate

$$\begin{aligned} \mathbb{H}_G(X, \mathcal{R}) = f_* \mathbb{1}_{\mathcal{R}_{X//G}} &\simeq f_* \lim_i (\pi_i)_* \pi_i^* \mathbb{1}_{\mathcal{R}_{X//G}} \simeq \lim_i f_* (\pi_i)_* \mathbb{1}_{\mathcal{R}_{X_i//G}} \\ &\simeq \lim_i (f_i)_* \mathbb{1}_{\mathcal{R}_{X_i//G}} \simeq \lim_i \mathbb{H}_G(X_i, \mathcal{R}). \quad \square \end{aligned}$$

As the main theme of this work, we show that a naive global 2-ring, in fact, endows its decategorification with significantly more structure. Before we dive into this, let us first give some examples of naive global 2-rings.

Example 6.8. Given any presentably symmetric monoidal stable ∞ -category $\mathcal{C}$, we may define a functor

$$\mathcal{C}_{\text{bor}}: \text{Glo}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}), \quad \mathbf{BG} \mapsto \mathcal{C}^{BG}.$$

The limit extension of this functor defines a naive global 2-ring, which we call a *Borel* 2-ring.

Example 6.9. Consider the functor $\text{Glo}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ which sends $\mathbf{BG}$ to the ∞ -category of genuine G -spectra Sp_G , and a morphism $\mathbf{B}\alpha: \mathbf{BH} \rightarrow \mathbf{BG}$ to the restriction–inflation functor $\alpha^*: \text{Sp}_G \rightarrow \text{Sp}_H$. Limit extending, we obtain a naive global 2-ring

$$\text{Sp}_{\bullet}: \mathcal{S}_{\text{gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}).$$

The resulting decategorified naive global ring is equivalent to the underlying naive global ring associated to the global sphere spectrum $\mathbb{S}_{\text{gl}}$.

Example 6.10. We may also consider the limit extension of the functor

$$\text{Sp}_{\bullet\text{-gl}}: \text{Glo}_{\text{fin}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}), \quad \mathbf{BG} \mapsto \text{Sp}_{G\text{-gl}}$$

defined for the family of finite groups in [CLL25a]. This functor sends $\mathbf{BG}$ to the ∞ -category of G -global spectra in the sense of [Len25]. Once again, the associated decategorified global ring is equivalent to the global sphere spectrum.

Example 6.11. Generalizing Example 6.9, one can associate to any $R \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$ the functor

$$\mathrm{Mod}_{R_\bullet} : \mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}), \quad \mathbf{B}G \mapsto \mathrm{Mod}_{\mathrm{res}_G R}(\mathrm{Sp}_G).$$

We refer to [LNP25, Theorem 5.10] for a construction of this functor. Taking the limit extension, we obtain a naive global 2-ring. Its decategorification agrees with the underlying naive global ring of R .

The following two examples form the motivation for this article. We introduce two multiplicative global cohomology theories, and then categorifications thereof. Here we are brief; we refer the reader to Part IV for a more extensive account of these examples.

Example 6.12. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$ -divisible group over a commutative ring spectrum R , in the sense of [Lur19, Definition 2.6.1]. Given these data, Lurie [Lur19, Construction 4.0.3] constructs a multiplicative global cohomology theory

$$R_{\mathbf{G}}^{\bullet} : \mathcal{S}_{\mathrm{fab}\text{-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg},$$

called $\mathbf{G}$ -tempered cohomology. Here, *fab* refers to the family of finite abelian groups. Given a finite abelian group H , there is a noncanonical equivalence

$$\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}H}) \simeq \mathbf{G}[\widehat{H}]$$

between the spectrum of $R_{\mathbf{G}}^{\mathbf{B}H}$ and the $\widehat{H}$ -torsion points in $\mathbf{G}$, where $\widehat{H}$ denotes the Pontryagin dual of H .

Tempered cohomology is categorified by the notion of tempered local systems.

Example 6.13. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$ -divisible group over R . As discussed in [Lur19, Remark 5.2.11], there exists a limit-preserving functor

$$\mathrm{LocSys}_{\mathbf{G}} : \mathcal{S}_{\mathrm{fab}\text{-gl}}^{\mathrm{op}} \rightarrow \widehat{\mathrm{Cat}}, \quad \mathcal{X} \mapsto \mathrm{LocSys}_{\mathbf{G}}(\mathcal{X}),$$

which sends a global space $\mathcal{X}$ to the ∞ -category of $\mathbf{G}$ -tempered local systems on $\mathcal{X}$, and a morphism of global spaces $f : \mathcal{X} \rightarrow \mathcal{Y}$ to the pullback functor $f^* : \mathrm{LocSys}_{\mathbf{G}}(\mathcal{Y}) \rightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathcal{X})$ (see § 15 for more details). If, in addition, $\mathbf{G}$ is *oriented* in the sense of [Lur18b, Proposition 4.3.23], then Proposition 15.8 below implies that $\mathrm{LocSys}_{\mathbf{G}}$ is a naive global 2-ring.

So far, we have only given examples of naive global 2-rings coming from limit-preserving functors. The following example motivates our more general definition.

Example 6.14. Let $\mathbf{S}$ be a nonconnective spectral Deligne–Mumford stack and let $\mathbf{E}$ be an oriented elliptic curve in nonconnective spectral Deligne–Mumford stacks over $\mathbf{S}$, in the sense of [GM23, Definition 3.1]. Following [GM23], we can associate to $\mathbf{E}$ a global cohomology theory

$$\mathrm{H}_{\mathrm{gl}}(-, \mathbf{E}) : \mathcal{S}_{\mathrm{ab}\text{-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Sp}), \quad \mathbf{B}A \mapsto \Gamma(\mathbf{E}[\widehat{A}], \mathcal{O}_{\mathbf{E}[\widehat{A}]}).$$

Similarly to the case of tempered cohomology, if A is a compact abelian Lie group, we denote by $\mathbf{E}[\widehat{A}]$ the $\widehat{A}$ -torsion points of $\mathbf{E}$.

Example 6.15. As we explain in § 14, there is also a way to associate to $\mathbf{E} \rightarrow \mathbf{S}$ a naive global 2-ring:

$$\mathcal{Q}_{\bullet}^{\mathbf{E}} : \mathcal{S}_{\mathrm{ab}\text{-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}).$$

Morally, this comes about by taking the categories of quasi-coherent sheaves on a suitable geometric extension of $\mathbf{E}$ to compact ab-global spaces. This assignment will not, however, be

limit-preserving, so this is an example of a naive global 2-ring not coming from Lemma 6.7. Considering the natural equivalence $\mathrm{End}_{\mathrm{QCoh}(X)}(\mathbb{1}) \simeq \Gamma(X, \mathcal{O}_X)$, we see that $\mathcal{Q}_{\bullet}^{\mathbf{E}}$ categorifies $H_{\mathrm{gl}}(-, \mathbf{E})$.

7. Relative global sections

In this section, we construct the functoriality which is required to prove Theorem 6.2. To motivate the steps of our construction, we first sketch a construction of the functor

$$\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathcal{R}_{\mathbf{B}G}), \quad [X//G \xrightarrow{p} \mathbf{B}G] \mapsto p_* \mathbb{1}_{X//G}.$$

- (1) First we claim that there exists an oplax cone which sends a G -space X to the unit $\mathbb{1}_{X//G}$ of the symmetric monoidal ∞ -category $\mathcal{R}_{X//G}$.

$$\begin{array}{ccc} & & \mathrm{CAlg}(\mathcal{R}_{X//G}) \\ & \nearrow \mathbb{1}_{X//G} & \downarrow f_* \\ * & & \mathrm{CAlg}(\mathcal{R}_{Y//G}) \\ & \searrow \mathbb{1}_{Y//G} & \end{array}$$

The triangles are filled by the map $\mathbb{1}_{Y//G} \rightarrow f_* \mathbb{1}_{X//G}$ coming from the lax monoidal structure of f_* .

- (2) Next, we observe that because the ∞ -category of G -spaces admits a final object, there exists a unique cone $\mathrm{CAlg}(\mathcal{R}_{\bullet//G}) \Rightarrow \Delta(\mathrm{CAlg}(\mathcal{R}_{\mathbf{B}G}))$ which sends a G -space $p: X//\mathbf{B}G \rightarrow \mathbf{B}G$ to the functor $p_*: \mathrm{CAlg}(\mathcal{R}_{X//\mathbf{B}G}) \rightarrow \mathrm{CAlg}(\mathcal{R}_{\mathbf{B}G})$.
- (3) Given the previous two constructions, we define the functor $\mathbb{H}_G(-, \mathcal{R})$ by pasting the previous two oplax natural transformations together

$$\Delta(*) \xRightarrow{\mathbb{1}_{\bullet}} \mathrm{CAlg}(\mathcal{R}_{\bullet//G}) \Rightarrow \Delta(\mathrm{CAlg}(\mathcal{R}_{\mathbf{B}G})),$$

using the identifications of functors $\mathcal{S}_G^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathcal{R}_{\mathbf{B}G})$ with $\mathcal{S}_G$ -shaped oplax cones $\Delta(*) \Rightarrow \Delta(\mathrm{CAlg}(\mathcal{R}_{\mathbf{B}G}))$ (see Example 2.8).

Clearly, the resulting functor will send $p: X//G \rightarrow \mathbf{B}G$ to $p_* \mathbb{1}_{X//G}$ and so agrees with (1) of Definition 6.3. Similarly, considering the definition, one sees that it will agree on morphisms with the desideratum (2) of Definition 6.3.

We construct the oplax transformations sketched in step (1) and step (2) above in §§ 7.1 and 7.2, respectively. In both cases, our task is made more difficult by the fact that we require a construction which is functorial enough to allow us to deduce Theorem 6.2. Finally, in § 7.3, we combine these two constructions together, as in step (3).

7.1 Constructing unit sections

In this section, we construct the oplax cone $\mathbb{1}_{\bullet}: \Delta(*) \Rightarrow \mathcal{R}_{-//G}$, in the guise of a section of the cartesian unstraightening of the functor $\mathrm{CAlg}(\mathcal{R}_{-//G})$. We begin with a simplification of the task. Note that the unit $\mathbb{1}_{\mathcal{R}_X}$ is the initial object of $\mathrm{CAlg}(\mathcal{R}_X)$. In particular, given $f: X \rightarrow Y$ in $\mathcal{S}_{\mathcal{E}\text{-gl}}$, the map $\mathrm{lax}_f: \mathbb{1}_{\mathcal{R}_Y} \rightarrow f_* \mathbb{1}_{\mathcal{R}_X}$ can be characterized as the unique map of commutative algebras from $\mathbb{1}_{\mathcal{R}_Y}$ to $f_* \mathbb{1}_{\mathcal{R}_X}$. Therefore, the construction of a unit section as above is subsumed by the construction of initial object sections.

DEFINITION 7.1. We write Cart for the subcategory of $\text{Ar}(\text{Cat})$ spanned on objects by the cartesian fibrations and on morphisms by those squares such that F sends p -cartesian edges to q -cartesian edges.

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow p & & \downarrow q \\ S & \xrightarrow{f} & T \end{array}$$

We define $\text{Cart}^{(\emptyset)}$ to be the full subcategory of Cart on those cartesian fibrations $p: X \rightarrow S$ such that, for each object $s \in S$, the fiber X_s of p over s admits an initial object.

PROPOSITION 7.2. *The inclusion $\text{Cart}^{(\emptyset)} \subset \text{Ar}(\text{Cat})$ factors through the subcategory $\text{Fun}_R^{\text{oplax}}([1], \text{Cat})$. In particular, postcomposing with the first mate equivalence (Theorem 2.15), we obtain a functor*

$$s_{\emptyset}: \text{Cart}^{(\emptyset)} \rightarrow \text{Fun}_L^{\text{lax}}([1], \text{Cat}).$$

Proof. Consider a cartesian fibration $p: X \rightarrow S$, which classifies a functor $F: S^{\text{op}} \rightarrow \text{Cat}$. It suffices to show that $p: X \rightarrow S$ has a left adjoint. We claim that the object $(\emptyset, s)$ is a left-adjoint object of $s \in S$, where $\emptyset$ is the initial object of X_s . To see this, consider the map

$$\phi: \text{Map}_X((\emptyset, s), (x, t)) \rightarrow \text{Map}_S(s, t)$$

induced by p . By [Lur09b, Proposition 2.4.4.2], the fiber over $f: s \rightarrow t$ is equivalent to

$$\text{Map}_{X_s}(\emptyset, F(f)(x)) \simeq *.$$

Thus, ϕ is an equivalence. $\square$

Remark 7.3. The argument given in the proof of Proposition 7.2 also shows that the left adjoint s to $p: X \rightarrow S$ is fully faithful. In particular, the composite ps is canonically equivalent to the identity on S via the unit of the adjunction $s \dashv p$, and so s is canonically a section of p .

Example 7.4. Consider a cartesian fibration $p \in \text{Cart}^{(\emptyset)}$, which classifies a functor $F: S^{\text{op}} \rightarrow \text{Cat}$, and let s be a section constructed by this procedure, that is, $s = s_{\emptyset}(p)$. Then, s may be interpreted as oplax cones $* \Rightarrow F$ (see the dual of Remark 2.7). The component at $i \in \mathcal{I}$ is given by the functor $\{\emptyset_{F(i)}\}: * \rightarrow F(i)$. If we begin with a functor $F: \mathcal{I} \rightarrow \text{Cat}^{\otimes, \text{lax}}$, then applying the previous construction to the functor $\text{CAlg}(F): \mathcal{I} \rightarrow \text{Cat}^{(\emptyset)}$, we obtain a lax cocone, which we denote by $\mathbb{1}_{\bullet}: * \Rightarrow \text{CAlg}(F)$.

7.2 Cartesian transport to the initial object

Having constructed the unit section, we now build the natural transformation $\text{CAlg}(\mathcal{R}_{\bullet//G}) \Rightarrow \Delta(\text{CAlg}(\mathcal{R}_{\text{BG}}))$. Note that such a natural transformation unstraightens to a functor $\text{Un}^{\text{ct}}(\text{CAlg}(\mathcal{R}_{\bullet//G})) \rightarrow \text{CAlg}(\mathcal{R}_{\text{BG}}) \times \mathcal{S}_G^{\text{op}}$, which by adjunction is, in turn, equivalent to a functor $\text{Un}^{\text{ct}}(\text{CAlg}(\mathcal{R}_{\bullet//G})) \rightarrow \text{CAlg}(\mathcal{R}_{\text{BG}})$. It is in this guise that we build the natural transformation above. Of course, we define it more generally for any functor $F: \mathcal{I} \rightarrow \text{Cat}$ such that $\mathcal{I}$ admits a final object. In this case, it will be a functor with signature $\text{Un}^{\text{ct}}(F) \rightarrow F(*)$.

To be able to deduce Theorem 6.2, we require the functor above to itself be functorial in both natural transformations $F \Rightarrow G$ and precomposition by an *arbitrary* functor $\mathcal{J} \rightarrow \mathcal{I}$, not just one which preserves the final object. This is the case because the induction functors generally do not preserve the final object. To make this precise, we introduce the following definition.

DEFINITION 7.5. Let $\text{Cart}_{(\emptyset)}$ be the *full* subcategory of Cart spanned by those cartesian fibrations $p: X \rightarrow \mathcal{I}$ such that $\mathcal{I}$ has an initial object.

Note that the base of a cartesian fibration $p: X \rightarrow \mathcal{I}$ is given by the opposite of the source of its straightening $\mathcal{I}^{\text{op}} \rightarrow \text{Cat}$, explaining the switch from final to initial objects. With the previous definition, we may state the goal of this section.

PROPOSITION 7.6. *There exists a functor $\text{Cart}_{(\emptyset)} \rightarrow \text{Fun}([1], \text{Cat})$ which sends a cartesian fibration $p: X \rightarrow \mathcal{I}$ to the functor*

$$\text{Tr}_{\emptyset}: X \rightarrow X_{\emptyset}, \quad (x, i) \mapsto F(\emptyset \rightarrow i)(x).$$

To prove this statement, we appeal to the functoriality of cartesian lifts in a cartesian fibration. There is a convenient formulation of this, which goes via the notion of free fibrations, in the sense of [GHN17], which we recall now.

PROPOSITION 7.7. *The inclusion $\text{Cart} \subseteq \text{Ar}(\text{Cat})$ admits a left adjoint $\text{F}^{\text{ct}}: \text{Ar}(\text{Cat}) \rightarrow \text{Cart}$.*

Proof. Both Cart and $\text{Ar}(\text{Cat})$ are cartesian fibrations over Cat , and the inclusion is clearly a map of cartesian fibrations. Therefore, by [Lur17, Proposition 7.3.2.6], it suffices to prove that the restriction of the inclusion to each fiber

$$\text{incl}: \text{Cart}(\mathcal{I}) \rightarrow \text{Cat}_{/\mathcal{I}}$$

has a left adjoint. This is precisely [GHN17, Theorem 4.5]. $\square$

DEFINITION 7.8. Concretely, the functor F^{ct} sends a functor $p: X \rightarrow \mathcal{I}$ to the cartesian fibration whose total space is given by the following pullback square.

$$\begin{array}{ccc} \text{F}_{\mathcal{I}}^{\text{ct}}(X) & \longrightarrow & \text{Ar}(\mathcal{I}) \\ \downarrow & \lrcorner & \downarrow \text{ev}_1 \\ X & \xrightarrow{p} & \mathcal{I} \end{array}$$

We view $\text{F}_{\mathcal{I}}^{\text{ct}}(X)$ as living over $\mathcal{I}$ via the composite $\text{F}_{\mathcal{I}}^{\text{ct}}(X) \rightarrow \text{Ar}(\mathcal{I}) \xrightarrow{\text{ev}_0} \mathcal{I}$. When the base ∞ -category $\mathcal{I}$ is clear from the context, we drop it from the notation and simply write $\text{F}^{\text{ct}}(X)$. We refer to $\text{F}^{\text{ct}}(X) \rightarrow \mathcal{I}$ as the *free fibration* of p .

DEFINITION 7.9. We write

$$\Theta: \text{Cart} \rightarrow \text{Fun}([1] \times [1], \text{Cat})$$

for the functor which sends a cartesian fibration $p: X \rightarrow S$ to the counit of the adjunction $\text{F}^{\text{ct}}(X) \dashv \text{incl}$.

$$\begin{array}{ccc} \text{F}^{\text{ct}}(X) & \xrightarrow{\text{Tr}} & X \\ \downarrow & & \downarrow p \\ \mathcal{I} & \xlongequal{\quad} & \mathcal{I} \end{array}$$

We write $\text{Tr}: \text{Cart} \rightarrow \text{Ar}(\text{Cat})$ for the projection to the top of the square. Write $G: S^{\text{op}} \rightarrow \text{Cat}$ for the straightening of $p: X \rightarrow S$. Unwinding the definitions, we find that Tr sends the pair $(x \in X_i, f: i \rightarrow j)$ to $G(f)(x) \in X_j \subset X$. The functor Tr encodes the functoriality of cartesian lifts. Therefore, we call it the *cartesian transport functor*.

We now apply the cartesian transport functor to construct the functor $\text{Tr}_{\emptyset}: \text{Un}^{\text{ct}}(F) \rightarrow F(\emptyset)$. Recall that we require this construction to be functorial in maps in the base which do not

necessarily preserve the initial object. Nevertheless, we begin by assuming that it does, and then later reduce to this case.

CONSTRUCTION 7.10. Let $\text{Cat}_\emptyset$ be the subcategory of Cat spanned by those ∞ -categories which have initial objects and those functors which preserve the initial object. We build a functor $\text{Cat}_\emptyset \rightarrow \text{Fun}([1], \text{Cat})$ as follows. First, consider the functor $\text{Cat} \rightarrow \text{Fun}([1], \text{Cat})$ which sends an ∞ -category $\mathcal{C}$ to the functor $\text{ev}_1: \mathcal{C} \times [1] \rightarrow \mathcal{C}$. Now note that when $\mathcal{C}$ has an initial object, this functor admits a right adjoint, given by the unique functor

$$c_\emptyset: \mathcal{I} \rightarrow \text{Ar}(\mathcal{I}), \quad x \mapsto [\emptyset \rightarrow x].$$

Therefore, we may pass through the first mate equivalence (2.15) to obtain a functor

$$c_\emptyset: \text{Cat}^* \rightarrow \text{Fun}^{\text{lax}}([1], \text{Cat}).$$

A quick check shows that the Beck–Chevalley transformations are in fact equivalences, and so this functor factors through $\text{Fun}([1], \text{Cat})$.

CONSTRUCTION 7.11. Consider the subcategory $\text{Cart}_\emptyset = \text{Cart} \times_{\text{Cat}} \text{Cat}_\emptyset$ of Cart spanned by those cartesian fibrations whose base admits an initial object and those maps of cartesian fibrations such that the functor in the base preserves the initial object. We build a functor $\text{Cart}_\emptyset \rightarrow \text{Fun}([1] \times \Lambda_2^2, \text{Cat})$ by sending an object $p: X \rightarrow \mathcal{I}$ to the following diagram.

$$\begin{array}{ccccc} \mathcal{I} & \xlongequal{\quad} & \mathcal{I} & \xleftarrow{p} & X \\ \downarrow c_\emptyset & & \parallel & & \parallel \\ \text{Ar}(\mathcal{I}) & \xrightarrow{\text{ev}_1} & \mathcal{I} & \xleftarrow{p} & X \end{array}$$

Taking the limit along Λ_2^2 results in a functor $\Psi: \text{Cart}_\emptyset \rightarrow \text{Fun}([1], \text{Cat})$. Note that the composite $\text{ev}_1 \circ \Psi$ is equivalent to the restriction of the functor $\text{F}^{\text{ct}}: \text{Cart} \rightarrow \text{Cat}$ to $\text{Cart}_\emptyset$ and $\text{ev}_0 \circ \Psi$ is equivalent to the identity functor on $\text{Cart}_\emptyset$. In particular, Ψ defines a natural transformation from the identity of $\text{Cart}_\emptyset$ to F^{ct} .

CONSTRUCTION 7.12. As observed before, the source of Tr agrees with the target of Ψ , and so we may paste them to obtain a new functor

$$\text{Tr}_\emptyset: \text{Cart}_\emptyset \rightarrow \text{Fun}([1], \text{Cat}).$$

By definition, $\text{Tr}_\emptyset$ sends a cartesian fibration $p: X \rightarrow \mathcal{I}$ to the functor

$$\text{Tr}_\emptyset(X): X \xrightarrow{(\text{id}, c_\emptyset)} \text{F}^{\text{ct}}(X) \xrightarrow{\text{Tr}} X.$$

Thus, we find that $\text{Tr}_\emptyset(X)(x, i) \simeq (\emptyset \rightarrow i)_*(x)$. In particular, the composite factors through the fiber $X_\emptyset$. We implicitly restrict the target of the functor to the fiber over the final object from hereon. Similarly, applying this functor to a map in $\text{Cart}_\emptyset$ as follows, returns a commutative square.

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ \downarrow p & & \downarrow q \\ \mathcal{I} & \xrightarrow{h} & \mathcal{J} \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Tr}_\emptyset(X)} & X_\emptyset \\ H \downarrow & & \downarrow H(\emptyset) \\ Y & \xrightarrow{\mathrm{Tr}_\emptyset(Y)} & Y_\emptyset \end{array}$$

We have built the functor $\mathrm{Tr}_\emptyset$ of Proposition 7.6. However, we have so far only exhibited its functoriality in natural transformations and maps in the base which preserve the initial object. Next, we explain how to proceed in the case that the map in the base does *not* necessarily preserve the final object.

CONSTRUCTION 7.13. Consider the functor

$$(-)^\triangleleft: \mathrm{Cat} \rightarrow \mathrm{Cat}, \quad \mathcal{I} \mapsto \mathcal{I}^\triangleleft := \{\mathrm{pt}\} * \mathcal{I},$$

where $*$ refers to the join of two ∞ -categories. Note that including the second component of the join induces a natural transformation $b: \mathrm{id} \Rightarrow (-)^\triangleleft$. Consider the induced natural transformation

$$b^*: \mathrm{Fun}((-)^\triangleleft, \mathrm{Cat}) \Rightarrow \mathrm{Fun}(-, \mathrm{Cat})$$

and note that each restriction functor admits a right adjoint, given by right Kan extension. Passing through the first mate equivalence (2.15), we obtain an oplax natural transformation

$$\mathrm{Fun}(-, \mathrm{Cat}) \Rightarrow \mathrm{Fun}((-)^\triangleleft, \mathrm{Cat})$$

which, evaluated at an ∞ -category $\mathcal{I}$, is given by right Kan extension along $b: \mathcal{I} \rightarrow \mathcal{I}^\triangleleft$. Observe that the right Kan extension of a diagram $F: \mathcal{I} \rightarrow \mathrm{Cat}$ along b is given by the universal limit cone $\mathcal{I}^\triangleright \rightarrow \mathrm{Cat}$ associated to F . Such an oplax natural transformation induces an endofunctor

$$\Phi: \mathrm{Cart} \rightarrow \mathrm{Cart}$$

of the cartesian unstraightening of the functor $\mathrm{Fun}((-)^\mathrm{op}, \mathrm{Cat})$.

CONSTRUCTION 7.14. Suppose we now restrict Φ to the *full* subcategory $\mathrm{Cart}_{(\emptyset)} \subseteq \mathrm{Cart}$ spanned by those fibrations whose base has an initial object. We emphasize that the functors do not necessarily preserve the initial object. The crucial observation, however, is that the induced functor $h^\triangleleft: \mathcal{I}^\triangleleft \rightarrow \mathcal{J}^\triangleleft$ does preserve the initial object. Therefore, Φ restricts to a functor

$$\Phi: \mathrm{Cart}_{(\emptyset)} \rightarrow \mathrm{Cart}_{\emptyset}.$$

It is worth unraveling the effect of the functor Φ . Note that the limit of a functor $F: \mathcal{I} \rightarrow \mathrm{Cat}$ such that $\mathcal{I}$ has a initial object is simply given by $F(\emptyset)$, and so $\Phi(F): \mathcal{I}^\triangleleft \rightarrow \mathrm{Cat}$ agrees with F on $\mathcal{I}$ and sends the cone point to $F(\emptyset)$ again. Therefore, the effect of Φ on objects is not so interesting. Nevertheless, the effect of Φ on morphisms is useful. Suppose that the following is a morphism in $\mathrm{Cart}_{(\emptyset)}$ which represents the natural transformation $\eta: F \Rightarrow G \circ h$.

$$\begin{array}{ccc} X & \xrightarrow{H} & Y \\ p \downarrow & & \downarrow q \\ \mathcal{I} & \xrightarrow{h} & \mathcal{J} \end{array}$$

Then, the induced morphism

$$\begin{array}{ccc} \Phi(X) & \xrightarrow{\Phi(H)} & \Phi(Y) \\ p^\triangleleft \downarrow & & \downarrow q^\triangleleft \\ \mathcal{I}^\triangleleft & \xrightarrow{h^\triangleleft} & \mathcal{J}^\triangleleft \end{array}$$

represents the natural transformation

$$\Phi(\eta): \Phi(F) \rightarrow \Phi(G)$$

which agrees with η on every object except at the cone point, where it is given by the composite

$$F(\emptyset) \xrightarrow{\eta_\emptyset} G(h(\emptyset)) \xrightarrow{G(h(\emptyset) \rightarrow \emptyset)} G(\emptyset).$$

Proof of Proposition 7.6. We may consider the composite

$$\mathrm{Cart}_{(\emptyset)} \xrightarrow{\Phi} \mathrm{Cart}_\emptyset \xrightarrow{\mathrm{Tr}_\emptyset} \mathrm{Fun}([1], \mathrm{Cat}).$$

On objects, $\Phi \circ \mathrm{Tr}_\emptyset$ sends the cartesian fibration $p: \mathrm{Un}^{\mathrm{ct}}(F) \rightarrow \mathcal{I}$ to the functor $\mathrm{Tr}_\emptyset: \mathrm{Un}^{\mathrm{ct}}(F^\triangleleft) \rightarrow F(\emptyset)$. From now on, we restrict this functor in the source to the subcategory $\mathrm{Un}^{\mathrm{ct}}(F)$. This is natural in $\mathcal{I}$ and so constitutes a new functor

$$\mathrm{Tr}_{(\emptyset)}: \mathrm{Cart}_{(\emptyset)} \rightarrow \mathrm{Fun}([1], \mathrm{Cat}).$$

Note that, on objects, this functor agrees with $\mathrm{Tr}_\emptyset$. However, on morphisms, $\mathrm{Tr}_{(\emptyset)}$ sends a square, where now the map h does not necessarily preserve the initial object, to a commutative square.

$$\begin{array}{ccc} \mathrm{Un}^{\mathrm{ct}}(F) & \xrightarrow{H} & \mathrm{Un}^{\mathrm{ct}}(G) \\ \downarrow p & & \downarrow q \\ \mathcal{I} & \xrightarrow{h} & \mathcal{J} \end{array}$$

$$\begin{array}{ccc} \mathrm{Un}^{\mathrm{ct}}(F) & \xrightarrow{H} & \mathrm{Un}^{\mathrm{ct}}(G) \\ \downarrow & & \downarrow \\ F(\emptyset) & \xrightarrow{G(\emptyset \rightarrow h(\emptyset))H(\emptyset)} & G(\emptyset) \end{array}$$

□

7.3 Combining the previous two constructions

We now combine the previous two sections to obtain the required result. To state it, we define $\mathbf{Cat}_{(*)}$ to be the *full* subcategory of $\mathbf{Cat}$ spanned by those ∞ -categories which admit a final object. Given an ∞ -category $\mathcal{I}$ with a terminal object and an object $i \in \mathcal{I}$, we write $p_i: i \rightarrow *$ for the unique map from i to $*$. Combining the constructions of this section, we obtain the following functor.

THEOREM 7.15. *There exists a functor*

$$\mathbb{P}(-): \mathbf{Cat}_{(*)} \downarrow^{\mathrm{opl}} \mathrm{Cat}^{\otimes, \mathrm{lax}} \rightarrow \mathrm{Fun}^{\mathrm{lax}}([1], \mathbf{Cat})$$

with the following properties.

On objects, $\mathbb{P}(-)$ sends a functor $F: \mathcal{I} \rightarrow \mathrm{Cat}^{\otimes, \mathrm{lax}}$ to the functor $\mathbb{P}(F): \mathcal{I}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(F(*))$ such that:

- (1) $\mathbb{P}(F)(i) \simeq F(p_i) \mathbb{1}_{F(i)}$; and
- (2) given a map $f: X \rightarrow Y$ in $\mathcal{I}$, the map $\mathbb{P}(F)(f): F(p_j) \mathbb{1}_{F(j)} \rightarrow F(p_i) \mathbb{1}_{F(i)}$ is given by the composite

$$F(p_j) \mathbb{1}_{F(j)} \xrightarrow{F(p_j) \mathrm{lax}_f} F(p_j) F(f) \mathbb{1}_{F(i)} \simeq F(p_i) \mathbb{1}_{F(i)},$$

where lax_f is the lax unit map of the lax monoidal functor $F(f)$.

On morphisms, $\mathbb{P}(-)$ sends the following oplaxly commuting triangle to the square below, where:

(3) $\mathbb{P}(\eta)$ is given at an object $i \in \mathcal{I}$ by the composite

$$\begin{aligned} G(p_{F(i)})\mathbb{1}_{GF(i)} &\xrightarrow{G(p_{F(i)})(\text{lax}_{\eta_i})} G(p_{F(i)})\eta_i\mathbb{1}_{H(i)} \simeq G(p_{F(*)})GF(p_i)\eta_i\mathbb{1}_{H(i)} \\ &\simeq G(p_{F(*)})\eta_*H(p)(\mathbb{1}_{H(i)}), \end{aligned}$$

where lax_{η_i} is the lax unit map of η_i .

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{F} & \mathcal{J} \\ & \searrow H & \swarrow G \\ & \text{Cat}^{\otimes, \text{lax}} & \end{array}$$

$$\begin{array}{ccc} \mathcal{I}^{\text{op}} & \xrightarrow{\mathbb{P}(H)} & \text{CAlg}(H(*)) \\ F \downarrow & \nearrow \mathbb{P}(\eta) & \downarrow G(p_{F(*)})\circ\eta_* \\ \mathcal{J}^{\text{op}} & \xrightarrow{\mathbb{P}(G)} & \text{CAlg}(G(*)) \end{array}$$

Proof. Consider the composite

$$\mathbf{Cat}_{(*)} \downarrow^{\text{opl}} \text{Cat}^{\otimes, \text{lax}} \xrightarrow{\text{CAlg}(-)\circ-} \mathbf{Cat} \downarrow^{\text{opl}} \text{Cat} \xrightarrow[\sim]{\text{Un}^{\text{ct}}(-)} \mathbf{Cart},$$

where the last equivalence is Corollary 2.12. Because the ∞ -category $\text{CAlg}(\mathcal{D})$ always has an initial object given by the unit $\mathbb{1}_{\mathcal{D}}$ and $\mathcal{I}^{\text{op}}$ has an initial object by assumption, this composite lands in the intersection $\mathbf{Cart}^{(\emptyset)} \cap \mathbf{Cart}_{(\emptyset)}$. We may therefore apply the functor

$$(s_{\emptyset}, \text{Tr}_{(\emptyset)}): \mathbf{Cart}^{(\emptyset)} \cap \mathbf{Cart}_{(\emptyset)} \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat}) \times_{\mathbf{Cat}} \text{Fun}^{\text{lax}}([1], \mathbf{Cat}).$$

Here, the pullback is taken along evaluation at source and target, respectively. The fact that $(s_{\emptyset}, \text{Tr}_{(\emptyset)})$ factors through the pullback is justified by the fact that the target of $s_{\emptyset}$ agrees with the source of $\text{Tr}_{(\emptyset)}$: both are the identity on $\mathbf{Cart}^{(\emptyset)} \cap \mathbf{Cart}_{(\emptyset)}$. Now, we may paste these two lax natural transformations to obtain the functor $\mathbb{P}$. More formally, this means we postcompose by the functor

$$\text{paste}: \text{Fun}^{\text{lax}}([1], \mathbf{Cat}) \times_{\mathbf{Cat}} \text{Fun}^{\text{lax}}([1], \mathbf{Cat}) \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat}).$$

We define the result to be $\mathbb{P}(-)$. Unwinding the construction, one finds it has the properties required. $\square$

Remark 7.16. We note the similarity between the properties of $\mathbb{P}(-)$ and the desiderata of Definition 6.3.

8. Families of equivariant cohomology theories

After the technical work of the last section, we can cash in and prove Theorem 6.2.

PROPOSITION 8.1. *Consider a functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \text{Cat}^{\otimes, \text{lax}}$. Then there exists a functor*

$$\mathbb{H}_{\bullet}(-, \mathcal{R}): \text{Glo}_{\mathcal{E}} \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})$$

such that the following hold.

(1) *The functor $\mathbb{H}_{\bullet}(-, \mathcal{R})$ sends $\mathbf{BG}$ to a functor*

$$\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\text{op}} \rightarrow \text{CAlg}(\mathcal{R}_{\mathbf{BG}})$$

which agrees on objects and morphisms with (1) and (2) of Definition 6.3, respectively.

- (2) The functor $\mathbb{H}_\bullet(-, \mathcal{R})$ sends a morphism $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ to the following lax square, filled by a natural transformation

$$T_\alpha: \mathbb{H}_G(\alpha_!(-), \mathcal{R}) \Rightarrow \alpha_* \mathbb{H}_H(-, \mathcal{R})$$

which agrees pointwise with (3) of Definition 6.3.

$$\begin{array}{ccc} \mathcal{S}_H^{\text{op}} & \xrightarrow{\mathbb{H}_H(-, \mathcal{R})} & \text{CAlg}(\mathcal{R}_{\mathbf{B}H}) \\ \alpha_! \downarrow & \nearrow T_\alpha & \downarrow \alpha_* \\ \mathcal{S}_G^{\text{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \text{CAlg}(\mathcal{R}_{\mathbf{B}G}) \end{array} \quad (8.1)$$

Remark 8.2. Note that T_α is an equivalence when α is injective. Indeed, borrowing the notation of Definition 6.3(3), in this case h is necessarily an equivalence and so its associated lax structure map $\mathbb{1}_{\mathcal{R}(\alpha_! Z)} \rightarrow h_* \mathbb{1}_{\mathcal{R}(Z)}$ is also.

Remark 8.3. Let $\mathbf{Cat}^{\text{II}, \text{lax}}$ be the $(\infty, 2)$ -category of cocartesian monoidal ∞ -categories and lax monoidal functors. We note that $\mathbf{Cat}^{\text{II}, \text{lax}}$ is equivalent to the full subcategory $\mathbf{Cat}^{\text{II}} \subseteq \mathbf{Cat}$ spanned by those ∞ -categories which admit finite coproducts (see [Lur17, Proposition 2.4.3.8]).

Proof. We first pass $\mathcal{R}$ through the composite

$$\text{Fun}(\mathcal{S}_{\mathcal{E}\text{-gl}}, \text{Cat}^{\otimes, \text{lax}}) \xrightarrow{3.22} \text{oplaxlim}^\dagger \text{Fun}(\mathcal{S}_G, \text{Cat}^{\otimes, \text{lax}}) \xrightarrow{\text{forget}} \text{Fun}(\text{Glo}_{\mathcal{E}}, \mathbf{Cat} \downarrow^{\text{opl}} \text{Cat}^{\otimes, \text{lax}}),$$

where the second functor is justified by the identification of Construction 2.13. Next, we observe that the resulting object factors through the subcategory $\mathbf{Cat}_{(*)} \downarrow^{\text{opl}} \text{Cat}^{\otimes, \text{lax}}$. Therefore, we may postcompose by the functor

$$\mathbb{P}: \mathbf{Cat}_{(*)} \downarrow^{\text{opl}} \text{Cat}^{\otimes, \text{lax}} \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat})$$

from Theorem 7.15 to obtain

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}} \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat}).$$

Finally, we observe that this functor factors through the subcategory $\text{Fun}^{\text{lax}}([1], \mathbf{Cat}^{\text{II}})$, and so, by Remark 8.3, we can equivalently view $\mathbb{H}_\bullet(-, \mathcal{R})$ as a functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}} \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat}^{\otimes, \text{lax}}).$$

This functor has the properties advertised by Theorem 7.15. □

If we now assume that the diagram $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \text{Cat}^{\otimes, \text{lax}}$ is obtained from a naive global 2-ring by passing to right adjoints, then one can pass the functor $\mathbb{H}_\bullet(-, \mathcal{R})$ constructed above through the mate equivalence, a fact which we record here.

COROLLARY 8.4. *Let $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ be a functor. Then, there is a functor*

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}}^{\text{op}} \rightarrow \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})$$

which agrees with the functor from Proposition 8.1 on objects, but sends a map $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ in $\text{Glo}_{\mathcal{E}}$ to the following oplax square, which is filled by the Beck–Chevalley transformation of T_α .

$$\begin{array}{ccc}
 \mathcal{S}_G^{\text{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \text{CAlg}(\mathcal{R}_{BG}) \\
 \alpha^* \downarrow & \nearrow Q_\alpha & \downarrow \alpha^* \\
 \mathcal{S}_H^{\text{op}} & \xrightarrow{\mathbb{H}_H(-, \mathcal{R})} & \text{CAlg}(\mathcal{R}_{BH})
 \end{array} \quad (8.2)$$

Proof. Consider the functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \text{Cat}^{\otimes, \text{lax}}$ given by passing to adjoints. Applying the previous proposition to this, we obtain a functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}} \rightarrow \text{Fun}([1], \mathbf{Cat}).$$

Observe that, because α^* is strong monoidal, the adjunction $\alpha^* \dashv \alpha_*$ induces an adjunction on commutative algebra objects. Therefore, $\mathbb{H}_\bullet(-, \mathcal{R})$ factors through the subcategory $\text{Fun}^{\text{R, lax}}([1], \text{Cat})$ from Definition 2.17. We may therefore pass $\mathbb{H}_\bullet(-, \mathcal{R})$ through the second mate equivalence (2.18) to obtain the functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}}^{\text{op}} \rightarrow \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}).$$

Once again, we observe that the diagram factors through $\mathbf{Cat}^{\text{II}}$, and so lifts uniquely to a functor into $\text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})$. $\square$

The unfurling construction is functorial in the following sense.

PROPOSITION 8.5. *The previous construction constitutes a functor*

$$\mathbb{H}: 2\text{CAlg}_{\mathcal{E}\text{-gl}}^{\text{naive}} \rightarrow \text{Fun}(\text{Glo}_{\mathcal{E}}^{\text{op}}, \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})).$$

Proof. The construction is given by the following composite of functors:

$$\begin{aligned}
 2\text{CAlg}_{\mathcal{E}\text{-gl}}^{\text{naive}} &\subset \text{Fun}(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}}, \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})) \xrightarrow{\text{adj}} \text{Fun}(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}}, (\text{Pr}_{\text{st}}^{\text{R, lax}})^{\text{op}}) \simeq \text{Fun}(\mathcal{S}_{\mathcal{E}\text{-gl}}, \text{Pr}_{\text{st}}^{\text{R, lax}})^{\text{op}} \subset \dots \\
 &\dots \subset \text{Fun}(\mathcal{S}_{\mathcal{E}\text{-gl}}, \text{Cat}^{\otimes, \text{lax}})^{\text{op}} \xrightarrow{3.22} (\text{oplaxlim}^\dagger \text{Fun}(\mathcal{S}_G, \text{Cat}^{\otimes, \text{lax}}))^{\text{op}} \rightarrow \dots \\
 &\dots \xrightarrow{\text{forget}} \text{Fun}(\text{Glo}_{\mathcal{E}}, \mathbf{Cat}_{(*)}) \downarrow^{\text{opl}} \text{Cat}^{\otimes, \text{lax}})^{\text{op}} \xrightarrow{\mathbb{P}^*} \text{Fun}(\text{Glo}_{\mathcal{E}}, \text{Fun}^{\text{R, lax}}([1], \mathbf{Cat}^{\text{II}}))^{\text{op}} \rightarrow \dots \\
 &\dots \xrightarrow{2.18} \text{Fun}(\text{Glo}_{\mathcal{E}}, \text{Fun}^{\text{L, oplax}}([1], \mathbf{Cat}^{\text{II}}))^{\text{op}} \simeq \text{Fun}(\text{Glo}_{\mathcal{E}}^{\text{op}}, \text{Fun}^{\text{L, oplax}}([1], \mathbf{Cat}^{\text{II}})) \subset \dots \\
 &\dots \subset \text{Fun}(\text{Glo}_{\mathcal{E}}^{\text{op}}, \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}})),
 \end{aligned}$$

and so functorial. $\square$

Remark 8.6. Unwinding the effect of the construction above, we find that it sends a map $F: \mathcal{R} \rightarrow \mathcal{R}'$ of naive global rings to a $\text{Glo}_{\mathcal{E}}^{\text{op}}$ -indexed collection of oplaxly commuting squares as follows.

$$\begin{array}{ccc}
 \mathcal{S}_G^{\text{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \mathcal{R}_{BG} \\
 \parallel & \nearrow u_{F, G} & \downarrow F_{BG} \\
 \mathcal{S}_G^{\text{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R}')} & \mathcal{R}'_{BG}
 \end{array}$$

The transformation $u_{F, G}$ is given at a G -space X by the map

$$F\mathbb{H}_G(X, \mathcal{R}) := F_{BG} p_* \mathbb{1}_{\mathcal{R}(X//G)} \xrightarrow{BC} p_* F_{X//G} \mathbb{1}_{\mathcal{R}(X//G)} \simeq p_* \mathbb{1}_{\mathcal{R}'(X//G)} \simeq \mathbb{H}_G(X, \mathcal{R}'),$$

where BC denotes the Beck–Chevalley transformation of the naturality equivalence $F_{BG} p^* \simeq p^* F_{X//G}$.

All in all, we have now proven Theorem 6.2.

Proof of Theorem 6.2. The natural transformation constructed in Corollary 8.4 has all of the required properties by Proposition 8.1. $\square$

9. Global sections and equivariant cohomology theories

In this section, we show that the family of equivariant cohomology theories constructed before recovers the naive global ring $H_{\text{gl}}(-, \mathcal{R})$. More precisely, we exhibit an equivalence

$$H_{\text{gl}}(-//\bullet, \mathcal{R}) \simeq \Gamma \mathbb{H}_{\bullet}(-, \mathcal{R}) \quad (9.1)$$

as objects of $\text{laxlim}^{\dagger} \text{Fun}(\mathcal{S}_{\bullet}^{\text{op}}, \text{CAlg})$. Of course, we should begin by explaining the notation used and how precisely we interpret both sides of this expression as objects of the partially lax limit. We begin with the left-hand side, which we have previously seen.

CONSTRUCTION 9.1. We may apply the equivalence of Proposition 5.18(2),

$$\Phi: \text{Fun}^{\text{R}}(\mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}}, \text{CAlg}) \rightarrow \text{laxlim}^{\dagger} \text{Fun}^{\text{R}}(\mathcal{S}_{\bullet}^{\text{op}}, \text{CAlg}),$$

to the functor $H_{\text{gl}}(-, \mathcal{R})$ to obtain the object $H_{\text{gl}}(-//\bullet, \mathcal{R})$ of $\text{laxlim}^{\dagger} \text{Fun}^{\text{R}}(\mathcal{S}_{\bullet}^{\text{op}}, \text{CAlg})$.

Before we can explain how to view $\Gamma \mathbb{H}_{\bullet}(-, \mathcal{R})$ as an object of the partially lax limit, we must reinterpret the global section functor Γ .

CONSTRUCTION 9.2. Recall that Sp is initial in $\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$, and so there exists a unique natural transformation $i: \Delta(\text{Sp}) \Rightarrow \text{id}_{\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})}$ from the constant functor at Sp to the identity on $\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$. We write $\Gamma: \mathcal{C} \rightarrow \text{Sp}$ for the right adjoint of the component of i at $\mathcal{C}$, and call this the *global sections functor* of $\mathcal{C}$. The global section functors form a natural transformation $\Gamma: \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})^{\text{op}} \rightarrow \text{Fun}([1], \text{Pr}_{\text{st}}^{\text{R, lax}})$ by passing to right adjoints. By uniqueness of adjoints, one finds that Γ agrees with the natural transformation of functors $\text{map}(\mathbb{1}, -): \mathcal{C} \rightarrow \text{Sp}$. Postcomposing with the functor $\text{CAlg}(-)$, we obtain another natural transformation $\text{CAlg}(\Gamma): \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})^{\text{op}} \rightarrow \text{Fun}([1], \text{Cat})$ which sends $\mathcal{C}$ to $(\Gamma: \text{CAlg}(\mathcal{C}) \rightarrow \text{CAlg})$.

Remark 9.3. Pasting the natural transformation $\text{CAlg}(\Gamma)$ with the oplax cocone $\mathbb{1}_{\bullet}$ of units constructed in Example 7.4, we obtain an $(\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}))^{\text{op}}$ -shaped oplax cocone $\Gamma \mathbb{1}: * \Rightarrow \Delta(\text{CAlg})$. By Example 2.8, this is equivalent to a functor in $\text{Fun}(\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}), \text{CAlg})$. As such, it agrees with $\text{End}_{\bullet}(\mathbb{1}): \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}) \rightarrow \text{CAlg}$ of Example 6.6. This follows from the identification in the previous construction.

CONSTRUCTION 9.4. Let $\mathcal{R}$ be a naive global 2-ring and consider the functor $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}} \rightarrow \text{Pr}_{\text{st}}^{\text{R, lax}}$ given by passing to right adjoints. We may paste the lax transformation $\mathbb{H}_{\bullet}(-, \mathcal{R})$ from Proposition 8.1 with the natural transformation $\text{CAlg}(\Gamma)$ to obtain a new lax natural transformation

$$\mathbb{H}_{\bullet}(-, \mathcal{R}): \text{Glo}_{\mathcal{E}} \rightarrow \text{Fun}^{\text{lax}}([1], \mathbf{Cat})$$

which sends $\mathbf{BG} \in \text{Glo}_{\mathcal{E}}$ to the composite

$$H_{\mathcal{G}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{G}}^{\text{op}} \xrightarrow{\mathbb{H}_{\mathcal{G}}(-, \mathcal{R})} \text{CAlg}(\mathcal{R}_{\mathbf{BG}}) \xrightarrow{\Gamma} \text{CAlg}. \quad (9.2)$$

Because the target of this lax natural transformation is constant at CAlg , $\mathbb{H}_{\bullet}(-, \mathcal{R})$ defines a functor $\text{Glo}_{\mathcal{E}} \rightarrow \mathbf{Cat} \downarrow^{\text{lax}} \text{CAlg}$ into the lax slice category. As a consequence of the dual of Construction 2.13, we may equivalently view this as an object of $\text{laxlim} \text{Fun}(\mathcal{S}_{\bullet}^{\text{op}}, \text{CAlg})$ as before. Furthermore, by Remark 8.2 and by definition of a naive global 2-ring, we find it is contained in the full subcategory $\text{laxlim}^{\dagger} \text{Fun}^{\text{R}}(\mathcal{S}_{\bullet}^{\text{op}}, \text{CAlg})$.

Having constructed our two objects of $\text{laxlim}^\dagger \text{Fun}(\mathcal{S}_\bullet^{\text{op}}, \text{CAlg})$, we may now compare them.

PROPOSITION 9.5. *The objects $H_{\text{gl}}(-//\bullet, \mathcal{R})$ and $H_\bullet(-, \mathcal{R})$ of $\text{laxlim}^\dagger \text{Fun}(\mathcal{S}_\bullet^{\text{op}}, \text{CAlg})$ are equivalent.*

Proof. This is essentially true by definition. However, since the definitions are slightly involved, this deserves elaboration. On the one hand, the object $H_\bullet(-, \mathcal{R})$ is given by the partially lax diagram of composite oplax natural transformations

$$* \xrightarrow{\mathbb{1}_{-//G}} \text{CAlg}(\mathcal{R}_{-//G}) \xrightarrow{\text{Tr}_\emptyset} \text{CAlg}(\mathcal{R}_{\mathbf{B}G}) \xrightarrow{\text{CAlg}(\Gamma)} \text{CAlg}.$$

Note that the second two oplax natural transformations compose to $\text{CAlg}(\mathcal{R}_{-//G}) \xrightarrow{\text{CAlg}(\Gamma)} \text{CAlg}$ by uniqueness of the transformation $\text{CAlg}(\Gamma)$. On the other hand, $H_{\text{gl}}(-//\bullet, \mathcal{R})$ is given by applying the equivalence of Proposition 5.18(2) to the composite oplax natural transformation

$$* \xrightarrow{\mathbb{1}_{-}} \text{CAlg}(\mathcal{R}) \xrightarrow{\text{CAlg}(\Gamma)} \text{CAlg}.$$

Therefore, it is also equivalent to the partially lax diagram of oplax natural transformations

$$* \xrightarrow{\mathbb{1}_{-//G}} \text{CAlg}(\mathcal{R}_{-//G}) \xrightarrow{\text{CAlg}(\Gamma)} \text{CAlg}.$$

We conclude that the two objects agree. $\square$

PART III. Genuine global 2-rings

We now introduce a refinement of the notion of a naive global 2-ring, which we call (*genuine*) *global 2-rings*. We show that for such global 2-rings, the family of equivariant cohomology theories constructed before are strongly related and computable. We then explain how to use this to construct a canonical genuine refinement $\Gamma(\mathcal{R})_{\text{gl}} \in \text{CAlg}(\text{Sp}_{\text{gl}})$ for the associated naive global ring $H_{\text{gl}}(-, \mathcal{R})$, thus accomplishing the main goal of this article. Finally, given a global 2-ring $\mathcal{R}$ and a group G , we construct a genuine global section functor $\Gamma_G: \mathcal{R}_G \rightarrow \text{Sp}_G$. These are best behaved in the case of a *rigid* global 2-ring, as we explain in the final section of this part.

10. The Ginzburg–Kapranov–Vasserot axioms

So far, we have associated to any naive global 2-ring $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ a family

$$\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\text{op}} \rightarrow \text{CAlg}(\mathcal{R}_{\mathbf{B}G})$$

of multiplicative G -equivariant $\mathcal{R}_{\mathbf{B}G}$ -valued cohomology theories. These cohomology theories are related by two *change-of-group transformations*

$$Q_\alpha: \mathbb{H}_H(\alpha^*(-), \mathcal{R}) \Rightarrow \alpha^* \mathbb{H}_G(-, \mathcal{R}) \quad \text{and} \quad T_\alpha: \mathbb{H}_G(\alpha_!(-), \mathcal{R}) \Rightarrow \alpha_* \mathbb{H}_H(-, \mathcal{R})$$

which are associated to any group homomorphism $\alpha: H \rightarrow G$. Moreover, these data are coherently functorial in the group homomorphism, as expressed by the existence of a functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \text{Glo}_{\mathcal{E}}^{\text{op}} \rightarrow \text{Fun}^{\text{oplax}}([1], \mathbf{Cat}^{\otimes, \text{lax}}).$$

Such data are reminiscent of the axiomatics for equivariant elliptic cohomology, as expressed by [GKV95]. In particular, our choice of the name T_α for the first transformation above is inspired by the notation there. In this section, we introduce a condition on $\mathcal{R}$ such that these data satisfy the change-of-group axioms of [GKV95].

DEFINITION 10.1. Let $\mathcal{E}$ be a multiplicative global family of compact Lie groups and let $\mathcal{T} \subset \text{Glo}_{\mathcal{E}}$ be a subcategory of enough injective objects in the sense of Definition 4.2. We say a naive global 2-ring $\mathcal{R}$ is $\mathcal{T}$ -pregenuine if the following conditions hold.

- (1) Take the following pullback square in $\mathcal{S}_{\mathcal{E}\text{-gl}}$, where f' is faithful and $\mathbf{B}G \in \mathcal{T}$.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathbf{B}K \\ f \downarrow & \lrcorner & \downarrow f' \\ \mathbf{B}H & \xrightarrow{g'} & \mathbf{B}G \end{array}$$

Then, the Beck–Chevalley transformation filling the following square is an equivalence.

$$\begin{array}{ccc} \mathcal{R}_{\mathbf{B}K} & \xrightarrow{(g')^*} & \mathcal{R}_{\mathcal{X}} \\ f'_* \downarrow & \nearrow & \downarrow f_* \\ \mathcal{R}_{\mathbf{B}G} & \xrightarrow{g^*} & \mathcal{R}_{\mathbf{B}H} \end{array}$$

- (2) Given a faithful morphism $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ in $\text{Orb}_{\mathcal{E}}$, the canonical map

$$\mathcal{F} \otimes \alpha_* \mathcal{G} \rightarrow \alpha_*(\alpha^* \mathcal{F} \otimes \mathcal{G})$$

adjoint to the composite

$$\alpha^*(\mathcal{F} \otimes \alpha_* \mathcal{G}) \simeq \alpha^* \mathcal{E} \otimes \alpha^* \alpha_* \mathcal{G} \xrightarrow{\alpha^* \mathcal{F} \otimes \epsilon} \alpha^* \mathcal{F} \otimes \mathcal{G}$$

is an equivalence for all $\mathcal{F} \in \mathcal{R}_{\mathbf{B}G}$ and $\mathcal{G} \in \mathcal{R}_{\mathbf{B}H}$. In other words, we require that the adjunction (α^*, α_*) satisfies the right projection formula.

Remark 10.2. If $\mathcal{T} = \mathcal{E}$ is the whole family, then the previous definition admits a recasting in the language of global ∞ -categories developed in [CLL25a] and [CLL25b]. Namely a $\mathcal{E}$ -pregenuine global 2-ring $\mathcal{R}$ is equivalently a fiberwise presentable, fiberwise stable, equivariantly complete $\mathcal{E}$ -global ∞ -category equipped with a symmetric monoidal structure which commutes with fiberwise colimits and finite equivariant limits in each variable. We will not use this perspective in this article, except in Lemma 11.21.

Given these assumptions, we prove the following theorem.

THEOREM 10.3. Let $\mathcal{R}$ be a $\mathcal{T}$ -pregenuine global 2-ring.

- (1) Induction. Let $\alpha: G \rightarrow G/N$ be a surjective group homomorphism with kernel N and let X be a G -space such that the action of N on X is free. Write $p: X \rightarrow X/N$ for the canonical projection map, which is G equivariant. Then, the composite

$$\mathbb{H}_{G/N}(\alpha_!(X/N), \mathcal{R}) \xrightarrow{T_\alpha} \alpha_* \mathbb{H}_G(X/N, \mathcal{R}) \xrightarrow{\alpha_* \mathbb{H}_G(p, \mathcal{R})} \alpha_* \mathbb{H}_G(X, \mathcal{R})$$

is an equivalence.

- (2) Base change. Let $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ be a map in $\text{Glo}_{\mathcal{E}}$. If $\mathbf{B}G \in \mathcal{T}$, then the natural transformation

$$Q_\alpha: \alpha^* \mathbb{H}_G(X, \mathcal{R}) \rightarrow \mathbb{H}_H(\alpha^* X, \mathcal{R})$$

of Equation (8.2) is an equivalence for all compact G -spaces X .

- (3) Künneth. Let G and H be two groups in $\mathcal{T}$, X a compact G -space, and Y a compact H -space. Then, there is an equivalence

$$\pi_G^* \mathbb{H}_G(X, \mathcal{R}) \otimes \pi_H^* \mathbb{H}_H(Y, \mathcal{R}) \simeq \mathbb{H}_{G \times H}(X \times Y, \mathcal{R}),$$

where π_H and π_G denote the two projections $G \times H \rightarrow H, G$. Moreover, the functor $\mathbb{H}_G(-, \mathcal{R}): (\mathcal{S}_G^\omega)^{\text{op}} \rightarrow \mathcal{R}_{\mathbf{B}G}$ is strong monoidal.

We consider each point in turn, beginning with the induction axiom. This axiom, in fact, holds for the unraveling of an arbitrary naive global 2-ring.

PROPOSITION 10.4. *Consider a naive global 2-ring $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$. Let $\alpha: G \rightarrow G/N$ be a surjective group homomorphism with kernel N and let X be a G -space such that the action of N on X is free. Write $p: X \rightarrow X/N$ for the canonical projection map, which is G -equivariant. Then, the composite*

$$\mathbb{H}_{G/N}(\alpha!(X/N), \mathcal{R}) \xrightarrow{T_\alpha} \alpha_* \mathbb{H}_G(X/N, \mathcal{R}) \xrightarrow{\alpha_* \mathbb{H}_G(p, \mathcal{R})} \alpha_* \mathbb{H}_G(X, \mathcal{R})$$

is an equivalence.

Proof. We note that the G/N -space X/N is equivalent to $\alpha!(X)$ and so the G -space $\alpha^*(X/N)$ is equivalent to $\alpha^* \alpha! X$. Furthermore, the map $p: X \rightarrow X/N$ identifies with the unit of the adjunction $\alpha! \dashv \alpha^*$. Therefore, applying Remark 3.17, we obtain the following diagram of global spaces.

$$\begin{array}{ccccc} & & q & & \\ & \nearrow & & \searrow & \\ X//G & \xrightarrow{p} & \alpha^*(X/N)//G & \xrightarrow{h} & X/N//G/N \\ & \searrow f & \downarrow & \lrcorner & \downarrow f/N \\ & & \mathbf{B}G & \xrightarrow{\alpha} & \mathbf{B}(G/N) \end{array}$$

Recall that the maps T_α and $\alpha_* \mathbb{H}_G(p, \mathcal{R})$ are both given by the image of the lax unit comparisons of h and $p//G$ in $\mathcal{R}_{\mathbf{B}(G/N)}$ (see Definition 6.3(3) and (2)). Since lax unit maps compose, we find that the composite in the statement is equivalent to the map

$$(f/N)_* \mathbb{1}_{\mathcal{R}_{X/N//G/N}} \xrightarrow{(f/N)_*(\text{lax}_q)} (f/N)_* q_* \mathbb{1}_{\mathcal{R}_{X//G}} \simeq \alpha_* f_* \mathbb{1}_{\mathcal{R}_{X//G}}.$$

We have to show that this map is an equivalence when X is a G -space on which N acts freely. Note that the subcategory of G -spaces such that the action of N on X is free is generated under colimits by the G -orbits G/H such that $H \cap N = e$. Because the condition of the proposition is closed under limits, it suffices to consider such a G -orbit. In this case, $X//G \simeq \mathbf{B}H$ and $f \simeq \mathbf{B}\iota$ for an inclusion $\iota: H \rightarrow G$. Note that the composite $\mathbf{B}\alpha \circ \mathbf{B}\iota: \mathbf{B}H \rightarrow \mathbf{B}(G/N)$ is clearly faithful since $H \cap N = e$, and so q is an equivalence. From this, the conclusion follows immediately. $\square$

Next, we consider the base-change axiom.

LEMMA 10.5. *Let $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ be a $\mathcal{T}$ -pregenuine global 2-ring, and let $\alpha: H \rightarrow G$ be a group homomorphism such that $\mathbf{B}G \in \mathcal{T}$. The natural transformation $Q_\alpha: \alpha^* \mathbb{H}_G(X, \mathcal{R}) \Rightarrow \mathbb{H}_H(\alpha^* X, \mathcal{R})$ from Equation (8.2) is an equivalence on all compact G -spaces.*

Proof. Note that, because $\mathbb{H}_G(-, \mathcal{R})$ and $\mathbb{H}_H(-, \mathcal{R})$ preserve all limits by definition of a naive global 2-ring, the condition that Q_α is an equivalence is closed under finite limits and retracts of G -spaces. Therefore, it suffices to check that Q_α is an equivalence at every orbit. Hence, we fix a G -orbit $X \simeq G/K$. As discussed in Remark 3.17, $\alpha^* X$ fits into the following pullback, where we have furthermore fixed a factorization of the map $h: \alpha^* X//H \rightarrow X//G$ into a quotient map followed by a faithful morphism.

$$\begin{array}{ccc}
 \alpha^* X // H & \xrightarrow{\gamma} & \alpha! \alpha^* X // G \xrightarrow{\epsilon} X // G \\
 \downarrow \alpha^* f & \lrcorner & \downarrow f \\
 \mathbf{B}H & \xrightarrow{\mathbf{B}\alpha} & \mathbf{B}G
 \end{array}$$

The natural transformation Q_α is, by definition, given by the composite

$$\alpha^* \mathbb{H}_G(X, \mathcal{R}) \xrightarrow{\alpha^* \mathbb{H}_G(\epsilon, \mathcal{R})} \alpha^* \mathbb{H}_G(\alpha! \alpha^* X, \mathcal{R}) \xrightarrow{T_\alpha} \alpha^* \alpha_* \mathbb{H}_H(\alpha^* X, \mathcal{R}) \xrightarrow{\epsilon'} \mathbb{H}_H(\alpha^* X, \mathcal{R}),$$

where ϵ' is the counit of the adjunction $\alpha^* \dashv \alpha_*$. Recall from Definition 6.3 that the map $\mathbb{H}_G(\epsilon, \mathcal{R})$ is given by applying f_* to the lax unit map of ϵ while T_α is given by applying $f_* \epsilon_*$ to the lax unit map of γ . Therefore, the composite is given by applying f to the lax unit map of the composite $h = \epsilon \circ \gamma$. Recall that the lax unit map of h is induced by h_* being a right adjoint of h^* and therefore is equivalent to the composite

$$\mathbb{1}_{\mathcal{R}_{X//G}} \xrightarrow{\eta} h_* h^* \mathbb{1}_{\mathcal{R}_{X//G}} \simeq h_* \mathbb{1}_{\mathcal{R}_{\alpha^* X // H}}.$$

We find that Q_α is given by going the right-most way around the following diagram.

$$\begin{array}{ccc}
 \alpha^* f_* \mathbb{1}_{\mathcal{R}_{X//G}} & & \\
 \eta \downarrow & & \\
 \alpha^* f_* h_* h^* \mathbb{1}_{\mathcal{R}_{X//G}} & \xrightarrow{\sim} & \alpha^* f_* h_* \mathbb{1}_{\mathcal{R}_{\alpha^* X // H}} \\
 \sim \downarrow & & \downarrow \sim \\
 \alpha^* \alpha_* (\alpha^* f)_* h^* \mathbb{1}_{\mathcal{R}_{X//G}} & \xrightarrow{\sim} & \alpha^* \alpha_* (\alpha^* f)_* \mathbb{1}_{\mathcal{R}_{\alpha^* X // H}} \\
 \epsilon' \downarrow & & \downarrow \epsilon' \\
 (\alpha^* f)_* h^* \mathbb{1}_{\mathcal{R}_{X//G}} & \xrightarrow{\sim} & (\alpha^* f)_* \mathbb{1}_{\mathcal{R}_{\alpha^* X // H}}
 \end{array}$$

Note that both squares commute by naturality. In particular, Q_α is an equivalence if and only if the left composite in the diagram above is an equivalence. However, this is the Beck–Chevalley transformation associated to the following square.

$$\begin{array}{ccc}
 \mathcal{R}_{\alpha^* X // H} & \xrightarrow{h_*} & \mathcal{R}_{X//G} \\
 (\alpha^* f)_* \downarrow & & \downarrow f_* \\
 \mathcal{R}_{\mathbf{B}H} & \xrightarrow{\alpha_*} & \mathcal{R}_{\mathbf{B}G}
 \end{array}$$

This is an equivalence by (1) of Definition 10.1, finishing the proof. $\square$

We now turn to the Künneth axiom. We find it simpler to first prove that $\mathbb{H}_G(-, \mathcal{R})$ is strong monoidal on compact G -spaces.

LEMMA 10.6. *Let $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ be a $\mathcal{T}$ -pregenuine global 2-ring. Then, for all $G \in \mathcal{T}$, the functor $\mathbb{H}_G(-, \mathcal{R}): (\mathcal{S}_G^{\omega})^{\text{op}} \rightarrow \mathcal{R}_{\mathbf{B}G}$ is strong monoidal.*

Proof. Once again, the result of the lemma is closed under finite limits and retracts in each variable, and so it suffices to prove that the lax symmetric monoidal structure map

$$\mathbb{H}_G(X, \mathcal{R}) \otimes \mathbb{H}_G(Y, \mathcal{R}) \rightarrow \mathbb{H}_G(X \times Y, \mathcal{R})$$

is an equivalence when X and Y are both orbits, that is, $X = G/K$ and $Y = G/H$. Note as well that under the equivalence of Theorem 3.15, the cartesian product of G -spaces corresponds to the pullback of global spaces over $\mathbf{B}G$. Therefore, we consider the following pullback square of

global spaces and compute

$$\begin{aligned}\mathbb{H}_G(G/K \times G/H, \mathcal{R}) &= (f\pi_1)_* \mathbb{1}_{\mathcal{R}_X} \simeq f_*(\pi_1)_*(\pi_1^* \mathbb{1}_{\mathcal{R}_{BK}} \otimes \pi_2^* \mathbb{1}_{\mathcal{R}_{BH}}) \\ &\simeq f_*(\mathbb{1}_{\mathcal{R}_{BK}} \otimes (\pi_1)_* \pi_2^* \mathbb{1}_{\mathcal{R}_{BH}}) \\ &\simeq f_*(\mathbb{1}_{\mathcal{R}_{BK}} \otimes f^* g_* \mathbb{1}_{\mathcal{R}_{BH}}) \\ &\simeq f_* \mathbb{1}_{\mathcal{R}_{BK}} \otimes g_* \mathbb{1}_{\mathcal{R}_{BH}} \\ &\simeq \mathbb{H}_G(G/K, \mathcal{R}) \otimes \mathbb{H}_G(G/H, \mathcal{R}).\end{aligned}$$

The second and fourth equivalence are an application of the right projection formula for f and π_1 , and so guaranteed by (2) of Definition 10.1. The third equivalence is base-change for the pullback square defining $\mathcal{X}$, and so follows from (1) of Definition 10.1.

$$\begin{array}{ccc}\mathcal{X} & \xrightarrow{\pi_2} & \mathbf{B}H \\ \pi_1 \downarrow & \lrcorner & \downarrow g \\ \mathbf{B}K & \xrightarrow{f} & \mathbf{B}G\end{array}$$

The remaining equivalences are clear. A diagram chase shows that the equivalence constructed agrees with the lax monoidal structure map of $\mathbb{H}_G(-, \mathcal{R})$. $\square$

We now show that this internal Künneth formula implies the external analog.

COROLLARY 10.7. *Consider two groups $H, G \in \mathcal{T}$ and X and Y compact H - and G -spaces, respectively. Let $\mathcal{R}$ be a $\mathcal{T}$ -pregenuine global 2-ring. Then, there is an equivalence*

$$\pi_H^* \mathbb{H}_H(X, \mathcal{R}) \otimes \pi_G^* \mathbb{H}_G(Y, \mathcal{R}) \simeq \mathbb{H}_{H \times G}(X \times Y, \mathcal{R}),$$

where π_H and π_G denote the two projections $G \times H \rightarrow H, G$.

Proof. The equivalence is given by the composite

$$\pi_H^* \mathbb{H}_H(X, \mathcal{R}) \otimes \pi_G^* \mathbb{H}_G(Y, \mathcal{R}) \xrightarrow{Q_{\pi_H} \otimes Q_{\pi_G}} \mathbb{H}_{H \times G}(\pi_H^* X, \mathcal{R}) \otimes \mathbb{H}_{H \times G}(\pi_G^* Y, \mathcal{R}) \simeq \mathbb{H}_{H \times G}(X \times Y, \mathcal{R}),$$

where we have applied both Lemma 10.5 and Lemma 10.6. $\square$

In total, we have proven Theorem 10.3.

Proof of Theorem 10.3. This follows immediately by combining all the results of this section. $\square$

For later use, we observe the following consequence of the theorem for the functor $\mathbb{H}_\bullet(-, \mathcal{R})$.

COROLLARY 10.8. *Suppose $\mathcal{R}: \mathcal{S}_{\mathcal{E}\text{-gl}}^{\text{op}} \rightarrow \text{CA}(\text{Pr}_{\text{st}}^{\text{L}})$ is a $\mathcal{T}$ -pregenuine global 2-ring. Then, the functor*

$$\mathbb{H}_\bullet(-, \mathcal{R}): \mathcal{T}^{\text{op}} \rightarrow \text{Fun}^{\text{oplax}}([1], \text{Cat}^{\otimes, \text{lax}}), \quad \mathbf{B}G \mapsto [\mathbb{H}_G(-, \mathcal{R}): (\mathcal{S}_G^\omega)^{\text{op}} \rightarrow \mathcal{R}_{\mathbf{B}G}],$$

obtained from Corollary 8.4, factors through the subcategory

$$\text{Fun}([1], \text{Cat}_{\text{lex}}^{\otimes}) \subset \text{Fun}^{\text{oplax}}([1], \text{Cat}^{\otimes, \text{lax}})$$

of limit-preserving strong symmetric monoidal functors and strict natural transformations.

Proof. Point (2) of Theorem 10.3 shows that Q_α is always an equivalence, while point (3) shows that each functor is strong monoidal. Finally, each functor is limit-preserving, since $\mathcal{R}$ is in particular a naive global 2-ring. $\square$

Let us once again consider the functoriality of the unraveling construction on $\mathcal{T}$ -pregenuine rings.

DEFINITION 10.9. We say that a morphism $F: \mathcal{R} \rightarrow \mathcal{R}'$ of $\mathcal{T}$ -pregenuine global 2-rings is $\mathcal{T}$ -pregenuine if, for every faithful morphism $\iota: \mathbf{B}H \rightarrow \mathbf{B}G$ such that $\mathbf{B}G \in \mathcal{T}$, the Beck–Chevalley transformation $F_{\mathbf{B}G} \circ \iota_*(X) \rightarrow \iota_* \circ F_{\mathbf{B}H}(X)$, associated to the natural equivalence $F_{\mathbf{B}H}^* \simeq \iota^* F_{\mathbf{B}G}$, is an equivalence on $X = \mathbb{1}_{\mathbf{B}H}$.

We write $2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}\text{-pre}}$ for the ∞ -category of $\mathcal{T}$ -genuine global 2-rings and $\mathcal{T}$ -genuine morphisms.

PROPOSITION 10.10. Consider the functor

$$\mathbb{H}: 2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathrm{naive}} \rightarrow \mathrm{Fun}(\mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{oplax}}([1], \mathrm{Cat}^{\mathrm{lax}, \otimes}))$$

of Proposition 8.5. After restricting the source to $2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}\text{-pre}}$, and restricting the result in the target to $\mathcal{T}$ and compact equivariant spaces (as in Corollary 10.8), $\mathbb{H}$ restricts to a functor

$$\mathbb{H}: 2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}\text{-pre}} \rightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Fun}([1], \mathrm{Cat}_{\mathrm{lex}}^{\otimes})).$$

Proof. Because

$$\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Fun}([1], \mathrm{Cat}_{\mathrm{lex}}^{\otimes})) \subset \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{oplax}}([1], \mathrm{Cat}^{\mathrm{lax}, \otimes}))$$

is a subcategory, it suffices to prove the statement on objects and morphisms. The claim on objects is precisely Corollary 10.8. On morphisms, it follows immediately from the assumptions placed on $\mathcal{T}$ -pregenuine morphisms, by Remark 8.6. $\square$

11. Cohomology theories on equivariant spectra

In this section, we impose a further condition on a pregenuine global 2-ring $\mathcal{R}$ such that $\mathbb{H}_G(-, \mathcal{R})$ extends to a cohomology theory on Sp_G , the ∞ -category of genuine G -spectra. First, we extend our cohomology theories to pointed spaces. Namely, observe that because the categories $\mathcal{R}_{\mathbf{B}G}$ are all pointed, the lax symmetric monoidal functors $\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\mathrm{op}} \rightarrow \mathcal{R}_{\mathbf{B}G}$ canonically lift to symmetric monoidal functors out of the ∞ -category of pointed G -spaces, in such a way that $\mathbb{H}_G(X_+, \mathcal{R}) := \mathbb{H}_G(X, \mathcal{R})$ for any G -space X . In the following proposition, we ensure such an assignment can be made coherent.

PROPOSITION 11.1. Let $\mathcal{R}$ be a naive global 2-ring. The diagram

$$\mathbb{H}_{\bullet}(-, \mathcal{R}): \mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{oplax}}([1], \mathrm{Cat}^{\otimes, \mathrm{lax}}), \quad \mathbf{B}G \mapsto [\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_G^{\mathrm{op}} \rightarrow \mathcal{R}_{\mathbf{B}G}]$$

extends to a functor

$$\mathbb{H}_{\bullet}(-, \mathcal{R}): \mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{oplax}}([1], \mathrm{Cat}^{\otimes, \mathrm{lax}}), \quad \mathbf{B}G \mapsto [\mathbb{H}_G(-, \mathcal{R}): \mathcal{S}_{G,*}^{\mathrm{op}} \rightarrow \mathcal{R}_{\mathbf{B}G}]$$

such that, precomposing with $(-)_+: \mathcal{S}_{\bullet} \rightarrow \mathcal{S}_{\bullet,*}$, one recovers the original functor.

Proof. For this proof, we instead view $\mathbb{H}_G(-, \mathcal{R})$ as a functor from $\mathcal{S}_G$ to $\mathcal{R}_G^{\mathrm{op}}$, in which case it is a colimit-preserving functor of cocomplete categories. Consider $[1]$ as a symmetric monoidal ∞ -category via the coproduct, also known as $\max$. Given a cocomplete ∞ -category $\mathcal{C}$, one may identify $\mathcal{C}_*$ as a symmetric monoidal subcategory of $\mathrm{Fun}([1], \mathcal{C})$, equipped with the Day convolution symmetric monoidal structure. Since Day convolution is a 2-functor

$$\mathrm{Fun}([1], -): \mathrm{Cat}_{\mathrm{L}}^{\otimes, \mathrm{lax}} \rightarrow \mathrm{Cat}_{\mathrm{L}}^{\otimes, \mathrm{lax}},$$

we conclude that $(-)_*: \mathrm{Cat}_{\mathrm{L}}^{\otimes, \mathrm{lax}} \rightarrow \mathrm{Cat}_{\mathrm{L}}^{\otimes, \mathrm{lax}}$ is also 2-functorial. The composite

$$\mathrm{Glo}_{\mathcal{E}}^{\mathrm{op}} \xrightarrow{\mathbb{H}_{\bullet}(-, \mathcal{R})} \mathrm{Fun}^{\mathrm{oplax}}([1], \mathrm{Cat}_{\mathrm{L}}^{\otimes, \mathrm{lax}}) \xrightarrow{\mathrm{Fun}^{\mathrm{oplax}}([1], (-)_*)} \mathrm{Fun}^{\mathrm{oplax}}([1], \mathrm{Cat}_{\mathrm{L}}^{\otimes, \mathrm{lax}})$$

is our desired extension. $\square$

We would like to extend our cohomology theory further to take values in G -spectra. This involves constructing representation sphere deloopings for the functor $\mathbb{H}_G(-, \mathcal{R})$. To explain how we accomplish this, we recall the process of inverting objects in a symmetric monoidal ∞ -category and its connection to genuine G -spectra.

11.1 Inverting representation spheres

DEFINITION 11.2. We define the ∞ -category $\text{Cat}^{\otimes, \text{aug}}$ to be the ∞ -category of symmetric monoidal ∞ -categories $\mathcal{C}$ equipped with an augmentation, i.e. a set S of objects of $\mathcal{C}$. The morphisms of $\text{Cat}^{\otimes, \text{aug}}$ are strong monoidal functors which preserve the augmentations. Formally, we may define $\text{Cat}^{\otimes, \text{aug}}$ as the following pullback.

$$\begin{array}{ccc} \text{Cat}^{\otimes, \text{aug}} & \longrightarrow & \text{Ar}(\text{Cat}) \\ \downarrow & & \downarrow \text{ev}_0 \times \text{ev}_1 \\ \text{Set} \times \text{Cat}^{\otimes} & \xrightarrow{\text{incl} \times \text{fgt}} & \text{Cat} \times \text{Cat} \end{array}$$

Recall that an object X in a symmetric monoidal ∞ -category is called invertible if $X \otimes (-): \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence.

DEFINITION 11.3. We define $\text{Cat}^{\otimes, \text{aug}^{-1}}$ to be the full subcategory of $\text{Cat}^{\otimes, \text{aug}}$ spanned by those pairs $(\mathcal{C}, S)$ such that every object $X \in S$ is an invertible object of $\mathcal{C}$.

PROPOSITION 11.4. *The inclusion $\text{Cat}^{\otimes, \text{aug}^{-1}} \subset \text{Cat}^{\otimes, \text{aug}}$ admits a left adjoint $\mathbb{I}$.*

Proof. Note that $\text{Cat}^{\otimes, \text{aug}}$ is a pullback in Pr_{L} and so is itself a presentable ∞ -category. Observe that the subcategory $\text{Cat}^{\otimes, \text{aug}^{-1}}$ is equivalent to the subcategory of objects $(\mathcal{C}, S)$ which are right orthogonal to the group completion map $\text{FinSet}^{\simeq} \rightarrow \Omega^{\infty} \mathbb{S}$, viewed as a map of augmented symmetric monoidal ∞ -categories by augmenting every object of both $\text{FinSet}^{\simeq}$ and $\Omega^{\infty} \mathbb{S}$. Therefore, the result follows from [Lur09b, Proposition 5.5.4.15]. $\square$

DEFINITION 11.5. We define $\text{Cat}_{\text{rex}, +}^{\otimes, \text{aug}}$ to be the (nonfull) subcategory of $\text{Cat}^{\otimes, \text{aug}}$ spanned on objects by those pairs $(\mathcal{C}, S)$ such that $\mathcal{C}$ is idempotent complete, admits finite colimits, and the tensor product commutes with finite colimits in each variable. On morphisms, it is spanned by those strong monoidal functors which preserve finite colimits. We define $\text{Cat}_{\text{rex}, +}^{\otimes, \text{aug}^{-1}}$ to be the full subcategory spanned by those objects $(\mathcal{C}, S)$ in $\text{Cat}^{\otimes, \text{aug}^{-1}}$. Similarly, we may define $\text{Pr}_{\text{L}}^{\otimes, \text{aug}}$ and $\text{Pr}_{\text{L}}^{\otimes, \text{aug}^{-1}}$, as well as $\text{Cat}_{\text{rex}}^{\otimes, \text{aug}}$ and $\text{Cat}_{\text{rex}}^{\otimes, \text{aug}^{-1}}$.

Using Proposition 11.4, one can easily prove the following.

PROPOSITION 11.6. *The inclusions*

$$\text{Cat}_{\text{rex}, +}^{\otimes, \text{aug}^{-1}} \rightarrow \text{Cat}_{\text{rex}, +}^{\otimes, \text{aug}}, \quad \text{Cat}_{\text{rex}}^{\otimes, \text{aug}^{-1}} \rightarrow \text{Cat}_{\text{rex}}^{\otimes, \text{aug}}, \quad \text{and} \quad \text{Pr}_{\text{L}}^{\otimes, \text{aug}} \rightarrow \text{Pr}_{\text{L}}^{\otimes, \text{aug}^{-1}}$$

admits left adjoints, which we denote by $\mathbb{I}^{\text{rex}, +}$, $\mathbb{I}^{\text{rex}}$, and $\mathbb{I}^{\text{L}}$, respectively.

Proof. We first argue the presentable case. Write $\mathbb{P}(\mathcal{C})$ for the free symmetric monoidal ∞ -category on an ∞ -category $\mathcal{C}$. We then define

$$\mathbb{I}^{\text{L}}(\mathcal{C}, S) := \mathcal{C} \coprod_{\text{PSh}(\mathbb{P}(S))} \text{PSh}(\mathbb{I}(\mathbb{P}(S), S))$$

as a pushout in $\text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$. By the symmetric monoidal universal property of taking presheafs (see [Lur17, Proposition 4.8.1.10(3)]), this clearly has the required universal property. For

the other cases, the same proof applies by replacing the presheaf category by the relevant cocompletion functor. $\square$

DEFINITION 11.7. Let G be a compact Lie group. Given a G -representation V , we define $S^V = V \cup \{\infty\}$ to be the one-point compactification of V , viewed as a pointed G -space by choosing $\{\infty\}$ as a basepoint. We enhance the ∞ -category $\mathcal{S}_{G,*}^\omega$ of compact pointed G -spaces to an augmented symmetric monoidal ∞ -category by the set $\text{RepSph}_G \subset \mathcal{S}_{G,*}^\omega$ of pointed G -spaces which are weak homotopy-equivalent to a representation sphere S^V .

Key for us will be the following calculation.

PROPOSITION 11.8. Let G be a compact Lie group. Then, $\text{Sp}_G^\omega \simeq \mathbb{I}^{\text{rex},+}(\mathcal{S}_{G,*}^\omega, \text{RepSph}_G)$.

Proof. By [GM23, Appendix C], the large ∞ -category Sp_G is the initial presentably symmetric monoidal ∞ -category under $\mathcal{S}_{G,*}$ such that the representation spheres are invertible in Sp_G , that is, $\text{Sp}_G \simeq \mathbb{I}^{\text{L}}(\mathcal{S}_{G,*}, \text{RepSph}_G)$. However, Sp_G is compactly generated, and so is equivalent to $\text{Ind}(\text{Sp}_G^\omega)$ as a symmetric monoidal ∞ -category. Similarly, $\mathcal{S}_{G,*} \simeq \text{Ind}(\mathcal{S}_{G,*}^\omega)$. Using the symmetric monoidal universal property of the Ind-functor, we conclude that $\mathbb{I}^{\text{rex},+}(\mathcal{S}_{G,*}^\omega, \text{RepSph}_G)$ agrees with Sp_G^ω . $\square$

We can apply the previous proposition levelwise to extend families of equivariant cohomology theories.

COROLLARY 11.9. Let $\mathcal{T} \subset \text{Glo}_\mathcal{E}$ be some full subcategory and let $\Lambda_\bullet: \mathcal{S}_{\bullet,*}^\omega \Rightarrow \mathcal{C}_\bullet$ be a natural transformation of functors $\mathcal{T}^{\text{op}} \rightarrow \text{Cat}_{\text{rex},+}^\otimes$ such that $\Lambda_G(S^V)$ is invertible for every group $G \in \mathcal{T}$ and representation sphere $S^V \in \mathcal{S}_G^\omega$. Then, there exists a unique dotted natural transformation in $\text{Cat}_{\text{rex},+}^\otimes$ such that the following triangle commutes.

$$\begin{array}{ccc} \mathcal{S}_{\bullet,*}^\omega & & \\ \Sigma_\bullet^\infty \downarrow & \searrow \Lambda_\bullet & \\ \text{Sp}_\bullet^\omega & \xrightarrow{\quad \cong \quad} & \mathcal{C}_\bullet \end{array}$$

Proof. Note that, for every group homomorphism $\alpha: H \rightarrow G$, the restriction functor $\alpha^*: \mathcal{S}_{G,*} \rightarrow \mathcal{S}_{H,*}$ preserves representation spheres. Therefore, $\mathcal{S}_{\bullet,*}^\omega: \text{Glo}^{\text{op}} \rightarrow \text{Cat}_{\text{rex},+}^\otimes$ lifts to a functor into $\text{Cat}_{\text{rex},+}^{\otimes, \text{aug}}$. Hence, the corollary follows immediately from the previous proposition by viewing $\mathcal{C}_\bullet$ as a functor into $\text{Cat}_{\text{rex},+}^{\otimes, \text{aug}^{-1}}$ by augmenting $\mathcal{C}_{\mathbf{B}G}$ by $\text{Pic}(\mathcal{C}_{\mathbf{B}G})$, the set of invertible objects in $\mathcal{C}_{\mathbf{B}G}$. $\square$

11.2 Extending to equivariant spectra

As mentioned, we would like to apply the previous results to extend the cohomology theories $\mathbb{H}_G(-, \mathcal{R})$ to cohomology theories defined on genuine G -spectra. For this, we have to assume that the representation spheres are inverted by our cohomology theories.

DEFINITION 11.10. We say that a naive 2-ring $\mathcal{R}$ is $\mathcal{T}$ -genuine if it is $\mathcal{T}$ -pregenuine and:

- (3) for every group $G \in \mathcal{T}$ and every irreducible G -representation V , the object $\mathbb{H}_G(S^V, \mathcal{R}) \in \mathcal{R}_{\mathbf{B}G}$ is invertible.

For certain statements, such as Theorem 12.7, it is in fact not relevant exactly which subcategory $\mathcal{T}$ of enough injective objects a naive global 2-ring is $\mathcal{T}$ -genuine for, and so we make the following definition.

DEFINITION 11.11. We say a naive global 2-ring $\mathcal{R}$ is *genuine* if there is a subcategory $\mathcal{T}$ of enough injective objects such that $\mathcal{R}$ is $\mathcal{T}$ -genuine. We will typically refer to a genuine global 2-ring simply as a *global 2-ring*.

We now make the object $\mathbb{H}_G(S^V, \mathcal{R})$, and so the condition of the previous definition, more explicit.

LEMMA 11.12. Let V be a G -representation and let $p: S(V)//G \rightarrow \mathbf{BG}$ be the canonical faithful map of global spaces exhibiting $S(V)$, the unit sphere in V , as a G -space. Then, the fiber of the unit map $\eta: \mathbb{1}_{\mathcal{R}_{\mathbf{BG}}} \rightarrow p_*p^*\mathbb{1}_{\mathcal{R}_{\mathbf{BG}}}$ is equivalent to $\mathbb{H}_G(S^V, \mathcal{R})$.

Proof. We may consider the cofiber sequence $S(V)_+ \rightarrow S^0 \rightarrow S^V$ of pointed G -spaces. Because the extension of $\mathbb{H}_G(-, \mathcal{R})$ to pointed spaces is again limit-preserving, we obtain a fiber sequence

$$\mathbb{H}_G(S^V, \mathcal{R}) := \mathrm{fib}(\mathbb{H}_G(S^0, \mathcal{R}) \rightarrow \mathbb{H}_G(S(V)_+, \mathcal{R})) \in \mathcal{R}_{\mathbf{BG}}.$$

Unraveling the definition, we find that $\mathbb{H}_G(\mathrm{pt}, \mathcal{R}) = \mathbb{1}_{\mathcal{R}_{\mathbf{BG}}}$ and $\mathbb{H}_G(S(V), \mathcal{R}) = p_*\mathbb{1}_{\mathcal{R}_{S(V)//G}} = p_*p^*\mathbb{1}_{\mathcal{R}_{\mathbf{BG}}}$. Moreover, the map $\mathbb{H}_G(\mathrm{pt}, \mathcal{R}) \rightarrow \mathbb{H}_G(S(V), \mathcal{R})$ is induced by the lax unit map of the lax monoidal functor p_* , which is in turn induced by p_* being a right adjoint of p^* . It then follows that $\mathbb{H}_G(\mathrm{pt}, \mathcal{R}) \rightarrow \mathbb{H}_G(S(V), \mathcal{R})$ can be identified with the unit map $\eta: \mathbb{1}_{\mathcal{R}_{\mathbf{BG}}} \rightarrow p_*p^*\mathbb{1}_{\mathcal{R}_{\mathbf{BG}}}$. $\square$

The key result of this section is the following corollary, which, by a representability argument that we carry out in the following section, allows us to deduce Theorem C.

COROLLARY 11.13. Let $\mathcal{R}$ be a $\mathcal{T}$ -genuine global 2-ring. There exists a unique functor

$$\mathbb{H}_\bullet(-, \mathcal{R}): \mathcal{T}^{\mathrm{op}} \rightarrow \mathrm{Fun}([1], \mathrm{Cat}_{\mathrm{lex},+}^\otimes), \quad \mathbf{BG} \mapsto [\mathbb{H}_G(-, \mathcal{R}): (\mathrm{Sp}_G^\omega)^{\mathrm{op}} \rightarrow \mathcal{R}_{\mathbf{BG}}]$$

which agrees with the functor $\mathbb{H}_\bullet(-, \mathcal{R})$ of Corollary 10.8 when restricted to suspension spectra.

Proof. Let us show that all representation spheres are inverted by $\mathbb{H}_\bullet(-, \mathcal{R})$. First, we note that the set of invertible objects in $\mathcal{R}_{\mathbf{BG}}$ is closed under two out of three, i.e. given two objects $X, Y \in \mathcal{R}_{\mathbf{BG}}$, if two of X , Y , and $X \otimes Y$ are invertible, then so is the third. If V and W are irreducible G -representations, then we know by axiom (3) that $\mathbb{H}_G(S^V, \mathcal{R})$ and $\mathbb{H}_G(S^W, \mathcal{R})$ are invertible objects. Using Lemma 10.6, we conclude that

$$\mathbb{H}_G(S^{V \oplus W}, \mathcal{R}) = \mathbb{H}_G(S^V \otimes S^W, \mathcal{R}) \simeq \mathbb{H}_G(S^V, \mathcal{R}) \otimes \mathbb{H}_G(S^W, \mathcal{R})$$

is also an invertible object. Because every G -representation is a direct sum of irreducible representations, the result follows. Given this, the result follows immediately from Corollary 11.9 applied to (the pointwise opposite of) the functor of Corollary 10.8. $\square$

REMARK 11.14. Let $\mathcal{R}$ be a $\mathcal{T}$ -genuine global 2-ring. It is unlikely that the cohomology theory $\mathbb{H}_G(-, \mathcal{R}): (\mathcal{S}_G^\omega)^{\mathrm{op}} \rightarrow \mathcal{R}_{\mathbf{BG}}$ extends to a cohomology theory on Sp_G^ω when $G \notin \mathcal{T}$ in general.

However, there is a specific case where this is possible. Recall that equivariant elliptic cohomology defines an ab-global naive 2-ring $\mathcal{Q}_\bullet$. We will see that $\mathcal{Q}_\bullet$ is *Tori-genuine*, but *not* *Glo_{ab}-genuine*, in both cases in the sense of Definition 11.10. Therefore, the previous theorem only guarantees the lift of equivariant elliptic cohomology to a cohomology theory on equivariant spectra at tori. Nevertheless, it is in fact possible to define such extensions for any compact abelian Lie group. The general construction would follow the strategy of [Chu22, Lemma 2.1] for the group C_n . In particular, it would in principle depend on a choice of embedding of G into a torus, however, we expect that one can show that it is coherently independent of this choice.

Of course, we may also observe the effect of this construction on morphisms of $\mathcal{T}$ -genuine global 2-rings.

DEFINITION 11.15. We define $2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}} \subset 2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}\text{-pre}}$ to be the full subcategory spanned by the $\mathcal{T}$ -genuine 2-rings.

The following result is immediate.

PROPOSITION 11.16. *The functor*

$$\mathbb{H}: 2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}\text{-pre}} \rightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Fun}([1], \mathrm{Cat}_{\mathrm{lex}}^{\otimes}))$$

of Proposition 10.10 induces a functor

$$\mathbb{H}: 2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}} \rightarrow \mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Fun}([1], \mathrm{Cat}_{\mathrm{lex},+}^{\otimes}))$$

which agrees on objects with the construction of Corollary 11.13.

Let us finally discuss various simplifications of (3) above which apply in specific instances. One case is when $\mathcal{E}$ is the family of abelian compact Lie groups. For what follows, we denote by $\mathbb{T}$ the torus of rank 1 and τ for the tautological representation, which is represented by the identity group homomorphism $\mathrm{id}: \mathbb{T} \rightarrow \mathbb{T}$.

PROPOSITION 11.17. *Let Ab be the global family of abelian compact Lie groups, let $\mathrm{Tori} \subset \mathcal{E}$ be the full subcategory of enough injective objects spanned by the tori, and let $\mathcal{R}$ be a naive global 2-ring which is Tori -pregenuine. Then axiom (3) is equivalent to the following.*

(3') *The object $\mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathcal{R})$ is invertible in $\mathcal{R}_{\mathbf{BT}}$.*

Proof. Clearly, (3) implies (3') so let us prove the converse. First, we note that the set of invertible objects in $\mathcal{R}_{\mathbf{BG}}$ is closed under two out of three, i.e. given two objects $X, Y \in \mathcal{R}_{\mathbf{BG}}$, if two of X , Y , and $X \otimes Y$ are invertible, then so is the third. Because every real representation is a direct summand of a complex representation, it therefore suffices to prove that $\mathbb{H}_G(S^V, \mathcal{R})$ is invertible for V an irreducible complex G -representation. Because G is abelian, every irreducible complex G -representation is one-dimensional, i.e. given by a character $\alpha: G \rightarrow \mathbb{T}$. By Lemma 10.5, we have equivalences

$$\mathbb{H}_G(S^{\alpha}, \mathcal{R}) = \mathbb{H}_G(S^{\alpha^* \tau}, \mathcal{R}) \simeq \mathbb{H}_G((\alpha^* S^{\tau}, \mathcal{R}) \xrightarrow{Q_{\alpha}} \alpha^* \mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathcal{R})).$$

Because $\alpha^*: \mathcal{R}_{\mathbf{BT}} \rightarrow \mathcal{R}_{\mathbf{BG}}$ is strong monoidal (see Lemma 10.6), it preserves invertible objects. Therefore, it suffices to prove that $\mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathcal{R})$ is an invertible object, which is precisely (3'). $\square$

REMARK 11.18. Let $\mathbb{T}$ and τ be as above; then $S(\tau) \simeq \mathbb{T}$ and the map $p: S(\tau) // \mathbb{T} \rightarrow \mathbf{BT}$ identifies with the map $i: \mathrm{pt} \rightarrow \mathbf{BT}$ induced by the inclusion $e \rightarrow \mathbb{T}$. Because tensoring in $\mathcal{R}_{\mathbf{BT}}$ is exact, we find that tensoring with $\mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathcal{R})$ is equivalent to the functor

$$\mathrm{fib}((-) \otimes \mathbb{1}_{\mathcal{R}_{\mathbf{BT}}} \rightarrow (-) \otimes i_* \mathbb{1}_{\mathcal{R}_{\mathrm{pt}}}),$$

which is in turn equivalent to the functor $\mathrm{fib}(\eta: (-) \rightarrow i_* i^*(-))$ by an application of the right projection formula for p (which holds by axiom (2) of a global 2-ring). By assumption, $\mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathcal{R})$ is invertible, and so this functor is an equivalence. We conclude that $i^* \dashv i_*$ is a spherical adjunction, in the sense of [DKS⁺24].

Similarly, if $\mathcal{E}$ is a family of finite abelian groups and $\mathcal{T} = \mathcal{E}$, then we can again simplify axiom (3). We write $\iota_n: C_n \rightarrow \mathbb{T}$ for the canonical inclusion, which defines a one-dimensional complex C_n -representation.

PROPOSITION 11.19. *Let $\mathcal{E}$ be a global family of finite abelian groups and let $\mathcal{R}$ be a naive global 2-ring which is $\mathcal{E}$ -pregenuine. Then axiom (3) is equivalent to the following.*

(3') *The object $\mathbb{H}_{C_n}(S^{\iota_n}, \mathcal{R})$ is invertible in $\mathcal{R}_{\mathbf{B}C_n}$ for all $C_n \in \mathcal{E}$.*

Proof. Clearly, (3) implies (3'). Conversely, as before, we can reduce to showing that for all $G \in \mathcal{E}$, $\mathbb{H}_G(-, \mathcal{R})$ inverts spheres associated to irreducible complex representations $V: G \rightarrow \mathbb{T}$ of G . We factor V into a surjection p followed by an injection, which is necessarily of the form $\iota_n: C_n \rightarrow \mathbb{T}$ for some n . As before, we obtain an equivalence

$$\mathbb{H}_G(S^V, \mathcal{R}) = \mathbb{H}_G(S^{p^* \iota_n}, \mathcal{R}) \simeq \mathbb{H}_G(p^* S^{\iota_n}, \mathcal{R}) \xrightarrow{Q_\alpha} p^* \mathbb{H}_{C_n}(S^{\iota_n}, \mathcal{R}).$$

Because p^* is strong monoidal, we conclude that $\mathbb{H}_G(S^V, \mathcal{R})$ is invertible. $\square$

11.3 A canonical class of examples

Finally, before we discuss more interesting examples in Part IV, let us first discuss a canonical class of examples of global 2-rings.

LEMMA 11.20. *Let $\mathcal{E}$ be any multiplicative global family. Then, the naive global 2-ring $\mathrm{Sp}_\bullet$ from Example 6.9 is $\mathcal{E}$ -pregenuine.*

Proof. The fact that $\mathrm{Sp}_\bullet$ is $\mathcal{E}$ -pregenuine expresses a collection of rather standard properties of the diagram of genuine equivariant spectra; nevertheless, let us give references and arguments here. Restricting the diagram $\mathrm{Sp}_\bullet$ to the subcategory $\mathrm{Orb}_\mathcal{E}$, we obtain a diagram which is equivalent to OrbSp , as defined in [Cno23]. It follows from Proposition 4.17 of [Cno23] that $\mathrm{Sp}_\bullet$ satisfies condition (2), and that it satisfies (1) for faithful g . Let us argue the case when $\mathbf{B}g: \mathbf{B}G \rightarrow \mathbf{B}K$ is a quotient map. By applying the Wirthmüller isomorphism, we reduce to the analogous Beck–Chevalley statement for induction, which is discussed in [Lin24, Example 2.18]. $\square$

From Lemma 10.5, we obtain a functor $\mathbb{H}_\bullet(-, \mathrm{Sp}_\bullet): \mathrm{Glo}_\mathcal{E}^{\mathrm{op}} \rightarrow \mathrm{Fun}([1], \mathrm{Cat}_\infty)$.

LEMMA 11.21. *The functor $\mathbb{H}_\bullet(-, \mathrm{Sp}_\bullet)$ agrees with the $\mathrm{Glo}_\mathcal{E}^{\mathrm{op}}$ -indexed natural transformation*

$$(\mathcal{S}_G^\omega)^{\mathrm{op}} \xrightarrow{\Sigma_G^\infty} \mathrm{Sp}_G^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-, S_G^0)} \mathrm{Sp}_G.$$

Remark 11.22. This lemma shows that the unraveling associated to $\mathrm{Sp}_\bullet$ agrees with the family of equivariant Spanier–Whitehead duality functors.

Proof. To prove this statement, it is easiest to apply the parameterized perspective. As noted in Remark 10.2, $\mathrm{Sp}_\bullet$ admits finite equivariant limits. Therefore, by the dual of [CLL25b, Corollary 4.27] applied to $T = \mathrm{Glo}_\mathcal{E}$ and $S = \mathrm{Orb}_\mathcal{E}$, a natural transformation $F_\bullet: (\mathcal{S}_\bullet^\omega)^{\mathrm{op}} \Rightarrow \mathrm{Sp}_\bullet$ such that:

- each functor F_G preserves finite limits; and
- for each faithful morphism $f: \mathbf{B}H \rightarrow \mathbf{B}G$ the Beck–Chevalley transformation $F_G f_! \Rightarrow f_* F_H$ is an equivalence

is determined by the object $F_e(*) \in \mathrm{Sp}$. By Remark 8.2 and the definition of a naive global 2-ring, the transformation $\mathbb{H}_\bullet(-, \mathrm{Sp}_\bullet)$ satisfies both properties, while a simple calculation shows that $\mathrm{Hom}(\Sigma_\bullet^\infty(-), S_G^0)$ does as well. In both cases, $*$ is sent to the sphere spectrum $S^0 \in \mathrm{Sp}$, and so the two natural transformations must agree. $\square$

PROPOSITION 11.23. *The global 2-ring $\mathrm{Sp}_\bullet$ is $\mathcal{E}$ -genuine.*

Proof. By Lemma 11.20, $\mathrm{Sp}_\bullet$ is $\mathcal{E}$ -pregenuine. Moreover, by the previous lemma, $\mathbb{H}_G(-, \mathrm{Sp}_\bullet)$ agrees with $\underline{\mathrm{Hom}}(\Sigma^\infty -, S_G^0)$. In particular, $\mathbb{H}_G(S^V, \mathrm{Sp}_\bullet)$ is equivalent to S^{-V} and so clearly invertible, showing that $\mathrm{Sp}_\bullet$ is $\mathcal{E}$ -genuine. $\square$

PROPOSITION 11.24. *Suppose $E \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$ is a global ring. Then, the naive global 2-ring $\mathrm{Mod}_{E_\bullet}$ defined in Example 6.11 is $\mathcal{E}$ -genuine and the unraveling $\mathbb{H}_\bullet(-, \mathrm{Mod}_{E_\bullet})$ agrees with the natural transformation*

$$(\mathcal{S}_\bullet^\omega)^{\mathrm{op}} \xrightarrow{\Sigma_\bullet^\infty} \mathrm{Sp}_\bullet^{\mathrm{op}} \xrightarrow{\underline{\mathrm{Hom}}(-, E)} \mathrm{Sp}_\bullet.$$

Proof. Observe that axioms (1) and (2) of a global 2-ring reduce to showing that certain canonical maps are equivalences, and therefore we can check these axioms, forgetting the module structures. However, we note that for any continuous group homomorphism $\alpha: H \rightarrow G$, we have the following commutative diagrams.

$$\begin{array}{ccc} \mathrm{Mod}_{E_G}(\mathrm{Sp}_G) & \xrightarrow{\alpha^*} & \mathrm{Mod}_{E_H}(\mathrm{Sp}_H) \\ \mathrm{fgt} \downarrow & & \downarrow \mathrm{fgt} \\ \mathrm{Sp}_G & \xrightarrow{\alpha^*} & \mathrm{Sp}_H \end{array} \qquad \begin{array}{ccc} \mathrm{Mod}_{E_G}(\mathrm{Sp}_G) & \xrightarrow{\alpha_*} & \mathrm{Mod}_{E_H}(\mathrm{Sp}_H) \\ \mathrm{fgt} \downarrow & & \downarrow \mathrm{fgt} \\ \mathrm{Sp}_G & \xrightarrow{\alpha_*} & \mathrm{Sp}_H \end{array}$$

See, for instance, [BCH⁺25, Recollection 2.29]. Therefore, after forgetting, the resulting maps agree with those of Lemma 11.20 and are in particular equivalences. The remaining arguments are exactly as for $\mathrm{Sp}_\bullet$. $\square$

12. Genuine refinements of naive global rings

Recall that, given a naive global 2-ring $\mathcal{R}$, its decategorification was a naive global ring which represented the multiplicative cohomology theory $\mathrm{H}_{\mathrm{gl}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{E}\text{-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}$. In this section, we show that if $\mathcal{R}$ is genuine, then this naive global ring admits a canonical genuine refinement. In other words, we explain how to decategorify a global 2-ring to a global ring. Recall from Proposition 5.18(1) that a genuine refinement was a global ring $\Gamma_{\mathrm{gl}}(\mathcal{R}) \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$ together with an equivalence

$$\mathrm{H}_{\mathrm{gl}}(-, \mathcal{R}) \simeq \mathrm{Hom}_{\mathrm{Sp}_{\mathcal{E}\text{-gl}}}(\Sigma_{\mathrm{gl}}^\infty(-), \Gamma_{\mathrm{gl}}(\mathcal{R}))$$

between $\mathrm{H}_{\mathrm{gl}}(-, \mathcal{R})$ and the multiplicative cohomology theory represented by $\Gamma_{\mathrm{gl}}(\mathcal{R})$. To begin, we first construct the global ring $\Gamma_{\mathrm{gl}}(\mathcal{R})$. Therefore, let us now assume that $\mathcal{R}$ is $\mathcal{T}$ -genuine for $\mathcal{T} \subset \mathcal{E}$ some family of enough injective objects.

CONSTRUCTION 12.1. Passing the natural transformation of Lemma 9.2 along the second mate equivalence (2.18) gives an oplax natural transformation as follows.

$$\begin{array}{ccc} \mathcal{R}_{\mathrm{BG}} & \xrightarrow{\Gamma} & \mathrm{Sp} \\ \alpha^* \downarrow & \swarrow & \parallel \\ \mathcal{R}_{\mathrm{BH}} & \xrightarrow{\Gamma} & \mathrm{Sp} \end{array}$$

In fact, applying the generalized mate equivalence of [AGH25, Theorem 5.3.6] to the $(\infty, 2)$ -category $\mathbf{Cat}^{\otimes, \mathrm{lax}}$, we obtain Γ as an oplax natural transformation of lax monoidal functors. Pasting this with the natural transformation $\mathbb{H}_\bullet(-, \mathcal{R})$ of strong monoidal functors from Corollary 11.13, we obtain a new oplax natural transformation in $\mathbf{Cat}^{\otimes, \mathrm{lax}}$, which we denote

by $H_\bullet(-, \mathcal{R})$.

$$\begin{array}{ccccc} (\mathrm{Sp}_G^\omega)^{\mathrm{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \mathcal{R}_{\mathbf{B}G} & \xrightarrow{\Gamma} & \mathrm{Sp} \\ \alpha^* \downarrow & & \alpha^* \downarrow & \swarrow & \parallel \\ (\mathrm{Sp}_H^\omega)^{\mathrm{op}} & \xrightarrow{\mathbb{H}_H(-, \mathcal{R})} & \mathcal{R}_{\mathbf{B}H} & \xrightarrow{\Gamma} & \mathrm{Sp} \end{array}$$

The target of this oplax natural transformation is constant at Sp and so $H_\bullet(-, \mathcal{R})$ is equivalently an object of $\mathrm{oplaxlim}_{\mathcal{T}}(\mathrm{Fun}^{\mathrm{lex}, \otimes - \mathrm{lax}}((\mathrm{Sp}_\bullet^\omega)^{\mathrm{op}}, \mathrm{Sp}))$ (see Construction 2.13). Finally, if we equip the functor categories with the localized Day convolution symmetric monoidal structure of Lemma A.1, then this equivalently defines an object $\mathrm{oplaxlim}_{\mathcal{T}}(\mathrm{CAlg}(\mathrm{Fun}^{\mathrm{lex}}((\mathrm{Sp}_\bullet^\omega)^{\mathrm{op}}, \mathrm{Sp}))$ via the universal property of Day convolution (see [Lur17, Example 2.2.6.9]).

We now reinterpret this construction by applying higher Brown representability, in the following form.

PROPOSITION 12.2. *The Yoneda embedding induces an equivalence $\mathrm{Sp}_\bullet \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathrm{Sp}_\bullet^\omega)^{\mathrm{op}}, \mathrm{Sp})$, which is moreover symmetric monoidal if the right-hand side is given the localized Day convolution symmetric monoidal structure of Lemma A.1. Moreover, this equivalence is natural. For $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$, the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Sp}_H & \xrightarrow[\sim]{y} & \mathrm{Fun}^{\mathrm{lex}}((\mathrm{Sp}_H^\omega)^{\mathrm{op}}, \mathrm{Sp}) \\ \downarrow \alpha_* & & \downarrow (\alpha^*)^* \\ \mathrm{Sp}_G & \xrightarrow[\sim]{y} & \mathrm{Fun}^{\mathrm{lex}}((\mathrm{Sp}_G^\omega)^{\mathrm{op}}, \mathrm{Sp}) \end{array}$$

Proof. This is the content of Corollary A.4, restricted to the diagram $\mathrm{Sp}_\bullet$. $\square$

CONSTRUCTION 12.3. The natural equivalence $\mathrm{Sp}_\bullet \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathrm{Sp}_\bullet^\omega)^{\mathrm{op}}, \mathrm{Sp})$ induces an equivalence

$$\mathrm{oplaxlim}_{\mathcal{T}} \mathrm{CAlg}(\mathrm{Sp}_\bullet) \simeq \mathrm{oplaxlim}_{\mathcal{T}} \mathrm{CAlg}(\mathrm{Fun}^{\mathrm{lex}}((\mathrm{Sp}_\bullet^\omega)^{\mathrm{op}}, \mathrm{Sp})).$$

We pass the object $H_\bullet(-, \mathcal{R})$ along this equivalence to obtain an object of $\mathrm{oplaxlim}_{\mathcal{T}} \mathrm{CAlg}(\mathrm{Sp}_\bullet)$. We denote the resulting object by $\{\Gamma_G(\mathcal{R})\}$.

Next, we may pass from the oplax limit of $\mathrm{Sp}_\bullet$ along coinduction to the lax limit of $\mathrm{Sp}_\bullet$ along restriction.

CONSTRUCTION 12.4. Applying Proposition 2.21 to the diagram $\mathrm{CAlg}(\mathrm{Sp}_\bullet): \mathcal{T} \rightarrow \mathrm{Cat}$, we obtain an equivalence

$$\mathrm{oplaxlim}_{\mathcal{T}} \mathrm{CAlg}(\mathrm{Sp}_\bullet) \simeq \mathrm{laxlim}_{\mathcal{T}^{\mathrm{op}}} \mathrm{CAlg}(\mathrm{Sp}_\bullet),$$

where the functoriality on the right-hand side is given by restriction. Passing $\{\Gamma_G(\mathcal{R})\}$ along this equivalence gives an object in $\mathrm{laxlim} \mathrm{CAlg}(\mathrm{Sp}_\bullet)$, which we again denote by $\{\Gamma_G(\mathcal{R})\}$.

PROPOSITION 12.5. *The object $\Gamma_\bullet(\mathcal{R})$ lies in the full subcategory $\mathrm{laxlim}_{\mathcal{T}^{\mathrm{op}}}^{\dagger} \mathrm{CAlg}(\mathrm{Sp}_\bullet)$, where we mark the faithful maps in $\mathcal{T}^{\mathrm{op}}$.*

Proof. Let $\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ be a faithful map in $\mathcal{T}$. We need to verify that the lax structure map f_α associated to α is an equivalence. After passing to represented functors, the adjoint of the lax structure map (i.e. the structure map in the oplax limit) is given by the following composite

of the natural transformations.

$$\begin{array}{ccccc}
 (\mathrm{Sp}_H^\omega)^{\mathrm{op}} & \xrightarrow{\mathbb{H}_H(-, \mathcal{R})} & \mathcal{R}_{\mathbf{B}H} & \xrightarrow{\Gamma} & \mathrm{Sp} \\
 \alpha^* \uparrow & \swarrow Q_\alpha & \alpha^* \uparrow & \swarrow \Gamma & \\
 (\mathrm{Sp}_G^\omega)^{\mathrm{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \mathcal{R}_{\mathbf{B}G} & &
 \end{array}$$

Note that when α is faithful, α^* admits a left adjoint $\alpha_!$. Some yoga with adjoint functors implies that the lax structure map is equivalent to the natural transformation obtained by passing to adjoints in the vertical direction. Doing so, we obtain the following diagram, where the right triangle commutes by Construction 9.2.

$$\begin{array}{ccccc}
 (\mathrm{Sp}_H^\omega)^{\mathrm{op}} & \xrightarrow{\mathbb{H}_H(-, \mathcal{R})} & \mathcal{R}_{\mathbf{B}H} & \xrightarrow{\Gamma} & \mathrm{Sp} \\
 \alpha_! \downarrow & \nearrow T_\alpha & \downarrow \alpha_* & \nearrow \Gamma & \\
 (\mathrm{Sp}_G^\omega)^{\mathrm{op}} & \xrightarrow{\mathbb{H}_G(-, \mathcal{R})} & \mathcal{R}_{\mathbf{B}G} & &
 \end{array}$$

The commutativity of the left square follows from Remark 8.2, using that every functor in the diagram is exact and that Sp_H^ω is generated as a stable ∞ -category by the image of $\Sigma_+^\infty: \mathcal{S}_H^\omega \rightarrow \mathrm{Sp}_H^\omega$. We conclude that the composite natural transformation is an equivalence, finishing the proof. $\square$

DEFINITION 12.6. Recall that Theorem 4.7 gave a symmetric monoidal equivalence

$$\mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}}) \simeq \mathrm{laxlim}_{\mathcal{T}^{\mathrm{op}}}^{\dagger} \mathrm{CAlg}(\mathrm{Sp}_\bullet).$$

Passing $\Gamma_\bullet(\mathcal{R})$ through this equivalence, we obtain a global ring $\Gamma_{\mathrm{gl}}(\mathcal{R}) \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$.

We can now conclude the main theorem of the paper.

THEOREM 12.7. *Let $\mathcal{R}$ be a global 2-ring. Then, the naive global ring $\mathbb{H}_{\mathrm{gl}}(-, \mathcal{R}): \mathcal{S}_{\mathcal{E}\text{-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}$ admits a canonical genuine refinement $\Gamma_{\mathrm{gl}}(\mathcal{R}) \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$.*

Proof. We have to exhibit an equivalence between

$$\mathbb{H}_{\mathrm{gl}}(-, \mathcal{R}) \simeq \mathrm{Map}_{\mathrm{Sp}_{\mathcal{E}\text{-gl}}}(\Sigma^\infty(-), \Gamma_{\mathrm{gl}}(\mathcal{R})).$$

However, by Proposition 5.18 it suffices to exhibit an equivalence between the family of equivariant cohomology theories associated to these global theories, which we note are $\{\mathbb{H}_{\mathrm{gl}}(-//\bullet, \mathcal{R})\}$ and $\{\mathbb{H}_\bullet(-, \mathcal{R})\}$, respectively. The second of these identifications again uses Proposition 5.18 and the definition of $\Gamma_{\mathrm{gl}}(\mathcal{R})$. However, these families agree by Proposition 9.5, and so we conclude. $\square$

For the final time, let us note the functoriality of this construction.

PROPOSITION 12.8. *There exists a functor $\Gamma_{\mathrm{gl}}(-): 2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}} \rightarrow \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$, which agrees with the construction of Theorem 12.7 on objects.*

Proof. The constructions of this section are evidently functorial, and take as input the family of cohomology theories $\mathbb{H}_\bullet(-, G)$ on genuine G -spectra. This is a functor in $2\mathrm{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}}$ by Proposition 11.16. $\square$

EXAMPLE 12.9. Recall that, given a global ring $R \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{E}\text{-gl}})$, there was a canonical genuine categorification of R given by

$$\mathcal{R} := \mathrm{Mod}_{R_\bullet}, \quad \mathbf{B}G \mapsto \mathrm{Mod}_{\mathrm{res}_G R}(\mathrm{Sp}_G)$$

(see Proposition 11.24). One can easily show that $\Gamma_{\text{gl}}(\text{Mod}_{R_\bullet}) \simeq R$. In fact, the assignment $R \mapsto \text{Mod}_{R_\bullet}$ defines a functor $\text{CAlg}(\text{Sp}_{\mathcal{E}\text{-gl}}) \rightarrow 2\text{CAlg}_{\mathcal{E}\text{-gl}}^{\mathcal{T}}$, which is a section of the decategorification $\Gamma_{\text{gl}}(-)$.

13. Genuine global section functors

Suppose $\mathcal{R}$ is a global 2-ring with respect to a family $\mathcal{T}$ of enough injective objects. We have associated to $\mathcal{R}$ a family of equivariant spectra $\Gamma_G(\mathcal{R})$ for all $G \in \mathcal{T}$. In this section, we more generally construct a genuine G -spectrum $\mathbb{T}_G(\mathcal{F})$ associated to any object $\mathcal{F} \in \mathcal{R}_{\mathbf{BG}}$.

CONSTRUCTION 13.1. Suppose $\mathcal{R}$ is a global 2-ring which is genuine with respect to $\mathcal{T}$. Then, by Corollary 11.13, we have a natural transformation

$$\mathbb{H}_\bullet(-, \mathcal{R}): (\text{Sp}_\bullet^\omega)^{\text{op}} \rightarrow \mathcal{R}$$

of functors $\mathcal{T}^{\text{op}} \rightarrow \text{Cat}_{\text{lex}}^\otimes$. We may precompose this natural transformation with equivariant Spanier–White duality to obtain a natural transformation of finite limit-preserving functors

$$\mathbb{H}_\bullet(\mathbb{D}(-), \mathcal{R}): \text{Sp}_\bullet^\omega \rightarrow \mathcal{R}.$$

However, the source and target are stable, and so they are also finite colimit-preserving. Passing to Ind-categories in the source gives a natural transformation of colimit-preserving functors $F_\bullet: \text{Sp}_\bullet \Rightarrow \mathcal{R}$, which one may interpret as $\mathcal{R}$ -valued homology. We may further pass to right adjoints pointwise (via Theorem 2.18) to obtain a lax natural transformation

$$\mathbb{T}_\bullet: \mathcal{R} \rightarrow \text{Sp}_\bullet.$$

We write $R_f: f^*\mathbb{T}_G(-) \Rightarrow \mathbb{T}_H(f^*(-))$ for the natural transformation associated to $f: \mathbf{BH} \rightarrow \mathbf{BG}$.

$$\begin{array}{ccc} \mathcal{R}_{\mathbf{BG}} & \xrightarrow{\mathbb{T}_G} & \text{Sp}_G \\ f^* \downarrow & \swarrow R_f & \downarrow f^* \\ \mathcal{R}_{\mathbf{BH}} & \xrightarrow{\mathbb{T}_H} & \text{Sp}_H \end{array}$$

We record a few observations about this construction.

Remark 13.2. Consider $\mathcal{F} \in \mathcal{R}_{\mathbf{BG}}$. We note that

$$\mathbb{T}_G(\mathcal{F})^G \simeq \text{map}_{\text{Sp}_G}(\mathbb{1}, \mathbb{T}_G(\mathcal{F})) \simeq \text{Map}_{\mathcal{R}_{\mathbf{BG}}}(\mathbb{H}_G(\mathbb{D}\mathbb{1}, \mathcal{R}), \mathcal{F}) \simeq \text{Map}_{\mathcal{R}_{\mathbf{BG}}}(\mathbb{1}, \mathcal{F}) \simeq \Gamma(\mathcal{F}).$$

In other words, the genuine fixed points of $\mathbb{T}_G(\mathcal{F})$ agree with the global sections of $\mathcal{F}$.

Remark 13.3. Let $G \in \mathcal{T}$. Then, we claim that $\mathbb{T}_G(\mathbb{1}_{\mathbf{BG}}) \simeq \Gamma_G(\mathcal{R})$. This follows from the following sequence of equivalences, where X is a compact G -spectrum:

$$\begin{aligned} \text{map}_{\text{Sp}_G}(X, \mathbb{T}_G(\mathbb{1}_{\mathbf{BG}})) &\simeq \text{map}_{\mathcal{R}_{\mathbf{BG}}}(\mathbb{H}_G(\mathbb{D}X, \mathcal{R}), \mathbb{1}_{\mathbf{BG}}) \simeq \text{map}_{\mathcal{R}_{\mathbf{BG}}}(\mathbb{D}\mathbb{H}_G(X, \mathcal{R}), \mathbb{1}_{\mathbf{BG}}) \\ &\simeq \Gamma(\mathbb{D}\mathbb{H}_G(X, \mathcal{R})) \simeq \Gamma(\mathbb{H}_G(X, \mathcal{R})). \end{aligned}$$

The second equivalence follows from the fact that $\mathbb{H}_G(-, \mathcal{R})$ is strong monoidal, and the fourth from the fact that $\mathbb{H}_G(X, \mathcal{R})$ is therefore a dualizable object of $\mathcal{R}_{\mathbf{BG}}$.

Furthermore, one can show that, under these equivalences, the map

$$R_f: f^*\mathbb{T}_G(\mathbb{1}_{\mathbf{BG}}) \rightarrow \mathbb{T}_H(f^*\mathbb{1}_{\mathbf{BG}}) \simeq \mathbb{T}_H(\mathbb{1}_{\mathbf{BH}})$$

agrees with the lax structure map of the object $\Gamma_{\text{gl}}(\mathcal{R}) \in \text{Sp}_{\mathcal{E}\text{-gl}} \simeq \text{laxlim}^\dagger \text{Sp}_\bullet$. In particular, we conclude that R_f is an equivalence on the unit when f is faithful, but not generally. We would

like to generalize this observation to arbitrary objects $X \in \mathcal{R}_G$, but it is not clear to the authors if we should always expect this. However, under stronger assumptions, we can show that R_f is a natural equivalence when f is faithful. We introduce these stronger assumptions in the following.

DEFINITION 13.4. A naive global 2-ring $\mathcal{R}$ is *rigid* if the following hold.

- (1) Each ∞ -category $\mathcal{R}_{\mathbf{B}G}$ is rigidly compactly generated, which is to say that $\mathcal{R}_{\mathbf{B}G}$ is a compactly generated ∞ -category in which an object is compact if and only if it is dualizable.
- (2) For every faithful map $f: \mathbf{B}H \rightarrow \mathbf{B}G$, the functor $f_*: \mathcal{R}_{\mathbf{B}G} \rightarrow \mathcal{R}_{\mathbf{B}H}$ preserves compact objects.

Remark 13.5. If $\mathcal{R}$ is rigid, then axiom (2) of Definition 10.1 is automatically satisfied (see [BDS16, Theorem 1.3]).

Example 13.6. The global 2-ring $\mathrm{Sp}_\bullet$, and more generally $\mathrm{Mod}_{E_\bullet}$, is rigid.

Example 13.7. Let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group. Then, the naive global 2-ring $\mathrm{LocSys}_{\mathbf{G}}(-)$ is rigid (see Proposition 15.8(6)).

In the following lemma, we introduce the crucial additional structure afforded to a rigid naive global 2-ring.

LEMMA 13.8. Suppose $\mathcal{R}$ is rigid, and let $f: \mathbf{B}H \rightarrow \mathbf{B}G$ be a faithful map. Then, there exists an object $\omega_f \in \mathcal{R}_{\mathbf{B}H}$ such that f^* is right adjoint to $f_\# := f_*(- \otimes \omega_f)$. Moreover, ω_f is uniquely characterized by the existence of a natural equivalence

$$\underline{\mathrm{Hom}}_{\mathcal{R}_{\mathbf{B}G}}(f_*(-), \mathbb{1}_{\mathbf{B}G}) \simeq f_* \underline{\mathrm{Hom}}_{\mathcal{R}_{\mathbf{B}H}}(-, \omega_f).$$

Proof. This is [BDS16, Theorem 1.7]. □

With this, we can prove that R_f is an equivalence whenever f is faithful.

THEOREM 13.9. Suppose $\mathcal{R}$ is a rigid $\mathcal{T}$ -genuine global 2-ring and let $f: \mathbf{B}H \rightarrow \mathbf{B}G$ be a faithful map in $\mathcal{T}$. Then, the natural transformation

$$R_f: f^* \Gamma_G(-) \Rightarrow \Gamma_H(f^*(-))$$

is an equivalence.

Proof. Because $\mathcal{R}$ is rigid, all of the functors involved have left adjoints. Therefore, we may instead show that the total mate

$$U_f: f_\# F_H(X) \Rightarrow F_G f_!(X)$$

of R_f is an equivalence for all H -spectra. In fact, because all of the functors preserve colimits, it suffices to prove this for X compact. In this case, we may compute

$$\begin{aligned} f_\# F_H(X) &:= f_\# \mathbb{H}_H(\mathbb{D}X, \mathcal{R}) \simeq f_\# \mathbb{D}\mathbb{H}_H(X, \mathcal{R}) \\ &\simeq f_* (\underline{\mathrm{Hom}}_{\mathcal{R}_{\mathbf{B}H}}(\mathbb{H}_H(X, \mathcal{R}), \mathbb{1}_{\mathbf{B}H}) \otimes \omega_f) \\ &\simeq f_* \underline{\mathrm{Hom}}_{\mathcal{R}_{\mathbf{B}H}}(\mathbb{H}_H(X, \mathcal{R}), \omega_f) \\ &\simeq \underline{\mathrm{Hom}}_{\mathcal{R}_{\mathbf{B}G}}(f_* \mathbb{H}_H(X, \mathcal{R}), \mathbb{1}_{\mathbf{B}G}) \\ &\xrightarrow[\sim]{\mathbb{D}(T_f)} \mathbb{D}\mathbb{H}_G(f_! X, \mathcal{R}) \\ &\simeq \mathbb{H}_G(\mathbb{D}f_! X, \mathcal{R}) =: F_G(f_! X). \end{aligned}$$

The second and fourth equivalences use Lemma 13.8, the third that $\mathbb{H}_H(X, \mathcal{R})$ is dualizable, and the first and fifth that $\mathbb{H}_H(-, \mathcal{R})$ and $\mathbb{H}_G(-, \mathcal{R})$ are strong monoidal functors. Finally, T_f is an equivalence by Remark 8.2. A tedious diagram chase shows that this equivalence agrees with U_f . $\square$

Remark 13.10. Recall that, when $\mathcal{R}$ is $\mathcal{T}$ -genuine, the functors $\mathbb{H}_G(-, \mathcal{R})$, and therefore also F_G , are strong monoidal for all $G \in \mathcal{T}$. We conclude that the functors $\mathbb{T}_G$ are therefore canonically lax monoidal, and that if $\mathcal{R}$ is rigid, then $\mathbb{T}_\bullet$ is a natural transformation of lax monoidal functors.

Remark 13.11. Rigidity is a sufficient but not necessary condition for the result of the previous theorem to hold. For example, the naive global 2-ring $\mathrm{Sp}_{\bullet\text{-gl}}$ of Example 6.10 is *not* rigid. Indeed, in the ∞ -category $\mathrm{Sp}_{\mathrm{gl}}$, the object $\Sigma_{\mathrm{gl}}^\infty \mathbf{B}G$ is compact but not dualizable. Nevertheless, one can show that R_f is still an equivalence when f is faithful. The reason for this is that $\mathrm{Sp}_{\bullet\text{-gl}}$ nevertheless admits Wirthmüller isomorphisms, and so in particular satisfies the first part of Lemma 13.8. This is, in fact, the only input for the proof of the previous theorem which does not hold in an arbitrary $\mathcal{T}$ -genuine global 2-ring.

PART IV. Examples

We will now apply the theory developed in previous parts to two families of examples. The first are oriented spectral abelian group objects. Specializing to the universal example of an oriented elliptic curve, we obtain a global spectrum enhancing the spectrum of topological modular forms. The second are $\mathbf{P}$ -divisible groups over a commutative ring spectrum R , as defined in [Lur19]. We conclude that tempered cohomology is canonically represented by a global spectrum $\underline{R}^{\mathrm{gl}}$. Finally, using the notion of genuine global sections, we show that for an oriented $\mathbf{P}$ -divisible group, the ∞ -category $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ of tempered local systems on $\mathbf{B}A$ is equivalent to modules over the restriction of $\underline{R}^{\mathrm{gl}}$ to an A -spectrum.

14. Globally equivariant elliptic cohomology

We start this section by giving a brief recollection on (pre)oriented strict abelian group objects in a general ∞ -category. As we will see, these are very closely connected to diagrams out of the global orbit ∞ -category. We then fix a nonconnective spectral Deligne–Mumford stack $\mathbf{S}$ and focus attention on oriented elliptic curves $\mathbf{E}$ in spectral Deligne–Mumford stacks over $\mathbf{S}$, and explain how these data give rise to a genuine ab-global 2-ring $\mathcal{Q}_\bullet^{\mathbf{E}}$ (see Theorem 14.17). Finally, we discuss how to obtain global refinements for the spectra of topological K -theory, elliptic cohomology, and topological modular forms.

Let us start by recalling some material on (pre)oriented strict abelian group objects. For more detail, we refer the reader to [GM23, Section 3] (see also [Lur18a, § 1]). Let Lat denote the full subcategory of the 1-category of abelian groups spanned by the finitely generated free abelian groups and let $\mathcal{C}$ be a complete and cocomplete category.

DEFINITION 14.1. We let $\mathrm{Ab}(\mathcal{C})$ denote the ∞ -category of *abelian group objects in $\mathcal{C}$* , that is, the ∞ -category of product-preserving functors from $\mathrm{Lat}^{\mathrm{op}}$ to $\mathcal{C}$.

Remark 14.2. Objects of $\mathrm{Ab}(\mathcal{C})$ are often called *strict abelian group objects in $\mathcal{C}$* , to distinguish them from group-like commutative monoids in $\mathcal{C}$. For example, $\mathrm{CGrp}(\mathcal{S}) \simeq \mathrm{Sp}^{\geq 0}$, while, by [Lur18a, Remark 1.2.10], $\mathrm{Ab}(\mathcal{S}) \simeq \mathrm{Mod}_{\mathbb{Z}}^{\geq 0}$.

Example 14.3. Note that every compact abelian Lie group is an abelian group object of the topological category $\mathbf{AbCptLie}$, by [GM23, Example 3.3]. Because the functor $\mathbf{B}(-): \mathbf{AbCptLie} \rightarrow \mathbf{Glo}_{\mathbf{ab}} \rightarrow \mathcal{S}_{\mathbf{ab-gl}}$ preserves products (see Remark 3.5), we conclude that every orbit stack $\mathbf{B}A$ is canonically an abelian group object in $\mathcal{S}_{\mathbf{ab-gl}}$. Furthermore, for every group homomorphism $\alpha: H \rightarrow G$ of abelian groups, $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ is canonically a morphism of abelian group objects.

Example 14.4. Let $\mathcal{S}$ be a nonconnective spectral Deligne–Mumford stack, keeping in mind our convention (item (7) in Section 1.8). By definition, any (strict) elliptic curve over $\mathcal{S}$ is an abelian group object in the ∞ -category of nonconnective spectral Deligne–Mumford stacks over $\mathcal{S}$ (see [GM23, Definition 5.6]).

Example 14.5. Every object $A \in \mathbf{Ab}(\mathcal{S})$ canonically defines an object $A := A \otimes \mathrm{pt}$ in $\mathbf{Ab}(\mathcal{C})$ by taking the constant A -shaped colimit of the point. For example, $B\mathbb{T}$ is canonically an abelian group object in spaces (corresponding to $\Sigma^2(\mathbb{Z})$ in $\mathrm{Mod}_{\mathbb{Z}}^{\geq 0}$), and so $B\mathbb{T}$ is also an object of $\mathbf{Ab}(\mathcal{C})$ for any presentable $\mathcal{C}$.

DEFINITION 14.6. A preorientation of an abelian group object X in $\mathcal{C}$ is a map $B\mathbb{T} \rightarrow X$ of abelian group objects in $\mathcal{C}$. We define the ∞ -category of *preoriented abelian group objects* in $\mathcal{C}$ as $\mathrm{PreAb}(\mathcal{C}) := \mathbf{Ab}(\mathcal{C})_{B\mathbb{T}/}$.

CONSTRUCTION 14.7 [GM23, Construction 3.8]. Consider the strict abelian group object $\mathbf{B}\mathbb{T}$ in $\mathcal{S}_{\mathbf{ab-gl}}$. Mapping into $\mathbf{B}\mathbb{T}$ defines a functor $\mathcal{S}_{\mathbf{ab-gl}}^{\mathrm{op}} \rightarrow \mathbf{Ab}(\mathcal{S})$. Restricting this along the subcategory $\mathbf{Glo}_{\mathbf{ab}}$, we obtain a functor $(\hat{-}): \mathbf{Glo}_{\mathbf{ab}}^{\mathrm{op}} \rightarrow \mathbf{Ab}(\mathcal{S})$. The final object pt of $\mathbf{Glo}_{\mathbf{ab}}$ is sent to $\mathrm{Map}_{\mathbf{Glo}_{\mathbf{ab}}}(\mathrm{pt}, \mathbf{B}\mathbb{T}) \simeq B\mathbb{T}$ by $(\hat{-})$, and so the previous functor lifts to a functor

$$(\hat{-}): \mathbf{Glo}_{\mathbf{ab}}^{\mathrm{op}} \rightarrow \mathrm{PreAb}(\mathcal{S}).$$

We call this construction *shifted Pontryagin duality*. Note that $\widehat{\mathbf{B}G} \simeq \widehat{G} \times B\mathbb{T}$, where $\widehat{G}$ is the ordinary Pontryagin dual of G , explaining the name.

CONSTRUCTION 14.8. By [GM23, Proposition 3.10], there is an equivalence

$$\mathrm{PreAb}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathrm{PreAb}(\mathcal{S})^{\mathrm{op}}, \mathcal{C}).$$

Given $\mathbf{G} \in \mathrm{PreAb}(\mathcal{C})$, we restrict the associated functor $\mathrm{PreAb}(\mathcal{S})^{\mathrm{op}} \rightarrow \mathcal{C}$ along shifted Pontryagin duality to obtain

$$\mathbf{G}[\hat{-}]: \mathbf{Glo}_{\mathbf{ab}} \rightarrow \mathcal{C}, \quad \mathbf{B}A \mapsto \mathbf{G}[\hat{A}].$$

Clearly, this defines a functor

$$\mathrm{PreAb}(\mathcal{C}) \rightarrow \mathrm{Fun}(\mathbf{Glo}_{\mathbf{ab}}, \mathcal{C}), \quad \mathbf{G} \mapsto \mathbf{G}[\hat{-}]. \quad (14.1)$$

Remark 14.9. As observed in the proof of [GM23, Proposition 3.15], the functor $\mathbf{G}[\hat{-}]: \mathbf{Glo}_{\mathbf{ab}} \rightarrow \mathcal{C}$ preserves finite products and pullbacks in which at least one of the maps is a quotient map. Moreover, if $\mathcal{C}$ is cartesian closed, it also follows that $\mathbf{G}_{\bullet}: \mathcal{S}_{\mathbf{ab-gl}} \rightarrow \mathcal{C}$ preserves finite products.

Example 14.10. Unraveling the definitions, one finds that $\mathbf{G}_{B\mathbb{T}} \simeq \mathbf{G}$. Moreover, using the previous remark, one sees that $\mathbf{G}_{B\mathbb{C}_n}$ sits in the following pullback square, and so can be identified with the n -torsion points of $\mathbf{G}$.

$$\begin{array}{ccc} \mathbf{G}_{B\mathbb{C}_n} & \longrightarrow & \mathrm{pt} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{G} & \xrightarrow{n \cdot -} & \mathbf{G} \end{array}$$

As mentioned, the functor $\mathbf{G}[\widehat{-}]$ preserves finite products, and so we conclude that the value of $\mathbf{G}[\widehat{-}]$ on all abelian compact Lie groups is precisely the $\hat{A}$ -valued points of $\mathbf{G}$, as the notation suggests.

We now explain how to apply this construction to define the global 2-ring enhancing elliptic cohomology. We begin with the affine case, where it is induced by a ‘geometric’ incarnation of elliptic cohomology

$$\mathbf{E}_\bullet: \mathcal{S}_{\text{ab-gl}}^\omega \rightarrow \text{Shv}(\text{Aff}_{\mathbf{S}})$$

by passing to ∞ -categories of quasi-coherent sheaves. We first make precise what we mean by quasi-coherent sheaves.

DEFINITION 14.11. Consider the functor $\text{Mod}: \text{Aff}_{\mathbf{S}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ which assigns to every affine scheme its category of modules. This satisfies descent with respect to Zariski covers, and so we obtain a unique limit-preserving extension

$$\text{QCoh}: \text{Shv}(\text{Aff}_{\mathbf{S}})^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}).$$

CONSTRUCTION 14.12 [GM23, Construction 6.1]. Let $\text{Top}_{\text{CAlg}}^{\text{loc}}$ denote the ∞ -category of locally spectrally ringed spaces, as defined in [Lur16]. Observe that fully faithful inclusion $\text{Aff}_{\mathbf{S}} \rightarrow \text{Top}_{\text{CAlg}}^{\text{loc}}$ induces a restricted Yoneda embedding functor

$$\text{Top}_{\text{CAlg}}^{\text{loc}} \rightarrow \text{Shv}(\text{Aff}_{\mathbf{S}}),$$

where we endow $\text{Aff}_{\mathbf{S}}$ with the Zariski topology. This is because pushouts of affine spectral schemes along Zariski open immersions can be computed in the ∞ -category of spectrally locally ringed spaces. Then, given an oriented elliptic curve $\mathbf{E} \rightarrow \text{Spec } R$ over an affine base, we may left Kan extend the functor $\mathbf{E}[\widehat{-}]: \text{Glo}_{\text{ab}} \rightarrow \text{Top}_{\text{CAlg}}^{\text{loc}}$ along the fully faithful inclusion $\text{Glo}_{\text{ab}} \rightarrow \mathcal{S}_{\text{ab-gl}}^\omega$ in order to obtain a functor

$$\mathbf{E}_\bullet: \mathcal{S}_{\text{ab-gl}}^\omega \rightarrow \text{Top}_{\text{CAlg}}^{\text{loc}}.$$

We then define $\mathcal{Q}^{\mathbf{E}}$ to be right Kan extension to all global spaces of the composite

$$(\mathcal{S}_{\text{ab-gl}}^\omega)^{\text{op}} \rightarrow (\text{Top}_{\text{CAlg}}^{\text{loc}})^{\text{op}} \rightarrow \text{Shv}(\text{Aff}_{\mathbf{S}})^{\text{op}} \xrightarrow{\text{QCoh}(-)} \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}}).$$

By definition, this functor preserves cofiltered limits, but it will typically not preserve finite limits, as finite colimits are not respected by the functor of points $\text{Top}_{\text{CAlg}}^{\text{loc}} \rightarrow \text{Shv}(\text{Aff}_{\mathbf{S}})$.

Remark 14.13. Observe that, by Construction 14.7, we may identify $\mathbf{E}_{\mathbf{B}A}$ with the $\hat{A}$ -torsion points of $\mathbf{G}$. By definition, the value of $\mathbf{E}_\bullet$ on any finite ab-global space is the Zariski sheaf represented by a locally spectrally ringed space glued together from copies of $\mathbf{E}[\hat{A}]$ for various compact abelian Lie groups A . We note that if A is a finite abelian global quotient, then this locally ringed space is in fact a spectral scheme, but we do not make use of this fact. Nevertheless, it provides a justification for taking finite colimits of the $\mathbf{E}[\hat{A}]$ as spectrally locally ringed spaces instead of Zariski sheaves, since the latter would not be represented by schemes in general.

As we prove in the following, the previous construction provides the global 2-ring associated to an elliptic curve over an affine base. We now explain how to globalize this construction.

CONSTRUCTION 14.14. The functor $\mathcal{Q}^{\mathbf{E}}: \mathcal{S}_{\text{ab-gl}}^{\text{op}} \rightarrow \text{CAlg}(\text{Pr}_{\text{st}}^{\text{L}})$ constructed above took as input an oriented elliptic curve $\mathbf{E} \rightarrow \text{Spec } R$ over an affine scheme. Observe that this is evidently contravariantly functorial in affine spectral schemes equipped with an oriented elliptic curve, or equivalently, with a map to $\mathcal{M}_{\text{el}}^{\text{or}}$, the moduli stack of oriented elliptic curves. Thus, the previous

construction refines to a functor

$$\mathrm{Aff}_{/\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}}^{\mathrm{op}} \rightarrow \mathrm{Fun}(\mathcal{S}_{\mathrm{ab-gl}}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})).$$

The global 2-ring associated to the universal oriented elliptic curve $\mathbf{E}^{\mathrm{univ}} \rightarrow \mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}$ over the moduli stack itself is obtained by taking the limit of this functor.

More generally, an arbitrary nonconnective spectral Deligne–Mumford equipped $\mathbf{S}$ with an oriented spectral elliptic curve $\mathbf{E} \rightarrow \mathbf{S}$ determines a functor $\mathrm{Aff}_{/\mathbf{S}} \rightarrow \mathrm{Aff}_{/\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}}$ and hence, by precomposition, a functor

$$\mathrm{Aff}_{/\mathbf{S}}^{\mathrm{op}} \rightarrow \mathrm{Fun}(\mathcal{S}_{\mathrm{ab-gl}}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})).$$

Taking the limit of this functor, we obtain the elliptic cohomology global 2-ring $\mathcal{Q}^{\mathbf{E}}: \mathcal{S}_{\mathrm{ab-gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$ associated to an oriented elliptic curve $\mathbf{E} \rightarrow \mathbf{S}$. Observe that, if $\mathbf{S} \simeq \mathrm{Spec} R$ is affine, then the ∞ -category $\mathrm{Aff}_{/\mathbf{S}}$ admits a final object, recovering the original definition in the affine case.

LEMMA 14.15. *Let $\mathbf{E}$ be an oriented elliptic curve in spectral Deligne–Mumford stacks over $\mathbf{S}$. Then, $\mathcal{Q}^{\mathbf{E}}$ is a naive global 2-ring.*

Proof. Since $\mathcal{Q}^{\mathbf{E}}$ is extended via limits from the affine case, we may immediately reduce to the case that $\mathbf{S}$ is affine and locally 2-periodic. Consider the unraveling of $\mathcal{Q}^{\mathbf{E}}$ at a compact abelian Lie group A :

$$\mathbb{H}_G(-, \mathcal{Q}^{\mathbf{E}}): \mathcal{S}_A^{\mathrm{op}} \rightarrow \mathrm{QCoh}(\mathbf{E}_{\mathbf{B}A}), \quad (X//A \xrightarrow{f} \mathbf{B}A) \mapsto f_* \mathbb{1}_{\mathrm{QCoh}(\mathbf{E}_{X//A})}.$$

If we restrict to compact A -spaces, then the resulting functor is limit-preserving by the argument of [GM23, Lemma 6.5]. For a general A -space X , we may write X as a filtered colimit of compact A -spaces X_i . Then, $\mathrm{QCoh}(\mathbf{E}_X) = \lim_i \mathrm{QCoh}(\mathbf{E}_{X_i})$, and so $f_* \mathbb{1}_X$ is isomorphic to $\lim_i (f_i)_* \mathbb{1}_{X_i//A}$ by [HY17, Theorem B]. It follows that $\mathbb{H}_G(-, \mathcal{Q}^{\mathbf{E}})$ is Kan extended from the orbits, and so preserves all limits.

We finally want to verify that the decategorification is limit-preserving. Since we have right Kan extended from compact global spaces, it suffices to check that the restricted functor

$$(\mathcal{S}_{\mathrm{ab-gl}}^{\omega})^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}) \xrightarrow{\mathrm{End}(\mathbb{1})} \mathrm{CAlg}$$

is limit-preserving. Since taking endomorphisms of the unit in quasi-coherent sheaves is naturally equivalent to taking global sections, this composite is equivalent to the functor

$$(\mathcal{S}_{\mathrm{ab-gl}}^{\omega})^{\mathrm{op}} \rightarrow \mathrm{LRS}^{\mathrm{op}} \xrightarrow{\Gamma} \mathrm{Sp}.$$

The first functor is limit-preserving by definition, while the second is a right adjoint, and so we conclude that the composite preserves limits. $\square$

We now proceed to show that $\mathcal{Q}^{\mathbf{E}}$ is a genuine global 2-ring. We begin with a lemma.

LEMMA 14.16. *Let $\mathbf{E}$ be an oriented elliptic curve in spectral Deligne–Mumford stacks over $\mathbf{S}$. Consider the following pullback square in $\mathcal{S}_{\mathrm{ab-gl}}$ such that G is a torus.*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbf{B}K \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{B}H & \xrightarrow{f} & \mathbf{B}G \end{array}$$

Then,

$$\begin{array}{ccc} \mathbf{E}_{\mathcal{X}} & \longrightarrow & \mathbf{E}_{\mathbf{B}K} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{E}_{\mathbf{B}H} & \xrightarrow{f} & \mathbf{E}_{\mathbf{B}G} \end{array}$$

is a pullback square in $\mathrm{Shv}(\mathrm{Aff}_{\mathbf{S}})$.

The previous lemma is adapted from a forthcoming work of the first author and Lennart Meier, *Equivariant elliptic cohomology with integral coefficients*. The argument below is a sketch of the key points in the proof.

Proof. We immediately reduce to the case where $\mathbf{S}$ is an affine scheme. We observe that up to homotopy, the diagram $\mathbf{B}H \rightarrow \mathbf{B}G \leftarrow \mathbf{B}K$ lifts to a diagram of pointed objects, and so by Remark 3.4 to a diagram of abelian compact Lie groups. It follows that the diagram lifts to a diagram in $\mathrm{Ab}(\mathcal{S}_{\mathrm{ab-gl}})$. Since the forgetful functor $\mathrm{Ab}(\mathcal{S}_{\mathrm{ab-gl}}) \rightarrow \mathcal{S}_{\mathrm{ab-gl}}$ preserves limits, it suffices to check that the diagram is a pullback in abelian group objects. Since we are now working in an additive category and $\mathbf{E}_{\bullet}$ preserves finite products, it suffices to show that $\mathbf{E}_{\bullet}$ preserves fibers of maps of the form $f: \mathbf{B}H \rightarrow \mathbf{B}G$, where G is a torus and H is an abelian compact Lie group (but not necessarily a subgroup).

First, consider the case in which the map $f: \mathbf{B}H \rightarrow \mathbf{B}G$ is faithful. Then, the fiber of f is the loop space $\mathrm{pt} \times_{\mathbf{B}(G/H)} \mathrm{pt}$, which is G/H . Because G is a torus, G/H is again a torus. Computing the colimit in locally ringed spaces defining $\mathbf{E}_{\mathrm{pt} \times_{\mathbf{B}(G/H)} \mathrm{pt}}$, one easily sees that it is affine over $\mathbf{S}$. It follows that the map

$$\mathbf{E}_{\mathrm{pt} \times_{\mathbf{B}(G/H)} \mathrm{pt}} \rightarrow \mathbf{E}_{\mathrm{pt}} \times_{\mathbf{E}_{\mathbf{B}(G/H)}} \mathbf{E}_{\mathrm{pt}}$$

is an equivalence by [GM23, Theorem 8.1].

Now, consider a general map $\mathbf{B}H \rightarrow \mathbf{B}G$ such that G is a torus. We may factor this as a composite

$$\mathbf{B}H \twoheadrightarrow \mathbf{B}G' \hookrightarrow \mathbf{B}G$$

such that $H \rightarrow G'$ is surjective and $G' \rightarrow G$ injective. Let $H' \subset H$ denote the kernel of the map $H \rightarrow G$.

The fiber F of the map $\mathbf{B}H \rightarrow \mathbf{B}G$ is then a $\mathbf{B}H'$ -bundle over G/G' in $\mathrm{Ab}(\mathcal{S}_{\mathrm{ab-gl}})$. Using that G/G' is also a torus and $\mathbf{B}H'$ is connected, one computes that, in fact, $F \simeq G/G' \times \mathbf{B}H'$. Since $\mathbf{E}_{\bullet}$ preserves finite products and pullbacks along quotient maps (see Remark 14.9), we deduce that the map $\mathbf{E}_Q \rightarrow \mathbf{E}_{\mathbf{B}H} \times_{\mathbf{E}_{\mathbf{B}G}} \mathbf{E}_{\mathrm{pt}}$ is an equivalence. $\square$

THEOREM 14.17. *Suppose that $\mathbf{E} \rightarrow \mathbf{S}$ is an oriented elliptic curve in spectral Deligne–Mumford stacks over $\mathbf{S}$. Then, the naive ab-global 2-ring*

$$\mathcal{Q}^{\mathbf{E}}: \mathcal{S}_{\mathrm{ab}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$$

is genuine with respect to the subcategory of tori.

Proof. We show that $\mathcal{Q}^{\mathbf{E}}$ is genuine with respect to the family of tori in $\mathrm{Glo}_{\mathrm{ab}}$. By Lemma 14.16, given a pullback square as in condition (1) of Definition 10.1, the resulting square is a pullback square in $\mathrm{Shv}(\mathrm{Aff}_{\mathbf{S}})$.

$$\begin{array}{ccc} \mathbf{E}_{\mathcal{X}} & \longrightarrow & \mathbf{E}_{\mathbf{B}K} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{E}_{\mathbf{B}H} & \xrightarrow{\alpha} & \mathbf{E}_{\mathbf{B}G} \end{array}$$

Furthermore, the map $\mathbf{E}_{\mathbf{B}K} \hookrightarrow \mathbf{E}_{\mathbf{B}G}$ is equivalent to the homomorphism of abelian group schemes $\mathbf{E}[\hat{K}] \rightarrow \mathbf{E}[\hat{G}]$ obtained by taking torsion points indexed by the Pontryagin dual groups, and so affine and proper by [GM23, Proposition 3.15] and the proof of Proposition 6.3 of [GM23]. Therefore, the resulting square given by applying $\mathrm{QCoh}(-)$ is right adjointable by the push–pull formula for quasi-coherent sheaves (see [Lur11, Corollary 3.2.6]). Condition (2) also follows immediately from the previous observations and [Lur11, Proposition 3.2.11]. Finally, condition (3) follows from [GM23, Lemma 9.1], which computes that $\mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathcal{Q}^{\mathbf{E}})$ is equivalent to the canonical line bundle $\mathcal{O}_{\mathbf{E}}(-e_1)$ of the elliptic curve $\mathbf{E}$, by an application of Proposition 11.17. $\square$

Remark 14.18. It is *not* true that the diagram $\mathcal{Q}^{\mathbf{E}}$ satisfies right base-change for arbitrary pullback squares of global spaces as follows.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathbf{B}H \\ \downarrow & \lrcorner & \downarrow \alpha \\ \mathbf{B}K & \longrightarrow & \mathbf{B}G \end{array}$$

For example, as explained in a forthcoming work by Gepner and Meier, *Equivariant elliptic cohomology with integral coefficients*, Remark 2.10, if α is injective and $\mathbf{B}G$ is not connected, then right base-change is not satisfied. Therefore, it is crucial in this example that we have the flexibility to restrict to a family of enough injective objects. In this case, the family of tori.

Having shown that $\mathcal{Q}^{\mathbf{E}}$ is genuine, we obtain a family of equivariant cohomology theories which satisfy the axioms of Ginzburg, Kapranov, and Vasserot.

THEOREM 14.19. *Let $\mathbf{E}$ be an oriented elliptic curve in spectral Deligne–Mumford stacks over $\mathbf{S}$. Then, there exists an equivariant elliptic cohomology theory*

$$\mathbb{H}_G(-, \mathbf{E}): \mathcal{S}_G^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{QCoh}(\mathbf{E}[\hat{G}]))$$

for every abelian compact Lie group G , as well as coherently functorial change-of-group transformations

$$Q_{\alpha}: \mathbb{H}_H(\alpha^*(-), \mathbf{E}) \Rightarrow \alpha^* \mathbb{H}_G(-, \mathbf{E})$$

for any group homomorphism $\alpha: H \rightarrow G$. Furthermore, these data satisfy the Ginzburg–Kapranov–Vasserot axioms.

- (1) Induction. Let $\alpha: G \rightarrow G/N$ be a surjective group homomorphism with kernel N and let X be a G -space such that the action of N on X is free. Write $p: X \rightarrow X/N$ for the canonical projection map, which is G equivariant. Then, the composite

$$\mathbb{H}_{G/N}(\alpha_!(X/N), \mathbf{E}) \xrightarrow{T_{\alpha}} \alpha_* \mathbb{H}_G(X/N, \mathbf{E}) \xrightarrow{\alpha_* \mathbb{H}_G(p, \mathbf{G})} \alpha_* \mathbb{H}_G(X, \mathbf{E})$$

is an equivalence.

- (2) Base-change. Let $\mathbf{B}\alpha: \mathbf{B}H \rightarrow \mathbf{B}G$ be a map in $\mathrm{Glo}_{\mathcal{E}}$ such that G is a torus. Then, the natural transformation

$$Q_{\alpha}: \alpha^* \mathbb{H}_G(X, \mathbf{E}) \Rightarrow \mathbb{H}_H(\alpha^* X, \mathbf{E})$$

is an equivalence for all compact G -spaces.

- (3) Künneth. Let G and H be two tori, X a compact G -space, and Y a compact H -space. Then, there is an equivalence

$$\pi_G^* \mathbb{H}_G(X, \mathbf{E}) \otimes \pi_H^* \mathbb{H}_H(Y, \mathbf{E}) \simeq \mathbb{H}_{G \times H}(X \times Y, \mathbf{E}),$$

where π_H and π_G denote the two projections $G \times H \rightarrow H, G$. Moreover, the functor $\mathbb{H}_G(-, \mathbf{E}): (\mathcal{S}_G^{\omega})^{\mathrm{op}} \rightarrow \mathrm{QCoh}(\mathbf{E}[\hat{G}])$ is strong monoidal.

Proof. Applying Theorem 6.2 to the naive global 2-ring $\mathcal{Q}^{\mathbf{E}}$, we obtain the data as in the theorem. Because $\mathcal{Q}^{\mathbf{E}}$ is genuine with respect to the family of tori, the Ginzburg–Kapranov–Vasserot axioms follow from Theorem 10.3. $\square$

Remark 14.20. Let us, more explicitly, compare these axioms to those of Ginzburg–Kapranov–Vasserot. Firstly we note that, as observed in Remark 14.18, $\mathcal{Q}^{\mathbf{E}}$ is *not* genuine with respect to the family of all groups. In particular, the base-change axiom only holds for group homomorphisms into a torus, and the Künneth axiom only holds for tori (see the previously mentioned forthcoming work by the first author and Meier for details and counterexamples of abelian groups which are not tori).

Secondly, let us address the periodicity axiom, which we have so far not discussed. We compute that

$$\mathbb{H}_{\mathbb{T}}(X \wedge S^{\tau}, \mathbf{E}) \simeq \mathbb{H}_{\mathbb{T}}(X, \mathbf{E}) \otimes \mathbb{H}_{\mathbb{T}}(S^{\tau}, \mathbf{E}) \simeq \mathbb{H}_{\mathbb{T}}(X, \mathbf{E}) \otimes \mathcal{O}_{\mathbf{E}}(-e_1).$$

We interpret this as stating that the periodicity axiom holds in $\mathrm{RO}(\mathbb{T})$ -grading. Obtaining the periodicity axiom precisely, as stated in [GKV95], would require the object $\mathcal{O}_{\mathbf{E}}(-e_1)$ to be equivalent to $\Sigma^2 \mathcal{O}_{\mathbf{E}}$, which is typically not the case.

From our results, we also obtain a global spectrum representing globally equivariant elliptic cohomology.

DEFINITION 14.21. Let $\mathbf{E} \rightarrow \mathbf{S}$ be an oriented elliptic curve in spectral Deligne–Mumford stacks over $\mathbf{S}$. Applying Theorem 12.7, we conclude that the naive ab-global ring $\mathrm{H}_{\mathrm{gl}}(-, \mathbf{E}): \mathcal{S}_{\mathrm{ab-gl}}^{\mathrm{op}} \rightarrow \mathrm{Sp}$ is represented by an ab-global spectrum, which we denote by $\Gamma_{\mathbf{E}}(\mathbf{S}; \mathcal{O}_{\mathbf{S}})$.

Remark 14.22. The underlying spectrum of the global spectrum $\Gamma_{\mathbf{E}}(\mathbf{S}; \mathcal{O}_{\mathbf{S}})$ is simply the global sections $\Gamma(\mathbf{S}) = \Gamma(\mathbf{S}; \mathcal{O}_{\mathbf{S}})$ of the spectral Deligne–Mumford stack $\mathbf{S} = (\mathbf{S}, \mathcal{O}_{\mathbf{S}})$. However, the globally equivariant structure depends on the oriented abelian group object $\mathbf{E}$ over $\mathbf{S}$.

Example 14.23. Most of the arguments of this section can be adapted and carried out for an arbitrary oriented abelian group object over a spectral Deligne–Mumford stack. In particular, we may apply the construction for the multiplicative group scheme $\mathbb{G}_m$ over $\mathrm{Spec}(\mathrm{KU})$, oriented as in [Lur09a, § 3.1] (see also [GM23, § 4]). By our previous construction, we obtain a global spectrum $\Gamma_{\mathbb{G}_m}(\mathrm{KU}; \mathcal{O}_{\mathrm{KU}})$, which is a global form of equivariant complex K-theory.

Remark 14.24. We note that a global spectrum $\mathrm{KU}_{\mathrm{gl}}$ enhancing globally equivariant complex K-theory has previously been constructed by [Sch18]. In fact, using [GM23, § 4], one can show that the global cohomology theory associated to $\Gamma_{\mathbb{G}_m}(\mathrm{KU}; \mathcal{O}_{\mathrm{KU}})$ agrees with that associated to Schwede’s global complex K-theory spectrum. However, it is less clear how to construct an equivalence of these objects as global spectra. In other words, a coherent identification of the representation sphere deloopings provided by both constructions does not seem to follow formally. While we certainly expect this to be possible, we do not attempt a rigorous comparison here.

Example 14.25. Suppose $\mathbf{E}$ is the universal elliptic curve lying over the moduli stack of oriented elliptic curves $(\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}, \mathcal{O}_{\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}})$ (see [Lur18b, Proposition 7.2.10] and [Mei22, § 4.1]). Then, $\mathrm{TMF}_{\mathrm{gl}} := \Gamma_{\mathbf{E}}(\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}, \mathcal{O}_{\mathcal{M}_{\mathrm{ell}}^{\mathrm{or}}}) \in \mathrm{CAlg}(\mathrm{Sp}_{\mathrm{ab-gl}})$ is a global refinement of the spectrum TMF of topological modular forms.

Example 14.26. Let $\mathbf{E}$ be an oriented elliptic curve over $\mathbf{S}$. Restricting the ab-global spectrum $\Gamma_{\mathbf{E}}(\mathbf{S}; \mathcal{O}_{\mathbf{S}})$ to any compact abelian Lie group, we obtain a genuine A-spectrum representing

A -equivariant elliptic cohomology. By construction, the restriction of $\Gamma_{\mathbf{E}}(\mathcal{S}; \mathcal{O}_{\mathcal{S}})$ to a $\mathbb{T}$ -spectrum agrees with the $\mathbb{T}$ -spectrum R of [GM23, Construction 9.3]. Moreover, given an *arbitrary* compact Lie group G , our construction also gives a definition of a genuine G -spectrum representing G -equivariant elliptic cohomology. Namely, we may first right induce $\Gamma_{\mathbf{E}}(\mathcal{S}; \mathcal{O}_{\mathcal{S}})$, in the sense of [Sch18, Theorem 4.5.1], to a fully global spectrum and then restrict this to a G -spectrum. This definition is motivated by the case of equivariant K-theory, which is also right induced from abelian groups, and implements the suggestion of [Lur09a].

15. Tempered cohomology

We start this section by recalling the theory of $\mathbf{G}$ -tempered cohomology and $\mathbf{G}$ -tempered local systems for an oriented $\mathbf{P}$ -divisible group $\mathbf{G}$ over a commutative ring spectrum R following [Lur19]. We then apply our main results to canonically refine tempered cohomology to a global spectrum (see Theorem 15.24). In the final part of this section, we identify Lurie's ∞ -category of tempered local systems in terms of equivariant stable homotopy theory (see Theorem 15.27).

15.1 Tempered cohomology and tempered local systems

Recall that, by [Lur19, Remark 3.5.2], the data of a $\mathbf{P}$ -divisible group $\mathbf{G}$ over a commutative ring spectrum R are equivalent to the data of a functor

$$\mathbf{G}[-]: \mathbf{Ab}_{\text{fin}} \rightarrow \mathbf{CAlg}_R$$

from the subcategory of finite abelian groups to commutative R -algebras such that the following hold.

- (1) $\mathbf{G}[-]$ preserves finite coproducts. In particular, $\mathbf{G}[e] \simeq R$.
- (2) For every short exact sequence $A \rightarrow B \rightarrow C$ of finite abelian groups, the following square is a pushout square.

$$\begin{array}{ccc} \mathbf{G}[A] & \longrightarrow & \mathbf{G}[B] \\ \downarrow & & \downarrow \\ \mathbf{G}[e] & \longrightarrow & \mathbf{G}[C] \end{array}$$

- (3) Applied to an injective group homomorphism, $\mathbf{G}[-]$ is finite flat of positive degree.

Notation 15.1. In this section, we exclusively consider the global family of *finite abelian groups*. To simplify notation, we make the convention that, in this section, $\mathbf{Glo}$ denotes the global orbit category with isotropy in the family of finite abelian groups. We therefore write $\mathcal{S}_{\text{gl}}$ for the associated ∞ -category of global spaces and $\mathbf{Orb}$ for the wide subcategory of $\mathbf{Glo}$ spanned by the faithful morphisms.

By [Lur19, Theorem 3.5.5], a *preorientation* for $\mathbf{G}$ can equivalently be defined as a lift of $\mathbf{G}[-]$ in the following diagram, where $(\hat{})$ denotes the Pontryagin duality functor.

$$\begin{array}{ccc} \mathbf{Ab}_{\text{fin}} & \xrightarrow{\mathbf{G}[-]} & \mathbf{CAlg}_R \\ \mathbf{B}(\hat{}) \downarrow & \nearrow R_{\mathbf{G}}^{\bullet} & \\ \mathbf{Glo}^{\text{op}} & & \end{array} \quad (15.1)$$

A preoriented $\mathbf{P}$ -divisible group $\mathbf{G}$ is *oriented* if, roughly, at every prime p the canonical map from $R_{(p)}^{B(\mathbb{Q}_p/\mathbb{Z}_p)}$ to the formal part of $\mathbf{G}_{(p)}$ is an equivalence of formal groups. See [Lur19, Definition 2.6.12] for a precise definition.

DEFINITION 15.2. Let $\mathbf{G}$ be a preoriented $\mathbf{P}$ -divisible group over a commutative ring spectrum R . Limit extending the composite

$$\mathrm{Glo}^{\mathrm{op}} \xrightarrow{R_{\mathbf{G}}^{\bullet}} \mathrm{CAlg}_R \xrightarrow{\Gamma} \mathrm{Sp}$$

from (15.1), we obtain a global cohomology theory $R_{\mathbf{G}}^{\bullet}$ which, following [Lur19], we call *$\mathbf{G}$ -tempered cohomology*.

To investigate the properties of $\mathbf{G}$ -tempered cohomology, Lurie introduced the notion of $\mathbf{G}$ -tempered local systems. We now recall the definition and list some of the most important properties that this construction satisfies.

DEFINITION 15.3 [Lur19, Construction 5.1.3]. Let R be a commutative ring spectrum and let $\mathbf{G}$ be a preoriented $\mathbf{P}$ -divisible group. For any global space $\mathcal{X}$, we let $\mathrm{Glo}/_{\mathcal{X}}$ denote the fiber product $\mathrm{Glo} \times_{\mathcal{S}_{\mathrm{gl}}} \mathcal{S}_{\mathrm{gl}}/\mathcal{X}$. More informally, $\mathrm{Glo}/_{\mathcal{X}}$ is the ∞ -category whose objects are pairs $(\mathbf{B}A, \eta)$, where $\mathbf{B}A \in \mathrm{Glo}$ and $\eta: \mathbf{B}A^{(-)} \rightarrow \mathcal{X}$ is a map of global spaces (see Notation 3.6).

We also let $\underline{R}_{\mathcal{X}}$ denote the composite

$$(\mathrm{Glo}/_{\mathcal{X}})^{\mathrm{op}} \xrightarrow{\mathrm{fgt}} \mathrm{Glo}^{\mathrm{op}} \xrightarrow{R_{\mathbf{G}}^{\bullet}} \mathrm{CAlg}_R,$$

which we can view as a commutative algebra in $\mathrm{Fun}((\mathrm{Glo}/_{\mathcal{X}})^{\mathrm{op}}, \mathrm{Sp})$, equipped with the pointwise tensor product. A *$\mathbf{G}$ -pretempered local system* $\mathcal{F}$ on $\mathcal{X}$ is an $\underline{R}_{\mathcal{X}}$ -module object of the functor ∞ -category $\mathrm{Fun}((\mathrm{Glo}/_{\mathcal{X}})^{\mathrm{op}}, \mathrm{Sp})$ satisfying the following.

- (A) For any morphism $\alpha: \mathbf{B}A \rightarrow \mathbf{B}A_0$ in $\mathrm{Glo}/_{\mathcal{X}}$ which is represented by a surjective group homomorphism, the map $R_{\mathbf{G}}^{\mathbf{B}A} \otimes_{R_{\mathbf{G}}^{\mathbf{B}A_0}} \mathcal{F}(\mathbf{B}A_0) \rightarrow \mathcal{F}(\mathbf{B}A)$ induced by $\mathcal{F}(\alpha)$ is an equivalence of $R_{\mathbf{G}}^{\mathbf{B}A}$ -modules.

We denote by $\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathcal{X})$ the full subcategory of $\underline{R}_{\mathcal{X}}$ -modules spanned by the $\mathbf{G}$ -pretempered systems on $\mathcal{X}$.

DEFINITION 15.4 [Lur19, Definition 5.2.4]. In the situation of Definition 15.3, let $\mathcal{F}$ be a $\mathbf{G}$ -pretempered system on a global space $\mathcal{X}$. We say that $\mathcal{F}$ is a *$\mathbf{G}$ -tempered local system* if it satisfies the following additional condition.

- (B) For any $\mathbf{B}A \in (\mathrm{Glo}^{\mathrm{op}})_{/\mathcal{X}}$ and faithful morphism $\alpha: \mathbf{B}A_0 \rightarrow \mathbf{B}A$, the canonical map

$$\mathcal{F}(\mathbf{B}A) \rightarrow \mathcal{F}(\mathbf{B}A_0)^{hA/A_0}$$

exhibits the target as a $I(A_0/A)$ -completion of $\mathcal{F}(\mathbf{B}A)$, where $I(A_0/A) := \ker(R_{\mathbf{G}}^{\mathbf{B}A} \rightarrow R_{\mathbf{G}}^{\mathbf{B}A_0})$ is the relative augmentation ideal.

We denote by $\mathrm{LocSys}_{\mathbf{G}}(\mathcal{X})$ the full subcategory spanned by the $\mathbf{G}$ -tempered local systems on $\mathcal{X}$.

Example 15.5. Let X be a space which we identify with a constant global space, as in Notation 3.8. By [Lur19, Variant 5.1.15], there is an equivalence of categories

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(X) \simeq \mathrm{Fun}(X, \mathrm{Mod}_R).$$

If, furthermore, $\mathbf{G}$ is oriented, then by [Lur19, Corollary 5.4.3] every pretempered local system on X is tempered, and so there is also an equivalence

$$\mathrm{LocSys}_{\mathbf{G}}(X) \simeq \mathrm{Fun}(X, \mathrm{Mod}_R).$$

The next example shows that the ∞ -category of the (pre)-local system over $\mathbf{B}A$ is controlled by the faithful maps into $\mathbf{B}A$.

Example 15.6. If $\mathbf{G}$ is a preoriented $\mathbf{P}$ -divisible group over a commutative ring spectrum R , we let $\underline{R}_{\mathbf{G}, \mathrm{fth}}$ denote the composite functor

$$(\mathrm{Orb}/\mathbf{B}A)^{\mathrm{op}} \hookrightarrow (\mathrm{Glo}/\mathbf{B}A)^{\mathrm{op}} \rightarrow \mathrm{Glo}^{\mathrm{op}} \xrightarrow{R_{\mathbf{G}}} \mathrm{CAlg}$$

which we can regard as a commutative algebra of objects in $\mathrm{Fun}((\mathrm{Orb}/\mathbf{B}A), \mathrm{Sp})$. Then, by [Lur19, Proposition 5.1.12], there is an equivalence

$$\mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathbf{B}A) \simeq \mathrm{Mod}_{\underline{R}_{\mathbf{G}, \mathrm{fth}}}(\mathrm{Fun}((\mathrm{Orb}/\mathbf{B}A)^{\mathrm{op}}, \mathrm{Sp})).$$

If $\mathbf{G}$ is oriented, then one can also describe the ∞ -category $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ as a full subcategory of $\mathrm{Mod}_{\underline{R}_{\mathbf{G}, \mathrm{fth}}}(\mathrm{Fun}((\mathrm{Orb}/\mathbf{B}A)^{\mathrm{op}}, \mathrm{Sp}))$ satisfying the analog of condition (B). We refer the reader to [Lur19, Proposition 5.4.2] for more details.

Notation 15.7. For any A there is an evaluation functor

$$\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A) \subseteq \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathbf{B}A) \rightarrow \mathrm{Mod}_{R^{\mathbf{B}A}}, \quad \mathcal{F} \mapsto \mathcal{F}(A),$$

which we often denote by $\mathcal{F}^A$. By Example 15.6, the collection of functors $\{\mathcal{F} \mapsto \mathcal{F}^{A_0}, A_0 \subseteq A\}$ is jointly conservative on pretempered local systems.

Given a morphism of global spaces $f: \mathcal{X} \rightarrow \mathcal{Y}$, consider the functor $(\mathrm{Glo}/\mathcal{X})^{\mathrm{op}} \rightarrow (\mathrm{Glo}/\mathcal{Y})^{\mathrm{op}}$ given by composition with f . This induces a functor on $\mathbf{G}$ -pretempered local systems

$$f^*: \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathcal{Y}) \rightarrow \mathrm{LocSys}_{\mathbf{G}}^{\mathrm{pre}}(\mathcal{X}).$$

Furthermore, the pullback functor clearly preserves $\mathbf{G}$ -tempered local systems and so restricts to a functor $f^*: \mathrm{LocSys}_{\mathbf{G}}(\mathcal{Y}) \rightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathcal{X})$. In fact, by [Lur19, Remark 5.2.11], this construction determines a limit-preserving functor

$$\mathrm{LocSys}_{\mathbf{G}}: \mathcal{S}_{\mathrm{gl}}^{\mathrm{op}} \rightarrow \widehat{\mathrm{Cat}}_{\infty}. \quad (15.2)$$

In the next result, we record some important facts about this functor that we use throughout the section.

PROPOSITION 15.8. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group over a commutative ring spectrum R .*

- (1) *For any $\mathbf{P}$ -global space $\mathcal{X}$, the ∞ -category $\mathrm{LocSys}_{\mathbf{G}}(\mathcal{X})$ is stable and presentably symmetric monoidal with unit object given by $\underline{R}_{\mathcal{X}}$.*
- (2) *For any morphism of global spaces $f: \mathcal{X} \rightarrow \mathcal{Y}$, the functor $f^*: \mathrm{LocSys}_{\mathbf{G}}(\mathcal{Y}) \rightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathcal{X})$ preserves limits and colimits and is symmetric monoidal. In particular, f^* admits a left adjoint $f_!$ and a right adjoint f_* .*
- (3) *Let $f: \mathbf{B}A_0 \rightarrow \mathbf{B}A$ be a faithful map in Glo . There is an equivalence of functors $f_! \simeq f_*$.*
- (4) *For any finite abelian group A , the evaluation functor $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \mathrm{Mod}_{R^{\mathbf{B}A}}$ preserves all limits and colimits.*

- (5) For any finite abelian group A , the ∞ -category $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ is rigidly compactly generated. A set of compact generators is given by $\{f_! \mathbb{1} \mid f \in \mathrm{Orb}/_{\mathbf{B}A}\}$.
- (6) The functor $\mathrm{LocSys}_{\mathbf{G}}$ is a rigid naive global 2-ring.

Proof. For part (1), combine Proposition 5.2.12, Corollary 5.8.6 and Remark 5.8.8 from [Lur19]. Part (2) follows by combining Corollary 5.2.13, Corollary 5.3.2, and Proposition 5.8.13 from [Lur19]. Part (3) follows from the fact that $f: \mathbf{B}H \rightarrow \mathbf{B}G$ is $v_{\mathbf{G}}$ -ambidextrous (see the discussion before [Lur19, Definition 7.2.4]). We first observe that f is relative π -finite by [Lur19, Proposition 7.2.7] together with Example 3.7. It then follows from [Lur19, Theorem 7.2.10] that the map f is $v_{\mathbf{G}}$ -ambidextrous as needed. Part (4) follows by combining Corollaries 5.2.13 and 5.3.2 of [Lur19]. The previous point implies that the unit of $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ is compact for all $\mathbf{B}A$. Note that $f_!$ preserves compact objects as f^* preserves all colimits, and so the object $f_! \mathbb{1}$ associated to a faithful map $f: \mathbf{B}A_0 \hookrightarrow \mathbf{B}A$ is compact and corepresents evaluation at A_0 . Notation 15.7 therefore implies that $\{f_! \mathbb{1} \mid f \in \mathrm{Orb}/_{\mathbf{B}A}\}$ is a set of compact generators for $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$. These objects are even self-dual. This is a consequence of part (3), as is explained in [Lur19, Proposition 7.3.15]. It then follows that $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ is rigidly compactly generated as the unit is compact and we have a set of compact and dualizable generators, which concludes part (5). Finally, part (6) easily follows from parts (1), (2), and (4). $\square$

Remark 15.9. Part (3) of the theorem above shows that the categories of tempered local systems satisfy an analog of the Wirthmüller isomorphisms, also known as ambidexterity with respect to faithful maps of orbits. In [Lur19], Lurie in fact shows that the categories of tempered local systems satisfy ambidexterity for a much larger class of maps, namely all relatively π -finite maps of global spaces. For the purposes of our article, we only require the special case of Wirthmüller isomorphisms.

Decategorifying $\mathrm{LocSys}_{\mathbf{G}}$, we obtain a naive global ring with associated global cohomology theory

$$H_{\mathrm{gl}}(-, \mathrm{LocSys}_{\mathbf{G}}): \mathcal{S}_{\mathrm{gl}}^{\mathrm{op}} \rightarrow \mathrm{CAlg}.$$

By [Lur19, Example 7.1.5], it agrees with $R_{\mathbf{G}}^{\bullet}$. We conclude that $\mathbf{G}$ -tempered local systems function as a categorification of tempered cohomology. We would like to exhibit LocSys as a genuine global 2-ring. To accomplish this, we have to take a digression and first discuss a notion of geometric fixed points for tempered local systems.

15.2 Geometric fixed points

To begin, we follow [Lur19, Notation 5.6.8] by making the following definition.

DEFINITION 15.10. Let R be a commutative ring spectrum, $\mathbf{G}$ be a preoriented $\mathbf{P}$ -divisible group over R , and A a finite abelian group. We define $\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\mathrm{deg}} \subset \mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})$ to be the union of the images of all maps $\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A_0}) \rightarrow \mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})$ associated to all proper subgroups A_0 of A . Then, $\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\mathrm{deg}} \subset \mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})$ is the vanishing locus of some finitely generated ideal $I \subset \pi_0(R_{\mathbf{G}}^{\mathbf{B}A})$. We say an $R_{\mathbf{G}}^{\mathbf{B}A}$ -module M is $\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\mathrm{deg}}$ -nilpotent/local/complete if and only if it is I -nilpotent/local/complete in the sense of Definitions 7.1.1.1, 7.2.4.1, and 7.3.1.1 of [Lur16], respectively. We write $\mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\mathrm{deg}-\mathrm{loc}}$ for the subcategory of $\mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}$ spanned by the $\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\mathrm{deg}}$ -local modules and write ι_{loc} for the inclusion of $\mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\mathrm{deg}-\mathrm{loc}}$ into $\mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}$. By [Lur16, Proposition 7.2.4.4], this inclusion admits a left adjoint $L: \mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}} \rightarrow \mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\mathrm{deg}-\mathrm{loc}}$.

DEFINITION 15.11. Let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group over a commutative ring spectrum R . Let A be a finite abelian group and let $\mathcal{F} \in \text{LocSys}_{\mathbf{G}}(\mathbf{B}A)$. We define $\Phi^A(\mathcal{F}) := L(\mathcal{F}^A)$, that is, as the localization of $\mathcal{F}^A \in \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}$ at $\text{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\text{deg}}$. We refer to this as the A -geometric fixed points of $\mathcal{F}$. This gives a functor

$$\text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\text{deg}-\text{loc}} \simeq \text{Mod}_{\Phi^A(\mathbb{1})}.$$

For any subgroup $\iota: A_0 \subset A$, we define $\Phi^{A_0}(\mathcal{F}) := \Phi^{A_0}(\iota^* \mathcal{F})$.

We now show that our definition of a geometric fixed points functor enjoys similar properties as the geometric fixed points functor in equivariant stable homotopy theory. We start with the following proposition.

PROPOSITION 15.12. For any finite abelian group A , the functor $\Phi^A: \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\text{deg}-\text{loc}}$ is a smashing localization, with right adjoint given by $U_A \circ \iota_{\text{loc}}$, where U_A sends an $R_{\mathbf{G}}^{\mathbf{B}A}$ -module M to the tempered local system determined by the assignment

$$A_0 \subset A \mapsto \begin{cases} M, & A_0 = A, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the functor $\Phi^{A_0}: \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A_0}}^{\text{deg}-\text{loc}}$ is symmetric monoidal for all $A_0 \subset A$.

Proof. Consider the functor

$$L \circ \text{ev}_A: \text{LocSys}_{\mathbf{G}}^{\text{pre}}(\mathbf{B}A) \rightarrow \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}} \rightarrow \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\text{deg}-\text{loc}},$$

where the first functor is the evaluation at A , and the second functor is localization with respect to $\text{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\text{deg}}$. Note that the evaluation functor ev_A admits a fully faithful right adjoint U_A , given by relative right Kan extension, which is defined as in the proposition. We claim that the right adjoint $U_A \circ \iota_{\text{loc}}$ of $L \circ \text{ev}_A$ takes values in *tempered* local systems. Note that tempered local systems form the right class in an orthogonal decomposition on $\text{LocSys}^{\text{pre}}(\mathbf{B}A)$ by [Lur16, Proposition 7.2.1.4], and therefore it suffices to show that $L \circ \text{ev}_A$ kills those pretempered local systems which are left orthogonal to tempered local systems. However, by [Lur19, Proposition 5.7.7], these are precisely the *null* local systems, defined as those $\mathcal{F} \in \text{LocSys}^{\text{pre}}(\mathbf{B}A)$ such that $\mathcal{F}^{A_0}$ is $\text{Spec}(R_{\mathbf{G}}^{\mathbf{B}A_0})^{\text{deg}}$ -nilpotent for all faithful maps $A_0 \hookrightarrow A$. Such local systems are clearly sent to zero by $L \circ \text{ev}_A$. We conclude that $U_A \circ \iota_{\text{loc}}$ restricts to a fully faithful right adjoint to Φ^A . The essential image of $U_A \circ \iota_{\text{loc}}$ is closed under colimits by Proposition 15.8(5), and so we conclude that Φ^A is even a smashing localization (see [HPS97, Definition 3.3.2], for example). In particular, it is canonically strong monoidal. As the pullback functors are symmetric monoidal by Proposition 15.8(2), we conclude that Φ^B is also symmetric monoidal for all $B \subset A$. $\square$

We next recognize geometric fixed points as a localization procedure.

REMARK 15.13. Let A be a finite abelian group and let $\text{Orb}_{/A}^{\circ}$ denote the full subcategory of $\text{Orb}_{/A}$ obtained by removing the terminal object. By [Lur19, Theorem 5.6.9], if $\mathcal{F} \in \text{LocSys}_{\mathbf{G}}(\mathbf{B}A)$, then

$$\mathcal{F}^A \rightarrow \lim_{A_0 \in (\text{Orb}_{/A}^{\circ})^{\text{op}}} \mathcal{F}^{A_0}$$

exhibits the right-hand side as the completion of $\mathcal{F}^A$ at $\text{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\text{deg}}$. Since local and complete objects form an orthogonal decomposition, we conclude that the functor

$$\text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}} \rightarrow \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\text{deg}-\text{loc}} \times \text{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\text{deg}-\text{cpl}}, \quad \mathcal{F}^A \mapsto \Phi^A \mathcal{F} \times \lim_{A_0 \in (\text{Orb}_{/A}^{\circ})^{\text{op}}} \mathcal{F}^{A_0}$$

is conservative.

PROPOSITION 15.14. *Fix a finite abelian group A and let I_A denote the localizing ideal of $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ spanned by the objects of the form $\iota_! \mathbb{1}$, where $\iota: A_0 \hookrightarrow A$ is an inclusion of a proper subgroup. Consider $\mathcal{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$. Then, $\Phi^A \mathcal{F} = 0$ if and only if $\mathcal{F} \in I_A$.*

Proof. The statement in the proposition will follow from the following three claims:

- (i) $I_A = {}^\perp(I_A^\perp)$;
- (ii) $I_A^\perp$ is precisely the essential image of $U_A \iota_{\mathrm{loc}}: \mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\mathrm{deg}-\mathrm{loc}} \rightarrow \mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$;
- (iii) $\Phi^A \mathcal{F} = 0$ if and only if $\mathcal{F}$ lies in the left orthogonal of the essential image of $U_A \iota_{\mathrm{loc}}$.

Claim (i) is [Sta24, Lemma 0CQS] (we use that I_A is localizing, so the canonical inclusion functor admits a right adjoint). For claim (ii), note that if $\mathcal{F} = U_A \iota_{\mathrm{loc}}(M)$, then $\iota^* \mathcal{F} = \iota^* U_A \iota_{\mathrm{loc}}(M) \simeq 0$ for any proper inclusion $\iota: A_0 \hookrightarrow A$. This, together with a simple adjunction argument, shows that $\mathcal{F} \in I_A^\perp$. For the converse, suppose $\mathcal{F}$ is right orthogonal to the induced objects. Given a faithful map $\iota: A_0 \rightarrow A$, mapping out of $\iota_! \mathbb{1}$ corepresents evaluation at A_0 by adjunction. Therefore, $\mathcal{F}^{A_0}$ is necessarily zero for all proper subgroups A_0 . However, by Remark 15.13, the tempered local systems with this property are clearly all of the form $U_A(M)$, where M is a $\mathrm{Spec}(R_{\mathbf{G}}^{\mathbf{B}A})^{\mathrm{deg}-\mathrm{local}}$ object. This proves claim (ii). Finally, claim (iii) easily follows from the equivalence

$$\mathrm{Map}(\Phi^A \mathcal{F}, M) \simeq \mathrm{Map}(\mathcal{F}, U_A \iota_{\mathrm{loc}}(M)). \quad \square$$

Remark 15.15. We conclude from the previous result that the quotient $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)/I_A$ is equivalent to $\mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A}}^{\mathrm{deg}-\mathrm{loc}}$ and that the geometric fixed points functor Φ^A is equivalent to the canonical functor from $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ to the quotient.

We next prove that the geometric fixed points functor forms a jointly conservative family.

LEMMA 15.16. *Let A be a finite abelian group. For all subgroups $A_0 \subseteq A$, the functors*

$$\Phi^{A_0}: \mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \mathrm{Mod}_{R_{\mathbf{G}}^{\mathbf{B}A_0}}^{\mathrm{deg}-\mathrm{loc}}$$

form a jointly conservative family.

Proof. We argue by induction on the order of the group A . If A is the trivial group, then $\Phi^A \mathcal{F} = \mathcal{F}^A$ and $(-)^A$ is an equivalence by [Lur19, Example 5.1.13], and so conservative. Now, suppose A is an arbitrary abelian group, and let $\mathcal{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ be a tempered local system such that $\Phi^{A_0}(\mathcal{F}) \simeq 0$ for all $A_0 \subseteq A$. By induction, we know that for every proper subgroup inclusion $\iota: A_0 \subset A$, we have $\iota^* \mathcal{F} \simeq 0$ and so also $\mathcal{F}^{A_0} \simeq 0$. As noted in Remark 15.13, there is a conservative functor

$$\mathcal{F}^A \mapsto \Phi^A \mathcal{F} \times \lim_{A_0 \in (\mathrm{Orb}_{/A}^{\circ})^{\mathrm{op}}} \mathcal{F}^{A_0}.$$

The right-hand side is trivial by assumption and by induction. Therefore, $\mathcal{F}^A \simeq 0$. By Notation 15.7, we conclude that $\mathcal{F} \simeq 0$. $\square$

Given the above, we have all the pieces required to show that geometric fixed points detect invertible dualizable objects.

PROPOSITION 15.17. *An object $\mathcal{F}$ is an invertible object of $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ if and only if*

- (1) $\mathcal{F}$ is a dualizable object of $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}A)$ and
- (2) $\Phi^{A_0} \mathcal{F}$ is an invertible object for all $A_0 \subset A$.

Proof. This follows immediately from the fact that the functors Φ^{A_0} form a jointly conservative family of strong monoidal functors. $\square$

We would like to apply this to the object $\mathbb{H}_{C_n}(S^{t_n}, \mathbf{G})$. For this, we need to understand the geometric fixed points of ‘Borel tempered local systems’. To this end, we introduce the proper Tate construction and explain its connection to the geometric fixed points functor.

Notation 15.18. Consider $R \in \mathbf{CAlg}$ and a finite group A . We let $J_A \subset \mathrm{Fun}(BA, \mathrm{Mod}_R)$ denote the thick ideal generated by the objects $i_! R$, for every proper subgroup inclusion $i: BA_0 \rightarrow BA$. Following [AMR21], we define the *proper Tate construction* as the composite

$$(-)^{\tau A}: \mathrm{Fun}(BA, \mathrm{Mod}_R) \xrightarrow{p} \mathrm{Fun}(BA, \mathrm{Mod}_R)/J_A \xrightarrow{\mathrm{Map}(\mathbb{1}, -)} \mathrm{Sp},$$

where p is the quotient functor.

Recall, from Example 3.11, that there is a canonical map $\nu: BA \rightarrow \mathbf{BA}$ of global spaces. The next result relates the proper Tate construction to the geometric fixed points functor.

THEOREM 15.19. *The following triangle commutes.*

$$\begin{array}{ccc} \mathrm{Fun}(BA, \mathrm{Mod}_R) & \xrightarrow{\nu_*} & \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}) \\ & \searrow (-)^{\tau A} & \swarrow \Phi^A \\ & \mathrm{Sp} & \end{array}$$

The proof is an adaptation of the argument given in [AMR21, Proposition 5.9] for genuine G -spectra. For the convenience of the reader, we record a version of the argument here after some preparation.

Remark 15.20. Since ν^* preserves colimits and $\mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA})$ is compactly generated, we get an extension $\bar{\nu}^*$ as in the following diagram.

$$\begin{array}{ccccc} \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA})^\omega & \hookrightarrow & \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}) & \xrightarrow{\nu^*} & \mathrm{Fun}(BA, \mathrm{Mod}_R) \\ \downarrow & & & & \downarrow i \\ \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}) & \xrightarrow{\bar{\nu}^*} & \mathrm{Ind}(\mathrm{Fun}(BA, \mathrm{Mod}_R)) & & \end{array}$$

In other words, for any $\mathcal{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA})^\omega$, we have the formula

$$\bar{\nu}^*(\mathcal{F}) = i\nu^*(\mathcal{F}). \quad (15.3)$$

Moreover, by construction, the functor $\bar{\nu}^*$ fits into the following commutative diagram.

$$\begin{array}{ccc} \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}) & \xrightarrow{\nu^*} & \mathrm{Fun}(BA, \mathrm{Mod}_R) \\ \bar{\nu}^* \downarrow & \nearrow \mathrm{colim} & \\ \mathrm{Ind}(\mathrm{Fun}(BA, \mathrm{Mod}_R)) & & \end{array}$$

Passing to right adjoints, we also deduce the following.

$$\begin{array}{ccc} \mathrm{Fun}(BA, \mathrm{Mod}_R) & \xleftarrow{i} & \mathrm{Ind}(\mathrm{Fun}(BA, \mathrm{Mod}_R)) \\ & \searrow \nu_* & \downarrow \bar{\nu}_* \\ & & \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}) \end{array}$$

Let $\mathrm{Ind}(J_A)$ denote the localizing ideal in $\mathrm{Ind}(\mathrm{Fun}(BA, \mathrm{Mod}_R))$ generated by $i(J_A)$.

LEMMA 15.21. *The localizing ideal generated by $\bar{\nu}^*(I_A)$ is equal to $\text{Ind}(J_A)$ in $\text{Ind}(\text{Fun}(BA, \text{Mod}_R))$.*

Proof. We note that $\text{Ind}(J_A)$ can equivalently be described as the localizing ideal generated by the objects $i\iota_!\mathbb{1}$, where $\iota: A_0 \rightarrow A$ is the inclusion of a subgroup (see [AMR21, Observation 3.8]). Since I_A is generated by the objects $\iota_!\mathbb{1} \in \text{LocSys}(\mathbf{B}A)$, it suffices to observe the following chain of equivalences:

$$\bar{\nu}_A^*(\iota_!\mathbb{1}) \simeq i\nu_A^*\iota_!\mathbb{1} \simeq i\iota_!\nu_{A_0}^*\mathbb{1} \simeq i\iota_!\mathbb{1}.$$

The first equivalence follows from Equation (15.3) and compactness of $\iota_!\mathbb{1}$. For the second, we observe that there is a pullback square of global spaces as follows.

$$\begin{array}{ccc} BA_0 & \xrightarrow{\nu_{A_0}} & \mathbf{B}A_0 \\ B\iota \downarrow & & \downarrow \mathbf{B}\iota \\ BA & \xrightarrow{\nu_A} & \mathbf{B}A \end{array}$$

Therefore, we obtain, by [Lur19, Corollary 7.1.7], an equivalence $\nu_A^* \circ \iota_! \simeq \iota_! \circ \nu_{A_0}^*$. $\square$

Remark 15.22. There is a commutative diagram of Verdier quotients as follows, where p, q , and r are symmetric monoidal.

$$\begin{array}{ccccc} \text{Fun}(BA, \text{Mod}_R) & \xhookrightarrow{i} & \text{Ind}(\text{Fun}(BA, \text{Mod}_R)) & \xrightarrow{\bar{\nu}_*} & \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \\ \downarrow p & & \downarrow j \uparrow q & & \downarrow k \uparrow r \\ \text{Fun}(BA, \text{Mod}_R)/J_A & \hookrightarrow & \text{Ind}(\text{Fun}(BA, \text{Mod}_R))/\text{Ind}(J_A) & \xrightarrow{\bar{\nu}_*} & \text{LocSys}_{\mathbf{G}}(\mathbf{B}A)/I_A \end{array}$$

The existence of right adjoints j and k is a formal consequence of the fact that $\text{Ind}(J_A)$ and I_A are localizing ideals. It follows from [AMR24, Corollary 4.2.15], together with Lemma 15.21, that

$$\bar{\nu}^*(kr\mathbb{1}) = jq\mathbb{1}. \quad (15.4)$$

Proof of Theorem 15.19. Consider the following diagram, where the left vertical composite agrees with $(-)^{\tau^A}$ by definition, the right vertical composite agrees with the forgetful functor composed with Φ^A by Remark 15.15, and the top horizontal composite agrees with ν^* by Remark 15.20.

$$\begin{array}{ccccc} \text{Fun}(BA, \text{Mod}_R) & \xhookrightarrow{i} & \text{Ind}(\text{Fun}(BA, \text{Mod}_R)) & \xrightarrow{\bar{\nu}_*} & \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \\ \downarrow p & & \downarrow j \uparrow q & & \downarrow k \uparrow r \\ \text{Fun}(BA, \text{Mod}_R)/J_A & \xhookrightarrow{\bar{i}} & \text{Ind}(\text{Fun}(BA, \text{Mod}_R))/\text{Ind}(J_A) & \xrightarrow{\bar{\nu}_*} & \text{LocSys}_{\mathbf{G}}(\mathbf{B}A)/I_A \\ \text{Map}(\mathbb{1}, -) \downarrow & & & & \downarrow \text{Map}(\mathbb{1}, -) \\ \text{Sp} & \xrightarrow{\quad \quad \quad = \quad \quad \quad} & & & \text{Sp} \end{array}$$

In other words, we can rephrase the proposition as claiming that the outer diagram commutes. We first note that the commutativity of the top part of the diagram follows from Remark 15.22. Moreover, since p and r are symmetric monoidal, we can rewrite the global section functor as $\text{Map}(p\mathbb{1}, -)$ and $\text{Map}(r\mathbb{1}, -)$. Therefore, we ought to prove that for all $X \in \text{Fun}(BA, \text{Mod}_R)$, there is a natural equivalence

$$\text{Map}(r\mathbb{1}, r\bar{\nu}_*iX) \simeq \text{Map}(p\mathbb{1}, pX).$$

Applying various adjunction equivalences, together with Equation (15.4), yields

$$\mathrm{Map}(r\mathbb{1}, r\bar{\nu}_*iX) \simeq \mathrm{Map}(\bar{\nu}^*kr\mathbb{1}, iX) \simeq \mathrm{Map}(jq\mathbb{1}, iX) \simeq \mathrm{Map}(q\mathbb{1}, qiX).$$

Finally, using that i is symmetric monoidal, the commutativity of the top right part of the diagram, and the fact that $\bar{i}$ is fully faithful, we see that

$$\mathrm{Map}(q\mathbb{1}, qiX) \simeq \mathrm{Map}(qi\mathbb{1}, qiX) = \mathrm{Map}(\bar{i}p\mathbb{1}, \bar{i}pX) \simeq \mathrm{Map}(p\mathbb{1}, pX)$$

as required. $\square$

15.3 Tempered local systems are genuine

We may unravel $\mathrm{LocSys}_{\mathbf{G}}$ into a family of equivariant cohomology theories

$$\mathbb{H}_A(-, \mathrm{LocSys}_{\mathbf{G}}) : \mathcal{S}_A^{\mathrm{op}} \rightarrow \mathrm{CAlg}(\mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}))$$

using Theorem 6.2.

Notation 15.23. We abbreviate the functor $\mathbb{H}_{\mathrm{gl}}(-, \mathrm{LocSys}_{\mathbf{G}})$ by $\mathbb{H}_{\mathrm{gl}}(-, \mathbf{G})$ and write $\mathbb{H}_A(-, \mathbf{G})$ instead of $\mathbb{H}_A(-, \mathrm{LocSys}_{\mathbf{G}})$.

As shown in [Lur19], tempered local systems function as an incredibly rich and powerful categorification of tempered cohomology. Some of this richness is captured by the following theorem.

THEOREM 15.24. *Let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group over a commutative ring spectrum R . Then, $\mathrm{LocSys}_{\mathbf{G}}$ is genuine with respect to all finite abelian groups, and so a rigid global 2-ring. In particular, tempered cohomology is represented by a global spectrum $\underline{R}_{\mathbf{G}}^{\mathrm{gl}}$.*

Proof. We have already noted in Proposition 15.8 that $\mathrm{LocSys}_{\mathbf{G}}(-)$ is a rigid naive global 2-ring. We only need to show that $\mathrm{LocSys}_{\mathbf{G}}(-)$ satisfies the three conditions of Definition 10.1 for $\mathcal{T} = \mathrm{Glo}$. The first two conditions are already contained in [Lur19].

- (1) This follows from [Lur19, Theorem 7.1.6].
- (2) This is a consequence of [Lur19, Theorem 7.3.10], where we note that any faithful map $f : \mathbf{B}H \rightarrow \mathbf{B}G$ is $v_{\mathbf{G}}$ -ambidextrous (see the proof of Proposition 15.8(3)).

Condition (3) of a global 2-ring requires more work. By Proposition 11.19, it suffices to show that the object

$$\mathcal{F}_n := \mathbb{H}_{C_n}(S^{\iota_n}, \mathbf{G}) \in \mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}C_n)$$

is invertible for all $1 \leq n < \infty$. To this end, we show that $\mathcal{F}_n$ verifies conditions (1) and (2) of Proposition 15.17.

For condition (1), we note that S^{ι_n} is a finite colimit of C_n -orbits. It follows that $\mathcal{F}_n$ is a finite limit of objects of the form $i_*\mathbb{1}$ associated to subgroup inclusions $i : \mathbf{B}H \rightarrow \mathbf{B}G$. Since each object $i_*\mathbb{1}$ is dualizable by Proposition 15.8(3) and (4), we conclude that $\mathcal{F}_n$ is dualizable.

For condition (2), we have to show that $\Phi^{C_m}\mathcal{F}_n$ is an invertible object for all $1 \leq m \leq n$. However, note that for all subgroups $\alpha : \mathbf{B}C_m \rightarrow \mathbf{B}C_n$, we have $\alpha^*S^{\iota_n} \simeq S^{\iota_m}$. Therefore, by Lemma 10.5 with $\mathcal{T} = \mathrm{Glo}$, it suffices to prove that $\Phi^{C_n}\mathcal{F}_n$ is invertible for all $n \geq 1$. Recall from Lemma 11.12 that there is a fiber sequence

$$\mathcal{F}_n \rightarrow \mathbb{1}_{\mathrm{LocSys}_{\mathbf{G}}(\mathbf{B}C_n)} \rightarrow p_*\mathbb{1}_{\mathrm{LocSys}_{\mathbf{G}}(S(\iota_n)//C_n)} =: \mathcal{G}_n$$

where $p : S(\iota_n)//C_n \rightarrow \mathbf{B}C_n$ is the structure map of the C_n -space $S(\iota_n)$.

If $n = 1$, then $\text{LocSys}_{\mathbf{G}}(\mathbf{BC}_1) = \text{Mod}_R$. Since $S(\iota_1)//C_1$ is the constant global space S^1 , we also know that $\text{LocSys}_{\mathbf{G}}(S(\iota_1)//C_1) \simeq \text{Fun}(S^1, \text{Mod}_R)$. In this case, p_* is given right Kan extension. We can then calculate that $\Phi^{C_1}\mathcal{G}_1 = R^{S^1} \simeq R \oplus \Sigma^{-1}R$, and so $\Phi^{C_1}\mathcal{F}_1 \simeq \text{fib}(R \rightarrow R \oplus \Sigma^{-1}R) = \Sigma^{-2}R$, which is invertible.

Now, suppose $n > 1$; we prove that $\Phi^{C_n}(\mathcal{G}_n)$ is trivial. To do this, we have to understand the tempered local system $\mathcal{G}_n$ better. Note that the action of C_n on $S(\iota_n)$ is free, and so by Proposition 3.18, $S(\iota_n)//C_n$ is equivalent as a global space to the constant global space on the space $S(\iota_n)_{hC_n}$. Furthermore, using Proposition 3.18 again, $S(\iota_n)//C_n$ lives over $\mathbf{BC}_n$ via the composite

$$p: S(\iota_n)_{hC_n} \xrightarrow{f} BC_n \xrightarrow{\nu} \mathbf{BC}_n.$$

Thus, we may compute p_* as the composite $\nu_* f_*$. Now we note that the source and target of f are constant global spaces, so

$$f_*: \text{LocSys}_{\mathbf{G}}(S(\iota_n)_{hC_n}) = \text{Fun}(S(\iota_n)_{hC_n}, \text{Mod}_R) \rightarrow \text{LocSys}_{\mathbf{G}}(BC_n) = \text{Fun}(BC_n, \text{Mod}_R)$$

is given by right Kan extension. Applying this to the unit in $\text{Fun}(S(\iota_n)_{hC_n}, \text{Mod}_R)$, we obtain $\mathbb{D}(S(\iota_n)) \otimes R \in \text{Fun}(BC_n, \text{Mod}_R)$, the Spanier–Whitehead dual of $S(\iota_n)$ with its induced C_n -action tensored by the base R . Note that $\mathbb{D}(S(\iota_n)) \otimes R \in J_{C_n}$ since $S(\iota_n)$ is a finite free C_n -space. Therefore, we conclude that

$$\Phi^{C_n} \nu_*(\mathbb{D}(S(\iota_n)) \otimes R) \simeq (\mathbb{D}(S(\iota_n)) \otimes R)^{\tau_{C_n}} \simeq 0$$

by Theorem 15.19. □

15.4 Tempered local systems as genuine spectra

Write $\underline{R}_{\mathbf{G}}^{\mathbf{BA}}$ for the underlying A -spectrum of $\underline{R}_{\mathbf{G}}^{\text{gl}}$. In this section, we identify $\text{LocSys}_{\mathbf{G}}(\mathbf{BA})$ with the ∞ -category of $\underline{R}_{\mathbf{G}}^{\mathbf{BA}}$ -modules in A -spectra.

CONSTRUCTION 15.25. To begin with, we recall that the naive global 2-ring $\text{LocSys}_{\mathbf{G}}(-)$ is genuine and rigid by Theorem 15.24. Therefore, we obtain a natural transformation

$$\mathbb{T}_{\bullet}: \text{LocSys}_{\mathbf{G}} \Rightarrow \text{Sp}_{\bullet}$$

of functors $\text{Orb}^{\text{op}} \rightarrow \text{Cat}$ by Theorem 13.9. As observed in Remark 13.10, $\mathbb{T}_{\bullet}$ is a natural transformation of lax monoidal functors, and so lifts to a natural transformation

$$\mathbb{T}_{\bullet}: \text{LocSys}_{\mathbf{G}}(-) \Rightarrow \text{Mod}_{\underline{R}_{\mathbf{G}}^{\bullet}},$$

valued in modules over the image of the unit in $\text{LocSys}_{\mathbf{G}}(\bullet)$, which agrees with $\underline{R}_{\mathbf{G}}^{\bullet}$ by Remark 13.3. We note that this is again a transformation of lax monoidal right adjoints.

LEMMA 15.26. *Let A be a finite abelian group and let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group. Then for every subgroup $B \subset A$, the following triangle commutes.*

$$\begin{array}{ccc} \text{LocSys}_{\mathbf{G}}(\mathbf{BA}) & \xrightarrow{\mathbb{T}_A} & \text{Sp}_A \\ & \searrow (-)^B & \swarrow (-)^B \\ & \text{Sp} & \end{array}$$

Proof. Because $\mathbb{T}_A$ commutes with restriction to subgroups, it suffices to prove the lemma for all A with $B = A$. In both cases, the functor $(-)^A$ is given by mapping out of the unit. Therefore, the result follows from the fact that the left adjoint F_A of $\mathbb{T}_A$ is strong monoidal (see Remark 13.10). □

THEOREM 15.27. *Let A be a finite abelian group and let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group. Then,*

$$\Gamma_A: \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Mod}_{\underline{R}_{\mathbf{G}}^{\mathbf{B}A}}$$

is a symmetric monoidal equivalence.

Proof. Recall that $\Gamma_A: \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Sp}_A$ admits a symmetric monoidal left adjoint, which we denote by F_A . We will show that the adjunction $F_A \dashv \Gamma_A$ satisfies the criteria of [MNN17, Proposition 5.29] and so induces an equivalence as in the statement. For simplicity, we recall the conditions of the cited proposition:

- (1) $F_A \dashv \Gamma_A$ satisfies the left projection formula;
- (2) Γ_A commutes with colimits;
- (3) Γ_A is conservative.

For condition (1), we may appeal to [BDS16, Theorem 1.3]. To show condition (2), we observe that it suffices to show that the composite $(-)^B \circ \Gamma_A$ preserves colimits for every subgroup $B \subset A$, because the genuine fixed points functors jointly detect colimits. However, by Lemma 15.26, this agrees with the functor $(-)^B: \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Sp}$, which commutes with colimits by Proposition 15.8(5). Finally, for condition (3), let $\mathcal{F}$ be a tempered local system such that $\Gamma_A(\mathcal{F})$ is zero. Then, $\mathcal{F}^B \simeq \Gamma_A(\mathcal{F})^B$ is zero for all $B \subset A$, and we conclude that $\mathcal{F}$ is zero because the evaluation functors $(-)^A$ are jointly conservative on tempered local systems (see Notation 15.7). $\square$

For completeness, we record the fact that both notions of geometric fixed points agree.

PROPOSITION 15.28. *Let A be a finite abelian group and let $\mathbf{G}$ be an oriented $\mathbf{P}$ -divisible group. Then the following triangle commutes.*

$$\begin{array}{ccc} \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) & \xrightarrow{\Gamma_A} & \text{Mod}_{\underline{R}_{\mathbf{G}}^{\mathbf{B}A}} \\ & \searrow \Phi^A \quad \swarrow \Phi^A & \\ & \text{Mod}_{\Phi^A \underline{R}_{\mathbf{G}}^{\mathbf{B}A}} & \end{array}$$

Proof. It suffices to observe that in both cases, geometric fixed points are given by localizing away from the localizing subcategory generated by objects induced from proper subgroups $B \subset A$. $\square$

Finally, we also show that the equivalence $\Gamma_A: \text{LocSys}_{\mathbf{G}}(\mathbf{B}A) \rightarrow \text{Mod}_{\underline{R}_{\mathbf{G}}^{\mathbf{B}A}}$ is natural in Glo^{op} , in particular with inflation along a surjective group homomorphism. Recall that $\Gamma_{\bullet}: \text{LocSys}_{\mathbf{G}}(\bullet) \Rightarrow \text{Sp}_{\bullet}$ was constructed in Construction 13.1 by passing a natural transformation $F_{\bullet}$ of functors $\text{Glo}^{\text{op}} \rightarrow \text{Cat}$ through the mate equivalence, and so canonically laxly commutes. By the 2-functoriality of taking modules, the lax transformation $\Gamma_{\bullet}$ lifts to a lax natural transformation

$$\text{LocSys}_{\mathbf{G}}(\bullet) \Rightarrow \text{Mod}_{\underline{R}_{\mathbf{G}}^{\bullet}}$$

taking values in modules over the unit. We write $R_{\alpha}: \alpha^* \Gamma_C(-) \Rightarrow \Gamma_A(\alpha^*(-))$ for the natural transformation associated to a map $\mathbf{B}\alpha: \mathbf{B}A \rightarrow \mathbf{B}C$ of global spaces.

PROPOSITION 15.29. *Let $\mathbf{B}\alpha: \mathbf{B}A \rightarrow \mathbf{B}C$ be an arbitrary morphism. Then, the natural transformation*

$$R_{\alpha}: \alpha^* \Gamma_C(-) \Rightarrow \Gamma_A(\alpha^*(-))$$

is an equivalence. In other words, the following square canonically commutes.

$$\begin{array}{ccc} \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BC}) & \xrightarrow{\mathbb{T}_C} & \mathrm{Mod}_{\underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \\ \alpha^* \downarrow & & \downarrow \alpha^* \\ \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BA}) & \xrightarrow{\mathbb{T}_A} & \mathrm{Mod}_{\underline{R}_{\mathbf{G}}^{\mathbf{BA}}} \end{array}$$

Proof. Recall that every map of global spaces factors into a faithful map followed by a quotient map, and therefore we can consider these two cases separately. The case of a faithful map is already shown in Theorem 13.9. Therefore, we may assume that $\mathbf{B}\alpha: \mathbf{BA} \rightarrow \mathbf{BC}$ is a quotient map, that is, $\alpha: A \rightarrow C$ is a surjective group homomorphism. To show the result, it suffices to show that R_α is an equivalence after taking geometric fixed points for all subgroups $A_0 \subset A$. However, because $\mathbb{T}_\bullet$ commutes with faithful maps, we may reduce to the case $A_0 = A$.

We compute that, for all $\mathcal{F} \in \mathrm{LocSys}_{\mathbf{G}}(\mathbf{BC})$,

$$\begin{aligned} \Phi^A \alpha^*(\mathbb{T}_C \mathcal{F}) &= \Phi^A(\alpha^* \mathbb{T}_C \mathcal{F} \otimes_{\alpha^* \underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \underline{R}_{\mathbf{G}}^{\mathbf{BA}}) \simeq \Phi^A \alpha^* \mathbb{T}_C \mathcal{F} \otimes_{\Phi^A \alpha^* \underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \Phi^A \underline{R}_{\mathbf{G}}^{\mathbf{BA}} \\ &\simeq \Phi^C \mathbb{T}_C \mathcal{F} \otimes_{\Phi^C \underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \Phi^A \underline{R}_{\mathbf{G}}^{\mathbf{BA}} \\ &\simeq \Phi^C \mathcal{F} \otimes_{\Phi^C \underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \Phi^A \underline{R}_{\mathbf{G}}^{\mathbf{BA}}, \end{aligned}$$

where in the last step we used Proposition 15.28. Next, we note that because α is surjective, the degenerate ideal of $\underline{R}_{\mathbf{G}}^{\mathbf{BA}}$ is generated by the image of the degenerate ideal of $\underline{R}_{\mathbf{G}}^{\mathbf{BC}}$ (see [Lur19, Remark 5.2.3]). Writing L_A and L_C for the localization at $\mathrm{Spec}(\underline{R}_{\mathbf{G}}^{\mathbf{BA}})^{\mathrm{deg}}$ and $\mathrm{Spec}(\underline{R}_{\mathbf{G}}^{\mathbf{BC}})^{\mathrm{deg}}$, respectively, we can continue computing

$$\Phi^C \mathcal{F} \otimes_{\Phi^C \underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \Phi^A \underline{R}_{\mathbf{G}}^{\mathbf{BA}} \simeq L_C \mathcal{F}^C \otimes_{L_C \underline{R}_{\mathbf{G}}^{\mathbf{BC}}} L_A \underline{R}_{\mathbf{G}}^{\mathbf{BA}} \simeq L_A(\mathcal{F}^C \otimes_{\underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \underline{R}_{\mathbf{G}}^{\mathbf{BA}}).$$

On the other hand,

$$\Phi^A \mathbb{T}_A \alpha^* \mathcal{F} \simeq \Phi^A \alpha^* \mathcal{F} \simeq L_A(\alpha^* \mathcal{F})^A \simeq L_A(\mathcal{F}(\mathbf{BA})) \simeq L_A(\mathcal{F}^C \otimes_{\underline{R}_{\mathbf{G}}^{\mathbf{BC}}} \underline{R}_{\mathbf{G}}^{\mathbf{BA}}),$$

where we used Proposition 15.28, the definition of A -geometric fixed points, the definition of α^* , and finally condition (A) of Definition 15.3. $\square$

Appendix A. Symmetric monoidal higher Brown representability

Suppose that $\mathcal{C}$ is a compactly generated stable ∞ -category. Then, the restricted spectral Yoneda embedding gives an equivalence $\mathcal{C} \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathcal{C}^\omega)^{\mathrm{op}}, \mathrm{Sp})$. In this short appendix, we show that this equivalence is symmetric monoidal, where the right-hand side is endowed with a localized form of Day convolution. We begin by showing that this localization of Day convolution exists.

LEMMA A.1 [CHL⁺25, Proposition 3.3.4]. *Let $\mathcal{C}$ be a presentable ∞ -category and let I be a symmetric monoidal small ∞ -category with κ -small limits, such that the tensor product commutes with κ -small limits in each variable. Then, the localization $L: \mathrm{Fun}(I, \mathcal{C}) \rightarrow \mathrm{Fun}^{\mathrm{R}\kappa}(I, \mathcal{C})$ is compatible with Day convolution.*

We call the induced symmetric monoidal structure on $\mathrm{Fun}^{\mathrm{lex}}(I, \mathcal{C})$ the *localized Day convolution* symmetric monoidal structure.

DEFINITION A.2. Suppose $\mathcal{C}$ and $\mathcal{D}$ are presentably symmetric monoidal ∞ -categories. Suppose, furthermore, that $\mathcal{C}$ is κ -compactly generated for some κ . Consider the Day convolution operad $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\otimes$. Applying the equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{R}\kappa}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{D})$$

and the previous result, we find that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})^{\otimes}$ localizes to a symmetric monoidal structure on $\mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$. We again call this the localized Day convolution symmetric monoidal structure. We note that one can easily show that the resulting symmetric monoidal structure is independent of the cardinal chosen above.

Recall that the Lurie tensor product defines a symmetric monoidal functor $\mathrm{Pr}^L \times \mathrm{Pr}^L \rightarrow \mathrm{Pr}^L$ and so passes to commutative algebra objects $\mathrm{CAlg}(\mathrm{Pr}^L) \times \mathrm{CAlg}(\mathrm{Pr}^L) \rightarrow \mathrm{CAlg}(\mathrm{Pr}^L)$. In particular, we obtain a natural symmetric monoidal structure on $\mathcal{C} \otimes \mathcal{D}$ for any pair $(\mathcal{C}, \mathcal{D})$ of presentably symmetric monoidal ∞ -categories.

THEOREM A.3. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are presentably symmetric monoidal ∞ -categories. Then, the natural equivalence*

$$\mathcal{C} \otimes \mathcal{D} \xrightarrow{\sim} \mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

of [Lur17, Proposition 4.8.1.17] is symmetric monoidal, where the right-hand side is equipped with the Day convolution symmetric monoidal structure.

Proof. Let $\mathcal{C}$ be κ -compactly generated. By definition, there is an equivalence

$$\mathrm{Fun}^R(\mathcal{C}^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{R_\kappa}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{D})$$

of symmetric monoidal categories, and so it suffices to prove that $\mathcal{C} \otimes \mathcal{D}$ is equivalent to the right-hand side. We first do this in the case that $\mathcal{D} \simeq \mathcal{S}$, the unit of Pr^L . In this case, the equivalence is given by the restricted Yoneda embedding. In particular, it is defined by currying the hom functor, and so is lax symmetric monoidal by the universal property of Day convolution. It suffices to show that it is, in fact, strong monoidal. We first observe that, on tensor products of objects in $\mathcal{C}^\kappa$, this follows immediately from [Lur17, Corollary 4.8.1.12]. However, the symmetric monoidal structure on $\mathrm{Fun}^{R_\kappa}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S})$ commutes with colimits in each variable, and so we deduce the statement for arbitrary objects in $\mathcal{C}$.

Now consider the following diagram.

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{D} & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lex}}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{D} & \longrightarrow & \mathrm{Fun}^{\mathrm{lex}}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{D}) \\ & & \uparrow L \otimes \mathrm{Sp} & & \uparrow L \\ & & \mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{D} & \xrightarrow{\sim} & \mathrm{Fun}((\mathcal{C}^\kappa)^{\mathrm{op}}, \mathcal{D}) \end{array}$$

The composite along the top is equivalent to the equivalence $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}((\mathcal{C}^\omega)^{\mathrm{op}}, \mathcal{D})$. Note that the vertical maps are symmetric monoidal localizations by Lemma A.1. Therefore, because the bottom equivalence is symmetric monoidal by [BS24, Proposition 3.10], the top rightmost horizontal map is again symmetric monoidal. We have previously shown that the first map is symmetric monoidal, and therefore the composite is also symmetric monoidal. $\square$

COROLLARY A.4. *Let $\mathcal{C}$ be a compactly generated symmetric monoidal stable ∞ -category. Then, the restricted spectral Yoneda embedding*

$$y: \mathcal{C} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{C}^\omega)^{\mathrm{op}}, \mathrm{Sp})$$

is canonically symmetric monoidal, where the right-hand side is given the localized Day convolution structure.

Proof. The equivalence of the corollary factors as the composite

$$\mathcal{C} \xrightarrow{\sim} \mathcal{C} \otimes \mathrm{Sp} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}((\mathcal{C}^\omega)^{\mathrm{op}}, \mathrm{Sp}).$$

The first map is symmetric monoidal by [Lur17, Proposition 4.8.2.10] and the second by the previous theorem. $\square$

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CONFLICTS OF INTEREST

None.

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